An Introduction to K-theory

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0 Introduction

These notes are a reasonably faithful transcription of lectures which I gave in Trieste in May 2007. My objective was to provide participants of the Algebraic K-theory Summer School an overview of various aspects of algebraic K-theory, with the intention of making these lectures accessible to participants with little or no prior knowledge of the subject. Thus, these lectures were intended to be the most elementary as well as the most general of the six lecture series of our summer school.

At the end of each lecture, various references are given. For example, at the end of Lecture 1 the reader will find references to several very good expositions of aspects of algebraic K-theory which present their subject in much more detail than I have given in these lecture notes. One can view these present notes as a "primer" or a "course outline" which offer a guide to formulations, results, and conjectures of algebraic K-theory found in the literature.

The primary topic of each of my six lectures is reflected in the title of each lecture:

1. $K_0(-)$, $K_1(-)$, and $K_2(-)$
2. Classifying spaces and higher K-theory
3. Topological K-theory
4. Algebraic K-theory and Algebraic Geometry
5. Some Difficult Problems
6. Beilinson's vision partially fulfilled

Taken together, these lectures emphasize the connections between algebraic K-theory and algebraic geometry, saying little about connections with number theory and nothing about connections with non-commutative geometry. Such omissions, and many others, can be explained by the twin factors of the ignorance of the lecturer and the constraints imposed by the brevity of these lectures. Perhaps what is somewhat novel, especially in such brief format, is the emphasis on the algebraic K-theory of not necessarily affine schemes. Another attribute of these lectures is the continual reference to topological K-theory and algebraic topology as a source of inspiration and intuition.

We very briefly summarize the content of each of these six lectures. Lecture 1 introduces low dimensional K-theory, with emphasis on $K_0(X)$, the Grothendieck group of finitely generated projective R-modules for a (commutative) ring R if $\operatorname{Spec} R = X$, of topological vector vector bundles over X if X is a finite dimensional C.W. complex, and of coherent, locally free $\mathcal{O}_X$-modules if X is a scheme. Without a doubt, a primary goal (if not the primary goal) of K-theory is the understanding of K_0.

The key concept discussed in Lecture 2 is that of "homotopy theoretic group completion", an enriched extension of the process introduced by Alexander Grothendieck of taking the group associated to a monoid. We briefly consider three versions of such a group completion, all due to Daniel Quillen: the plus-construction, the $S^{-1}S$-construction, and the Q-construction. In this lecture, we remind the reader of simplicial sets, abelian categories, and the nerve of a category.

The early development of topological K-theory by Michael Atiyah and Fritz Hirzebruch has been a guide for many algebraic K-theorists during the past 45 years. Lecture 3 presents some of machinery of topological K-theory (spectra in the sense of algebraic topology, the Atiyah-Hirzebruch spectral sequence, and operations in K-theory) which reappear in more recent developments of algebraic K-theory.

In Lecture 4 we discuss the relationship of algebraic K-theory to the study of algebraic cycles on (smooth) quasi-projective varieties. In particular, we remind the reader of the definition of Chow groups of algebraic cycles modulo rational equivalence. The relationship between algebraic K-theory and algebraic cycles was realized by Alexander Grothendieck when he first introduced algebraic K-theory; indeed, algebraic K_0 figures in the formulation of Grothendieck's Riemann-Roch theorem. As we recall, one beautiful consequence of this theorem is that the Chern character from $K_0(X)$ to $CH^*(X)$ of a smooth, quasi-projective variety X is a rational equivalence.

In order to convince the intrigued reader that there remain many fundamental questions which await solutions, we discuss in Lecture 5 a few difficult open problems. For example, despite very dramatic progress in recent years, we still do not have a complete computation of the algebraic K-theory of the integers $\mathbb{Z}$. This lecture concludes somewhat idiosyncratically with a discussion of integral analogues of famous questions formulated in terms of the "semi-topological K-theory" constructed by Mark Walker and the author.

The final lecture could serve as an introduction to Professor Weibel's lectures on the proof of the Bloch-Kato Conjecture. The thread which organizes the effort of many mathematicians is a list of 7 conjectures by Alexander Beilinson which proposes to explain to what extent algebraic K-theory possesses properties analogous to those enjoyed by topological K-theory. We briefly discuss the status of these conjectures (all but the Beilinson-Soulé vanishing conjecture appear to be verified) and discuss briefly the organizational features of the motivic spectral sequence. We conclude this Lecture 6, and thus our series of lectures, with a very cursory discussion of etale cohomology and Grothendieck sites introduced by Vladimir Voevodsky in his dazzling proof of the Milnor Conjecture.

1 $K_0(-)$, $K_1(-)$, and $K_2(-)$

Perhaps the first new concept that arises in the study of K-theory, and one which recurs frequently, is that of the group completion of an abelian monoid.

The basic example to keep in mind is that the abelian group of integers $\mathbb{Z}$ is the group completion of the monoid $\mathbb{N}$ of natural numbers. Recall that an abelian monoid M is a set together with a binary, associative, commutative operation $+ : M \times M \to M$ and a distinguished element $0 \in M$ which serves as an identify (i.e., $0 + m = m$ for all $m \in M$). Then we define the group completion $\gamma : M \to M^+$ by setting M^+ equal to the quotient of the free abelian group with generators $[m], m \in M$ modulo the subgroup generated by elements of the form $[m] + [n] - [m + n]$ and define $\gamma : M \to M^+$ by sending $m \in M$ to $[m]$. We frequently refer to M^+ as the *Grothendieck group* of M.

The group completion map $\gamma : M \to M^+$ satisfies the following *universal property*. For any homomorphism $\phi : M \to A$ from M to a group A, there exists a unique homomorphism $\phi^+ : M^+ \to A$ such that $\phi^+ \circ \gamma = \phi : M \to A$.

1.1 Algebraic K_0 of rings

This leads almost immediately to K-theory. Let R be a ring (always assumed associative with unit, but not necessarily commutative). Recall that an (always assumed left) R-module P is said to be projective if there exists another R-module Q such that $P \oplus Q$ is a free R-module.

Definition 1.1. Let $\mathcal{P}(R)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of finitely generated projective R-modules. Then we define $K_0(R)$ to be $\mathcal{P}(R)^+$.

Warning: The group completion map $\gamma : \mathcal{P}(R) \to K_0(R)$ is frequently not injective.

Exercise 1.2. *Verify that if $j : R \to S$ is a ring homomorphism and if P is a finitely generated projective R-module, then $S \otimes_R P$ is a finitely generated projective S-module. Using the universal property of the Grothendieck group, you should also check that this construction determines $j_* : K_0(R) \to K_0(S)$.*

Indeed, we see that $K_0(-)$ is a (covariant) functor from rings to abelian groups.

Example 1.3. If $R = F$ is a field, then a finitely generated F-module is just a finite dimensional F-vector space. Two such vector spaces are isomorphic if and only if they have the same dimension. Thus, $\mathcal{P}(F) \simeq \mathbb{N}$ and $K_0(F) = \mathbb{Z}$.

Example 1.4. Let $K/\mathbb{Q}$ be a finite field extension of the rational numbers (K is said to be a *number field*) and let $\mathcal{O}_K \subset K$ be the ring of algebraic integers in K. Thus, $\mathcal{O}$ is the subring of those elements $\alpha \in K$ which satisfy a monic polynomial $p_\alpha(x) \in \mathbb{Z}[x]$. Recall that $\mathcal{O}_K$ is a Dedekind domain. The theory of Dedekind domains permits us to conclude that

$$K_0(\mathcal{O}_K) \;=\; \mathbb{Z} \oplus Cl(K)$$

where $Cl(K)$ is the ideal class group of K.

A well-known theorem of Minkowski asserts that $Cl(K)$ is finite for any number field K (cf. [5]). Computing class groups is devilishly difficult. We do know that there only finitely many cyclotomic fields (i.e., of the form $\mathbb{Q}(\zeta_n)$ obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$) with class group $\{1\}$. The smallest n with non-trivial class group is $n = 23$ for which $Cl(Q(\zeta_{23})) = \mathbb{Z}/3$. A check of tables shows, for example, that $Cl(\mathbb{Q}(\zeta_{100})) = \mathbb{Z}/65$.

The reader is referred to the book [4] for an accessible introduction to this important topic.

The K-theory of integral group rings has several important applications in topology. For a group π, the integral group ring $\mathbb{Z}[\pi]$ is defined to be the ring whose underlying abelian group is the free group on the set $[g], g \in \pi$ and whose ring structure is defined by setting $[g] \cdot [h] = [g \cdot h]$. Thus, if π is not abelian, then $\mathbb{Z}[\pi]$ is not a commutative ring.

Application 1.5. *Let X be a path-connected space with the homotopy type of a C.W. complex and with fundamental group π. Suppose that X is a* *retract of a finite C.W. complex. Then the Wall finiteness obstruction is an element of $K_0(\mathbb{Z}[\pi])$ which vanishes if and only if X is homotopy equivalent to a finite C.W. complex.*

1.2 Topological K_0

We now consider topological K-theory for a topological space X. This is also constructed as a Grothendieck group and is typically easier to compute than algebraic K-theory of a ring R. Moreover, results first proved for topological K-theory have both motivated and helped to prove important theorems in algebraic K-theory. A good introduction to topological K-theory can be found in [1].

Definition 1.6. Let $\mathbb{F}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. An $\mathbb{F}$-vector bundle on a topological space X is a continuous open surjective map $p : E \to X$ satisfying

(a) For all $x \in X$, $p^{-1}(x)$ is a finite dimensional $\mathbb{F}$-vector space.

(b) There are continuous maps $E \times E \to E, \mathbb{F} \times E \to E$ which provide the vector space structure on $p^{-1}(x)$, all $x \in X$.

(c) For all $x \in X$, there exists an open neighborhood $U_x \subset X$, an $\mathbb{F}$-vector space V, and a homeomorphism $\psi_x : V \times U_x \to p^{-1}(U_x)$ over U_x (i.e., $pr_2 = p \circ \psi_x : V \times U_x \to U_x$) compatible with the structure in (b).

Example 1.7. Let $X = S^1$, the circle. The projection of the Möbius band M to its equator $p : M \to S^1$ is a rank 1, real vector bundle over S^1.

Let $X = S^2$, the 2-sphere. Then the projection $p : T_{S^2} \to S^2$ of the tangent bundle is a non-trivial vector bundle.

Let $X = S^2$, but now view X as the complex projective line, so that points of X can be viewed as complex lines through the origin in $\mathbb{C}^2$ (i.e., complex subspaces of $\mathbb{C}^2$ of dimension 1). Then there is a natural rank 1, complex line bundle $E \to X$ whose fibre above $x \in X$ is the complex line parametrized by x; if $E - o(X) \to X$ denotes the result of removing the origin of each fibre, then we can identify $E - o(X)$ with $\mathbb{C}^2 - \{0\}$.

Definition 1.8. Let $Vect_{\mathbb{F}}(X)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of $\mathbb{F}$-vector bundles of X. We define

$$K^0_{top}(X) = Vect_{\mathbb{C}}(X)^+, \quad KO^0_{top}(X) = Vect_{\mathbb{R}}(X)^+.$$

(This definition will agree with our more sophisticated definition of topological K-theory given in a later lecture provided that the X has the homotopy type of a finite dimensional C.W. complex.)

The reason we use a superscript 0 rather than a subscript 0 for topological K-theory is that it determines a contravariant functor. Namely, if $f : X \to Y$ is a continuous map of topological spaces and if $p : E \to Y$ is an $\mathbb{F}$-vector bundle on Y, then $pr_2 : E \times_Y X \to X$ is an $\mathbb{F}$-vector bundle on X. This determines

$$f^* : K^0_{top}(Y) \to K^0_{top}(X).$$

Example 1.9. Let n_{S^2} denote the "trivial" rank n, real vector bundle over S^2 (i.e., $pr_2 : \mathbb{R}^n \times S^2 \to S^2$) and let T_{S^2} denote the tangent bundle of S^2. Then $T_{S^2} \oplus 1_{S^2} \simeq 3_{S^2}$. We conclude that $Vect_{\mathbb{R}}(S^2) \to K\mathcal{O}^0_{top}(S^2)$ is *not injective* in this case.

Here is one of the early theorems of K-theory, a theorem proved by Richard Swan. You can find a full proof, for example, in [5].

Theorem 1.10. *(Swan) Let $\mathbb{F} = \mathbb{R}$ (respectively, $= \mathbb{C}$), let X be a compact Hausdorff space, and let $\mathcal{C}(X, \mathbb{F})$ denote the ring of continuous functions $X \to \mathbb{F}$. For any $E \in Vect_{\mathbb{F}}(X)$, define the $\mathbb{F}$-vector space of global sections $\Gamma(X, E)$ to be*

$$\Gamma(X, E) = \{s : X \to E \ \textit{continuous}; p \circ s = id_X\}.$$

Then sending E to $\Gamma(X, E)$ determines isomorphisms

$$KO^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbb{R})), \quad K^0_{top}(X) \to K_0(\mathcal{C}(X, \mathbb{C})).$$

1.3 Quasi-projective Varieties

We briefly recall a few basic notions of classical algebraic geometry; a good basic reference is the first chapter of [3]. Let us assume our ground field k is algebraically closed, so that we need only consider k-rational points. For more general fields k, we could have to consider "points with values in some finite field extension L/k."

Recollection 1.11. Recall projective space $\mathbb{P}^N$, whose k-rational points are equivalence classes of $N + 1$-tuple, $\langle a_0, \ldots, a_N \rangle$, some entry of which is non-zero. Two $N + 1$-tuples $(a_0, \ldots, a_N), (b_0, \ldots, b_N)$ are equivalent if there exists some $0 \neq c \in k$ such that $(a_0, \ldots, a_N) = (cb_0, \ldots, cb_N)$.

If $F(X_0, \dots, X_N)$ is a homogeneous polynomial, then the zero locus $Z(F) \subset \mathbb{P}^N$ is well defined.

Recall that $\mathbb{P}^N$ is covered by standard affine opens $U_i = \mathbb{P}^N \backslash Z(X_i)$.

Recall the Zariski topology on $\mathbb{P}^N$, a base of open sets for which are the subsets of the form $U_G = \mathbb{P}^N \backslash Z(G)$.

Recollection 1.12. Recall the notion of a presheaf on a topological space T: a contravariant functor from the category whose objects are open subsets of T and whose morphisms are inclusions.

Recall that a sheaf is a presheaf satisfying the sheaf axiom: for T compact, this axiom can be simply expressed as requiring for each pair of open subsets U, V that

$$F(U \cup V) = F(U) \times_{F(U \cap V)} F(V).$$

Recall the structure sheaf of "regular functions" $\mathcal{O}_{\mathbb{P}^N}$ on $\mathbb{P}^N$, sections of $\mathcal{O}_{\mathbb{P}^N}(U)$ on any open U are given by quotients $\frac{P(X_0,\dots,X_N)}{Q(X_0,\dots,X_N)}$ of homogeneous polynomials of the same degree satisfying the condition that Q has no zeros in U. In particular,

$$\mathcal{O}_{\mathbb{P}^N}(U_G) = \{F(X)/G^j, j \geq 0; F \text{ homgeneous} \quad \text{of} \quad \deg = j \cdot deg(G)\}.$$

Definition 1.13. A projective variety X is a space with a sheaf of commutative rings $\mathcal{O}_X$ which admits a closed embedding into some $\mathbb{P}^N$, $i : X \subset \mathbb{P}^N$, so that $\mathcal{O}_X$ is the quotient of the sheaf $\mathcal{O}_{\mathbb{P}^N}$ by the ideal sheaf of those regular functions which vanish on X.

A quasi-projective variety U is once again a space with a sheaf of commutative rings $\mathcal{O}_U$ which admits locally a closed embedding into some $\mathbb{P}^N$, $j : U \subset \mathbb{P}^N$, so that the closure $\overline{U} \subset \mathbb{P}^N$ of U admits the structure of a projective variety and so that $\mathcal{O}_U$ equals the restriction of $\mathcal{O}_{\overline{U}}$ to $U \subset \overline{U}$.

A quasi-projective variety U is said to be affine if U admits a closed embedding into some $\mathbb{A}^N = \mathbb{P}^N \backslash Z(X_0)$ so that $\mathcal{O}_U$ is the quotient of $\mathcal{O}_{\mathbb{A}^N}$ by the sheaf of ideals which vanish on U.

Remark 1.14. Any quasi-projective variety U has a base of (Zariski) open subsets which are affine.

Most quasi-projective varieties are neither projective nor affine.

There is a bijective correspondence between affine varieties and finitely generated commutative k-algebras. If U is an affine variety, then $\Gamma(\mathcal{O}_U)$ is the corresponding finitely generated k-algebra. Conversely, if A is written as a quotient $k[x_1, \ldots, x_N] \to A$, then $SpecA \to Spec(k[x_1, \ldots, x_N]) = \mathbb{A}^N$ is the corresponding closed embedding of the affine variety $SpecA$.

Example 1.15. Let F be a polynomial in variables $X_0, \ldots, X_N$ homogeneous of degree d (i.e., $F(ca_0, \ldots, ca_N) = c^d F(a_0, \ldots, a_N)$. Then the zero locus $Z(F) \subset \mathbb{P}^N$ is called a hypersurface of degree d. For example if $N = 2$, then $Z(F)$ is 1-dimensional (i.e., a curve). If $k = \mathbb{C}$ and if the Jacobian of F does not vanish anywhere on $C = Z(F)$ (i.e., if C is *smooth*), then C is a projective, smooth, algebraic curve of genus $\frac{(d-1)(d-2)}{2}$.

1.4 Algebraic vector bundles

Definition 1.16. Let X be a quasi-projective variety. A *quasi-coherent sheaf* $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules (i.e., an abelian sheaf equipped with a pairing $\mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ of sheaves satisfying the condition that for each open $U \subset X$ this pairing gives $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module) with the property that there exists an open covering $\{U_i \subset X; i \in I\}$ by affine open subsets so that $\mathcal{F}_{|U_i}$ is the sheaf associated to an $O_X(U_i)$-module M_i for each i.

If each of the M_i can be chosen to be finitely generated as an $O_X(U_i)$-module, then such a quasi-coherent sheaf is called *coherent*.

Definition 1.17. Let X be a quasi-projective variety. A coherent sheaf $\mathcal{E}$ on X is said to be an algebraic vector bundle if $\mathcal{E}$ is locally free. In other words, if there exists a (Zariski) open covering $\{U_i; i \in I\}$ of X such that $\mathcal{E}_{|U_i} \simeq \mathcal{O}^{e_i}_{X|U_i}$ for each i.

Remark 1.18. If a quasi-projective variety is affine, then an algebraic vector bundle on X is equivalent to a projective $\Gamma(\mathcal{O}_X)$-module.

Construction 1. If M is a free A-module of rank r, then the symmetric algebra $Sym^\bullet_A(M)$ is a polynomial algebra of r generators over A and the structure map $\pi : \operatorname{Spec} Sym^\bullet_A(M) \to \operatorname{Spec} A$ is just the projection $\mathbb{A}^r \times \operatorname{Spec} A \to \operatorname{Spec} A$. This construction readily globalizes: if $\mathcal{E}$ is an algebraic vector bundle over X, then

$$\pi_{\mathcal{E}} : \mathbb{V}(\mathcal{E}) \equiv \operatorname{Spec} Sym^\bullet_{\mathcal{O}_X}(\mathcal{E})^* \ \to \ X$$

is locally in the Zariski topology a product projection: if $\{U_i \subset X; i \in\}$ is an open covering restricted to which $\mathcal{E}$ is trivial, then the restriction of $\pi_{\mathcal{E}}$ above each U_i is isomorphic to the product projection $\mathbb{A}^r \times U_i \to U_i$. In the above definition of $\pi_{\mathcal{E}}$ we consider the symmetric algebra on the dual $\mathcal{E}^* = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, so that the association $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E}^*)$ is covariantly functorial.

Thus, we may alternatively think of an algebraic vector bundle on X as a map of varieties

$$\pi_{\mathcal{E}} : \mathbb{V}(\mathcal{E}^*) \ \rightarrow \ X$$

satisfying properties which are the algebraic analogues of the properties of the structure map of a topological vector bundle over a topological space.

