# Math puzzles

Puzzles published in *High school teachers' mathematics blog*

*Comments by:*

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*Last updated:*

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# Contents

# Preface

The puzzles in this volume were discussed in a Malayalam blog for Mathematics teachers in Kerala[**7**]. Many of these were solved by the readers of that blog. This volume attempts to provide detailed explanation how these can be solved mathematically.

There is another reason behind this volume. many of these problems can be solved using a computer program. While agreeing that is indeed a good method, I believe we should not transform a mathematical problem to a programming problem. Some problems appear to have no mathematical solution other than trial and error (which is what computer programs are essentially doing), but surprisingly that is not true for many cases. Some do require a little more than high school mathematics, but providing those will give the students the urge to learn more. Some puzzles are solved in more than one way as well.

I have added a chapter with some theory mentioned in the other chapters. This is not intended to be a comprehensive reference–it just gives explanations to some techniques used elsewhere. It may grow to be more comprehensive when puzzles of different topics are discussed.

This is a living document. I'll add more chapters when I solve more problems from that blog. Check the date on the cover page to ensure whether you have downloaded the latest version.

Please send your comments and corrections to `umesh.p.nair@gmail.com`.

Umesh P. N.

CHAPTER 1

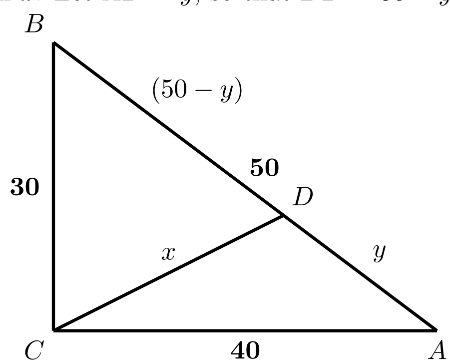
# Fence in the farm

## 1. Question

A farm is in triangular shape with sides 30 metres, 40 metres and 50 metres. From the biggest angle, there is a fence to the opposite side. The two small triangles thus formed had equal perimeters. Find the length of the fence.*

## 2. Solution

In $\triangle ABC$, let $BC = 30, AC = 40$ and $AB = 50$. Since $BC^2 + AC^2 = AB^2$, C is a right angle.
Let the fence be $\overline{CD}$, with length $x$. Let $AD = y$, so that $BD = 50 - y$.




Now,

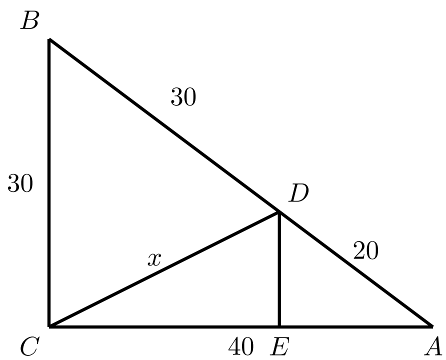
$$\begin{aligned} 40 + y + x &= 30 + (50 - y) + x \\ 2y &= 30 + 50 - 40 = 40 \\ y &= 20 \end{aligned} \tag{1.1}$$

Thus we wanted to find $CD$, but instead got $AD = 20$ and $BD = 30$.
To find $CD$, there are many ways, three of them are given below.

**2.1. Using geometry.** Draw $\overline{DE}$ perpendicular to $\overline{CA}$.
AED and ACB are similar, so,

$$\begin{aligned} \frac{DE}{BC} &= \frac{AD}{AB} \\ \frac{DE}{30} &= \frac{20}{50} \\ DE &= \frac{30 \cdot 20}{50} = 12 \end{aligned}$$

*http://mathematicsschool.blogspot.com/2009/11/fence-answer.html




Now,

$$\begin{array}{rcccccl} AE & = & \sqrt{AD^2 - DE^2} & = & \sqrt{20^2 - 12^2} & = & \mathbf{16} \\ CE & = & AC - AE & = & 40 - 16 & = & \mathbf{24} \\ CD & = & \sqrt{CE^2 + DE^2} & = & \sqrt{24^2 + 12^2} & = & \sqrt{\mathbf{720}} \end{array}$$

**2.2. Using Trigonometry.** Since $30^2 + 40^2 = 50^2$, C is a right angle. Hence, $\cos B = \frac{30}{50} = \frac{3}{5}$. So, using Equation (38.17) on page 78,

$$\begin{aligned} CD^2 &= BC^2 + BD^2 - 2 \cdot BC \cdot BD \cdot \cos B \\ &= 30^2 + 30^2 - 2 \cdot 30 \cdot 30 \cdot \frac{3}{5} \\ &= 900 + 900 - 1080 \\ &= 720 \end{aligned} \tag{1.2a}$$

Alternately,

$$\begin{aligned} CD^2 &= AC^2 + AD^2 - 2 \cdot AC \cdot AD \cdot \cos A \\ &= 40^2 + 20^2 - 2 \cdot 40 \cdot 20 \cdot \frac{4}{5} \\ &= 1600 + 400 - 1280 \\ &= 720 \end{aligned} \tag{1.2b}$$

So, $CD = \sqrt{\mathbf{720}}$.

**2.3. Using Analytical Geometry.** Let the co-ordinates of $A$, $B$, $C$ be $(40, 0)$, $(0, 30)$ and $(0, 0)$. Let the co-ordinates of $D$ be $(x, y)$.
$D$ lies on $BD$. So, using Equation (38.13) on page 77,

$$\frac{x - 0}{40 - 0} = \frac{y - 30}{0 - 30}$$

So,

$$3x + 4y = 120 \tag{1.3}$$

$BD = 30$. So,

$$(x - 0)^2 + (y - 30)^2 = 30^2$$

Means

$$x^2 + y^2 - 60y = 0 \tag{1.4}$$

Similarly, $AD = 20$.

$$
\begin{aligned}
(x-40)^2+(y-0)^2 &= 20^2 \\
x^2+y^2-80x &= -1200 \qquad (1.5)
\end{aligned}
$$

(1.4) - (1.5) gives

$$
\begin{aligned}
80x-60y &= 1200 \\
4x-3y &= 60 \qquad (1.6)
\end{aligned}
$$

Solving (1.3) and (1.6) using Equation (38.12) on page 77,

$$
x = \frac{-360-240}{-9-16} = 24 \qquad (1.7a)
$$

$$
y = \frac{180-480}{-9-16} = 12 \qquad (1.7b)
$$

So, $CD = \sqrt{240^2+120^2} = \sqrt{\mathbf{720}}$.

### 3. Answer

The length of the fence is $\sqrt{720} = \mathbf{26.83}\cdots$.

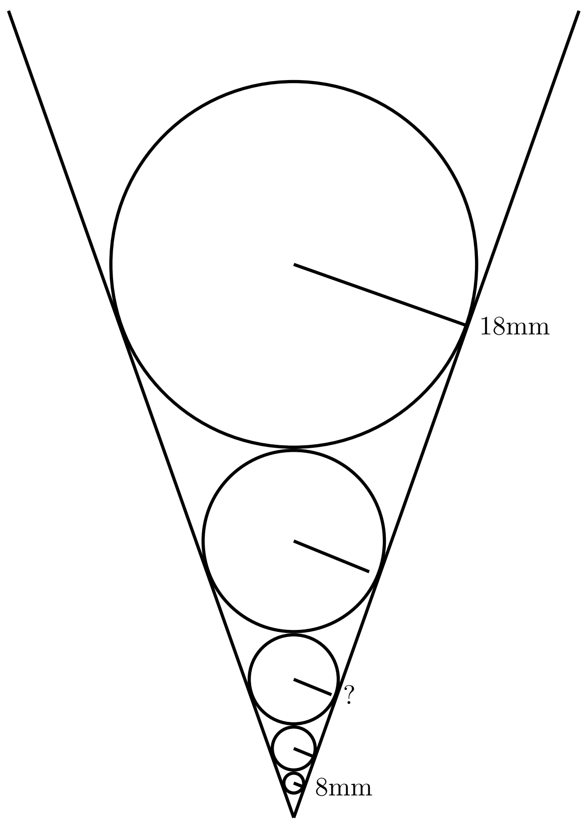
CHAPTER 2

# Five marbles in a funnel

## 1. Question

Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall.

The smallest marble has a radius of 8 mm. The largest marble has a radius of 18 mm. What is the radius of the middle marble?*




## 2. Solution

Let $r_1 = 8, r_2, r_3, r_4, r_5 = 18$ be the five radii. We need to find $r_3$.

Consider any adjacent marbles. Let the radii of the two marbles be $r_i = a$ and $r_{i+1} = b$ and their centers at $A$ and $B$. Let the vertex of the funnel is $O$. Join $\overline{OB}$. $A$ lies on $OB$.

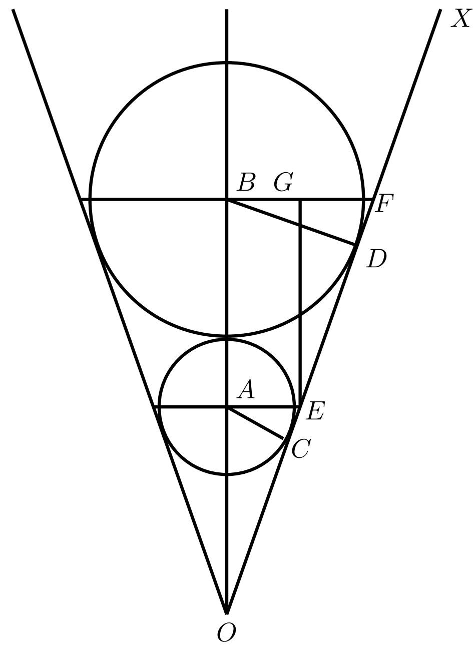
Draw the tangent line $\overline{OX}$ that touches the circles at $C$ and $D$. Join $\overline{AC}$ and $\overline{BD}$.

*http://mathematicsschool.blogspot.com/2009/11/coordinate-geometry.html

Draw $\overline{AE}$ and $\overline{BF}$ perpendicular to $\overline{OB}$, with $E$ and $F$ on $\overline{OX}$ .

Let $\angle BOX = \alpha$. It is clear that $\angle CAE = \alpha$ and $\angle DBF = \alpha$, because they are the angles between the perpendiculars of $\overline{OB}$ and $\overline{OX}$.

Draw $\overline{EG}$ perpendicular to $\overline{BF}$. $\angle GEF = \alpha$.




Now,

$$AE = \frac{a}{\cos\alpha}$$
$$BF = \frac{b}{\cos\alpha}$$
$$FG = BF - BG = BF - AE = \frac{b-a}{\cos\alpha}$$
$$EG = AB = a + b$$
$$\tan\alpha = \frac{FG}{EG} = \frac{b-a}{(b+a)\cos\alpha}$$
$$\sin\alpha = \frac{b-a}{b+a}$$

This means $\frac{b-a}{a+b}$ is the same for every pair of touching marbles because they share the same funnel and hence the same $\alpha$.

$$\frac{b-a}{a+b} = k$$
$$b - a = ka + kb$$
$$b(1-k) = a(1+k)$$
$$\frac{b}{a} = \frac{1+k}{1-k}$$

So, the ratio of the radii of consecutive marbles is the same. Let $r$ be the ratio.

$$
\begin{aligned}
r_1 &= 8 \\
r_2 &= 8r \\
r_3 &= 8r^2 \\
r_4 &= 8r^3 \\
r_5 &= 8r^4 = 18
\end{aligned}
$$

So,

$$
\begin{aligned}
r^4 &= \frac{18}{8} = \frac{9}{4} \\
r^2 &= \sqrt{\frac{9}{4}} = \frac{3}{2}
\end{aligned}
$$

Radius of the middle marble $= r_3 = 8r^2 = 8 \cdot \frac{3}{2} = \mathbf{12}$.[†]

## 3. Answer

Radius of the middle marble is **12** mm.

[†]We can find the radius of all marbles: $r_1 = 8, r_2 = \frac{8\sqrt{3}}{\sqrt{2}} = 9.798, r_3 = 12, r_4 = \frac{12\sqrt{3}}{\sqrt{2}} = 14.697, r_5 = 18$.

CHAPTER 3

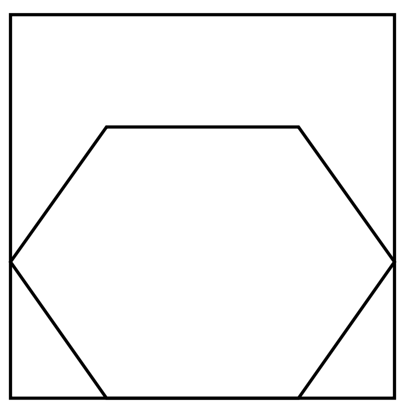
# Hexagon in square

## 1. Question

What is the size of the largest regular hexagon that can be constructed inside a square with side $x$?*

## 2. Solution

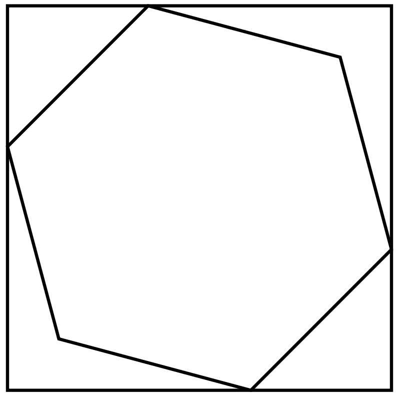
The obvious solution, with one side of the hexagon on a side of the square and two corners on two other sides, doesn't give the biggest hexagon.



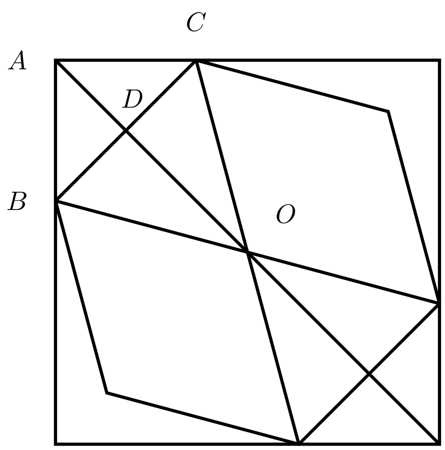
The hexagon in the above solution has a side of $\frac{x}{2}$. The size of the hexagon can be increased by placing it symmetrical to all sides of the square (with both centroids coinciding) and rotating and increasing the size of the hexagon in such a way that at least four corners touch the square.

It is clear that the optimal solution is obtained when it is rotated through $\frac{\pi}{4}(45^o)$, as in the following figure.

*http://mathematicsschool.blogspot.com/2009/11/unit-square.html



To solve the problem, one-sixth of the figure is detailed below.




$O$ is the centroid of the square and the hexagon. $A$ is a corner of the square. $\overline{BC}$ is a side of the hexagon, so that $B$ and $C$ lie on the square's two sides. $\overline{OA}$ and $\overline{BC}$ meet at $D$.

