Electromagnetic field theory for physicists and engineers:Fundamentals and Applications

R. Gómez Martín

Contents

0.1 Prefacio

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Part I

Electromagnetic field: radiation and propagation

Chapter 1

Electromagnetic field fundamentals

1.1 Introduction

This chapter starts with a brief review of Maxwell's equations, which are the fundamental laws that, together with the theory of electromagnetic behavior of matter, explain on a macroscopic scale the properties of the electromagnetic field, the relationships of this field with its sources, and its interaction with matter. The reader is assumed to be familiar with these equations at least at an undergraduate level. Next, after reviewing other fundamental topics such as constitutive parameters and boundary conditions, we apply the energy-conservation law to a bounded volume, limited by a surface S, inside of which there exists a time-variable electromagnetic field. We shall see that when the energy balance is formulated, there appears a term representing a flow of energy carried by the electromagnetic field through the surface S that limits V. This term leads us to the definition of Poynting's vector. Similarly, when the law of conservation of momentum is applied to the same region, we find that the electromagnetic field also carries a momentum density, which can also be expressed in terms of Poynting's vector.

1.2 Review of Maxwell's equations

The general theory of electromagnetic phenomena is based on Maxwell's equations, which constitute a set of four coupled first-order vector partial-differential equations relating the space and time changes of electric and magnetic fields to their scalar source densities (divergence) and vector source densities (curl) [1].

[1] According to the Helmholtz theorem a vector field $\vec{K}$ is uniquely determined by its divergence and curl if they are given throughout the entire space and if they approach zero at infinity at least as $1/r^n$ with $n > 1$. A proof of this theorem is given in Appendix **??**

Maxwell's equations are usually formulated in differential form (i.e., as relationships between quantities at the same point in space and at the same instant in time) or in integral form where, at a given instant, the relations of the fields with their sources are considered over an extensive region of space. The two formulations are related by the divergence (**??**) and Stokes' (**??**) theorems.

For stationary media[2], Maxwell's equations in differential and integral forms are:

Differential form of Maxwell's equations

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \text{ (Gauss' law)} \tag{1.1a}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \text{ (Gauss' law for magnetic fields)} \tag{1.1b}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \text{ (Faraday's law)} \tag{1.1c}$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t} \text{ (Generalized Ampère's law)} \tag{1.1d}$$

Integral form of Maxwell's equations

$$\oint_S \vec{D}(\vec{r},t) \cdot d\vec{s} = Q_T(t) \text{ (Gauss' law)} \tag{1.2a}$$

$$\oint_S \vec{B}(\vec{r},t) \cdot d\vec{s} = 0 \text{ (Gauss' law for magnetic fields)} \tag{1.2b}$$

$$\oint_\Gamma \vec{E}(\vec{r},t) \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \cdot d\vec{s} \text{ (Faraday's law)} \tag{1.2c}$$

$$\oint_\Gamma \vec{H}(\vec{r},t) \cdot d\vec{l} = \int_S (\vec{J}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}) \cdot d\vec{s} \text{ (Generalized Ampère's law)} \tag{1.2d}$$

Maxwell's equations, involve only macroscopic electromagnetic fields and, explicitly, only macroscopic densities of free-charge, $\rho(\vec{r},t)$, which are free to move within the medium, giving rise to the free-current densities, $\vec{J}(\vec{r},t)$. The effect of the macroscopic charges and current densities bound to the medium's molecules is implicitly included in the auxiliary magnitudes $\vec{D}$ and $\vec{H}$ which are related to the electric and magnetic fields, $\vec{E}$ and $\vec{B}$ by the so-called constitutive equations that describe the behavior of the medium (see Subsection 1.2.2). In general, the quantities in these equations are arbitrary functions of the position ($\vec{r}$) and time[3] (t). The definitions and units of these quantities are

$\vec{E}$ = electric field intensity (volts/meter; $V\ m^{-1}$)

[2] In a stationary medium all quantities are evaluated in a reference frame in which the observer and all the surfaces and volumes are assumed to be at rest. Maxwell's equations for moving media can be considered in terms of the special theory of relativity, as shown in chapter **??**.

[3] Throughout the book, in most cases, in order to make the notation more concise, we will not explicitly indicate the arguments, $(\vec{r},t)$, of the magnitudes unless we consider it convenient to emphasize the dependence on any of the variables.

$\vec{B}$ = magnetic flux density (teslas[4] or webers/square meter; T or $Wb\ m^{-2}$)
$\vec{D}$ = electric flux density (coulombs/square meter; $C\ m^{-2}$)
$\vec{H}$ = magnetic field intensity (amperes/meter; $A\ m^{-1}$)
ρ = free electric charge density (coulombs/ cubic meter; $C\ m^{-3}$)
Q_T = net free charge, in coulombs (C), inside any closed surface S
$\vec{J}$= free electric current density (amperes/square meter $A\ m^{-2}$).

Three of Maxwell's equations (1.1a), (1.1c), (1.1d), or their alternative integral formulations (1.2a), (1.2c), (1.2d), are normally known by the names of the scientists who deduced them. For its similarity with (1.1a), equation (1.1b) is usually termed the Gauss' law for magnetic fields, for which the integral formulation is given by (1.2b). These four equations as a whole are associated with the name of Maxwell because he was responsible for completing them, adding to Ampère's original equation, $\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t)$, the displacement current density term or, in short, the displacement current, $\partial\vec{D}/\partial t$, as an additional vector source for the field $\vec{H}$. This term has the same dimensions as the free current density but its nature is different because no *free* charge movement is involved. Its inclusion in Maxwell's equations is fundamental to predict the existence of electromagnetic waves which can propagate through empty space at the constant velocity of light c. The concept of displacement current is also fundamental to deduce from (1.1d) the principle of charge conservation by means of the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{1.3}$$

or, in integral form,

$$\oint \vec{J}.d\vec{s} = -\frac{dQ_T}{dt} \tag{1.4}$$

With his equations, Maxwell validated the concept of "field" previously introduced by Faraday to explain the remote interactions of charges and currents, and showed not only that the electric and magnetic fields are interrelated but also that they are in fact two aspects of a single concept, the electromagnetic field.

The link between electromagnetism and mechanics is given by the empirical Lorenz force equation, which gives the electromagnetic force density, $\vec{f}$ (in $N\ m^{-3}$), acting on a volume charge density ρ moving at a velocity $\vec{u}$ (in $m\ s^{-1}$) in a region where an electromagnetic field exists,

$$\vec{f} = \rho(\vec{E} + \vec{u} \times \vec{B}) = \rho\vec{E} + \vec{J} \times \vec{B} \tag{1.5}$$

where $\vec{J} = \rho\vec{u}$ is the current density in terms of the mean drift velocity of the particles[5], which is independent of any random velocity due to collisions. The

[4] Given that the tesla is an excessivelly high magnitude to express the values of the magnetic field usually found in practice, the cgs unit (gauss, G) is often used instead, $1T = 10^4G$.

[5] In general, when there is more than one type of particle the current density its defined as $\vec{J} = \sum_i \rho_i\vec{u}_i$ where ρ_i and $\vec{u}_i$ represent the volume charge density and drift velocity of the charges of class i.

total force $\vec{F}$ exerted on a volume of charge is calculated by integrating $\vec{f}$ in this volume. For a single particle with charge q the Lorentz force is

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) \tag{1.6}$$

Maxwell's equations together with Lorenz's force constitute the basic mathematical formulation of the physical laws that at a macroscopic level explain and predict all the electromagnetic phenomena which basically comprise the remote interaction of charges and currents taking place via the electric and/or magnetic fields that they produce. From Eq. (1.6) the work done by an electromagnetic field acting on a volume charge density ρ inside a volume dv during a time interval dt is

$$dW = \vec{f} \cdot \vec{u}dtdv = \rho(\vec{E} + \vec{u} \times \vec{B}) \cdot \vec{u}dtdv = \rho\vec{E} \cdot \vec{u}dtdv = \vec{E} \cdot \vec{J}dtdv \tag{1.7}$$

This work is transformed into heat. The corresponding power density P_v (Wm^{-3}) that the electromagnetic field supplies to the charge distribution is

$$P_v = \frac{dP}{dv} = \frac{dW}{dtdv} = \vec{E} \cdot \vec{J} \tag{1.8}$$

This equation is known as the point form of Joule's law.

In applications, Maxwell's equations have to be complemented by appropriate initial and boundary conditions. The initial conditions involve values or derivatives of the fields at $t = 0$, while the boundary conditions involve the values or derivatives of the fields on the boundary of the spatial region of interest. Usually, we consider the initial conditions as a form of boundary conditions and refer to the solution of Maxwell´s equations, with all these conditions, as a boundary-value problem.

Next, we briefly describe the physical meaning of Maxwell's equations.

1.2.1 Physical meaning of Maxwell's equations

Gauss' law, (1.1a) or (1.2a), is a direct mathematical consequence of Coulomb's law, which states that the interaction force between electric charges depends on the distance, r, between them, as r^{-2}. According to Gauss' law, the divergence of the vector field $\vec{D}$ is the volume density of free electric charges which are sources or sinks of the field $\vec{D}$, i.e. the lines of $\vec{D}$ begin on positive charges ($\rho > 0$) and end on negative charges ($\rho < 0$). In its integral form, Gauss' law relates the flux of the vector $\vec{D}$ through a closed surface S (which can be imaginary; Fig. 1.1), to the total free charge within that surface.

Gauss' law for magnetic fields, (1.1b) or (1.2b), states that the $\vec{B}$ field does not have scalar sources, i.e., it is divergenceless or solenoidal. This is because no free magnetic charges or monopoles have been found in nature (see Section 2.5) which would be the magnetic analogues of electric charges for $\vec{E}$. Hence, there are no sources or sinks where the field lines of $\vec{B}$ start or finish, i.e., the field lines of $\vec{B}$ are closed. In its integral form, this indicates that the flux of the $\vec{B}$ field through any closed surface S is null.

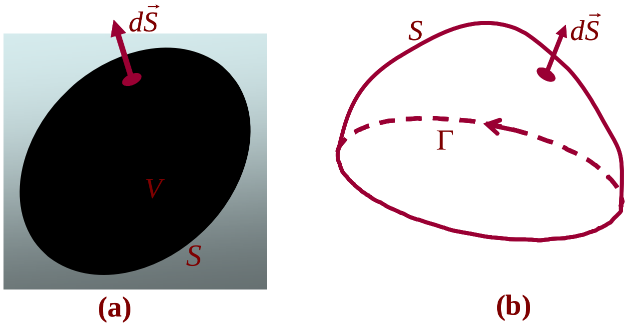

Figure 1.1: (a) Closed surface S bounding a volume V. (b) Open surface S bounded by the closed loop Γ. The direction of the surface element $d\vec{S}$ is given by the right-hand rule: the thumb of the right hand is pointed in the direction of $d\vec{S}$ and the fingertips give the sense of the line integral over the contour Γ. ¡¡ ¡Atención: las $d\vec{S}$ debe ser $d\vec{s}$

Faraday's law, (1.1c) or (1.2c), establishes that a time-varying $\vec{B}$ field produces a nonconservative electric field whose field lines are closed. In its integral form, Faraday's law states that the time variation of the magnetic flux ($\int \vec{B} \cdot d\vec{s}$) through any surface S bounded by an arbitrary closed loop Γ, (Fig. 1.1), induces an electromotive force given by the integral of the tangential component of the induced electric field around Γ. The line integration over the contour Γ must be consistent with the direction of the surface vector $d\vec{s}$ according to the right-hand rule. The minus sign in (1.1c) and (1.2c) represents the feature by which the induced electric field, when it acts on charges, would produce an induced current that opposes the change in the magnetic flux (Lenz's law).

Ampère's generalized law, (1.1d) or (1.2d), constitutes another connection, different from Faraday's law, between $\vec{E}$ and $\vec{B}$. It states that the vector sources of the magnetic field may be free currents, $\vec{J}$, and/or displacement currents, $\partial\vec{D}/\partial t$. Thus, the displacement current performs, as a vector source of $\vec{H}$, a similar role to that played by $\partial\vec{B}/\partial t$ as a source of $\vec{E}$. In its integral form the left-hand side of the generalized Ampere's law equation represents the integral of the magnetic field tangential component along an arbitrary closed loop Γ and the right-hand side is the sum of the flux, through any surface S bounded by a closed loop Γ (Fig. 1.1), of both currents: the free current $\vec{J}$ and the displacement current $\partial\vec{D}/\partial t$.

1.2.2 Constitutive equations

Maxwell's equations (1.1) can be written without using the artificial fields $\vec{D}$ and $\vec{H}$, as

$$\nabla \cdot \vec{E}(\vec{r},t) = \frac{\rho_{all}}{\varepsilon_0}(\vec{r},t) \tag{1.9a}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{1.9b}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \tag{1.9c}$$

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}_{all}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t} \tag{1.9d}$$

where $\varepsilon_0 = 10^{-9}/(36\pi)$ (farad/meter; $F\ m^{-1}$) and $\mu_0 = 4\pi 10^{-7}$ (henry/meter; $H\ m^{-1}$) are two constants called electric permittivity and magnetic permeability of free space, respectively. The subscript *all* indicates that all kinds of charges (free and bound) must be individually included in ρ and $\vec{J}$. These equations are, within the limits of classical electromagnetic theory, absolutely general. Nevertheless, in order to make it possible to study the interaction between an electromagnetic field and a medium and to take into account the discrete nature of matter, it is absolutely necessary to develop macroscopic models to extend equations (**??**) and (**??**) and to obtain Maxwell's macroscopic equations (1.1), in which only macroscopic quantities are used and in which only the densities of free charges and currents explicitly appear as sources of the fields. To this end, the atomic and molecular physical properties, which fluctuate greatly over atomic distances, are averaged over microscopically large-volume elements, Δv, so that these contain a large number of molecules but at the same time are macroscopically small enough to represent accurate spatial dependence at a macroscopic scale. As a result of this average, the properties of matter related to atomic and molecular charges and currents are described by the macroscopic parameters, electric permittivity ε, magnetic permeability μ, and electrical conductivity σ. These parameters, called constitutive parameters, are in general smoothed point functions. The derivation of the constitutive parameters of a medium from its microscopic properties is, in general, an involved process that may require complex models of molecules as well as quantum and statistical theory to describe their collective behavior. Fortunately, in most of the practical situations, it is possible to achieve good results using simplified microscopic models. Appendices **??** and **??** present a brief introduction to the microscopic theory of electric and magnetic media, respectively.

To define the electric permittivity and describe the behaviour of the electric field in the presence of matter, we must introduce a new macroscopic field quantity, $\vec{P}$ ($C\ m^{-2}$), called electric polarization vector, such that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{1.10}$$

and defined as the average dipole moment per unit volume

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{n=1}^{N\Delta v} \vec{p}_n}{\Delta v} \tag{1.11}$$

where N is the number of molecules per unit volume and the numerator is the vector sum of the individual dipolar moments, $\vec{p}_n$, of atoms and molecules contained in a macroscopically infinitessimal volume Δv. For many materials, called linear isotropic media, $\vec{P}$ can be considered colinear and proportional to the electric field applied. Thus we have

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{1.12}$$

where the dimensionless parameter χ_e, called the electric susceptibility of the medium, describes the capability of a dielectric to be polarized. Expression (1.10) can be written in a more compact form as

$$\vec{D} = (1 + \chi_e)\varepsilon_0 \vec{E} \tag{1.13}$$

so that

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \tag{1.14}$$

where

$$\varepsilon_r = 1 + \chi_e \tag{1.15}$$

and

$$\varepsilon = \varepsilon_0 \varepsilon_r \tag{1.16}$$

are the relative permittivity and the permittivity of the medium, respectively.

To define the magnetic permeability and describe the behaviour of the magnetic field in the presence of magnetic materials, we must introduce another new macroscopic field quantity, called magnetization vector $\vec{M}$ $(A\ m^{-1})$, such that

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \tag{1.17}$$

where $\vec{M}$ is defined, in a similar way to that of the electric polarization vector, as the average magnetic dipole moment per unit volume

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{n=1}^{N\Delta v} \vec{m}_n}{\Delta v} \tag{1.18}$$

where N is the number of atomic current elements per unit volume and the numerator is the vector sum of the individual magnetic moments, $\vec{m}_n$ contained in a macroscopically infinitessimal volume Δv.

In general, $\vec{M}$ is a function of the history of $\vec{B}$ or $\vec{H}$, which is expressed by the hysteresis curve. Nevertheless, many magnetic media can be considered isotropic and linear, such that

$$\vec{M} = \chi_m \vec{H} \tag{1.19}$$

where χ_m is the adimensional magnetic susceptibility magnitude, being negative and small for diamagnets, positive and small for paramagnets, and positive and large for ferromagnets. Thus

$$\vec{B} = (1+\chi_m)\mu_0\vec{H} = \mu_r\mu_0\vec{H} = \mu\vec{H} \tag{1.20}$$

where

$$\mu_r = (1+\chi_m) \tag{1.21}$$

and

$$\mu = \mu_r\mu_0 \tag{1.22}$$

are the relative magnetic permeability and the permeability of the medium, respectively, which can reach very high values in magnetic materials such as iron and nickel.

The concept of μ_r requires a careful definition when working with magnetic materials with strong hysteresis, such as ferromagnetic media. The phenomenon of hysteresis may also occur in certain dielectric materials called ferroelectric (see Appendix **??**).

In a vacuum, or free space, $\varepsilon_r = 1$; $\mu_r = 1$, and therefore the fields vectors $\vec{D}$ and $\vec{E}$, as well as $\vec{B}$ and $\vec{H}$, are related by

$$\vec{D} = \varepsilon_0\vec{E} \tag{1.23a}$$
$$\vec{B} = \mu_0\vec{H} \tag{1.23b}$$

Very often the relation between an electric field and the conduction current density $\vec{J}_c$ that it generates is given, at any point of the conducting material, by the phenomenological relation, called Ohm's law

$$\vec{J}_c = \sigma\vec{E} \tag{1.24}$$

so that $\vec{J}$ is linearly related to $\vec{E}$ trough the proportionality factor σ called the conductivity of the medium. Conductivity is measured in siemens per meter $(S\ m^{-1} \equiv \Omega^{-1}m^{-1})$ or mhos per meter $(mho\ m^{-1})$. Media in which (1.24) is valid are called ohmic media. A typical example of ohmic media are metals where (1.24) holds in a wide range of circumstances. However, in other materials, such as semiconductors, (1.24) it may not be applicable. For most metals σ is a scalar with a magnitude that depends on the temperature and that, at room temperature, has a very high value of the order of $10^7 mho$ m^{-1}.Then very often metals are considered as perfect conductors with an infinite conductivity.

The relations between macroscopic quantities, (1.13), (1.20) and (1.24), are called constitutive relations. Depending on the characteristics of the constitutive macroscopic parameters ε, μ and σ, which are associated with the microscopic response of atoms and molecules in the medium, this medium can classified as:

Nonhomogeneous or homogeneous: according to whether or not the constitutive parameter of interest is a function of the position, $\varepsilon = \varepsilon(\vec{r})$, $\mu = \mu(\vec{r})$, or $\sigma = \sigma(\vec{r})$.

Anisotropic or isotropic: according to whether or not the response of the medium depends on the orientation of the field. In isotropic media all the magnitudes of interest are parallel, i.e., $\vec{E}$ and $\vec{D}$; and/or $\vec{E}$ and $\vec{J}_c$; and/or

$\vec{B}$ and $\vec{H}$. In anisotropic materials the constitutive parameter of interest is a tensor (see Chapter **??**)

Nonlinear or linear: according to whether or not the constitutive parameters depend on the magnitude of the applied fields. For instance $\varepsilon(E)$, $\mu(H)$ or $\sigma(E)$ *en general función de $\vec{E}$ y $\vec{B}$??*

Time-invariant: if the constitutive parameters do not vary with time $\varepsilon \neq \varepsilon(t)$, $\mu \neq \mu(t)$ or $\sigma \neq \sigma(t)$

Dispersive: according to whether or not, for time-harmonic fields, the constitutive parameters depend on the frequency, $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$ or $\sigma = \sigma(\omega)$. The materials in which these parameters are functions of the frequency are called dispersive[6].

Magnetic medium: if $\mu \neq \mu_0$. Otherwise the medium is called nonmagnetic because its only significant reaction to the electromagnetic field is polarization.

Fortunately, in many cases the medium in which the electromagnetic field exists can be considered homogeneous, linear and isotropic, time-invariant, nondispersive and nonmagnetic. Indeed, this assumption is not very restrictive since many electromagnetic phenomena can be studied using this simplification. In fact, even practical cases of the propagation of electromagnetic waves through nonlinear media (semiconductors, ferrites, nonlinear crystals, etc.) are analysed with linear models using the so-called small-signal approach. Most of this book concerns homogeneous, linear, isotropic and nonmagnetic media, except in Chapter **??** where anisotropic and magnetic materials (ferrites) are considered.

The effect of the properties of a medium on the macroscopic field can be emphasized by expressing $\vec{E}$ and $\vec{B}$ in Maxwell's equations (1.1a) and (1.1d) by (1.10) and (1.17). Thus we have

$$\nabla \cdot \vec{E} = \frac{\rho_{all}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\rho - \nabla \cdot \vec{P}\right) \tag{1.25a}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}_{all} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0(\vec{J} + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{1.25b}$$

---

[6] Eqs (1.12), (1.19) and (1.24) are strictly valid only for nondispersive media Effectively, for example, because of the dependence of the electric permittivity with frequency we generally have $\vec{P}(\omega) = \varepsilon_0 \chi_e(\omega) \vec{E}(\omega)$. Thus, according to the convolution theorem, for arbitrary time dependence this expression becomes

$$\vec{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi_e(t - t')\vec{E}(t')dt'$$

Similarly for magnetization and Ohms' law we have

$$\vec{M}(t) = \int_{-\infty}^{t} \chi_m(t - t')\vec{H}(t')dt'$$

$$\vec{J}(t) = \int_{-\infty}^{t} \sigma(t - t')\vec{E}(t')dt'$$

These expressions indicate that, as for any physical system, the response of the medium to an applied field is not instantaneous.

In (1.25a) we have explicitly as scalar sources of $\vec{E}$ both the free charge ρ and the polarization or bounded density of charge, $-\nabla \cdot \vec{P}$. Then in (1.9a) we have

$$\rho_{all} = \rho - \nabla \cdot \vec{P} \tag{1.26}$$

Similarly, in (1.25b), we have, explicitly as vector sources of $\vec{B}$, besides the free current density $\vec{J}$ (which includes the conduction current density $\vec{J}_c = \sigma\vec{E}$), the polarization current $\partial\vec{P}/\partial t$ (which results from the motion of the bounded charges in dielectrics), the displacement current in the vacuum, $\varepsilon_0 \partial\vec{E}/\partial t$ and the magnetization current, $\nabla \times \vec{M}$ (which takes place when a non-uniformly magnetized medium exists). Then in (1.9d) we have

$$\vec{J}_{all}(\vec{r}, t) = \vec{J} + \frac{\partial\vec{P}}{\partial t} + \nabla \times \vec{M} \tag{1.27}$$

In the following we will assume that there is no magnetization current.

1.2.3 Boundary conditions

As is evident from (1.1a)-(1.1d) and (1.13), (1.20), (1.24), in general the fields $\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$ are discontinuous at points where ε, μ and σ also are. Hence the field vectors will be discontinuous at a boundary between two media with different constitutive parameters.

The integral form of Maxwell's equations can be used to determine the *relations*, called boundary conditions, of the normal and tangential components of the fields at the interface between two regions with different constitutive parameters ε, μ and σ where surface density of sources may exist along the boundary.

The boundary condition for $\vec{D}$ can be calculated using a very thin, small pillbox that crosses the interface of the two media, as shown in Fig. 1.2. Applying the divergence theorem[7] to (1.1a) we have

$$\oint \vec{D}.d\vec{s} = \int_{\text{Base 1}} \vec{D}_1.d\vec{s} + \int_{\text{Curved surface}} \vec{D}.d\vec{s} + \int_{\text{Base 2}} \vec{D}_2.d\vec{s} = \int \rho dv \tag{1.28}$$

where $\vec{D}_1$ denotes the value of $\vec{D}$ in medium 1, and $\vec{D}_2$ the value in medium 2. Since both bases of the pillbox can be made as small as we like, the total outward flux of $\vec{D}$ over them is $(D_{n1} - D_{n2})ds = (\vec{D}_1 - \vec{D}_2) \cdot \hat{n} ds$, where these D_n are the normal components of $\vec{D}$, ds is the area of each base, and $\hat{n}$ is the unit normal drawn from medium 2 to medium 1. At the limit, by taking a shallow enough pillbox, we can disregard the flux over the curved surface, whereupon the sources of $\vec{D}$ reduce to the density of surface free charge ρ_s on the interface

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \tag{1.29}$$

[7] El teorema de la divergencia requiere que las propiedades del medio varíen de forma contínua, pero puede suponerse una transición rápida pero contínua del medio 1 al 2

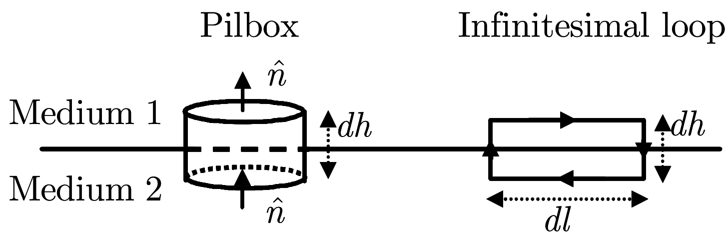

Figure 1.2: Derivation of boundary conditions at the interface of two media. Pintar solo la $\hat{n}$ hacia arriba y las $d\vec{s}$ una Hcia arriba y la de abajo hacia abajo Cuidado pilbox es con dos l

Hence the normal component of $\vec{D}$ changes discontinously across the interface by an amount equal to the free charge surface density ρ_s on the surface boundary.

Similarly the boundary condition for $\vec{B}$ can be established using the Gauss' law for magnetic fields (1.1b). Since the magnetic field is solenoidal, it follows that the normal components of $\vec{B}$ are continuous across the interface between two media

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \tag{1.30}$$

The behavior of the tangential components of $\vec{E}$ can be determined using a infinitesimal rectangular loop at the interface which has sides of lengh dh, normal to the interface, and sides of lengh dl parallel to it (Fig. 1.2). From the integral form of the Faraday's law, (1.2c) and defining $\hat{t}$ as the unit tangent vector parallel to the direction of integration on the upper side of the loop, we have

$$\begin{aligned} &(\vec{E}_1 \cdot \hat{t} - \vec{E}_2 \cdot \hat{t})dl + \text{ contributions of sides } dh \\ = \quad & -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \end{aligned} \tag{1.31}$$

In the limit, as $dh \to 0$, the area $ds = dldh$ bounded by the loop approaches zero and, since $\vec{B}$ is finite, the flux of $\vec{B}$ vanishes. Hence$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0$ and we conclude that the tangential components of $\vec{E}$ are continuous across the interface between two media. In terms of the normal $\hat{n}$ to the boundary, this can be written as

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \tag{1.32}$$

Analogously, using the same infinitesimal rectangular loop, it can be deduced from the generalized Ampère's law, (1.2d), that

$$\begin{aligned} &(\vec{H}_1 \cdot \hat{t} - \vec{H}_2 \cdot \hat{t})dl + \text{ contributions of sides } dh \\ = \quad & -\left(\frac{\partial \vec{D}}{\partial t} + \vec{J}\right) \cdot d\vec{s} \end{aligned} \tag{1.33}$$

where, since $\vec{D}$ is finite, its flux vanishes. Nevertheless, the flux of the surface current can have a non-zero value when the integration loop is reduced to zero, if the conductivity σ of the medium 2, and consequently $\vec{J}_s$, is infinite. This requires the surface to be a perfect conductor. Thus

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \tag{1.34}$$

the tangential component of $\vec{H}$ is discontinuous by the amount of surface current density $\vec{J}_s$. For finite conductivity, the tangential magnetic field is continuous across the boundary.

A summary of the boundary conditions, given in (1.35), are particularized in (1.36) for the case when the medium 2 is a perfect conductor ($\sigma_2 \to \infty$).

General boundary conditions

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \tag{1.35a}$$
$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \tag{1.35b}$$
$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \tag{1.35c}$$
$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \tag{1.35d}$$

Boundary conditions when the medium 2 is a perfect conductor ($\sigma_2 \to \infty$)

$$\hat{n} \times \vec{E}_1 = 0 \tag{1.36a}$$
$$\hat{n} \times \vec{H}_1 = \vec{J}_s \tag{1.36b}$$
$$\hat{n} \cdot \vec{D}_1 = \rho_s \tag{1.36c}$$
$$\hat{n} \cdot \vec{B}_1 = 0 \tag{1.36d}$$

1.3 The conservation of energy. Poynting's theorem

Poynting's theorem represents the electromagnetic energy-conservation law. To derive the theorem, let us calculate the divergence of the vector field $\vec{E} \times \vec{H}$ in a homogeneous, linear and isotropic finite region V bounded by a closed surface S. If we assume that V contains power sources (generators) generating currents $\vec{J}$, then, from Maxwell's equations (1.1c) and (1.1d), we get

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{E} \cdot (\sigma \vec{E} + \vec{J}) \tag{1.37}$$

where $\vec{J}$ represents the source current density distribution which is the primary origin of the electromagnetic fields[8], while the induced conduction current density is written as $\vec{J}_c = \sigma \vec{E}$ (1.24).

[8] The source current may be maintained by external power sources or generators (this current is often called driven or impressed current).

As the medium is assumed to be linear, the derivates with respect to time can be written as

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right) \tag{1.38a}$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}\cdot\vec{H}\right) \tag{1.38b}$$

By introducing the equalities (1.38a) and (1.38b) into (1.37), integrating over the volume V, applying the divergence theorem, and then rearranging terms, we have

$$\int_V \vec{J}\cdot\vec{E}dv = -\frac{\partial}{\partial t}\int_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})dv - \int_V \sigma E^2 dv - \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} \tag{1.39}$$

To interpret this result we accept that

$$U_{ev} = \frac{1}{2}\vec{D}\cdot\vec{E} \tag{1.40}$$

and

$$U_{mv} = \frac{1}{2}\vec{B}\cdot\vec{H} \tag{1.41}$$

represent, as a generalization of their expression for static fields, the instantaneous electric energy density, U_{ev}, and magnetic energy density, U_{mv}, stored in the respective fields. Thus according to (1.8) the left side of (1.39) represents the total electromagnetic power supplied by all the sources within the volume V. Regarding the right side of (1.39), the first term represents the change rate of the stored electromagnetic energy within the volume; the second term represents the dissipation rate of electromagnetic energy within the volume; and the third term represents the flow of electromagnetic energy per second (power) through the surface S that bounds volume V. Defining Poynting's vector $\vec{\mathcal{P}}$ as

$$\vec{\mathcal{P}} = \vec{E}\times\vec{H} \quad (W/m^2) \tag{1.42}$$

we can write

$$\oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} = \oint_S \vec{\mathcal{P}}\cdot d\vec{s} \tag{1.43}$$

This equation represents the total flow of power passing through the closed surface S and, consequently, we conclude that $\vec{\mathcal{P}} = \vec{E}\times\vec{H}$ represents the power passing through a unit area perpendicular to the direction of $\vec{\mathcal{P}}$. This conclusion may seem questionable because it could be argued that any vector with an integral of zero over the closed surface S could be added to $\vec{\mathcal{P}}$ without affecting the total flow. Nevertheless, this is a natural interpretation that does not contradict any experience. Only when we try to particularize (1.39) to steady fields do we find ambiguous results, because, in static, the location of the electric and magnetic energy has no physical significance.

Note that Eq. (1.39) was deduced by assuming a linear medium and that the losses occur only through conduction currents. Otherwise the equation should be modified to include other kinds of losses such as those due to hysteresis or possible transformations of the electromagnetic energy into mechanical energy, etc. When there are no sources within V, (1.39) represents an energy balance of that flowing through S versus that stored and dissipated in V.

1.4 Momentum of the electromagnetic field

As we have seen in the previous section, when we apply the law of conservation of electromagnetic energy to a finite volume V bounded by a surface S, it is necessary to include a term that, by means of the Poynting vector $\vec{\mathcal{P}}$, takes into account the flow of power through S. We shall now see that when an electromagnetic field interacts with the charges and currents in V, it is also necessary to consider a momentum associated with the electromagnetic field in order to guarantee the conservation of momentum. To calculate this momentum, we will begin by expressing, only in terms of the fields, the Lorentz force density, (1.5), exerted by the electromagnetic field on the distribution of charges and current, which we assume to be in free space. For this purpose, let us consider Maxwell's equations (1.1a) and (1.1d) to express ρ and $\vec{J}$ as

$$\rho = \nabla \cdot \vec{D} \tag{1.44}$$

$$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \tag{1.45}$$

so that

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} = \left(\nabla \cdot \vec{D}\right) \vec{E} - \vec{B} \times (\nabla \times \vec{H}) + \vec{B} \times \frac{\partial \vec{D}}{\partial t} \tag{1.46}$$

which, taking into account that

$$\begin{aligned} \vec{B} \times \frac{\partial \vec{D}}{\partial t} &= -\frac{\partial}{\partial t}(\vec{D} \times \vec{B}) + \vec{D} \times \frac{\partial \vec{B}}{\partial t} = \\ &\quad -\frac{\partial}{\partial t}(\vec{D} \times \vec{B}) - \vec{D} \times (\nabla \times \vec{E}) \end{aligned} \tag{1.47}$$

becomes

$$\vec{f} = (\nabla \cdot \vec{D})\vec{E} - \vec{B} \times (\nabla \times \vec{H}) - \frac{\partial}{\partial t}(\vec{D} \times \vec{B}) - \vec{D} \times (\nabla \times \vec{E}) \tag{1.48}$$

By adding the term $\vec{H}(\nabla \cdot \vec{B}) = 0$ to this equality to make the final expressions symmetrical, and by reordering, we can write the Lorentz force density as

$$\vec{f} = \vec{E}(\nabla \cdot \vec{D}) - \vec{D} \times (\nabla \times \vec{E}) + \vec{H} \nabla \cdot \vec{B} - \vec{B} \times (\nabla \times \vec{H}) - \frac{\partial}{\partial t}(\vec{D} \times \vec{B}) \tag{1.49}$$

The component α of Lorentz force density can be written, taking into account the definition of the Poynting vector $\vec{\mathcal{P}}$, as

$$f_\alpha = \varepsilon_o \frac{\partial}{\partial_\beta}\left[E_\beta E_\alpha - \frac{1}{2}\delta_{\beta\alpha}E^2\right] + \mu_0 \frac{\partial}{\partial_\beta}\left[H_\beta H_\alpha - \frac{1}{2}\delta_{\beta\alpha}H^2\right] - \frac{1}{c^2}\frac{\partial}{\partial t}\mathcal{P}_\alpha \quad (1.50)$$

where $\delta_{\beta\alpha}$ is the Kronecker delta ($\delta_{\beta\alpha} = 1$ if $\beta = \alpha$ and zero if $\beta \neq \alpha$) and the indices $\alpha, \beta = 1, 2, 3$ correspond to the coordinates x, y, z, respectively, and we have made use of the Einstein's summation convention (i.e., the repetition of an index automatically implies a summation over it). To obtain (1.50) we have made use of the following equalities

$$\begin{aligned} \vec{E}_\alpha \nabla \cdot \vec{D} - \vec{D} \times (\nabla \times \vec{E})\Big|_\alpha &= \varepsilon_o \frac{\partial}{\partial_\beta}\left[E_\beta E_{\alpha -} \frac{1}{2}\delta_{\beta\alpha}E^2\right] \\ \vec{B}_\alpha \nabla \cdot \vec{B} - \vec{B} \times (\nabla \times \vec{H})\Big|_\alpha &= \mu_0 \frac{\partial}{\partial_\beta}\left[H_\beta H_\alpha - \frac{1}{2}\delta_{\beta\alpha}H^2\right] \\ \vec{D} \times \vec{B}\Big|_\alpha &= \frac{P_\alpha}{c^2} \end{aligned} \quad (1.51)$$

The first two summands in (1.50) constitute the α component of the divergence of a tensor quantity, T^{em}, such that

$$\left(\nabla \cdot T^{em}\right)_\alpha = \frac{\partial T^{em}_{\beta\alpha}}{\partial_\beta} \quad (1.52)$$

where T^{em} is a symmetric tensor, known as the Maxwell stress tensor, defined by

$$T^{em}_{\beta\alpha} = \varepsilon_o \left[E_\beta E_{\alpha -} \frac{1}{2}\delta_{\beta\alpha}E^2\right] + \mu_0 \left[H_\beta H_\alpha - \frac{1}{2}\delta_{\beta\alpha}H^2\right] \quad (1.53)$$

Therefore, from (1.50) and (1.52), we have

$$\vec{f} = \nabla \cdot T^{em} - \frac{1}{c^2}\frac{\partial \vec{\mathcal{P}}}{\partial t} \quad (1.54)$$

with

$$\nabla \cdot T^{em} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \begin{bmatrix} T^{em}_{xx} & T^{em}_{xy} & T^{em}_{xz} \\ T^{em}_{yx} & T^{em}_{yy} & T^{em}_{yz} \\ T^{em}_{zx} & T^{em}_{zy} & T^{em}_{zz} \end{bmatrix} \quad (1.55)$$

The components of the electromagnetic tensor $T^{em}_{\beta\alpha}$ can be written as

$$T^{em}_{\beta\alpha} = T^e_{\beta\alpha} + T^m_{\beta\alpha} = D_\beta E_\alpha - \frac{1}{2}\delta_{\beta\alpha}E_\gamma D_\gamma + B_\beta H_\alpha - \frac{1}{2}\delta_{\beta\alpha}H_\gamma B_\gamma \quad (1.56)$$

where $T^m_{\beta\alpha}$ and $T^e_{\beta\alpha}$ represent, respectively, the electric and magnetic tensors defined by

$$T^e_{\beta\alpha} = D_\beta E_\alpha - \frac{1}{2}\delta_{\beta\alpha}E_\gamma D_\gamma \quad (1.57)$$

$$T^m_{\beta\alpha} = B_\beta H_\alpha - \frac{1}{2}\delta_{\beta\alpha}H_\gamma B_\gamma \quad (1.58)$$

Integrating (1.50) over the volume V the total electromagnetic force $\vec{F}$ exerted on the volume is

$$\vec{F} = \int_V \vec{f} dv = \int_V (\rho\vec{E} + \vec{J} \times \vec{B}) dv = \int_S \vec{f}_s ds - \frac{1}{c^2}\frac{\partial}{\partial t}\int_V \vec{\mathcal{P}} dv \tag{1.59}$$

where $\vec{f}_s$ is the force per unit of area on S

$$\vec{f}_s = T^{em} \cdot \hat{n} \tag{1.60}$$

and we have applied the theorem of divergence to the tensor T^{em} i.e.

$$\int_V \nabla \cdot T^{em} dv = \int_S T^{em} \cdot d\vec{s} = \int_S T^{em} \cdot \hat{n}\ ds = \int_S \vec{f}_s\ ds \tag{1.61}$$

Thus

$$\int_S \vec{f}_s ds = \vec{F} + \frac{1}{c^2}\frac{\partial}{\partial t}\int_V \vec{\mathcal{P}}\ dv \tag{1.62}$$

Note that the term

$$\frac{1}{c^2}\frac{\partial}{\partial t}\int_V \vec{\mathcal{P}} dv \tag{1.63}$$

is not null even in the absence of charges and currents. Since the only electromagnetic force possible due to the interaction of the field with charges and currents is $\vec{F}$, the term (1.63) must represent another physical quantity with the same dimensions as a force, i.e., the rate of momentum transmitted by the electromagnetic field to the volume V. This is equivalent to associating a momentum density $\vec{g}$ with the electromagnetic field, given by $1/c^2$ times the Poynting vector,

$$\vec{g} = \frac{\vec{\mathcal{P}}}{c^2} \tag{1.64}$$

which propagates in the same direction as the flow of energy. Thus, Eq. 1.62 represents the formulation for the momentum conservation in the presence of electromagnetic fields.