Remark 1.19. We should be looking at the maximal ideal spectrum of a variety over a field k, rather than simply the k rational points, whenever k is not algebraically closed. We suppress this point, for we will soon switch to prime ideal spectra (i.e., work with schemes of finite type over k). However, we do point out that the reason it suffices to consider the maximal ideal spectrum rather the spectrum of all prime ideals is the validity of the Hilbert Nullstellensatz. One form of this important theorem is that the subset of maximal ideals constitute a dense subset of the space of prime ideals (with the Zariski topology) of a finitely generated commutative k-algebra.

1.5 Examples of Algebraic Vector Bundles

Example 1.20. Rank 1 vector bundles $O_{\mathbb{P}^N}(k), k \in \mathbb{Z}$ on $\mathbb{P}^N$. The sections of $O_{\mathbb{P}^N}(j)$ on the basic open subset $U_G = \mathbb{P}^N \backslash Z(G)$ are given by the formula

$$O_{\mathbb{P}^N}(k)(U_G) = k[X_0, \dots, X_N, 1/G]_{(j)}$$

(i.e., ratios of homogeneous polynomials of total degree j).

In terms of the trivialization on the open covering $U_i, 0 \leq i \leq N$, the patching functions are given by $X_i^j / X_{i'}^j$.

$\Gamma(O_{\mathbb{P}^N}(j))$ has dimension $\binom{N+j}{j}$ if $j > 0$, dimension 1 if $j = 0$, and 0 otherwise. Thus, using the fact that $O_{\mathbb{P}^N}(j) \otimes_{\mathcal{O}_X} O_{\mathbb{P}^N}(j') = O_{\mathbb{P}^N}(j + j')$, we conclude that $\Gamma(O_{\mathbb{P}^N}(j))$ is not isomorphic to $\Gamma(O_{\mathbb{P}^N}(j'))$ provided that $j' \neq j$.

Proposition 1.21. *(Grothendieck) Each vector bundle on $\mathbb{P}^1$ has a unique decomposition as a finite direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(k), k \in \mathbb{Z}$.*

Example 1.22. Serre's conjecture (proved by Quillen and Suslin) asserts that every algebraic vector bundle on $\mathbb{A}^N$ (or any affine open subset of $\mathbb{A}^N$) is trivial. In more algebraic terms, every finitely generated projective $k[x_1, \dots, x_n]$-module is free.

Example 1.23. Let $X = Grass_{n,N}$, the Grassmann variety of $n-1$-planes in P^N (i.e., n-dimensional subspaces of k^{N+1}). We can embed $Grass_{n,N}$ as a Zariski closed subset of $\mathbb{P}^{M-1}$, where $M = \binom{N+1}{n}$, by sending the subspace $V \subset k^{N+1}$ to its n-th exterior power $\Lambda^n V \subset \Lambda^n(k^{N+1})$. There is a natural rank n algebraic vector bundle $\mathcal{E}$ on X provided with an embedding in the trivial rank $N+1$ dimensional vector bundle $\mathcal{O}_X^{N+1}$ (in the special case $n = 1$, this is $\mathcal{O}_{\mathbb{P}^N}(-1) \subset \mathcal{O}_{\mathbb{P}^N}^{N+1}$) whose fibre above a point in X is the corresponding subspace. Of equal importance is the natural rank $N-n$-dimensional quotient bundle $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^N}^{N+1}/\mathcal{E}$.

This example readily generalizes to flag varieties.

Example 1.24. Let A be a commutative k-algebra and recall the module $\Omega_{A/k}$ of Kähler differentials. These globalize to a quasi-coherent sheaf Ω_X on a quasi-projective variety X over k. If X is *smooth* of dimension r, then Ω_X is an algebraic vector bundle over X of rank r.

1.6 Picard Group $Pic(X)$

Definition 1.25. Let X be a quasi-projective variety. We define $Pic(X)$ to be the abelian group whose elements are isomorphism classes of rank 1 algebraic vector bundles on X (also called "invertible sheaves"). The group structure on $Pic(X)$ is given by tensor product.

So defined, $Pic(X)$ is a generalization of the construction of the Class Group (of fractional ideals modulo principal ideal) for $X = \operatorname{Spec} A$ with A a Dedekind domain.

Example 1.26. By examining patching data, we readily verify that

$$H^1(X, \mathcal{O}_X^*) = Pic(X)$$

where $\mathcal{O}_X^*$ is the sheaf of abelian groups on X with sections $\Gamma(U, \mathcal{O}_X^*)$ defined to be the invertible elements of $\Gamma(U, \mathcal{O}_X)$ (with group structure given by multiplication).

If $k = \mathbb{C}$, then we have a short exact sequence of *analytic sheaves* of abelian sheaves on the analytic space $X(\mathbb{C})^{an}$,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \stackrel{exp}{\to} \mathcal{O}_X^* \to 0.$$

We use identification due to Serre of analytic and algebraic vector bundles on a projective variety. If $X = C$ is a smooth curve, this identification enables us to conclude the short exact sequence

$$0 \to \mathbb{C}^g/\mathbb{Z}^{2g} \to Pic(C) \to H^2(C,\mathbb{Z})$$

since $H^1(C,\mathcal{O}_C) \simeq H^0(C,\Omega_C) = \mathbb{C}^g$ (where g is the genus of C). In particular, we conclude that for a curve of positive genus, $Pic(C)$ is very large, having a "continuous part" (which is an abelian variety).

Example 1.27. A $K3$ surface S over the complex numbers is characterized by the conditions that $H^0(S,\Lambda^2(\Omega_S)) = 0 = H^1_{sing}(S,\mathbb{Q})$. Even though the homotopy type of a smooth $K3$ surface does not depend upon the choice of such a surface S, the rank of $Pic(S)$ can vary from 1 to 20. [The dimension of $H^2_{sing}(S,\mathbb{Q})$ is 22.]

1.7 K_0 of Quasi-projective Varieties

Definition 1.28. Let X be a quasi-projective variety. We define $K_0(X)$ to be the quotient of the free abelian group generated by isomorphism classes $[\mathcal{E}]$ of (algebraic) vector bundles $\mathcal{E}$ on X modulo the equivalence relation generated pairs $([\mathcal{E}],[\mathcal{E}_1]+[\mathcal{E}_2])$ for each short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of vector bundles.

Remark 1.29. Let A be a finitely generated k-algebra. Observe that every short exact sequence of projective A-modules splits. Thus, the equivalence relation defining $K_0(A)$ is generated by pairs $([\mathcal{E}_1 \oplus \mathcal{E}_2],[\mathcal{E}_1]+[\mathcal{E}_2])$. Every element of $K_0(A)$ can be written as $[P]-[m]$ for some non-negative integer m; moreover, projective modules P,Q determine the same class in $K_0(A)$ if and only if there exists some non-negative integer m such that $P \oplus A^m \simeq Q \oplus A^m$.

Proposition 1.30. *$K_0(\mathbb{P}^N)$ is a free abelian group of rank $N+1$. Moreover, for any $k \in \mathbb{Z}$, the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(k),\ldots,\mathcal{O}_{\mathbb{P}^N}(k+N)$ generate $K_0(\mathbb{P}^N)$.*

Proof. One obtains a relation among $N+2$ invertible sheaves from the Koszul complex on $N+1$ dimensional vector space V:

$$0 \to \Lambda^{N+1}V \otimes S^{*-N-1}(V) \to \cdots \to V \otimes S^{*-1}(V) \to S^*(V) \to k \to 0.$$

One shows that the invertible sheaves $\mathcal{O}_{\mathbb{P}^N}(j), j \in \mathbb{Z}$ generate $K_0(\mathbb{P}^N)$ using Serre's theorem that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ there exist integers $m,n > 0$ and a surjective map of $\mathcal{O}_{\mathbb{P}^N}$-modules $\mathcal{O}_{\mathbb{P}^N}(m)^n \to \mathcal{F}$.

One way to show that the rank of $K_0(\mathbb{P}^N)$ equals $N+1$ is to use Chern classes. $\square$

1.8 K_1 of rings

So far, we have only considered degree 0 algebraic and topological K-theory. Before we consider $K_n(R), n \in \mathbb{N}, K^n_{top}(X), n \in \mathbb{Z}$, we look explicitly at $K_1(R)$. This was first investigated in depth in the classic book by Bass [2].

Definition 1.31. Let R be a ring (assumed associative, as always and with unit). We define $K_1(R)$ by the formula

$$K_1(R) \ \equiv \ GL(R)/[GL(R), GL(R)],$$

where $GL(R) = \varinjlim_n GL(n, R)$ and where $[GL(r), GL(R)]$ is the commutator subgroup of the group $GL(R)$. Thus, $K_1(R)$ is the maximal abelian quotient of $GL(R)$,

$$K_1(R) \ = \ H_1(GL(R), \mathbb{Z}).$$

The commutator subgroup $[GL(R), GL(R)]$ equals the subgroup $E(R) \subset GL(R)$ defined as the subgroup generated by elementary matrices $E_{i,j,}(r), r \in R, i \neq j$ (i.e., matrices which differ by the identity matrix by having r in the (i, j) position). This group is readily seen to be *perfect* (i.e., $E(R) = [E(R), E(R)]$); indeed, it is an elementary exercise to verify that $E(n, R) = E(R) \cap GL(n, R)$ is perfect for $n \geq 3$.

Proposition 1.32. *If R is a commutative ring, then the determinant map*

$$det : K_1(R) \to R^{\times}$$

from $K_1(R)$ to the multiplicative group of units of R provides a natural splitting of $R^{\times} = GL(1, R) \to GL(R) \to K_1(R)$. Thus, we can write

$$K_1(R) \ = \ R^{\times} \times SL(R)$$

where $SL(R) = ker\{det\}$.

If R is a field or more generally a local ring, then $SK_1(R) = 0$.

The following theorem is not at all easy, but it does tell us that nothing surprising happens for rings of integers in number fields.

Theorem 1.33. *(Bass-Milnor-Serre) If $\mathcal{O}_K$ is the ring of integers in a number field K, then $SK_1(\mathcal{O}_K) = 0$.*

Application 1.34. *The work of Bass-Milnor-Serre was dedicated to solving the following question: is every subgroup $H \subset SL(\mathcal{O}_K)$ of finite index a "congruent subgroup" (i.e., of the form $ker\{SL(\mathcal{O}_K) \to SL(\mathcal{O}_K/p^n)\}$ for some prime ideal $p \subset \mathcal{O}_K$. The answer is yes if the number field F admits a real embedding, and no otherwise.*

The Bass-Milnor-Serre theorem is complemented by the following classical result due to Dirichlet (cf. [5]).

Theorem 1.35. *(Dirichlet's Theorem) Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then*

$$O_K^* = \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1}$$

where $\mu(K) \subset K$ denotes the finite subgroup of roots of unity and where r_1 (respectively, r_2) denotes the number of embeddings of K into $\mathbb{R}$ (resp., number of conjugate pairs of embeddings of K into $\mathbb{C}$).

We conclude this brief commentary on K_1 with the following early application to topology.

Application 1.36. *Let π be a finitely generated group and consider the Whitehead group*

$$Wh(\pi) = K_1(R)/\{\pm g; g \in \pi\}.$$

A homotopy equivalence of finite complexes with fundamental group π has an invariant (its "Whitehead torsion") in $Wh(\pi)$ which determines whether or not this is a simple homotopy equivalence (given by a chain of "elementary expansions" and "elementary collapses").

The interested reader can find a wealth of information about K_0 and K_1 in the books [2] and [6].

1.9 K_2 of rings

One can think of $K_0(R)$ as the "stable group" of projective modules "modulo trivial projective modules" and of $K_1(R)$ of the stabilized group of automorphisms of the trivial projective module modulo "trivial automorphisms" (i.e., the elementary matrices up to isomorphism. This philosophy can be extended to the definition of K_2, but has not been extended to $K_i, i > 2$. Namely, $K_2(R)$ can be viewed as the relations among the trivial automorphisms (i.e., elementary matrices) modulo those relations which hold universally.

Definition 1.37. Let $St(R)$, the Steinberg group of R, denote the group generated by elements $X_{i,j}(r), i \neq j, r \in R$ subject to the following commutator relations:

$$[X_{i,j}(r), X_{k,\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq \ell \\ X_{i,\ell}(rs) & \text{if } j = k, i \neq \ell \\ X_{k,j}(-rs) & \text{if } j \neq k, i = \ell \end{cases}$$

We define $K_2(R)$ to be the kernel of the map $St(R) \to E(R)$, given by sending $X_{i,j}(r)$ to the elementary matrix $E_{i,j}(r)$, so that we have a short exact sequence

$$1 \to K_2(R) \to St(R) \to E(R) \to 1.$$

Proposition 1.38. *The short exact sequence*

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

is the universal central extension of the perfect group $E(R)$. Thus, $K_2(R) = H_2(E(R), \mathbb{Z})$, the Schur multiplier of $E(R)$.

Proof. One can show that a universal central extension of a group E exists if and only E is perfect. In this case, a group S mapping onto E is the universal central extension if and only if S is also perfect and $H_2(S, \mathbb{Z}) = 0$. □

Example 1.39. If R is a field, then $K_1(F) = F^\times$, the non-zero elements of the field viewed as an abelian group under multiplication. By a theorem of Matsumoto, $K_2(F)$ is characterized as the target of the "universal Steinberg symbol". Namely, $K_2(F)$ is isomorphic to the free abelian group with generators "Steinberg symbols" $\{a, b\},\ a, b \in F^\times$ and relations

i. {ac,b} = {a,b} {c,b},

ii. {a,bd} = {a,b} {a,d},

iii. $\{a, 1-a\} = 1, \quad a \neq 1 \neq 1-a.$ (*Steinberg relation*)

Observe that for $a \in F^\times,\ -a = \frac{1-a}{1-a^{-1}}$, so that

$$\{a, -a\} = \{a, 1-a\}\{a, 1-a^{-1}\}^{-1} = \{a, 1-a^{-1}\}^{-1} = \{a^{-1}, 1-a^{-1}\} = 1.$$

Then we conclude the skew symmetry of these symbols:

$$\{a, b\}\{b, a\} = \{a, -a\}\{a, b\}\{b, a\}\{b, -b\} = \{a, -ab\}\{b, -ab\} = \{ab, -ab\} = 1.$$

Milnor used this presentation of $K_2(F)$ as the starting point of his definition of the *Milnor K-theory* K_*^{Milnor} of a field F, discussed briefly in Lecture 5.

2 Classifying spaces and higher K-theory

2.1 Recollections of homotopy theory

Much of our discussions will require some basics of homotopy theory. Two standard references are [8] and [14].

Definition 2.1. Two continuous maps $f, g : X \to Y$ between topological spaces are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ with $F_{|X\times\{0\}} = f, F_{|X\times\{1\}} = g$ (where I denotes the unit interval $[0, 1]$).

If $x \in X, y \in Y$ are chosen ("base points"), then two ("pointed") maps $f, g : (X, \{x\}) \to (Y, \{y\})$ are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ such that $F|_{X\times\{0\}} = f, F_{|X\times\{1\}} = g$, and $F_{|\{x\}\times I} = \{y\}$ (i.e., F must project $\{x\} \times I$ to $\{y\}$. We use the notation $[(X, x), (Y, y)]$ to denote the pointed homotopy classes of maps from (X, x) (previously denoted $(X, \{x\})$) to $(Y, \{y\})$.

We shall employ the usual notation, $[X, Y]$ to denote homotopy classes of continuous maps from X to Y.

Another basic definition is that of the homotopy groups of a topological space.

Definition 2.2. For any $n \geq 0$ and any pointed space (X, x),

$$\pi_n(X, x) \equiv [(S^n, \infty), (X, x)].$$

For $n = 0$, $\pi_n(X, x)$ is a pointed set; for $n \geq 1$, a group; for $n \geq 2$, an abelian group. If (X, x) is "nice", then $\pi_n(X, x) \simeq [S^n, X]$; moreover, if X is path connected, then the isomorphism class of $\pi_n(X, x)$ is independent of $x \in X$.

A **relative C.W. complex** is a topological pair (X, A) (i.e., A is a subspace of X) such that there exists a sequence of subspaces $A = X_{-1} \subset X_0 \subset \cdots \subset X_n \subset \cdots$ of X with union equal to X such that X_n is obtained from X_{n-1} by "attaching" n-cells (i.e., possibly infinitely many copies of the closed unit disk in $\mathbb{R}^n$, where "attachment" means that the boundary of the disk is identified with its image under a continuous map $S^{n-1} \to X_{n-1}$) and such that a subset $F \subset X$ is closed if and only if $X \cap X_n \subset X_n$ is closed for all n. A space X is a C.W. complex if $(X, \emptyset)$ is a relative C.W. complex. A pointed C.W. complex (X, x) is a relative C.W. complex for $(X, \{x\})$.

C.W. complexes have many good properties, one of which is the following.

Theorem 2.3. *(Whitehead theorem) If $f : X \to Y$ is a continuous map of connected C.W. complexes such that $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence.*

Moreover, C.W. complexes are quite general: If (T, t) is a pointed topological space, then there exists a pointed C.W. complex (X, x) and a continuous map $g : (X, x) \to (T, t)$ such that $g_* : \pi_*(X, x) \to \pi_*(T, t)$ is an isomorphism.

Recall that a continuous map $f : X \to Y$ is said to be a fibration if it has the homotopy lifting property: given any commutative square of continuous maps

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & Y \end{array}$$

then there exits a map $A \times I \to X$ whose restriction to $A \times \{0\}$ is the upper horizontal map and whose composition with the right vertical map equals the lower horizontal map. A very important property of fibrations is that if $f : X \to Y$ is a fibration, then there is a long exact sequence of homotopy groups for any $x_o \in X, y \in Y$:

$$\cdots \to \pi_n(f^{-1}(y), x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \pi_{n-1}(f^{-1}(y), x_0) \to \cdots$$

If $f : (X, x) \to (Y, y)$ is any pointed map of spaces, we can naturally construct a fibration $\tilde{f} : \tilde{X} \to Y$ together with a homotopy equivalence $X \to \tilde{X}$ over Y. We denote by $htyfib(f)$ the fibre $\tilde{f}^{-1}(y)$ of $\tilde{f}$.

2.2 *BG*

Definition 2.4. Let G be a topological group and X a topological space. Then a **G-torsor** over X (or principal G-bundle) is a continuous map $p : E \to X$ together with a continuous action of G on E over X such that there exists an open covering $\{U_i\}$ of X homeomorphisms $G \times U_i \to E_{|U_i}$ for each i respecting G-actions (where G acts on $G \times U_i$ by left multiplication on G).

Example 2.5. Assume that G is a discrete group. Then a G-torsor $p : E \to X$ is a normal covering space with covering group G.

Theorem 2.6. *(Milnor) Let G be a topological group with the homotopy type of a C.W. complex. There there exists a connected C.W. complex BG and a*

G-torsor $\pi : EG \to BG$ such that sending a continuous function $X \to BG$ to the G-torsor $X \times_{BG} EG \to X$ over X determines a 1-1 correspondence

$$[X, BG] \overset{\simeq}{\to} \{\textit{isom classes of G-torsors over } X\}$$

Moreover, the homotopy type of BG is thereby determined; furthermore, EG is contractible.

The topology on G when considering the classifying space BG is crucial. One interesting construction one can consider is the map on classifying spaces induced by the continuous, bijective function $G^\delta \to G$ where G is a topological group and G^δ is the same group but provided with the discrete topology.

Corollary 2.7. *If G is discrete, then $\pi_1(BG, *) = G$ and $\pi_n(BG, *) = 0$ for all $n > 0$ (where $*$ is some choice of base point). Moreover, these properties characterize the C.W. complex BG up to homotopy type.*

Proof. Sketch of proof. If $n > 0$, then the facts that $\pi_1(S^n) = 0$ and EG is contractible imply that $[S^n, BG] = \{0\}$. The fact that $\pi_1(BG, *) = G$ is classical covering space theory. □

The proof of the following proposition is fairly elementary, using a standard projection resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\pi]$-module.