Let the side of the regular hexagon be $y$, so $OB = OC = BC = y$.

$$OA = \frac{\sqrt{2}x}{2} = \frac{x}{\sqrt{2}}$$
$$BC = y$$
$$BD = CD = \frac{y}{2}$$
$$AD = \frac{y}{2}, \text{because } \angle ACD = \angle ADC = 45^o$$
$$OD = OC \cdot \sin 60^o = \frac{\sqrt{3}y}{2}$$

Now,

$$\begin{aligned}
OA &= OD + AD \\
\frac{x}{\sqrt{2}} &= y\frac{\sqrt{3}}{2} + \frac{y}{2} \\
&= \frac{1+\sqrt{3}}{2}y \\
y &= \frac{2x}{\sqrt{2}(1+\sqrt{3})}
\end{aligned}$$

So, the side of the hexagon

$$y = \frac{\sqrt{2}}{1+\sqrt{3}} \cdot x = \mathbf{0.51763}x$$

### 3. Answer

Size of the regular hexagon $= \frac{\sqrt{2}}{1+\sqrt{3}} \cdot x = \mathbf{0.51763}x$

CHAPTER 4

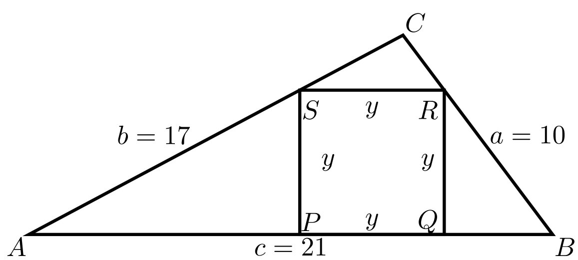
# Square inside triangle

## 1. Question

A triangle has sides 10, 17, and 21. A square is inscribed in the triangle. One side of the square lies on the longest side of the triangle. The other two vertices of the square touch the two shorter sides of the triangle. What is the length of the side of the square?

## 2. Solution

Let ABC be the triangle, with $AB = c = 21$, $BC = a = 10$ and $AC = b = 17$.




**2.1. Using Analytical Geometry.** Let $A(0, 0)$ and $B(21, 0)$ be the co-ordinates so that $\overline{AB}$ is along the $X$-axis.
Let $(p, q)$ be the co-ordinates of $C$.

$$
\begin{aligned}
(p-0)^2 + (q-0)^2 &= 17^2 = 289 \\
p^2 + q^2 &= 289 && \text{(4.1a)} \\
(p-21)^2 + (q-0)^2 &= 10^2 = 100 \\
p^2 + q^2 - 42p &= -341 && \text{(4.1b)}
\end{aligned}
$$

(4.1a) - (4.1b) gives

$$
\begin{aligned}
42p &= 630 \\
p &= 15 && \text{(4.2a)} \\
q^2 &= 289 - 15^2 \\
q &= 8 && \text{(4.2b)}
\end{aligned}
$$

So, $C(15, 8)$ are the co-ordinates.*
Let PQRS be the square inscribed in the triangle, with PQ along the X-axis. Let $PQ = QR = RS = PS = y$. Let $AP = x$.
$S(x, y)$ should be on $\overline{AC}$. So,

*Since $q^2 = 64$ implies $q = \pm 8$, This value also is valid, with the triangle on the other side.

$$\frac{y-0}{x-0} = \frac{8-0}{15-0}$$

$$8x - 15y = 0 \tag{4.3}$$

Similarly, $R(x+y, y)$ should be on $\overline{BC}$. So,

$$\begin{aligned}
\frac{y-0}{(x+y)-21} &= \frac{8-0}{15-21} \\
8x + 8y - 168 &= -6y \\
8x + 14y &= 168 \qquad (4.4)
\end{aligned}$$

(4.4) - (4.4) gives

$$\begin{aligned}
29y &= 168 \\
y &= \mathbf{\frac{168}{29}} \qquad (4.5)
\end{aligned}$$

**2.2. Using Trigonometry.** Using the results in Equation (38.17) on page 78,

$$\begin{aligned}
\cos A &= \frac{17^2 + 21^2 - 10^2}{2 \cdot 17 \cdot 21} = \frac{15}{17} \\
\sin A = \sqrt{1 - \cos^2 A} &= \sqrt{1 - \frac{15^2}{17^2}} = \frac{8}{17} \\
\cot A &= \frac{\cos A}{\sin A} = \frac{15}{8} \qquad (4.6)
\end{aligned}$$

Also,

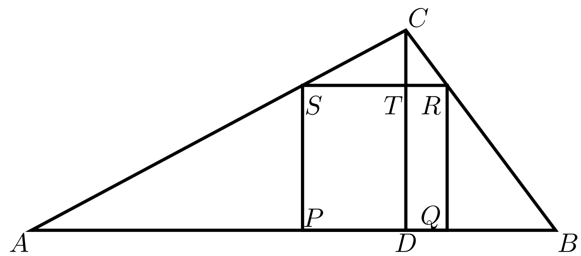
$$\begin{aligned}
\cos B &= \frac{10^2 + 21^2 - 17^2}{2 \cdot 10 \cdot 21} = \frac{3}{5} \\
\sin B = \sqrt{1 - \cos^2 B} &= \sqrt{1 - \frac{3^2}{5^2}} = \frac{4}{5} \\
\cot B &= \frac{\cos B}{\sin B} = \frac{3}{4} \qquad (4.7)
\end{aligned}$$

Now,

$$\begin{aligned}
AP + PQ + QB &= AB \\
y \cot A + y + y \cot B &= 21 \\
y \left( \frac{15}{8} + 1 + \frac{3}{4} \right) &= 21 \\
y\,(15 + 8 + 6) &= 21 \cdot 8
\end{aligned}$$

$$y = \frac{21 \cdot 8}{29} = \mathbf{\frac{168}{29}} \tag{4.8}$$

**2.3. Using plane geometry.** $s = \frac{a+b+c}{2} = 24$ and the area of the triangle is $\sqrt{24(24-21)(24-10)(24-17)} = 84$.
Draw $\overline{CD}$ perpndiculat to $\overline{AB}$. Let $\overline{CD}$ meets $\overline{RS}$ at $T$.




$$\text{Area of the triangle} = \frac{1}{2} \cdot AB \cdot CD = \frac{21 \cdot CD}{2} = 84$$

$$\begin{aligned} CD &= \frac{84 \cdot 2}{21} = 8 \\ AD &= \sqrt{17^2 - 8^2} = 15 \\ BD &= 21 - 15 = 6 \end{aligned}$$

Let $ST = z$. Triangles $\triangle RCT$ and $\triangle BCT$ are similar.

$$\begin{aligned} \frac{CT}{CD} &= \frac{RT}{BD} \\ \frac{8-y}{8} &= \frac{y-z}{6} \\ 48 - 6y &= 8y - 8z \\ 7y - 4z &= 24 \end{aligned} \tag{4.9}$$

Similarly, triangles $\triangle SCT$ and $\triangle ACD$ are similar.

$$\begin{aligned} \frac{CT}{CD} &= \frac{ST}{AD} \\ \frac{8-y}{8} &= \frac{z}{15} \\ 120 - 15y &= 8z \\ 15y + 8z &= 120 \end{aligned} \tag{4.10}$$

Solving (4.9) and (4.10) using Equation (38.12) on page 77,

$$\begin{aligned} y &= \frac{8 \cdot 24 + 4 \cdot 120}{7 \cdot 8 + 4 \cdot 15} \\ &= \frac{672}{116} = \mathbf{\frac{168}{29}} \end{aligned} \tag{4.11}$$

## 3. Answer

The side of the square is $\frac{168}{29} == \mathbf{5.7931}\cdots$

CHAPTER 5

# The hundred thousandth number

## 1. Question

There are $9! = 362880$ nine-digit numbers that has all the nine digits 1–9. If we write that in the ascending order, which will be the $100000^{th}$ (hundred-thousandth) one?*

## 2. Solution

There are $8! = 40320$ numbers starting with each of the digits 1–9. So, the first 40320 start with 1, 40321–80640 with 2, and 80641–120960 with 3. So, the $100000^{th}$ number should start with the $\lceil \frac{100000}{40320} \rceil = 3^{rd}$ number $= 3$.

The required number is the $19360^{th}$ $(100000 - 80640)$ number in the numbers starting with 3. Among the numbers starting with 3, there are $7! = 5040$ numbers with the second digit is one of $1, 2, 4, 5, 6, 7, 8, 9$. So, the $19360^{th}$ number will start with the $\lceil \frac{19360}{5040} \rceil = 4^{th}$ number in the list, 5.

The required number is the $4240^{th}$ $(100000 - 3 \cdot 5040)$ number among the numbers starting with "35". There are $6! = 720$ numbers for each of the digits $1, 2, 4, 6, 7, 8, 9$. So, the third digit is the $6^{th}$ one, 8.

Continuing this and putting this as a table,

| **Remaining digits** | | **Prefix** | **Index** ($r$) | $x$ | **Digit index** ($k$) | **Digit** |
|---|---|---|---|---|---|---|
| $n$ | **Digits** | | $r - (k-1)x$ | $(n-1)!$ | $\lceil \frac{i}{x} \rceil$ | |
| 9 | $1, 2, 3, 4, 5, 6, 7, 8, 9$ | | 100000 | 40320 | 3 | 3 |
| 8 | $1, 2, 4, 5, 6, 7, 8, 9$ | $3 \cdots$ | 19360 | 5040 | 4 | 5 |
| 7 | $1, 2, 4, 6, 7, 8, 9$ | $35 \cdots$ | 4240 | 720 | 6 | 8 |
| 6 | $1, 2, 4, 6, 7, 9$ | $358 \cdots$ | 640 | 120 | 6 | 9 |
| 5 | $1, 2, 4, 6, 7$ | $3589 \cdots$ | 40 | 24 | 2 | 2 |
| 4 | $1, 4, 6, 7$ | $35892 \cdots$ | 16 | 6 | 3 | 6 |
| 3 | $1, 4, 7$ | $358926 \cdots$ | 4 | 2 | 2 | 4 |
| 2 | $1, 7$ | $3589264 \cdots$ | 2 | 1 | 2 | 7 |
| 1 | 1 | $35892647 \cdots$ | 1 | 1 | 1 | 1 |

## 3. Answer

The $100000^{th}$ number is **358926471**.

*http://mathematicsschool.blogspot.com/2009/11/one-lakh.html

CHAPTER 6

# 1000 bulbs

## 1. Question

There are 1000 bulbs in a room numbered 1 to 1000, each one having a numbered switch to turn the bulb on or off. Also, there are 1000 people numbered 1 to 1000.

Each of the 1000 persons go into the room, at random, once and only once, and toggles (i.e., if the bulb was off, turns it on; if it was on, turns it off.) all the switches that are multiples of his/her number.

for example, the person numbered 150 toggles bulbs numbered 150, 300, 450, 600, 750 and 900 as these are its multiples of 150 .

All the bulbs are off at the start, and each person goes exactly once to the room.

What will be the state at the end, i.e., which bulbs will be on and which will be off?*

## 2. Solution

Let us take the case of that $30^{th}$ bulb. The following people will toggle it. $1, 2, 3, 5, 6, 10, 15, 30$. The following people occur in pairs and nullify each other's action.

- 1 and 30
- 2 and 15
- 3 and 10
- 5 and 6

So, the $30^{th}$ bulb will remain off.

Now let us take the $100^{th}$ switch. The following occur in pairs.

- 1 and 100
- 2 and 50
- 4 and 25
- 5 and 20

but 10 will not have a pair, so the effect will be the switch ending up on.

A close examination will show that bulbs with number as a perfect square will be on, because its square root doesn't have a pair, while all others will be off.

## 3. Another explanation

A bulb with number $n$ will be toggled $m$ times, where $m$ is the number of factors of $n$.

Let us say the prime factorization of $n$ is

$$n = p_1^{k_1} p_2^{k_2} \cdots p_q^{k_q}$$

The number of factors $n$ has is given by

$$m = (k_1 - 1)(k_2 - 1) \cdots (k_q - 1)$$

*`http://mathematicsschool.blogspot.com/2009/11/ratio-of-radii.html`

The bulb will be on if this is odd, and off if it is even. To make it odd, *all* of $k_1, k_2, k_3, \cdots k_q$ should be even, means every prime factor has an even power, or in other words, $n$ is a perfect square. In all other cases, one of the $k_i$s will be odd, making $m$ even. So, the bulbs with square numbers will be on, and others off.

## 4. Answer

The bulbs whose numbers are perfect squares will be on, and others off.

CHAPTER 7

# The car

## 1. Question

A car travels downhill at 72 mph (miles per hour), on the level at 63 mph, and uphill at only 56 mph. The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes. Find the distance between the two towns.*

## 2. Solution

Let $a$, $b$ and $c$ be the distances in miles that are downhill, level and uphill in the onward journey.

$$\frac{a}{72} + \frac{b}{63} + \frac{c}{56} = 4$$

Multiplying by $7 \cdot 8 \cdot 9$,

$$7a + 8b + 9c = 4 \cdot 7 \cdot 8 \cdot 9 \tag{7.1}$$

In the reverse journey, the distance $a$ is uphill and $c$ is downhill. So,

$$\frac{a}{56} + \frac{b}{63} + \frac{c}{72} = 4\frac{40}{60} = \frac{14}{3}$$

Multiplying by $7 \cdot 8 \cdot 9$,

$$9a + 8b + 7c = \frac{14}{3} \cdot 7 \cdot 8 \cdot 9 = 14 \cdot 3 \cdot 7 \cdot 8 \tag{7.2}$$

Adding (7.1) and (7.2),

$$16a + 16b + 16c = 3 \cdot 7 \cdot 8 \cdot (4 \cdot 3 + 14)$$

So,

$$\begin{aligned} a + b + c &= \frac{1}{16} \cdot 3 \cdot 7 \cdot 8 \cdot 26 \\ &= 3 \cdot 7 \cdot 13 \\ &= 273 \end{aligned}$$

So, distance between the two towns = 273 miles. Note that we cannot find the individual $a$, $b$ and $c$ because there are three unknowns and only two equations.

Also note that it would be much easier if we do not multiply the numbers but keep only the calculations. Otherwise the final calculation will be

$$\frac{2016 + 2352}{16} = \frac{4368}{16} = 273$$

*http://mathematicsschool.blogspot.com/2009/10/blog-post_29.html

## 3. Answer

Distance between the two towns = **273 miles**.

## 4. Extended Question

**4.1. Question.** If the uphill, downhill and level distances are integer values in miles, find those. Find all such solutions.

**4.2. Solution.** (7.2) - (7.1) gives

$$2a - 2c = 3 \cdot 7 \cdot 8(14 - 12) = 6 \cdot 7 \cdot 8$$

$$a - c = 168 \tag{7.3}$$

So, we can assume a value to $c$ and then calculate $a$ and $c$ by

$$\begin{aligned} a &= c + 168 \\ b &= 273 - (c + c + 168) \\ &= 105 - 2c \end{aligned}$$

So, the solutions are

| **c** | **a** $(c + 168)$ | **b** $(105 - 2c)$ |
|---|---|---|
| 0 | 168 | 105 |
| 1 | 169 | 103 |
| 2 | 170 | 101 |
| ... | ... | ... |
| 51 | 219 | 3 |
| 52 | 220 | 1 |

There are a total of 53 solutions.

CHAPTER 8

# Adding instead of Multiplication

## 1. Question

A problem child in my class does not know how to find the area of a rectangle. He adds the length of the sides instead of multiplying them. But he got the correct answer today even with this mistake. What were the sides?