The momentum of an electromagnetic field, which can be determined experimentally, is inappreciable under normal conditions and its value is often below the limits of the measurement error. However, in the domain of atomic phenomena, the momentum of an electromagnetic field can be comparable to that of particles, and plays a crucial role in all the processes of interaction with matter. The transfer of momentum to a system of charges and currents implies a reduction in the field momentum, and the loss of momentum by the system, for example by radiation, leaves to an increase in the momentum of the field.

1.5 Time-harmonic electromagnetic fields

A particular case of great interest is one in which the sources vary sinusoidally in time. In linear media the time-harmonic dependence of the sources gives rise to fields which, once having reached the steady state, also vary sinusoidally in time. However, time-harmonic analysis is important not only because many electromagnetic systems operate with signals that are practically harmonic, but also because arbitrary periodic time functions can be expanded into Fourier series of harmonic sinusoidal components while transient nonperiodic functions can be expressed as Fourier integrals. Thus, since the Maxwell's equations are linear differential equations, the total fields can be synthesized from its Fourier components.

Analytically, the time-harmonic variation is expressed using the complex exponential notation based on Euler's formula, where it is understood that the physical fields are obtained by taking the real part, whereas their imaginary part is discarded. For example, an electric field with time-harmonic dependence given by $\cos(\omega t + \varphi)$, where ω is the angular frequency, is expressed as

$$\vec{E} = \mathrm{Re}\{\vec{\boldsymbol{E}} e^{j\omega t}\} = \frac{1}{2}(\vec{\boldsymbol{E}} e^{j\omega t} + (\vec{\boldsymbol{E}} e^{j\omega t})^*) = \vec{E}_0 \cos(\omega t + \varphi) \tag{1.65}$$

where $\vec{\boldsymbol{E}}$ is the complex phasor,

$$\vec{\boldsymbol{E}} = \vec{E}_0 e^{j\varphi} \tag{1.66}$$

of amplitude E_0 and phase φ, which will in general be a function of the angular frequency and coordinates. The asterisk * indicates the complex conjugate, and Re {} represents the real part of what is in curly brackets.

Throughout the book, we will represent both complex phasor magnitudes (either scalar or vector) by symbols in bold, e.g. $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}(\vec{r}, \omega)$, and $\boldsymbol{\rho} = \boldsymbol{\rho}(\vec{r}, \omega)$. In this way, time-dependent (real) quantities, which are represented by mathematical symbols not in bold, such as $\vec{E} = \vec{E}(\vec{r}, t)$, and $\rho = \rho(\vec{r}, t)$, can be distinguished from complex phasors which do not depend on time. In general, as indicated, these complex phasors may depend on the angular frequency. The real time-dependent quantity associated with a complex phasor is calculated, as in (1.65), by multiplying it by $e^{j\omega t}$ and taking the real part.

1.5.1 Maxwell's equations for time-harmonic fields

Assuming $e^{j\omega t}$ time dependence, we can get the phasor form or time-harmonic form of Maxwell's equations simply by changing the operator $\partial/\partial t$ to the factor $j\omega$ in (1.1a)-(1.2d) and eliminating the factor $e^{j\omega t}$. Maxwell's equations in differential and integral forms for time-harmonic fields are given below.

Differential form of Maxwell's equations for time-harmonic fields

$$\nabla \cdot \vec{\boldsymbol{D}} = \boldsymbol{\rho} \text{ (Gauss' law)} \tag{1.67a}$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \text{ (Gauss' law for magnetic fields)} \tag{1.67b}$$

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\, \vec{\boldsymbol{B}} \text{ (Faraday's law)} \tag{1.67c}$$

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + j\omega \vec{\boldsymbol{D}} \text{ (Generalized Ampère's law)} \tag{1.67d}$$

Integral form of Maxwell's equations for time harmonic fields

$$\oint_S \vec{\boldsymbol{D}} \cdot d\vec{s} = \boldsymbol{Q}_T \text{ (Gauss' law)} \tag{1.68a}$$

$$\oint_S \vec{\boldsymbol{B}} \cdot d\vec{s} = 0 \text{ (Gauss' law for magnetic fields)} \tag{1.68b}$$

$$\oint_\Gamma \vec{\boldsymbol{E}} \cdot d\vec{l} = -j\omega \int_S \vec{\boldsymbol{B}} \cdot d\vec{s} \text{ (Faraday's law)} \tag{1.68c}$$

$$\oint_\Gamma \vec{\boldsymbol{H}} \cdot d\vec{l} = \int_S (\vec{\boldsymbol{J}} + j\omega\vec{\boldsymbol{D}}) \cdot d\vec{s} \text{ (Generalized Ampère's law)} \tag{1.68d}$$

For time-harmonic fields, expressions (1.25a) and (1.25b) become

$$\nabla \cdot \vec{\boldsymbol{E}} = \frac{\boldsymbol{\rho}_{all}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\boldsymbol{\rho} - \nabla \cdot \vec{\boldsymbol{P}}\right) \tag{1.69a}$$

$$\nabla \times \vec{\boldsymbol{B}} = j\omega\varepsilon_0\mu_0 \vec{\boldsymbol{E}} + \mu_0\vec{\boldsymbol{J}}_{all} = j\omega\varepsilon_0\mu_0\vec{\boldsymbol{E}} + \mu_0(\vec{\boldsymbol{J}} + j\omega\vec{\boldsymbol{P}} + \nabla \times \vec{\boldsymbol{M}}) \tag{1.69b}$$

1.5.2 Complex dielectric constant.

Over certain frequency ranges, due to the atomic and molecular processes involved in the macroscopic response of a medium to an electromagnetic field, there appear relatively strong damping forces that give rise to a delay between the polarization vector $\vec{P}$ and $\vec{E}$ (a phase shift between $\vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{E}}$), and consequently between $\vec{E}$ and $\vec{D}$, and to a loss of electromagnetic energy as heat in overcoming the damping forces (see Appendix **??**). At the macroscopic level this effect is analytically expressed by means of a complex permittivity, ε_c as

$$\vec{\boldsymbol{D}} = \varepsilon_c\vec{\boldsymbol{E}} \tag{1.70}$$

with

$$\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon_0\varepsilon_{cr} \tag{1.71}$$

where ε_{cr}

$$\varepsilon_{cr} = 1 + \chi_{ce} = \varepsilon_r' - j\varepsilon_r'' \tag{1.72}$$

is the relative complex permittivity and $\chi_{ce} = \chi'_{cer} - j\chi''_{cer}$ is the complex electric susceptibility. In general both ε' and ε'' present a strong frequency dependence and they are closely related to one another by the Kramer-Kronig relations as is shown in Appendix **??**, where the dependence with the frequency of the dielectric constant is studied.

Similar processes occur in magnetic and conducting media, and, within a given frequency range, there may be a phase shift between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{J}}_c$ or between $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{H}}$ which, at the macroscopic level, is reflected in the corresponding complex constitutive parameters $\sigma_c = \sigma' - j\sigma''$ and $\mu_c = \mu' - j\mu''$.

For a medium with complex permittivity, the complex phasor form of the displacement current is

$$j\omega\vec{\boldsymbol{D}} = j\omega\varepsilon_c\vec{\boldsymbol{E}} = \omega\varepsilon''\vec{\boldsymbol{E}} + j\omega\varepsilon'\vec{\boldsymbol{E}} \tag{1.73a}$$

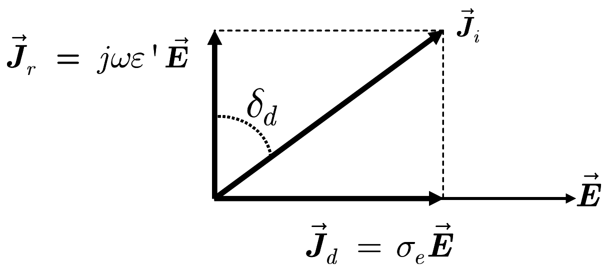

Figure 1.3: Induced current density in the complex plane.

while the sum, of the displacement and conduction current, called total induced current, $\vec{J}_i$, is

$$\vec{J}_i = \sigma\vec{E} + j\omega\varepsilon_c\vec{E} = (\sigma + \omega\varepsilon'')\vec{E} + j\omega\varepsilon'\vec{E} = \vec{J}_d + \vec{J}_r \quad (1.74)$$

where $\vec{J}_d$, called the dissipative current,

$$\vec{J}_d = (\sigma + \omega\varepsilon'')\vec{E} \quad (1.75)$$

in phase with the electric field, is the real part of the induced current $\vec{J}_i$ (Fig. 1.3) while $\vec{J}_r$, called the reactive current,

$$\vec{J}_r = j\omega\varepsilon'\vec{E} \quad (1.76)$$

is the imaginary part of the induced current which is in phase quadrature with the electric field. The dissipative current can be expressed in a more compact form as

$$\vec{J}_d = \sigma_e\vec{E} \quad (1.77)$$

where σ_e is the effective or equivalent conductivity

$$\sigma_e = \sigma + \omega\varepsilon'' \quad (1.78)$$

which includes the ohmic losses due to σ and the damping losses due to $\omega\varepsilon''$. Thus the induced current, (1.74), can be written as

$$\vec{J}_i = \sigma_e\vec{E} + j\omega\varepsilon'\vec{E} = \sigma_{ec}\vec{E} \quad (1.79)$$

where σ_{ec} is the complex effective conductivity, defined as

$$\sigma_{ec} = \sigma_e + j\omega\varepsilon' \quad (1.80)$$

Thus a medium with conductivity σ_{ec} and null permittivity is formally equivalent to one with conductivity and permittivity, σ and ε_c, respectively.

On the other hand, the phase angle δ_d between the induced and reactive currents, (Fig. 1.3), is called the loss or dissipative angle, and its tangent (i.e., the ratio of the dissipative and reactive currents) is called the loss tangent

$$\tan\delta_d = \frac{\sigma_e}{\omega\varepsilon'} \tag{1.81}$$

and the induced current, (1.79), can be written in terms of the loss tangent as

$$\vec{\boldsymbol{J}}_i = \sigma_{ec}\,\vec{\boldsymbol{E}} = j\omega\varepsilon'(1 - j\frac{\sigma_e}{\omega\varepsilon'})\vec{\boldsymbol{E}} = j\omega\varepsilon'\,(1 - j\tan\delta_d)\,\vec{\boldsymbol{E}} = j\omega\varepsilon_{ec}\vec{\boldsymbol{E}} \tag{1.82}$$

where ε_{ec} is defined as the effective complex permittivity

$$\varepsilon_{ec} = \varepsilon'\,(1 - j\tan\delta_d) = \varepsilon_0\varepsilon_{er} \tag{1.83}$$

and

$$\varepsilon_{er} = (1 - j\tan\delta_d)\varepsilon'_r \tag{1.84}$$

denotes the effective relative permittivity. Thus, according to (1.79) and (1.82), a medium can be formally considered alternatively either as a medium of permittivity ε' and effective conductivity σ_e, or as a dielectric medium of effective permittivity ε_{ec} or as a conducting medium of effective conductivity σ_{ec}. In summary, this possibilities are

	Permittivity	Conductivity
Original medium	$\varepsilon_c = \varepsilon' - j\varepsilon''$	σ
Equivalent medium 1	ε'	$\sigma_e = \sigma + \omega\varepsilon''$
Equivalent medium 2	$\varepsilon_{ec} = \varepsilon' - j(\varepsilon'' + \frac{\sigma}{\omega})$	0
Equivalent medium 3	0	$\sigma_{ec} = \sigma + \omega\varepsilon'' + j\omega\varepsilon'$

(1.85)

The loss tangent is equal to the inverse of the quality factor Q of the dielectric which is a dimensionless quantity defined as

$$Q = \omega\frac{\text{Maximun energy stored per unit volume}}{\text{Time average power lost per unit volume}} = \omega\frac{W_v}{P'_{dv}} \tag{1.86}$$

The average power dissipated per cycle and unit volume, P'_{dv}, due both to the Joule effect and to that of dielectric polarization, is given, according to (1.8), by

$$\begin{aligned} P'_{dv} &= \frac{1}{T}\int_0^T \vec{E}\cdot\vec{J}_i dt = \frac{1}{T}\int_0^T \vec{E}_0\cos\omega t\cdot(\sigma_e\vec{E}_0\cos\omega t + \omega\varepsilon'\vec{E}_0\sin\omega t)dt \\ &= \frac{1}{T}\int_0^T \sigma_e E_0^2\cos^2\omega t dt = \frac{1}{T}\int_0^T \vec{E}\cdot\vec{J}_d dt = \frac{\sigma_e E_0^2}{2} \end{aligned} \tag{1.87}$$

where $T = 2\pi/\omega$ is the period of the signal. Note that only the dissipative part of J_i contributes to the average power. Of this power, the part corresponding to polarization losses is

$$\frac{1}{T}\int_0^T \omega\varepsilon'' E_0^2 \cos^2 \omega t dt = \frac{\omega\varepsilon'' E_0^2}{2} \tag{1.88}$$

The maximum electric field energy stored per unit of volume is

$$W_v = \frac{1}{2}\varepsilon' E_0^2 \tag{1.89}$$

Thus, dividing (1.89) by (1.87), we have

$$Q = \frac{\omega\varepsilon'}{\sigma_e} = \frac{1}{\tan\delta_d} \tag{1.90}$$

Although both dimensionless quantities, Q and $\tan\delta_d$, can be used to define the characteristics of a dielectric, we will use the loss tangent throughout this book.

Depending on whether the reactive or the dissipative current is predominant at the operating frequency, a medium is classified as a weakly lossy or a strongly lossy medium respectively. Thus for weakly lossy media, usually called good dielectrics or insulators, we have, $\omega\varepsilon' >> \sigma_e$, so that

$$\tan\delta_d = \frac{\sigma_e}{\omega\varepsilon'} << 1 \tag{1.91}$$

Or, if $\sigma = 0$,

$$\tan\delta_d = \frac{\varepsilon''}{\varepsilon'} << 1 \tag{1.92}$$

If $\sigma_e = 0$ (i.e. $\tan\delta_d = 0$), the medium is termed a perfect or ideal dielectric, in which case the reactive current coincides with the displacement current, and the dielectric is characterized by a real permittivity ε.

If the medium is strongly lossy we have $\omega\varepsilon' << \sigma_e$, so that

$$\tan\delta_d = \frac{\sigma_e}{\omega\varepsilon'} >> 1 \tag{1.93}$$

which for good conductors where $\varepsilon'' = 0$; $\varepsilon' = \varepsilon$ simplifies to

$$\tan\delta_d = \frac{\sigma}{\omega\varepsilon} >> 1 \tag{1.94}$$

being practically $\varepsilon = \varepsilon_0$. If $\sigma = \infty$ (i.e. $\tan\delta_d = \infty$) the medium is termed a perfect conductor.

For a homogeneous conducting medium where ε' and σ_e do not depend on the position, Gauss' law (1.1a) and the continuity equation (1.3) can be writen as

$$\nabla \cdot \vec{E} = \rho/\varepsilon' \tag{1.95}$$

and

$$\sigma_e \nabla \cdot \vec{E} = -\frac{\partial \rho}{\partial t} \tag{1.96}$$

respectively. Hence we have

$$\frac{\sigma_e \rho}{\varepsilon'} + \frac{\partial \rho}{\partial t} = 0 \tag{1.97}$$

so that the expression for the decay of a charge distribution in a conductor is given by

$$\rho = \rho_0 e^{-\left(\sigma_e/\varepsilon'\right)t} \tag{1.98}$$

where ρ_0 is the charge density at time $t = 0$. The characteristic time

$$\tau = \frac{\varepsilon'}{\sigma_e} \tag{1.99}$$

required for the charge at any point to decay to $1/e$ of its original value is called the relaxation time.

For most metals $\tau = 10^{-14}s$, signifying that in good conductors the charge distribution decays exponentially so quickly that it may be assumed that $\rho = 0$ at any time. In terms of the relaxation time, the loss tangent can be written as

$$\tan \delta_d = \frac{\sigma}{\varepsilon\omega} = (\tau\omega)^{-1} \tag{1.100}$$

Thus the classification of a medium as a good or poor conductor depends on whether the relaxation time is short or long compared with the period of the signal.

1.5.3 Boundary conditions for harmonic signals

For harmonic signals the boundary conditions of the normal and tangential components of the fields at the interface between two regions with different constitutive parameters ε, μ and σ, (1.35a)-(1.36d), become

General boundary conditions

$$\hat{n} \times (\vec{\boldsymbol{E}}_1 - \vec{\boldsymbol{E}}_2) = 0 \tag{1.101a}$$

$$\hat{n} \times (\vec{\boldsymbol{H}}_1 - \vec{\boldsymbol{H}}_2) = \vec{\boldsymbol{J}}_s \tag{1.101b}$$

$$\hat{n} \cdot (\vec{\boldsymbol{D}}_1 - \vec{\boldsymbol{D}}_2) = \boldsymbol{\rho}_s \tag{1.101c}$$

$$\hat{n} \cdot (\vec{\boldsymbol{B}}_1 - \vec{\boldsymbol{B}}_2) = 0 \tag{1.101d}$$

Boundary conditions when the medium 2 is a perfect conductor ($\sigma_2 \to \infty$)

$$\hat{n} \times \vec{\boldsymbol{E}}_1 = 0 \tag{1.102a}$$

$$\hat{n} \times \vec{\boldsymbol{H}}_1 = \vec{\boldsymbol{J}}_s \tag{1.102b}$$

$$\hat{n} \cdot \vec{\boldsymbol{D}}_1 = \boldsymbol{\rho}_s. \tag{1.102c}$$

$$\hat{n} \cdot \vec{\boldsymbol{B}}_1 = 0 \tag{1.102d}$$

1.5.4 Complex Poynting vector

In formulating the conservation-energy equation for time-harmonic fields, it is convenient to find, first, the time-average Poynting vector over a period, i.e. the time-average power passing through a unit area perpendicular to the direction of $\vec{\mathcal{P}}$. From (1.65) we have

$$\vec{E} = \mathrm{Re}\left\{\vec{\boldsymbol{E}}e^{j\omega t}\right\} = \frac{1}{2}\left(\vec{\boldsymbol{E}}e^{j\omega t} + (\vec{\boldsymbol{E}}e^{j\omega t})^*\right) \tag{1.103a}$$

$$\vec{H} = \mathrm{Re}\left\{\vec{\boldsymbol{H}}e^{j\omega t}\right\} = \frac{1}{2}\left(\vec{\boldsymbol{H}}e^{j\omega t} + (\vec{\boldsymbol{H}}e^{j\omega t})^*\right) \tag{1.103b}$$

Thus, the instantaneous Poynting vector (1.42) can be written as

$$\begin{aligned}\vec{\mathcal{P}} &= \vec{E}\times\vec{H} = \mathrm{Re}\{\vec{\boldsymbol{E}}e^{j\omega t}\}\times\mathrm{Re}\{\vec{\boldsymbol{H}}e^{j\omega t}\}\\ &= \frac{1}{2}\mathrm{Re}\{\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^* + \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}e^{2j\omega t}\}\end{aligned} \tag{1.104}$$

where we have made use of the general relation for any two complex vectors $\vec{A}$ and $\vec{B}$

$$\begin{aligned}\mathrm{Re}\{\vec{\boldsymbol{A}}\}\times\mathrm{Re}\{\vec{\boldsymbol{B}}\} &= \frac{1}{2}(\vec{\boldsymbol{A}}+\vec{\boldsymbol{A}}^*)\times\frac{1}{2}(\vec{\boldsymbol{B}}+\vec{\boldsymbol{B}}^*)\\ &= \frac{1}{4}(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}^* + \vec{\boldsymbol{A}}^*\times\vec{\boldsymbol{B}}) + \frac{1}{4}(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}^*\times\vec{\boldsymbol{B}}^*)\\ &= \frac{1}{4}\left(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}^* + \left(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}^*\right)^*\right) + \frac{1}{4}\left(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} + \left(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}\right)^*\right)\\ &= \frac{1}{2}\mathrm{Re}\{\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}^* + \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}\}\end{aligned} \tag{1.105}$$

The time-average value of the instantaneous Poynting vector can be calculated integrating (1.104) over a period , i.e.,

$$\begin{aligned}\vec{\mathcal{P}}_{av} &= \frac{1}{T}\int_0^T \vec{\mathcal{P}}dt = \frac{1}{2T}\int_0^T \mathrm{Re}\{\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^* + \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}e^{2j\omega t}\}dt\\ &= \frac{1}{2}\mathrm{Re}\{\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\vec{\mathcal{P}}_c\}\end{aligned} \tag{1.106}$$

since the time average of $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}e^{2j\omega t}$ vanishes. The magnitude

$$\vec{\mathcal{P}}_c = \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^* \tag{1.107}$$

is termed the complex Poynting vector. Thus the time-average of the Poynting vector is equal to one-half the real part of the complex Poynting vector

For a more complete view of the meaning of the complex Poynting vector, let us again formulate Poynting's theorem particularized for sources with time-harmonic dependence. From Faraday's law, (1.67c), and from Ampère's general law, (1.67d), in its conjugate complex form, we have

$$\nabla\times\vec{\boldsymbol{E}} = -j\omega\mu\vec{\boldsymbol{H}} \tag{1.108a}$$

$$\nabla\times\vec{\boldsymbol{H}}^* = -j\omega\varepsilon\vec{\boldsymbol{E}}^* + \vec{\boldsymbol{J}}^* + \sigma\vec{\boldsymbol{E}}^* \tag{1.108b}$$

where $\vec{J}^*$ represents the complex conjugate of the current supplied by the sources. Performing a scalar multiplication of Eq. 1.108a by $\vec{H}^*$ and of Eq. 1.108b by $\vec{E}$, and subtracting the results, we get

$$\begin{aligned}\nabla \cdot \left(\vec{E} \times \vec{H}^*\right) &= \vec{H}^* \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}^* \\ &= -j\omega\left(\mu H_0^2 - \varepsilon E_0^2\right) - \vec{E} \cdot (\vec{J}^* + \sigma \vec{E}^*) \qquad (1.109)\end{aligned}$$

where it has been taken into account that $\vec{H} \cdot \vec{H}^* = H_0^2$ and $\vec{E} \cdot \vec{E}^* = E_0^2$, with H_0 and E_0 being the amplitude of the two harmonic fields. After dividing (1.109) by 2 we get

$$\nabla \cdot \left(\frac{1}{2}\vec{E} \times \vec{H}^*\right) = -2j\omega\left(\mu\frac{H_0^2}{4} - \varepsilon\frac{E_0^2}{4}\right) - \frac{\sigma E_0^2}{2} - \frac{1}{2}\vec{J}^* \cdot \vec{E} \qquad (1.110)$$

The terms $\mu H_0^2/4$ and $\varepsilon E_0^2/4$ represent, respectively, the mean density of the magnetic and electric energy, while $\sigma E_0^2/2$ is the the mean power transformed into heat[9] within V, since the mean value of the square of a sine or cosine function is 1/2.

By multiplying Equation (1.110) by the volume element dv, integrating over an arbitrary volume V and applying the divergence theorem, we obtain the complex version of the Poynting theorem

$$\begin{aligned}\int_V \frac{1}{2}\left(\vec{J}^* \cdot \vec{E}\right) dv &= -\int_V \frac{\sigma E_0^2}{2} dv - 2j\omega \int_V \left(\mu\frac{H_0^2}{4} - \varepsilon\frac{E_0^2}{4}\right) dv \\ &\quad - \int_S \frac{1}{2}\left(\vec{E} \times \vec{H}^*\right) \cdot d\vec{s} \qquad (1.111)\end{aligned}$$

which is the expression corresponding to (1.39) in complex notation and where the first member represents the power supplied by external sources. By separating the real and imaginary parts, we obtain the following two equalities

$$\int_V \text{Re}\,\frac{1}{2}(\vec{J}^* \cdot \vec{E})dv = -\int_V \frac{\sigma E_0^2}{2} dv - \int_S \text{Re}\,\frac{1}{2}(\vec{E} \times \vec{H}^*) \cdot d\vec{s} \qquad (1.112a)$$

$$\int_V \text{Im}\,\frac{1}{2}(\vec{J}^* \cdot \vec{E})dv = -2\omega \int_V \left(\mu\frac{H_0^2}{4} - \varepsilon\frac{E_0^2}{4}\right) dv - \int_S \text{Im}\,\frac{1}{2}(\vec{E} \times \vec{H}^*) \cdot d\vec{s} \qquad (1.112b)$$

The first member of (1.112a)

$$P_a = \int_V \text{Re}\,\frac{1}{2}(\vec{J}^* \cdot \vec{E})dv \qquad (1.113)$$

represents the active mean power supplied by all the sources within V. On the right-hand side of (1.112a) the first integral, as commented above, gives the

[9] Expression (1.112b) can be easily extended to the case of lossy dielectric just substituting σ by the equivalent conductivity σ_e defined in (1.78)and ε by ε' defined in (1.71).

power transformed into heat within V, while the surface integral represents the mean flow of power through the surface S.

Regarding to expression (1.112b), the first member

$$P_r = \int_V \mathrm{Im}\left(\frac{1}{2}\vec{\boldsymbol{J}}^{\,*} \cdot \vec{\boldsymbol{E}}\right) dv \qquad (1.114)$$

is called the reactive power of the sources. On the right-hand side the first summand is 2ω times the difference of the average energies stored in the electric and magnetic fields, while the second represents the flow of reactive power that is exchanged with the external medium through S. If the surface integral in (1.112a) is non-zero, the external region is said to be an active charge for the sources within V. Similarly, if the surface integral of Eq. (1.112b) is non-zero, the external region is said to be a reactive charge for the sources within V. In general, both of these surface integrals are non-zero and the external region becomes both an active and a reactive charge for the sources.

1.6 On the solution of Maxwell's equations

Despite their apparent simplicity, Maxwell's equations are in general not easy to solve. In fact, even in the most favorable situation of homogeneous, linear and isotropic media, there are not many problems of interest that can be analytically solved except for those presenting a high degree of geometrical symmetry. Moreover, the frequency range of scientific and technological interest can vary by many orders of magnitude, expanding from frequency values of zero (or very low) to roughly 10^{14} $Hertz$. The behavior and values of the constitutive parameters can change very significantly in this frequency. range. Conductivity, for example, can vary from 0 to 10^7 $S\ m^{-1}$. It is even possible to build artificial materials, called metamaterials, which present electromagnetic properties that are not found in nature. Examples of such as metamaterials are those characterized with both negative permittivity ($\varepsilon < 0$) and negative permeability ($\mu < 0$). These media are called DNG (double-negative) metamaterials and, owing to their unusual electromagnetic properties, they present many potential technological applications.

Another important factor to study the interaction of an electromagnetic field with an object is the electrical size of the body, i.e., the relationship between the wavelength and the body size, which can also vary by several orders of magnitude. All these circumstances make it in general necessary to use analytical, semi-analytical or numerical methods appropriate to each situation. In particular, numerical methods are fundamental for simulating and solving complex problems that do not admit analytical solutions. Today numerical methods make up the so-called computational electromagnetics, which together, with experimental and theoretical or analytical electromagnetics, constitute the three pillars supporting research in Electromagnetics. Of course, both the development of analitycal, numerical or experimental tools, as well as the interpretation of the results, require theoretical knowledge of electromagnetic phenomena

Chapter 2

Fields created by a source distribution: retarded potentials

In this chapter, we introduce the scalar electric and magnetic vector potentials as magnitudes that facilitate the calculation of the fields created by a bounded-source distribution, paying special attention to the radiation field. Finally, we extend Maxwell's equations, in order to make them symmetric, by introducing the concept of magnetic charges and currents.

2.1 Electromagnetic potentials

A basic problem in electromagnetism is that of finding the fields created for a time-varying source distribution of finite size, which we assume to be in a non-magnetic, lossless, homogeneous, time-invariant, linear and isotropic medium. Figure 2.1 represents such a distribution, where, as usual, the coordinates associated with source points, $\vec{J} = \vec{J}(\vec{r}', t')$, $\rho = \rho(\vec{r}', t')$, are designated by primes, while those associated with field points or observation points $P(\vec{r}, t)$ are without primes. In the following, we will assume the medium surrounding the source distribution to be free space, i.e. $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, although of course all the resulting formulas remain valid for media of constant permittivity and permeability, provided that ε_0 is replaced by $\varepsilon_r \varepsilon_0$ and μ by $\mu_r \mu_0$. While the expressions for the fields can be derived directly from their sources, the task can often be facilitated by calculating first two auxiliary functions, the scalar electric potential $\Phi = \Phi(\vec{r}, t)$ and the magnetic vector potential $\vec{A} = \vec{A}(\vec{r}, t)$ (Fig. 2.2). Once the potentials are obtained, it is a simple matter to calculate the fields from them. In this section, we formulate the general expressions for these potentials.

Since, according to (1.1b), the divergence of the magnetic field $\vec{B}$ is always

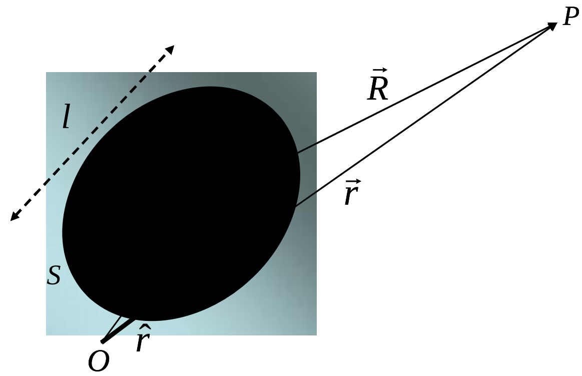

Figure 2.1: Time-varying source bounded distribution V' of maximun dimension l. The coordinates associated with source points of currents and charges $\vec{J} = \vec{J}(\vec{r}', t')$, and $\rho = \rho(\vec{r}', t')$, respectivelly are designated by primes, while the associate with field points, $P(\vec{r}, t)$, are without primes.

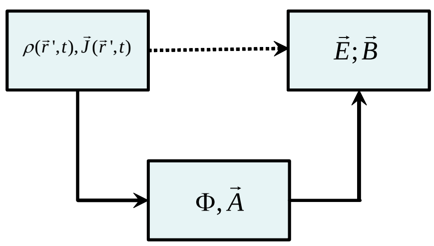

Figure 2.2: Quitar o poner argumentos pero unificar

zero, we can express it as the curl of an electromagnetic vector potential $\vec{A}$ as

$$\vec{B} = \nabla \times \vec{A} \tag{2.1}$$

Inserting this expression into (1.1c) we get

$$\nabla \times \left(\vec{E} + \frac{\partial}{\partial t} \vec{A} \right) = 0 \tag{2.2}$$

Since any vector with a zero curl can be expressed as the gradient of a scalar function Φ, called the scalar potential, we can write

$$\vec{E} + \frac{\partial}{\partial t} \vec{A} = -\nabla \Phi \tag{2.3}$$

or

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \tag{2.4}$$

where $\partial \vec{A}/\partial t$ is the nonconservative part of the electric field with a non-vanishing curl. When the vector potential $\vec{A}$ is independent of time, expression (2.4) reduces to the familiar $\vec{E}(\vec{r}) = -\nabla \Phi(\vec{r})$.

According to the relations (2.1) and, (2.4) the fields $\vec{B}$ and $\vec{E}$ are completely determined by the vector and scalar potentials $\vec{A}$ and Φ. However, the fields do not uniquely determine the potentials. For instance, it is clear that the transformation

$$\vec{A} = \vec{A}' + \nabla \Psi \tag{2.5}$$

where $\Psi = \Psi(\vec{r}, t)$ is any arbitrary, single-valued, continuously differentiable, scalar function of position and time that vanishes at infinity, leaves $\vec{B}$ unchanged

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}' + \nabla \times \nabla \Psi = \nabla \times \vec{A}' \tag{2.6}$$

Inserting (2.5) into (2.4), it follows that

$$\vec{E} = -\nabla \left(\Phi + \frac{\partial \Psi}{\partial t} \right) - \frac{\partial \vec{A}'}{\partial t} \tag{2.7}$$

so that the value of $\vec{E}$, obtained from $\vec{A}'$, also remains unchanged provided that Φ is replaced by the scalar potential

$$\Phi' = \Phi + \frac{\partial \Psi}{\partial t} \tag{2.8}$$

Thus different sets of potentials $\vec{A}$ and Φ give rise to the same set of fields[1] $\vec{B}$ and $\vec{E}$. The joint transformation (2.5) and (2.8) leaves the electromagnetic field

[1] The liberty to select the value of $\vec{A}$ is understandable taking into account that by (2.1) the magnetic field fixes only $\nabla \times \vec{A}$. However, Helmholtz's theorem posits that, to determine the (spatial) behavior of $\vec{A}$ completely, $\nabla . \vec{A}$ (which is still undetermined) must also be specified. Thus, we can choose it in any way we consider suitable for facilitating the calculation of the electromagnetic fields.

invariant. The different forms of choosing the potentials $\vec{A}$ and Φ leaving the fields unchanged are called gauge transformations, and the function Ψ is called the gauge function. The degree of freedom provided by the gauge transformations facilitates the calculation of the potentials and hence of the fields because, once the potentials are known, the fields are easily derived by differentiation from (2.1) and (2.4). An example of gauge transformation is the Lorenz gauge, also called the Lorenz condition.

2.1.1 Lorenz gauge

Inserting (2.1) and (2.4) into the generalized Ampère's law, (1.1d), and Gauss' law, (1.1a), using (**??**) and rearranging terms, we get two, coupled, second-order partial-differential equations

$$\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \mu_0\vec{J} - \nabla\left[\nabla\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t}\right] \tag{2.9a}$$

$$\mu_0\varepsilon_0\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t}\left[\nabla\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t}\right] \tag{2.9b}$$

These equations could be considerably simplified if we could force (without changing the fields) the potentials to satisfy the auxiliary relation

$$\boxed{\nabla\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t} = 0} \tag{2.10}$$

called the Lorenz gauge (or Lorenz condition)[2]. Fortunately, as we will show below, we can always take advantage of the freedom in choosing the potentials so that they fulfil the Lorenz condition and consequently simplify Eqs (2.9) *to the inhomogeneous Helmholtz wave equations*

$$\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \mu_0\vec{J} \tag{2.11a}$$

$$\mu_0\varepsilon_0\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \frac{\rho}{\varepsilon_0} \tag{2.11b}$$

The advantage of having applied the Lorenz condition is that the equations (2.11) for the potentials are uncoupled and each one depends on only one type of source. This makes it easier to calculate the potentials than the fields (see Section **??**).

It remains to be shown that it is always possible to force the potentials to satisfy the Lorenz condition (2.10). To this end, let us consider two potentials, $\vec{A}'$ and Φ', which fulfil Equations (**??**) and (**??**) and check whether it is possible

[2] A very interesting property of the Lorenz condition is that, as shown in (**??**), it is covariant, i.e., if it holds in one particular inertial frame then it automatically holds in all other inertial frames.

to select them so that they satisfy equations (2.11). By inserting (2.5) and (2.8) into (2.9) and by rearranging, we get

$$\nabla^2 \vec{A}' - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}'}{\partial t^2} = -\mu_0 \vec{J} + \nabla\left(\nabla\cdot\vec{A}' + \nabla^2\Psi + \mu_0\varepsilon_0\frac{\partial \Phi'}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\Psi}{\partial t^2}\right) \tag{2.12a}$$

$$\nabla^2 \Phi' - \mu_0\varepsilon_0 \frac{\partial^2 \Phi'}{\partial t^2} = -\frac{\rho}{\varepsilon_0} - \frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}' + \nabla^2\Psi + \mu_0\varepsilon_0\frac{\partial \Phi'}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\Psi}{\partial t^2}\right) \tag{2.12b}$$

and, given that the scalar function Ψ is arbitrary, we can choose it as the solution to the differential equation

$$\nabla^2\Psi - \mu_0\varepsilon_0\frac{\partial^2\Psi}{\partial t^2} = -\nabla\cdot\vec{A}' - \mu_0\varepsilon_0\frac{\partial\Phi'}{\partial t} \tag{2.13}$$

Thus (2.12a) and (2.12b) become (2.11a) and (2.11b), respectively, meaning that $\vec{A}'$ and Φ' fulfil the Lorenz condition.

Expressions (2.11a) and (2.11b) are the inhomogeneous wave equations for the potentials, and their solutions, which are provided in the next section, represent waves propagating at the velocity $c = 1/\sqrt{\mu_0\varepsilon_0} \simeq 3\times 10^8 m/s$ of light in free space. They take the form

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \Box\vec{A} = -\mu_0\vec{J} \tag{2.14a}$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \Box\Phi = -\frac{\rho}{\varepsilon_0} \tag{2.14b}$$

where the symbol $\Box$ represents the D'Alembertian operator defined by

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{2.15}$$

The Lorenz gauge (2.10) for harmonic fields simplifies to

$$\nabla\cdot\boldsymbol{\vec{A}} + \frac{j\omega}{c^2}\boldsymbol{\Phi} = 0 \tag{2.16}$$

such that

$$\boldsymbol{\Phi} = \frac{jc^2\nabla\cdot\boldsymbol{\vec{A}}}{\omega} \tag{2.17}$$

while (2.14a) and (2.14b) simplify to

$$\nabla^2\boldsymbol{\vec{A}} + \frac{\omega^2}{c^2}\boldsymbol{\vec{A}} = -\mu_0\boldsymbol{\vec{J}} \tag{2.18a}$$

$$\nabla^2\boldsymbol{\Phi} + \frac{\omega^2}{c^2}\boldsymbol{\Phi} = -\frac{\boldsymbol{\rho}}{\varepsilon_0} \tag{2.18b}$$

In addition to Lorenz's gauge, other gauge conditions may sometimes be useful. For instance, in quantum field theory, where the potentials are used to describe the interaction of the charges with the electromagnetic field instead of being used to calculate the fields, it is useful to use Coulomb's gauge, in which $\nabla \cdot \vec{A} = 0$. By taking the divergence of (2.4) and the curl of (2.1), and taking into account the generalized Ampère's law and Gauss' law , we can easily see that with Coulomb's gauge the expressions for the potential wave equations are

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \tag{2.19}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2}\nabla\frac{\partial \Phi}{\partial t} = -\mu_0 \vec{J} \tag{2.20}$$

As can be seen from (2.19), in Coulomb's gauge the scalar potential is determined by the instantaneous value of the charge distribution, using an equation similar to Poisson's expression in electrostatics. The vector potential, however, is considerably more difficult to calculate. According (2.19), a time change in ρ implies an instantaneous change in Φ. This fact denotes the non-physical nature of Φ since real physical magnitudes can change only after a delay determined by the propagation time between the perturbation and the measurement point.

