PROJECT REPORT

On

ABELIAN THEOREMS ASSOCIATED WITH CONTINUOUS WATSON WAVELET TRANSFORM

in partial fulfilment of the requirement for the degree of
B. Sc (H) Mathematics

submitted by

Bijoy Kumar Singh (19SBAS1130013)

Om Singh (19SBAS1130005)

Pawan (19SBAS1130003)

B. Sc(H) Mathematics (Sem VI)

under the supervision of

Dr. Alok Tripathi

Division of Mathematics

School of Basic and Applied Sciences

Uttar Pradesh

JULY 2022

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GALGOTIAS UNIVERSITY
(Established under Galgotias University Uttar Pradesh Act No. 14 of 2011)

School of Basic and Applied Sciences

CERTIFICATE

This is to certify that **Bijoy Kumar Singh, Om Singh** and **Pawan** have carried out their project work **"Abelian Theorems Associated with Continuous Watson Wavelet Transform"** under my supervision. This work is now fit for the submission for the award of Bachelor's Degree in Mathematics (Hons).

Signature
Dr. Alok Tripathi
Supervisor
Division of Mathematics
Dept. of Basic Sciences
School of Basic & Applied Sciences

Signature
Dr. Varsha Gautam
Division Chair
Division of Mathematics
Dept. of Basic Sciences
School of Basic & Applied Sciences

Signature
Dr. Diwakar Chauhan
Head, Dept. of Basic Sciences
School of Basic & Applied Sciences

Signature
Dr. A. K. Jain
Dean,
School of Basic and Applied Sciences
Galgotias University

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Thanking You

Bijoy Kumar Singh

Om Singh

Pawan

CANDIDATE DECLARATION

We, hereby declare that the project work entitled **"Abelian Theorems Associated with Continuous Watson Wavelet Transform"** submitted by us in partial fulfilment for the degree of B. Sc (H) Mathematics to the Division of Mathematics, School of Basic and Applied Sciences, Galgotias University, Greater Noida, UP, India is our original work. It has not been submitted in part or full to this university or any other university for the award of degree.

Signature

Bijoy Kumar Singh
19SBAS1130013

Signature

Om Singh
19SBAS1130005

Signature

Pawan
19SBAS1130003

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Abstract

Abelian theorems play an important role in the solution of boundary value problems. Many researchers have studied abelian theorems related to several integral transforms. Using the fact that Watson Transform is the generalization of Hankel transform, authors developed the theory of continuous Watson wavelet transform as the generalization of Bessel Wavelet transform. Keeping this view in mind, here we are generalizing the results proved by Upadhyay and Singh [6].

1. Introduction

With applications in mathematics, physics, and engineering, the concept of the Fourier series and transform is regarded as one of the most significant discoveries in science and engineering. In a lot of respects, Fourier series and Fourier transforms are connected. The Fourier transform in time and frequency domains is used in a variety of applications, including signal analysis and real-time signal processing. Let $f(t)$ is representing a signal then the Fourier transform of it is defined as according to [7].

$$\mathcal{F}\{f(t)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} \exp(-i\,\omega\, t) f(t)\, dt = \langle f, e^{i\omega t} \rangle$$

As a result, the Fourier transform theory has been proven to be extremely beneficial for the analysis of harmonic signals or signals that do not require local information.

Fourier transform analysis, on the other hand, has proven to be extremely valuable in a number of other fields like quantum mechanics, wave motion, and turbulence etc. In these fields, the space and wavenumber domains are used to define the Fourier transform $\hat{f}(k)$ of a function f(x), where x represents the space variable and k represents the wavenumber. The kernel of Fourier transformation $exp(-ikx)$ fluctuates continuously, and so the localized knowledge of the signal f(x) in the x-space is broadly spread among $\hat{f}(k)$ in the frequency space, which is one of the significant properties. Despite the fact that $\hat{f}(k)$ contains all of the information from the signal f(x), it spreads out in k-space.

Despite some notable successes, Fourier transform analysis appears to be insufficient for analyzing signals for at least two reasons. First, the Fourier transform of a signal lacks local information since it does not account for variations in frequency with time or wavenumber with space. Second, we cannot analyse problems simultaneously in the frequency (wavenumber) and time (space) domains using the Fourier transform approach. These are most likely the Fourier transform analyses' key flaws. It is generally required to create a single transform to analyse the signal in time and frequency (or space and wavenumber) domain both.