Find five answers to this question.*

## 2. Solution

Let the sides be $a$ and $b$.

$$a + b = ab$$
$$a = b(a - 1)$$

$$b = \frac{a}{a-1} \tag{8.1}$$

Any values of $a$ and $b$ satisfying (8.1) provide a solution. Since $(a - 1)$ does not divide $a$ except when $a = 2$, the only integer solution is $a = b = 2$.

Some other solutions:

1) $a = 3, b = \frac{3}{2}$

2) $a = 4, b = \frac{4}{3}$

3) $a = 5, b = \frac{5}{4}$

4) $a = 6, b = \frac{6}{5}$

etc.

*http://mathematicsschool.blogspot.com/2009/11/achu-teaschers.html

CHAPTER 9

# How many ants?

## 1. Question

A red eyed teacher was advised to take bed rest by the doctor and her husband was forced to participate the kitchen dance. In the evening, she went to the kitchen to see husband's preparation. Suddenly an ant bit her leg.

"What is this?" she shouted, "How many ants are here in the kitchen?"

The husband felt ashamed to see full of ants below his foot. He replied, "I don't know the exact number. If the ants walk in rows of 7,11 or 13, there are 2 ants left over, while in rows of 10, there are 6 left over. Calculate the smallest number of ants, that are in the kitchen."

The husband went back to his salt and vegetables and the wife, to bed, without getting the number of ants.

Will you help her to find the smallest number of ants that will obey this condition?

## 2. Solution

Since rows of 7, 11 or 13 leaves a remainder of 2, the number of ants should be in the form of $7 \cdot 11 \cdot 13 \cdot a + 2$, where $a$ is a positive integer. At the same time, it should be in the form of $10b + 6$ as well.

$$1001a + 2 = 10b + 6$$

which reduces to

$$1001a - 10b = 4 \tag{9.1}$$

This can be solved using the techniques given in Section 8.2 on page 81.

**2.1. Using continued fractions.** See Section 8.2.1 on page 81. To solve (9.1), let us first solve the equation

$$1001a - 10b = 1 \tag{9.2}$$

It is clear that $a = 1, b = 100$ is a solution to this.* Hence $a = 4, b = 400$ should be a solution to (9.1).

The general solution of it is given by (See Section 8.2.3 on page 81)

$$a = 4 + 10k \tag{9.3a}$$
$$b = 400 + 1001k \tag{9.3b}$$

*This is evident by inspection, but this may not be the case always. In such cases, the following method (see the theory at Section 8.1 on page 80) will suffice: Express $\frac{1001}{10}$ as a continued fraction and determine its convergents. Here, $\frac{1001}{10} = 100 + \frac{1}{10}$, and the convergents are $\frac{100}{1}$ and $\frac{1001}{10}$. Take the one before the last convergent, which is $\frac{100}{1}$. This will give a solution, i.e., $a = 1, b = 100$.

**2.2. Using congruences.** Since one of the coefficients of (9.1) is a small value (10), we can use the technique given in Section 8.2.2 on page 81 as well. From (9.1),

$$1001a \equiv 4 \pmod{10} \tag{9.4}$$

Since $1001 \equiv 1 \pmod{10}$, this is equivalent to

$$a \equiv 4 \pmod{10} \tag{9.5}$$

Hence $a = 10k + 4$.
From (9.1),

$$\begin{aligned} b &= \frac{1001(10k+4) - 4}{10} \\ &= 1001k + \frac{1001 \cdot 4 - 4}{10} \\ &= 1001k + 400 \end{aligned}$$

So, the general solution is

$$a = 4 + 10k \tag{9.6a}$$
$$b = 400 + 1001k \tag{9.6b}$$

**2.3. Particular solutions.** To get positive values, $k$ can start at 0. First few solutions are:

$k = 0$, $a = 4$, $b = 400$, $N = 4006$
$k = 1$, $a = 14$, $b = 1401$, $N = 14016$
$k = 2$, $a = 24$, $b = 2402$, $N = 24026$
$\ldots$

## 3. Answer

The minimum number of ants that satisfy the requirements is **4006**, but there are infinite number of solutions.

CHAPTER 10

# Hari's house, Brahmagupta, Pell, Ramanujan...

## 1. Question

Hari lives on a long street and has noticed that the sum of the house numbers up to his own house, but excluding, is equal to the sum of the numbers of his house to the end of the road. If the houses are numbered starting from 1, what is the number of Hari's house? Assume that there are less than 1000 houses on the road.*

## 2. Solution

**2.1. Formulation.** Let $(n+1)$ be Hari's house number (so that there are $n$ houses before it), and there are a total of $n+k$ houses in the street. The requirement is†

$$1 + 2 + 3 + \cdots + n = (n+1) + (n+2) + \cdots + (n+k)$$

which means

$$2(1 + 2 + 3 + \cdots + n) = 1 + 2 + \cdots + (n+k)$$

Since $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$,

$$2 \cdot \frac{n(n+1)}{2} = \frac{(n+k)(n+k+1)}{2}$$

Simplifying,

$$n^2 - (2k-1)n - k(k+1) = 0$$

Solving for $n$ using Equation (38.2) on page 76,

$$n = \frac{(2k-1) \pm \sqrt{(2k-1)^2 + 4k(k+1)}}{2} \tag{10.1}$$

To get an integer for $n$, the descriminant should be a perfect square. Let it be $p^2$.

$$\begin{aligned} p^2 &= (2k-1)^2 + 4k(k+1) \\ &= 4k^2 - 4k + 1 + 4k^2 + 4k \\ &= 8k^2 + 1 \end{aligned}$$

That is,

$$p^2 - 8k^2 = 1 \tag{10.2}$$

---

*`http://mathematicsschool.blogspot.com/2009/11/unit-square.html`

†Note that Hari's house is counted in the second sum. If it is not counted in the either sum, the problem is surprisingly different. That puzzle is famous because of a story connected with the legendary mathematical genious Srinivasa Ramanujan. See Chapter 1 on page 64.

So, the problem reduces to finding solutions to (10.2) in integers, and then compute $n$ by

$$n = \frac{(2k-1) \pm p}{2} \tag{10.3}$$

Hari's house number will be $n+1$, and total number of houses will be $(n+p)$.

**2.2. Archimedes, Brahmagupta, Pell...** Equation (10.2) is a very famous one. Archimedes' famous cattle problem

> *The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown. Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown; the number of the black, one quarter plus one fifth the number of the spotted greater than the brown; the number of the spotted, one sixth and one seventh the number of the white greater than the brown. Among the cows, the number of white ones was one third plus one quarter of the total black cattle; the number of the black, one quarter plus one fifth the total of the spotted cattle; the number of spotted, one fifth plus one sixth the total of the brown cattle; the number of the brown, one sixth plus one seventh the total of the white cattle. What was the composition of the herd?*

reduces to

$$x^2 - 4729424y^2 = 1$$

It is doubtful whether anybody could solve it during Archimedes' time.

This type of equation was first solved by the seventh century Indian mathematician Brahmagupta, using the *chakravaala* (cyclic) method. Later Bhaskara II simplified the method and included it in his famous book *Lilavati*. This equation was incorrectly called *Pell's equation* by the 17th century mathematician L. Euler, thinking it was solved by John Pell (1611–1685). In the western world, this was first completely solved by Lord Brouncker. The first published proof was by Lagrange in 1766. The modern method uses the continued fraction expansion of $\sqrt{D}$ in $x^2 - Dy^2 \pm 1$. For details of the history of this problem, see [**3**].

**2.3. Back to the problem.** To solve (10.2), we need to find the first solution. It can be obtained by inspection, or by expanding $\sqrt{8}$ into a continued fraction. I am using the latter here. The continued fraction expansion of $\sqrt{8}$ is

$$\begin{aligned} \sqrt{8} &= 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\cdots}}}}} \\ &= [2; \overline{1, 4}] \end{aligned} \tag{10.4}$$

with convergents (See Equation (38.18) on page 80) $\frac{2}{1}, \frac{3}{2}, \frac{9}{4}, \frac{17}{6}, \cdots$

Since the period of the continued fraction is 2, the convergent after the first term $(\frac{3}{2})$ should give the primitive solution. (See [**2**] for a detailed analysis and proof.) That is,

$$3^2 - 8 \cdot 1^2 = 1$$

is the primitive solution.

To get all the solutions, we can find the second, fourth, sixth,... convergents of (10.4). An alternative is to solve the equation

$$p + k\sqrt{8} = (3 + \sqrt{8})^i \tag{10.5}$$

for $i = 1, 2, 3, \cdots$.

In other words, if we have a solution $(p_i, k_i)$ for (10.2), the next solution is given by

$$\begin{aligned} p_{i+1} + k_{i+1}\sqrt{8} &= (p_i + k_i\sqrt{8})(3 + \sqrt{8}) \\ &= 3p_i + p_i\sqrt{8} + 3k_i\sqrt{8} + 8k_i \\ &= (3p_i + 8k_i) + (p_i + 3k_i)\sqrt{8} \end{aligned}$$

means

$$p_{i+1} = 3p_i + 8k_i \tag{10.6a}$$

$$k_{i+1} = p_i + 3k_i \tag{10.6b}$$

From $p_0 = 3, k_0 = 1$, and using (10.6) and (10.3), we get the following solutions:

| $i$ | **Calculation** | $p_i$ | $k_i$ | $n$ | $(n+1)$ | $(n+k)$ |
|---|---|---|---|---|---|---|
| 0 | | 3 | 1 | 2 | 3 | 3 |
| 1 | $p = 3 \cdot 3 + 8 \cdot 1, k = 3 + 3 \cdot 1$ | 17 | 6 | 14 | 15 | 20 |
| 2 | $p = 3 \cdot 17 + 8 \cdot 6, q = 17 + 3 \cdot 6$ | 99 | 35 | 84 | 85 | 119 |
| 3 | $p = 3 \cdot 99 + 8 \cdot 35, q = 99 + 3 \cdot 35$ | 577 | 204 | 492 | 493 | 696 |
| 4 | $p = 3 \cdot 577 + 8 \cdot 204, q = 577 + 3 \cdot 204$ | 3363 | 1189 | 2870 | 2871 | 4059 |

Table 1. First five solutions.

## 3. Answer

The first five solutions are

$$\begin{aligned} 1 + 2 &= \mathbf{3} \\ 1 + 2 + \cdots + 14 &= \mathbf{15} + 16 + \cdots + \mathbf{20} \\ 1 + 2 + \cdots + 84 &= \mathbf{85} + 86 + \cdots + \mathbf{119} \\ 1 + 2 + \cdots + 492 &= \mathbf{493} + 494 + \cdots + \mathbf{696} \\ 1 + 2 + \cdots + 2870 &= \mathbf{2871} + 2872 + \cdots + \mathbf{4059} \end{aligned}$$

We need only solutions less than 1000, so can stop here.

## 4. A similar puzzle

What if we don't count Hari's house in both additions? That is, what if the sum of the numbers of the houses on the left of Hari's house is equal to the sum of numbers of the houses on the right?

This question was asked to Srinivasa Ramanujan by a hundred years back. The story goes like this: Ramanujan was stirring vegetables in a frying pan when P.C. Mahalanobis asked this question, which he found in a magazine (It asked to find a solution between 50 and 500). Ramanujan asked Mahalanobis to write down the solution on a piece of paper, and narrated the general solution while stirring the vegetables.

Even though many people have given this story, nobody has provided the solution, other than it was a continued fraction.

That is another problem, and I will deal with that in another chapter.[‡]

[‡]See Chapter 1 on page 64.

CHAPTER 11

# D'Morgan's age

## 1. Question

In the year 1871 a mathematician named Augustus D'Morgan died. D'Morgan made a statement about his age. He said that he was $x$ years in the year $x^2$. Can you make a reasonable year in which he was born?

## 2. Solution

Let he was born in the year $k$. The requirement states that

$$k + x = x^2$$

or

$$x^2 - x - k = 0$$

Solving for $x$ using Equation (38.2) on page 76,

$$x = \frac{1 \pm \sqrt{1 + 4k}}{2}$$

We need to consider $k$ between 1750 and 1860 only, means $4k+1$ between 7001 and 7440. Since $\sqrt{7001} = 83.67$ and $\sqrt{7440} = 86.26$, the squares between those numbers are $84^2$, $85^2$ and $86^2$. Since the square should be in the form $4k + 1$, it must be odd. So, it must be $85^2$. That means

$$x = \frac{1 + 85}{2} = 43$$
$$k = \frac{85^2 - 1}{4} = 1806$$

## 3. Answer

D'Morgan was born in 1806, and he was 43 years old in 1849.

CHAPTER 12

# The dictator and the soldiers

## 1. Question

A sadistic dictator gathered 1000 soldiers, numbered them 1 to 1000 and made them stand in a circle so that the soldiers numbered 1-2-3-4-…-999-1000- 1 form a circle. He handed a sword to soldier number 1, who kills soldier number 2 and hands over the sword to soldier number 3, who kills soldier number 4 and hands over the sword to soldier number 5 and so on… The process is repeated through the circle on and on till only one soldier is alive, who is made the commander of the army.

If you were one of these 1000 soldiers, which soldier number would you wish you were?*

## 2. Solution

This problem is called *The Josephus problem*. [**4**] has an interesting account of the problem. Its connection with a popular kids' game is discussed in [**8**].

The general problem is to find the last person ($L_{(n,k)}$) when there are $n$ soldiers, and every $k^{th}$ soldier is killed at a time. No single formula has been discovered for the answer, though algorithms with complexities $O(k \log n)$ and $O(n)$ have been devised. Depending on which of $n$ or $k$ is big, one of these algorithms can be used.

A particular case, when $k = 2$, has been studied in depth. When $n = 1, 2, 3, \cdots$, $L_{(n,2)} = 1, 1, 3, 1, 3, 5, 1, 3, 5, 7, \cdots$. Using this fact, we can find $L_{1000,2}$.

An easier method is represent $n$ as a binary number (without leading zeros), then do a *cyclic left shift*, and the resulting number will be the answer.

In this case, 1000 in binary is 1111101000. A cyclic left shift will give 1111010001, which is equivalent to the decimal number 977. So, the $977^{th}$ soldier will survive.

## 3. Answer

The **977^th^** soldier.

*`http://mathematicsschool.blogspot.com/2009/11/ratio-of-radii.html`

CHAPTER 13

# Sum of digit cubes

## 1. Question

Sum of the cubes of some three digit numbers are exactly equal to that number itself. How many such numbers are there in between 300 and 400?

## 2. Solution

Let number be $300 + 10a + b$. We need to solve

$$300 + 10a + b = 3^3 + a^3 + b^3$$

Simplifying,

$$a^3 + b^3 - 10a - b = 273 \tag{13.1}$$

Here, $a$ cannot be even, because the LHS of (13.1) will be even in that case. So, $a$ must be odd, and $b$ can be odd or even.

At least one of the digits should be more than 5, so that the result will reach to 273. Also, the digits must be less than or equal to 7, to get the cube less than 400.