In this book we will use only the Lorenz condition, but it should be made clear that the $\vec{E}$ and $\vec{H}$ fields calculated from the potentials with the Coulomb or Lorenz gauges must be identical.

The complete solutions of the inhomogeneous wave equations for the potentials (2.14) are linear combinations of the particular solutions and of the general solutions for the corresponding homogeneous wave equations. The next section is devoted to finding these particular solutions, which express the potentials in terms of integrals over the source distributions $\vec{J}$ and ρ.

2.2 Solution of the inhomogeneous wave equation for potentials

Let us now calculate the expression of the potentials created by an arbitrary bounded source distribution (charges and currents) in an unbounded homogeneous, time-invariant, linear and isotropic medium of conductivity zero that we assume to be free space (Fig. 2.1). From (2.14) we see that the scalar potential Φ as well as each of the three components A_i, $(i = 1, 2, 3)$, of the vector potential $\vec{A}$ satisfy inhomogeneous scalar wave equations with the general form

$$\Box\Psi(\vec{r}, t) = \nabla^2\Psi(\vec{r}, t) - \frac{1}{c^2}\frac{\partial^2\Psi(\vec{r}, t)}{\partial t^2} = -g\,(\vec{r}, t) \tag{2.21}$$

where the operator $\Box$ acts on the coordinates r, t of the field point, while the sources coordinates are r', t'.

To facilitate the solution of this equation, we can use, owing to the linearity of the problem, the superposition principle and consider a source distribution

$g(\vec{r},t)$ as constructed from a sum of weighted space-time Dirac delta function sources, i.e.,

$$g(\vec{r},t) = \int_{t'=-\infty}^{t} \int_{V'} g(\vec{r}\,',t')\,\delta(\vec{r}-\vec{r}\,')\delta(t-t')dv'dt' \tag{2.22}$$

where V' is a volume containing all the sources. Thus, (2.21) can be solved in two steps, using Green's method in the time domain, as follows.

a) The first step is to calculate the response, $G(\vec{r},\vec{r}\,',t,t')$, generated by the space-time Dirac $\delta-$function source, $\delta(\vec{r}-\vec{r}\,')\delta(t-t')$, located at position $\vec{r}\,'$ and applied at time t' which obeys the inhomogeneous wave equation

$$\Box G(\vec{r},\vec{r}\,',t,t') = \nabla^2 G(\vec{r},\vec{r}\,',t,t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r},\vec{r}\,',t,t')}{\partial t^2} = -\delta(\vec{r}-\vec{r}\,')\delta(t-t') \tag{2.23}$$

and satisfies the boundary conditions of the problem. The function $G(\vec{r},\vec{r}\,',t,t')$ is called Green's free-space function, which, because of the homogeneity of the space, must be a spherical wave centred at position $\vec{r}\,'$ at time t'. This function depends on the relative distance, $R = |\vec{r}-\vec{r}\,'|$, between the point source and the observation or field point and on the time difference $\tau = t - t'$. Thus $G(\vec{r},\vec{r}\,',t,t') = G(R,\tau)$ and (2.23) can be written, using spherical coordinates, as

$$\Box G(R,\tau) = \frac{1}{R}\frac{\partial^2 (RG)}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 G}{\partial \tau^2} = -\delta(\vec{R})\delta(\tau) \tag{2.24}$$

where, (**??**),

$$\nabla^2 G = \frac{1}{R}\frac{\partial^2 (RG)}{\partial R^2} \tag{2.25a}$$

$$\frac{\partial^2 G}{\partial \tau^2} = \frac{\partial^2 G}{\partial t^2} \tag{2.25b}$$

b) The second step is to find $\Psi(\vec{r},t)$ from Green's function. Owing to the linearity of the problem and, from (2.22), if the solution of (2.23) is G, then the solution of (2.21) is [3]

$$\Psi = \int_{t'=-\infty}^{t} \int_{V'} g(\vec{r}\,',t')\,G(R,\tau)dv'dt'. \tag{2.26}$$

Because G fulfils the boundary conditions, so too does $\Psi(\vec{r},t)$.

To find the Green's function let us consider first a general point $R \neq 0$ such that equation (2.24) simplifies to

$$\Box G(R,\tau) = \frac{1}{R}\frac{\partial^2 (RG)}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 G}{\partial \tau^2} = 0. \tag{2.27}$$

Multiplying this equation by R and defining $G' = RG$ we have the homogeneous wave equation

$$\frac{\partial^2 G'}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 G'}{\partial \tau^2} = 0 \tag{2.28}$$

[3] Note that Eq. (2.27) represents the spatial and temporal convolution of $g(r,t)$ and $G(r,t)$

The general solution of the above expression, as can be verified by direct substitution, is

$$G'(R,\tau) = f(\tau - R/c) + h(\tau + R/c) \tag{2.29}$$

where $f(\tau - R/c)$ and $h(\tau + R/c)$ are two arbitrary functions of their respective arguments and they represent waves propagating along R in the positive and negative directions, respectively. Therefore

$$G(R,\tau) = \frac{f(\tau - R/c)}{R} + \frac{h(\tau + R/c)}{R} \tag{2.30}$$

The potential that results from substituting Green's function $h(\tau + R/c)/R$ in (2.26) is termed the advanced potential and is a function of the value of the sources at the future observation instant. This advanced potential is clearly not consistent with our ideas about causality, according to which the potential at $(t, \vec{r})$ can depend only on sources at earlier times. Thus, in (2.30) we must consider only the retarded $f(\tau - R/c)/R$ solution as physically meaningful.

To determine $f(\tau - R/c)/R$, we integrate the differential equation (2.23) in a very small volume around the singular point $R = 0$. Thus, taking into account that for $R \to 0$ the function G behaves as $f(\tau)/R$, we have

$$\int_{V'} \left(\nabla^2 G(R,\tau) - \frac{1}{c^2}\frac{\partial^2 G(R,\tau)}{\partial \tau^2} \right)_{R\to 0} dv' \tag{2.31}$$

$$= \int_{V'} \left(\nabla^2 \left(\frac{f(\tau)}{R} \right) - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\left(\frac{f(\tau)}{R} \right) \right) dv' \tag{2.32}$$

$$= -\int_{V'} \delta(\vec{r} - \vec{r}\,')\delta(\tau) dv' = -\delta(\tau) \tag{2.33}$$

or, since $\nabla^2 (1/R) = -4\pi\delta(R)$ and $dv' = 4\pi R^2 dR$,

$$-\int_{V'} 4\pi f(\tau)\delta(R) dv' + \frac{4\pi}{c^2}\int_{V'} R\frac{\partial^2 f(\tau)}{\partial \tau^2} dR = -\delta(\tau) \tag{2.34}$$

As $R \to 0$, the second integral can be eliminated and therefore

$$f(\tau) = \frac{\delta(\tau)}{4\pi} \tag{2.35}$$

As the function f depends on $\tau - R/c$ and $f(\tau) = f(\tau - R/c)|_{R=0}$, we have

$$f(\tau - R/c) = \frac{\delta(\tau - R/c)}{4\pi} \tag{2.36}$$

and the solution of (2.24) is given by

$$G(R,\tau) = \frac{\delta(\tau - R/c)}{4\pi R} = \frac{\delta(t - t' - R/c)}{4\pi R} \tag{2.37}$$

This is Green's time-dependent retarded function, which takes into account the time needed for the electromagnetic perturbation to reach the observation point from the point source. Substituting this function in (2.26), we have

$$\Psi(\vec{r},t)=\int_{t'=-\infty}^{t}\int_{V'}g\left(\vec{r}\,',t'\right)\frac{\delta(\tau-R/c)}{4\pi R}dv'dt'. \tag{2.38}$$

and, integrating in t', we finally find that, under the assumption of causality, the solution of the inhomogeneous wave equation for potentials is given by

$$\Psi(\vec{r},t)=\frac{1}{4\pi}\int_{V'}\frac{g\left(\vec{r}\,',t-R/c\right)}{R}dv'=\frac{1}{4\pi}\int_{V'}\frac{[g]}{R}dv'. \tag{2.39}$$

where

$$[g]=g(\vec{r}\,',t-\frac{R}{c})=g(\vec{r}\,',t') \tag{2.40}$$

is the value of the source densities evaluated at the retarded times $t'=t-R/c$, which in general are different for each source point, R/c being the delay time due to the finite propagation velocity of the electromagnetic perturbations. In the following the physical magnitudes evaluated in retarded times are shown in brackets.

By analogy with (2.39) the solutions to the inhomogeneous equations for the potentials are

$$\Phi\left(\vec{r},t\right) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{[\rho]}{R}dv' \tag{2.41a}$$

$$\vec{A}\left(\vec{r},t\right) = \frac{\mu_0}{4\pi}\int_{V'}\frac{[\vec{J}]}{R}dv' \tag{2.41b}$$

where the bracket symbol [ ] indicates that the enclosed magnitude must be evaluated at the retarded time $t'=t-R/c$. That is

$$[\rho] = \rho(\vec{r}\,',t')=\rho(\vec{r}\,',t-R/c) \tag{2.42}$$

$$[\vec{J}] = \vec{J}(\vec{r}\,',t')=\vec{J}(\vec{r}\,',t-R/c) \tag{2.43}$$

are the charge and current densities, respectively, evaluated in the retarded times t'.

Expressions (2.41a) and (2.41b), which are called retarded potentials, indicate that the potentials created by a distribution at the field point P are determined, at a given time t, by the values of the the charge and current densities at the source points evaluated at previous times t', which generally differ for each source poin. It is easy to check that these potentials, together with the continuity equation, verify Lorenz's condition (2.10).

It should be noted that (2.39) is a particular solution of (2.21), to which a complementary solution of the homogeneous wave equation $\Box\Psi(\vec{r},t)=0$ can be added in order to arrive at other possible solutions of (2.39). Thus, other conditions must be imposed to ensure that the only possible solution of (2.21) is (2.39). These conditions, establishing the uniqueness of (2.39), can be found in Appendix **??**.

For sources with time-harmonic dependence

$$\rho(\vec{r}', t) = \mathrm{Re}\left\{\boldsymbol{\rho}(r')e^{j\omega t})\right\} \tag{2.44}$$

$$\vec{J}(\vec{r}', t) = \mathrm{Re}\{\vec{\boldsymbol{J}}(\vec{r}')e^{j\omega t}\} \tag{2.45}$$

the expressions of the retarded potentials Φ and $\vec{A}$ simplify to

$$\vec{A}(\vec{r}, t) = \frac{\mu_o}{4\pi}\mathrm{Re}\left\{\int_{V'}\frac{1}{R}\vec{\boldsymbol{J}}(\vec{r}')e^{j\omega(t-\frac{R}{c})}dv'\right\} = \mathrm{Re}\{\vec{\boldsymbol{A}}(\vec{r})e^{j\omega t}\} \tag{2.46a}$$

$$\Phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\mathrm{Re}\left\{\int_{V'}\frac{1}{R}\boldsymbol{\rho}(\vec{r}')e^{j\omega(t-\frac{R}{c})}dv'\right\} = \mathrm{Re}\left\{\boldsymbol{\Phi}(\vec{r})e^{j\omega t}\right\} \tag{2.46b}$$

where

$$\vec{\boldsymbol{A}}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{V'}\frac{1}{R}\vec{\boldsymbol{J}}(\vec{r}')e^{-jkR}dv' \tag{2.47a}$$

$$\boldsymbol{\Phi}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{V'}\frac{1}{R}\boldsymbol{\rho}(\vec{r}')e^{-jkR}dv' \tag{2.47b}$$

where $k = \omega/c = 2\pi/\lambda$ is the wavenumber in the unbounded medium and λ is the wavelength in the medium. For harmonic signals the time delay R/c, when multiplied by ω, becomes a phase shift given by kR.

2.3 Electromagnetic fields from a bounded source distribution

The fields created by a bounded source distribution (charges and currents in free space) of arbitrary time dependence can be determined by inserting (2.41a) and (2.41b) into (2.1) and (2.4). Next, we find the expression for the magnetic field first and for the electric field afterwards[4].

Magnetic field

Starting from the equation

$$\vec{B} = \nabla\times\vec{A} = \frac{\mu_o}{4\pi}\int_{V'}\nabla\times\frac{[\vec{J}]}{R}dv' \tag{2.48}$$

and transforming the integrand by the vector analysis formulas (**??**) and (**??**) of Appendix **??**, with $\Psi = 1/R$ and $\vec{A} = \vec{J}$, we can directly find the magnetic field equation

$$\vec{B}(\vec{r}, t) = \frac{\mu_o}{4\pi}\int_{V'}\left(\frac{[\vec{J}]\times\vec{R}}{R^3} + \frac{1}{c}\frac{\left[\frac{\partial\vec{J}}{\partial t}\right]\times\vec{R}}{R^2}\right)dv' \tag{2.49}$$

[4] An alternative way of obtaining the electromagnetic fields is indicated in Section ??.

where $[\vec{J}]$ is the retarded current density at the source point $\vec{r}'$ and $\left[\partial\ \vec{J}/\partial t\right] = \partial[\vec{J}]/\partial t' = \partial[\vec{J}]/\partial t$ is its time derivative at the instant $t' = t - R/c$.

Expression (2.49) can be written as the sum of the two components $\vec{B} = \vec{B}_{bs} + \vec{B}_{rad}$, which are defined below.

The Biot-Savart term, $\vec{B}_{bs}$:

$$\vec{B}_{bs} = \frac{\mu_o}{4\pi} \int_{V'} \frac{[\vec{J}] \times \vec{R}}{R^3} dv' \tag{2.50}$$

which is formally analogous to the Biot-Savart expression of magnetostatics, although here with the sources evaluated at the retarded times. As this term decreases with $1/R^2$, its contribution is appreciable only at short distances.

The radiation term, $\vec{B}_{rad}$:

$$\vec{B}_{rad} = \frac{\mu_o}{4\pi c} \int_{V'} \frac{\left[\frac{\partial \vec{J}}{\partial t}\right] \times \vec{R}}{R^2} dv' \tag{2.51}$$

which depends on $1/R$, and consequently its contribution to the magnetic field predominates at long distances from the sources.

At the static limit, when the sources do not change with time (i.e., for a stationary current distribution) equation (2.49) simplifies to the Biot-Savart expression of magnetostatics

$$\vec{B}_{bs} = \frac{\mu_o}{4\pi} \int_{V'} \frac{\vec{J} \times \vec{R}}{R^3} dv' \tag{2.52}$$

Electric field

From (2.4) and (2.41) we see that

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0} \int_{V'} \nabla \frac{[\rho]}{R} dv' - \frac{\mu_0}{4\pi} \int_{V'} \frac{\partial}{\partial t} \frac{[\vec{J}]}{R} dv' \tag{2.53}$$

Taking into account that $\partial/\partial t' = \partial/\partial t$ and that $\nabla\Psi\left(R\right) = (d\Psi/dR)\nabla R$ we have

$$\nabla \frac{[\rho]}{R} = [\rho]\,\nabla \frac{1}{R} + \frac{1}{R}\nabla\,[\rho] = [\rho]\left(-\frac{\vec{R}}{R^3}\right) + \frac{\vec{R}}{R^2}\frac{\partial\,[\rho]}{\partial R} \tag{2.54a}$$

$$\frac{\partial\,[\rho]}{\partial R} = \frac{\partial\,[\rho]}{\partial t'}\frac{dt'}{dR} = \left[\frac{\partial \rho}{\partial t}\right]\left(-\frac{1}{c}\right) \tag{2.54b}$$

$$\nabla \frac{[\rho]}{R} = [\rho]\left(-\frac{\vec{R}}{R^3}\right) - \frac{\vec{R}}{cR^2}\left[\frac{\partial \rho}{\partial t}\right] \tag{2.54c}$$

which, when substituted in (2.53), and taking into account the continuity equation $\nabla\cdot\vec{J}=-\partial\rho/\partial t$, gives

$$\vec{E}(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\int_{V'}\left(\frac{[\rho]\,\vec{R}}{R^3}-\frac{1}{c^2}\frac{\left[\frac{\partial\vec{J}}{\partial t}\right]}{R}-\frac{\vec{R}}{R^2c}\left[\nabla'\cdot\vec{J}\right]\right)dv' \qquad (2.55)$$

To get the exact form of the radiation term, which depends on the distance as $1/R$, we need to transform the integrand of this expression by developing $\nabla'\cdot[\vec{J}]$ as[5]

$$[\nabla'\cdot\vec{J}]=\nabla'\cdot[\vec{J}]-\frac{\vec{R}\cdot\left[\frac{\partial\vec{J}}{\partial t}\right]}{cR} \qquad (2.56)$$

thus we can rewrite the third term on the right-hand side of (2.55) as

$$-\int_{V'}\frac{\vec{R}}{R^2c}\left[\nabla'\cdot\vec{J}\right]dv'=-\int_{V'}\frac{\vec{R}}{R^2c}\nabla'\cdot[\vec{J}]dv'+\int_{V'}\frac{\left(\left[\frac{\partial\vec{J}}{\partial t}\right]\cdot\vec{R}\right)\vec{R}}{c^2R^3}dv' \qquad (2.57)$$

The calculation of the first term on the right-hand side can be facilitated by calculating just one component, for example the x component

$$\begin{aligned}
-\int_{V'}\frac{R_x}{R^2c}\nabla'\cdot[\vec{J}]dv' &= \int_{V'}\frac{[\vec{J}]}{c}\cdot\nabla'\frac{R_x}{R^2}dv'-\int_{V'}\nabla'\cdot\left(\frac{R_x}{R^2c}[\vec{J}]\right)dv'\\
&= \int_{V'}\frac{[\vec{J}]}{c}\cdot\nabla'\frac{R_x}{R^2}dv'\\
&= \int_{V'}\frac{[\vec{J}]}{R^2c}\cdot\nabla' R_x dv'+\int_{V'}R_x\frac{[\vec{J}]}{c}\cdot\nabla'\frac{1}{R^2}dv'\\
&= \int_{V'}\left(-\frac{[J_x]}{cR^2}+\frac{2\left([\vec{J}]\cdot\vec{R}\right)R_x}{cR^4}\right)dv' \qquad (2.58)
\end{aligned}$$

where we have used (**??**), applied the divergence theorem, and integrated over an external surface that encloses the sources in which $[\vec{J}]=0$. Therefore, generalizing to three dimensions and inserting the result in (2.55), we get

$$\begin{aligned}
4\pi\varepsilon_o\vec{E} &= \int_{V'}\frac{[\rho]\,\vec{R}}{R^3}dv'+\int_{V'}\left(\frac{2\left([\vec{J}]\cdot\vec{R}\right)\vec{R}-[\vec{J}]\left(\vec{R}\cdot\vec{R}\right)}{cR^4}\right)dv'+\\
&+\frac{1}{c^2}\int_{V'}\frac{\left(\left[\frac{\partial\vec{J}}{\partial t}\right]\times\vec{R}\right)\times\vec{R}}{R^3}dv' \qquad (2.59)
\end{aligned}$$

[5] $\nabla'\cdot[\vec{J}]=(\nabla'\cdot\vec{J})_{t'}+\frac{\partial[\vec{J}]}{\partial t'}\cdot\nabla' t'=(\nabla'\cdot\vec{J})_{t'}-\frac{\partial[\vec{J}]}{\partial t'}\cdot\nabla t'=(\nabla'\cdot\vec{J})_{t'}+\frac{\vec{R}}{cR}\cdot\frac{\partial[\vec{J}]}{\partial t'}$
Thus
$(\nabla'\cdot\vec{J})_{t'}=[\nabla'\cdot\vec{J}]=\nabla'\cdot[\vec{J}]-\frac{\vec{R}\cdot\frac{\partial[\vec{J}]}{\partial t'}}{cR}$

which can be expressed as the sum of the three components $\vec{E} = \vec{E}_c + \vec{E}_i + \vec{E}_{rad}$, which are defined below.

Coulomb's term, $\vec{E}_c$,

$$\vec{E}_c = \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{[\rho]\,\vec{R}}{R^3} dv' \tag{2.60}$$

This term is similar to the static Coulomb's expression except concerning the time delay.

Induction term, $\vec{E}_i$,

$$\vec{E}_i = \frac{1}{4\pi\varepsilon_o} \int_{V'} \left(\frac{2\left([\vec{J}] \cdot \vec{R}\right)\vec{R}}{cR^4} - \frac{[\vec{J}]}{cR^2} \right) dv' \tag{2.61}$$

Because of their dependence on $1/R^2$, the contribution to the field of the terms (2.60) and (2.61) decrease quickly with distance.

Radiation term, $\vec{E}_{rad}$,

$$\vec{E}_{rad} = \frac{1}{4\pi\varepsilon_o c^2} \int_{V'} \frac{\left(\left[\frac{\partial \vec{J}}{\partial t}\right] \times \vec{R}\right) \times \vec{R}}{R^3} dv' = \frac{\mu_0}{4\pi} \int_{V'} \frac{\left(\left[\frac{\partial \vec{J}}{\partial t}\right] \times \vec{R}\right) \times \vec{R}}{R^3} dv' \tag{2.62}$$

This term, which depends on $1/R$, is the electric field component that predominates for long distances. Together with (2.51), this component is of interest in radiation phenomena (see next subsection) .

At the static limit, expression (2.59) simplifies to Coulomb's expression of electrostatics

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\rho\vec{R}}{R^3} dv' \tag{2.63}$$

Alternatively, the electric field can be expressed only in terms of the current density, by using the continuity equation. In fact, from (2.56) we have

$$[\rho] = -\int_{-\infty}^{t} [\nabla' \cdot \vec{J}] dt' = -\int_{-\infty}^{t} \left(\nabla' \cdot [\vec{J}] - \frac{\vec{R} \cdot [\frac{\partial \vec{J}}{\partial t}]}{cR} \right) dt' \tag{2.64}$$

Inserting (2.64) into (2.59) and operating in a similar way to (2.58), we obtain another alternative expression for the electric field created by a bounded source distribution

$$\begin{aligned} \vec{E} &= \frac{1}{4\pi\varepsilon_o} \int_{V'} \int_{-\infty}^{t} \left(\frac{3\left([\vec{J}] \cdot \vec{R}\right)\vec{R}}{R^5} - \frac{[\vec{J}]}{R^3} \right) dt' dv' \\ &+ \frac{1}{4\pi\varepsilon_o} \int_{V'} \left(\frac{3\left([\vec{J}] \cdot \vec{R}\right)\vec{R}}{cR^4} - \frac{[\vec{J}]}{cR^2} \right) dv' \\ &+ \frac{1}{4\pi\varepsilon_o} \frac{1}{c^2} \int_{V'} \frac{\left(\left[\frac{\partial \vec{J}}{\partial t}\right] \times \vec{R}\right) \times \vec{R}}{R^3} dv' \end{aligned} \tag{2.65}$$

Fields created by a time-harmonic source distribution

For time-harmonic dependence of the sources, the field expressions (2.4) and (2.1) simplify to

$$\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}} \tag{2.66a}$$
$$\vec{\boldsymbol{E}} = -\nabla \boldsymbol{\Phi} - j\omega \vec{\boldsymbol{A}} \tag{2.66b}$$

and equation (2.49) for the magnetic field becomes

$$\vec{\boldsymbol{B}} = \frac{\mu_o}{4\pi} \int_{V'} (\vec{\boldsymbol{J}} \times \vec{R}) \left(\frac{1}{R^3} + \frac{jk}{R^2} \right) e^{-jkR} dv'$$

while the different expressions for the electric field, (2.55), (2.59) and (2.65) become, respectively,

$$\begin{aligned} \vec{\boldsymbol{E}} &= \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\boldsymbol{\rho}\, e^{-jkR} \vec{R}}{R^3} dv' + \\ &\quad \frac{jk}{4\pi\varepsilon_o} \int_{V'} \left(\frac{\boldsymbol{\rho}\, \vec{R}}{R} - \frac{\vec{\boldsymbol{J}}(\vec{r}')}{c} \right) \frac{e^{-jkR}}{R} dv' \end{aligned} \tag{2.68a}$$

$$\begin{aligned} \vec{\boldsymbol{E}} &= \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\boldsymbol{\rho}\vec{R}}{R^3} e^{-jkR} dv' + \frac{1}{4\pi\varepsilon_o} \int_{V'} \left(\frac{2\left(\vec{\boldsymbol{J}} \cdot \vec{R}\right) \vec{R}}{cR^4} - \frac{\vec{\boldsymbol{J}}}{cR^2} \right) e^{-jkR} dv' + \\ &\quad + \frac{jk}{4\pi\varepsilon_o c} \int_{V'} \frac{\left(\vec{\boldsymbol{J}} \times \vec{R}\right) \times \vec{R}}{R^3} e^{-jkR} dv' \end{aligned} \tag{2.68b}$$

$$\begin{aligned} \vec{\boldsymbol{E}} &= \frac{j}{4\pi\omega\varepsilon_o} \int_{V'} \left(\frac{\vec{\boldsymbol{J}}}{R^3} - \frac{3\left(\vec{\boldsymbol{J}} \cdot \vec{R}\right) \vec{R}}{R^5} \right) e^{-jkR} dv' + \\ &\quad \frac{1}{4\pi\varepsilon_o} \int_{V'} \left(\frac{3\left(\vec{\boldsymbol{J}} \cdot \vec{R}\right) \vec{R}}{cR^4} - \frac{\vec{\boldsymbol{J}}}{cR^2} \right) e^{-jkR} dv' + \\ &\quad + \frac{jk}{4\pi\varepsilon_o c} \int_{V'} \frac{\left(\vec{\boldsymbol{J}} \times \vec{R}\right) \times \vec{R}}{R^3} e^{-jkR} dv' \end{aligned} \tag{2.68c}$$

and the radiation fields (2.51) and (2.62) become

$$\vec{\boldsymbol{B}} = \frac{j\omega\mu_o}{4\pi c} \int_{V'} \frac{\vec{\boldsymbol{J}} \times \vec{R}}{R^2} e^{-jkR} dv' \tag{2.69a}$$
$$\vec{\boldsymbol{E}} = \frac{j\omega\mu_o}{4\pi} \int_{V'} \frac{\left(\vec{\boldsymbol{J}} \times \vec{R}\right) \times \vec{R}}{R^3} e^{-jkR} dv' \tag{2.69b}$$

2.3.1 Radiation fields

Examining the total fields (2.49) and (2.65) generated by a bounded distribution of sources with arbitrary time dependence, we find that in general the near-zone terms, which depend on $1/R^n$ $(n > 1)$, are negligible compared to the radiation terms, (2.51) and (2.62), which depend on $1/R$, when the condition

$$R >> c\frac{\left|[\vec{J}]\right|}{\left|d[\vec{J}]/dt\right|} \tag{2.70}$$

is fulfilled for any of the infinitesimal volume elements into which the source can be subdivided. For time-harmonic fields, this condition becomes

$$R >> \lambda \tag{2.71}$$

Hence, the radiation term predominates when distances from the sources are great compared to any wave-length involved. The zone where the radiation fields predominate can be called by several names: far zone, wave zone and Fraunhofer zone. Note that the far zone is farther away from the sources at lower time dependence (i.e., at lower frequencies) and there is no far zone at the static limit.

Let us select the reference origin close to or within the source distribution, (Fig. 2.1). If the field point is far away from any source point such that $r >> r'$, or equivalently $r >> l$, where l is the largest dimension of the source distribution, then it is possible to make some general approximations in the expressions (2.51) and (2.62) which greatly simplify the calculations. To confirm this, let us write R in Fig. 2.1 as

$$R = |\vec{r} - \vec{r}'| = \left(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2\right)^{1/2} \tag{2.72}$$

Since the reference origin is close to or within the source distribution, we can calculate the radiation fields at distances $r >> r'$ by expanding the binomial (2.72) as a series in powers of the small parameter r'/r and take only the linear terms of the expansion

$$R \;=\; r\left(1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2} = r - \frac{\vec{r}\cdot\vec{r}'}{r} + ... \simeq r - \vec{r}'\cdot\hat{r} = r - r'\cos\theta \tag{2.73}$$

where θ is the angle between $\hat{r}$ and $\vec{r}'$. This approximation is equivalent to considering that, far away from the sources, r and R become parallel.

Thus, as $r'/r << 1$, in the expressions (2.51) and (2.62), we can make the approximation

$$R \simeq r \tag{2.74}$$

in the denominator. This is equivalent to ignoring, in the modulus of the contribution of each source point to the total field, the difference in the distance travelled by the signal. Thus (2.70) becomes

$$r >> c\frac{\left|[\vec{J}]\right|}{\left|d[\vec{J}]/dt\right|} \tag{2.75}$$

and (2.71) becomes

$$r >> \lambda \tag{2.76}$$

In the retarded time, $t' = t - R/c$, the approximation (2.74) is not valid because the sources can be very sensitive to small changes in the delay time R/c. Thus, for the delay time, at distances $r >> r'$ we need to keep at least the two linear terms of the expansion (2.73). Therefore

$$t' = t - \frac{R}{c} = t - \frac{r}{c} + \frac{\vec{r}' \cdot \hat{r}}{c} = t_0' + \frac{r' \cos\theta}{c} \tag{2.77}$$

where $t_0' = t - r/c$.

Therefore, from (2.77), the retarded time has two components. One, r/c, is the time needed for the electromagnetic field to reach the field point from the origin of the coordinates. The other, $\vec{r}' \cdot \hat{r}/c$, represents the time necessary for the propagation of the electromagnetic perturbation within the geometric limits of the source distribution. This term, given that the largest dimension of the source distribution is l, (Fig. 2.1), has a magnitude of

$$\vec{r}' \cdot \hat{r}/c \sim l/c << r/c \tag{2.78}$$

Hence, using the approximations (2.74) and (2.77) the integrands of the radiation fields (2.51) and (2.62) simplify to

$$\vec{B}_{rad} = \frac{\mu_o}{4\pi cr}\int_{V'} \frac{\partial \vec{J}(\vec{r}', t_0' + \frac{\vec{r}' \cdot \hat{r}}{c})}{\partial t} \times \hat{r} dv' \tag{2.79a}$$

$$\vec{E}_{rad} = \frac{1}{4\pi\varepsilon_o c^2 r}\int_{V'} \left(\frac{\partial \vec{J}(\vec{r}', t_0' + \frac{\vec{r}' \cdot \hat{r}}{c})}{\partial t} \times \hat{r}\right) \times \hat{r} dv' \tag{2.79b}$$

or, for time-harmonic dependence,

$$\boldsymbol{\vec{E}}_{rad} = \frac{j\omega\mu_o}{4\pi r}\int_{V'} \left(\boldsymbol{\vec{J}} \times \hat{r}\right) \times \hat{r}\ e^{-jkR} dv' \tag{2.80a}$$

$$\boldsymbol{\vec{B}}_{rad} = \frac{j\omega\mu_o}{4\pi cr}\int_{V'} \boldsymbol{\vec{J}} \times \hat{r}\ e^{-jkR} dv' = \frac{jk\eta_o}{4\pi cr}\int_{V'} \boldsymbol{\vec{J}} \times \hat{r}\ e^{-jkR} dv' \tag{2.80b}$$

A comparison of Eqs. (2.79a) and (2.79b), shows that the radiation fields are perpendicular to each other and to the direction of propagation. They are related by

$$\vec{E} = \eta_0 \vec{H} \times \hat{r} \tag{2.81}$$

where the ratio η_0 is defined as

$$\eta_0 = \frac{E}{H} \quad = (\mu_o/\varepsilon_o)^{\frac{1}{2}} = 120\pi \quad \Omega \tag{2.82}$$

and is called the intrinsic impedance of free space.

According to Poynting's theorem the total radiated energy passing through the unit area perpendicular to the direction of the vector $\vec{E}_{rad} \times \vec{H}_{rad}$ is given by

$$\int_{-\infty}^{t} \vec{\mathcal{P}}_{rad} dt = \int_{-\infty}^{t} (\vec{E}_{rad} \times \vec{H}_{rad}) dt \tag{2.83}$$

and the total flow of power passing through the closed surface S situated in the far-field zone is

$$\int_{-\infty}^{t} \int_{S} \vec{\mathcal{P}}_{rad} \cdot d\vec{s} dt = \int_{-\infty}^{t} \int_{S} (\vec{E}_{rad} \times \vec{H}_{rad}) \cdot d\vec{s} dt \tag{2.84}$$

In summary, the assumptions involved in using (2.79) and (2.80) to calculate the radiation fields created by a bounded source distribution in the far-field zone are:

a) $r >> (c\,|[J]|\,/\,|d[J]/dt|)$ or, equivalently, $r >> \lambda$ for any wavelength of the radiation spectrum which allows us to neglect $1/r^2$ terms.

b) $r >> l$, where l is the largest dimension of the source distribution which allows us to make the approximations (2.74) and (2.77).

2.3.2 Fields created by an infinitesimal current element

The simplest case of a bounded source distribution is that of an infinitesimal current element $i(t)$, which is assumed to be oriented on the z axis (Fig. 2.3) and to have arbitrary time dependence. This current is mathematically defined, in terms of the Dirac delta function, as

$$\vec{J}(\vec{r}, t) = \; i(t)\delta(x')\delta(y')\hat{z} \qquad -\frac{\Delta z}{2} < z' < \frac{\Delta z}{2} \tag{2.85}$$

The fields of this current element can be easily calculated by substituting (2.85) in (2.49) and (2.65). Thus, we have

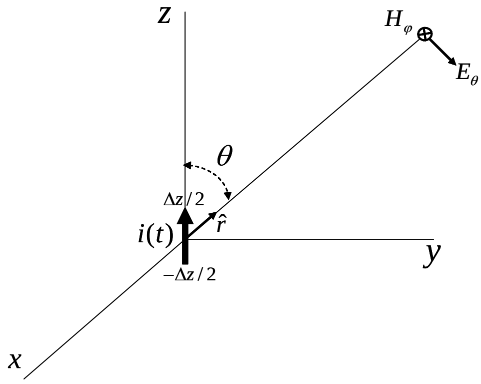

Figure 2.3: Infinitesimal current element solo campos de radiacion ¡ ¡¡falta la r del radio vector del punto campo

Fields created by an infinitesimal current element with arbitrary-time dependence:

$$\vec{H}(\vec{r},t) = \frac{\Delta z}{4\pi}\left(\frac{1}{cr}\frac{d\,[i]}{dt}+\frac{[i]}{r^2}\right)(\hat{z}\times\hat{r}) = \frac{\Delta z}{4\pi}\left(\frac{1}{cr}\frac{d\,[i]}{dt}+\frac{[i]}{r^2}\right)\sin\theta\;\hat{\varphi} \tag{2.86a}$$

$$\vec{E}(\vec{r},t) = \frac{\Delta z}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\int_{-\infty}^{t}[i]\,dt+\frac{[i]}{cr^2}\right)(3\,(\hat{z}\cdot\hat{r})\,\hat{r}-\hat{z}) +$$

$$\frac{\Delta z}{4\pi\varepsilon_o}\frac{1}{c^2 r}\frac{d\,[i]}{dt}\,(\hat{r}\times(\hat{r}\times\hat{z})) = \frac{\Delta z}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\int_{-\infty}^{t}[i]\,dt+\frac{[i]}{cr^2}\right)(2\cos\theta\;\hat{r}+\sin\theta\hat{\theta}\;) + \frac{\Delta z}{4\pi\varepsilon_o}\frac{1}{c^2 r}\frac{d\,[i]}{dt}\sin\theta\hat{\theta} \tag{2.86b}$$

where $[i] = i(t-r/c)$.

For time-harmonic dependence of the current element, $i = \mathrm{Re}\left\{\boldsymbol{I}e^{j\omega t}\right\}$, equations (2.86a) and (2.86b) simplify to

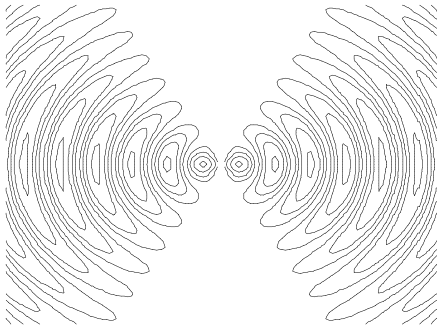

Figure 2.4: Radiation field separates from the source and propagates to infinity Dibujar el dipolo. Note that there is not radiation in the direction in which the current element is pointing. pp 259 del panofsky :-Como puede verse en la figura las lineas de campo de raciación representa una familia de lazos, atravezados por las líneas de campo magnético, que se propagan hacia el infinito (i.e. waves see chapter tal)

Fields created by an infinitesimal current element with time-harmonic dependence:

$$\vec{\boldsymbol{H}}(\vec{r}) = \frac{\boldsymbol{I}\Delta z}{4\pi} jk \left(1 + \frac{1}{jkr}\right) \frac{e^{-jkr}}{r} \sin\theta \; \hat{\varphi} \tag{2.87a}$$

$$\begin{aligned}\vec{\boldsymbol{E}}(\vec{r}) &= \frac{\boldsymbol{I}\Delta z}{4\pi} jk\eta_0 \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) \frac{e^{-jkr}}{r} \sin\theta \; \hat{\theta} + \\ &\quad \frac{\boldsymbol{I}\Delta z}{2\pi} jk\eta_0 \left(\frac{1}{jkr} - \frac{1}{(kr)^2}\right) \frac{e^{-jkr}}{r} \cos\theta \; \hat{r}\end{aligned} \tag{2.87b}$$

These expressions can be also derived directly from the vector potential (2.41b), which in this case simplifies to

$$\vec{A} = \hat{z}\frac{\mu_0}{4\pi} \int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} \frac{[i]}{r} dz' \simeq \hat{z}\frac{[i]\mu_0}{4\pi r} \Delta z \tag{2.88}$$

Thus, the magnetic field is given by[6]

$$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} = \frac{1}{\mu_0}\nabla\times(A\hat{z}) = \frac{1}{\mu_0}(\nabla A\times\hat{z} + A(\nabla\times\hat{z})) = \frac{1}{\mu_0}\nabla A\times\hat{z} \tag{2.89}$$

where we have applied the vector identity (??) and taken into account that the curl of a constant vector is zero. Hence, using spherical coordinates, we get

$$\begin{aligned}\vec{H} &= \frac{1}{\mu_0}\nabla A\times\hat{z} = \frac{\Delta z}{4\pi}\frac{\partial}{\partial r}\left(\frac{[i]}{r}\right)\hat{r}\times\hat{z}\\ &= \frac{\Delta z}{4\pi}\left(-\frac{1}{cr}\frac{d\,[i]}{dt} - \frac{[i]}{r^2}\right)\hat{r}\times\hat{z} = \frac{\Delta z}{4\pi}\left(\frac{1}{cr}\frac{d\,[i]}{dt} + \frac{[i]}{r^2}\right)\sin\theta\hat{\varphi}\end{aligned} \tag{2.90}$$

which of course coincides with (2.86a).

