Lecture Notes

Geometric Wave Equations

Stefan Waldmann

Department Mathematik
Friedrich-Alexander Universität Erlangen-Nürnberg
Cauerstrasse 11
91058 Erlangen
Germany

Contact: Stefan.Waldmann@math.fau.de

In these lecture notes we discuss the solution theory of geometric wave equations as they arise in Lorentzian geometry: for a normally hyperbolic differential operator the existence and uniqueness properties of Green functions and Green operators is discussed including a detailed treatment of the Cauchy problem on a globally hyperbolic manifold both for the smooth and finite order setting. As application, the classical Poisson algebra of polynomial functions on the initial values and the dynamical Poisson algebra coming from the wave equation are related. The text contains an introduction to the theory of distributions on manifolds as well as detailed proofs.

Preface

These lecture notes grew out of a two-semester course on wave equations on Lorentz manifolds which I gave in Freiburg at the physics department in the winter term 2008/2009 and the following summer term 2009. This lecture originated from a long term project on the deformation quantization of classical field theories started some nine years before: the aim was to understand recent developments on quantization following the works of Dütsch and Fredenhagen [18–20]. As time passed, the beautiful book of Bär, Ginoux, and Pfäffle [4] on the global theory of wave equations appeared and provided the basis for a revival of that old project. So the idea of presenting the results of [4] to a larger audience of students was born. The resulting lectures aimed at master and PhD students in mathematics and mathematical physics with some background in differential geometry and a lively interest in the analysis of hyperbolic partial differential equations. Though both, the lecture and these notes, followed essentially the presentation of [4], I added more detailed proofs and some background material which hopefully make this material easily accessible already for students.

During the preparation of these lecture notes many colleagues and friends gave me their help and support. To all of them I am very grateful: First of all, I would like to thank Frank Pfäffle for his continuous willingness to explain many details of [4] to me. Without his help, neither the lecture nor these lecture notes would have been possible in the present form. Also, I would like to thank Michael Dütsch and Klaus Fredenhagen for continuing discussions concerning their works as well as on related questions on deformation quantization of classical field theories, thereby constantly raising my interest in the whole subject. Florian Becher helped not only with the exercise and discussion group for the students but is ultimately responsible for this project by pushing me to "give a lecture on the book of Bär, Ginoux, and Pfäffle". I am also very grateful to Domenico Giulini who helped me out in many questions on general relativity and gave me access and guidance to various references. Moreover, I am indebted to Stefan Suhr for helping me in many questions on Lorentz geometry and improving various arguments during the lecture. I would like to thank also all the participant of the course who brought the lecture to success by their constant interest, their questions, and their remarks on the manuscript of these notes, in particular Jan Paki. Finally, I am very much obliged to Jan-Hendrik Treude for taking care of the LaTeX-files, the Xfig-pictures, and all the typing as well as for his numerous comments and remarks. Without his help, the manuscript would have never been finished.

Most grateful I am for my children Silvia, Richard, Sonja, and Robert, as a source of unlimited inspiration and for Viola, for her continuous support, both morally and scientifically.

Erlangen, August 2012 *Stefan Waldmann*

Contents

Introduction and Overview

The theory of linear partial differential equations can be divided into three principal parts: the first is the elliptic theory of equations like the Laplace equation, the next is the parabolic theory being the habitat of the heat equation, and the third is the hyperbolic theory. All three differ in their behaviour, concepts, and applications.

It will be the hyperbolic theory where the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1}$$

on $\mathbb{R}^n$ provides the first and most important example. While for the elliptic theory the boundary problem is characteristic, for the hyperbolic situation the main task is to understand an initial value problem: for time $t = 0$ one specifies the solution $u(0, x)$ and its first time derivative $\frac{\partial u}{\partial t}(0, x)$ for all $x \in \mathbb{R}^{n-1}$ and seeks a solution of the wave equation with these prescribed initial values. Of course, also for the wave equation one can pose boundary condition on top of the initial value problem. Together with the question of how (continuous) the solution depends on the initial conditions this becomes the Cauchy problem for hyperbolic equations.

The relevance of the wave equation as coming from the science and in particular from physics is overwhelming; we indicate just two major occurrences: on a phenomenological level it describes propagating waves in elastic media in a linearized approximation. This approximation is typically well justified as long as the displacements are not too big. Then the wave equation provides a good model for many everyday situations like water waves, elastic vibrations of solids, or propagation of sound. The constant c in the wave equation is then the speed of propagation and a characteristic quantity of the material. On a more fundamental level, and more important for our motivation, is the appearance of the wave equation in various physical theories of fundamental interactions. Most notable here is Maxwell's theory of electromagnetic fields. In this context, the wave equation appears as an exact and fundamental equation describing the propagation of electromagnetic waves (light, radio waves, etc.) in the vacuum. Remarkably, it is a field equation not relying on any sort of carrier material like the hypothetical ether. The constant c becomes the speed of light, one of the few truly fundamental constants in physics. But even beyond Maxwell's theory the wave equation and its generalizations like the Klein-Gordon equation provide the linear part of all known fundamental field theories.

Needless to say, it is worth studying such wave equations. But which framework should be taken to formulate the problem in a mathematically meaningful and yet still interesting way?

A short look at the wave equation shows that it is invariant under the affine pseudo-orthogonal group $\mathrm{O}(1, n-1) \ltimes \mathbb{R}^n$ in the sense that the natural affine action of $\mathrm{O}(1, n-1) \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$ pulls back solutions of the wave equation to solutions again. In more physical terms we have the invariance group of special relativity, the Poincaré group. This already indicates to take a *geometric* point of view and interpret the wave equation as coming from the d'Alembert operator $\Box$ corresponding to the Minkowski metric $\eta = \mathrm{diag}(+1, -1, \ldots, -1)$. Indeed, this point of view opens the door for various generalizations if we replace η and $\mathbb{R}^n$ by an arbitrary Lorentz metric g on an arbitrary manifold M: we still have a d'Alembert operator (coming from g) and hence a wave equation. In more physical terms we pass from special to general relativity. But even if one is not interested in geometry a priori, generalizations of the wave equation like

$$\sum_{i,j} A^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_i B^i \frac{\partial u}{\partial x^i} + Cu = 0, \tag{2}$$

with coefficient *functions* A^{ij}, B^i, and C on $\mathbb{R}^n$ such that the matrix $(A^{ij}(x))$ has signature $(+, -, \ldots, -)$ at every point $x \in \mathbb{R}^n$, can be treated best only after a geometric interpretation of the functions A^{ij}. Otherwise, it will be almost impossible to get hands on the Cauchy problem of such a wave equation with non-constant coefficients. In fact, the first naive idea would be to find adapted coordinates on order to bring (2) to the form (1), at least concerning the second order derivatives. However, generically this has to fail since the typically non-zero curvature of the metric corresponding to the coefficients A^{ij} is precisely the obstruction to get constant coefficients in front of the leading orders of differentiation by a change of coordinates. This brings us back to a geometric point of view which we will take in the following.

The Geometric Framework

The wave equations we will discuss are located on a Lorentz manifold, i.e. on a smooth n-dimensional manifold M equipped with a smooth Lorentz metric g. We choose the signature $(+, -, \ldots, -)$ as common in (quantum) field theory but probably less common in general relativity. The notions of light-, time-, and spacelike vectors, future and past, causality, etc. which we will develop in the sequel, have their origin in the theory of general relativity which is the main source of inspiration in Lorentz geometry. In particular, the notion of a spacetime will be used synonymously for a Lorentz manifold.

The metric allows to speak of the d'Alembert operator $\Box$ acting on the smooth functions on M. While this gives already many interesting wave equations there are still two directions of generalization: first, we would like to incorporate also lower order terms of differentiation as in (2). Second, many application like e.g. Maxwell's theory require to go beyond the *scalar* wave equations and need "multicomponent" functions u^α instead of a single, scalar one. These components may even be coupled in a non-trivial way.

Both situations can be combined into the following framework. We take a vector bundle $E \longrightarrow M$ over M and consider a linear second order differential operator D acting on the sections of E with leading symbol being the same as for the scalar d'Alembert operator. Such a *normally hyperbolic* differential operator will have the local form

$$Du = \left(g^{ij} \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} + B^{i\alpha}_\beta \frac{\partial u^\beta}{\partial x^i} + C^\alpha_\beta u^\beta \right) e_\alpha, \tag{3}$$

where the section $u = u^\alpha e_\alpha$ is expressed locally in terms of a local frame $\{e_\alpha\}$ of E and we use local coordinates $\{x^i\}$ on M. Here g^{ij} are the coefficients of the (inverse) metric tensor g while $B^{i\alpha}_\beta$ and C^α_β are coefficient functions determined by D. In this expression and from now on we shall use Einstein's summation convention that pairs of matching coordinate or frame indexes are automatically summed over their range.

A differential operator D like in (3) makes sense even on any semi-Riemannian manifold. For the formulation of the Cauchy problem we need the Lorentz signature and two extra structures beside the metric. The first is a time orientation which separates future from past. This will allow for notions of causality and thus for the notions of advanced and retarded solutions of the wave equation. From a physics point of view such a time orientation is absolutely necessary to have a true interpretation of (M, g) as a spacetime. The second ingredient is that of a hypersurface Σ in M on which we can specify the initial values. Thus Σ corresponds to "$t = 0$" in this geometric context. At first sight any spacelike hypersurface might be suitable. However, already in $\mathbb{R}^n$ the $t = 0$ hypersurface has additional properties: it divides $\mathbb{R}^n$ into two disjoint pieces, the future and the past of $t = 0$. Moreover, *every* inextensible causal curve has to pass through this $t = 0$ hypersurface in precisely one point. Physically speaking, this means that knowing things on Σ allows to compute the entire time evolution in a deterministic way. This is the main idea behind an initial value problem. Thus we can already anticipate that this feature will turn out to be crucial for a good Cauchy problem. In general, a spacelike hypersurface Σ will be called a *Cauchy hypersurface* if it satisfies this condition: every inextensible causal curve passes in exactly one point through Σ. It is a non-trivial and in fact quite recent theorem that the existence of such a *smooth* Cauchy hypersurface is equivalent to the notion of a globally hyperbolic Lorentz manifold. Moreover, having one such Cauchy hypersurface allows already to split M into a time axis and spacelike directions, i.e. $M \cong \mathbb{R} \times \Sigma$, in such a way that also the metric becomes block-diagonal. We will have to explain all these notions in more detail.

The Analytic Framework

After setting the geometric stage we also have to specify the analytic aspects properly in order to obtain a complete formulation of the Cauchy problem. Handling linear partial differential equations allows for various approaches. Most notably, one can use Sobolev space techniques or distribution theory. In the sequel, we will exclusively use the distributional approach for reasons which are not even that easy to explain. Nevertheless, let us try to motivate our choice:

At first, physicists are usually more adapted to the notions of distributions, at least on a heuristic level, than to Sobolev spaces and their usage. Moreover, and more important, the solution to the Cauchy problem using distribution theory relies on the notion of *Green functions* also called *fundamental solutions*. These are particular distributional solutions of the wave equation with a δ-distribution as inhomogeneity. The collection of all these Green functions can be combined into a single operator, the Green operator. Very informally, this will be an "inverse" of the differential operator D. Now these Green operators allow for a very efficient description of the solutions to the Cauchy problem and are hence worth to be studied. Finally, and this might be the most important reason to choose the distributional approach, these Green operators appear as fundamental ingredients, the *propagators*, for every quantum field theory build on top of the classical field theory described by the wave equation. Even though we do not enter the discussion of quantizing the classical field theory we at least provide the starting point by constructing the Poisson algebra of the classical theory. The Poisson bracket is then defined by means of the Green operators and will allow us to view the time evolution of the initial values as a "Hamiltonian system" with infinitely many degrees of freedom. The interest in this Hamiltonian picture is the ultimate reason for us to favour the distributional approach over the Sobolev one. Even though we do not discuss this here, there is yet another reason why the distributional approach is interesting: it is within this framework where one can discuss the propagation of singularities most naturally by means of wavefront analysis.

Within the distributional approach we will have an interplay of very singular objects, the distributional sections of vector bundles, and very regular ones, the smooth sections of the corresponding dual bundles. Here smooth stands for $\mathcal{C}^\infty$, i.e. infinitely often differentiable. However, at many places we will pay attention to the number of derivatives which are actually needed. This will result in certain "finite order" statements. Even though there is also a well-developed theory of real analytic wave equations and their solutions we will exclusively stick to the $\mathcal{C}^\infty$- and $\mathcal{C}^k$-case.

Throughout this work, we will avoid techniques from Fourier analysis and stay exclusively in "coordinate space". It is clear that in a geometric framework there is no intrinsic definition of a *global* Fourier transform. In principle, one can pass to a *microlocal* version of Fourier transform between tangent and cotangent spaces. However, we shall not need this more sophisticated approach here, even though this will lead to some deeper insights in the nature of the singularities of the Green operators by means of a wavefront analysis. As this text should serve as a first reading in this area we decided to concentrate on the more basic formulations.

A User's Guide for Reading

This text addresses mainly master and PhD students who want to get a fast but yet detailed access to an important research topic in global analysis and partial differential equations on manifolds of great recent interest. The reader should have some background knowledge in differential geometry. We use the language of manifolds, vector bundles, and tensor calculus without further explanations. Some previous exposure to locally convex analysis and distribution theory on $\mathbb{R}^n$ might be useful but will not be required: all relevant notions will either be explained in detail or accompanied with explicit references to other textbooks for detailed proofs. Knowledge in Lorentz geometry is of course useful as well, but we will develop those parts of the theory which are relevant for our purposes, essentially the notions of causality. We assume that the reader has at least some vague interest in the physical applications of the theory as we will take this often as motivation.

The presented material is entirely standard and can also be found in various other sources. We mainly follow the beautiful exposition of Bär, Ginoux, and Pfäffle [4] but rely also on the textbooks [23, 27, 31] for certain details and further aspects on distributions on manifolds and geometric wave equations not discussed in [4]. Concerning Lorentz geometry we refer to the textbook of O'Neill [46] and the recent review article of Minguzzi and Sanchez [45] on the causal structure. Other resources on Lorentz geometry and general relativity are the classical texts [6, 29, 56, 59]. More details on distribution theory and locally convex analysis can be found in the standard textbooks [34,51,58]. For further reading one should consult the recent booklet [3] as well as the articles [14, 15] for approaches to (quantum) field theories on curved spacetimes based on the construction of Green functions for geometric wave equations. Though we do not touch this subject, background information on axiomatic approaches to quantum field theory might be helpful and can be found in the classical textbooks [28,57]. Beside these general references we will provide more detailed ones throughout the text.

The material is divided into four chapters and two supplementary appendices:

In the first chapter we set the stage for the relevant analysis on manifolds. In Section 1.1 we introduce test function and test section spaces and investigate their locally convex topologies. The central result will be Theorem 1.1.11 establishing the LF topology for compactly supported smooth sections as well as important properties like completeness of this topology. Moreover, we study continuous linear maps between test section spaces: on one hand pull-backs with respect to bundle maps and on the other hand various multilinear pairings between sections. Finally, we show that the smooth sections with compact support are sequentially dense in all other $\mathcal{C}^k$- and $\mathcal{C}^k_0$-sections. Then in Section 1.2 we discuss differential operators and their symbols. In particular, we introduce a global symbol calculus based on the usage of covariant derivatives. Differential operators are then shown to be continuous linear maps for the test section spaces. We show that differential operators have adjoints for various natural pairings and compute the adjoints explicitly by using the global symbol calculus in Theorem 1.2.21. We arrive in Section 1.3 at the definition of distributions or, more precisely, of generalised sections. Here we first present the intrinsic definition. Later on, we interpret distributions always with respect to a fixed reference density: this way, one can avoid carrying around the additional density bundle everywhere. We define the weak* topology and explain the support and singular support of generalized sections. Important for later use will be the characterization of generalized sections with compact support in Theorem 1.3.18. We introduce the push-forward, the action of differential operators as well as the external tensor product of distributional sections. Parallel to the smooth case we develop the $\mathcal{C}^k$-case, both for test sections and distributions of finite order.

Chapter 2 contains a rough overview on Lorentz geometry where we focus on particular topics rather than on a general presentation. In Section 2.1 we recall some basic concepts from semi-Riemannian geometry like parallel transport and the exponential map of a connection, the Levi-Civita connection and the d'Alembert operator. Still for general semi-Riemannian manifolds we introduce the notion of a connection d'Alembertian and provide a definition and characterization of normally hyperbolic differential operators. We pass to true Lorentz geometry in Section 2.2 where we mainly focus on aspects related to the causal structure. As motivation, also for the wave equations, we recall some features of general relativity. This gives us the notions of time orientability, causality, and ultimately, of Cauchy hypersurfaces. Here we discuss the characterization of globally hyperbolic spacetimes by the existence of smooth Cauchy hypersurfaces in Theorem 2.2.31 and present some important consequences of this "splitting theorem". Throughout this section our proofs are rather sketchy but illustrated by simple geometric (counter-) examples. Even without explicit proofs this should help to develop the right intuition. We conclude this chapter with some general remarks on wave equations, the Cauchy problem, and advanced and retarded Green functions in Section 2.3.

Even though Chapter 3 deals with the local construction of Green functions we need already here geometric concepts like parallel transport and the exponential map. As warming up we start in Section 3.1 with the wave equation (1) on flat Minkowski spacetime and obtain the advanced and retarded Green functions by constructing an entirely holomorphic family $\{R^{\pm}(\alpha)\}_{\alpha\in\mathbb{C}}$ of distributions, the *Riesz distributions*. For $\alpha = 2$ one obtains the Green functions of $\Box$. We examine these Riesz distributions in great detail as they will be the crucial tool to construct local Green functions in general. The case of spacetime dimensions $n = 1$ (only time) and $n = 2$ is discussed explicitly as one obtains a drastically simpler approach here. In Section 3.2 we use the exponential map to transfer the Riesz distributions also to the curved situation, at least in a small normal neighborhood of a given point. However, the curvature will now cause slightly different features of the Riesz distributions which results in the failure of $R^{\pm}(p, 2)$ being a Green function of the scalar d'Alembert operator. Nevertheless, the defect can be computed explicitly enough to use the Riesz distributions in Section 3.3 to formulate an heuristic Ansatz for the true Green function, now for a general normally hyperbolic differential operator, as a series expansion in the "degree of singularity". This Ansatz leads to transport equations similar to the WKB approximation whose solutions will be the *Hadamard coefficients*. Even though working on a small coordinate patch the construction of the Hadamard coefficients in Theorem 3.3.10 requires the full machinery of differential geometry and would be hard to understand without the usage of covariant derivatives and their parallel transports. As an application of this general approach we compute the Hadamard coefficients for the Klein-Gordon equation in flat spacetime explicitly and obtain an explicit formula for the advanced and retarded Green functions in Theorem 3.3.18. Back in the general situation we show in the rather technical Section 3.4 how a true Green function with good causal properties can be obtained from the Hadamard coefficients. Here one first enforces the convergence of the above Ansatz thereby destroying the property of a Green function. The result is a parametrix which can be modified in a second step to obtain the Green functions in Theorem 3.4.42. As a first application we use the local Green functions to construct particular solutions of the inhomogeneous wave equation for distributional and smooth inhomogeneities in Section 3.5 in Theorem 3.5.17.

Chapter 4 is now devoted to the global situation. First we have to recall the notion of the time separation on a Lorentz manifold in Section 4.1 which is then used to prove uniqueness of solutions in Theorem 4.1.11 with either future or past compact support provided the global causal structure is well-behaved enough. Section 4.2 contains the precise formulation of the global Cauchy problem as well as its solution for globally hyperbolic spacetimes. We discuss both the smooth situation as well as certain finite differentiability versions of the Cauchy problem in Theorem 4.2.16. The continuous dependence on the initial values in the Cauchy problem follows from general arguments using the open mapping theorem. This feature is then used in Section 4.3 to obtain global Green functions and the corresponding global Green operators. The difference of the advanced and retarded Green operator provides an "inverse" to the wave operator in the sense of a specific exact sequence discussed in Theorem 4.3.18. Moreover, it constitutes the core ingredient for the classical Poisson algebra of the field theory corresponding to the wave equation as discussed then in Section 4.4. We give two alternative definitions of the Poisson algebra: one as polynomial algebra on the initial conditions depending on the choice of the Cauchy hypersurface with the canonical "symplectic" Poisson bracket. The other version is obtained as quotient of the polynomial algebra on all field configurations with Poisson bracket coming from the Green operators. The equivalence of both is shown in Theorem 4.4.22 and gives an easy proof of the "time-slice" axiom of the classical field theory in Theorem 4.4.29, analogously to the quantum field theoretic formulation. Also a classical analog of the "locality" axiom is proved in Theorem 4.4.27.

Appendix A contains background information on parallel transports and the Taylor expansion of various geometric objects like the exponential map and the volume density. In Appendix B we recall some basic applications of Stokes' theorem.

The text does not contain exercises. However, it is understood that students who really want to learn these topics in a profound way have to delve deep into the text. Some of the proofs are sketched and require some extra thoughts, others contain rather long computations which can and should be repeated.

Chapter 1

Distributions and Differential Operators on Manifolds

In this chapter we discuss the basic ingredients for analysis on smooth manifolds: first we introduce the canonical locally convex topologies for the smooth functions (with compact support) on M as well as for smooth sections of vector bundles. These spaces will constitute the spaces of test functions and test sections, respectively. We have to discuss convergence of test functions as well as the completeness of the test function spaces. In a second step we consider differential operators acting on test functions and test sections. After discussing elementary algebraic and topological properties we compute the adjoint of a differential operator with respect to a given positive density explicitly: here a symbol calculus is introduced and basic properties are shown. Finally, we introduce distributions as the continuous linear functionals on the various test function spaces. This allows to dualize all operations on test functions in an appropriate way. In particular, differential operators will act on distributions as well. We discuss the module structure of distributions, give first basic examples and define the support, and singular support of distributions.

1.1 Test Functions and Test Sections

A good understanding of the topological properties of test sections of vector bundles is crucial. The manifold M will be n-dimensional. In the following, we shall use Einstein's summation convention: the summation over dual pairs of indexes in multilinear expressions is automatic.

1.1.1 The Locally Convex Topologies of Test Functions and Test Sections

In this subsection, we give several different but equivalent descriptions of the locally convex topology of test functions and test sections. Let $E \longrightarrow M$ be a vector bundle of rank N. The first collection of seminorms is obtained as follows. For a chart (U, x) we consider a compact subset $K \subseteq U$ together with a collection $\{e_\alpha\}_{\alpha=1,\ldots,N}$ of local sections $e_\alpha \in \Gamma^\infty(E\big|_U)$ such that $\{e_\alpha(p)\}_{\alpha=1,\ldots,N}$ is a basis of the fiber E_p. We always assume that U is sufficiently small or e.g. contractible such that local base sections exist. The collection $\{e_\alpha\}_{\alpha=1,\ldots,N}$ will also be called a *local frame*. The dual frame will then be denoted by $\{e^\alpha\}_{\alpha=1,\ldots,N}$ where $e^\alpha \in \Gamma^\infty(E^*\big|_U)$ are the local sections with $e^\alpha(e_\beta) = \delta^\alpha_\beta$. For $s \in \Gamma^\infty(E)$ we have unique functions $s^\alpha = e^\alpha(s) \in \mathcal{C}^\infty(U)$ such that

$$s\big|_U = s^\alpha e_\alpha. \tag{1.1.1}$$

We define the seminorms

$$\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}(s) = \sup_{\substack{p\in K\\ |I|\leq \ell\\ \alpha=1,\ldots,N}} \left| \frac{\partial^{|I|} s^\alpha}{\partial x^I}(p) \right|, \tag{1.1.2}$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ denotes a multiindex of total length $|I| = i_1 + \cdots + i_n$. Clearly, the seminorm depends on the choice of the chart, the compactum, the integer $\ell \in \mathbb{N}_0$ as well as on the choice of the local base sections. In case we have just functions, i.e. sections of the trivial vector bundle $E = M \times \mathbb{C}$, we can use the canonical trivialization which results in the simpler form

$$\mathrm{p}_{U,x,K,\ell}(f) = \sup_{\substack{p\in K\\ |I|\leq \ell}} \left| \frac{\partial^{|I|} f}{\partial x^I}(p) \right| \tag{1.1.3}$$

of the seminorm for $f \in \mathcal{C}^\infty(M)$.

Lemma 1.1.1 *For all choices of a chart (U, x), a compact subset $K \subseteq U$, an integer $\ell \in \mathbb{N}_0$ and local base sections $\{e_\alpha\}$ of E on U, the map*

$$\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}} : \Gamma^\infty(E\big|_U) \longrightarrow \mathbb{R}_0^+ \tag{1.1.4}$$

is a well-defined seminorm.

Proof. Clearly, the supremum over K is finite as all partial derivatives are continuous. The remaining properties of a seminorm are checked easily. □

An alternative construction of seminorms is as follows. On $E \longrightarrow M$ we choose a covariant derivative ∇^E and on $TM \longrightarrow M$ a torsion-free covariant derivative ∇, e.g. the Levi-Civita connection for some (semi-) Riemannian metric. Moreover, on E we choose a Riemannian fiber metric if E is a real vector bundle or a Hermitian fiber metric if E is complex, respectively. Finally, we shall use a Riemannian metric on M. Then the two metric structures give rise to fiber metrics on all bundles constructed out of TM and E via tensor products etc. Moreover, we have the following operator of symmetrized covariant differentiation:

Definition 1.1.2 (Symmetrized covariant differentiation) *Let ∇^E be a covariant derivative for a vector bundle $E \longrightarrow M$ and let ∇ a torsion-free covariant derivative on M. Then*

$$\mathsf{D}^E : \Gamma^\infty(\mathrm{S}^k T^*M \otimes E) \longrightarrow \Gamma^\infty(\mathrm{S}^{k+1} T^*M \otimes E) \tag{1.1.5}$$

is defined by

$$\mathsf{D}^E(\alpha \otimes s)(X_1, \ldots, X_{k+1}) = \sum_{\ell=1}^{k+1} \left(\nabla_{X_\ell} \alpha \otimes s + \alpha \otimes \nabla^E_{X_\ell} s \right) (X_1, \ldots, \overset{\ell}{\wedge}, \ldots, X_{k+1}), \tag{1.1.6}$$

where $\alpha \in \Gamma^\infty(\mathrm{S}^k T^*M)$, $s \in \Gamma^\infty(E)$, and $X_1, \ldots, X_{k+1} \in \Gamma^\infty(TM)$.

Proposition 1.1.3 *The operator D^E is linear, well-defined, and satisfies the following properties:*

i.) For $E = M \times \mathbb{C}$ with the canonical flat covariant derivative and $f \in \mathcal{C}^\infty(M)$ we have

$$\mathsf{D} f = \mathrm{d} f. \tag{1.1.7}$$

ii.) For $\alpha \in \Gamma^\infty(\mathrm{S}^k T^*M)$ and $\beta \otimes s \in \Gamma^\infty(\mathrm{S}^\ell T^*M \otimes E)$ we have

$$\mathsf{D}^E\left((\alpha \vee \beta) \otimes s\right) = (\mathsf{D}\,\alpha \vee \beta) \otimes s + \alpha \vee \mathsf{D}^E(\beta \otimes s). \tag{1.1.8}$$

iii.) Locally in a chart (U, x) we have

$$\mathsf{D}^E(\alpha \otimes s)\Big|_U = \left(\mathrm{d}\, x^i \vee \nabla_{\frac{\partial}{\partial x^i}} \alpha\right) \otimes s + \mathrm{d}\, x^i \vee \alpha \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s. \tag{1.1.9}$$

Proof. Clearly, (1.1.6) gives a well-defined E-valued symmetric $(k+1)$-form. On the trivial line bundle the flat connection is $\nabla_X f = \mathscr{L}_X f = (\mathrm{d}\, f)(X)$ from which (1.1.7) is obvious. The Leibniz rule (1.1.8) is a direct consequence from (1.1.9) but can also be obtained in a coordinate free way. We prove (1.1.9) by an explicit computation.

$$\begin{aligned}
&\left(\mathsf{D}^E(\alpha \otimes s)\right)(X_1, \ldots, X_{k+1}) \\
&= \sum_{\ell=1}^{k+1} \left(\nabla_{X_\ell}\alpha \otimes s + \alpha \otimes \nabla^E_{X_\ell} s\right)(X_1, \ldots, \overset{\ell}{\wedge}, \ldots, X_{k+1}) \\
&= \sum_{\ell=1}^{k+1} \left(\mathrm{d}\, x^i(X_\ell)\nabla_{\frac{\partial}{\partial x^i}}\alpha \otimes s + \mathrm{d}\, x^i(X_\ell)\alpha \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s\right)(X_1, \ldots, \overset{\ell}{\wedge}, \ldots, X_{k+1}) \\
&= \left(\mathrm{d}\, x^i \vee \left(\nabla_{\frac{\partial}{\partial x^i}}\alpha \otimes s + \alpha \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s\right)\right)(X_1, \ldots, X_{k+1}).
\end{aligned}$$

□

Using this symmetrized covariant differentiation we can construct a seminorm for $s \in \Gamma^\infty(E)$ as follows. First we consider $(\mathsf{D}^E)^\ell s \in \Gamma^\infty(\mathrm{S}^\ell T^*M \otimes E)$. Then we can use the fiber metric h on $\mathrm{S}^\ell T^*M \otimes E$ to get a fiberwise norm $\|\cdot\|_h$. Then for every compact subset $K \subseteq M$ we consider

$$\mathrm{p}_{K,\ell}(s) = \sup_{p \in K} \left\| (\mathsf{D}^E)^\ell s\big|_p \right\|_h, \tag{1.1.10}$$

where we suppress the dependence of $\mathrm{p}_{K,\ell}$ on the choices of ∇, ∇^E and h to simplify our notation.

Lemma 1.1.4 *For all choices of a compactum $K \subseteq M$ and $\ell \in \mathbb{N}_0$ the map*

$$\mathrm{p}_{K,\ell} : \Gamma^\infty(E) \longrightarrow \mathbb{R}^+_0 \tag{1.1.11}$$

is a well-defined seminorm.

Proof. Thanks to the continuity of $\left\|(D^E)^\ell s\right\|$ the supremum is actually a maximum over the compact subset K. Thus $\mathrm{p}_{K,\ell}(s) \in \mathbb{R}^+_0$ is finite. The remaining properties of a seminorm follow at once. □

We can now use both types of seminorms to construct locally convex topologies for $\Gamma^\infty(E)$. Since neither the system of the $\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}$ nor the $\mathrm{p}_{K,\ell}$ are filtrating, we have to take maximums over finitely many of them in each of the following cases:

- **A** Choose an atlas (U, x) with local base sections $\{e_\alpha\}$ on each chart and consider *all* seminorms $\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}$ arising from the charts of this atlas, all $\ell \in \mathbb{N}_0$, and all compact subsets $K \subseteq U$.
- **B** Choose ∇, ∇^E and fiber metrics and consider *all* seminorms $\mathrm{p}_{K,\ell}$ arising from all compact subsets $K \subseteq M$ and all $\ell \in \mathbb{N}_0$.

As a slight variation of **A** we can also consider the locally convex topology where we only take countably many compacta:

- **A'** Take only at most countably many charts and in each chart (U, x) only an exhausting sequence $\ldots \subseteq K_n \subseteq \mathring{K}_{n+1} \subseteq K_{n+1} \subseteq \ldots \subseteq U$ of compacta.

Analogously we can use only an exhausting sequence of compacta in the second version:

B' Take the $\mathrm{p}_{K,\ell}$ seminorms for an exhausting sequence $\ldots \subseteq K_n \subseteq \mathring{K}_{n+1} \subseteq K_{n+1} \subseteq \ldots \subseteq M$ of M by compacta.

Note that for second countable manifolds we can indeed find a countable atlas together with a choice of countably many compacta, each contained in a chart, which cover the whole manifold M.

Theorem 1.1.5 *Let $E \longrightarrow M$ be a vector bundle over M.*

*i.) The four locally convex topologies induced by the choices **A**, **B**, **A'**, and **B'** of seminorms coincide. Thus $\Gamma^\infty(E)$ has an intrinsic locally convex topology not depending on any of the above choices.*

ii.) $\Gamma^\infty(E)$ is a Fréchet space with respect to the above natural topology.

iii.) When restricting to those seminorms with $\ell \leq k$ for a fixed $k \in \mathbb{N}_0$, then we obtain natural Fréchet topologies for $\Gamma^k(E)$.

Proof. First we note that the topologies induced by **B** and **B'** are the same: indeed **B** is clearly finer than **B'** as it contains all the seminorms of **B'**. Conversely, we have $\mathrm{p}_{K,\ell}(s) \leq \mathrm{p}_{K',\ell}(s)$ for $K \subseteq K'$. Now if K_n is an exhausting sequence of compacta, then $K \subseteq K_n$ for sufficiently large n, hence the seminorm $\mathrm{p}_{K,\ell}$ can be dominated by $\mathrm{p}_{K_n,\ell}$. Thus the induced topologies are equivalent.

For the first version it is clear that **A** induces a finer topology than **A'** as **A** contains all seminorms from **A'**. Now let (U, x) be a chart of the chosen atlas and U_n the sequence of charts which already cover M which works since M is assumed to be second countable. Moreover, let $K_{n,m} \subseteq U_n$ be the exhausting sequence of compacta and let $K \subseteq U$ be given. Since K is compact, finitely many $U_{n_1}, \ldots, U_{n_k}$ already cover K. Furthermore, since the $\mathring{K}_{n,m}$ cover U_n, already finitely many $\mathring{K}_{n,m}$ cover K. Thus the compactum K is covered by finitely many of the $K_{n,m}$'s. From the chain rule it is clear that there are smooth functions $\Phi_{IJ} \in \mathcal{C}^\infty(U \cap \widetilde{U})$ such that for $|I| \leq \ell$

$$\frac{\partial^{|I|} f}{\partial x^I}\Big|_{U\cap\widetilde{U}} = \sum_{|J|\leq\ell} \Phi_{IJ} \frac{\partial^{|J|} f}{\partial \widetilde{x}^J}\Big|_{U\cap\widetilde{U}}$$

on the overlap of two charts (U, x) and $(\widetilde{U}, \widetilde{x})$. In fact, the Φ_{IJ} are certain polynomials in the partial derivatives of the Jacobian of the coordinate change. It follows that there is a constant c with

$$\mathrm{p}_{U,x,K,\ell}(f) \leq c\, \mathrm{p}_{\widetilde{U},\widetilde{x},K,\ell}(f)$$

for all $f \in \mathcal{C}^\infty(U \cap \widetilde{U})$ and $K \subseteq U \cap \widetilde{U}$ compact. The constant depends on U, x, K, ℓ, $\widetilde{U}$, and $\widetilde{x}$ but not on f. The precise form of c is irrelevant, it can be obtained from the maximum of the Φ_{IJ} over K where the Φ_{IJ} can be obtained recursively from the chain rule. From this we see that

$$\mathrm{p}_{U,x,K,\ell} \leq \max_{n,m} c_{n,m}\, \mathrm{p}_{U_n,x_n,K_{n,m},\ell},$$

where the maximum is taken over the finitely many n, m such that the $K_{n,m}$ cover K. This shows that the topology induced by **A'** is finer than the one obtained by **A**. Thus all together, they coincide.

Finally, let $\ell \in \mathbb{N}_0$ be given. By induction and the local expressions

$$\nabla^E_{\frac{\partial}{\partial x^i}} e_\alpha = A^\beta_{i\alpha} e_\beta \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}\, x^j = -\Gamma^j_{ik} \mathrm{d}\, x^k$$

with the connection one-forms and Christoffel symbols of ∇^E and ∇, respectively, we see that there exist smooth functions $a^{J}_{i_1 \ldots i_\ell}{}^{\gamma}_{\alpha} \in \mathcal{C}^\infty(U)$ such that

$$(\mathsf{D}^E)^\ell s\Big|_U = \sum_{|J|\leq\ell} \frac{1}{\ell!} a^{J}_{i_1 \ldots i_\ell}{}^{\gamma}_{\alpha}\, \mathrm{d}\, x^{i_1} \vee \cdots \vee \mathrm{d}\, x^{i_l} \otimes e_\gamma \frac{\partial^{|J|} s^\alpha}{\partial x^J}. \tag{$*$}$$

The precise form of the $a^{J}_{i_1\ldots i_\ell}{}^{\gamma}_{\alpha}$ is irrelevant, they can be obtained recursively as polynomials in the partial derivatives of the $A^{\beta}_{i\alpha}$ and Γ^k_{ij}. Moreover, for the term with highest derivatives, i.e. where $|J| = \ell$, we have the following explicit expression

$$(\mathsf{D}^E)^\ell s\Big|_U = \mathrm{d}\, x^{i_1} \vee \cdots \vee \mathrm{d}\, x^{i_\ell} \otimes e_\alpha \frac{\partial^\ell s^\alpha}{\partial x^{i_1} \cdots \partial x^{x_\ell}} + \text{(lower order terms)}. \qquad (**)$$

This can easily be obtained by induction since the difference between partial derivatives and covariant derivatives is given by additional terms involving the $A^{\beta}_{i\alpha}$ and Γ^k_{ij}. But these terms do not involve derivatives of the functions s^α. Now let K be a compactum. Then we find finitely many compacta $K_n \subseteq U_n$ contained in charts (U_n, x_n) such that the K_n cover K. In each chart (U, x) we see that there are constants $c_U > 0$ with

$$\left\|(\mathsf{D}^E)^\ell s\Big|_p\right\| \le c_U \max_{\substack{|J|\le \ell \\ \alpha}} \left|\frac{\partial^{|J|} s^\alpha}{\partial x^J}(p)\right| \qquad \text{for} \quad p \in U,$$

where c_U is obtained from the maximum of the $a^{J}_{i_1\cdots i_\ell}{}^{\gamma}_{\alpha}$ and the norms of the $\mathrm{d}\, x^{i_1} \vee \cdots \vee \mathrm{d}\, x^{i_\ell} \otimes e_\gamma$ with respect to the chosen fiber metrics according to $(*)$. But this shows that

$$\mathrm{p}_{K,\ell} \le c \max_n \mathrm{p}_{U_n, x_n, K_n, \ell, \{e_\alpha\}},$$

where the maximum is taken over the finitely many n such that K_n cover K and $c = \max_n c_{U_n}$. This shows that the topology induced by **A** is finer than the one induced by **B**. Conversely, given a $\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}$ we see from $(**)$ that we can estimate the partial derivatives $\frac{\partial^{|J|} s^\alpha}{\partial x^J}$ with $|J| \le \ell$ by norms of $(\mathsf{D}^E)^\ell s$ and norms of partial derivatives $\frac{\partial^{|J'|} s^\alpha}{\partial x^{J'}}$ for $|J'| < \ell$. By induction on ℓ we conclude that we can estimate the partial derivatives $\frac{\partial^{|J|} s^\alpha}{\partial x^J}$ with $|J| = \ell$ by norms of $(\mathsf{D}^E)^{\ell'} s$ with $\ell' \le \ell$. Since the relative coefficient functions are all smooth this gives a constant $c > 0$ such that

$$\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}(s) \le c \max_{\ell' \le \ell} \mathrm{p}_{K,\ell'}(s).$$

This shows that the topology induced by **B** is finer than the one induced by **A**. Thus, we have shown that all four topologies coincide. Since the version **A** does not depend on the choices of ∇^E, ∇ and the fiber metrics and since the version **B** does not depend on an atlas and local trivializations we see that the topology itself does not depend on any of the chosen data. Note however, that the particular systems of seminorms certainly do depend on these choices, only the resulting topology is independent.

For the second part, we first notice that the topology is certainly Hausdorff: the seminorms $\mathrm{p}_{\{p\},0}$ with $p \in M$ are already separating. Moreover, the versions **A'** and **B'** consist of countably many seminorms which define the topology. Here it is crucial to have second countable manifolds. Thus we only have to show completeness and thanks to the countably many seminorms we only have to consider Cauchy sequences and not general Cauchy nets. Thus let $s_n \in \Gamma^\infty(E)$ be a Cauchy sequence with respect to e.g. **A**. Taking $K = \{p\}$ a point and $\ell = 0$ we see that the sequence $s^\alpha_n(p) \in \mathbb{C}$ (or $\mathbb{R}$) is a Cauchy sequence and hence a convergent sequence. Thus $s_n(p) \longrightarrow s(p) \in E_p$ for some unique vector $s(p)$. This shows that there is a section $s : M \longrightarrow E$ of which we have to show smoothness. However, smoothness is a local concept which we can check in a local chart. But then the seminorms $\mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}$ just define the usual $\mathcal{C}^\infty$-topology of functions on U with values in $\mathbb{R}^N$ or $\mathbb{C}^N$, respectively, via the trivialization $\{e_\alpha\}$. Hence we conclude that all functions $s^\alpha = e^\alpha(s)$ are smooth and thus $s \in \Gamma^\infty(E)$ is a smooth section everywhere since by **A** we can cover the whole manifold with charts (U, x). Again we can argue locally to show that $s_n \longrightarrow s$ in the sense of **A**. This shows that $\Gamma^\infty(E)$ is (sequentially) complete which gives the second part. The third part is clear, we have shown the most difficult part $k = +\infty$ already. □

In the following, we shall always endow $\Gamma^\infty(E)$ as well as $\Gamma^k(E)$ with these naturally defined topologies.

Definition 1.1.6 ($\mathcal{C}^\infty$-Topology) *The natural Fréchet topology of $\Gamma^\infty(E)$ is called the $\mathcal{C}^\infty$-topology. Analogously, we call the natural Fréchet topology of $\Gamma^k(E)$ the $\mathcal{C}^k$-topology.*

Remark 1.1.7 ($\mathcal{C}^\infty$-Topology)

i.) A sequence $s_n \in \Gamma^\infty(E)$ converges to s with respect to the $\mathcal{C}^\infty$-topology if and only if s_n converges uniformly on all compact subsets of M with all derivatives to s. Similar, the convergence in the $\mathcal{C}^k$-topology is the locally uniform convergence in the first k derivatives.

ii.) If M is compact, we can use $K = M$ in the seminorms of **A** and **B**. This shows that the $\mathcal{C}^k$-topology is even a *Banach topology* since we can also take the maximum $0 \leq \ell \leq k$. Thus for this particular case, techniques from Banach space analysis become available. However, the $\mathcal{C}^\infty$-topology is not Banach, even if M is compact. In the non compact situation, none of the $\mathcal{C}^k$-topologies is Banach.

iii.) The case of smooth functions instead of smooth sections is somewhat easier. Here we do not need the additional local base sections $\{e_\alpha\}$, hence from **A** we obtain seminorms $\mathrm{p}_{U,x,K,\ell}$. In the second version, we do not need the additional covariant derivative ∇^E nor the fiber metric on E but only ∇ and a Riemannian metric on M.

Remark 1.1.8 In the following we can use either types of seminorms to characterize the $\mathcal{C}^\infty$-topology. Since the main importance of the seminorms is to control derivatives of order up to ℓ on a compactum K we shall sometimes symbolically write $\mathrm{p}_{K,\ell}$ for the seminorms obtained from *either* the maximum of some finitely many $\mathrm{p}_{U_n,x_n,K_n,\ell,\{e_{n\alpha}\}}$ where the K_n are such that they cover K from the seminorms of type **A** *or* the maximum of the $\mathrm{p}_{K,\ell'}$ with $\ell' \leq \ell$ from the seminorms of type **B**. Clearly, the seminorms $\mathrm{p}_{K,\ell}$ obtained this way specify the topology already completely and are filtrating and Hausdorff. It should become clear from the context whether we apply these symbolic seminorms or the more concrete ones as in **A** or **B**.

On a non compact manifold the space $\mathcal{C}^\infty_0(M)$ is a proper subspace of all smooth functions $\mathcal{C}^\infty(M)$. Analogously, $\Gamma^\infty_0(M)$ is a proper subspace of $\Gamma^\infty(M)$ for every vector bundle of positive rank. The following proposition shows that we can use sections with compact support to approximate arbitrary ones.

Proposition 1.1.9 *For a vector bundle $E \longrightarrow M$ the subspace $\Gamma^\infty_0(M)$ of compactly supported sections is dense in $\Gamma^\infty(M)$ with respect to the $\mathcal{C}^\infty$-topology. Analogously, $\Gamma^k_0(E)$ is dense in $\Gamma^k(E)$ in the $\mathcal{C}^k$-topology for all $k \in \mathbb{N}_0$.*

Proof. We choose an exhausting sequence $\ldots K_n \subseteq \mathring{K}_{n+1} \subseteq K_{n+1} \subseteq \ldots \subset M$ of compacta and appropriate functions $\chi_n \in \mathcal{C}^\infty_0(M)$ with the property

$$\chi_n\big|_{K_n} = 1 \quad \text{and} \quad \operatorname{supp}(\chi_n) \subseteq K_{n+1}.$$

Clearly, such χ_n exists thanks to the $\mathcal{C}^\infty$-version of the Urysohn Lemma, see e.g. [60, Kor. A.1.5]. Then for $s \in \Gamma^\infty(E)$ we define $s_n = \chi_n s \in \Gamma^\infty_0(E)$ and have for all $\ell \in \mathbb{N}_0$

$$\mathrm{p}_{K_n,\ell}(s - s_m) = 0,$$

for $m \geq n$. This shows that $s_n \longrightarrow s$ in the $\mathcal{C}^\infty$-topology. For the $\mathcal{C}^k$-topology the argument is the same. □

While on one hand, the above statement will be very useful to approximate sections by compactly supported sections, it shows on the other hand that the $\mathcal{C}^\infty$-topology is not appropriate for $\Gamma^\infty_0(M)$

as this subspace is *not complete* in the $\mathcal{C}^\infty$-topology. Thus we are looking for a *finer* locally convex topology which makes $\Gamma_0^\infty(M)$ complete. The construction is based on the following observation:

Lemma 1.1.10 *Let $A \subseteq M$ be a closed subset and let*

$$\Gamma_A^k(E) = \left\{ s \in \Gamma^k(E) \,\middle|\, \operatorname{supp}(s) \subseteq A \right\}. \tag{1.1.12}$$

Then $\Gamma_A^k(E) \subseteq \Gamma^k(E)$ is a closed subspace with respect to the $\mathcal{C}^k$-topology for all $k \in \mathbb{N}_0 \cup \{+\infty\}$.

Proof. Since we are in a Fréchet situation, it is sufficient to consider sequences in order to approach the closure. Thus let $s_n \in \Gamma_A^k(E)$ with $s_n \longrightarrow s \in \Gamma^k(E)$ be given. Since $\mathcal{C}^k$-convergence implies pointwise convergence we see that for $p \in M \setminus A$

$$0 = s_n(p) \longrightarrow s(p),$$

whence $s(p) = 0$. Thus $\operatorname{supp}(s) \subseteq A$ as desired and $s \in \Gamma_A^k(E)$ follows. □

This way, the $\Gamma_A^k(E)$ become Fréchet spaces themselves being closed subspaces of the Fréchet space $\Gamma^k(E)$. We call the resulting topology the $\mathcal{C}_A^k$-topology. With respect to their induced topology, the inclusion maps

$$\Gamma_A^k(E) \hookrightarrow \Gamma_{A'}^k(E) \tag{1.1.13}$$

for $A \subseteq A'$ are continuous and have closed image. This is clear as the seminorms $\mathrm{p}_{K,\ell}$ needed for $\Gamma_A^k(E)$ are also continuous seminorms on $\Gamma_{A'}^k(E)$. Moreover, the induced topology on $\Gamma_A^k(E)$ by (1.1.13) is again the $\mathcal{C}_A^k$-topology. Thus (1.1.13) is an embedding and not just an injective continuous map. We shall now focus on compact subsets $K \subseteq M$ and choose an exhausting sequence K_n as before. Then the corresponding sequence

$$\Gamma_{K_0}^k(E) \hookrightarrow \Gamma_{K_1}^k(E) \hookrightarrow \cdots \hookrightarrow \Gamma_{K_n}^k(E) \hookrightarrow \Gamma_{K_{n+1}}^k(E) \hookrightarrow \cdots \hookrightarrow \Gamma_0^k(E) \tag{1.1.14}$$

allows to endow the "limit" $\Gamma_0^k(E)$ with the *inductive limit topology*. Since all the inclusions are embeddings and since we only need countably many compacta, we have a *countable strict inductive limit topology* (or *LF topology*) for $\Gamma_0^k(E)$. By general nonsense on such limit topologies, see e.g. [34, Sect. 4.6], we obtain the following characterization of a locally convex topology on $\Gamma_0^k(E)$, which we call the $\mathcal{C}_0^\infty$-topology:

Theorem 1.1.11 ($\mathcal{C}_0^\infty$-topology) *Let $k \in \mathbb{N}_0 \cup \{+\infty\}$. The inductive limit topology on $\Gamma_0^k(E)$ obtained from* (1.1.14) *enjoys the following properties:*

i.) *$\Gamma_0^k(E)$ is a Hausdorff locally convex complete and sequentially complete topological vector space. The topology does not depend on the chosen sequence of exhausting compacta.*

ii.) *All the inclusion maps*

$$\Gamma_K^k(E) \hookrightarrow \Gamma_0^k(E) \tag{1.1.15}$$

are continuous and the $\mathcal{C}_0^k$-topology is the finest locally convex topology on $\Gamma_0^k(E)$ with this property. Every $\Gamma_K^k(E)$ is closed in $\Gamma_0^k(E)$ and the induced topology on $\Gamma_K^k(E)$ is the $\mathcal{C}_K^k$-topology.

iii.) *A sequence $s_n \in \Gamma_0^k(E)$ is a $\mathcal{C}_0^k$-Cauchy sequence if and only if there exists a compact subset $K \subseteq M$ with $s_n \in \Gamma_K^k(E)$ for all n and s_n is a $\mathcal{C}_K^k$-Cauchy sequence. An analogous statement holds for convergent sequences.*

iv.) *If V is a locally convex vector space, then a linear map $\Phi : \Gamma_0^k(E) \longrightarrow V$ is $\mathcal{C}_0^k$-continuous if and only if each restriction $\Phi\big|_{\Gamma_K^k(E)} : \Gamma_K^k(E) \longrightarrow V$ is $\mathcal{C}_K^k$ continuous. It suffices to consider an exhausting sequence of compacta.*

v.) *If M is non compact $\Gamma_0^k(E)$ is not first countable and hence not metrizable.*

Proof. We shall only sketch the arguments and refer to [34, Sect. 4.6] for details on strict inductive limit topologies. The first part follows from general nonsense on countable strict inductive limit topologies since all the constituents $\Gamma^k_K(E)$ are Fréchet spaces. The second part is an alternative characterization of inductive limit topologies. Part *iii.)* and *iv.)* are also general facts on inductive limit topologies. The last part follows essentially from Baire's theorem. □

Remark 1.1.12 In the sequel, we only need the properties *i.) – iv.)* of the $\mathcal{C}^k_0$-topology, not its precise definition. In fact, it will turn out that the actual handling of this rather complicated LF topology is fairly easy. We refer to the literature for more background information on LF topologies, see e.g. [34, Sect. 4.6] or [36, 37, 58]. Of course, we are mainly interested in the case $k = \infty$.

Remark 1.1.13 We also remark that the inclusion maps $\Gamma^k_0(E) \hookrightarrow \Gamma^k(E)$ are continuous for all $k \in \mathbb{N}_0 \cup \{+\infty\}$.

1.1.2 Continuous Maps between Test Section Spaces

In this subsection we shall collect some basic examples of maps between test function and test section spaces which on one hand have a geometric origin, and which on the other hand are continuous in the $\mathcal{C}^k$- and $\mathcal{C}^k_0$-topologies, respectively. We start with the following situation:

Proposition 1.1.14 *Let $\phi : M \longrightarrow N$ be a smooth map. Then the pull-back $\phi^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M)$ is a continuous linear map with respect to the $\mathcal{C}^\infty$-topology.*

Proof. Let $K \subseteq M$ be a compact subset and $\ell \in \mathbb{N}_0$ be given. Moreover, let (U, x) be a chart with $K \subseteq U$. Then we consider the compact subset $\phi(K) \subseteq N$. This will be covered by finitely many charts (V, y) of N and we can assume that already one chart will do the job. Then we compute by the chain rule

$$\mathrm{p}_{U,x,K,\ell}(\phi^* f) = \sup_{\substack{p\in K\\ |I|\le\ell}} \left| \frac{\partial^{|I|}\phi^* f}{\partial x^I}(p) \right| = \sup_{\substack{p\in K\\ |I|\le\ell}} \left| \sum_{J\le I} \Phi_{IJ}(p) \frac{\partial^{|J|} f}{\partial y^J}(\phi(p)) \right|,$$

where again the Φ_{IJ} are smooth functions on U obtained from polynomials in the derivatives of the Jacobi matrix of the map ϕ with respect to the charts (V, y) and (U, x). Since ϕ is smooth the maps Φ_{IJ} turn out to be smooth, too, hence on K they are bounded. Moreover, the partial derivatives of f on $\phi(K)$ are bounded as well so we finally obtain an estimate

$$\mathrm{p}_{U,x,K,\ell}(\phi^* f) \le c\, \mathrm{p}_{V,y,\phi(K),\ell}(f),$$

where the constant c depends on the maxima of the functions Φ_{IJ} over K and thus on ϕ but not on f. But this is the desired continuity. □

Remark 1.1.15 Since in the proof we estimated a seminorm with order of differentiation ℓ again by a seminorm with order of differentiation ℓ, the statement remains true for a $\mathcal{C}^k$-map $\phi : M \longrightarrow N$: the pullback $\phi^* : \mathcal{C}^k(N) \longrightarrow \mathcal{C}^k(M)$ is $\mathcal{C}^k$-continuous.

For functions with compact support the pull-back $\phi^* f$ with an arbitrary map $\phi : M \longrightarrow N$ will no longer have compact support in general. Take e.g. any smooth map $\phi : M \longrightarrow N$ from a non compact manifold M into a compact one, then $\phi^* 1_N = 1_M$ but $1_N \in \mathcal{C}^\infty(N) = \mathcal{C}^\infty_0(N)$ and $1_M \notin \mathcal{C}^\infty_0(M)$. Thus we need an extra condition to assure that ϕ^* maps $\mathcal{C}^\infty_0(N)$ into $\mathcal{C}^\infty_0(M)$:

Definition 1.1.16 (Proper map) *A smooth map $\phi : M \longrightarrow N$ is called proper if $\phi^{-1}(K) \subseteq M$ is compact for all compact $K \subseteq N$.*

The above definition makes perfect sense in a general topological context, the smoothness and the manifold structure of M, N are not needed. Note that a continuous map maps compact subsets to compact subsets, but inverse images of compact subsets need not be compact as the above example shows.

Proposition 1.1.17 *Let $\phi : M \longrightarrow N$ be a smooth proper map. Then*

$$\phi^* : \mathcal{C}^\infty_0(N) \longrightarrow \mathcal{C}^\infty_0(M) \tag{1.1.16}$$

is continuous in the $\mathcal{C}^\infty_0$-topology.

Proof. Let $K \subseteq N$ be compact. By Theorem 1.1.11, *iv.)* we have to show that the restriction $\phi^*\big|_{\mathcal{C}^\infty_K(N)} : \mathcal{C}^\infty_K(N) \longrightarrow \mathcal{C}^\infty_0(M)$ is continuous. Now $\phi^{-1}(K)$ is compact since ϕ is proper and thus we know

$$\phi^* : \mathcal{C}^\infty_K(N) \longrightarrow \mathcal{C}^\infty_{\phi^{-1}(K)}(M),$$

since in general $\operatorname{supp}(\phi^* f) = \phi^{-1}(\operatorname{supp}(f))$. The proof of Proposition 1.1.14 shows that the $\mathrm{p}_{\phi^{-1}(K),\ell}$-seminorms of the images of ϕ^* can be estimated by the $\mathrm{p}_{K,\ell}$-seminorms. Thus ϕ^* is continuous. Finally, we know that

$$\mathcal{C}^\infty_{\phi^{-1}(K)}(M) \hookrightarrow \mathcal{C}^\infty_0(M)$$

is continuous by Theorem 1.1.11, *ii.)*. Thus the criterion for the continuity of ϕ^* is fulfilled. □

Remark 1.1.18 Again, there is a $\mathcal{C}^k_0$-version of this statement since we only used the same ℓ for the estimation in the proof of Proposition 1.1.14.

In a last step, we shall treat test sections of vector bundles. Let $E \longrightarrow M$ and $F \longrightarrow M$ be vector bundles. Since a smooth map $\phi : M \longrightarrow N$ alone does not yield any map between $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$ by itself, we need a vector bundle morphism. Recall that a *vector bundle morphism* $\Phi : E \longrightarrow F$ is a smooth map such that Φ maps fibers of E into fibers of F and Φ is linear on each fiber. Thus Φ induces a smooth map ϕ such that

$$\begin{array}{ccc} E & \xrightarrow{\ \Phi\ } & F \\ {\scriptstyle \iota_E}\uparrow\downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[\ \phi\ ]{} & N \end{array} \tag{1.1.17}$$

commutes. Indeed, $\phi = \pi_F \circ \Phi \circ \iota_E$, where $\iota_E : M \longrightarrow E$ denotes the zero section.

Lemma 1.1.19 *Let $\Phi : E \longrightarrow F$ be a vector bundle morphism and $\omega \in \Gamma^\infty(F^*)$. Then*

$$(\Phi^*\omega)\big|_p(s_p) = \omega\big|_{\phi(p)}(\Phi(s_p)) \tag{1.1.18}$$

for $s_p \in E_p$ and $p \in M$ defines a smooth section $\Phi^*\omega \in \Gamma^\infty(E^*)$ called the pull-back of ω by Φ.

Proof. It is easy to check that for $s \in \Gamma^\infty(E)$ the function $p \mapsto \Phi^*\omega\big|_p(s(p))$ is smooth whence $\Phi^*\omega$ is smooth itself. Moreover, $\Phi^*\omega\big|_p : E_p \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is clearly linear hence the statement follows. □

The pull-back indeed obeys the usual properties of a pull-back, i.e. for vector bundle morphisms $E \xrightarrow{\Phi} F \xrightarrow{\Psi} G$ we have

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \qquad \text{and} \qquad (\mathsf{id}_E)^* = \mathsf{id}_{\Gamma^\infty(E^*)}\,. \tag{1.1.19}$$

We claim that this gives again continuous maps.

Proposition 1.1.20 *Let $\Phi: E \longrightarrow F$ be a vector bundle morphism. Then $\Phi^*: \Gamma^\infty(F^*) \longrightarrow \Gamma^\infty(E^*)$ is continuous with respect to the $\mathcal{C}^\infty$-topology.*

Proof. We first need some local expressions. Let $e_\alpha \in \Gamma^\infty(E\big|_U)$ and $f_\beta \in \Gamma^\infty(F\big|_V)$ be local base sections defined over open subsets $U \subseteq M$ and $V \subseteq N$. We assume that on V we have local coordinates y and x on $U \subseteq \phi^{-1}(V)$. By choosing V and U sufficiently small this is possible. Then $\Phi\big|_{E|_U}$ can be written as follows. For $s_p = s_p^\alpha e_\alpha(p) \in E_p$ there exist coefficients $\Phi_\alpha^\beta(p)$ such that

$$\Phi(s_p) = s_p^\alpha \Phi_\alpha^\beta(p) f_\beta(\phi(p)),$$

since $\Phi(s_p) \in F_{\phi(p)}$ for all $s_p \in E_p$ and $p \in M$. The smoothness of Φ gives the smoothness of the locally defined functions $\Phi_\alpha^\beta \in \mathcal{C}^\infty(U)$. Now let $\omega \in \Gamma^\infty(F^*)$ be given as

$$\omega\big|_V = \omega_\beta f^\beta,$$

where $f^\beta \in \Gamma^\infty(F^*\big|_V)$ are the dual base sections of the f_β as usual. Then

$$(\Phi^*\omega)(s)\big|_U = (\omega \circ \phi)(\Phi(s)) = \Big((\omega_\beta \circ \phi)(f^\beta \circ \phi)\Big)\left(\Phi_\alpha^\gamma (f_\gamma \circ \phi) s^\alpha\right) = \phi^*(\omega_\beta)\Phi_\alpha^\beta s^\alpha.$$

Hence we have $\Phi^*\omega\big|_U = \phi^*(\omega_\beta)\Phi_\alpha^\beta e^\alpha$. Now we can estimate

$$\begin{aligned} \mathrm{p}_{U,x,K,\ell,\{e_\alpha\}}(\Phi^*\omega) &= \sup_{\substack{|I|\le \ell \\ p\in K \\ \alpha=1,\dots,\mathrm{rank}(E)}} \left| \frac{\partial^{|I|}}{\partial x^I} (\Phi^*\omega)_\alpha(p) \right| \\ &= \sup_{\substack{|I|\le \ell \\ p\in K \\ \alpha=1,\dots,\mathrm{rank}(E)}} \left| \frac{\partial^{|I|}}{\partial x^I} \left(\phi^*(\omega_\gamma)\Phi_\alpha^\gamma\right)(p) \right| \\ &\le c\, \mathrm{p}_{V,y,\phi(K),\ell,\{f^\beta\}}(\omega), \end{aligned}$$

by the same kind of computation as for the proof of Proposition 1.1.14. The constant c involves the maxima of polynomials in the partial derivatives of the Jacobi matrix of ϕ as well as of Φ_α^β, again by the chain rule and the Leibniz rule. But then the continuity is clear. □

Remark 1.1.21 (Pull-back of sections)

i.) For the support of $\Phi^*\omega$ we obtain

$$\operatorname{supp}\Phi^*\omega \subseteq \phi^{-1}(\operatorname{supp}\omega), \tag{1.1.20}$$

which is immediate from the definition. Note that due to possible degeneration in the fiberwise maps $\Phi\big|_{E_p}$ the support may be strictly smaller than the right hand side.

ii.) Again, for a vector bundle morphism $\Phi: E \longrightarrow F$ of class $\mathcal{C}^k$ we obtain a continuous map

$$\Phi^*: \Gamma^k(F^*) \longrightarrow \Gamma^k(E^*) \tag{1.1.21}$$

with respect to the $\mathcal{C}^k$-topology.

Example 1.1.22 (Tangent map) Let $\phi : M \longrightarrow N$ be a smooth map. Then $T\phi : TM \longrightarrow TN$ is a smooth vector bundle morphism over ϕ. Thus the pull-back gives $(T\phi)^* : \Gamma^\infty(T^*N) \longrightarrow \Gamma^\infty(T^*M)$. Clearly, the pull-back $(T\phi)^*$ in the sense of Lemma 1.1.19 coincides with the usual pull-back ϕ^* of one-forms in this case. Note that if ϕ is $\mathcal{C}^k$ then $T\phi$ is only of class $\mathcal{C}^{k-1}$. Moreover, $T\phi$ extends to vector bundle morphisms $(T\phi)^{\otimes r} : \otimes^r TM \longrightarrow \otimes^r TN$ hence we also obtain pull-backs $\phi^* : \Gamma^\infty(\otimes^r T^*N) \longrightarrow \Gamma^\infty(\otimes^r T^*M)$ being continuous linear maps with respect to the $\mathcal{C}^\infty$-topology.

The case of compactly supported sections is treated analogously to the case of $\mathcal{C}^\infty_0(N)$. Using (1.1.20) we can copy the proof of Proposition 1.1.17 and obtain the following result:

Proposition 1.1.23 *Let $\Phi : E \longrightarrow F$ be a vector bundle morphism such that the induced map $\phi : M \longrightarrow N$ is proper. Then the pull-back*

$$\Phi^* : \Gamma^\infty_0(F^*) \longrightarrow \Gamma^\infty_0(E^*) \tag{1.1.22}$$

is continuous with respect to the $\mathcal{C}^\infty_0$-topology. Analogous statements hold for the $\mathcal{C}^k$ case.

We conclude this section with yet another type of maps, namely the module structures and various tensor products.

Proposition 1.1.24 *Let $E \longrightarrow M$ and $F \longrightarrow M$ be vector bundles.*

i.) The pointwise multiplication

$$\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \ni (f, g) \mapsto fg \in \mathcal{C}^\infty(M) \tag{1.1.23}$$

is continuous with respect to the $\mathcal{C}^\infty$-topology, hence $\mathcal{C}^\infty(M)$ becomes a Fréchet algebra.

ii.) The module structure

$$\mathcal{C}^\infty(M) \times \Gamma^\infty(E) \ni (f, s) \mapsto f \cdot s \in \Gamma^\infty(E) \tag{1.1.24}$$

is continuous with respect to the $\mathcal{C}^\infty$-topology, hence $\Gamma^\infty(E)$ becomes a Fréchet module over the Fréchet algebra $\mathcal{C}^\infty(M)$.

iii.) The tensor product

$$\Gamma^\infty(E) \times \Gamma^\infty(F) \ni (s, t) \mapsto s \otimes t \in \Gamma^\infty(E \otimes F) \tag{1.1.25}$$

is continuous with respect the $\mathcal{C}^\infty$-topology.

iv.) The natural pairing

$$\Gamma^\infty(E^*) \times \Gamma^\infty(E) \ni (\omega, s) \mapsto \omega(s) \in \mathcal{C}^\infty(M) \tag{1.1.26}$$

is continuous with respect to the $\mathcal{C}^\infty$-topology.

Analogous statements hold for the $\mathcal{C}^k$ case.

Proof. All the above statements rely only on the Leibniz rule for differentiation of products. Let $K \subseteq U$ be compact and let x be local coordinates on U, then

$$\mathrm{p}_{U,x,K,\ell}(fg) = \sup_{\substack{p \in K \\ |I| \le \ell}} \left| \frac{\partial^{|I|}}{\partial x^I}(fg)\Big|_p \right| = \sup_{\substack{p \in K \\ |I| \le \ell}} \left| \sum_{J \le I} \binom{I}{J} \frac{\partial^{|J|} f}{\partial x^J}(p) \frac{\partial^{|I-J|} g}{\partial x^{I-J}}(p) \right| \le c\, \mathrm{p}_{U,x,K,\ell}(f)\, \mathrm{p}_{U,x,K,\ell}(g),$$

with a constant only depending on the combinatorics of the multinomial coefficients $\binom{I}{J}$ and hence only on ℓ. This shows the first part. Writing out the local expressions for all the other parts in terms of coefficient functions and local base sections shows that all other parts can be reduced to part *i.)* and hence the above computation. □

Remark 1.1.25 As usual there are $\mathcal{C}^k_0$-versions of this statement. Moreover, we have analogous statements for various multilinear pairings and applications of endomorphisms to sections etc.

1.1.3 Approximations

In this subsection we shall sketch some approximation results of how less differentiable functions can be approximated by smooth ones. This rather technical section will turn out to be useful in many places.

Theorem 1.1.26 *Let $E \longrightarrow M$ be a vector bundle. Then $\Gamma^\infty_0(E)$ is (sequentially) dense in $\Gamma^k(E)$ for all $k \in \mathbb{N}_0$ with respect to the $\mathcal{C}^k$-topology.*

Proof. First we know from Proposition 1.1.9 that $\Gamma^k_0(E)$ is dense in $\Gamma^k(E)$. Thus we only have to show that $\Gamma^\infty_0(E)$ is dense in $\Gamma^k_0(E)$ with respect to the $\mathcal{C}^k$-topology thanks to the continuous embedding of $\mathcal{C}^k_0(E)$ into $\mathcal{C}^k(E)$ according to Remark 1.1.13. Let $s \in \Gamma^k_0(E)$ be given. Then we choose charts (U_i, x_i) of M together with local base sections $e_{\alpha i} \in \Gamma^\infty(E\big|_{U_i})$. Moreover, we choose a partition of unity $\varphi_i \in \mathcal{C}^\infty_0(M)$ with $\operatorname{supp}\varphi_i \subseteq U_i$ being compact and $\sum \varphi_i = 1$. The compactness of $\operatorname{supp} s$ guarantees that finitely many U_i already cover $\operatorname{supp} s$ and hence

$$s = \sum\nolimits_i \varphi_i s,$$

with a finite sum. Thus we only have to approximate a $\varphi_i s \in \Gamma^k_0(E)$ where $\operatorname{supp}(\varphi_i s) \subseteq U_i$ is in the domain of a chart. Now

$$\varphi_i s = s^\alpha_i e_{\alpha i}$$

with $s^\alpha_i \in \mathcal{C}^k_0(U_i)$. From the local theory we know that we find functions $s^\alpha_{im} \in \mathcal{C}^\infty_0(U_i)$ with $s^\alpha_{im} \to s^\alpha_i$ in the $\mathcal{C}^k$-topology: e.g. one can use a convolution of the s^α_i with a function $\chi_m(x) = m^n \chi(mx)$, where $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ is a function with $\int \chi(x)\, \mathrm{d}^n x = 1$. Then for sufficiently large m

$$(s^\alpha_i \circ x_i) * \chi_m \in \mathcal{C}^\infty_0(x(U)),$$

hence

$$s^\alpha_{im} = ((s^\alpha_i \circ x_i) * \chi_m) \circ x^{-1} \in \mathcal{C}^\infty_0(U)$$

is smooth and fulfills $s^\alpha_{im} \longrightarrow s^\alpha_i$ in the $\mathcal{C}^k$-topology. For details see e.g. [31, Thm. 1.3.2]. Since we can approximate each s^α_i we also can approximate $s_i = s^\alpha_i e_{\alpha i}$ and thus $s = \sum_i \varphi_i s = \sum_i s_i$ since the sums are always finite. □

1.2 Differential Operators

In this section we introduce differential operators on sections of vector bundles and discuss their continuity properties with respect to the various $\mathcal{C}^k$- and $\mathcal{C}^k_0$-topologies.

1.2.1 Differential Operators and their Symbols

There are several equivalent definitions of differential operators on manifolds. We present here the most pragmatic one. Let $E \longrightarrow M$ and $F \longrightarrow M$ be vector bundles over M.