Proposition 2.8. *Let π be a discrete group and let A be a $\mathbb{Z}[\pi]$-module. Then*

$$H^*(B\pi, A) = Ext^*_{\mathbb{Z}[\pi]}(\mathbb{Z}, A) \equiv H^*(\pi, A)$$

$$H_*(B\pi, A) = Tor_*^{\mathbb{Z}[\pi]}(\mathbb{Z}, A) \equiv H_*(\pi, A).$$

Now, vector bundles are not G-torsors but rather fibre bundles for the topological groups $O(n)$ (respectively, $U(n)$) in the case of a real (resp., complex) vector bundle of rank n. Nevertheless, because $O(n)$ (resp., $U(n)$) acts faithfully and transitively on $\mathbb{R}^n$ (resp., $\mathbb{C}^n$), we can readily conclude using Theorem 2.6

$$[X, BO(n)] = \{\text{isom classes of real rank n vector bundles over X}\}$$

$$[X, BU(n)] = \{\text{isom classes of complex rank n vector bundles over X}\}.$$

2.3 Quillen's plus construction

Daniel Quillen's original definition of $K_i(R), i > 0$, was in terms of the following "Quillen plus construction". A detailed exposition of this construction can be found in [7].

Theorem 2.9. *(Plus construction) Let G be a discrete group and $H \subset G$ be a perfect normal subgroup. Then there exists a C.W. complex BG^+ and a continuous map*

$$\gamma : BG \to BG^+$$

such that $ker\{\pi_1(BG) \to \pi_1(BG^+)\} = H$ and such that $\tilde{H}_*(htyfib(\gamma), \mathbb{Z}) = 0$. Moreover, γ is unique up to homotopy.

The classical "Whitehead Lemma" implies that the commutator subgroup $[GL(R), GL(R)]$ of $GL(R)$ is perfect. (One verifies that an $n \times n$ elementary matrix is itself a commutator of elementary matrices provided that $n \geq 4$.)

Definition 2.10. For any ring R, let

$$\gamma : BGL(R) \to BGL(R)^+$$

denote the Quillen plus construction with respect to $[GL(R), GL(R)] \subset GL(R)$. We define

$$K_i(R) \equiv \pi_i(BGL(R)^+), \quad i > 0.$$

This construction is closely connected to the group completions of our first lecture. In some sense, $\coprod_n BGL(n, R)$ is "up to homotopy, a commutative topological monoid" and $BGL(R)^+ \times \mathbb{Z}$ is a group completion in an appropriate sense. There are several technologies which have been introduced in part to justify this informal description (e.g., the "$S^{-1}S$ construction" discussed below).

Remark 2.11. Essentially by definition, $K_1(R)$ as defined in the first lecture agrees with that of Definition 2.10. Moreover, for any $K_1(R)$-module A,

$$H^*(BGL(R)^+, A) = H^*(BGL(R), A).$$

Moreover, one can verify that $K_2(R)$ as introduced in the first lecture agrees with that of Definition 2.10 for any ring R by identifying this second homotopy group with the second homology group of the perfect group $[GL(R), GL(R)]$.

When Quillen formulated his definition of $K_*(R)$, he also made the following fundamental computation. Indeed, this computation was a motivating factor for Quillen's definition (cf. [10]).

Theorem 2.12. *(Quillen's computation for finite fields) Let $\mathbb{F}_q$ be a finite field. Then the space $BGL(\mathbb{F}_q)^+$ can be described as the homotopy fibre of a computable map. This leads to the following computation for $i > 0$:*

$$K_i(\mathbb{F}_q) = \mathbb{Z}/q^j - 1 \qquad \text{if } \ i = 2j - 1$$

$$K_i(\mathbb{F}_q) = 0 \qquad \text{if } \ i = 2j.$$

As you probably know, homotopy groups are notoriously hard to compute. So Quillen has played a nasty trick on us, giving us very interesting invariants with which we struggle to make the most basic calculations. For example, a fundamental problem which is still not fully solved is to compute $K_i(\mathbb{Z})$.

Early computations of higher K-groups of a ring R often proceeded by first computing the group homology groups of $GL(n, R)$ for n large, then relating these homology groups to the homotopy groups of $BGL(R)^+$.

2.4 Abelian and exact categories

Much of our discussion in these lectures will require the language and concepts of category theory. Indeed, working with categories will give us a method to consider various kinds of K-theories simultaneously.

I shall assume that you are familiar with the notion of an abelian category. Recall that in an abelian category $\mathcal{A}$, the set of morphisms $Hom_{\mathcal{A}}(B, C)$ for any $A, B \in Obj\ \mathcal{A}$ has the natural structure of an abelian group; moreover, for each $A, B \in Obj\ \mathcal{A}$, there is an object $B \oplus C$ which is both a product and a coproduct; moreover, any $f : A \to B$ in $Hom_{\mathcal{A}}(A, B)$ has both a kernel and a cokernel. In an abelian category, we can work with exact sequences just as we do in the category of abelian groups.

Example 2.13. Here are a few "standard" examples of abelian categories.

- the category $Mod(R)$ of (left) R-modules.
- the category $mod(R)$ of finitely generated R-modules (in which case we must take R to be Noetherian).
- the category $QCoh(X)$ of quasi-coherent sheaves on a variety X.

- the category $Coh(X)$ of coherent sheaves on a Notherian variety X.

Warning. The full subcategory $\mathcal{P}(R) \subset mod(R)$ is not an abelian category. For example, if $R = \mathbb{Z}$, then $n : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism of projective R-modules whose cokernel is not projective and thus is not in $\mathcal{P}(\mathbb{Z})$.

Definition 2.14. An exact category $\mathcal{P}$ is a full additive subcategory of some abelian category $\mathcal{A}$ such that

(a) There exists some set $S \subset Obj\ \mathcal{A}$ such that every $A \in Obj\ \mathcal{A}$ is isomorphic to some element of S.

(b) If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence in $\mathcal{A}$ with both $A_1, A_3 \in Obj\ \mathcal{P}$, then $A_2 \in Obj\ \mathcal{P}$.

An admissible monomorphism (respectively, epimorphism) in $\mathcal{P}$ is a monomorphism $A_1 \to A_2$ (resp., $A_2 \to A_3$) in $\mathcal{P}$ which fits in an exact sequence of the form of (b).

Definition 2.15. If $\mathcal{P}$ is an exact category, we define $K_0(\mathcal{P})$ to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of objects of $\mathcal{P}$ (with respect to $\oplus$) modulo the equivalence relation $[A_2] - [A_1] - [A_3]$ for every exact sequence of the form (I.5.b).

Exercise 2.16. *Show that $K_0(R)$ equals $K_0(\mathcal{P}(R))$, where $\mathcal{P}(R)$ is the exact category of finitely generated projective R-modules.*

More generally, show that $K_0(X)$ equals $K_0(\mathcal{V}ect(X))$, where $\mathcal{V}ect(X)$ is the exact category of algebraic vector bundles on the quasi-projective variety X.

Definition 2.17. Let $\mathcal{P}$ be an exact category in which all exact sequences split. Consider pairs (A, α) where $A \in Obj\ \mathcal{P}$ and α is an automorphism of A. Direct sums and exact sequences of such pairs are defined in the obvious way. Then $K_1(\mathcal{P})$ is defined to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of such pairs modulo the relations given by short exact sequences.

2.5 The $S^{-1}S$ construction

Recall that a *symmetric monoidal category* S is a (small) category with a unit object $e \in S$ and a functor $\Box : S \times S \to S$ which is associative and commutative up to coherent natural isomorphisms. For example, if we consider the category $\mathcal{P}$ of finitely generated projective R-modules, then the direct sum $\oplus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative but only commutative up to natural isomorphism. The symmetric monoidal category relevant for the K-theory of a ring R is the category $Iso(\mathcal{P})$ whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms between projective R-modules.

Quillen's construction of $S^{-1}S$ for a symmetric monoidal category S is appealing, modelling one way we would introduce inverses to form the group completion of an abelian monoid. A good reference for this is [13].

Definition 2.18. Let S be a symmetric monoidal category. The category $S^{-1}S$ is the category whose objects are pairs $\{a, b\}$ of objects of S and whose maps from $\{a, b\}$ to $\{c, d\}$ are equivalence classes of compositions of the following form:

$$\{a, b\} \stackrel{s\Box-}{\to} \{s\Box a, s\Box b) \stackrel{(f,g)}{\to} \{c, d\}$$

where s is some object of S, f, g are morphisms in S. Another such composition

$$\{a, b\} \stackrel{s'\Box-}{\to} \{s'\Box a, s'\Box b) \stackrel{(f',g')}{\to} \{c, d\}$$

is declared to be the same map in $S^{-1}S$ from $\{a, b\}$ to $\{c, d\}$ if and only if there exists some isomorphism $\theta : s \to s'$ such that $f = f' \circ (\theta\Box a), g = g' \circ (\theta\Box b)$.

Heuristically, we view $\{a, b\} \in S^{-1}S$ as representing $a - b$, so that $\{s\Box a, s\Box b\}$ also represents $a - b$. Moreover, we are forcing morphisms in S to be invertible in $S^{-1}S$. If we were to apply this construction to the natural numbers $\mathbb{N}$ viewed as a discrete category with addition as the operation, then we get $\mathbb{N}^{-1}\mathbb{N} = \mathbb{Z}$.

The following theorem of Quillen shows how the $S^{-1}S$ construction can provide a homotopy-theoretic group completion

Theorem 2.19. *(Quillen) Let S be a symmetric monoidal category with the property that for all $s, t \in S$ the map $s\Box- : Aut(t) \to Aut(s\Box t)$ is injective. Then the natural map $BS \to B(S^{-1}S)$ of classifying spaces (see the next section) is a homotopy-theoretic group completion.*

In particular, if S denotes the category whose objects are finite dimensional projective R-modules and whose maps are isomorphisms (so that $BS = \coprod_{[P]} BAut(P)$), then $\mathcal{K}(R)$ is homotopy equivalent to $B(S^{-1}S)$.

2.6 Simplicial sets and the Nerve of a Category

The reader is referred to [9] for a detailed introduction to simplicial sets.

Definition 2.20. The category of standard simplices, Δ, has objects $\underline{n} = \langle 0, 1, \ldots, n \rangle$ indexed by $n \in \mathbb{N}$ and morphisms given by

$$Hom_\Delta(\underline{m}, \underline{n}) = \{\text{non-decreasing maps } \langle 0, 1, \ldots, n \rangle \to \langle 0, 1, \ldots, m \rangle\}.$$

The special morphisms

$$\partial_i : \underline{\text{n-1}} \to \underline{\text{n}} \ \ (\textit{skip } i); \quad \sigma_j : \underline{\text{n+1}} \to \underline{\text{n}} \ \ (\text{repeat j})$$

in Δ generate (under composition) all the morphisms of Δ and satisfy certain standard relations which many topologists know by heart.

A simplicial set $S_\bullet$ is a functor $\Delta^{op} \to (sets)$.

In other words, $S_\bullet$ consists of a set S_n for each $n \geq 0$ and maps $d_i : S_n \to S_{n-1}, s_j : S_n \to S_{n+1}$ satisfying the relations given by the relations satisfied by $\partial_i, \sigma_j \in \Delta$.

Example 2.21. Let T be a topological space. Then the **singular complex** $Sing_\bullet T$ is a simplicial set. Recall that $Sing_n T$ is the set of continuous maps $\Delta^n \to T$, where $\Delta^n \subset \mathbb{R}^{n+1}$ is the standard n-simplex: the subspace consisting of those points $\underline{x} = (x_0, \ldots, x_n)$ with each $x_i \geq 0$ and $\sum x_i = 1$. Since any map $\mu : \underline{n} \to \underline{m}$ determines a (linear) map $\Delta^n \to \Delta^m$, it also determines $\mu : Sing_m T \to Sing_n T$, so that we may easily verify that

$$Sing_\bullet T : \Delta^{op} \to (sets)$$

is a well-defined functor.

Definition 2.22. (Milnor's geometric realization functor) For any simplicial set $X_\bullet$, we define its geometric realization as the topological space $|X_.|$ given as follows:

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is given by $(x, \mu \circ t) \simeq (\mu \circ x, t)$ whenever $x \in X_m, t \in \Delta^n, \mu : \underline{n} \to \underline{m}$ a map of Δ. This quotient is given the quotient topology, where each $X_n \times \Delta^n$ is topologized as a disjoint union indexed by $x \in X_n$ of copies of $\Delta^n \subset \mathbb{R}^{n+1}$.

Now, simplicial sets are a very good combinatorial model for homotopy theory as the next theorem reveals.

Theorem 2.23. *(Homotopy category) The categories of topological spaces and simplicial sets satisfy the following relationships.*

- *Milnor's geometric realization functor is left adjoint to the singular functor; in other words, for every simplicial set $X_\bullet$ and every topological space T,*

$$Hom_{(s.sets)}(X_\bullet, Sing_\bullet T) = Hom_{(spaces)}(|X_\bullet|, T).$$

- *For any simplicial set $X_\bullet$, $|X_\bullet|$ is a C.W. complex; moreover, for any topological space T, $Sing._\bullet(T)$ is a particularly well behaved type of simplicial set called a Kan complex.*

- *For any topological space T and any point $t \in T$, the adjunction morphism*

$$(|Sing_\bullet T|, t) \to (T, t)$$

induces an isomorphism on homotopy groups.

- *The adjunction morphisms above induce an equivalence of categories*

$$(Kan\ cxes)/\sim hom.equiv \quad \simeq \quad (C.W.\ cxes)/\sim hom.equiv\ .$$

Now we can define the classifying space of a (small) category.

Definition 2.24. Let $\mathcal{C}$ be a small category. We define the **nerve** $N\mathcal{C} \in (s.sets)$ to be the simplicial set whose set of n-simplices is the set of composable n-tuples of morphisms in $\mathcal{C}$:

$$N\mathcal{C}_n = \{C_n \stackrel{\gamma_n}{\to} C_{n-1} \to \cdots \stackrel{\gamma_1}{\to} C_0\}.$$

For $\partial_i : \underline{\text{n-1}} \to \underline{\text{n}}$, we define $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ to send the n-tuple $C_n \to \cdots \to C_0$ to that $n-1$-tuple given by composing γ_{i+1} and γ_i whenever $0 < i < n$, by dropping $\stackrel{\gamma_1}{\to} C_0$ if $i = 0$ and by dropping $C_n \stackrel{\gamma_n}{\to}$ if $i = n$. For $\sigma_j : \underline{\text{n}} \to \underline{\text{n+1}}$, we define $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ by repeating C_j and inserting the identity map.

We define the **classifying space** $B\mathcal{C}$ of the category $\mathcal{C}$ to be $|N\mathcal{C}|$, the geometric realization of the nerve of $\mathcal{C}$.

The reader is encouraged to consult [12] for a discussion and insight into this construction.

Example 2.25. Let G be a (discrete) group and let $\mathcal{G}$ denote the category with a single object (denoted $*$) and with $Hom_{\mathcal{G}}(*,*) = G$. Then $B\mathcal{G}$ is a model for BG (i.e., $B\mathcal{G}$ is a connected C.W. complex with $\pi_1(B\mathcal{G},*) = G$ and all higher homotopy groups 0).

Example 2.26. Let X be a polyhedron and let $\mathcal{S}(X)$ denote the category whose objects are simplices of X and maps are the inclusions of simplices. Then $B\mathcal{S}(X)$ can be identified with the first barycentric subdivision of X.

2.7 Quillen's Q-construction

What are the higher K-groups of an exact category? In particular, what are the higher K-groups of a quasi-projective variety X (i.e., of the exact category $\mathcal{V}ect(X)$) or more generally of a scheme?

Quillen defines these in terms of another construction, the "Quillen Q-construction." This construction as well as many fundamental applications can be found in Quillen's remarkable paper [11].

Definition 2.27. Let $\mathcal{P}$ be an exact category and let $Q\mathcal{P}$ be the category obtained from $\mathcal{P}$ by applying the Quillen Q-construction (as discussed below). Then

$$K_i(\mathcal{P}) = \pi_{i+1}(BQ\mathcal{P}), \quad i \geq 0,$$

the homotopy groups of the geometric realization of the nerve of $Q\mathcal{P}$.

Theorem 2.28. *Let X be a scheme and let $\mathcal{V}ect(X)$ denote the exact category of finitely presented, locally free $\mathcal{O}_X$-modules. Then*

$$K_i(X) \;\equiv\; \pi_i(\mathcal{V}ect(X)) \;\equiv\; \pi_{i+1}(BQ\mathcal{V}ect(X))$$

agrees for $i = 0$ with the Grothendieck group of $\mathcal{V}ect(X)$ and for $X = SpecA$ an affine scheme agrees with $K_i(A) = \pi_i(BLG(A)^+)$ provided that $i > 0$.

Quillen proves this theorem using the $S^{-1}S$ construction as an intermediary.

Here is the formulation of Quillen's Q-construction.

Definition 2.29. Let $\mathcal{P}$ be an exact category. We define the category $Q\mathcal{P}$ as follows. We set $Obj\ Q\mathcal{P}$ equal to $Obj\ \mathcal{P}$. For any $A, B \in Obj\ Q\mathcal{P}$, we define

$$Hom_{Q\mathcal{P}}(A,B) = \{A \overset{p}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B; \text{p (resp. i) admissible epi (resp. mono)}/\sim\}$$

where the equivalence relation is generated by pairs

$$A \twoheadleftarrow X \rightarrowtail B, A \twoheadleftarrow X' \rightarrowtail B$$

which fit in a commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow[p]{} & X & \xrightarrow{i} & B \\ \downarrow{=} & & \downarrow & & \downarrow{=} \\ A & \xleftarrow[p']{} & X & \xrightarrow{i'} & B \end{array}$$

Waldhausen in [15] gives a somewhat more elaborate construction of Quillen's Q construction which produces "n-fold deloopings" of $BQ\mathcal{P}$ for every $n \geq 0$: pointed spaces T_n with the property that $\Omega^n(T_n)$ is homotopy equivalent to $BQ\mathcal{P}$.

3 Topological K-theory

In this lecture, we will discuss some of the machinery which makes topological K-theory both useful and computable. Not only does topological K-theory play a very important role in topology, but also it has played the most important guiding role in the development of algebraic K-theory. As general references, the books [17], [18] and [14] are recommended.

3.1 The Classifying space $BU \times \mathbb{Z}$

The following statements about topological vector bundles are not valid (in general) for algebraic vector bundles. These properties suggest that topological K-theory is better behaved than algebraic K-theory.

Proposition 3.1. *(cf. [1]) Let T be a compact Hausdorff space. If $p : E \to T$ is a topological vector bundle on T, then for some $N > 0$ there is a surjective map of bundles on T, $(\mathbb{C}^{N+1} \times T) \to E$.*

Any surjective map $E \to F$ of topological vector bundles on T admits a splitting over T.

The set of homotopy classes of maps $[T, BU(n)]$ is in natural 1-1 correspondence with the set of isomorphism classes of rank n topological vector bundles on T.

Proof. The first statement is proved using a partition of unity argument.

The proof of the second statement is by establishing a Hermitian metric on E (so that $E \simeq F \oplus F^{\perp}$), which is achieved by once again using a partition of unity argument.

To prove the last statement, one verifies that if $T \times I \to G$ is a homotopy relating continuous maps $f, g : T \to G$ and if E is a topological vector bundle on G, then $f^*E \simeq g^*E$ as topological vector bundles on T. Once again, a partition of unity argument is the key ingredient in the proof. □

Proposition 3.2. *For any space T, the set of homotopy classes of maps*

$$[T, BU \times \mathbb{Z}], \quad BU = \varinjlim_n BU_n$$

admits a natural structure of an abelian group induced by block sum of matrices $U_n \times U_m \to U_{n+m}$. We define

$$K^0_{top}(T) \equiv [T, BU \times \mathbb{Z}].$$

For any compact, Hausdorff space T, $K^0_{top}(T)$ is naturally isomorphic to the Grothendieck group of topological vector bundles on T:

$$K^0_{top}(T) \simeq \frac{\mathbb{Z}[\text{iso classes of top vector bundles on } T]}{[E] = [E_1] + [E_2], \text{ whenever } E \simeq E_1 \oplus E_2}.$$

Proof. (External) direct sum of matrices gives a monoid structure on $\sqcup_n BU_n$ which determines a (homotopy associative and commutative) H-space structure on $BU \times Z$ which we view as the mapping telescope of the self map

$$\sqcup_n BU_n \to \sqcup_n BU_n, \quad BU_i \times \{\star \in BU_1\} \to BU_{i+1}.$$

The (abelian) group structure on $[T, BU \times \mathbb{Z}]$ is then determined.