That reduces the number of possibilities. Since $a$ has only odd numbers, let us consider them one by one. From (13.1),

$$b(b^2 - 1) = 273 - a(a^2 - 10) \tag{13.2}$$

We can calculate $273 - a(a^2 - 10)$, and consider only those digits that divide it.

| $a$ | $273 - a(a^2 - 10)$ | $b$ | $b(b^2 - 1)$ | |
|---|---|---|---|---|
| 7 | 0 | 0 | 0 | |
| | | 1 | 0 | |
| 5 | 198 | 6 | 210 | |

So, the only solutions are

$$3^3 + 7^3 + 0^3 = 370$$
$$3^3 + 7^3 + 1^3 = 371$$

## 3. Answer

**370** and **371**.

CHAPTER 14

# Two cubes

## 1. Question

Two cubes have integer sides. The sum of their volumes (cubic units) is numerically equal to the sum of their total length of edges. What are their sides?

## 2. Solution

Let the cubes have sides $a$ and $b$.

A cube has 12 edges. So, the question states

$$\begin{aligned} a^3 + b^3 &= 12a + 12b \\ (a+b)(a^2 - ab + b^2) &= 12(a+b) \\ a^2 - ab + b^2 &= 12 \end{aligned}$$

Solving for $a$ using (38.2) on page 76,

$$a = \frac{b \pm \sqrt{b^2 - 4(b^2 - 12)}}{2} = \frac{b \pm \sqrt{48 - 3b^2}}{2}$$

So, the descriminant $48 - 3b^2$ should be non-negative and a perfect square. So, $1 \leq b \leq 4$.

For $b = 1, 2, 3, 4$, $48 - 3b^2 = 45, 36, 21, 0$. So, only $b = 2$ and $b = 4$ give perfect squares.

$$\begin{aligned} b = 2, a &= \frac{2 \pm 6}{2} = 4 \text{ or } -2 \\ b = 4, a &= \frac{4 \pm 0}{2} = 2 \end{aligned}$$

## 3. Answer

4 and 2.

CHAPTER 15

# Number of children

## 1. Question

All the students in the school are made to stand in rows so as to form an equlateral triangle - the first row consists of 1 student, $2^{nd}$ row 2 students and so on. If there were 90 more students, the total number of students could have been arranged in the shape of a square, so that each side of the square has 5 students less than the number students in the side of the triangle.

What is the total number of students in the school?

## 2. Solution

Let the size of the equilateral triangle is $n$. So the number of students is given by

$$N = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

The question states that

$$N + 90 = (n-5)^2$$

That is

$$\begin{aligned}\frac{n(n+1)}{2} + 90 &= n^2 - 10n + 25 \\ n^2 + n + 180 &= 2n^2 - 20n + 50\end{aligned}$$

$$n^2 - 21n - 130 = 0$$

Solving for $n$ using (38.2) on page 76,

$$\begin{aligned} n &= \frac{21 \pm \sqrt{21^2 + 4 \cdot 130}}{2} \\ &= \frac{21 \pm 31}{2} \\ &= 26 \text{ or } -5 \end{aligned}$$

So, $n = 26$, and $N = \frac{26 \cdot 27}{2} = 351$.

## 3. Answer

**351** children.

CHAPTER 16

# The cone

## 1. Question

The height of a cone is 24 cm and its curved surface area is 550 sq. cm. Find the volume of the cone.

## 2. Solution

Given

$$h = 24 \tag{16.1}$$
$$\pi r l = 550 \tag{16.2}$$

We need to find the value of $\frac{1}{3}\pi r^2 h$.

$$\begin{aligned} \pi^2 r^2 l^2 &= 550^2 \\ \pi^2 r^2 (r^2 + h^2) &= 550^2 \\ r^4 + 24^2 r^2 - \left(\frac{550}{\pi}\right)^2 &= 0 \end{aligned}$$

Solving for $r^2$ using Equation (38.2) on page 76,

$$r^2 = \frac{-24^2 \pm \sqrt{24^4 + 4(\frac{550}{\pi})^2}}{2} \tag{16.3}$$
$$r = \sqrt{\frac{-24^2 \pm \sqrt{24^4 + 4(\frac{550}{\pi})^2}}{2}} \tag{16.4}$$

the volume of the cone is given by

$$\begin{aligned} \frac{1}{3}\pi r^2 h &= \frac{\pi}{3}(l^2 - h^2)h \\ &= \frac{\pi}{3}\left(\left(\frac{550}{\pi r}\right)^2 - 24^2\right) \cdot 24 \end{aligned} \tag{16.5}$$

**2.1. Using $\pi = \frac{22}{7}$.** It seems that this puzzle is designed to use $\pi = \frac{22}{7}$ rather than more accurate value. So, the calculations above becomes

$$
\begin{aligned}
\frac{550}{\pi} &= 175 \\
r^2 &= \frac{-576 \pm \sqrt{24^4 + 4 \cdot 175^2}}{2} \\
&= \frac{-576 \pm 674}{2} \\
&= 49 \text{ or } -625 \\
r &= 7 \\
V &= \frac{22}{21}\left(\left(\frac{175}{7}\right)^2 - 576\right) \cdot 24 \\
&= \frac{22}{21} \cdot (625 - 576) \cdot 24 \\
&= 1232
\end{aligned}
$$

**2.2. Using more accurate $\pi$.** If we use more accurate value of $\pi$, we get

$$
\begin{aligned}
r &= 7.00261256 \\
V &= 1232.4237427596495
\end{aligned}
$$

## 3. Answer

The volume is around **1232** cubic centimeters.

CHAPTER 17

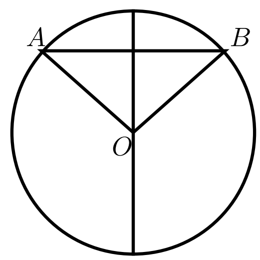
# Length of chord

## 1. Question

Find the radius of the circle, if a chord of length 8 cm divides the area of circle 1:8.*

## 2. Solution

This means the circular segment has an area of $\frac{\pi r^2}{9}$.
In the figure, let $\overline{AB}$ be the chord and $O$, the center. $OA = OB = r$, and $AB = 8cm$. Let $\angle AOB = \theta$.




$$\begin{aligned}
\text{Area of segment} &= \text{Area of sector} - \text{Area of } \triangle ABO \\
&= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\
&= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{\pi r^2}{9} \\
\theta - \sin\theta &= \frac{2\pi}{9}
\end{aligned}$$

So, we need to solve the equation

$$f(\theta) = \theta - \sin\theta - \frac{2\pi}{9} = 0 \tag{17.1}$$

I don't know any analytical method to solve this trigonomeric equation. There are several methods to solve it numerically, two of which are shown below:

**2.1. Bisection method.** We know

$$\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2} - 1 - \frac{2\pi}{9} = -0.1273 \\
f\left(\frac{2\pi}{3}\right) &= \frac{2\pi}{3} - 1 - \frac{2\pi}{9} = 0.5302
\end{aligned}$$

*`http://mathematicsschool.blogspot.com/2009/12/to-check-whether-divisible-by-7.html#comment-5816648343522349283`

and the function is an increasing continuous function between $\frac{\pi}{2}$, i.e., 1.57079632679 and $\frac{2\pi}{3}$, i.e., 2.09439510239. We can recursively bisect this range to get the value of $\theta$ where $f(\theta) = 0$.

Successive bisection yields 1.83259571459, 1.70169602069, 1.63624617374, 1.66897109722, 1.68533355896, 1.69351478983, 1.68942417439, 1.69146948211, 1.69044682825, 1.69095815518, 1.69070249171, 1.69083032345, 1.69089423931, 1.69092619725 and $f(1.69092619725) = 1.416 \times 10^{-6}$.

So, $theta = 1.69092619725\, radians = 96.882934^o$.

**2.2. Newton-Raphson Method.** According to this method, successive approximations can be obtained by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{17.2}$$

Here,

$$\begin{aligned} f(x) &= x - \sin x - \frac{2\pi}{9} \\ f'(x) &= 1 - \cos x \end{aligned}$$

Starting with $x_0 = \frac{2\pi}{3}$,

$$\begin{aligned} x_0 &= \frac{2\pi}{3} = 2.09439510239 \\ x_1 &= \frac{0.530237997811}{1.5} = 1.74090310385 \\ x_2 &= \frac{0.0572047065571}{1.16928758594} = 1.69198040319 \\ x_3 &= \frac{0.0011825108797}{1.12088768436} = 1.69092542605 \end{aligned}$$

And $f(1.69092542605) = 5.5248 \times 10^{-7}$

So, $theta = 1.69092542605\, radians = 96.882890^o$.

**2.3. Computing radius.** The chord length is $2r \sin\left(\frac{\theta}{2}\right)$. So,

$$\begin{aligned} r &= \frac{8}{2\sin\left(\frac{\theta}{2}\right)} \\ &= 5.3456 \end{aligned}$$

## 3. Answer

The radius is 5.3456 cm.

CHAPTER 18

# Cutting a right-angled triangle

## 1. Question

A paper is folded in the form of right angled triangle. A portion of the triangle is cut by drawing a line parallel to the hypotenuse of the triangle.The length of the hypotenuse is reduced by 35% after the cut.

If the area of the original triangle was 34 what is the area of the new triangle?*

## 2. Solution

The length of the hypetenuse got reduced by 35%, means it became $\frac{65}{100} = \frac{13}{20}$ of the original.

**2.1. Quick solution.** Since the side became $\frac{13}{20}$ of the original, the area will become $\left(\frac{13}{20}\right)^2$ of the orignal, that is $34 \cdot \frac{169}{400} = 14.365$.

**2.2. Detailed solution.** Let the original lengths of the sides enclosing the right angle be $x$ and $y$.

$$A_{old} = \frac{xy}{2} = 34 \tag{18.1}$$

Since the hypotenuse is reduced to $\frac{13}{20}$ of the original, the two other sides $x$ and $y$ also must have been reduced by that, because the two triangles are similar.

So, the new area must be

$$\begin{aligned} A_{new} &= \frac{1}{2} \cdot \frac{13x}{20} \cdot \frac{13y}{20} && (18.2)\\ &= \frac{1}{2} xy \cdot \frac{169}{400} && (18.3)\\ &= 34 \cdot \frac{169}{400} && (18.4)\\ &= 14.365 && (18.5) \end{aligned}$$

## 3. Answer

New area = 14.365 sq. units.

*http://mathematicsschool.blogspot.com/2009/12/to-check-whether-divisible-by-7.html#comment-8461563791037329077

CHAPTER 19

# Volume of the box

## 1. Question

Akhil received a parcel from his uncle on last x'mas day and a note on it "guess what?". He was surprised and opened the packet. He was again puzzled–"Why did his uncle send a cheap gift with a note?"

After one week, his uncle asked what the volume of that rectangular box was. He replied that the packet was spoilt and the remaining part was only the thread that is used to tie it.

"How can I calculate the volume?", he asked. His uncle demanded the gift back if he failed to calculate the volume.

All he remembers is this much:

(1) The box was tied once lengthwise and twice breadthwise.
(2) The thread was 360 cm long, ignoring the length of the knot.

Please help Akhil to calculate the length, breadth and height of the box if it had the *maximum volume* possible with the above specifications.

## 2. Answer

Let $l$, $b$ and $h$ be the length, breadth and height of the box respectively.

Each lengthwise loop will use $(2l + 2h)$ thread, and each breadthwise loop will use $(2b + 2h)$ thread, so one lengthwise loops and two breadthwise loops will take $(2l + 4b + 6h)$ string.

That means

$$\begin{aligned} 2l + 4b + 6h &= 360 \\ l + 2b + 3h &= 180 \end{aligned} \tag{19.1}$$

The volume is

$$\begin{aligned} V &= lbh \\ &= (180 - 2b - 3h)bh \\ &= 180bh - 2b^2h - 3bh^2 \end{aligned} \tag{19.2}$$

To find the maximum value of $V$, differentiate $V$ partially by $b$ and $h$ and equate to zero.

Partially differentiating with respect to $b$,

$$\begin{aligned} \frac{\partial V}{\partial b} &= 0 \\ 180h - 4bh - 3h^2 &= 0 \\ 3h + 4b &= 180 \end{aligned} \tag{19.3}$$

Partially differentiating with respect to $h$,

$$\frac{\partial V}{\partial h} = 0$$
$$180b - 2b^2 - 6bh = 0$$
$$3h + b = 90 \tag{19.4}$$

(19.3) - (19.4) gives

$$3b = 90$$
$$b = 30 \tag{19.5}$$
$$h = \frac{90 - 30}{3} = 20 \tag{19.6}$$
$$l = 180 - 2 \cdot 30 - 3 \cdot 20 = 60 \tag{19.7}$$

So, the maximum volume $= 60 \cdot 30 \cdot 20 = 36000$ sq. cm.

### 3. Answer

Length = **60** cm, Breadth = **30** cm, Height = **20** cm, Volume = **36000** sq. cm.

CHAPTER 20

# The ant and the rubber band

## 1. Question

An ant is crawling at a rate of one foot per minute along a rubber band which can be infinitely and uniformly stretched. The strip is initially one yard long and is stretched an additional yard at the end of each minute. If the ant starts at one end of the rubber band, will it ever reach the other end, and if so when?*

(One yard = 3 feet)

## 2. Solution

At first, it seems the ant will never reach the end, because it is travelling only one feet per minute while the rubber band stretches three feet in a minute.

But the fact is that, the band stretches *behind the ant* as well, so it moves more than one feet every minute. How much it moves in a minute will give a clue to this puzzle.

The stretching occurs in one minute intervals, and the ant keeps the relative position in the band before and after the stretch. For example, if the ant is at the one-fourth length of the band before the stretch, it will be at one-fourth the length after the stretch.

Let us compute this for first few minutes.

| **Minute** | **Stretch?** | **Length** | **Ant** | **Relative position** |
|---|---|---|---|---|
| 0 | - | 3 | 0 | 0 |
| 1 | B | 3 | 1 | $\frac{1}{3}$ |
| 1 | A | 6 | 2 | - do - |
| 2 | B | 6 | 3 | $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ |
| 2 | A | 9 | $\frac{9}{2}$ | - do - |
| 3 | B | 9 | $\frac{11}{2}$ | $\frac{11}{18} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9}$ |
| 3 | A | 12 | $\frac{22}{3}$ | - do - |
| 4 | B | 12 | $\frac{25}{3}$ | $\frac{25}{36} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12}$ |
| 4 | A | 15 | $\frac{125}{12}$ | - do - |

So, at the $n^{th}$ minute, the ant will reach the

$$\frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) \tag{20.1}$$

of the total length.

This is a divergent series and approaches infinity when $n$ approaches infinity, which means the ant will be able to reach the end of a band of any length given enough time.

Since there is no formula available to sum the harmonic series, we need to manually calculate.

*http://mathematicsschool.blogspot.com/2009/12/to-check-whether-divisible-by-7.html#comment-4011568688148997326

$$
\begin{aligned}
1+\frac{1}{2} &= 1\frac{1}{2}\\
1\frac{1}{2}+\frac{1}{3} &= 1\frac{5}{6}\\
1\frac{5}{6}+\frac{1}{4} &= 2\frac{1}{12}\\
2\frac{1}{12}+\frac{1}{5} &= 2\frac{17}{60}\\
2\frac{17}{60}+\frac{1}{6} &= 2\frac{9}{20}\\
2\frac{9}{20}+\frac{1}{7} &= 2\frac{83}{140}\\
2\frac{83}{140}+\frac{1}{8} &= 2\frac{201}{280}\\
2\frac{201}{280}+\frac{1}{9} &= 2\frac{2089}{2520}\\
2\frac{2089}{2520}+\frac{1}{10} &= 2\frac{2341}{2520}\\
2\frac{2341}{2520}+\frac{1}{11} &= 3\frac{551}{27720}
\end{aligned}
$$

Thus, in 11 minutes, this sum will exceed 3, so (20.1) will exceed 1. So, the ant will reach the end between 10 and 11 minutes.