Definition 1.2.1 (Differential operators) *Let $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ be a linear map. Then D is called differential operator of order $k \in \mathbb{N}_0$ if the following conditions are fulfilled.*

i.) D can be restricted to open subsets $U \subseteq M$, i.e. for any open subset $U \subseteq M$ there exists a linear map $D_U : \Gamma^\infty(E\big|_U) \longrightarrow \Gamma^\infty(F\big|_U)$ such that

$$D_U(s\big|_U) = (Ds)\big|_U \tag{1.2.1}$$

for all sections $s \in \Gamma^\infty(E)$.

ii.) *In any chart* (U, x) *of* M *and for every local base sections* $e_\alpha \in \Gamma^\infty(E\big|_U)$ *and* $f_\beta \in \Gamma^\infty(F\big|_U)$ *we have*

$$Ds\big|_U = \sum_{r=0}^{k} \frac{1}{r!} D_U^{i_1 \ldots i_r}{}^\beta_\alpha f_\beta \frac{\partial^r s^\alpha}{\partial x^{i_1} \cdots \partial x^{i_r}} \tag{1.2.2}$$

with locally defined functions $D_U^{i_1 \ldots i_r}{}^\beta_\alpha \in \mathcal{C}^\infty(U)$*, totally symmetric in* $i_1, \ldots, i_r$*.*

The set of differential operators $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ of order $k \in \mathbb{N}_0$ is denoted by $\mathrm{DiffOp}^k(E; F)$ and we define

$$\mathrm{DiffOp}^\bullet(E; F) = \bigcup_{k=0}^{\infty} \mathrm{DiffOp}^k(E; F). \tag{1.2.3}$$

Remark 1.2.2 (Differential operators)

i.) Clearly, $\mathrm{DiffOp}^k(E; F)$ is a vector space and we have

$$\mathrm{DiffOp}^k(E; F) \subseteq \mathrm{DiffOp}^{k+1}(E; F) \tag{1.2.4}$$

for all $k \in \mathbb{N}_0$. Thus $\mathrm{DiffOp}(E; F)$ is a filtered vector space. Note however that (1.2.3) does not yield a *graded* vector space.

ii.) The restriction of a differential operator D is important since we also want to apply D to sections which are only locally defined.

iii.) If we are given an atlas of charts and local bases and locally defined functions $D_U^{i_1 \ldots i_r}{}^\beta_\alpha$, then we can define a global differential operator D by specifying its local form as in (1.2.2), *provided* the functions $D_U^{i_1 \ldots i_r}{}^\beta_\alpha$ transform in such a way that two definitions agree on the overlap of any two charts in that atlas. In fact, the precise transformation law of the $D_U^{i_1 \ldots i_r}{}^\beta_\alpha$ is rather complicated thanks to the complicated form of the chain rule for multiple partial derivatives.

iv.) Differential operators are *local*, i.e. $\mathrm{supp}(Ds) \subseteq \mathrm{supp}(s)$.

Lemma 1.2.3 (Leading symbol) *If* $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ *is a differential operator of order* $k \in \mathbb{N}_0$*, locally given by* (1.2.2)*, then the definition*

$$\sigma_k(D)\big|_U = \frac{1}{k!} D_U^{i_1 \ldots i_k}{}^\beta_\alpha \frac{\partial}{\partial x^{i_1}} \vee \cdots \vee \frac{\partial}{\partial x^{i_k}} \otimes f_\beta \otimes e^\alpha \tag{1.2.5}$$

yields a globally well-defined tensor field, called the leading symbol of D

$$\sigma_k(D) \in \Gamma^\infty(\mathrm{S}^k TM \otimes F \otimes E^*). \tag{1.2.6}$$

Proof. This is a straightforward computation since the terms with maximal number of derivatives of s^α in (1.2.2) transform nicely. □

Note that there is no intrinsic way to define "sub-leading" symbols of a differential operator of order $k \geq 2$. The functions $D_U^{i_1 \ldots i_r}{}^\beta_\alpha$ do not have a tensorial transformation law. In fact, terms with different r even mix. This is also the reason that we can only speak of the maximal number of partial derivatives appearing in (1.2.2) as "order". There is no intrinsic way to characterize differential operators "with exactly k partial derivatives": this would be a chart dependent statement.

Since canonically $F \otimes E^* \simeq \mathsf{Hom}(E, F)$, we can interpret the leading symbol $\sigma_k(D)$ also as a section

$$\sigma_k(D) \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F)). \tag{1.2.7}$$

We shall sketch now another, more conceptual approach to differential operators, see [26, Def. 16.8.1]: it is essentially based on the observation that for a differential operator D the commutator $[D, f]$ with a left multiplication by $f \in \mathcal{C}^\infty(M)$ is a differential operator of at least one order less than D because of the Leibniz rule. We consider an associative, commutative algebra $\mathcal{A}$ over some ground field $\Bbbk$. Of course, we are mainly interested in $\mathcal{A} = \mathcal{C}^\infty(M)$ and $\Bbbk = \mathbb{C}$. Next we consider two $\mathcal{A}$-modules $\mathcal{E}, \mathcal{F}$ and set for $k < 0$

$$\mathrm{DiffOp}^k(\mathcal{E}; \mathcal{F}) = \{0\} \tag{1.2.8}$$

and for $k \geq 0$ inductively

$$\mathrm{DiffOp}^k(\mathcal{E}; \mathcal{F}) = \left\{ D \in \mathsf{Hom}_{\Bbbk}(\mathcal{E}, \mathcal{F}) \;\middle|\; [D, L_a] \in \mathrm{DiffOp}^{k-1}(\mathcal{E}; \mathcal{F}) \; \forall a \in \mathcal{A} \right\}, \tag{1.2.9}$$

where L_a denotes the left multiplication of elements in the module with a. As before we set

$$\mathrm{DiffOp}^\bullet(\mathcal{E}; \mathcal{F}) = \bigcup_{k \in \mathbb{Z}} \mathrm{DiffOp}^k(\mathcal{E}; \mathcal{F}). \tag{1.2.10}$$

By general considerations it is rather easy to show that $\mathrm{DiffOp}^k(\mathcal{E}, \mathcal{F}) \subseteq \mathrm{DiffOp}^{k+1}(\mathcal{E}, \mathcal{F})$ whence (1.2.10) is again filtered. Moreover, $\mathrm{DiffOp}^k(\mathcal{E}; \mathcal{F})$ is a $\Bbbk$-vector space and a left $\mathcal{A}$-module via

$$(a \cdot D)(e) = a \cdot D(e), \tag{1.2.11}$$

where $a \in \mathcal{A}$, $D \in \mathrm{DiffOp}^k(\mathcal{E}; \mathcal{F})$, and $e \in \mathcal{E}$. If $\mathcal{G}$ is yet another $\mathcal{A}$-module then the composition of differential operators is defined and yields again differential operators. In fact,

$$\mathrm{DiffOp}^k(\mathcal{F}; \mathcal{G}) \circ \mathrm{DiffOp}^\ell(\mathcal{E}; \mathcal{F}) \subseteq \mathrm{DiffOp}^{k+\ell}(\mathcal{E}; \mathcal{G}) \tag{1.2.12}$$

holds for all $k, \ell \in \mathbb{Z}$. It follows that

$$\mathrm{DiffOp}^\bullet(\mathcal{E}) = \mathrm{DiffOp}^\bullet(\mathcal{E}; \mathcal{E}) \tag{1.2.13}$$

is a filtered subalgebra of all $\Bbbk$-linear endomorphisms $\mathsf{End}_{\Bbbk}(\mathcal{E})$ of $\mathcal{E}$. Moreover, by definition we have

$$\mathrm{DiffOp}^0(\mathcal{E}; \mathcal{F}) = \mathsf{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}). \tag{1.2.14}$$

Theorem 1.2.4 *For $\mathcal{A} = \mathcal{C}^\infty(M)$ and $\mathcal{E} = \Gamma^\infty(E)$, $\mathcal{F} = \Gamma^\infty(F)$ the algebraic definition of* $\mathrm{DiffOp}^\bullet(\mathcal{E}; \mathcal{F})$ *yields the usual differential operators* $\mathrm{DiffOp}^\bullet(E; F)$.

The proof is contained e.g. in [60, App. A.5]. We omit it here as we shall mainly work with the local description of differential operators.

1.2.2 A Global Symbol Calculus for Differential Operators

The leading symbol of a differential operator is in many aspects a much nicer object as it is a tensor field. The problem of having no canonical definition of sub-leading symbols can be cured at the price of a covariant derivative. We choose a torsion-free covariant derivative ∇ for the tangent bundle as well as a covariant derivative ∇^E for E. Then for the operator of symmetrized covariant differentiation D^E as in Definition 1.1.2 we have in any chart (U, x) and with respect to any local base sections e_α

$$\left(\mathsf{D}^E\right)^\ell s\Big|_U = \frac{\partial^\ell s^\alpha}{\partial x^{i_1} \dots \partial x^{i_\ell}} \, \mathrm{d} x^{i_1} \vee \dots \vee \mathrm{d} x^{i_\ell} \otimes e_\alpha + (\text{lower order terms}), \tag{1.2.15}$$

for every section $s \in \Gamma^\infty(E)$. This was used in the proof of Theorem 1.1.5 and is an easy consequence of the local expression $D^E\big|_U = \mathrm{d} x^i \vee \nabla_{\frac{\partial}{\partial x^i}}$ together with a simple induction on ℓ.

Now let $X \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F))$ be given. Then locally we can write

$$X\big|_U = \frac{1}{k!} X^{i_1 \ldots i_k \beta}_{\alpha} \frac{\partial}{\partial x^{i_1}} \vee \cdots \vee \frac{\partial}{\partial x^{i_k}} \otimes f_\beta \otimes e^\alpha. \tag{1.2.16}$$

This indicates how we can define a differential operator out of X and D^E. We use the natural pairing of the $\mathrm{S}^k TM$-part of X with the $\mathrm{S}^k T^* M$-part of $\left(\mathsf{D}^E\right)^k s$ and apply the $\mathsf{Hom}(E, F)$-part of X to the E-part of $\left(\mathsf{D}^E\right)^k s$. This gives a well-defined section of F. In the literature, different conventions concerning the pairing of symmetric tensor fields are used. We adopt the following convention, best expressed locally as

$$\left\langle X, \left(\mathsf{D}^E\right)^k s \right\rangle = k! X^{i_1 \ldots i_k \beta}_{\alpha} \frac{\partial^k s^\alpha}{\partial x^{i_1} \ldots \partial x^{i_k}} f_\beta + (\text{lower order terms}). \tag{1.2.17}$$

With other words, this is the natural pairing of $\underbrace{V \otimes \cdots \otimes V}_{k-\text{times}}$ with $\underbrace{V^* \otimes \cdots \otimes V^*}_{k-\text{times}}$ restricted to symmetric tensors *without* additional pre-factors. Indeed, note that the tensor indexes of $\left(\mathsf{D}^E\right)^k s$ are given by

$$\left(\mathsf{D}^E\right)^k s\Big|_U = k! \frac{\partial^\ell s^\alpha}{\partial x^{i_1} \ldots \partial x^{i_\ell}} \, \mathrm{d} x^{i_1} \otimes \cdots \otimes \mathrm{d} x^{i_\ell} \otimes e_\alpha + (\text{lower order terms}) \tag{1.2.18}$$

according to our convention for the symmetrized tensor product $\vee$.

Definition 1.2.5 (Standard ordered quantization) *Let $X \in \Gamma^\infty(\mathrm{S}^\bullet TM \otimes \mathsf{Hom}(E, F))$ be a not necessarily homogeneous section and let $\hbar > 0$. Then the standard ordered quantization $\varrho_{\mathrm{Std}}(X) : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ of X is defined by*

$$\varrho_{\mathrm{Std}}(X) s = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \left\langle X^{(r)}, \frac{1}{r!} \left(\mathsf{D}^E\right)^r s \right\rangle, \tag{1.2.19}$$

for $s \in \Gamma^\infty(E)$, where $X = \sum_r X^{(r)}$ with $X^{(r)} \in \Gamma^\infty(\mathrm{S}^r TM \otimes \mathsf{Hom}(E, F))$ are the homogeneous parts of X.

Note that by definition of the direct sum there are only finitely many $X^{(r)}$ different from zero whence the sum in (1.2.19) is always *finite*.

Theorem 1.2.6 (Global symbol calculus) *The standard ordered quantization provides a filtration preserving $\mathcal{C}^\infty(M)$-linear isomorphism*

$$\varrho_{\mathrm{Std}} : \bigoplus_{k=0}^{\infty} \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F)) \longrightarrow \mathrm{DiffOp}^\bullet(E; F), \tag{1.2.20}$$

such that for $X \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F))$ we have

$$\sigma_k(\varrho_{\mathrm{Std}}(X)) = \left(\frac{\hbar}{\mathrm{i}}\right)^k X. \tag{1.2.21}$$

Proof. From the local expression of $\left(\mathsf{D}^E\right)^\ell s$ as in the proof of Theorem 1.1.5 it is clear that $\varrho_{\mathrm{Std}}(X)$ is indeed a differential operator. Note that the sum is finite and for $X = X^{(k)} \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F))$ the differential operator $\varrho_{\mathrm{Std}}(X)$ has order k. For $f \in \mathcal{C}^\infty(M)$ we clearly have $\varrho_{\mathrm{Std}}(fX) = f \, \varrho_{\mathrm{Std}}(X)$

since the natural pairing is $\mathcal{C}^\infty(M)$-bilinear. This shows that ϱ_{Std} is a filtration preserving $\mathcal{C}^\infty(M)$-linear map. Let $X \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E, F))$ be homogeneous of degree $k \in \mathbb{N}_0$. Then locally

$$
\begin{aligned}
\varrho_{\mathrm{Std}}(X)s\big|_U &= \frac{1}{k!}\left(\frac{\hbar}{\mathrm{i}}\right)^k \left\langle X, \frac{1}{k!}\left(\mathsf{D}^E\right)^k s\right\rangle\Big|_U \\
&= \frac{1}{k!k!}\left(\frac{\hbar}{\mathrm{i}}\right)^k X^{i_1\ldots i_k}{}_\alpha^\beta k! \frac{\partial^k s^\alpha}{\partial x^{i_1}\ldots\partial x^{i_k}} f_\beta + (\text{lower order terms}),
\end{aligned}
$$

hence (1.2.21) is clear by the definition of σ_k as in (1.2.5). Now let $D \in \mathrm{DiffOp}^k(E; F)$ be given. Then

$$
\sigma_k\left(D - \left(\frac{\mathrm{i}}{\hbar}\right)^k \varrho_{\mathrm{Std}}(\sigma_k(D))\right) = 0,
$$

hence $D - \left(\frac{\mathrm{i}}{\hbar}\right)^k \varrho_{\mathrm{Std}}(\sigma_k(D))$ is a differential operator of order at most $k-1$. By induction we can find $D_k = \sigma_k(D), D_{k-1}, \ldots, D_0$ with $D_\ell \in \Gamma^\infty(\mathrm{S}^\ell TM \otimes \mathsf{Hom}(E, F))$ such that

$$
D = \varrho_{\mathrm{Std}}\left(\sum_{r=0}^k \left(\frac{\mathrm{i}}{\hbar}\right)^r D_r\right), \tag{1.2.22}
$$

which proves surjectivity. The injectivity is also clear, as $\sigma_k(D)$ is uniquely determined by D and by induction the above $D_{k-1}, \ldots, D_0$ are unique as well. □

Remark 1.2.7 (Global symbol calculus)

i.) The standard ordered quantization and its inverse map $\varrho_{\mathrm{Std}}{}^{-1}$, i.e. the *global symbol calculus*, come indeed from quantization theory, where $E = F = M \times \mathbb{C}$ is the trivial line bundle and $\Gamma^\infty(\mathrm{S}^\bullet TM)$ is identified in the usual, canonical way with functions on T^*M being polynomial in the fibers. Indeed, there is a unique algebra isomorphism

$$
\mathcal{J} : \bigoplus_{k=0}^\infty \Gamma^\infty(\mathrm{S}^k TM) \ni X \mapsto \mathcal{J}(X) \in \mathrm{Pol}^\bullet(T^*M) \tag{1.2.23}
$$

with $\mathcal{J}(f) = \pi^* f$ and $\mathcal{J}(X)(\alpha_p) = \alpha_p(X(p))$ for $f \in \mathcal{C}^\infty(M) = \Gamma^\infty(\mathrm{S}^0 TM)$ and $X \in \Gamma^\infty(TM)$, where $\alpha_p \in T_p^* M$. The pre-factor $\frac{\hbar}{\mathrm{i}}$ in (1.2.19) is due to the physical conventions since we can interpret functions in $\mathrm{Pol}^1(T^*M)$ to be linear in the momenta on the phase space T^*M corresponding to the configuration space M. In the case $M = \mathbb{R}^n$ with the flat covariant derivative ∇, the map ϱ_{Std} is indeed the standard ordered quantization on $T^*M = \mathbb{R}^{2n}$, i.e. first all "momenta to the right". A more detailed discussion can be found in [60, Sect. 5.4].

ii.) For $X \otimes A \in \Gamma^\infty(TM \otimes \mathsf{Hom}(E, F))$ with $X \in \Gamma^\infty(TM)$ and $A \in \Gamma^\infty(\mathsf{Hom}(E, F))$ we simply have

$$
\varrho_{\mathrm{Std}}(X \otimes A)s = \frac{\hbar}{\mathrm{i}} A(\nabla^E_X s). \tag{1.2.24}
$$

In particular, the choice of ∇ does not yet enter. This is of course no longer the case for higher symmetric degrees. Also

$$
\varrho_{\mathrm{Std}}(A) = A \tag{1.2.25}
$$

is just a $\mathcal{C}^\infty(M)$-linear operator, not yet differentiating.

1.2.3 Continuity Properties of Differential Operators

From the local form of differential operators we immediately obtain the following continuity statement:

Theorem 1.2.8 (Continuity of differential operators) *Let $D \in \operatorname{DiffOp}^k(E; F)$ be a differential operator of order k. Then for all $\ell \in \mathbb{N}_0$ the map*

$$D : \Gamma^{k+\ell}(E) \longrightarrow \Gamma^{\ell}(E) \tag{1.2.26}$$

is well-defined and continuous with respect to the $\mathcal{C}^{k+\ell}$- and $\mathcal{C}^{\ell}$-topology.

Proof. Clearly, if $s \in \Gamma^{k+\ell}(E)$ then $\left(\mathsf{D}^E\right)^k s \in \Gamma^{\ell}(E)$ is still ℓ times continuously differentiable. Since the natural pairing does not lower the degree of differentiability, we can define $\varrho_{\mathrm{Std}}(X)s$ in the obvious way. Since furthermore every differential operator D of order k is of the form $\varrho_{\mathrm{Std}}(X)$ with X having at most tensorial degree k, the extension (1.2.26) is defined in a unique way. If (U, x) is a chart and $e_\alpha \in \Gamma^\infty(E\big|_U)$ and $f_\beta \in \Gamma^\infty(F\big|_U)$ are local base sections then

$$\begin{aligned}
\mathrm{p}_{U,x,K,\ell,\{f_\beta\}}(Ds) &= \sup_{\substack{p\in K\\ |I|\le\ell\\ \beta}} \left| \frac{\partial^{|I|}}{\partial x^I} \sum_{r=0}^{\ell} \frac{1}{r!} D_U^{i_1\ldots i_r\beta}{}_{\alpha}(p) \frac{\partial^r s^\alpha}{\partial x^{i_1}\cdots\partial x^{i_r}}(p) \right| \\
&\le c \sum_{\substack{i_1,\ldots,i_r\\ r}} \sup_{\substack{p\in K\\ |I|\le\ell\\ \beta,\alpha}} \left| \frac{\partial^{|I|}}{\partial x^I} D_U^{i_1\ldots i_r\beta}{}_{\alpha}(p) \right| \sup_{\substack{p\in K\\ |J|\le\ell\\ \alpha}} \left| \frac{\partial^{|J|}}{\partial x^J} \frac{\partial^r s^\alpha}{\partial x^{i_1}\cdots\partial x^{i_r}}(p) \right| \\
&\le c' \max_{\substack{i_1,\ldots,i_r\\ \beta,\alpha}} \mathrm{p}_{U,x,K,\ell}(D_U^{i_1\ldots i_r\beta}{}_{\alpha}) \max_r \mathrm{p}_{U,x,K,\ell+r,\{e_\alpha\}}(s) \\
&\le c'\, \mathrm{p}_{U,x,K,\ell,\{e_\alpha\},\{f_\beta\}}(D)\, \mathrm{p}_{U,x,K,\ell+k,\{e_\alpha\}}(s),
\end{aligned}$$

where c' is a combinatorial factor depending only on ℓ and k and

$$\mathrm{p}_{U,x,K,\ell,\{e_\alpha\},\{f_\beta\}}(D) = \sup_{\substack{p\in K\\ \alpha,\beta\\ |I|\le\ell\\ i_1,\ldots,i_r}} \left| \frac{\partial^{|I|} D_U^{i_1\ldots i_r\beta}{}_{\alpha}}{\partial x^I}(p) \right|.$$

But this is the desired estimate to conclude the continuity with respect to the $\mathcal{C}^{k+\ell}$- and $\mathcal{C}^{\ell}$-topology. $\square$

Corollary 1.2.9 *A differential operator $D \in \operatorname{DiffOp}^\bullet(E; F)$ is continuous with respect to the $\mathcal{C}^\infty$-topology.*

In the proof of Theorem 1.2.8 we have made use of the quantities

$$\mathrm{p}_{U,x,K,\ell,\{e_\alpha\},\{f_\beta\}}(D) = \sup_{\substack{p\in K\\ \alpha,\beta\\ |I|\le\ell\\ i_1,\ldots,i_r}} \left| \frac{\partial^{|I|} D_U^{i_1\ldots i_r\beta}{}_{\alpha}}{\partial x^I}(p) \right|, \tag{1.2.27}$$

which are easily shown to be seminorms on $\mathrm{DiffOp}^\bullet(E;F)$. For a fixed $k \in \mathbb{N}_0$, these make $\mathrm{DiffOp}^k(E;F)$ again a Fréchet space, a simple fact which we shall not prove here. Moreover, the standard ordered quantization is then a *continuous* isomorphism with continuous inverse

$$\varrho_{\mathrm{Std}} : \bigoplus_{\ell=0}^{k} \Gamma^\infty(\mathrm{S}^\ell TM \otimes \mathsf{Hom}(E,F)) \longrightarrow \mathrm{DiffOp}^k(E;F). \tag{1.2.28}$$

However, *all* differential operators $\mathrm{DiffOp}^\bullet(E;F)$ will have to be equipped with an inductive limit topology similar to the construction of the $\mathcal{C}_0^\infty$-topology. In any case, we shall not need these aspects here.

Instead, we consider now the restriction of $D \in \mathrm{DiffOp}^k(E;F)$ to compactly supported sections $\Gamma_K^{k+\ell}(E)$. Since $\mathrm{supp}(Ds) \subseteq \mathrm{supp}\, s$ we have

$$D : \Gamma_A^{k+\ell}(E) \longrightarrow \Gamma_A^\ell(F) \tag{1.2.29}$$

for all closed subsets $A \subseteq M$. Since in the estimate

$$\mathrm{p}_{U,x,K,\ell,\{f_\beta\}}(Ds) \leq c\, \mathrm{p}_{U,x,K,\ell,\{e_\alpha\},\{f_\beta\}}(D)\, \mathrm{p}_{U,x,K,\ell+k,\{e_\alpha\}}(s) \tag{1.2.30}$$

we have the same compactum on both sides, we find that

$$D : \Gamma_K^{k+\ell}(E) \longrightarrow \Gamma_K^\ell(F) \tag{1.2.31}$$

is continuous in the $\mathcal{C}_K^{k+\ell}$- and $\mathcal{C}_K^\ell$-topology. From this we immediately obtain the following continuity statement:

Theorem 1.2.10 *Let $D \in \mathrm{DiffOp}^k(E;F)$ be a differential operator of order $k \in \mathbb{N}_0$. Then for all $\ell \in \mathbb{N}_0$ the restriction*

$$D : \Gamma_0^{k+\ell}(E) \longrightarrow \Gamma_0^\ell(F) \tag{1.2.32}$$

is continuous in the $\mathcal{C}_0^{k+\ell}$- and the $\mathcal{C}_0^\ell$-topology. Moreover

$$D : \Gamma_0^\infty(E) \longrightarrow \Gamma_0^\infty(F) \tag{1.2.33}$$

is continuous in the $\mathcal{C}_0^\infty$-topology.

Proof. This follow immediately from (1.2.31) and the characterization of continuous maps as in Theorem 1.1.11, *iv.)*. □

1.2.4 Adjoints of Differential Operators

For a section $s \in \Gamma^\infty(E)$ and $\mu \in \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ the natural pairing of E and E^* gives a density $\mu(s) \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ which we can integrate, provided the support is compact. Therefore we define

$$\langle s, \mu \rangle = \int_M \mu(s) = \int_M s \cdot \mu, \tag{1.2.34}$$

whenever the support of at least one of s or μ is compact.

Lemma 1.2.11 *The pairing* (1.2.34) *is bilinear and non-degenerate. Moreover $\langle s, f\mu \rangle = \langle fs, \mu \rangle$ for $f \in \mathcal{C}^\infty(M)$.*

Proof. Let $s \in \Gamma^\infty(E)$ be not the zero section and let $p \in M$ be such that $s(p) \neq 0$. Then we find an open neighborhood U of p and a section $\varphi \in \Gamma_0^\infty(E^*)$ with compact support $\operatorname{supp}\varphi \subseteq U$ such that

$$\varphi(s) \geq 0 \quad \text{and} \quad \varphi(s)\big|_p > 0.$$

Using local base sections this is obvious. Now choose a positive density $\nu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$, then $\varphi \otimes \nu \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ will satisfy $\langle s, \varphi \otimes \nu \rangle \neq 0$. This shows that (1.2.34) is non-degenerate in the first argument. The other non-degeneracy is shown analogously. The second statement is clear. □

In particular, $\langle \cdot\,, \cdot \rangle$ restricts to a non-degenerate pairing

$$\langle \cdot\,, \cdot \rangle : \Gamma_0^\infty(E) \times \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}. \tag{1.2.35}$$

As an immediate consequence we obtain the following statement. First recall that an operator

$$D : V \longrightarrow W \tag{1.2.36}$$

is *adjointable* with respect to bilinear pairings

$$\langle \cdot\,, \cdot \rangle_{V,\widetilde{V}} : V \times \widetilde{V} \longrightarrow \mathbb{C} \quad \text{and} \quad \langle \cdot\,, \cdot \rangle_{W,\widetilde{W}} : W \times \widetilde{W} \longrightarrow \mathbb{C}, \tag{1.2.37}$$

if there is a map $D^{\mathrm{T}} : \widetilde{W} \longrightarrow \widetilde{V}$ such that

$$\langle Dv, \widetilde{w} \rangle_{W,\widetilde{W}} = \langle v, D^{\mathrm{T}}\widetilde{w} \rangle_{V,\widetilde{V}}\,. \tag{1.2.38}$$

If the pairings are non-degenerate then an adjoint D^{T} is necessarily unique (if it exists at all) and both maps D, D^{T} are linear maps. Clearly, D^{T} is adjointable, too, with $(D^{\mathrm{T}})^{\mathrm{T}} = D$. Thus in our situation, adjointable maps with respect to the pairing (1.2.34) or (1.2.35) have unique adjoints and are necessarily linear.

Proposition 1.2.12 *Let $D \in \operatorname{DiffOp}^k(E; F)$ be a differential operator of order k. Then $D : \Gamma_0^\infty(E) \longrightarrow \Gamma_0^\infty(F)$ is adjointable with respect to* (1.2.34) *and the (unique) adjoint*

$$D^{\mathrm{T}} : \Gamma^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \tag{1.2.39}$$

is again a differential operator of order k.

Proof. Let $\{(U_i, x_i)\}_{i\in I}$ be a locally finite atlas and let $e_{i\alpha} \in \Gamma^\infty(E\big|_{U_i})$ and $f_{i\beta} \in \Gamma^\infty(F\big|_{U_i})$ be local base sections. Moreover let $\{\chi_i\}_{i\in I}$ be a locally finite partition of unity subordinate to the atlas with $\operatorname{supp}\chi_i$ being compact. As usual, we write

$$Ds\big|_{U_i} = \sum_{r=0}^{k} \frac{1}{r!} D_{U_i}{}^{i_1\ldots i_r\beta}_{\alpha} f_\beta \frac{\partial^r s_i^\alpha}{\partial x_i^{i_1}\cdots\partial x_i^{i_r}},$$

where $s\big|_{U_i} = s_i^\alpha e_{i\alpha}$ with $s_i^\alpha = e_i^\alpha(s) \in \mathcal{C}^\infty(U_i)$. For $\mu \in \Gamma^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ we write

$$\mu\big|_{U_i} = \mu_{i\beta} f^\beta |\mathrm{d}\,x_i^1 \wedge \cdots \wedge \mathrm{d}\,x_i^n|,$$

with $\mu_{i\beta} \in \mathcal{C}^\infty(U_i)$. Here $|\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n|$ denotes the unique locally defined density with value 1 when evaluated on the coordinate base vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. Then we compute

$$\langle Ds, \mu \rangle = \int_M \mu(Ds) = \sum_i \int_{x_i(U_i)} (\chi_i \mu(Ds)) \circ x_i^{-1}\, \mathrm{d}^n\, x_i$$

$$= \sum_i \int_{x_i(U_i)} \left(\chi_i \mu_{i\beta} \sum_{r=0}^{k} \frac{1}{r!} D_{U_i}^{i_1 \ldots i_r \beta}{}_{\alpha} \frac{\partial^r s_i^\alpha}{\partial x_i^{i_1} \cdots \partial x_i^{i_r}} \right) \circ x_i^{-1} \, \mathrm{d}^n x_i.$$

Note that the integrand consists of compactly supported functions only. Thus we can integrate by parts and obtain

$$\langle Ds, \mu \rangle = \sum_i \int_{x_i(U_i)} \left(\sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x^{i_1} \cdots \partial x^{i_r}} \left(\chi_i \mu_{i\beta} D_{U_i}^{i_1 \ldots i_r \beta}{}_{\alpha} \right) s_i^\alpha \right) \circ x_i^{-1} \, \mathrm{d}^n x_i.$$

Now the function $\chi_i \mu_{i\beta} D_{U_i}^{i_1 \ldots i_r \beta}{}_{\alpha}$ has compact support in U_i thanks to the choice of the χ_i. Thus it defines a global function in $\mathcal{C}_0^\infty(M)$. It follows that

$$\mu_i = \sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x^{i_1} \cdots \partial x^{i_r}} \left(\chi_i \mu_{i\beta} D_{U_i}^{i_1 \ldots i_r \beta}{}_{\alpha} \right) e_i^\alpha \otimes |\, \mathrm{d} x_i^1 \wedge \cdots \wedge \mathrm{d} x_i^n|$$

is a global section in $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}| T^*M)$ with compact support in U_i. Since the χ_i are locally finite, the sum

$$D^{\mathrm{T}} \mu = \sum_i \mu_i$$

is well-defined and yields a global section $D^{\mathrm{T}}\mu \in \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}| T^*M)$ such that

$$\langle Ds, \mu \rangle = \langle s, D^{\mathrm{T}} \mu \rangle \, .$$

This shows that D is adjointable. From the actual computation above it is clear that D^{T} differentiates again k times. Thus $D^{\mathrm{T}} \in \mathrm{DiffOp}^k(F^* \otimes |\Lambda^{\mathrm{top}}| T^*M, E^* \otimes |\Lambda^{\mathrm{top}}| T^*M)$ follows. However, there is also another nice argument based on the algebraic definition of differential operators: Let $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ be a differential operator of order zero. Thus D can be viewed as a section of $\mathsf{Hom}(E, F)$, i.e. $D \in \Gamma^\infty(\mathsf{Hom}(E, F))$. Then in $\mu(Ds)$ we can simply apply the pointwise transpose of D to the F^*-part of μ. This defines $D^{\mathrm{T}}\mu$ pointwise in such a way that $(D^{\mathrm{T}}\mu)(s) = \mu(Ds)$. Clearly $\langle Ds, \mu \rangle = \langle s, D^{\mathrm{T}}\mu \rangle$ follows. Now we proceed by induction. We assume that the adjoint always exists (what we have shown already) and for differential operators of order $\ell \leq k - 1$ the adjoint has order ℓ, too. Thus let $D \in \mathrm{DiffOp}^k(E; F)$ and $f \in \mathcal{C}^\infty(M)$. Then we have

$$\langle f D s, \mu \rangle = \langle Ds, f\mu \rangle = \langle s, D^{\mathrm{T}} f \mu \rangle \, ,$$

and on the other hand

$$\begin{aligned} \langle f D s, \mu \rangle &= \langle [f, D] s, \mu \rangle + \langle D(fs), \mu \rangle \\ &= \langle s, [f, D]^{\mathrm{T}} \mu \rangle + \langle fs, D^{\mathrm{T}} \mu \rangle \\ &= \langle s, [f, D]^{\mathrm{T}} \mu \rangle + \langle s, f D^{\mathrm{T}} \mu \rangle \, . \end{aligned}$$

Hence by the non-degeneracy of $\langle \,\cdot\,, \,\cdot\, \rangle$ we conclude that

$$[f, D^{\mathrm{T}}] = [f, D]^{\mathrm{T}} \in \mathrm{DiffOp}^{k-1}(F^* \otimes |\Lambda^{\mathrm{top}}| T^*M, E^* \otimes |\Lambda^{\mathrm{top}}| T^*M)$$

by induction. But this shows $D^{\mathrm{T}} \in \mathrm{DiffOp}^k(F^* \otimes |\Lambda^{\mathrm{top}}| T^*M, E^* \otimes |\Lambda^{\mathrm{top}}| T^*M)$ as wanted. □

Corollary 1.2.13 *Let $D \in \mathrm{DiffOp}^k(E; F)$. Then for the leading symbol $\sigma_k(D^{\mathrm{T}}) \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(F^* \otimes |\Lambda^{\mathrm{top}}| T^*M, E^* \otimes |\Lambda^{\mathrm{top}}| T^*M))$ we have*

$$\sigma_k(D^{\mathrm{T}}) = (-1)^k \sigma_k(D)^{\mathrm{T}} \otimes \mathsf{id}_{|\Lambda^{\mathrm{top}}| T^*M}, \tag{1.2.40}$$

where $\sigma_k(D)^{\mathrm{T}}$ denotes the pointwise transpose from $\mathsf{Hom}(E, F)$ to $\mathsf{Hom}(F^*, E^*)$.

Proof. From the local computations in the proof of Proposition 1.2.12 we obtained

$$\begin{aligned}\mu_i &= \sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x^{i_1} \cdots \partial x^{i_r}} \left(\chi_i \mu_{i\beta} D_{U_i}^{i_1 \ldots i_r \beta}{}_{\alpha}\right) e_i^\alpha \otimes |\,\mathrm{d}\, x_i^1 \wedge \cdots \wedge \mathrm{d}\, x_i^n| \\ &= \frac{(-1)^k}{k!} \chi_i D_{U_i}^{i_1 \ldots i_k \beta}{}_{\alpha} \frac{\partial^k \mu_{i\beta}}{\partial x^{i_1} \cdots \partial x^{i_k}} e_i^\alpha \otimes |\,\mathrm{d}\, x_i^1 \wedge \cdots \wedge \mathrm{d}\, x_i^n| + (\text{lower order terms}).\end{aligned}$$

Since $D^{\mathrm{T}}\mu = \sum_i \mu_i$ and $\sum_i \chi_i = 1$, we conclude that

$$\begin{aligned}D^{\mathrm{T}}\mu\big|_{U_i} &= \frac{(-1)^k}{k!} D_{U_i}^{i_1 \ldots i_k \beta}{}_{\alpha} \frac{\partial^k \mu_{i\beta}}{\partial x_i^{i_1} \cdots \partial x_i^{i_k}} e_i^\alpha \otimes |\,\mathrm{d}\, x_i^1 \wedge \cdots \wedge \mathrm{d}\, x_i^n| + (\text{lower order terms}) \\ &= (-1)^k \sigma_k(D)^{\mathrm{T}} \otimes \mathsf{id}_{|\Lambda^{\mathrm{top}}|T^*M}(\mu) + (\text{lower order terms}).\end{aligned}$$

□

Remark 1.2.14 (Other pairings)

i.) There are several variations of the above proposition. On one hand one can consider the natural pairing of α- and $(1-\alpha)$-densities for any $\alpha \in \mathbb{C}$ to obtain

$$\langle\,\cdot\,,\,\cdot\,\rangle : \Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^\alpha T^*M) \times \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|^{1-\alpha} T^*M) \longrightarrow \mathbb{C} \tag{1.2.41}$$

via pointwise natural pairing and integration of the remaining 1-density. This is again non-degenerate. Thus we can also compute the adjoints of differential operators

$$D : \Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^\alpha T^*M) \longrightarrow \Gamma_0^\infty(F \otimes |\Lambda^{\mathrm{top}}|^\beta T^*M) \tag{1.2.42}$$

and obtain differential operators

$$D^{\mathrm{T}} : \Gamma^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|^{1-\beta} T^*M) \longrightarrow \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|^{1-\alpha} T^*M) \tag{1.2.43}$$

by the same kind of computation as in Proposition 1.2.12. There, we considered the case $\alpha = 0 = \beta$.

ii.) Another important case is for complex bundles E with a (pseudo-) Hermitian fiber metric h_E. Then we can use the $\mathbb{C}$-sesquilinear pairings

$$\langle s, t \otimes \mu \rangle = \int_M h(s,t)\, \mu, \tag{1.2.44}$$

where $s, t \in \Gamma^\infty(E)$ and $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ and at least one has compact support. Clearly, this extends to

$$\langle\,\cdot\,,\,\cdot\,\rangle : \Gamma^\infty(E) \times \Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C} \tag{1.2.45}$$

in a $\mathbb{C}$-sesquilinear way. While $D \mapsto D^{\mathrm{T}}$ is $\mathbb{C}$-linear, now the adjoint D^* depends on D in an *antilinear* way.

iii.) A very important situation is obtained by merging the above possibilities. For a Hermitian vector bundle $E \longrightarrow M$ with Hermitian fiber metric h we consider the sections $\Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^{\frac{1}{2}} T^*M)$. On factorizing sections we can define

$$\langle s \otimes \mu, t \otimes \nu \rangle = \int_M h(s,t)\, \overline{\mu}\nu, \tag{1.2.46}$$

since $\overline{\mu}\nu$ is a 1-density. Then the pairing extends to a $\mathbb{C}$-sesquilinear pairing

$$\langle \cdot\,, \cdot \rangle : \Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^{\frac{1}{2}} T^*M) \times \Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^{\frac{1}{2}} T^*M) \longrightarrow \mathbb{C}, \tag{1.2.47}$$

which is not only non-degenerate but *positive definite*. Thus $\Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|^{\frac{1}{2}} T^*M)$ becomes a *pre-Hilbert space*. Moreover, taking E to be the trivial line bundle with the canonical fiber metric gives a pre-Hilbert space $\Gamma_0^\infty(|\Lambda^{\mathrm{top}}|^{\frac{1}{2}} T^*M)$ of half densities. Its completion to a Hilbert space is the so-called *intrinsic Hilbert space* on M.

While the above constructions are always slightly asymmetric unless we take half-densities, we obtain a more symmetric situation if we integrate with respect to a given positive density. Thus we choose once and for all a positive density $\mu > 0$ on M. Later on, this will be the (pseudo-) Riemannian volume density, but for now we do not need this additional property. For a vector bundle $E \longrightarrow M$ we then have the pairing

$$\langle s, \varphi \rangle_\mu = \int_M \varphi(s)\, \mu, \tag{1.2.48}$$

for $s \in \Gamma^\infty(E)$ and $\varphi \in \Gamma^\infty(E^*)$, at least one having compact support. Clearly,

$$\langle s, \varphi \rangle_\mu = \langle s, \varphi \otimes \mu \rangle \tag{1.2.49}$$

with the original version (1.2.34) of the pairing $\langle \cdot\,, \cdot \rangle$. Since $\mu > 0$ it easily follows that (1.2.49) is non-degenerate and satisfies

$$\langle fs, \varphi \rangle_\mu = \langle s, f\varphi \rangle_\mu \tag{1.2.50}$$

for all $f \in \mathcal{C}^\infty(M)$. For the action of differential operators we again have adjoints:

Theorem 1.2.15 *Let $D \in \mathrm{DiffOp}^k(E; F)$ be a differential operator of order $k \in \mathbb{N}_0$. Then there exists a differential operator $D^{\mathrm{T}} \in \mathrm{DiffOp}^k(F^*; E^*)$ such that*

$$\langle Ds, \varphi \rangle_\mu = \langle s, D^{\mathrm{T}}\varphi \rangle_\mu \tag{1.2.51}$$

for all $s \in \Gamma^\infty(E)$ and $\varphi \in \Gamma^\infty(F^*)$, at least one having compact support.

Proof. The proof is now fairly simple. Since D has an adjoint, denoted by $\widetilde{D}$ for a moment, with respect to (1.2.34) we have

$$\langle Ds, \varphi \rangle_\mu = \langle Ds, \varphi \otimes \mu \rangle = \left\langle s, \widetilde{D}(\varphi \otimes \mu) \right\rangle,$$

and locally

$$\begin{aligned}
\widetilde{D}(\varphi \otimes \mu)\Big|_U &= \sum_{r=0}^k \frac{1}{r!} \widetilde{D}_U^{i_1\ldots i_r}{}^\beta_\alpha \frac{\partial^r}{\partial x^{i_1}\cdots\partial x^{i_r}} (\varphi_\beta \mu_U)\, e^\alpha \otimes |\,\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n| \\
&= \sum_{r=0}^k \frac{1}{r!} \widetilde{D}_U^{I}{}^\beta_\alpha \sum_{J \le I} \binom{I}{J} \frac{\partial^{|J|}\varphi_\beta}{\partial x^J} \frac{\partial^{|I-J|}\mu_U}{\partial x^{I-J}} e^\alpha \otimes |\,\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n| \\
&= \sum_{\substack{r=0\\|I|=r\\J\le I}}^k \frac{1}{r!} \binom{I}{J} \widetilde{D}_U^{I}{}^\beta_\alpha \frac{\partial^{|J|}\varphi_\beta}{\partial x^J} \frac{1}{\mu_U} \frac{\partial^{|I-J|}\mu_U}{\partial x^{I-J}} e^\alpha \otimes \mu_U |\,\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n|
\end{aligned}$$

$$= \Bigg(\sum_{\substack{r=0\\|I|=r\\J\leq I}}^{k} \frac{1}{r!} \binom{I}{J} \widetilde{D}_{U\alpha}^{I\,\beta} \frac{\partial^{|J|}\varphi_\beta}{\partial x^J} \frac{1}{\mu_U} \frac{\partial^{|I-J|}\mu_U}{\partial x^{I-J}} e^\alpha \Bigg) \otimes \mu\big|_U,$$

since $\mu_U > 0$ thanks to $\mu > 0$. This shows that with

$$D^{\mathrm{T}}\varphi\Big|_U = \sum_{\substack{r=0\\|I|=r\\J\leq I}}^{k} \frac{1}{r!} \binom{I}{J} \widetilde{D}_{U\alpha}^{I\,\beta} \frac{\partial^{|J|}\varphi_\beta}{\partial x^J} \frac{1}{\mu_U} \frac{\partial^{|I-J|}\mu_U}{\partial x^{I-J}} e^\alpha \tag{$*$}$$

we obtain a locally defined differential operator D^{T} such that

$$\widetilde{D}(\varphi \otimes \mu)\Big|_U = (D^{\mathrm{T}}\varphi) \otimes \mu\Big|_U.$$

Now the left hand side is globally well-defined and hence the right hand side is chart independent as well. This shows that D^{T} is indeed a global object, locally given by $(*)$. Obviously, it is a differential operator of order k. □

Remark 1.2.16

i.) Note that D^{T} as in Theorem 1.2.15 depends on the choice of $\mu > 0$ while the adjoint as in Proposition 1.2.12 is intrinsically defined, though of course between different vector bundles. However, we shall not emphasize the dependence of D^{T} on μ in our notation. It should become clear from the context which version of adjoint we use.

ii.) Analogously to Corollary 1.2.13 we see that the leading symbol of D^{T} is given by

$$\sigma_k(D^{\mathrm{T}}) = (-1)^k \sigma_k(D)^{\mathrm{T}}, \tag{1.2.52}$$

where again $\sigma_k(D)^{\mathrm{T}} \in \Gamma^\infty(\mathsf{Hom}(F^*, E^*))$ is the pointwise adjoint of $\sigma_k(D) \in \Gamma^\infty(\mathsf{Hom}(E, F))$. This is obvious from the local computations in the proof as we have to collect those terms with all k derivatives hitting the φ_β instead of the μ_U.

Sometimes it will be important to compute the adjoint of D^{T} more explicitly. Here we can use our global symbol calculus developed in Section 1.2.2. To this end, we introduce the following divergence operators. If $X \in \Gamma^\infty(TM)$ is a vector field then its *covariant divergence* is defined by

$$\operatorname{div}_\nabla(X) = \mathsf{tr}(Y \mapsto \nabla_Y X), \tag{1.2.53}$$

where the trace is understood to be the pointwise trace: indeed $Y \mapsto \nabla_Y X$ is a $\mathcal{C}^\infty(M)$-linear map $\Gamma^\infty(TM) \longrightarrow \Gamma^\infty(TM)$ which therefor can be identified with a section in $\Gamma^\infty(\mathsf{End}(TM))$. Thus the trace is well-defined. More explicitly, in local coordinates (U, x) we have

$$\operatorname{div}_\nabla(X)\big|_U = \mathrm{d}\,x^i \left(\nabla_{\frac{\partial}{\partial x^i}} X\right). \tag{1.2.54}$$

Clearly, we have for $f \in \mathcal{C}^\infty(M)$ and $X \in \Gamma^\infty(TM)$ the relation

$$\operatorname{div}_\nabla(fX) = X(f) + f \operatorname{div}_\nabla(X) \tag{1.2.55}$$

This Leibniz rule suggests to extend the covariant divergence to higher symmetric multivector fields as follows.

Definition 1.2.17 (Covariant divergence) *Let ∇ be a torsion-free covariant derivative for M and let ∇^E be a covariant derivative for E. For $X \in \Gamma^\infty(\mathrm{S}^\bullet TM \otimes E)$ we define*

$$\operatorname{div}^E_\nabla(X) = \mathrm{i_s}(\mathrm{d}\, x^i)\nabla_{\frac{\partial}{\partial x^i}} X. \tag{1.2.56}$$

Lemma 1.2.18 *By* (1.2.56) *we obtain a globally well-defined operator*

$$\operatorname{div}^E_\nabla : \Gamma^\infty(\mathrm{S}^\bullet TM \otimes E) \longrightarrow \Gamma^\infty(\mathrm{S}^{\bullet-1} TM \otimes E), \tag{1.2.57}$$

which is given on factorizing sections by

$$\operatorname{div}^E_\nabla(X_1 \vee \cdots \vee X_k \otimes s) = \sum_{\ell=1}^k X_1 \vee \cdots \overset{\ell}{\wedge} \cdots \vee X_k \otimes \left(\operatorname{div}_\nabla(X_\ell)s + \nabla^E_{X_\ell} s\right) \tag{1.2.58}$$

$$+ \sum_{\substack{\ell,m=1\\ \ell\neq m}}^k (\nabla_{X_\ell} X_m) \vee X_1 \vee \cdots \overset{\ell}{\wedge} \cdots \vee X_k \otimes s, \tag{1.2.59}$$

where $X_1, \ldots, X_k \in \Gamma^\infty(TM)$ and $s \in \Gamma^\infty(E)$.

Proof. First it is clear that the transformation properties of $\frac{\partial}{\partial x^i}$ and $\mathrm{d}\, x^i$ under a change of local coordinates guarantee that $\operatorname{div}^E_\nabla$ is indeed well-defined and independent of the chart. Thus $\operatorname{div}^E_\nabla$ is a globally defined operator lowering the symmetric degree by one. Now let $X_1, \ldots, X_k \in \Gamma^\infty(TM)$ and $s \in \Gamma^\infty(E)$ be given. Then we compute

$$\begin{aligned}
&\operatorname{div}^E_\nabla(X_1 \vee \cdots \vee X_k \otimes s)\\
&\quad= \mathrm{i_s}(\mathrm{d}\, x^i)\nabla_{\frac{\partial}{\partial x^i}}(X_1 \vee \cdots \vee X_k \otimes s)\\
&\quad= \mathrm{i_s}(\mathrm{d}\, x^i)\left(\sum_{\ell=1}^k X_1 \vee \cdots \vee \nabla_{\frac{\partial}{\partial x^i}} X_\ell \vee \cdots X_k \otimes s + X_1 \vee \cdots \vee X_k \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s\right)\\
&\quad= \sum_{\substack{\ell,m=1\\ \ell\neq m}} X_1 \vee \cdots \vee \nabla_{\frac{\partial}{\partial x^i}} X_\ell \vee \cdots \vee \mathrm{d}\, x^i(X_m) \vee \cdots \vee X_k \otimes s\\
&\qquad+ \sum_{\ell=1}^k X_1 \vee \cdots \vee \mathrm{d}\, x^i\left(\nabla_{\frac{\partial}{\partial x^i}} X_\ell\right) \vee \cdots \vee X_k \otimes s + \sum_{\ell=1}^k X_1 \vee \cdots \vee \mathrm{d}\, x^i(X_\ell) \vee \cdots \vee X_k \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s\\
&\quad= \sum_{\substack{\ell,m=1\\ \ell\neq m}} X_1 \vee \cdots \vee \nabla_{X_m} X_\ell \vee \cdots \overset{m}{\wedge} \cdots \vee X_k \otimes s\\
&\qquad+ \sum_{\ell=1}^k X_1 \vee \cdots \vee \operatorname{div}_\nabla(X_\ell) \vee \cdots \vee X_k \otimes s + \sum_{\ell=1}^k X_1 \vee \cdots \overset{\ell}{\wedge} \cdots \vee X_k \otimes \nabla^E_{X_\ell} s.
\end{aligned}$$

□

The covariant derivative ∇ also acts on densities hence we can compute the derivative $\nabla_X \mu$ of the positive density μ. This defines a function

$$\alpha(X) = \frac{\nabla_X \mu}{\mu}, \tag{1.2.60}$$

depending $\mathcal{C}^\infty(M)$-linearly on X. Thus we obtain a one-form $\alpha \in \Gamma^\infty(T^*M)$ which measures how much μ is *not* covariantly constant. Similarly, we can define the μ-divergence of a vector field by

$$\operatorname{div}_\mu(X) = \frac{\mathscr{L}_X \mu}{\mu}. \tag{1.2.61}$$

Lemma 1.2.19 *For $X \in \Gamma^\infty(TM)$ we have*

$$\operatorname{div}_\mu(X) = \operatorname{div}_\nabla(X) + \alpha(X). \tag{1.2.62}$$

Proof. This can be obtained from a simple computation in local coordinates which we omit here, see e.g. [60, Sect. 2.3.4]. □

Writing this as

$$\operatorname{div}_\mu(X) = \operatorname{div}_\nabla(X) + \mathrm{i_s}(\alpha)X, \tag{1.2.63}$$

we can motivate the following definition. For $X \in \Gamma^\infty(\mathrm{S}^\bullet TM \otimes E)$ we set

$$\operatorname{div}^E_\mu(X) = \operatorname{div}^E_\nabla(X) + \mathrm{i_s}(\alpha)X, \tag{1.2.64}$$

where $\mathrm{i_s}(\alpha)$ acts on the $\mathrm{S}^\bullet TM$-part as usual.

Lemma 1.2.20 *On factorizing section we have*

$$\operatorname{div}^E_\mu(X_1 \vee \cdots \vee X_k \otimes s) = \sum_{\ell=1}^{k} X_1 \vee \cdots \overset{\ell}{\wedge} \cdots \vee X_k \otimes \left(\operatorname{div}_\mu(X_\ell)s + \nabla^E_{X_\ell} s\right) \tag{1.2.65}$$

$$+ \sum_{\substack{\ell,m=1\\ \ell\neq m}}^{k} \nabla_{X_\ell} X_m \vee X_1 \vee \cdots \overset{\ell}{\wedge} \cdots \overset{m}{\wedge} \cdots \vee X_k \otimes s. \tag{1.2.66}$$

Proof. The proof of (1.2.65) is completely analogous to the proof of Lemma 1.2.18. □

We can now use the divergence operator to compute the adjoint of a differential operator in a symbol calculus explicitly:

Theorem 1.2.21 (Neumaier) *Let $X \in \Gamma^\infty(\mathrm{S}^k TM \otimes \mathsf{Hom}(E,F))$ and let ∇ and ∇^E, ∇^F be given. Then the adjoint operator to $\varrho_{\mathrm{Std}}(X)$ with respect to $\langle \cdot, \cdot \rangle_\mu$ is explicitly given by*

$$\varrho_{\mathrm{Std}}(X)^{\mathrm{T}} = (-1)^k \, \varrho_{\mathrm{Std}}(N^2 X^{\mathrm{T}}), \tag{1.2.67}$$

where

$$N = \exp\left(\frac{\hbar}{2\mathrm{i}} \operatorname{div}^{\mathsf{Hom}(E,F)}_\mu\right) \tag{1.2.68}$$

and where we use the induced covariant derivative on $\mathsf{Hom}(E,F)$ *and* $\mathsf{Hom}(F^*,E^*)$.

Proof. By a partition of unity argument we can reduce the problem to the case where the involved tensor fields have compact support in a chart (U,x). In this chart we first note that from the definition of the covariant derivative of a density we obtain the local expression

$$\alpha = \left(\frac{\mathscr{L}_{\frac{\partial}{\partial x^i}} \mu}{\mu} - \Gamma^\ell_{i\ell}\right) \mathrm{d}\, x^i$$

for the one-form α. Now let $\omega \in \Gamma^\infty(\mathrm{S}^\ell T^*M \otimes F^*)$ and $s \in \Gamma^\infty(E)$. In the following of this proof, we shall simply write div_μ for all divergences instead of specifying the vector bundle explicitly, just to simplify our notation. For $X \in \Gamma^\infty(\mathrm{S}^{k+\ell}TM \otimes \mathsf{Hom}(E, F))$ we compute

$$
\begin{aligned}
&\mathscr{L}_{\frac{\partial}{\partial x^i}} \left(\left\langle \mathrm{i_s}(\mathrm{d}\, x^i) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \right) \\
&= \frac{\partial}{\partial x^i} \left(\left\langle \mathrm{i_s}(\mathrm{d}\, x^i) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \right) \mu + \left\langle \mathrm{i_s}(\mathrm{d}\, x^i) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mathscr{L}_{\frac{\partial}{\partial x^i}} \mu \\
&= \left\langle \nabla_{\frac{\partial}{\partial x^i}} \left(\mathrm{i_s}(\mathrm{d}\, x^i) X\right), \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle \mathrm{i_s}(\mathrm{d}\, x^i) X, \nabla_{\frac{\partial}{\partial x^i}} \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&\quad + \left\langle \mathrm{i_s} \left(\frac{\mathscr{L}_{\frac{\partial}{\partial x^i}} \mu}{\mu} \mathrm{d}\, x^i \right) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&= \left\langle \mathrm{i_s}(\mathrm{d}\, x^i) \nabla_{\frac{\partial}{\partial x^i}} X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle \mathrm{i_s} \left(-\Gamma^i_{i\ell} \,\mathrm{d}\, x^\ell \right) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&\quad + \left\langle \mathrm{i_s}(\mathrm{d}\, x^i) X, \nabla_{\frac{\partial}{\partial x^i}} \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s + \omega \otimes \nabla_{\frac{\partial}{\partial x^i}} \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&\quad + \left\langle \mathrm{i_s} \left(\frac{\mathscr{L}_{\frac{\partial}{\partial x^i}} \mu}{\mu} \mathrm{d}\, x^i \right) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&= \left\langle \operatorname{div}_\nabla(X), \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle \mathrm{i_s}(\alpha) X, \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu \\
&\quad + \left\langle X, \mathsf{D}^{F^*} \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^{k} s \right\rangle \mu \\
&= \left\langle \operatorname{div}_\mu(X), \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle X, \mathsf{D}^{F^*} \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu + \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^{k} s \right\rangle \mu.
\end{aligned}
$$

Integrating this equality over M gives immediately

$$
\int_M \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^k s \right\rangle \mu = - \int_M \left\langle X, \mathsf{D}^{F^*} \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu - \int_M \left\langle \operatorname{div}_\mu(X), \omega \otimes \left(\mathsf{D}^E\right)^{k-1} s \right\rangle \mu. \quad (*)
$$

This result is now again true for general compactly supported sections by the above partition of unity argument. We claim now that for all $\ell \leq k$ we have

$$
\int_M \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^k s \right\rangle \mu = (-1)^\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell - r} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell} s \right\rangle \mu.
$$

Indeed, a simple induction gives this formula as we can successively apply $(*)$

$$
\begin{aligned}
&(-1)^\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell - r} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell} s \right\rangle \mu \\
&= (-1)^\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \Bigg(\int_M \left\langle \operatorname{div}_\mu^{r+1}(X), (\mathsf{D}^{F^*})^{\ell - r} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell-1} s \right\rangle \mu \\
&\qquad + \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell - r + 1} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell+1} s \right\rangle \mu \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{\ell+1} (-1)^{\ell+1} \binom{\ell}{r-1} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell-r+1} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell-1} s \right\rangle \mu \\
&\quad + \sum_{r=0}^{\ell} \binom{\ell}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell+1-r} \omega \otimes \left(\mathsf{D}^E\right)^{k-(\ell+1)} s \right\rangle \mu \\
&= (-1)^{\ell+1} \int_M \left\langle \operatorname{div}_\mu^{\ell+1}(X), \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell+1} s \right\rangle \mu \\
&\quad + \sum_{r=1}^{\ell} (-1)^{\ell+1} \left(\binom{\ell}{r-1} + \binom{\ell}{r} \right) \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell+1-r} \omega \otimes \left(\mathsf{D}^E\right)^{k-(\ell+1)} s \right\rangle \mu \\
&\quad + (-1)^{\ell+1} \int_M \left\langle \operatorname{div}_\mu(X), (\mathsf{D}^{F^*})^{\ell+1} \omega \otimes \left(\mathsf{D}^E\right)^{k-\ell+1} s \right\rangle \mu \\
&= (-1)^{\ell+1} \sum_{r=0}^{\ell+1} \binom{\ell+1}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{\ell+1-r} \omega \otimes \left(\mathsf{D}^E\right)^{k-(\ell+1)} s \right\rangle \mu.
\end{aligned}
$$

In particular, for $k = \ell$ we obtain the formula

$$
\int_M \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^k s \right\rangle \mu = (-1)^k \sum_{r=0}^{k} \binom{k}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{k-r} \omega \otimes s \right\rangle \mu
$$

with no derivatives acting on s anymore. Thus we have computed the adjoint of $\varrho_{\mathrm{Std}}(X)$. Indeed, collecting the pre-factors gives

$$
\begin{aligned}
\int_M (\varrho_{\mathrm{Std}}(X) s)\, \mu &= \frac{1}{k!} \left(\frac{\hbar}{\mathrm{i}}\right)^k \int_M \left\langle X, \omega \otimes \left(\mathsf{D}^E\right)^k s \right\rangle \mu \\
&= \frac{(-1)^k}{k!} \left(\frac{\hbar}{\mathrm{i}}\right)^k \sum_{r=0}^{k} \binom{k}{r} \int_M \left\langle \operatorname{div}_\mu^r(X), (\mathsf{D}^{F^*})^{k-r} \omega \otimes s \right\rangle \mu \\
&= \frac{(-1)^k}{k!} \left(\frac{\hbar}{\mathrm{i}}\right)^k \sum_{r=0}^{k} \binom{k}{r} \int_M (k-r)! \left(\frac{\hbar}{\mathrm{i}}\right)^{k-r} \left(\varrho_{\mathrm{Std}}(\operatorname{div}_\mu^r(X^{\mathrm{T}}))\omega\right)(s)\, \mu \\
&= (-1)^k \int_M \left(\varrho_{\mathrm{Std}} \left(\sum_{r=0}^{k} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \operatorname{div}_\mu^r(X^{\mathrm{T}}) \right) \omega \right)(s)\, \mu \\
&= (-1)^k \int_M \left(\varrho_{\mathrm{Std}}(N^2 X^{\mathrm{T}})\omega\right)(s)\, \mu,
\end{aligned}
$$

with N as in (1.2.68). □

Remark 1.2.22 The reason for the unpleasant prefactor $(-1)^k$ is that we have not used a sesquilinear pairing. Indeed, if we have the situation as in Remark 1.2.14 then we would have the following result: For simplicity we consider the scalar case only, i.e. $E = F = M \times \mathbb{C}$ are both the trivial line bundle hence $\Gamma^\infty(E) = \mathcal{C}^\infty(M)$. Then consider

$$
\langle \varphi, \psi \rangle_\mu = \int_M \overline{\varphi} \psi \, \mu \tag{1.2.69}
$$

for $\varphi, \psi \in \mathcal{C}^\infty_0(M)$ instead of (1.2.48). The additional complex conjugation uses the sign $(-1)^k$ to obtain

$$\varrho_{\mathrm{Std}}(X)^{\mathrm{T}} = \varrho_{\mathrm{Std}}(N^2(\overline{X})^{\mathrm{T}}) \tag{1.2.70}$$

for $X \in \Gamma^\infty(\mathrm{S}^k TM)$ in this case. This also generalizes to the case of Hermitian vector bundles, see [11] for an additional discussion.

1.3 Distributions on Manifolds

In this section we introduce distributions as continuous linear functionals and discuss several of their basic properties. In particular, the behaviour under smooth maps and differential operators will be discussed.

1.3.1 Distributions and Generalized Sections

As in the well-known case of $M = \mathbb{R}^n$ we define distributions as continuous linear functionals on the test function spaces:

Definition 1.3.1 (Distribution) *A distribution u on M is a continuous linear functional*

$$u : \mathcal{C}^\infty_0(M) \longrightarrow \mathbb{C}. \tag{1.3.1}$$

The space of all distributions is denoted by $\mathcal{C}^\infty_0(M)'$ or $\mathcal{D}'(M)$.

Remark 1.3.2 (Distributions)

i.) The continuity of course refers to the LF topology of $\mathcal{C}^\infty_0(M)$ as introduced in Theorem 1.1.11. In particular, a linear functional is continuous if and only if for all compacta $K \subseteq M$ the restriction

$$u\big|_{\mathcal{C}^\infty_K(M)} : \mathcal{C}^\infty_K(M) \longrightarrow \mathbb{C} \tag{1.3.2}$$

is continuous in the $\mathcal{C}^\infty_K$-topology. This is the case if and only if for all $\varphi \in \mathcal{C}^\infty_K(M)$ we have a constant $c > 0$ and $\ell \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le c \max_{\ell' \le \ell} \mathrm{p}_{K,\ell'}(\varphi). \tag{1.3.3}$$

Analogously, we could have used the seminorms $p_{U,x,K,\ell}$ avoiding the usage of a covariant derivative but taking a maximum over finitely many compacta in the domain of a chart. With the symbolic seminorms of Remark 1.1.8 we can combine this to

$$|u(\varphi)| \le c\, \mathrm{p}_{K,\ell}(\varphi). \tag{1.3.4}$$

In the following, we shall mainly use this version of the continuity. Since each $\mathcal{C}^\infty_K(M)$ is a Fréchet space, u restricted to $\mathcal{C}^\infty_K(M)$ is continuous iff it is sequentially continuous. This gives yet another criterion: A linear functional $u : \mathcal{C}^\infty_0(M) \longrightarrow \mathbb{C}$ is continuous iff for all $\varphi_n \in \mathcal{C}^\infty_0(M)$ with $\varphi_n \longrightarrow \varphi$ in the $\mathcal{C}^\infty_0$-topology we have

$$u(\varphi_n) \longrightarrow u(\varphi). \tag{1.3.5}$$

ii.) The minimal $\ell \in \mathbb{N}_0$ such that (1.3.3) is valid is called the *local order* $\mathrm{ord}_K(u)$ of u on K. Clearly, this is a quantity independent of the connection used for $\mathrm{p}_{K,\ell}$ and can analogously be obtained from the seminorms $\mathrm{p}_{U,x,K,\ell}$ as well. The independence follows at once from the various estimates between the seminorms as in the proof of Theorem 1.1.5. The *total order* of u is defined as

$$\operatorname{ord}(u) = \sup_K \operatorname{ord}_K(u) \in \mathbb{N}_0 \cup \{+\infty\}, \tag{1.3.6}$$

and the distributions of total order $\leq k$ are sometimes denoted by $\mathcal{D}'^k(M)$. Their union is denoted by $\mathcal{D}'_F(M)$ and called *distributions of finite order.*

iii.) The distributions $\mathcal{D}'(M)$ as well as $\mathcal{D}'^k(M)$ and $\mathcal{D}'_F(M)$ are vector spaces. We have $\mathcal{D}'^k(M) \subseteq \mathcal{D}'^\ell(M)$ for $k \leq \ell$. It can be shown that already for $M = \mathbb{R}^n$ all the inclusions $\mathcal{D}'^k(M) \subseteq \mathcal{D}'^\ell(M) \subseteq \mathcal{D}'_F(M) \subseteq \mathcal{D}'(M)$ are proper.

iv.) If u has order $\leq k$ it can be shown that u extends uniquely to a continuous linear function

$$u : \mathcal{C}^\ell_0(M) \longrightarrow \mathbb{C} \tag{1.3.7}$$

with respect to the $\mathcal{C}^\ell_0$-topology provided $\ell \geq k$. This follows essentially from the approximation Theorem 1.1.26, see e.g [31, Thm 2.16].

Example 1.3.3 (δ-functional) For $p \in M$ the evaluation functional

$$\delta_p : \mathcal{C}^\infty_0(M) \ni \varphi \mapsto \varphi(p) \in \mathbb{C} \tag{1.3.8}$$

is clearly continuous and has order zero. More generally, if $v_p \in T_pM$ is a tangent vector then

$$v_p : \varphi \mapsto v_p(\varphi) \tag{1.3.9}$$

is again continuous and has order one.

Example 1.3.4 (Locally integrable densities) Let $\mu : M \longrightarrow |\Lambda^{\mathrm{top}}|T^*M$ be a not necessarily continuous section. Then μ is called *locally integrable* if for all charts (U, x) and all $K \subseteq U$ the function μ_U in $\mu\big|_U = \mu_U |\,\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n|$ is integrable over K with respect to the Lebesgue measure on $x(U)$. Since the $|\,\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n|$ transform with the *smooth* absolute value of the Jacobian of the change of coordinates, it follows at once that local integrability is intrinsically defined and it is sufficient to check it for an atlas and an exhausting sequence of compacta. It is then easy to see that

$$\mu : \mathcal{C}^\infty_0(M) \ni \varphi \mapsto \int_M \varphi\, \mu \in \mathbb{C} \tag{1.3.10}$$

is continuous. Indeed, if $K \subseteq M$ is compact then $\operatorname{vol}_\mu(K) = \int_K |\mu| < \infty$ is well-defined and we have

$$\left| \int_M \varphi\, \mu \right| \leq \operatorname{vol}_\mu(K)\, \mathrm{p}_{K,0}(\varphi). \tag{1.3.11}$$

Note that $|\mu| = \sqrt{\overline{\mu}\mu}$ is well-defined as 1-density and still locally integrable. In particular, (1.3.10) is a distribution of order zero.

Remark 1.3.5 (Generalized densities) The last example shows that we can identify densities of quite general type (locally integrable) with certain distributions. For this reason, we call distributions also "*generalized densities*", following e.g. [23,27]. Note however that e.g. Hörmander takes a different point of view and treats distributions as "*generalized functions*". In [31] a distribution is *not* a continuous linear functional on $\mathcal{C}^\infty_0(M)$ but has a slightly different transformation behaviour under local diffeomorphisms. In fact, his generalized functions can be viewed as continuous linear functionals on $\Gamma_0^\infty(|\Lambda^{\mathrm{top}}|T^*M)$. To emphasize the generalized density aspect from now on we adopt the notation of [27] and write

$$\Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M) = \left\{ u : \mathcal{C}_0^\infty(M) \longrightarrow \mathbb{C} \;\middle|\; u \text{ is linear and continuous} \right\}. \tag{1.3.12}$$

This point of view will be very useful when we discuss the transformation properties of distributions. Later on, both versions will be combined anyway since we consider distributional sections of arbitrary vector bundles. Thus speaking of generalized functions will be non ambiguous.

We can now generalize the notion of distributions to test sections instead of test functions.

Definition 1.3.6 (Generalized section) *Let $E \longrightarrow M$ be a smooth vector bundle. Then a generalized section (or: distributional section) of E is a continuous linear functional*

$$s : \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}. \tag{1.3.13}$$

The generalized sections will be denoted by $\Gamma^{-\infty}(E)$.

Remark 1.3.7 Note that here we have some mild clash of notations since we defined a distribution already as a generalized density $u \in \Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)$ while a generalized density according to Definition 1.3.6 is a continuous linear functional

$$u : \Gamma_0^\infty\left(\left(|\Lambda^{\mathrm{top}}|T^*M\right)^* \otimes |\Lambda^{\mathrm{top}}|T^*M \right) \longrightarrow \mathbb{C}, \tag{1.3.14}$$

and not $u : \mathcal{C}_0^\infty(M) \longrightarrow \mathbb{C}$. However, for any line bundle L we have canonically $L^* \otimes L \simeq M \times \mathbb{C}$ hence we can (and will) canonically identify $\Gamma_0^\infty((|\Lambda^{\mathrm{top}}|T^*M)^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ with $\mathcal{C}_0^\infty(M)$. Thus Definition 1.3.6 and Definition 1.3.1 are consistent.

Moreover, a section of E is always a generalized section of E since for $s \in \Gamma^\infty(E)$ we can integrate $\omega(s)$ with $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ over M and obtain a continuous linear functional which we can identify with an element in $\Gamma^{-\infty}(E)$. In fact, the section s is uniquely determined be the values $\int_M \omega(s)$ for all $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ hence this is indeed an injection. Therefor we have

$$\Gamma^\infty(E) \subseteq \Gamma^{-\infty}(E). \tag{1.3.15}$$

More generally, we also have

$$\Gamma^k(E) \subseteq \Gamma^{-\infty}(E) \tag{1.3.16}$$

for all $k \in \mathbb{N}_0$ by the same argument.