Only in the early 1980s did the term "wavelets" become more widely used. This novel idea is a synthesis of ideas from several fields, including engineering, physics, and mathematics. The wavelet transform was invented by Jean Morlet, a French geophysical engineer, in 1982. It provided a new mathematical tool for seismic wave research. In the beginning, wavelets were described by translations and dilations of a single function known as the "mother wavelet" $\Psi(t)$ by Morlet in the following way

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\Psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R}, a \neq 0$$

Where a control the dilation and b controls translation. The wavelet is a compressed version of the mother wavelet with a strong correlation to higher frequencies if $|a| < 1$. When $|a| > 1$, $\Psi_{a,b}(t)$ has a wider temporal window and correlates to lower frequencies than $\Psi(t)$. Wavelets have time-widths that are adjusted to their frequencies as a result. This is the primary factor behind the Morlet wavelets' effectiveness in time-frequency signal analysis and signal processing.

It was Alex Grossman, who noticed the importance of the Morlet wavelet transform and defined the wavelet transform as

$$W_\Psi[f](a,b) = \langle f, \Psi_{(a,b)} \rangle = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f(t)\overline{\Psi\left(\frac{t-b}{a}\right)}\, dt$$

where $\Psi_{(a,b)}(t)$ will play the role of the kernel as $\exp(i\,\omega\,t)$ in the Fourier transform. The continuous wavelet transformation W_Ψ is linear. For the reconstruction of the function f(t), the inverse wavelet transform is defined by the following way:

$$f(t) = C_\Psi^{-1} \iint_{-\infty}^{\infty} W_\Psi[f](a,b)\, \Psi_{(a,b)}(t)(a^{-2}da)db$$

provided C_Ψ is satisfying the following admissibility condition

$$C_\Psi = 2\pi \int_{-\infty}^{\infty} \frac{|\Phi(\omega)|^2}{|\omega|} d\omega < \infty$$

Where $\Phi(\omega)$ is the Fourier transform of the $\Psi(t)$.

2. Watson Wavelet Transform

Watson transform is an integral transform whose kernel is of general nature and therefore, several integral transforms like Hankel transform, H_ν, K_ν etc. are the particular examples of it. The theory of this integral transform can be extensively found in the classical monograph [5] and we have taken it from there.

$L^p(0,\infty)$ – Space: Any $f \in L^p(0,\infty)$, if it satisfies the following norm

$$\|f\|_{L^p} = \left[\int_0^\infty |f(x)|^p dx\right]^{\left(\frac{1}{p}\right)} < \infty$$

$L^{\infty}(0,\infty)$ **−Space:** Any $f \in L^{\infty}(0,\infty)$ if it satisfies the following norm

$$\|f\|_{L^{\infty}} = ess\ sup|f(x)| < \infty,\ 0 < x < \infty$$

From [1, 2], The Watson transform of a function $\Phi \in L^1(I), I = (0,\infty)$ is defined by,

$$\Psi(x) = (W\Phi)(x) = \int_0^{\infty} k(xt)\Phi(t)dt \quad (2.1)$$

The inverse of Watson transform is defined as,

$$\Phi(x) = (W^{-1}\Psi)(x) = \int_0^{\infty} k(xt)\Psi(t)dt \quad (2.2)$$

From [1], we define

$$w(x,y,z) = \int_0^{\infty} k(xt)k(yt)k(zt)dz \quad (2.3)$$

provided the given integral exists under the assumption that kernel is in $L^1(I) \cap L^{\infty}(I), I \in (0,\infty)$ and $k(0) = 1$ and $w(x,y,z) > 0\ \forall\ x,y,z \in (0,\infty)$.

Using inversion formula of Watson transform, we get

$$k(xt)k(yt) = \int_0^{\infty} w(x,y,z)k(zt)dz \quad (2.4)$$

For $t = 0$ in (4), we get

$$\int_0^{\infty} w(x,y,z)dz = 1 \quad (2.5)$$

Watson Translation: Let $\psi \in L^1(I), I = (0,\infty)$, then

$$\Psi(x,y) = \tau_y\Psi(x) = \tau_y\Psi(y) = \int_0^{\infty} \Psi(x)w(x,y,z)dz, \quad 0 < x,y < \infty \quad (2.6)$$

represents Watson translation.

From [1], the Watson convolution of two functions $f,g \in L^1(0,\infty)$ can be written as

$$(f\#g)(x) = \int_0^{\infty} (\tau_x g)(y)f(y)dy \quad (2.7)$$

or

$$(f\#g)(x) = \int_0^{\infty}\int_0^{\infty} g(z)f(y)\,w(x,y,z)\,dydz,\ 0 < x < \infty \quad (2.8)$$