After 10 minutes, the ant would have covered $\frac{1}{3}\cdot 2\frac{2341}{2520} = \frac{7381}{7560}$ of the rubber band. In 10 minutes, the band will be $3+10\cdot 3 = 33$ feet long, so the ant would have covered $33\times\frac{7381}{7560} = 32\frac{551}{2520}$ feet. The remaining $\frac{1969}{2520} = 0.7813$ feet will be covered in 0.7813 minutes.

### 3. Answer

The ant will reach the end of the rubber band in **10.7813** minutes.

CHAPTER 21

# Buying animals

## 1. Question

An elephant costs 50 rupees, a horse 10 rupees and a goat 50 paise.[*] How can one buy 1000 animals with exactly 1000 rupees?[†]
(1 rupee = 100 paise)

## 2. Solution

The basic equations are

$$50e + 10h + \frac{g}{2} = 1000 \tag{21.1}$$
$$e + h + g = 1000 \tag{21.2}$$

Multiplying the first equation by 2 and subtracting the second equation,

$$99e + 19h = 1000 \tag{21.3}$$

The techniques given in Section 8.2 on page 81 can be used to solve this. Here, the technique using continued fractions is used.
First of all, solve the equation

$$99e + 19h = 1 \tag{21.4}$$

using the technique given in Section 8.1 on page 80.
The continued fraction expansion of $\frac{99}{19}$ is $[5; 4, 1, 3]$, giving convergents $\frac{5}{1}$, $\frac{21}{4}$, $\frac{26}{5}$ and $\frac{99}{19}$.
Taking the previous convergent $\frac{26}{5}$, $e = 5, h = -26$ is a solution to (21.4), so (see Section 8.2.1 on page 81),

$$e = 5000 - 19k \tag{21.5a}$$
$$h = -26000 + 99k \tag{21.5b}$$

will give all solutions. The only value for $k$ giving positive $e$ and $h$ is 263, giving

$$e = 3 \tag{21.6a}$$
$$h = 37 \tag{21.6b}$$
$$g = 960 \tag{21.6c}$$

[*] Must be toy animals!
[†] http://mathematicsschool.blogspot.com/2009/12/1000-animals.html

### 3. Answer

**3 elephants, 37 horses and 960 goats**. Total cost $= 150 + 370 + 480 = 1000$.

CHAPTER 22

# Seven digit phone number

## 1. Question

My new telephone number has 7 digits. I can never remember it. My wife told me that she can remember it easily because if the last four digits are placed in front of the remaining three we get one more than twice that number. What is that number?

## 2. Solution

Let the first 3 digits form a three-digit number $x$ and the last four-digits form a four-digit number $y$. The phone number is $10^4x + y$. Now,

$$10^3y + x = 2(10^4x + y) + 1$$

That is,

$$998y - 19999x = 1 \tag{22.1}$$

To solve this (see the theory at Section 8.1 on page 80), express $\frac{998}{19999}$ as a continued fraction. It is $[0; 20, 25, 1, 1, 2, 3]$, with convergents $\frac{0}{1}$, $\frac{1}{20}$, $\frac{25}{501}$, $\frac{26}{521}$, $\frac{51}{1022}$, $\frac{128}{2565}$, $\frac{435}{8717}$ and $\frac{998}{19999}$.
So, $x = 435, y = 8717$ is a solution to (22.1), and the general solution is given by (See Section 8.2.3 on page 81)

$$x = 435 + 998k \tag{22.2a}$$
$$y = 8717 + 19999k \tag{22.2b}$$

But only $k = 0$ provides $x$ and $y$ in the domain of this puzzle.

## 3. Answer

The phone number is **4358717**. Verify: $2 \cdot 4358717 + 1 = 8717435$.

CHAPTER 23

# The bus

## 1. Question

A bus takes 3 hours less to cover a distance of 680 km when its speed is increased by 6 km/h. Find its usual speed.

## 2. Solution

Let its usual speed be $x$ km/h.

$$\frac{680}{x} - \frac{680}{x+6} = 3 \tag{23.1}$$

$$680 \cdot \frac{x+6-x}{x(x+6)} = 3 \tag{23.2}$$

$$x^2 + 6x - 1360 = 0 \tag{23.3}$$

$$\tag{23.4}$$

Solving for $x$ using Equation (38.2) on page 76,

$$x = \frac{-6 \pm \sqrt{6^2 + 4 \cdot 1360}}{2} \tag{23.5}$$

$$= \frac{-6 \pm 74}{2} \tag{23.6}$$

$$= 34 \text{ or } -40 \tag{23.7}$$

## 3. Answer

The usual speed is **34 km/h**. It covers 680 km in 20 hours. When the speed becomes $34 + 6 = 40$ km/h, it covers the same distance in 17 hours, which is 3 hours less than before. 1

CHAPTER 24

# Mis-taken cheque

## 1. Question

A man goes to bank with a cheque for encashing. Amount written on cheque was some rupees and some paise. Total amount was less than rupees one hundred.

By the mistake of the cashier, he was paid paise in place of rupees and rupees in place of paise.

While returning to home, he spends 20 paise for some item.

After reaching home, he found that the total amount remaining with him is exactly double of the amount written on cheque.

Can you tell cheque amount?*

## 2. Finding a particular solution

Let $r$ be the rupees and $p$ the paise portion in the cheque. The puzzle states

$$100p + r - 20 = 2(100r + p)$$

That is,

$$98p - 199r = 20 \tag{24.1}$$

We need to solve (24.1) in integers such that $0 \leq p, r \leq 99$. Section 8.2 on page 81 gives two techniques to solve this equation.

**2.1. Using congruences.** To solve this using congruences (See Section 8.2.2 on page 81), we need to solve one of the following two equations.

$$98p \equiv 20 \pmod{199} \tag{24.2a}$$
$$199r \equiv 78 \pmod{98} \tag{24.2b}$$

Both are difficult to solve: the first needs to try 199 values, and the second requires 98 values.

However, this can be simplified by a small transformation. By inspection, we can see that $p$ is very close to double of $r$. Let

$$p = 2r + z \tag{24.3}$$

Now, (24.1) becomes

$$\begin{aligned} 98(2r + z) - 199r &= 20 \\ 98z - 3r &= 20 \end{aligned} \tag{24.4}$$

(24.4) is simpler to solve. Write it as

*http://mathematicsschool.blogspot.com/2009/12/1000-animals.html#comment-7333207366942660378

$$
\begin{aligned}
98z &\equiv 20 \pmod 3 \\
98z &\equiv 2 \pmod 3 \qquad (24.5)
\end{aligned}
$$

Putting $z = 0, 1, 2$, we find

$$
\begin{aligned}
98 \cdot 0 &\equiv 0 \pmod 3 \\
\mathbf{98 \cdot 1} &\equiv \mathbf{2} \pmod{\mathbf{3}} \\
98 \cdot 2 &\equiv 1 \pmod 3
\end{aligned}
$$

So, $z = 1$.

$$r = \frac{98z - 20}{3} = 26 \qquad (24.6a)$$

$$p = \frac{199r + 20}{98} = 53 \qquad (24.6b)$$

The general solution is

$$r = 26 + 98k \qquad (24.7a)$$

$$p = 53 + 199k \qquad (24.7b)$$

for any integer $k$. These values will be positive and less than 100 only for $k = 0$.
So the cheque amount is 26.53.

**2.2. A simpler explanation.** Rewrite (24.4) as

$$96z + 2z - 3r = 6 \cdot 3 + 2 \qquad (24.8)$$

$$2(z - 1) = 3(r + 6 - 32z) \qquad (24.9)$$

So, $z - 1$ should be divisible by 3. So, $z = 1, 4, 7, 10, \cdots$.

**2.3. Using continued fractions.** (See Section 8.2.1 on page 81 and Section 8.1 on page 80.)
The coninued fraction for $\frac{98}{199}$ is $[0; 2, 32, 1, 1]$, with convergents $\frac{0}{1}$, $\frac{1}{2}$, $\frac{32}{65}$, $\frac{33}{67}$, $\frac{65}{132}$, $\frac{98}{199}$.
So, $p = 132, r = 65$ is a solution to

$$98p - 199r = 1 \qquad (24.10)$$

So, the solution to (24.1) is $p = 2640, r = 1300$.
The general solution is

$$r = 1300 + 98k \qquad (24.11a)$$

$$p = 2640 + 199k \qquad (24.11b)$$

Putting $k = -13$ gives these values between 0 and 100:

$$r = 26 \qquad (24.12a)$$

$$p = 53 \qquad (24.12b)$$

## 3. Answer

The check amount was 26.53, the cashier gave 53.26, he spent 0.20, and the remaining 53.06 is the double of the original amount.

CHAPTER 25

# Thieves and gold coins

## 1. Question

Three thieves stole some gold coins from a jewel shop and decided to spend the night near a temple. In the night, one of the thieves put one gold coin in the temple's *hundi*, divided the rest of the coins exactly into three, and hid one portion. During the night, the second and the third thieves also did the same.

In the morning, the three thieves woke up, gave one coin to the temple, and divided the rest of the coins into three, and took one portion.

How many gold coins were stolen from the jewel shop?*

## 2. Solution

Let $n$ be the number of gold coins each one got at the last division. That means there were $3n+1$ coins before that division. So,

- Just before the third thief's division, there were $\frac{3}{2}(3n+1)+1 = \frac{9n+3}{2}+1 = \frac{9n+5}{2}$ coins.
- Just before the second thief's division, there were $\frac{3}{2}\cdot\frac{9n+5}{2}+1 = \frac{27n+15}{4}+1 = \frac{27n+19}{4}$ coins.
- Just before the first thief's division, there were $\frac{3}{2}\cdot\frac{27n+19}{4}+1 = \frac{81n+57}{8}+1 = \frac{81n+65}{8}$ coins.

This is the original number. So, the original number of gold coins should be in the form $\frac{81n+65}{8}$. Let us say this number $m$. So, we get†

$$\frac{81n+65}{8} = m$$
$$8m - 81n = 65 \tag{25.1}$$

The integer equation (25.1) can be solved using the techniques given in Section 8.2 on page 81. Here, the first technique (using congruences) is used.

Since the coefficient of $m$ is only 8, we can easily find the solution by putting $n = 0, 1, \cdots 7$, in the equation

$$81n \equiv -65 \pmod 8$$
$$81n \equiv 7 \pmod 8 \tag{25.2}$$

Since $81n = 8\cdot 10n + n$, $81n \equiv n \pmod 8$. So, the above modular equation becomes

$$n \equiv 7 \pmod 8 \tag{25.3}$$

So, $n = 7$ is a solution, and $n = 7 + 8k$, where $k$ is any integer, is the general solution.

---

*`http://mathematicsschool.blogspot.com/2010/01/gold-coins.html`

†We can obtain the same equation by forward computation as well: Let the total number of coins be $m$. After the first thief divided, it reduced to $\frac{2}{3}(m-1) = \frac{2m-2}{3}$. After the second thief divided, it became $\frac{2}{3}\left(\frac{2m-2}{3}-1\right) = \frac{4m-10}{9}$. After the third thief divided, it became $\frac{2}{3}\left(\frac{4m-10}{9}-1\right) = \frac{8m-38}{27}$. So, $n = \frac{1}{3}\left(\frac{8m-38}{27}-1\right) = \frac{8m-65}{81}$, leading to the same equation.

$$m = \frac{81(7+8k)+65}{8} = \frac{632+648k}{8} = 79+81k$$

The general solution is

$$n = 7+8k \tag{25.4}$$
$$m = 79+81k \tag{25.5}$$

with the smallest solution $n = 7, m = 79$.

### 3. Answer

They stole $79+81k$ gold coins, where $k$ can be any integer.

The first thief put 1 coin in the temple, kept $26+27k$ coins and left $52+54$ coins.

The second thief put 1 coin in the temple, kept $17+18k$ coins and left $34+36$ coins.

The third thief put 1 coin in the temple, kept $11+12k$ coins and left $22+24$ coins.

The thieves together put 1 coin in the temple, and divided the rest into three, so that each one gets $7+8k$ coins.

There are infinite number of solutions, the smallest being 79 coins.

### 4. Additional check

| | | | | |
|---|---|---|---|---|
| First thief | = | $26+27k+7+8k$ | = | $33+35k.$ |
| Second thief | = | $17+18k+7+8k$ | = | $24+26k.$ |
| Third thief | = | $11+12k+7+8k$ | = | $18+20k.$ |
| Temple | = | $1+1+1+1$ | = | $4$ |
| Total | | | = | $79+81k.$ |

CHAPTER 26

# Mice and cats

## 1. Question

Some mice went into a hole. When they came out, their number doubled. Some cats were waiting outside, and each cat ate one mouse. The remaining mice again went into the hole. When they came back, their number doubled. The same cats ate one more mouse each. The remaining mice again went inside. Their number doubled when they came back. The same cats ate one more mouse each. Now, there were no mice left.

What is the relation between the initial number of mice and the number of cats?*

## 2. Solution

Let the initial number of mice be $m$ and the number of cats be $c$.

When the mice went inside for the first time and came back, the number of mice is $2m$. After the cats ate one mouse each, it got reduced to $2m - c$.

For the second time when the mice went inside, their number doubled, and the cats ate $c$ of them. The number of remaining mice is $2(2m - c) - c = 4m - 3c$.

Similarly, in the third time, the remaining mice is $2(4m - 3c) - c = 8m - 7c$.

This is zero. So, $8m - 7c = 0$, so

$$m = \frac{7}{8}c \tag{26.1}$$

## 3. Answer

The initial number of mice is $\frac{7}{8}^{th}$ of the number of cats.

## 4. General Solution

What if the mice got exhausted after the $n^{th}$ time, instead of the third time in the question?

From the above example, the number of mice after 1, 2 and 3 times are $2m - c$, $4m - 3c$ and $8m - 7c$. So, the number of mice after the $n^{th}$ time is $2^n m - (2^m - 1)c$.† So, the general expression is

$$m = \frac{2^m - 1}{2^m}c \tag{26.2}$$

For example, if the mice went inside 5 times and got exhausted, the number of mice should be $\frac{31}{32}$ of the cats' number.

---

*`http://mathematicsschool.blogspot.com/2010/01/gold-coins.html#comment-4630416664027901030`

†This can be derived easily, or proved by induction, based on the fact that $2(2^k m - (2^k - 1)c) - c = 2^{k+1}m - 2^{k+1}c + 2c - c = 2^{k+1}m - 2^{k+1}c + c = 2^{k+1}m - (2^{k+1} - 1)c$.

## 5. Solution by back-calculation

Sometimes the number of cats is given as a number (rather than a variable here) and the initial number of mice needs to be determined. Such problems can be solved using simple arithmetic rather than algebra.

For example, let us say it is given that there were 16 cats. Now, instead of using algebra, we can find the initial number of mice as follows:

(1) The mice were exhausted the third time, so the number of mice that came out the third time must be 16.
(2) That means the number of mice that went in after the second time is $\frac{16}{2} = 8$.
(3) That means the number of mice that came out before the second time must be $8 + 16 = 24$.
(4) That means the number of mice that went in after the first time is $\frac{24}{2} = 12$.
(5) That means the number of times that came out the first time is $12 + 16 = 28$.
(6) That means the number of mice that went in initially must be $\frac{28}{2} = 14$.