Remark 1.3.8 If we choose a smooth positive density $\mu > 0$ then we can also identify $\Gamma^{-\infty}(E)$ with the topological dual of $\Gamma_0^\infty(E^*)$. Indeed, if $s \in \Gamma^{-\infty}(E)$ then we can define

$$I_\mu(s) : \Gamma_0^\infty(E^*) \ni \omega \;\mapsto\; s(\omega \otimes \mu) \in \mathbb{C}, \tag{1.3.17}$$

and clearly obtain an element $I_\mu(s) \in \Gamma_0^\infty(E^*)'$ in the topological dual. The reason is that the map

$$\Gamma_0^\infty(E^*) \ni \omega \;\mapsto\; \omega \otimes \mu \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \tag{1.3.18}$$

is continuous in the $\mathcal{C}_0^\infty$-topology according to Proposition 1.1.24 and Remark 1.1.25. Moreover, since (1.3.18) is even a bijection with continuous inverse, we obtain an isomorphism

$$I_\mu : \Gamma^{-\infty}(E) \longrightarrow \Gamma_0^\infty(E^*)'. \tag{1.3.19}$$

In case of $M = \mathbb{R}^n$ one uses the Lebesgue measure $\mathrm{d}^n x \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*\mathbb{R}^n)$ to provide such an identification. Note however that (1.3.19) does not behave well under vector bundle morphisms as we shall see later since μ needs not to be invariant. Finally, if the choice of μ is clear from the context, we shall omit the symbol I_μ and identify $\Gamma^{-\infty}(E)$ directly with the dual space $\Gamma^\infty(E^*)'$ to simplify our notation. This will frequently happen starting from Chapter 3.

Remark 1.3.9 (Module structure) The generalized sections $\Gamma^{-\infty}(E)$ become a $\mathcal{C}^\infty(M)$-module via the definition

$$(f \cdot s)(\omega) = s(f\omega). \tag{1.3.20}$$

Indeed $\omega \mapsto f\omega$ is $\mathcal{C}_0^\infty$-continuous and hence (1.3.20) is indeed a continuous linear functional $f \cdot s \in \Gamma^{-\infty}(E)$. The module property is clear.

Remark 1.3.10 (Order of generalized sections) The continuity of $s \in \Gamma^{-\infty}(E)$ is again expressed using the seminorms of $\Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ in the following way. For every compactum $K \subseteq M$ there are constants $c > 0$ and $\ell \in \mathbb{N}_0$ such that

$$|s(\omega)| \leq c \max_{\ell' \leq \ell} \mathrm{p}_{K,\ell'}(\omega), \tag{1.3.21}$$

for all $\omega \in \Gamma_K^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$. Again, the *local order* of s on K is defined to be the smallest ℓ such that (1.3.21) holds. This also defines the *global order*

$$\mathrm{ord}(s) = \sup_K \mathrm{ord}_K(s) \tag{1.3.22}$$

as before. As in the scalar case, a generalized section $s \in \Gamma^{-\infty}(E)$ with global order $\mathrm{ord}(s) \leq k$ extends uniquely to a $\mathcal{C}_0^\ell$-continuous functional

$$s : \Gamma_0^\ell(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C} \tag{1.3.23}$$

for all $\ell \geq k$. We shall denote the distributional sections of order $\leq \ell$ by $\Gamma^{-\ell}(E)$. Note that $\Gamma^{-0}(E)$ are *not* just the continuous sections.

We also want to topologize the distributions. Here we use the most simple locally convex topology: the weak* topology:

Definition 1.3.11 (Weak* topology) *The weak* topology for* $\Gamma^{-\infty}(E)$ *is the locally convex topology obtained from all the seminorms*

$$\mathrm{p}_\omega(s) = |s(\omega)|, \tag{1.3.24}$$

where $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$.

In the following we always use the weak* topology for $\Gamma^{-\infty}(E)$. We have the following properties:

Theorem 1.3.12 (Weak* topology of $\Gamma^{-\infty}(E)$)

i.) A sequence $s_n \in \Gamma^{-\infty}(E)$ *converges to* $s \in \Gamma^{-\infty}(E)$ *if and only if for all* $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$

$$s_n(\omega) \longrightarrow s(\omega). \tag{1.3.25}$$

ii.) $\Gamma^{-\infty}(E)$ *is sequentially complete, i.e. every weak* Cauchy sequence converges.*

iii.) The inclusions $\Gamma^k(E) \subseteq \Gamma^{-\infty}(E)$ *are continuous in the* $\mathcal{C}^k$*- and weak* topology for all* $k \in \mathbb{N}_0 \cup \{+\infty\}$.

iv.) The map $\Gamma^{-\infty}(E) \ni s \mapsto fs \in \Gamma^{-\infty}(E)$ *is weak* continuous for all* $f \in \mathcal{C}^\infty(M)$.

v.) The sections $\Gamma_0^\infty(E)$ *are sequentially weak* dense in* $\Gamma^{-\infty}(E)$.

Proof. The first part is clear since $s_n \longrightarrow s$ means for every seminorm p_ω we have

$$\mathrm{p}_\omega(s_n - s) \longrightarrow 0,$$

which is (1.3.25). Thus the notion of convergence in $\Gamma^{-\infty}(E)$ is pointwise convergence on the test sections $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$. The second part is non-trivial but follows from general arguments:

first one shows that the topological dual V' of a Fréchet space V is sequentially complete by a Banach-Steinhaus argument. Here Fréchet is crucial. Second, one extends this result to LF spaces like our $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$, see e.g. [31, Thm. 2.1.8] or [51, Thm. 6.17] for details. Note however that $\Gamma^{-\infty}(E)$ is *not* complete; in fact, the completion is the full algebraic dual [34, p.147]. The third part is easy since for a $\mathcal{C}^k$-section $s \in \Gamma^k(E)$ we have for all $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$

$$\mathrm{p}_\omega(s) = |s(\omega)| = \left| \int_M \omega(s) \right| \le c\, \mathrm{p}_{K,0}(s),$$

with some constant $c > 0$ depending on ω but not on s and a compactum $\operatorname{supp} \omega \subseteq K$. Essentially, c is the volume of K times the maximum of ω with respect to the metrics used to define $\mathrm{p}_{K,0}$. From this the continuity is obvious. For the fourth part we compute

$$\mathrm{p}_\omega(fs) = |fs(\omega)| = |s(f\omega)| = \mathrm{p}_{f\omega}(s),$$

which already shows the continuity. The last part is slightly more tricky. We have to construct a sequence $s_n \in \Gamma_0^\infty(E)$ with $s_n \longrightarrow s$ in the weak* topology using of course the identification of s_n with an element of $\Gamma^{-\infty}(E)$. We choose a countable atlas of charts (U_n, x_n) and a partition of unity χ_n subordinate to this atlas. Then we consider the distributions $\chi_n s \in \Gamma^{-\infty}(E)$. We claim that

$$\sum_{n=0}^{\infty} \chi_n s = s$$

in the weak* topology. To prove this, let $\omega \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ be given and let $K = \operatorname{supp} \omega$. Then only finitely many χ_n are nonzero on K, hence

$$\sum_n (\chi_n s)(\omega) = \sum_n^{\text{finite}} s(\chi_n \omega) = s\left(\sum_n^{\text{finite}} \chi_n \omega \right) = s(\omega).$$

This proves convergence. Since the $\chi_n s$ are countable, it is sufficient to prove that each $\chi_n s$ can be approximated by a sequence of sections in $\Gamma_0^\infty(E)$. Since $\operatorname{supp} \chi_n \subseteq U_n$ we also conclude that $(\chi_n s)(\omega) = 0$ if $\operatorname{supp} \omega \cap U_n = \emptyset$. Thus we are left with the problem to approximate a distribution on a chart which can be done by some appropriate convolution, see e.g. [51, Thm. 6.32]. □

Remark 1.3.13 (Weak* topology of $\Gamma^{-\infty}(E)$)

i.) It should be noted that $\Gamma^{-\infty}(E)$ is not Fréchet, in fact it is not metrizable. Thus sequential completeness is weaker than completeness: $\Gamma^{-\infty}(E)$ is not complete and its completion is the full *algebraic* dual of $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$.

ii.) The importance of continuity of the inclusion is that for sections $s_n \in \Gamma^k(E)$ with $s_n \longrightarrow s$ in the $\mathcal{C}^k$-topology we also have $s_n \longrightarrow s$ in the weak* topology of $\Gamma^{-\infty}(E)$ for all $k \in \mathbb{N}_0 \cup \{+\infty\}$.

iii.) The last part shows that $\Gamma^{-\infty}(E)$ is, on one hand, a large extension of $\Gamma_0^\infty(E)$ and also $\Gamma^\infty(E)$ which, on the other hand, is still not "too large": continuous operations with distributions are already determined by their restrictions to $\Gamma_0^\infty(E)$. This justifies the name "generalized section".

1.3.2 Calculus with Distributions

In this subsection we shall extend various constructions with sections to generalized sections. The main idea is to "dualize" continuous linear operations on test sections in an appropriate way.

We begin with the definition of the support of a distribution and its restriction to open subsets.

Definition 1.3.14 (Restriction and support) *Let $U \subseteq M$ be open and $s \in \Gamma^{-\infty}(E)$.*

i.) The restriction $s\big|_U$ *is defined by*

$$s\big|_U(\omega) = s(\omega) \tag{1.3.26}$$

for $\omega \in \Gamma^\infty_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M\big|_U)$, *i.e. for* $\omega \in \Gamma^\infty_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ *with* $\operatorname{supp}\omega \subseteq U$.

ii.) The support of s is defined by

$$\operatorname{supp} s = \bigcap_{\substack{A \subseteq M \, closed \\ s|_{M\setminus A} = 0}} A. \tag{1.3.27}$$

Remark 1.3.15 (Restriction and Support)

i.) It is easy to show that $s\big|_U \in \Gamma^{-\infty}(E\big|_U)$. Moreover, we clearly have

$$(s\big|_U)\big|_V = s\big|_V, \tag{1.3.28}$$

for $V \subseteq U$. In more sophisticated terms this means that $\Gamma^{-\infty}(E)$ has the structure of a *presheaf* over M with values in locally convex vector spaces.

ii.) If $U_\alpha \subseteq M$ is an open cover of M and if we have $s_\alpha \in \Gamma^{-\infty}(E\big|_{U_\alpha})$ given such that

$$s_\alpha\big|_{U_\alpha \cap U_\beta} = s_\beta\big|_{U_\alpha \cap U_\beta}, \tag{1.3.29}$$

whenever $U_\alpha \cap U_\beta \neq \emptyset$ then there exists a unique $s \in \Gamma^{-\infty}(E)$ with $s\big|_{U_\alpha} = s_\alpha$. The proof of this fact uses a partition of unity argument to glue together the locally defined s_α. In fact, if χ_α is a subordinate partition of unity one checks that the definition

$$s(\omega) = \sum_\alpha s_\alpha(\chi_\alpha \omega) \tag{1.3.30}$$

indeed gives the desired s, independent of the choice of the partition of unity. Moreover, if $s, t \in \Gamma^{-\infty}(E)$ are given then

$$s\big|_{U_\alpha} = t\big|_{U_\alpha} \tag{1.3.31}$$

for all α implies $s = t$. This is obvious. Again, with more high-tech language this means that $\Gamma^{-\infty}(E)$ is in fact a *sheaf* and not only a presheaf.

iii.) The support $\operatorname{supp} s$ of $s \in \Gamma^{-\infty}(E)$ is the smallest closed subset with $s\big|_{M\setminus \operatorname{supp} s} = 0$ and we have $p \in \operatorname{supp} s$ if and only if for every open neighborhood U of p we find $\omega \in \Gamma^\infty_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ with $\operatorname{supp}\omega \subseteq U$ and $s(\omega) \neq 0$.

iv.) For $s \in \Gamma^{-\infty}(E)$, $f \in \mathcal{C}^\infty(M)$, $t \in \Gamma^0(E)$ and $\omega \in \Gamma^\infty_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ we have

$$\operatorname{supp}(fs) \subseteq \operatorname{supp} f \cap \operatorname{supp} s \tag{1.3.32}$$

$$s(\omega) = 0 \quad \text{if} \quad \operatorname{supp} s \cap \operatorname{supp}\omega = \emptyset, \tag{1.3.33}$$

and the support of t as a *continuous section* in $\Gamma^0(E)$ coincides with the support of t viewed as distribution. Thus the notion of support has the usual properties as known from continuous or smooth sections.

After the support we also have a more refined notion, namely the *singular support*. It characterizes where a generalized section is not just a smooth section but actually "singular".

Definition 1.3.16 (Singular support) *Let* $s \in \Gamma^{-\infty}(E)$.

i.) s is called regular in $p \in M$ *if there is an open neighborhood* $U \subseteq M$ *of* p *such that*

$$s\big|_U \in \Gamma^\infty(E\big|_U). \tag{1.3.34}$$

ii.) The singular support of s is

$$\operatorname{sing\,supp} s = \{p \in M \mid s \textit{ is not regular in } p\}\,. \tag{1.3.35}$$

The singular support of s indeed behaves similar to the support.

Remark 1.3.17 (Singular support) Let $s \in \Gamma^{-\infty}(E)$, $t \in \Gamma^{\infty}(E)$ and $f \in \mathcal{C}^{\infty}(M)$.

i.) The singular support $\operatorname{sing\,supp} s$ is the smallest closed subset of M with

$$s\big|_{M \setminus \operatorname{sing\,supp} s} \in \Gamma^{\infty}(E). \tag{1.3.36}$$

This follows easily from the fact that smooth sections are determined by their restrictions to open subsets and by (1.3.29) in Remark 1.3.15.

ii.) We have

$$\operatorname{sing\,supp} s \subseteq \operatorname{supp} s, \tag{1.3.37}$$

$$\operatorname{sing\,supp}(fs) \subseteq \operatorname{sing\,supp} s, \tag{1.3.38}$$

and

$$\operatorname{sing\,supp} t = \emptyset. \tag{1.3.39}$$

Again these properties follow in a rather straightforward way from the very definition.

Having a notion of support of distributions it is interesting to consider those elements of $\Gamma^{-\infty}(E)$ with *compact* support. The following theorem gives a full description:

Theorem 1.3.18 (Generalized sections with compact support) *Let $s \in \Gamma^{-\infty}(E)$ have compact support. Then we have:*

i.) s has finite global order $\operatorname{ord}(s) < \infty$.

ii.) s has a unique extension to a linear functional

$$s : \Gamma^{\infty}(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}, \tag{1.3.40}$$

which is continuous in the $\mathcal{C}^{\infty}$-topology.

Conversely, if $s : \Gamma^{\infty}(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}$ is a continuous linear functional then its restriction to $\Gamma^{\infty}_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ is a generalized section of E with compact support.

Proof. Thanks to the compactness of $\operatorname{supp} s$ we can find an open neighborhood U of $\operatorname{supp} s$ such that $U^{\mathrm{cl}} \subseteq M$ is still compact. Hence there is a $\chi \in \mathcal{C}^{\infty}_0(M)$ with $\chi\big|_{U^{\mathrm{cl}}} = 1$. It follows from (1.3.32) that

$$\chi s = s.$$

For $K = \operatorname{supp} \chi$ we find some $\ell \in \mathbb{N}_0$ and $c > 0$ such that for all $\omega \in \Gamma^{\infty}_K(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ we have

$$|s(\omega)| \le c\, \mathrm{p}_{K,\ell}(\omega),$$

since s is continuous with the seminorms of Remark 1.1.8. If $\omega \in \Gamma^{\infty}_0(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ is arbitrary we have $\chi\omega \in \Gamma^{\infty}_K(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$, hence

$$|s(\omega)| = |s(\chi\omega)| \le c\, \mathrm{p}_{K,\ell}(\chi\omega) \le c'\, \mathrm{p}_{K,\ell}(\omega) \le c'\, \mathrm{p}_{M,\ell}(\omega)$$

by the Leibniz rule and the compactness of $\operatorname{supp} \omega$. From this we immediately see that s has global order $\operatorname{ord}(s) \le \ell$. For the second part consider $\omega \in \Gamma^{\infty}(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ then $\chi\omega$ has compact support and we can set

$$s(\omega) = s(\chi\omega).$$

This clearly provides a linear extension of s and since $\operatorname{supp}(\chi\omega) \subseteq K$ we have

$$|s(\omega)| = |s(\chi\omega)| \leq c\, \mathrm{p}_{K,\ell}(\chi\omega) \leq c'\, \mathrm{p}_{K,\ell}(\omega),$$

which is the continuity in the $\mathcal{C}^\infty$-topology. Thus s is a continuous extension. Since

$$\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \subseteq \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$$

is dense by Proposition 1.1.9, such an extension is necessarily unique. Now let $s : \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}$ be linear and continuous in the $\mathcal{C}^\infty$-topology. Then there exists a compactum $K \subseteq M$ and $\ell \in \mathbb{N}_0$, $c > 0$ with

$$|s(\omega)| \leq c\, \mathrm{p}_{K,\ell}(\omega)$$

for all $\omega \in \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$. From this it follows easily that $s\big|_{\Gamma^\infty_{K'}(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)}$ is continuous in the $\mathcal{C}^\infty_{K'}$-topology for all compacta K'. Moreover, for $\operatorname{supp}\omega \cap K = \emptyset$ we have $s(\omega) = 0$, hence $\operatorname{supp} s \subseteq K$ follows. ☐

Definition 1.3.19 *The generalized sections of E with compact support are denoted by $\Gamma_0^{-\infty}(E)$.*

After having identified the distributions with compact support we can extend this construction under slightly milder assumptions: if only the overlap $\operatorname{supp} s \cap \operatorname{supp}\omega$ is compact then the pairing $s(\omega)$ is already well-defined:

Proposition 1.3.20 *Let $s \in \Gamma^{-\infty}(E)$ be a generalized section. Then there exists a unique extension $\widetilde{s}$ of s to a linear functional*

$$\widetilde{s} : \left\{\omega \in \Gamma^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \;\middle|\; \operatorname{supp}\omega \cap \operatorname{supp} s \text{ is compact}\right\} \longrightarrow \mathbb{C}, \tag{1.3.41}$$

such that

i.) $\widetilde{s}$ *coincides with* s *on* $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$,

ii.) $\widetilde{s}(\omega) = 0$ *if* $\operatorname{supp} s \cap \operatorname{supp}\omega = \emptyset$.

Proof. Assume first that $\widetilde{s}'$ is another such extension and let ω be a test sections as in (1.3.41). Then we choose a cut-off function $\chi \in \mathcal{C}^\infty(M)$ with $\chi = 1$ on an open neighborhood U of $K = \operatorname{supp} s \cap \operatorname{supp}\omega$. Thus $\omega = \chi\omega + (1-\chi)\omega$ with $\chi\omega$ having compact support and $\operatorname{supp}(1-\chi)\omega \cap \operatorname{supp} s = \emptyset$. Hence for the extension $\widetilde{s}$ we get by linearity and *i.)* and *ii.)*

$$\widetilde{s}(\omega) = \widetilde{s}(\chi\omega + (1-\chi)\omega) = \widetilde{s}(\chi\omega) + \widetilde{s}((1-\chi)\omega) = s(\chi\omega).$$

The same arguments hold for $\widetilde{s}'$ whence $\widetilde{s}'(\omega) = s(\chi\omega) = \widetilde{s}(\omega)$ follows. This shows that such an extension is necessarily unique. To show existence we simply define $\widetilde{s}(\omega) = s(\chi\omega)$ where χ is chosen as above. Clearly, two different choices of χ lead to the same extension by the above uniqueness argument. Since for ω, ω' we can find a common χ, satisfying the requirements with respect to both ω and ω', we see that the above definition is linear. For $\operatorname{supp}\omega$ compact we find a χ with $\chi\omega = \omega$ whence *i.)* follows. Finally, if $\operatorname{supp} s \cap \operatorname{supp}\omega = \emptyset$ then $\chi = 0$ will do the job and so *ii.)* holds. ☐

One can also put a certain locally convex topology on the vector space of such test functions such that the extension is actually continuous. In the following we will denote this extension simply by s.

Remark 1.3.21 A slight variation of this proposition is the following. If $\operatorname{ord}_K s \leq \ell$ for some compact subset K then s extends uniquely to a linear functional

$$s : \left\{\omega \in \Gamma^\ell(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \;\middle|\; \operatorname{supp}\omega \cap \operatorname{supp} s \subseteq K\right\} \longrightarrow \mathbb{C}, \tag{1.3.42}$$

such that

i.) $\widetilde{s}$ coincides with the continuous extension of s to $\Gamma^{\ell}_{K}(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ on those ω with $\operatorname{supp}\omega \subseteq K$.

ii.) $s(\omega) = 0$ if $\operatorname{supp}\omega \cap \operatorname{supp}s = \emptyset$.

After the discussion of supports we can now move distributions around by using smooth maps between manifolds and vector bundle morphisms. The latter one clearly includes the case of smooth maps by viewing smooth functions as sections of the trivial line bundle and extending a smooth map in the unique way to a vector bundle morphism of the trivial line bundles.

Thus let $E \longrightarrow M$ and $F \longrightarrow M$ by vector bundles and let $\Phi : E \longrightarrow F$ be a smooth vector bundle morphism over the smooth map $\phi : M \longrightarrow N$. We can now obtain pull-backs and push-forwards of distributions by dualizing the statements of the Propositions 1.1.20 and 1.1.23 appropriately. We start with the scalar case:

Definition 1.3.22 (Push-forward of distributions) *Let $\phi : M \longrightarrow N$ be a smooth map. The push-forward of compactly supported generalized densities*

$$\phi_* : \Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*N) \tag{1.3.43}$$

is defined on $f \in \mathcal{C}^\infty(M)$ by

$$(\phi_*\mu)(f) = \mu(\phi^* f). \tag{1.3.44}$$

Proposition 1.3.23 (Push-forward of distributions) *Let $\phi : M \longrightarrow N$ be a smooth map.*

i.) The push-forward $\phi_*\mu$ of $\mu \in \Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*M)$ is a well-defined generalized density with compact support

$$\phi_*\mu \in \Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*N). \tag{1.3.45}$$

The map ϕ_* is linear and continuous with respect to the weak topologies.*

ii.) Assume ϕ is in addition proper. Then the push-forward extends uniquely to $\Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)$ and gives a linear continuous map

$$\phi_* : \Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*N) \tag{1.3.46}$$

with respect to the weak topologies. Explicitly, for all $\varphi \in \mathcal{C}^\infty_0(N)$ the push-forward $\phi_*\mu$ of μ is given by*

$$(\phi_*\mu)(\varphi) = \mu(\phi^*\varphi). \tag{1.3.47}$$

iii.) We have

$$(\mathsf{id}_M)_* = \mathsf{id}_{\Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)} \quad \textit{and} \quad (\phi \circ \psi)_* = \phi_* \circ \psi_*. \tag{1.3.48}$$

Proof. Since by Proposition 1.1.14 the pull-back $\phi^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty$-continuous, by (1.3.44) one obtains a well-defined transpose map of ϕ^* which —consequently— is denoted by ϕ_*. Clearly, ϕ_* is linear and

$$\mathrm{p}_f(\phi_*\mu) = |\phi_*\mu(f)| = |\mu(\phi^* f)| = \mathrm{p}_{\phi^* f}(\mu)$$

shows immediately that ϕ_* is weak* continuous. The second part follows analogously, now using Proposition 1.1.17 instead. The uniqueness of this extension follows since $\phi_*\mu$ is continuous and since the compactly supported distributions $\Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*M)$ are sequentially dense in $\Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)$. The later follows from Theorem 1.3.12, *v.)* since already $\Gamma^{\infty}_0(|\Lambda^{\mathrm{top}}|T^*M) \subseteq \Gamma^{-\infty}_0(|\Lambda^{\mathrm{top}}|T^*M) \subseteq \Gamma^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)$ is sequentially dense. The last part is obvious and follows immediately from the corresponding properties of the pull-back of functions. ☐

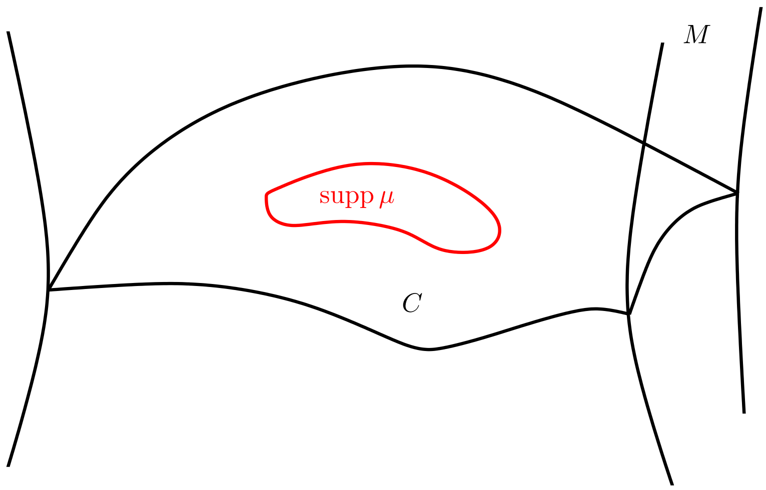

Figure 1.1: The push-forward has now singular support.

Remark 1.3.24 (Push-forward of smooth densities) Since by Remark 1.3.15, *iv.)* we have $\Gamma_0^\infty(|\Lambda^{\mathrm{top}}|T^*M) \subseteq \Gamma_0^{-\infty}(|\Lambda^{\mathrm{top}}|T^*M)$, we can always push-forward compactly supported smooth densities in the sense of generalized densities by (1.3.44). However, even though μ is smooth, $\phi_*\mu$ needs not to be smooth at all. A simple example is obtained as follows: Let $\iota : C \longrightarrow M$ be a submanifold of positive codimension and let $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C)$ be a smooth density on C. Then for $f \in \mathcal{C}_0^\infty(M)$ we have

$$\iota_*\mu(f) = \int_C \iota^* f \, \mu, \tag{1.3.49}$$

which can not be written as $\int_M f \, \nu$ with some *smooth* $\nu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$. In fact, one can show rather easily that

$$\operatorname{supp} \iota_*\mu = \operatorname{sing\,supp} \iota_*\mu = \iota(\operatorname{supp} \mu) \tag{1.3.50}$$

in this case, see also Figure 1.1. The simplest case of this class of examples is given by $C = \{\mathrm{pt}\}$ and $\mu = \delta_{\mathrm{pt}}$ the evaluation functional on $\mathcal{C}^\infty(\mathrm{pt}) = \mathbb{C}$. On C, the δ-functional is actually a *smooth* density but on any higher dimensional manifold this is of course no longer the case.

Remark 1.3.25 There is also a vector-valued version of push-forward. Since for a vector bundle morphism $\Phi : E \longrightarrow F$ over $\phi : M \longrightarrow N$ we have a continuous pull-back

$$\Phi^* : \Gamma^\infty(F^*) \longrightarrow \Gamma^\infty(E^*), \tag{1.3.51}$$

this dualizes to a push-forward

$$\Phi_* : \Gamma_0^{-\infty}(E \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma_0^{-\infty}(F \otimes |\Lambda^{\mathrm{top}}|T^*N) \tag{1.3.52}$$

being again linear and weak* continuous. In case ϕ is proper we get an extension

$$\Phi_* : \Gamma^{-\infty}(E \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma^{-\infty}(F \otimes |\Lambda^{\mathrm{top}}|T^*N), \tag{1.3.53}$$

which is again linear, unique and weak* continuous. In general, a smooth section of $E \otimes |\Lambda^{\mathrm{top}}|T^*M$ is pushed forward to a singular section of $F \otimes |\Lambda^{\mathrm{top}}|T^*N$. Note however, that there are conditions on Φ and ϕ such that Φ_*s is again smooth for a smooth s, see e.g. the discussion in [27, p. 307].

Analogously to the pull-backs we shall now dualize the action of differential operators to find an extension to distributional sections. As we had (at least) two versions of dualizing differential operators, we again obtain several possibilities for distributions.

We start with the "intrinsic" version. Thus let $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ be a differential operator. Then its adjoint is a differential operator

$$D^{\mathrm{T}} : \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \tag{1.3.54}$$

of the same order as D. This motivates the following definition:

Definition 1.3.26 (Differentiation of generalized sections) *Let $D \in \mathrm{DiffOp}^\bullet(E; F)$ then*

$$D : \Gamma^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(F) \tag{1.3.55}$$

is defined by

$$(Ds)(\mu) = s(D^{\mathrm{T}}\mu) \tag{1.3.56}$$

for all $s \in \Gamma^{-\infty}(E)$ and $\mu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$.

This definition indeed gives a reasonable notion of differentiation of generalized sections as the following theorem shows:

Theorem 1.3.27 *Let $D \in \mathrm{DiffOp}^k(E; F)$.*

i.) For all $s \in \Gamma^{-\infty}(E)$ the definition (1.3.56) *gives a well-defined generalized section $Ds \in \Gamma^{-\infty}(F)$ and the map*

$$D : \Gamma^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(F) \tag{1.3.57}$$

is linear and weak continuous. Moreover, we have for all $\ell \in \mathbb{N}_0$*

$$D : \Gamma^{-\ell}(E) \longrightarrow \Gamma^{-\ell-k}(F). \tag{1.3.58}$$

ii.) The map D is the unique extension of $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ which is linear and weak continuous.*

iii.) With respect to the $\mathcal{C}^\infty(M)$-module structure of $\Gamma^{-\infty}(E)$ and $\Gamma^{-\infty}(F)$, the map D as in (1.3.57) *is a differential operator of order k in the sense of the algebraic definition of differential operators, i.e.*

$$D \in \mathrm{DiffOp}^k(\Gamma^{-\infty}(E), \Gamma^{-\infty}(F)). \tag{1.3.59}$$

iv.) We have

$$\mathrm{supp}(Ds) \subseteq \mathrm{supp}\, s \tag{1.3.60}$$

and

$$\mathrm{sing\,supp}(Ds) \subseteq \mathrm{sing\,supp}\, s. \tag{1.3.61}$$

v.) For every open subset $U \subseteq M$ we have

$$Ds\big|_U = D\big|_U(s\big|_U). \tag{1.3.62}$$

Proof. Since $D^{\mathrm{T}} : \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ is again a differential operator of order k by Proposition 1.2.12 and since differential operators are $\mathcal{C}_0^\infty$-continuous by Theorem 1.2.10, the definition (1.3.56) yields indeed a continuous linear functional $Ds \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)' = \Gamma^{-\infty}(F)$. Clearly, D is linear and we have

$$\mathrm{p}_\mu(Ds) = |Ds(\mu)| = |s(D^{\mathrm{T}}\mu)| = \mathrm{p}_{D^{\mathrm{T}}\mu}(s),$$

from which we obtain the weak*-continuity at once. The claim (1.3.58) is clear by counting. The second part follows easily since by Theorem 1.3.12, *v.)* the space $\Gamma_0^\infty(E) \subseteq \Gamma^{-\infty}(E)$ is weak* dense hence any weak* continuous extensions is necessarily unique. For $s \in \Gamma^\infty(E)$ the definition (1.3.56) coincides with the usual application of D by Proposition 1.2.12: the definition (1.3.56) was made precisely that way to have an *extension* of $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$. For the third part, we first consider a differential operator $D \in \mathrm{DiffOp}^0(E;F) = \mathsf{Hom}_{\mathcal{C}^\infty(M)}(\Gamma^\infty(E), \Gamma^\infty(F)) = \Gamma^\infty(\mathsf{Hom}(E,F))$ of order zero. For $s \in \Gamma^{-\infty}(E)$ we have then for all $\mu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ the relation

$$D(f\cdot s)(\mu) = (f\cdot s)(D^{\mathrm{T}}\mu) = s(fD^{\mathrm{T}}(\mu)) = s(D^{\mathrm{T}}(f\mu)) = (f\cdot Ds)(\mu),$$

hence $D(f\cdot s) = f\cdot D(s)$ follows. Thus D as in (1.3.56) is a $\mathcal{C}^\infty(M)$-linear map and hence a differential operator of order zero in the sense of definition (1.2.9). Now we can proceed by induction on the order: assume that $D \in \mathrm{DiffOp}^k(E;F)$ yields a differential operator $D \in \mathrm{DiffOp}^k(\Gamma^{-\infty}(E), \Gamma^{-\infty}(F))$ of the same order k for all $k \leq \ell$. Then for $D \in \mathrm{DiffOp}^{\ell+1}(E,F)$ we have

$$\left(D(f\cdot s) - f\cdot D(s)\right)(\mu) = (f\cdot s)(D^{\mathrm{T}}\mu) - D(s)(f\mu) = s\left(fD^{\mathrm{T}}\mu - D^{\mathrm{T}}(f\mu)\right) = s([f, D^{\mathrm{T}}]\mu).$$

Since for $A = [f,D] \in \mathrm{DiffOp}^\ell(E,F)$ we have $A^{\mathrm{T}} = [f, D^{\mathrm{T}}]$, we see that $[f,D] : \Gamma^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(F)$ is a differential operator of order ℓ by induction. Thus D is again a differential operator of order $\ell+1$, since f was arbitrary. This shows the third part. Now let $\mu \in \Gamma_0^\infty(F^* \otimes T^*M)$ with $\operatorname{supp}\mu \subseteq M\setminus\operatorname{supp}s$ then $\operatorname{supp}D^{\mathrm{T}}\mu \subseteq M\setminus\operatorname{supp}s$ as well hence $(Ds)(\mu) = s(D^{\mathrm{T}}\mu) = 0$ by Remark 1.3.15, *iv.)*. Thus (1.3.60) follows. Let $t \in \Gamma^\infty(E\big|_{M\setminus\operatorname{sing\,supp}s})$ be the smooth section such that for all $\mu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ with $\operatorname{supp}\mu \subseteq M\setminus\operatorname{sing\,supp}s$ we have $s(\mu) = \int_M t\,\mu$. Then for those μ we have

$$(Ds)(\mu) = s(D^{\mathrm{T}}\mu) = \int_M t\,D^{\mathrm{T}}\mu = \int_M (Dt)\mu,$$

since $\operatorname{supp}D^{\mathrm{T}}\mu \subseteq \operatorname{supp}\mu$. Thus Ds is regular on $M\setminus\operatorname{sing\,supp}s$, too, hence for the singular support we get $\operatorname{sing\,supp}(Ds) \subseteq M\setminus(M\setminus\operatorname{sing\,supp}s) = \operatorname{sing\,supp}s$. For the last part let $\mu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ be a test section with $\operatorname{supp}\mu \subseteq U$. Then

$$Ds\big|_U(\mu) = Ds(\mu) = s(D^{\mathrm{T}}\mu) = s(D^{\mathrm{T}}\big|_U\mu) = s\big|_U(D^{\mathrm{T}}\big|_U\mu) = \left(D\big|_U(s\big|_U)\right)(\mu),$$

since $D^{\mathrm{T}}\big|_U(\mu)$ has still support in U by the locality of differential operators. □

Remark 1.3.28 In Theorem 1.2.15 we have defined a different adjoint $D^{\mathrm{T}} \in \mathrm{DiffOp}(E^*;F^*)$ of $D \in \mathrm{DiffOp}(E;F)$ with respect to an a priori chosen positive density $\mu > 0$. We can use this adjoint to extend D to distributional sections as well. To this end we first observe that every section in $\Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ is a tensor product $\omega\otimes\mu$ of a uniquely determined section $\omega \in \Gamma_0^\infty(F^*)$ and the positive density μ, since μ provides a trivialization of $|\Lambda^{\mathrm{top}}|T^*M$. Thus it is sufficient to consider $\omega\otimes\mu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ in the following. For $s \in \Gamma^{-\infty}(E)$ we define $Ds : \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \mathbb{C}$ by

$$(Ds)(\omega\otimes\mu) = s((D^{\mathrm{T}}\omega)\otimes\mu), \tag{1.3.63}$$

which gives a well-defined linear map. Since D^{T} is continuous and since the tensor product is continuous too, $Ds \in \Gamma^{-\infty}(F)$. Moreover,

$$\mathrm{p}_{\omega\otimes\mu}(Ds) = |Ds(\omega\otimes\mu)| = \mathrm{p}_{D^{\mathrm{T}}\omega\otimes\mu}(s) \tag{1.3.64}$$

shows that $D : \Gamma^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(F)$ is weak* continuous. Since by construction D coincides with $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$ on the smooth sections $\Gamma^\infty(E) \subseteq \Gamma^{-\infty}(E)$, we conclude that the definition (1.3.63) and the intrinsic definition from Definition 1.3.26 actually *coincide*. In particular, even though D^{T} in (1.3.63) depends on μ explicitly, the combination $s(D^{\mathrm{T}}\omega\otimes\mu)$ only depends on the combination $\omega\otimes\mu$. In [4, Sect. 1.1.2] the approach (1.3.63) was used to define the extension of D to generalized sections.

1.3.3 Tensor Products

In this section we consider various tensor product constructions for distributions. The first one is about the values of a distribution and provides a rather trivial extension of our previous considerations.

Definition 1.3.29 (Vector-valued generalized sections) *Let $E \longrightarrow M$ be a vector bundle and V a finite-dimensional vector space. Then a V-valued generalized section of E is a continuous linear map*

$$s : \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow V. \tag{1.3.65}$$

The set of all V-valued generalized sections of E is denoted by $\Gamma^{-\infty}(E; V)$.

Since we always assume that the target vector space V is finite-dimensional, all Hausdorff locally convex topologies on V coincide. Thus the notion of continuity of (1.3.65) is non-ambiguous. It is clear that all the previous operations on distributions can be carried over to the vector-valued case since they were constructed from operations on the *arguments* of s.

Proposition 1.3.30 *For a finite-dimensional vector space V and a vector bundle $E \longrightarrow M$ we have the canonical isomorphism*

$$\Gamma^{-\infty}(E) \otimes V \ni s \otimes v \;\mapsto\; (\omega \;\mapsto\; s(\omega)v) \in \Gamma^{-\infty}(E; V). \tag{1.3.66}$$

Proof. First we note that the map $\omega \mapsto s(\omega)v$ is linear and continuous with respect to the $\mathcal{C}_0^\infty$-topology of $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$. Indeed, if $|s(\omega)| \le c\, \mathrm{p}_{K,\ell}(\omega)$ for $K \subseteq M$ compact, $c > 0$ and $\ell \in \mathbb{N}_0$ and all $\omega \in \Gamma_K^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ then

$$\|s(\omega)v\| \le c\, \mathrm{p}_{K,\ell}(\omega)\, \|v\|\,,$$

where $\|\cdot\|$ is any norm on V. Thus the right hand side of (1.3.66) is a vector-valued distribution. Clearly, the map is bilinear in s and v hence it indeed defines a linear map

$$\Gamma^{-\infty}(E) \otimes V \longrightarrow \Gamma^{-\infty}(E; V).$$

Let $e_1, \ldots, e_k \in V$ be a vector space basis. For a V-valued distribution $s \in \Gamma^{-\infty}(E; V)$ we have scalar distributions $s^\alpha = e^\alpha \circ s$ since for finite-dimensional vector spaces the algebraic and topological duals coincide. Thus $s = s^\alpha e_\alpha$ in the sense that $s(\omega) = s^\alpha(\omega) e_\alpha$. Moreover, the $s^\alpha(\omega)$ are uniquely determined hence the s^α are unique. It follows that $s^\alpha \otimes e_\alpha$ is a pre-image of s under (1.3.66), hence (1.3.66) is surjective. Injectivity is clear since the s^α are unique. □

In the following we shall use this isomorphism to identify $\Gamma^{-\infty}(E) \otimes V$ with $\Gamma^{-\infty}(E; V)$. In particular, the weak* topology of $\Gamma^{-\infty}(E; V)$ is just the component-wise weak* topology of $\Gamma^{-\infty}(E)$. One can endow $\Gamma^{-\infty}(E) \otimes V$ with a tensor product topology such that (1.3.66) is even an isomorphism of locally convex vector spaces. However, we shall not need this here. Note also that for arbitrary locally convex V the map (1.3.66) is still defined and injective, but usually no longer surjective.

The next tensor product is based on the tensor product of the arguments. We consider a product manifold $M \times N$ with the canonical projections

$$M \stackrel{\mathrm{pr}_M}{\longleftarrow} M \times N \stackrel{\mathrm{pr}_N}{\longrightarrow} N. \tag{1.3.67}$$

For this situation, we first prove the following statement which is of independent interest:

Theorem 1.3.31 *Let M, N be manifolds. Then for all $k \in \mathbb{N}_0 \cup \{+\infty\}$ the map*

$$\mathcal{C}_0^k(M) \otimes \mathcal{C}_0^k(N) \ni f \otimes g \;\mapsto\; \mathrm{pr}_M^* f\, \mathrm{pr}_N^* g \in \mathcal{C}_0^k(M \times N) \tag{1.3.68}$$

is a continuous injective algebra homomorphism with sequentially dense image with respect to the $\mathcal{C}^k_0$*-topologies. In more detail, we have estimates*

$$\mathrm{p}_{K\times L,k}(\mathrm{pr}_M^* f \mathrm{pr}_N^* g) \leq c \max_{\ell\leq k} \mathrm{p}_{K,\ell}(f) \max_{\ell \leq k} \mathrm{p}_{L,\ell}(g), \tag{1.3.69}$$

if we use factorizing data to define the seminorms $\mathrm{p}_{K\times L,k}$ *on* $M \times N$*.*

Proof. First we discuss the linear algebra aspects. Since the algebraic tensor product of two associative algebras is canonically an associative algebra, we can indeed speak of an algebra homomorphism. It follows immediately that (1.3.68) is bilinear in f and g and thus well-defined on the tensor product. Then the homomorphism property is clear. The injectivity is clear as for linear independent f_α and linear independent g_β the images of $f_\alpha \otimes g_\beta$ are still linear independent. This can be seen by evaluating at appropriate points $(x, y) \in M \times N$. Thus we can identify $f \otimes g$ with $\mathrm{pr}_M^* f \mathrm{pr}_N^* g$ and avoid the latter, more clumsy notation. We come now to the continuity property. Thus let ∇^M and ∇^N be torsion-free covariant derivatives and let $\nabla^{M\times N}$ be the corresponding covariant derivative on $M \times N$. By D_M, D_N, and $\mathsf{D}_{M\times N}$ we denote the corresponding symmetrized covariant derivatives. Now let $K \subseteq M$ and $L \subseteq N$ be compact. Then $K \times L \subseteq M \times N$ is compact, too, and every compact subset of $M \times N$ is contained in such a compactum for appropriate K and L. Thus it suffices to consider $K \times L \subseteq M \times N$. For $f \in \mathcal{C}^k_K(M)$ and $g \in \mathcal{C}^k_L(N)$ we compute

$$\begin{aligned} \mathsf{D}^k_{M\times N}(\mathrm{pr}_M^*(f)\mathrm{pr}_N^*(g)) &= \sum_{\ell=0}^{k} \binom{k}{\ell} \mathsf{D}^\ell_{M\times N}(\mathrm{pr}_M^* f) \vee \mathsf{D}^{k-\ell}_{M\times N}(\mathrm{pr}_N^* g) \\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} \mathrm{pr}_M^* \left(\mathsf{D}^\ell_M f\right) \vee \mathrm{pr}_N^* \left(\mathsf{D}^{k-\ell}_N g\right), \end{aligned}$$

since $\mathsf{D}_{M\times N}$ is a derivation and since $\mathsf{D}_{M\times N} \mathrm{pr}_M^* = \mathrm{pr}_M^* \mathsf{D}_M$ as well as $\mathsf{D}_{M\times N} \mathrm{pr}_N^* = \mathrm{pr}_N^* \mathsf{D}_N$. If we also choose the Riemannian metric on $M \times N$ to be the product metric of g_M on M and g_N on N we obtain for the $\mathrm{p}_{K\times L,k}$ seminorm

$$\begin{aligned} \mathrm{p}_{K\times L,k}(\mathrm{pr}_M^*(f)\mathrm{pr}_N^*(g)) &= \sup_{(x,y)\in K\times L} \left\| \mathsf{D}^k_{M\times N}(\mathrm{pr}_M^*(f)\mathrm{pr}_N^*(g))\Big|_{(x,y)} \right\|_{M\times N} \\ &\leq \sup_{\substack{x\in K \\ 0\leq \ell\leq k}} \sup_{y\in L} c \left\| \mathsf{D}^\ell_M f\big|_x \right\|_M \left\| \mathsf{D}^{k-\ell}_N g\big|_y \right\|_N \\ &= c \max_{\ell\leq k} \mathrm{p}_{K,\ell}(f)\, \mathrm{p}_{L,\ell}(g), \end{aligned}$$

which shows the continuity property of (1.3.68). We are left with the task to show that finite sums of factorizing functions are sequentially dense. Thus let $F \in \mathcal{C}^k_0(M \times N)$ be given. We choose atlases $\{(U_\alpha, x_\alpha)\}$ of M and $\{(V_\beta, y_\beta)\}$ of N together with subordinate partitions of unity $\{\chi_\alpha\}$ and $\{\psi_\beta\}$, respectively. Then the $\{(U_\alpha \times V_\beta, x_\alpha \times y_\beta, \chi_\alpha \otimes \psi_\beta)\}$ provides an atlas of $M \times N$ with a corresponding partition of unity. Since $\operatorname{supp} F$ is compact, it follows that

$$F = \sum_{\alpha,\beta} \chi_\alpha \otimes \psi_\beta \cdot F$$

is a finite sum and each term $\chi_\alpha \otimes \psi_\beta \cdot F$ has compact support in $U_\alpha \otimes V_\beta$. Thus it will be sufficient to find a sequence in $\mathcal{C}^k_0(U_\alpha) \otimes \mathcal{C}^k_0(V_\beta)$ which approximates a function in $\mathcal{C}^k_0(U_\alpha \times V_\beta)$. This reduces the problem to the following *local* problem: We have to show that $\mathcal{C}^k_0(\mathbb{R}^n) \otimes \mathcal{C}^k_0(\mathbb{R}^m)$ is sequentially dense in $\mathcal{C}^k_0(\mathbb{R}^{n+m})$. We will need the following technical lemma:

Lemma 1.3.32 *Let $f \in \mathcal{C}_0^k(\mathbb{R}^n)$ and $K \subseteq \mathbb{R}^n$ compact. For every $\epsilon > 0$ there exists a polynomial $p_\epsilon \in \operatorname{Pol}(\mathbb{R}^n)$ such that for all $\ell \leq k$*

$$\mathrm{p}_{K,\ell}(p_\epsilon - f) < \epsilon. \tag{1.3.70}$$

Proof. We only sketch the proof which uses some convolution tricks. We consider the normalized Gaussian

$$G_\delta(x) = \frac{1}{\sqrt{\pi\delta}^n} \mathrm{e}^{-\frac{1}{\delta}x^2}$$

for $\delta > 0$. Then the integral of G_δ equals one for all δ. For $f \in \mathcal{C}_0^k(\mathbb{R}^n)$ the convolution

$$(G_\delta * f)(x) = \int_{\mathbb{R}^n} G_\delta(x-y) f(y) \, \mathrm{d}^n y$$

is a smooth function $G_\delta * f \in \mathcal{C}^\infty(\mathbb{R}^n)$ and we have $\frac{\partial^{|I|}}{\partial x^I}(G_\delta * f) = G_\delta * \frac{\partial^{|I|} f}{\partial x^I}$ for all multiindexes I with $|I| \leq k$. It is now a well-known fact that $G_\delta * f$ approximates f uniformly on $\mathbb{R}^n$, i.e.

$$\|G_\delta * f - f\|_\infty \longrightarrow 0 \quad \text{for} \quad \delta \longrightarrow 0$$

in the sup-norm $\|\cdot\|_\infty$ over $\mathbb{R}^n$. If $k \geq 1$ we can repeat the argument and obtain that

$$\sup_{x \in \mathbb{R}^n} \left| \frac{\partial^{|I|}}{\partial x^I}(G_\delta * f) - \frac{\partial^{|I|} f}{\partial x^I} \right| \longrightarrow 0$$

for $\delta \longrightarrow 0$ and for all multiindexes with $|I| \leq k$. In a second step we approximate the Gaussian by its Taylor series. Since on every compact subset $K \subseteq \mathbb{R}^n$ the Taylor series converges to G_δ in the $\mathcal{C}^\infty_K$-topology we find a polynomial

$$p_{\epsilon,K,k,\delta}(x) = \frac{1}{\sqrt{\pi\delta}^n} \sum_{r \geq 0}^{\text{finite}} \frac{1}{r!} \left(-\frac{x^2}{\delta}\right)^r$$

such that for $\ell \leq k$

$$\mathrm{p}_{K,\ell}(G_\delta - p_{\epsilon,K,k,\delta}) < \epsilon.$$

The convolution

$$f_{\epsilon,K,k,\delta}(x) = \int_{\mathbb{R}^n} p_{\epsilon,K,k,\delta}(x-y) f(y) \, \mathrm{d}^n y$$

is again a polynomial of x of the same order as $p_{\epsilon,K,k,\delta}$ and we use this to approximate f on a compact subset. Thus let $K \subseteq \mathbb{R}^n$ be fixed and consider $x \in K$. Then

$$\begin{aligned}
&\left| \frac{\partial^{|I|}}{\partial x^I} f_{\epsilon,B_R(0),k,\delta}(x) - \frac{\partial^{|I|}}{\partial x^I}(G_\delta * f)(x) \right| \\
&\quad = \left| \int \left(p_{\epsilon,B_R(0),k,\delta}(x-y) \frac{\partial^{|I|} f}{\partial x^I}(y) - G_\delta(x-y) \frac{\partial^{|I|} f}{\partial x^I}(y) \right) \mathrm{d}^n y \right| \\
&\quad \leq \int_{\operatorname{supp} f} \left| p_{\epsilon,B_R(0),k,\delta}(x-y) - G_\delta(x-y) \right| \left| \frac{\partial^{|I|} f}{\partial x^I}(y) \right| \\
&\quad \leq \epsilon \int_{\operatorname{supp} f} \left| \frac{\partial^{|I|} f}{\partial x^I}(y) \right| \mathrm{d}^n y,
\end{aligned}$$

if we choose $B_R(0)$ large enough such that $K - \operatorname{supp} f \subseteq B_R(0)$. This is clearly possible since both K and $\operatorname{supp} f$ are compact. It follows that on a compact subset $K \subseteq \mathbb{R}^n$ the polynomial $f_{\epsilon,B_R(0),k,\delta}$ approximates $G_\delta * f$ in the $\mathcal{C}^k_K$-topology. Thus we obtain that $f_{\epsilon,B_R(0),k,\delta}$ also approximates f in the $\mathcal{C}^k_K$-topology as well. Rescaling ϵ appropriately gives the polynomials p_ϵ as desired. ▽

Using this lemma we can proceed as follows: Let $F \in \mathcal{C}^k_0(\mathbb{R}^{n+m})$ be given and choose $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^m)$ such that their tensor product $\chi \otimes \psi$ is equal to one on $\operatorname{supp} F$. This is clearly possible. Then $F = \chi \otimes \psi \cdot F$ can be approximated by polynomials on every compact subset. Let $K \times L \subseteq \mathbb{R}^{n+m}$ be a compactum such that $\chi \otimes \psi \in \mathcal{C}^\infty_{K\times L}(\mathbb{R}^{n+m})$ and choose $p_r \in \operatorname{Pol}(\mathbb{R}^{n+m})$ such that on $K \times L$ the polynomials p_r converge to F in the $\mathcal{C}^k_{K\times L}$-topology by Lemma 1.3.32. Since for polynomials we have

$$\operatorname{Pol}(\mathbb{R}^{n+m}) = \operatorname{Pol}(\mathbb{R}^n) \otimes \operatorname{Pol}(\mathbb{R}^m),$$

we find that $\chi \otimes \psi \cdot p_r \in \mathcal{C}^\infty_{K\times L}(\mathbb{R}^{n+m})$ is actually in $\mathcal{C}^\infty_K(\mathbb{R}^n) \otimes \mathcal{C}^\infty_L(\mathbb{R}^m)$. Now

$$\mathrm{p}_{K\times L,k}(F - \chi \otimes \psi \cdot p_r) = \mathrm{p}_{K\times L,k}(\chi \otimes \psi \cdot F - \chi \otimes \psi \cdot p_r) \leq c\, \mathrm{p}_{K\times L,k}(F - p_r) \longrightarrow 0$$

for $r \longrightarrow \infty$, which shows the density with respect to the $\mathcal{C}^k_0$-topology since all members of the sequence are in one fixed compactum $K \times L$. □

Remark 1.3.33 In fact, the proof even shows that

$$\mathcal{C}^\infty_0(M) \otimes \mathcal{C}^\infty_0(N) \subseteq \mathcal{C}^k_0(M \times N) \tag{1.3.71}$$

is sequentially dense in the $\mathcal{C}^k_0$-topology for all $k \in \mathbb{N}_0 \cup \{+\infty\}$. Note that this gives an independent proof of Theorem 1.1.26 at least for the scalar case as we can choose $N = \{\mathrm{pt}\}$ hence $\mathcal{C}^\infty_0(N) = \mathbb{C}$ and $\mathcal{C}^k_0(M \times N) \simeq \mathcal{C}^k_0(M)$. Thus we recover that

$$\mathcal{C}^\infty_0(M) \subseteq \mathcal{C}^k_0(M) \tag{1.3.72}$$

is dense in the $\mathcal{C}^k_0$-topology.

Corollary 1.3.34 *For all $k \in \mathbb{N}_0 \cup \{+\infty\}$ the map*

$$\mathcal{C}^k(M) \otimes \mathcal{C}^k(N) \ni f \otimes g \;\mapsto\; ((x,y) \;\mapsto\; f(x)g(y)) \in \mathcal{C}^k(M \times N) \tag{1.3.73}$$

extends to a linear injective continuous algebra homomorphism with dense image with respect to the $\mathcal{C}^k$-topology.

Proof. The estimates (1.3.69) also show that (1.3.73) is continuous. The fact that the image is dense follows from Theorem 1.3.31 since it contains the images of $\mathcal{C}^k_0(M) \otimes \mathcal{C}^k_0(N)$ which is dense in $\mathcal{C}^k_0(M\times N)$ in the $\mathcal{C}^k_0$-topology. By Proposition 1.1.9 the subspace $\mathcal{C}^k_0(M\times N)$ is dense in $\mathcal{C}^k(M\times N)$ in the $\mathcal{C}^k$-topology. Since $\mathcal{C}^k_0$-convergence implies $\mathcal{C}^k$-convergence, the statement follows. The remaining statements are clear. □

We can also extend the above statements to vector bundles. To this end we recall the following construction of the *external tensor product* of two vector bundles $E \longrightarrow M$ and $F \longrightarrow N$. Over the Cartesian product $M \times N$ we consider the vector bundle

$$E \boxtimes F = \mathrm{pr}_M^\#(E) \otimes \mathrm{pr}_N^\#(F), \tag{1.3.74}$$

where pr_M and pr_N are the usual projections and $\mathrm{pr}_M^\#(E) \longrightarrow M \times N$ as well as $\mathrm{pr}_N^\#(F) \longrightarrow M \times N$ denote the pull-backs of the vector bundles E and F, respectively. More informally, $E \boxtimes F$ is the vector bundle with fiber $E_x \otimes F_y$ over $(x,y) \in M \times N$ and vector bundle structure coming from (1.3.74). If

$e_\alpha \in \Gamma^\infty(E\big|_U)$ and $f_\beta \in \Gamma^\infty(F\big|_V)$ are local base sections then $\mathrm{pr}_M^\#(e_\alpha) \otimes \mathrm{pr}_N^\#(f_\beta) \in \Gamma^\infty\left(E \boxtimes F\big|_{U\times V}\right)$ are local base sections, too. To simplify our notation we shall write

$$s \boxtimes t = \mathrm{pr}_M^\#(s) \otimes \mathrm{pr}_N^\#(t) \tag{1.3.75}$$

for $s \in \Gamma^\infty(E)$ and $t \in \Gamma^\infty(F)$ in the sequel. Without going into the details, the local trivializations of E and F allow to use Theorem 1.3.31 and Corollary 1.3.34 to obtain the following analogue for vector bundles:

Theorem 1.3.35 *Let $k \in \mathbb{N}_0 \cup \{+\infty\}$ and let $E \longrightarrow M$ and $F \longrightarrow N$ be vector bundles. Then*

$$\Gamma_0^k(E) \otimes \Gamma_0^k(F) \ni s \otimes t \ \mapsto \ s \boxtimes t \in \Gamma_0^k(E \boxtimes F) \tag{1.3.76}$$

is an injective continuous $\mathcal{C}_0^k(M) \otimes \mathcal{C}_0^k(N)$-module morphism with sequentially dense image in the $\mathcal{C}_0^k$-topology. Analogously,

$$\Gamma^k(E) \otimes \Gamma^k(F) \ni s \otimes t \ \mapsto \ s \boxtimes t \in \Gamma^k(E \boxtimes F) \tag{1.3.77}$$

is an injective continuous $\mathcal{C}^k(M) \otimes \mathcal{C}^k(N)$-module morphism with dense image in the $\mathcal{C}^k$-topology.

Note that on the left hand side the tensor product is taken over $\mathbb{R}$ or $\mathbb{C}$, depending on the type of the vector bundles. The module structures on both sides are the canonical ones.

Remark 1.3.36 It should be noted that for $s \in \Gamma^\infty(E)$ and $t \in \Gamma^\infty(F)$ we have

$$\mathrm{supp}(s \boxtimes t) = \mathrm{supp}\, s \times \mathrm{supp}\, t. \tag{1.3.78}$$

Remark 1.3.37 For the density bundles we have canonically

$$|\Lambda^{\mathrm{top}}|T^*M \boxtimes |\Lambda^{\mathrm{top}}|T^*N \cong |\Lambda^{\mathrm{top}}|T^*(M \times N), \tag{1.3.79}$$

where the isomorphism is defined by

$$|\Lambda^{\mathrm{top}}|T_x^*M \otimes |\Lambda^{\mathrm{top}}|T_y^*N \ni \mu_x \otimes \nu_y \ \mapsto \ \mu_x \boxtimes \nu_y \in |\Lambda^{\mathrm{top}}|T_{(x,y)}^*(M \times N), \tag{1.3.80}$$

with

$$(\mu_x \boxtimes \nu_y)(v_1, \ldots, v_m, w_1, \ldots, w_n) = \mu_x(v_1, \ldots, v_m)\nu_y(w_1, \ldots, w_n), \tag{1.3.81}$$

where $v_1, \ldots, v_m \in T_xM$ and $w_1, \ldots, w_n \in T_yN$. Moreover, for $E_i \longrightarrow M$ and $F_i \longrightarrow N$ with $i = 1, 2$ we have the compatibility

$$(E_1 \otimes E_2) \boxtimes (F_1 \otimes F_2) \simeq (E_1 \boxtimes F_1) \otimes (E_2 \boxtimes F_2), \tag{1.3.82}$$

and in particular

$$(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \boxtimes (F^* \otimes |\Lambda^{\mathrm{top}}|T^*N) \simeq (E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N), \tag{1.3.83}$$

which we shall frequently use in the following.

In order to define the tensor product of distributions we need the following technical lemma:

Lemma 1.3.38 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open and let $\phi \in \mathcal{C}^\infty(X \times Y)$ be smooth. Assume that there is a compact subset $K \subseteq X$ such that* $\operatorname{supp}\phi \subseteq K \times Y$. *Let $u \in \mathcal{C}_0^\infty(X)'$ be a scalar distribution. Then the function*

$$y \mapsto u(\phi(\,\cdot\,, y)) \tag{1.3.84}$$

is smooth on Y. Moreover, for all multiindexes $I \in \mathbb{N}_0^m$ we have

$$\frac{\partial^{|I|}}{\partial y^I} u(\phi(\,\cdot\,, y)) = u\left(\frac{\partial^{|I|}}{\partial y^I}\phi(\,\cdot\,, y)\right). \tag{1.3.85}$$

Finally, for $f \in \mathcal{C}^\infty(Y)$ we have

$$u(f(y)\phi(\,\cdot\,, y)) = f(y)u(\phi(\,\cdot\,, y)), \tag{1.3.86}$$

i.e. the map $\phi \mapsto (y \mapsto u(\phi(\,\cdot\,, y)))$ is $\mathcal{C}^\infty(Y)$-linear.

Proof. In (1.3.84) we apply u to the function $x \mapsto \phi(x, y)$ for fixed $y \in Y$. By assumption, this function has compact support in $K \subseteq X$ with respect to the x-variables for every fixed $y \in Y$, hence (1.3.84) is a well-defined function. We shall now consider a slightly more detailed statement. On the compact subset K the distribution u has some finite order $\ell = \operatorname{ord}_K(u)$. Thus for all test functions $\varphi \in \mathcal{C}_K^\infty(X)$

$$|u(\varphi)| \le c\, \mathrm{p}_{K,\ell}(\varphi), \tag{$*$}$$

and we can extend u to a continuous linear functional on $\mathcal{C}_K^\ell(X)$ such that $(*)$ still holds for $\varphi \in \mathcal{C}_K^\ell(X)$. We refine the claim as follows: for $\phi \in \mathcal{C}^k(X \times Y)$ with $\operatorname{supp}\phi \subseteq K \times Y$ and $k \ge \ell$ the function $y \mapsto u(\phi(\,\cdot\,, y))$ is in $\mathcal{C}^{k-\ell}(Y)$ and (1.3.85) holds for all $|I| \le k - \ell$. Clearly, this statement includes (1.3.85) and (1.3.84) for the smooth case $k = \infty$. Let $y_0 \in Y$ be fixed and consider some $B_r(y_0)^{\mathrm{cl}} \subseteq Y$. Then on the compact subset $K \times B_r(y_0)^{\mathrm{cl}} \subseteq X \times Y$ the function $\frac{\partial^{|J|}\phi}{\partial x^J}$ is uniformly continuous as long as $|J| \le k$. Thus for $\epsilon > 0$ there is a $\delta > 0$ such that for (x, y_0), $(x, y_0 + h) \in K \times B_r(y_0)^{\mathrm{cl}}$ with $|h| \le \delta$ we have

$$\left|\frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0) - \frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0 + h)\right| < \epsilon.$$

It follows that

$$\mathrm{p}_{K,\ell}(\phi(\,\cdot\,, y_0) - \phi(\,\cdot\,, y_0 + h)) < \epsilon$$

for those h since $\ell \le k$. Thus the continuity of u with respect to the norm $\mathrm{p}_{K,\ell}$ on $\mathcal{C}_K^\ell(X)$ as in $(*)$ yields

$$u(\phi(\,\cdot\,, y_0 + h)) \longrightarrow u(\phi(\,\cdot\,, y_0))$$

for $h \longrightarrow 0$ for all $y_0 \in Y$. Thus (1.3.84) is continuous, This proves the case $k = \ell$. Now assume $k \ge \ell + 1$ hence we have some orders of differentiation for "free". Thus let $e \in \mathbb{R}^n$ be a unit vector and $y_0 \in Y$ together with a sufficiently small ball $B_r(y_0)^{\mathrm{cl}} \subseteq Y$ as before. Then for $|J| \le \ell$ the partial derivatives $\frac{\partial^{|J|}\phi}{\partial x^J}$ are at least once continuously differentiable. Hence for $0 < |t| < r$

$$\frac{1}{t}\left(\frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0 + te) - \frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0)\right) = e^i \frac{\partial^{|J|+1}\partial\phi}{\partial x^J \partial y^i}(x, y_0 + t_0 e)$$

with some appropriate $t_0 \in [0, t]$. Since the $(|J|+1)$-st derivatives are still continuous, on $K \times B_r(y_0)^{\mathrm{cl}}$ they are uniformly continuous. Thus for all $0 < |t| \le \delta$

$$\left|\frac{1}{t}\left(\frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0 + te) - \frac{\partial^{|J|}\phi}{\partial x^J}(x, y_0) - \frac{\partial^{|J|+1}\phi}{\partial x^J \partial y^i}(x, y_0)e^i\right)\right| < \epsilon$$

with some appropriately chosen $\delta > 0$. This means that

$$\mathrm{p}_{K,\ell}\left(\frac{1}{t}\left(\phi(\,\cdot\,, y_0 + te) - \phi(\,\cdot\,, y_0)\right) - \partial_e \phi(\,\cdot\,, y_0)\right) < \epsilon.$$

Hence again by the continuity of u we get for the directional derivative in direction e

$$\partial_e u(\phi(\,\cdot\,, y_0)) = u(\partial_e \phi(\,\cdot\,, y_0)).$$

Since y_0 was arbitrary and since $\partial_e \phi$ is $\mathcal{C}^{k-1}$ with $k - 1 \geq \ell$ we see that all directional derivatives at all $y \in Y$ exist and are continuous. This proves that (1.3.84) is in $\mathcal{C}^1(Y)$ and (1.3.85) is valid for $|I| = 1$. By induction we can proceed as long as $k \geq \ell$. The last statement is clear since u acts only on the x-variables and not on the y-variables. □

In a geometric context the above lemma, in its refined version, becomes the following statement:

Proposition 1.3.39 *Let $E \longrightarrow M$ and $F \longrightarrow N$ be vector bundles and let $\mu \in \Gamma^k((E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ be a density such that there exists a compact subset $K \subseteq M$ with $\operatorname{supp} \mu \subseteq K \times N$. Let $s \in \Gamma^{-\infty}(E)$ be a generalized section such that $\operatorname{ord}_K(s) \leq \ell$. Then the map*

$$(s \otimes \mathsf{id})(\mu) : y \;\mapsto\; s(\mu(\,\cdot\,, y)) \tag{1.3.87}$$

defines a $\mathcal{C}^{k-\ell}$-section of $F^* \otimes |\Lambda^{\mathrm{top}}|T^*N$. If $F' \longrightarrow N$ is another vector bundle and $D \in \mathrm{DiffOp}^m(F \otimes |\Lambda^{\mathrm{top}}|T^*N; F' \otimes |\Lambda^{\mathrm{top}}|T^*N)$ a differential operator of order $m \leq k - \ell$ then D applied to (1.3.87) coincides with the section

$$y \;\mapsto\; s((\mathsf{id} \boxtimes D)(\mu)(\,\cdot\,, y)), \tag{1.3.88}$$

where $\mathsf{id} \boxtimes D$ means that D acts only on the y-variables. For the support of $(s \otimes \mathsf{id})(\mu)$ we have

$$\operatorname{supp}(s \otimes \mathsf{id})(\mu) \subseteq \mathrm{pr}_N(\operatorname{supp} \mu). \tag{1.3.89}$$

Proof. By the usual partition of unity argument with the usual local trivialization of the involved bundles we can reduce the above statements to the local and scalar case. Thus Lemma 1.3.38 yields that (1.3.87) is a well-defined $\mathcal{C}^{k-\ell}$-section and the combination of (1.3.85) and (1.3.86) gives (1.3.88). It remains to show (1.3.89). Thus let $y \in N \backslash \mathrm{pr}_N(\operatorname{supp} \mu)$. Thus for all $x \in M$ we have $\mu(x, y) = 0$. This gives immediately $s(\mu(\,\cdot\,, y)) = 0$. Since $N \backslash \mathrm{pr}_N(\operatorname{supp} \mu)$ is open, (1.3.89) follows. □

Remark 1.3.40 In particular, for all $s \in \Gamma^{-\infty}(E)$ and $\mu \in \Gamma_0^\infty((E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ we have $(s \otimes \mathsf{id})(\mu) \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*N)$.

We use this proposition now to prove the following statement on the (external) tensor product of distributions.

Theorem 1.3.41 (Tensor product of generalized sections) *Let $E \longrightarrow M$ and $F \longrightarrow N$ be vector bundles and let $s \in \Gamma^{-\infty}(E)$ and $t \in \Gamma^{-\infty}(F)$ be generalized sections. Then there exists a unique generalized section $s \boxtimes t \in \Gamma^{-\infty}(E \boxtimes F)$ such that*

$$(s \boxtimes t)(\mu \boxtimes \nu) = s(\mu)t(\nu) \tag{1.3.90}$$

for $\mu \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ and $\nu \in \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*N)$. Moreover, for $\omega \in \Gamma_0^\infty((E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ we have

$$(s \boxtimes t)(\omega) = t((s \otimes \mathsf{id})(\omega)) = s((\mathsf{id} \otimes t)(\omega)). \tag{1.3.91}$$

Proof. Since $\Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \otimes \Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*N)$ is dense in $\Gamma_0^\infty((E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ by Theorem 1.3.35 and the identification (1.3.83) of Remark 1.3.37, the uniqueness of $s \boxtimes t$ with the property (1.3.90) is clear. The idea is now to use the feature (1.3.91) to actually construct $s \boxtimes t$: Thus let $\omega \in \Gamma_0^\infty(E^* \boxtimes F^* \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ be given. We can assume that $\operatorname{supp} \omega \subseteq K \times L$ with compact subsets $K \subseteq M$ and $L \subseteq N$, respectively. For s and t we have estimates of the form

$$|s(\mu)| \le c \, \mathrm{p}_{K,k}(\mu) \tag{$*$}$$

$$|t(\nu)| \le c' \, \mathrm{p}_{L,\ell}(\nu), \tag{$**$}$$

for the seminorms of Remark 1.1.8 whenever $\operatorname{supp} \mu \subseteq K$ and $\operatorname{supp} \nu \subseteq L$. By Proposition 1.3.39 we know that

$$(s \otimes \mathsf{id})(\omega) : y \ \mapsto \ s(\omega(\,\cdot\,, y))$$

is a smooth section of $F^* \otimes |\Lambda^{\mathrm{top}}|T^*N$. Moreover, since the application of s is $\mathcal{C}^\infty(N)$-linear and commutes with differentiation in N-direction we immediately conclude that

$$\mathrm{p}_{L,\ell}((s \otimes \mathsf{id})(\omega)) \le c'' \, \mathrm{p}_{K \times L,\ell}(\omega). \tag{$***$}$$

Finally, by Remark 1.3.40 we have

$$\operatorname{supp}((s \otimes \mathsf{id})(\omega)) \subseteq \mathrm{pr}_N(\operatorname{supp} \omega) \subseteq L,$$

hence $(s \otimes \mathsf{id})(\omega)$ has compact support. Thus we can apply t and obtain by $(**)$

$$|t((s \otimes \mathsf{id})(\omega))| \le c' \, \mathrm{p}_{L,\ell}((s \otimes \mathsf{id})(\omega)) \le c' c'' \, \mathrm{p}_{K \times L,\ell}(\omega).$$

Thus $\omega \mapsto t((s \otimes \mathsf{id})(\omega))$ is a continuous linear functional on $\Gamma_{K \times L}^\infty((E^* \boxtimes F^*) \otimes |\Lambda^{\mathrm{top}}|T^*(M \times N))$ for all $K \times L$ with respect to the $\mathcal{C}_{K \times L}^\infty$-topology. Hence it defines a generalized section in $\Gamma^{-\infty}(E \boxtimes F)$ by the characterization of Theorem 1.1.11, *iv.)*. If $\omega = \mu \boxtimes \nu$ is an external tensor product itself, we obtain

$$\begin{aligned} t((s \otimes \mathsf{id})(\mu \boxtimes \nu)) &= t(y \ \mapsto \ s((\mu \boxtimes \nu)(\,\cdot\,, y))) \\ &= t(y \ \mapsto \ s(\mu(\,\cdot\,)\nu(y))) \\ &= t(y \ \mapsto \ \nu(y)s(\mu)) \\ &= t(\nu)s(\mu). \end{aligned}$$

This shows that the distribution $t \circ (s \otimes \mathsf{id})$ satisfies (1.3.90). Hence it is the *unique* solution $s \boxtimes t$ we are looking for. This proves existence of $s \boxtimes t$ and the first half of (1.3.91). However, we could have constructed $s \boxtimes t$ by taking $s \circ (\mathsf{id} \otimes t)$ as well which gives, by uniqueness, the same $s \boxtimes t$. Thereby we have (1.3.91). ☐

Remark 1.3.42 For the external tensor product

$$\boxtimes : \Gamma^{-\infty}(E) \otimes \Gamma^{-\infty}(F) \longrightarrow \Gamma^{-\infty}(E \boxtimes F) \tag{1.3.92}$$

one immediately obtains

$$\operatorname{supp}(s \boxtimes t) = \operatorname{supp} s \times \operatorname{supp} t, \tag{1.3.93}$$

whence we also have

$$\boxtimes : \Gamma_0^{-\infty}(E) \otimes \Gamma_0^{-\infty}(F) \longrightarrow \Gamma_0^{-\infty}(E \boxtimes F). \tag{1.3.94}$$

It can be shown that for compactly supported s and t the conclusions of Theorem 1.3.41 remain valid for μ, ν, ω not necessarily compactly supported.