To show that this mapping telescope is actually an H-space, one must verify that it has a 2-sided identity up to *pointed* homotopy: one must verify that product on the left with $\star \in BU_1$ gives a self map of $BU \times \mathbb{Z}$ which is related to the identity via a base-point preserving homotopy. (Such a verification is not difficult, but the analogous verification fails if we replace the topological groups U_n by discrete groups $GL_n(A)$ for some unital ring A.) □

Example 3.3. Since the Lie groups U_n are connected, the spaces BU_n are simply connected and thus

$$K^0_{top}(S^1) = \pi_1(BU \times \mathbb{Z}) = 0.$$

It is useful to extend $K^0_{top}(-)$ to a relative theory which applies to pairs (T, A) of spaces (i.e., T is a topological space and $A \subset T$ is a closed subset). In the special case that $A = \emptyset$, then $T/\emptyset = T_+/\star$, the pointed space obtained by taking the disjoint union of T with a point $\star$ which we declare to be the basepoint.

Definition 3.4. If T is a pointed space with basepoint t_0, we define the reduced K-theory of T by

$$\tilde{K}^*_{top}(T) \equiv K^*_{top}(T, t_0).$$

For any pair (T, A), we define

$$K^0_{top}(T, A) \equiv \tilde{K}^0_{top}(T/A)$$

thereby extending our earlier definition of $K^0_{top}(T)$.

For any $n > 0$, we define

$$K^n_{top}(T, A) \equiv \tilde{K}^0_{top}(\Sigma^n(T/A)).$$

In particular, for any $n \geq 0$, we define

$$K^{-n}_{top}(T) \equiv K^{-n}_{top}(T, \emptyset) \equiv \tilde{K}^0_{top}(\Sigma^n(T_+)).$$

Observe that

$$\tilde{K}^0_{top}(S \wedge T) = ker\{K^0_{top}(S \times T) \to K^0_{top}(S) \oplus K^0_{top}(T)\},$$

so that (external) tensor product of bundles induces a natural pairing

$$K^{-i}_{top}(S) \otimes K^{-j}_{top}(T) \to K^{-i-j}_{top}(S \times T).$$

Just to get the notation somewhat straight, let us take T to be a single point $T = \{t\}$. Then $T_+ = \{t, \star\}$, the 2-point space with new point $\star$ as base-point. Then $\Sigma^2(T_+)$ is the 2-sphere S^2, and thus

$$K_{top}^{-2}(\{t\}) = ker\{K_{top}^0(S^2)) \to K_{top}^0(\star)\}.$$

We single out a special element, *the Bott element*

$$\beta = [\mathcal{O}_{\mathbb{P}^1}(1)] - [\mathcal{O}_{\mathbb{P}^1}] \in K_{top}^{-2}(pt)),$$

where we have abused notation by identifying $(\mathbb{P}^1)^{an}$ with S^2 and the images of algebraic vector bundles on $\mathbb{P}^1$ in $K_{top}^0((\mathbb{P}^1)^{an})$ have the same names as in $K_0(\mathbb{P}^1)$.

3.2 Bott periodicity

Of fundamental importance in the study of topological K-theory is the following theorem of Raoul Bott. Recall that if (X, x) is pointed space, then the **loop space** ΩX is the function complex (with the compact-open topology) of continuous maps from (S^1, ∞) to (X, x). The loop space functor $\Omega(-)$ on pointed spaces is adjoint to the suspension functor $\Sigma(-)$: there is a natural bijection

$$Maps(\Sigma(X), Y) \;\simeq\; Maps(X, \Omega(Y))$$

of sets of continuous, pointed (i.e, base point preserving) maps. An extensive discussion of Bott periodicity can be found in [17].

Theorem 3.5. *(Bott Periodicity) There are the following homotopy equivalences.*

- *From $BO \times \mathbb{Z}$ to its 8-fold loop space:*

$$BO \times \mathbb{Z} \;\sim\; \Omega^8(BO \times \mathbb{Z})$$

Moreover, the homotopy groups $\pi_i(BO \times \mathbb{Z})$ are given by

$$\mathbb{Z},\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ 0,\ \mathbb{Z},\ 0,\ 0,\ 0$$

depending upon whether i is congruent to $0, 1, 2, 3, 4, 5, 6, 7$ modulo 8.

- *From $BU \times \mathbb{Z}$ to its 2-fold loop space:*

$$BU \times \mathbb{Z} \sim \Omega^2(BU \times \mathbb{Z})$$

Moreover, $\pi_i(BU \times \mathbb{Z})$ is $\mathbb{Z}$ if i is even and equals 0 if i is odd.

Atiyah interprets this 2-fold periodicity in terms of K-theory as follows.

Theorem 3.6. *(Bott Periodicity) For any space T and any $i \geq 0$, multiplication by the Bott element induces a natural isomorphism*

$$\beta : K^{-i}_{top}(T) \to K^{-i-2}_{top}(T).$$

Using the above theorem, we define $K^i_{top}(X)$ for any topological space X and any integer i as $K^{\bar{i}}_{top}(X)$, where $\bar{i}$ is 0 if i is even and $\bar{i}$ is -1 if i is odd.

In particular, taking T to be a point, we conclude that $\tilde{K}^0_{top}(S^2) = \mathbb{Z}$, generated by the Bott element.

Example 3.7. Let S^0 denote $\{*, \star\} = *_+$. According to our definitions, the K-theory $K_{top}(*)$, of a point equals the reduced K-theory of S^0. In particular, for $n > 0$,

$$K^{-n}_{top}(*) = \tilde{K}^{-n}_{top}(S^0) = \tilde{K}^0_{top}(S^n) = \pi_n(BU).$$

Thus, we conclude

$$K^n_{top}(*) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

We can reformulate this by writing

$$K^i_{top}(S^n) = \begin{cases} \mathbb{Z} & \text{if } i+n \text{ is even} \\ 0 & \text{if } i+n \text{ is odd} \end{cases}$$

3.3 Spectra and Generalized Cohomology Theories

Thus, both $BO \times \mathbb{Z}$ and $BU \times \mathbb{Z}$ are “infinite loop spaces” naturally determining Ω-spectra in the following sense.

Definition 3.8. A spectrum $\underline{\mathrm{E}}$ is a of pointed spaces $\{E^0, E^1, \ldots\}$, each of which has the homotopy type of a pointed C.W. complex, together with continuous structure maps $\Sigma(E^i) \to E^{i+1}$.

The spectrum $\underline{\mathrm{E}}$ is said to be an Ω-spectrum if the *adjoint* $E^i \to \Omega(E^{i+1})$ of each map is a homotopy equivalence; in other words, a sequence of pointed homotopy equivalences

$$E^0 \stackrel{\simeq}{\to} \Omega E^1 \stackrel{\simeq}{\to} \Omega^2 E^2 \stackrel{\simeq}{\to} \cdots \stackrel{\simeq}{\to} \Omega^n E^n \to \cdots$$

Each spectrum $\underline{\mathrm{E}}$ determines an Ω-spectrum $\underline{\tilde{\mathrm{E}}}$ defined by setting

$$\tilde{E}_n \;=\; \varinjlim_j \Omega^j \Sigma^{j-n}(E_n).$$

The importance of Ω-spectra is clear from the following theorem which asserts that an Ω-spectrum determines a "generalized cohomology theory".

Theorem 3.9. *(cf. [14]) Let $\underline{E}$ be an Ω-spectrum. For any topological space X with closed subspace $A \subset X$, set*

$$h^n_{\underline{E}}(X,A) = [(X,A), E^n], \quad n \geq 0$$

Then $(X,a) \mapsto h^*_{\underline{E}}(X,A)$ is a generalized cohomology theory; namely, this satisfies all of the Eilenberg-Steenrod axioms except that its value at a point (i.e., $(*,\emptyset)$) may not be that of ordinary cohomology:

(a) $h^*_{\underline{E}}(-)$ is a functor from the category of pairs of spaces to graded abelian groups.

(b) for each $n \geq 0$ and each pair of spaces (X,A), there is a functorial connecting homomorphism $\partial : h^n_{\underline{E}}(A) \to h^{n+1}_{\underline{E}}(X,A)$.

(c) the connecting homomorphisms of (b) determine long exact sequences for every pair (X,A).

(d) $h^*_{\underline{E}}(-)$ satisfies excision: i.e., for every pair (X,A) and every subspace $U \subset A$ whose closure lies in the interior of A, $h^*_{\underline{E}}(X,A) \simeq h^*_{\underline{E}}(X-U, A-U)$.

Observe that in the above definition we use the notation $h^*_{\underline{\mathrm{E}}}(X)$ for $h^*_{\underline{\mathrm{E}}}(X,\emptyset) = h^*_{\underline{\mathrm{E}}}(X_+,*)$, where X_+ is the disjoint union of X and a point $*$.

Definition 3.10. The (periodic) topological K-theories $KO^*_{top}(-)$, $K^*_{top}(-)$ are the generalized cohomology theories associated to the Ω-spectra given by $BO \times \mathbb{Z}$ and $BU \times \mathbb{Z}$ with their deloopings given by Bott periodicity.

In particular, whenever X is a finite dimensional C.W. complex,

$$K^{2j}_{top}(X) = [X, BU \times \mathbb{Z}], \quad K^{2j-1}_{top}(X) = [X, U],$$

so that we recover our definition of $K^0_{top}(X)$ (and similarly $KO^0_{top}(X)$).

Let us restrict attention to $K^*_{top}(X)$ which suffices to motivate our further discussion in algebraic K-theory. ($K0^*_{top}(X)$ motivates Hermetian algebraic K-theory.) There are also other interesting generalized cohomology theories (e.g., cobordism theory represented by the infinite loop space MU) which play a role in algebraic K-theory, and there are also more sophisticated equivariant K-theories, none of which will we discuss in these lectures.

Tensor product of vector bundles induces a multiplication

$$K^0_{top}(X) \otimes K^0_{top}(X) \to K^0_{top}(X)$$

for any finite dimensional C.W. complex X. This can be generalized by observing that tensor product induces group homomorphisms $U(m) \times U(n) \to U(n+m)$ and thereby maps of classifying spaces

$$BU(m) \times BU(n) \to BU(n+m).$$

With a little effort, one can show that these multiplication maps are compatible up to homotopy with the standard embeddings $U(m) \subset U(m+1), U(n) \subset U(n+1)$ and thereby give us a pairing

$$(BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z}$$

(factoring through the smash product). In this way, $BU \times \mathbb{Z}$ has the structure of an H-space which induces a pairing of spectra and thus a multiplication for the generalized cohomology theory $K^*_{top}(-)$. (A completely similar argument applies to $KO^*_{top}(-)$).

Remark: Each of the topological K-groups, $K^{-i}_{top}(X),\ i \in \mathbb{N}$, is given as $K^0_{top}(\Sigma^i X)$ where $\Sigma^i X$ is the i^{th} suspension of X. On the other hand, algebraic K-groups in non-zero degree are not easily related to the algebraic K_0 of some associated ring.

As an example of how topological K-theory inspired even the early (very algebraic) effort in algebraic K-theory we mention the following classical theorem of Hyman Bass. The analogous result in topological K-theory for rank e vector bundles over a finite dimension C.W. complex of dimension $d < e$ can be readily proved using the standard method of "obstruction theory".

Theorem 3.11. *(Bass stability theorem) Let A be a commutative, noetherian ring of Krull dimension d. Then for any two projective A-modules P, P' of rank $e > d$, if $[P] = [P'] \in K_0(A)$ then P must be isomorphic to P'.*

3.4 Skeleta and Postnikov towers

If X is a C.W. complex then we can define its **p-skeleton** $sk_p(X)$ for each $p \geq 0$ as the subspace of X consisting of the union of those cells of dimension $\leq p$. Thus, the C.W. complex can be written as the union (or colimit) of its skeleta,

$$X \;=\; \cup_p sk_p(X).$$

There is a standard way to "chop off" the bottom homotopy groups of a space (or an Ω-spectrum) using an analogue of the universal covering space of a space (which "chops off" the fundamental group).

Definition 3.12. Let X be a C.W. complex. For each $n \geq 0$, construct a map $X \to X[n]$ by attaching cells (proceeding by dimension) to kill all homotopy groups of X above dimension $n-1$. Define

$$X^{(n)} \; to \; X, \quad htyfib\{X \to X[n]\}.$$

So defined, $X^{(n)} \to X$ induces an isomorphism on homotopy groups $\pi_i, \; i \geq n$ and $\pi_j(X^{(n)}) = 0, \; j \leq n$.

The **Postinov tower** of X is the sequence of spaces

$$X \;\cdots\; \to X^{(n+1)} \;\to X^{(n)} \;\to \cdots$$

Thus, X can be viewed as the "homotopy inverse limit" of its Postnivkov tower.

Algebraic K-theory corresponds most closely the topological K-theory which is obtained by replacing the Ω-spectrum $\underline{\mathrm{K}} = \underline{\mathrm{BU}} \times \mathbb{Z}$ by $\underline{\mathrm{kU}} = \underline{\mathrm{bu}} \times \mathbb{Z}$ obtained by taking at stage i the i^{th} connected cover of $BU \times \mathbb{Z}$ starting at stage 0. The associated generalized cohomology theory is denoted $kU^*(-)$ and satisfies

$$kU^i(X) \;\simeq K^i_{top}(X), \quad i \leq 0.$$

In studying the mapping complex $Map_{cont}(X,Y)$ continuous maps from a C.W. complex X to a space Y, one typically filters this mapping complex using the skeleton filtration of X by its skeleta or the "coskeleton" filtration of Y by its Postnikov tower. We refer to [14] for details of these complementary approaches.

3.5 The Atiyah-Hirzebruch Spectral sequence

The Atiyah-Hirzebruch spectral sequence for topological K-theory has been a strong motivating factor in recent developments in algebraic K-theory. Indeed, perhaps the fundamental criterion for *motivic cohomology* is that it should satisfy a relationship to algebraic K-theory strictly analogous to the relationship of singular cohomology to topological K-theory.

Theorem 3.13. *(Atiyah-Hirzebruch spectral sequence [16]) For any generalized cohomology theory $h^*_{\underline{E}}(-)$ and any topological space X, there exists a right half-plane spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(X, h^q(*)) \Rightarrow h^{p+q}_{\underline{E}}(X).$$

The filtration on $h^*_{\underline{E}}(X)$ is given by

$$F^p E^*_\infty = ker\{h^*_{\underline{E}}(X) \to h^*_{\underline{E}}(sk_p(X)\}.$$

In the special case of $K^*_{top}(-)$, this takes the following form

$$E_2^{p,q} = H^p(X, \mathbb{Z}(q/2)) \Rightarrow K^{p+q}_{top}(X)$$

where $\mathbb{Z}(q/2) = \mathbb{Z}$ if q is even and 0 otherwise.

In the special case of $kU^*(-)$, this takes the following form

$$E_2^{p,q} = H^p(X, \mathbb{Z}(q/2)) \Rightarrow kU^{p+q}(X)$$

where $\mathbb{Z}(q/2) = \mathbb{Z}$ if q is an even non-positive integer and 0 otherwise.

Proof. There are two basic approaches to proving this spectral sequence. The first is to assume T is a cell complex, then consider T as a filtered space with $T_n \subset T$ the union of cells of dimension $\leq n$. The properties of $K^*_{top}(-)$ stated in the previous theorem give us an exact couple associated to the long exact sequences

$$\cdots \to \oplus K^q_{top}(S^n) \simeq K^q_{top}(T_n/T_{n-1}) \to K^q_{top}(T_n) \to K^q_{top}(T_{n-1}) \to \oplus K^{q+1}_{top}(S^n) \to \cdots$$

where the direct sum is indexed by the n-cells of T.

The second approach applies to a general space T and uses the Postnikov tower of $BU \times \mathbb{Z}$. This is a tower of fibrations whose fibers are Eilenberg-MacLane spaces for the groups which occur as the homotopy groups of $BU \times \mathbb{Z}$.

What is a spectral sequence of cohomological type? This is the data of a 2-dimensional array $E_r^{p,q}$ of abelian groups for each $r \geq r_0$ (typically, r_0 equals 0, or 1 or 2; in our case $r_0 = 2$) and homomorphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that the next array $E_{r+1}^{p,q}$ is given by the cohomology of these homomorphisms:

$$E_{r+1}^{p,q} = ker\{d_r^{p,q}\}/im\{d_r^{p-r,q+r-1}\}.$$

To say that the spectral sequence is "right half plane" is to say $E_r^{p,q} = 0$ whenever $p < 0$. We say that the spectral sequence **converges to the abutment** E_∞^* (in our case $h_{\underline{\mathrm{E}}}^*(X)$) if at each spot (p,q) there are only finitely many non-zero homomorphisms going in and going out and if there exists a decreasing filtration $\{F^pE_\infty^n\}$ on each E_∞^n so that

$$E_\infty^n \;=\; \bigcup_p F^pE_\infty^n, \quad 0 = \bigcap_p F^pE_\infty^n,$$

$$F^pE_\infty^n/F^{p+1}E_\infty^n = E_R^{p,n-p}, \quad R >> 0.$$

□

The Postnikov tower argument together with a knowledge of the k-invariants of $BU \times \mathbb{Z}$ shows that after tensoring with $\mathbb{Q}$ this Atiyah-Hirzebruch spectral sequence collapses; in other words, that $E_2^{*,*} \otimes \mathbb{Q} = E_\infty^{*,*} \otimes \mathbb{Q}$.

Theorem 3.14. *([16]) Let X be a C.W. complex. Then there are isomorphisms*

$$kU^0(X)) \otimes \mathbb{Q} \;\simeq\; H^{ev}(X, \mathbb{Q}), \quad kU^{-1}(X) \otimes Q \simeq H^{odd}(X, \mathbb{Q}).$$

These isomorphisms are induced by the *Chern character*

$$ch = \sum_i ch_i : K_0(-) \;\to\; H^{ev}(-, \mathbb{Q})$$

discussed in Lecture 4.

While we are discussing spectral sequences, we should mention the following:

Theorem 3.15. *(Serre spectral sequence; cf. [14]) Let (B, b) be a connected, pointed C.W. complex. For any fibration $p : E \to B$ of topological spaces with fibre $F = p^{-1}(b)$ and for any abelian group A, there exists a convergent first quadrant spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(B, H^q(F, A)) \Rightarrow H^{p+q}(E, A)$$

provided that $\pi_1(B, b)$ acts trivially on $H^*(F, A)$.

The non-existence of an analogue of the Serre spectral sequence in algebraic geometry (for cohomology theories based on algebraic cycles or algebraic K-theory) presents one of the most fundamental challenges to computations of algebraic K-groups.

3.6 K-theory Operations

There are several reasons why topological K-theory has sometimes proved to be a more useful computational tool than singular cohomology.

- $K^0_{top}(-)$ can be torsion free, even though $H^{ev}(-, \mathbb{Z})$ might have torsion. This is the case, for example, for compact Lie groups.
- $K^*_{top}(-)$ is essentially $\mathbb{Z}/2$-graded rather than graded by the natural numbers.
- $K^*_{top}(-)$ has interesting cohomology operations not seen in cohomology. These operations originate from the observation that the exterior products $\Lambda^i(P)$ of a projective module P are likewise projective modules and the exterior products $\Lambda^i(E)$ of a vector bundle E are likewise vector bundles.

A good introduction to K-theory operations can be found in the appendix of [1].