Of course, this can be computed from (26.1): $m = \frac{7}{8} \cdot 16 = 14$.

CHAPTER 27

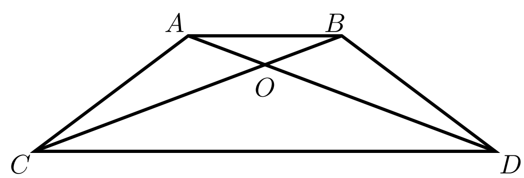
# Area of a trapezium

## 1. Question

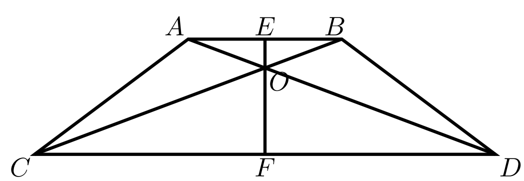
In a trapezium* $ABCD$, $\overline{AB}$ and $\overline{CD}$ are the parallel lines. The diagonals $\overline{AC}$ and $\overline{BD}$ meet at $O$. Express the area of the trapezium $ABCD$ in terms of the areas of the triangles $AOB$ and $COD$.†

## 2. Solution

See the figure. We need to find the area of $ABCD$ in terms of $OAB$ and $OCD$.




Let $AB = p$ and $CD = q$.
Draw a line $\overline{EF}$ perpendicular to $\overline{AB}$ and passing through $O$, with $E$ in $\overline{AB}$ and F in $\overline{CD}$. Let $EF = h$.




By the property of trapeziums,

$$\frac{OE}{OF} = \frac{AB}{CD} = \frac{p}{q}$$

Also, $OE + OF = h$. So,

$$OE = h\frac{p}{p+q} \tag{27.1a}$$
$$OF = h\frac{q}{p+q} \tag{27.1b}$$

Area of the trapezium is

*Trapezoid* in America.
†`http://mathematicsschool.blogspot.com/2010/01/find-area-of-trapezium.html`

$$A_{ABCD} = \frac{h}{2}(p+q) \tag{27.2}$$

while area of the triangles are

$$\begin{aligned} A_{OAB} &= \frac{AB \cdot OE}{2} \\ &= \frac{p^2 h}{2(p+q)} && \text{(27.3a)} \\ A_{OCD} &= \frac{CD \cdot OF}{2} \\ &= \frac{q^2 h}{2(p+q)} && \text{(27.3b)} \end{aligned}$$

We need to express the RHS of (27.2) in terms of the RHS of the equations in (27.3).

$$\begin{aligned} \sqrt{A_{OAB}} + \sqrt{A_{OCD}} &= (p+q)\sqrt{\frac{h}{2(p+q)}} \\ &= \sqrt{\frac{h(p+q)}{2}} \\ &= \sqrt{A_{ABCD}} \end{aligned}$$

So,

$$A_{ABCD} = (\sqrt{A_{OAB}} + \sqrt{A_{OCD}})^2 \tag{27.4}$$

### 3. Answer

The area of the trapezium is the sum of the square roots of the areas of the triangles.

CHAPTER 28

# Double and triple a circle

## 1. Question

Given a circle, construct two concentric circles with double and triple area of the original circle.

## 2. Solution

(1) Call the given circle $C_1$.
(2) Draw two chords of the circle. Draw their perpendicular bisectors (see Section 5.1 on page 78). The point of intersection of the bisectors is the center of the circle. Let it be $O$.
(3) Draw one diameter $\overline{PQ}$. Draw its perpendiculr bisector (see Section 5.1 on page 78), which is another diameter $\overline{RS}$. Join $\overline{PR}$. Now, $PR = \sqrt{2}$.
(4) With $P$ as center and that $PR$ as radius, draw a circle $C_2$. This circle has double the area of $C_1$.
(5) Using the construction given in Section 5.6 on page 79, draw circle $C_3$ with center as $P$ and radius same as $C_2$.
(6) **$\mathbf{C_3}$ is concentric to and has the double the area of $\mathbf{C_1}$.**
(7) Extend $\overline{RS}$ to meet $C_3$ at $T$.
(8) With center as $P$ and radius as $PT$, draw a circle $C_4$.
(9) Using the construction given in Section 5.6 on page 79, draw circle $C_5$ with center as $P$ and radius same as $C_4$.
(10) **$\mathbf{C_5}$ is concentric to and has the three the area of $\mathbf{C_1}$.**

CHAPTER 29

# Construct circle with area equal to the sum of area of two given circles

## 1. Question

Draw a circle with area equal to the sum of areas of two given circles, using only straight edge and compass.

## 2. Solution

(1) Let $C_1$ and $C_2$ be the two circles. Mark an external point $P$.
(2) Draw a circle $C_3$ with the same area as $C_1$ with center $P$, using the technique in Section 5.6 on page 79.
(3) Draw another circle $C_4$ with the same area as $C_2$ with center $P$, using the technique in Section 5.6 on page 79.
(4) Draw a line $\overline{AB}$ passing through $P$ and meeting $C_1$ at some points $A$ and $B$.
(5) Draw the perpendicular bisector of $\overline{AB}$, using the technique given in Section 5.1 on page 78. Extend it is necessary to meet $C_4$ at $C$ and $D$.
(6) Join $\overline{AC}$. With $A$ as center and $AC$ as radius, draw a circle $C_5$.
(7) $C_5$ is the required circle.

CHAPTER 30

# Average of weights

## 1. Question

We have unlimited supply of 3 kg and 8 kg weights.

(1) How can we make an average weight of 6 kg?
(2) What are the integer averages that can be made from these weights?
(3) What is the biggest integer average with these weights?
(4) What is the smallest integer average with these weights?
(5) Solve these problems with weights (7 kg, 2 kg).
(6) Solve these problems with weights (17 kg, 57 kg).
(7) Solve these problems with weights ($a$ kg, $b$ kg).

## 2. Solution

Let us solve the last question first and make others particular cases of it.

Let the two weights be $a$ and $b$, with $a < b$. The condition is

$$\frac{ma + nb}{m + n} = k \tag{30.1}$$

where $m$, $n$ and $k$ are non-negative integers.

$$\begin{aligned} ma + nb &= km + kn \\ (k - a)m &= (b - k)n \\ \frac{m}{n} &= \frac{b - k}{k - a} \end{aligned}$$

To get this non-negative, $a \leq k \leq b$. By examining these values, we can solve particular cases.

Questions 1–4 deals with $a = 3$, $b = 8$. So, $3 \leq k \leq 8$.

| **k** | **m : n** | **Weights** |
|---|---|---|
| 3 | 5 : 0 | $n = 0$ |
| 4 | 4 : 1 | $m = 4n$ |
| 5 | 3 : 2 | $m = 3p, n = 2p$ |
| 6 | 2 : 3 | $m = 2p, n = 3p$ |
| 7 | 1 : 4 | $n = 4m$ |
| 8 | 0 : 5 | $m = 0$ |

## 3. Answer

(1) Take $2p$ number of 3 kg, and $3p$ number of 8 kg, where $p$ is any integer.
(2) 3, 4, 5, 6, 7 and 8.
(3) 8, when number of 3 kg weights is zero.
(4) 3, when number of 8 kg weights is zero.
(5) Instead of 3–8, it will be 2–7.

(a) Any weight in the range 2–7 can be obtained by selecting $m$ and $n$ in such a way that $\frac{m}{n} = \frac{7-k}{k-2}$, and then selecting $m$ number of 2 kg weights and $n$ number of 7 kg weights.
(b) 2 to 7, both inclusive.
(c) 7, when number of 2 kg weights is zero.
(d) 3, when number of 7 kg weights is zero.

(6) Instead of 3–8, it will be 17–57.
  (a) Any weight in the range 17–57 can be obtained by selecting $m$ and $n$ in such a way that $\frac{m}{n} = \frac{57-k}{k-17}$, and then selecting $m$ number of 17 kg weights and $n$ number of 57 kg weights.
  (b) 17 to 57, both inclusive.
  (c) 57, when number of 17 kg weights is zero.
  (d) 17, when number of 57 kg weights is zero.

(7) All weights in the range $a$–$b$ can be the average.

CHAPTER 31

# Fifth Fermat's number

## 1. Question

Find whether $2^{(2^5)}+1$ is a prime. If not, factorize.

## 2. Solution

Fermat conjectured that all numbers in the form $F_n = 2^{(2^n)}+1$ are primes. It is true for $n = 0, 1, 2, 3, 4$ but false for many values after that. All Fermat numbers from $F_5$ to $F_{11}$ have been factorized, and all upto $F_{32}$ are proved to be composite. Many other Fermat's numbers also are proved composite. The largest Fermat number proved to be composite, as of January 16, 2010, is $F_{2478782}$. No prime $F_n$s with $n > 4$ has been discovered yet; and it is not proved that all $F_n$s for $n > 4$ are composite, either.

Fermat's conjecnture was proved wrong by Euler, by factorizing $F_5$.

**2.1. Euler's method.** Euler proved that if $a$ and $b$ are relative prime, every factor of $a^{2^n}+b^{2^n}$ is 2 or in the form $2^{n+1}k+1$. So, he found that if $F_5$ has a factor, it should be in the form $64k+1$. So he tried the numbers in that form* and found that $64 \cdot 10 + 1 = 641$ divides it[**3**].

**2.2. Another method.** It is clear that

$$2^32+1 = (5^4 \cdot 2^{28} + 2^{32}) - (5^4 \cdot 2^{28} - 1)$$

Also, $641 = 5^4 + 2^4 = 5 \cdot 2^7 + 1$. It is clear that $5^4 + 2^4$ divides $(5^4 \cdot 2^{28} + 2^{32})$ while $5 \cdot 2^7 + 1$ divides† $(5^4 \cdot 2^{28} - 1)$, so 641 should divide their difference as well, which is $F_5$[**5**].

*According to [**1**], Euler's first trial was 193 and 641 was the third number he considered, after rejecting other numbers with obvious reasoning.

†Apply $a^2 - b^2 = (a+b)(a-b)$ repeatedly.

CHAPTER 32

# Maximum numbers without a given difference

## 1. Question

Given two numbers $n$ and $k$, find the maximum number of different numbers that can be chosen in the interval $1 \cdots n$ such that the difference of no two numbers is $k$.

In particular, solve this puzzle for $n = 100$ and $k = 9$.*

## 2. Solution

If we take $2k$ consecutive numbers $x+1, x+2, \cdots x+2k$, there are $k$ pairs – $(x+1, x+k+1), (x+2, x+k+2), \cdots (x+k, x+2k)$ – that differ by $k$, so the maximum number of numbers that can be chosen in this interval without a pair with the difference $k$ is $k$, by selecting one number from each of these pairs.

So, we can choose $k$ numbers from each non-overlapping group of $2k$ consecutive numbers. We have a total of

$$m = \left\lfloor \frac{n}{2k} \right\rfloor$$

such groups, giving $mk$ distinct numbers.

There will be $n - 2mk$ numbers leftover, which is less than $2k$, and we can take maximum $k$ numbers from that too. So, the final expression is

$$M = mk + \min(n - 2mk, k) \tag{32.1a}$$

where

$$m = \left\lfloor \frac{n}{2k} \right\rfloor \tag{32.1b}$$

When $n = 100$, the value for $M$ for all $k = 1, 2, \cdots 100$ are tabulated below.

*This is a generalized for of the puzzle given at `http://mathematicsschool.blogspot.com/2010/02/exams-or-experiments.html?showComment=1265648302794#comment-6788806971579542839`.

| **k** | **M** | **k** | **M** |
|---|---|---|---|
| 1 | 50 | 26 | 52 |
| 2 | 50 | 27 | 54 |
| 3 | 51 | 28 | 56 |
| 4 | 52 | 29 | 58 |
| 5 | 50 | 30 | 60 |
| 6 | 52 | 31 | 62 |
| 7 | 51 | 32 | 64 |
| 8 | 52 | 33 | 66 |
| **9** | **54** | 34 | 66 |
| 10 | 50 | 35 | 65 |
| 11 | 55 | 36 | 64 |
| 12 | 52 | 37 | 63 |
| 13 | 52 | 38 | 62 |
| 14 | 56 | 39 | 61 |
| 15 | 55 | 40 | 60 |
| 16 | 52 | 41 | 59 |
| 17 | 51 | 42 | 58 |
| 18 | 54 | 43 | 57 |
| 19 | 57 | 44 | 56 |
| 20 | 60 | 45 | 55 |
| 21 | 58 | 46 | 54 |
| 22 | 56 | 47 | 53 |
| 23 | 54 | 48 | 52 |
| 24 | 52 | 49 | 51 |
| 25 | 50 | 50 | 50 |

Obviously, when $k \geq 50$, $M = k$.

### 3. Answer

The general solution to this problem is given by equations (32.1). For $n = 100$ and $k = 9$, the maximum number of distinct numbers that can be chosen in the interval $1, 2, \cdots 100$ so that no two numbers differ by 9, is 54.

CHAPTER 33

# Ramanujan's house problem

## 1. Question

In a certain street, there are more than fifty but less than five hundred houses in a row, numbered from 1, 2, 3 etc. consecutively. There is a house in the street, the sum of all the house numbers on the left side of which is equal to the sum of all house numbers on its right side. Find the number of this house.*

In addition to this, find a general solution to this problem, and find all solutions where the total number of houses is less than 1000.

## 2. Some background

This is a very famous puzzle. This is quoted in [**6**], the best biography of Ramanujan so far.

> *This problem was first published in the English magazine 'Strand' in December 1914. A King's college student, P.C. Mahalanobis, saw this puzzle in the magazine, solved it by trial and error, and decided to test the legendary mathematician Srinivasa Ramanujan. Ramanujan was stirring vegetables in a frying pan over the kitchen fire when Mahalanobis read this problem to him. After listening to this problem, still stirring vegetables, Ramanujan asked Mahalanobis to take down the solution, and gave the general solution to the problem, not just the one with the given constraints.*

Even though this puzzle is very famous, I am yet to see† any reference where a systematic mathematic solution (not just the answer) is given. I found that the solution of this puzzle does not require the genius of a Ramanujan, but can be done by elementary number theory.

## 3. Solution

First of all, let us assume that there were $n$ houses in the street, and the number of the house in question is $m$. Given that

$$1 + 2 + ... + (m - 1) = (m + 1) + (m + 2) + ... + n \tag{33.1}$$

the problem is to find $m$ and $m$‡

Using the well-known result that the sum of the first $k$ natural numbers is $\frac{k(k+1)}{2}$, we can rewrite the equation as

$$\frac{(m-1)m}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}$$
$$n(n+1) = m(m - 1 + m + 1) = 2m^2$$
$$\frac{n(n+1)}{2} = m^2$$

So, if we get $n$,

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*A variation of this problem can be found at `http://mathematicsschool.blogspot.com/2010/02/orange.html#comment-2345379607225731971`.

†I wrote the original article giving this solution in 1996.