Remark 1.3.43 ("Internal" tensor product of distributions) For vector bundles $E \longrightarrow M$ and $F \longrightarrow M$ over the same manifold, one may wonder whether there is an "internal" tensor product of generalized sections, i.e. a map

$$\otimes : \Gamma^{-\infty}(E) \otimes \Gamma^{-\infty}(F) \longrightarrow \Gamma^{-\infty}(E \otimes F), \tag{1.3.95}$$

extending the tensor product of smooth sections, which is now $\mathcal{C}^\infty(M)$-bilinear with respect to the $\mathcal{C}^\infty(M)$-module structures of generalized sections. If such an extension of the usual tensor product of smooth section would exist in general, this would result in an algebra structure on $\mathcal{C}_0^\infty(M)'$ if we take $E = F$ to be the trivial line bundles. Here on meets serious problems: such a multiplication (obeying the usual properties) can be shown to be impossible. A "definition" of $s \otimes t$ like

$$\text{``}(s \otimes t)(\mu \otimes \nu) = s(\mu)t(\nu)\text{''} \tag{1.3.96}$$

is *not* well-defined since the tensor product $\mu \otimes \nu$ of sections is $\mathcal{C}^\infty(M)$-bilinear while the right hand side of (1.3.96) is certainly not $\mathcal{C}^\infty(M)$-bilinear.

Note however, that under certain circumstances the tensor product $s \otimes t$ can indeed be defined in a reasonable way. However, a much more sophisticated analysis of the singularities of s and t is needed.

Chapter 2

Elements of Lorentz Geometry and Causality

In this second chapter we set the stage for wave equations on spacetime manifolds. First we recall some basic properties and notions for manifolds with covariant derivative, positive densities and semi-Riemannian metrics. We shall discuss their relations and introduce concepts like parallel transport as well as certain canonical differential operators arising from the choice of a semi-Riemannian metric. In particular, the d'Alembert operator will provide the prototype of a wave operator. We generalize this to arbitrary vector bundles and discuss several physical examples of wave equations resulting from these differential operators.

After discussing the basics of semi-Riemannian and Lorentz metrics we introduce the notions of causality on Lorentz manifolds. To this end we first have to endow the Lorentz manifold with a time orientation which then gives rise to the notions of future and past. The most important notion in this context for us will be that of Cauchy hypersurfaces. On one hand, the existence of a Cauchy hypersurface will yield a particularly nice causal structure of the Lorentz manifold. On the other hand, they will serve as the natural starting point where we can pose initial value problems for a wave equation.

Such initial value problems for wave equations will then be the subject of the last part of this chapter. Closely related will be the notion of Green functions of advanced and retarded type. They are particular elementary solutions of the wave equations subject to "boundary conditions" referring to the causal structure of the spacetime.

For several theorems we will not provide proofs in this chapter as this would lead us too far into the realm of Lorentz geometry. Instead we refer to the literature, in particular to the textbooks [6, 23, 46] as well as to the review article [45].

2.1 Preliminaries on Semi-Riemannian Manifolds

In this section we collect some further properties of covariant derivatives on vector bundles and their curvature, specializing to the Levi-Civita connection of a semi-Riemannian metric. All of the material is very much standard and can be found in textbooks like [6, 39, 46].

2.1.1 Parallel Transport and Curvature

Let ∇^E be a covariant derivative for a vector bundle $E \longrightarrow M$ as before. Recall that the curvature tensor R of ∇^E is defined by

$$R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{2.1.1}$$

for $X, Y \in \Gamma^\infty(TM)$ and $s \in \Gamma^\infty(E)$. A simple computation shows that R is $\mathcal{C}^\infty(M)$-linear in each argument and thus defines a tensor field

$$R \in \Gamma^\infty(\mathsf{End}(E) \otimes \Lambda^2 T^*M). \tag{2.1.2}$$

There are certain contractions we can build out of R. The most important one is the pointwise trace of the $\mathsf{End}(E)$-part of R. This gives a two-from

$$\mathsf{tr}\, R(X, Y) = \mathsf{tr}(s \mapsto R(X, Y)s), \tag{2.1.3}$$

i.e. a section $\mathsf{tr}\, R \in \Gamma^\infty(\Lambda^2 T^*M)$. The following lemma gives an interpretation of $\mathsf{tr}\, R$:

Lemma 2.1.1 *Let ∇^E be a covariant derivative for a vector bundle $E \longrightarrow M$.*

i.) The two-form $\mathsf{tr}\, T \in \Gamma^\infty(\Lambda^2 T^*M)$ is closed, $\mathrm{d}\, \mathsf{tr}\, R = 0$.

ii.) The two-form $\mathsf{tr}\, R$ is exact. In fact,

$$\mathsf{tr}\, R = -\,\mathrm{d}\,\alpha, \tag{2.1.4}$$

where $\alpha \in \Gamma^\infty(T^*M)$ is defined by

$$\alpha(X) = \frac{\nabla^E_X \mu}{\mu}, \tag{2.1.5}$$

with respect to any chosen positive density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|E^*)$.

Proof. Clearly, we only have to show *ii.)*. Note that *i.)* would also follow rather easily from the Bianchi identity. Let $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|E^*)$ be a positive density. Then the covariant derivative ∇^E is extended as usual to $|\Lambda^{\mathrm{top}}|E^*$ and α is a well-defined one-form. A simple computation shows that the curvature of $\nabla^{|\Lambda^{\mathrm{top}}|E^*}$ is given by $\mathrm{d}\,\alpha$. On the other hand, the curvature of $\nabla^{|\Lambda^{\mathrm{top}}|E^*}$ is given by $-\,\mathsf{tr}\, R$, see e.g. [60, Prop. 2.2.43]. □

With other words, $\mathsf{tr}\, R = 0$ is a necessary condition for the existence of a covariantly constant density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|E^*)$. In fact, the condition is locally also sufficient and globally the deRham class $[\alpha] \in \mathrm{H}^1_{\mathrm{dR}}(M)$ might be an obstruction.

Definition 2.1.2 (Unimodular covariant derivative) *A covariant derivative ∇^E is called unimodular if $\mathsf{tr}\, R^E = 0$.*

Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve defined on an open interval I and let $a, b \in I$. In general, the fibers of E at $\gamma(a)$ and $\gamma(b)$ are not related in a canonical way. Using the covariant derivative, this can be done as follows. We are looking at a section s along γ such that s is covariantly constant in the direction $\dot{\gamma}$. More precisely, we consider the pull-back bundle $\gamma^\# E \longrightarrow I$ together with the pull-back $\nabla^\#$ of ∇^E. Then we want to find a section $s \in \Gamma^\infty(\gamma^\# E)$ with

$$\nabla^\#_{\frac{\partial}{\partial t}} s = 0. \tag{2.1.6}$$

If $\{e_\alpha\}$ are local base sections of E over some open subset $U \subseteq M$ and $\gamma(I) \subseteq U$ then (2.1.6) is equivalent to

$$0 = \nabla^\#_{\dot{\gamma}}(s^\alpha(t) e_\alpha(\gamma(t))) = \dot{s}^\alpha(t) e_\alpha(\gamma(t)) + s^\alpha(t) A^\beta_\alpha(\dot{\gamma}(t)) e_\beta(\gamma(t)), \tag{2.1.7}$$

i.e.

$$\dot{s}^\beta(t) + A^\beta_\alpha(\dot{\gamma}(t)) s^\alpha(t) = 0. \tag{2.1.8}$$

Since (2.1.8) is an ordinary linear differential equation for the coefficient functions $s^\alpha : I \longrightarrow \mathbb{R}$, they have unique solutions $s^\alpha(t)$ for all t and all initial conditions $s^\alpha(a)$. Moreover, the resulting time evolution $s^\alpha(a) \mapsto s^\alpha(t)$ is a *linear* map and by uniqueness even an isomorphism. If the image of γ is not within the domain of a single bundle chart we can cover it with several ones (finitely many for compact time intervals) and use the uniqueness statement to glue the local solutions together in the usual way. The uniqueness will then guarantee that the result will not depend on the choice how we covered the curve with bundle charts. Finally, this gives the following result:

Proposition 2.1.3 *Let ∇^E be a covariant derivative for $E \longrightarrow M$ and let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve. Let $a, b \in I$.*

i.) For every initial condition $s_{\gamma(a)} \in E_{\gamma(a)}$ there exists a unique solution $s(t) \in E_{\gamma(t)}$ of (2.1.6).

ii.) The map $s_{\gamma(a)} \mapsto s(b)$ is a linear isomorphism $E_{\gamma(a)} \longrightarrow E_{\gamma(b)}$ which is denoted by

$$P_{\gamma,a\to b} : E_{\gamma(a)} \longrightarrow E_{\gamma(b)}. \tag{2.1.9}$$

Definition 2.1.4 (Parallel transport) *The linear isomorphism $P_{\gamma,a\to b} : E_{\gamma(a)} \longrightarrow E_{\gamma(b)}$ is called the parallel transport along γ with respect to ∇^E.*

Remark 2.1.5 (Parallel transport)

i.) In general, $P_{\gamma,a\to b}$ depends very much on the choice of the curve γ connecting $\gamma(a)$ and $\gamma(b)$.

ii.) We can define $P_{\gamma,a\to b}$ also for piecewise smooth curves by composing the parallel transports of the smooth pieces appropriately.

iii.) If the curvature R^E is zero then the parallel transport $P_{\gamma,a\to b}$ is *independent* of the curve γ but only depends on $\gamma(a)$ and $\gamma(b)$, *provided* the points are close enough. More precisely, if γ and $\widetilde{\gamma}$ are two curves with $\gamma(a) = \widetilde{\gamma}(a)$ and $\gamma(b) = \widetilde{\gamma}(b)$ such that there is a smooth homotopy between γ and $\widetilde{\gamma}$ then $P_{\gamma,a\to b} = P_{\widetilde{\gamma},a\to b}$. Note however that $R^E = 0$ is a rather strong condition which implies certain strong topological properties of the vector bundle $E \longrightarrow M$.

iv.) If $\gamma : I \longrightarrow M$ is a smooth curve and $\sigma : J \longrightarrow I$ is a smooth reparametrization then the parallel transports along γ and $\tilde{\gamma} = \gamma \circ \sigma$ coincide. More precisely, for $a', b' \in J$ we have

$$P_{\gamma,\sigma(a')\to\sigma(b')} = P_{\gamma\circ\sigma,a'\to b'}. \tag{2.1.10}$$

Since the parallel transport "connects" the fibers of E at different points, a covariant derivative is also called *connection*. Some further properties of the parallel transport are collected in the Appendix A.1.

2.1.2 The Exponential Map

In the case $E = TM$ a covariant derivative has additional features we shall discuss now. First, we have another contraction of the curvature tensor R given by

$$\mathrm{Ric}(X, Y) = \mathsf{tr}(Z \mapsto R(Z, X)Y) \tag{2.1.11}$$

for $X, Y \in \Gamma^\infty(TM)$. The resulting tensor field

$$\mathrm{Ric} \in \Gamma^\infty(T^*M \otimes T^*M) \tag{2.1.12}$$

is called the *Ricci tensor* of ∇. Note that the trace in (2.1.11) only can be defined for $E = TM$. The third contraction $\mathsf{tr}(Z \mapsto R(X, Z)Y)$ would give again the Ricci tensor up to a sign. Thus (2.1.11) is the only additional interesting contraction.

For a covariant derivative ∇ on TM we have yet another tensor field, the *torsion*

$$\mathrm{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{2.1.13}$$

which gives a tensor field

$$\mathrm{Tor} \in \Gamma^\infty(\Lambda^2 T^*M \otimes TM). \tag{2.1.14}$$

Then ∇ is called *torsion-free* if $\mathrm{Tor} = 0$. The relation between R and Tor is encoded in the first Bianchi identity, see e.g. [35, Chap. III]:

Lemma 2.1.6 (First Bianchi identity) *For any covariant derivative for TM we have*

$$R(X,Y)Z + cycl.(X,Y,Z) = (\nabla_X Tor)(Y,Z) + \mathrm{Tor}(\mathrm{Tor}(X,Y),Z) + cycl.(X,Y,Z). \tag{2.1.15}$$

In particular, for a torsion-free ∇ we have

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2.1.16}$$

for all $X, Y, Z \in \Gamma^\infty(TM)$.

Proof. The proof consists in a straightforward algebraic manipulation using only the definitions. □

Corollary 2.1.7 *Let ∇ be torsion-free. Then*

$$\mathrm{Ric}(X,Y) - \mathrm{Ric}(Y,X) + (\mathsf{tr}\, R)(X,Y) = 0, \tag{2.1.17}$$

whence Ric *is symmetric if in addition ∇ is unimodular.*

In case of the tangent bundle the parallel transport can be used to motivate the following question. For a starting point $p \in M$ and a starting velocity $v_p \in T_pM$, is there a curve γ with $\dot{\gamma}(0) = v_p$ such that $\dot{\gamma}$ is parallel along γ? Such an *auto-parallel* curve will be called a *geodesic*. To get an idea we consider this condition, which globally reads

$$\nabla^{\#}_{\frac{\partial}{\partial t}} \dot{\gamma} = 0, \tag{2.1.18}$$

in a local chart (U, x). We denote by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \tag{2.1.19}$$

the locally defined *Christoffel symbols* $\Gamma^k_{ij} \in \mathcal{C}^\infty(U)$. Then (2.1.18) means for the curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ with $\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$ and $\gamma^i = x^i \circ \gamma \in \mathcal{C}^\infty(I)$ explicitly

$$\ddot{\gamma}^i(t) + \Gamma^i_{k\ell}(\gamma(t)) \dot{\gamma}^k(t) \dot{\gamma}^\ell(t) = 0. \tag{2.1.20}$$

This is a (highly nonlinear) ordinary second order differential equation. Hence we have unique solutions for every initial condition $\dot{\gamma}^i(0) \frac{\partial}{\partial x^i}\Big|_{\gamma(0)} = v_p$, where $p = \gamma(0)$, at least for small times. Since locally

$$\mathrm{Tor}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}, \tag{2.1.21}$$

we see that the torsion Tor of ∇ does not enter the geodesic equation (2.1.20). We collect a few well-known facts about the solution theory of the geodesic equation:

Theorem 2.1.8 (Geodesics) *Let ∇ be a covariant derivative for $TM \longrightarrow M$.*

i.) For every $v_p \in T_pM$ there exists a unique solution $\gamma : I_{v_p} \subseteq \mathbb{R} \longrightarrow M$ of (2.1.20) with $\dot{\gamma}(0) = v_p$ and maximal open interval $I_{v_p} \subseteq \mathbb{R}$ around 0.

ii.) Let $\lambda \in \mathbb{R}$ and $v_p \in T_pM$. If γ denotes the geodesic with $\dot{\gamma}(0) = v_p$ then $\gamma_\lambda(t) = \gamma(\lambda t)$ is the geodesic with $\dot{\gamma}_\lambda(0) = \lambda v_p$.

iii.) There exists an open neighborhood $\mathcal{V} \subseteq TM$ of the zero section such that for all $v_p \in \mathcal{V}$ the geodesic with $\dot{\gamma}(0) = v_p$ is defined for all $t \in [0,1]$. We set $\exp_p(v_p) = \gamma(1)$ for this geodesic.

iv.) For $v_p \in \mathcal{V} \subseteq TM$ the curve $t \mapsto \exp_p(tv_p)$ is the geodesic γ with $\dot{\gamma}(0) = v_p$.

v.) The map $\exp : \mathcal{V} \subset TM \longrightarrow M$ is smooth.

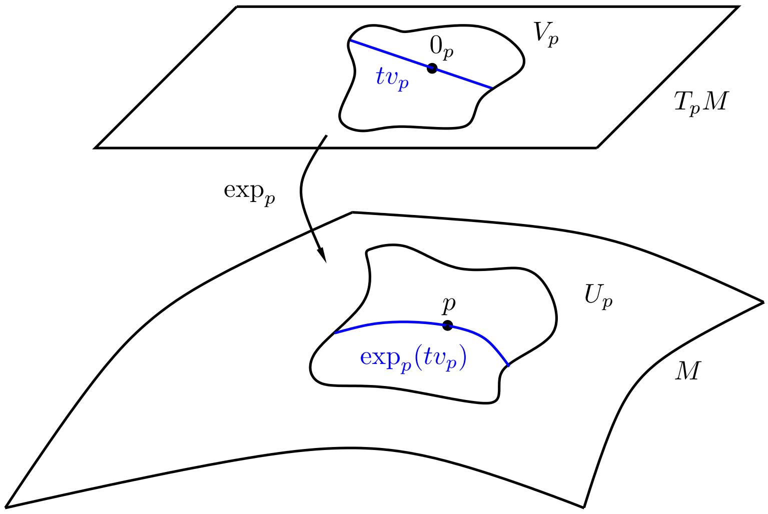

Figure 2.1: The exponential map gives a normal chart.

vi.) *The map*

$$\pi \times \exp : \mathcal{V} \subseteq TM \ni v_p \ \mapsto \ (p, \exp_p(v_p)) \in M \times M \tag{2.1.22}$$

is a local diffeomorphism around the zero-section. It maps the zero section onto the diagonal and for all $p \in M$

$$T_{0_p} \exp_p = \mathsf{id}_{T_pM} . \tag{2.1.23}$$

Proof. The proof can be found e.g. in [39, Chap. VIII, §5] or [12, §11 and §12]. □

Definition 2.1.9 (Exponential map) *For a given covariant derivative* ∇*, the map* $\exp : \mathcal{V} \subseteq TM \longrightarrow M$ *given by v.) of Theorem 2.1.8 is called the exponential map of* ∇*.*

Remark 2.1.10 (Exponential map) Let ∇ be a covariant derivative on M.

i.) Since the geodesic equation does not depend on the antisymmetric part of the Γ^k_{ij} we can safely pass from ∇ to a torsion-free covariant derivative by adding the appropriate multiple of the torsion tensor. The geodesics do not change and neither does the exponential map.

ii.) The exponential map is best understood in terms of *spray vector fields* on TM, see e.g. [39, Chap. VIII, §5] or [12, §11 and §12]. In fact, exp is just the projection of the time-one-flow of the spray vector field associated to ∇ by the bundle projection.

iii.) It follows from (2.1.23) that the exponential map $\exp_p$ at a given point $p \in M$ induces a diffeomorphism

$$\exp_p : V_p \subseteq T_pM \longrightarrow U_p \subseteq M \tag{2.1.24}$$

between a sufficiently small open neighborhood $V_p \subseteq T_pM$ of 0_p and its image $U_p \subseteq M$ which becomes an open neighborhood of

$$p = \exp_p(0_p) \in U_p \subseteq M. \tag{2.1.25}$$

Thus the map $(\exp_p\at{V_p})^{-1} : U_p \longrightarrow V_p$ yields a chart of M centered around p which is called a *normal* or *geodesic chart* with respect to ∇. The choice of linear coordinates on $V_p \subset T_pM$ induces then *normal coordinates* on $U_p \subseteq M$, see also Figure 2.1.

More details on properties of the exponential map can be found in the Appendix A.2 where we compute, among other things, the Taylor expansions of various objects in normal coordinates. The following definition is motivated by the flat situation where the notions of "star-shaped" and "convex" have an immediate meaning.

Definition 2.1.11 *An open subset $U \subseteq M$ is called*

i.) geodesically star-shaped with respect to $p \in M$ if there is a star-shaped $V \subseteq V_p$ with $\exp_p\big|_V : V \xrightarrow{\simeq} U = \exp_p(V)$.

ii.) geodesically convex if it is geodesically star-shaped with respect to any point $p \in U$.

Usually, we simply speak of star-shaped and convex open subsets of M if the reference to ∇ is clear. Note that the properties described in Definition 2.1.11 depend on the choice of ∇ and are *not* invariant under an arbitrary change of coordinates.

For a general covariant derivative it might well be that the domain of definition of exp is a proper open subset: geodesics need not be defined for all times but can "fall of the manifold". The simplest example is obtained from $\mathbb{R}^2\backslash\{0\}$ with the flat connection. Geodesics are straight lines. Thus the geodesic starting at $(-1, 0)$ with tangent vector $(1, 0)$ stops being defined at $t = 1$ since it would reach 0 which is not a part of $\mathbb{R}^2\backslash\{0\}$. While this example looks rather artificial there are more difficult situations where one can not just "add a few points". These considerations motivate the following definition:

Definition 2.1.12 (Geodesic completeness) *The covariant derivative ∇ is called geodesically complete if all geodesics are defined for all times.*

2.1.3 Levi-Civita-Connection and the d'Alembertian

We shall now specialize the connection ∇ further and add one more structure, namely a semi-Riemannian metric:

Definition 2.1.13 (Semi-Riemannian metric) *A section $g \in \Gamma^\infty(\mathrm{S}^2 T^*M)$ is called semi-Riemannian metric if the bilinear form $g_p \in \mathrm{S}^2 T_p^*M$ on T_pM is non-degenerate for all $p \in M$. If in addition g_p is positive definite for all $p \in M$ then g is called Riemannian metric. If g_p has signature $(+, -, \ldots, -)$ then g is called Lorentz metric.*

Remark 2.1.14 (Semi-Riemannian metrics)

i.) The signature of a semi-Riemannian metric is locally constant and hence constant on a connected manifold, since it depends continuously on p and has only discrete values.

ii.) For Lorentz metrics also the opposite signature $(-, +, \ldots, +)$ is used in the literature. This causes some confusions and funny signs. So be careful here! Our convention is the more common one in quantum field theory, while the other one is preferred in general relativity.

A semi-Riemannian metric specifies a unique covariant derivative and a unique positive density:

Proposition 2.1.15 *Let g be a semi-Riemannian metric on M.*

i.) There exists a unique torsion-free covariant derivative ∇, the Levi-Civita connection, such that

$$\nabla g = 0. \tag{2.1.26}$$

ii.) There exists a unique positive density $\mu_g \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ such that

$$\mu_g\big|_p(v_1, \ldots, v_n) = 1, \tag{2.1.27}$$

whenever $v_1, \ldots, v_n$ form a basis of T_pM with $|g_p(v_i, v_j)| = \delta_{ij}$. In a chart (U, x) we have

$$\mu_g\big|_U = \sqrt{|\det(g_{ij})|}\, |\mathrm{d}\, x^1 \wedge \cdots \wedge \mathrm{d}\, x^n|, \tag{2.1.28}$$

with $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

iii.) The density μ_g is covariantly constant with respect to the Levi-Civita connection,

$$\nabla \mu_g = 0. \tag{2.1.29}$$

Thus ∇ is unimodular.

Proof. The proof is very much standard and will be omitted here, see e.g. [60, Aufgabe 3.7 and 5.10]. □

Remark 2.1.16 (Semi-Riemannian metrics) Let g be a semi-Riemannian metric on M.

i.) For a semi-Riemannian metric we have a notion of geodesics, namely those with respect to the corresponding Levi-Civita connection.

ii.) The covariant divergence $\mathrm{div}_\nabla(X)$ of a vector field $X \in \Gamma^\infty(TM)$ and the divergence with respect to the density μ_g, i.e.

$$\mathrm{div}_{\mu_g}(X) = \frac{\mathscr{L}_X\, \mu_g}{\mu_g} \tag{2.1.30}$$

coincide: We have

$$\mathrm{div}_\nabla(X) = \mathrm{div}_{\mu_g}(X), \tag{2.1.31}$$

which follows immediately from Lemma 1.2.19, see also [60, Sect. 2.3.4], since $\nabla \mu_g = 0$. Thus we shall speak of *the* divergence and simply write

$$\mathrm{div}(X) = \mathrm{div}_\nabla(X) = \mathrm{div}_{\mu_g}(X) \tag{2.1.32}$$

on a semi-Riemannian manifold.

iii.) Since $g \in \Gamma^\infty(\mathrm{S}^2 T^*M)$ is non-degenerate it induces a *musical isomorphism*

$$\flat : T_pM \ni v_p \mapsto v_p^\flat = g(v_p, \cdot\,) \in T_p^*M, \tag{2.1.33}$$

which gives a vector bundle isomorphism

$$\flat : TM \longrightarrow T^*M. \tag{2.1.34}$$

The inverse of $\flat$ is usually denoted by

$$\sharp : T^*M \longrightarrow TM. \tag{2.1.35}$$

Extending $\flat$ and $\sharp$ to higher tensor powers we get musical isomorphisms also between all corresponding contravariant and covariant tensor bundles. If locally in a chart (U, x)

$$g\big|_U = \frac{1}{2} g_{ij}\, \mathrm{d}\, x^i \vee \mathrm{d}\, x^j, \tag{2.1.36}$$

then $v^\flat = g_{ij} v^i\, \mathrm{d}\, x^j$, where $v = v^i \frac{\partial}{\partial x^i}$. If g^{ij} denotes the inverse matrix to the g_{ij} from (2.1.36), i.e. $g^{ij} g_{jk} = \delta_{ik}$, then

$$\alpha^\sharp = g^{ij} \alpha_i \frac{\partial}{\partial x^j} \tag{2.1.37}$$

for a one-form $\alpha = \alpha_i\, \mathrm{d}\, x^i$. This motivates the notion "musical" as $\flat$ lowers the indexes while $\sharp$ raises them. Finally, we have the dual metric locally given by

$$g^{-1}\big|_U = \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \vee \frac{\partial}{\partial x^j}, \tag{2.1.38}$$

which is a global section $g^{-1} \in \Gamma^\infty(\mathrm{S}^2 TM)$.

iv.) The metric $g \in \Gamma^\infty(\mathrm{S}^2 T^*M)$ can equivalently be interpreted as a homogeneous quadratic function on TM via the usual canonical isomorphism from Remark 1.2.7. The function

$$T = \mathcal{J}(g) \in \mathrm{Pol}^2(TM) \tag{2.1.39}$$

is then usually called the *kinetic energy function* in the Lagrangian picture of mechanics. Analogously, $g^{-1} \in \Gamma^\infty(\mathrm{S}^2 TM)$ gives a homogeneous quadratic function

$$T = \mathcal{J}(g^{-1}) \in \mathrm{Pol}^2(T^*M) \tag{2.1.40}$$

on T^*M, the kinetic energy in the Hamiltonian picture of mechanics. It turns out that all notions of geodesics etc. can be understood in this geometric mechanical framework. For example, geodesics are just the base point curves of solutions of the Euler-Lagrange equations and Hamilton's equations with respect to the Lagrangian $L = T$ and Hamiltonian $H = T$, respectively. Thus geodesic motion is motion *without* additional forces induced by some addition potentials. The exponential map exp is then just the Hamiltonian flow of T at time $t = 1$ projected back to M. For more on this mechanical point of view, see e.g. [60, Sect. 3.2.2].

v.) Using the inverse matrix g^{ij} we have the following local Christoffel symbols of the Levi-Civita connection

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right). \tag{2.1.41}$$

Since by Corollary 2.1.7 and Proposition 2.1.15, *iii.)* for a semi-Riemannian manifold (M, g) the Ricci tensor Ric is in fact symmetric

$$\mathrm{Ric} \in \Gamma^\infty(\mathrm{S}^2 T^*M), \tag{2.1.42}$$

we can compute a further "trace" by using the metric g. Note that while Ric can be defined for every covariant derivative this further contraction requires g. One calls the function

$$\mathrm{scal} = \left\langle g^{-1}, \mathrm{Ric} \right\rangle \in \mathcal{C}^\infty(M) \tag{2.1.43}$$

the *scalar curvature*. Locally, scal is just

$$\mathrm{scal}\big|_U = g^{ij}\, \mathrm{Ric}_{ij}. \tag{2.1.44}$$

In the literature, there are many other notations for scal, e.g. R (without indexes) or s or S.

We come now to differential operators defined by means of a semi-Riemannian metric. We have already seen the divergence operator div which acts on vector fields and which can be extended as in Lemma 1.2.18 to all sections $\Gamma^\infty(\mathrm{S}^\bullet TM)$. We have two other important operators.

Definition 2.1.17 (Gradient and d'Alembertian) *On a semi-Riemannian manifold (M, g) the gradient of a function is defined by*

$$\mathrm{grad}\, f = (\mathrm{d} f)^\sharp \in \Gamma^\infty(TM) \tag{2.1.45}$$

and the d'Alembertian of a function $f \in \mathcal{C}^\infty(M)$ is

$$\Box f = \mathrm{div}(\mathrm{grad}\, f) \in \mathcal{C}^\infty(M). \tag{2.1.46}$$

In case of a Riemannian manifold we write $\Delta f = \mathrm{div}(\mathrm{grad}\, f)$ instead and call Δ the Laplacian.

Remark 2.1.18 There are different sign conventions in the definition of the Laplacian and the d'Alembertian. In particular, sometimes $-\Delta$ is favoured instead of our Δ since Δ as we defined it turns out to be a *negative* essentially selfadjoint operator on $\mathcal{C}^\infty(M)$ for compact M.

We discuss now a couple of local formulas which allow to handle the operators div, grad and $\Box$ more explicitly.

Proposition 2.1.19 *Let (M, g) be a semi-Riemannian manifold and let (U, x) be a chart of M.*

i.) The gradient of $f \in \mathcal{C}^\infty(M)$ is locally given by

$$\operatorname{grad}(f)\big|_U = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{2.1.47}$$

ii.) The divergence of $X \in \Gamma^\infty(TM)$ is locally given by

$$\operatorname{div}(X)\big|_U = \frac{\partial X^i}{\partial x^i} + \Gamma^k_{ki} X^i. \tag{2.1.48}$$

iii.) The d'Alembertian of $f \in \mathcal{C}^\infty(M)$ is locally given by

$$\Box f\big|_U = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right). \tag{2.1.49}$$

iv.) The d'Alembertian is a second order differential operator with leading symbol

$$\sigma_2(\Box) = 2g^{-1} \in \Gamma^\infty(\mathrm{S}^2 TM). \tag{2.1.50}$$

Moreover, with respect to the global symbol calculus induced by the Levi-Civita connection we have

$$\Box = \left(\frac{\mathrm{i}}{\hbar} \right)^2 \varrho_{\mathrm{Std}}(2g^{-1}), \tag{2.1.51}$$

whence

$$\Box f = \frac{1}{2} \left\langle g^{-1}, \mathsf{D}^2 f \right\rangle. \tag{2.1.52}$$

Proof. The local formulas (2.1.47) and (2.1.48) are clear. Then (2.1.49) follows from some straightforward computation using the precise form of (2.1.41) for the Christoffel symbols. Then (2.1.50) is clear by definition of the leading symbol. For (2.1.51) and (2.1.52) we compute

$$\begin{aligned} \mathsf{D}^2 f = \mathsf{D}\,\mathrm{d} f &= \mathrm{d} x^i \vee \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial f}{\partial x^j} \,\mathrm{d} x^j \right) \\ &= \mathrm{d} x^i \vee \frac{\partial^2 f}{\partial x^i \partial x^j} \,\mathrm{d} x^j + \mathrm{d} x^i \vee \frac{\partial f}{\partial x^j} \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d} x^j \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} \,\mathrm{d} x^i \vee \mathrm{d} x^j - \Gamma^j_{ik} \frac{\partial f}{\partial x^j} \,\mathrm{d} x^i \vee \mathrm{d} x^k, \end{aligned}$$

which gives

$$\mathsf{D}^2 f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) \mathrm{d} x^i \vee \mathrm{d} x^j$$

for a general connection ∇. For $g^{-1} = \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \vee \frac{\partial}{\partial x^j}$ we find

$$\left\langle g^{-1}, \mathsf{D}^2 f \right\rangle = 2g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) = 2\Box f.$$

$\Box$

Remark 2.1.20 (Hessian) Sometimes $\frac{1}{2}\,\mathsf{D}^2 f \in \Gamma^\infty(\mathrm{S}^2 T^* M)$ is also called the *Hessian*

$$\mathrm{Hess}(f) = \frac{1}{2}\,\mathsf{D}^2 f \in \Gamma^\infty(\mathrm{S}^2 T^* M). \tag{2.1.53}$$

Then the d'Alembertian is the trace of the Hessian with respect to g^{-1}. Moreover, the gradient $\mathrm{grad} : \mathcal{C}^\infty(M) \longrightarrow \Gamma^\infty(TM)$ is a differential operator of order one, the same holds for the divergence $\mathrm{div} : \Gamma^\infty(TM) \longrightarrow \mathcal{C}^\infty(M)$.

Remark 2.1.21 For later use we also mention the following Leibniz rules

$$\mathrm{grad}(fg) = g\,\mathrm{grad}(f) + f\,\mathrm{grad}(g), \tag{2.1.54}$$

$$\mathrm{div}(fX) = f\,\mathrm{div}(x) + X(f), \tag{2.1.55}$$

$$\Box(fg) = g\Box f + \mathrm{grad}(g) f + \mathrm{grad}(f) g + f\Box g = g\Box f + 2\,\langle \mathrm{grad}(f), \mathrm{grad}(g)\rangle + f\Box g, \tag{2.1.56}$$

for $f, g \in \mathcal{C}^\infty(M)$ and $X \in \Gamma^\infty(TM)$. They can easily be obtained from the definitions.

Example 2.1.22 (Minkowski spacetime) We consider the n-dimensional Minkowski spacetime. As a manifold we have $M = \mathbb{R}^n$ with canonical coordinates $x^0, x^1, \ldots, x^{n-1}$. Then the Minkowski metric η on M is the *constant* metric

$$\eta = \frac{1}{2}\eta_{ij}\,\mathrm{d}\,x^i \vee \mathrm{d}\,x^j \tag{2.1.57}$$

with $(\eta_{ij}) = \mathrm{diag}(+1, -1, \ldots, -1)$. One easily computes that in this global chart all Christoffel symbols vanish: (M, η) is *flat*. Moreover, we have for the above differential operators

$$\mathrm{grad}\, f = \frac{\partial f}{\partial x^0}\frac{\partial}{\partial x^0} - \sum_{i=1}^{n-1} \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}, \tag{2.1.58}$$

$$\mathrm{div}\, X = \frac{\partial X^0}{\partial x^0} + \sum_{i=1}^{n-1} \frac{\partial X^i}{\partial x^i}, \tag{2.1.59}$$

$$\Box f = \frac{\partial^2 f}{\partial (x^0)^2} - \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial (x^i)^2}. \tag{2.1.60}$$

This shows that $\Box$ is indeed the usual wave operator or d'Alembertian as known from the theory of special relativity, see e.g. [50]. Finally, the Lorentz density with respect to η is just the usual Lebesgue measure

$$\mu_\eta = |\,\mathrm{d}\,x^0 \wedge \cdots \wedge \mathrm{d}\,x^{n-1}|. \tag{2.1.61}$$

2.1.4 Normally Hyperbolic Differential Operators

The aim of this subsection is to generalize the d'Alembertian to more general fields than scalar fields. As it will turn out later, the most important feature of $\Box$ is the fact that the leading symbol is given by the metric. This motivates the following definition:

Definition 2.1.23 (Normally hyperbolic operator) *Let $E \longrightarrow M$ be a vector bundle over a Lorentz manifold (M, g). A differential operator $D : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$ is called normally hyperbolic if it is of second order and*

$$\sigma_2(D) = 2g^{-1} \otimes \mathsf{id}_E\,. \tag{2.1.62}$$

Recall that $\sigma_2(D) \in \Gamma^\infty(\mathrm{S}^2 TM \otimes \mathsf{End}(E))$ which explains the second tensor factor in (2.1.62). Usually, we simply write $\sigma_2(D) = 2g^{-1}$ with some slight abuse of notation. Note also that, as already for the d'Alembertian itself, the factor 2 in the symbol comes from our convention for symbols. Here also other conventions are used in the literature. However, this will not play any role later. The important fact is that D has a symbol being just a *constant nonzero* multiple of g^{-1}.

The following construction will always lead to a normally hyperbolic operator:

Example 2.1.24 (Connection d'Alembertian) Let ∇^E be a covariant derivative for $E \longrightarrow M$ and let ∇ be the Levi-Civita connection. This yields a global symbol calculus whence by

$$\Box^\nabla = \left(\frac{\mathrm{i}}{\hbar}\right)^2 \varrho_{\mathrm{Std}}(2g^{-1} \otimes \mathsf{id}_E) = \frac{1}{2}\left\langle 2g^{-1} \otimes \mathsf{id}_E, \frac{1}{2}(\mathsf{D}^E)^2 \cdot \right\rangle \tag{2.1.63}$$

a second order differential operator is given with leading symbol

$$\sigma_2(\Box^\nabla) = \left(\frac{\mathrm{i}}{\hbar}\right)^2 \sigma_2(\varrho_{\mathrm{Std}}(2g^{-1} \otimes \mathsf{id}_E)) = 2g^{-1} \otimes \mathsf{id}_E \tag{2.1.64}$$

by Theorem 1.2.6. Thus $\Box^\nabla$ is normally hyperbolic for any choice of ∇^E. An operator of this type is called the *connection d'Alembertian* with respect to ∇^E.

Lemma 2.1.25 (Connection d'Alembertian) *Let ∇^E be a covariant derivative for $E \longrightarrow M$ and $\Box^\nabla$ the corresponding connection d'Alembertian.*

i.) For $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma^\infty(E)$ we have

$$\Box^\nabla(fs) = (\Box f)s + 2\nabla^E_{\mathrm{grad}(f)} s + f \Box^\nabla s. \tag{2.1.65}$$

ii.) Let $A^\alpha_{i\beta} = e^\alpha\left(\nabla^E_{\frac{\partial}{\partial x^i}} e_\beta\right) \in \mathcal{C}^\infty(U)$ denote the local Christoffel symbols with respect to a chart (U, x) and local base sections $e_\alpha \in \Gamma^\infty(E\big|_U)$. Then locally

$$\Box^\nabla s = \left(g^{ij}\frac{\partial^2 s^\alpha}{\partial x^i \partial x^j} + 2g^{ij}\frac{\partial s^\gamma}{\partial x^i}A^\alpha_{j\gamma} - g^{ij}\Gamma^k_{ij}\frac{\partial s^\alpha}{\partial x^k} + g^{ij}\left(\frac{\partial A^\alpha_{i\beta}}{\partial x^j} - A^\alpha_{k\beta}\Gamma^k_{ij} + A^\gamma_{i\beta}A^\alpha_{j\gamma}\right)s^\beta\right)e_\alpha. \tag{2.1.66}$$

Proof. For the first part we use Proposition 1.1.3 to compute

$$(\mathsf{D}^E)^2(fs) = \mathsf{D}^E(\mathrm{d}f \otimes s + f\,\mathsf{D}^E s) = \mathsf{D}\,\mathrm{d}f \otimes s + 2\,\mathrm{d}f \vee \mathsf{D}^E s + f(\mathsf{D}^E)^2 s.$$

Then for the natural pairing we have

$$\begin{aligned}
\Box^\nabla(fs) &= \frac{1}{2}\left\langle g^{-1}, (\mathsf{D}^E)^2(f \cdot s)\right\rangle \\
&= \frac{1}{2}\left\langle g^{-1}, \mathsf{D}\,\mathrm{d}f\right\rangle \cdot s + \left\langle g^{-1}, \mathrm{d}f \vee \mathsf{D}^E s\right\rangle + \frac{1}{2}f\left\langle g^{-1}, (\mathsf{D}^E)^2 s\right\rangle \\
&= \Box f \cdot s + 2g^{ij}\frac{\partial f}{\partial x^i}\nabla^E_{\frac{\partial}{\partial x^j}} s + f\Box^\nabla s \\
&= \Box f \cdot s + 2\nabla^E_{\mathrm{grad}(f)} s + f\Box^\nabla s,
\end{aligned}$$

proving the first part. For the second, let $A^\alpha_{i\beta}$ be the local Christoffel symbols. Then first we have

$$\mathsf{D}^E s = \mathrm{d}x^i \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s = \mathrm{d}x^i \otimes \left(\frac{\partial s^\alpha}{\partial x^i}e_\alpha + s^\alpha \nabla^E_{\frac{\partial}{\partial x^i}} e_\alpha\right) = \mathrm{d}x^i \otimes \frac{\partial s^\alpha}{\partial x^i}e_\alpha + \mathrm{d}x^i \otimes s^\alpha A^\beta_{i\alpha}e_\beta$$

$$= \left(\frac{\partial s^\beta}{\partial x^i} + s^\alpha A^\beta_{i\alpha}\right) \mathrm{d}\, x^i \otimes e_\beta.$$

Consequently, we have

$$\begin{aligned}
(\mathsf{D}^E)^2 s &= \mathrm{d}\, x^j \vee \nabla^{E\otimes T^*M}_{\frac{\partial}{\partial x^j}} \left(\left(\frac{\partial s^\beta}{\partial x^i} + s^\alpha A^\beta_{i\alpha}\right) \mathrm{d}\, x^i \otimes e_\beta\right) \\
&= \mathrm{d}\, x^j \vee \left(\frac{\partial^2 s^\beta}{\partial x^i \partial x^j} + \frac{\partial s^\alpha}{\partial x^j} A^\beta_{i\alpha} + s^\alpha \frac{\partial A^\beta_{i\alpha}}{\partial x^j}\right) \mathrm{d}\, x^i \otimes e_\beta \\
&\quad + \mathrm{d}\, x^j \vee \left(\frac{\partial s^\beta}{\partial x^i} + s^\alpha A^\beta_{i\alpha}\right) \left(-\Gamma^i_{jk}\, \mathrm{d}\, x^k \otimes e_\beta + A^\gamma_{j\beta}\, \mathrm{d}\, x^i \otimes e_\gamma\right) \\
&= \frac{\partial^2 s^\beta}{\partial x^i \partial x^j} \mathrm{d}\, x^i \vee \mathrm{d}\, x^j \otimes e_\beta + 2 \frac{\partial s^\alpha}{\partial x^i} A^\beta_{j\alpha}\, \mathrm{d}\, x^i \vee \mathrm{d}\, x^j \otimes e_\beta - \frac{\partial s^\beta}{\partial x^i} \Gamma^i_{ji}\, \mathrm{d}\, x^j \vee \mathrm{d}\, x^k \otimes e_\beta \\
&\quad + s^\alpha \frac{\partial A^\beta_{i\alpha}}{\partial x^j} \mathrm{d}\, x^i \vee \mathrm{d}\, x^j \otimes e_\beta - s^\alpha A^\beta_{i\alpha} \Gamma^i_{jk}\, \mathrm{d}\, x^j \vee \mathrm{d}\, x^k \otimes e_\beta + s^\alpha A^\gamma_{i\alpha} A^\beta_{j\gamma}\, \mathrm{d}\, x^i\, \mathrm{d}\, x^j \otimes e_\beta.
\end{aligned}$$

The natural pairing with g^{-1} means replacing $\frac{1}{2}\, \mathrm{d}\, x^i \vee \mathrm{d}\, x^j$ with g^{ij} everywhere. This gives the result. □

We now prove that every normally hyperbolic operator is actually a connection d'Alembertian up to a $\mathcal{C}^\infty(M)$-linear operator. We have the following result, sometimes called a generalized Weitzenböck formula, see e.g. [5, Prop. 3.1]:

Proposition 2.1.26 (Weitzenböck formula) *Let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator. Then there exists a unique covariant derivative ∇^E for E and a unique $B \in \Gamma^\infty(\mathsf{End}(E))$ such that*

$$D = \Box^\nabla + B. \tag{2.1.67}$$

Proof. First we show uniqueness. Assume that ∇^E and B exist such that (2.1.67) holds. Then from Lemma 2.1.25 we know that

$$D(f \cdot s) - f D(s) = \Box^\nabla (f \cdot s) + B(f \cdot s) - f \Box^\nabla (s) - f B(s) = (\Box f) \cdot s + 2 \nabla^E_{\mathrm{grad}(f)} s,$$

since B is $\mathcal{C}^\infty(M)$-linear. Thus we have

$$\nabla^E_{\mathrm{grad}(f)} s = \frac{1}{2} \left(D(f \cdot s) - f D(s) - (\Box f) \cdot s\right) \tag{$*$}$$

for all $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma^\infty(E)$. Since gradients of functions span every T_pM for all $p \in M$, the covariant derivative ∇^E is uniquely determined by D via $(*)$. But then also $B = D - \Box^\nabla$ is uniquely determined. Let us now turn to the existence: to this end we compute the right hand side of $(*)$ locally in order to show that it actually *defines* a connection. Let locally

$$Ds\big|_U = g^{ij} \frac{\partial^2 s^\alpha}{\partial x^i \partial x^j} e_\alpha + D^{i\beta}_{\ \alpha} \frac{\partial s^\alpha}{\partial x^i} e_\beta + D^\beta_\alpha s^\alpha e_\beta$$

with local coefficients $D^{i\beta}_{\ \alpha}, D^\beta_\alpha \in \mathcal{C}^\infty(U)$. Then we have

$$\begin{aligned}
&\frac{1}{2}(D(f \cdot s) - f D(s) - (\Box f) \cdot s) \\
&= \frac{1}{2} \Bigg(g^{ij} \frac{\partial^2 (f s^\alpha)}{\partial x^i \partial x^j} e_\alpha + D^{i\beta}_{\ \alpha} \frac{\partial (f s^\alpha)}{\partial x^i} e_\beta + f D^\beta_\alpha s^\alpha e_\beta - f g^{ij} \frac{\partial^2 s^\alpha}{\partial x^i \partial x^j} e_\alpha - f D^{i\beta}_{\ \alpha} \frac{\partial s^\alpha}{\partial x^i} e_\beta - f D^\beta_\alpha s^\alpha e_\beta
\end{aligned}$$

$$
\begin{aligned}
&-g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} s^\alpha e_\alpha + g^{ij}\Gamma^k_{ij}\frac{\partial f}{\partial x^k} s^\alpha e_\alpha\Bigg) \\
&= \frac{1}{2}\left(2g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial s^\alpha}{\partial x^j} e_\alpha + D^{i\beta}_{\;\;\alpha}\frac{\partial f}{\partial x^i} s^\alpha e_\beta + g^{ij}\Gamma^k_{ij}\frac{\partial f}{\partial x^k} s^\alpha e_\alpha\right) \\
&= (\operatorname{grad} f)^j \frac{\partial s^\alpha}{\partial x^j} e_\alpha + \frac{1}{2}\left(D^{i\beta}_{\;\;\alpha} g_{ij} (\operatorname{grad} f)^j s^\alpha e_\beta + g^{ij}\Gamma^k_{ij} g_{k\ell}(\operatorname{grad} f)^\ell s^\alpha e_\alpha\right).
\end{aligned}
$$

On one hand we know that the right hand side of $(*)$ is globally defined. On the other hand, we see from the local expression that replacing $\operatorname{grad} f$ by an arbitrary vector field X defines locally a connection with connection one-forms

$$
A^\beta_{i\alpha} = \frac{1}{2} D^{j\beta}_{\;\;\alpha} g_{ij} + \frac{1}{2} g^{rs}\Gamma^j_{rs} g_{ij}\delta^\beta_\alpha, \tag{$**$}
$$

i.e. a connection ∇^E such that on U

$$
\nabla^E_X s = (\mathscr{L}_X\, s^\alpha) e_\alpha + A^\beta_{i\alpha} X^i s^\alpha e_\beta.
$$

This is clear from the local expression. Together we see that we indeed have a global connection ∇^E with local connection one-forms $A^\beta_{i\alpha}$ as in $(**)$. It remains to show that this connection ∇^E yields (2.1.67). So we have to show that $D - \Box^\nabla$ is $\mathcal{C}^\infty(M)$-linear. Using the explicit expression $(**)$ for $A^\beta_{i\alpha}$ together with Lemma 2.1.25, *ii.)* this is a straightforward computation. We have

$$
\begin{aligned}
Ds - \Box^\nabla s &= D^{i\beta}_{\;\;\alpha}\frac{\partial s^\alpha}{\partial x^i} e_\beta + D^\beta_\alpha s^\alpha e_\beta - 2g^{ij}\frac{\partial s^\alpha}{\partial x^i} A^\beta_{i\alpha} e_\beta + g^{ij}\Gamma^k_{ij}\frac{\partial s^\alpha}{\partial x^k} e_\alpha \\
&\quad - g^{ij}\left(\frac{\partial A^\beta_{i\alpha}}{\partial x^j} s^\alpha - A^\beta_{k\alpha}\Gamma^k_{ij} s^\alpha + A^\gamma_{i\alpha} A^\beta_{j\gamma} s^\alpha\right) e_\beta \\
&= \frac{\partial s^\alpha}{\partial x^i}\left(D^{i\beta}_{\;\;\alpha} - 2g^{ij}\left(\frac{1}{2} D^{\ell\beta}_{\;\;\alpha} g_{j\ell} + \frac{1}{2} g^{rs}\Gamma^\ell_{rs} g_{j\ell}\delta^\beta_\alpha\right) + g^{jk}\Gamma^i_{jk}\delta^\gamma_\alpha\right) e_\beta \\
&\quad + D^\beta_\alpha s^\alpha e_\beta - g^{ij}\left(\frac{\partial A^\beta_{i\alpha}}{\partial x^j} s^\alpha - A^\beta_{k\alpha}\Gamma^k_{ij} s^\alpha + A^\gamma_{i\alpha} A^\beta_{j\gamma} s^\alpha\right) e_\beta \\
&= \left(D^\beta_\alpha - g^{ij}\frac{\partial A^\beta_{i\alpha}}{\partial x^j} + g^{ij} A^\beta_{k\alpha}\Gamma^k_{ij} - g^{ij} A^\gamma_{i\alpha} A^\beta_{j\gamma}\right) s^\alpha e_\beta.
\end{aligned}
$$

This is clearly $\mathcal{C}^\infty(M)$-linear and hence the local expression for an endomorphism field $B \in \Gamma^\infty(\mathsf{End}(E))$. Since $D - \Box^\nabla$ is globally defined, B is indeed a globally defined section. Of course, taking the explicit but complicated transformation laws for coefficients of second order differential operators, connection one-forms and Christoffel symbols, this can also be checked by hand (though it is not very funny). □

Remark 2.1.27 (Normally hyperbolic operators)

i.) If D is normally hyperbolic and ∇^E is the corresponding covariant derivative then D satisfies the Leibniz rule

$$
D(f \cdot s) = fD(s) + 2\nabla^E_{\operatorname{grad}(f)} s + (\Box f)\cdot s \tag{2.1.68}
$$

for all $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma^\infty(E)$. This follows from the above proof. The connection ∇^E is also called the *D-compatible connection*. In the following, we can safely assume that D is of the form $\Box^\nabla + B$ as above.

ii.) While in general every $B \in \Gamma^\infty(\mathsf{End}(E))$ gives a normally hyperbolic $\Box^\nabla + B$, in specific contexts there are sometimes more geometrically motivated choices for both, the connection ∇^E and the additional tensor field B.

iii.) Even though we formulated the above proposition and the definition of normally hyperbolic differential operators with respect to a Lorentz signature, it is clear that the above considerations apply also to the general semi-Riemannian case. In the Riemannian case, the corresponding operators are called connection Laplacians and normally elliptic operators, respectively.

2.2 Causal Structure on Lorentz Manifolds

While most of the material up to now was applicable for general semi-Riemannian manifolds we shall now discuss the causal structure referring to the Lorentz signature exclusively.

2.2.1 Some Motivation from General Relativity

In general relativity the spacetime is described by a four-dimensional manifold M equipped with a Lorentz metric g subject to Einstein's equation. One defines the *Einstein tensor*

$$G = \mathrm{Ric} - \frac{1}{2}\,\mathrm{scal}\cdot g, \tag{2.2.1}$$

which is a symmetric covariant tensor field

$$G \in \Gamma^\infty(\mathrm{S}^2 T^* M). \tag{2.2.2}$$

It can be shown that the covariant divergence of G vanishes,

$$\operatorname{div} G = 0, \tag{2.2.3}$$

while G itself needs not to be covariant constant at all. Physically, (2.2.3) is interpreted as a *conservation law.* Einstein's equation is then given by

$$G = \kappa T, \tag{2.2.4}$$

where $T \in \Gamma^\infty(\mathrm{S}^2 T^* M)$ is the so-called *energy-momentum tensor* of all matter and interaction fields on M *excluding* gravity. The constant κ is up to numerical constants Newton's constant of gravity. The precise form of T is complicated and depends on the concrete realization of the matter content of the spacetime under consideration. More generally, Einstein's equation with *cosmological constant* are

$$G + \lambda g = \kappa T, \tag{2.2.5}$$

where $\lambda \in \mathbb{R}$ is a constant, additional parameter of the theory, the cosmological constant.

The nature of these equations is that for a given functional expression for T usually coming from a variational principle, the metric g has to be found in such a way that (2.2.4) or (2.2.5) is satisfied. However, this is rather complicated as (2.2.4) and (2.2.5) turn out to be quadratic partial differential equations of second order in the coefficients of the metric which are of a rather complicated type. On one hand they are "hyperbolic" and therefor ask for an "initial value problem". On the other hand, when formulating (2.2.4) or (2.2.5) as initial value problem for a metric on a 3-dimensional submanifold, the Equations (2.2.4) or (2.2.5) have a certain gauge freedom thanks to the diffeomorphism invariance of the condition (2.2.4) and (2.2.5), respectively. This yields "constraints" which have to be satisfied. For more details on this initial value problem in general relativity see e.g. [16, 17, 22].

All this makes general relativity quite complicated, both from the conceptual and practical point of view. We refer to textbooks on general relativity for a more detailed and sophisticated discussion, see e.g. [6, 29, 54, 56].

For $T = 0$ one speaks of a *vacuum solution* to Einstein's equation: only those degrees of freedom are relevant which come directly from geometry and hence from gravity. Already this particular case is very complicated as it is still highly non-linear. Nevertheless, there are solutions which look like propagating waves or black holes.

In the following, we take the point of view that a certain energy and momentum content of the spacetime results in a certain metric g. Then we assume that there is a slight perturbation by some additional field ϕ on M which on one hand has only a minor contribution to T and thus does not influence g. On the other hand, the field is subject to field equations determined by g. With other words, we neglect the *back-reaction* of the field on g but investigate the field equations in a *fixed* background metric g.

Thus we arrive at field equations for ϕ on a given spacetime (M, g). It turns out that the question whether g is a solution to Einstein's equation or not, is of minor importance when we want to understand the field equations for ϕ. In fact, the geometric features of g which guarantee a "good behaviour" of ϕ are rather independent of Einstein's equation.

In general, physically relevant field equations for ϕ can be quite complicated: if we are interested in "interacting fields" then the field equations are non-linear. Thus all the technology of distributions etc. does *not* apply, at least not in a naive way. For this reason we restrict to *linear* field equations: one motivation is that even if the original field equations for ϕ are non-linear, a linearization around a solution ϕ_0 might be interesting. Assuming that ϕ_0 is a solution one considers $\phi = \phi_0 + \psi$ and rewrites the (non-linear) equations for ϕ as field equations for the perturbation ψ and *neglects* higher order terms in ψ. This way one obtains an approximation in form of a linear field equation for ψ.

We shall now discuss some typical examples. The prototype of a field equation is the *Klein-Gordon equation* for a scalar field $\phi \in \mathcal{C}^\infty(M)$ of mass $m \in \mathbb{R}$

$$\Box\phi + m^2\phi = 0. \tag{2.2.6}$$

On non-trivial geometries there are physical arguments suggesting that the Klein-Gordon equation should be modified in a way incorporating the scalar curvature, i.e. one considers

$$\Box\phi + \xi \operatorname{scal} \phi + m^2\phi = 0, \tag{2.2.7}$$

where $\xi \in \mathbb{R}$ is a parameter. While (2.2.7) is still linear, a self-interacting modification of the Klein-Gordon equation is e.g.

$$\Box\phi + m^2\phi + \lambda\phi^2 + \mu\phi^3 = 0, \tag{2.2.8}$$

where again $\lambda, \mu \in \mathbb{R}$ are parameters of the theory. If ϕ_0 is a solution of (2.2.8) then a linearized version of (2.2.8) for $\phi = \phi_0 + \psi$ is given by

$$\Box\psi + m^2\psi + 2\lambda\phi_0\psi + 3\phi_0^2\psi = 0. \tag{2.2.9}$$

By this procedure we obtain a rather general linear equation with leading symbol being the metric but fairly general $\mathcal{C}^\infty(M)$-linear part, in our case either $\xi \operatorname{scal} + m^2$ or $m^2 + 2\lambda\phi_0 + 3\mu\phi_0^2$ or a combination of both.

This motivates that one should consider linear second order differential equations of normal hyperbolic type, i.e.

$$(\Box + B)\phi = 0, \tag{2.2.10}$$

with $B \in \mathcal{C}^\infty(M)$. Finally, the step towards general vector bundles and sections $\phi \in \Gamma^\infty(E)$ is only a mild generalization: in many physical field theories the fields have more than one component. This way we arrive at field equations of the form

$$(\Box^\nabla + B)\phi = 0 \tag{2.2.11}$$

for $\phi \in \Gamma^\infty(E)$ with a connection d'Alembertian $\square^\nabla$ and some $B \in \Gamma^\infty(\mathsf{End}(E))$. Note once more that in our approximation to general relativity we have a fixed background metric g used in the definition of $\square^\nabla$.

2.2.2 Future and Past on a Lorentz Manifold

Having a fixed Lorentz metric g on a spacetime manifold M we can now transfer the notions of special relativity, see e.g. [50], to (M, g). In fact, each tangent space (T_pM, g_p) is isometrically isomorphic to Minkowski spacetime $(\mathbb{R}^n, \eta)$ with $\eta = \mathrm{diag}(+1, -1, \ldots, -1)$, by choosing a Lorentz frame: there exist tangent vectors $e_i \in T_pM$ with $i = 1, \ldots, n$ such that

$$g_p(e_i, e_j) = \eta_{ij} = \pm\delta_{ij}. \tag{2.2.12}$$

Remark 2.2.1 (Local Lorentz frame) The pointwise isometry from (T_pM, g_p) to $(\mathbb{R}^n, \eta)$ can be made to depend smoothly on p at least in a local neighborhood: For every $p \in M$ there exists a small open neighborhood U of p and local sections $e_1, \ldots, e_n \in \Gamma^\infty(E\big|_U)$ such that for all $q \in U$

$$g_q(e_i(q), e_j(q)) = \eta_{ij}. \tag{2.2.13}$$

In general, the frame $\{e_i\}_{i=1,\ldots,n}$ can *not* be chosen to come from a chart x on U, i.e. e_i is not $\frac{\partial}{\partial x^i}$. Here the curvature of g is the obstruction. Nevertheless, such *local Lorentz frames* will simplify certain computations. We note that for two local Lorenz frames $\{e_i\}_{i=1,\ldots,n}$ and $\{\widetilde{e}_i\}_{i=1,\ldots,n}$ on U there exists a unique smooth function $\Lambda : U \longrightarrow \mathrm{O}(1, n-1)$ such that

$$e_i(p) = \Lambda_i^j(p)\widetilde{e}_j(p), \tag{2.2.14}$$

since the Lorentz transformations $\mathrm{O}(1, n-1)$ are precisely the linear isometries of $(\mathbb{R}^n, \eta)$.

As in special relativity one states the following definition:

Definition 2.2.2 *Let (M, g) be a Lorentz manifold and $v_p \in T_pM$ a non-zero vector. Then v_p is called*

i.) timelike if $g_p(v_p, v_p) > 0$,

ii.) lightlike or null if $g_p(v_p, v_p) = 0$,

iii.) spacelike if $g_p(v_p, v_p) < 0$.

Non-zero vectors with $g_p(v_p, v_p) \geq 0$ are sometimes also called causal. To the zero vector, no attribute is assigned.

In a fixed tangent space we have two open convex cones of timelike vectors whose boundaries consists of the lightlike vectors together with the zero vector, see Figure 2.2. Already in Minkowski spacetime there are Lorentz transformations which exchange the two connected components of the timelike vectors. Thus there is no intrinsic definition of "future-" and "past-directed" vectors in $(\mathbb{R}^n, \eta)$. Clearly, for physical purposes it is crucial to have such a distinction: we choose once and for all a *time-orientation* on Minkowski spacetime $(\mathbb{R}^n, \eta)$, i.e. a choice of one of the interiors of the light-cones to be future directed. We symbolize this choice by $(\mathbb{R}^n, \eta, \uparrow)$. Now only the *orthochronous Lorentz transformations*

$$\mathrm{L}^\uparrow(1, n-1) = \left\{\Lambda \in \mathrm{O}(1, n, 1) \;\middle|\; \Lambda_0^0 > 0\right\} \tag{2.2.15}$$

preserve the time-orientation $(\mathbb{R}^n, \eta, \uparrow)$. Clearly, $\mathrm{L}^\uparrow(1, n-1)$ is a closed subgroup of $\mathrm{O}(1, n-1)$ of the same dimension.

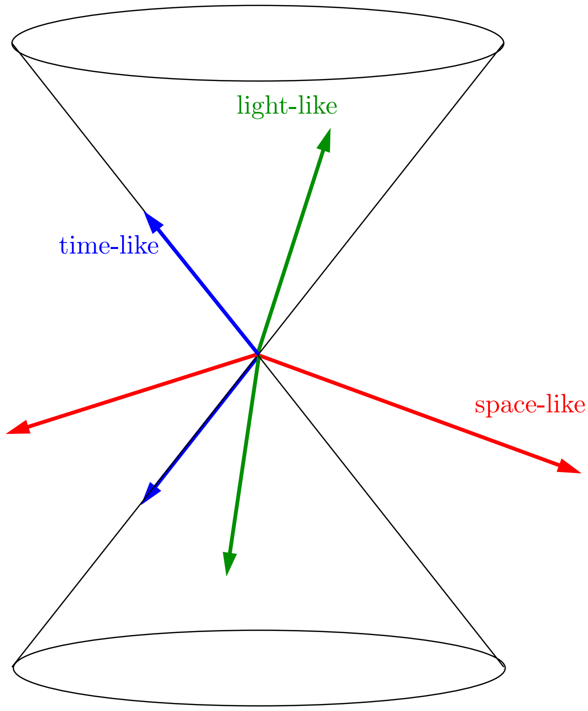

Figure 2.2: Light cone structure in Minkowski spacetime

Analogously, Lorentz transformations do not preserve the space-orientation in general. For a spacelike sub vector space $\Sigma \subseteq \mathbb{R}^n$ (of dimension $n-1$), there are orientation preserving and reversing Lorentz transformations. Choosing one orientation of Σ we obtain an additional structure on Minkowski spacetime which we symbolize as $(\mathbb{R}^n, \eta, +)$ or $(\mathbb{R}^n, \eta, \uparrow, +)$ in the case where we have chosen a time-orientation as well. One can check that "$+$" does not depend on the particular choice of Σ. The subgroups preserving $+$ or $+$ and $\uparrow$ are the *proper* and the *proper and orthochronous* Lorentz transformations denoted by $\mathrm{L}_+(1, n-1)$ and $\mathrm{L}_+^\uparrow(1, n-1)$, respectively. It is a standard fact that $\mathrm{L}_+^\uparrow(1, n-1)$ is the connected component of the identity and hence a normal subgroup. The discrete resulting quotient group is

$$\mathrm{L}(1, n-1)/\mathrm{L}_+^\uparrow(1, n-1) = \{\mathsf{id}, \mathrm{P}, \mathrm{T}, \mathrm{PT}\} \tag{2.2.16}$$

with relations $\mathrm{P}^2 = \mathrm{T}^2 = \mathsf{id}$ and $\mathrm{PT} = \mathrm{TP}$. Then T is the time-reversal while P is the parity operation.

We want to use now the time- and space-oriented Minkowski spacetime $(\mathbb{R}^n, \eta, \uparrow, +)$ in order to obtain time and space orientations for (M, g) as well. Here we meet the usual obstructions analogously to the obstructions for orientability in general. In the following the time-orientability will be crucial while the space-orientability is not that important. Thus we focus on the time-orientability. Here one has the following result:

Proposition 2.2.3 *Let (M, g) be a Lorentz manifold. Then the following statements are equivalent:*

i.) There exists a timelike vector field $X \in \Gamma^\infty(TM)$, i.e. $X(p)$ is timelike for all $p \in M$.

ii.) There exists an open cover $\{U_\alpha\}_\alpha$ of M with local Lorentz frames $\{e_{\alpha i}\}_{i=1,\ldots,n} \in \Gamma^\infty(TU_\alpha)$ such that on $U_\alpha \cap U_\beta \neq \emptyset$ the transition matrix $\Lambda_{\alpha\beta} \in \mathrm{O}(1, n-1)$ with

$$e_{\alpha i} = \Lambda_{\alpha\beta}{}^j_i e_{\beta j} \tag{2.2.17}$$

takes values in $\mathrm{L}^\uparrow(1, n-1)$.

Proof. Assume that $X \in \Gamma^\infty(TM)$ is timelike. Then we choose an open cover $\{U_\alpha\}$ of M with locally defined Lorentz frames $\{e_{\alpha i}\}$ on U_α. Without restriction we can choose the U_α to be connected. Then on U_α either the timelike vector $e_{\alpha 1}$ or the timelike vector $-e_{\alpha 1}$ is in the same connected component of the timelike vectors as X. Changing $e_{\alpha 1}$ to $-e_{\alpha 1}$ if necessary yields a local Lorentz frame on U_α with $e_{\alpha 1}$ in the same connected component as X. Since this holds for all α we obtain transition matrices $\Lambda_{\alpha\beta}$ in $\mathcal{C}^\infty(U_\alpha \cap U_\beta, \mathrm{L}^\uparrow)$ as wanted.

Conversely, let such an open cover and local Lorentz frames be given. We choose a partition of unity χ_α subordinate to U_α with $\chi_\alpha \geq 0$. Then we define

$$X = \sum_\alpha \chi_\alpha e_{\alpha 1} \tag{$*$}$$

which is clearly a globally defined smooth vector field $X \in \Gamma^\infty(TM)$. At $p \in M$ only finitely many α contribute to $(*)$. Moreover, since by (2.2.17) all the $e_{\alpha 1}(p)$ are in the *same* connected component of the timelike vectors and since this connected component is *convex*, also $X(p)$ is in this connected component. It follows that $X(p)$ is timelike. □

There are still alternative formulations of the property described by *i.)* and *ii.)* in Proposition 2.2.3. However, for the time being we take the result of Proposition 2.2.3 as definition of time-orientability:

Definition 2.2.4 (Time-orientability) *Let (M, g) be a Lorentz manifold.*

i.) *(M, g) is called time-orientable if there exists a timelike vector field $X \in \Gamma^\infty(TM)$.*

ii.) *The choice of a timelike vector field $X \in \Gamma^\infty(TM)$ is called a time-orientation.*

iii.) *With respect to a time-orientation, a timelike vector $v_p \in T_pM$ is called future directed if v_p is in the same connected component as $X(p)$. It is called past directed if $-v_p$ is future directed.*

Remark 2.2.5 (Time-orientability) Note that time-orientability of (M, g) is rather independent of (topological) orientability of M. One can find easily a Lorentz metric on the Möbius strip which is time-orientable and, conversely, a Lorentz metric on the cylinder $\mathbb{S}^1 \times \mathbb{R}$ which is *not* time-orientable. We leave it as an exercise to figure out the details of these examples.

In the following, we shall always assume that (M, g) is time orientable. Moreover, we assume that a time-orientation has been chosen once and for all. This will be important for a consistent interpretation of (M, g) as a spacetime manifold. Using the time orientation we can define the future and past of a given point in M. More precisely, one calls a curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ *timelike*, *lightlike*, *spacelike* or *causal* if $\dot\gamma(t)$ is timelike, lightlike, spacelike, or causal for all $t \in I$, respectively. A causal vector $v_p \in T_pM$ is called future or past directed if it is contained in the closure of the future or past directed timelike vectors at p. Then a curve γ is called future or past directed if $\dot\gamma(t)$ is causal and future or past directed at every t. By continuity we see that a causal curve is either future or past directed. In a time-oriented spacetime it cannot change the causal direction. Clearly a $\mathcal{C}^1$-curve is sufficient for this argument.