Let $f(x), g(x) \in L^1(I), I = (0,\infty)$ be two functions, then their convolution $(f\#g)(x)$ converges for almost all $x, 0 < x < \infty$ and the following inequality holds

$$\|f\#g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1} \quad (2.9)$$

Moreover, from [2], we have

$$W(f\#g) = (Wf)(wg). \quad (2.10)$$

If $f(x) \in L^1(I), I = (0,\infty)$ and if $(Wf)(t) \in L^1(I), I = (0,\infty)$, then by inversion formula, we have

$$f(x) = \int_0^\infty k(xt)(Wf)(t)dt, \quad 0 < x < \infty \tag{2.11}$$

If f(x) and g(x) are in $L^1(I) \cap L^2(I), I \in (0, \infty)$, then the following Parseval formula holds:

$$\int_0^\infty (Wf)(t)(Wg)(t)dt = \int_0^\infty f(x)g(x)dx. \tag{2.12}$$

Dilation D_a is defined as

$$D_a f(x, y) = \left(\frac{1}{a}\right) f\left(\frac{x}{a}, \frac{y}{a}\right), \quad a > 0 \tag{2.13}$$

Upadhyay and Tripathi [3] developed the theory of continuous Watson wavelet transform. Some important results are as follows:

Theorem 1. If f and g $\in L^1(I) \cap L^2(I)$ and $(W_g f)(b, a)$ is the continuous Watson wavelet transform, then

$$(W_g f)(b, a) = \int_0^\infty k(\omega b)(Wf)(\omega)\overline{(Wg)(a\omega)}\, d\omega. \tag{2.14}$$

Relation between Watson Wavelet Transform with Watson convolution is given by

$$(W_g f)(b, a) = (f \# \overline{g_a})(b). \tag{2.15}$$

Theorem 2. Let $\varphi \in L^2(I), I = (0, \infty)$ be the Watson wavelet which satisfies the admissibility condition

$$A_\varphi = \int_0^\infty \frac{|W_\varphi)(\omega)|^2}{\omega} d\omega > 0. \tag{2.16}$$

Then the Parseval formula of the continuous Watson wavelet transform is given by

$$\int_0^\infty \int_0^\infty (W_\varphi f)(b, a)\overline{(W_\varphi g)(b, a)} \frac{dbda}{a} = A_\varphi < f, g > \tag{2.17}$$

$\forall\ f, g \in L^1(I) \cap L^2(I), I = (0, \infty)$.

3. Abelian theorems for the Watson wavelet transform of functions:

The abelian theorems for wavelet transform in ordinary and distributional sense both have been studied by Prof. R. S. Pathak [4]. The main use of these theorems is to solve boundary value problems. Many researchers studied the abelian theorems for several types of wavelet transforms. In particular, abelian theorem for Bessel wavelet transform have been studied by

Upadhyay and Singh [6]. Upadhyay and Tripathi [3] proved that under certain conditions, the generalization of the Bessel wavelet transform is the Watson wavelet transform. Using this fact, we are proving the abelian theorems for continuous Watson wavelet transform which will be the generalization of the result proved by Upadhyay and Singh in their paper.

In this section initial and final value theorems for the Watson wavelet transform have been proved.

We will assume that $|(W\Psi)(\omega)| \leq A(\omega)^{\mu} \quad \omega \to 0^{+}$ (3.1)

And $\int_0^{\infty}(W\Psi)(\omega)\omega^{-\eta+\frac{1}{2}}d\omega = H(\eta), \qquad \frac{3}{2} < \eta < \mu + 2$ (3.2)

Theorem 3.1: Let $\frac{3}{2} < \eta < \mu + 2$. Assume that $\omega^{-\eta+\frac{1}{2}}(W\Psi)(\omega) \in L^1(I) \cap L^{\infty}(I), I = (0,\infty)$, $(W\Psi)(\omega)$ *is bounded i.e.* $\exists\, K > 0$ *s.t* $|(W\Psi)(\omega)| < K$ and $\omega^{-\mu-\frac{1}{2}}(W\Phi)(\omega) \in L^1(\delta,\infty) \cap L^{\infty}(\delta,\infty)$ *for all* $\delta > 0$.

If
$$\lim_{\omega \to 0} \omega^{-\eta+\frac{1}{2}}(W\Phi)(\omega) = \alpha \tag{3.3}$$

Then
$$\lim_{a \to \infty} a^{\frac{3}{2}-\eta}(W_{\psi}\Phi)(b,a) = \alpha H(\eta). \tag{3.4}$$

Proof: Using (2.14), we have

$$\left|a^{\frac{3}{2}-\eta}(W_{\psi}\Phi)(b,a) - \alpha H(\eta)\right|$$
$$= \left|a^{\frac{3}{2}-\eta}\int_0^{\infty} k(b\omega)\,(W\Phi)(\omega)(W\Psi)(a\omega)d\omega - \alpha\int_0^{\infty}(W\Psi)(\omega)\omega^{-\eta+\frac{1}{2}}\,d\omega\right|$$