‡Note that $m$ is not counted in either sum. If it is counted in the second sum, the problem is surprisingly different. See Chapter 10 on page 26.

$$m = \sqrt{\frac{n(n+1)}{2}} \tag{33.2}$$

Or if we get m,

$$n = -1 + \sqrt{\frac{1+8m^2}{2}} \tag{33.3}$$

Thus, this problem can be stated in two other ways.

(1) Find a triangular number that is a perfect square.
(2) Find $n$, such that the sum of the first $n$ natural numbers is a perfect square.
   This will give $n$, from which $m$ can be found out using eq. 33.2.

**3.1. Solution by iteration.** Starting from 1, add each natural number and check whether the sum is a perfect square. If it is, the last number added is a solution giving n.

*Example:* **1**, 3, 6, 10, 15, 21, 28, **36**, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, ... and only two of them so far, 1 and 36, are perfect squares, giving $(m, n)$ as $(1, 1)$ and $(6, 8)$.

This method is too laborious.

**3.2. Solving Pell's equation.** Look at (33.2). Either $n$ or $(n+1)$ is an even number. If $n$ is even, let $n = 2k$. Then (33.2) becomes $k(2k+1) = m^2$. Now, since $k$ and $(2k+1)$ cannot have a common factor greater than 1, each of them must be a perfect square in order that their product is a perfect square. Let $(2k+1) = a^2$ and $k = b^2$. This means

$$a^2 - 2b^2 = 1 \tag{33.4}$$

Similarly, if $(n+1)$ is even and $(n+1) = 2k$, then $(2k-1)k = m^2$; $(2k-1)$ and $k$ must be squares. Let $(2k-1) = a^2$ and $k = b^2$,

$$a^2 - 2b^2 = -1 \tag{33.5}$$

Combining (33.4) and (33.5),

$$a^2 - 2b^2 = \pm 1 \tag{33.6}$$

After solving this, $m = ab$, and $n$ is given by (33.3).

(33.6) is an equation wrongly called *Pell's equation*, first solved by Brahmagupta and Bhaskara II in India. See Section 8.3 on page 81 for details and theory.

So, the problem can be re-stated as

$$\boxed{\textbf{\textit{Solve the integer equation }} a^2 - 2b^2 = \pm 1.}$$

See Section 8.3 on page 81 for a detailed theory of this type of equations.

We know that $(3, 2)$ is the fundamental solution to $x - 2y^2 = 1$. So, if we know any solution $(p, q)$, the next solution is obtained by $(x + y\sqrt{2}) = (p + q\sqrt{2})(3 + q\sqrt{2})$, which means

$$\begin{aligned} x &= (3p + 4q) \\ y &= (2p + 3q) \end{aligned} \tag{33.7}$$

So, the following method could be used:

3.2.1. *Half solution.* Starting from $(p,q)=(1,0)$, find the next solution $(a,b)=(3p+4q,2p+3q)$. This set will give half of the solutions.

$$\begin{aligned}(a,b)&=(3.1+4.0,2.1+3.0)=(3,2)\\ m&=6,n=8\\ \mathbf{(a,b)}&=\mathbf{(3.3+4.2,2.3+3.2)=(17,12)}\\ \mathbf{m}&=\mathbf{204,n=288}\\ (a,b)&=(3.17+4.12,2.17+3.12)=(99,70)\\ m&=6930,n=9800\\ (a,b)&=(3.99+4.70,2.99+3.70)=(577,408)\\ m&=235416,n=332928\end{aligned} \tag{33.8}$$

So, we got half the solutions where $n<10000$. We can continue it any further.

$x-Dy^2=-1$, where D is not a perfect square, is not always soluble. (For example, $x-2y^2=-1$ doesn't have a non-trivial solution). When it is soluble, it also has infinite number of solutions. If $(p,q)$ is the smallest positive solution (the fundamental solution) to $x-Dy^2=-1$, then all solutions are given by (38.22).

By inspection, we can find $(1,1)$ is the fundamental solution to $x-2y^2=-1$. So, if we know one solution $(p,q)$, the next solution is given by (33.7). So,

3.2.2. *The other half solution.* Starting from $(p,q)=(1,1)$, find the next solution $(a,b)=(3p+4q,2p+3q)$. This set will give the other half of the solutions.

Using (33.7),

$$\begin{aligned}(a,b)&=(3.1+4.1,2.1+3.1)=(7,5)\\ m&=35,n=49\\ (a,b)&=(3.7+4.5,2.7+3.5)=(41,29)\\ m&=1189,n=1681\\ (a,b)&=(3.41+4.29,2.41+3.29)=(239,169)\\ m&=40391,n=57121\end{aligned} \tag{33.9}$$

So, we got the other half solutions where $n<10000$.

3.2.3. *Total solution.* Combining the two halves, the final solution is

$(m,n)=(1,1)$, $(6,8)$, $(35,49)$, $(\mathbf{204},\mathbf{288})$, $(1189,1681)$, $(6930,9800)$,...

This means

$$\begin{aligned}0&=0\\ 1+2+\cdots+5&=7+8\\ 1+2+\cdots+34&=36+37+\cdots+49\\ \mathbf{1+2+\cdots+203}&=\mathbf{205+206+\cdots+288}\\ 1+2+\cdots+1188&=1190+1191+\cdots+1681\\ 1+2+\cdots+6929&=6931+6932+\cdots+9800\end{aligned} \tag{33.10}$$

3.2.4. *Combined solution.* I the combined solution, the complete solution to (33.6), when put in the ascending order, $(1,0)$, $(1,1)$, $(3,2)$, $(7,5)$, $(17,12)$, $(41,29)$, $(99,70)$, $(239,160)$, $(577,408)$,...

There is a simple relation between subsequent solutions. Do you find any relation between these values? We may observe that these values $(a,b)$ are the solutions to the equation

$$(x+y\sqrt{2})=(1+\sqrt{2})k$$

where $k=1,2,\cdots$

In other words, if $(p, q)$ is any solution to (33.6), then the next solution is given by $(p + 2q, p + q)$.

**3.3. Using Lagrange's method and continued fractions.** Using the theory given in Section 8.3.3 on page 82, the solutions can be directly computed.
Here, when $D = 2$,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Since $n = 1$, all convergents are solutions. The convergents of these continued fraction are

$$1 = \frac{1}{1}$$
$$1 + \frac{1}{2} = \frac{3}{2}$$
$$1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$$
$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12}$$

giving $(1/1), (3/2), (7/5), (17/12), (41/29), \dots$, giving the same numbers we found above. The other results also are obtained from this easily.

## 4. Answer

Between 50 and 500, there is only one solution: There are 288 houses, and the given house is the $204^{th}$.

CHAPTER 34

# The bus and stops

## 1. Question

A bus has 12 stops, numbered 1–12. The bus has place for 20 passengers. Each day the bus arrives empty at stop 1, and the last passenger leaves at stop 12 (or before that). If passengers A and B get on the bus at the same stop, they are not allowed to both leave at the same stop. What is the maximal number of passengers that can use this bus each day?

## 2. Solution

The optimal strategy is this: Get as many people into the bus, so that the number of people is the maximum (20 if possible), as early as possible; and get each one out as early as possible (so that more people can get in).

Now,

- 11 people can enter the bus at stop 1, one person can exit at each of the subsequent 11 stops.
- At stop 2, one of the 11 people will get down, so that there will be 10 people in the bus; so, 10 people can enter. These 10 people can exit at the next 10 stops, 1 at each stop.
- At stop 3, there are 20 people in the bus, but 2 will get down at the stop (one entered at stop 1 and the other, stop 2.) So, 2 more can enter. They should exit at stops 4 and 5.

This is better solved if we write down in a tabular form. In the table below, the positive number shows the number of people entered and the negative number shows the number exited.

| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| | 10 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| | | 2 | -1 | -1 | | | | | | | |
| | | | 3 | -1 | -1 | -1 | | | | | |
| | | | | 4 | -1 | -1 | -1 | -1 | | | |
| | | | | | 4 | -1 | -1 | -1 | -1 | | |
| | | | | | | 5 | -1 | -1 | -1 | -1 | -1 |
| | | | | | | | 4 | -1 | -1 | -1 | -1 |
| | | | | | | | | 3 | -1 | -1 | -1 |
| | | | | | | | | | 2 | -1 | -1 |
| | | | | | | | | | | 1 | -1 |

So, the total number of people who used the bus $= 11 + 10 + 2 + 3 + 4 + 4 + 5 + 4 + 3 + 2 + 1 = 49$.

## 3. Answer

49 people.

## 4. Extended question

What if the number of maximum passengers in the bus is $p$ and the number of stops is $s$. Can this general problem be solved and a formula for the maximum number of passengers found? There may be three cases to consider–when $p < s$, $p = s$ and $p > s$.

Want to try?

CHAPTER 35

# Puzzle of five person's weights

## 1. Question

The weight of five persons, when measured in pairs, are 129 kg, 125 kg, 124 kg, 123 kg, 122 kg, 121 kg, 120 kg, 118 kg, 116 kg and 114 kg on a weighing machine. What was the weight of each of the five persons?

## 2. Solution

Five persons when weighed in pairs can have a total of $\binom{5}{2} = 10$ different pairs. Those 10 values are given in the question. Since there is no repetition, we can conclude that no two persons have the same weight.

Let $a$, $b$, $c$, $d$, $e$ be the five weights. Adding the five pairwise weights given in the question, and observing that each weight is added with four other wights, we get

$$\begin{aligned} 4a + 4b + 4c + 4d + 4e &= 1212 \\ \therefore\ a + b + c + d + e &= 303 \end{aligned} \tag{35.1}$$

WLOG, assume $a > b > c > d > e$. Obviously, we can make out the following.

$$a + b = 129 \tag{35.2a}$$

$$a + c = 125 \tag{35.2b}$$

$$d + e = 114 \tag{35.2c}$$

$$c + e = 116 \tag{35.2d}$$

Now, (35.1) - (35.2a) gives

$$c + d + e = 174 \tag{35.3}$$

(35.3) - (35.2c) gives

$$c = 60 \tag{35.4}$$

Similarly, (35.3) - (35.2d) gives

$$d = 58 \tag{35.5}$$

(35.2b) - (35.4) gives

$$a = 65 \tag{35.6}$$

And, (35.2a) - (35.6) gives

$$b = 64 \tag{35.7}$$

Finally, (35.2d) - (35.4) gives

$$e = 56 \tag{35.8}$$

So, $a = 65$, $b = 64$, $c = 60$, $d = 58$ and $e = 56$, so that

$$\begin{aligned}
a + b &= 129 \\
a + c &= 125 \\
a + d &= 123 \\
a + e &= 121 \\
b + c &= 124 \\
b + d &= 122 \\
b + e &= 120 \\
c + d &= 118 \\
c + e &= 116 \\
d + e &= 114
\end{aligned}$$

### 3. Answer

The weights are 65, 64, 60, 58 and 56.

CHAPTER 36

# Four people and a ditch

## 1. Question

Four people are digging a ditch of some pre-specified size, one after another, and finished a ditch. These four might have different speed in their work. Each of them might have worked for a different time and finished some portion of the work.

It is observed that each of them dug for such time that, during that time the other three, working together, could have finished half the ditch. This is true for each of the workers.

Question: If they worked together, instead of working one after another, how faster they would have finished the ditch?

## 2. Solution

Let $A$, $B$, $C$ and $D$ be the four people; $t_A$, $t_B$, $t_C$ and $t_D$ be the times they digged; and $s_A$, $s_B$, $s_C$ and $s_D$ be the speed (amout digged in unit time) of digging. Let $V$ ve the volume of the ditch.

The question can be transformed into:

$$s_A \cdot t_A + s_B \cdot t_B + s_C \cdot t_C + s_D \cdot t_D = V \tag{36.1a}$$

$$(s_B + s_C + s_D)t_A = \frac{V}{2} \tag{36.1b}$$

$$(s_A + s_C + s_D)t_B = \frac{V}{2} \tag{36.1c}$$

$$(s_A + s_B + s_D)t_C = \frac{V}{2} \tag{36.1d}$$

$$(s_A + s_B + s_C)t_D = \frac{V}{2} \tag{36.1e}$$

We need to find the value of

$$\frac{(s_A + s_B + s_C + s_D)(t_A + t_B + t_C + t_D)}{V} \tag{36.2}$$

Adding (36.1a), (36.1b), (36.1c), (36.1d) and (36.1e) and simplifying,

$$(s_A + s_B + s_C + s_D)(t_A + t_B + t_C + t_D) = 3V \tag{36.3}$$

So, the value of (36.2) is 3.

So, working together is 3 times faster than working sequantially.

## 3. Simple explanation

This question was asked for the Mathematical Olympiad competion for second grade kids. Obviously, this can be solved without using algebra at all.

The question states that while one guy works, the other three together can dig half the ditch during the same time. So, in addition to the first ditch, if the dig another dig another ditch, with the assigned person digging the first, while the other three working on the other, we know that they will dig half the ditch four times, which means they will dig $4 \times \frac{1}{2} = 2$ times the volume of the ditch.

This means they will dig three times the vaolume of the original ditch. So, they will be three times faster working together.

### 4. Answer

Working together will be **three times** faster than working one after another.

CHAPTER 37

# Horses and total grazing area

## 1. Question

Consider a triangular field ABC of sides $AB = 100m$, $BC = 120m$, $AC = 125m$. To a pole at each vertex A,B,C, a horse is tethered, with a rope of length 10m. Find the total area available to the three horses to graze.

## 2. Solution

Let $\alpha$, $\beta$ and $\gamma$ be the three angles in radians. We need not find these angles. The first horse can graze $\frac{10 \cdot 10 \cdot \alpha}{2} = 50\alpha$. Similar case with the other horses.
Since these three areas do not overlap, the total area is the sum of these, which is

$$50\alpha + 50\beta + 50\gamma = 50(\alpha + \beta + \gamma) = 50\pi,$$

since $\alpha + \beta + \gamma = \pi$, as they are the angles of a triangle.

## 3. The hard way

While a simple solution like the above exists, this can be solved in the hard way also, using trigonometry.
Using the cosine formula (Equation (38.17) on page 78),

$$\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{125^2 + 100^2 - 120^2}{2 \cdot 125 \cdot 100} \\
&= \frac{11225}{25000} \\
&= 0.449 \\
\therefore A &= 63.32^o
\end{aligned}$$

$$\begin{aligned}
\cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\
&= \frac{120^2 + 100^2 - 125^2}{2 \cdot 120 \cdot 100} \\
&= \frac{8775}{24000} \\
&= 0.365625 \\
\therefore B &= 68.55^o
\end{aligned}$$

$$
\begin{aligned}
\cos C &= \frac{a^2+b^2-c^2}{2ab} \\
&= \frac{120^2+125^2-100^2}{2 \cdot 120 \cdot 125} \\
&= \frac{20025}{30000} \\
&= 0.6675 \\
\therefore C &= 48.13^o
\end{aligned}
$$

$$
\begin{aligned}
\text{Total area} &= \frac{1}{2} \cdot 10^2 \cdot \frac{\pi}{180}(63.32 + 68.55 + 48.13) \\
&= \frac{1}{2} \cdot 10^2 \cdot \frac{\pi}{180} \cdot 180.00 \\
&= 50\pi
\end{aligned}
$$

### 4. Answer

The total area the horses can graze is $50\pi = 157.08$ sqm.