Definition 2.2.6 *Let (M, g) be a time-oriented Lorentz manifold and $p, q \in M$. The we define*

i.) *$p \ll q$ if there exists a future directed, timelike smooth curve from p to q.*

ii.) *$p \leq q$ if either $p = q$ or there exists a future directed, causal smooth curve from p to q.*

iii.) *$p < q$ if $p \leq q$ but $p \neq q$.*

Clearly the relations $\ll$ and $\leq$ are transitive. We use these relations to define the chronological and causal future and past of a point:

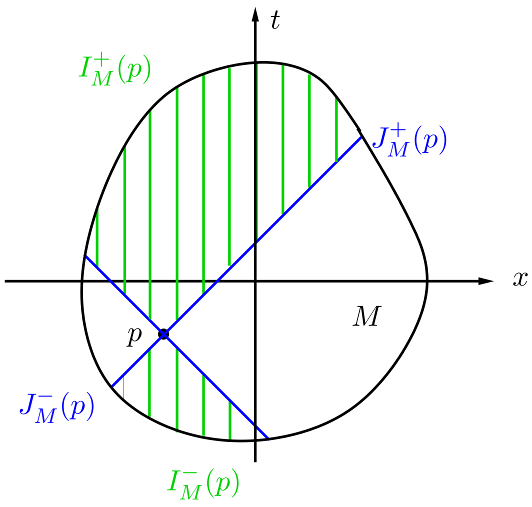

Figure 2.3: Future and past for a convex subset of Minkowski spacetime.

Definition 2.2.7 (Chronological and causal future and past) *Let (M, g) be a time-oriented Lorentz manifold and $p \in M$.*

i.) The chronological future of p is

$$I^+(p) = \{q \in M \mid p \ll q\}. \tag{2.2.18}$$

ii.) The chronological past of p is

$$I^-(p) = \{q \in M \mid q \ll p\}. \tag{2.2.19}$$

iii.) The causal future of p is

$$J^+(p) = \{q \in M \mid p \leq q\}. \tag{2.2.20}$$

iv.) The causal past of p is

$$J^-(p) = \{q \in M \mid q \leq p\}. \tag{2.2.21}$$

Sometimes we indicate the ambient spacetime M in the definitions by $I^\pm_M(p)$ and $J^\pm_M(p)$ since they will play a crucial role. The definitions of $I^\pm_M(p)$ and $J^\pm_M(p)$ reflect *global* properties of M which are not necessarily preserved under isometric embeddings. We illustrate the meaning of $I^\pm_M(p)$ and $J^\pm_M(p)$ by some examples:

Example 2.2.8 The spacetime (M, g) in Figure 2.3 and the following pictures are open subsets of the usual Minkowski spacetime $(\mathbb{R}^2, \eta)$ with future direction being "upward". Figure 2.4 shows that $I^+_M(p)$ and $J^+_M(p)$ are not just the intersections of M with $I^+_{\mathbb{R}^2}(p)$ and $J^+_{\mathbb{R}^2}(p)$, but actually smaller. Figure 2.5 illustrates that $J^+_M(p)$ needs not to be the closure of $I^+_M(p)$. In fact, $J^+_M(p)$ is not closed at all in this example.

Without proof we state the following result, see e.g. [46, Chap. 14]:

Proposition 2.2.9 *Let (M, g) be a time-oriented Lorentz manifold. Then for every $p \in M$ the chronological future and past $I^\pm_M(p)$ of p is an open subset of M.*

The intuition behind this proposition is clear and is visualized in Figure 2.6. Since the sets $I^\pm_M(p)$ are open, we can use them to define a collection of open subsets of M. In particular, we consider the intersections $I^+_M(p) \cap I^-_M(q)$ for $p, q \in M$. These subsets are sometimes called (chronological) open *diamonds* as Figure 2.7 suggests. In flat Minkowski space the sets $I^+_M(p) \cap I^-_M(q)$ are diamond-shaped.

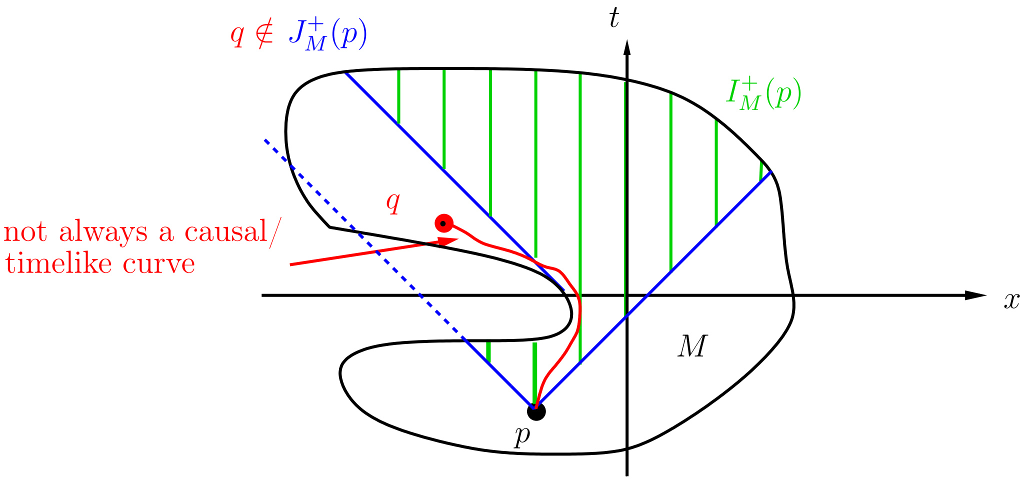

Figure 2.4: Future and past for a spacetime M with “notch”.

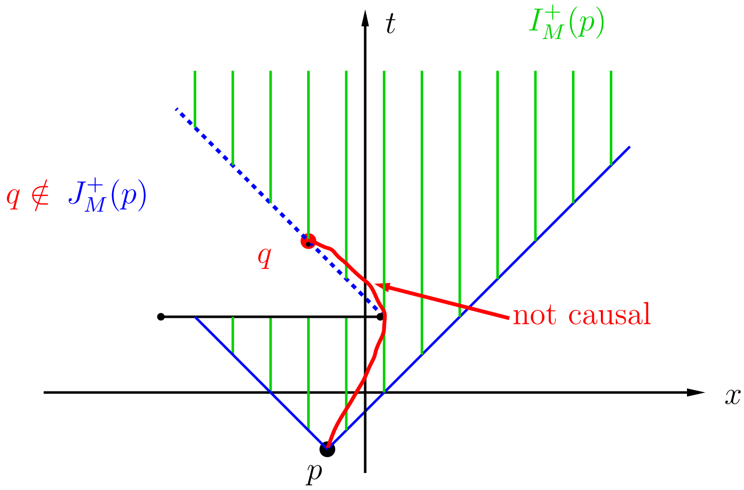

Figure 2.5: Future and past for a spacetime with an excluded line segment.

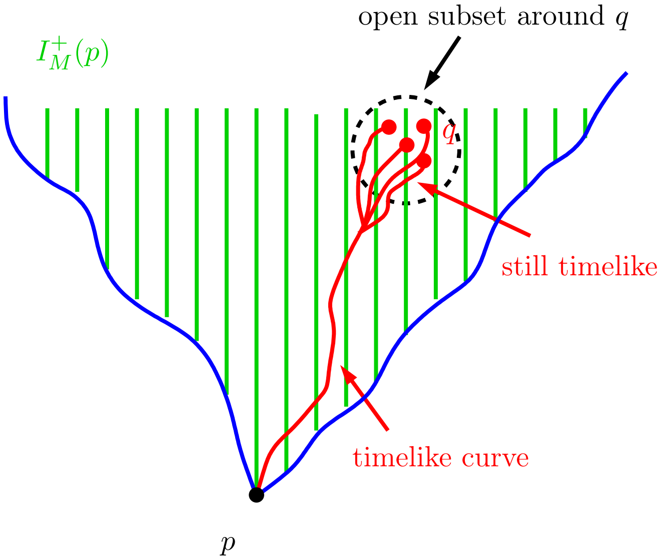

Figure 2.6: The chronological future is open.

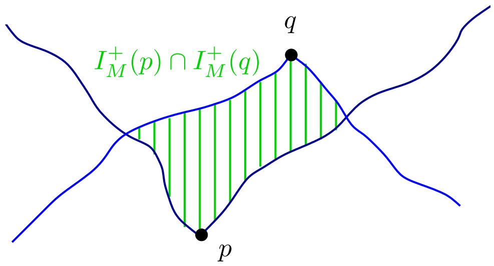

Figure 2.7: An chronological open diamond in a spacetime.

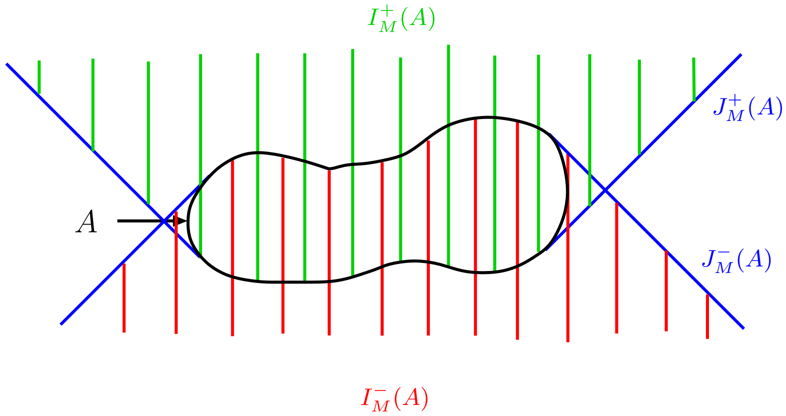

Figure 2.8: Chronological and causal future and past of A in Minkowski spacetime $(\mathbb{R}^2, \eta)$.

These open diamonds can be used to define a *new* topology on M: they form a basis of a topology sometimes called the *Alexandrov topology* of (M, g). By Proposition 2.2.9 it is coarser than the original topology. We will come back to the question whether the Alexandrov topology actually coincides with the usual one; a case which is of course physically interesting: in this case the topological structure of M is determined by the causal structure. Analogously to the chronological open diamonds, we define the diamonds

$$J_M(p, q) = J^+_M(p) \cap J^-_M(q). \tag{2.2.22}$$

Finally, we can extend Definition 2.2.7 to arbitrary subsets $A \subseteq M$. One defines the chronological future and past as well as the causal future and past of A by

$$I^\pm_M(A) = \bigcup_{p \in A} I^\pm_M(p) \tag{2.2.23}$$

and

$$J^\pm_M(A) = \bigcup_{p \in A} J^\pm_M(p), \tag{2.2.24}$$

respectively. Again, $J^\pm_M(A)$ needs not to be closed but is contained in the closure of $I^\pm_M(A)$ which is always open by Proposition 2.2.9.

Definition 2.2.10 (Future and past compactness) *Let (M, g) be a time-oriented Lorentz manifold. Then a subset $A \subseteq M$ is called future compact if $J^+_M(p) \cap A$ is compact for all $p \in M$ and past compact if $J^-_M(p) \cap A$ is compact for all $p \in M$.*

The geometric interpretation is clear and can be visualized again in Minkowski spacetime as in Figure 2.9. Clearly, A needs not be compact in the topological sense. However, if all the $J^\pm_M(p)$ are

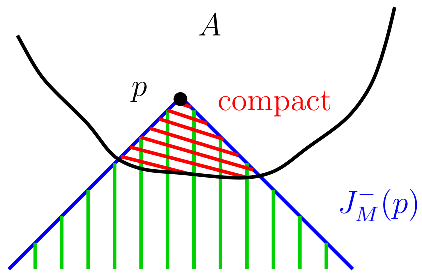

Figure 2.9: A past compact subset A in Minkowski spacetime.

closed then every compact subset $A \subseteq M$ is future and past compact.

The phenomenon in Figure 2.4 motivates the following definition:

Definition 2.2.11 (Causal compatibility) *Let (M, g) be a time-oriented Lorentz manifold and $U \subseteq M$ open. Then U is called causally compatible if for all $p \in M$ we have*

$$J_U^{\pm}(p) = J_M^{\pm}(p) \cap U. \tag{2.2.25}$$

More generally, a time-orientation preserving isometric embedding $\iota : (N, h) \hookrightarrow (M, g)$ of a time-oriented Lorentz manifold (N, h) into (M, g) is called causally compatible if $\iota(N) \subseteq M$ is causally compatible.

Remark 2.2.12 Let (M, g) be a time-oriented Lorentz manifold.

i.) $U \subseteq M$ is causally compatible if for every causal curve from $p \in U$ to $q \in U$ in M one also finds a causal curve from p to q which lies entirely in U. In Figure 2.4 this is not the case for the subset $M \subseteq \mathbb{R}^2$.

ii.) If $V \subseteq U \subseteq M$ are open subset such that $V \subseteq U$ is causally compatible in the Lorentz manifold $(U, g\big|_U)$ and U is causally compatible in M, then also $V \subseteq M$ is causally compatible.

iii.) If $U \subseteq M$ is causally compatible and $A \subseteq U$ the clearly

$$J_U^{\pm}(A) = J_M^{\pm}(A) \cap U. \tag{2.2.26}$$

iv.) Since the relation "causally compatible" is transitive with respect to inclusion, we obtain a category of n-dimensional time-oriented Lorentz manifolds $\mathsf{Lorentz}_n$ as follows: the objects will be n-dimensional time-oriented Lorentz manifolds and the morphisms $\iota : (N, h) \hookrightarrow (M, g)$ will be isometric embeddings preserving the time-orientations which are causally compatible. Even though there are usually not many morphisms between two objects in this category, it will turn out to be a very useful notion. In recent approaches to axiomatic quantum field theory on generic spacetimes this point of view becomes important, see e.g. [15, 30] and references therein.

2.2.3 Causality Conditions and Cauchy-Hypersurfaces

We continue our investigation of the causality structure of a time-oriented Lorentz manifold (M, g). We start with the following definition:

Definition 2.2.13 (Causal subsets) *Let $U \subseteq M$ be an open subset. Then U is called causal if there is a geodesically convex open subset $U' \subseteq M$ such that $U^{\mathrm{cl}} \subseteq U'$ and for any two points $p, q \in U^{\mathrm{cl}}$ the diamond $J_{U'}(p, q)$ is compact and contained in U^{cl}.*

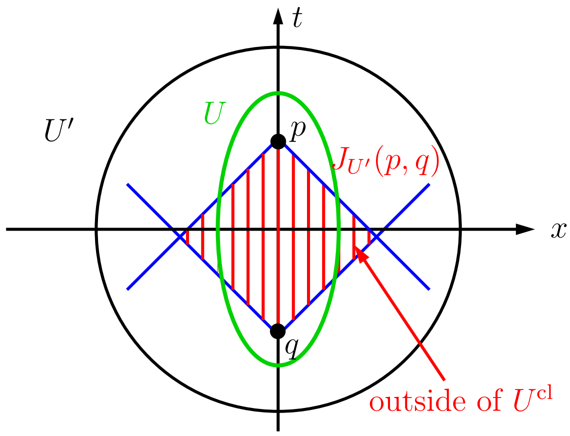

Figure 2.10: A subset U which is convex but not causal.

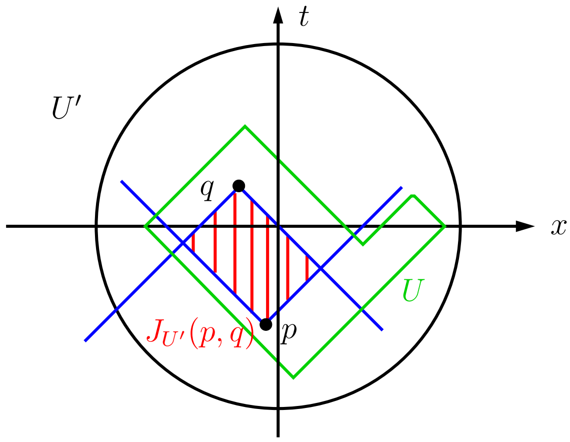

Figure 2.11: A subset U which is causal but not convex.

Figure 2.10 to Figure 2.12 show the relations between the notions of geodesically convex and causal subsets. Again, the ambient spacetime is the Minkowski spacetime $(\mathbb{R}^2, \eta)$. Since the geodesics are still the straight lines, open convex subsets $U' \subseteq \mathbb{R}^2$ in the usual sense coincide with the geodesically convex subsets. The "opposite" of a causal domain are the acausal subsets of M.

Definition 2.2.14 (Acausal and achronal subsets) *Let $A \subseteq M$ be a subset of a time-oriented Lorentz manifold. Then A is called*

i.) achronal if every timelike curve intersects A in at most one point.

ii.) acausal if every causal curve intersects A in at most one point.

Clearly, acausal subsets are achronal but the reverse is not true. Already the light cones in Minkowski spacetime are achronal but not acausal, as Figure 2.13 illustrates. Using the causal structure of (M, g) we obtain a refined notion of boundary and closure of a subset $A \subseteq M$. One defines $p \in A^{\mathrm{cl}}$ to be an *edge point* if for all open neighborhoods U of p there exists a timelike curve from $I_U^-(p)$ to $I_U^+(p)$ which does not meet A. In Figure 2.14 the point q is an edge point of the segment while p is not. In Figure 2.15, the line segment A is considered as subset of 3-dimensional Minkowski spacetime $(\mathbb{R}^3, \eta)$. Then all points in A^{cl} are edge points. Thus the notion of edge points is finer than the notion of a (topological) boundary point. We want to get as large achronal or acausal subsets as possible: they will be good candidates for Cauchy hypersurfaces where we can impose initial conditions. The following theorem states that we can expect at least $\mathcal{C}^0$-submanifolds.

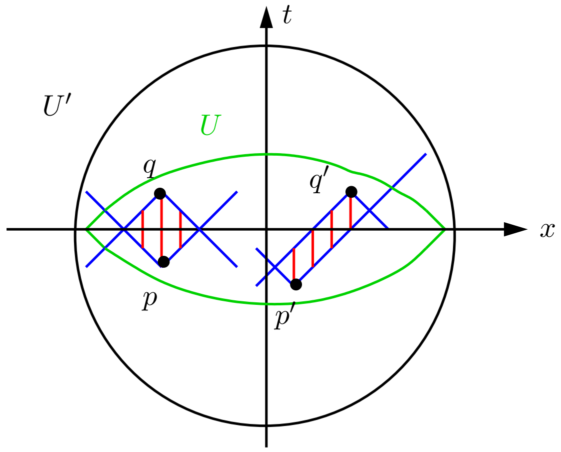

Figure 2.12: A subset U which is both convex and causal.

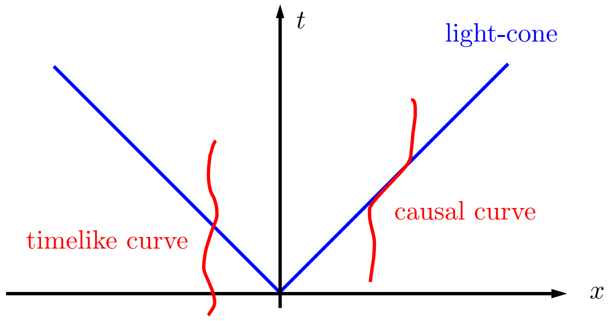

Figure 2.13: The light cones in Minkowski spacetime are achronal but not acausal.

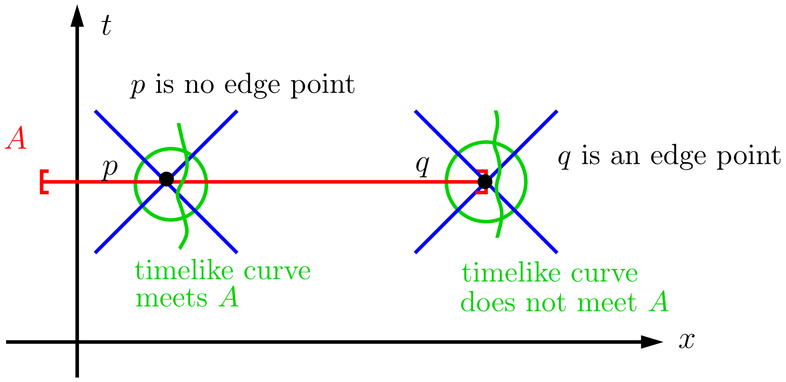

Figure 2.14: Examples of edge points of a line segment in 2-dimensional Minkowski spacetime.

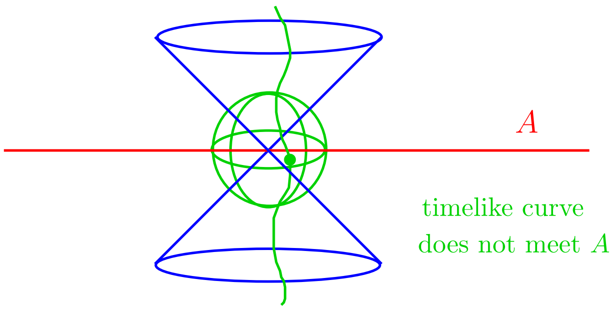

Figure 2.15: Examples of edge points of a line segment A in 3-dimensional Minkowski spacetime.

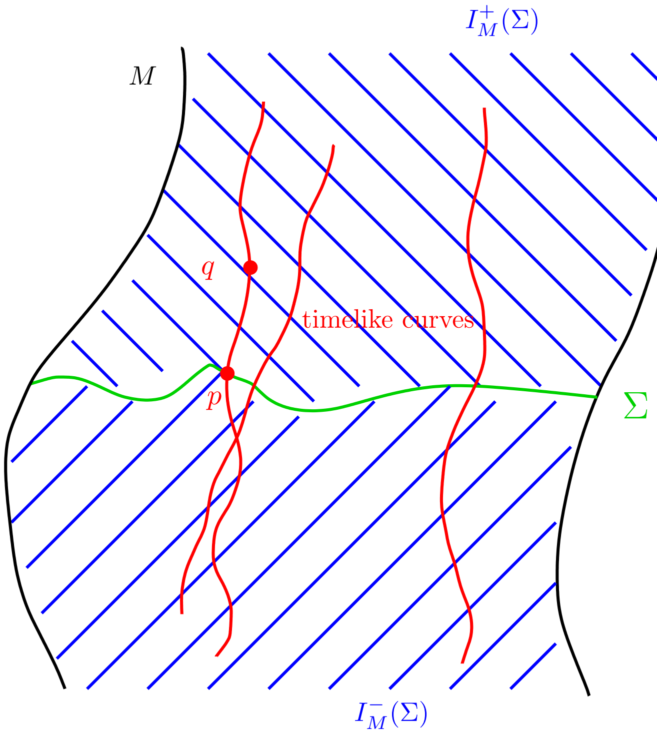

Figure 2.16: A Cauchy hypersurface Σ in a spacetime M.

Theorem 2.2.15 (Achronal hypersurfaces) *Let (M, g) be a time-oriented Lorentz manifold and $A \subseteq M$ achronal. Then A is a topological hypersurface in M if and only if A does not contain any of its edge points.*

Recall that a topological hypersurface Σ of M is a $\mathcal{C}^0$-manifold Σ together with a $\mathcal{C}^0$-embedding $i : \Sigma \hookrightarrow M$ with codimension one. In general, we can not expect more than a $\mathcal{C}^0$-hypersurface as the example of the light cone shows. For a proof we refer to [46, Prop. 24 in Chap 14]. The following corollary is a straightforward consequence of Theorem 2.2.15.

Corollary 2.2.16 *An achronal subset A is a closed topological hypersurface if and only if A is edgeless.*

The extreme case of an achronal hypersurface will be a Cauchy hypersurface. First recall that a timelike curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called *inextensible* if there is no "reparametrization" $\widetilde{\gamma} : J \subset \mathbb{R} \longrightarrow M$ of γ such that $\widetilde{\gamma}(J) \supsetneq \gamma(I)$ is strictly larger. Then we can formulate the following definition:

Definition 2.2.17 (Cauchy hypersurface) *Let (M, g) be a time-oriented Lorentz manifold. A subset $\Sigma \subseteq M$ is called a Cauchy hypersurface if every inextensible timelike curve meets Σ in exactly one point.*

Remark 2.2.18 (Cauchy hypersurface) Clearly, a Cauchy hypersurface Σ is achronal. Moreover, by the very definition of an edge point, Σ has no edge points. Thus Σ is a closed topological hypersurface by Theorem 2.2.15. Finally, if $q \in M$ there exists a timelike curve through q, say a timelike geodesic. Thus such a timelike curve has an extension which meets Σ in one point $p \in \Sigma$. It follows that either $q \ll p$, $p = q$, or $p \ll q$. Thus M is the disjoint union of the non-empty open subsets $I^{\pm}_M(\Sigma)$ and Σ. Hence Σ is the topological boundary of $I^{\pm}_M(\Sigma)$, i.e. we have the disjoint union

$$M = I^+_M(\Sigma) \mathbin{\dot{\cup}} \Sigma \mathbin{\dot{\cup}} I^-_M(\Sigma). \tag{2.2.27}$$

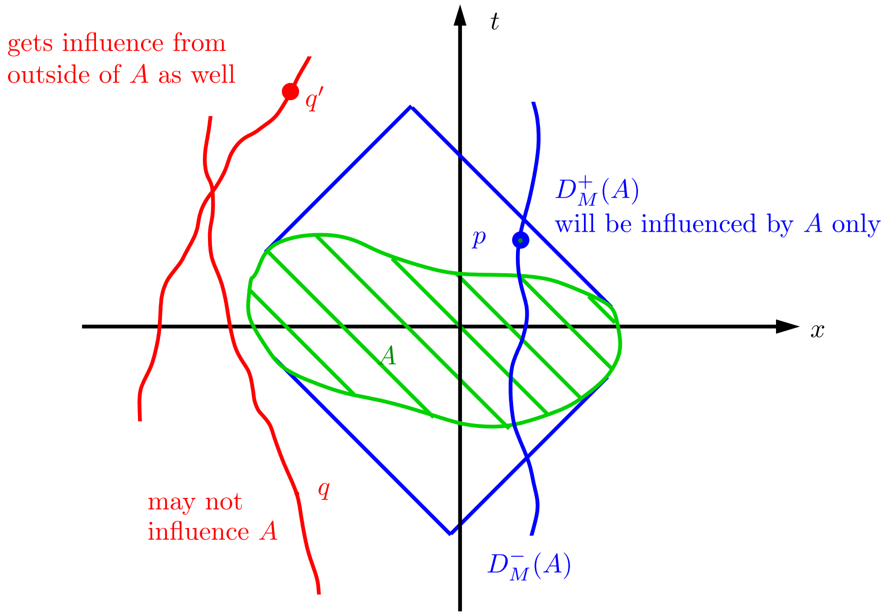

Figure 2.17: Cauchy development of a subset A in the 2-dimensional Minkowski spacetime.

Furthermore, on can show that a Cauchy hypersurface is met by every inextensible causal curve at least once, see e.g. [46, Lem. 29 in Chap. 14].

The physical interpretation of a Cauchy hypersurface is that the whole future of the spacetime, viewed from Σ is predictable in the sense that every particle or light ray being in the future $I_M^+(\Sigma)$ of Σ has passed through Σ at earlier times. Analogously, viewed from Σ, the whole past of M is already known.

For an arbitrary subset $A \subseteq M$ we can still ask which part of M is predictable from A. This motivates the following definition of the Cauchy development of A:

Definition 2.2.19 (Cauchy development) *Let $A \subseteq M$ be a subset. The future Cauchy development $D_M^+(A) \subseteq M$ of A is the set of all those points $p \in M$ for which every past-inextensible causal curve through p also meets A. Analogously, one defines the past Cauchy development $D_M^-(A)$ and we call*

$$D_M(A) = D_M^+(A) \cup D_M^-(A) \tag{2.2.28}$$

the Cauchy development of A.

Remark 2.2.20 (Cauchy development) Let $A \subseteq M$ be a subset. The physical interpretation of $D_M^+(A)$ is that $D_M^+(A)$ is predictable from A. Analogously, $D_M^-(A)$ consists of those points which certainly influence A in their future. We have $A \subseteq D_M^\pm(A)$.

Remark 2.2.21 For $A \subseteq M$ we clearly have

$$D_M^\pm(D_M^\pm(A)) = D_M^\pm(A) \tag{2.2.29}$$

and hence

$$D_M(D_M(A)) = D_M(A). \tag{2.2.30}$$

Moreover, for $A \subseteq B \subseteq M$ we have

$$D_M^\pm(A) \subseteq D_M^\pm(B) \tag{2.2.31}$$

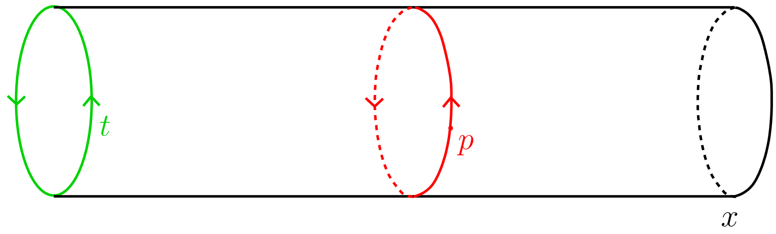

Figure 2.18: Periodic timelike geodesic on a cylinder.

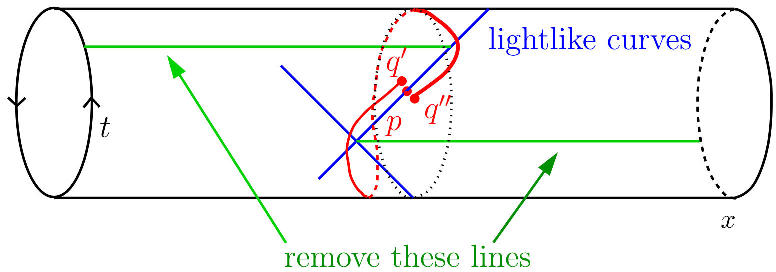

Figure 2.19: Almost periodic timelike curves on a cylinder with removed line segments.

and

$$D_M(A) \subseteq D_M(B). \tag{2.2.32}$$

Thus the three operations $D_M^{\pm}(\,\cdot\,)$ and $D_M(\,\cdot\,)$ behave similar as the topological closure $A \mapsto A^{\mathrm{cl}}$.

Remark 2.2.22 If $A \subseteq M$ is achronal then A is a Cauchy hypersurface if and only if $D_M(A) = M$. Thus for an achronal hypersurface, $D_M(A)$ can be viewed as the largest subset of M for which A is a Cauchy hypersurface. In fact, one can show that $D_M(A)$ is open for an acausal topological hypersurface, see e.g. [46, Lem. 43 in Chap. 14].

While the existence of a Cauchy hypersurface is from the physical point of view very appealing, it is by far not evident. In fact, not every time-oriented Lorentz manifold has a Cauchy hypersurface. Quite contrary to the existence of a Cauchy hypersurface is the following example:

Example 2.2.23 We consider the cylinder $M = \mathbb{S}^1 \times \mathbb{R}$ with Lorenz metric $\mathrm{d}\,t^2 - \mathrm{d}\,x^2$ where the time variable is in $\mathbb{S}^1$-direction. The global vector field $\frac{\partial}{\partial t}$ is timelike and defines the time-orientation. Then through every point $p \in M$ there is a timelike geodesic which is *periodic*. Thus there cannot be any Cauchy hypersurface. Figure 2.18 illustrates this situation. A slight variation is obtained by removing two lines in Figure 2.19. Then there are no longer closed timelike curves. However, starting arbitrarily close to the point p at q' there is a timelike curve (no longer geodesic of course) which ends again arbitrarily close to p in q''.

Both situations are of course very bad for physical interpretations: in the first case one could travel into ones own past with all the funny paradoxa appearing. In the second case one could do so at least approximately. This motivates the following definition:

Definition 2.2.24 (Causality condition) *Let (M, g) be a time-oriented Lorentz manifold.*

i.) M is called causal if there are no closed causal curves in M.

ii.) An open subset $U \subseteq M$ is called causally convex if no causal curve intersects with U in a disconnected subset of U.

iii.) *M is called strongly causal at $p \in M$ if every open neighborhood of p contains an open causally convex neighborhood.*

iv.) *M is called strongly causal if M is strongly causal at every point $p \in M$.*

Without proof we mention the following interpretation of the strong causality condition, see e.g. [6, Prop. 3.11]:

Theorem 2.2.25 (Kronheimer, Penrose) *A time-oriented Lorentz manifold (M, g) is strongly causal if and only if the Alexandrov topology coincides with the original topology of M.*

The last ingredient we need is the following: In Example 2.2.8 we have seen examples of time-oriented spacetimes where the sets $J_M^{\pm}(p)$ are not closed and hence not the closure of the $I_M^{\pm}(p)$. To cure this effect one demands that the diamonds $J_M(p, q) = J_M^+(p) \cap J_M^-(q)$ are *compact* for all $p, q \in M$. Here one has the following nice consequence, see e.g. [45]:

Proposition 2.2.26 *Assume that $J_M(p, q) = J_M^+(p) \cap J_M^-(q)$ is compact for all $p, q \in M$ on a time-oriented spacetime (M, g). Then the causal past and future $J_M^{\pm}(p)$ of any point $p \in M$ are closed subsets of M.*

Remark 2.2.27 In this section we only introduced some of the characteristic features of a time-oriented Lorentz manifold. There are many other notions of causality with increasing strength. Remarkably, many fundamental insights have been obtained only recently. We refer to the very nice review article of Minguzzi and Sánchez [45] for an additional discussion.

2.2.4 Globally Hyperbolic Spacetimes

We are now in the position to define a globally hyperbolic spacetime according to [10]:

Definition 2.2.28 (Globally hyperbolic spacetime) *A time-oriented Lorentz manifold (M, g) is called globally hyperbolic if*

i.) *(M, g) is causal,*

ii.) *all diamonds $J_M(p, q)$ are compact for $p, q \in M$.*

Note that in earlier works the notion of globally hyperbolic spacetimes involved a strongly causal (M, g) instead of just a causal one. It was observed only recently that these two notions actually coincide, see [10].

The relevance of this condition comes from the relation to Cauchy hypersurfaces. To this end, we first introduce the notion of a time function:

Definition 2.2.29 (Time function) *Let (M, g) be a time-oriented Lorentz manifold and $t : M \longrightarrow \mathbb{R}$ a continuous function. Then t is called a*

i.) *time function if t is strictly increasing along all future directed causal curves.*

ii.) *temporal function if t is smooth and $\operatorname{grad} t$ is future directed and timelike.*

iii.) *Cauchy time function if t is a time function whose level sets are Cauchy hypersurfaces.*

iv.) *Cauchy temporal function if t is a temporal function such that all level sets are Cauchy hypersurfaces.*

Remark 2.2.30 (Time functions)

i.) With the other sign convention for the metric a temporal function has *past* directed gradient.

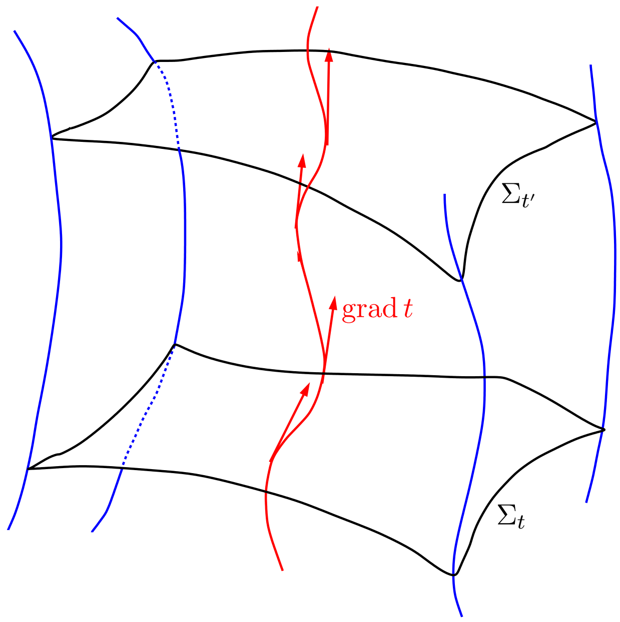

Figure 2.20: The gradient flow of a Cauchy temporal function.

ii.) If t is temporal, its level sets are (if nonempty) embedded smooth submanifolds since the gradient is non-zero everywhere and hence every value is a regular value. Note that they do not need to be Cauchy hypersurfaces at all: In fact, remove a single point from Minkowski spacetime then the usual time function is temporal but there is no Cauchy hypersurface at all.

iii.) The gradient flow of t gives a diffeomorphism between the different level sets of t. Since every timelike curve intersects a Cauchy hypersurface precisely once we see that this gives a diffeomorphism

$$M \simeq t(M) \times \Sigma_{t_0}, \tag{2.2.33}$$

and all Cauchy hypersurfaces are diffeomorphic to a given reference Cauchy hypersurface Σ_{t_0}. This gives a very strong implication on the structure of M.

iv.) By rescaling t we can always assume that the image of t is the whole real line $\mathbb{R}$. This follows as the image of t is necessarily open and connected (for connected M).

The following celebrated and non-trivial theorem brings together the notions of globally hyperbolic spacetimes and the existence of Cauchy temporal functions.

Theorem 2.2.31 *Let (M, g) be a connected time-oriented Lorentz manifold. Then the following statements are equivalent:*

i.) (M, g) is globally hyperbolic.

ii.) There exists a topological Cauchy hypersurface.

iii.) There exists a smooth spacelike Cauchy hypersurface.

In this case there even exists a Cauchy temporal function t and (M, g) is isometrically diffeomorphic to the product manifold

$$\mathbb{R} \times \Sigma \quad \textit{with metric} \quad g = \beta \, \mathrm{d} t^2 - g_t, \tag{2.2.34}$$

where $\beta \in \mathcal{C}^\infty(\mathbb{R} \times \Sigma)$ is positive and $g_t \in \Gamma^\infty(\mathrm{S}^2 T^* \Sigma)$ is a Riemannian metric on Σ depending smoothly on t. Moreover, each level set

$$\Sigma_t = \{(t, \sigma) \in \mathbb{R} \times \Sigma\} \subseteq M \tag{2.2.35}$$

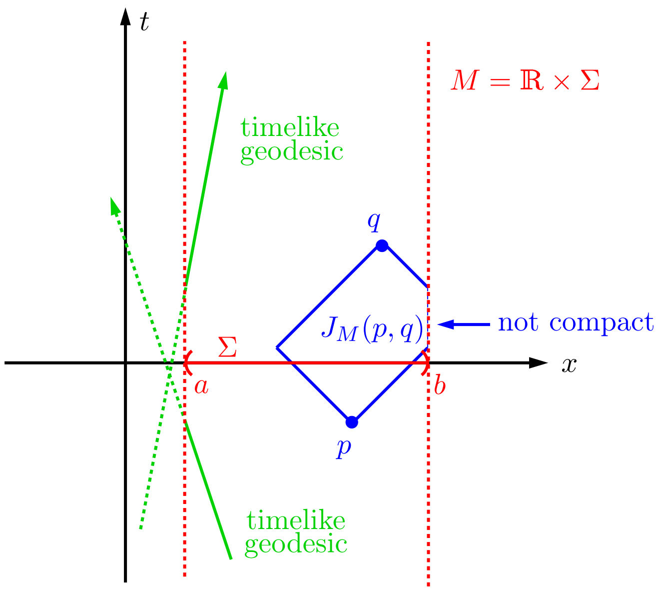

Figure 2.21: The Minkowski strip

of the temporal function t is a smooth spacelike Cauchy hypersurface.

Remark 2.2.32 The equivalence of *i.)* and *ii.)* is the celebrated theorem of Geroch [25]. The enhancement to the smooth setting is due to Bernal and Sánchez [7–10]. Conversely, having a metric of the form $\beta \, \mathrm{d}\, t^2 - g_t$ on $\mathbb{R} \times \Sigma$ it is trivial to see that all level sets Σ_t are spacelike hypersurfaces diffeomorphic to Σ. Note however, that the form (2.2.34) alone does not guarantee that the Σ_t are Cauchy hypersurfaces.

Example 2.2.33 (Minkowski strip) We consider $\Sigma = (a, b)$ an open interval with $-\infty < a < b < +\infty$ and $M = \mathbb{R} \times \Sigma \subseteq \mathbb{R}^2$ as open subset of Minkowski space. Then Σ_t is not a Cauchy hypersurface for any t. This is clear from the observation that there are inextensible timelike geodesics not passing through Σ_t. In fact, M is not globally hyperbolic at all: while M is causal (and even strongly causal) it fails to satisfy the second condition of global hyperbolicity: there are diamonds $J_M(p, q)$ which are not compact, see Figure 2.21. Thus by Theorem 2.2.31 there cannot exist any Cauchy hypersurface. Nevertheless, the metric is of the very simple form

$$g = \mathrm{d}\, t^2 - \mathrm{d}\, x^2. \tag{2.2.36}$$

The problem with this example comes from the geometric feature of the open interval $\Sigma = (a, b) \subseteq \mathbb{R}$ of being "too short". The following proposition gives now a sufficient condition such that this can not happen:

Proposition 2.2.34 *Let $M = \mathbb{R} \times \Sigma$ with Lorentz metric*

$$g = \frac{1}{2} \mathrm{d}\, t \vee \mathrm{d}\, t - f(t) g_\Sigma, \tag{2.2.37}$$

where g_Σ is a Riemannian metric on Σ and $f \in \mathcal{C}^\infty(\mathbb{R})$ is positive. The time-orientation is such that $\frac{\partial}{\partial t}$ is future directed. Then (M, g) is globally hyperbolic if and only if g_Σ is geodesically complete.

For a proof see e.g. [4, Lem. A.5.14]. Many of the physically interesting examples of spacetimes from general relativity can be brought to the form (2.2.37) whence the above Proposition can be used to discuss the global hyperbolicity of (M, g).

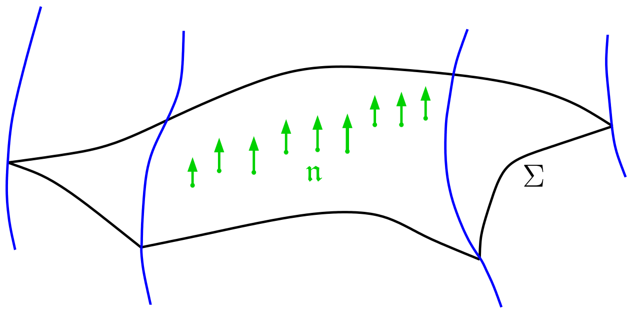

Figure 2.22: The future directed normal vector field of a Cauchy hypersurface Σ.

For later use we mention the following result which still enhances Theorem 2.2.31, see [9, Thm. 1.2].

Theorem 2.2.35 *Let (M, g) be globally hyperbolic and let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface. Then there exists a Cauchy temporal function t such that the $t = 0$ Cauchy hypersurface coincides with Σ.*

2.3 The Cauchy Problem and Green Functions

Having the notion of a Cauchy hypersurface we are now in the position to formulate the Cauchy problem for a normally hyperbolic differential operator. Here we still be rather informal only fixing the principal ideas. The precise formulation of the Cauchy problem will be given and discussed in detail in Section 4.2.

Thus let (M, g) be globally hyperbolic and $\Sigma \subseteq M$ a smooth Cauchy hypersurface which we assume to be spacelike throughout the following. At a given point $p \in \Sigma \subseteq M$ the tangent plane $T_p\Sigma \subseteq T_pM$ is spacelike whence there exists a unique vector $\mathfrak{n}_p \in T_pM$ which satisfies

$$g_p(\mathfrak{n}_p, T_p\Sigma) = 0, \tag{2.3.1}$$

$$g_p(\mathfrak{n}_p, \mathfrak{n}_p) = 1, \tag{2.3.2}$$

$$\mathfrak{n}_p \text{ is future directed.} \tag{2.3.3}$$

This vector is called the *future directed normal vector* of Σ at p. Taking all points $p \in \Sigma$ we obtain the future directed normal vector field of Σ, i.e. the vector field

$$\mathfrak{n} \in \Gamma^\infty\left(TM\big|_\Sigma\right), \tag{2.3.4}$$

such that (2.3.1), (2.3.2), and (2.3.3) hold for every $p \in \Sigma$. Since Σ is a smooth submanifold, $\mathfrak{n}$ is smooth itself. We consider now a normally hyperbolic differential operator $D \in \mathrm{DiffOp}(E)$ on some vector bundle $E \longrightarrow M$. Then this operator gives the homogeneous wave equation

$$Du = 0, \tag{2.3.5}$$

or more generally

$$Du = v, \tag{2.3.6}$$

where $v \in \Gamma^\infty(E)$ is a given *inhomogeneity* and $u \in \Gamma^\infty(E)$ is the field we are looking for. Having specified the inhomogeneity which physically corresponds to a *source term*, we can try to find a solution u which has specified initial values and initial velocities on Σ. More precisely, we want

$$u\Big|_\Sigma = u_0 \in \Gamma^\infty(E\big|_\Sigma) \tag{2.3.7}$$

and

$$\nabla^E_{\mathfrak{n}} u\Big|_{\Sigma} = \dot{u}_0 \in \Gamma^\infty(E\big|_{\Sigma}) \tag{2.3.8}$$

with a priori given u_0 and $\dot{u}_0$. The hope is that this *Cauchy problem* has a unique solution, probably after considering compactly supported v, u_0, and $\dot{u}_0$. Moreover, one hopes that the solution u depends in a reasonably continuous way on the initial values u_0 and $\dot{u}_0$ and perhaps also on v.

More generally, one can try to find solutions $u \in \Gamma^{-\infty}(E)$ for distributional initial values $u_0, \dot{u}_0 \in \Gamma^{-\infty}(E\big|_{\Sigma})$ and distributional $v \in \Gamma^{-\infty}(E)$. In general, however, we meet difficulties with this Cauchy problem. Namely, we can not just restrict a distribution u to a submanifold Σ in order to make sense out of (2.3.7) and (2.3.8): this is only possible if u behaves nicely enough around Σ. Clearly, the restriction is not problematic as soon as u is at least $\mathcal{C}^1$.

As a last comment we note that the Cauchy problem still makes sense if Σ is just a spacelike hypersurface which is not necessarily a Cauchy hypersurface. In this case we still can hope to get a solution to the Cauchy problem but we have to expect non-uniqueness for obvious reasons.

The main idea to attack this problem is to construct particular distributional solutions, the *fundamental solutions* $F_p \in \Gamma^{-\infty}(E) \otimes E^*_p \otimes |\Lambda^{\mathrm{top}}| T^*_p M$ such that

$$DF_p = \delta_p, \tag{2.3.9}$$

where δ_p is the δ-distribution at $p \in M$ viewed as $E^*_p \otimes |\Lambda^{\mathrm{top}}| T^*_p M$-valued generalized section of E, i.e. for a test section $\mu \in \Gamma^\infty_0(E^* \otimes |\Lambda^{\mathrm{top}}| T^* M)$ we have

$$\delta_p(\mu) = \mu(p) \in E^*_p \otimes |\Lambda^{\mathrm{top}}| T^*_p M. \tag{2.3.10}$$

Definition 2.3.1 (Green function) *Let $p \in M$. A generalized section F_p of E which satisfies* (2.3.9) *is called fundamental solution of D at p. If a fundamental solution $F^{\pm}_p$ in addition satisfies*

$$\operatorname{supp} F^{\pm}_p \subseteq J^{\pm}_M(p), \tag{2.3.11}$$

then $F^{\pm}_p$ is called advanced or retarded Green function of D at p, respectively.

Remark 2.3.2 (Green function) Note that the notion of a fundamental solution makes sense for every differential operator on any manifold. The notion of advanced and retarded Green functions makes sense for any differential operator on a time-oriented Lorentz manifold, see also Figure 2.23.

The remaining part of these notes are now devoted to the study of existence and uniqueness of Green functions $F^{\pm}_p$. Moreover, we have to relate the Green functions to the Cauchy problem for D. Here it will be important not only to have a Green function F_p for every $p \in M$. We also will need a reasonable dependence of F_p on p.

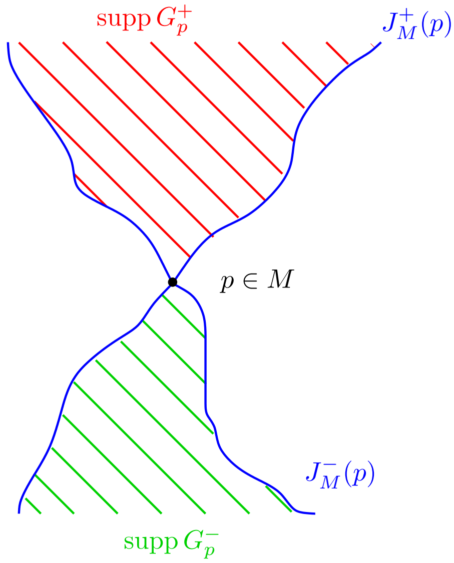

Figure 2.23: The support of a Green function at a point $p \in M$.

Chapter 3

The Local Theory of Wave Equations

The purpose of this chapter is to discuss the existence and uniqueness of fundamental solutions for the wave equation determined by a normally hyperbolic differential operator at least on small enough open subsets of M. Thus the global structure of M does not yet play a role in this chapter. Nevertheless, already locally the geometry enters in form of non-trivial curvature terms and resulting non-trivial parallel transports. Thus already at this stage we will be beyond the usual flat situation of the wave equation in $\mathbb{R}^{2n}$.

We basically follow [4] and construct the fundamental solution first in the flat case of Minkowski spacetime. Here we use the approach of Riesz [49] by specifying the fundamental solutions using holomorphic function techniques. Then one constructs a formal solution on a domain as a series with certain coefficients, the *Hadamard coefficients*. This solution will be a series with no good control of convergence and in fact, no convergence in general. Thus an additional step is needed to find the "true" fundamental solutions. To this end certain cut-off parameters are introduced yielding a convergent series which is however no longer a fundamental solution but only a parametrix. With some convolution tricks this can be cured in the last step. The fundamental solution will have nice causal properties allowing to find solutions to the inhomogeneous wave equation with good causal properties as well.

3.1 The d'Alembert Operator on Minkowski Spacetime

As warming up we consider the most simple case of a normally hyperbolic differential operator, the *d'Alembert operator* on flat Minkowski spacetime.

3.1.1 The Riesz Distributions

We shall not only construct the fundamental solutions of the d'Alembert operator

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta \tag{3.1.1}$$

in n dimensions but a local family of distributions associated to $\Box$. Sometimes we will set $t = x^0$ and $\vec{x} = (x^1, \ldots, x^{n-1})$ for abbreviation. In more physical terms, we set the speed of light c to 1 by choosing appropriate units. Here we follow essentially the approach of Riesz [49], see [4, Sect. 1.2] for a modern presentation of this approach.

Using the Minkowski metric η we have the following function, also denoted by η,

$$\eta(x) = \eta(x, x) \tag{3.1.2}$$

on $\mathbb{R}^n$. Clearly $\eta \in \mathrm{Pol}^2(\mathbb{R}^2)$ is a homogeneous quadratic polynomial. Explicitly, in the standard coordinates we have

$$\eta(x^0, \ldots, x^{n-1}) = (x^0)^2 - \sum_{i=1}^{n-1} (x^i)^2 = t^2 - (\vec{x})^2. \tag{3.1.3}$$

We consider the following family of continuous functions on Minkowski spacetime:

Definition 3.1.1 *Let $\alpha \in \mathbb{C}$ have $\mathsf{Re}(\alpha) > n$. Then one defines*

$$R^{\pm}(\alpha)(x) = \begin{cases} c(\alpha, n)\eta(x)^{\frac{\alpha-n}{2}} & \text{for } \alpha \in I^{\pm}(0) \\ 0 & \text{else,} \end{cases} \tag{3.1.4}$$

where the coefficient is

$$c(\alpha, n) = \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1)}. \tag{3.1.5}$$

Remark 3.1.2 (Gamma function) The *Gamma function*

$$\Gamma : \mathbb{C}\backslash\{0, -1, -2, \ldots\} \longrightarrow \mathbb{C} \tag{3.1.6}$$

is known to be a holomorphic function with simple poles at $-n$ for $n \in \mathbb{N}_0$. One has the following properties:

i.) The residue at $-n \in \mathbb{N}_0$ is given by

$$\mathrm{res}_{-n}\Gamma = \frac{(-1)^n}{n!}. \tag{3.1.7}$$

ii.) For $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ one has the functional equation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{with} \quad \Gamma(1) = 1. \tag{3.1.8}$$

iii.) For $n \in \mathbb{N}_0$ one obtains from (3.1.8) immediately

$$\Gamma(n+1) = n!. \tag{3.1.9}$$

iv.) For $\mathsf{Re}(z) > 0$ one has Euler's integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}\,t \tag{3.1.10}$$

in the sense of an improper Riemann integral.

v.) For all $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ one has *Legendre's duplication formula*

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{3.1.11}$$

For more details and proofs of the above properties of Γ we refer to any textbook on complex function theory like e.g. [48, Chap. 2]. The graph of the Gamma function along the real axis can be seen in Figure 3.1.

Since the Gamma function Γ has no zeros we conclude that the prefactor $c(\alpha, n)$ is holomorphic for all $\alpha \in \mathbb{C}$: indeed, for those $\alpha \in \mathbb{C}$ where $\Gamma(\frac{\alpha}{2})$ or $\Gamma(\frac{\alpha-n}{2}+1)$ has a pole the inverse is well-defined and has a zero of the same (first) order as the pole of the Γ function. This happens for

$$\frac{\alpha}{2} = 0, -1, -2, \ldots \quad \text{and} \quad \frac{\alpha-n}{2} + 1 = 0, -1, -2, \ldots$$

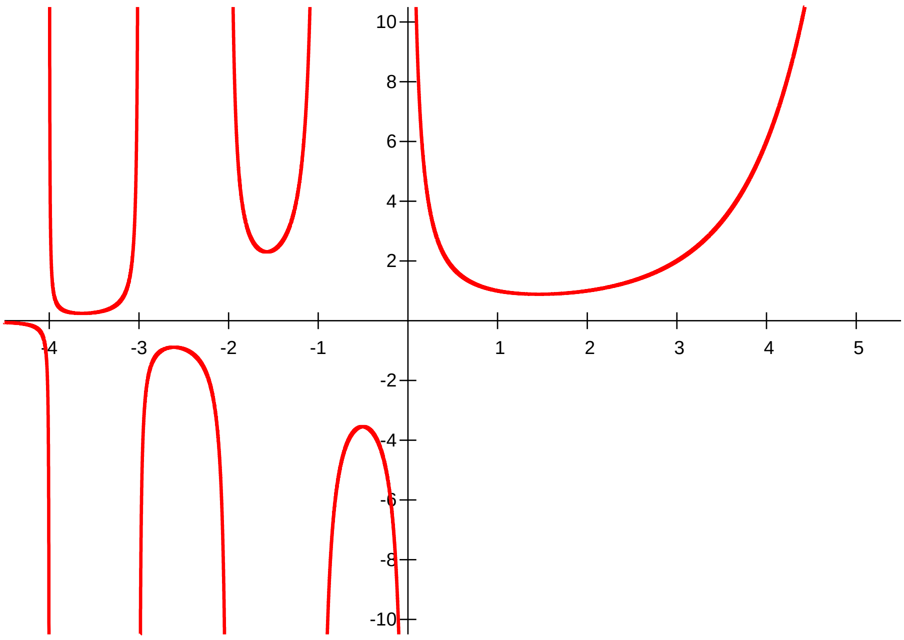

Figure 3.1: The Gamma function along the real axis

Thus we conclude

$$c(\alpha, n) = 0 \quad \text{iff} \quad \alpha \in \left\{-2k \,\middle|\, k \in \mathbb{N}_0\right\} \cup \left\{n - 2k \,\middle|\, k \in \mathbb{N}_0\right\}, \tag{3.1.12}$$

since the nominator has clearly no zeros. For α not being in the above special set but still with $\mathsf{Re}(\alpha) > n$, the function $R^{\pm}(\alpha)$ is continuous but not smooth on $\mathbb{R}^n$:

Lemma 3.1.3 *For* $\mathsf{Re}(\alpha) > n$ *the function* $R^{\pm}(\alpha)$ *is continuous on* $\mathbb{R}^n$*. It is smooth in* $I^{\pm}(0)$ *and in* $\mathbb{R}^n \setminus J^{\pm}(0)$*.*

Proof. The function $x \mapsto c(\alpha, n)\eta(x)^{\frac{\alpha-n}{2}}$ is clearly smooth for $x \in I^{\pm}(0)$ since here $\eta(x) > 0$. Conversely, on the open subset $\mathbb{R}^n \setminus J^{\pm}(0)$ the function $R^{\pm}(\alpha)$ is zero and hence smooth, too. The continuity follows as $\eta(x) \longrightarrow 0$ for $x \in I^{\pm}(0)$ with $x \longrightarrow \partial I^{\pm}(0)$ and $\alpha > n$ guarantees that the function $0 \leq \xi \mapsto \xi^{\frac{\alpha-n}{2}}$ is at least continuous at 0. □

The next lemma clarifies the behaviour under Lorentz transformations.

Lemma 3.1.4 *Let* $\Lambda \in \mathrm{L}^{\uparrow}(1, n-1)$ *be an orthochronous Lorentz transformation and* $\mathsf{Re}(\alpha) > n$*. Then*

$$\Lambda^* R^{\pm}(\alpha) = R^{\pm}(\alpha). \tag{3.1.13}$$

If $\mathrm{T} \in \mathrm{L}(1, n-1)$ *is the time-reversal* $x^0 \mapsto -x^0$ *then*

$$\mathrm{T}^* R^{\pm}(\alpha) = R^{\mp}(\alpha). \tag{3.1.14}$$

Proof. If $\Lambda \in \mathrm{L}(1, n-1)$ is an arbitrary Lorentz transformation then by the very definition of $\mathrm{L}(1, n-1)$ we have

$$(\Lambda^*\eta)(x) = \eta(\Lambda x) = \eta(\Lambda x, \Lambda x) = \eta(x, x) = \eta(x),$$

and thus $\Lambda^*\eta = \eta$. But then (3.1.13) and (3.1.14) are obvious since the light cones $J^{\pm}(0)$ are mapped to $J^{\pm}(0)$ and to $J^{\mp}(0)$ under $\Lambda \in \mathrm{L}^{\uparrow}(1, n-1)$ and under T, respectively. □

In particular, it would be sufficient to consider $R^+(\alpha)$ alone since we can recover every information about $R^-(\alpha)$ from $R^+(\alpha)$ via (3.1.14).

Since $R^\pm(\alpha) \in \mathcal{C}^0(\mathbb{R}^n)$ we can consider $R^\pm(\alpha)$ also as a distribution (of order zero) via the usual identification, i.e.

$$R^\pm(\alpha) : \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x) R^\pm(\alpha)(x) \, \mathrm{d}^n x \tag{3.1.15}$$

for test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Here and in the following we use the Lebesgue measure $\mathrm{d}^n x$ for integration. Note that this coincides with the Lorentz density induced by η.

Lemma 3.1.5 *Let* $\mathsf{Re}(\alpha) > n$.

i.) For every $x \in \mathbb{R}^n$ *the function*

$$\alpha \mapsto R^\pm(\alpha)(x) \tag{3.1.16}$$

is holomorphic.

ii.) For every test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *the function*

$$\alpha \mapsto R^\pm(\alpha)(\varphi) \tag{3.1.17}$$

is holomorphic.

Proof. The first part is clear as the Gamma function and hence the coefficient $c(\alpha, n)$ is holomorphic. Moreover, for $x \in J^\pm(0)$ the map $\alpha \mapsto \eta(x)^{\frac{\alpha-n}{2}}$ is holomorphic. However, this pointwise holomorphy of $R^\pm(\alpha)$ is not the relevant feature for the following. Instead, we need the second part. To prove this, we consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Then

$$R^\pm(\alpha)(\varphi) = \int_{\mathbb{R}^n} \varphi(x) R^\pm(\alpha)(x) \, \mathrm{d}^n x = \int_{\operatorname{supp} \varphi} \varphi(x) R^\pm(\alpha)(x) \, \mathrm{d}^n x.$$

Since $\operatorname{supp} \varphi$ is compact we can exchange the orders of integration for every closed triangle path Δ in $\{\alpha \in \mathbb{C} \mid \mathsf{Re}(\alpha) > n\}$ by Fubini's theorem. Thus

$$\int_\Delta R^\pm(\alpha)(\varphi) \, \mathrm{d}\alpha = \int_\Delta \int_{\operatorname{supp} \varphi} \varphi(x) R^\pm(\alpha)(x) \, \mathrm{d}^n x \, \mathrm{d}\alpha = \int_{\operatorname{supp} \varphi} \varphi(x) \int_\Delta R^\pm(\alpha)(x) \, \mathrm{d}\alpha \, \mathrm{d}^n x = 0,$$

since $R^\pm(\alpha)(x)$ is holomorphic for every $x \in \mathbb{R}^n$. It follows by Morera's theorem that (3.1.17) is holomorphic, too. □

In this sense we have a holomorphic map

$$\{\alpha \in \mathbb{C} \mid \mathsf{Re}(\alpha) > n\} \ni \alpha \mapsto R^\pm(\alpha) \in \mathcal{D}'(\mathbb{R}^n) \tag{3.1.18}$$

with values in the distributions. The key idea is now to investigate (3.1.18) in detail to show that, as a holomorphic map, it has a unique extension to the whole complex plane $\mathbb{C}$. To this end we need the following technical lemma:

Lemma 3.1.6 *In the sense of continuous functions we have:*

i.) For $\mathsf{Re}(\alpha) > n$ *we have*

$$\eta R^\pm(\alpha) = \alpha(\alpha - n + 2) R^\pm(\alpha + 2).$$

ii.) For $\mathsf{Re}(\alpha) > n + 2k$ *the function* $R^\pm(\alpha)$ *is* $\mathcal{C}^k$ *and we have*

$$\frac{\partial}{\partial x^i} R^\pm(\alpha) = \frac{1}{\alpha - 2} R^\pm(\alpha - 2) \eta_{ij} x^j. \tag{3.1.19}$$

iii.) For $\mathsf{Re}(\alpha) > n$ *we have*

$$\operatorname{grad}\eta \cdot R^{\pm}(\alpha) = 2\alpha \operatorname{grad} R^{\pm}(\alpha+2). \tag{3.1.20}$$

iv.) For $\mathsf{Re}(\alpha) > n+2$ *we have*

$$\Box R^{\pm}(\alpha+2) = R^{\pm}(\alpha). \tag{3.1.21}$$

Proof. The first part is a simple calculation. We have

$$\begin{aligned}
\alpha(\alpha+2-n)R^{\pm}(\alpha+2) &= \alpha(\alpha+2-n)c(\alpha+2,n)\eta^{\frac{\alpha+2-n}{2}} \\
&= \alpha(\alpha+2-n)c(\alpha+2,n)\eta^{\frac{\alpha-n}{2}}\eta \\
&= \frac{\alpha(\alpha+2-n)c(\alpha+2,n)}{c(\alpha,n)}\eta R^{\pm}(\alpha),
\end{aligned}$$

and

$$\frac{c(\alpha+2,n)}{c(\alpha,n)} = \frac{2^{1-2-\alpha}\pi^{\frac{2-n}{2}}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1)}{\Gamma(\frac{\alpha+2}{2})\Gamma(\frac{\alpha+2-n}{2}+1)2^{1-\alpha}\pi^{\frac{2-n}{2}}} = \frac{2^{-2}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1)}{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})\frac{\alpha+2-n}{2}\Gamma(\frac{\alpha-n}{2}+1)} = \frac{1}{\alpha(\alpha+2-n)}. \tag{$*$}$$

For the second part we recall that in $I^{\pm}(0)$ the function $R^{\pm}(\alpha)$ is smooth as well as in $\mathbb{R}^n \setminus J^{\pm}(0)$. On the latter, the function and hence all its derivatives are zero. In $I^{\pm}(0)$ we compute

$$\begin{aligned}
\frac{\partial}{\partial x^i}R^{\pm}(\alpha)\Big|_{I^{\pm}(0)} &= c(\alpha,n)\frac{\partial}{\partial x^i}\eta(x)^{\frac{\alpha-n}{2}} = c(\alpha,n)\frac{\alpha-n}{2}\eta(x)^{\frac{\alpha-n}{2}-1}\frac{\partial}{\partial x^i}\eta(x) \\
&= c(\alpha,n)\frac{\alpha-n}{2}\eta(x)^{\frac{\alpha-2-n}{2}}2\eta_{ij}x^j = c(\alpha,n)(\alpha-n)\eta^{\frac{\alpha-2-n}{2}}\eta_{ij}x^j \\
&= \frac{c(\alpha,n)}{c(\alpha-2,n)}(\alpha-n)R^{\pm}(\alpha-2)\eta_{ij}x^j \overset{(*)}{=} \frac{1}{(\alpha-2)(\alpha-n)}(\alpha-n)R^{\pm}(\alpha-2)\eta_{ij}x^j \\
&= \frac{1}{(\alpha-2)}R^{\pm}(\alpha-2)\eta_{ij}x^j.
\end{aligned}$$

Now if $\mathsf{Re}(\alpha) > n+2k$ then $\mathsf{Re}(\alpha-2) > n+2k-2$ is still larger than n for positive $k \in \mathbb{N}$. Thus the partial derivative $\frac{\partial}{\partial x^i}R^{\pm}(\alpha)\big|_{I^{\pm}(0)}$ is the continuous function $\frac{1}{(\alpha-2)}R^{\pm}(\alpha-2)\eta_{ij}x^j$ in $I^{\pm}(0)$ which continuously extends to $\mathbb{R}^n$ by setting it zero outside of $I^{\pm}(0)$. Indeed, since $R^{\pm}(\alpha-2)$ has this as continuous extension, we obtain a continuous extension of $\frac{\partial}{\partial x^i}R^{\pm}(\alpha)\big|_{I^{\pm}(0)}$. But this matches the partial derivative of $R^{\pm}(\alpha)$ outside of $J^{\pm}(0)$. Thus we obtain a continuous partial derivative $\frac{\partial}{\partial x^i}R^{\pm}(\alpha)$ on all of Minkowski space $\mathbb{R}^n$ which shows that $R^{\pm}(\alpha)$ is at least $\mathcal{C}^1$. By induction we can proceed as long as $\alpha - 2k > n$. The third part is now a simple consequence of the first and second part. We have

$$\operatorname{grad}\eta = \left(\frac{\partial\eta}{\partial x^i}\,\mathrm{d}\,x^i\right)^{\sharp} = \frac{\partial\eta}{\partial x^i}\eta^{ij}\frac{\partial}{\partial x^j} = 2\eta_{ik}x^k\eta^{ij}\frac{\partial}{\partial x^j} = 2x^j\frac{\partial}{\partial x^j} = 2\xi.$$

Thus $\operatorname{grad}\eta$ is twice the *Euler vector field* on $\mathbb{R}^n$, which, remarkably, does not depend on the metric η but only on the vector space structure. Using (3.1.19) we compute for $\mathsf{Re}(\alpha) > n$

$$\begin{aligned}
2\alpha \operatorname{grad} R^{\pm}(\alpha+2) &= 2\alpha\eta^{ij}\frac{\partial R^{\pm}(\alpha+2)}{\partial x^i}\frac{\partial}{\partial x^j} = 2\alpha\eta^{ij}\frac{1}{\alpha+2-2}R^{\pm}(\alpha)\eta_{ik}x^k\frac{\partial}{\partial x^j} \\
&= R^{\pm}(\alpha)2x^k\frac{\partial}{\partial x^k} = R^{\pm}(\alpha)\operatorname{grad}\eta.
\end{aligned}$$

For the last part we use (3.1.19) twice and obtain

$$\begin{aligned}
\Box R^{\pm}(\alpha+2) &= \eta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}R^{\pm}(\alpha+2)\\
&= \eta^{ij}\frac{\partial}{\partial x^i}\left(\frac{1}{\alpha+2-2}R^{\pm}(\alpha+2-2)\eta_{jk}x^k\right)\\
&= \eta^{ij}\frac{1}{\alpha}\left(\frac{\partial}{\partial x^i}R^{\pm}(\alpha)\right)\eta_{jk}x^k + \eta^{ij}\frac{1}{\alpha}R^{\pm}(\alpha)\eta_{jk}\frac{\partial}{\partial x^i}x^k\\
&= \frac{1}{\alpha}\eta^{ij}\frac{1}{\alpha-2}\eta_{il}x^l R^{\pm}(\alpha-2)\eta_{jk}x^k + \frac{1}{\alpha}R^{\pm}(\alpha)\eta^{ij}\eta_{jk}\delta_i^k\\
&= \frac{1}{\alpha}\frac{1}{\alpha-2}\eta_{il}x^l x^i R^{\pm}(\alpha-2) + \frac{n}{\alpha}R^{\pm}(\alpha)\\
&= \frac{1}{\alpha(\alpha-2)}\eta\cdot R^{\pm}(\alpha-2) + \frac{n}{\alpha}R^{\pm}(\alpha)\\
&\stackrel{i.)}{=} \frac{(\alpha-2)(\alpha-2-n+2)}{\alpha(\alpha-2)}R^{\pm}(\alpha) + \frac{n}{\alpha}R^{\pm}(\alpha)\\
&= \frac{\alpha-n+n}{\alpha}R^{\pm}(\alpha) = R^{\pm}(\alpha).
\end{aligned}$$

□

The above relations hold in the "strong sense", i.e. they are equalities of continuous or even $\mathcal{C}^k$-functions valid point by point. Since $\mathcal{C}^k(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is injectively embedded via (3.1.15) we conclude that the above relations also hold in the sense of distributions. This gives us now the idea how one can *define* $R^{\pm}(\alpha)$ for arbitrary $\alpha \in \mathbb{C}$ at least in the sense of distributions. On one hand, we want to obtain a holomorphic family of distributions $R^{\pm}(\alpha)$ for all $\alpha \in \mathbb{C}$ extending the already given ones as in Lemma 3.1.5, *ii.)*. Since a holomorphic function is already determined by its values on the non-empty open half space of $\mathsf{Re}(\alpha) > n$, such an extension is necessarily unique if it exists at all. On the other hand, we can make use of the relations in Lemma 3.1.6, in particular the one in *iv.)*, to define such an extension. Indeed, we can express $R^{\pm}(\alpha)$ as the d'Alembert operator acting on $R^{\pm}(\alpha+2)$ for $\mathsf{Re}(\alpha) > n+2$. Now if $\mathsf{Re}(\alpha) > n$ we *define* $R^{\pm}(\alpha)$ as *distribution* by

$$R^{\pm}(\alpha) = \Box R^{\pm}(\alpha+2). \tag{3.1.22}$$

Since $\alpha \mapsto R^{\pm}(\alpha)$ is a holomorphic family of distributions for $\mathsf{Re}(\alpha) > n$ by Lemma 3.1.5 *ii.)* the definition (3.1.22) and the previous Definition 3.1.1 coincide as they coincide for $\mathsf{Re}(\alpha) > n+2$ by Lemma 3.1.6, *iv.)*. Thus we can define inductively for $\mathsf{Re}(\alpha+2k) > n$

$$R^{\pm}(\alpha) = \Box R^{\pm}(\alpha+2k) \tag{3.1.23}$$

for $k \in \mathbb{N}$. We need the following Lemma:

Lemma 3.1.7 *Let $\alpha \in \mathbb{C}$ and define $R^{\pm}(\alpha)$ by*

$$R^{\pm}(\alpha) = \Box^k R^{\pm}(\alpha+2k), \tag{3.1.24}$$

where $k \in \mathbb{N}_0$ is such that $\mathsf{Re}(\alpha+2k) > n$. Then (3.1.24) does not depend on the choice of k and yields an entirely holomorphic family of distributions which extends the family $\{R^{\pm}(\alpha)\}_{\mathsf{Re}(\alpha)>n}$.