Definition 3.16. Let X be a finite dimensional C.W. complex and $E \to X$ be a topological vector bundle of rank r. Define

$$\lambda_t(E) = \sum_{i=0}^{r} [\Lambda^i E] t^i \ \in \ K^0_{top}(X)[t],$$

a polynomial with constant term 1 and thus an invertible element in $K^0_{top}(X)[[t]]$. Extend this to a homomorphism

$$\lambda_t : K^0_{top}(X) \to (1 + K^0_{top}(X)[[t]])^*,$$

(using the fact that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$) and define $\lambda^i : K^0_{top}(T) \to K^0_{top}(T)$ to be the coefficient of t^i of λ_t.

For a general topological space X, define these λ operations on $K^0_{top}(X)$ for by defining them first on the universal vector bundles over Grassmannians and using the functoriality of $K^0_{top}(-)$.

In particular, J. Frank Adams introduced operations

$$\psi^k(-) : K^0_{top}(-) \to K^0_{top}(-), \quad k > 0$$

(called **Adams operations**) which have many applications and which are similarly constructed for algebraic K-theory.

Definition 3.17. For any topological space T, define

$$\psi_t(x) = \sum_{i \geq 0} \psi^i(X) t^i \equiv rank(x) - t \cdot \frac{d}{dt}(log\lambda_{-t}(x))$$

for any $x \in K^0_{top}(T)$.

The *Adams operations* ψ^k satisfy many good properties, some of which we list below.

Proposition 3.18. *For any topological space T, any $x, y \in K^0_{top}(T)$, any $k > 0$*

- $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$.
- $\psi^k(xy) = \psi^k(x)\psi^k(y)$.
- $\psi^k(\psi^\ell(x) = \psi^{k\ell}(x)$.
- $ch_q(\psi^k(x)) = k^q ch_q(x) \in H^{2q}(T, \mathbb{Q})$.
- $\psi^p(x)$ *is congruent modulo p to x^p if p is a prime number.*
- $\psi^k(x) = x^k$ *whenever x is a line bundle*

In particular, if E is a sum of line bundles $\oplus_i L_i$, then $\psi^k(E) = \oplus((L_i)^k)$, the k-th power sum. By the splitting principle, this property alone uniquely determines ψ^k.

We introduce further operations, the γ-operations on $K_0^{top}(T)$.

Definition 3.19. For any topological space T, define

$$\gamma_t(x) = \sum_{i \geq 0} \gamma^i(X)t^i \equiv \lambda_{t/1-t}(x)$$

for any $x \in K^0_{top}(T)$.

Basic properties of these γ-operations include the following

1. $\gamma_t(x + y) = \gamma_t(x)\gamma_t(y)$
2. $\gamma([L] - 1) = 1 + t([L] - 1)$.
3. $\lambda_s(x) = \gamma_{s/1+s}(x)$

Using these γ operations, we define the γ filtration on $K^0_{top}(T)$ as follows.

Definition 3.20. For any topological space T, define $K^{\gamma,1}_{top}(T)$ as the kernel of the rank map

$$K^{\gamma,1}_{top}(T) \equiv ker\{\text{rank} : K^0_{top}(T) \to K^0_{top}(\pi_0(T))\}.$$

For $n > 1$, define

$$K^0_{top}(T)^{\gamma,n} \subset K^{\gamma,0}_{top}(T) \equiv K^0_{top}(T)$$

to be the subgroup generated by monomials $\gamma^{i_1}(x_1) \cdots \gamma^{i_k}(x_k)$ with $\sum_j i_j \geq n, x_i \in K^{\gamma,1}_{top}(T)$.

3.7 Applications

We can use the Adams operations and the γ-filtration to describe in the following theorem the relationship between $K^0_{top}(T)$, a group which has no natural grading, and the graded group $H^{ev}(T, \mathbb{Q})$.

Theorem 3.21. *Let T be a finite cell complex. Then for any $k > 0$, ψ^k restricts to a self-map of each $K^{\gamma,n}_{top}(T)$ and satisfies the property that it induces multiplication by k^n on the quotient*

$$\psi^k(x) = k^n \cdot x, \quad x \in K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)).$$

Furthermore, the Chern character ch induces an isomorphism

$$ch_n : K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)) \otimes Q \simeq H^{2n}(T, \mathbb{Q}).$$

In particular, the preceding theorem gives us a K-theoretic way to define the grading on $K^0_{top}(T) \otimes \mathbb{Q}$ induced by the Chern character isomorphism. The graded piece of (the associated graded of) $K^0_{top}(T) \otimes \mathbb{Q}$ corresponding to $H^{2n}(T, \mathbb{Q})$ consists of those classes x for which $\psi^k(x) = k^n x$ for some (or all) $k > 0$.

Here is a short list of famous theorems of Adams using topological K-theory and Adams operations:

Application 3.22. *Adams used his operations in topological K-theory to solve fundamental problems in algebraic topology. Examples include:*

- *Determination of the number of linearly independent vector fields on the n-sphere S^n for all $n > 1$.*
- *Determination of the only dimensions (namely, $n = 1, 2, 4, 8$) for which $\mathbb{R}^n$ admits the structure of a division algebra. (The examples of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions, and the Cayley numbers gives us structures in these dimensions.)*
- *Determination of those (now well understood) elements of the homotopy groups of spheres associated with $KO^0_{top}(S^n)$.*

4 Algebraic K-theory and Algebraic Geometry

4.1 Schemes

Although our primary interest will be in the K-theory of smooth, quasi-projective algebraic varieties, for completeness we briefly recall the more general context of schemes. (A good basic reference is [3].) A quasi-projective variety corresponds to a globalization of a finitely generated commutative algebra over a field; a scheme similarly corresponds to the globalization of a general commutative ring.

Recall that if A is a commutative ring we denote by $\operatorname{Spec} A$ the set of prime ideals of A. The set $X = SpecA$ is provided with a topology, the **Zariski topology** defined as follows: a subset $Y \subset X$ is closed if and only if there exists some ideal $I \subset A$ such that $Y = \{p \in X; I \subset p\}$. We define the **structure sheaf** $\mathcal{O}_X$ of commutative rings on $X = \operatorname{Spec} A$ by specifying its value on the basic open set $X_f = \{p \in SpecA, f \notin p\}$ for some $f \in A$ to be the ring A_f obtained from A by adjoining the inverse to f. (Recall that $A \to A_f$ sends to 0 any element $a \in A$ such that $f^n \cdot a = 0$ for some n).

We now use the sheaf axiom to determine the value of $\mathcal{O}_X$ on any arbitrary open set $U \subset X$, for any such U is a finite union of basic open subsets. The stalk $\mathcal{O}_{X,p}$ of the structure sheaf at a prime ideal $p \subset A$ is easily computed to be the local ring $A_p = \{f \notin p\}^{-1}A$.

Thus, $(X = \operatorname{Spec} A, \mathcal{O}_X)$ has the structure of a **local ringed space**: a topological space with a sheaf of commutative rings each of whose stalks is a local ring. A map of local ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \to Y$ of topological spaces and a map of sheaves $O_Y \to f_*O_X$ on Y, where $f_*O_X(V) = O_X(f^{-1}(V))$ for any open $V \subset Y$.

If M is an A-module for a commutative ring A, then M defines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$. Namely, for each basic open subset $X_f \subset X$, we define $\tilde{M}(X_f) \equiv A_f \otimes_A M$. This is easily seen to determine a sheaf of abelian groups on X with the additional property that for every open $U \subset X$, $\tilde{M}(U)$ is a sheaf of $\mathcal{O}_X(U)$-modules with structure compatible with restriction to smaller open subsets $U' \subset U$.

Definition 4.1. A local ringed space $(X, \mathcal{O}_X)$ is said to be an **affine scheme** if it is isomorphic (as local ringed spaces) to $(X = \operatorname{Spec} A, \mathcal{O}_X)$ as defined above. A **scheme** $(X, \mathcal{O}_X)$ is a local ringed space for which there exists a finite open covering $\{U_i\}_{i \in I}$ of X such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine scheme.

If k is a field, a k-**variety** is a scheme $(X, \mathcal{O}_X)$ with the property there is a finite open covering $\{U_i\}_{i \in I}$ by affine schemes with the property that each $(U_i, \mathcal{O}_{X|U_i}) \simeq (SpecA_i, \mathcal{O}_{SpecA_i})$ with A_i a finitely generated k-algebra without nilpotents. The $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$ are affine varieties admitting a locally closed embedding in $\mathbb{P}^N$, where $N + 1$ is the cardinality of some set of generators of A_i over k.

Example 4.2. The scheme $\mathbb{P}^1_{\mathbb{Z}}$ is a non-affine scheme defined by patching together two copies of the affine scheme $Spec\mathbb{Z}[t]$. So $\mathbb{P}^1_{\mathbb{Z}}$ has a covering $\{U_1, U_2\}$ corresponding to rings $A_1 = \mathbb{Z}[u], A_2 = \mathbb{Z}[v]$. These are "patched together" by identifying the open subschemes $Spec(A_1)_u \subset SpecA_1$, $Spec(A_2)_v \subset SpecA_2$ via the isomorphism of rings $(A_1)_u \simeq (A_2)_v$ which sends u to v^{-1}.

Note that we have used $SpecR$ to denote the local ringed space ($SpecR$, $\mathcal{O}_{SpecR}$); we will continue to use this abbreviated notation.

Definition 4.3. Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $\mathcal{V}ect(X)$ the exact category of sheaves F of $\mathcal{O}_X$-modules with the property that there exists an open covering $\{U_i\}$ of X by affine schemes $U_i = SpecA_i$ and free, finitely generated A_i-modules M_i such that the restriction $F_{|U_i}$ of F to U_i is isomorphic to the sheaf $\tilde{M}_i$ on $Spec A_i$. In other words, $\mathcal{V}ect(X)$ is the exact category of coherent, locally free $\mathcal{O}_X$-modules (i.e., of vector bundles over X).

We define the algebraic K-theory of the scheme X by setting

$$K_*(X) = K_*(\mathcal{V}ect(X)).$$

4.2 Algebraic cycles

For simplicity, we shall typically restrict our attention to quasi-projective varieties. In some sense, the most intrinsic objects associated to an algebraic variety are the (algebraic) vector bundles $E \to X$ and the algebraic cycles $Z \subset X$ on X. As we shall see, these are closely related.

Definition 4.4. Let X be a scheme. An *algebraic r-cycle* on X if a formal sum

$$\sum_Y n_Y[Y], \quad Y \text{ irreducible of dimension } r, \quad n_Y \in \mathbb{Z}$$

with all but finitely many n_Y equal to 0.

Equivalently, an algebraic r-cycle is a finite integer combination of (not necessarily closed) points of X of dimension r. (This is a good definition even for X a quite general scheme.)

If $Y \subset X$ is a reduced subscheme each of whose irreducible components $Y_1, \ldots, Y_m$ is r-dimensional, then the algebraic r-cycle

$$Z = \sum_{i=1}^{m} [Y_i]$$

is called the *cycle associated* to Y.

The group of (algebraic) r-cycles on X will be denoted $Z_r(X)$.

For example, if X is an integral variety of dimension d (i.e., the field of fractions of X has transcendence d over k), then a Weil divisor is an algebraic $d-1$-cycle. In the following definition, we extend to r-cycles the equivalence relation we impose on locally principal divisor when we consider these modulo principal divisors. As motivation, observe that if C is a smooth curve and $f \in frac(C)$, then f determines a morphism $f : C \to \mathbb{P}^1$ and

$$(f) = f^{-1}(0) - f^{-1}(\infty),$$

where $f^{-1}(0), f^{-1}(\infty)$ are the scheme-theoretic fibres of f above $0, \infty$.

Definition 4.5. Two r-cycles Z, Z' on a quasi-projective variety X are said to be *rationally equivalent* if there exist algebraic $r+1$-cycles $W_0, \dots, W_n$ on $X \times \mathbb{P}^1$ for some $n > 0$ with the property that each component of each W_i projects onto an open subvariety of $\mathbb{P}^1$ and that $Z = W_0[0], Z' = W_n[\infty]$, and $W_i[\infty] = W_{i+1}[0]$ for $0 \leq i < n$. Here, $W_i[0]$ (respectively, $W_i[\infty]$ denotes the cycle associated to the scheme theoretic fibre above $0 \in \mathbb{P}^1$ (resp., $\infty \in \mathbb{P}^1$) of the restriction of the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ to (the components of) W_i.

The *Chow group* $CH_r(X)$ is the group of r-cycles modulo rational equivalence.

Observe that in the above definition we can replace the role of $r+1$-cycles on $X \times \mathbb{P}^1$ and their geometric fibres over $0, \infty$ by $r+1$-cycles on $X \times U$ for any non-empty Zariski open $U \subset X$ and geometric fibres over any two k-rational points $p, q \in U$.

Remark 4.6. Given some $r+1$ dimensional irreducible subvariety $V \subset X$ together with some $f \in k(V)$, we may define $(f) = \sum_S ord_S(f)[S]$ where S runs through the codimension 1 irreducible subvarieties of V. Here, $ord_S(-)$ is the valuation centered on S if V is regular at the codimension 1 point corresponding to S; more generally, $ord_S(f)$ is defined to be the length of the $O_{V,S}$-module $O_{V,S}/(f)$.

We readily check that (f) is rationally equivalent to 0: namely, we associate to (V, f) the closure $W = \Gamma_f \subset X \times \mathbb{P}^1$ of the graph of the rational map $V \dashrightarrow \mathbb{P}^1$ determined by f. Then $(f) = W[0] - W[\infty]$.

Conversely, given an $r+1$-dimensional irreducible subvariety W on $X \times \mathbb{P}^1$ which maps onto $\mathbb{P}^1$, the composition $W \subset X \times \mathbb{P}^1 \stackrel{pr_2}{\to} \mathbb{P}^1$ determines $f \in frac(W)$ such that

$$(f) \;=\; W[0] - W[\infty].$$

Thus, the definition of rational equivalence on r-cycles of X can be given in terms of the equivalence relation generated by

$$\{(f), f \in frac(W); W \text{ irreducible of dimension } r+1\}$$

In particular, we conclude that the subgroup of principal divisors inside the group of all locally principal divisors consists precisely of those locally principal divisors which are rationally equivalent to 0.

The reader is referred to the beginning of [20] for a discussion of algebraic cycles and equivalence relations on cycles.

4.3 Chow Groups

One should view $CH_*(X)$ as a homology/cohomology theory. These groups are covariantly functorial for proper maps $f : X \to Y$ and contravariantly functorial for flat maps $W \to X$, so that they might best be viewed as some sort of Borel-Moore homology theory.

Construction 1. Assume that X is integral and regular in codimension 1. Let $\mathcal{L} \in Pic(X)$ be a locally free sheaf of rank 1 (i.e., a "line bundle" or "invertible sheaf") and assume that $\Gamma(\mathcal{L}) \neq 0$. Then any $0 \neq s \in \Gamma(\mathcal{L})$ determines a well defined locally principal divisor on X, $Z(s) \subset X$. Namely, if $\mathcal{L}_{|U} \simeq \mathcal{O}_{X|U}$ is trivial when restricted to some open $U \subset X$, then $s_U \in \mathcal{L}(U)$ determines an element of $\mathcal{O}_X(U)$ well defined up to a unit in $\mathcal{O}_X(U)$ (i.e., an element of $\mathcal{O}_X^*(U)$) so that the valuation $v_x(s)$ is well defined for every $x \in U^{(1)}$. We define $Z(s)$ by the property that $Z(s)_U = (s_U)_{|U}$ for any open $U \subset X$ restricted to which $\mathcal{L}$ is trivial, and where (s_U) denotes the divisor of an element of $\mathcal{O}_X(U)$ corresponding to s_U under any $(\mathcal{O}_X)_{|U}$ isomorphism $\mathcal{L}_{|U} \simeq (\mathcal{O}_X)_{|U}$.

Theorem 4.7. *(cf. [3]) Assume that X is an integral variety regular in codimension 1. Let $\mathcal{D}(X)$ denote the group of locally principal divisors on X modulo principal divisors. Then the above construction determines a well defined isomorphism*

$$Pic(X) \;\simeq\; \mathcal{D}(X).$$

Moreover, if $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, then $D(X)$ equals the group $CH^1(X)$ of codimension 1 cycles modulo rational equivalence.

Proof. If $s, s' \in \Gamma(\mathcal{L})$ are non-zero global sections, then there exists some $f \in K = frac(\mathcal{O}_X)$ such that with respect to any trivialization of $\mathcal{L}$ on some open covering $\{U_i \subset X\}$ of X the quotient of the images of s, s' in $\mathcal{O}_X(U_i)$ equals f. A line bundle $\mathcal{L}$ is trivial if and only if it is isomorphic to $\mathcal{O}_X$ which is the case if and only if it has a global section $s \in \Gamma(X)$ which never vanishes if and only if $(s) = 0$. If $\mathcal{L}_1, \mathcal{L}_2$ are two such line bundles with non-zero global sections s_1, s_2, then $(s_1 \otimes s_2) = (s_1) + (s_2)$.

Thus, the map is a well defined homomorphism on the monoid of those line bundles with a non-zero global section. By Serre's theorem concerning coherent sheaves generated by global sections, for any line bundle $\mathcal{L}$ there exists a positive integer n such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is generated by global sections (and in particular, has non-zero global sections), where we have implicitly chosen a locally closed embedding $X \subset \mathbb{P}^M$ and taken $\mathcal{O}_X(n)$ to be the pull-back via this embedding of $\mathcal{O}_{\mathbb{P}^M}(n)$. Thus, we can send such an $\mathcal{L} \in Pic(X)$ to $(s) - (w)$, where $s \in \Gamma(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$ and $w \in \Gamma(\mathcal{O}_X(n))$.

The fact that $Pic(X) \to \mathcal{D}(X)$ is an isomorphism is an exercise in unravelling the formulation of the definition of line bundle in terms of local data.

Recall that a domain A is a unique factorization domain if and only every prime of height 1 is principal. Whenever $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, every codimension 1 subvariety $Y \subset X$ is thus locally principal, so that the natural inclusion $D(X) \subset CH^1(X)$ is an equality. □

Remark 4.8. This is a first example of relating bundles to cycles, and moreover a first example of duality. Namely, $Pic(X)$ is the group of rank 1 vector bundles; the group $CH^1(X)$ of is a group of cycles. Moreover, $Pic(X)$ is contravariant with respect X whereas $Z^1(X)$ is covariant with respect to equidimensional maps. To relate the two as in the above theorem, some smoothness conditions are required.

Example 4.9. Let $X = \mathbb{A}^N$. Then any $N-1$-cycle (i.e., Weil divisor) $Z \in CH_{N-1}(\mathbb{A}^N)$ is principal, so that $CH_{N-1}(\mathbb{A}^N) = 0$.

More generally, consider the map $\mu : \mathbb{A}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ which sends $(x_1, \dots, x_n), t$ to $\langle t \cdot x_1, \dots, t \cdot x_n, 1\rangle, t$. Consider an irreducible subvariety $Z \subset \mathbb{A}^N$ of dimension $r > N$ not containing the origin and $\overline{Z} \subset \mathbb{P}^N$ be its closure. Let $W = \mu^{-1}(\overline{Z} \times \mathbb{A}^1)$. Then $W[0] = \emptyset$ whereas $W[1] = Z$. Thus, $CH_r(\mathbb{A}^N) = 0$ for any $r < N$.

Example 4.10. Arguing in a similar geometric fashion, we see that the inclusion of a linear plane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ induces an isomorphism $CH_r(\mathbb{P}^{N-1}) = CH_r(\mathbb{P}^N)$ provided that $r < N$ and thus we conclude by induction that $CH_r(\mathbb{P}^N) = \mathbb{Z}$ if $r \leq N$. Namely, consider $\mu : \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ sending $\langle x_0, \dots, x_N\rangle, t$ to $\langle x_), \dots, x_{N-1}, t \cdot x_N\rangle, t$ and set $W = \mu^{-1}(Z \times \mathbb{A}^1)$ for any Z not containing $\langle 0, \dots, 0, 1\rangle$. Then $W[0] = pr_{N*}(Z), W[1] = Z$.

Example 4.11. Let C be a smooth curve. Then $Pic(C) \simeq CH_0(X)$.