Replace ω by $a\omega$ in the second integral, we get

$$= \left|a^{\frac{3}{2}-\eta}\int_0^{\infty} k(b\omega)\,(W\Phi)(\omega)(W\Psi)(a\omega)d\omega - \alpha\int_0^{\infty}(W\Psi)(a\omega)(a\omega)^{-\eta+\frac{1}{2}}\,a d\omega\right|$$

$$= \left|\int_0^{\infty}\{a^{\frac{3}{2}-\eta}k(b\omega)\,(W\Phi)(\omega)(W\Psi)(a\omega) - \alpha(W\Psi)(a\omega)\left(a\omega\right)^{-\eta+\frac{1}{2}}a\right\}d\omega\,\Big|$$

$$= \left|\int_0^{\infty} a(W\Psi)(a\omega)(a\omega)^{-\eta+\frac{1}{2}}\{a^{\frac{1}{2}-\eta}k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\}d\omega\right|$$

$$\leq \int_0^{\infty}\left|(W\Psi)(a\omega)(a\omega)^{-\eta+\frac{1}{2}}\right|\left|(a)^{\frac{1}{2}-\eta}k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right|a dx$$

$$\leq \sup_{0<\omega<\delta} \left|(a)^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right| \int_0^\delta \left|(a\omega)^{\frac{1}{2}-\eta}(W\Phi)(a\omega)\right| a d\omega + K a^{\frac{3}{2}-\eta} \int_\delta^\infty \left|k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right| d\omega \quad (3.5)$$

In (3.5), both integrals are convergent for sufficiently large values of a and $\eta > \frac{3}{2}$.

Therefore, for all $\varepsilon > 0$, we have

$$\lim_{a\to\infty} \left|a^{\frac{3}{2}-\eta}(W_\psi\Phi)(b,a) - \alpha H(\eta)\right| < \varepsilon.$$

Theorem 3.2: *Let* $\frac{3}{2} < \eta < \mu + 2,\ \mu > 0$. *Assume that* $\omega^{\frac{1}{2}-\eta}(W\Psi)(\omega) \in L^1(I) \cap L^\infty(I) = (0,\infty)$ *and* $\omega^{-\mu-\frac{1}{2}}(W\Psi)(\omega) \in L^1(0,X), X > 0.$

If
$$\lim_{\omega\to\infty} k(b\omega)\omega^{\eta-\frac{1}{2}}(W\Psi)(\omega) = \alpha$$

Then

$$\lim_{a\to 0} a^{\frac{3}{2}-\eta}(W_\psi\Phi)(b,a) = \alpha H(\eta).$$

Proof: From the previous theorem we have

$$\left|a^{\frac{3}{2}-\eta}(W_\psi\Phi)(b,a) - \alpha H(\eta)\right|$$

$$\leq a \int_0^\infty |(W\Psi)(a\omega)(a\omega)^{-\eta+\frac{1}{2}}| |a^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha| d\omega$$

$$= a \int_{\omega<X}^\infty \left|a^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right| \left|(a\omega)^{-\eta+\frac{1}{2}}(W\Psi)(a\omega)\right| d\omega$$

$$a \int_{\omega>X}^\infty \left|a^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right| \left|(a\omega)^{-\eta+\frac{1}{2}}(W\Psi)(a\omega)\right| d\omega$$

$$\leq a^{\frac{3}{2}-\eta} \int_{\omega<X}^\infty \left|a^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right| \int_0^\infty \left|(\omega)^{-\eta+\frac{1}{2}}(W\Psi)(a\omega)\right| d\omega$$

$$+ \sup_{\omega>X} \left|a^{\frac{1}{2}-\eta} k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}}\alpha\right| \int_0^\infty \left|(\omega)^{-\eta+\frac{1}{2}}(W\Psi)(a\omega)\right| d\omega$$

Using (3.5) we get

$$\left|a^{\frac{3}{2}-\eta}(W_\psi\Phi)(b,a) - \alpha H(\eta)\right|$$

$$\leq Aa^{\frac{3}{2}-\eta+\mu}\int_0^X\left|a^{\frac{1}{2}-\eta}k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}}-\alpha\right|\omega^{\mu}d\omega +$$

$$\sup_{\omega>X}\left|a^{\frac{1}{2}-\eta}k(b\omega)(W\Phi)(\omega)\omega^{\eta-\frac{1}{2}}-\alpha\right|\left|\int_0^{\infty}\left|(\omega)^{-\eta+\frac{1}{2}}(W\Psi)(a\omega)\right|d\omega\right. \qquad (3.6)$$

Since both the integrals in right hand side of (2.6) are convergent and for large X; $\exists B > 0$ $s.t$ $when$ $\eta < \frac{3}{2} + \mu$ the first term can be made less than ε/2 for any ε >0.

Conclusion: We have proved the initial and final value abelian theorems associated with continuous Watson Wavelet Transform in the ordinary sense in this work, which are the generalization of the result proved by Upadhyay and Singh [7]. These results would be useful in developing solutions of certain boundary value problems.

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