Proof. First we note that (3.1.24) yields a well-defined distribution as $R^{\pm}(\alpha+2k)$ is even a continuous function for all $k \in \mathbb{N}_0$ with $\mathsf{Re}(\alpha+2k) > n$ and derivatives of distributions yield distributions. Thus $R^{\pm}(\alpha) \in \mathcal{D}'(\mathbb{R}^n)$ is well-defined. If $k' \in \mathbb{N}_0$ is another number with $\mathsf{Re}(\alpha+2k') > n$, say $k' > k$, then $\Box^k R^{\pm}(\alpha+2k) = \Box^{k'} R^{\pm}(\alpha+2k')$ since by Lemma 3.1.6 *iv.)* we have $R^{\pm}(\alpha+2k) = \Box^{k'-k} R^{\pm}(\alpha+2k')$. This shows that (3.1.24) does not depend on k. In particular, if already $\mathsf{Re}(\alpha) > n$ then $k = 0$ would suffice and $R^{\pm}(\alpha)$ coincides with the previous definition in this case. Thus (3.1.24) extends our previous definition. Finally, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function, then

$$R^{\pm}(\alpha)(\varphi) = \left(\Box^k R^{\pm}(\alpha+2k)\right)(\varphi) = R^{\pm}(\alpha+2k)(\Box^k \varphi)$$

depends holomorphically on α since $\Box^k \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is again a test function and $R^{\pm}(\alpha+2k)$ depends holomorphically on α by Lemma 3.1.5 *ii.)* in the distributional sense. Thus (3.1.24) is a holomorphic extension of our previous definition. □

Corollary 3.1.8 *The family $\{R^{\pm}(\alpha)\}_{\alpha\in\mathbb{C}}$ of distributions as in* (3.1.24) *is the unique holomorphic family of distributions extending the family from* (3.1.17)*.*

After these preparations we are now in the position to state the main definition of this section:

Definition 3.1.9 (Riesz distributions) *For $\alpha \in \mathbb{C}$ the distributions $R^+(\alpha)$ are called the advanced Riesz distributions and the $R^-(\alpha)$ are called the retarded Riesz distributions.*

3.1.2 Properties of the Riesz Distributions

Having a definition of $R^{\pm}(\alpha)$ for all complex numbers $\alpha \in \mathbb{C}$ we can start to collect some properties of the Riesz distributions. In particular, they will turn out to provide Green functions for $\Box$ on Minkowski spacetime. We start with the following observation:

Proposition 3.1.10 *Let $\alpha \in \mathbb{C}$. Then we have:*

i.) *For all orthochronous Lorentz transformations $\Lambda \in \mathrm{L}^\uparrow(1, n-1)$ we have*

$$\Lambda^* R^{\pm}(\alpha) = R^{\pm}(\alpha), \tag{3.1.25}$$

and for the time-reversal $\mathrm{T} \in \mathrm{L}(1, n-1)$ we have

$$\mathrm{T}^* R^{\pm}(\alpha) = R^{\mp}(\alpha). \tag{3.1.26}$$

ii.) *One has*

$$\eta R^{\pm}(\alpha) = \alpha(\alpha - n + 2) R^{\pm}(\alpha+2). \tag{3.1.27}$$

iii.) *For all $i = 1, \ldots, n$ one has*

$$(\alpha-2)\frac{\partial}{\partial x^i} R^{\pm}(\alpha) = R^{\pm}(\alpha-2)\eta_{ij}x^j. \tag{3.1.28}$$

iv.) *Let $\lambda > 0$. Then for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ one has*

$$\lambda^{\alpha-n} R^{\pm}(\alpha)(\varphi_\lambda) = R^{\pm}(\alpha)(\varphi), \tag{3.1.29}$$

where $\varphi_\lambda(x) = \lambda^n \varphi(\lambda x)$. Infinitesimally, this means for the Lie derivative with respect to the Euler vector field

$$\mathscr{L}_\xi R^{\pm}(\alpha) = (\alpha - n) R^{\pm}(\alpha), \tag{3.1.30}$$

i.e. $R^{\pm}(\alpha)$ is homogeneous of degree $\alpha - n$.

v.) One has

$$\operatorname{grad}\eta \cdot R^{\pm}(\alpha) = 2\alpha \operatorname{grad} R^{\pm}(\alpha+2) \tag{3.1.31}$$

and

$$\square R^{\pm}(\alpha+2) = R^{\pm}(\alpha). \tag{3.1.32}$$

Proof. For the first part we first note that the Jacobi determinant of the diffeomorphism $x \mapsto \Lambda x$ is ± 1 for $\Lambda \in \mathrm{L}^{\uparrow}(1, n-1)$ whence it preserves the Lorentz volume density $|\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n| = \mathrm{d}^n\,x$. Thus the general definition of $\Lambda^* R^{\pm}(\alpha)$ simplifies in this case and is compatible with (3.1.15) for $\mathsf{Re}(\alpha) > n$. In fact, we have for $\mathsf{Re}(\alpha) > n$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}
\int \varphi(x)(\Lambda^* R^{\pm}(\alpha))(x)\,\mathrm{d}^n\,x &= \int \varphi(x) R^{\pm}(\alpha)(\Lambda x)\,\mathrm{d}^n\,x \\
&= \int \varphi(\Lambda^{-1}y) R^{\pm}(\alpha)(y)\,\mathrm{d}^n\,y \\
&= \int (\Lambda_*\varphi)(y) R^{\pm}(\alpha)(y)\,\mathrm{d}^n\,y.
\end{aligned}$$

Since the continuous function $R^{\pm}(\alpha)$ for $\mathsf{Re}(\alpha) > n$ is $\mathrm{L}^{\uparrow}(1, n-1)$-invariant by Lemma 3.1.4, and since

$$\alpha \ \mapsto\ (\Lambda^* R^{\pm}(\alpha))(\varphi) = R^{\pm}(\alpha)(\Lambda\varphi)$$

as well as $\alpha \mapsto R^{\pm}(\alpha)(\varphi)$ are both holomorphic for all $\alpha \in \mathbb{C}$, these holomorphic functions coincide for all $\alpha \in \mathbb{C}$. The second and third part follow by the same arguments as both sides are holomorphic functions of α when evaluated on $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and they coincide for $\mathsf{Re}(\alpha)$ sufficiently large by Lemma 3.1.5. Now let $\lambda > 0$. Then $\alpha \mapsto \lambda^\alpha$ is holomorphic on $\mathbb{C}$ and thus $\alpha \mapsto \lambda^\alpha R^{\pm}(\alpha)(\varphi_\lambda)$ is holomorphic on $\mathbb{C}$ for any fixed $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Thus we have to show (3.1.29) only for sufficiently large $\mathsf{Re}(\alpha)$ in order to apply the uniqueness arguments. But for $\mathsf{Re}(\alpha) > n$ we have

$$\begin{aligned}
R^{\pm}(\alpha)(\lambda x) &= \begin{cases} c(\alpha,n)\eta(\lambda x)^{\frac{\alpha-n}{2}} & x \in I^{\pm}(0) \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} c(\alpha,n)(\lambda^2)^{\frac{\alpha-n}{2}}\eta(x)^{\frac{\alpha-n}{2}} & x \in I^{\pm}(0) \\ 0 & \text{else} \end{cases} \\
&= \lambda^{\alpha-n} R^{\pm}(\alpha)(x)
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Then, in the sense of distributions,

$$\begin{aligned}
\lambda^{\alpha-n} R^{\pm}(\alpha)(\varphi_\lambda) &= \lambda^{\alpha-n} \int R^{\pm}(\alpha)(x)\lambda^n \varphi(\lambda x)\,\mathrm{d}^n\,x \\
&= \int R^{\pm}(\alpha)(\lambda x)\varphi(\lambda x)\lambda^n\,\mathrm{d}^n\,x \\
&= \int R^{\pm}(\alpha)(y)\varphi(y)\,\mathrm{d}^n\,y \\
&= R^{\pm}(\alpha)(\varphi).
\end{aligned}$$

Thus we conclude that (3.1.29) holds for all $\alpha \in \mathbb{C}$. To prove the infinitesimal version (3.1.30) one can either use (3.1.28) and (3.1.27) or differentiate (3.1.29): Indeed, since $(\lambda, x) \mapsto \lambda^n \varphi(\lambda x)$ is smooth and compactly supported in x "locally uniform in λ", a slight variation of Lemma 1.3.38 shows that $\lambda \mapsto \lambda^{\alpha-n} R^{\pm}(\alpha)(\varphi_\lambda)$ is smooth in λ and the derivatives can be computed by differentiating "under the integral sign" as in Lemma 1.3.38. We find

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(\lambda^{\alpha-n} R^{\pm}(\alpha)(\varphi_\lambda)\right) &= (\alpha-n)\lambda^{\alpha-n-1} R^{\pm}(\alpha)(\varphi_\lambda) + \lambda^{\alpha-n} R^{\pm}(\alpha)\left(\frac{\partial}{\partial\lambda}(x \mapsto \lambda^n \varphi(\lambda x))\right) \\ &= (\alpha-n)\lambda^{\alpha-n-1} R^{\pm}(\alpha)(\varphi_\lambda) + \lambda^{\alpha-n}\lambda^n R^{\pm}(\alpha)\left(\frac{\partial\varphi}{\partial x^i}(\lambda x) x^i\right) \\ &\quad + \lambda^{\alpha-n} n \lambda^{n-1} R^{\pm}(\alpha)(x \mapsto \varphi(\lambda x)).\end{aligned}$$

Since the left hand side does *not* depend on λ, this has to vanish for all $\lambda > 0$. Setting $\lambda = 1$ yields

$$\begin{aligned}0 &= (\alpha-n) R^{\pm}(\alpha)(\varphi) + R^{\pm}(\alpha)\left(x^i \frac{\partial\varphi}{\partial x^i}\right) + n R^{\pm}(\alpha)(\varphi) \\ &= \alpha R^{\pm}(\alpha)(\varphi) + R^{\pm}(\alpha)\left(\frac{\partial}{\partial x^i}\left(x \mapsto x^i\varphi(x)\right)\right) - R^{\pm}(\alpha)(n\varphi) \\ &= (\alpha-n) R^{\pm}(\varphi) - \frac{\partial R^{\pm}(\alpha)}{\partial x^i}(x^i\varphi) \\ &= (\alpha-n) R^{\pm}(\alpha)(\varphi) - (\mathscr{L}_\xi R^{\pm}(\alpha))(\varphi),\end{aligned}$$

and thus (3.1.30). The last part again follows from Lemma 3.1.6, *iii.)* and *iv.)* as well as the uniqueness argument: clearly both sides evaluated on a test function give holomorphic functions of α which coincide for large $\mathsf{Re}(\alpha)$. □

Remark 3.1.11 (Homogeneous distributions) In general, a distribution $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is called *homogeneous of degree* $\alpha \in \mathbb{C}$ if for all test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ one has

$$\lambda^\alpha u(\varphi_\lambda) = u(\varphi) \tag{3.1.33}$$

for all $\lambda > 0$, where $\varphi_\lambda(x) = \lambda^n \varphi(\lambda x)$ as before. By the same argument as in the proof one can show that (3.1.33) implies

$$\mathscr{L}_\xi u = \alpha u. \tag{3.1.34}$$

In fact, (3.1.34) turns out to be equivalent to its integrated form (3.1.33). It is then a non-trivial but interesting question whether a homogeneous distribution $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ of some degree α can be extended to a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ such that the homogeneity is preserved. A detailed discussion of homogeneous distributions can be found in [31, Sect. 3.2]. As a final remark we mention that many problems in renormalization theory of quantum field theories can be reformulated mathematically as the question whether certain homogeneous distributions on $\mathbb{R}^n \setminus \{0\}$ have homogeneous extensions to $\mathbb{R}^n$, see e.g. [52, 55].

In a next step we want to understand the support and singular support of the Riesz distributions $R^{\pm}(\alpha)$. Here we can build on the results from Lemma 3.1.3 and 3.1.4: the support and singular support have to be Lorentz invariant subsets under the orthochronous Lorentz group $\mathrm{L}^{\uparrow}(1, n-1)$. We denote by

$$C^{\pm}(0) = \left\{x \in \mathbb{R}^n \;\middle|\; x \in J^{\pm}(0) \text{ and } \eta(x,x) = 0\right\} \tag{3.1.35}$$

the boundary of $I^{\pm}(0)$. The particular values $\alpha \in \mathbb{C}$ where $c(\alpha, n)$ vanishes play an exceptional role for the support of $R^{\pm}(\alpha)$. We call them *exceptional*, i.e. $\alpha \in \mathbb{C}$ is exceptional if

$$\alpha \in \{n-2k, -2k \mid k \in \mathbb{N}_0\}. \tag{3.1.36}$$

Then we have the following result:

Proposition 3.1.12 (Support of $R^{\pm}(\alpha)$) *Let $\alpha \in \mathbb{C}$.*

i.) If α is not exceptional then

$$\operatorname{supp} R^{\pm}(\alpha) = J^{\pm}(0), \tag{3.1.37}$$

and the singular support

$$\operatorname{sing\,supp} R^{\pm}(\alpha) \subseteq \partial I^{\pm}(0) = C^{\pm}(0) \tag{3.1.38}$$

is either $\{0\}$ or $C^{\pm}(0)$.

ii.) If α is exceptional then

$$\operatorname{supp} R^{\pm}(\alpha) = \operatorname{sing\,supp} R^{\pm}(\alpha) \subseteq C^{\pm}(0). \tag{3.1.39}$$

iii.) Let $n \geq 3$. For $\alpha \in \{n - 2k \mid k \in \mathbb{N}_0, k < \frac{n}{2}\}$ we have

$$\operatorname{supp} R^{\pm}(\alpha) = \operatorname{sing\,supp} R^{\pm}(\alpha) = C^{\pm}(0). \tag{3.1.40}$$

Proof. Let $\alpha \in \mathbb{C}$ be arbitrary. Since by definition of $R^{\pm}(\alpha)$ we have

$$R^{\pm}(\alpha) = \square^k R^{\pm}(\alpha + 2k)$$

for k sufficiently large such that $\mathsf{Re}(\alpha + 2k) > n$, we have by Theorem 1.3.27, *v.)*

$$\begin{aligned} R^{\pm}(\alpha)\Big|_{\mathbb{R}^n \setminus C^{\pm}(0)} &= \square^k \left(R^{\pm}(\alpha + 2k)\Big|_{\mathbb{R}^n \setminus C^{\pm}(0)} \right) \\ &= \begin{cases} \square^k c(\alpha + 2k, n)\eta^{\frac{\alpha+2k-n}{2}} & \text{on} \quad I^{\pm}(0) \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} c(\alpha, n)\eta^{\frac{\alpha-n}{2}} & \text{on} \quad I^{\pm}(0) \\ 0 & \text{else,} \end{cases} \end{aligned}$$

using the explicit computation of $\square\eta$ as in the proof of Lemma 3.1.3. Thus on the open subset $\mathbb{R}^n \setminus C^{\pm}(0) = I^{\pm}(0) \cup (\mathbb{R}^n \setminus J^{\pm}(0))$ we have a smooth function

$$R^{\pm}(\alpha)\Big|_{\mathbb{R}^n \setminus C^{\pm}(0)} = \begin{cases} c(\alpha, n)\eta^{\frac{\alpha-n}{2}} & \text{on} \quad I^{\pm}(0) \\ 0 & \text{else,} \end{cases}$$

for all $\alpha \in \mathbb{C}$. From this we immediately conclude that for all $\alpha \in \mathbb{C}$

$$\operatorname{supp} R^{\pm}(\alpha) \subseteq J^{\pm}(0) \tag{$*$}$$

and

$$\operatorname{sing\,supp} R^{\pm}(\alpha) \subseteq C^{\pm}(0), \tag{$**$}$$

since $J^{\pm}(0)$ and the light cone $C^{\pm}(0)$ are already closed. Then the Lorentz invariance $\Lambda_* R^{\pm}(\alpha) = R^{\pm}(\alpha)$ for all $\Lambda \in \mathrm{L}^{\uparrow}(1, n-1)$ yields that the support and the singular support have to be Lorentz invariant subsets. Indeed, in general one has

$$\operatorname{supp} \Lambda_* R^{\pm}(\alpha) = \Lambda(\operatorname{supp} R^{\pm}(\alpha))$$

$$\operatorname{sing\,supp} \Lambda_* R^{\pm}(\alpha) = \Lambda(\operatorname{sing\,supp} R^{\pm}(\alpha))$$

for every diffeomorphism Λ. Thus $\operatorname{supp} R^{\pm}(\alpha)$ and $\operatorname{sing\,supp} R^{\pm}(\alpha)$ are closed Lorentz invariant subsets of Minkowski space. In particular, $\operatorname{sing\,supp} R^{\pm}(\alpha)$ is either $\{0\}$ or $C^{\pm}(0)$ as these are the

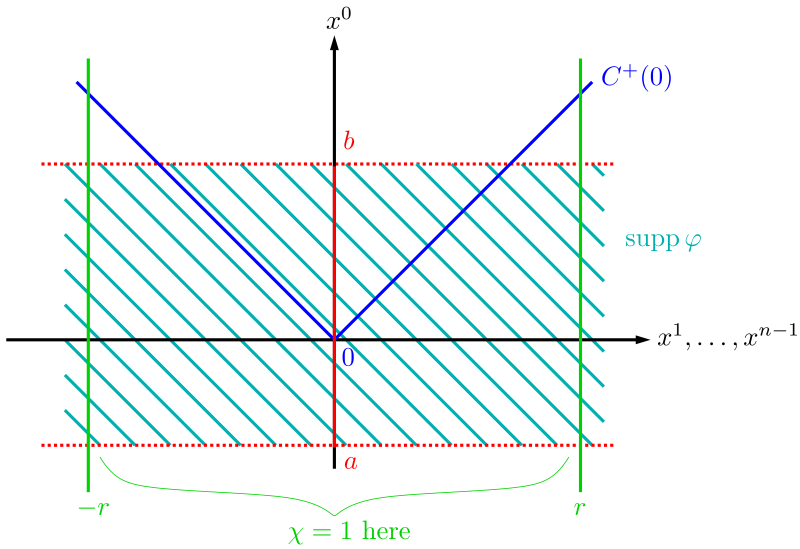

Figure 3.2: The test function constructed in the proof of Proposition 3.1.12.

only Lorentz invariant subsets of $C^{\pm}(0)$. Now let α be not exceptional. Then $c(\alpha, n)$ is non-zero and hence $R^{\pm}(\alpha)\big|_{I^{\pm}(0)}$ is non-zero and even smooth. Thus

$$\operatorname{supp} R^{\pm}(\alpha) \supseteq I^{\pm}(0).$$

On the other hand, by $(*)$, we note $\operatorname{supp} R^{\pm}(\alpha) \subseteq J^{\pm}(0)$, whence (3.1.38) follows. This shows the first part. For the second part, let α be exceptional. Then $c(\alpha, n) = 0$ whence $R^{\pm}(\alpha)\big|_{\mathbb{R}^n \setminus C^{\pm}(0)}$ vanishes identically. Thus

$$\operatorname{supp} R^{\pm}(\alpha) \subseteq C^{\pm}(0)$$

follows. Now $C^{\pm}(0)$ has empty open interior whence the support of $R^{\pm}(\alpha)$ is either empty or necessarily entirely singular. Thus

$$\operatorname{supp} R^{\pm}(\alpha) = \operatorname{sing\,supp} R^{\pm}(\alpha)$$

follows, proving the second part. For the last part we follow [4, Prop. 1.2.4.] and prove first the following technical statement. We consider a test function $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} \psi \subseteq [a, b]$ and a bump function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ such that $\chi\big|_{B_r(0)} = 1$ for some $r > b$. Then the test function $\varphi(x^0, x^1, \ldots, x^{n-1}) = \psi(x^0)\chi(x^1, \ldots, x^{n-1})$ has the property that

$$\varphi\Big|_{J^+(0)}(x^0, \ldots, x^{n-1}) = \psi(x^0) \tag{$***$}$$

for all $x \in J^+(0)$, see also Figure 3.2. Then the claim is that for all $\mathsf{Re}(\alpha) > 0$ one has

$$R^+(\alpha)(\varphi) = \frac{1}{\Gamma(\alpha)} \int_0^\infty (x^0)^{\alpha-1} \psi(x^0) \, \mathrm{d}\, x^0. \tag{$****$}$$

Indeed, we first note that both sides are holomorphic in α. For the left hand side this is true for all $\alpha \in \mathbb{C}$ and for the right hand side this follows as $\frac{1}{\Gamma(\alpha)}$ is entire and the integral is holomorphic by the same Morera type argument as in the proof of Lemma 3.1.5. Thus it will be sufficient to show $(****)$ for $\mathsf{Re}(\alpha) > n$ where we can use the explicit form of $R^+(\alpha)$ as continuous function. We compute

$$R^+(\alpha)(\varphi) = c(\alpha, n) \int_{J^+(0)} \eta(x)^{\frac{\alpha-n}{2}} \varphi(x) \, \mathrm{d}^n \, x$$

$$= c(\alpha, n) \int_0^\infty \mathrm{d}\, x^0 \int_{|\vec{x}|\leq x^0} \left((x^0)^2 - (\vec{x})^2\right)^{\frac{\alpha-n}{2}} \psi(x^0)\chi(\vec{x})\, \mathrm{d}^{n-1}\, x$$

$$\stackrel{(***)}{=} c(\alpha, n) \int_0^\infty \mathrm{d}\, x^0 \psi(x^0) \int_{|\vec{x}|\leq x^0} \left((x^0)^2 - (\vec{x})^2\right)^{\frac{\alpha-n}{2}} \mathrm{d}^{n-1}\, x.$$

For the $\vec{x}$-integration we use $(n-1)$-dimensional polar coordinates r and $\vec{\Omega}$, i.e. the radius $r = |\vec{x}|$ and the remaining point $\vec{\Omega} = \frac{\vec{x}}{|\vec{x}|}$ on the unit sphere $\mathbb{S}^{n-2}$. We evaluate for fixed x^0 the inner integral

$$\int_{|\vec{x}|\leq x^0} \left((x^0)^2 - (\vec{x})^2\right)^{\frac{\alpha-n}{2}} \mathrm{d}^{n-1}\, x = \int_0^{x^0} r^{n-2}\, \mathrm{d}\, r \int_{\mathbb{S}^{n-2}} \left((x^0)^2 - r^2\right)^{\frac{\alpha-n}{2}} \mathrm{d}\, \Omega$$

$$= \operatorname{vol}(\mathbb{S}^{n-2}) \cdot \int_0^{x^0} \left((x^0)^2 - r^2\right)^{\frac{\alpha-n}{2}} r^{n-2}\, \mathrm{d}\, r.$$

The remaining integral can be brought to the following form. First we substitute $\rho = \frac{r}{x^0}$ and then $\rho = \cos\theta$. This yields

$$\int_0^{x^0} \left((x^0)^2 - r^n\right)^{\frac{\alpha-n}{2}} r^{n-2}\, \mathrm{d}\, r = \int_0^1 (1-\rho^2)^{\frac{\alpha-n}{2}} \rho^{n-2} (x^0)^{\alpha-n+n-2+1}\, \mathrm{d}\, \rho$$

$$= (x^0)^{\alpha-1} \int_0^1 (1-\rho^2)^{\frac{\alpha-n}{2}} \rho^{n-2}\, \mathrm{d}\, \rho$$

$$= (x^0)^{\alpha-1} \int_0^{\pi/2} (\sin^2\theta)^{\frac{\alpha-n}{2}} (\cos\theta)^{n-2} \sin \mathrm{d}\, \theta$$

$$= (x^0)^{\alpha-1} \int_0^{\pi/2} (\sin\theta)^{\alpha-n+1} (\cos\theta)^{n-2}\, \mathrm{d}\, \theta.$$

The last integral is *Bronstein-integrable*, see e.g. [13, Sect. 1.1.3.4, Integral 10] and gives

$$(x^0)^{\alpha-1} \int_0^{\pi/2} (\sin\theta)^{\alpha-n+1} (\cos\theta)^{n-2}\, \mathrm{d}\, \theta = \frac{1}{2} \frac{\Gamma(\frac{\alpha-n}{2}+1)\Gamma(\frac{n-3}{2}+1)}{\Gamma(\frac{\alpha-n}{2}+\frac{n-3}{2}-2)} = \frac{1}{2} \frac{\Gamma(\frac{\alpha-n}{2}+1)\Gamma(\frac{n-3}{2}+1)}{\Gamma(\frac{\alpha-1}{2}+1)}.$$

Since finally the surface of the $(n-2)$-dimensional unit sphere is known to be

$$\operatorname{vol}(\mathbb{S}^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})},$$

see e.g. [24, p. 142], we obtain

$$R^+(\alpha)(\varphi) = c(\alpha, n) \cdot \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{2} \frac{\Gamma(\frac{\alpha-n}{2}+1)\Gamma(\frac{n-3}{2}+1)}{\Gamma(\frac{\alpha-1}{2}+1)} \cdot \int_0^\infty (x^0)^{\alpha-1} \psi(x^0)\, \mathrm{d}\, x^0$$

$$= \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1)} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{2} \frac{\Gamma(\frac{\alpha-n}{2}+1)\Gamma(\frac{n-3}{2}+1)}{\Gamma(\frac{\alpha-1}{2}+1)} \cdot \int_0^\infty (x^0)^{\alpha-1} \psi(x^0)\, \mathrm{d}\, x^0$$

$$= \frac{2^{1-\alpha}\sqrt{\pi}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha}{2}+\frac{1}{2})} \cdot \int_0^\infty (x^0)^{\alpha-1} \psi(x^0)\, \mathrm{d}\, x^0$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty (x^0)^{\alpha-1} \psi(x^0)\, \mathrm{d}\, x^0,$$

where the last equality is valid thanks to Legendre's duplication formula (3.1.11). This finally establishes the claim $(****)$. In particular, for $\alpha = 2$ we obtain

$$R^+(2)(\varphi) = \frac{1}{\Gamma(2)} \int_0^\infty (x^0)^{\alpha-1} \psi(x^0) \, \mathrm{d}\, x^0 = \int_0^\infty (x^0)^{\alpha-1} \psi(x^0) \, \mathrm{d}\, x^0,$$

from which it follows that the support of $R^+(2)$ cannot be $0 \in \mathbb{R}^n$ alone as we get a non-trivial result for a φ with $0 \notin \operatorname{supp} \varphi$ by taking a ψ with support away from zero. Thus by the previous arguments the support is at least $C^+(0)$. So if n is even then 2 is an exceptional value whence $\operatorname{supp} R^+(2) = \operatorname{sing\,supp} R^+(2) \subseteq C^+(0)$ and thus

$$\operatorname{supp} R^+(2) = \operatorname{sing\,supp} R^+(2) = C^+(0)$$

follows. Since in this case also $2, 4, \ldots, n-2, n$ are exceptional and

$$R^+(2) = \Box^k R^+(2 + 2k)$$

for all $k \in \mathbb{N}_0$ we conclude from the locality (1.3.60) of differential operators by Theorem 1.3.27 that

$$C^+(0) = \operatorname{supp} R^+(2) \subseteq \operatorname{supp} R^+(2 + 2k) \subseteq C^+(0)$$

for all those k with $2 + 2k \leq n$. But then again $\operatorname{supp} R^+(2 + 2k) = C^+(0)$ follows. Now let n be odd. Since $R^+(\alpha)(\varphi)$ is holomorphic for all α and since the limit $\alpha \longrightarrow 1$ of $(****)$ exists, we conclude

$$R^+(1)(\varphi) = \int_0^\infty \psi(x^0) \, \mathrm{d}\, x^0,$$

whence the support of $R^+(1)$ is again not only $\{0\}$. Thus we can repeat the argument with $R^+(1)$ instead of $R^+(2)$ and obtain (3.1.40) also in this case. Of course the result for $R^-(\alpha)$ is completely analogous or can be deduced from the time reversal symmetry (3.1.26). □

The following counting of the order of the Riesz distributions $R^\pm(\alpha)$ is straightforward:

Proposition 3.1.13 (Order of $R^\pm(\alpha)$) *Let $\alpha \in \mathbb{C}$.*

i.) If $\mathsf{Re}(\alpha) > n$ then the global order of $R^\pm(\alpha)$ is zero

$$\operatorname{ord}(R^\pm(\alpha)) = 0. \tag{3.1.41}$$

ii.) The global order of $R^\pm(\alpha)$ is bounded by $2k$ where $k \in \mathbb{N}_0$ is such that $\mathsf{Re}(\alpha) + 2k > n$.

iii.) If $\mathsf{Re}(\alpha) > 0$ then the global order of $R^\pm(\alpha)$ is bounded by n if n is even and by $n+1$ if n is odd.

Proof. The first part is clear since for $\mathsf{Re}(\alpha) > n$ the distribution $R^\pm(\alpha)$ is even a continuous function. For the second part let $k \in \mathbb{N}_0$ be such that $\mathsf{Re}(\alpha) + 2k > n$. Then

$$\operatorname{ord}(R^\pm(\alpha)) = \operatorname{ord}(\Box^k R^\pm(\alpha + 2k)) \leq \operatorname{ord}(R^\pm(\alpha + 2k)) + 2k = 0 + 2k,$$

since $\operatorname{ord}(R^\pm(\alpha + 2k)) = 0$ by the first part. Finally, let $\mathsf{Re}(\alpha) > 0$ and $n = 2k$ be even. Then by the second part $\operatorname{ord}(R^\pm(\alpha)) \leq 2k = n$ since $\mathsf{Re}(\alpha) + n > n$. If on the other hand $n = 2k + 1$ is odd then by the second part $\operatorname{ord}(R^\pm(\alpha)) \leq 2(k+1) = n+1$ since $\mathsf{Re}(\alpha) + 2(k+1) > n$. □

The next statement is on the reality of $R^\pm(\alpha)$ for real $\alpha \in \mathbb{R}$. In fact, one has the following statement:

Proposition 3.1.14 (Reality of $R^\pm(\alpha)$) *Let $\alpha \in \mathbb{C}$. Then one has*

$$\overline{R^\pm(\alpha)} = R^\pm(\overline{\alpha}). \tag{3.1.42}$$

In particular, for $\alpha \in \mathbb{R}$ one has

$$\overline{R^\pm(\alpha)} = R^\pm(\alpha). \tag{3.1.43}$$

Proof. First we consider $\mathsf{Re}(\alpha) > n$. Then we have

$$\overline{R^{\pm}(\alpha)(x)} = \begin{cases} \overline{c(\alpha, n)}\,\overline{\eta(x)^{\frac{\alpha-n}{2}}} & \text{for } x \in I^{\pm}(0) \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} c(\overline{\alpha}, n)\eta(x)^{\frac{\overline{\alpha}-n}{2}} & \text{for } x \in I^{\pm}(0) \\ 0 & \text{else} \end{cases}$$

$$= R^{\pm}(\overline{\alpha})(x)$$

for all $x \in \mathbb{R}^n$ since $\overline{\Gamma(\alpha)} = \Gamma(\overline{\alpha})$ and hence $\overline{c(\alpha, n)} = c(\overline{\alpha}, n)$. For arbitrary $\alpha \in \mathbb{C}$ let $k \in \mathbb{N}_0$ be such that $\mathsf{Re}(\alpha) + 2k > n$. Then

$$\overline{R^{\pm}(\alpha)} = \overline{\Box^k R^{\pm}(\alpha + 2k)} = \Box^k \overline{R^{\pm}(\alpha + 2k)} = \Box^k R^{\pm}(\overline{\alpha} + 2k) = R^{\pm}(\overline{\alpha}),$$

since $\Box$ is a real differential operator and $\overline{R^{\pm}(\alpha + 2k)} = R^{\pm}(\overline{\alpha} + 2k)$ for $\mathsf{Re}(\alpha + 2k) > n$. □

The next statement is the key observation why the Riesz distributions are actually what we are looking for.

Proposition 3.1.15 *One has*

$$R^{\pm}(0) = \delta_0. \tag{3.1.44}$$

Proof. We have to compute $R^{\pm}(0)(\varphi)$ for $\varphi \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$. Let $\varphi \in \mathcal{C}^{\infty}_K(\mathbb{R}^n)$ with some compact $K \subseteq \mathbb{R}^n$ and choose $\chi \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$ with $\chi\big|_K = 1$. Then we have $\varphi = \chi\varphi$. Moreover, by the usual Hadamard trick we have smooth functions $\varphi_i \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ such that

$$\varphi(x) = \varphi(0) + \sum\nolimits_{i=1}^{n} x^i \varphi_i(x).$$

In fact,

$$\varphi_i(x) = \int_0^1 \frac{\partial \phi}{\partial x^i}(tx)\,\mathrm{d}\,t$$

will do the job. Note that $\operatorname{supp} \varphi_i$ is *not* compact. In any case, we have

$$\varphi = \chi\varphi = \chi\varphi(0) + \sum\nolimits_{i=1}^{n} x^i \chi\varphi_i$$

with compactly supported $\chi\varphi(0)$ and $x^i\chi\varphi_i$. Only now we can apply the distribution $R^{\pm}(0)$ to both terms giving

$$R^{\pm}(0)(\varphi) = R^{\pm}(0)\left(\chi\varphi(0) + \sum_i x^i \chi\varphi_i\right) = \varphi(0)R^{\pm}(0)(\chi) + \sum_i \left(x^i R^{\pm}(0)\right)(\chi\varphi_i).$$

Now $2x^i$ is the i-th component of $\operatorname{grad}\eta$ whence by Proposition 3.1.10, *v.)* for $\alpha = 0$ we obtain

$$2\left(x^i R^{\pm}(0)\right)(\chi\varphi_i) = 2 \cdot 0 \cdot \eta^{ij}\frac{\partial R^{\pm}(2)}{\partial x^j}(\chi\varphi_i) = 0.$$

This shows

$$R^{\pm}(0)(\varphi) = \varphi(0)R^{\pm}(0)(\chi).$$

Since $R^{\pm}(0)(\varphi)$ does not depend on the choice of the cut-off function χ the constant $R^{\pm}(0)(\chi)$ does neither. However, it might still depend on the chosen compactum K which is easy to see to be not the case. This shows that

$$R^{\pm}(0) = R^{\pm}(0)(\chi)\delta_0 = c \cdot \delta_0$$

is a multiple of the δ-functional at zero. We are left with the computation of $c = R^{\pm}(0)(\chi)$. To this end it is obviously sufficient to compute $R^{\pm}(0)(\varphi)$ for one function with $\varphi(0) \neq 0$. Thus we again use a factorizing function

$$\varphi(x) = \psi(x^0)\chi(x^1, \ldots, x^{n-1})$$

with $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ such that χ is equal to 1 on a large enough ball around 0 in order to have

$$\varphi\Big|_{J^+(0)}(x^0, \ldots, x^{n-1}) = \psi(x^0).$$

Recall that we constructed such a function in the proof of Proposition 3.1.12, *iii.)*. Then we have

$$\Box\varphi\Big|_{J^+(0)}(x) = \ddot{\psi}(x^0),$$

since the x^1-, ..., x^{n-1}-derivatives in $\Box$ do not contribute. From the above proof we know that

$$R^{\pm}(0)(\varphi) = R^{\pm}(2)(\Box\varphi) = \int_0^\infty x^0\ddot{\psi}(x^0)\,\mathrm{d}\,x^0 = -\int_0^\infty \dot{\psi}(x^0)\,\mathrm{d}\,x^0 = \psi(0) = \varphi(0),$$

by integration by parts and using that ψ has compact support. Thus $R^{\pm}(0)(\varphi) = \varphi(0)$ whence the multiple is 1 and the proof is finished for dimensions $n \geq 3$. The two remaining cases $n = 1, 2$ are indeed much simpler. Either, one can modify the above argument to work also in this simpler situation. Or, as we shall do in Subsection 3.1.3, one uses a direct computation. □

The last proposition allows us to formulate the following main result of this subsection: we have found the advanced and retarded Green functions of the scalar wave equation on Minkowski spacetime.

Theorem 3.1.16 (Green function of $\Box$) *The Riesz distributions $R^{\pm}(2)$ are advanced and retarded Green functions for the scalar d'Alembert operator $\Box$ on Minkowski spacetime.*

Proof. First we know by Proposition 3.1.10, *v.)* that $\Box R^{\pm}(2) = R^{\pm}(0)$ which is δ_0 by Proposition 3.1.15. Thus the $R^{\pm}(2)$ are fundamental solutions of $\Box$. Moreover, by Proposition 3.1.12 we know that $\operatorname{supp} R^{\pm}(2) \subseteq J^{\pm}(0)$ whence we indeed have advanced and retarded Green functions. □

Remark 3.1.17 For the later use we mention that for $\varphi \in \mathcal{C}_0^k(\mathbb{R}^n)$ the distribution $R^{\pm}(\alpha)$ can still be applied to φ as long as $\operatorname{ord}(R^{\pm}(\alpha)) \leq k$ by Remark 1.3.10. This is the case for

$$\mathsf{Re}(\alpha) > n - 2 \cdot \left[\frac{k}{2}\right] \tag{3.1.45}$$

by Proposition 3.1.13, *ii.)*. In this case $R^{\pm}(\alpha)(\varphi) = R^{\pm}(\alpha + 2\ell)(\Box^\ell\varphi)$ for $2\ell \leq k$ and since $R^{\pm}(\alpha)(\varphi)$ is still holomorphic for $\mathsf{Re}(\alpha) > n$ and $\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)$ we obtain the result that $R^{\pm}(\alpha)(\varphi)$ is holomorphic for $\mathsf{Re}(\alpha) > n - 2 \cdot \left[\frac{k}{2}\right]$ and $\varphi \in \mathcal{C}_0^k(\mathbb{R}^n)$.

3.1.3 The Riesz Distributions in Dimension $n = 1, 2$

In this small section we compute the Riesz distributions $R^{\pm}(\alpha)$ and in particular $R^{\pm}(2)$ for low dimensions explicitly.

We start with the most trivial case $n = 1$. In this case $\mathbb{R}^1$ is equipped with the *Riemannian* metric $\eta = \mathrm{d}\,t^2$, where we denote the canonical coordinate simply by t. Though we do not even have an honest Lorentz spacetime in this case the results from the preceding sections are nevertheless valid.

In this case, the advanced and retarded Green functions $R^{\pm}(2)$ are even defined as *continuous functions* since $\mathsf{Re}(2) = 2 > 1 = n$.

Proposition 3.1.18 *Let $n = 1$. Then the advanced and retarded Green functions of $\square = \frac{\partial^2}{\partial t^2}$ are explicitly given as the continuous functions*

$$R^+(2)(t) = \begin{cases} t & \textit{for} \quad t > 0 \\ 0 & \textit{else} \end{cases} \tag{3.1.46}$$

and

$$R^-(2)(t) = \begin{cases} |t| & \textit{for} \quad t < 0 \\ 0 & \textit{else.} \end{cases} \tag{3.1.47}$$

Moreover, for $\mathsf{Re}(\alpha) > 1$ we have

$$R^\pm(\alpha)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} |t|^{\alpha-1} & \textit{for} \quad t \in \mathbb{R}^\pm \\ 0 & \textit{else.} \end{cases} \tag{3.1.48}$$

Proof. First we compute for all $\alpha \in \mathbb{C}$

$$c(\alpha, 1) = \frac{2^{1-\alpha} \pi^{\frac{2-1}{2}}}{\Gamma(\frac{\alpha}{2}) \Gamma(\frac{\alpha-1}{2} + 1)} = \frac{2^{1-\alpha}\sqrt{\pi}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha}{2} + \frac{1}{2})} = \frac{1}{\Gamma(\alpha)}$$

by Legendre's duplication formula. Since $\eta(t) = t^2$ and $I^\pm(0) = \mathbb{R}^\pm$ we have (3.1.48). Finally, $\Gamma(2) = 1$ whence (3.1.46) and (3.1.47) follow. $\square$

Remark 3.1.19 (Riesz distribution in one dimension)

i.) It is an easy exercise to compute $\frac{\partial^2}{\partial t^2} R^\pm(2)$ in the sense of distributions directly to show that

$$\frac{\partial^2}{\partial t^2} R^\pm(2) = \delta_0. \tag{3.1.49}$$

In fact, we have done this implicitly in the proof of Proposition 3.1.15.

ii.) The functions $R^\pm(\alpha)$ for $\mathsf{Re}(\alpha) > 1$ coincide with the functions $\chi_\pm^{\alpha-1}$ of Hörmander in [31, Sect. 3.2., (3.2.17)]. In fact, even though the function $R^\pm(\alpha)$ defined by (3.1.48) is no longer continuous for $\mathsf{Re}(\alpha) > 0$, it is still *locally integrable*. Thus it defines a distribution also in this case, depending holomorphically on α. Hence we conclude

$$R^\pm(\alpha)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} |t|^{\alpha-1} & \text{for } t \in \mathbb{R}^\pm \\ 0 & \text{else} \end{cases} \tag{3.1.50}$$

is valid for $\mathsf{Re}(\alpha) > 0$ in the sense of locally integrable functions. The functions $\chi_\pm^\alpha$ are at the heart of the study of homogeneous distributions and can be used to obtain fundamental solutions of much more general second order differential operators with constant coefficients than just for $\square$, see [31, Sect. 3.2].

We turn now to the case $n = 1 + 1$. Here it is convenient to use the coordinates $(t, x) \in \mathbb{R}^2$ with

$$\eta(t, x) = t^2 - x^2. \tag{3.1.51}$$

First we compute the prefactor $c(\alpha, n)$ for $n = 2$. We have

$$c(\alpha, 2) = \frac{2^{1-\alpha}}{\Gamma(\frac{\alpha}{2})^2} \tag{3.1.52}$$

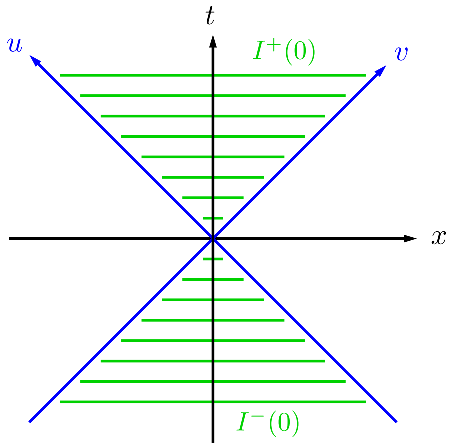

Figure 3.3: Light cone coordinates.

as one immediately obtains from the definition. In order to evaluate $\eta^{\frac{\alpha-2}{2}}$ we introduce new coordinates on $\mathbb{R}^2$. We pass to the *light cone coordinates*

$$u = \frac{1}{\sqrt{2}}(t-x) \qquad \text{and} \qquad v = \frac{1}{\sqrt{2}}(t+x), \tag{3.1.53}$$

i.e.

$$t = \frac{1}{\sqrt{2}}(u+v) \qquad \text{and} \qquad x = \frac{1}{\sqrt{2}}(v-u). \tag{3.1.54}$$

Since this is clearly a global diffeomorphism we can evaluate $R^{\pm}(\alpha)$ in these new coordinates. The prefactors are chosen in such a way that the diffeomorphism is orientation preserving and has Jacobi determinant equal to one: It is just the counterclockwise rotation by 45° in the (t,x)-plane, see Figure 3.3. First we note that the function η in these coordinates is

$$\eta(u,v) = \frac{1}{2}(u+v)^2 - \frac{1}{2}(v-u)^2 = \frac{1}{2}(u^2+2uv+v^2-u^2+2uv-v^2) = 2uv. \tag{3.1.55}$$

Moreover, the future and past $I^{\pm}(0)$ of 0 can be described by

$$I^+(0) = \left\{(u,v) \in \mathbb{R}^2 \;\middle|\; u,v > 0\right\} \tag{3.1.56}$$

and

$$I^-(0) = \left\{(u,v) \in \mathbb{R}^2 \;\middle|\; u,v < 0\right\}, \tag{3.1.57}$$

see again Figure 3.3. Thus we have for $\mathsf{Re}(\alpha) > 2$

$$R^{\pm}(\alpha)(u,v) = \begin{cases} \frac{2^{1-\alpha}}{\Gamma(\frac{\alpha}{2})^2}(2uv)^{\frac{\alpha-2}{2}} & \text{for } u,v \in \mathbb{R}^{\pm} \\ 0 & \text{else} \end{cases} \tag{3.1.58}$$

$$= \begin{cases} \frac{2^{1-\alpha}}{\Gamma(\frac{\alpha}{2})^2}|\sqrt{2}u|^{\frac{\alpha-2}{2}}|\sqrt{2}v|^{\frac{\alpha-2}{2}} & \text{for } u,v \in \mathbb{R}^{\pm} \\ 0 & \text{else,} \end{cases} \tag{3.1.59}$$

whence $R^{\pm}(\alpha)$ is *factorizing* in these coordinates. This suggests to consider the following functions

$$r^{\pm}(\alpha)(u) = \begin{cases} \frac{2^{-\frac{\alpha}{4}}}{\Gamma(\frac{\alpha}{2})}|u|^{\frac{\alpha-2}{2}} & \text{for } u \in \mathbb{R}^{\pm} \\ 0 & \text{else} \end{cases} \tag{3.1.60}$$

for $\mathsf{Re}(\alpha) > 2$. Since the prefactor is still holomorphic for all $\alpha \in \mathbb{C}$ and since $|u|^z$ is locally integrable for $\mathsf{Re}(z) > -1$ we can extend this definition to the case $\mathsf{Re}(\alpha) > 0$.

Proposition 3.1.20 *Let* $\mathsf{Re}(\alpha) > 0$.

i.) The functions $r^{\pm}(\alpha)$ *on* $\mathbb{R}$ *are locally integrable and thus define distributions of order zero with*

$$\operatorname{supp} r^{\pm}(\alpha) = \mathbb{R}^{\pm} \cup \{0\}. \tag{3.1.61}$$

ii.) For every $\varphi \in \mathcal{C}_0^{\infty}(\mathbb{R})$ *the function*

$$\alpha \mapsto \int r^{\pm}(\alpha)(u)\varphi(u)\, \mathrm{d}\, u \tag{3.1.62}$$

is holomorphic for $\mathsf{Re}(\alpha) > 0$.

iii.) For $\alpha = 2$ *we have*

$$r^{\pm}(2)(u) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } u \in \mathbb{R}^{\pm} \\ 0 & \text{else,} \end{cases} \tag{3.1.63}$$

i.e. a multiple of the Heaviside distribution.

Proof. Let $\mathsf{Re}(\alpha) > 0$. Clearly, the only interesting thing about the local integrability of $r^{\pm}(\alpha)$ is around zero since on $\mathbb{R} \setminus \{0\}$ the function is clearly smooth and thus in L^1_{loc}. Thus we consider $\alpha = \beta + \mathrm{i}\gamma$ with $\beta > 0$ and $\gamma \in \mathbb{R}$. Then

$$|r^{\pm}(\alpha)(u)| = \begin{cases} \left| \frac{2^{\frac{\beta+\mathrm{i}\gamma}{4}}}{\Gamma(\frac{\beta+\mathrm{i}\gamma}{2})} \right| \left| |u|^{\frac{\beta+\mathrm{i}\gamma-2}{2}} \right| & u \in \mathbb{R}^{\pm} \\ 0 & \text{else} \end{cases} \leq c_{\beta,\gamma} |u|^{\frac{\beta-2}{2}}.$$

But the function $u \mapsto |u|^{\frac{\beta-2}{2}}$ is locally integrable for $\beta > 0$. This shows the first part as (3.1.61) is obvious. Now let $\varphi \in \mathcal{C}_0^{\infty}(\mathbb{R})$ or, which would be sufficient, $\varphi \in \mathcal{C}_0^0(\mathbb{R})$. Let $\operatorname{supp} \varphi \subseteq [a, b]$ and without restriction $b > 0$, then

$$\int_{\mathbb{R}} r^{+}(\alpha)(u)\varphi(u)\, \mathrm{d}\, u = \int_0^{\infty} \frac{2^{-\frac{\alpha}{4}}}{\Gamma(\frac{\alpha}{2})} u^{\frac{\alpha-2}{2}} \varphi(u)\, \mathrm{d}\, u = \frac{2^{-\frac{\alpha}{4}}}{\Gamma(\frac{\alpha}{2})} \int_0^b u^{\frac{\alpha-2}{2}} \varphi(u)\, \mathrm{d}\, u.$$

Since the function $u \mapsto u^{\frac{\alpha-2}{2}}\varphi(u)$ is integrable over $[a, b]$ we can again exchange the integration over a triangle path $\int_{\Delta} \mathrm{d}\, \alpha$ in the complex half space with $\mathsf{Re}(\alpha) > 0$ and the integral $\int_0^b \mathrm{d}\, u$. Thus Morera's theorem again yields the statement that (3.1.62) is holomorphic. The last part is clear. □

Lemma 3.1.21 *In the lightcone coordinates the d'Alembert operator is*

$$\square = 2\frac{\partial^2}{\partial u \partial v}. \tag{3.1.64}$$

Proof. This is a trivial computation. □

Proposition 3.1.22 *Let* u, v *be the light cone coordinates on* $\mathbb{R}^2$. *Then the distributions*

$$R^{\pm}(2)(u, v) = r^{\pm}(2)(u)r^{\pm}(2)(v) \tag{3.1.65}$$

are advanced and retarded Green functions of $\square$ *of order zero.*

Proof. Of course, we know this from the general Theorem 3.1.16, but here we can give a more elementary proof. We consider the + case where we have

$$r^{+}(2)(u) = \frac{1}{\sqrt{2}}\Theta(u)$$

with the Heaviside function Θ. Since $r^+(2)$ is locally integrable, we have a distribution $r^+(2) \in \mathcal{D}'(\mathbb{R})$ of order zero. Moreover, one knows

$$\frac{\partial}{\partial u}\Theta = \delta_0.$$

The same holds for the v-dependence. Thus we can interpret (3.1.65) as *external* tensor product

$$R^{\pm}(2) = \frac{1}{2}\Theta_u \boxtimes \Theta_v,$$

whence

$$2\frac{\partial^2}{\partial u \partial v} R^{\pm}(2) = 2\frac{\partial}{\partial u} \boxtimes \frac{\partial}{\partial v}\left(\frac{1}{2}\Theta_u \boxtimes \Theta_v\right) = \frac{\partial}{\partial u}\Theta_u \boxtimes \frac{\partial}{\partial v}\Theta_v = \delta_{(0,0)}.$$

Since the Jacobi determinant of the coordinate change is one, the δ-distribution in (u, v) is the same as the one in (t, x). Thus the claim follows. Note that this formulation is of course more elementary and can almost be "guessed". □

Remark 3.1.23 In $n = 1 + 1$ all the Riesz distributions are factorizing as external tensor products of the distributions $r^{\pm}(\alpha)$ of one variable. This simplifies the discussion considerably. Note however, that this is a particular feature of $n = 1 + 1$ and no longer true in higher dimensions. Note also that from Proposition 3.1.13 we only get the estimate $\operatorname{ord}(R^{\pm}(2)) \leq 2$ which is clearly not optimal: The Riesz distributions $R^{\pm}(2)$ in $n = 1 + 1$ are locally integrable and hence of order zero.

It is a good exercise to work out the cases $n = 1 + 2$ and $n = 1 + 3$ explicitly.

3.2 The Riesz Distributions on a Convex Domain

We pass now from Minkowski spacetime to a general Lorentz manifold (M, g) and try to find analogs of the Riesz distributions at least locally around a point $p \in M$. The main idea is to use the Riesz distributions on the tangent space T_pM, which is isometric to Minkowski space, and push forward the Riesz distributions via the exponential map.

3.2.1 The Functions ϱ_p and η_p

Since on M we have a canonical positive density, namely the Lorentz volume density μ_g from Proposition 2.1.15, *ii.)*, we can use this density to identify functions and densities once and for all. In particular, this results in an identification of the generalized sections $\Gamma^{-\infty}(E)$ of a vector bundle $E \longrightarrow M$ with the topological dual of $\Gamma_0^\infty(E^*)$ and not of $\Gamma_0^\infty(E \otimes |\Lambda^{\mathrm{top}}|T^*M)$ as we did before. In more detail, for $s \in \Gamma^{-\infty}(E)$ and a test section $\varphi \in \Gamma_0^\infty(E^*)$ we first map φ to $\varphi \otimes \mu_g \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M)$ and then apply s, i.e. we set

$$s(\varphi) = s(\varphi \otimes \mu_g), \tag{3.2.1}$$

and drop the explicit reference to μ_g to simplify our notation. Since

$$\Gamma_0^\infty(E^*) \ni \varphi \;\mapsto\; \varphi \otimes \mu_g \in \Gamma_0^\infty(E^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \tag{3.2.2}$$

is indeed an isomorphism of LF spaces as discussed in Remark 1.3.8, we have an induced isomorphism of the topological duals which is (3.2.1).

If we now want to push forward the $R^{\pm}(\alpha)$ from T_pM to M we have to take care of the two different notion of volume densities. On T_pM we have the *constant* density coming from the Minkowski scalar product g_p while one M we have μ_g. In general, the push-forward of $\mu_g(p)$ via $\exp_p$ to M *does not* coincide with μ_g whence we need a way to compare the two densities. This is done by the following construction. Let $V_p \subseteq T_pM$ be a suitable open star-shaped neighborhood of 0_p and let $U_p = \exp_p(V_p) \subseteq M$ be the corresponding open neighborhood of p such that

$$\exp_p : V_p \longrightarrow U_p \tag{3.2.3}$$

is a diffeomorphism. Then we define the function

$$\varrho_p = \frac{\mu_g\big|_{U_p}}{\exp_{p*}(\mu_g(p))\big|_{U_p}}. \tag{3.2.4}$$

Lemma 3.2.1 *The function ϱ_p is well-defined and smooth on U_p. We have $\varrho_p > 0$ and*

$$\varrho_p \exp_{p*}(\mu_g(p)) = \mu_g \tag{3.2.5}$$

on U_p.

Proof. Since on V_p the exponential map is a diffeomorphism, the push-forward of the constant density $\mu_g(p) \in |\Lambda^{\mathrm{top}}|T_p^*M$ gives a smooth density on U_p. Clearly, it is still positive whence the quotient (3.2.4) is well-defined and a smooth function. Since also $\mu_g > 0$ it follows that $\varrho_p > 0$ everywhere. □

Sometimes it will be convenient to work on V_p instead of U_p. Thus we can pull-back everything to V_p by $\exp_p$ and obtain

$$\exp_p^*(\varrho_p) \underbrace{\exp_p^*(\exp_{p*} \mu_g(p))}_{\mu_g(p)} = \exp_p^*(\varrho_p)\mu_g(p) = \exp_p^*(\mu_g) \tag{3.2.6}$$

on V_p. To simplify our notation we abbreviate

$$\widetilde{\varrho}_p = \exp_p^*(\varrho_p) \in \mathcal{C}^\infty(V_p) \tag{3.2.7}$$

and have

$$\widetilde{\varrho}_p \mu_g(p) = \exp_p^*(\mu_g). \tag{3.2.8}$$

Thus $\widetilde{\varrho}_p$ is the function which measures how much $\exp_p^*(\mu_g)$ is *not* constant.

To effectively compute ϱ_p or $\widetilde{\varrho}_p$ one proceeds as follows. Let $e_1, \ldots, e_n \in T_pM$ be a basis. Then we can evaluate both densities on $e_1, \ldots, e_n$ to get ϱ_p and $\widetilde{\varrho}_p$. More precisely, by the definition of the pull-back we have for $v \in V_p$

$$\widetilde{\varrho}_p(v) = \frac{\mu_g(\exp_p(v)) \big(T_v \exp_p e_1, \ldots, T_v \exp_p e_n\big)}{\mu_g(p)(e_1, \ldots, e_n)}. \tag{3.2.9}$$

Thus we have to compute "determinants" of the tangent map of $\exp_p$ in order to obtain $\widetilde{\varrho}_p$. This can indeed be done rather explicitly by using Jacobi vector fields at least in a formal power series expansion in v. We give here the result without going into details, but refer to Appendix A.3 for more background information.

Proposition 3.2.2 *The Taylor expansion of $\widetilde{\varrho}_p$ up to second order is explicitly given by*

$$\widetilde{\varrho}_p(v) = 1 - \frac{1}{6} \operatorname{Ric}_p(v, v) + \cdots, \tag{3.2.10}$$

where Ric_p is the Ricci tensor at p and $v \in T_pM$.

Proof. First we note that μ_g is covariantly constant and hence parallel along all curves. Thus we do not get any contributions of covariant derivatives of μ_g. This simplifies the general result from Theorem A.3.5 and yields the result (3.2.10). $\square$

Corollary 3.2.3 *At $p \in M$ we have*

$$\square \varrho_p\big|_p = -\frac{1}{3}\operatorname{scal}(p). \tag{3.2.11}$$

Proof. By general results from Appendix A.1 we know that for any function $f \in \mathcal{C}^\infty(M)$ one has the formal Taylor expansion

$$(\exp_p^* f)(v) \sim_{v\to 0} \sum_{r=0}^{\infty} \frac{1}{r!}\frac{1}{r!}\,\mathsf{D}^r f\big|_p(v,\ldots,v),$$

where D is the symmetrized covariant derivative. By Proposition 3.2.2 we have for $\varrho_p = \exp_{p*}(\widetilde{\varrho}_p)$

$$\frac{1}{4}\,\mathsf{D}^2\,\varrho_p\big|_p(v,v) = -\frac{1}{6}\operatorname{Ric}_p(v,v),$$

whence the Hessian of ϱ_p at p is given by

$$\operatorname{Hess}\varrho_p = \frac{1}{2}\,\mathsf{D}^2\,\varrho_p = -\frac{1}{3}\operatorname{Ric}_p.$$

Thus we conclude

$$\square\varrho_p\big| = \frac{1}{2}\left\langle g^{-1}, \mathsf{D}^2\,\varrho_p\right\rangle\big|_p = -\frac{1}{3}\left\langle g^{-1}, \operatorname{Ric}\right\rangle\big|_p = -\frac{1}{3}\operatorname{scal}(p)$$

by the definition of the scalar curvature as in (2.1.43) as well as by Proposition 2.1.19, *iv.)*. $\square$

With the general techniques from the appendix it is also possible to obtain the higher orders in the Taylor expansion of ϱ_p in a rather explicit and systematic way. They turn out to be universal algebraic combinations of the curvature tensor and its covariant derivatives. However, we shall not need this here. Instead, we mention that by the usual expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + \cdots$ we immediately find

$$\square\sqrt{\varrho_p}\Big|_p = -\frac{1}{6}\operatorname{scal}(p) \tag{3.2.12}$$

and

$$\square\frac{1}{\sqrt{\varrho_p}}\Big|_p = \frac{1}{6}\operatorname{scal}(p). \tag{3.2.13}$$

For the Riesz distributions we needed the quadratic function $\eta(x) = \eta(x,x)$ as basic ingredient. Clearly, we have this on every tangent space whence we can define

$$\widetilde{\eta}_p(v) = g_p(v,v) \tag{3.2.14}$$

for every $v \in T_pM$. Analogously to the relation between ϱ_p and $\widetilde{\varrho}_p$ we set

$$\eta_p(q) = \widetilde{\eta}_p(\exp_p^{-1}(q)) \tag{3.2.15}$$

for $q \in U_p$. With other words, $\eta_p \in \mathcal{C}^\infty(U_p)$ is the function with

$$\exp_p^*(\eta_p) = \widetilde{\eta}_p. \tag{3.2.16}$$

We collect now some properties of the functions η_p and $\widetilde{\eta}_p$.

Proposition 3.2.4 *Let (M, g) be a time-oriented Lorentz manifold and $p \in M$. Moreover, let $U \subseteq M$ be geodesically star-shaped with respect to p.*

i.) The gradient of $\eta_p \in \mathcal{C}^\infty(N)$ is given by

$$\operatorname{grad} \eta_p\big|_q = 2T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)) \tag{3.2.17}$$

for $q \in U$.

ii.) One has

$$g(\operatorname{grad} \eta_p, \operatorname{grad} \eta_p) = 4\eta_p. \tag{3.2.18}$$

iii.) On $I_U^\pm(p)$ the gradient of η_p is a future resp. past directed timelike vector field.

iv.) One has

$$\square \eta_p = 2n + g(\operatorname{grad} \log \varrho_p, \operatorname{grad} \eta_p). \tag{3.2.19}$$

Proof. For the first part we need the Gauss Lemma which says

$$g_{\exp_p(v)}\left(T_v \exp_p(v), T_v \exp_p(w)\right) = g_p(v, w)$$

for $v \in V_p \subseteq T_pM$ and $w \in T_pM$ arbitrary, see Proposition A.2.11. Using this we compute for $q \in U$ and $w_q \in T_qM$

$$\begin{aligned}
\mathrm{d}\, \eta_p\big|_q(w_q) &= w_q(\widetilde{\eta}_p \circ \exp_p^{-1}) \\
&= \mathrm{d}\, \widetilde{\eta}_p\Big|_{\exp_p^{-1}(q)} \left(T_q \exp_p^{-1}(w_q)\right) \\
&= 2g_p\left(\exp_p^{-1}(q), T_q \exp_p^{-1}(w_q)\right) \\
&= 2g_q\left(T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)), T_{\exp_p^{-1}(q)} \exp_p T_q \exp_p^{-1}(w_q)\right) \\
&= 2g_q\left(T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)), w_q\right)
\end{aligned}$$

by the Gauss Lemma for $v = \exp_p^{-1}(q)$ and the chain rule. By the very definition of the gradient this gives (3.2.17). For the second part we again use the Gauss Lemma and get with $v = \exp_p^{-1}(q)$ for $q \in U$

$$\begin{aligned}
g_q(\operatorname{grad} \eta_p\big|_q, \operatorname{grad} \eta_p\big|_q) &= 4g_{\exp_p(v)}\left(T_v \exp_p(v), T_v \exp_p(v)\right) \\
&= 4g_p(v, v) \\
&= 4g_p(\exp_p^{-1}(q), \exp_p^{-1}(q)) \\
&= 4\eta_p(q)
\end{aligned}$$

as claimed. For the third part we first notice that the points in $I^\pm(0_p) \subseteq T_pM$ are mapped under $\exp_p$ to points in $I_U^\pm(p)$ since there is a timelike curve joining p and such a point $q = \exp_p(v)$, namely the geodesic $t \mapsto \exp_p(tv)$. This is indeed a timelike curve for all t thanks to the Gauss Lemma. Thus for $q \in I_U^\pm(p)$ we have $q = \exp_p(v)$ with $v \in I^\pm(0_p) \subseteq T_pM$ whence $\eta_p(q) = g_p(\exp_p^{-1}(q), \exp_p^{-1}(q)) > 0$. This shows $\eta_p > 0$ on $I_U^\pm(p)$. By the second part we conclude

$$g(\operatorname{grad} \eta_p, \operatorname{grad} \eta_p) > 0$$

on $I_U^{\pm}(p)$ whence $\operatorname{grad}\eta_p$ is timelike on $I_U^{\pm}(p)$. Now let $v \in I^+(0_p) \subseteq T_pM$ be future directed. Then $t \mapsto \exp_p(tv)$ is a future directed geodesic with tangent vector

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\exp_p(tv) = T_{tv}\exp_p(v) = \frac{1}{2t}\operatorname{grad}\eta_p\Big|_{\exp_p(tv)}.$$

Thus for $t > 0$ the gradient of η_p is a positive multiple of the tangent vector of $\exp_p(tv)$ and hence future directed itself at $\exp_p(tv)$. Since every point in $I_U^+(p)$ can be reached this way, $\operatorname{grad}\eta_p$ is future directed on all of $I_U^+(p)$. With the same argument we see that $\operatorname{grad}\eta_p$ is past directed on $I_U^-(p)$. The last part is again a computation. First we note that thanks to $\varrho_p > 0$ everywhere, we have a smooth real-valued logarithm $\log\varrho_p \in \mathcal{C}^\infty(U)$. The Leibniz rule (1.2.55) for div gives

$$\operatorname{div}\left(\frac{1}{\varrho_p}\operatorname{grad}\eta_p\right) = \mathscr{L}_{\operatorname{grad}\eta_p}\left(\frac{1}{\varrho_p}\right) + \frac{1}{\varrho_p}\operatorname{div}\operatorname{grad}\eta_p = g\left(\operatorname{grad}\frac{1}{\varrho_p}, \operatorname{grad}\eta_p\right) + \frac{1}{\varrho_p}\Box\varrho_p$$

and thus

$$\begin{aligned}
\Box\varrho_p &= \varrho_p\operatorname{div}\left(\frac{1}{\varrho_p}\operatorname{grad}\eta_p\right) - g\left(\varrho_p\operatorname{grad}\frac{1}{\varrho_p}, \operatorname{grad}\eta_p\right)\\
&= \varrho_p\operatorname{div}\left(\frac{1}{\varrho_p}\operatorname{grad}\eta_p\right) - g\left(\operatorname{grad}\log\frac{1}{\varrho_p}, \operatorname{grad}\eta_p\right)\\
&= \varrho_p\operatorname{div}\left(\frac{1}{\varrho_p}\operatorname{grad}\eta_p\right) + g\left(\operatorname{grad}\log\varrho_p, \operatorname{grad}\eta_p\right).
\end{aligned}$$

We still have to compute the first divergence. Since div here is always the divergence with respect to μ_g we consider on U

$$\begin{aligned}
\operatorname{div}_{\mu_g}\left(\frac{1}{\varrho_p}X\right) &= \operatorname{div}_{\varrho_p\exp_{p*}(\mu_g(p))}\left(\frac{1}{\varrho_p}X\right)\\
&= \operatorname{div}_{\exp_{p*}(\mu_g(p))}\left(\frac{1}{\varrho_p}X\right) + \mathscr{L}_X(\log\varrho_p)\\
&= \mathscr{L}_X\left(\frac{1}{\varrho_p}\right) + \frac{1}{\varrho_p}\operatorname{div}_{\exp_{p*}(\mu_g(p))}(X) + \frac{1}{\varrho_p}\mathscr{L}_X\,\varrho_p\\
&= \frac{1}{\varrho_p}\operatorname{div}_{\exp_{p*}(\mu_g(p))}(X)
\end{aligned}$$

by the chain rule and the behaviour of the divergence operator under the change of the reference density, see e.g. [60, Lemma 2.3.45]. Thus we have for a general vector field X

$$\varrho_p\operatorname{div}\left(\frac{1}{\varrho_p}X\right) = \operatorname{div}_{\exp_{p*}(\mu_g(p))}(X)$$

on U. Since the definition of the divergence operator is natural with respect to diffeomorphisms we have

$$\operatorname{div}_{\exp_{p*}(\mu_g(p))}(X) = \exp_{p*}\left(\operatorname{div}_{\mu_g(p)}(\exp_p^* X)\right).$$

Now we consider again $X = \operatorname{grad}\eta_p$ whence

$$\exp_p^*(\operatorname{grad}\eta_p)\Big|_v = T_{\exp_p(v)}\exp_p^{-1}\left(\operatorname{grad}\eta_p\Big|_{\exp_p(v)}\right)$$

$$= 2T_{\exp_p(v)} \exp_p^{-1} \left(T_{\exp_p^{-1}(\exp_p(v))} \exp_p(\exp_p^{-1}(\exp_p(v))) \right)$$

$$= 2v.$$

With other words

$$\exp_p^*(\operatorname{grad} \eta_p) = 2\xi_{T_pM}$$

is twice the Euler vector field on the tangent space T_pM. But the divergence of ξ_{T_pM} with respect to the constant density is easily seen to be $n = \dim M$. Thus we end up with $\Box\eta_p = 2n + g(\operatorname{grad}\log\varrho_p, \operatorname{grad}\eta_p)$, finishing the proof. □

Remark 3.2.5 In fact, it will be the last statement of the last proposition which causes new complications compared to the trivial, flat case. Here we have of course

$$\Box_{\text{flat}}\eta_p^{\text{flat}} = 2n \tag{3.2.20}$$

without the additional term as in (3.2.19). Clearly $\varrho_p^{\text{flat}} = 1$ whence this additional contribution vanishes. However, (3.2.20) was essential for the correct functional equation of the (flat) Riesz distributions in Section 3.1.

3.2.2 Construction of the Riesz Distributions $R_U^{\pm}(\alpha, p)$

For $\mathsf{Re}(\alpha) > n$ the Riesz distributions $R^{\pm}(\alpha)$ are even continuous functions on Minkowski space. As such we can simply push-forward them via $\exp_p$, at least on the star-shaped $V \subseteq T_pM$, to a continuous function on $U \subseteq M$. There, a continuous function defines a distribution after multiplying with the density μ_g.