Definition 4.12. If $f : X \to Y$ is a proper map of quasi-projective varieties, then the proper push-forward of cycles determines a well defined homomorphism

$$f_* : CH_r(X) \ \to \ CH_r(Y), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to $d \cdot [f(Z)] \in CH_r(Y)$ where $[k(Z) : k(f(Z))] = d$ if $dim\ Z = dim\ f(Z)$ and is sent to 0 otherwise.

If $g : W \to X$ is a flat map of quasi-projective varieties of relative dimension e, then the flat pull-back of cycles induces a well defined homomorphism

$$g^* : CH_r(X) \ \to \ CH_{r+e}(W), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to the cycle on W associated to $Z \times_X W \subset W$.

Proposition 4.13. *Let Y be a closed subvariety of X and let $U = X \backslash Y$. Let $i : Y \to X, j : U \to X$ be the inclusions. Then the sequence*

$$CH_r(Y) \stackrel{i_*}{\to} CH_r(X) \stackrel{j^*}{\to} CH_r(U) \to 0$$

is exact for any $r \geq 0$.

Proof. If $V \subset U$ is an irreducible subvariety of U of dimension r, then the closure of V in X, $\overline{V} \subset X$, is an irreducible subvariety of X of dimension r with the property that $j^*([\overline{V}]) = [V]$. Thus, we have an exact sequence

$$Z_r(Y) \stackrel{i_*}{\to} Z_r(X) \stackrel{j^*}{\to} Z_r(U) \to 0.$$

If $Z = \sum_i n_i[Y_i]$ is a cycle on X with $j^*(Z) = 0 \in CH_r(U)$, then $j^*Z = \sum_{W,f}(f)$ where each $W \subset U$ is an irreducible subvarieties of U of dimension $r+1$ and $f \in k(W)$. Thus, $Z' = \sum_i n_i[\overline{Y}_i] - \sum_{\overline{W},f}(f)$ is an r-cycle on Y with the property that $i_*(Z')$ is rationally equivalent to Z. Exactness of the asserted sequence of Chow groups is now clear. □

Corollary 4.14. *Let $H \subset \mathbb{P}^N$ be a hypersurface of degree d. Then $CH_{N-1}(\mathbb{P}^N \backslash H) = \mathbb{Z}/d\mathbb{Z}$.*

The following "examples" presuppose an understanding of "smoothness" briefly discussed in the next section.

Example 4.15. Mumford shows that if S is a projective smooth surface with a non-zero global algebraic 2-form (i.e., $H^0(S, \Lambda^2(\Omega_S)) \neq 0$), then $CH_0(S)$ is *not finite dimensional* (i.e., must be very large).

Bloch's Conjecture predicts that if S is a projective, smooth surface with geometric genus equal to 0 (i.e., $H^0(S, \Lambda^2(\Omega_S)) = 0$), then the natural map from $CH_0(S)$ to the (finite dimensional) Albanese variety is injective.

4.4 Smooth Varieties

We restrict our attention to quasi-projective varieties over a field k.

Definition 4.16. A quasi-projective variety X is smooth of dimension n at some point $x \in X$ if there exists an open neighborhood $x \in U \subset X$ and k polynomials $f_1, \ldots, f_k$ in $n+k$ variables (viewed as regular functions on $\mathbb{A}^{n+k}$) vanishing at $0 \in \mathbb{A}^{n+k}$ with Jacobian $|\frac{\partial f_i}{\partial x_j}|(0)$ of rank k and an isomorphism of U with $Z(f_1, \ldots, f_k) \subset \mathbb{A}^{n+k}$ sending x to 0.

In more algebraic terms, a point $x \in X$ is smooth if there exists an open neighborhood $x \in U \subset X$ and a map $p : U \to \mathbb{A}^n$ sending x to 0 which is flat and unramified at x.

Definition 4.17. Let X be a quasi-projective variety. Then $K'_0(X)$ is the Grothendieck group of isomorphism classes of coherent sheaves on X, where the equivalence relation is generated pairs $([\mathcal{E}], [\mathcal{E}_1] + [\mathcal{E}_2])$ for short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of $\mathcal{O}_X$-modules.

Example 4.18. Let $A = k[x]/x^2$. Consider the short exact sequence of A-modules

$$0 \to k \to A \to k \to 0$$

where k is an A-module via the augmentation map (i.e., x acts as multiplication by 0), where the first map sends $a \in k$ to $ax \in A$, and the second map sends x to 0. We conclude that the class $[A]$ of the rank 1 free module equals $2[k]$.

On the other hand, because A is a local ring, $K_0(A) = \mathbb{Z}$, generated by the class $[A]$. Thus, the natural map $K_0(\operatorname{Spec} A) \to K'_0(\operatorname{Spec} A)$ is not surjective. The map is, however, injective, as can be seen by observing that $dim_k(-) : K'_0(\operatorname{Spec} A) \to \mathbb{Z}$ is well defined.

Theorem 4.19. *If X is smooth, then the natural map $K_0(X) \to K'_0(X)$ is an isomorphism.*

Proof. Smoothness implies that every coherent sheaf has a finite resolution by vector bundles, This enables us to define a map

$$K'_0(X) \ \to \ K_0(X)$$

by sending a coherent sheaf $\mathcal{F}$ to the alternating sum $\Sigma_{i=1}^N (-1)^i \mathcal{E}_i$, where $0 \to \mathcal{E}_N \to \cdots \mathcal{E}_0 \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ by vector bundles.

Injectivity follows from the observation that the composition

$$K_0(X) \ \rightarrow \ K_0'(X) \ \rightarrow \ K_0(X)$$

is the identity. Surjectivity follows from the observation that $\mathcal{F} = \Sigma_{i=1}^N (-1)^i \mathcal{E}_i$ so that the composition

$$K_0'(X) \ \rightarrow \ K_0(X) \ \rightarrow \ K_0'(X)$$

is also the identity. □

Perhaps the most important consequence of this is the following observation. Grothendieck explained to us how we can make $K_0'(-)$ a *covariant* functor with respect to proper maps. (Every morphism between projective varieties is proper.) Consequently, restricted to smooth schemes, $K_0(-)$ is not only a contravariant functor but also a covariant functor for proper maps.

"Chow's Moving Lemma" is used to give a ring structure on $CH^*(X)$ on smooth varieties as made explicit in the following theorem. The role of the moving lemma is to verify for an r-cycle Z on X and an s-cycle W on X that Z can be moved within its rational equivalence class to some Z' such that Z' meets W "properly". This means that the intersection of any irreducible component of Z' with any irreducible component of W is either empty or of codimension $d - r - s$, where $d = dim(X)$.

Theorem 4.20. *Let X be a smooth quasi-projective variety of dimension d. Then there exists a pairing*

$$CH_r(X) \otimes CH_s(X) \stackrel{\bullet}{\rightarrow} CH_{d-r-s}(X), \quad d \geq r + s,$$

with the property that if $Z = [Y], Z' = [W]$ are irreducible cycles of dimension r, s respectively and if $Y \cap W$ has dimension $\leq d - r - s$, then $Z \bullet Z'$ is a cycle which is a sum with positive coefficients (determined by local data) indexed by the irreducible subvarieties of $Y \cap W$ of dimension $d - r - s$.

Write $CH^s(X)$ for $CH_{d-s}(X)$. With this indexing convention, the intersection pairing has the form

$$CH^s(X) \otimes CH^t(X) \ \stackrel{\bullet}{\rightarrow} \ CH^{s+t}(X).$$

Proof. Classically, this was proved by showing the following geometric fact: given a codimension r cycle Z and a codimension s cycle $W = \sum_j m_j R_j$ with $r + s \leq d$, then there is another codimension r cycle $Z' = \sum_i n_i Y_i$

rationally equivalent to Z (i.e., determining the same element in $CH^r(X)$) such that Z' meets W "properly"; in other words, every component $C_{i,j,k}$ of each $Y_i \cap R_j$ has codimension $r+s$. One then defines

$$Z' \bullet W = \sum_{i,j,k} n_i m_j \cdot int(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}$$

where $int(Y_i \cap R_j, C_{i,j,k})$ is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a Z'' rationally equivalent to both Z, Z' and also intersecting W properly, then $Z' \bullet W$ is rationally equivalent to $Z" \bullet W$.

A completely different proof is given by William Fulton and Robert MacPherson (cf. [20]). They use a powerful geometric technique discovered by MacPherson called *deformation to the normal cone*. For $Y \subset X$ closed, the deformation space $M_Y(X)$ is a variety mapping to $\mathbb{P}^1$ defined as the complement in the *blow-up* of $X \times \mathbb{P}^1$ along $Y \times \infty$ of the blow-up of $X \times \infty$ along $Y \times \infty$. One readily verifies that $Y \times \mathbb{P}^1 \subset M(X, Y)$ restricts above $\infty \neq p \in \mathbb{P}^1$ to the given embedding $Y \subset X$; and above ∞, restricts to the inclusion of Y into the normal cone $C_Y(X) = Spec(\oplus_{n \geq 0} \mathcal{I}_Y^n / \mathcal{I}_Y^{n_1})$, where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf defining $Y \subset X$. When $Y \subset X$ is a regular closed embedding, then this normal cone is a bundle, the normal bundle $N_Y(X)$.

This enables a regular closed embedding (e.g., the diagonal $\delta : X \to X \times X$ for X smooth) to be deformed into the embedding of the 0-section of the normal bundle $N_{\delta(X)}(X \times X)$. One defines the intersection of Z, W as the intersection of $\delta(X), Z \times W$ and thus one reduces the problem of defining intersection product to the special case of intersection of the 0-section of the normal bundle $N_X(X \times X)$ with the normal cone $N_{(Z \times W) \cap \delta(X)}(Z \times W)$. $\square$

4.5 Chern classes and Chern character

The following construction of Chern classes is due to Grothendieck (cf. [19]); it applies equally well to topological vector bundles (in which case the Chern classes of a topological vector bundle over a topological space T are elements of the singular cohomology of T).

If $\mathcal{E}$ is a rank $r+1$ vector bundle on a quasi-projective variety X, we define $\mathbb{P}(\mathcal{E}) = \text{Proj}\ (Sym_{O_X} \mathcal{E}) \to X$ to be the projective bundle of lines in $\mathcal{E}$. Then $\mathbb{P}(\mathcal{E})$ comes equipped with a canonical line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$; for X a point, $\mathbb{P}(\mathcal{E}) = \mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}^r}(1)$.

Construction 2. Let $\mathcal{E}$ be a rank r vector bundle on a smooth, quasi-projective variety X of dimension d. Then $CH^*(\mathbb{P}(\mathcal{E}))$ is a free module over $CH^*(X)$ with generators $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$, where $\zeta \in CH^1(\mathbb{P}(\mathcal{E}))$ denotes the divisor class associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. .

We define the i-th Chern class $c_i(\mathcal{E}) \in CH^i(X)$ of $\mathcal{E}$ by the formula

$$CH^*(\mathbb{P}(\mathcal{E})) \;=\; CH^*(X)[\zeta]/\sum_{i=0}^{r}(-1)^i\pi^*(c_i(\mathcal{E}))\cdot\zeta^{r-i}.$$

We define the total Chern class $c(\mathcal{E})$ by the formula

$$c(\mathcal{E}) \;=\; \sum_{i=0}^{r} c_i(\mathcal{E})$$

and set $c_t(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E})t^i$. Then the *Whitney sum* formula asserts that $c_t(\mathcal{E}\oplus\mathcal{F}) = c_t(\mathcal{E})\cdot c_t(\mathcal{F})$.

We define the *Chern roots*, $\alpha_1, \ldots, \alpha_r$ of $\mathcal{E}$ by the formula

$$c_t(\mathcal{E}) \;=\; \prod_{i=1}^{r}(1+\alpha_i t)$$

where the factorization can be viewed either as purely formal or as occurring in $\mathbb{F}(\mathcal{E})$. Observe that $c_k(\mathcal{E})$ is the k-th elementary symmetric function of these Chern roots.

In other words, the Chern classes of the rank r vector bundle $\mathcal{E}$ are given by the expression for $\zeta^r \in CH^r(\mathbb{P}(\mathcal{E}))$ in terms of the generators $1, \zeta, \ldots, \zeta^{r-1}$. Thus, the Chern classes depend critically on the identification of the first Chern class ζ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the multiplicative structure on $CH^*(X)$. The necessary structure for such a definition of Chern classes is called an *oriented multiplicative cohomology theory.* The splitting principle guarantees that Chern classes are uniquely determined by the assignment of first Chern classes to line bundles.

Grothendieck introduced many basic techniques which we now use as a matter of course when working with bundles. The following *splitting principle* is one such technique, a technique which enable one to frequently reduce constructions for arbitrary vector bundles to those which are a sum of line bundles.

Proposition 4.21. *(Splitting Principle) Let $\mathcal{E}$ be a rank $r+1$ vector bundle on a quasi-projective variety X. Then $p_1^* : CH_*(X) \to CH_{*+r}(\mathbb{P}(\mathcal{E}))$ is split injective and $p_1^*(\mathcal{E}) = \mathcal{E}_1$ is a direct sum of a rank r bundle and a line bundle.*

Applying this construction to $\mathcal{E}_1$ on $\mathbb{P}(\mathcal{E})$, we obtain $p_2 : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$; proceeding inductively, we obtain

$$p = p_r \circ \cdots \circ p_1 : \mathbb{F}(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{r-1}) \to X$$

with the property that $p^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$ is split injective and $p^*(\mathcal{E})$ is a direct sum of line bundles.

One application of the preceding proposition is the following definition (due to Grothendieck) of the Chern character.

Construction 3. Let X be a smooth, quasi-projective variety, let $\mathcal{E}$ be a rank r vector bundle over X, and let $\pi : \mathbb{F}(\mathcal{E}) \to X$ be the associated bundle of flags of $\mathcal{E}$. Write $\pi^*(\mathcal{E}) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where each $\mathcal{L}_i$ is a line bundle on $\mathbb{F}(\mathcal{E})$. Then $c_t(\pi^*(\mathcal{E})) = \prod_{i=1}^r (1 \oplus c_1(\mathcal{L}_i))t$.

We define the Chern character of $\mathcal{E}$ as

$$ch(\mathcal{E}) = \sum_{i=1}^r \{1 + c_1(\mathcal{L}_i) + \frac{1}{2} c_1(\mathcal{L}_i)^2 + \frac{1}{3!} c_1(\mathcal{L}_i)^3 + \cdots\} = \sum_{i=1}^r exp(c_t(\mathcal{L}_i)),$$

where this expression is verified to lie in the image of the injective map $CH^*(X) \otimes \mathbb{Q} \to CH^*(\mathbb{F}(\mathcal{E})) \otimes \mathbb{Q}$. (Namely, one can identify $ch_k(\mathcal{E})$ as the k-th power sum of the Chern roots, and therefore expressible in terms of the Chern classes using Newton polynomials.)

Since $\pi^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$, $\pi^* : CH^*(X) \to CH^*(\mathbb{F}(\mathcal{E}))$ are ring homomorphisms, the splitting principle enables us to immediately verify that ch is also a ring homomorphism (i.e., sends the direct sum of bundles to the sum in $CH^*(X)$ of Chern characters, sends the tensor product of bundles to the product in $CH^*(X)$ of Chern characters).

4.6 Riemann-Roch

Grothendieck's formulation of the Riemann-Roch theorem is an assertion of the behaviour of the Chern character ch with respect to push-forward maps induced by a proper smooth map $f : X \to Y$ of smooth varieties. It is not the case that ch commutes with the these push-forward maps; one must modify the push forward map in K-theory by multiplication by the Todd class.

This modification by multiplication by the Todd class is necessary even when consideration of the push-forward of a divisor. Indeed, the Todd class

$$td : K_0(X) \to CH^*(X)$$

is characterized by the properties that

i. $td(L) \;=\; c_1(L)/(1 - exp(-c_1(L)) \;=\; 1 + \frac{1}{2}c_1(L) + \cdots;$

ii. $td(E_1 \oplus E_2) = td(E_1) \cdot td(E_2)$; and

iii. $td \circ f^* \;=\; f^* \circ td.$

The reader is recommended to consult [19] for an excellent exposition of Grothendieck's Riemann-Roch Theorem.

Theorem 4.22. *(Grothendieck's Riemann-Roch Theorem)*

Let $f : X \to Y$ be a projective map of smooth varieties. Then for any $x \in K_0(X)$, we have the equality

$$ch(f_!(x)) \cdot td(T_Y) \;=\; f_*(ch(x) \cdot td(T_X))$$

where T_X, T_Y are the tangent bundles of X, Y and $td(T_X), td(T_Y)$ are their Todd classes.

Here, $f_! : K_0(X) \to K_0(Y)$ is defined by identifying $K_0(X)$ with $K_0'(X)$, $K_0(Y)$ with $K_0'(Y)$, and defining $f_! : K_0'(X) \to K_0'(Y)$ by sending a coherent sheaf $\mathcal{F}$ on X to $\sum_i (-1)^i R^i f_*(F)$. The map $f_* : CH_*(X) \to CH_*(Y)$ is proper push-forward of cycles.

Just to make this more concrete and more familiar, let us consider a very special case in which X is a projective, smooth curve, Y is a point, and $x \in K_0(X)$ is the class of a line bundle $\mathcal{L}$. (Hirzebruch had earlier proved a version of Grothendieck's theorem in which the target Y was a point.)

Example 4.23. Let C be a projective, smooth curve of genus g and let $f : C \to Spec\mathbb{C}$ be the projection to a point. Let $\mathcal{L}$ be a line bundle on C with first Chern class $D \in CH^1(C)$. Then

$$f_!([\mathcal{L}]) = dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) \in \mathbb{Z},$$

and $ch : K_0(Spec\mathbb{C}) = \mathbb{Z} \to A^*(Spec\mathbb{C}) = \mathbb{Z}$ is an isomorphism. Let $K \in CH^1(C)$ denote the "canonical divisor", the first Chern class of the line bundle Ω_C, the dual of T_C. Then

$$td(T_C) = \frac{-K}{1-(1+K+\frac{1}{2}K^2)} = 1 - \frac{1}{2}K.$$

Recall that $deg(K) = 2g-2$. Since $ch([\mathcal{L}]) = 1 + D$, we conclude that

$$f_*(ch([\mathcal{L}]) \cdot td(T_C)) = f_*((1+D)\cdot(1-\frac{1}{2}K)) = deg(D) - \frac{1}{2}deg(K).$$

Thus, in this case, Riemann-Roch tell us that

$$dim\mathcal{L}(C) - dimH^1(C,\mathcal{L}) = deg(D) + 1 - g.$$

For our purpose, Riemann-Roch is especially important because of the following consequence.

Corollary 4.24. *Let X be a smooth quasi-projective variety. Then*

$$ch : K_0(X) \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$$

is a ring isomorphism.

Proof. The essential ingredient is the Riemann-Roch theorem. Namely, we have a natural map $CH^*(X) \to K_0'(X)$ sending an irreducible subvariety W to the $\mathcal{O}_X$-module $\mathcal{O}_W$. We show that the composition with the Chern character is an isomorphism by applying Riemann-Roch to the closed immersion $W \backslash W_{sing} \to X \backslash W_{sing}$. □

5 Some Difficult Problems

As we discuss in this lecture, many of the basic problems formulated years ago for algebraic K-theory remain unsolved. This remains a subject in which much exciting work remains to be done.

5.1 $K_*(\mathbb{Z})$

Unfortunately, there are few examples (rings or varieties) for which a complete computation of the K-groups is known. As we have seen earlier, one such complete computation is the K-theory of an arbitrary finite field,

$K_*(\mathbb{F}_q)$. Indeed, general theorems of Quillen give us the complete computations

$$K_*(\mathbb{F}_q[t]) \ = \ K_*(\mathbb{F}_q), \quad K_*(\mathbb{F}_q([t,t^{-1}]) \ = \ K_*(\mathbb{F}_q) \oplus K_{*-1}(\mathbb{F}_q).$$

Perhaps the first natural question which comes to mind is the following: "what is the K-theory of the integers."

In recent years, great advances have been made in computing $K_*(\mathcal{O}_K)$ of a ring of integers in a number field K (e.g., $\mathbb{Z}$ inside $\mathbb{Q}$).