The electric field (2.86b) can be calculated from (2.90), taking into account that from (1.1d), in source-free regions, we have[7]

$$\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\int_{-\infty}^{t}\nabla\times\vec{H}(\vec{r},t)\ dt \tag{2.93}$$

From the relation between the charge and current, $i(t) = dq(t)/dt$, we have

$$i\triangle z = \frac{dq}{dt}\triangle z = \frac{dp}{dt} = \dot{p} \tag{2.94}$$

where $p = q\triangle z$ is the dipole moment of a time-varying electric dipole[8], the so-called Hertzian dipole, formed by two point charges with values of $+q(t)$

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[6] For a given vector field $\vec{A}$, the field lines are defined by the condition that, at any point, the line element $d\vec{l}$ and the field are parallel i.e. $d\vec{l}\times\vec{A} = 0$. For the field created by a current element, from Eqs (2.87a), the magnetic field has only $\hat{\varphi}$ component and consequently their field lines are closed around the Z axis. The radiation electric field has $\hat{\theta}$ and $\hat{r}$ components, although the radiation electric field has only $\hat{\theta}$ component which becomes null in the region $\theta\to 0$. Then in this region predominates the $E_r = \vec{E}\cdot\hat{r}$ component and consequently the electric field lines close(see Fig 2.4) as would be expected from Maxwell's equations since, outside the sources, there only exist curl sources.

[7] Note that once calculed $\vec{H} = 1/\mu_0\nabla\times\vec{A}$ we can obtain $\vec{E}$ using (1.1d) or (1.67d) and taking into account that, in source-free regions, we have

$$\vec{E} = \frac{1}{\varepsilon_0}\int\nabla\times\vec{H}\ dt\ = c\int\nabla\times(\nabla\times\boldsymbol{\vec{A}})\ dt \tag{2.91}$$

or

$$\boldsymbol{\vec{E}} = \frac{1}{j\varepsilon_0\omega}\nabla\times\boldsymbol{\vec{H}} = \frac{1}{jk}\nabla\times(\nabla\times\boldsymbol{\vec{A}}) \tag{2.92}$$

for arbitrary or harmonic time dependence respectively. Thus we do not need necessarily to calculate Φ to obtain the fields.

[8] The time varying electric dipole is defined as two time varying charges of opposite magnitude $\pm q(t)$ separated by a constant distance Δz much less than the field point r. The dipole moment $\vec{p}(t)$ is given by the magnitude of the charge times the distance Δz between them and the defined direction is toward the positive charge i.e. $\vec{p}(t) = q(t)\Delta z$.. Alternatively it would be possible to model the oscillating dipole as two constant point charges of opposite sign separated by oscillating distance $\Delta z(t)$. However, the fields created for such accelerated charges need from the theory developed in Chapter ??.

and $-q(t)$ and the dot indicates differentiation with respect to time. Thus the time-varying current element is equivalent to

$$i(t) = \frac{1}{\triangle z}\frac{dp}{dt} \tag{2.95}$$

or, for time-harmonic dependence,

$$\boldsymbol{I} = \frac{j\omega \boldsymbol{p}}{\triangle z} \tag{2.96}$$

Introducing (2.95) into (2.86a) and (2.86b), and (2.96) into (2.87a) and (2.87b), we get the field created by an infinitesimal current element (hertzian dipole) in terms of its dipole moment as:

Fields created by a Hertzian dipole with arbitrary-time dependence:

$$\vec{H} = \frac{1}{4\pi r}\left(\frac{[\dot{p}]}{r} + \frac{[\ddot{p}]}{c}\right)\sin\theta\hat{\varphi} \tag{2.97a}$$

$$\vec{E} = \frac{1}{4\pi r\varepsilon_0}\left(\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc} + \frac{[\ddot{p}]}{c^2}\right)\sin\theta\hat{\theta} + \frac{1}{2\pi r\varepsilon_o}\left(\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right)\cos\theta\hat{r} \tag{2.97b}$$

Fields created by a Hertzian dipole with time-harmonic dependence:

$$\vec{\boldsymbol{H}} = \frac{j\omega \boldsymbol{p}}{4\pi}\left(\frac{1}{r} + jk\right)\frac{e^{-jkr}}{r}\sin\theta\hat{\varphi} \tag{2.98a}$$

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{p}}{4\pi\varepsilon_0}\left(\frac{1}{r^2} + \frac{jk}{r} - k^2\right)\frac{e^{-jkr}}{r}\sin\theta\hat{\theta} + \frac{\boldsymbol{p}}{2\pi\varepsilon_o}\left(\frac{1}{r^2} + \frac{jk}{r}\right)\frac{e^{-jkr}}{r}\cos\theta\hat{r} \tag{2.98b}$$

The radiation fields created by an infinitesimal current element can be expressed, from (2.86a) to (2.87b), in terms of its current amplitude or of its equivalent dipolar moment.

Radiation fields created by an infinitesimal current element:

For arbitrary-time dependence

$$\vec{H}_{rad} = \frac{\Delta z}{4\pi}\frac{1}{cr}\frac{d\,[i]}{dt}\sin\theta\hat{\varphi} \tag{2.99a}$$

$$\vec{E}_{rad} = \frac{\Delta z}{4\pi\varepsilon_o}\frac{1}{c^2 r}\frac{d\,[i]}{dt}\sin\theta\hat{\theta} \tag{2.99b}$$

For time-harmonic dependence

$$\vec{\boldsymbol{H}}_{rad} = \frac{\boldsymbol{I}\Delta z}{4\pi} jk \frac{e^{-jkr}}{r} \sin\theta\hat{\varphi} \tag{2.100a}$$
$$\vec{\boldsymbol{E}}_{rad} = \frac{\boldsymbol{I}\Delta z}{4\pi} jk\eta_0 \frac{e^{-jkr}}{r} \sin\theta\hat{\theta} \tag{2.100b}$$

and from (2.97a) to (2.98b), we have the radiation fields in terms of its equivalent Hertzian dipole:

Radiation fields created by an electric dipole:

For arbitrary-time dependence

$$\vec{H}_{rad} = \frac{1}{4\pi r}\frac{[\ddot{p}]}{c} \sin\theta\hat{\varphi} \tag{2.101a}$$
$$\vec{E}_{rad} = \frac{1}{4\pi r\varepsilon_0}\frac{[\ddot{p}]}{c^2} \sin\theta\hat{\theta} \tag{2.101b}$$

For time-harmonic dependence

$$\vec{\boldsymbol{H}}_{rad} = -\frac{\omega \boldsymbol{p} k}{4\pi}\frac{e^{-jkr}}{r} \sin\theta\hat{\varphi} \tag{2.102a}$$
$$\vec{\boldsymbol{E}}_{rad} = \frac{-\boldsymbol{p}k^2}{4\pi\varepsilon_0}\frac{e^{-jkr}}{r} \sin\theta\hat{\theta} \tag{2.102b}$$

More details about this elemental radiators and how they can be physical approximated are given in subsubsection **??** of chapter **??**.

2.3.3 Far-zone approximations for the potentials

The general expressions (2.49) and (2.59) for the fields due to an arbitrary source distribution of finite size are of theoretical and sometimes of practical interest. However, except for the case of the infinitesimal current element, it is much easier to calculate the fields created by a given source distribution via the potentials, as indicated in Fig. 2.2. This can be seen simply by comparing the complexity of the expressions for these fields, (2.49) and (2.59), with those for the potentials (2.41a) and (2.41b). Because of the vector product in the integrand of (2.51) and (2.62), this argument continues being true even when we are interested only in the radiation fields. In the far zone, we can make the approximations (2.74) and (2.77) for the potentials. Hence, the integrands of the retarded potentials (2.41) simplify to

$$\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}',t')}{R}dv' \simeq \frac{1}{4\pi\varepsilon_0 r}\int_{V'}\rho(\vec{r}',t'_0+\frac{\vec{r}'\cdot\hat{r}}{c})dv' \tag{2.103a}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}',t')}{R}dv' \simeq \frac{\mu_0}{4\pi r}\int_{V'}\vec{J}(\vec{r}',t'_0+\frac{\vec{r}'\cdot\hat{r}}{c})dv' \tag{2.103b}$$

The magnetic field can now be calculated from (2.1), using (??), as

$$\begin{aligned}\vec{B} &= \nabla\times\vec{A} = \frac{\mu_0}{4\pi}\int_{V'}\nabla\times\frac{\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})}{r}dv' \\ &= \frac{\mu_0}{4\pi}\int_{V'}\frac{\nabla\times\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})}{r}dv' - \frac{\mu_0}{4\pi}\int_{V'}\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})\times\nabla\frac{1}{r}dv'\end{aligned} \tag{2.104}$$

where, if we are interested only in the radiation field, the second term can be ignored since it depends on $1/r^2$, and therefore

$$\vec{H} = \frac{\nabla\times\vec{A}}{\mu_0} = \frac{1}{4\pi}\int_{V'}\frac{\nabla\times\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})}{r}dv' \tag{2.105}$$

Furthermore, from (??), we have $\nabla\times\vec{J}(\Psi) = \nabla\Psi\times d\vec{J}/d\Psi$ with $\Psi = t_0'+\vec{r}'\cdot\hat{r}/c$. Thus, it follows that

$$\begin{aligned}\nabla\times\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c}) &= -\nabla\frac{r}{c}\times\frac{\partial\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})}{\partial t} \\ &= -\frac{\hat{r}}{c}\times\frac{\partial\vec{J}(\vec{r}',t_0'+\frac{\vec{r}'\cdot\hat{r}}{c})}{\partial t}\end{aligned} \tag{2.106}$$

and therefore

$$\vec{H} = -\frac{1}{\mu_0 c}\hat{r}\times\frac{\partial\vec{A}}{\partial t} \tag{2.107}$$

which, as would be expected, leads to (2.79a). If the time variations of the sources are harmonic the expressions (2.103a) , (2.103b) and (2.107) become

$$\boldsymbol{\Phi} = \frac{1}{4\pi\varepsilon_0 r}e^{-jkr}\int_{V'}\boldsymbol{\rho}(\vec{r}')e^{j\vec{k}\cdot\vec{r}'}dv' \tag{2.108a}$$

$$\vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi r}e^{-jkr}\int_{V'}\vec{\boldsymbol{J}}(\vec{r}')e^{j\vec{k}\cdot\vec{r}'}dv' \tag{2.108b}$$

$$\vec{\boldsymbol{H}} = -\frac{j\omega}{\mu_0 c}\hat{r}\times\vec{\boldsymbol{A}} \tag{2.108c}$$

The radiation electric field can be calculated from (2.107) or (2.108c) simply using (2.81).

2.4 Multipole expansion for potentials

In many cases, such as the study of most antennas, in order to calculate the radiation fields, we cannot make any approximation concerning the potentials other than those assumed above. For example, we need to carry out the integration in (2.103b) or (2.108b) in order to calculate the vector potential. However, if we assume that the charge distribution does not change appreciably over time

l/c, we may expand the integrands of (2.103) in a Taylor series about t'_0 in terms of the parameter $\vec{r}' \cdot \hat{r}/c$. For example, for the vector potential, we have

$$\vec{J}(\vec{r}', t'_0 + \frac{\vec{r}' \cdot \hat{r}}{c}) = \vec{J}(\vec{r}', t'_0) + \left.\frac{\partial \vec{J}(\vec{r}', t')}{\partial t'}\right|_{t'=t'_0} \frac{\vec{r}' \cdot \hat{r}}{c} + ... \tag{2.109}$$

where we have omitted higher-order terms in $\vec{r}' \cdot \hat{r}/c$. Thus after inserting (2.109) in (2.103b), we can write $\vec{A}$ as the power-series expansion

$$\begin{aligned} \vec{A} &\simeq \vec{A}_1 + \vec{A}_2 + ... = \\ &\frac{\mu_0}{4\pi r} \int_{V'} \vec{J}(\vec{r}', t'_0) dv' + \frac{\mu_0}{4\pi c r} \int_{V'} \left.\frac{\partial \vec{J}(\vec{r}', t')}{\partial t'}\right|_{t'=t'_0} \vec{r}' \cdot \hat{r} dv' + ... \end{aligned} \tag{2.110}$$

Therefore the first two terms of the expansion (2.110), $\vec{A}_1$ and $\vec{A}_2$, are given by

$$\vec{A}_1 = \frac{\mu_0}{4\pi r} \int_{V'} \vec{J}(\vec{r}', t'_0) dv' \tag{2.111a}$$

$$\begin{aligned} \vec{A}_2 &= \frac{\mu_0}{4\pi c r} \int_{V'} \left.\frac{\partial \vec{J}(\vec{r}', t')}{\partial t'}\right|_{t'=t'_0} \vec{r}' \cdot \hat{r} dv' \\ &= \frac{\mu_0}{4\pi c r} \int_{V'} \frac{\partial}{\partial t'} \vec{J}(\vec{r}', t') \vec{r}' \cdot \hat{r} dv' \Big|_{t'=t'_0} \end{aligned} \tag{2.111b}$$

If the time dependence of the sources is sinusoidal the condition that the source distribution does not change appreciably over time l/c is equivalent to assuming that $l/c << T$ (where T is the period of the signal) or equivalently $l/\lambda \; << 1$, i.e., that the dimension of wavelength is much greater than that of the source distribution

$$\lambda \; >> l \tag{2.112}$$

In this case, we can perform the series expansion

$$e^{j\vec{k}\cdot\vec{r}'} = e^{jk\hat{r}\cdot\vec{r}'} \approx 1 + jk\hat{r} \cdot \vec{r}' - \frac{1}{2} k^2 (\hat{r} \cdot \vec{r}')^2 + ... \tag{2.113}$$

which, after substituting in (2.108b), leads to

$$\boldsymbol{\vec{A}} = \boldsymbol{\vec{A}}_1 + \boldsymbol{\vec{A}}_2 + ... \tag{2.114}$$

where

$$\boldsymbol{\vec{A}}_1 = \frac{\mu_0}{4\pi} \frac{e^{-jkr}}{r} \int_{V'} \boldsymbol{\vec{J}}(r') dv' \tag{2.115a}$$

$$\boldsymbol{\vec{A}}_2 = jk \frac{\mu_0}{4\pi} \frac{e^{-jkr}}{r} \int_{V'} \boldsymbol{\vec{J}}(r') \hat{r} \cdot \vec{r}' dv' \tag{2.115b}$$

which are the Fourier transforms of (2.111a) and (2.111b), respectively.

Of course, there are analogous expressions for the terms of Φ

$$\begin{aligned}\Phi &= \Phi_1 + \Phi_2 + ... \\ &= \frac{1}{4\pi\varepsilon_0 r}\int_{V'} \rho(\vec{r}', t_0')dv' + \frac{1}{4\pi\varepsilon_0 cr}\int_{V'} \left.\frac{\partial\rho(\vec{r}', t')}{\partial t'}\right|_{t'=t_0'} \vec{r}'\cdot\hat{r}dv' + ...\end{aligned} \tag{2.116}$$

where

$$\Phi_1 = \frac{1}{4\pi\varepsilon_0 r}\int_{V'} \rho(\vec{r}', t_0')dv' \tag{2.117a}$$

$$\Phi_2 = \frac{1}{4\pi\varepsilon_0 cr}\int_{V'} \left.\frac{\partial\rho(\vec{r}', t')}{\partial t'}\right|_{t'=t_0'} \vec{r}'\cdot\hat{r}dv' \tag{2.117b}$$

Note that, since the contribution of each point source to the integral in (2.117a) is evaluated at the same time t_0', this integral represents the total charge of the source distribution. Thus, if the net charge of the distribution is zero, we have $\Phi_1 = 0$. If the net charge is not zero, the constant, the electrostatic potential Φ_1 created by that charge depends on r^{-2} and consequently it does not contribute to the radiation.

The expansion (2.110) allows us to decompose the electromagnetic field created by a time-varying source distribution of finite dimension in terms of elementary time-varying source distributions, called electric and magnetic multipoles, located at the origin. This is similar to the well-known multipolar expansion of the electrostatics (or magnetostatics) to decompose the field created by a stationary source distribution of charge (or current) in terms of electric (or magnetic) multipoles. However, now the original distribution is time-varying and produces both electric and magnetic fields. Thus, as result of the expansion, we will obtain both, electric and magnetic multipoles. To verify this, we next analyze the first two terms, (2.111a) and (2.111b), of (2.110).

2.4.1 Electric dipolar radiation

The evaluation of the term (2.111a) of the power-series expansion of $\vec{A}$ can be facilitated by calculating just one component of $\int_{V'} \vec{J}(\vec{r}', t_0')dv'$, for example the x component

$$\begin{aligned}\int_{V'} J_x(\vec{r}', t_0')dv' &= \int_{V'} \vec{J}(\vec{r}', t_0')\cdot\hat{x}dv' = \int_{V'} \vec{J}(\vec{r}', t_0')\cdot\nabla' x' dv' \\ &= \int_{V'} \nabla'\cdot\left(x'\vec{J}(\vec{r}', t_0')\right)dv' - \int_{V'} x'\nabla'\cdot\vec{J}(\vec{r}', t_0')dv' \\ &= -\int_{V'} x'\nabla'\cdot\vec{J}(\vec{r}', t_0')dv'\end{aligned} \tag{2.118}$$

since

$$\int_{V'} \nabla'\cdot\left(x'\vec{J}(\vec{r}', t_0')\right)dv' = 0 \tag{2.119}$$

as can be seen by applying the divergence theorem and by integrating over an external surface, where $\vec{J}(\vec{r}', t_0') = 0$, that encloses the sources. Therefore, generalizing to three dimensions we have

$$\int_{V'} \vec{J}(\vec{r}', t_0')dv' = -\int_{V'} \vec{r}'\nabla' \cdot \vec{J}(\vec{r}', t_0')dv' \tag{2.120}$$

and using the equation of continuity

$$\nabla' \cdot \vec{J}(\vec{r}', t_0') = -\frac{\partial\rho(\vec{r}', t_0')}{\partial t} \tag{2.121}$$

we get

$$\int_{V'} \vec{J}(\vec{r}', t_0')dv' = \int_{V'} \vec{r}'\frac{\partial\rho(\vec{r}', t_0')}{\partial t}dv' \tag{2.122}$$

which, when substituted in (2.111a), gives

$$\vec{A}_1 = \frac{\mu_0}{4\pi r}\int_{V'} \vec{r}'\frac{\partial\rho(\vec{r}', t_0')}{\partial t}dv' = \frac{\mu_0}{4\pi r}\frac{\partial}{\partial t}\int_{V'} \vec{r}'\rho(\vec{r}', t_0')dv' \tag{2.123}$$

The integral $\int_{V'} \vec{r}'\rho(\vec{r}', t_0')dv'$ is by definition the electric dipole moment, $[\vec{p}]$, evaluated at the retarded time t_0', of the time-varying source distribution, i.e.,

$$[\vec{p}] = \int_{V'} \vec{r}'\rho(\vec{r}', t_0')dv' = \int_{V'} \vec{r}'\rho(\vec{r}', t - \frac{r}{c})dv' \tag{2.124}$$

Thus we have

$$\vec{A}_1 = \frac{\mu_0}{4\pi r}\frac{\partial[\vec{p}]}{\partial t} = \frac{\mu_0[\dot{\vec{p}}]}{4\pi r} \tag{2.125}$$

The magnetic radiation field, from (2.107), is given by

$$\vec{H}_{rad} = -\frac{\hat{r} \times [\ddot{\vec{p}}]}{4\pi rc} = \frac{[\ddot{p}]\sin\theta}{4\pi rc}\hat{\varphi} \tag{2.126}$$

where we have assumed the direction of $\vec{p}$ parallel to the polar z axis. This expression, as might be expected, coincides with the radiation term, (2.101a), of (2.97a). From (2.126), the electric radiation field, given by (2.101b), can be obtained using (2.81). Of course the corresponding expressions for time-harmonic fields are given by (2.102a) and (2.102b). Therefore, in a preliminary approximation, the original source distribution can be replaced by an electric dipole located at the origin of coordinates.

2.4.2 Magnetic dipolar radiation

The analysis of the term (2.111b), can be facilitated by expressing the integrand as follows

$$\begin{aligned}\vec{J}(\vec{r}',t_0')(\hat{r}\cdot\vec{r}') &= \frac{1}{2}\left(\vec{J}(\vec{r}',t_0')(\vec{r}'\cdot\hat{r})-\vec{r}'\left(\vec{J}(\vec{r}',t_0')\cdot\hat{r}\right)\right)\\ &\quad+\frac{1}{2}\left(\vec{J}(\vec{r}',t_0')(\vec{r}'\cdot\hat{r})+\vec{r}'\left(\vec{J}(\vec{r}',t_0')\cdot\hat{r}\right)\right)\\ &= \frac{1}{2}\hat{r}\times\left(\vec{J}(\vec{r}',t_0')\times\vec{r}'\right)+\frac{1}{2}\left(\vec{J}(\vec{r}',t_0')(\hat{r}\cdot\vec{r}')+\vec{r}'\left(\vec{J}(\vec{r}',t_0')\cdot\hat{r}\right)\right)\end{aligned} \tag{2.127}$$

Then, substituting in $\vec{A}_2$, we get

$$\vec{A}_2 = \vec{A}_{2m} + \vec{A}_{2q} \tag{2.128}$$

where

$$\vec{A}_{2m} = \frac{\mu_0}{8\pi cr}\int_{V'}\frac{\partial}{\partial t}\hat{r}\times\left(\vec{J}(\vec{r}',t_0')\times\vec{r}'\right)dv' \tag{2.129}$$

and

$$\vec{A}_{2q} = \frac{\mu_0}{8\pi cr}\frac{\partial}{\partial t}\int_{V'}\left(\vec{J}(\vec{r}',t_0')(\hat{r}\cdot\vec{r}')+\vec{r}'\left(\vec{J}(\vec{r}',t_0')\cdot\hat{r}\right)\right)dv' \tag{2.130}$$

The integral (2.129) can be written as

$$\vec{A}_{2m} = \frac{\mu_0}{4\pi cr}\frac{\partial[\vec{m}]}{\partial t}\times\hat{r} \tag{2.131}$$

where

$$[\vec{m}] = \int_{V'}\frac{\vec{r}'\times\vec{J}(\vec{r}',t_0')}{2}dv' \tag{2.132}$$

is by definition the magnetic dipolar moment about O, evaluated at the retarded time t_0', of the source distribution. Thus, under the assumption that $\vec{m} = m\hat{z}$, the magnetic radiation field given by (2.107) is

$$\vec{H}_{rad} = \frac{1}{4\pi c^2 r}\hat{r}\times(\hat{r}\times[\ddot{\vec{m}}]) = \frac{1}{4\pi r}\frac{[\ddot{m}]}{c^2}\sin\theta\hat{\theta} \tag{2.133}$$

From (2.81) the electric radiation field is given by

$$\vec{E}_{rad} = -\frac{\mu_0}{4\pi r}\frac{[\ddot{m}]}{c}\sin\theta\hat{\varphi} \tag{2.134}$$

For time-harmonic dependence, we have

$$\vec{\boldsymbol{H}}_{rad} = \frac{-k^2\boldsymbol{m}\sin\theta}{4\pi r}e^{-jkr}\hat{\theta} \tag{2.135}$$

and

$$\vec{\boldsymbol{E}}_{rad} = \frac{k^2 \boldsymbol{m} \sin\theta}{4\pi r} \eta_0 e^{-jkr} \hat{\varphi} \tag{2.136}$$

where

$$\vec{\boldsymbol{m}} = \int_{V'} \frac{\vec{r}' \times \vec{\boldsymbol{J}}(r')}{2} dv' \tag{2.137}$$

These expressions are similar to (2.101a)-(2.102b), which were obtained for the electric field of the radiation of the electric dipole. In fact, as we will see in the next section, there exists a duality in the analysis of the electric and magnetic dipoles.

In the particular case of a current loop of radius a , Fig. **??**, for which the current i does not change appreciably over time a/c (or equivalently $a << \lambda$ for any frequency involved), (2.132), becomes

$$\vec{m} = i \int_{\Gamma} \frac{\vec{r}' \times d\vec{l}}{2} = i\vec{S} \tag{2.138}$$

where Γ is the countour of loop and $\vec{S}$ is the vector area of the surface subtended by the contour Γ. In this expression, $\vec{J}dv$ has been changed to $id\vec{l}$. The surface vector $\vec{S}$ is directed normal to the loop according to the right-hand rule for the direction of the current in the loop. Thus, for the circular current loop, the radiation fields (2.133)-(2.136), can be written, for arbitrary time dependence, as

$$\vec{H}_{rad} = \frac{S}{4\pi r} \frac{[\ddot{i}]}{c^2} \sin\theta \hat{\theta} \tag{2.139a}$$

$$\vec{E}_{rad} = -\frac{\mu_0 S}{4\pi r} \frac{[\ddot{i}]}{c} \sin\theta \hat{\varphi} \tag{2.139b}$$

where i is evaluated at t_0'. These equations, for time-harmonic dependence become

$$\vec{\boldsymbol{H}}_{rad} = \frac{-k^2 \boldsymbol{I} S \sin\theta}{4\pi r} e^{-jkr} \hat{\theta} \tag{2.140a}$$

$$\boldsymbol{E}_{rad} = \frac{k^2 \boldsymbol{I} S \sin\theta}{4\pi r} \eta_0 e^{-jkr} \hat{\varphi} \tag{2.140b}$$

It should be mentioned that the magnetic moment is important only when there exists no radiation of the electric moment of the system. Otherwise the one due to the magnetic moment may be ignored. Effectively, comparing Eqs. (2.101b) and (2.134), and using E_p and E_m to indicate the amplitudes of the electric radiation fields from an electric and a magnetic dipole, respectively, we have

$$\frac{E_{p_{rad}}}{E_{m_{rad}}} = \frac{c\ddot{p}}{\ddot{m}} \tag{2.141}$$

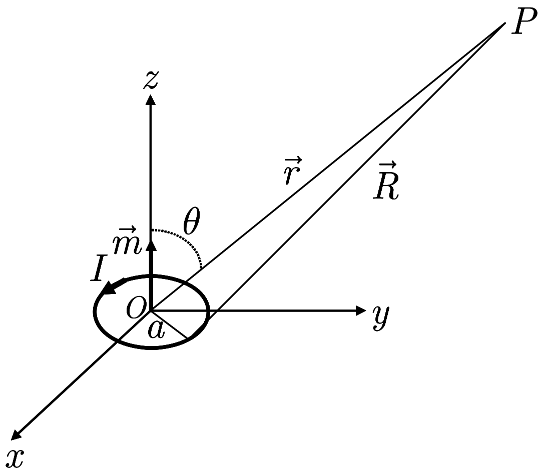


Figure 2.5: Poner $\vec{m} = i\vec{S}$, poner i en vez de I y el contour Γ. A circular loop of current in the x-y plane y dibujar campos como en el elemento de corriente. Hacer el dibujo igual que el del electrico

or for time-harmonic variation with both dipoles oscillating at the same frequency,

$$\frac{E_{p_{rad}}}{E_{m_{rad}}} = \frac{cp_0}{m_0} \tag{2.142}$$

Since from (2.137) we have

$$\vec{m}_0 = \int_{V'} \frac{\vec{r}' \times \vec{J}_0}{2} dv' = \frac{1}{2} \int_{V'} \rho_0 \vec{r}' \times \vec{u} dv' \tag{2.143a}$$

$$\vec{p}_0 = \int_{V'} \rho_0 \vec{r}' dv' \tag{2.143b}$$

and consequently

$$m_0 \sim up_0 \tag{2.144}$$

where u is the velocity of motion of the charges. Thus from (2.142) we have, for $u << c$,

$$E_{p_{rad}} >> E_{m_{rad}} \tag{2.145}$$

i.e., the magnetic dipolar radiation may be ignored in comparison with the electric dipolar radiation.

2.4.3 Electric quadrupole radiation

The second term, $\vec{A}_{2q}$, of $\vec{A}_2$ in (2.130), is associated with the electric quadrupole radiation, but to see this we must transform it further. To this end let us consider the x component of the first summand of the integral $\int_{V'}\left(\vec{J}(\vec{r}\,',t_0')\hat{r}\cdot\vec{r}\,'\right)dv'$ i.e.

$$\begin{aligned}\int_{V'}\hat{r}\cdot\vec{r}\,'\left(\vec{J}(\vec{r}\,',t_0')\cdot\hat{x}'\right)dv' &= \int_{V'}\hat{r}\cdot\vec{r}\,'\left(\vec{J}(\vec{r}\,',t_0')\cdot\nabla' x'\right)dv' \\ = \int_{V'}\nabla'\cdot\left(x'\left(\hat{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,',t_0')\right)dv' &- \int_{V'}x'\nabla'\cdot\left(\left(\hat{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,',t_0')\right)dv'\end{aligned} \tag{2.146}$$

where the first integral is null, as can be seen using the divergence theorem to convert the volume integral in a surface integral with the surface of integration outside of the source distribution. Thus

$$\begin{aligned}\int_{V'}\hat{r}\cdot\vec{r}\,'\left(\vec{J}(\vec{r}\,',t_0')\cdot\hat{x}'\right)dv' &= -\int_{V'}x'\nabla'\cdot\left(\left(\hat{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,',t_0')\right)dv' \\ &= -\int_{V'}x'\nabla'\left(\hat{r}\cdot\vec{r}\,'\right)\cdot\vec{J}(\vec{r}\,',t_0')dv' \\ &\quad -\int_{V'}x'\left(\hat{r}\cdot\vec{r}\,'\right)\nabla'\cdot\vec{J}(\vec{r}\,',t_0')dv'\end{aligned} \tag{2.147}$$

but

$$\nabla'\left(\hat{r}\cdot\vec{r}\,'\right) = \hat{r} \tag{2.148a}$$

$$\nabla'\cdot\vec{J}(\vec{r}\,',t_0') = -\frac{\partial\rho(\vec{r}\,',t_0')}{\partial t} \tag{2.148b}$$

therefore

$$\begin{aligned}&\int_{V'}\vec{J}_x(\vec{r}\,',t_0')\hat{r}\cdot\vec{r}\,'dv' \\ = &-\int_{V'}x'\left(\vec{J}(\vec{r}\,',t_0')\cdot\hat{r}\right)dv' + \int_{V'}x'\left(\hat{r}\cdot\vec{r}\,'\right)\frac{\partial\rho(\vec{r}\,',t_0')}{\partial t}dv'\end{aligned} \tag{2.149}$$

Generalizing to three dimensions

$$\int_{V'}\vec{J}(\vec{r}\,',t_0')\left(\hat{r}\cdot\vec{r}\,'\right)dv' = -\int_{V'}\vec{r}\,'\left(\vec{J}(\vec{r}\,',t_0')\cdot\hat{r}\right)dv' + \int_{V'}\vec{r}\,'\left(\hat{r}\cdot\vec{r}\,'\right)\cdot\frac{\partial\rho(\vec{r}\,',t_0')}{\partial t}dv' \tag{2.150}$$

and therefore, substituting in (2.130), we have

$$\vec{A}_{2q} = \frac{\mu_0}{8\pi cr}\frac{\partial^2}{\partial t^2}\int_{V'}\vec{r}\,'\left(\hat{r}\cdot\vec{r}\,'\right)\rho(\vec{r}\,',t_0')dv' \tag{2.151}$$

The magnetic radiation field, given by (2.107), is

$$\vec{H}_{2q_{rad}} = -\frac{1}{8\pi c^2 r}\hat{r}\times\frac{\partial^3}{\partial t^3}\int_{V'}\vec{r}\,'\left(\hat{r}\cdot\vec{r}\,'\right)\rho(\vec{r}\,',t_0')dv' \tag{2.152}$$

The above expression can be written in a more useful form by adding the term $\hat{r}r'^2\rho(\vec{r}', t'_0)$ to the integrand

$$\vec{H}_{2q_{rad}} = -\frac{1}{24\pi c^2 r}\hat{r} \times \frac{\partial^3}{\partial t^3}\int_{V'}\left(3\vec{r}'\left(\hat{r}\cdot\vec{r}'\right) - \hat{r}r'^2\right)\rho(\vec{r}', t'_0)dv' \tag{2.153}$$

Note that, since $\hat{r} \times \hat{r}r'^2 = 0$, the added term do no affect to the value of the integral. The advantage of including this term is that, now, the integrand can be written as the product of a second rank tensor Q, called electric quadrupole-moment tensor of the source distribution, and the vector $\hat{r}$

$$\int_{V'}\left(3\vec{r}'\left(\hat{r}\cdot\vec{r}'\right) - \hat{r}r'^2\right)\rho(\vec{r}', t'_0)dv' = [Q]\hat{r} \tag{2.154}$$

The elements of $[Q]$ are

$$[Q_{\alpha\beta}] = \int_{V'}\left(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}\right)\rho(\vec{r}', t'_0)dv' \tag{2.155}$$

and $[Q]\hat{r}$ is a vector with components

$$\sum_\alpha [Q_{\alpha\beta}]\hat{r}_\beta \tag{2.156}$$

Therefore the radiation magnetic field from a varying electric quadrupole is given by

$$\vec{H}_{2q_{rad}} = -\frac{1}{24\pi c^2 r}\hat{r} \times \frac{\partial^3[Q]\hat{r}}{\partial t^3} = -\frac{1}{24\pi c^2 r}\hat{r} \times [\dddot{Q}]\hat{r} \tag{2.157}$$

or, for, time-harmonic dependence,

$$\boldsymbol{\vec{H}}_{2q_{rad}} = \frac{jck^3}{24\pi r}e^{j(\omega t - kr)}\hat{r} \times \boldsymbol{Q}\hat{r} \tag{2.158}$$

The radiation electric field can be calculated as usual by (2.81). It can be shown that quadrupole radiation fields are of the same order as the magnetic dipole moment and thus much less than that corresponding to the Hertzian dipole (Ejercicio)..

Of course, if we continued analyzing other terms in the expansion tal, we would find other multipole moments, such as magnetic quadrupole radiation, electric octupole radiation, etc. However, for this, other more complex mathematical methods provide the results more systematically.

2.5 Maxwell's symmetric equations

It can be observed from (1.1a)-(1.1d) that Maxwell's equations present a certain symmetry that, except in free space and with no source terms, is not complete because of the absence of magnetic charges and currents. Indeed, despite many experimental attempts, no free magnetic charges or monopoles have been found in nature nor, therefore, would magnetic currents be created[9]. Nevertheless, from a purely theoretical standpoint, nothing prevents us from assuming the existence of magnetic monopoles; therefore, to complete Maxwell's equations we must add the necessary magnetic source terms in order to achieve complete symmetry between electric and magnetic quantities. To this end, we can reformulate Faraday's law (1.1c) and Gauss' law for magnetic fields (1.1b) by introducing, on their right-hand side, hypothetical magnetic current densities $\vec{J}_m$ (Vm^{-2}) and magnetic charge densities ρ_m (Wb/m^3), respectively, as additional source terms. With these new quantities included, we can rewrite Maxwell's equations for the case that both, electric as well as magnetic sources, exist in free space, in the following completely symmetric manner:

Differential form of Maxwell's symmetric equations

$$\nabla \cdot \vec{D} = \rho \tag{2.159a}$$

$$\nabla \cdot \vec{B} = \rho_m \tag{2.159b}$$

$$\nabla \times \vec{E} = -\vec{J}_m - \mu_0 \frac{\partial \vec{H}}{\partial t} \tag{2.159c}$$

$$\nabla \times \vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2.159d}$$

Integral form of Maxwell's symmetric equations

$$\oint_S \vec{D} \cdot d\vec{s} = Q_T \tag{2.160a}$$

$$\oint_S \vec{B} \cdot d\vec{s} = Q_m \tag{2.160b}$$

$$\oint_\Gamma \vec{E} \cdot d\vec{l} = -\int_S \vec{J}_m \cdot d\vec{s} - \frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{s} \tag{2.160c}$$

$$\oint_\Gamma \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{s} \tag{2.160d}$$

It should be emphasized that the symmetrization of Maxwell's equations is a powerful mathematical tool which greatly facilitates the solution of many practical problems such as the radiation and scattering from aperture antennas or permeable bodies.

Taking the divergence of (2.159c) and using (2.159b)

$$\nabla \cdot \nabla \times \vec{E} = -\nabla \cdot \vec{J}_m - \frac{\partial \nabla \cdot \vec{B}}{\partial t} = 0 \tag{2.161}$$

[9] It should be emphasized that, although there is no experimental evidence for the existence of magnetic charges, such existence does not violate any known principle of physics. In fact, from a purely theoretical viewpoint, Dirac showed [P.A.M. Dirac, Proc Roy. Soc.Lond. A133, 60 (1931)] that the existence of magnetic monopoles with magnetic charge g would explain the quantization of the electric charge e. We refer to the magnetically charged particles as magnetic monopoles or simply monopoles.

we get the equation of continuity

$$\nabla \cdot \vec{J}_m = -\frac{\partial \rho_m}{\partial t} \tag{2.162}$$

which expresses the conservation of magnetic monopoles and has the same form as that for the electric charges (1.3).