Remark 3.2.6 Let $f \in \mathcal{C}^0(T_pM)$ be a continuous function on the tangent space of p. We view f as a distribution as usual via

$$f(\varphi) = \int_{T_pM} f(v)\varphi(v)\,\mu_g(p) \tag{3.2.21}$$

for $\varphi \in \mathcal{C}_0^\infty(T_pM)$. Using $\exp_p$ we can write this as follows. Let $\varphi \in \mathcal{C}_0^\infty(M)$ with $\operatorname{supp}\varphi \subseteq U$ then the continuous function $\exp_{p*}(f\big|_V) \in \mathcal{C}^0(U)$ can be viewed as a distribution on U

$$\exp_{p*}(f\big|_V)(\varphi) = \int_M \exp_{p*}(f\big|_V)(q)\varphi(q)\,\mu_g(q) \tag{3.2.22}$$

according to our convention. This equals

$$\exp_{p*}(f\big|_V)(\varphi) = \int_M \exp_{p*}(f\big|_V)\exp_{p*}(\exp_p^*\varphi)(q)\varrho_p(q)(\exp_{p*}\mu_g(p))(q) \tag{3.2.23}$$

$$= \int_M \exp_{p*}\left(f\big|_V \exp_p^*\varphi \exp_p^*\varrho_p\mu_g(p)\right)(q) \tag{3.2.24}$$

$$= \int_{T_pM} f\exp_p^*\varphi\widetilde{\varrho}_p\,\mu_g(p) \tag{3.2.25}$$

$$= (\widetilde{\varrho}_p f)(\exp_p^*\varphi). \tag{3.2.26}$$

Thus, if we want to have a consistent definition of the push-forward of a distribution on T_pM to a distribution on M we should include the prefactor $\widetilde{\varrho}_p$: let $u \in \mathcal{D}'(T_pM) = \mathcal{C}_0^\infty(T_pM)'$ be a distribution. Then one defines $\exp_{p*} u$ as the distribution $\exp_{p*}(u\big|_V) \in \mathcal{D}'(U) = \mathcal{C}_0^\infty(U)'$ via

$$\exp_{p*}(u\big|_V)(\varphi) = u(\widetilde{\varrho}_p \exp_p^*\varphi), \tag{3.2.27}$$

which is a well-defined distribution as the restriction of u to V is a well-defined distribution on V and $\operatorname{supp}(\widetilde{\varrho}_p \exp_p^* \varphi) \subseteq V$ thanks to $\operatorname{supp} \varphi \subseteq U$. Note that this definition *differs* from the entirely intrinsic definition of the push-forward of distributions in Proposition 1.3.23 in so far as we have modified our notion of distributions itself.

We apply this construction of the push-forward now to the Riesz distributions $R^{\pm}(\alpha)$. First we note that $R^{\pm}(\alpha)$ is intrinsically defined on T_pM *without* specifying a particular isometric isomorphism $(T_pM, g_p) \simeq (\mathbb{R}^n, \eta)$. The reason is that $R^{\pm}(\alpha)$ on Minkowski spacetime is invariant under orthochronous Lorentz transformations. We still denote the Riesz distribution on T_pM by $R^{\pm}(\alpha)$. Then the following definition makes sense:

Definition 3.2.7 (Riesz distributions on U) *Let $p \in M$ and let $U \subseteq M$ be a geodesically star-shaped open neighborhood of p. Moreover, let $V = \exp_p^{-1}(U) \subseteq T_pM$ be the corresponding star-shaped open neighborhood of $0 \in T_pM$. Then the advanced and retarded Riesz distributions $R^{\pm}_U(\alpha, p) \in \mathcal{C}^\infty_0(U)'$ are defined by*

$$R^{\pm}_U(\alpha, p)(\varphi) = \exp_{p*}\left(R^{\pm}(\alpha)\big|_V\right)(\varphi) = R^{\pm}(\alpha)\big|_V \left(\widetilde{\varrho}_p \exp_p^* \varphi\right) \tag{3.2.28}$$

for $\alpha \in \mathbb{C}$ and $\varphi \in \mathcal{C}^\infty_0(U)$.

We collect now the properties of $R^{\pm}(\alpha, p)$ in complete analogy to those of $R^{\pm}(\alpha)$. In fact, most properties can be transferred immediately using (3.2.28). However, when it comes to differentiation, the additional prefactor $\widetilde{\varrho}_p$ has to be taken into account properly.

Proposition 3.2.8 *Let $U \subseteq M$ be geodesically star-shaped around $p \in M$. Then the Riesz distributions $R^{\pm}_U(\alpha, p)$ have the following properties:*

i.) If $\mathsf{Re}(\alpha) > n$ then $R^{\pm}_U(\alpha, p)$ is continuous on U and given by

$$R^{\pm}_U(\alpha, p)(q) = \begin{cases} c(\alpha, n) \left(\eta_p(q)\right)^{\frac{\alpha - n}{2}} & \text{for} \quad q \in I^{\pm}_U(p) \\ 0 & \text{else.} \end{cases} \tag{3.2.29}$$

ii.) For $\mathsf{Re}(\alpha) > n + 2k$ the function $R^{\pm}_U(\alpha, p)$ is even $\mathcal{C}^k$ on U.

iii.) For all α we have $R^{\pm}_U(\alpha, p)\big|_{I^{\pm}_U(p)} = c(\alpha, n)\eta_p^{\frac{\alpha-n}{2}} \in \mathcal{C}^\infty\left(I^{\pm}_U(p)\right)$ and $0 = R^{\pm}(\alpha, p)\big|_{U \setminus J^{\pm}_U(p)} \in \mathcal{C}^\infty\left(U \setminus J^{\pm}_U(p)\right)$.

Proof. By definition of $\exp_{p*}\left(R^{\pm}(\alpha)\big|_V\right)$ the singularities of $R^{\pm}(\alpha)$ correspond one-to-one to the singularities of $R^{\pm}_U(\alpha, p)$ under $\exp_p$ since $\exp_p$ is a *diffeomorphism* and the function $\widetilde{\varrho}_p$ is smooth and nonzero on V. In particular, for $q \notin J^{\pm}_U(p)$ we have $\exp_p^{-1}(q) \notin J^{\pm}(0) \subseteq T_pM$. Thus on this open subset, $R^{\pm}(\alpha)$ coincides with the smooth function being identically zero. This shows $R^{\pm}_U(\alpha, p)\big|_{U \setminus J^{\pm}_U(p)} = 0$. Inside the light cone, i.e. for $q \in I^{\pm}_U(p)$ and hence $\exp_p^{-1}(q) \in I^{\pm}(0)$, we have that $R^{\pm}(\alpha)$ is the smooth function $c(\alpha, n)\widetilde{\eta}_p^{\frac{\alpha-n}{2}}$. Thus by (3.2.27) we have for $\varphi \in \mathcal{C}^\infty_0\left(I^{\pm}_U(p)\right)$

$$\begin{aligned} R^{\pm}_U(\alpha, p)(\varphi) &= R^{\pm}_U(\alpha, p)\Big|_{I^{\pm}_U(p)}(\varphi) \\ &= R^{\pm}(\alpha)\Big|_{I^{\pm}(0)} \left(\widetilde{\varrho}_p \exp_p^* \varphi\right) \\ &= \int_{I^{\pm}(0)} R^{\pm}(\alpha)(v)\widetilde{\varrho}_p(v)(\exp_p^* \varphi)(v) \, \mathrm{d}^n v \end{aligned}$$

$$= \int_{I_U^\pm(p)} c(\alpha, n) \left(\widetilde{\eta}_p \circ \exp_p^{-1}\right)^{\frac{\alpha-n}{2}} (q)\varphi(q)\, \mu_g(q)$$

$$= \int_{I_U^\pm(p)} c(\alpha, n) \left(\eta_p(q)\right)^{\frac{\alpha-n}{2}} \varphi(q)\, \mu_g(q)$$

$$= \left(c(\alpha, n)\eta_p^{\frac{\alpha-n}{2}}\right)(\varphi),$$

since $\widetilde{\eta}_p \circ \exp_p^{-1} = \eta_p$ by definition of η_p. This shows the third part. The first and second part follow from the continuity properties of $R^\pm(\alpha)$ as in Lemma 3.1.3 and Lemma 3.1.6, *ii.)*. □

The analogue of Lemma 3.1.5 and Lemma 3.1.7 is the following statement:

Proposition 3.2.9 *Let $U \subseteq M$ be star-shaped around $p \in M$. Then for every fixed test function $\varphi \in \mathcal{C}_0^\infty(U)$ the map $\alpha \mapsto R_U^\pm(\alpha, p)(\varphi)$ is entirely holomorphic on $\mathbb{C}$.*

Proof. Since for $\varphi \in \mathcal{C}_0^\infty(U)$ the function $\widetilde{\varrho}_p \exp_p^* \varphi$ is a test function on $V \subseteq T_pM$ and hence on T_pM, Lemma 3.1.5 and Lemma 3.1.7 guarantee that $\alpha \mapsto R^\pm(\alpha)(\widetilde{\varrho}_p \exp_p^* \varphi)$ is holomorphic. □

Proposition 3.2.10 *Let $U \subseteq M$ be geodesically star-shaped around $p \in M$.*

i.) For all $\alpha \in \mathbb{C}$ we have

$$\eta_p R_U^\pm(\alpha, p) = \alpha(\alpha - n + 2)R_U^\pm(\alpha + 2, p). \tag{3.2.30}$$

ii.) For all $\alpha \in \mathbb{C}$ we have

$$\operatorname{grad} \eta_p \cdot R_U^\pm(\alpha, p) = 2\alpha \operatorname{grad} R_U^\pm(\alpha + 2, p). \tag{3.2.31}$$

iii.) For all $\alpha \in \mathbb{C} \setminus \{0\}$ we have

$$\Box R_U^\pm(\alpha + 2, p) = \left(\frac{\Box\eta_p - 2n}{2\alpha} + 1\right) R_U^\pm(\alpha, p). \tag{3.2.32}$$

iv.) For $\alpha = 0$ we have

$$R_U^\pm(0, p) = \delta_p. \tag{3.2.33}$$

Proof. The first part is the literal translation of Proposition 3.1.10 *ii.)* together with the fact that $\eta_p = \widetilde{\eta}_p \circ \exp_p^{-1}$. For the second part we have to be slightly more careful: in general, the gradient operator grad on M with respect to g is *not* intertwined into the gradient operator on T_pM with respect to the flat metric g_p via $\exp_p$. This is only true for arbitrary functions if the metric g is *flat*. Nevertheless we have for $\mathsf{Re}(\alpha) > n$ on $I_U^\pm(p)$

$$\begin{aligned} 2\alpha \operatorname{grad} R_U^\pm(\alpha + 2, p) &= 2\alpha c(\alpha + 2, n) \operatorname{grad} \left(\eta_p^{\frac{\alpha+2-n}{2}}\right) \\ &= 2\alpha c(\alpha + 2, n)\frac{\alpha + 2 - n}{2}\eta_p^{\frac{\alpha-n}{2}} \operatorname{grad} \eta_p \\ &= c(\alpha, n)\eta_p^{\frac{\alpha-n}{2}} \operatorname{grad} \eta_p \\ &= \operatorname{grad} \eta_p \cdot R_U^\pm(\alpha, p). \end{aligned}$$

Since for $\mathsf{Re}(\alpha) > n$ the distribution $R_U^\pm(\alpha + 2, p)$ is actually a $\mathcal{C}^1$-function and since on $U \setminus I_U^\pm(p)$ the relation (3.2.31) is trivially fulfilled, (3.2.31) holds on U in the sense of $\mathcal{C}^0$-functions and thus also in the sense of distributions. The usual holomorphy argument shows that (3.2.31) holds for all

$\alpha \in \mathbb{C}$. For the third part we repeat our considerations from Lemma 3.1.6, *iii.)*. We first consider $\mathsf{Re}(\alpha) > n+2$ whence $R_U^\pm(\alpha+2,p)$ is $\mathcal{C}^2$, $R_U^\pm(\alpha,p)$ is $\mathcal{C}^1$, and we can compute $\Box$ in the sense of functions. On $I_U^\pm(p)$ we have

$$
\begin{aligned}
\Box R_U^\pm(\alpha+2,p) &= \operatorname{div}(\operatorname{grad} R_U^\pm(\alpha+2,p)) \\
&\overset{(3.2.31)}{=} \operatorname{div}\left(\frac{1}{2\alpha}\operatorname{grad}\eta_p \cdot R_U^\pm(\alpha,p)\right) \\
&= \frac{1}{2\alpha} g(\operatorname{grad} R_U^\pm(\alpha,p), \operatorname{grad}\eta_p) + \frac{1}{2\alpha} R_U^\pm(\alpha,p)\Box\eta_p \\
&\overset{(3.2.31)}{=} \frac{1}{2\alpha} g\left(\frac{1}{2(\alpha-2)}\operatorname{grad}\eta_p \cdot R_U^\pm(\alpha-2,p), \operatorname{grad}\eta_p\right) + \frac{1}{2\alpha}\Box\eta_p \cdot R_U^\pm(\alpha,p) \\
&\overset{(3.2.18)}{=} \frac{1}{2\alpha}\frac{1}{2(\alpha-2)} 4\eta_p R_U^\pm(\alpha-2,p) + \frac{1}{2\alpha}\Box\eta_p \cdot R_U^\pm(\alpha,p) \\
&\overset{(3.2.30)}{=} \frac{1}{\alpha(\alpha-2)}(\alpha-2)(\alpha-2-n+2) R_U^\pm(\alpha,p) + \frac{1}{2\alpha} R_U^\pm(\alpha,p)\Box\eta_p \\
&= \left(\frac{\alpha-n}{\alpha} + \frac{1}{2\alpha}\Box\eta_p\right) R_U^\pm(\alpha,p) \\
&= \left(\frac{\Box\eta_p - 2n}{2\alpha} + 1\right) R_U^\pm(\alpha,p).
\end{aligned}
$$

Since for $\mathsf{Re}(\alpha) > n+2$ Equation (3.2.32) is an equality between at least continuous functions, we have shown (3.2.32) since on $U \setminus J_U^\pm(p)$ we trivially have (3.2.32) as both sides are identically zero. Thus (3.2.32) holds for $\mathsf{Re}(\alpha) > n+2$ and by the obvious holomorphy in $\alpha \in \mathbb{C}\setminus\{0\}$ of both sides it holds for all $\alpha \neq 0$. Finally, we have

$$
R_U^\pm(0,p)(\varphi) = R^\pm(0)(\widetilde{\varrho}_p \exp_p^* \varphi) = \delta_0(\widetilde{\varrho}_p \exp_p^* \varphi) = \widetilde{\varrho}_p(0)\cdot\varphi(\exp_p(0)) = 1\cdot\varphi(p) = \delta_p(\varphi),
$$

since $\widetilde{\varrho}_p(0) = 1$. $\Box$

Note that in the flat case we have $\Box\eta_p = 2n$ whence (3.2.32) simplifies to $\Box_{\mathrm{flat}} R_{\mathrm{flat}}^\pm(\alpha+2,p) = R_{\mathrm{flat}}^\pm(\alpha,p)$ from which we deduced that $R_{\mathrm{flat}}^\pm(2,p)$ is the Green function to $\Box_{\mathrm{flat}}$ in Theorem 3.1.16. However, in the general situation we have

$$
\Box\eta_p = 2n + g(\operatorname{grad}\log\varrho_p, \operatorname{grad}\eta_p) \tag{3.2.34}
$$

by our computation in Proposition 3.2.8, *iv.)*. This additional term is responsible for the failure of $R_U^\pm(2,p)$ to be a Green function at p.

In order to determine the support and singular support of $R_U^\pm(\alpha,p)$ we recall that under $\exp_p$ the chronological future and past $I^\pm(0)$ of $0 \in T_pM$ are mapped to $I_U^\pm(p)$. The same holds for $J^\pm(0)$ and $J_U^\pm(p)$ since $\exp_p$ is assumed to be a diffeomorphism on the neighborhood U of p. Then the following statement is again a direct consequence of Proposition 3.1.12.

Proposition 3.2.11 (Support and singular support of $R_U^\pm(\alpha,p)$) *Let $U \subseteq M$ be star-shaped around $p \in M$ and let $\alpha \in \mathbb{C}$.*

i.) If α is not exceptional then

$$
\operatorname{supp} R_U^\pm(\alpha,p) = J_U^\pm(p) \tag{3.2.35}
$$

and

$$
\operatorname{sing\,supp} R_U^\pm(\alpha,p) \subseteq \partial I_U^\pm(p). \tag{3.2.36}
$$

ii.) If α is exceptional then

$$\operatorname{sing\,supp} R_U^{\pm}(\alpha, p) = \operatorname{supp} R_U^{\pm}(\alpha, p) \subseteq \partial I_U^{\pm}(p). \tag{3.2.37}$$

iii.) If $n \geq 3$ and $\alpha \in \left\{ n - 2k \middle| k \in \mathbb{N}_0, k < \frac{n}{2} \right\}$ we have

$$\operatorname{sing\,supp} R_U^{\pm}(\alpha, p) = \operatorname{supp} R_U^{\pm}(\alpha, p) = \partial I_U^{\pm}(p). \tag{3.2.38}$$

Proof. This follows from Proposition 3.1.12 and the general behaviour of supp and sing supp under push-forwards with diffeomorphisms and multiplication with positive smooth functions. □

Proposition 3.2.12 (Order of $R_U^{\pm}(\alpha, p)$) *Let $U \subseteq M$ be star-shaped around $p \in M$ and let $\alpha \in \mathbb{C}$.*

i.) If $\mathsf{Re}(\alpha) > n$ then $\operatorname{ord}_U(R_U^{\pm}(\alpha, p)) = 0$.

ii.) The global order of $R_U^{\pm}(\alpha, p)$ is bounded by $2k$ where $k \in \mathbb{N}_0$ is such that $\mathsf{Re}(\alpha) + 2k > n$.

iii.) If $\mathsf{Re}(\alpha) > 0$ then the global order of $R_U^{\pm}(\alpha, p)$ is bounded by n if n is even and by $n + 1$ if n is odd.

Proof. The order of a distribution does not change under push-forwards with diffeomorphisms and multiplication with positive smooth functions. Thus the result follows directly from Proposition 3.1.13. □

Proposition 3.2.13 (Reality of $R_U^{\pm}(\alpha, p)$) *Let $U \subseteq M$ be star-shaped around $p \in M$ and let $\alpha \in \mathbb{C}$. Then we have*

$$\overline{R_U^{\pm}(\alpha, p)} = R_U^{\pm}(\overline{\alpha}, p). \tag{3.2.39}$$

Proof. Since $\widetilde{\varrho}_p = \overline{\widetilde{\varrho}_p} > 0$ this follows from Proposition 3.1.14. □

In a next step we need to understand how the Riesz distribution $R_U^{\pm}(\alpha, p)$ depends on the point $p \in M$. To this end we have to be slightly more specific with our definition of $R_U^{\pm}(\alpha, p)$. In order to compare (3.2.28) for different p it is convenient to choose a common reference Minkowski spacetime. Thus we consider the following situation: assume that U is not only star-shaped with respect to p but also with respect to $p' \in O$ where $O \subseteq U$ is a small open neighborhood of p. In particular, if U is even geodesically convex then we can choose $O = U$. Moreover, let $e_1, \ldots, e_n$ be a smooth Lorentz frame on U inducing isometric isomorphisms

$$I_{p'} : (T_{p'}M, g_{p'}, \uparrow) \longrightarrow (\mathbb{R}^n, \eta, \uparrow) \tag{3.2.40}$$

preserving the time orientation. Clearly, $I_{p'}$ depends smoothly on p' in this case. Then for $\varphi \in \mathcal{C}_0^\infty(U)$ we have for all $p' \in O$

$$R_U^{\pm}(\alpha, p')(\varphi) = R^{\pm}(\alpha) \left(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi) \right) \tag{3.2.41}$$

with $R^{\pm}(\alpha)$ being the Riesz distributions on $\mathbb{R}^n$, independent of p'.

Lemma 3.2.14 *Let $K \subseteq U$ be compact. Then for every compact subset $L \subseteq O$ there exists a compactum $\widetilde{K} \subseteq \mathbb{R}^n$ such that*

$$\operatorname{supp} \left(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi) \right) \subseteq \widetilde{K} \tag{3.2.42}$$

for all $\varphi \in \mathcal{C}_K^\infty(U)$ and all $p' \in L$.

Proof. For all $p' \in O$ the function $x \mapsto \left(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi) \right)(x)$ is a compactly supported smooth function on $\mathbb{R}^n$. Since $I_{p'}$ is a linear isomorphism and $\widetilde{\varrho}_{p'}$ is strictly positive,

$$K_{p'} = \operatorname{supp} \left(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi) \right) = I_{p'}(\exp_{p'}^{-1}(\operatorname{supp} \varphi))$$

by the general behaviour of supports under diffeomorphisms. The various compacta $K_{p'}$ depend on p' in a continuous way. More precisely, there is a map $\Phi_{p'} : \widetilde{V} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $K_{p'} = \Phi_{p'}(K_p)$ which depends continuously on p'. In fact, define

$$\Phi(p', x) = I_{p'}(\exp_{p'}^{-1}(\exp_p(I_p^{-1}(x))))$$

for $x \in I_p(V) \subseteq \mathbb{R}^n$. Then $\Phi : O \times V \longrightarrow \mathbb{R}^n$ is even smooth. Now for $\{p'\} \subseteq O$ compact we have $K_{p'} \subseteq \Phi(\{p'\} \times K_p)$ and thus $\bigcup_{p'} K_{p'} \subseteq \Phi\left(\bigcup_{p'}\{p'\} \times K_p\right)$. If $p' \in L$ runs through a compact subset $L \subseteq O$ then the union of the $K_{p'}$ is contained in a compactum itself since Φ is continuous. This is the $\widetilde{K}$ we are looking for. □

Using this lemma we see that the support of $I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'} \varphi)$ is uniformly contained in some compactum in $\mathbb{R}^n$. This allows to use the continuity of the distributions $R^{\pm}(\alpha)$ to obtain the following result:

Proposition 3.2.15 *Let $U \subseteq M$ be star-shaped around $p \in M$ and let $O \subseteq U$ be an open neighborhood of U such that U is star-shaped around every $p' \in O$.*

i.) For every compacta $K \subseteq U$ and $L \subseteq O$ there exists a constant $c_{K,L,\alpha} > 0$ such that

$$|R_U^{\pm}(\alpha, p')(\varphi)| \leq c_{K,L,\alpha}\, \mathrm{p}_{K,2k}(\varphi) \tag{3.2.43}$$

for all $\varphi \in \mathcal{C}_K^\infty(U)$ and $p' \in L$ where $k \in \mathbb{N}_0$ is such that $\mathsf{Re}(\alpha) + 2k > n$.

ii.) In particular, for $\mathsf{Re}(\alpha) > 0$ and every compacta $K \subseteq U$ and $L \subseteq O$ there exists a constant $c_{K,L,\alpha} > 0$ such that

$$|R_U^{\pm}(\alpha, p')(\varphi)| \leq c_{K,L,\alpha}\, \mathrm{p}_{K,n+1}(\varphi) \tag{3.2.44}$$

for all $\varphi \in \mathcal{C}_K^\infty(U)$.

iii.) Let $k \in \mathbb{N}_0$ satisfy $\mathsf{Re}(\alpha) + 2k > n$. Then for every $\Phi \in \mathcal{C}_0^{2k+\ell}(O \times U)$ the map

$$O \ni p' \;\mapsto\; R_U^{\pm}(\alpha, p')(\Phi(p', \cdot\,)) \in \mathbb{C} \tag{3.2.45}$$

is $\mathcal{C}^\ell$ on O.

iv.) Again, for $\mathsf{Re}(\alpha) > 0$ and $\Phi \in \mathcal{C}^{n+1+\ell}(O \times U)$ the corresponding map (3.2.45) is $\mathcal{C}^\ell$ on O.

v.) Let $\varphi \in \mathcal{C}_0^k(U)$ then the map

$$\alpha \;\mapsto\; R_U^{\pm}(\alpha, p)(\varphi) \tag{3.2.46}$$

is holomorphic for $\mathsf{Re}(\alpha) > n - 2\left[\frac{k}{2}\right]$.

vi.) If $\Phi \in \mathcal{C}^\infty(O \times U)$ is even smooth and has support $\operatorname{supp}\Phi \subseteq O \times K$ with some compact K, then the function

$$O \ni p' \;\mapsto\; R_U^{\pm}(\alpha, p')(\Phi(p', \cdot\,)) \tag{3.2.47}$$

is smooth on O.

Proof. By Lemma 3.2.14 we have a compact subset $\widetilde{K} \subseteq \mathbb{R}^n$ such that

$$\operatorname{supp}\left(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi)\right) \subseteq \widetilde{K}$$

for all $p' \in L$ and $\varphi \in \mathcal{C}_K^\infty(U)$. Thus by continuity of $R^{\pm}(\alpha)$ and the fact that $R^{\pm}(\alpha)$ has order $\leq 2k$ whenever $\mathsf{Re}(\alpha) + 2k > n$, see Proposition 3.1.13, *ii.)*, we have

$$\left|R_U^{\pm}(\alpha, p')(\varphi)\right| = \left|R^{\pm}(\alpha)(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi))\right| \leq c\, \mathrm{p}_{\widetilde{K},2k}(I_{p'*}(\widetilde{\varrho}_{p'} \exp_{p'}^* \varphi)) = c'\, \mathrm{p}_{K,2k}(\varphi),$$

since $I_{p'*}\widetilde{\varrho}_{p'}$ is bounded with all its derivatives on the compactum $\widetilde{K}$ as it is smooth anyway, and $I_{p'*} \exp_{p'}^* \varphi$ is also smooth on $\widetilde{K}$. Since the exponential map $\exp_{p'}$ also depends smoothly on p' all its derivatives up to order $2k$ are bounded as long as $p' \in L$, the same holds for $I_{p'}$. This gives the new constant c' independent of p' but only depending on L. This proves the first part. The second follows since for $\mathsf{Re}(\alpha) > 0$ the order of $R^{\pm}(\alpha)$ is bounded by $n+1$ by Proposition 3.1.13, *iii.)*. The third part follows immediately from the technical Lemma 3.2.14 and a careful counting of the number of derivatives needed in the proof of that lemma, see also Proposition 1.3.39. The fourth part is a particular case thereof. The holomorphy follows immediately from Remark 3.1.17. For the last part note that by definition of $R^{\pm}_U(\alpha, p')$ we have

$$R^{\pm}_U(\alpha, p')(\Phi(p', \cdot)) = R^{\pm}(\alpha)\left(I_{p'*}(\widetilde{\varrho}_{p'} \exp^*_{p'} \Phi(p', \cdot))\right),$$

and the function

$$(p', x) \mapsto I_{p'*}(\widetilde{\varrho}_{p'} \exp^*_{p'} \Phi(p', \cdot))\Big|_x$$

has support in $O \times \widetilde{K}$ with $\widetilde{K} \subseteq \mathbb{R}^n$ compact. Moreover, by the smooth choice of $I_{p'}$ and the smoothness of $\widetilde{\varrho}$ and exp we conclude that it is smooth in *both* variables. Thus we can apply Lemma 1.3.38 to obtain the smoothness of (3.2.47). □

In particular, it follows from the fourth part that the map $p' \mapsto (\mathsf{id} \otimes R^{\pm}_U)(\alpha, p')(\Phi)$ is smooth on O for $\Phi \in \mathcal{C}^\infty_0(O \times U)$.

Let us now discuss an additional symmetry property of the Riesz distributions. In the flat case the exponential map

$$\exp_p : T_pM \longrightarrow M \tag{3.2.48}$$

is just the translation, i.e. for $(M, g) = (\mathbb{R}^n, \eta)$ we have

$$\exp_p(v) = p + v. \tag{3.2.49}$$

Thus in this case for $\mathsf{Re}(\alpha) > 0$ we have

$$R^{\pm}(\alpha, p)(q) = (\exp_{p*} R^{\pm}(\alpha))(q) = R^{\pm}(\alpha)\left(\exp_p^{-1}(q)\right) = R^{\pm}(\alpha)(q - p). \tag{3.2.50}$$

In particular,

$$R^{\pm}(\alpha, p)(q) = R^{\mp}(\alpha, q)(p) \tag{3.2.51}$$

follows since $q - p \in I^+(0)$ iff $p - q \in I^-(0)$ and the function η is invariant under total inversion $x \mapsto -x$. While the phrase "$R^{\pm}(\alpha, p)(q)$ depends only on the difference $q - p$" clearly only makes sense on a vector space, the symmetry feature (3.2.51) remains to be true also in the geometric context. Of course, now we have to take care that the points p and q enter equally in (3.2.51) whence the domain U has to be star-shaped with respect to both. But then we have the following statement:

Proposition 3.2.16 (Symmetry of $R^{\pm}_U(\alpha, p)$) *Let $U \subseteq M$ be geodesically convex and $\alpha \in \mathbb{C}$.*

i.) If $\mathsf{Re}(\alpha) > n$ then

$$R^{\pm}_U(\alpha, p)(q) = R^{\mp}_U(\alpha, q)(p) \tag{3.2.52}$$

for all $p, q \in U$.

ii.) For all $\Phi \in \mathcal{C}^\infty_0(U \times U)$ one has

$$\int_U R^{\pm}(\alpha, p)(\Phi(p, \cdot))\, \mu_g(p) = \int_U R^{\mp}(\alpha, q)(\Phi(\cdot, q))\, \mu_g(q). \tag{3.2.53}$$

Proof. First we note that thanks to the convexity of U the Riesz distributions $R^{\pm}_U(\alpha, p)$ are defined for all $p \in U$. For $\mathsf{Re}(\alpha) > n$ the Riesz distributions are continuous functions explicitly given by (3.2.29) in Proposition 3.2.8, *i.)*. We compute

$$\eta_p(q) = g_p\left(\exp_p^{-1}(q), \exp_p^{-1}(q)\right)$$

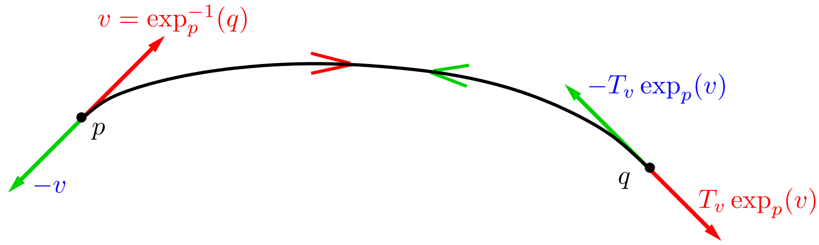

Figure 3.4: A geodesic running backwards.

$$= g_{\exp_p(\exp_p^{-1}(q))}\left(T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)), T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q))\right)$$
$$= g_q\left(T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)), T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q))\right)$$

by the Gauss Lemma. Now $v = \exp_p^{-1}(q)$ is the tangent vector of the geodesic $t \mapsto \exp_p(tv)$ which starts at p and reaches q at $t = 1$. Reversing the time the curve $\tau \mapsto \exp_p((1-\tau)v)$ is still a geodesic which now starts at q for $\tau = 0$ and reaches p at $\tau = 1$. Thus the tangent vector of this geodesic is uniquely fixed to be $\exp_p^{-1}(q)$ since in the convex U the exponential map $\exp_p$ is a diffeomorphism. On the other hand, by the chain rule it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,\tau}\right|_{\tau=0} \exp_p((1-\tau)v) = T_v \exp_p(-v) = -T_v \exp_p(v),$$

whence we have shown

$$\exp_q^{-1}(p) = -T_{\exp_p^{-1}(q)} \exp_p(\exp_p^{-1}(q)). \tag{$*$}$$

It follows that

$$\eta_p(q) = g_q(\exp_q^{-1}(p), \exp_q^{-1}(p)) = \eta_q(p).$$

Since $\eta_p(q)$ is something like the “Lorentz distance square” it is not surprising that this quantity is symmetric in p and q: everything else would be rather disturbing. Since we have a relative sign in $(*)$ we see that if $\exp_p^{-1}(q) \in I^+(0_p)$ then the geodesic $t \mapsto \exp_p(t \exp_p^{-1}(q))$ is future directed for all times whence $\exp_p^{-1}(q) \in I^-(0_q)$ is past directed. From Figure 3.4 this is clear. But then (3.2.52) follows directly from (3.2.29) since the prefactors $c(\alpha, n)$ are the same for the advanced and retarded Riesz distributions. For the second part we first consider $\mathsf{Re}(\alpha) > n$. Then $R_U^{\pm}(\alpha, p)(q)$ is a continuous function on $U \times U$ since $\eta_p(q)$ is smooth in both variables. Thus $R_U^{\pm}(\alpha, p)(\,\cdot\,)$ is locally integrable and hence the function

$$(p, q) \;\mapsto\; R_U^{\pm}(\alpha, p)(q)\Phi(p, q)$$

has compact support and is continuous. Thus we apply Fubini's theorem and interchange the q- and p-integrations

$$\begin{aligned}
\int_U R_U^{\pm}(\alpha, p)(\Phi(p, \cdot))\, \mu_g(p) &= \int_U \int_U R_U^{\pm}(\alpha, p)(q)\Phi(p, q)\, \mu_g(q)\mu_g(p) \\
&= \int_U \int_U R_U^{\pm}(\alpha, p)(q)\Phi(p, q)\, \mu_g(p)\mu_g(q) \\
&\overset{(3.2.52)}{=} \int_U \int_U R_U^{\mp}(\alpha, q)(p)\Phi(p, q)\, \mu_g(p)\mu_g(q) \\
&= \int_U R_U^{\mp}(\alpha, q)(\Phi(\,\cdot\,, q))\, \mu_g(q),
\end{aligned}$$

which proves (3.2.53) for $\mathsf{Re}(\alpha) > n$. For general $\alpha \in \mathbb{C}$ we notice that the integrands of both sides are compactly supported smooth function on U thanks to Proposition 1.3.39 and Remark 1.3.40. Thus the usual Morera type argument shows that both sides are holomorphic functions of α since the integrands are holomorphic in α and we exchange the integrations $\int_U$ and $\int_\Delta \mathrm{d}\,\alpha$ as usual: by holomorphy we conclude that the equality (3.2.53) holds for all α as it holds for $\mathsf{Re}(\alpha) > n$. □

3.3 The Hadamard Coefficients

Differently from the flat situation, the Riesz distribution $R_U^{\pm}(2,p)$ does not yield a fundamental solution for $\Box$. Indeed, we cannot evaluate $\Box R_U^{\pm}(2,p)$ as we did in the flat case since in Proposition 3.2.10 we had to exclude the value of α needed for $\Box R_U^{\pm}(2,p)$ explicitly. Instead, from

$$\Box R_U^{\pm}(\alpha+2,p) = \left(\frac{\Box\eta_p - 2n}{2\alpha} + 1\right) R_U^{\pm}(\alpha,p), \tag{3.3.1}$$

valid for $\alpha \neq 0$ we only see the following: The limit $\alpha \longrightarrow 0$ of the right hand side, which would be the interesting point, is problematic. One has $R_U^{\pm}(0,p) = \delta_p$ but the prefactor itself is singular, at least on first sight. However, the simple pole in $\frac{\Box\eta_p - 2n}{2\alpha}$ is not as dangerous as it seems. In fact, we *know* that $\alpha \mapsto R_U^{\pm}(\alpha+2,p)(\Box\varphi)$ is holomorphic on the whole complex plane. Hence the limit $\alpha \longrightarrow 0$ of the left hand side certainly exists. Thus we *do* have an analytic continuation of the right hand side for $\alpha = 0$, the singularity was not present after all. However, the precise value at $\alpha = 0$ is hard to obtain and not just δ_p. Of course, we know it is $\Box R_U^{\pm}(2,p)$, but this does not help.

Thus one proceeds differently. The Ansatz is to use all Riesz distributions $R_U^{\pm}(2+2k,p)$ and approximate the true Green function by a series in the $R_U^{\pm}(2+2k,p)$ for $k \in \mathbb{N}_0$ with appropriate coefficients. These coefficients are the Hadamard coefficients we are going to determine now. The expansion we obtain can be thought of as an expansion of the Green functions in increasing regularity as the $R_U^{\pm}(2+2k,p)$ become more and more regular for $k \longrightarrow \infty$.

3.3.1 The Ansatz for the Hadamard Coefficients

The setting will the following. We consider a normally hyperbolic differential operator $D = \Box^{\nabla} + B$ on some vector bundle $E \longrightarrow M$ over M with induced connection ∇^E and $B \in \Gamma^\infty(\mathsf{End}(E))$ as in Section 2.1.4. Moreover, for $p \in M$ we choose a geodesically star-shaped open neighborhood $U \subseteq M$ on which $R_U^{\pm}(\alpha,p)$ is defined as before. According to our convention for distributions, the Green functions are now generalized sections

$$\mathcal{R}^{\pm}(p) \in \Gamma^{-\infty}(E) \otimes E_p^*, \tag{3.3.2}$$

as we take care of the density part using μ_g. The pairing with a test section $\varphi \in \Gamma_0^\infty(E^*)$ yields then an element in E_p^*. The equation to solve is

$$D\mathcal{R}^{\pm}(p) = \delta_p, \tag{3.3.3}$$

where δ_p is viewed as E_p^*-valued distribution on $\Gamma_0^\infty(E^*)$ and $D\mathcal{R}^{\pm}(p)$ is defined as usual.

The Ansatz for $\mathcal{R}^{\pm}(p)$ is now the following. Since the $R_U^{\pm}(\alpha,p)$ have increasing regularity for increasing $\mathsf{Re}(\alpha)$ we try a series

$$\mathcal{R}^{\pm}(p) = \sum_{k=0}^{\infty} V_p^k R_U^{\pm}(2+2k,p) \tag{3.3.4}$$

with *smooth* sections

$$V_p^k \in \Gamma^\infty\left(E\big|_U\right) \otimes E_p^*. \tag{3.3.5}$$

Then (3.3.4) should be thought of as an expansion with respect to regularity. The starting point for $k = 0$ will be the most singular term coming from $R_U^{\pm}(2, p)$. Of course, such an Ansatz can hardly be expected to work just like that. Even if we can find reasonable V_p^k such that (3.3.3) holds "in each order of regularity", the series (3.3.4) has to be shown to converge: In fact, this will not be the case (except for some very particular cases) whence we have to go a step beyond (3.3.4). However, for the time being we shall investigate the Ansatz (3.3.4).

First we note that a scalar distribution like $R_U^{\pm}(\alpha, p)$ can be multiplied with a smooth section like V_p^k and yields a distributional section

$$V_p^k R_U^{\pm}(2 + 2k, p) \in \Gamma_0^{\infty}(E^*)' \otimes E_p^* = \Gamma^{-\infty}(E) \otimes E_p^*. \tag{3.3.6}$$

In Remark 1.3.9 it is only necessary that one factor of the product is actually smooth. We compute now (3.3.3). First we assume that the series (3.3.4) converges at least in the weak* topology so that we can apply D componentwise. This yields

$$\begin{aligned}
D\mathcal{R}^{\pm}(p) &= D \sum_{k=0}^{\infty} V_p^k R_U^{\pm}(2 + 2k, p) \\
&= \sum_{k=0}^{\infty} D\left(V_p^k R_U^{\pm}(2 + 2k, p)\right) \\
&= \sum_{k=0}^{\infty} \left(D(V_p^k) R_U^{\pm}(2 + 2k, p) + 2\nabla^E_{\operatorname{grad} R_U^{\pm}(2+2k,p)} V_p^k + V_p^k \Box R_U^{\pm}(2 + 2k, p)\right)
\end{aligned} \tag{3.3.7}$$

by the Leibniz rule of a normally hyperbolic differential operator as in Remark 2.1.27, *i.)*. Note that in (2.1.68) it is sufficient that one of the factors is smooth. Inserting the properties of $R_U^{\pm}(\alpha, p)$ from Proposition 3.2.8 yields then

$$\begin{aligned}
D\mathcal{R}^{\pm}(p) &= D(V_p^0) R_U^{\pm}(2, p) + 2\nabla^E_{\operatorname{grad} R_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) \\
&\quad + \sum_{k=1}^{\infty} \left(D(V_p^k) R_U^{\pm}(2 + 2k, p) + 2\nabla^E_{\frac{1}{4k} R_U^{\pm}(2k,p) \operatorname{grad} \eta_p} V_p^k + V_p^k \left(\frac{\Box\eta_p - 2n}{4k} + 1\right) R_U^{\pm}(2k, p)\right) \\
&= 2\nabla^E_{\operatorname{grad} R_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) + \sum_{k=0}^{\infty} D(V_p^k) R_U^{\pm}(2 + 2k, p) \\
&\quad + \sum_{k=1}^{\infty} \left(2\nabla^E_{\frac{1}{4k} \operatorname{grad} \eta_p} V_p^k + V_p^k \left(\frac{\Box\eta_p - 2n}{4k} + 1\right)\right) R_U^{\pm}(2k, p) \\
&= 2\nabla^E_{\operatorname{grad} R_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) \\
&\quad + \sum_{k=1}^{\infty} \left(D(V_p^{k-1}) + 2\nabla^E_{\frac{1}{4k} \operatorname{grad} \eta_p} V_p^k + \left(\frac{\Box\eta_p - 2n}{4k} + 1\right) V_p^k\right) R_U^{\pm}(2k, p).
\end{aligned} \tag{3.3.8}$$

We view (3.3.8) as an expansion with respect to regularity. Thus, we ask for (3.3.7) in each "order", i.e. (3.3.7) should be fulfilled for each component in front of the $R_U^{\pm}(2k, p)$. This yields the following equations. In lowest order we have for V_p^0 the equation

$$2\nabla^E_{\operatorname{grad} R_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) = \delta_p, \tag{3.3.9}$$

while for $k \geq 1$ we have the recursive equations

$$\frac{1}{2k} \nabla^E_{\operatorname{grad} \eta_p} V^k_p + \left(\frac{\Box \eta_p - 2n}{4k} + 1 \right) V^k_p = -D(V^{k-1}_p) \tag{3.3.10}$$

for V^k_p. Equivalently, we can write this for $k \geq 1$ as

$$\nabla^E_{\operatorname{grad} \eta_p} V^k_p + \left(\frac{1}{2} \Box \eta_p - n + 2k \right) V^k_p = -2kD(V^{k-1}_p). \tag{3.3.11}$$

Since (3.3.11) also makes sense for $k = 0$ it seems tempting to unify (3.3.9) and (3.3.11). To this end, we take (3.3.11) for $k = 0$ and multiply this by $R^{\pm}_U(\alpha, p)$ yielding

$$\nabla^E_{\operatorname{grad} \eta_p R^{\pm}_U(\alpha,p)} V^0_p + \left(\frac{1}{2} \Box \eta_p - n \right) V^0_p R^{\pm}_U(\alpha, p) = 0, \tag{3.3.12}$$

which is equivalent to

$$\nabla^E_{2\alpha \operatorname{grad} R^{\pm}_U(\alpha+2,p)} V^0_p + \alpha \left(\Box R^{\pm}_U(\alpha + 2, p) - R^{\pm}_U(\alpha, p) \right) V^0_p = 0, \tag{3.3.13}$$

by Proposition 3.2.10. Now we can divide by α and obtain the condition

$$2\nabla^E_{\operatorname{grad} R^{\pm}_U(\alpha+2,p)} V^0_p + \left(\Box R^{\pm}_U(\alpha + 2, p) - R^{\pm}_U(\alpha, p) \right) V^0_p = 0, \tag{3.3.14}$$

whose limit $\alpha \longrightarrow 0$ exists and is given by

$$2\nabla^E_{\operatorname{grad} R^{\pm}_U(2,p)} V^0_p + \left(\Box R^{\pm}_U(2, p) - R^{\pm}_U(0, p) \right) V^0_p = 0, \tag{3.3.15}$$

since $R^{\pm}_U(\alpha, p)$ is holomorphic in α for all $\alpha \in \mathbb{C}$. Since moreover $R^{\pm}_U(0, p) = \delta_p$ we can evaluate the condition (3.3.15) further and obtain

$$2\nabla^E_{\operatorname{grad} R^{\pm}_U(2,p)} V^0_p + V^0_p \Box R^{\pm}_U(2, p) = V^0_p \delta_p. \tag{3.3.16}$$

Thus we conclude that (3.3.11) for $k = 0$ implies (3.3.9) iff $V^0_p(p) = \mathsf{id}_{E_p}$. This motivates that we want to solve (3.3.9) with the additional requirement

$$V^0_p(p) = \mathsf{id}_{E_p}, \tag{3.3.17}$$

which we can view as an *initial condition*. Indeed, all the gradients $\operatorname{grad} R^{\pm}_U(\alpha, p)$ are pointing in "radial" direction parallel to $\operatorname{grad} \eta_p$ by Proposition 3.2.10. Thus a differential equation like (3.3.9) should have a unique solution once the value is fixed in the "center", i.e. at p. Then one has just to follow the flow of $\operatorname{grad} \eta_p$ in order to determine the value elsewhere. Of course, this geometric intuition has to be justified more carefully. In any case, we take these heuristic considerations as motivation for the following definition:

Definition 3.3.1 (Transport equations) *Let $k \in \mathbb{N}_0$ and let $D \in \operatorname{DiffOp}^2(E)$ be normally hyperbolic. Then the recursive equations*

$$\nabla^E_{\operatorname{grad} \eta_p} V^k_p + \left(\frac{1}{2} \Box \eta_p - n + 2k \right) V^k_p = -2kDV^{k-1}_p \tag{3.3.18}$$

together with the initial condition

$$V^0_p(p) = \mathsf{id}_{E_p} \tag{3.3.19}$$

are called the transport equations for $V^k_p \in \Gamma^\infty(E\big|_U) \otimes E^*_p$ corresponding to D.

Remark 3.3.2 (Transport equations) Let $D \in \mathrm{DiffOp}^2(E)$ be normally hyperbolic.

i.) According to our above computation, the transport equation for $k = 0$ implies

$$2\nabla^E_{\operatorname{grad} R^{\pm}_U(2,p)} V^0_p + V^0_p \Box R^{\pm}_U(2,p) = \delta_p. \tag{3.3.20}$$

ii.) The transport equations are the same for the advanced and retarded $\mathcal{R}^{\pm}(p)$. Thus we only have to solve them once and can us the *same* coefficients V^k_p for both Green functions.

Definition 3.3.3 (Hadamard coefficients) *Let $D \in \mathrm{DiffOp}^2(E)$ be normally hyperbolic and $U \subseteq M$ geodesically star-shaped around $p \in M$ as before. Solutions $V^k_p \in \Gamma^\infty(E\big|_U) \otimes E^*_p$ of the transport equations are then called Hadamard coefficients for D at the point p.*

In the following we shall now explicitly construct the Hadamard coefficients and show their uniqueness. Note however, that even having the V^k_p does not yet solve the problem of finding a Green function since the convergence of (3.3.4) is still delicate.

3.3.2 Uniqueness of the Hadamard Coefficients

We shall now prove that the Hadamard coefficients are necessarily unique. To this end we need the parallel transport in E with respect to the covariant derivative ∇^E induced by D. Since on U we have unique geodesics joining p with any other point $q \in U$, namely

$$\gamma_{p\to q}(t) = \exp_p(t \exp_p^{-1}(q)), \tag{3.3.21}$$

we shall always use these paths for parallel transport. For abbreviation, we set

$$P_{p\to q} = P_{\gamma_{p\to q}, 0\to 1} : E_p \longrightarrow E_q. \tag{3.3.22}$$

From the explicit definition of the parallel transport we find the following technical statement:

Lemma 3.3.4 *The parallel transport along geodesics in U yields a smooth map*

$$U \ni q \mapsto P_{p\to q} \in E_q \otimes E^*_p, \tag{3.3.23}$$

which we can view as a smooth section

$$P_{p\to\cdot} \in \Gamma^\infty(E\big|_U) \otimes E^*_p. \tag{3.3.24}$$

Proof. Let $e_\alpha \in \Gamma^\infty(E\big|_U)$ be a locally defined smooth frame and let A^β_α be the corresponding smooth connection one-forms. Then the parallel transport is determined by the equation

$$\dot{s}^\beta(t) + A^\beta_\alpha\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\exp_p(t\exp_p^{-1}(q))\right) s^\alpha(t) = 0. \tag{$*$}$$

Since the map $(q,t) \mapsto \exp_p(t\exp_p^{-1}(q))$ is smooth on an open neighborhood of $U \times [0,1] \subseteq U \times \mathbb{R}$ the solutions to $(*)$ also depend smoothly on q and t on this neighborhood. Thus, the solutions depend smoothly on q when evaluated at $t = 1$, which implies the smoothness of (3.3.24). □

Using this smoothness of the parallel transport we can obtain the following result:

Theorem 3.3.5 (Uniqueness of the Hadamard coefficients) *Let $U \subseteq M$ be geodesically star-shaped around p and let $D \in \mathrm{DiffOp}^2(E)$ be normally hyperbolic. Then the Hadamard coefficients for D at p are necessarily unique. In fact, they satisfy*

$$V^0_p = \frac{1}{\sqrt{\varrho_p}} P_{p\to\cdot}. \tag{3.3.25}$$

and for $k \geq 1$ and $q \in U$

$$V_p^k(q) = -k \frac{1}{\sqrt{\varrho_p(q)}} P_{p \to q} \left(\int_0^1 \sqrt{\varrho_p}(\gamma_{p\to q}(\tau)) \tau^{k-1} P^{-1}_{\gamma_{p\to q}, 0 \to \tau} \left(D(V_p^{k-1})(\gamma_{p\to q})(\tau) \right) \right) \mathrm{d}\tau. \tag{3.3.26}$$

Proof. We consider the "Lorentz radius" function $r_p = \sqrt{|\eta_p|} \in \mathcal{C}^0(U)$ which is continuous but not differentiable. However, on $U \setminus C_U(p)$ where $C_U(p) = C_U^+(p) \cup C_U^-(p)$ with

$$C_U^{\pm}(p) = \exp_p(C^{\pm}(0) \cap V),$$

the function η_p is non-zero and hence $r_p \in \mathcal{C}^\infty(U \setminus C_U(p))$ is smooth. On $U \setminus C_U(p)$ we have

$$\eta_p = \epsilon r_p^2$$

with $\epsilon(q) = +1$ for $\exp_p^{-1}(q)$ timelike and $\epsilon(q) = -1$ for $\exp_p^{-1}(q)$ spacelike, respectively. Using our results from Proposition 3.2.4 we find

$$\frac{1}{2}\Box\eta_p - n = \frac{1}{2} g(\operatorname{grad}\log\varrho_p, \operatorname{grad}\eta_p) = \frac{1}{2}\mathscr{L}_{\operatorname{grad}\eta_p} \log\varrho_p = \mathscr{L}_{\operatorname{grad}\eta_p} \log\sqrt{\varrho_p},$$

valid on U since $\varrho_p > 0$. Moreover, by (3.2.18) we get on $U \setminus C_U(p)$

$$\begin{aligned} \mathscr{L}_{\operatorname{grad}\eta_p}(\log r_p^k) &= k\,\mathscr{L}_{\operatorname{grad}\eta_p}(\log r_p) = k\frac{1}{r_p}\mathscr{L}_{\operatorname{grad}\eta_p}(r_p) = k\frac{1}{\sqrt{\epsilon\eta_p}}\mathscr{L}_{\operatorname{grad}\eta_p}\left(\sqrt{\epsilon\eta_p}\right) \\ &= k\frac{1}{\sqrt{\epsilon\eta_p}}\frac{\epsilon}{2\sqrt{\epsilon\eta_p}}\mathscr{L}_{\operatorname{grad}\eta_p}\eta_p = \frac{k}{2\eta_p}\langle\operatorname{grad}\eta_p, \operatorname{grad}\eta_p\rangle = 2k. \end{aligned}$$

Since $\sqrt{\varrho_p} r_p^k > 0$ on $U \setminus C_U(p)$ we can rewrite the transport equation (3.3.18) equivalently as

$$-2kD(V_p^{k-1}) = \nabla^E_{\operatorname{grad}\eta_p} V_p^k + \left(\frac{1}{2}\Box\eta_p - n + 2k\right) V_p^k = \nabla^E_{\operatorname{grad}\eta_p} V_p^k + \frac{1}{\sqrt{\varrho_p} r_p^k}\mathscr{L}_{\operatorname{grad}\eta_p}\left(\sqrt{\varrho_p} r_p^k\right) V_p^k,$$

and thus as

$$\nabla^E_{\operatorname{grad}\eta_p}\left(\sqrt{\varrho_p} r_p^k V_p^k\right) = -\sqrt{\varrho_p} r_p^k 2kD(V_p^{k-1}). \tag{$*$}$$

Since on $U \setminus C_U(p)$ the additional factor $\sqrt{\varrho_p} r_p^k$ is both positive and smooth, $(*)$ is equivalent to the transport equation on $U \setminus C_U(p)$.

Now we consider first $k = 0$. Then $(*)$ means that

$$\nabla^E_{\operatorname{grad}\eta_p}\left(\sqrt{\varrho_p} V_p^0\right)\Big|_{U \setminus C_U(p)} = 0. \tag{$**$}$$

Since the gradient $\operatorname{grad}\eta_p$ is at every point q just twice the tangent vector of the geodesic $\gamma_{p\to q}(t) = \exp_p(t\exp_p^{-1}(q))$ we conclude from $(**)$ that the local section $\sqrt{\varrho_p} V_p^0 \in \Gamma^\infty(E\big|_U) \otimes E_p^*$ is covariantly constant in direction of all geodesics $\gamma_{p\to q}$ as long as $q \in U \setminus C_U(p)$, i.e. as long as $\exp_p^{-1}(q)$ is either timelike or spacelike. But $\sqrt{\varrho_p} V_p^0$ is smooth and thus by continuity we conclude that $(**)$ holds on all of U. But this shows that $\sqrt{\varrho_p} V_p^0$ is parallel along all geodesics starting at p whence it is given by means of the parallel transport, i.e.

$$\sqrt{\varrho_p} V_p^0\big|_q = P_{p\to q}\left(\sqrt{\varrho_p} V_p^0\big|_p\right) = P_{p\to q}\left(1 \cdot \mathsf{id}_{E_p}\right) = P_{p\to q},$$

since by assumption $V_p^0(p) = \mathsf{id}_{E_p}$ and $\varrho_p(p) = 1$ by Proposition 3.2.2. Indeed, if $e_\alpha \in E_p$ is a basis then $P_{p\to q}(\mathsf{id}_{E_p}) = P_{p\to q}(e_\alpha \otimes e^\alpha) = P_{p\to q}(e_\alpha) \otimes e^\alpha = P_{p\to q}$ since the parallel transport only acts on the E_p-part of id_{E_p} and not on the E_p^*-part which is considered as values in all of our considerations up to now. But this shows (3.3.25) and hence the uniqueness of V_p^0.

Now let $k \geq 1$. Then we again consider $(*)$ on $U \setminus C_U(p)$. To this end we first note that since $\operatorname{grad} \eta_p$ is twice the push-forward of the Euler vector field ξ_{T_pM} on T_pM its flow can be computed explicitly. In fact, let $c(t) = \exp_p(\mathrm{e}^{2t} \exp_p^{-1}(q))$ then for small t around 0 we have by Proposition 3.2.4

$$\dot{c}(t) = T_{\exp_p(\mathrm{e}^{2t}\exp_p^{-1}(q))} \exp_p \left(2\mathrm{e}^{2t} \exp_p^{-1}(q)\right) = 2T_{c(t)} \exp_p(\exp_p^{-1}(c(t))) = \operatorname{grad} \eta_p\big|_{c(t)},$$

whence $c(t)$ is the integral curve of $\operatorname{grad} \eta_p$ through $c(0) = q$. Thus $(*)$ implies

$$\nabla^{\#}_{\frac{\partial}{\partial t}} \left(c^{\#} \sqrt{\varrho_p} r_p^k V_p^k\right) = -2kc^{\#} \left(\sqrt{\varrho_p} r_p^k D(V_p^{k-1})\right), \tag{$***$}$$

where $\nabla^{\#}$ is the pull-back connection with respect to the curve c. Thus $\sqrt{\varrho_p} r_p^k V_p^k$ satisfies the perturbed parallel transport equation along c with perturbation given by the right hand side of $(***)$. The solutions of such equations are obtained in terms of the parallel transport as follows:

Lemma 3.3.6 *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve on an open interval and let $f \in \Gamma^\infty(\gamma^{\#} E)$ be a smooth section. Then the perturbed parallel transport equation*

$$\nabla^{\#}_{\frac{\partial}{\partial t}} s = f \tag{3.3.27}$$

has

$$s(t) = P_{\gamma, t_0 \to t} \left(s(t_0) + \int_{t_0}^t P^{-1}_{\gamma, t_0 \to \tau}(f(\tau)) \,\mathrm{d}\tau\right) \tag{3.3.28}$$

as unique and smooth solution $s \in \Gamma^\infty(\gamma^{\#} E)$ with initial condition $s(t_0) \in E_{\gamma(t_0)}$ for $t_0 \in I$.

Proof. We choose a frame $e_\alpha(t_0) \in E_{\gamma(t_0)}$ at $\gamma(t_0)$ and parallel transport it to $e_\alpha(t) = P_{\gamma, t_0 \mapsto t}(e_\alpha(t_0))$. This yields a covariantly constant frame $e_\alpha \in \Gamma^\infty(\gamma^{\#} E)$, i.e. we have $\nabla^{\#}_{\frac{\partial}{\partial t}} e_\alpha = 0$, see also the proof of Lemma A.1.1. Then $s(t) = s^\alpha(t) e_\alpha(t)$ for any section $s \in \Gamma^\infty(\gamma^{\#} E)$ with $s^\alpha \in \mathcal{C}^\infty(I)$. Thus (3.3.27) becomes

$$\dot{s}^\alpha(t) = f^\alpha(t)$$

for all α with initial conditions $s^\alpha(t_0)$ for $t = t_0$. The unique solution to this system of ordinary first order differential equations is

$$s^\alpha(t) = s^\alpha(t_0) + \int_{t_0}^t f^\alpha(\tau) \,\mathrm{d}\tau. \tag{$\diamond$}$$

Now we compute

$$P_{\gamma, t_0 \to t} \circ P^{-1}_{\gamma, t_0 \to \tau}(f^\alpha(\tau) e_\alpha(\tau)) = f^\alpha(\tau) P_{\gamma, t_0 \to t}(e_\alpha(t_0)) = f^\alpha(\tau) e_\alpha(t),$$

whence

$$\begin{aligned} P_{\gamma, t_0 \to t} \left(s(t_0) + \int_{t_0}^t P^{-1}_{\gamma, t_0 \to \tau}(f(\tau)) \,\mathrm{d}\tau\right) &= P_{\gamma, t_0 \to t}(s^\alpha(t_0) e_\alpha(t_0)) + \int_{t_0}^t f^\alpha(\tau) e_\alpha(t) \,\mathrm{d}\tau \\ &= \left(s^\alpha(t_0) + \int_{t_0}^t f^\alpha(\tau) \,\mathrm{d}\tau\right) e_\alpha(t) \\ &= s^\alpha(t) e_\alpha(t) \\ &= s(t) \end{aligned}$$

as wanted. The uniqueness is clear from specifying the initial conditions and smoothness follows from the smoothness of the f^α and the explicit form ($\diamond$). $\triangledown$

We apply the lemma to the curve $c(t) = \exp_p\left(\mathrm{e}^{2t}\exp_p^{-1}(q)\right)$ where $v = \exp_p^{-1}(q)$ is either spacelike or timelike and $t \in (-\infty, \epsilon)$ with some small $\epsilon > 0$ such that $\mathrm{e}^{2t}v$ is still in the domain $V \subseteq T_pM$ of $\exp_p$. Then the *homogeneous* transport equation for $k \geq 1$ implies

$$\nabla^{\#}_{\frac{\partial}{\partial t}} c^{\#}\left(\sqrt{\varrho_p} r_p^k V_p^k\right) = 0, \tag{☹}$$

and hence

$$\sqrt{\varrho_p} r_p^k V_p^k\big|_{c(t)} = P_{c,t_0\to t}\left(\sqrt{\varrho_p} r_p^k V_p^k\big|_{c(t_0)}\right).$$

Taking e.g. $t_0 = 0$ we obtain for all $t \in (-\infty, \epsilon)$

$$V_p^k(\exp_p(\mathrm{e}^{2t}v)) = \frac{1}{\left(\sqrt{\varrho_p} r_p^k\right)\left(\exp_p(\mathrm{e}^{2t}v)\right)} P_{c,0\to t}\left(\sqrt{\varrho_p} r_p^k V_p^k\big|_q\right).$$

Suppose $V_p^k(q) \neq 0$ then also $\sqrt{\varrho_p} r_p^k V_p^k\big|_q \neq 0$. Since the parallel transport is reparametrization invariant we can write this equally well as

$$V_p^k(\exp_p(tv)) = \frac{1}{\left(\sqrt{\varrho_p} r_p^k\right)\left(\exp_p(\mathrm{e}^{2t}v)\right)} P_{\gamma,t_0\to t}\left(\sqrt{\varrho_p} r_p^k V_p^k\big|_q\right), \tag{☺}$$

with $\gamma(t) = \exp_p(tv)$ being the geodesic reparametrization by the "arc length". Now the limit $t \longrightarrow 0$ of $P_{\gamma,t_0\to t}$ exists and is given by $P^{-1}_{p\to q}$. Thus the limit of $t \longrightarrow 0$ of $P_{\gamma,t_0\to t}\left(\sqrt{\varrho_p} r_p^k V_p^k\big|_q\right)$ exists and is a certain non-zero vector. But for $k \geq 1$ the limit of $r_p^k(\exp_p(tv))$ for $t \longrightarrow 0$ is 0 whence the prefactor in (☺) becomes singular. Thus V_p^k can not be continuous at p. Thus $V_p^k(q) \neq 0$ for *some* $q \in U \setminus C_U(p)$ already implies that V_p^k is non-continuous at p. We conclude that the homogeneous equation (☹) has only the trivial solution $V_p^k = 0$ as everywhere *smooth* solution. This implies that the inhomogeneous transport equation ($***$) can have at most *one* everywhere smooth solution which shows uniqueness of the V_p^k for $k \geq 1$. It remains to shows that they necessarily satisfy Equation (3.3.26). According to Lemma 3.3.6, a particular solution along the curve c is given by

$$\left(\sqrt{\varrho_p} r_p^k V_p^k)(c(t)\right) = P_{c,t_0\to t}\left(\left(\sqrt{\varrho_p} r_p^k V_p^k\right)(c(t_0)) + \int_{t_0}^t \left(P_{\gamma,t_0\to\tau}\right)^{-1}\left(-2k\sqrt{\varrho_p} r_p^k D(V_p^{k-1})\right)(c_\tau)\,\mathrm{d}\tau\right),$$

where $t, t_0 \in (-\infty, \epsilon)$ with some suitable $\epsilon > 0$. Since we are interested in a solution V_p^k which is still defined at $q = p$, the limit of $V_p^k(c(t_0))$ for $t_0 \longrightarrow -\infty$ should exist. But then the limit $t_0 \longrightarrow -\infty$ yields $(\sqrt{\varrho_0} r_p^k)(c(t_0)) \longrightarrow (\sqrt{\varrho_0} r_p^k)(p) = 0$ whence the first term on the right hand side does not contribute in this limit. Thus under the regularity assumption we have

$$\begin{aligned}
(\sqrt{\varrho_p} r_p^k V_p^k)(c(t)) &= P_{c,-\infty\to t}\left(\int_{-\infty}^t \left(P_{\gamma,-\infty\to\tau}\right)^{-1}\left(-2k\sqrt{\varrho_p} r_p^k D(V_p^{k-1})\right)(c_\tau)\,\mathrm{d}\tau\right) \\
&= -2kP_{c,-\infty\to t}\int_{-\infty}^t \sqrt{\varrho_p}(c(\tau)) r_p^k(c(\tau)) P^{-1}_{c,-\infty\to\tau}\left(D(V_p^{k-1})(c(\tau))\right)\mathrm{d}\tau.
\end{aligned}$$

Now we have by the isometry properties of the exponential map in radial direction

$$\begin{aligned}
r_p(c(t)) &= \sqrt{|\eta_p|}\exp_p\left(\mathrm{e}^{2t}\exp_p^{-1}(q)\right) \\
&= \sqrt{\left|g_p\left(\mathrm{e}^{2t}\exp_p^{-1}(q), \mathrm{e}^{2t}\exp_p^{-1}(q)\right)\right|}
\end{aligned}$$

$$= \mathrm{e}^{2t}\sqrt{\left|g_p\left(\exp_p^{-1}(q), \exp_p^{-1}(q)\right)\right|},$$

and by assumption $g_p\left(\exp_p^{-1}(q), \exp_p^{-1}(q)\right) \neq 0$. After dividing by $\sqrt{\left|g_p\left(\exp_p^{-1}(q), \exp_p^{-1}(q)\right)\right|}$ we obtain

$$\begin{aligned}
\mathrm{e}^{2kt}\sqrt{\varrho_p}(c(t))V_p^k(c(t)) &= -2kP_{c,-\infty\to t}\int_{-\infty}^{t}\sqrt{\varrho_p}(c(\tau))\mathrm{e}^{2k\tau}P^{-1}_{c,-\infty\to\tau}\left(D(V_p^{k-1})(c(\tau))\right)\mathrm{d}\,\tau \\
&= -2kP_{\alpha,0\to\mathrm{e}^{2t}}\int_0^{\mathrm{e}^{2t}}\sqrt{\varrho_p}(c(\sigma))\sigma^k P^{-1}_{\gamma,0\to\sigma}\left(D(V_p^{k-1})(\gamma(\sigma))\right)\frac{\mathrm{d}\,\sigma}{2\sigma},
\end{aligned}$$

with the substitution $\sigma = \mathrm{e}^{2\tau}$ and thus $\mathrm{d}\,\tau = \frac{\mathrm{d}\,\sigma}{2\sigma}$. Note that with this substitution $c(\tau(\sigma)) = \exp_p\left(\mathrm{e}^{2\tau(\sigma)}\exp_p^{-1}(q)\right) = \exp_p(\sigma\exp_p^{-1}(q)) = \gamma(\sigma)$ is indeed the geodesic from p to q. By the invariance under reparametrization of the parallel transport we get $P_{c,-\infty\to\tau} = P_{\gamma,0\to\sigma}$ which explains the above formula. Taking $t = 0$ we find $c(0) = \gamma(1) = q$ and thus

$$\sqrt{\varrho_p(q)}V_p^k(q) = -kP_{p\to q}\int_0^1\sqrt{\varrho_p}(\gamma_{p\to q}(\tau))\tau^{k-1}P^{-1}_{\gamma_{p\to q},0\to\tau}\left(D(V_p^{k-1})(\exp_p(\tau\exp_p^{-1}(q)))\right)\mathrm{d}\,\tau$$

after replacing σ by τ again. Since $\sqrt{\varrho_p} > 0$ this gives (3.3.26). Indeed, (3.3.26) only follows for $q \in U \setminus C_U(p)$ but the continuity of the right hand side makes (3.3.26) correct everywhere. □

Remark 3.3.7 Note that the additional r_p^k in the higher transport equations yields a completely different behaviour of the solution for $q \longrightarrow p$. While for $k = 0$ no singularities arise the case $k \geq 1$ behaves much more singular. In fact, only one solution is everywhere smooth. This is the reason why for $k = 0$ we have to specify an initial condition $V_p^0(p) = \mathsf{id}_{E_p}$ while for $k \geq 1$ the boundary condition of being smooth at $q = p$ fixes the solution.

3.3.3 Construction of the Hadamard Coefficients

In Theorem 3.3.5 we have not only shown the uniqueness of the Hadamard coefficients which was essentially a consequence of the desired smoothness at p but we also obtained a rather explicit recursive formula for the V_p^k. Using (3.3.25) and (3.3.26) we recursively *define* V_p^k for $k \geq 0$ by

$$V_p^0(q) = \frac{1}{\sqrt{\varrho_p(q)}}P_{p\to q} \tag{3.3.29}$$

and

$$V_p^k(q) = -\frac{k}{\sqrt{\varrho_p(q)}}P_{p\to q}\int_0^1\sqrt{\varrho_p}(\gamma_{p\to q}(\tau))\tau^{k-1}P^{-1}_{\gamma_{p\to q},0\to\tau}\left(D(V_p^{k-1})\exp_p(\tau\exp_p^{-1}(q))\right)\mathrm{d}\,\tau \tag{3.3.30}$$

for $q \in U$. Thus it remains to show that these V_p^k indeed define smooth sections satisfying the transport equations. The smoothness is guaranteed from the following proposition which even handles the smooth dependence on p. We again formulate it for a situation as in Proposition 3.2.15.

Proposition 3.3.8 (Smoothness of V^k) *Let $O \subseteq U \subseteq M$ be open subsets such that U is geodesically star-shaped around all $p \in O$. Then the recursive definitions* (3.3.29) *and* (3.3.30) *yield smooth sections*

$$V^k \in \Gamma^\infty(E^* \boxtimes E\big|_{O\times U}) \tag{3.3.31}$$

via the definition

$$V^k(p, q) = V_p^k(q) \tag{3.3.32}$$

for $(p, q) \in O \times U$ and $k \geq 0$.

Proof. First we note that $\varrho(p, g) = \varrho_p(q)$ is actually a smooth function $\varrho \in \mathcal{C}^\infty(O \times U)$ with $\varrho > 0$ everywhere. This follows from Lemma A.3.2. From Lemma 3.3.4 we deduce that the dependence of $P_{p \to q}$ on q is smooth and a similar argument shows that also the dependence on p is smooth. In fact, the parallel transport depends smoothly on $(p, q) \in O \times U$ yielding thereby a smooth section

$$P \in \Gamma^\infty\left(E^* \boxtimes E\big|_{O\times U}\right).$$

It follows that V^0 is smooth on $O \times U$. We rewrite the recursive definition (3.3.30) in terms of ϱ,

$$\gamma(p, q, \tau) = \gamma_{p\to q}(\tau) = \gamma_\tau(p, q)$$

and the V^k. Then (3.3.30) becomes

$$V^k = \frac{k}{\sqrt{\varrho}} P \int_0^1 \sqrt{\varrho} \circ \gamma_\tau P \circ \gamma_\tau \left((\mathsf{id} \boxtimes D(V^{k-1})) \circ \gamma_\tau\right) \tau^{k-1} \,\mathrm{d}\,\tau.$$

By induction we assume that V^{k-1} is smooth. Now γ is smooth on $O \times U \times [0, 1]$ and thus the integrand is smooth with a compact domain of integration. This results in a smooth V^k. □

As already in Proposition 3.2.15 we can e.g. take a *convex* $U \subseteq M$ and set $O = U$ in order to meet the conditions of Proposition 3.3.8. It remains to show that the V_p^k actually satisfy the transport equations with the correct initial condition.