- $K_0(\mathcal{O}_K) \otimes \mathbb{Q}$ is 1 dimensional by the finiteness of the class number of K (Minkowski).

- $K_1(\mathcal{O}_K) \otimes \mathbb{Q}$ has dimension $r_1 + r_2 - 1$, where $r_1,\ r_2$ are the numbers of real and complex embeddings of K. (Dirichlet).

- Quillen proved that $K_i(\mathcal{O}_K)$ is a finitely generated abelian group for any i.

- For $i > 1$, Borel determined

$$K_i(\mathcal{O}_K) \otimes \mathbb{Q} \ = \begin{cases} 0, \quad i \equiv 0\ (mod\ 4) \\ r_1 + r_2, \quad i \equiv 1\ (mod\ 4) \\ 0, \quad i \equiv 2\ (mod\ 4) \\ \ r_2, \quad i \equiv 3\ (mod\ 4) \end{cases} \tag{1}$$

  in terms of the numbers $r_1,\ r_2$.

- $K_*(O_K, \mathbb{Z}/2)$ has been computed by Rognes-Weibel as a corollary of Voevodsky's proof of the Milnor Conjecture.

- $K_*(\mathbb{Z}, \mathbb{Z}/p)$ follows in all degrees not divisible by 4 from the Bloch-Kato Conjecture, now seemingly proved by Rost and Voevodsky.

Here is a table of the values of $K_*(\mathbb{Z})$ whose likely inaccuracy is due to my confusion of indexing of Bernoulli numbers. Many more details can be found in [27].

Theorem 5.1. *The K-theory of $\mathbb{Z}$ is given by (according to Weibel's survey paper):*

$$\begin{cases} K_{8k} \;=\; ?0?, \qquad 0 < k \\ K_{8k+1} = \mathbb{Z} \oplus \mathbb{Z}/2, \quad 0 < k \\ K_{8k+2} \;=\; \mathbb{Z}/2c_{2k+1} \oplus \mathbb{Z}/2 \\ K_{8k+3} = \mathbb{Z}/2d_{4k+2}, \quad i \equiv 3 \\ K_{8k+4} \;=\; ?0? \\ K_{8k+5} \;=\; \mathbb{Z} \\ K_{8k+6} \;=\; \mathbb{Z}/c_{2k+2} \\ K_{8k+7} \;=\; Z/d_{4k+4} \end{cases} \tag{2}$$

Here, c_k/d_k is defined to be the reduced expression for $B_k/4k$, where B_k is the k-th Bernoulli number (defined by

$$\frac{t}{e^t-1} = 1 + \sum_{k=1}^{\infty} \frac{B_k}{(2k)!} t^{2k} \ .$$

Challenge 5.2. *Prove the vanishing of $K_{4i}(\mathbb{Z}),\ i > 0$.*

5.2 Bass Finiteness Conjecture

This is one of the most fundamental and oldest conjectures in algebraic K-theory. Very little progress has been made on this in the past 35 years.

Conjecture 5.3. *(Bass finiteness) Let A be a commutative ring which is finitely generated as an algebra over $\mathbb{Z}$. Is $K'_n(A)$ (i.e., the Quillen K-theory of mod(A)) finitely generated for all n?*

In particular, if A is regular as well as commutative and finitely generated over $\mathbb{Z}$, is each $K_n(A)$ a finitely generated abelian group?

This conjecture seems to be very difficult, even for $n = 0$. There are similar finiteness conjectures for the K-theory of projective varieties over finite fields.

Example 5.4. Here is an example of Bass showing that we must assume A is regular or consider $G_*(A)$. Let $A = \mathbb{Z}[x,y]/x^2$. Then the ideal (x) is infinitely additively generated by $x, xy, xy^2, \ldots$. On the other hand, if $t \in (x)$, then $1 + t \in A^*$, so that we see that $K_1(A)$ is not finitely generated.

Example 5.5. As pointed out by Bass, it is elementary to show (using general theorems of Quillen and Quillen's computation of the K-theory of finite fields) that if A is finite, then $G_n(A) \simeq G_n(A/radA)$ is finite for every $n \geq 0$. Subsequently, Kuku proved that $K_n(A)$ is also finite whenever A is finite (see [32]).

There are many other finiteness conjectures involving smooth schemes of finite type over a finite field, $\mathbb{Z}$ or $\mathbb{Q}$. Even partial solutions to these conjectures would represent great progress.

5.3 Milnor K-theory

We recall Milnor K-theory, a major concept in Professor Vishik's lectures. This theory is motivated by Matsumoto's presentation of $K_2(F)$ (mentioned in Lecture 1),

Definition 5.6. (Milnor) Let F be a field with multiplicative group of units $F^\times$. The Milnor K-group $K_n^{Milnor}(F)$ is defined to be the n-th graded piece of the graded ring defined as the tensor algebra $\bigoplus_{n\geq 0}(F^\times)^{\otimes n}$ modulo the ideal generated by elements $\{a, 1-a\} \in F^* \otimes F^*, a \neq 1 \neq 1-a$.

In particular, $K_1(F) = K_1^{Milnor}(F), K_2(F) = K_2^{Milnor}(F)$ for any field F, and $K_n^{Milnor}(F)$ is a quotient of $\Lambda^n(F^\times)$. For F an infinite field, Suslin in [24] proved that there are natural maps

$$K_n^{Milnor}(F) \to K_n(F) \to K_n^{Milnor}(F)$$

whose composition is $(-1)^{n-1}(n-1)!$. This immediately implies, for example, that the higher K-groups of a field of high transcendence degree are very large.

Remark 5.7. It is difficult to even briefly mention K_2 of fields without also mentioning the deep and import theorem of Mekurjev and Suslin [23]: for any field F and positive integer n,

$$K_2(F)/nK_2(F) \;\simeq\; H^2(F, \mu_n^{\otimes 2}).$$

In particular, $H^2(F, \mu_n^{\otimes 2})$ is generated by products of elements in $H^1(F, \mu_n) = \mu_n(F)$.

Moreover, if F contains the n^{th} roots of unity, then

$$K_2(F)/nK_2(F) \;\simeq\; {}_nBr(F),$$

where ${}_nBr(F)$ denotes the subgroup of the Brauer group of F consisting of elements which are n-torsion. In particular, ${}_nBr(F)$ is generated by "cyclic central simple algebras".

The most famous success of K-theory in recent years is the following theorem of Voevodsky [26], establishing a result conjectured by Milnor.

Theorem 5.8. *Let F be a field of characteristic $\neq 2$. Let $W(F)$ denote the Witt ring of F, the quotient of the Grothendieck group of symmetric inner product spaces modulo the ideal generated by the hyperbolic space $\langle 1\rangle \oplus \langle -1\rangle$ and let $I = ker\{W(F) \to \mathbb{Z}/2\}$ be given by sending a symmetric inner product space to its rank (modulo 2). Then the map*

$$K_n^{Milnor}(F)/2 \cdot K_n^{Milnor}(F) \to I^n/I^{n+1}, \quad \{a_1, \dots, a_n\} \mapsto \prod_{i=1}^{n}(\langle a_i\rangle - 1)$$

is an isomorphism for all $n \geq 0$. Here, $\langle a\rangle$ is the 1 dimensional symmetric inner product space with inner product $(-,-)_a$ defined by $(c,d)_a = acd$.

Suslin also proved the following theorem, the first confirmation of a series of conjectures which now seem to be on the verge of being settled.

Theorem 5.9. *Let F be an algebraic closed field. If F has characteristic 0 and $i > 0$, then $K_{2i}(F)$ is a $\mathbb{Q}$ vector space and $K_{2i-1}(F)$ is a direct sum of $\mathbb{Q}/\mathbb{Z}$ and a rational vector space. If F has characteristic $p > 0$ and $i > 0$, then $K_{2i}(F)$ is a $\mathbb{Q}$ vector space and $K_{2i-1}(F)$ is a direct sum of $\oplus_{\ell \neq p}\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a rational vector space.*

Question 5.10. *What information is reflected in the uncountable vector spaces $K_n(\mathbb{C}) \otimes \mathbb{Q}$? Are there interesting structures to be obtained from these vector spaces?*

5.4 Negative K-groups

Bass introduced *negative* algebraic K-groups, groups which vanish for regular rings or, more generally, smooth varieties. These negative K-groups measure the failure of K-theory in positive degree to obey "homotopy invariance" and "localization" (i.e.,

$$K_*(X) \stackrel{?}{=} K_*(X \times \mathbb{A}^1), \quad K_*(X) \oplus K_{*-1}(X) \stackrel{?}{=} K_*(X \times \mathbb{A}^1 \backslash \{0\}).$$

Very recently, there has been important progress in computing these negative K-groups by Cortinas, Haesemeyer, Schlicting, and Weibel.

Question 5.11. *Can negative K-groups give useful invariants for the geometric study of singularities?*

5.5 Algebraic versus topological vector bundles

Let X be a complex projective variety, and let X^{an} denote the topological space of complex points of X equipped with the analytic topology. Then any algebraic vector bundle $E \to X$ naturally determines a topological vector bundle $E^{an} \to X^{an}$. This determines a natural map

$$K_0(X) \ \to \ K^0_{top}(X^{an}).$$

Challenge 5.12. *Understand the kernel and image of the above map, especially after tensoring with $\mathbb{Q}$:*

$$CH^*(X) \otimes \mathbb{Q} \ \simeq \ K_0(X) \otimes \mathbb{Q} \ \to \ K^0_{top}(X^{an}) \otimes \ \simeq \ H^{ev}(X^{an}, \mathbb{Q}). \tag{3}$$

The kernel of (3) can be identified with the subspace of $CH^*(X) \otimes \mathbb{Q}$ consisting of rational equivalence classes of algebraic cycles on X which are homologically equivalent to 0.

The image of (3) can be identified with those classes in $H^*(X^{an}, \mathbb{Q})$ represented by algebraic cycles – the subject of the Hodge Conjecture!

In positive degree, the analogue of our map is uninteresting.

Proposition 5.13. *If X is a complex projective variety, then the natural map*

$$K_i(X) \otimes \mathbb{Q} \ \to \ K^{-i}_{top}(X^{an}), \quad i > 0$$

is the 0-map.

5.6 K-theory with finite coefficients

Although the map in positive degrees

$$K_i(X) \ \to \ K^{-i}_{top}(X^{an})$$

is typically of little interest, the situation changes drastically if we consider K-theory mod-n.

As an example, we give the following special case of a theorem of Suslin. Recall that $(\operatorname{Spec} \mathbb{C})^{an}$ is a point, which we denote by $\star$.

Theorem 5.14. *(cf. [25]) The map*

$$K_i(\operatorname{Spec}\mathbb{C}) \;\to\; K^{-i}_{top}(\star)$$

is the 0-map for $i > 0$. On the other hand, for any positive integer n and any integer $i \geq 0$, the map

$$K_i(\operatorname{Spec}\mathbb{C}, \mathbb{Z}/n) \;\to\; K^{-i}_{top}(\star, \mathbb{Z}/n)$$

is an isomorphism.

How can the preceding theorem be possibly correct? The point is that $K_{2i-1}(\operatorname{Spec}\mathbb{C})$ is a divisible group with torsion subgroup $\mathbb{Q}/\mathbb{Z}$. Then, we see that this $\mathbb{Q}/\mathbb{Z}$ in odd degree integral homotopy determines a $\mathbb{Z}/n$ in even degree mod-n homotopy. This is exactly what $K^{-*}_{top}(\star)$ determines in even mod-n homotopy degree.

The K-groups modulo n are defined to be the homotopy groups modulo n of the K-theory space (or spectrum).

Definition 5.15. For positive integers $i, n > 1$, let $M(i, \mathbb{Z}/n)$ denote the C.W. complex obtained by attaching an i-cell D^i to S^{i-1} via the map $\partial(D^i) = S^{i-1} \to S^{i-1}$ given by multiplication by n.

For any connected C.W. complex, we define

$$\pi_i(X, \mathbb{Z}/n) \equiv [M(i, \mathbb{Z}/n), X], \quad i, n > 1.$$

If $X = \Omega^2 Y$, we define

$$\pi_i(X, \mathbb{Z}/n) \equiv [M(i+2, \mathbb{Z}/n), Y], \quad i \geq 0, n > 1.$$

Since $S^{i-1} \to M(i, \mathbb{Z}/n)$ is the cone on the multiplication by n map $S^{i-1} \xrightarrow{n} S^{i-1}$, we have long exact sequences

$$\cdots \to \pi_i(X) \xrightarrow{n} \pi_i(X) \to \pi_i(X/\mathbb{Z}/n) \to \pi_{i-1}(X) \to \cdots$$

Perhaps this is sufficient to motivate our next conjecture, which we might call the Quillen-Lichtenbaum Conjecture for smooth complex algebraic varieties. The special case in which $X = \operatorname{Spec}\mathbb{C}$ is the theorem of Suslin quoted above.

Conjecture 5.16. *(Q-L for smooth **C** varieties) If X is a smooth complex variety of dimension d, then is the natural map*

$$K_i(X, \mathbb{Z}/n) \to K_i^{top}(X^{an}, \mathbb{Z}/n)$$

an isomorphism provided that $i \geq d - 1 \geq 0$?

Remark In "low" degrees, $K_*(X, \mathbb{Z}/n)$ should be more interesting and will not be periodic. For example, $K_{ev}^{\text{-}top}(X, \mathbb{Z}/n)$ has a contribution from the Brauer group of X whereas $K_0(X, \mathbb{Z}/n)$ does not.

5.7 Etale K-theory

It is natural to try to find a good "topological model" for the mod-n algebraic K-theory of varieties over fields other than the complex numbers. Suslin's Theorem in its full generality can be formulated as follows

Theorem 5.17. *If k is an algebraically closed field of characteristic $p \geq 0$, then there is a natural isomorphism*

$$K_*(k, \mathbb{Z}/n) \stackrel{\simeq}{\rightarrow} K_*^{et}(\operatorname{Spec} k, \mathbb{Z}/n), \quad (n, p) = 1.$$

Moreover, if the characteristic of k is a positive integer p, then $K_i(k, \mathbb{Z}/p) = 0$, for all $i > 0$.

We have stated the previous theorem in terms of *etale K-theory* although this is not the way Suslin formulated his theorem. We did this in order to introduce the etale topology, a Grothendieck topology associated to the etale site. For this site, the distinguished morphisms $\mathcal{E}$ are etale morphisms of schemes. A map of schemes $f : U \to V$ is said to be etale (or "smooth of relative dimension 0) if there exist affine open coverings $\{U_i\}$ of U, $\{V_j\}$ of V such that the restriction to U_i of f lies in some V_j and such that the corresponding map of commutative rings $A_i \leftarrow R_j$ is unramified (i.e., for all homomorphisms from R to a field k, $A \otimes_R k \leftarrow k$ is a finite separable k algebra) and flat.

The etale topology was introduced by Grothendieck partly to reinterpret Galois cohomology of fields and partly to algebraically realize singular cohomology of complex algebraic varieties. The following "comparison theorem" proved by Michael Artin and Alexander Grothendieck is an important property of the etale topology. (See, for example, [21].)

Theorem 5.18. *(Artin, Grothendieck) If X is a complex algebraic variety, then*

$$H_{et}^*(X, \mathbb{Z}/n) \simeq H_{sing}^*(X^{an}, \mathbb{Z}/n).$$

Here, $H_{et}^*(X, \mathbb{Z}/n)$ denotes the derived functors of the global section functor applied to the constant sheaf $\mathbb{Z}/n$ on the etale site.

The etale topology not only enables us to define etale cohomological groups, but also etale homotopy types. Using the etale homotopy type, etale K-theory (defined by Bill Dwyer and myself) can be defined in a manner similar to topological K-theory.

For this theory, there is an Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H_{et}^p(X, K_{et}^q(\star)) \Rightarrow K_{et}^{p+q}(X, \mathbb{Z}/n)$$

provided that $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}[1/n]$-modules. If we let μ_n denote the etale sheaf of n-th roots of unity and let $\mu_n^{\otimes q/2}$ denote $\mu_n^{\otimes j}$ if $q = 2j$ and 0 if j is odd, then this spectral sequence can be rewritten

$$E_2^{p,q} = H_{et}^p(X, \mu^{\otimes q/2}) \Rightarrow K_{q-p}^{et}(X, \mathbb{Z}/n).$$

Using etale K-theory, we can reformulate and generalize the Quillen-Lichtenbaum Conjecture (originally stated for $\operatorname{Spec} K$, where K is a number field), putting this conjecture in a quite general context.

Conjecture 5.19. *(Quillen-Lichtenbaum) Let X be a smooth scheme of finite type over a field k, and assume that n is a positive integer with $1/n$ in k or A. Then the natural map*

$$K_i(X, \mathbb{Z}/n) \to K_i^{et}(X, \mathbb{Z}/n)$$

is an isomorphism for $i-1$ greater or equal to the mod-n etale cohomological dimension of X.

This conjecture appears to be proven, or near-proven, thanks to the work of Rost and Voevodsky on the Bloch-Kato Conjecture.

5.8 Integral conjectures

There has been much progress in understanding K-theory with finite coefficients, but much less is known about the result of tensoring the algebraic K-groups with $\mathbb{Q}$.

The following theorem of Soulé is proved by investigating the group homology of general linear groups over fields. Soulé proves a vanishing theorem for more general rings R with a range depending upon the "stable range" of R.

Theorem 5.20. *(Soulé) For any field F,*

$$K_n(F)^{(s)}_{\mathbb{Q}} = 0, \quad s > n.$$

Here $K_n(F)^{(s)}_{\mathbb{Q}}$ is the s-eigenspace with respect to the action of the Adams operations on $K_n(F)$.

This motivates the following Beilinson-Soulé vanishing conjecture, part of the Beilinson Conjectures discussed in the next lecture. This conjecture is now known if we replace the coefficients $\mathbb{Z}(n)$ by their finite coefficients analogue $\mathbb{Z}/\ell(n)$.

Conjecture 5.21. *(Beilinson-Soulé) For any field F, the motivic cohomology groups $H^p(\operatorname{Spec} F, \mathbb{Z}(n))$ equal 0 for $p < 0$.*

Yet another auxillary K-theory has been developed to investigate K-theory of complex varieties, especially some aspects involving rational coefficients (cf. [22]).

Theorem 5.22. *(Friedlander-Walker) Let X be a complex quasi-projective variety. The map from the algebraic K-theory spectrum $\mathcal{K}(X)$ to the topological K-theory spectrum $\mathcal{K}_{top}(X^{an})$ factors through the "semi-topological K-theory spectrum $\mathcal{K}^{sst}(X)$.*

$$\mathcal{K}(X) \;\to\; \mathcal{K}^{sst}(X) \;\to\; \mathcal{K}_{top}(X^{an}).$$

The first map induces an isomorphism in homotopy groups modulo n, whereas the second map induces an isomorphism for certain special varieties and typically induces an isomorphism after "inverting the Bott element."

This semi-topological K-theory is related to cycles modulo algebraic equivalence is much the same way as usual algebraic K-theory is related to Chow groups (cycles modulo rational equivalence).

One important aspect of this semi-topological K-theory is that leads to conjectures which are "integral" whose reduction modulo n give the familiar Quillen-Lichtenbaum Conjecture.

We state one precise form of such a conjecture, essentially due to Suslin.

Conjecture 5.23. *Let X be a smooth, quasi-projective complex variety. Then the natural map*

$$K_i^{sst}(X) \;\to\; K_{top}^{-i}(X^{an})$$

is an isomorphism for $i \geq dim(X) - 1$ and a monomorphism for $i = dim(X) - 2$.

Now, we also have a "good semi-topological model" for the K-theory of quasi-projective varieties over $\mathbb{R}$, the real numbers. This is closely related to "Atiyah Real K-theory rather than the topological K-theory we have discussed at several points in these lectures.

Challenge 5.24. *Develop a semi-topological K-theory for varieties over an arbitrary field.*

5.9 K-theory and Quadratic Forms

another topic of considerable interest is *Hermetian K-theory* in which we take into account the presence of quadratic forms. Perhaps this topic is best left to Professor Vishik!