In linear media, we can apply the superposition principle and split each one of the field quantities, $\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$, into the sum of two components

$$\vec{D} = \vec{D}_e + \vec{D}_m = \varepsilon_0 \left(\vec{E}_e + \vec{E}_m\right) = \varepsilon_0 \vec{E} \tag{2.163a}$$

$$\vec{B} = \vec{B}_e + \vec{B}_m = \mu_0 \left(\vec{H}_e + \vec{H}_m\right) = \mu_0 \vec{H} \tag{2.163b}$$

where the quantities with the e subscript depend only on the "true" electric sources ρ and $\vec{J}$ while the quantities with the m subscript depend only on the "hypothetical" magnetic sources ρ_m and $\vec{J}_m$. In this way, we divide Maxwell's equations into two groups corresponding to the field components associated with the electrical and magnetic sources, respectively; that is

$$\nabla \cdot \vec{D}_e = \rho \tag{2.164a}$$

$$\nabla \cdot \vec{B}_e = 0 \tag{2.164b}$$

$$\nabla \times \vec{E}_e = -\mu_0 \frac{\partial \vec{H}_e}{\partial t} \tag{2.164c}$$

$$\nabla \times \vec{H}_e = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}_e}{\partial t} \tag{2.164d}$$

$$\nabla \cdot \vec{D}_m = 0 \tag{2.165a}$$

$$\nabla \cdot \vec{B}_m = \rho_m \tag{2.165b}$$

$$\nabla \times \vec{E}_m = -\vec{J}_m - \mu_0 \frac{\partial \vec{H}_m}{\partial t} \tag{2.165c}$$

$$\nabla \times \vec{H}_m = \varepsilon_0 \frac{\partial \vec{E}_m}{\partial t} \tag{2.165d}$$

Note that the sum of each expression (2.164), added to its equivalent (2.165), gives (2.159) and that the set (2.164) coincides with the conventional Maxwell's equations (2.159), and that Eqs. (2.164) are formally identical to Eqs. (1.1a)-(1.1d) and therefore can be solved as in the previous sections by means of the scalar and vector potentials Φ and $\vec{A}$. Thus, from (2.4) and (2.1), we have

$$\vec{B}_e = \nabla \times \vec{A} \tag{2.166}$$

$$\vec{E}_e = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \tag{2.167}$$

where

$$\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} = 0 \tag{2.168}$$

and where $\vec{A}$ and Φ fulfil the wave equations (2.14a) and (2.14b)

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{2.169}$$

$$\nabla^2 \Phi - \mu_0 \varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{2.170}$$

the solutions to which are the retarded potentials (2.41a) and (2.41b)

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{[\rho]}{R} dv' \tag{2.171}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{[\vec{J}]}{R} dv' \tag{2.172}$$

The fields created by the magnetic sources ρ_m and $\vec{J}_m$ can be deduced by observing that equations (2.164) are transformed into (2.165) and vice versa with the simultaneous replacement of the following quantities, called duals

$$\begin{array}{ccc} \vec{E}_e & \text{dual of} & \vec{H}_m \\ \vec{H}_e & \text{dual of} & -\vec{E}_m \\ \varepsilon_0 & \text{dual of} & \mu_0 \\ \mu_0 & \text{dual of} & \varepsilon_0 \\ \rho & \text{dual of} & \rho_m \\ \vec{J} & \text{dual of} & \vec{J}_m \end{array} \tag{2.173}$$

The fields $\vec{E}_e$ and $\vec{H}_e$ associated with the electric sources can be calculated from the magnetic vector potential $\vec{A}$ and the electric scalar potential Φ by means of (2.171) and (2.172). To calculate the fields $\vec{H}_m$ and $\vec{E}_m$ we can use the same formalism defining two new potentials, termed "electric vector potential" $\vec{F}$ and "magnetic scalar potential" ψ, such that

$$\boxed{\begin{array}{ccc} \vec{A} & \text{dual of} & \vec{F} \\ \Phi & \text{dual of} & \psi \end{array}} \tag{2.174}$$

Hence

$$\psi = \frac{1}{4\pi\mu_0} \int_{V'} \frac{[\rho_m]}{R} dv' \tag{2.175}$$

$$\vec{F} = \frac{\varepsilon_0}{4\pi} \int_{V'} \frac{[\vec{J}_m]}{R} dv' \tag{2.176}$$

which are the dual expressions of (2.171) and (2.172).

By substituting the magnitudes in the first column of (2.173 and 2.174) for their duals in the equations from (2.166) to (2.172) we get

$$\vec{D}_m = \varepsilon_o \vec{E}_m = -\nabla \times \vec{F} \tag{2.177a}$$

$$\vec{H}_m = -\nabla\psi - \frac{\partial \vec{F}}{\partial t} \tag{2.177b}$$

$$\nabla \cdot \vec{F} + \mu_0 \varepsilon_0 \frac{\partial \psi}{\partial t} = 0 \tag{2.177c}$$

in which ψ and $\vec{F}$ satisfy wave equations that are analogous to (2.169) and (2.170):

$$\nabla^2 \vec{F} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{F}}{\partial t^2} = -\varepsilon_0 \vec{J}_m \tag{2.178}$$

$$\nabla^2 \psi - \mu_0 \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} = -\frac{\rho_m}{\mu_0} \tag{2.179}$$

Thus, by the superposition principle, if both current densities $\vec{J}$ and $\vec{J}_m$ exist simultaneously in a region of free space, the total field $\vec{E}$ produced at any point is the sum of $\vec{E}_e$ and $\vec{E}_m$ given by (2.167) and (2.177a). Hence

$$\begin{aligned} \vec{E} &= \vec{E}_e + \vec{E}_m = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} - \frac{1}{\varepsilon_0}\nabla \times \vec{F} = \\ & \frac{1}{c^2}\nabla \int \nabla \cdot \vec{A} dt - \frac{\partial \vec{A}}{\partial t} - \frac{1}{\varepsilon_0}\nabla \times \vec{F} \end{aligned} \tag{2.180}$$

where Lorenz gauge Eq. (2.10) has been used to express $\vec{E}$ in terms of $\vec{A}$ and $\vec{F}$.

The total field $\vec{H}$ is determined analogously from (2.166) and (2.177b)

$$\vec{H} = \vec{H}_e + \vec{H}_m = -\nabla\psi - \frac{\partial \vec{F}}{\partial t} + \frac{1}{\mu_0}\nabla \times \vec{A} \tag{2.181}$$

In practice, it is not necessary to use the latter expression, because once $\vec{E}$ has been calculated using (2.180), by substituting the result in (2.159c), with $\vec{J}_m = 0$ we obtain $\vec{H}$.

2.5.1 Boundary conditions

It is easy to show, ejercicio, that the boundary conditions corresponding to Maxwell's symmetric equations are a logical extension of (1.35); that is,

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \tag{2.182a}$$

$$\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = \rho_{sm} \tag{2.182b}$$

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = -\vec{J}_{sm} \tag{2.182c}$$

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s \tag{2.182d}$$

in which $\hat{n}$ is the normal unit vector that goes from region 2 to region 1. Equations (2.182b) and (2.182c) show the additional effects of the imaginary surface magnetic charges and currents, ρ_{sm} and $\vec{J}_{sm}$, at the interface. According to (2.182c) and (2.182d), the tangential components of the fields on a real or imaginary surface S can be written in terms of surface distributions of electric currents

$$\hat{n} \times \vec{H}\Big|_S = \vec{J}_s \tag{2.183}$$

and magnetic ones

$$-\hat{n} \times \vec{E}\Big|_S = \vec{J}_{sm} \tag{2.184}$$

2.5.2 Harmonic variations

For harmonic variations, the symmetric equations (2.159) simplify to

$$\nabla \cdot \vec{\boldsymbol{D}} = \boldsymbol{\rho} \tag{2.185a}$$

$$\nabla \cdot \vec{\boldsymbol{B}} = \boldsymbol{\rho}_m \tag{2.185b}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\vec{\boldsymbol{J}}_{\boldsymbol{m}} - j\mu_0\omega\vec{\boldsymbol{H}} \tag{2.185c}$$

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + j\varepsilon_0\omega\vec{\boldsymbol{E}} \tag{2.185d}$$

and the wave equations for the magnetic scalar potential ψ, the electric vector potential $\vec{F}$, and the Lorenz relations are

$$\nabla^2\boldsymbol{\psi} + \omega^2\mu_0\varepsilon_0\boldsymbol{\psi} = -\frac{\boldsymbol{\rho}_m}{\mu_0} \tag{2.186a}$$

$$\nabla^2\vec{\boldsymbol{F}} + \omega^2\mu_0\varepsilon_0\vec{\boldsymbol{F}} = -\varepsilon_0\vec{\boldsymbol{J}}_{\boldsymbol{m}} \tag{2.186b}$$

$$\boldsymbol{\psi} = \frac{j\nabla \cdot \vec{\boldsymbol{F}}}{\omega\varepsilon_0\mu_0} \tag{2.186c}$$

with the solutions to (2.186a) and (2.186b) being

$$\boldsymbol{\psi} = \frac{1}{4\pi\mu_0}\int_{V'} \frac{\boldsymbol{\rho}_m e^{-jkR}}{R} dv' \tag{2.187}$$

$$\vec{\boldsymbol{F}} = \frac{\varepsilon_0}{4\pi}\int_{V'} \frac{\vec{\boldsymbol{J}}_{\boldsymbol{m}} e^{-jkR}}{R} dv' \tag{2.188}$$

The total field $\vec{\boldsymbol{E}}$ produced at any point is the sum of $\vec{\boldsymbol{E}}_e$ and $\vec{\boldsymbol{E}}_m$, and is given by

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_e + \vec{\boldsymbol{E}}_m = -j\frac{c^2}{\omega}\nabla\left(\nabla \cdot \vec{\boldsymbol{A}}\right) - j\omega\vec{\boldsymbol{A}} - \frac{1}{\varepsilon_0}\nabla \times \vec{\boldsymbol{F}} \tag{2.189}$$

while for the total field $\vec{\boldsymbol{H}}$ we have

$$\vec{\boldsymbol{H}} = -j\frac{c^2}{\omega}\nabla\left(\nabla \cdot \vec{\boldsymbol{F}}\right) - j\omega\vec{\boldsymbol{F}} + \frac{1}{\mu_0}\nabla \times \vec{\boldsymbol{A}}. \tag{2.190}$$

where $\vec{A}$ is given by (2.47a).

2.5.3 Fields created by an infinitesimal magnetic current element

From (2.86a) and (2.86b), using the dual equations (2.173), we deduce that the fields generated by an infinitesimal magnetic current element,

$$\vec{J}_m(\vec{r},t) = i_m(t)\delta(x')\delta(y')\hat{z} \qquad -\frac{\Delta z}{2} < z' < \frac{\Delta z}{2} \tag{2.191}$$

are given by, (Fig. 2.6),

$$\vec{E} = -\frac{\Delta z}{4\pi}\left(\frac{1}{cr}\frac{d\,[i_m]}{dt} + \frac{[i_m]}{r^2}\right)\sin\theta\;\hat{\varphi} \tag{2.192a}$$

$$\vec{H} = \frac{\Delta z}{4\pi\mu_o}\left(\frac{1}{r^3}\int_{-\infty}^{t}[i_m]\,dt + \frac{[i_m]}{cr^2}\right)(2\cos\theta\;\hat{r} + \sin\theta\hat{\theta}\;) + \frac{\Delta z}{4\pi\mu_o}\frac{1}{c^2 r}\frac{d\,[i_m]}{dt}\sin\theta\hat{\theta} \tag{2.192b}$$

or, for time-harmonic variation

$$\vec{\boldsymbol{E}} = -\frac{\Delta z \boldsymbol{I}_m}{4\pi} jk\left(1 + \frac{1}{jkr}\right)\frac{e^{-jkr}}{r}\sin\theta\;\hat{\varphi} \tag{2.193a}$$

$$\vec{\boldsymbol{H}} = \frac{\boldsymbol{I}_m\Delta z}{4\pi} j\omega\varepsilon_0\left(1 + \frac{1}{jkr} - \frac{1}{k^2r^2}\right)\frac{e^{-jkr}}{r}\sin\theta\hat{\theta} + \frac{\boldsymbol{I}_m\Delta z}{2\pi} j\omega\varepsilon_0\left(-\frac{1}{k^2r^2} + \frac{1}{jkr}\right)\frac{e^{-jkr}}{r}\cos\theta\hat{r} \tag{2.193b}$$

Comparing the radiation terms of these equations to (2.139a)-(2.140b), we find that

$$i_m\Delta z = \mu_0 S\frac{di}{dt} \tag{2.194}$$

or for time-harmonic dependence.

$$\boldsymbol{I}_m\Delta z = j\omega\mu_0\boldsymbol{I}S \tag{2.195}$$

2.6 Theorem of uniqueness

Whenever we have to resolve a differential equation, it is desirable to know the conditions that must be fulfilled in order to state that a unique solution is possible. In our context, this means to seek the conditions for which we can state that there exists a single electromagnetic field that satisfies, simultaneously, Maxwell's equations and the given boundary conditions.

Next, we establish these conditions for non-harmonic and time-harmonic electromagnetic fields.

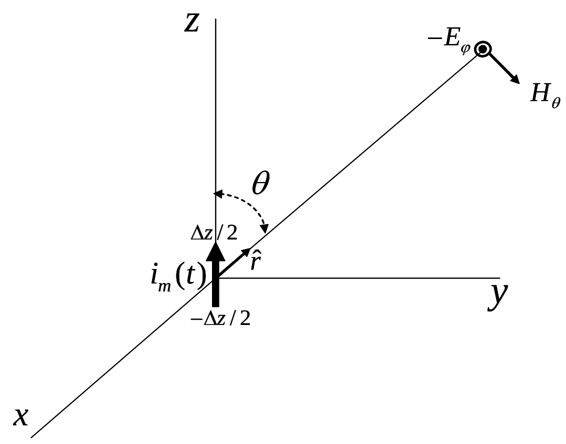


Figure 2.6: solo los campos de radiacion se representan"'

2.6.1 Non-harmonic electromagnetic field

A non-harmonic electromagnetic field that varies in a linear region V bounded by a surface S is uniquely determined from an initial time, $t = t_0$, if the following are known:

i) The values of the sources at each point and at each time for every $t > t_0$ within the region.

ii) The values of the electromagnetic field ($\vec{E}$ and $\vec{H}$) at each point of V at the initial time $t = t_0$.

iii) The tangential components of the electric field $\vec{E}$ or of the magnetic field $\vec{H}$ on the entire the surface S for all $t > t_0$, or, alternatively, the tangential components of the electric field $\vec{E}$ in any part of S and of the magnetic field $\vec{H}$ in the remaining part of S, for all $t > t_0$.

Proof

This theorem can be proven by a reduction to absurdity– that is, by showing that to assume the opposite of what is postulated would lead to a contradiction. Let us assume that having defined the three above conditions within a volume V, there exist two different electromagnetic fields, ($\vec{E}_1$ and $\vec{H}_1$) and ($\vec{E}_2$ and $\vec{H}_2$), respectively, which are solutions to the problem. Given the linearity of Maxwell's equations, any linear combination of these two solutions must in itself be a solution. In particular, the difference between the two aforementioned fields, i.e. the field defined by ($\vec{E}' = \vec{E}_1 - \vec{E}_2$ and $\vec{H}' = \vec{H}_1 - \vec{H}_2$), must also be a solution to the problem. Given that, from the hypothesis, the sources are the same for the fields ($\vec{E}_1$ and $\vec{H}_1$) and ($\vec{E}_2$ and $\vec{H}_2$), the field ($\vec{E}', \vec{H}'$) is source-free in V. Thus, if we apply the Poynting theorem (1.39) to ($\vec{E}', \vec{H}'$), we get

$$0 = \frac{\partial}{\partial t}\int_V \frac{1}{2}(\vec{E}'\cdot\vec{D}' + \vec{B}'\cdot\vec{H}')dv + \int_V \sigma E'^2 dv + \oint_S (\vec{E}'\times\vec{H}')\cdot d\vec{s} \qquad (2.196)$$

It is straightforward to show that if the tangential components of the electric field $\vec{E}$ and/or of the magnetic field $\vec{H}$ are uniquely determined on surface S, the final term in (2.196) is null. By integrating this expression with respect to the time from t_0 to t and, taking into account that the initial values for $t = t_0$ are defined for all V, we find that

$$0 = \int_V \frac{1}{2}(\vec{E}'\cdot\vec{D}' + \vec{B}'\cdot\vec{H}')dv + \int_{t_0}^{t}\left(\int_V \sigma E'^2 dv\right)dt \qquad (2.197)$$

As both of the terms on the second member in (2.197) are positive, this equality can be fulfilled only when both $\vec{E}'$ and $\vec{H}'$ are null (i.e. when $\vec{E}_1 = \vec{E}_2$ and $\vec{H}_1 = \vec{H}_2$), which is what we set out to prove.

2.6.2 Time-harmonic fields

In the case of harmonic variations, the uniqueness theorem states that a field in a lossy ($\sigma \neq 0$) [10] region is uniquely determined by the sources within the region together with the tangential components of the electric field $\vec{E}$ or of the magnetic field $\vec{H}$ on S, or, alternatively, the tangential components of the electric field $\vec{E}$ in any part of S and of the magnetic field $\vec{H}$ in the remaining part of S.

Proof By a reasoning similar to that used for the above case, but using the expression (1.111), we get

$$0 = \int_V \frac{\sigma E_0'^2}{2}dv + 2j\omega\int_V\left(\frac{\mu H_0'^2}{4} - \frac{\varepsilon E_0'^2}{4}\right)dv \qquad (2.198)$$

By making the real and the imaginary parts equal to zero, we see that these two equalities imply that H_0' and E_0' are both equal to zero only if $\sigma \neq 0$. This is why we started from the premise that the medium occupying the volume has a conductivity that may be arbitrarily small but which is non-zero at all points. The field in a lossless region can be considered the limit to the lossy case when such losses tend to zero.

[10] *The reason why we need the extra condition of the space to be lossy for time-harmonic signals is that, by definition, a pure harmonic signal has an infinite duration.*

Chapter 3

??Electromagnetic waves

In chapter 2 the fields created by a bounded time-variyng source distribution were calculated and in particular we found that the radiation field propagates energy far away from the sources. Of all the possible solutions for the wave equation, we will examine primarily the properties of their plane-wave solutions, i.e., waves for which the wave-front are planes[1]. Plane waves constitute a good approximation to actual waves in many situations because at sufficiently large distances from the sources, in a sufficiently small region, any wave front can be treated as a plane wave. For example, a great deal of optics is founded on the plane-wave approximation and, similarly, in radiocommunications the radiated field at sufficient distance from the antenna can be considered to be a plane wave. Moreover, it is possible to demostrate that, in general, an electromagnetic field can puede descomponerse como suma lineal de ondas planas (see Appendix **??**) In this Chapter we consider this kind of waves in a linear homogeneous isotropic medium libre de fuentes. Then incidencia normal y oblicua. Ondas esféricas , desarrollo en ondas planas?

Harmonic..Electromagnetic waves are not limited in wavelength and in fact cover the spectrum from gamma rays (wavelengths of ¿¿¿¿ ¿¿ 10-12 cm???????) through X-rays, visible light, microwaves, and radio waves, to long waves (hundreds of kilometers long).

3.1 Wave equation

For time-varying electromagnetic fields it is possible to combine Maxwell's equations to eliminate one of the fields, $\vec{H}$ or $\vec{E}$, to obtain two uncoupled second-order differential equations, one in $\vec{E}$ and the other in $\vec{H}$, known as wave equations. To formulate these wave equations, let us consider a non-magnetic ($\mu = \mu_0$), homogeneous, linear and isotropic region where, in general, source terms $\vec{J}$ and ρ may exist. Taking the curl of (1.1c) and using the vector relation (**??**) we

[1] Wave-front is defined as a surface that, at any time t, is orthogonal to the propagation vector $\hat{n}$ at all the points on the surface.

have

$$\begin{aligned}
\nabla\times\nabla\times\vec{E} &= \nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\frac{\partial\nabla\times\vec{B}}{\partial t}\\
&= -\mu_0\frac{\partial}{\partial t}(\vec{J}_c+\vec{J}+\frac{\partial\vec{D}}{\partial t})\Rightarrow\\
\nabla^2\vec{E} &= \nabla(\nabla\cdot\vec{E})+\mu_0\frac{\partial}{\partial t}(\vec{J}_c+\vec{J}+\frac{\partial\vec{D}}{\partial t})\\
&= \frac{1}{\varepsilon}\nabla\rho+\mu_0\sigma\frac{\partial\vec{E}}{\partial t}+\mu_0\frac{\partial\vec{J}}{\partial t}+\mu_0\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}
\end{aligned} \tag{3.1}$$

where $\vec{J}$ and $\vec{J}_c$ are the source and induced conduction density of the currents, respectively. Thus, rearranging terms, we get

$$\nabla^2\vec{E}-\mu_0\sigma\frac{\partial\vec{E}}{\partial t}-\mu_0\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=\frac{\nabla\rho}{\varepsilon}+\mu_0\frac{\partial\vec{J}}{\partial t} \tag{3.2}$$

which is known as the inhomogeneous vector-wave equation for the electric field.

A similar equation can be written for the magnetic field $\vec{H}$ by taking the curl of (1.1d),

$$\nabla^2\vec{H}-\mu_0\sigma\frac{\partial\vec{H}}{\partial t}-\mu_0\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=-\nabla\times\vec{J} \tag{3.3}$$

For a *lossless* media (3.2) and (3.3) reduce to

$$\nabla^2\vec{E}-\mu_0\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\nabla\rho+\mu_0\frac{\partial\vec{J}}{\partial t} \tag{3.4a}$$

$$\nabla^2\vec{H}-\mu_0\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} = -\nabla\times\vec{J} \tag{3.4b}$$

These Eqs are analogous to the inhomogeneous wave equation for the vector potential (2.14a), and consequently their solutions take the form of the retarded vector potential given by Eq. (2.41b), i.e.

$$\vec{E}(\vec{r},t) = -\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\nabla[\rho]+\frac{1}{c^2}\left[\frac{\partial\vec{J}}{\partial t}\right]}{R}dv' \tag{3.5a}$$

$$\vec{H}(\vec{r},t) = \frac{1}{4\pi}\int_{V'}\frac{\nabla\times\left[\vec{J}\right]}{R}dv' \tag{3.5b}$$

from which, by means of straightforward operations, we can obtain the expressions (2.49) and (2.55) for the fields created by a bounded distribution of finite densities of charges and currents with arbitrary space and time dependence.

In source-free regions ($\vec{J}=0$; $\rho=0$, *except the charge and current densities induced by the presence of the fields, which are expressed in terms of the* *constitutive parameters) the equations (3.2) and (3.3) simplify to*

$$\nabla^2 \vec{E} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} = 0 \quad (3.6a)$$

$$\nabla^2 \vec{H} - \mu_0 \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu_0 \sigma \frac{\partial \vec{H}}{\partial t} = 0 \quad (3.6b)$$

which are the homogeneous wave equations that determine the propagation of the fields $\vec{E}$ and $\vec{H}$ in a sourceless homogeneous, linear and isotropic medium. The solutions to these wave equations must be compatible with Maxwell's equations and the coefficients of the solutions must be derived from the boundary conditions.

Uniform plane waves are defined as waves with a field amplitude that, at any instant, is the same at all points of the wave-front plane. Thus, the field amplitude depends only on the distance ξ from the origin to the plane *(fig.6.1)*. Therefore, if $\hat{n} = \vec{\xi}/\xi$ is the unit vector that is normal to the plane, the *del* operator ∇ becomes $\nabla = \partial/\partial\xi \ \hat{n}$ and Maxwell's equations simplify to

$$\hat{n} \cdot \frac{\partial \vec{D}}{\partial \xi} = 0 \quad (3.7a)$$

$$\hat{n} \cdot \frac{\partial \vec{B}}{\partial \xi} = 0 \quad (3.7b)$$

$$\hat{n} \times \frac{\partial \vec{E}}{\partial \xi} = -\frac{\partial \vec{B}}{\partial t} \quad (3.7c)$$

$$\hat{n} \times \frac{\partial \vec{H}}{\partial \xi} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \quad (3.7d)$$

and the wave equations become

$$\frac{\partial^2 \vec{E}}{\partial \xi^2} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} = 0 \quad (3.8a)$$

$$\frac{\partial^2 \vec{H}}{\partial \xi^2} - \mu_0 \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu_0 \sigma \frac{\partial \vec{H}}{\partial t} = 0 \quad (3.8b)$$

These equations, which describe the propagation of plane waves in a homogeneous conducting medium, are called the "telegrapher's equations". For nondissipative media, for example the free space, these equations simplify to

$$\frac{\partial^2 \vec{E}}{\partial \xi^2} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial^2 \vec{E}}{\partial \xi^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (3.9a)$$

$$\frac{\partial^2 \vec{H}}{\partial \xi^2} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = \frac{\partial^2 \vec{H}}{\partial \xi^2} - \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad (3.9b)$$

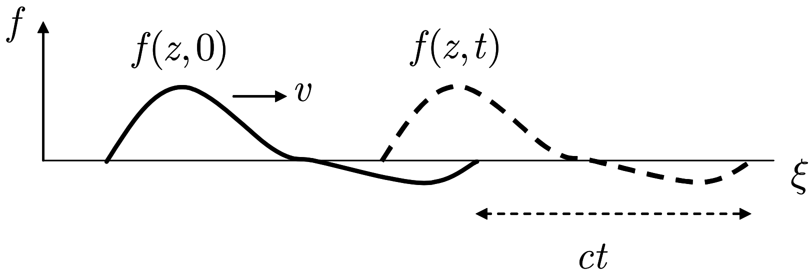


Figure 3.1: The wave *tal*, in a lossless medium, propagates at velocity *tal* to the right without changing shape

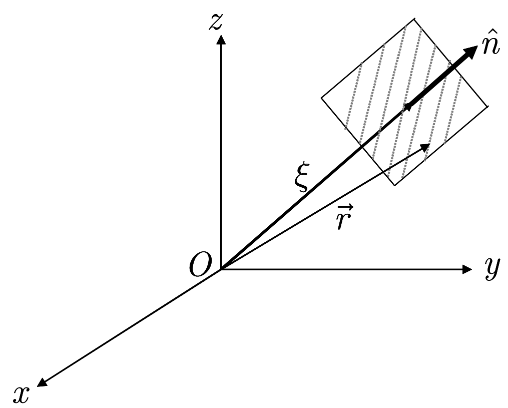


Figure 3.2: poner plane wave front

3.2 Harmonic waves

For time-harmonic fields, when the medium presents a conductivity σ and, at *the operating frequency,* a *complex* dielectric constant, $\varepsilon_c = \varepsilon' - j\varepsilon''$,(1.71), the wave equation (3.6a) can be written as a time-independent wave equation

$$\begin{aligned}\nabla^2\vec{\boldsymbol{E}} - j\omega\mu_0\sigma\vec{\boldsymbol{E}} + \mu_0\omega^2\varepsilon_c\vec{\boldsymbol{E}} &= \nabla^2\vec{\boldsymbol{E}} - j\omega\mu_0\sigma_e\vec{\boldsymbol{E}} + \mu_0\omega^2\varepsilon'\vec{\boldsymbol{E}} \\ &= \nabla^2\vec{\boldsymbol{E}} + \omega^2\mu_0\varepsilon'\left(1 - j\tan\delta_d\right)\vec{\boldsymbol{E}} \\ &= (\nabla^2 + \omega^2\mu_0\varepsilon_{ec})\vec{\boldsymbol{E}} = 0 \qquad (3.10)\end{aligned}$$

where $\sigma_e = \sigma + \omega\varepsilon''$, $\tan\delta_d = \sigma_e/\omega\varepsilon'$, and $\varepsilon_{ec} = \varepsilon'(1 - j\tan\delta_d)$, are the *effective* conductivity, the loss tangent and the effective complex permittivity defined in (1.78), (1.81), and (1.83) respectively.

According to Subsection (**??**), depending on the characteristics of the medium, the values of the term $j\tan\delta_d$ in Eq. (3.10) may range from $<< 1$ (zero for a perfect dielectric or lossless medium) to $>> 1$ (infinite for a perfect conductor). In a highly conductive medium $\tan\delta_d >> 1$ and $1 - j\tan\delta_d \simeq -j\tan\delta_d$, and thus Eq. (3.10) becomes the so-called time-independent diffusion equation for the electric field $\vec{\boldsymbol{E}}$

$$\nabla^2\vec{\boldsymbol{E}} - j\omega\mu_0\sigma\vec{\boldsymbol{E}} = 0 \qquad (3.11)$$

which is of the same type as the one that determines the propagation of heat by conduction or by diffusion. As commented in Subsection (**??**), for most metals the relaxation time τ is $10^{-14}s$, which is a low value compared with the period for all frequencies lower than the optical ones. Thus, since $\tan\delta_d = (\tau\omega)^{-1}$, the diffusion equation is adequate for metals at all these frequencies.

Equation (3.10) can be written more concisely as

$$\nabla^2\vec{\boldsymbol{E}} - \gamma^2\vec{\boldsymbol{E}} = 0 \qquad (3.12)$$

where γ is in general a complex quantity called the complex propagation constant, which, from (3.10) and (3.12), is given by

$$\begin{aligned}-\gamma^2 &= \omega^2\mu_0(\varepsilon_c - j\frac{\sigma}{\omega}) \\ &= \omega^2\mu_0(\varepsilon' - j(\varepsilon'' + \frac{\sigma}{\omega})) \\ &= \omega^2\mu_0\varepsilon'\left(1 - j\tan\delta_d\right) = k^2\left(1 - j\tan\delta_d\right) = \omega^2\mu_0\varepsilon_{ec}\end{aligned} \qquad (3.13)$$

where

$$k = \omega\sqrt{\mu_o\varepsilon'} \qquad (3.14)$$

is the wavenumber corresponding to an *unbounded* lossless medium with a real dielectric constant ε'.

Analogously, for the magnetic field, we have

$$\nabla^2\vec{\boldsymbol{H}} - \gamma^2\vec{\boldsymbol{H}} = 0 \qquad (3.15)$$

3.2.1 Uniform plane harmonic waves

For uniform plane waves, we have $\nabla^2 = \partial^2/\partial^2\xi$ and Eqs. (3.12) and (3.15) simplify to

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial \xi^2} - \gamma^2 \vec{\boldsymbol{E}} = 0 \tag{3.16a}$$

$$\frac{\partial^2 \vec{\boldsymbol{H}}}{\partial \xi^2} - \gamma^2 \vec{\boldsymbol{H}} = 0 \tag{3.16b}$$

The complex propagation constant γ is usually written as[2]

$$\gamma = jk\,(1 - j\tan\delta_d)^{1/2} = \alpha + j\beta \tag{3.17}$$

where the imaginary part, β, is termed the phase constant, whereas the real part, α, is called the attenuation constant of the wave. Thus, from, (3.13) and (3.17), we can easily calculate the explicit expressions for β and α

$$\begin{aligned}\beta &= \omega\left(\frac{\mu_0\varepsilon'}{2}\right)^{\frac{1}{2}}\left[(1+\tan^2\delta_d)^{1/2}+1\right]^{1/2}\\ &= \frac{\omega\sqrt{\mu_o\varepsilon'}}{\sqrt{2}}\left(\sqrt{1+\left(\frac{\sigma_e}{\omega\varepsilon'}\right)^2}+1\right)^{1/2}\\ &= \frac{k}{\sqrt{2}}\left(\sqrt{1+\left(\frac{\sigma_e}{\omega\varepsilon'}\right)^2}+1\right)^{1/2}\end{aligned} \tag{3.18a}$$

$$\begin{aligned}\alpha &= \omega\left(\frac{\mu_0\varepsilon'}{2}\right)^{\frac{1}{2}}\left[(1+\tan^2\delta_d)^{1/2}-1\right]^{1/2}\\ &= \frac{\omega\sqrt{\mu_o\varepsilon'}}{\sqrt{2}}\left(\sqrt{1+\left(\frac{\sigma_e}{\omega\varepsilon'}\right)^2}-1\right)^{1/2}\\ &= \frac{k}{\sqrt{2}}\left(\sqrt{1+\left(\frac{\sigma_e}{\omega\varepsilon'}\right)^2}-1\right)^{1/2}\end{aligned} \tag{3.18b}$$

The dimensions of α and β are m^{-1} and they are referred to as neper and radian, respectively, to indicate their attenuative and phase meanings in wave expressions. For lossless media we have $\sigma_e = 0$, $\alpha = 0$ and the phase constant becomes $\gamma = j\beta = j\ k$.

[2] *Recordemos que los valores del factor de atenuación, tal como se han calculado, vienen expresados en nepers/metro y que multiplicados por* $8'868$ *se convierten en* dB/m.

Equations (3.16) have solutions of the form $\vec{\boldsymbol{E}}e^{\gamma\xi}$ and $\vec{\boldsymbol{H}}e^{\gamma\xi}$ so that the instantaneous values for the fields are given by wave equations

$$\vec{E} = \mathrm{Re}\{\vec{\boldsymbol{E}}e^{(j\omega t-\gamma\xi)}\} = \mathrm{Re}\{\vec{\boldsymbol{E}}e^{(j\omega t-\vec{\gamma}\cdot\vec{r})}\} = \mathrm{Re}\{\vec{\boldsymbol{E}}e^{-\alpha\xi}e^{j(\omega t-\beta\xi)}\} \tag{3.19a}$$

$$\vec{H} = \mathrm{Re}\{\vec{\boldsymbol{H}}e^{(j\omega t-\gamma\xi)}\} = \mathrm{Re}\{\vec{\boldsymbol{H}}e^{(j\omega t-\vec{\gamma}\cdot\vec{r})}\} = \mathrm{Re}\{\vec{\boldsymbol{H}}e^{-\alpha\xi}e^{j(\omega t-\beta\xi)}\} \tag{3.19b}$$

where the so-called complex propagation vector $\vec{\gamma} = \gamma\hat{n}$ (with module γ and direction of the *unit vector* $\hat{n}$ *normal to the* wave-front *planes) has been introduced and* $\vec{r}$ is the position of any point on the wave-front plane so that $\hat{n}\cdot\vec{r} = \xi$.

Equations (3.19) represent waves traveling at a speed given by the phase velocity v_p

$$v_p = \frac{\omega}{\beta} \tag{3.20}$$

which in general, as β *is given by (3.18a), depends on the frequency (dispersive media).*

The penetration factor δ is defined as

$$\delta = \frac{1}{\alpha} \tag{3.21}$$

This is the distance at which, due to the attenuation α, the field module decreases from an initial given value to $1/e$ of this value.

From (3.7) the following equalities may be deduced

$$\vec{\gamma}\cdot\vec{\boldsymbol{E}} = 0 \tag{3.22a}$$

$$\vec{\gamma}\cdot\vec{\boldsymbol{H}} = 0 \tag{3.22b}$$

$$\vec{\gamma}\times\vec{\boldsymbol{E}} = j\mu_0\omega\vec{\boldsymbol{H}} \tag{3.22c}$$

$$\vec{\gamma}\times\vec{\boldsymbol{H}} = -j\varepsilon_{ec}\omega\vec{\boldsymbol{E}} \tag{3.22d}$$

From these equations, we see that $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{H}}$ and $\hat{n}$ are perpendicular to one another and that they form a right-handed system in the order $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{H}}$, $\hat{n}$. For this reason these waves are often referred to as transverse electromagnetic (TEM) waves. The magnitudes of $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{H}}$ are related by

$$\boldsymbol{H} = \frac{\boldsymbol{E}}{\eta_c} = \frac{\gamma\boldsymbol{E}}{j\omega\mu_0} \tag{3.23}$$

where the quantity η_c, known as the complex characteristic impedance of the medium, is given, taking into account (3.13) and (3.17), by

$$\eta_c = \frac{\boldsymbol{E}}{\boldsymbol{H}} = \frac{j\omega\mu_0}{\gamma} = \left(\frac{\mu_0}{\varepsilon_{ec}}\right)^{1/2} = \frac{\omega\mu_0}{\alpha^2+\beta^2}(\beta+j\alpha) = \mid \eta_c \mid e^{j\theta} \tag{3.24}$$

Thus, its module and phase is given by

$$|\eta_c| = \frac{\left(\frac{\mu_0}{\varepsilon'}\right)^{1/2}}{[1+(\frac{\sigma_e}{\omega\varepsilon'})^2]^{1/4}} \tag{3.25a}$$

$$\theta = \tan^{-1}\frac{\alpha}{\beta} = \frac{1}{2}\tan^{-1}\frac{\sigma_e}{\varepsilon'\omega} = \frac{\delta_d}{2} \tag{3.25b}$$

Therefore, in general there is a phase shift θ between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$.

3.2.2 Propagation in lossless media

By particularizing the above expressions for a lossless medium where, $\varepsilon' = \varepsilon = \varepsilon_r\varepsilon_0$, $\varepsilon'' = 0$, and $\sigma = 0$, we thus have $\tan\delta_d = 0$; $\gamma = jk$; and $\vec{\gamma} = \vec{k} = k\,\hat{n}$, and consequently equations (3.12) and (3.15) simplify to

$$\nabla^2\vec{\boldsymbol{H}} + k^2\vec{\boldsymbol{H}} = 0 \tag{3.26a}$$

$$\nabla^2\vec{\boldsymbol{E}} + k^2\vec{\boldsymbol{E}} = 0 \tag{3.26b}$$

and the complex characteristic impedance of the medium, (3.25), simplifies to

$$\eta_c = \eta = \left(\frac{\mu_0}{\varepsilon}\right)^{1/2} = \left(\frac{\mu_0}{\varepsilon_0\varepsilon_r}\right)^{1/2} = \frac{\eta_0}{\varepsilon_r^{1/2}} = \frac{120\pi}{\varepsilon_r^{1/2}}$$

$$\theta = 0 \tag{3.27}$$

so that the impedance is real and constant. In particular, when the medium is free space, η simplifies to the impedance of free space

$$\eta = \eta_0 = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} = 120\pi \tag{3.28}$$

Consequently, in unbounded lossless media, there is no phase shift between $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ and the attenuation is null ($\alpha = 0$). Thus $\gamma = jk$ and $\delta = \infty$ and Eqs (3.22) simplify to

$$\vec{k}\cdot\vec{\boldsymbol{E}} = 0 \tag{3.29a}$$

$$\vec{k}\cdot\vec{\boldsymbol{H}} = 0 \tag{3.29b}$$

$$\vec{k}\times\vec{\boldsymbol{E}} = \mu_0\omega\vec{\boldsymbol{H}} \tag{3.29c}$$

$$\vec{k}\times\vec{\boldsymbol{H}} = -j\varepsilon\omega\vec{\boldsymbol{E}} \tag{3.29d}$$

3.2.3 Propagation in good dielectrics or insulators

In a good dielectric (see Subsection ??) the reactive current predominates on the dissipative current and according to (1.91), $\tan\delta_d = \sigma_e/\omega\varepsilon' << 1$. In this

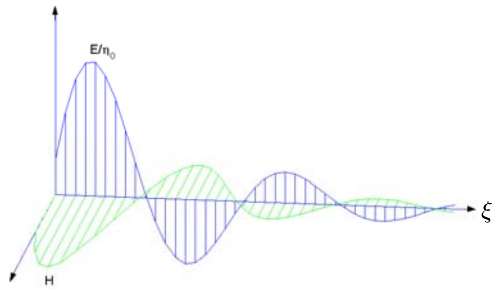


Figure 3.3: Cuidado¡¡¡ estan normalizada a η_0 possitive ξ traveling fields of a uniform plane in dissipative medium

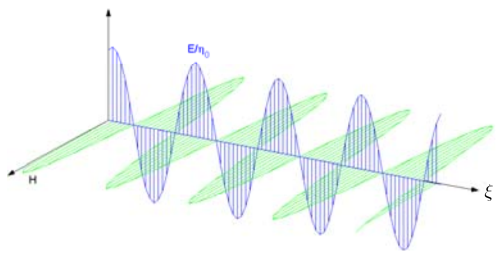


Figure 3.4: Uniform plane wave propagating in the $+\xi$ direction in a lossless medium

case, we can develop the complex propagation constant (3.17) to get

$$\gamma = jk\left(1 - j\tan\delta_d\right)^{1/2} = j\omega\left(\mu_0\varepsilon'\right)^{1/2}\left(1 - \frac{j\tan\delta_d}{2} + \frac{\tan^2\delta_d}{8} + ..\right) \simeq j\omega\left(\mu_0\varepsilon'\right)^{1/2}\left(1 - \frac{j\tan\delta_d}{2}\right) \tag{3.30}$$

and therefore

$$\alpha \simeq \frac{\omega\left(\mu_0\varepsilon'\right)^{1/2}\tan\delta_d}{2} = \frac{\sigma_e}{2}\left(\frac{\mu_0}{\varepsilon'}\right)^{1/2} \tag{3.31a}$$

$$\beta \simeq k = \omega\left(\mu_0\varepsilon'\right)^{1/2} \tag{3.31b}$$

Thus the propagation velocity can be approximated by

$$\frac{\omega}{\beta} \simeq \frac{1}{\left(\mu_0\varepsilon'\right)^{1/2}} \tag{3.32}$$

From (3.31a) it can be seen that α is small and therefore so is the wave attenuation. Moreover, since $\sigma_e/\omega\varepsilon' << 1$, the intrinsic impedance of the medium (3.25) is usually simplified to

$$\eta_c \simeq \eta = \left(\frac{\mu_0}{\varepsilon'}\right)^{1/2} \tag{3.33a}$$

$$\theta = 0 \tag{3.33b}$$

3.2.4 Propagation in good conductors

For a good conductor (see Subsection **??**) the dissipative current predominates on the reactive current and according to (1.94), $\tan\delta_d = \sigma/\omega\varepsilon >> 1$. In this case, from (1.93) and (3.13) we have

$$\gamma = jk\left(1 - j\tan\delta_d\right)^{1/2} \simeq jk\left(-j\tan\delta_d\right)^{1/2} = jk\left(\frac{\sigma}{2\varepsilon\omega}\left(1-j\right)\left(1-j\right)\right)^{1/2} = \left(1+j\right)\left(\frac{\mu_0\sigma\omega}{2}\right)^{1/2} \tag{3.34}$$

and consequently from (3.17),

$$\alpha = \beta = \left(\frac{\mu_0\sigma\omega}{2}\right)^{1/2} \tag{3.35}$$

Thus the electric field from (3.19a), simplifies to

$$\vec{E} = \mathrm{Re}\{\vec{\boldsymbol{E}}e^{-\xi/\delta}e^{j(\omega t - \xi/\delta)}\} \tag{3.36}$$

where δ

$$\delta = \frac{1}{\alpha} = \left(\frac{2}{\mu_0\omega\sigma}\right)^{1/2} \tag{3.37}$$

is the penetration factor (3.21) particularized by a good conductor. Thus, for good conductors, the penetration factor δ has a very low value which decreases as the frequency increases. Thus the fields are confined within a very short distance from the surface of the conductor. For a perfect conductor, $\sigma \to \infty$ and $\delta = 0$. Furthermore, the dielectric constant and the complex impedance are reduced to

$$\varepsilon_{ec} = \varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right) \simeq -j\frac{\sigma}{\omega} \tag{3.38}$$

and, respectively

$$\eta_c = \left(\frac{\mu_0}{\varepsilon_{ec}}\right)^{1/2} = \left(-j\frac{\mu_0\omega}{\sigma}\right)^{1/2} = (1+j)\left(\frac{\mu_0\omega}{2\sigma}\right)^{1/2} = (1+j)\,\frac{\omega\mu_0\delta}{2} \tag{3.39}$$

Thus the phase shift between $\vec{E}$ and $\vec{H}$ is 45^o.

