

# School of Basic and Applied Sciences

Mathematics  
ETE - May 2023

Time : 3 Hours

Marks : 50

## Sem IV - MSCM327 - Measures Theory

Your answer should be specific to the question asked  
Draw neat labeled diagrams wherever necessary

1. Define  $\sigma$ -Algebra on an arbitrary set  $X$ . K2 CO1 (2)
2. Let  $X = \{a, b, c, d\}$ . Find a nontrivial Algebra and  $\sigma$ -Algebra on  $X$ . K2 CO2 (2)
3. Give example of two measures on an arbitrary measurable space  $(X, P(X))$ . K3 CO1 (2)
4. Define measurable functions on the measure space  $(\mathbb{N}, P(\mathbb{N}), \mu)$ , where  $\mu$  is counting measure. K3 CO2 (2)
5. Let  $(X, \mathbb{M}, \mu)$  be a measure space and  $\phi : X \rightarrow [0, \infty]$  be a simple function. Define integration of the simple function  $\phi$ . K4 CO2 (2)
6. Let  $(X, \mathbb{M})$  be a measure space and  $A \subset X$ . The characteristic function  $\chi_A$  is measurable if and only if  $A \in \mathbb{M}$ . K3 CO2 (5)
7. Let the  $\sigma$ -algebra  $\mathbb{N}$  be generated by  $\mathfrak{R} \subseteq P(Y)$ . Prove that a function  $f : X \rightarrow Y$  is measurable with respect to  $(\mathbb{M}, \mathbb{N})$  if and only if  $f^{-1}(E) \in \mathbb{M}$  for all  $E \in \mathfrak{R}$ . K4 CO3 (5)
8. Let  $X$  be an uncountable set. A set function  $\mu : P(X) \rightarrow [0, \infty]$  is defined by 
$$\mu(A) = \begin{cases} n & \text{If A has n elements,} \\ \infty & \text{If A has infinite elements.} \end{cases}$$
 Prove that  $\mu$  defines a measure on  $P(X)$ . K6 CO2 (6)
9. State and discuss Fatou's Lemma. Give an example of a sequence of functions satisfying all the theorem's assumptions. K4 CO3 (8)
10. Calculate  $\lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} \frac{\cos mn}{2^{mn}}$  using dominated convergence theorem. K4 CO4 (8)
11. Let  $(X, \mathbb{M})$  be a measurable space and let  $f, g : X \rightarrow \mathbb{R}$  be measurable functions. Then define  $h : X \rightarrow \mathbb{R}^2$  with  $h(x) = (f(x), g(x))$ . Prove that  $h$  is a measurable function. K5 CO3 (8)