CHAPTER 38

# Theory

## 1. Solving equations of one variable

**1.1. Quadratic equation.** The solution to the quadratic equation

$$ax^2 + bx + c = 0 \tag{38.1}$$

is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{38.2}$$

Sometimes factorizing my be faster than using the formula above. For factorizing,

(1) Find two numbers with sum $b$ and product $ac$. Let them be $p$ and $q$
(2) Write the above equation as
$$ax^2 + px + qx + c = 0$$
(3) Group two terms together.
(4) There will be a common factor among the two groups. Express them in that terms.
(5) Now represent the left part as the product of two terms. Equating each to zero will give the two solutions.

Example: Solve $5x^2 - 12x + 4 = 0$.

$$5x^2 - 10x - 2x + 4 = 0 \tag{38.3}$$
$$5x(x-2) - 2(x-2) = 0 \tag{38.4}$$
$$(5x-2)(x-2) = 0x = \frac{2}{5} \text{ and } x = 2 \tag{38.5}$$

We can separate in the other way as well.

$$5x^2 - 2x - 10x + 4 = 0 \tag{38.6}$$
$$x(5x-2) - 2(5x-2) = 0 \tag{38.7}$$
$$(x-2)(5x-2) = 0 \tag{38.8}$$
$$x = 2 \text{ and } \frac{2}{5} \tag{38.9}$$

## 2. Simultaneous linear equations

**2.1. Cramer's rule for two simultaneous equations.** The pair of simultaneous linear equations in two variables

$$a_1x + b_1y = c_1 \tag{38.10a}$$
$$a_2x + b_2y = c_2 \tag{38.10b}$$

is solved by

$$\frac{x}{b_2c_1 - b_1c_2} = \frac{y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \tag{38.11}$$

so that

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \tag{38.12a}$$
$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \tag{38.12b}$$
$$\tag{38.12c}$$

## 3. Lines

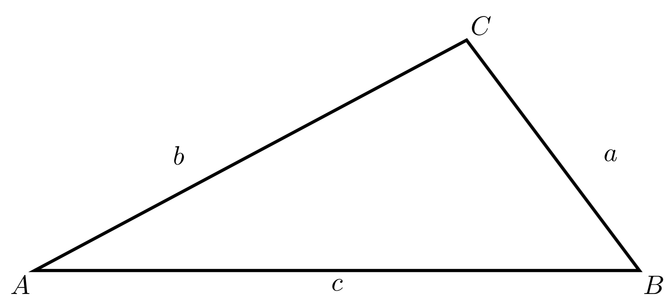
**3.1. General formula of a line passing through two points.** If a point $(x, y)$ lies on a line joining $(x_1, y_1)$ and $(x_2, y_2)$, then

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \tag{38.13}$$

This forumla* can be used to find the general formula for a line (keeping $x$ and $y$ as variables) or checking/solving for a given third point.

## 4. Triangles

Consider a triangle $\triangle ABC$ with $BC = a, AC = b, AB = b$.




**4.1. Area.** Area of the triangle is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{38.14}$$

where $s = \frac{a+b+c}{2}$.†

Also,

*It is just making use of the fact that the slope of the line joining any two points in a line is the same.

†$s$ is the radius of the inscribed circle as well. This formula is a particular case of the area of a cyclic quadrilateral $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{a+b+c+d}{2}$, with $d = 0$.

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B \tag{38.15}$$

### 4.2. The Sine Formula.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{38.16}$$

### 4.3. The Cosine Formula.

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{38.17a}$$
$$b^2 = c^2 + a^2 - 2ac\cos B \tag{38.17b}$$
$$c^2 = a^2 + b^2 - 2ab\cos C \tag{38.17c}$$
$$\tag{38.17d}$$

## 5. Construction with straight edge and compass

There are set of geometrical constructions that are known with the name *primary construction.* They can be done using a stright edge and a compass.

A *straight edge* is a straight, unmarked, infinitely long ruler with only one edge. This means

(1) You can draw a straight line using it.
(2) You can draw a straight line passing through two given points.
(3) You *cannot* use it measure lengths (no markings on it) or draw another line segment with the same length as a given one.
(4) You *cannot* use it to draw a line segment of some fixed length (it is infinitely long).
(5) You *cannot* use it to draw two parallel lines (it has only one edge).
(6) You *cannot* use it to draw a perpendicular (or any fixed angle) line to a given line. (it has no side edge.)

A *compass* is a devise with two infinitely long hands joined on top, and a pivot and a pencil at the edges of the hands. The pivot can be pivoted at a point, and the compass can freely turn around that point while it is pivoted. When it is pivoted, the other end (with the pencil) can move so that the distance between the pivot and the pencil is any arbitary length. When the pivot is raised from the paper, the compass *collapses*, so that the distance between the pivot and the pencil is no longer reproduced.

A compass can be used to

(1) draw arcs and circles with arbitary center and radius.
(2) pivot it at one point, adjust the pencil to another point, and draw arcs with that center and radius.

A compass cannot be used to

(1) transfer a fixed length from one place to another (the compass collapses when unpivoted.) However, if one endpoint is common, this is possible by pivoting it on that point, adjusting the pencil to the other point, and drawing an arc that pass in some other direction.
(2) draw arcs of a given length, angle or radius.

There are anumber of constructions possible with these two divises. Euclid gives many such constructions. Some of such constructions are given below.

### 5.1. Draw a perpendicular bisector of a given line segment.

(1) Let the line segment be $\overline{AB}$.
(2) Draw a circle $C_1$ with $A$ as center and $AB$ as radius.
(3) Draw a circle $C_2$ with $B$ as center and $BA$ as radius.

(4) Circles $C_1$ and $C_2$ meet at two points. Join these two points. This line is the perpendicular bisector of $\overline{AB}$.

**5.2. Draw a line perpendicular to a given line.**

(1) Mark any two points $A$ and $B$ on the line.
(2) Draw a perpendicular bisector of $\overline{AB}$ as given in Section 5.1 on the facing page.
(3) The line now constructed is perpendicular to the original line.

**5.3. Draw a line perpendicular to a given line passing through a given point on the line.**

(1) Let the point be $A$.
(2) Draw a circle with $A$ as center and any length as radius. It meets the line at two points. Let them be $B$ and $C$.
(3) Draw the perpendicular bisector of $\overline{BC}$ as given in Section 5.1 on the preceding page. This is the perpendicular line through $A$.

**5.4. Draw a line perpendicular to a given line passing through a given point not on the line.**

(1) With the point as center, draw a circle with radius big enough to cut the line on two points. Let those points be $A$ and $B$.
(2) Draw the perpendicular bisector of $\overline{AB}$ as given in Section 5.1 on the facing page. This is the required line.

**5.5. Draw a line parallel to another line and passing through a given point.**

(1) Let the line be $\overline{AB}$ and the point be $P$.
(2) Draw a line perpendicular to $\overline{AB}$ and passing through $P$, as given in Section 5.4.
(3) Draw a line perpendicular to the line constructed in the previous step and passing through $P$, as given in Section 5.3. This is the required line.

**5.6. Draw a circle with the same radius as a given circle centered at a given point.**

(1) Let there be a circle $C_1$ centered at $A$, and $B$ be any arbitrary point. We need to draw a circle with center as $B$ and radius same as $C_1$.
(2) Draw circle $C_2$ with center as $A$ and radius as $AB$.
(3) Draw circle $C_3$ with center as $B$ and radius as $BA$.
(4) Let $C_2$ and $C_3$ meet at $C$.[‡]
(5) Let $C_1$ and $C_3$ meet at $D$.[§]
(6) With $C$ as center and $CD$ as radius, draw circle $C_4$.
(7) Let $C_4$ and $C_2$ meet at $E$.
(8) With $B$ as center and $BE$ as radius, draw circle $C_5$.
(9) $C_5$ is the required circle.

## 6. Prime factorization

**6.1. Number of factors.** If the prime factorization a positive integer $n$ is

$$n = p_1^{k_1} p_2^{k_2} \cdots p_q^{k_q}$$

then the number of factors $n$ has is given by

$$m = (k_1 - 1)(k_2 - 1) \cdots (k_q - 1)$$

[‡]They meet at two points, but we need to take only one.
[§]They meet at two points. Take the one at the same side of $C$ with respect to $\overline{AB}$.

## 7. Continued fractions

A good reference is in wikipedia[**9**].

**7.1. Finding the next convergent.** If the $(n-2)^{th}$ convergent is $\frac{a_{n-2}}{b_{n-2}}$, the $(n-1)^{th}$ convergent is $\frac{a_{n-1}}{b_{n-1}}$, and the $n^{th}$ term is $p_n$, then, the $n^{th}$ convergent $\frac{a_n}{b_n}$ is given by

$$a_n = p_n a_{n-1} + a_{n-2} \tag{38.18a}$$
$$b_n = p_n b_{n-1} + b_{n-2} \tag{38.18b}$$

For example, let us evaluate

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}$$

or

$$\pi = [3; 7, 15, 1, 292, 1, 1, ...]$$

$$\begin{aligned}
&\frac{a_0}{b_0} = \frac{3}{1} = 3 \\
&\frac{a_1}{b_1} = \frac{22}{7} = 3.142857\cdots \\
&\frac{a_2}{b_2} = \frac{15 \cdot 22 + 3}{15 \cdot 7 + 1} = \frac{\mathbf{333}}{\mathbf{106}} = 3.141509433962264\cdots \\
&\frac{a_3}{b_3} = \frac{1 \cdot 333 + 22}{1 \cdot 106 + 7} = \frac{\mathbf{355}}{\mathbf{113}} = 3.141592920353982\cdots \\
&\frac{a_4}{b_4} = \frac{292 \cdot 355 + 333}{292 \cdot 113 + 106} = \frac{\mathbf{103993}}{\mathbf{33102}} = 3.14159292653011903\cdots \\
&\frac{a_5}{b_5} = \frac{1 \cdot 103993 + 355}{1 \cdot 33102 + 113} = \frac{\mathbf{104348}}{\mathbf{33215}} = 3.141592653921421\cdots \\
&\frac{a_6}{b_6} = \frac{1 \cdot 104348 + 102993}{1 \cdot 33215 + 33102} = \frac{\mathbf{208341}}{\mathbf{66317}} = 3.141592653467437\cdots \\
&\cdots
\end{aligned}$$

## 8. Solving integer equations

**8.1. Solving $ax - by = 1$.** One method is (another method in the next section) to express $\frac{a}{b}$ as a continued fraction that has an even number of terms including the integer part[¶] and finding the successive convergents. Get $\frac{p}{q}$, the convergent just before $\frac{a}{b}$. Now, $qx - py = 1$.

[¶]If it has an odd number of terms, it is easy to make it even. $[\cdots p] = [\cdots p-1, 1]$. For example, $[2; 1, 2, 3, 4] = [2; 1, 2, 3, 3, 1]$.

**Example:** Solve $3x - 4y = 1$.

$\frac{3}{4} = [0; 1, 3] = [0; 1, 2, 1]$, with convergents $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{3}{4}$. Previous convergent is $\frac{2}{3}$, so $(3, 2)$ is a solution to $3x - 4y = 1$.

### 8.2. Solving $ax - by = c$.

8.2.1. *Using continued fractions.* Solve $ax - by = 1$ as explained in the previous section. Let $(p, q)$ be a solution. Now, $(cp, cq)$ is a solution of $ax - by = c$. The general solution is given by

$$x = cp + kb \tag{38.19a}$$

$$y = cq + ka \tag{38.19b}$$

By putting some suitable value for $k$, this general form can be simplified.

**Example:** Solve $3x - 4y = 5$.

Since $(3, 2)$ is a solution to $3x - 4y = 1$, $(15, 10)$ is a solution to $3x - 4y = 5$. The general solution is $(15 + 4k, 10 + 3k)$. Putting $k = -3$ and reducing, we can write $(3 + 4p, 1 + 3p)$ as another general and simpler solution.

8.2.2. *Using congruences.* If one of $a$ or $b$ is small, this can be solved using congruents. For illustration, let us solve $3x - 4y = 5$.

Write it as

$$\begin{aligned} 4y &\equiv -5 \pmod 3 \\ 4y &\equiv 1 \pmod 3 \end{aligned} \tag{38.20}$$

(38.20) will have one (and only one solution) solution in $y = 0, 1, 2$. It is easy to check one by one.

$$\begin{aligned} 4 \cdot 0 &\equiv 0 \pmod 3 \\ 4 \cdot 1 &\equiv 1 \pmod 3 \\ 4 \cdot 2 &\equiv 2 \pmod 3 \end{aligned}$$

So, $y = 1$ gives a solution. Solving, we get $x = 3$.

We can use it to solve problems like $2189x - 4y = 15$, because we need to consider only four values, but not good for $9999x - 1250y = 13$. Need to use the continued fraction method in such cases.

8.2.3. *General solution.* If $x = x_0, y = y_0$ is a solution to the integer equation $ax - by = c$, then the general solution is given by

$$x = x_0 + kb \tag{38.21a}$$

$$y = y_0 + ka \tag{38.21b}$$

where $k$ is any integer. This gives all solutions.

### 8.3. Solving $x^2 - Dy^2 \pm 1$.

8.3.1. *History.* This is *Brahmagupta-Bhaskara* equation, wrongly known as *Pell's equation* (In fact, Pell[‖] did nothing for this equation, as we see later) in the western world. The equation $x^2 - Dy^2 = 1$ was considered as a challenging problem since the beginning of arithmetic. Archimedes' famous cattle problem reduces to $x^2 - 4729494y^2 = \pm 1$. This type of equations was first solved by Brahmagupta[**] using his *Chakravala* method. Later, Bhaskara[††] simplified this method.

[‖] John Pell, 1611-1685

[**] Brahmagupta, AD 7th century

[††] Bhaskara II, AD 12th century

In the Western world, mathematicians like Fermat and Frenicle devised many such equations as a challenge to other mathematicians.(Brahmagupta states "A person who can solve the equation $x^2 - 92y^2 = 1$ within a year, is a mathematician"), and it was Wallis and Brouncker who solved it with all generality. In 1768, Lagrange gave the first complete proof of the solvability of $x^2 - Dy^2 = 1$, based on continued fractions. Euler4Euler, though discussed this equation in his famous book 'Algebra', did nothing to its theory other than giving its credit wrongly to Pell, thinking that Wallis' proof was Pell's.

8.3.2. *General properties.* We will ignore the trivial solution $(\pm 1, 0)$ to this equation.

$x^2 - Dy^2 = 1$, where D is not a perfect square, is always soluble and has infinite number of solutions. If $(p, q)$ is the smallest positive solution (called the fundamental solution), then all solutions are given by the equation

$$x + y \times \sqrt{D} = (p + q \times \sqrt{D})k \tag{38.22}$$

where , $k = 1, 2, 3, \ldots$.

If we get one solution, the general solution can be found using the equation above. In most cases, this is the simplest method.

8.3.3. *Lagrange's method using continued fractions.* In order to solve

$$x^2 - Dy^2 = 1 \tag{38.23}$$

express $\sqrt{D}$ as a periodic infinite continued fraction. Let $n$ be the periodic length of the expansion. Let $(p/q)$ be the $(n-1)^{th}$ convergent in any period. Then

$$p^2 - Dq^2 = (-1)^n$$

. This condition, giving one solution per period, will give all the solutions.

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