3.2.5 Surface resistance

Let us consider an area element perpendicular to the direction of propagation ξ. Since the wave amplitudes of $\vec{E}$ and $\vec{H}$ decrease exponentially according to the factor $e^{-\alpha\xi}$, the complex Poynting vector (1.107), and consequently the mean power per unit of area, (1.106), attenuates along the direction of propagation by the factor $e^{-2\alpha\xi}$. Therefore

$$\vec{\mathcal{P}}_{av} = \frac{1}{2}\,\mathrm{Re}\{\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}^*\} = \vec{\mathcal{P}}_{av}(0)e^{-2\alpha\xi} \tag{3.40}$$

where $\vec{\mathcal{P}}_{av}(0)$ is the mean power per unit area at $\xi = 0$. Thus the total power per unit area transmitted by the wave to the medium along the distance $\xi = l$ is given by

$$\frac{dP}{ds} = \vec{\mathcal{P}}_{av}(0) - \vec{\mathcal{P}}_{av}(l) = \vec{\mathcal{P}}_{av}(0)(1 - e^{-2\alpha l}) \tag{3.41}$$

This can be also calculated, according to (1.87), as

$$\frac{dP}{ds} = \frac{\sigma_e}{2}\left(\int_0^l (E_0^2 e^{-2\alpha\xi})d\xi\right) = \frac{\sigma_e E_0^2}{4\alpha}(1 - e^{-2\alpha l}) \tag{3.42}$$

This expression for $l = \infty$, or for a distance l such that the magnitude of the fields becomes negligible, simplifies to

$$\frac{dP}{ds} = \frac{\sigma_e E_0^2}{4\alpha} = \frac{1}{2}\,\mathrm{Re}\{\eta_c^{-1}\}E_0^2 = \frac{1}{2}\,\mathrm{Re}\{\eta_c\}H_0^2 = \tag{3.43}$$

since

$$\mathrm{Re}\{\eta_c^{-1}\} = \frac{\sigma_e}{2\alpha} \tag{3.44}$$

For a good conductor, expression (3.43) simplifies, from (3.39), to

$$\frac{dP}{ds} = \frac{H_0^2}{2}\left(\frac{\mu_0\omega}{2\sigma}\right)^{1/2} = \frac{1}{2}R_s H_0^2 \tag{3.45}$$

where R_s is the so-called surface resistance

$$R_s = \left(\frac{\mu_0\omega}{2\sigma}\right)^{\frac{1}{2}} = \frac{1}{\sigma\delta} \tag{3.46}$$

and δ is the penetration factor given by (3.37).

3.3 Group velocity

So far, we have considered the ideal case of a plane harmonic wave, i.e. one in which the wave number and the frequency are fixed. When this type of wave propagates through a dispersive medium, the propagation velocity (phase velocity) of a harmonic wave depends on its frequency. In practice, the ideal situation of a pure harmonic wave which extends to infinity both backward and forward in time never arises and, moreover, such a wave could not carry information. What in fact happens is that a transmitter emits a given signal $f(\xi, t)$ for a finite period of time that, according to Fourier's theorem, can be expanded into a continuous spectrum of amplitudes A_ω such that

$$f(\xi, t) = \int_{-\infty}^{\infty} A_\omega e^{j(\omega t - \beta\xi)} d\omega \tag{3.47}$$

When the signal propagates through a dispersive medium, i.e. a medium where the phase velocity depends on the frequency, each spectral component travels at a different velocity and, as a consequence, the signal will deform as it propagates. When, as commonly occurs in practice, the spectrum of the signal is narrow [3] and the transmission medium is only slightly dispersive, then a single velocity, termed the group velocity, may be assigned to the signal which is usually known as a wave group or wave package. The velocity with which the envelope or energy of the wave group propagates in the medium is called group velocity. To calculate this, let us consider a wave group centered on a frequency ω_0 such that $A_\omega \simeq 0$ except for $\omega = \omega_0 \pm \triangle\omega/2$ (Fig. 7.4). Under these conditions, Eq. (3.47) simplifies to

$$f(\xi, t) = \int_{\triangle\omega} \boldsymbol{A}_\omega e^{j(\omega t - \beta\xi)} d\omega \tag{3.48}$$

extended to the values of ω in which $\boldsymbol{A}_\omega \neq 0$. Given that $\beta = \beta(\omega)$, it can be developed into a Taylor series around the frequency ω_0

$$\beta(\omega) = \beta(\omega_0) + \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}(\omega - \omega_0) + \left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\omega_0}\frac{(\omega - \omega_0)^2}{2} \tag{3.49}$$

[3] *Note that a concentration of the field in space does not imply a concentration in the frequency spectrum, but just the opposite, in accordance with the scale change property of the Fourier transform, which indicates that an inverse relation exists between the duration of a signal and its bandwidth.*

If the dispersive medium is such that the dependence of the phase velocity v_p on the frequency is so slowly that we can consider (as a good approximation) that there exists a linear relation between β and ω, then (3.49) simplifies to

$$\beta(\omega) = \beta_0 + \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} (\omega - \omega_0) \tag{3.50}$$

where $\beta_0 = \beta(\omega_0)$.

By substituting (3.50) in (3.48) we get

$$f(\xi, t) = e^{j\left(\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}\omega_0\xi - \beta_0\xi\right)} \int_{\triangle\omega} A_\omega e^{j\omega\left(t - \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}\xi\right)} d\omega \tag{3.51}$$

which, taking into account (3.48), can be written as a function of $f(0,t)$ in the following way

$$f(\xi, t) = f\left(0, t - \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}\xi\right) e^{j\left(\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}\omega_0\xi - \beta_0\xi\right)} \tag{3.52}$$

This means that, at a point ξ, the signal has the same amplitude as at the origin after a time $t = \partial\beta/\partial\omega|_{\omega_0}\,\xi$ and a *phase shift* given by $\partial\beta/\partial\omega|_{\omega_0}\,\omega_0\xi - \beta_0\xi$. Consequently, the velocity at which the signal, and thus its associated energy, propagates is

$$\begin{aligned} v_g &= \frac{d\xi}{dt} = \left.\frac{d\omega}{d\beta}\right|_{\omega_0} \\ &= \left.\frac{d}{d\beta}(v_p\beta)\right|_{\omega_0} = v_p + \beta\left.\frac{dv_p}{d\beta}\right|_{\omega_0} \\ &= v_p - \lambda\left.\frac{dv_p}{d\lambda}\right|_{\omega_0} = \left.\frac{1}{d\beta/d\omega}\right|_{\omega_0} \end{aligned} \tag{3.53}$$

If the phase velocity varies slowly with the frequency, then a pulse may travel through a dispersive medium a certain distance without a significant change. If this condition is not satisfied and the medium is very dispersive the shape of signal changes rapidly and the concept of group velocity is not longer valid. The sign of $dv_p/d\omega$ determines whether v_g is greater or less than v_p. If the phase velocity v_p increases with the frequency, it is termed normal dispersion. On the contrary, when v_p decreases with the frequency, it is termed anomalous dispersion. In an ideal dielectric where $v_p \neq v_p(\beta)$, so that all the wavelengths propagate at the same velocity $v_p = v_g$, the signal propagates without deformation.

3.4 Polarization

As the wave equation is a linear differential equation, it fulfils the superposition principle and any sum of solutions is also a solution of the differential equation.

In particular, let us consider the sum of two plane waves propagating in direction z (one with the electric field lying along the x axis and the other along the y axis) at identical frequencies but, in general, with different amplitudes (a and b) and phases (δ_1 and δ_2), respectively. Each of these waves, because the direction of their electric field does not change with time, is said to be linearly polarized, one in the x direction and the other in the y direction. However, in an electromagnetic wave the direction of the electric field generally changes and *traces out* an ellipse as the wave propagates[4]. To see this, let us consider the total time-varying electric field, which is sum of the two linearly polarized waves, given by

$$\vec{\boldsymbol{E}}(z,t) = (ae^{j\delta_1}\hat{x} + be^{j\delta_2}\hat{y})e^{j(\omega t - kz)} \tag{3.54}$$

Let us determine the time evolution in a plane $z = cte$ of the electric field vector resulting from the composition of these two plane waves. We will assume a homogeneous, isotropic, lossless medium (although the effects of losses as an exponential factor common to all the field components do not influence the polarization).

At the plane $z = 0$, for example, we have

$$E_x = a\cos(\omega t + \delta_1) \tag{3.55a}$$

$$E_y = b\cos(\omega t + \delta_2) \tag{3.55b}$$

$$E_z = 0 \tag{3.55c}$$

Using the trigonometric identity for the sum of two angles, solving for $\cos\omega t$ and $\sin\omega t$ in terms of a and b, defining $\delta = \delta_1 - \delta_2$ as the relative phase difference between the two components and after some simplifications based on simply trigonometric identities, we find

$$\frac{E_x^2}{a^2} + \frac{E_y^2}{b^2} - \frac{2E_xE_y}{ab}\cos\delta = \sin^2\delta \tag{3.56}$$

which is the equation of an ellipse with its major axis tilted depending on the value of δ. This means that at a plane $z = cte$, as the time goes on, the electric field *delineates* an ellipse or, equivalently, that the electric field delineates an elliptical helix in the direction of propagation. The resulting polarization is referred to as elliptical polarization. The angular velocity of the vector $\overrightarrow{E_t} = E_x \hat{x} + E_y \hat{y}$ is given by

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{d}{dt}(\tan^{-1}\frac{E_y}{E_x}) = \frac{E_x\dot{E_y} - E_y\dot{E_x}}{|E_t|^2} \tag{3.57}$$

where φ *is ....*

The sense of rotation together with the direction of propagation define left-handed polarized versus right-handed polarized waves, according to the right-hand rule: the thumb of the right hand is pointed in the direction of propagation.

[4] In a unpolarized wave, the vector $\vec{E}$ is subject to random changes of amplitud and phase

Thus, if the fingertips are curling in the direction of the rotation of the electric field, the wave is right-handed polarized, and in the contrary case the wave is left polarized.

Particular cases occur depending on the values of a, b, δ, and the polarization ellipse may degenerate into a centred ellipse, a circle or a straight line.

When $a \neq b$ and $\delta = m\pi/2$, with $m = \pm 1, \pm 3, \pm 5, ..$ the polarization ellipse (3.56) becomes a centred ellipse with the major and minor axis oriented along the x, y directions, i.e.

$$\frac{E_x^2}{a^2} + \frac{E_y^2}{b^2} = 1 \tag{3.58}$$

If $a = b$, then

$$E_x^2 + E_y^2 = a^2 \tag{3.59}$$

which is the equation of a circumference.

When $\delta = \pm m\pi$, with m being an integer, the equation (3.56) becomes

$$\left[\frac{E_x}{a} \pm \frac{E_y}{b}\right]^2 = 0 \tag{3.60}$$

which represents the equation of a straight line

$$E_y = \mp \frac{b}{a} E_x \tag{3.61}$$

intersecting the origin. The wave is then linearly polarized and the components of $\vec{E}$ are

$$E_x = a\cos(\omega t - kz) \tag{3.62a}$$
$$E_y = b\cos(\omega t - kz \pm m\pi) \tag{3.62b}$$

The angle of the slope with the x axis is

$$\tan\varphi = \tan\frac{E_y}{E_x} = (-1)^m \frac{b}{a} \tag{3.63}$$

Chapter 4

Reflection and refraction of plane waves

In the previous chapter, we studied the characteristics of harmonic plane waves, and now consider what happens when such waves reach the interface (assumed to be plane and indefinite) separating two linear, nonmagnetic, homogeneous and isotropic dielectrics having different electromagnetic characteristics. The change in the constitutive parameters, as the wave passes from one medium to the other, is assumed to take place in an electrically very narrow region with a thickness much less than λ. In general, when a wave propagating through a medium strikes the interface (incident wave), part of its energy is reflected and propagates through the same medium (reflected wave), while another part is transmitted to the second medium (transmitted, or refracted wave). The characteristics of reflected and transmitted waves can be calculated from those of the incident wave by forcing the total field on the interface to fulfil the boundary conditions. We will consider first the simplest case of normal incidence, i.e. when the interface is perpendicular to the propagation direction of the wave, [1]. and then the more general case of oblique incidence. This study has extensive applications in optics where the interface of many optical devices, such as lenses and fiber-optic transmission lines, has a radius of curvature much larger than the wavelength of the incident wave. Thus the interface can be considered quite accurately as a plane interface. In the following, with no loss of generality, we will assume the interface to be parallel to the xy plane.

[1] La incidencia normal tiene muchas analogías con la líneas de transmisión que se estudiarán en el capítulo *Tal*

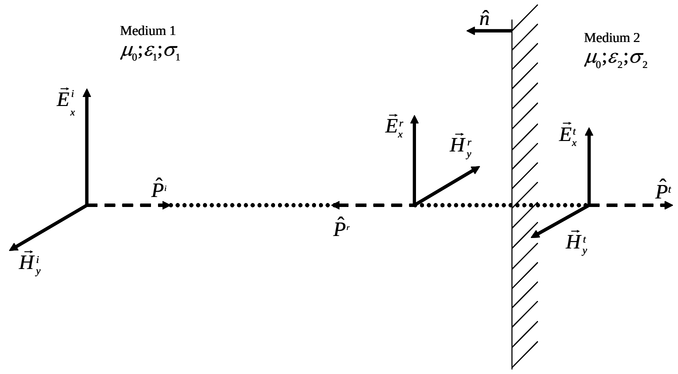


Figure 4.1: Poner los vectores de pynting $\mathcal{P}$ El subindice de campo electrico incidente ponerlo mejor

4.1 Normal incidence.

4.1.1 General case: interface between two lossy media

normally incident from a lossy media, characterized by the parameters $\mu_0, \varepsilon_1 = \varepsilon_1' - j\varepsilon_1'', \sigma_1$ *to the surface of another one with different constitutive parameters* $\mu_0, \varepsilon_2 = \varepsilon_2' - j\varepsilon_2'', \sigma_2$

Considering two semi-indefinite lossy media that are separated by the plane $z = 0$, see *figure 4.1*, let us assume that a harmonic plane wave propagates through the first medium in the positive sense of the z axis with the electric field parallel to the x axis. The wave impinges with normal incidence on this plane. Due to the discontinuity of the constitutive parameters, $\mu = \mu_0$, $\varepsilon_{ci} = \varepsilon_i' - j\varepsilon_i''$, and σ_i where subindex i $(i = 1, 2)$ refers to medium 1 or 2, part of the wave is propagated through medium 2 and part is reflected back through medium 1. Therefore, the total field in medium 1 (where $z < 0$) and medium 2 (where $z > 0$) is given by

Medium 1

$$\boldsymbol{E}_{x1} = \boldsymbol{E}^i_{x1} e^{-\alpha_1 z} e^{-j\beta_1 z} + \boldsymbol{E}^r_{x1} e^{\alpha_1 z} e^{j\beta_1 z} = \boldsymbol{E}^i_{x1} e^{-\gamma_1 z} + \boldsymbol{E}^r_{x1} e^{\gamma_1 z} \tag{4.1a}$$

$$\boldsymbol{H}_{y1} = \frac{\boldsymbol{E}^i_{x1}}{\eta_{c1}} e^{-\alpha_1 z} e^{-j\beta_1 z} - \frac{\boldsymbol{E}^r_{x1}}{\eta_{c1}} e^{\alpha_1 z} e^{j\beta_1 z} = \frac{\boldsymbol{E}^i_{x1}}{\eta_{c1}} e^{-\gamma_1 z} - \frac{\boldsymbol{E}^r_{x1}}{\eta_{c1}} e^{\gamma_1 z} \tag{4.1b}$$

Medium 2

$$\boldsymbol{E}_{x2} = \boldsymbol{E}^t_{x2} e^{-\alpha_2 z} e^{-j\beta_2 z} = \boldsymbol{E}^t_{x2} e^{-\gamma_2 z} \tag{4.1c}$$

$$\boldsymbol{H}_{y2} = \frac{\boldsymbol{E}^t_{x2}}{\eta_{c2}} e^{-\alpha_2 z} e^{-j\beta_2 z} = \frac{\boldsymbol{E}^t_{x2}}{\eta_{c2}} e^{-\gamma_2 z} \tag{4.1d}$$

The ¿¿¿superindices?? i, r, and t indicate the incident wave (medium 1), the reflected wave (medium 1) and the transmitted wave (medium 2), respectively. The minus sign for the reflected wave of the magnetic field is associated with the fact that the Poynting vector of the reflected wave propagates in the $-\hat{z}$ direction. In these expressions, $\eta_{ci} = \sqrt{\mu_0/\varepsilon_{eci}}$ represents the impedance (3.24) of medium i while γ_i is the complex propagation factor (3.17),

$$\gamma_i = \alpha_i + j\beta_i \tag{4.2}$$

where α_i and β_i are the attenuation and propagation constants (3.18a)and (3.18b), respectively

$$\beta_i = \frac{\omega\sqrt{\mu_o\varepsilon_i'}}{\sqrt{2}}\left[\sqrt{1+(\sigma_e/\omega\varepsilon_i')}+1\right]^{1/2} \tag{4.3}$$

$$\alpha_i = \frac{\omega\sqrt{\mu_o\varepsilon_i'}}{\sqrt{2}}\left[\sqrt{1+(\sigma_e/\omega\varepsilon_i')}-1\right]^{1/2} \tag{4.4}$$

The time dependence of the fields is achieved by adding the factor $e^{j\omega t}$ to (4.1).

For each instant of time, by imposing the boundary conditions in the plane $z = 0$ (2.182b) onto the tangential components of $\vec{E}$ and $\vec{H}$,

$$\begin{aligned} \vec{E}_{1t} &= \vec{E}_{2t} && (4.5a) \\ \vec{H}_{1t} &= \vec{H}_{2t} && (4.5b) \end{aligned}$$

we obtain

$$\begin{aligned} \omega^i &= \omega^t = \omega^r = \omega && (4.6a) \\ \boldsymbol{E}_{x1}^r &= \Gamma_L\, \boldsymbol{E}_{x1}^i && (4.6b) \\ \boldsymbol{E}_{x2}^t &= T_L \boldsymbol{E}_{x1}^i = (1+\Gamma_L)\boldsymbol{E}_{x1}^i && (4.6c) \\ \boldsymbol{H}_{y1}^r &= -\Gamma_L \boldsymbol{H}_{y1}^i && (4.6d) \\ \boldsymbol{H}_{y2}^t &= \frac{\eta_{c1}}{\eta_{c2}} T_L \boldsymbol{H}_{y1}^i && (4.6e) \end{aligned}$$

where Γ_L is the reflection coefficient in the plane $z = 0$ defined by

$$\Gamma_L = \frac{\eta_{c2}-\eta_{c1}}{\eta_{c2}+\eta_{c1}} = |\Gamma_L|\, e^{j\Phi_L} \tag{4.7}$$

and T_L is the transmission coefficient in the same plane, defined by

$$\begin{aligned} T_L &= \frac{2\eta_{c2}}{\eta_{c2}+\eta_{c1}} && (4.8a) \\ T_L &= 1+\Gamma_L = |T_L|\, e^{j\Psi_L} && (4.8b) \end{aligned}$$

If there is an impedance adaptation ($\eta_{c2} = \eta_{c1}$) then there is no reflected wave, and so all the incident energy is absorbed by the second medium.

From (4.1a) and (4.6b), the total electric field in the first medium can be expressed as

$$\begin{aligned} \boldsymbol{E}_{x1} &= \boldsymbol{E}^i_{x1} e^{-\alpha_1 z} e^{-j\beta_1 z} (1 + \Gamma_L e^{2\alpha_1 z} e^{2j\beta_1 z}) && (4.9) \\ &= \boldsymbol{E}^i_{x1} e^{-\alpha_1 z} e^{-j\beta_1 z} (1 + \Gamma(z)) = \boldsymbol{E}^i_{x1} e^{-\gamma_1 z} (1 + \Gamma(z)) && (4.10) \end{aligned}$$

where $\Gamma(z)$, defined as

$$\Gamma(z) = \Gamma_L e^{2\alpha_1 z} e^{2j\beta_1 z} \tag{4.11}$$

is the reflection coefficient in the plane $z = z$. Similarly, for the magnetic field, we have

$$\boldsymbol{H}_{y1} = \frac{\boldsymbol{E}^i_{x1}}{\eta_{c1}} e^{-\gamma_1 z} (1 - \Gamma(z)) \tag{4.12}$$

The impedance associated with the total field at a coordinate point z in the first medium is defined as

$$\eta_{inp}(z) = \left. \frac{\boldsymbol{E}_{x1}}{\boldsymbol{H}_{y1}} \right|_z = \eta_{c1} \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \eta_{c1} \frac{\eta_{c2} - \eta_{c1} \tanh(\gamma_1 z)}{\eta_{c1} - \eta_{c2} \tanh(\gamma_1 z)} \tag{4.13}$$

The impedance $\eta_{inp}(z)$ is continuous through the interface, because the tangential components E_{x1} and H_{y1} are similarly continuous, while the reflection coefficient Γ is discontinuous.

4.1.2 Perfect/Lossy dielectric interface

In the particular case in which the first dielectric is perfect, i.e. lossless ($\sigma_1 = 0$ and $\varepsilon''_1 = 0, \varepsilon_1 = \varepsilon'_1, \gamma_1 = jk_1$), the characteristic impedances reduce to

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \tag{4.14}$$

and the coefficient of reflection (4.7) at the interface ($z = 0$) becomes

$$\Gamma_L = \frac{1 - \sqrt{\frac{\varepsilon_{ec2}}{\varepsilon_1}}}{1 + \sqrt{\frac{\varepsilon_{ec2}}{\varepsilon_1}}} \tag{4.15}$$

Since $\sigma_1 = 0$ and $\varepsilon''_1 = 0$, it follows that $\alpha_1 = 0$ and therefore

$$\boldsymbol{E}_{x1} = \boldsymbol{E}^i_{x1} e^{-jk_1 z} (1 + \Gamma_L e^{2jk_1 z}) = \boldsymbol{E}^i_{x1} e^{-jk_1 z} (1 + \Gamma(z)) \tag{4.16}$$

and

$$\boldsymbol{H}_{y1} = \frac{\boldsymbol{E}^i_{x1}}{\eta_1} e^{-jk_1 z} (1 - \Gamma(z)) \tag{4.17}$$

where

$$\Gamma(z) = \Gamma_L e^{2jk_1 z} \tag{4.18}$$

The input impedance (4.13) simplifies to

$$\eta_{inp}(z) = \eta_1 \frac{\eta_{c2} - \eta_1 \tan(k_1 z)}{\eta_1 - \eta_{c2} \tan(k_1 z)} \tag{4.19}$$

4.1.3 Perfect dielectric/Perfect conductor interface

Another particular case arises when the second medium is a perfect conductor ($\eta_2 = 0$) and therefore $T_L = 0$ and $\Gamma_L = -1$. Then the fields in the first medium are

$$\boldsymbol{E}_{x1} = \boldsymbol{E}^i_{x1} e^{-jk_1 z}\left(1 - e^{2jk_1 z}\right) = \boldsymbol{E}^i_{x1}\left(e^{-jk_1 z} - e^{jk_1 z}\right) = -2j\boldsymbol{E}^i_{x1}\sin(k_1 z) \tag{4.20a}$$

$$\boldsymbol{H}_{y1} = \frac{\boldsymbol{E}^i_{x1}}{\eta_1}\left(e^{-jk_1 z} + e^{jk_1 z}\right) = 2\frac{\boldsymbol{E}^i_{x1}}{\eta_1}\cos(k_1 z) \tag{4.20b}$$

4.1.4 Standing waves

It is well known that two waves with the same frequency that are propagating in opposite directions interfere and form what are termed standing (or stationary) waves. To examine this concept, let us first consider the case in which the first medium is lossless, and then analyse the case in which the first medium is dissipative.

a) Lossless case

For the first medium, the expression of the total electric field is,

$$\begin{aligned}
\boldsymbol{E}_{x1} &= \boldsymbol{E}^i_{x1} e^{-jk_1 z} + \boldsymbol{E}^r_{x1} e^{jk_1 z} = (1+\Gamma_L)\boldsymbol{E}^i_{x1} e^{-jk_1 z} + \Gamma_L \boldsymbol{E}^i_{x1}(e^{jk_1 z} - e^{-jk_1 z}) \\
&= T_L \boldsymbol{E}^i_{x1} e^{-jk_1 z} + |\Gamma_L|\,\boldsymbol{E}^i_{x1}(e^{j(\Phi_L + k_1 z)} - e^{j(\Phi_L - k_1 z)}) \\
&= |T_L|\,\boldsymbol{E}^i_{x1} e^{j(\Psi_L - k_1 z)} + 2\,|\Gamma_L|\,\boldsymbol{E}^i_{x1}\sin(k_1 z)\; e^{j(\Phi_L + \pi/2)}
\end{aligned} \tag{4.21}$$

By including the time dependence, *and assuming an initial phase* $\varphi = 0$, we obtain the following expression for the total field

$$E_{x1}(z,t) = |T_L|\,E^i_{0x1}\cos(\omega t - k_1 z + \Psi_L) - 2\,|\Gamma_L|\,E^i_{0x1}\sin(k_1 z)\;\sin(\omega t + \Phi_L) \tag{4.22}$$

where the first summand of the second member corresponds to a wave that is propagating, while the second summand represents a standing wave, i.e., one in which the mean energy transported by the wave is null. The amplitude of the propagating wave is determined by the coefficient of transmission, while that of the standing wave depends on the coefficient of reflection. The envelope of the equation (4.22) is termed the diagram of the standing wave. If the coefficient of transmission T is null (which occurs when the second medium is a perfect conductor) the wave of the first medium becomes a pure standing wave.

From (4.16) and (4.17) the magnitudes of the fields are

$$E_{0x1} = E^i_{0x1}\,|1 + \Gamma(z)| \tag{4.23a}$$

$$H_{0y1} = \frac{1}{\eta_1}E^i_{0x1}\,|1 - \Gamma(z)| \tag{4.23b}$$

The maximum values of E_{0x1} (the minima of H_{0y1}) are given by

$$E_{0x1}(z)_{\max} = E^i_{0x1} + E^r_{0x1} \tag{4.24}$$

at the coordinate points

$$z_{\max} = -\frac{\Phi_L + 2n\pi}{2k_1} \qquad\qquad n = 0, 1, ... \tag{4.25}$$

and the minimum values of E_{0x} (the maxima of H_{0y}), assuming $E^i_{0x1} > E^r_{0x1}$, are given by

$$E_{0x}(z)_{\min} = E^i_{0x1} - E^r_{0x1} \tag{4.26}$$

at the points

$$z_{\min} = -\frac{\Phi_L + (2n+1)\,\pi}{2k_1}; \quad n = 0, 1, ... \tag{4.27}$$

Ratio of the standing wave

The relation between the maximum and minimum values of the diagram of the standing wave is called the ratio of the standing wave, and is described by

$$SWR = \frac{E_{0x1}(z)_{\max}}{E_{0x1}(z)_{\min}} = \frac{E^i_{0x1} + E^r_{0x1}}{E^i_{0x1} - E^r_{0x1}} = \frac{1 + |\Gamma(z)|}{1 - |\Gamma(z)|} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{4.28}$$

Its value ranges from 1 (no reflected wave) to infinity (pure standing wave), i.e.

$$1 \leq SWR \leq \infty \tag{4.29}$$

b) Lossy case

In this case, the expression of the total electric field in the first medium is

$$\begin{aligned} \boldsymbol{E}_{x1}(z) &= \boldsymbol{E}^i_{x1} e^{-\alpha_1 z} e^{-j\beta_1 z} + \boldsymbol{E}^r_{x1} e^{\alpha_1 z} e^{j\beta_1 z} = \boldsymbol{E}^i_{x1}(e^{-\gamma_1 z} + \Gamma_L e^{\gamma_1 z}) \\ &= T_L \boldsymbol{E}^i_{x1} e^{-\gamma_1 z} + \Gamma_L \boldsymbol{E}^i_{x1}(e^{\gamma_1 z} - e^{-\gamma_1 z}) \end{aligned} \tag{4.30}$$

and, by including the time dependence, we get

$$\begin{aligned} E_{x1}(z,t) &= |T_L|\, E^i_{0x1} e^{-\alpha_1 z} \cos(\omega t - \beta_1 z + \Psi_L) + \\ &\quad\; 2\,|\Gamma_L|\, E^i_{0x1} \sinh(\gamma_1 z) \cos(\omega t + \Phi_L) \end{aligned} \tag{4.31}$$

In these media, it makes no sense to define the SWR parameter because the maxima and minima are not constant.

4.1.5 Measures of impedances

Assuming that the first medium is lossless, from (4.27) the first field minimum occurs at $\phi_L + 2k_1 z_{\min} = \pi$. Consequently, we can determine the phase angle ϕ_L, assuming that k_1 is known and that $z_{\min}$ is determined experimentally (by using an appropriate device to detect the first field minimum). If $k_1$is not known, it can be calculated from the distance between two consecutive minima.

The value of $|\Gamma_L|$ can be found from the ratio between the maximum and minimum field values $E_{0x1\,\max}/E_{0x1\,\min} = SWR = (1 + |\Gamma_L|)/(1 - |\Gamma_L|)$.

Note that if the incident wave has an amplitude of one, then $|\Gamma_L|$ is identical to E_0^r and thus we need to measure only this amplitude. Thus η_2 is determined from this information and from expression (4.7).

4.2 Multilayer structures

Let us now consider the normal incidence of an electromagnetic wave on a structure in which there are more than two media separated by parallel planes. To simplify the analysis, we consider the case of three lossless dielectrics, as shown in Fig. tal. The generalization to more media, including the possibility of losses, is straightforward. Clearly, for a wave that is propagating to the right in medium 2, the problem is analogous to the two-layer cases discussed above. Therefore, the coefficient of reflection in the $z = 0$ plane is

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} = \Gamma(z = 0) \tag{4.32}$$

where subindex 23 refers to the surface that separates medium 2 from medium 3. Particularizing (4.13) for $z = -l$ we have the *load impedance* η_L

$$\eta_L = \eta_{inp}(z = -l) = \eta_2 \frac{1 + \Gamma_{23} e^{-2jkl}}{1 - \Gamma_{23} e^{-2jkl}} \tag{4.33}$$

Taking into account that $\eta_{inp}(z)$ is continuous at an interface, the coefficient of reflection (4.7) at $z = -l$, becomes

$$\Gamma_L = \Gamma(z = -l) = \frac{\eta_L - \eta_1}{\eta_L + \eta_1}$$

by introducing (4.33) into this equation and then operating, we get

$$\Gamma_L = \frac{\Gamma_{12} + \Gamma_{23} e^{-2jkl}}{1 + \Gamma_{12}\Gamma_{23} e^{-2jkl}} \tag{4.34}$$

where

$$\Gamma_{ij} = \frac{\eta_j - \eta_i}{\eta_j + \eta_i} \tag{4.35}$$

Thus, for an electromagnetic wave with an amplitude of one, impinging normally from the first medium onto the structure, the amplitude of the reflected wave is given by (4.34)

Quarter-wave layer

For a quarter-wave layer, $l = \lambda/4$ ($e^{-2jkl} = -1$), equation (4.34) becomes

$$\Gamma = \frac{\Gamma_{12} - \Gamma_{23}}{1 - \Gamma_{12}\Gamma_{23}} \tag{4.36}$$

Thus, to transmit the incident energy completely (adaptation of impedance), the coefficient of reflection must be null, and so

$$\Gamma_{12} - \Gamma_{23} = 0 \tag{4.37}$$

Taking into account equation (4.35) we have

$$\eta_2 = \sqrt{\eta_1\eta_3} \tag{4.38}$$

as a condition for impedance adaptation to exist.

Half-wave layer

For a half-wave layer, *i.e.* $l = \lambda/2$ ($e^{-2jkl} = 1$), expression (4.34) is reduced to

$$\Gamma = \frac{\Gamma_{12} + \Gamma_{23}}{1 + \Gamma_{12}\Gamma_{23}} \tag{4.39}$$

For impedance adaptation to exist, the following must be fulfilled

$$\Gamma_{12} + \Gamma_{23} = 0 \tag{4.40}$$

By replacing the coefficients by the values given in (4.35), we have

$$(\eta_3 = \eta_1) \tag{4.41}$$

and thus $\Gamma_L = 0$ irrespectively of η_2. Thus any material with a thickness of $\lambda/2$ is adapted so long as the impedances of media 1 and 3 are the same.

4.2.1 Stationary and transitory regimes

The above analyses are valid for monochromatic waves in a stationary regime. It should be noted that such a regime is the limit of a transitory process involving multiple reflected and transmitted waves within media 1 and 2. To illustrate this limit process, let us consider the normal incidence of a wave that impinges upon a structure formed of three perfect dielectrics, as shown in Fig. tal. From the process of multiple reflections and transmissions, we find that in medium 1 a reflected field is given by

$$\begin{aligned} \boldsymbol{E}^r_{x1} &= E^i_{0x1}(\Gamma_{12} + T_{12}\Gamma_{23}T_{21}e^{-2jk_2d} + T_{12}\Gamma^2_{23}\Gamma_{21}T_{21}e^{-4jk_2d} \\ &\quad + T_{12}\Gamma^3_{23}\Gamma^2_{21}T_{21}e^{-6jk_2d} + \ldots) \end{aligned} \tag{4.42}$$

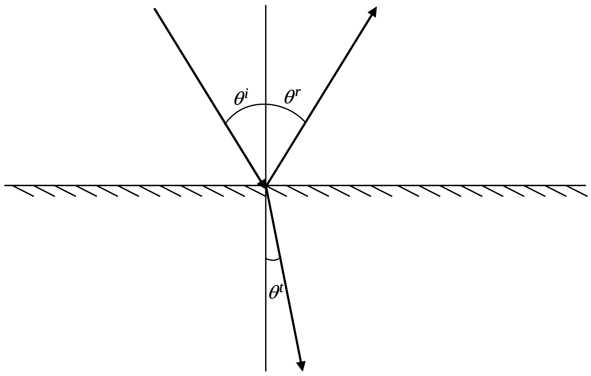


Figure 4.2: Poner sistemas de ejes

Observing the second member, we can see that the summands following the first one constitute a geometric progression of common ratio $\Gamma_{21}\Gamma_{23}e^{-2jk_2d}$. Thus the coefficient of reflection can be written as

$$\Gamma_L = \Gamma_{12} + \frac{T_{12}\Gamma_{23}T_{21}e^{-2jk_2d}}{1 - \Gamma_{23}\Gamma_{21}e^{-2jk_2d}} \tag{4.43}$$

which, taking into account the equalities

$$\Gamma_{21} = -\Gamma_{12} \tag{4.44}$$

$$T_{12} = 1 + \Gamma_{12} \tag{4.45}$$

$$T_{21} = 1 - \Gamma_{12} \tag{4.46}$$

is reduced to equation (4.34).

4.3 Oblique incidence

As a more general case than the normal incidence, let us now consider the oblique incidence of a plane wave on a plane interface separating two media. In general, in medium 1 there exists an incident and a reflected wave, while the transmitted (also called refracted) wave is in medium 2. To study the oblique incidence we will use the geometry shown in Fig.**??**, where the waves have been represented, as usual, by arrows (called rays) in the direction of propagation. These rays are perpendicular to the equiphase planes (wavefronts). The oblique incident has extensive applications in optics where the interface of many optical devices, such as lenses and fiber optic waveguides, has a radius of curvature much larger than the wavelength of the incident wave. Thus the interface can be considered very approximately as a plane interface.