6 Beilinson's vision partially fulfilled

6.1 Motivation

In this lecture, we will discuss Alexander Beilinson's vision of what algebraic K-theory should be for smooth varieties over a field k (cf. [28], [30], and [31]). In particular, we will provide some account of progress towards the solution of these conjectures. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type using "motivic complexes" $\mathbf{Z}(n)$ which satisfy various good properties and whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory.

Although our goal is to describe conjectures which would begin to "explain" algebraic K-theory, let me start by mentioning one (of many) reasons why algebraic K-theory is so interesting to algebraic geometers (and algebraic number theorists). It has been known for some time that there can not be an algebraic theory whose values on complex algebraic varieties is integral (or even rational) singular homology of the associated analytic space. Indeed, Jean-Pierre Serre observed that this is not possible even for smooth projective algebraic curves because some such curves have automorphism groups which do not admit a representation which would be implied by functoriality. On the other hand, algebraic K-theory is in some sense integral – we define it without inverting residue characteristics or considering only mod-n coefficients. Thus, if we can formulate a sensible Atiyah-Hirzebruch type spectral sequence converging to algebraic K-theory, then the E_2-term offers an algebraic formulation of integral cohomology.

Before we launch into a discussion of Beilinson's Conjectures, let us recall two results relating algebraic cycles and algebraic K-theory which precede these conjectures.

The first is the theorem of Grothendieck mentioned earlier relating algebraic $K_0(X)$ to the Chow ring of algebraic cycles modulo algebraic equivalence.

Theorem 6.1. *If X is a smooth variety over a field k, then the Chern character determines an isomorphism*

$$ch : K_0(X) \otimes \mathbb{Q} \;\simeq\; CH^*(X) \otimes \mathbb{Q}.$$

The second is *Bloch's formula* proved in degree 2 by Bloch and in general by Quillen.

Theorem 6.2. *Let X be a smooth variety over a field and let $\underline{K}_i$ denote the Zariski sheaf associated to the presheaf $U \mapsto K_i(U)$ for an open subset $U \subset X$. Then there is a convergent spectral sequence of the form*

$$E_2^{p,q} \;=\; H^p_{Zar}(X, \underline{K}_q) \;\Rightarrow K_{q-p}(X).$$

6.2 Statement of conjectures

We now state Beilinson's conjectures and use these conjectures as a framework to discuss much interesting mathematics. It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored.

Conjecture 6.3. *(Beilinson's Conjectures) For each $n \geq 0$ there should be complexes $\mathbb{Z}(n), n \geq 0$ of sheaves on the Zariski site of smooth quasi-projective varieties over a field k, $(Sm/k)_{Zar}$ which satisfy the following:*

1. $\mathbb{Z}(0) = \mathbb{Z}, \quad \mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$.
2. $H^n(\operatorname{Spec} F, \mathbb{Z}(n)) = K_n^{Milnor}(F)$ *for any field F finitely generated over k.*
3. $H^{2n}(X, \mathbb{Z}(n)) = CH^n(X)$ *whenever X is smooth over k.*

4. *Vanishing Conjecture:* $\mathbb{Z}(n)$ *is acyclic outside of* $[0,n]$*:*

$$H^p(X, \mathbb{Z}(n)) \;=\; 0, \quad p < 0.$$

5. *Motivic spectral sequences for* X *smooth over* k*:*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}/\ell(-q)) \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell), \quad if \;\; 1/\ell \in k.$$

6. *Beilinson-Lichtenbaum Conjecture:*

$$\mathbb{Z}(n) \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq n} \mathbf{R}\pi_* \mu_\ell^{\otimes n}, \quad if \;\; 1/\ell \in k$$

where π : *etale site* $\to$ *Zariski site is the natural "forgetful continuous map" and* $\tau_{\leq} n$ *indicates truncation.*

7. $H^i(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \simeq K_{2n-i}(X)^{(n)}_{\mathbb{Q}}$.

In other words, Beilinson conjectures that there should be a bigraded *motivic cohomology* groups $H^p(X, \mathbb{Z}(q))$ computed as the Zariski cohomology of *motivic complexes* $\mathbb{Z}(q)$ of sheaves which satisfy good properties and are related to algebraic K-theory as singular cohomology is related to topological K-theory.

6.3 Status of Conjectures

Bloch's higher Chow groups $CH^q(X,n)$ (cf. [29]) serve as motivic cohomology groups which are known to satisfy most of the conjectures, where the correspondence of indexing is as follows:

$$CH^q(X,n) \;\simeq\; H^{2q-n}(X, \mathbb{Z}(q)). \tag{1}$$

Furthermore, Suslin and Voevodsky have formulated complexes $\mathbb{Z}(q),\ q \geq 0$ and Voevodsky has proved that the (hyper-)cohomology groups of these complexes satisfy the relationship to Bloch's higher Chow groups as in (1).

Presumably, these constructions will be discussed in detail in the lectures of Professor Levine. For completeness, I sketch the definitions. Recall that the standard (algebro-geometric) n-simplex Δ^n over a field F (which we leave implicit) is given by $\operatorname{Spec} F[t_0, \ldots, t_n]/\Sigma_i t_i = 1$.

Definition 6.4. Let X be a quasi-projective variety over a field. For any $q, n \geq 0$, we define $z^q(X, n)$ to be the free abelian group on the set of cycles $W \subset X \times \Delta^n$ of codimension q which meet all faces $X \times \Delta^i \subset X \times \Delta^n$ properly. This admits the structure of a simplicial abelian group and thus a chain complex with boundary maps given by restrictions to (codimension 1) faces.

The *Bloch higher Chow group* $CH^q(X, n)$ is defined by

$$CH^q(X, n) \ = H^{2q-n}(z^q(X, *)).$$

The values of Bloch's higher Chow groups are "correct", but they are not given as (hyper)-cohomology of complexes of sheaves and they are so directly defined that abstract properties for them are difficult to prove. The Suslin-Voevodsky motivic cohomology groups fit in a good formalism as envisioned by Beilinson and agree with Bloch's higher Chow groups as verified by Voevodsky.

Definition 6.5. Let X be a quasi-projective variety over a field. For any $q \geq 0$, we define the complex of sheaves in the cdh topology (the Zariski topology suffices if X is smooth over a field of characteristic 0)

$$\mathbb{Z}(q) \ = \ \underline{C}_*(c_{equi}(\mathbb{P}^n, 0)/c_{equi}(\mathbb{P}^{n-1}, 0))[-2n]$$

defined as the shift $2n$ steps to the right of the complex of sheaves whose value on a Zariski open subset $U \ \subset \ X$ is the complex

$$j \ \mapsto \ c_{equi}(\mathbb{P}^n, 0)(\Delta^j)/c_{equi}(\mathbb{P}^{n-1})(U \times \Delta^j)$$

where $c_{equi}(\mathbb{P}^n, 0)(U \times \Delta^j)$ is the free abelian group on the cycles on $\mathbb{P}^n \times U \times \Delta^j$ which are equidimensional of relative dimension 0 over $U \times \Delta^j$.

Conjecture (1) is essentially a normalization, for it specifies what $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ must be. Bloch verified Conjecture 2 (essentially, a result of Suslin), Conjecture 3, and Conjecture 7 (the latter with help from Levine) for his higher Chow groups. Bloch and Lichtenbaum produced a motivic spectral sequence for $X = \operatorname{Spec} k$; this was generalized to a verification of the full Conjecture (5) by Friedlander and Suslin, and later proofs were given by Levine and then Suslin following work of Grayson.

The Beilinson-Lichtenbaum conjecture in some sense "identifies" mod-ℓ motivic cohomology in terms of etale cohomology. Suslin and Voevodsky proved that this Conjecture (6) follows from the following:

Conjecture 6.6. *(Bloch-Kato Conjecture) For fields F finitely generated over k,*

$$K_n^{Milnor} \otimes \mathbb{Z}/\ell \ \simeq H^n_{et}(\operatorname{Spec} F, \mu_\ell^{\otimes n}).$$

In particular, the Galois cohomology of the field F is generated multiplicatively by classes in degree 1.

For $\ell = 2$, the Bloch-Kato Conjecture is a form of *Milnor's Conjecture* which has been proved by Voevodsky. For $\ell > 2$, a proof of Bloch-Kato Conjecture has apparently been given by Rost and Voevodsky, although not all details have been made available. This conjecture will be the main focus of Professor Weibel's lectures.

This leaves Conjecture (4), one aspect of this is the following Vanishing Conjecture due to Beilinson and Soulé.

Conjecture 6.7. *For fields F,*

$$K_p(F)^{(q)}_{\mathbb{Q}} \ = \ 0, \quad 2q \leq p, p > 0.$$

Reindexing according to Conjecture (7), this becomes

$$H^i(\operatorname{Spec} F, \mathbb{Z}(q)) \ = \ 0, \quad i \leq 0, q \neq 0.$$

The status of this Conjecture (4), and in particular the Beilinson-Soué vanishing conjecture, is up in the air. Experts are not at all convinced that this conjecture should be true for a general field F. It is known to be true for a number field.

6.4 The Meaning of the Conjectures

Let us begin by looking a bit more closely at the statement

$$\mathbb{Z}(1) \simeq \mathcal{O}^*[-1]$$

of Conjecture (1).

Convention If C^* is a cochain complex (i.e., the differential increases degree by 1, $d : C^i \to C^{i+1}$), we define the chain complex $C^*[n]$ for any $n \in \mathbb{Z}$ as the shift of C^* "n places to the right". In other words, $(C^*[n])^j \ = \ C^{*-j}$.

In particular, $\mathcal{O}^*[-1]$ is the complex (of Zariski sheaves) with only one non-zero term, the sheaf $\mathcal{O}^*$ of units, placed in degree -1 (i.e., shifted 1 place to the left). In particular,

$$H^*_{Zar}(X, \mathcal{O}^*[-1]) \ = \ H^{*-1}_{Zar}(X, \mathcal{O}^*);$$

thus,

$$Pic(X) \; = \; H^1_{Zar}(X, \mathcal{O}^*_X) \; = \; H^2(X, \mathbb{Z}(1)).$$

This last equality is a special case of item (3).

Perhaps it would be useful to be explicit about what we mean by the cohomology of a complex C^* of Zariski sheaves on X. A quick way to define this is as follows: find a map of complexes $C^* \to I^*$ with each I^j an injective object in the category of sheaves (an injective sheaf) such that the map on cohomology sheaves is an isomorphism; in other words, for each j, the map of presheaves

$$ker\{d : C^j \to C^{j+1}\}/im\{d : C^{j-1} \to C^j\}$$

$$\to \; ker\{d : I^j \to I^{j+1}\}/im\{d : I^{j-1} \to I^j\}$$

induces an isomorphism on associated sheaves

$$\mathcal{H}^j(C^*) \; \simeq \; \mathcal{H}^j(I^*)$$

for each j. A fundamental property of this cohomology is the existence of "hypercohomology spectral sequences"

$$'E_1^{p,q} \; = \; H^p(X, C^q) \; \Rightarrow \; H^{p+q}(X, C^*)$$

$$E_2^{p,q} \; = \; H^q(X, \mathcal{H}^j(C^*)) \; \Rightarrow \; H^{p+q}(X, C^*)$$

Conjecture (2) helps to pin down motivic cohomology by specifying what the top dimensional motivic cohomology (thanks to Conjecture (4)) should be for a field. Since Milnor K-theory and algebraic K-theory of the field k are different, this difference must be reflected in the other motivic cohomology groups of the field and tied together with the spectral sequence of Conjecture (5).

Conjecture (2) can be viewed as "arithmetic" for it deals with subtle invariants of the field k. Conjecture (3) is "geometric", stating that motivic cohomology reflects global geometric properties of X. Observe that since we are taking Zariski cohomology, $H^n(\operatorname{Spec} k, -) = 0$ for $n > 0$ and this item simply says that $CH^0(\operatorname{Spec} k) = \mathbb{Z}, \; CH^n(\operatorname{Spec} k) = 0, n > 0$.

Bloch has also proved that the spectral sequence of Conjecture (5) collapses after tensoring with $\mathbb{Q}$; indeed, Conjecture (7) proved by Bloch is a refinement of this "rational collapse". Conjectures (3) and (5) together with this collapsing gives Grothendieck's isomorphism $K_(X)\mathbb{Q} \simeq CH^*(X)$. By simply re-indexing, one can write the spectral sequence of Conjecture (5) in the more familiar "Atiyah-Hirzebruch manner"

$$E_2^{p,q} = H^p(X, \mathbf{Z}(-q/2)) \Rightarrow K_{-p-q}(X)$$

where $\mathbf{Z}(-q/2) = 0$ if q is not an even non-positive integer and $\mathbf{Z}(-q/2) = \mathbf{Z}(i)$ is $-q = 2i \geq 0$.

Let me try to "draw" this spectral sequence, using the notation

$$K_{q-i}^{(q)} \;\equiv\; H^i(X, \mathbb{Z}(q)$$

as in Conjecture (7).

	$\mathbb{Z}$				
0	$\mathcal{O}^*$	$Pic(X)$			
0?	$K_2^{(2)}$	$K_1^{(2)}$	$CH^2(X)$		
0?	$K_3^{(3)}$	$K_2^{(3)}$	$K_1^{(3)}$	$CH^3(X)$	
0?	$K_4^{(4)}$	$K_3^{(4)}$	$K_2^{(4)}$	$K_1^{(4)}$	$CH^4(X)$

In this picture, the associated graded of K_0 is given by the right-most diagonal, then $gr(K_1)$ by the next diagonal to the left, etc. The top horizontal row is the "weight 0" part of K_*, the next row down is the "weight 1" part of K_*, etc. There is conjectured vanishing at and to the left of the positions with 0? in the picture – i.e., to the left.

6.5 Etale cohomology

Our final task is to introduce the etale topology and attempt to give some understanding why Conjecture (6) of the Beilinson Conjectures comparing mod-ℓ motivic cohomology with mod-ℓ etale cohomology makes motivic cohomology more understandable.

Grothendieck had the insight to realize that one could formulate sheaves and sheaf cohomology in a setting more general than that of topological spaces. What is essential in sheaf theory is the notion of a covering, but such a covering need not consist of open subsets.

Definition 6.8. A (Grothendieck) site is the data of a category $\mathcal{C}/X$ of schemes over a given scheme X which is closed under fiber products and a distinguished class of morphisms (e.g., Zariski open embeddings; or etale morphisms) closed under composition, base change and including all isomorphisms. A covering of an object $Y \in \mathcal{C}/X$ for this site is a family of distinguished morphisms $\{g_i : U_i \to Y\}$ with the property that $Y = \cup_i g_i(U_i)$.

The data of the site $\mathcal{C}/X$ together with its associated family of coverings is called a Grothendieck topology on X.

The reader is referred to [33] for a foundational treatment of etale cohomology and to [21] for an overview.

Example 6.9. Recall that a map $f : U \to X$ of schemes is said to be *etale* if it is flat, unramified, and locally of finite type. Thus, open immersions and covering space maps are examples of etale morphisms. If $f : U \to X$ is etale, then for each point $u \in U$ there exist affine open neighborhoods $SpecA \subset U$ of u and $SpecR \subset X$ of $f(u)$ so that A is isomorphic to $(R[t]/g(t))_h$ for some monic polynomial $g(t)$ and some h so that $g'(t) \in (R[t]/g(t))_h$ is invertible.

The (small) etale site X_{et} has objects which are etale morphisms $Y \to X$ and coverings $\{U_i \to Y\}$ consist of families of etale maps the union of whose images equals Y. The big etale site X_{ET} has objects $Y \to X$ which are locally of finite type over X and coverings $\{U_i \to Y\}$ defined as for X_{et} consisting of families of etale maps the union of whose images equals Y. If k is a field, we shall also consider the site $(Sm/k)_{et}$ which is the full subcategory of $(\mathrm{Spec}\, k)_{ET}$ consisting of smooth, quasi-projective varieties Y over k.

An instructive example is that of $X = SpecF$ for some field F. Then an etale map $Y \to X$ with Y connected is of the form $SpecE \to SpecF$, where E/F is a finite separable field extension.

Definition 6.10. A presheaf sets (respectively, groups, abelian groups, rings, etc) on a site $\mathcal{C}/X$ is a contravariant functor from $\mathcal{C}/X$ to $(sets)$ (resp., to groups, abelian groups, rings, etc). A presheaf $P : (\mathcal{C}/X)^{op} \to (sets)$ is said to be a sheaf if for every covering $\{U_i \to Y\}$ in $\mathcal{C}/X$ the following sequence is exact:

$$P(Y) \to \prod_i P(U_i) \stackrel{\to}{\to} \prod_{i,j} P(U_i \times_X U_j).$$

(Similarly, for presheaves of groups, abelian presheaves, etc.) In other words, if for every Y, the data of a section $s \in P(Y)$ is equivalent to the data of sections $s_i \in P(U_i)$ which are compatible in the sense that the restrictions of s_i, s_j to $U_i \times_X U_j$ are equal.

The category of abelian sheaves on a Grothendieck site $\mathcal{C}/X$ is an abelian category with enough injectives, so that we can define sheaf cohomology in the usual way. If $F : \mathcal{C}/X)^{op} \to (Ab)$ is an abelian sheaf, then we define

$$H^i(X_{\mathcal{C}/X}, F) = R^i\Gamma(X, F).$$

Etale cohomology has various important properties. We mention two in the following theorem.

Theorem 6.11. *Let X be a quasi-projective, complex variety. Then the etale cohomology of X with coefficients in (constant) sheaf $\mathbb{Z}/n$, $H^*(X_{et}, \mathbb{Z}/n)$, is naturally isomorphic to the singular cohomology of X^{an},*

$$H^*(X_{et}, \mathbb{Z}/n) \;\simeq\; H^*_{sing}(X^{an}, \mathbb{Z}/n).$$

Let $X = Speck$, the spectrum of a field. Then an abelian sheaf on X for the etale topology is in natural 1-1 correspondence with a (continuous) Galois module for the Galois group $Gal(\overline{k}/k)$. Moreover, the etale cohomology of X with coefficients in such a sheaf F is equivalent to the Galois cohomology of the associated Galois module,

$$H^*(k_{et}, F) \;\simeq\; H^*(Gal(\overline{F}/F), F(k)).$$

From the point of view of sheaf theory, the essence of a continuous map $g : S \to T$ of topological spaces is a mapping from the category of open subsets of T to the open subsets of S. In the context of Grothendieck topologies, we consider a map of sites $g : \mathcal{C}/X \to \mathcal{D}/Y$, a functor from $\mathcal{C}/Y$ to cC/X which takes distinguished morphisms to distinguished morphisms. In particular, for example, Conjecture (6) of Beilinson's Conjectures involves the map of sites

$$\pi : X_{et} \to X_{Zar}, \quad (U \subset X) \mapsto U \to X.$$

Such a map of sites induces a map on sheaf cohomology: if $F : (\mathcal{D}/Y)^{op} \to (Ab)$ is an abelian sheaf on $\mathcal{C}/Y$, then we obtain a map

$$H^*(Y_{\mathcal{D}/Y}, F) \to H^*(X_{\mathcal{C}/X}, g^*F).$$

6.6 Voevodsky's sites

We briefly mention two Grothendieck sites introduced by Voevodsky which are central to his approach to motivic cohomology. The reader can find details in [34].

Definition 6.12. The Nisnevich site on smooth quasi-projective varieties over a field k, $(Sm/k)_{Nis}$, is determined by specifying that a covering $\{U_i \to U\}$ of some $U \in (Sm/k)$ is an etale covering with the property that for each point $x \in U$ there exists some i and some point $\tilde{u} \in U_i$ such that the induced map on residue fields $k(u) \to k(\tilde{u})$ is an isomorphism.

Definition 6.13. The cdh (or completely decomposed, homotopy) site on smooth quasi-projective varieties over a field k, $(Sm/k)_{cdh}$, is determined as the site whose coverings of a smooth variety X are generated by Nisnevich coverings of X and coverings $\{Y \to X,\ X' \ \to \ X\}$ consisting of a closed immersion $i : Y \to X$ and a proper map $g : X' \to X$ with the property that the restriction of g to $g^{-1}(X \backslash i(Y))$ is an isomorphism.

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