In principle, we make no assumption that the three rays are coplanar, although they are shown as such in Fig Tal. The plane of incidence is defined by vector $\vec{\gamma}^i$ and by the z axis. Let us assume that $\vec{\gamma}^i$ is in the plane $y = 0$ and forms an angle θ_i with the z axis. In the general case of two lossy media, the electric fields of the incident, reflected, and refracted waves can be written, respectively, as

$$\vec{E}^i = \mathrm{Re}\{\vec{E}_0^i e^{(j\omega^i t - \vec{\gamma}^i \cdot \vec{r})}\} \quad (4.47a)$$
$$\vec{E}^r = \mathrm{Re}\{\vec{E}_0^r e^{(j\omega^r t - \vec{\gamma}^r \cdot \vec{r})}\} \quad (4.47b)$$
$$\vec{E}^t = \mathrm{Re}\{\vec{E}_0^t e^{(j\omega^t t - \vec{\gamma}^t \cdot \vec{r})}\} \quad (4.47c)$$

In $z = 0$, the tangential component of the electric field must be continuous, and thus we have

$$\vec{E}_x^i + \vec{E}_x^r = \vec{E}_x^t \quad (4.48)$$

so that

$$\mathrm{Re}\{\vec{E}_{0x}^i e^{(j\omega^i t - \vec{\gamma}^i \cdot \vec{r})}\} + \mathrm{Re}\{\vec{E}_{0x}^r e^{(j\omega^r t - \vec{\gamma}^r \cdot \vec{r})}\} = \mathrm{Re}\{\vec{E}_{0x}^t e^{(j\omega^t t - \vec{\gamma}^t \cdot \vec{r})}\} \quad (4.49)$$

A similar relation must be fulfilled between the components of the fields with respect to the y axis. These conditions can be satisfied only if

$$\omega^i = \omega^t = \omega^r = \omega \quad (4.50)$$

and

$$\vec{\gamma}^i \cdot \vec{r} = \vec{\gamma}^r \cdot \vec{r} = \vec{\gamma}^t \cdot \vec{r} \quad (4.51)$$

Since γ^i lies on the $y = 0$ plane, from (4.51) it follows that $\gamma_y^r = \gamma_y^t = 0$, signifying that the reflected and refracted waves are coplanar with the incident wave. Thus we have

$$\vec{\gamma}^i \cdot \vec{r} = \frac{N_{c1}\omega}{c}\left[x \sin\theta_i + z\cos\theta_i\right] \quad (4.52a)$$
$$\vec{\gamma}^r \cdot \vec{r} = \frac{N_{c1}\omega}{c}\left[x \sin\theta_r + z\cos\theta_r\right] \quad (4.52b)$$
$$\vec{\gamma}^t \cdot \vec{r} = \frac{N_{c2}\omega}{c}\left[x \sin\theta_t + z\cos\theta_t\right] \quad (4.52c)$$

where N_c is the complex index of refraction of the medium, such that

$$\gamma = \frac{N_c\omega}{c} \quad (4.53)$$

By substituting (4.52) in (4.49), and by making the coefficients of x equal, we get

$$\theta_i = \theta_r = \theta \quad (4.54a)$$
$$N_{c1}\sin\theta = N_{c2}\sin\theta_t \quad (4.54b)$$

These equations, together with the coplanarity of the rays, constitute Snell's laws.

For lossless media Eq. (4.54b) simplifies to

$$N_1 \sin\theta = N_2 \sin\theta_t \tag{4.55}$$

where

$$N_i = \frac{c}{v_{pi}} = (\mu_{ri}\varepsilon_{ri})^{\frac{1}{2}} \tag{4.56}$$

or, for nonmagnetic media,

$$N_i = (\varepsilon_{ri})^{\frac{1}{2}} \tag{4.57}$$

Next we study the relations between the amplitudes of the incident, transmitted and reflected waves by making use of the boundary conditions at the interface between the two media. For this we will assume lossless media although the generalization to lossy media is straightforward[2]. Let us analyze the problem in two stages, firstly where the electric field $\vec{\boldsymbol{E}}^i$ oscillates in the incidence plane, and then where it oscillates perpendicularly to the same plane. Any other case can be considered a superposition of these two situations.

4.4 Incident wave with the electric field contained in the plane of incidence

From the continuity of the tangential components of $\vec{E}$ and $\vec{H}$ (Eqs. (4.5a) and (4.5b), we obtain (Fig. 9.2)

$$\boldsymbol{E}_{\parallel}^i \cos\theta + \boldsymbol{E}_{\parallel}^r \cos\theta - \boldsymbol{E}_{\parallel}^t \cos\theta_t = 0 \tag{4.58a}$$

$$\frac{1}{\eta_1}(\boldsymbol{E}_{\parallel}^i - \boldsymbol{E}_{\parallel}^r) - \frac{1}{\eta_2}\boldsymbol{E}_{\parallel}^t = 0 \tag{4.58b}$$

where the subindex $\parallel$ indicates that the physical magnitude in question lies in the incidence plane and

$$\vec{\boldsymbol{E}}_{\parallel}^i = \vec{E}_{0\parallel}^i e^{-j\vec{k}^i\cdot\vec{r}} \tag{4.59a}$$

$$\vec{\boldsymbol{E}}_{\parallel}^r = \vec{E}_{0\parallel}^r e^{-j\vec{k}^r\cdot\vec{r}} \tag{4.59b}$$

From (4.58) we find that

$$\Gamma_{\parallel} = \frac{\boldsymbol{E}_{\parallel}^r}{\boldsymbol{E}_{\parallel}^i} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta}{\eta_2\cos\theta_t + \eta_1\cos\theta} \tag{4.60a}$$

$$\tau_{\parallel} = \frac{\boldsymbol{E}_{\parallel}^t}{\boldsymbol{E}_{\parallel}^i} = \frac{2\eta_2\cos\theta}{\eta_2\cos\theta_t + \eta_1\cos\theta} \tag{4.60b}$$

Where $\Gamma_{\parallel}$ and $\tau_{\parallel}$ are the coefficients of reflection and transmisison, respectively. If medium 2 is a perfect conductor ($\eta_2 = 0$) then $\Gamma_{\parallel} = -1$.

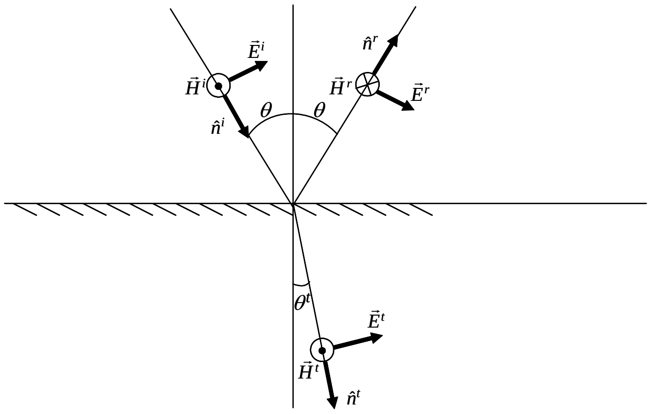


Figure 4.3: cuidado con superindices y subindices...

¿¿¿For lossless non-magnetic materials ?? ($\mu_1 = \mu_2 = \mu_0$) such that

cuidado notacion de vel fase.

$$N_{12} = \frac{\eta_2}{\eta_1} = \frac{v_2}{v_1} = \frac{N_1}{N_2} = \frac{\sin\theta_t}{\sin\theta} \tag{4.61}$$

with N_{12} being the ratio of the indices of refraction of medium 1 and medium 2, expressions (4.60a) and (4.60b) are reduced to

$$\frac{\boldsymbol{E}_{\|}^{r}}{\boldsymbol{E}_{\|}^{i}} = \frac{\tan(\theta_t - \theta)}{\tan(\theta_t + \theta)} \tag{4.62a}$$

$$\frac{\boldsymbol{E}_{\|}^{t}}{\boldsymbol{E}_{\|}^{i}} = \frac{2\sin\theta_t\cos\theta}{\sin(\theta_t + \theta)\cos(\theta_t - \theta)} \tag{4.62b}$$

The total electric field $\vec{\boldsymbol{E}}_{\|}^{1}$ in medium 1 is given by

$$\vec{\boldsymbol{E}}_{\|}^{1} = \vec{\boldsymbol{E}}_{\|}^{i} + \vec{\boldsymbol{E}}_{\|}^{r} \tag{4.63}$$

where

$$\vec{\boldsymbol{E}}_{\|}^{i} = \boldsymbol{E}_{\|}^{i}\cos\theta\hat{x} - \boldsymbol{E}_{\|}^{i}\sin\theta\hat{z} \tag{4.64a}$$

$$\vec{\boldsymbol{E}}_{\|}^{r} = \boldsymbol{E}_{\|}^{r}\cos\theta\hat{x} + \boldsymbol{E}_{\|}^{r}\sin\theta\hat{z} \tag{4.64b}$$

Thus we have

$$\vec{\boldsymbol{E}}_{\|}^{1} = \cos\theta\left(E_{0\|}^{i}e^{-j\vec{k}^i\cdot\vec{r}} + E_{0\|}^{r}e^{-j\vec{k}^r\cdot\vec{r}}\right)\hat{x} + \sin\theta\left(E_{0\|}^{r}e^{-j\vec{k}^r\cdot\vec{r}} - E_{0\|}^{i}e^{-j\vec{k}^i\cdot\vec{r}}\right)\hat{z} \tag{4.65}$$

[2] If the medium is lossy, we must replace $\{\eta, jk\}$ by (η_c, γ)

that is,

$$\begin{aligned}\boldsymbol{E}^1_{\|x} &= E^i_{0\|}\cos\theta\left(e^{-j\vec{k}^i\cdot\vec{r}}+\Gamma_{\|}e^{-j\vec{k}^r\cdot\vec{r}}\right) && (4.66a)\\ \boldsymbol{E}^1_{\|z} &= E^i_{0\|}\sin\theta\left(\Gamma_{\|}e^{-j\vec{k}^r\cdot\vec{r}}-e^{-j\vec{k}^i\cdot\vec{r}}\right) && (4.66b)\end{aligned}$$

Taking into account that

$$\begin{aligned}\vec{k}^r\cdot\vec{r} &= -k^r z\cos\theta+k^r x\sin\theta && (4.67a)\\ \vec{k}^i\cdot\vec{r} &= k^i z\cos\theta+k^i x\sin\theta && (4.67b)\end{aligned}$$

and by substituting these equations in (4.66), we find that

$$\begin{aligned}\boldsymbol{E}^1_{\|x} &= E^i_{0\|}e^{-jk^i(x\sin\theta+z\cos\theta)}\left(1+\Gamma_{\|}e^{2jk^i z\cos\theta}\right)\cos\theta && (4.68a)\\ \boldsymbol{E}^1_{\|z} &= -E^i_{0\|}e^{-jk^i(x\sin\theta+z\cos\theta)}\left(1-\Gamma_{\|}e^{2jk^i z\cos\theta}\right)\sin\theta && (4.68b)\end{aligned}$$

When the time factor $e^{j\omega t}$ is introduced, the term $e^{\left(j\omega t-jk^i x\sin\theta\right)}$ represents a wave that is propagating in the direction of the x axis, while the term

$$\left(e^{-jk^i z\cos\theta}+\Gamma_{\|}e^{jk^i z\cos\theta}\right)e^{j\omega t} \tag{4.69}$$

¿¿¿¿ ¿¿¿¿ ¿¿From (4.68a) or the corresponding one from (4.68b) gives us the superposition of two waves that are propagating with respect to the z axis, but in opposite directions. In other words, a stationary wave overlies a traveling one such that the energy that is transported in direction z, from medium 1 to medium 2, is transported by the traveling wave.????. For the case of a perfect conductor, $\Gamma_{\|}=-1$ and there exits only a standing wave along the z axis.

The total magnetic field $\vec{\boldsymbol{H}}^1_{\perp}$ in medium 1 is

$$\begin{aligned}\vec{\boldsymbol{H}}^1_{\perp} &= \vec{H}^i_{0\perp}e^{-j\vec{k}^i\cdot\vec{r}}+\vec{H}^r_{0\perp}e^{-j\vec{k}^r\cdot\vec{r}}=\frac{1}{\eta_1}\left(E^i_{0\|}e^{-j\vec{k}^i\cdot\vec{r}}-E^r_{0\|}e^{-j\vec{k}^r\cdot\vec{r}}\right)\hat{y}=\\ &\quad \vec{H}^i_{0\perp}e^{-jk^i(x\sin\theta+z\cos\theta)}\left(1-\Gamma_{\|}e^{2jk^i z\cos\theta}\right) && (4.70)\end{aligned}$$

where the symbol $\perp$ indicates that the magnitude in question is perpendicular to the incidence plane

In the case of a perfect conductor, it is straightforward to show that there is no energy flow towards z, but there there is towards x, as the mean time value of Poynting's vector towards z is zero.

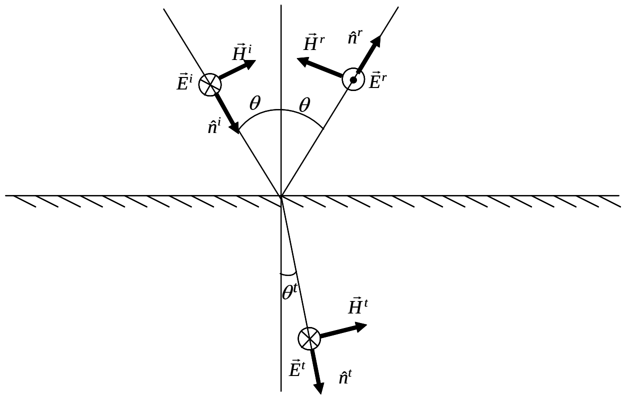


Figure 4.4: cuidado el reflejado tiene mal el sentido del campo electrico

4.5 Wave incident with the electric field perpendicular to the plane of incidence

As above, the continuity equations are used for the tangential components of $\vec{E}$ and $\vec{H}$ and thus (Fig.9.3):

$$\vec{\boldsymbol{E}}_{\perp}^{i} + \vec{\boldsymbol{E}}_{\perp}^{r} = \vec{\boldsymbol{E}}_{\perp}^{t} \tag{4.71}$$

$$\frac{\vec{\boldsymbol{E}}_{\perp}^{i} - \vec{\boldsymbol{E}}_{\perp}^{r}}{\eta_1}\cos\theta = \frac{\vec{\boldsymbol{E}}_{\perp}^{t}}{\eta_2}\cos\theta_t \tag{4.72}$$

where the subindex $\perp$ indicates that the physical magnitude in question corresponds to the case in which the electric field of the incident wave is perpendicular to the plane of incidence.

By resolving the two Eqs. (4.70) and (4.71), we get

$$\Gamma_{\perp} = \frac{\boldsymbol{E}_{\perp}^{r}}{\boldsymbol{E}_{\perp}^{i}} = \frac{\eta_2\cos\theta - \eta_1\cos\theta_t}{\eta_2\cos\theta + \eta_1\cos\theta_t} \tag{4.73a}$$

$$\tau_{\perp} = \frac{\boldsymbol{E}_{\perp}^{t}}{\boldsymbol{E}_{\perp}^{i}} = \frac{2\eta_2\cos\theta}{\eta_2\cos\theta + \eta_1\cos\theta_t} \tag{4.73b}$$

where the parameters $\Gamma_{\perp} = \boldsymbol{E}_{\perp}^{r}/\boldsymbol{E}_{\perp}^{i}$ and $\tau_{\perp} = \boldsymbol{E}_{\perp}^{t}/\boldsymbol{E}_{\perp}^{i}$ are the coefficients of reflection and transmission, respectively.

For lossless non-magnetic materials, Eqs. (4.73) are transformed into

$$\frac{\boldsymbol{E}_{\perp}^{r}}{\boldsymbol{E}_{\perp}^{i}} = \frac{\sin(\theta_t - \theta)}{\sin(\theta_t + \theta)} \tag{4.74a}$$

$$\frac{\boldsymbol{E}_{\perp}^{t}}{\boldsymbol{E}_{\perp}^{i}} = \frac{2\sin\theta_t\cos\theta}{\sin(\theta_t + \theta)} \tag{4.74b}$$

By operating in a similar way to that described for the case of $\vec{E}^i_{\parallel}$, we arrive at the following for the total electric and magnetic fields in a lossy medium 1

$$\vec{\boldsymbol{E}}^1_{\perp} = E^i_{0\perp} e^{-jk^i x \sin\theta} \left(e^{-jk^i z \cos\theta} + \Gamma_{\perp} e^{jk^i z \cos\theta} \right) \hat{y} \tag{4.75a}$$

$$\begin{aligned} \vec{\boldsymbol{H}}^1_{\parallel} &= \frac{E^i_{0\perp}}{\eta_1} \cos\theta \left(e^{-j\vec{k}^i \cdot \vec{r}} - \Gamma_{\perp} e^{-j\vec{k}^r \cdot \vec{r}} \right) \hat{x} \\ &\quad - \frac{E^i_{0\perp}}{\eta_1} \sin\theta \left(e^{-j\vec{k}^i \cdot \vec{r}} + \Gamma_{\perp} e^{-j\vec{k}^r \cdot \vec{r}} \right) \hat{z} \end{aligned} \tag{4.75b}$$

As in the case of $\vec{\boldsymbol{E}}_{\parallel}$, the field behaves as a travelling wave towards x and as a travelling wave overlying a standing one towards z. The formulas (4.62) and (4.74) are known as Fresnel's formulas, which give the relations between the amplitudes and phase of the incident, reflected, and tranmitted waves.

Chapter 5

Electromagnetic wave-guiding *structures*: Waveguides and transmission lines

5.1 Introduction

There are many engineering applications in which it is necessary to use devices to confine the propagation of the electromagnetic waves in order to transmit electromagnetic energy from one point to another with a minimum of interference, radiation, and heat losses. Although such transmission systems can take many different forms, a common characteristic is that they are uniform. That is, their cross-sectional geometry and constitutive parameters do not change in the direction of the wave propagation z for wavelengths numerous enough to make border effects negligible. In general, any device used to transmit confined electromagnetic waves can be considered a waveguide; however, when the transmission device contains two or more separate conductors the term "transmission line" is generally used instead of "waveguide". Figure (5.1) shows the cross-sectional shape of some guiding transmission systems: two-wire transmission lines; coaxial transmission lines formed by two concentric conductors separated by a dielectric; two hollow (or dielectric-filled) metal tubes of rectangular and circular cross section (i.e. a rectangular and a circular waveguide); two planar transmission lines (the stripline and microstrip); and two dielectric (without conducting parts) waveguides: the circular dielectric waveguide (or homogeneous dielectric rod) and the optical fiber.

In hollow conducting pipes waves propagate within the tube, whereas in transmission lines formed by two or more conductors, the waves propagate in the dielectric medium between the conductors. *In homogeneous dielectric* *waveguides the field decays exponentially away from the dielectric in the transverse plane, and consequently the electromagnetic waves are confined mainly within the dielectric medium. Optical fibers, used mostly at optical wavelengths, consist of a cylindrical core surrounded by a cladding and are usually circular in cross section. The light is essentially confined to the core (which has a larger refractive index than the cladding) by total internal reflection as it propagates along the fiber and the wave is confined without need of any conducting walls.*

The choice of a specific transmission system depends on the application and should take into account aspects such as frequency range, losses, power-transmission capacity, and production costs. For example, the two-wire transmission lines, which are usually covered by polyethylene, are relatively inexpensive to manufacture, but radiation losses (mainly at discontinuities and bends) make them inefficient for transferring electromagnetic energy farther than the lower range of microwaves. Coaxial lines and hollow metal pipe waveguides are more efficient than two-wire lines for transferring electromagnetic energy because the fields are completely confined by the conductors. For the transmission of large amounts of power at high frequencies, waveguides are the most appropriate means. In a coaxial cable, significant wave attenuation occurs at high frequencies because of the large current densities carried by the central conductor, which has a relatively small surface area. On the other hand, waveguides are intrinsically dispersive and consequently incapable of transmitting large bandwidth signals without distortion. However, coaxial lines can guide signals of much higher bandwidths than waveguides can.

As shown in the next chapter, the dimension of the cross section of a waveguide is related to the wavelength of the guided wave. Thus, for very low-frequency waveguides the cross section would be too large and thus impractical for frequencies lower than 1 GHz. On the other hand, at optical frequencies the size of a metal waveguide must be too small (in the range of the μm) and, moreover, at these frequencies the study of the interaction of the electromagnetic field with the metal walls requires of quantum mechanical theory.

As a result of the development in solid-state microwave and millimeter technology, planar transmission lines are used instead of waveguides in many applications because these lines are inexpensive, compact, and simple to match solid-state devices using printed-circuit technology. Planar lines allow different configurations, usually including a dielectric substrate material with a ground plane and one or more conducting strips on the upper surface. The most commonly used of these are striplines and microstrips, which are briefly described in Section **??**.

The field configurations that can be supported for any guiding structure must satisfy Maxwell's equations and the corresponding boundary conditions. The different field distributions that satisfy this requirement are termed modes. Although the electromagnetic field distribution in ideal guiding transmission systems (composed of perfect conductors separated by a lossless dielectric).can be expressed as a superposition of plane waves, the study of the propagation is greatly simplified when we seek other kinds of solutions called transverse magnetic (TM) modes, transverse electric (TE) modes, or transverse electromagnetic

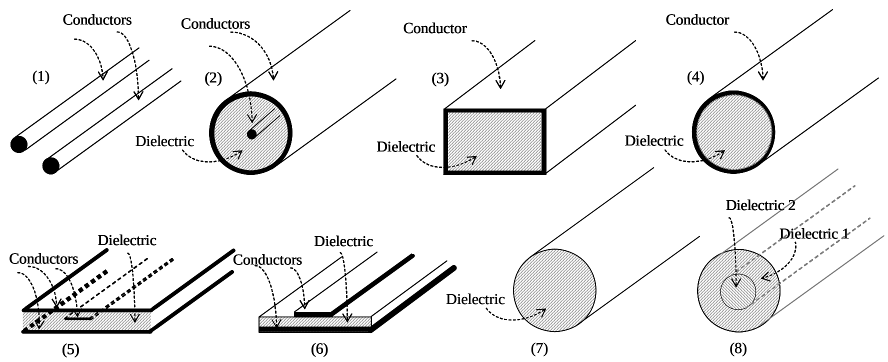


Figure 5.1: Examples of waveguides and transmission systems: (1) Two wire transmission line (2) Coaxial transmission line (3) Rectangular waveguide (4) Circular waveguide (5) Stripline (6) Microstrip (7) Circular dielectric waveguide (8) Optical fiver cable.

(TEM) modes. These terms indicate that, in the direction of propagation, the TM modes have no magnetic field component, the TE modes have no electric field component, and the TEM modes have neither electric nor magnetic field components. In practice, these modes form a complete set of orthogonal functions and, hence, any propagating electromagnetic field in the guiding structure can be expressed as a linear combination of these modes. As discussed below, there are two important properties that distinguish TEM from TE and TM modes:

1) TM and TE modes have a cutoff frequency below which they cannot propagate, which depends on the cross-sectional dimension of the guiding structure.

2) TEM modes cannot exist within a waveguide formed a single perfect conducting pipe while transmission lines can in general support TE, TM and TEM modes.

In this chapter, we present some general aspects of the propagation of time-harmonic electromagnetic waves in guiding systems formed by perfect conductors and only one homogeneous lossless dielectric in which the guided field propagates. Nevertheless, the results can serve as a basis for structures in which the cross-section contains more than one dielectric medium. The effect of lossy media is analyzed in the final section. The study of some specific geometries is left for the next chapter.

5.2 General relations between field components

Let us assume that a time-harmonic wave propagates along the z-axis, in the $+z$ direction, in a lossless guiding transmission system. Thus the dependence on z and time t is given by the factor $e^{j\left(\omega t-\beta_g z\right)}$and the fields are of the general form

$$\mathrm{Re}\left\{\begin{array}{c}\vec{E}_0 e^{j\left(\omega t-\beta_g z\right)}\\ \vec{H}_0 e^{j\left(\omega t-\beta_g z\right)}\end{array}\right\}=\mathrm{Re}\left\{\begin{array}{c}\vec{\boldsymbol{E}} e^{j\omega t}\\ \vec{\boldsymbol{H}} e^{j\omega t}\end{array}\right\} \tag{5.76}$$

where $\vec{\boldsymbol{E}}=\vec{E}_0 e^{-j\beta_g z}$and $\vec{\boldsymbol{H}}=\vec{H}_0 e^{-j\beta_g z}$ and β_g is the wavenumber of the guided wave. Because the geometry and constitutive parameters do not change along the z-axis, $\vec{E}_0$ and $\vec{H}_0$ are functions only of the transverse coordinates.

To determine $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$, we will first show that it is possible to express their transverse components, $\vec{\boldsymbol{E}}_t$ and $\vec{\boldsymbol{H}}_t$, in terms of their z-components, $\vec{\boldsymbol{E}}_z$ and $\vec{\boldsymbol{H}}_z$. For this, we divide the three dimensional Laplacian operator ∇^2 in the *homogeneous Helmholtz wave equations* (3.26) into two parts. One part, $\partial^2/\partial z^2$, acts only on the axial coordinate, z, and the other, ∇_t^2, on the transverse ones only[1], i.e.

$$\nabla^2=\frac{\partial^2}{\partial z^2}+\nabla_t^2 \tag{5.77}$$

Since $\partial/\partial z\equiv -j\beta_g$, the wave equations can be written as

$$\left(\nabla^2+k^2\right)\left\{\begin{matrix}\vec{\boldsymbol{E}}\\ \vec{\boldsymbol{H}}\end{matrix}\right\}=\left(\nabla_t^2+h^2\right)\left\{\begin{matrix}\vec{\boldsymbol{E}}\\ \vec{\boldsymbol{H}}\end{matrix}\right\}=0 \tag{5.78}$$

where

$$h^2=k^2-\beta_g^2 \tag{5.79}$$

and $k=\omega(\mu\varepsilon)^{\frac{1}{2}}$ is the wavenumber for the wave propagating in an unbounded medium of the matter which fills the transmission system. By particularizing (5.78) for the z field component, we have

$$\left(\nabla_t^2+h^2\right)\left\{\begin{matrix}\boldsymbol{E}_z\\ \boldsymbol{H}_z\end{matrix}\right\}=0 \tag{5.80}$$

This equation, when solved together with the boundary conditions of a given structure, has solutions for an infinite but discrete number, m, of characteristic values (eigenvalues) h_m, i.e.

$$\left(\nabla_t^2+h_m^2\right)\left\{\begin{matrix}\boldsymbol{E}_{zm}\\ \boldsymbol{H}_{zm}\end{matrix}\right\}=0 \tag{5.81}$$

where

$$h_m^2=k^2-\beta_{gm}^2 \tag{5.82}$$

with $\boldsymbol{E}_{zm}$, or $\boldsymbol{H}_{zm}$ being the corresponding functions characteristic (*eigenfunctions)* which satisfy the equations (5.81) and the corresponding boundary conditions, which are determined by the geometry of the system.

[1] For example in Cartesian coordinates we have $\nabla=\nabla_t+\hat{z}\frac{\partial}{\partial z}$ where $\nabla_t=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}$ so that $\nabla_t^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Now we are going to demonstrate that, once equation (5.80) has been solved, we can obtain $\vec{\boldsymbol{E}}_t$ or $\vec{\boldsymbol{H}}_t$ from $\boldsymbol{E}_z$ and $\boldsymbol{H}_z$. From Maxwell's equations (1.67c) and (1.67d), in a sourceless region, we have

$$\nabla \times \vec{\boldsymbol{E}} = -j\omega\mu\vec{\boldsymbol{H}} \tag{5.83a}$$

$$\nabla \times \vec{\boldsymbol{H}} = j\omega\varepsilon\vec{\boldsymbol{E}} \tag{5.83b}$$

The transverse components of these equations can be written as

$$(\nabla \times \vec{\boldsymbol{E}})_t = \nabla_t \times \vec{\boldsymbol{E}}_z + \nabla_z \times \vec{\boldsymbol{E}}_t = -j\omega\mu\vec{\boldsymbol{H}}_t \tag{5.84a}$$

$$(\nabla \times \vec{\boldsymbol{H}})_t = \nabla_t \times \vec{\boldsymbol{H}}_z + \nabla_z \times \vec{\boldsymbol{H}}_t = j\omega\varepsilon\vec{\boldsymbol{E}}_t \tag{5.84b}$$

Thus, as $\vec{\boldsymbol{E}}_z$ and $\vec{\boldsymbol{H}}_z$ are assumed to be known, we have a system of two equations and two unknowns, $\vec{\boldsymbol{E}}_t$ and $\vec{\boldsymbol{H}}_t$, the solutions to which are

$$\boxed{\vec{\boldsymbol{H}}_t = \frac{j}{h^2}\left(\omega\varepsilon\nabla_t \times \vec{\boldsymbol{E}}_z - \beta_g\nabla_t\boldsymbol{H}_z\right)} \tag{5.85a}$$

$$\boxed{\vec{\boldsymbol{E}}_t = -\frac{j}{h^2}\left(\omega\mu\nabla_t \times \vec{\boldsymbol{H}}_z + \beta_g\nabla_t\boldsymbol{E}_z\right)} \tag{5.85b}$$

According to (5.85), once the z components of the fields are known, the transverse components can also be calculated. Moreover, in ideal guiding ¿¿structures?? , we can express any field propagating in the homogeneous guiding transmission structure as a linear superposition of TE, TM and TEM waves or modes. Clearly, it is not possible to find specific expressions for the field distribution of any of these modes without previously knowing the geometry and characteristics of the transmission system. However, as shown below, we can study some of their general characteristics.

5.2.1 Transverse magnetic (TM) modes

Let us first consider TM modes so that $H_z = 0$ in (5.85). Thus we have

$$\boxed{\vec{\boldsymbol{E}}_t = -\frac{j\beta_g}{h^2}\nabla_t\boldsymbol{E}_z = \nabla_t\frac{1}{h^2}\frac{\partial\boldsymbol{E}_z}{\partial z} = \nabla_t\boldsymbol{\Phi}_{TM}} \tag{5.86a}$$

$$\boxed{\vec{\boldsymbol{H}}_t = \frac{j\omega\varepsilon}{h^2}\nabla_t \times \vec{\boldsymbol{E}}_z = -\frac{j\omega\varepsilon}{h^2}\hat{z} \times \nabla_t\boldsymbol{E}_z = \frac{\omega\varepsilon}{\beta_g}\hat{z} \times \vec{\boldsymbol{E}}_t = \frac{1}{Z_{TM}}\hat{z} \times \vec{\boldsymbol{E}}_t} \tag{5.86b}$$

where we have defined the scalar potential for the TM modes, $\boldsymbol{\Phi}_{TM}$, as

$$\boxed{\boldsymbol{\Phi}_{TM} = \frac{1}{h^2}\frac{\partial\boldsymbol{E}_z}{\partial z}} \tag{5.87}$$

To obtain (5.86b), we have used the equality $\nabla_t \times \vec{\boldsymbol{E}}_z = -\hat{z} \times \nabla_t\boldsymbol{E}_z$ and defined the frequency-dependent quantity Z_{TM}, as

$$\boxed{Z_{TM} = \frac{\beta_g}{\omega\varepsilon} = \frac{\eta\beta_g}{k}} \tag{5.88}$$

where $\eta = (\mu/\varepsilon)^{\frac{1}{2}}$ is the intrinsic impedance of the dielectric that fills the transmission system. The quantity Z_{TM}, which has the dimensions of impedance, is called the wave impedance for the TM modes. From (5.86b), we can see that $\vec{\boldsymbol{E}}_t$, $\vec{\boldsymbol{H}}_t$, and $\hat{z}$ form a right-handed system when the wave propagates in the z-positive direction.

Thus, from (5.86a), in TM modes, $\vec{\boldsymbol{E}}_t$ can be written as the gradient of a scalar function $\boldsymbol{\Phi}_{TM}$. This result could have been obtained by simple reasoning from Faraday's law (5.83a), taking into account that, since $\vec{\boldsymbol{H}}$ has only transverse components the same is true for $\nabla \times \vec{\boldsymbol{E}}$. Therefore, from Stokes' theorem, we have

$$\int_S (\nabla \times \vec{\boldsymbol{E}}) \cdot \hat{z} ds = \oint_\Gamma \vec{\boldsymbol{E}} \cdot d\vec{l} = 0 \tag{5.89}$$

where S is a transverse surface normal to the z axis. But E_z cannot contribute to the line integral because the integration path Γ lies on the transverse plane. Therefore

$$\oint_\Gamma \vec{\boldsymbol{E}} \cdot d\vec{l} = \oint_\Gamma \vec{\boldsymbol{E}}_t \cdot d\vec{l} = 0 \tag{5.90}$$

which implies that $\vec{E}_t$ is conservative and, therefore, can be written as the gradient of a scalar function $\boldsymbol{\Phi}_{TM}$.

5.2.2 Transverse electric (TE) modes

For TE modes, from equations (5.85), with $\boldsymbol{E}_z = 0$, we have

$$\boxed{\vec{\boldsymbol{H}}_t = -\frac{j\beta_g}{h^2}\nabla_t \boldsymbol{H}_z = \nabla_t \frac{1}{h^2}\frac{\partial \boldsymbol{H}_z}{\partial z} = \nabla_t \boldsymbol{\Phi}_{TE}} \tag{5.91a}$$

$$\boxed{\vec{\boldsymbol{E}}_t = -\frac{j\omega\mu}{h^2}\nabla_t \times \vec{\boldsymbol{H}}_z = \frac{j\omega\mu}{h^2}\hat{z} \times \nabla_t \boldsymbol{H}_z = -\frac{\omega\mu}{\beta_g}\hat{z} \times \vec{\boldsymbol{H}}_t = -Z_{TE}\hat{z} \times \vec{\boldsymbol{H}}_t} \tag{5.91b}$$

where

$$\boxed{\boldsymbol{\Phi}_{TE} = \frac{1}{h^2}\frac{\partial \boldsymbol{H}_z}{\partial z}} \tag{5.92}$$

is the scalar potential for TE waves and

$$\boxed{Z_{TE} = \frac{\omega\mu}{\beta_g} = \frac{\eta k}{\beta_g}} \tag{5.93}$$

is the wave impedance for the TE mode. From (5.91b) we can see that $\vec{\boldsymbol{E}}_t$, $\vec{\boldsymbol{H}}_t$, and $\hat{z}$ form a right-handed system when the wave propagates in the z-positive direction.

The fact that, according to (5.91a), $\vec{\boldsymbol{H}}_t$ can be expressed as the gradient of the scalar function Φ_{TE} can be explained by Ampere's law, (5.83b), following a reasoning similar to that used in the case of TE modes.

5.2.3 Transverse electromagnetic (TEM) modes

For TEM modes, since $\boldsymbol{E}_z = 0$ and $\boldsymbol{H}_z = 0$, substituting these values in (5.85), we can get no null or trivial solutions only if $h = 0$. Consequently, from (5.79), for TEM modes, we have

$$\beta_g^2 = k^2 = \omega^2 \mu \varepsilon \tag{5.94}$$

This means that a TEM mode in a transmission system has the same propagation constant as a uniform plane wave traveling in the unbounded dielectric between the conductors. Since $h = 0$ and $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_t$ and $\vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}_t$, (5.78) reduces to

$$\nabla_t^2 \vec{\boldsymbol{E}} = \nabla_t^2 \vec{\boldsymbol{E}}_t = 0 \tag{5.95a}$$

$$\nabla_t^2 \vec{\boldsymbol{H}} = \nabla_t^2 \vec{\boldsymbol{H}}_t = 0 \tag{5.95b}$$

Thus the distribution of the electric and magnetic fields on a transverse plane satisfies the same bidimensional Laplace's equation as for the static fields. This means that, for TEM modes, on a transverse plane, $\vec{E}$ is conservative, and derivable from a scalar function $\boldsymbol{\Phi}$ by means of the gradient function, i.e.

$$\vec{\boldsymbol{E}} = -\nabla \boldsymbol{\Phi} \tag{5.96}$$

Hence, the electric field distribution in the cross-sectional plane has the same spatial dependence as the electrostatic field created by static charges located on the conductors of the transmission system. Consequently, a TEM mode cannot exist within a waveguide formed by a single perfect conducting tube of any cross section since no electrostatic field can exist within a sourcesless region completely enclosed by a conductor. When two or more separated conductors exist, as for example in coaxial, two-wire or stripline transmission lines, TEM waves can be propagated along the dielectric separating the conductors.

It is straightforward from (5.84b) that

$$\vec{\boldsymbol{E}}_t = -Z_{TEM}\ \hat{z} \times \vec{\boldsymbol{H}}_t = -\eta \hat{z} \times \vec{\boldsymbol{H}}_t \tag{5.97}$$

where Z_{TEM}

$$Z_{TEM} = \eta = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \tag{5.98}$$

is the wave impedance for the TEM mode, which coincides with the characteristic impedance η of the dielectric that fills the transmission system.

cuidado en lo de lineas: usar o no negritas..?-

Now we will demonstrate that, for TEM modes, Maxwell's equations can be used to derive a pair of coupled differential equations which enable us to study the propagation of these modes in transmission lines as voltage and current waves (instead of electromagnetic waves), using elemental circuit theory.

From (5.96), according to the fundamental property of the gradient, in the transverse plane the line integral of the electric field is path independent and consequently voltage V and potential difference $\Phi_2 - \Phi_1$ will be the same.

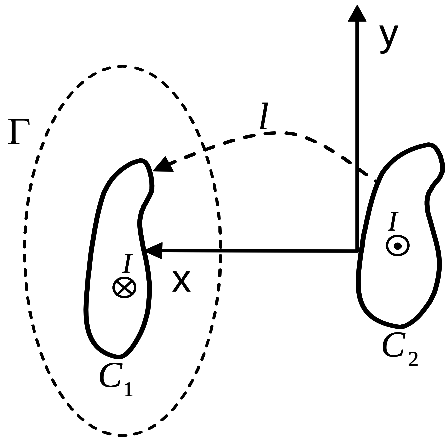


Figure 5.2: two conductor transmission linecambiar ejes¡ ¡ Comprobar si los ejes y el texto coincide. Los conductores se ponen en negro.entero no¿ decir que c1 son los conductores y que los sentidos de la corrinte in idreccion opuesta en cada conductor son indicadas. Poner origen coincidiendo con el conductor¡ ¡ver dibujos de Salva. en todo ccaso la flecha de l de acuerdo con libro de siempre es al revés

Then for TEM waves (using, without loss of generality, Cartesian coordinates) we have

$$V = \Phi(2) - \Phi(1) = -\int_l \vec{E}_t.d\vec{l} = -\int_l E_x dx + E_y dy \tag{5.99}$$

where $\Phi(2)$ *and* $\Phi(1)$ *are the values of the scalar function* Φ *at the the conductors* 1 *and* 2 *and where* l *is any line that joins the equipotential transverse sections of these conductors (fig.5.2).* Deriving with respect to z and taking into account Faraday's law, (1.1c), particularized for the source-free region, outside the conductors, we get

$$\frac{\partial V}{\partial z} = -\int_l \frac{\partial E_x}{\partial z} dx + \frac{\partial E_y}{\partial z} dy = -\frac{\partial}{\partial t}\int_l -B_y dx + B_x dy \tag{5.100}$$

Note that $\int_l -B_y dx + B_x dy$ is the magnetic flux through the area swept, along a unit of length in the direction z, by the line l joining the conducting surfaces. This flux can be expressed by using the magnetostatic definition of coefficient L of self-inductance per unit of length, as the product LI. Therefore we have

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \tag{5.101}$$

On the other hand, from Ampere's law, (1.1d), for the source-free dielectric region, we have

$$I = \oint_\Gamma \vec{H}_t.d\vec{l} = \oint_\Gamma H_x dx + H_y dy \tag{5.102}$$

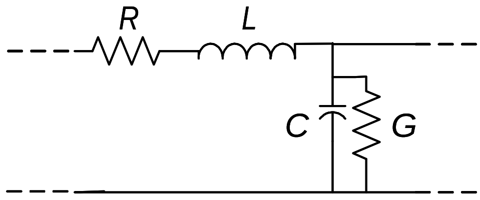


Figure 5.3:

where Γ is a closed path around one of the wires (see Fig. 5.2). Deriving with respect to z, we have

$$\frac{\partial I}{\partial z} = \oint \frac{\partial H_x}{\partial z} dx + \frac{\partial H_y}{\partial z} dy = -\frac{\partial}{\partial t} \oint_\Gamma D_x dy - D_y dx \tag{5.103}$$

where, in a similar way as above, $-D_y dy + D_x dx$ represents the flow of vector $\vec{D}$ per unit length in the direction z. Using the magnetostatic definition of capacitance C per unit length, this flux can be expressed as the product CV . Thus we have

$$\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t} \tag{5.104}$$

Note that, from (5.99) and (5.102), V and I must have the same z dependence as $\vec{E}$ and $\vec{H}$, respectively. Thus, V and I are also traveling waves.

In summary, according (5.101) and (5.104) we have

$$\boxed{\frac{\partial \Phi}{\partial z} = -L \frac{\partial I}{\partial t}} \tag{5.105a}$$

$$\boxed{\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}} \tag{5.105b}$$

which are the coupled differential equations that voltage and current satisfy at any z cross section of an ideal line composed of perfect conductors separated by a lossless dielectric. Equations (5.105) are called ideal "transmission line equations". The use of these equations to study the propagation of TEM waves in transmission lines is considered in Chapter **??**.

5.2.4 Boundary conditions for TE and TM modes on perfectly conducting walls

For a guiding transmission system with perfectly conducting walls, the general boundary conditions on the walls require that the tangential component $\vec{\boldsymbol{E}}_T$ of $\vec{\boldsymbol{E}}$ and the normal component $\boldsymbol{H}_n$ of $\vec{\boldsymbol{H}}$ be null, i.e.

$$\vec{\boldsymbol{E}}_T = \hat{n} \times \vec{\boldsymbol{E}} = 0 \tag{5.106a}$$

$$\boldsymbol{H}_n = \hat{n} \cdot \vec{\boldsymbol{H}} = 0 \tag{5.106b}$$

where $\hat{n}$ is the unit vector normal to the conducting walls. However, for TM and TE modes, as shown below, these conditions can be simplified and reduced to equivalent ones which are expressed only in terms of the z component of the fields. For example for TM modes, the requirement that

$$\boldsymbol{E}_z = 0 \tag{5.107}$$

on the perfectly conducting guide walls suffices to ensure that Eqs. (5.106) are fulfilled. From (5.86a) and the gradient properties, we can see that $\vec{\boldsymbol{E}}_t$ is normal to the lines where $\boldsymbol{E}_z = cte$ and, therefore, to the boundary of the conductor, since it represents a line with $\boldsymbol{E}_z = 0$. Given that $\vec{\boldsymbol{E}}_t$ and $\vec{\boldsymbol{H}}_t$ are perpendicular to each other, the magnetic field is tangential to the conductor and thus $\boldsymbol{E}_z = 0$ is equivalent to Eqs. (5.106).

For TE modes, the necessary and sufficient condition to ensure that Eqs. (5.106) are fulfilled is that the normal derivative of H_z be null on the perfect conducting parts of the guiding structure. That is

$$\frac{\partial \boldsymbol{H}_z}{\partial n} = \nabla \boldsymbol{H}_z \cdot \hat{n} = (\nabla_t + \nabla_z)\, \boldsymbol{H}_z \cdot \hat{n} = 0 \tag{5.108}$$

where we have divided ∇ into its transverse and axial components. Taking into account (5.91a), we see that

$$\nabla_t \boldsymbol{H}_z \cdot \hat{n} = \vec{\boldsymbol{H}}_t \cdot \hat{n} = 0 \tag{5.109a}$$

which means that $\vec{\boldsymbol{H}}_t$ is tangencial to the conductor and therefore, due to the perpendicularity of the fields, we have

$$\hat{n} \times \vec{\boldsymbol{E}}_t = 0 \tag{5.110}$$

In summary, the necessary and sufficient boundary conditions on the perfect conducting walls of the propagation system are

Boundary conditions on the perfect conducting walls

For TM modes
$$\boldsymbol{E}_z = 0 \tag{5.111a}$$

For TE modes
$$\frac{\partial \boldsymbol{H}_z}{\partial n} = 0 \tag{5.111b}$$

5.3 Cutoff frequency

From (5.76) and (5.79) we see that, for propagation to exist, β_g must be real, and consequently,

$$k^2 > h^2 \tag{5.112}$$

For this reason, β_c, defined as

$$\beta_c = h = \frac{2\pi}{\lambda_c} \tag{5.113}$$

is called the cutoff wavenumber, and λ_c is the cutoff wavelength. Thus, from Eq. (5.79), we have

$$\beta_g^2 = k^2 - \beta_c^2 \tag{5.114}$$

and, consequently

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} \tag{5.115}$$

where λ is the wavelength of a plane wave in the unbounded lossless dielectric medium filling the (¡ ¡*better guiding structure*¡¡ ¡) waveguide, and λ_g is that of the wave in the guide. Thus we have

$$k = \frac{2\pi}{\lambda}; \quad \beta_c = \frac{2\pi}{\lambda_c}; \quad \beta_g = \frac{2\pi}{\lambda_g} \tag{5.116}$$

The cutoff frequency f_c is defined[2] as

$$f_c = \frac{\omega_c}{2\pi} = \frac{\beta_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{v_p \beta_c}{2\pi} \tag{5.117}$$

where $v_p = \omega/k$ is the phase velocity in the unbounded medium filling the (¡¡*better guiding structure*¡¡¡) waveguide. Thus, from (5.79), the wavenumber β_g can be expressed in terms of f_c, as

$$\beta_g = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{5.118}$$

and the corresponding wavelength λ_g in the (¡¡*better guiding structure*¡ ¡¡) guide is

$$\lambda_g = \frac{2\pi}{\beta_g} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \tag{5.119}$$

which is greater than λ. According to (5.118) the wavenumber is imaginary for modes with frequencies below the cutoff frequency f_c, i.e. $f < f_c$ (or $\lambda > \lambda_c$). These modes, called evanescent modes, are attenuated and cannot propagate along the guide. Thus, (¡¡better guiding structure¡¡¡) waveguides behave as high-pass filters for the TE and TM modes since they cannot transmit any of these modes for which the wavelengths, in the unbounded medium filling the (¡¡better guiding structure¡¡¡) waveguide, exceed the value of the cutoff wavelength.

---

[2] For a guiding transmission system with more than one dielectric the cutoff frequency can be defined in a different manner than (5.117). See for example Section ??.

In terms of the cutoff frequency, the expressions of the wave impedances for the TM and TE modes (5.88) and (5.93) for Z_{TM} and Z_{TE} become, respectively

$$Z_{TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} \tag{5.120a}$$

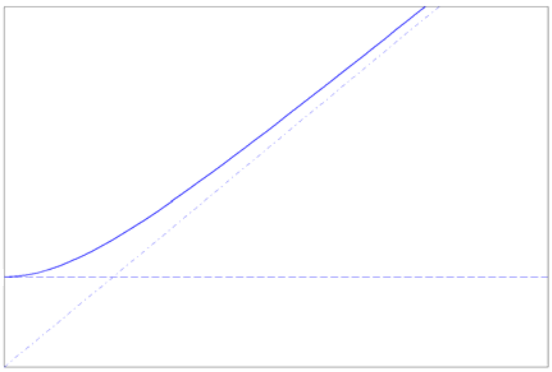
$$Z_{TE} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \tag{5.120b}$$

From (5.120a) and (5.120b), we can see that $Z_{TM} < \eta$ and $Z_{TE} > \eta$ and they become imaginary below the cutoff frequency. Thus, for $f < f_c$, the waveguide behaves, in this respect, as a reactive impedance.

From (5.79), we obtain the dispersion relation

$$\omega = \left(\omega_c^2 + v_p^2\beta_g^2\right)^{1/2} \tag{5.121}$$

which is analogous to that obtained in (**??**) for the transverse electromagnetic waves in a nonmagnetized plasma. The plot of the phase constant as a function of the frequency ω (dispersion diagram) is shown in Figure 5.2.3. The transversal broken line corresponds to $\omega_c = 0$, i.e. to an unbounded lossless, nondispersive medium in which the wave propagates at the phase velocity v_p regardless of its frequency. The solid-line curve represents Eq. (5.121) and shows that the waveguide is very dispersive close to the cutoff frequency ω_c. For frequencies $\omega >> \omega_c$ such that their wavelengths are much smaller than the transversal (¡¡*better guiding structure*¡ ¡¡) waveguide dimensions, the walls do not affect the propagation and the velocity tends to v_p.



JV: texto Dispersion diagram. Es la figura de plasmas. ver tb pp 444 del Jonk

The group velocity, v_{gg}, within the guide is given by

$$v_{gg} = \frac{d\omega}{d\beta_g} = v_p\sqrt{1-\left(\frac{f_c}{f}\right)^2} \tag{5.122}$$

which is smaller than the phase velocity v_p in the unbounded medium. The phase velocity within the waveguide(¡ ¡better guiding structure¡¡ ¡), v_{pg}, is given by

$$v_{pg} = \frac{\omega}{\beta_g} = f\lambda_g = \frac{v_p}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \tag{5.123}$$

which is always higher than that in the unbounded medium and is frequency dependent. Hence single conductor (¡¡better guiding structure¡¡¡) waveguides are dispersive transmission systems. Note that

$$v_{pg} \cdot v_{gg} = v_p^2 \tag{5.124}$$

For TEM modes, from (5.94), we have $\beta_g = k$ which is real and independent of the frequency. Thus, all frequencies propagate along a lossless transmission line at the same phase velocity v_p as that of the unbounded homogeneous dielectric filling the waveguide *and there is no cutoff frequency.*

5.4 Attenuation in guiding structures

For a propagating mode an attenuation constant α, owing to energy dissipation within the waveguide, can arise from losses in the non-perfect conducting walls (α_c) and in the non-perfect dielectric filling the waveguide (α_d). Thus, the attenuation constant α consists of two parts $\alpha = \alpha_d + \alpha_c$. Dielectric losses are generally negligible when (¡¡better guiding structure¡ ¡¡) waveguides are filled with air, which has a lower dielectric loss than do conventional dielectrics.

hay que decir TE y TM y TEM..

First, we analyze the losses for *TE and TM* due to a non-perfect dielectric and afterward the ones due to non-perfect walls. In any case, as generally occurs in practice, these losses are assumed to be very small.

5.4.1 TE and TM modes.

Dielectric Losses

When the dielectric filling the waveguide is lossy the attenuation can be easily taken into account if in the expressions obtained for ideal dielectrics the real propagation constants k and β_g are replaced by $-j\gamma$ and $-j\gamma_g$, respectively, where $\gamma = \alpha + j\beta$ and $\gamma_g = \alpha_d + j\beta_g$ are the complex propagation constants in the unbounded dielectric filling the waveguide and in the waveguide, respectively. Then, from equations (3.17), (5.79) and (5.113), we have

$$k^2(1 - j\tan\delta_d) = -\gamma^2 = -\gamma_g^2 + \beta_c^2 \tag{5.125}$$

Using the above expressions for γ and γ_g and neglecting the term α_d^2, because the attenuation constant α_d is very small, we find

$$\beta_c^2 = k^2 - \beta_g^2 \tag{5.126a}$$

$$\alpha_d = \frac{k^2}{2\beta_g}\tan\delta_d = \frac{{\beta_g}^2 + \beta_c^2}{2\beta_g}\tan\delta_d \tag{5.126b}$$

Thus, the attenuation factor is proportional to the loss tangent, $\tan\delta_d$, of the dielectric filling the waveguide. On the other hand, (5.126a) coincides with Eq. (5.114) for waveguides with ideal dielectric, and consequently the phase constant (and thus the wavelength) remains practically the same as those for a lossless waveguide. The dependence of the attenuation factor α_d on the frequency (assuming a range of frequencies in which the permittivity of the dielectric remain unchanged) can be deduced by substituting the expressions of $\tan\delta_d$ and β_g, given by (1.91) and (5.118), respectively, in (5.126b). Thus we get

$$\boxed{\alpha_d = \frac{\sigma_e\eta}{2\left(1-(f_c/f)^2\right)^{1/2}}} \tag{5.127}$$

where η is the intrinsic impedance of the dielectric given in (3.33a) and σ_e is its effective or equivalent conductivity (1.78). From (5.127) we can see that α_d becomes very high at frequencies close to the cutoff value, then decreases to a minimum value, and afterwards increases with the frequency, becoming almost proportional to it.

Wall losses

When the conductivity is finite the tangential magnetic field induces currents which are not restricted to the surface and, according to Ohm's law, are associated with a tangential electric field (i.e. $\vec{J} = \sigma\vec{E} = \hat{n}\times\vec{H}$) in the walls. The vector product of the fields $\vec{E}$ and $\vec{H}$ at the surface of the walls represents a flux of power directed towards the inner of the wall. This power coincides with the dissipation in the conductor caused by the Joule effect and is subtracted from the mode that propagates along the waveguide. As a consequence, the amplitude of the electric and magnetic fields of the mode are attenuated according to $e^{-\alpha_c z}$, where α_c is the attenuation constant due to wall losses. We can determine the value of α_c for a given propagating mode by taking into account that the time-average power P_{av} transmitted through the cross-section S of the guiding transmission system is

$$P_{av} = \int_S \vec{\mathcal{P}}_{av}\cdot d\vec{s} = \frac{1}{2}\int_S \mathrm{Re}(\vec{\boldsymbol{E}}_t \times \vec{\boldsymbol{H}}_t^*)\cdot d\vec{s} \tag{5.128}$$

Since, due to the losses, the amplitude of the field wave varies according to $e^{-\alpha_c z}$, then, P_{av} will vary according to $e^{-2\alpha_c z}$. Moreover, the law of energy conservation requires that the rate of the decrease of P_{av} with distance along the transmission system equals the time-average power loss on the surface of the walls per unit length, P_d', in the direction of propagation. Therefore, we have

$$P'_d = -\frac{dP_{av}}{dz} = 2\alpha_c P_{av} \tag{5.129}$$

and thus

$$\alpha_c = \frac{P'_d}{2P_{av}} \tag{5.130}$$

If $\vec{\boldsymbol{H}}$ is the *¿¿magnetic??* field existing near the walls, the time-average power dissipated per unit of length in the walls is given, according to (3.45), by

$$P'_d = \frac{1}{2} R_s \int_\Gamma H_0^2 dl = \frac{1}{2} R_s \int_\Gamma \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^* dl \tag{5.131}$$

where R_s is the surface resistance given by (3.46) and Γ is the cross-sectional contour of the non-perfect conducting walls. Thus the coefficient of attenuation of the n-th TE or TM mode is found to be

$$\boxed{\alpha_c = \frac{R_s \int_\Gamma \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^* dl}{4 \int_S \vec{\mathcal{P}}_{av} \cdot d\vec{s}} = \frac{R_s \int_\Gamma \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^* dl}{2 \int_S \mathrm{Re}(\vec{\boldsymbol{E}}_t \times \vec{\boldsymbol{H}}_t^*) \cdot d\vec{s}}} \tag{5.132}$$

This equation will be applied in next chapter to the calculation losses for TE and TM modes in waveguides. In strict terms, the modes we have found assuming perfect conducting walls are no longer valid since non-perfect conducting walls represent a change in the boundary conditions because in this case the tangential component of the electric field is not null. However, if the losses are small, we can make an approximate analysis (known in Mathematical Physics as "first order perturbation method") by assuming that the field configurations or modes in the waveguide coincide with those found for ideal-wall (*¡¡better guiding structure¡ ¡¡*) waveguides.

5.4.2 TEM modes

The coupled differential equations (5.105) for ideal transmission lines can be easily extended to lines with a non perfect of dielectric (constitutive parameters ε, μ, σ) separating the perfect conductors. In *this case, at any* z cross-section *of the line, an additional current increment* ΔI *leaves ..*

assuming that the dielectric has a conductivity σ *such that*

$$\Delta I = gV \tag{5.133}$$

in which g *deno tes the conductance .....*

pp 487 del Jonk: the series and shunt low-parameters r *and* g*...*

$$\frac{\partial \Phi}{\partial z} = -L\frac{\partial I}{\partial t} \tag{5.134a}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} - gV \tag{5.134b}$$

Chapter 6

Some types of waveguides and transmission lines

6.1 Introduction

In the previous chapter we examined some general properties of the propagation modes that may exist in an ideal guiding transmission system which has no sources and is constituted by perfect conductors and one ideal homogeneous dielectric. Specific expressions for such modes can be determined only when the particular geometry of the guide is given. In this chapter we will first analyze in some detail the homogeneously filled rectangular and circular metallic waveguides. After this, as a simple example of non homogeneous guiding structure in which the electromagnetic field propagates in more than one dielectric, we will study the dielectric slab waveguide. Then, we will give some basic ideas on propagation in strip and microstrip lines. Finally, we will consider cavity resonators which are basically constituted by a dielectric region totally enclosed by conducting walls. This region, when excited by an electromagnetic field, presents resonance with a very high-quality factor Q. In particular, we will study the common simple cases of rectangular and circular cavity resonators.

6.2 Rectangular waveguide

Figure 6.1 shows a rectangular waveguide of sides a and b, with $a > b$, and homogeneously filled with a perfect dielectric. Following the theory developed in the previous chapter, in order to calculate the TE and TM modes that can propagate in this waveguide, we start by solving the wave equation for the longitudinal components z of the field with the corresponding boundary conditions determined by the geometry of the system. The transverse components are then calculated from these longitudinal ones.

With axis chosen as shown in the figure, the expressions for the fields in (5.76), take the form

$$\vec{\boldsymbol{E}} = \vec{E}_0(x,y)e^{-j\beta_g z} \quad (6.135a)$$

$$\vec{\boldsymbol{H}} = \vec{H}_0(x,y)e^{-j\beta_g z} \quad (6.135b)$$

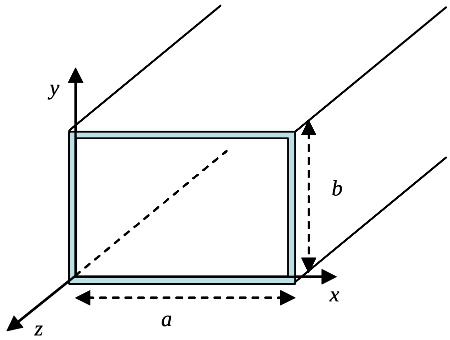


Figure 6.1: Rectangular waveguide of width a and height b Rellenar en negro.

Next, we are going to find the expression for these fields, first for TM modes and afterwards for the TE modes.

6.2.1 TM modes in rectangular waveguides

For the TM modes, the differential equation (5.80) for

$$\boldsymbol{E}_z = E_{0z}(x, y)e^{-j\beta_g z} \tag{6.136}$$

can be solved by using the standard method of separation of variables in rectangular Cartesian coordinates. For this, we assume, for E_{0z}, solutions in the form of the product

$$E_{0z}(x, y) = X(x)Y(y) \tag{6.137}$$

in which $X(x)$ and $Y(y)$ are, respectively, functions only of x and y.

By substituting (6.137) in (5.80) and dividing by E_{0z}, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + h^2 = 0 \tag{6.138}$$

As each summand depends on a different variable, it should be verified that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -h_x^2 \tag{6.139a}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -h_y^2 \tag{6.139b}$$

$$h^2 = \beta_c^2 = h_x^2 + h_y^2 \tag{6.139c}$$

where we have substituted, according to (5.113), h by the cutoff wavenumber β_c and where h_x and h_y are the separation constants to be determined from the boundary condition (5.111a) at the guide walls. This boundary condition for the geometry of Figure 6.1 implies

$$E_{0z} = 0 \quad \text{at} \begin{cases} x = \begin{cases} 0 \\ a \end{cases} \\ y = \begin{cases} 0 \\ b \end{cases} \end{cases} \tag{6.140}$$

The solution of the Eqs. (6.139a) and (6.139b) are, respectively,

$$X = C_1 \sin h_x x + C_2 \cos h_x x \tag{6.141a}$$

$$Y = C_3 \sin h_y y + C_4 \cos h_y y \tag{6.141b}$$

where the C_i coefficients are arbitrary constants to be determined from boundary conditions. Therefore the general solution (6.137) for E_{0z} takes the form

$$E_{0z} = (C_1 \sin h_x x + C_2 \cos h_x x)(C_3 \sin h_y y + C_4 \cos h_y y) \tag{6.142}$$

From the boundary conditions (6.140), we find that $C_2 = C_4 = 0$ and

$$h_x = \frac{\pi m}{a} \tag{6.143a}$$

$$h_y = \frac{\pi n}{b} \tag{6.143b}$$

and thus, from (6.139c),

$$\boxed{\beta_c^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2} \tag{6.144}$$

where m and n are integers. The different solutions achieved by giving values to m and n are termed TM_{mn} modes and each set of values of m and n indicates a specific mode. Thus, from (6.136), (6.142) and (6.143), for TM_{mn} modes, we have

$$\boldsymbol{E}_z = A_{mn} e^{-j\beta_g z} \sin \frac{\pi m}{a} x \sin \frac{\pi n}{b} y \tag{6.145}$$

where the product of the constants C_1 and C_3 has been replaced by a new constant A_{mn}.

Once we know the longitudinal component $\boldsymbol{E}_z$, we can calculate the transverse components $\vec{\boldsymbol{E}}_t$ by means of (5.86a) and then, by using (5.86b), which implies that

$$Z_{TM} = \frac{\boldsymbol{E}_x}{\boldsymbol{H}_y} = -\frac{\boldsymbol{E}_y}{\boldsymbol{H}_x} \tag{6.146}$$

we can obtain $\vec{\boldsymbol{H}}_t$. As a result, we get the following general expressions for the components of the TM modes in a rectangular waveguide

TM_{mn} modes in rectangular waveguides

$$
\begin{aligned}
&(\boldsymbol{E}_z)_{\mathrm{TM}_{mn}} = A_{mn} e^{-j\beta_g z} \sin\tfrac{\pi m}{a}x \sin\tfrac{\pi n}{b}y \\
&(\vec{\boldsymbol{E}}_t)_{\mathrm{TM}_{mn}} = \left(-jA_{mn}\tfrac{\beta_g}{\beta_c^2}\tfrac{\pi m}{a} e^{-j\beta_g z}\cos\tfrac{\pi m}{a}x \sin\tfrac{\pi n}{b}y\right)\hat{x} - \\
&\qquad \left(jA_{mn}\tfrac{\beta_g}{\beta_c^2}\tfrac{\pi n}{b} e^{-j\beta_g z}\sin\tfrac{\pi m}{a}x \cos\tfrac{\pi n}{b}y\right)\hat{y} \\
&(\vec{\boldsymbol{H}}_t)_{\mathrm{TM}_{mn}} = \left(jA_{mn}\eta^{-1}\tfrac{k}{\beta_c^2}\tfrac{\pi n}{b} e^{-j\beta_g z}\sin\tfrac{\pi m}{a}x \cos\tfrac{\pi n}{b}y\right)\hat{x} - \\
&\qquad \left(jA_{mn}\eta^{-1}\tfrac{k}{\beta_c^2}\tfrac{\pi m}{a} e^{-j\beta_g z}\cos\tfrac{\pi m}{a}x \sin\tfrac{\pi n}{b}y\right)\hat{y}
\end{aligned}
\tag{6.147}
$$

6.2.2 TE modes in rectangular waveguides

To analyze the TE modes, we can follow a procedure similar to that used for the TM modes but now solving for $\boldsymbol{H}_z$ and imposing the boundary condition (5.111b), $\partial \boldsymbol{H}_z/\partial n = 0$, on the guide walls. This, for the geometry of Figure 6.1, implies that

$$
\begin{aligned}
\frac{\partial \boldsymbol{H}_z}{\partial x} &= 0 \text{ at} \begin{cases} x = 0 \\ x = a \end{cases} \\
\frac{\partial \boldsymbol{H}_z}{\partial y} &= 0 \text{ at} \begin{cases} y = 0 \\ y = b \end{cases}
\end{aligned}
\tag{6.148}
$$

Then, using (5.91a) and (5.91b), and after steps analogous to those followed for TM modes, we get

TE_{mn} modes in rectangular waveguides

$$
\begin{aligned}
&(\boldsymbol{H}_z)_{\mathrm{TE}_{mn}} = B_{mn} e^{-j\beta_g z} \cos\tfrac{\pi m}{a}x \cos\tfrac{\pi n}{b}y \\
&(\vec{\boldsymbol{H}}_t)_{\mathrm{TE}_{mn}} = \left(jB_{mn}\tfrac{\beta_g}{\beta_c^2}\tfrac{\pi m}{a} e^{-j\beta_g z}\sin\tfrac{\pi m}{a}x \cos\tfrac{\pi n}{b}y\right)\hat{x} + \\
&\left(jB_{mn}\tfrac{\beta_g}{\beta_c^2}\tfrac{\pi n}{b} e^{-j\beta_g z}\cos\tfrac{\pi m}{a}x \sin\tfrac{\pi n}{b}y\right)\hat{y} \\
&(\vec{\boldsymbol{E}}_t)_{\mathrm{TE}_{mn}} = \left(jB_{mn}\eta\tfrac{k}{\beta_c^2}\tfrac{\pi n}{b} e^{-j\beta_g z}\cos\tfrac{\pi m}{a}x \sin\tfrac{\pi n}{b}y\right)\hat{x} - \\
&\left(jB_{mn}\eta\tfrac{k}{\beta_c^2}\tfrac{\pi m}{a} e^{-j\beta_g z}\sin\tfrac{\pi m}{a}x \cos\tfrac{\pi n}{b}y\right)\hat{y}
\end{aligned}
\tag{6.149}
$$

dedidir si subindexes or subindices.

Note that, for both, TM_{mn} and TE_{mn} modes, the subindexes m and n, indicate the number of half-wave variations of the field in the x and y directions, respectively. For a TM_{mn} mode with m or n equal to zero, from (6.145), we have $(\boldsymbol{E}_z)_{\mathrm{TE}_{00}} = 0$ and consequently, from Eqs (6.147), $(\vec{\boldsymbol{E}}_t)_{\mathrm{TE}_{00}} = 0$ and $(\vec{\boldsymbol{H}}_t)_{\mathrm{TE}_{00}} = 0$. Hence, there is no TM mode in which m or n is equal to zero. This was to be expected because a TM wave with $E_z = 0$ would degenerate to become a TEM wave which, as we saw in Subsection 5.2.3, cannot propagate within a waveguide.

For TE_{mn} modes, it is easy to see from (6.149) that either m or n may be equal to zero but not both at the same time, since in this case the expression of $(H_z)_{\text{TE}_{mn}}$ in (6.149) reduces to

$$(\boldsymbol{H}_z)_{\text{TE}_{00}} = B_{00} e^{-j\beta_g z} \tag{6.150}$$

while $(\vec{H}_t)_{\text{TE}_{00}} = 0$ and $\vec{E} = (\vec{E}_t)_{\text{TE}_{00}} = 0$, such that only $(H_z)_{\text{TE}_{00}}$ exists. This field does not fulfil Maxwell's equations, since a time-varying field $\vec{H}$ should generate an electric field $\vec{E}$. Therefore the TE_{00} mode cannot exist.

Cutoff frequencies in a rectangular waveguide

From (5.117), (6.139c), and (6.143), we see that the cutoff frequency for either a TE_{mn} or a TM_{mn} mode is given by

$$(f_c)_{mn} = \frac{v_p}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{\frac{1}{2}} \tag{6.151}$$

where v_p is the phase propagation velocity of the wave in the unbounded medium filling the waveguide. The wavelength and wavenumber in the waveguide are given, respectively, by

$$(\lambda_c)_{mn} = \frac{2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{\frac{1}{2}}} \tag{6.152a}$$

$$(\beta_g)_{mn} = \left[k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}} \tag{6.152b}$$

From (6.151) we see that the cutoff frequency of the modes depends on the dimensions of the cross-section of the waveguide. Values of the cutoff wavelengths and frequencies for several modes are

$$(\lambda_c)_{\text{TE}_{10}} = 2a;\; (f_c)_{\text{TE}_{10}} = \frac{v_p}{2a} \tag{6.153a}$$

$$(\lambda_c)_{\text{TE}_{01}} = 2b;\; (f_c)_{\text{TE}_{01}} = \frac{v_p}{2b} \tag{6.153b}$$

$$(\lambda_c)_{\text{TE}_{20}} = a;\; (f_c)_{\text{TE}_{20}} = \frac{v_p}{a} \tag{6.153c}$$

$$(\lambda_c)_{\text{TE}_{11}} = (\lambda_c)_{\text{TM}_{11}} = \frac{2ab}{(a^2+b^2)^{\frac{1}{2}}};\quad (f_c)_{\text{TE}_{11}} = (f_c)_{\text{TM}_{11}} = \frac{v_p\left(a^2+b^2\right)^{\frac{1}{2}}}{2ab} \tag{6.153d}$$

Note that if $a = b$ the cutoff frequencies of TE_{10} and TE_{01} and the two modes are equal except for a rotation of $\pi/2$.

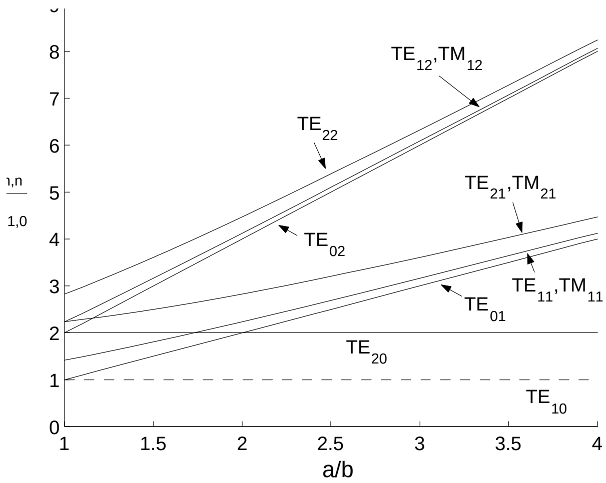


Figure 6.2: Rectangular waveguide: ratio of the cutoff frequency of several modes to that of the TE_{10} mode as a function of a/b. mas grande las letras de las coordenadas

The dominant TE_{10} mode

In practice, we usually wish to have only the mode which has the lowest cutoff frequency *(¿¿called fundamental or dominant mode??)* propagating through the guide. Thus, in the case of a rectangular waveguide, if $a > b$, such that $(f_c)_{TE_{10}} < (f_c)_{TE_{01}}$, the waveguide is usually designed so that only the TE_{10} mode can be propagated. The cutoff frequency of the dominant TE_{10} mode is selected by means of the dimension a. The ratio of the cutoff frequency of each mode to that of the TE_{10} mode as a function of a/b is plotted in Figure 6.2. We see that the separation of the cutoff frequencies for different modes is larger for higher values of the ratio of the a and b dimensions Note that if $a \simeq 2b$, then the cutoff frequencies of the modes TE_{01} and TE_{20} are nearly the same and in the frequency range $v/2a < f < v/2b$ only the TE_{10} mode can be propagated. Moreover, if $a > 2b$, then $(f_c)_{TE_{20}} < (f_c)_{TE_{01}}$. As we will see in the next Section, losses due to non-perfectly conducting walls increase as b decreases. Thus, to have the greatest frequency range in which only the TE_{10} mode can propagate and, at the same time, to have the smallest losses possible, we usually choose the dimensions of the guide such that $a \simeq 2b$. Under this condition, only TE_{10} modes will propagate in the frequency range $(f_c)_{TE_{10}} < f < 2(f_c)_{TE_{10}}$. For the dominant TE_{10} mode the general expressions (6.149) simplify to those given in (6.154) where the constant B_{10} has been replaced by H_0.

Rectangular TE_{10} mode

$$\boxed{\begin{array}{c} \beta_c = \frac{\pi}{a} \\ f_c = \frac{v_p}{2a} \\ \beta_g = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} \\ \boldsymbol{H}_z = H_0 e^{-j\beta_g z} \cos \frac{\pi}{a} x \\ \boldsymbol{H}_x = jH_0 \frac{\beta_g a}{\pi} e^{-j\beta_g z} \sin \frac{\pi}{a} x \\ \boldsymbol{H}_y = 0 \\ \boldsymbol{E}_z = 0 \\ \boldsymbol{E}_x = 0 \\ \boldsymbol{E}_y = -jH_0 \frac{\omega\mu a}{\pi} e^{-j\beta_g z} \sin \frac{\pi}{a} x \end{array}} \tag{6.154}$$

6.2.3 Attenuation in rectangular waveguides

Losses due to a non-perfect dielectric filling the waveguide and to non-perfect conducting walls can be calculated using the expressions (5.127) and (5.132), respectively. For a given mode, to obtain the attenuation due to dielectric losses, we simply need to use, in the formula (5.127), the value of the cutoff frequency of the mode, given by (6.151), and the values of the constitutive parameters of the dielectric at the work frequency. However, to find the the attenuation constant α_c due to wall losses for any TE or TM mode, though not complicated, is quite laborious. Here, to illustrate the procedure, we will consider the particular case of the dominant TE_{10} mode

Attenuation of the TE_{10} mode For the TE_{10} mode, the integrals of the formula (5.132) can be calculated from the general expressions for the field components (6.154). Thus, for the denominator, we have

$$\begin{aligned} P_{\mathrm{TE}_{10}} &= \int_S (\vec{\mathcal{P}}_{av})_{\mathrm{TE}_{10}} \cdot d\vec{s} = -\frac{1}{2} \int_0^b \int_0^a (\boldsymbol{E}_y \boldsymbol{H}_x^*)_{\mathrm{TE}_{10}}\, dxdy = \\ & \left(\frac{aH_0}{2\pi}\right)^2 \omega\mu ab\beta_g \end{aligned} \tag{6.155}$$

Regarding the integral of the numerator in (5.132), because in the dominant mode TE_{10} in a rectangular waveguide the magnetic field $\vec{H}$ has only H_x and

H_z components, this integral takes the form

$$\int_\Gamma \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^* dl = 2\left\{\int_0^a \left(|\boldsymbol{H}_x|^2 + |\boldsymbol{H}_z|^2\right) dx + \int_0^b \left(|\boldsymbol{H}_x|^2 + |\boldsymbol{H}_z|^2\right) dy\right\} \tag{6.156}$$

Using the expressions of $\boldsymbol{H}_x$ and $\boldsymbol{H}_z$ given in (6.154) and by operating, we obtain

$$\int_\Gamma \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{H}}^* dl = 2H_0^2 \left[\frac{a}{2}\left(1 + \frac{\beta_g^2}{\beta_c^2}\right) + b\right] = 2H_0^2 \left[\frac{a}{2}\left(\frac{f}{f_c}\right)^2 + b\right] \tag{6.157}$$

where the last expression is obtained from *(5.118).* By substituting (6.155) and (6.157) in (5.132) and after operating, we finally obtain the following expression for the attenuation factor $(\alpha_c)_{\mathrm{TE}_{10}}$

$$\boxed{(\alpha_c)_{\mathrm{TE}_{10}} = \frac{R_s\left(1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right)}{\eta b\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{1}{\eta b}\left(\frac{\mu\pi f}{\sigma(1-\left(\frac{f_c}{f}\right)^2)}\right)^{\frac{1}{2}}\left[1 + \frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right] \text{ Np/m}} \tag{6.158}$$

Following a similar analysis,we can show that the general expresions for the attenuation constant α_c due to wall losses for any TE$_{mn}$ mode is

$$\begin{aligned}(\alpha_c)_{\mathrm{TE}_{mn}} \quad = \quad & \frac{2\frac{R_s}{b\eta}}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}\left\{\left(1+\frac{b}{a}\right)\left(\frac{f_{c_{mn}}}{f}\right)^2 + \right. \\ & \left.\left(\frac{\delta_{0n}}{2} - \left(\frac{f_{c_{mn}}}{f}\right)^2\right)\frac{\frac{b}{a}\left\{(\frac{b}{a})m^2+n^2\right\}}{\left(\frac{b}{a}\right)^2 m^2 + n^2}\right\}\end{aligned} \tag{6.159}$$

where

$$\delta_{0n} = \begin{cases} 1 & q=0 \\ 2 & q\neq 0 \end{cases} \tag{6.160}$$

While for TM$_{mn}$ mode is

$$\alpha_{c_{\mathrm{TM}\,mn}} = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}\frac{(\frac{b}{a})^3 m^2 + n^2}{m^2\left(\frac{b}{a}\right)^2 + n^2} \tag{6.161}$$

These expressions show the dependence of the attenuation on the frequency. Reedactar: ¿ ¡¡Computed values of α_c for a few TE$_{mn}$ and TM$_{mn}$ modes are given in Figure 6.3In practice, surfaces imperfections, the value of α_c may be greater than the theoretical values. This effect can be reduced using well polished walls.

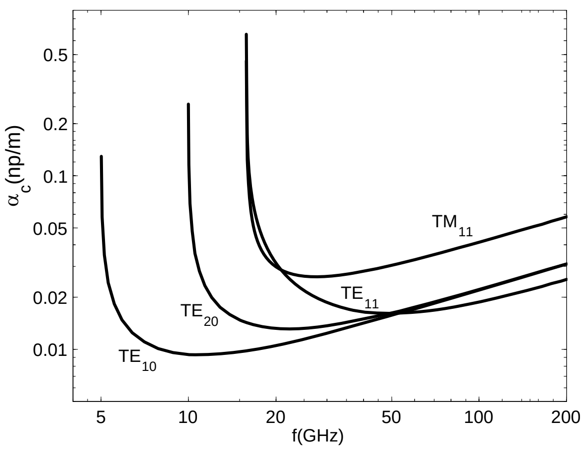


Figure 6.3: Atenuacion en guias rectangulares: a commom characteristic It tends to infinite when f is close to the cutoff frequency, decreases toward an optimum frequency (minimum value of α_c) an then increases almost linearly with f