Proposition 3.3.9 *Let $U \subseteq M$ be geodesically star-shaped around $p \in M$. Then the sections $V_p^k \in \Gamma^\infty(E\big|_U) \otimes E_p^*$ defined by* (3.3.29) *and* (3.3.30) *satisfy the transport equations* (3.3.18) *with initial condition $V_p^0(p) = \mathsf{id}_{E_p}$.*

Proof. Clearly $V_p^0(p) = \mathsf{id}_{E_p}$ since $\varrho_p(p) = 1$. In the proof of Theorem 3.3.5 we have seen that (3.3.18) is equivalent to

$$\nabla^E_{\operatorname{grad}\eta_p}\left(\sqrt{\varrho_p} r_p^k V_p^k\right) = -2k\sqrt{\varrho_p} r_p^k D(V_p^{k-1}) \tag{$*$}$$

on the open subset $U \setminus C_U(p)$. Since we already know that the section V^k are smooth on U by Proposition 3.3.8 we know that they satisfy (3.3.18) on U iff they satisfy (3.3.18) on $U \setminus C_U(p)$ by a continuity argument. Thus it suffices to show $(*)$ on $U \setminus C_U(p)$. In the proof of Theorem 3.3.5 we have shown that $(*)$ implies

$$\nabla^\#_{\frac{\partial}{\partial t}}\left(c^\#\left(\sqrt{\varrho_p} r_p^k V_p^k\right)\right) = -2kc^\#\left(\sqrt{\varrho_p} v_p^k D(V_p^{k-1})\right) \tag{$**$}$$

for the curve $c(t) = \exp_p(\mathrm{e}^{2t}\exp_p^{-1}(q))$ with $q \in U$ and $t \in (-\infty, \epsilon)$ and $\epsilon > 0$ sufficiently small. But if we have $(**)$ for *all* such curves c then we get back $(*)$ since $\operatorname{grad}\eta_p\big|_q = \dot{c}(0)$ and the left hand side of $(*)$ can be evaluated point by point as $\nabla^E_{\operatorname{grad}\eta_p}$ is tensorial in $\operatorname{grad}\eta_p$. Thus $(**)$ for *all* such curves is equivalent to $(*)$. But $V_p^k(q)$ was precisely the solution of $(**)$ at $t = 0$ by Lemma 3.3.6. But this means at q we have

$$\nabla^E_{\operatorname{grad}\eta_p}\left(\sqrt{\varrho_p} r_p^k V_p^k\right)\Big|_q = -2k\sqrt{\varrho_p} r_p^k D(V_p^{k-1})\big|_q.$$

Since $q \in U \setminus C_U(p)$ was arbitrary, $(*)$ follows which completes the claim. □

Theorem 3.3.10 (Hadamard Coefficients) *Let $O \subseteq U \subseteq M$ be open subsets such that U is geodesically star-shaped around all $p \in O$. Let $D \in \mathrm{DiffOp}^2(E)$ be normally hyperbolic. Then for each $p \in O$ the operator D has unique Hadamard coefficients $V_p^k \in \Gamma^\infty\left(E\big|_U\right) \otimes E_p^*$ explicitly given by $V_p^k(q) = V^k(p, q)$ where $V^k \in \Gamma^\infty\left(E^* \boxtimes E\big|_{O\times U}\right)$ is recursively determined by*

$$V^0 = \frac{1}{\sqrt{\varrho}} P \tag{3.3.33}$$

and

$$V^k = -\frac{k}{\sqrt{\varrho}} P \int_0^1 \left(\sqrt{\varrho} P \left(\mathsf{id} \boxtimes D(V^{k-1})\right)\right) \circ \gamma_\tau \, \tau^{k-1} \, \mathrm{d}\,\tau, \tag{3.3.34}$$

where $P \in \Gamma^\infty\left(E^* \boxtimes E\big|_{O\times U}\right)$ *is the parallel transport* $P(p,q) = P_{p\to q}$ *along* $\gamma_{p\to q}(\tau) = \gamma_\tau(p,q) = \exp_p(\tau \exp_p^{-1}(q))$. *On the diagonal we explicitly have the simplified recursion*

$$V^k(p,p) = -\left((\mathsf{id} \boxtimes D)(V^{k-1})\right)(p,p). \tag{3.3.35}$$

Proof. It remains to show the simplified recursion (3.3.35) for $p = q$. For $k \geq 1$ we have

$$\begin{aligned} V^k(p,p) &= -\frac{k}{\sqrt{\varrho(p,p)}} P_{p\to p} \int_0^1 \left(\sqrt{\varrho} P \left((\mathsf{id} \boxtimes D)(V^{k-1})\right)\right)(\gamma_\tau(p,p)) \tau^{k-1} \, \mathrm{d}\,\tau \\ &= -k \int_0^1 (\mathsf{id} \boxtimes D)(V^{k-1})\big|_{(p,p)} \tau^{k-1} \, \mathrm{d}\,\tau \\ &= -k (\mathsf{id} \boxtimes D)(V^{k-1})\big|_{(p,p)} \int_0^1 \tau^{k-1} \, \mathrm{d}\,\tau \\ &= -(\mathsf{id} \boxtimes D)(V^{k-1})\big|_{(p,p)}. \end{aligned}$$

□

We illustrate the recursion formula by computing the first non-trivial Hadamard coefficient along the diagonal.

Example 3.3.11 (First Hadamard coefficient) Let $D = \Box^\nabla + B$ be normally hyperbolic as usual. Thus let $s_p \in E_p$ be a vector in E_p and let

$$s(q) = P_{p\to q}(s_p), \tag{3.3.36}$$

which defines a vector field $s \in \Gamma^\infty\left(E\big|_U\right)$. We compute the covariant derivatives of s at p. At general points $q \in U$ this might be very complicated but at p we have by Proposition A.1.7 the formal Taylor expansion

$$\mathrm{i}(e_{i_1}) \cdots \mathrm{i}(e_{i_k}) \frac{1}{k!} \left(\mathsf{D}^E\right)^k s\Big|_p = \frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}} \left(P_{\gamma_v, 0\to 1}\right)^{-1} (s(\gamma_v(1))), \tag{3.3.37}$$

with a basis $e_1, \ldots, e_n \in T_pM$ and $\gamma_v(t) = \exp_p(tv)$ as usual. But

$$\left(P_{\gamma, 0\to 1}\right)^{-1} (s(\gamma_v(1))) = \left(P_{\gamma_v, 0\to 1}\right)^{-1} P_{p\to q=\exp_p(v)}(s_p) = s_p$$

is independent of v. Thus all partial derivatives vanish and we conclude $\left(\mathsf{D}^E\right)^k s\big|_p = 0$. But then $\Box^\nabla s\big|_p = \frac{1}{2}\left\langle g^{-1}, \left(\mathsf{D}^E\right)^2 s\right\rangle\big|_p = 0$ follows as well. From this we conclude by (3.3.35)

$$\begin{aligned} &V^1(p,p) \\ &= -(\mathsf{id} \boxtimes D)(V^0)(p,p) = -(\mathsf{id} \boxtimes D)\left(\frac{1}{\sqrt{\varrho}} P\right)(p,p) = -D\left(\frac{1}{\sqrt{\varrho_p}} P_{p\to\cdot}(e_\alpha)\right) \otimes e^\alpha\Big|_p \\ &= -\frac{1}{\sqrt{\varrho_p}} D\left(P_{p\to\cdot}(e_\alpha)\right) \otimes e^\alpha\Big|_p - 2\left(\nabla^E_{\mathrm{grad}\,\frac{1}{\sqrt{\varrho_p}}} P_{p\to\cdot}(e_\alpha)\right) \otimes e^\alpha\Big|_p - \Box \frac{1}{\sqrt{\varrho_p}}\Big|_p P_{p\to\cdot}(e_\alpha) \otimes e^\alpha\Big|_p \end{aligned}$$

$$
\begin{aligned}
&= -(\Box^{\nabla} + B)\,(P_{p\to\cdot}(e_\alpha))\Big|_p \otimes e^\alpha + 0 - \Box\frac{1}{\sqrt{\varrho_p}}\Big|_p e_\alpha \otimes e^\alpha \\
&= -B\Big|_p (e_\alpha) \otimes e^\alpha - \frac{1}{6}\operatorname{scal}(p)\,\mathsf{id}_{E_p},
\end{aligned}
$$

by (3.2.13) and $\mathsf{id}_{E_p} = e_\alpha \otimes e^\alpha$ with a basis e_α of E_p. Thus we have

$$
V^1(p,p) = -\frac{1}{6}\operatorname{scal}(p)\,\mathsf{id}_{E_p} - B(p). \tag{3.3.38}
$$

3.3.4 The Klein-Gordon Equation

Even though in general the convergence of (3.3.4) is hard to control and may even fail in general there is one example where we can compute the Hadamard coefficients explicitly and show weak* convergence of (3.3.4).

We consider again the flat Minkowski spacetime $(\mathbb{R}^n, \eta)$ but now the Klein-Gordon equation

$$
\left(\Box + m^2\right)\phi = 0 \tag{3.3.39}
$$

instead of $\Box$ alone. As usual m^2 denotes a positive constant. The physical meaning in quantum field theory of m is that of the mass of the particle described by (3.3.39).

Since the metric η is translation invariant and the operator $\Box + m^2$ is translation invariant as well, we only have to compute the Hadamard coefficients at a single point $p \in \mathbb{R}^n$ and can then translate everything. Thus we can choose $p = 0$. As already mentioned before, $\exp_p$ is just the addition with p whence

$$
\exp_0 : T_0\mathbb{R}^n = \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{3.3.40}
$$

is simply the identity map. Also the density function ϱ_p becomes very simple as we have

$$
\varrho_p = 1 \tag{3.3.41}
$$

for all p. Thus the recursion for the Hadamard coefficients simplifies drastically. Finally, we note that the Klein-Gordon operator $\Box + m^2$ has already the normal form with $B = m^2$. Thus the covariant derivative is the flat one and the parallel transport is the identity. Therefor we have

$$
V_p^0 = \frac{1}{\sqrt{\varrho_p}} P_{p\to\cdot} = \mathsf{id}
$$

and

$$
\begin{aligned}
V_p^k(q) &= -\frac{k}{\sqrt{\varrho_p(q)}} P_{p\to q} \int_0^1 \sqrt{\varrho_p}(\gamma_{p\to q}(\tau)) P^{-1}_{\gamma_{p\to q},0\to\tau} \left(D(V_p^{k-1})(\gamma_{p\to q}(\tau))\right) \tau^{k-1}\,\mathrm{d}\,\tau \\
&= -k\int_0^1 D(V_p^{k-1})(p + \tau(q-p))\tau^{k-1}\,\mathrm{d}\,\tau.
\end{aligned}
$$

Now V_p^0 is constant. We claim that, since m^2 is constant as well, all Hadamard coefficients are constant, too. Indeed, assuming this for $k-1$ shows that

$$
\begin{aligned}
V_p^k(q) &= -k\int_0^1 D(V_p^{k-1})(p + \tau(q-p))\tau^{k-1}\,\mathrm{d}\,\tau \\
&= -kD(V_p^{k-1})\int_0^1 \tau^{k-1}\,\mathrm{d}\,\tau
\end{aligned}
$$

$$= -D(V_p^{k-1})$$

$$= -m^2 V_p^{k-1},$$

which is again constant. Thus by induction we conclude the following:

Lemma 3.3.12 *The Hadamard coefficients for the Klein-Gordon operator $\Box + m^2$ on Minkowski spacetime are constant and explicitly given by*

$$V_p^k = (-m^2)^k \tag{3.3.42}$$

for $k \in \mathbb{N}_0$ and all points $p \in \mathbb{R}^n$.

This particularly simple form allows to determine the convergence of (3.3.4) explicitly. We consider large $k \in \mathbb{N}_0$ such that $R^\pm(2+2k)$ is actually a continuous function. More precisely, we fix $N \in \mathbb{N}_0$ then for $2k \geq n-2+2N$ the distribution $R^\pm(2+2k)$ is actually a $\mathcal{C}^N$ function according to Lemma 3.1.3, explicitly given by

$$R^\pm(2+2k)(x) = \frac{2^{1-(2+2k)}\pi^{\frac{2-n}{2}}}{\Gamma\left(\frac{2+2k}{2}\right)\Gamma\left(\frac{2+2k-n}{2}+1\right)}\,\eta(x)^{\frac{2+2k-n}{2}} = \frac{\pi^{\frac{2-n}{2}}}{2^{2k-1}k!\,\Gamma\left(k+2-\frac{n}{2}\right)}\,\eta(x)^{k+1-\frac{n}{2}} \tag{3.3.43}$$

for $x \in I^\pm(0)$ and 0 elsewhere. We want to estimate $R^\pm(2k)$ and its derivatives over a compactum $K \subseteq \mathbb{R}^n$. To this end we compute the first partial derivatives of $R^\pm(\alpha)$ explicitly. We know already

$$\frac{\partial}{\partial x^{i_1}} R^\pm(\alpha) = \frac{1}{\alpha-2} R^\pm(\alpha-2)\eta_{i_1 j}x^j = \frac{1}{\alpha-2} R^\pm(\alpha-2)x_{i_1}, \tag{3.3.44}$$

where we use the notation

$$x_i = \eta_{ij}x^j. \tag{3.3.45}$$

Thus we get

$$\frac{\partial^2}{\partial x^{i_1}\partial x^{i_2}} R^\pm(\alpha) = \frac{R^\pm(\alpha-4)}{(\alpha-2)(\alpha-4)} x_{i_1}x_{i_2} + \frac{R^\pm(\alpha-2)}{\alpha-2}\eta_{i_1 i_2}, \tag{3.3.46}$$

since clearly $\frac{\partial}{\partial x^{i_2}} x_{i_1} = \eta_{i_1 i_2}$. Moving on from this we get

$$\begin{aligned} \frac{\partial^3}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_4}} R^\pm(\alpha) &= \frac{R^\pm(\alpha-6)}{(\alpha-2)(\alpha-4)(\alpha-6)} x_{i_1}x_{i_2}x_{i_3} \\ &\quad + \frac{R^\pm(\alpha-4)}{(\alpha-2)(\alpha-4)} \left(\eta_{i_1 i_3}x_{i_2} + \eta_{i_2 i_3}x_{i_1} + \eta_{i_1 i_2}x_{i_4}\right) \end{aligned} \tag{3.3.47}$$

and

$$\begin{aligned} &\frac{\partial^4}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_4}\partial x^{i_4}} R^\pm(\alpha) \\ &= \frac{R^\pm(\alpha-8)}{(\alpha-2)(\alpha-4)(\alpha-6)(\alpha-8)} x_{i_1}x_{i_2}x_{i_3}x_{i_4} \\ &+ \frac{R^\pm(\alpha-6)}{(\alpha-2)(\alpha-4)(\alpha-6)} \left(\eta_{i_1 i_4}x_{i_2}x_{i_3} + \eta_{i_2 i_4}x_{i_1}x_{i_3} + \eta_{i_3 i_4}x_{i_1}x_{i_2} + \eta_{i_1 i_2}x_{i_3}x_{i_4} + \eta_{i_1 i_3}x_{i_2}x_{i_4}\right) \\ &+ \frac{R^\pm(\alpha-4)}{(\alpha-2)(\alpha-4)} \left(\eta_{i_1 i_2}\eta_{i_3 i_4} + \eta_{i_1 i_3}\eta_{i_2 i_4} + \eta_{i_1 i_4}\eta_{i_2 i_3}\right). \end{aligned} \tag{3.3.48}$$

Now we see how one can guess the general formula: For $\ell = 2r$ derivatives we have contributions of $\frac{1}{(\alpha-2)\cdots(\alpha-2(r+s))} R^{\pm}(\alpha - 2(r+s))$ with coefficients consisting of symmetrizations of s factors η and $2r - 2s$ factors of x where only those symmetrizations are done which are not automatic, i.e. $x_{i_1} x_{i_2}$ only occurs *once* and not twice. For $\ell = 2r + 1$ we have the analogous statement. Summarizing this in a more formalized way gives the following result:

Proposition 3.3.13 (Taylor coefficients of $R^{\pm}(\alpha)$) *Let $\ell \in \mathbb{N}_0$ and set $r = \left[\frac{\ell}{2}\right]$ whence $\ell = 2r$ or $\ell = 2r + 1$ depending on ℓ being even or odd. Then the partial derivatives of the Riesz distribution $R^{\pm}(\alpha)$ for $\alpha \notin \{2, 4, \ldots, 4r\}$ are given by*

$$\frac{\partial^{\ell}}{\partial x^{i_1} \cdots \partial x^{i_\ell}} R^{\pm}(\alpha) = \sum_{s=0}^{r} \frac{R^{\pm}(\alpha - 2\ell + 2(r+s))}{(\alpha-2)\cdots(\alpha - 2\ell + 2(r-s))} \sum_{\sigma \in \mathrm{S}_{r,s}} \eta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \eta_{i_{\sigma(2r-2s-1)} i_{\sigma(2(r-s))}} x_{i_{\sigma(2(r-s)+1)}} \cdots x_{i_{\sigma(\ell)}}, \tag{3.3.49}$$

where $\mathrm{S}_{r,s}$ denotes those permutations of $\{1, \ldots, \ell\}$ such that

$$\begin{aligned} &\sigma(1) < \sigma(2), \ldots, \sigma(2(r-s)) \\ &\sigma(3) < \sigma(4), \ldots, \sigma(2(r-s)) \\ &\vdots \\ &\sigma(2(r-s)-1) < \sigma(2(r-s)) \\ \textit{and} \quad &\sigma(2(r-s)+1) < \sigma(2(r-s)+2) < \ldots < \sigma(\ell). \end{aligned} \tag{3.3.50}$$

Proof. The proof consists in a rather boring and tedious understanding of the above symmetrization procedure. Since we only need some qualitative consequences of (3.3.49) we leave it as an exercise. □

Remark 3.3.14 The above result has again two possible interpretations. On one hand, (3.3.49) holds for all α except for the poles in the sense of distributions. Even for the singular α, the right hand side has an analytic continuation by the left hand side. On the other hand, for $\mathsf{Re}(\alpha)$ large enough, $R^{\pm}(\alpha)$ is a $\mathcal{C}^{\ell}$-function and (3.3.49) holds *pointwise* in the sense of functions. By Lemma 3.1.3 this is the case for $\mathsf{Re}(\alpha) > n + 2\ell$.

We consider now the case $\mathsf{Re}(\alpha) > n + 2\ell$ and want to use (3.3.49) to estimate the ℓ-th derivatives of the *function* $R^{\pm}(\alpha)$ over a compactum $K \subseteq \mathbb{R}^n$. Thus let $R > 0$ be large enough such that

$$K \subseteq B_R(0) \tag{3.3.51}$$

for some *Euclidean* ball around zero. The following is then obvious from the definition of $R^{\pm}(\alpha)$ and gives a (rather rough) estimate on the sup-norm of $R^{\pm}(\alpha)$ over K.

Lemma 3.3.15 *Let $K \subseteq \mathbb{R}^n$ be compact and let $R > 0$ with $K \subseteq B_R(0)$. Then for $\mathsf{Re}(\alpha) > n$ we have*

$$\mathrm{p}_{K,0}(R^{\pm}(\alpha)) \leq |c(\alpha, n)| R^{\mathsf{Re}(\alpha) - n}. \tag{3.3.52}$$

Proof. For those $x \in I^{\pm}(0)$ we have

$$|\eta(x, x)| = \left| (x^0)^2 - \sum_{i=1}^{n-1} (x^i)^2 \right| \leq |(x^0)^2 + \cdots + (x^n)^2| = R^2,$$

and outside of $I^{\pm}(0)$, the function $R^{\pm}(\alpha)$ vanishes anyway. □

Taking derivatives into account we have the following estimate for large $\mathsf{Re}(\alpha)$:

Proposition 3.3.16 *Let $K \subseteq \mathbb{R}^n$ be compact and let $R \geq 1$ with $K \subseteq B_R(0)$. Then for $\mathsf{Re}(\alpha) > n + 2\ell$ we have*

$$\mathrm{p}_{K,\ell}(R^{\pm}(\alpha)) \leq \ell \cdot \ell! \cdot R^{\mathsf{Re}(\alpha)-n} \cdot \max\left\{|c(\alpha)|, \frac{|c(\alpha-2)|}{|\alpha-2|}, \ldots, \frac{|c(\alpha-2\ell)|}{|(\alpha-2)\cdots(\alpha-2\ell)|}\right\}, \tag{3.3.53}$$

with $c(\alpha) = c(\alpha, n)$ for abbreviation.

Proof. From Proposition 3.3.13 we know that for precisely ℓ' derivatives we have for $x \in K$

$$\begin{aligned}\left|\frac{\partial^{\ell'}}{\partial x^{i_1}\cdots\partial x^{i_{\ell'}}} R^{\pm}(\alpha)(x)\right| &\leq \sum_{s=0}^{r=\left[\frac{\ell'}{2}\right]} \frac{|R^{\pm}(\alpha - 2\ell' + 2(r-s))(x)|}{|(\alpha-2)\cdots(\alpha-2\ell'+2(r-s))|} \sum |\eta\cdots\eta\cdot x\cdots x| \\ &\leq \sum_{s=0}^{r=\left[\frac{\ell'}{2}\right]} \frac{|c(\alpha-2\ell'+2(r-s))| R^{\mathsf{Re}(\alpha)-2\ell'+2(r-s)-n}}{|(\alpha-2)\cdots(\alpha-2\ell'+2(r-s))|} \ell'! R^{\ell'},\end{aligned}$$

since in the sum over all allowed permutations we have at most $\ell'!$ factors (In fact, we always have much less, but a rough estimate will do the job). Moreover, every factor in $\eta\cdots\eta\cdot x\cdots x$ is clearly $\leq R$ in absolute value. Now since we assumed $R > 1$ we have for $r = \left[\frac{\ell'}{2}\right]$ and $s = 0, \ldots, r$

$$R^{\mathsf{Re}(\alpha)-2\ell'+2(r-s)-n} R^{\ell'} \leq R^{\mathsf{Re}(\alpha)-\ell'+2\left[\frac{\ell'}{2}\right]-n} \leq R^{\mathsf{Re}(\alpha)-n},$$

since $-\ell' + 2\left[\frac{\ell'}{2}\right]$ is either -1 or 0. Thus we can simplify this to

$$\begin{aligned}\left|\frac{\partial^{\ell'}}{\partial x^{i_1}\cdots\partial x^{i_{\ell'}}} R^{\pm}(\alpha)(x)\right| &\leq R^{\mathsf{Re}(\alpha)-n}\ell'! \sum_{s=0}^{r=\left[\frac{\ell'}{2}\right]} \frac{|c(\alpha-2\ell'+2(r-s))|}{|(\alpha-2)\cdots(\alpha-2\ell'+2(r-s))|} \\ &\leq \ell'! R^{\mathsf{Re}(\alpha)-n}\ell' \max_{s=0}^{r=\left[\frac{\ell'}{2}\right]} \left\{\frac{|c(\alpha-2\ell'+2(r-s))|}{|(\alpha-2)\cdots(\alpha-2\ell'+2(r-s))|}\right\}.\end{aligned}$$

For $\mathrm{p}_{K,\ell}$ we finally have to take the maximum of this expression over all $\ell' = 0, \ldots, \ell$. In the maximum over s we can then simply take the largest of *all*, resulting in

$$\mathrm{p}_{K,\ell}(R^{\pm}(\alpha)) \leq \ell \cdot \ell! \cdot R^{\mathsf{Re}(\alpha)-n} \cdot \max\left\{|c(\alpha)|, \frac{|c(\alpha-2)|}{|\alpha-2|}, \ldots, \frac{|c(\alpha-2\ell)|}{|(\alpha-2)\cdots(\alpha-2\ell)|}\right\},$$

which is what we wanted to show. □

Note that we only gave a rather rough estimate, which will nevertheless be sufficient for the following. We specialize this now to the case $\alpha = 2 + 2k$ with k large enough such that $2 + 2k > n + 2\ell$. Then an even rougher estimate specializes (3.3.53) to the following estimate:

Corollary 3.3.17 *Let $\ell \in \mathbb{N}$ be fixed and $k \in \mathbb{N}$ such that $2 + 2k > 2\ell + n$ whence $R^{\pm}(2+2k)$ is $\mathcal{C}^{\ell}$. Then we have for any compactum $K \subseteq \mathbb{R}^n$ with $K \subseteq B_R(0)$ for a sufficiently large $R > 1$*

$$\mathrm{p}_{K,\ell}(R^{\pm}(2k)) \leq \ell\ell! R^{2+2k-n} \frac{\pi^{\frac{1-n}{2}}}{2^{k-2}k!}. \tag{3.3.54}$$

Proof. We compute explicitly by (3.3.43)

$$\frac{c(2+2k-2\ell)}{(2+2k-2)\cdots(2+2k-2\ell)} = \frac{\pi^{\frac{2-n}{2}}}{2^{2(k-\ell)-1}(k-\ell)!\,\Gamma\left(k-\ell+2-\frac{n}{2}\right)\cdot 2k\cdots 2(k-\ell+1)}$$

$$= \frac{\pi^{\frac{2-n}{2}}}{2^{2k-\ell-1}k!\,\Gamma\left(k-\ell+2-\frac{n}{2}\right)}.$$

Now by assumption $2+2k > 2\ell+n$ whence on one hand $\ell \leq k$ since $n \geq 1$. Thus $2^{2k-\ell-1} \geq 2^{k-1}$. Moreover, $k-\ell+2-\frac{n}{2} > 1$ whence by the monotonous growth of the Γ function, see Figure 3.1, the smallest contribution of $\Gamma\left(k-\ell+2-\frac{n}{2}\right)$ occurs at $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$. Thus we have

$$\left|\frac{c(2+2k-2\ell)}{(2+2k-2)\cdots(2+2k-2\ell)}\right| \leq \frac{\pi^{\frac{2-n}{2}}}{2^{k-1}k!\frac{1}{2}\sqrt{\pi}}$$

for all ℓ. Inserting this into (3.3.53) gives the result. □

Again, estimating $\frac{1}{\Gamma\left(k-\ell+2-\frac{n}{2}\right)}$ by $\frac{1}{2}\sqrt{\pi}$ is very rough, in particular as we are interested for fixed ℓ in the asymptotic behaviour for $k \longrightarrow \infty$. The additional Γ-factor behaves essentially like a $\frac{1}{k!}$ therefor improving the estimate (3.3.54) significantly. However, for the following theorem, already (3.3.54) is sufficient.

Theorem 3.3.18 (Green function of the Klein-Gordon operator) *Let $p \in \mathbb{R}^n$. Then the series*

$$\mathcal{R}^{\pm}(p) = \sum_{k=0}^{\infty}(-m^2)^k R^{\pm}(2+2k,p) \tag{3.3.55}$$

converges in the weak topology to the advanced and retarded Green function of the Klein-Gordon operator $\square + m^2$, respectively. More precisely, for $2+2k > 2\ell+n$ the series*

$$\sum_{2+2k>2\ell+n}(-m^2)^k R^{\pm}(2+2k,p) \tag{3.3.56}$$

converges in the $\mathcal{C}^\ell$-topology to a $\mathcal{C}^\ell$-function on $\mathbb{R}^n$. Finally, on $I^{\pm}(0)$ the series (3.3.55) converges in the $\mathcal{C}^\infty$-topology to a smooth function given by

$$\mathcal{R}^{\pm}(0)\Big|_{I^{\pm}(0)} = \sum_{k=0}^{\infty}\frac{\pi^{\frac{2-n}{2}}(-m^2)^k}{2^{2k-1}k!\,\Gamma\left(k+2-\frac{n}{2}\right)}\,\eta^{k+1-\frac{n}{2}} \tag{3.3.57}$$

for $p=0$ from which the other $\mathcal{R}^{\pm}(p)$ can be obtained by translation.

Proof. Clearly it suffices to show the convergence of (3.3.56) in the $\mathcal{C}^\ell$ topology: since $\mathcal{C}^\ell(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuously embedded, we can deduce the weak* convergence of (3.3.55) from that at once. To show (3.3.56), we even show absolute convergence: let $K \subseteq \mathbb{R}^n$ be compact with $K \subseteq B_R(0)$ for sufficiently large $R > 1$. Then

$$\sum_{2+2k>2\ell+n}\mathrm{p}_{K,\ell}\left((-m^2)^k R^{\pm}(2+2k,0)\right) \leq \sum_{2+2k>2\ell+n}(m^2)^k\ell\ell! R^{2+2k-n}\frac{\pi^{\frac{1-n}{2}}}{2^{k-2}k!} \leq c\sum_{2+2k>2\ell+n}\frac{(m^2R^2)^k}{k!}$$

with some constant $c > 0$ depending on ℓ, R. Since the series on the right is dominated by $\mathrm{e}^{m^2R^2}$ we see that we indeed have absolute convergence with respect to $\mathrm{p}_{K,\ell}$ for all K. This shows $\mathcal{C}^\ell$-convergence everywhere and hence weak* convergence. Finally, on $I^{\pm}(0)$ the functions $R^{\pm}(2+2k)\big|_{I^{\pm}(0)}$ are always smooth whence the above result shows that they converge in *all* $\mathcal{C}^\ell$-topologies. But this means convergence in the $\mathcal{C}^\infty$-topology, establishing the last claim (3.3.57). By translation invariance, the convergence results also hold for any other $p \in \mathbb{R}^n$. □

Remark 3.3.19 Of course, there are much more straightforward techniques to obtain the Green functions for $\square + m^2$ on Minkowski spacetime. The standard approach is to use Fourier transformation techniques and to construct $\mathcal{R}^{\pm}(0)$ as even *tempered* distribution on $\mathbb{R}^n$. In fact, for most applications in quantum field theory the momentum space representation of $\mathcal{R}^{\pm}(0)$ is needed anyway. However, our approach here is intrinsically geometric in the following sense: on a general spacetime Fourier transformation is not available, at least not in the naive way. Also, the above construction shows that $\mathcal{R}^{\pm}(0)$ depends *analytically* on m^2: the series (3.3.55) being precisely the weak* convergent Taylor expansion in the variable m^2 which may even be taken to be complex. This gives an entirely holomorphic family of distributions for $m^2 \in \mathbb{C}$. Finally, the series (3.3.57) can actually be expressed in terms of known transcendental functions, depending on the dimension n.

3.4 The Fundamental Solution on Small Neighborhoods

In this section we construct out of the local Riesz distributions $R^{\pm}(\alpha, p)$ and the corresponding Hadamard coefficients a fundamental solution on a small neighborhood of $p \in M$. One proceeds in two steps, first the formal series $\mathcal{R}^{\pm}(p)$ is made to converge by brutally modifying the higher order terms. The price paid is that the result is not yet a fundamental solution but differs from the fundamental solution by a "smoothing" kernel, i.e. one gets a *parametrix* for D. In a second step one shows how the parametrix can be changed to a fundamental solution by using an appropriate geometric series of the smooth kernel. Again, we follows essentially [4].

In the following we fix a geodesically convex open subset $U' \subseteq M$ and use the corresponding Riesz distributions $R^{\pm}_{U'}(\alpha, p)$ which are now available for all $p \in U'$. Moreover, by Theorem 3.3.10 the Hadamard coefficients are now smooth sections

$$V^k \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U' \times U'}\right), \tag{3.4.1}$$

out of which we obtain the *formal* fundamental solution

$$\mathcal{R}^{\pm}(p) = \sum_{k=0}^{\infty} V^k_p R^{\pm}_{U'}(2 + 2k, p) \tag{3.4.2}$$

on U'. Of course, there is no reason to believe that (3.4.2) converges in general, even not in the weak* sense. However, the Riesz distributions $R^{\pm}_{U'}(2 + 2k, p)$ are *continuous* functions if k is large enough. In fact, by Proposition 3.2.8 we know that $R^{\pm}_{U'}(2 + 2k, p)$ is at least continuous if $k > \frac{n}{2}$. Thus we fix $N \in \mathbb{N}_0$ with $N > \frac{n}{2}$ and split the sum (3.4.2) at $k = N$.

3.4.1 The Approximate Fundamental Solution

The idea is now that the finite sum

$$\sum_{k=0}^{N-1} V^k_p R^{\pm}_{U'}(2 + 2k, p) \in \Gamma^\infty_0\left(E^*\big|_{U'}\right)' \tag{3.4.3}$$

is a well-defined distribution. On the other hand, this contribution is believed to yield the most singular contribution to the yet to be found fundamental solution responsible for the δ-distribution in $D\mathcal{R}^{\pm}(p) = \delta_p$. Thus the hope is that the remaining, *infinite* sum can be modified and made to converge but yielding a less singular contribution than δ_p, in fact only a smooth one.

For technical reasons we will need a cutoff function $\chi \in \mathcal{C}^\infty_0(\mathbb{R})$ with

$$\operatorname{supp} \chi \subseteq [-1, 1], \quad 0 \le \chi \le 1, \quad \text{and} \quad \chi\big|_{[-\frac{1}{2}, \frac{1}{2}]} = 1. \tag{3.4.4}$$

For every choice of such a cutoff function, we have the following technical lemma:

Lemma 3.4.1 *Let $\ell \in \mathbb{N}$ and $\ell' > \ell + 1$. Then there are universal constants $c(\ell, \ell')$ such that for all $0 < \epsilon \leq 1$ one has*

$$\mathrm{p}_{K,0}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}\, t^\ell}\left(\chi\left(\frac{t}{\epsilon}\right) t^{\ell'}\right)\right) \leq \epsilon c(\ell, \ell')\, \mathrm{p}_{K,\ell}(\chi), \tag{3.4.5}$$

where K is any compactum containing $[-1, 1]$.

Proof. First note that $\chi\left(\frac{t}{\epsilon}\right) = 0$ for $\left|\frac{t}{\epsilon}\right| > 1$ and hence $|t| > \epsilon$. Thus the support of $t \mapsto \chi\left(\frac{t}{\epsilon}\right)$ is contained in $[-\epsilon, \epsilon] \subseteq [-1, 1]$. It follows that in (3.4.5) we can safely replace the supremum over K by a supremum over $\mathbb{R}$ everywhere. In any case, we have by the Leibniz rule and the chain rule

$$\begin{aligned}\frac{\mathrm{d}^\ell}{\mathrm{d}\, t^\ell}\left(\chi\left(\frac{t}{\epsilon}\right) t^{\ell'}\right) &= \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{\mathrm{d}^m}{\mathrm{d}\, t^m}\left(\chi\left(\frac{t}{\epsilon}\right)\right) \frac{\mathrm{d}^{\ell-m}\, t^{\ell'}}{\mathrm{d}\, t^{\ell-m}} \\ &= \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{1}{\epsilon^m} \frac{\mathrm{d}^m \chi}{\mathrm{d}\, t^m}\left(\frac{t}{\epsilon}\right) \ell'(\ell'-1)\cdots(\ell'-\ell+m+1) t^{\ell'-\ell+m}.\end{aligned}$$

Now for $|t| > \epsilon$ the factor $\frac{\mathrm{d}^m \chi}{\mathrm{d}\, t^m}\left(\frac{t}{\epsilon}\right)$ vanishes whence we find

$$\begin{aligned}\mathrm{p}_{K,0}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}\, t^\ell}\left(\chi\left(\frac{t}{\epsilon}\right) t^{\ell'}\right)\right) &\leq \sup_t \sum_{m=0}^{\ell} \binom{\ell}{m} \ell'(\ell'-1)\cdots(\ell'-\ell+m+1)\epsilon^{\ell'-\ell} \left|\frac{\mathrm{d}^m \chi}{\mathrm{d}\, t^m}\left(\frac{t}{\epsilon}\right)\right| \\ &\leq \epsilon \sum_{m=0}^{\ell} \binom{\ell}{m} \ell'(\ell'-1)\cdots(\ell'-\ell+m+1)\, \mathrm{p}_{K,\ell}(\chi),\end{aligned}$$

since only $|t| \leq \epsilon$ contribute and $\epsilon^{\ell'-\ell} \leq \epsilon$ for $\ell' \geq \ell + 1$. ☐

Since U' is assumed to be convex, the Lorentz distance square is defined on $U' \times U'$ and gives a smooth function $\eta \in \mathcal{C}^\infty(U' \times U')$ by setting

$$\eta(p, q) = \eta_p(q) = g_p\left(\exp_p^{-1}(q), \exp_p^{-1}(q)\right). \tag{3.4.6}$$

We know from the proof of Proposition 3.2.16 that η is even a symmetric function

$$\eta(p, q) = \eta(q, p). \tag{3.4.7}$$

Finally, since U' is assumed to be geodesically convex the geodesics joining $p, q \in U'$ in U' are unique. Thus we see that $\eta(p, q) = 0$ iff the geodesic joining p and q is *lightlike*. Since the points q which are in the image of $C(0) \subseteq T_pM$ under $\exp_p$ are just $C_{U'}(p)$ we see that

$$\eta^{-1}(\{0\}) = \bigcup_{p \in U'} \{p\} \times C_{U'}(p). \tag{3.4.8}$$

The idea is now to keep the series (3.4.2) unchanged in a small, and in fact only infinitesimal, neighborhood of the singular support, i.e. the light cones $\eta^{-1}(\{0\})$, and modify it outside to ensure convergence. To this end we will choose a sequence $\epsilon_j \in (0, 1]$ of cutoff parameters and consider the series

$$\begin{aligned}(p, q) &\mapsto \sum_{j=N}^{\infty} \chi\left(\frac{\eta(p, q)}{\epsilon_j}\right) V^j(p, q) R_{U'}^{\pm}(2 + 2j, p)(q) \\ &= \begin{cases} \sum_{j=N}^{\infty} \chi\left(\frac{\eta(p,q)}{\epsilon_j}\right) V^j(p, q) c(2 + 2j, n) \eta(p, q)^{j+1-\frac{n}{2}} & \text{for} \quad q \in I_{U'}^{\pm}(p) \\ 0 & \text{else.}\end{cases}\end{aligned} \tag{3.4.9}$$

Since $N \geq \frac{n}{2}$ all the terms in the modified (and truncated) series are at least $\mathcal{C}^0$. In fact, the j-th term is at least $(j-N)$-times continuously differentiable by Proposition 3.2.8, *ii.)* and by our choice of N. For estimating the derivatives of $\chi\left(\frac{\eta}{\epsilon_j}\right)$ in a suitable way, we first recall the following version of the chain rule:

Lemma 3.4.2 *Let $g : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be smooth, then for every multi-index $I \in \mathbb{N}_0^n$*

$$\frac{\partial^{|I|}}{\partial x^I}(f \circ g) = \sum_{\substack{r=1,\ldots,|I| \\ J_1,\ldots,J_r \leq I}} c^r_{J_1\cdots J_r} \frac{\mathrm{d}^r f}{\mathrm{d}\, t^r} \circ g \, \frac{\partial^{|J_1|}g}{\partial x^{J_1}} \cdots \frac{\partial^{|J_r|}g}{\partial x^{J_r}} \tag{3.4.10}$$

with some universal constants $c^r_{J_1\cdots J_r} \in \mathbb{Q}$.

Proof. This is clear by iterating the chain rule $|I|$ times. In fact, most of the $c^r_{J_1\cdots J_r}$ are zero anyway. □

We shall now use an exhausting series of compacta for U', i.e. we choose compact subsets

$$K_0 \subseteq \ldots K_\ell \subseteq \mathring{K}_{\ell+1} \subseteq \ldots \subseteq U' \tag{3.4.11}$$

with $U' = \bigcup_{\ell \geq 0} K_\ell$. This choice will give us seminorms $\mathrm{p}_{K_\ell,k}$ for all involved bundles satisfying a good estimate for natural pairings and

$$\mathrm{p}_{K_\ell,k} \leq \mathrm{p}_{K_{\ell'},k'} \tag{3.4.12}$$

for $\ell \leq \ell'$ and $k \leq k'$, see Remark 1.1.8. The filtration property (3.4.12) will turn out to be crucial. We shall use the same exhausting sequence of compacta (3.4.11) to obtain an exhausting sequence $K_\ell \times K_\ell$ of $U' \times U'$ as well.

We consider now the function

$$\chi_j^{\pm}(p,q) = \begin{cases} \chi\left(\frac{\eta(p,q)}{\epsilon_j}\right) \eta(p,q)^{j+1-\frac{n}{2}} & \text{for} \quad q \in I^{\pm}_{U'}(p) \\ 0 & \text{else,} \end{cases} \tag{3.4.13}$$

which is $\mathcal{C}^{j-N}$ for $j \geq N > \frac{n}{2}$ according to the properties of η as in Proposition 3.2.8, *ii.)*. We apply now Lemma 3.4.1 and Lemma 3.4.2 to obtain the following estimate:

Lemma 3.4.3 *Let $\ell, k \in \mathbb{N}_0$ and j large enough such that $j - N \geq k$. Then we have*

$$\mathrm{p}_{K_\ell \times K_\ell, k}(\chi_j^{\pm}) \leq \epsilon_j c(k, \ell, j), \tag{3.4.14}$$

with constants $c(k,\ell,j) > 0$ independent of ϵ_j satisfying

$$c(k,\ell,j) \leq c(k',\ell',j) \tag{3.4.15}$$

for $\ell \leq \ell'$ and $k \leq k'$.

Proof. We have by the chain rule as in Lemma 3.4.2

$$\begin{aligned} \mathrm{p}_{K_\ell\times K_\ell,k}(\chi_j^{\pm}) &\leq \sup_{\substack{(p,q)\in K_\ell\times K_\ell \\ |I|\leq k}} \sum_{\substack{r\leq|I| \\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \left| \frac{\mathrm{d}^r \left(\chi\left(\frac{t}{\epsilon_j}\right) t^{j+1-\frac{n}{2}}\right)}{\mathrm{d}\, t^r} \right| \left| \frac{\partial^{|J_1|}\eta}{\partial x^{J_1}} \right| \cdots \left| \frac{\partial^{|J_r|}\eta}{\partial x^{J_r}} \right| \\ &\leq \epsilon_j \sup_{|I|\leq k} \sum_{\substack{r\leq|I| \\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} c\Big(r, j+1-\frac{n}{2}\Big) c_r \, \mathrm{p}_{K_\ell\times K_\ell,k}(\eta)^r \end{aligned}$$

with $c_r = \max_{t\in\mathbb{R}} \left|\frac{\mathrm{d}^r \chi}{\mathrm{d}\, t^r}(t)\right| < \infty$. The maximum over $r \leq k$ is denoted by $\widetilde{c}_k$. The finitely many coefficients $c^r_{J_1\cdots J_r}$ have a maximum depending only on k and the sum has a certain maximal number of terms, again depending only on k. Thus there is a $\widetilde{\widetilde{c}}_k$ with

$$\mathrm{p}_{K_\ell\times K_\ell,k}(\chi_j^\pm) \leq \epsilon_j \widetilde{\widetilde{c}}_k \widetilde{c}_k \widetilde{c}\Big(k, j+1-\frac{n}{2}\Big) \max_{r\leq k} \mathrm{p}_{K_\ell\times K_\ell,k}(\eta)^r,$$

where $\widetilde{c}\big(k, j+1-\frac{n}{2}\big) = \max_{r\leq k} c(r, j+1-\frac{n}{2})$. But this is already the desired form since clearly $\widetilde{\widetilde{c}}_k$ increases with k, $\widetilde{c}_k$ increases with k and so does $\widetilde{c}(k, j+1-\frac{n}{2})$. Finally, the last maximum also increases with k and ℓ whence we can set

$$c(k,\ell,j) = \widetilde{\widetilde{c}}_k \widetilde{c}_k \widetilde{c}\big(k, j+1-\frac{n}{2}\big) \max_{r\leq k} \mathrm{p}_{K_\ell\times K_\ell,k}(\eta)^r,$$

which will do the job. □

Together with the usual product rule for the seminorms $\mathrm{p}_{K_\ell\times K_\ell,k}$ we obtain the following result:

Lemma 3.4.4 *Let $k, \ell \in \mathbb{N}_0$ and $j \geq N+k$. Then the j-th term of the series* (3.4.9) *satisfies the estimate*

$$\mathrm{p}_{K_\ell\times K_\ell,k}\left(\chi\left(\frac{\eta}{\epsilon_j}\right) V^j R^\pm_{U'}(2+2j,\,\cdot\,)\right) \leq \epsilon_j c(k,\ell,j) c(2+2j,n)\, \mathrm{p}_{K_\ell\times K_\ell,k}(V^j). \tag{3.4.16}$$

Proof. This is now easy from the product rule of the seminorms which gives a k-depending universal constant absorbed into the definition of $c(k,\ell,j)$ and the formula (3.4.9) for the j-th term. □

Choosing the ϵ_j appropriately, this can be made arbitrarily small in the following way:

Proposition 3.4.5

i.) For any $j \geq N$ and every $\epsilon_j \in (0,1]$ such that

$$\epsilon_j \max_k \Big\{ c(k,j,j) c(2+2j,n)\, \mathrm{p}_{K_j\times K_j,k}(V^j) \Big\} \leq \frac{1}{2^j} \tag{3.4.17}$$

the series (3.4.9) *converges absolutely in the $\mathcal{C}^0$-topology to a continuous section of $E^*\boxtimes E\big|_{U'\times U'}$.*

ii.) The series (3.4.9) *starting at $j \geq N+k$ converges absolutely in the $\mathcal{C}^k$-topology to a $\mathcal{C}^k$-section.*

iii.) The series (3.4.9) *restricted to the open subset $U'\times U' \setminus \eta^{-1}(\{0\})$ converges in the $\mathcal{C}^\infty$-topology to a smooth section of $E^*\boxtimes E\big|_{U'\times U'\setminus\eta^{-1}(\{0\})}$.*

Proof. For a fixed $j \geq N$ there are only finitely many $k \in \mathbb{N}_0$ with $j-N \geq k$ whence the maximum over the k's in (3.4.17) is well-defined. Thus we clearly can choose $\epsilon_j \in (0,1]$ to satisfy (3.4.17). Since we can take $k=0$, the second part implies the first as well. Thus let $k \in \mathbb{N}_0$ be arbitrary and consider the truncated series for $j \geq N+k$. First we note that every term is $\mathcal{C}^k$ whence we have to estimate their $\mathrm{p}_{K_\ell\times K_\ell,k}$-seminorms. We have for every $\ell \geq N+k$

$$\begin{aligned}
\mathrm{p}_{K_\ell\times K_\ell,k}\left(\sum_{j\geq N+k} \chi_j V^j c(2+2j,n)\right) &\leq \sum_{j\geq N+k} \epsilon_j c(k,\ell,j) c(2+2j,n)\, \mathrm{p}_{K_\ell\times K_\ell,k}(V^j) \\
&\leq \sum_{N+k\leq j\leq \ell} \epsilon_j c(k,\ell,j) c(2+2j,n)\, \mathrm{p}_{K_\ell\times K_\ell,k}(V^j) \\
&\quad + \sum_{j>\ell} \epsilon_j c(k,\ell,j) c(2+2j,n)\, \mathrm{p}_{K_\ell\times K_\ell,k}(V^j)
\end{aligned}$$

$$\leq \text{const.} + \sum_{j>\ell} \epsilon_j c(k,j,j) c(2+2j,n)\, \mathrm{p}_{K_j \times K_j, k}(V^j)$$

$$\leq \text{const.} + \sum_{j>\ell} \frac{1}{2^j} < \infty,$$

by the choice (3.4.17) and the fact that for $j \geq \ell$ we can replace $\mathrm{p}_{K_\ell \times K_\ell, k}(V^j)$ by $\mathrm{p}_{K_j \times K_j, k}(V^j)$ as well as $c(k,\ell,j) \leq c(k,j,j)$ according to (3.4.15). This shows absolute convergence with respect to $\mathrm{p}_{K_\ell \times K_\ell, k}$ for *all* $\ell \geq N + k$. But the compacta are increasing whence this shows absolute convergence in the $\mathcal{C}^k$-topology by the completeness of $\Gamma^k \left(E^* \boxtimes E\big|_{U' \times U'} \right)$. Finally, we note that every term in (3.4.9) is smooth on $U' \times U' \setminus \eta^{-1}(\{0\})$. Then we have $\mathcal{C}^k$-convergence by the second part for these restrictions, since omitting the first k terms does not change the convergence behaviour of the series. But this means that we have convergence in the $\mathcal{C}^\infty$-topology. □

We can thus define an *approximate fundamental solution* $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ by taking

$$\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \sum_{j=0}^{N-1} V_p^j R^{\pm}_{U'}(2+2j, p) + \sum_{j=N}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R^{\pm}_{U'}(2+2j,p), \tag{3.4.18}$$

after choosing the ϵ_j as in Proposition 3.4.5. From the support properties of the $R^{\pm}_{U'}(2+2j,p)$ and the above convergence statement, we obtain the following result:

Corollary 3.4.6 *Let the* $\epsilon_j \in (0,1]$ *be chosen to satisfy* (3.4.17). *Then* (3.4.18) *is weak** *convergent to a distributional section*

$$\widetilde{\mathcal{R}}^{\pm}_{U'}(p) \in \Gamma^{-(n+1)}\left(E\big|_{U'}\right) \otimes E_p^* \tag{3.4.19}$$

of global order $\leq n+1$ *with*

$$\operatorname{supp} \widetilde{\mathcal{R}}^{\pm}_{U'}(p) \subseteq J^{\pm}_{U'}(p), \tag{3.4.20}$$

$$\operatorname{sing\,supp} \widetilde{\mathcal{R}}^{\pm}_{U'}(p) \subseteq C^{\pm}_{U'}(p). \tag{3.4.21}$$

Proof. By Proposition 3.4.5 the series converges in the $\mathcal{C}^k$-topology and hence also in the weak* topology. Since the series is a continuous section it is of order 0, the finitely many extra terms for $j \leq N-1$ are all of order $\leq n+1$ by Proposition 3.2.12, *iii.)*. This shows (3.4.19). Since each term in (3.4.18) has support in $J^{\pm}_{U'}(p)$ also the limit has support in $J^{\pm}_{U'}(p)$ as this is already a *closed* subset of U' as we assume U' to be geodesically convex. Moreover, the singular support of the first terms with $j \leq N-1$ is in $C^{\pm}_{U'}(p)$. By Proposition 3.4.5 *iii.)*, the series is smooth inside $I^{\pm}_{U'}(p)$ whence (3.4.21) follows as well. □

Let us now determine in which sense $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ is an approximate solution. Since the series converges in the weak* sense we can apply $D = \square^\nabla + B$ term by term thanks to the continuity of differential operators, see Theorem 1.3.27, *i.)*. In our situation we can even argue in the sense of functions if we start the series at $N+2$ because then we have $\mathcal{C}^2$-convergence for which D is continuous as well. In any case we get

$$D\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \sum_{j=0}^{N-1} D\left(V_p^j R^{\pm}_{U'}(2+2j,p)\right) + \sum_{j=N}^{\infty} D\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R^{\pm}_{U'}(2+2j,p)\right)$$

$$= \delta_p + D(V_p^{N-1}) R^{\pm}_{U'}(2N,p) + \sum_{j=N}^{\infty} D\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R^{\pm}_{U'}(2+2j,p)\right), \tag{3.4.22}$$

thanks to the transport equations for V_p^j. Indeed, the transport equations, by their very construction, yield Hadamard coefficients V_p^j such that

$$
\begin{aligned}
&\sum_{j=0}^{N-1} D\left(V_p^j R_{U'}^{\pm}(2+2j,p)\right) \\
&\quad = D(V_p^0)R_{U'}^{\pm}(2,p) + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_{U'}^{\pm}(2,p) \\
&\qquad + D(V_p^1)R_{U'}^{\pm}(4,p) + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(4,p)} V_p^1 + V_p^1 \Box R_{U'}^{\pm}(4,p) \\
&\qquad + \cdots \\
&\qquad + D(V_p^{N-1})R_{U'}^{\pm}(2N,p) + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(2N,p)} V_p^{N-1} + V_p^{N-1} \Box R_{U'}^{\pm}(2N,p) \\
&\quad = \delta_p \\
&\qquad + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(4,p)} V_p^1 + V_p^1 \Box R_{U'}^{\pm}(4,p) + D(V_p^0)R_{U'}(2,p) \\
&\qquad + \cdots \\
&\qquad + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(2N,p)} V_p^{N-1} + V_p^{N-1} \Box R_{U'}^{\pm}(2N,p) + D(V_p^{N-2})R_{U'}^{\pm}(2N-2,p) \\
&\qquad + D(V_p^{N-1})R_{U'}^{\pm}(2N,p) \\
&\quad = \delta_p + 0 + \cdots + 0 + D(V_p^{N-1})R_{U'}^{\pm}(2N,p)
\end{aligned}
\tag{3.4.23}
$$

for arbitrary N by (3.3.9) and (3.3.10). We consider now the remaining sum over j in (3.4.22) and get by the Leibniz rule for D

$$
\begin{aligned}
D\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R_{U'}^{\pm}(2+2j,p)\right) &= \Box\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right)\right) V_p^j R_{U'}^{\pm}(2+2j,p) + 2\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta_p}{\epsilon_j}\right)}\left(V_p^j R_{U'}^{\pm}(2+2j,p)\right) \\
&\quad + \chi\left(\frac{\eta_p}{\epsilon_j}\right) D\left(V_p^j R_{U'}^{\pm}(2+2j,p)\right).
\end{aligned}
\tag{3.4.24}
$$

By the transport equations we have

$$
\begin{aligned}
D\left(V_p^j R_{U'}^{\pm}(2+2j,p)\right) &= D(V_p^j)R_{U'}^{\pm}(2+2j,p) + 2\nabla^E_{\operatorname{grad} R_{U'}^{\pm}(2+2j,p)} V_p^j + V_p^j \Box R_{U'}^{\pm}(2+2j,p) \\
&= D(V_p^j)R_{U'}^{\pm}(2+2j,p) - D(V_p^{j-1})R_{U'}^{\pm}(2j,p).
\end{aligned}
\tag{3.4.25}
$$

By shifting the summation index appropriately, we get

$$
\begin{aligned}
D\widetilde{\mathfrak{R}}_{U'}^{\pm}(p) - \delta_p = D(V_p^{N-1})R_{U'}^{\pm}(2N,p) &+ \sum_{j=N}^{\infty} \Box\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right)\right) V_p^j R_{U'}^{\pm}(2+2j,p) \\
&+ \sum_{j=N}^{\infty} 2\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta_p}{\epsilon_j}\right)}\left(V_p^j R_{U'}^{\pm}(2+2j,p)\right) + \sum_{j=N}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right) D(V_p^j)R_{U'}^{\pm}(2+2j,p) \\
&- \sum_{j=N}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right) D(V_p^{j-1})R_{U'}^{\pm}(2j,p)
\end{aligned}
$$

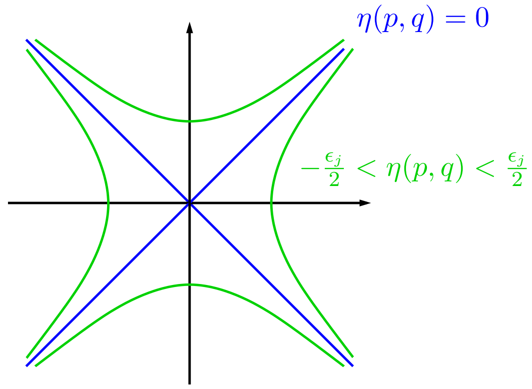

Figure 3.5: The open neighborhood S_j of $\eta^{-1}(\{0\})$ in the flat case.

$$
\begin{aligned}
&= \left(1 - \chi\left(\frac{\eta_p}{\epsilon_N}\right)\right) D(V_p^{N-1}) R^{\pm}_{U'}(2N, p) + \sum_{j=N}^{\infty} \Box\left(\chi\left(\frac{\eta_p}{\epsilon_j}\right)\right) V_p^j R^{\pm}_{U'}(2+2j, p) \\
&\quad + \sum_{j=N}^{\infty} 2\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta_p}{\epsilon_j}\right)} \left(V_p^j R^{\pm}_{U'}(2+2j, p)\right) \\
&\quad + \sum_{j=N}^{\infty} \left(\chi\left(\frac{\eta_p}{\epsilon_j}\right) - \chi\left(\frac{\eta_p}{\epsilon_{j+1}}\right)\right) D(V_p^j) R^{\pm}_{U'}(2+2j, p) \\
&= \left(1 - \chi\left(\frac{\eta_p}{\epsilon_N}\right)\right) D(V_p^{N-1}) R^{\pm}_{U'}(2N, p) + \Sigma_1 + \Sigma_2 + \Sigma_3,
\end{aligned}
\tag{3.4.26}
$$

where we abbreviated the last three series with Σ_1, Σ_2, and Σ_3, respectively. In order to investigate these three series we need the following technical lemma:

Lemma 3.4.7 *Let $\epsilon_j \in (0, 1]$ be chosen as in* (3.4.17)*.*

i.) The function $(p, q) \mapsto 1 - \chi\left(\frac{\eta(p,q)}{\epsilon_N}\right)$ vanishes on an open neighborhood of $\eta^{-1}(\{0\})$.

ii.) The vector field $U' \times U' \ni (p, q) \mapsto (\mathsf{id} \boxtimes \operatorname{grad})\left(\chi\left(\frac{\eta(p,q)}{\epsilon_j}\right)\right) \in T_qU'$ vanishes on an open neighborhood of $\eta^{-1}(\{0\})$.

iii.) The function $(\mathsf{id} \boxtimes \Box)\left(\chi\left(\frac{\eta}{\epsilon_j}\right)\right)$ vanishes on an open neighborhood of $\eta^{-1}(\{0\})$.

iv.) The function $\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)$ vanishes on an open neighborhood of $\eta^{-1}(\{0\})$.

v.) The section $\left(1 - \chi\left(\frac{\eta}{\epsilon_N}\right)\right) D(V^{N-1}) R^{\pm}_{U'}(2N, \cdot)$ as well as all the sections in the three series Σ_1, Σ_2, and Σ_3 are smooth on $U' \times U'$.

Proof. We consider the open neighborhood

$$
S_j = \left\{(p, q) \in U' \times U' \;\middle|\; -\frac{\epsilon_j}{2} < \eta(p, q) < \frac{\epsilon_j}{2}\right\} \subseteq U' \times U'
$$

of $\eta^{-1}(\{0\})$. Clearly, by continuity of η this is an open neighborhood, see Figure 3.5 for the flat analogue. Since the cutoff function χ is constant and equal to one on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we see that the function $\chi\left(\frac{\eta}{\epsilon_j}\right)$ is equal to one on the *open* S_j. From this *i.)* follows at once. Thus also the gradient vanishes on S_j whence *ii.)* and *iii.)* follow. Since $S_j \cap S_{j+1}$ is still an open neighborhood of $\eta^{-1}(\{0\})$,

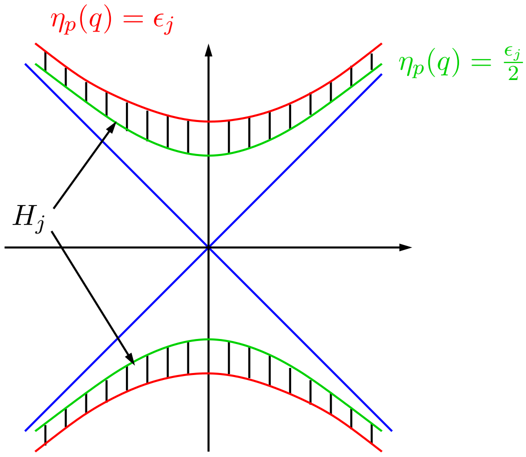

Figure 3.6: The set $H_j \cap \{p\} \times U'$.

we get *iv.)*. But this means that the prefactors in all the above terms vanish on an open neighborhood of $\eta^{-1}(\{0\})$ which was the only place where the Riesz distributions $R^{\pm}_{U'}(2+2j,\,\cdot\,)$ were non-smooth. Thus *v.)* follows, too. ☐

This lemma suggests that the weak* convergence of all the three sums Σ_1, Σ_2, and Σ_3, which we already know, can be sharpened to a $\mathcal{C}^\infty$-convergence: in this case the defect of $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ of being a fundamental solution would be just a *smooth section* and not a general, distributional section. After possibly redefining the ϵ_j this can indeed be achieved as we shall see now.

First we note that the functions $\chi\left(\frac{\eta}{\epsilon_j}\right)$ are only interesting in the following subset

$$H_j = \left\{(p,q) \in U' \times U' \;\middle|\; \frac{\epsilon_j}{2} \le \eta(p,q) \le \epsilon_j\right\}. \tag{3.4.27}$$

Indeed, for $\eta(p,q) > \epsilon_j$ the cutoff function produces a zero, for $\eta(p,q) < \frac{\epsilon_j}{2}$ the function is identically one until $\eta(p,q) < -\frac{\epsilon_j}{2}$. But for negative $\eta(p,q)$ the definition of $R^{\pm}_{U'}(2+2j,p)(q)$ gives already zero. Thus we only get contributions to each of the series Σ_1 and Σ_2 from H_j for the j-th term. Geometrically, $H_j \cap \{p\} \times U'$ looks like a thick mass shell, see Figure 3.6. It follows that for the j-th term in Σ_1 or Σ_2 we get only contributions from the compactum $K_\ell \times K_\ell \cap H_j$ for the seminorm $\mathrm{p}_{K_\ell \times K_\ell, k}$.

We start now estimating the $\mathrm{p}_{K_\ell \times K_\ell, k}$ of the j-th term in the sum Σ_2. To this end we first estimate the function η on $K_\ell \times K_\ell \cap H_j$ as follows.

Lemma 3.4.8 *Let $j \ge N$ and $k, \ell \in \mathbb{N}_0$ arbitrary. Then*

$$\mathrm{p}_{K_\ell \times K_\ell \cap H_j, k+1}\left(\eta^{j+1-\frac{n}{2}}\right) \le d(k,\ell,j)\epsilon_j^{j-\frac{n}{2}-k}, \tag{3.4.28}$$

with some constants $d(k,\ell,j) > 0$ such that

$$d(k,\ell,j) \le d(k',\ell',j) \tag{3.4.29}$$

for $k \le k'$ and $\ell \le \ell'$.

Proof. By the chain rule as in Lemma 3.4.2 we have

$$\mathrm{p}_{K_\ell \times K_\ell \cap H_j, k+1}\left(\eta^{j+1-\frac{n}{2}}\right)$$

$$
\begin{aligned}
&\leq \sup_{\substack{(p,q)\in K_\ell\times K_\ell\cap H_j\\ |I|\leq k+1}} \sum_{\substack{r\leq |I|\\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \left|\frac{\mathrm{d}\, t^{j+1-\frac{n}{2}}}{\mathrm{d}\, t^r}\right|_{t=\eta(p,q)} \left|\frac{\partial^{|J_1|}\eta}{\partial x^{J_1}}\right| \cdots \left|\frac{\partial^{|J_1|}\eta}{\partial x^{J_1}}\right| \\
&\leq \sup_{\substack{|I|\leq k+1\\ \frac{\epsilon_j}{2}\leq t\leq \epsilon_j}} \sum_{\substack{r\leq |I|\\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \left|\left(j+1-\frac{n}{2}\right)\cdots\left(j+1-\frac{n}{2}-r+1\right)t^{j+1-\frac{n}{2}-r}\right| \left(\mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(\eta)\right)^r \\
&\leq \sup_{|I|\leq k+1} \sum_{\substack{r\leq |I|\\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \left(\frac{\epsilon_j}{2}\right)^{j+1-\frac{n}{2}-(k+1)} \left(\mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(\eta)\right)^r \\
&\leq \epsilon_j^{j-\frac{n}{2}+k} \underbrace{\max_{|I|\leq k+1} \sum_{\substack{r\leq |I|\\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \frac{1}{2^{j+1-\frac{n}{2}-(k+1)}} \left(\mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(\eta)\right)^r}_{d(k,\ell,j)}.
\end{aligned}
$$

Note that the supremum over t and $r \leq k+1$ of $t^{j+1-\frac{n}{2}-r}$ is obtained for the smallest $t = \frac{\epsilon_j}{2}$ and the largest $r = k+1$. The constants $d(k,\ell,j)$ clearly grow if the compactum K_ℓ is replaced by the bigger one $K_{\ell'}$. They also grow if we allow larger k. □

This can now be used to estimate the j-th term of the series Σ_2. We have the following result:

Lemma 3.4.9 *Let $k, \ell \in \mathbb{N}_0$ and $j \geq N$. Then we have*

$$
\begin{aligned}
&\mathrm{p}_{K_\ell\times K_\ell,k}\left(\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}\left(V^j R^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\right) \\
&\leq c_k c(2+2j,n) d(k+1,\ell,j)\, \mathrm{p}_{K_\ell\times K_\ell,k+1}(V^j) \max_{r\leq k+1} \mathrm{p}_{K_\ell\times K_\ell,k+1}(\eta)^r \cdot \epsilon_j^{j-\frac{n}{2}-2k-1}.
\end{aligned}
\tag{3.4.30}
$$

Proof. We simply compute

$$
\begin{aligned}
&\mathrm{p}_{K_\ell\times K_\ell,k}\left(\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}\left(V^j R^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\right) \\
&\qquad = \mathrm{p}_{K_\ell\times K_\ell\cap H_j,k}\left(\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}\left(V^j R^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\right) \\
&\qquad \leq c_{k,\ell}\, \mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}\left(\chi\left(\frac{\eta}{\epsilon_j}\right)\right) \mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(V^j)\, \mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(R^{\pm}_{U'}(2+2j,\,\cdot\,)),
\end{aligned}
$$

since we need one order of differentiation for the gradient and one for the covariant derivative. In the constant $c_{k,\ell}$ the estimates of the derivatives of the metric, the connection, the Leibniz rule, etc. enter. Note that since these quantities are smooth everywhere, we can take the supremum over $K_\ell \times K_\ell$ whence $c_{k,\ell}$ does not depend on j. Now by the chain rule as in Lemma 3.4.2 we have

$$
\begin{aligned}
\mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}\left(\chi\left(\frac{\eta}{\epsilon_j}\right)\right) &\leq \sup_{\substack{(p,q)\in K_\ell\times K_\ell\cap H_j\\ |I|\leq k+1}} \sum_{\substack{r\leq |I|\\ J_1,\ldots,J_r\leq I}} c^r_{J_1\cdots J_r} \left|\frac{\mathrm{d}^r\chi}{\mathrm{d}\, t^r}\right|_{t=\frac{\eta}{\epsilon_j}} \frac{1}{\epsilon_j^r} \left|\frac{\partial^{|J_1|}\eta}{\partial x^{J_1}}\right| \cdots \left|\frac{\partial^{|J_r|}\eta}{\partial x^{J_r}}\right| \\
&\leq \frac{1}{\epsilon_j^{k+1}} c_k \max_{r\leq k+1} \mathrm{p}_{K_\ell\times K_\ell,k+1}(\eta)^r,
\end{aligned}
$$

where the sum over the $c^r_{J_1\cdots J_r}$ as well as the supremum over the r-th derivatives of χ are combined into the constant c_k. For the seminorm of $R^{\pm}_{U'}(2+2j,\,\cdot\,)$ we get

$$\begin{aligned}\mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}(R^{\pm}_{U'}(2+2j,\,\cdot\,)) &\le \mathrm{p}_{K_\ell\times K_\ell\cap H_j,k+1}\left(c(2+2j,n)\eta^{j+1-\frac{2}{2}}\right)\\ &\le c(2+2j,n)d(k+1,\ell,j)\epsilon_j^{j-\frac{n}{2}-k}\end{aligned}$$

by Lemma 3.4.8. Putting things together we obtain

$$\begin{aligned}&\mathrm{p}_{K_\ell\times K_\ell,k}\left(\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}V^jR^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\\ &\qquad\le c_kc(2+2j,n)d(k+1,\ell,j)\epsilon_j^{j-\frac{n}{2}-2k-1}\,\mathrm{p}_{K_\ell\times K_\ell,k+1}(V^j)\max_{r\le k+1}\mathrm{p}_{K_\ell\times K_\ell,k+1}(\eta)^r.\end{aligned}$$

□

Lemma 3.4.10 *Let $j\ge N$. Choose $\epsilon_j\in(0,1]$ such that in addition to* (3.4.17)

$$\epsilon_j\max_{\substack{\ell\le j\\ k\le\frac{1}{2}\left(j-\frac{n}{2}-1\right)}}c_kc(2+2j,n)d(k+1,\ell,j)\,\mathrm{p}_{K_\ell\times K_\ell,k+1}(V^j)\max_{r\le k+1}\mathrm{p}_{K_\ell\times K_\ell,k+1}(\eta)^r<\frac{1}{2^j}.\qquad(3.4.31)$$

Then the sum Σ_2 converges absolutely in the $\mathcal{C}^\infty$-topology to some $\Sigma_2\in\Gamma^\infty\left(E^*\boxtimes E\big|_{U'\times U'}\right)$.

Proof. First we note that we can indeed find $\epsilon_j\in(0,1]$ meeting the requirement (3.4.31). Then we have for fixed k,ℓ the estimate

$$\begin{aligned}&\mathrm{p}_{K_\ell\times K_\ell,k}\left(\sum_{j\ge N}2\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}V^jR^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\\ &\le\mathrm{p}_{K_\ell\times K_\ell,k}\left(\sum_{j=N}^{j_0-1}2\nabla^E_{\operatorname{grad}\chi\left(\frac{\eta}{\epsilon_j}\right)}V^jR^{\pm}_{U'}(2+2j,\,\cdot\,)\right)\\ &\quad+2\sum_{j\ge j_0}c_kc(2+2j,n)d(k+1,\ell,j)\,\mathrm{p}_{K_\ell\times K_\ell,k+1}(V^j)\max_{r\le k+1}\mathrm{p}_{K_\ell\times K_\ell,k+1}(\eta)^r\cdot\epsilon_j^{j-\frac{n}{2}-2k-1}\\ &\le\text{const.}+2\sum_{j\ge j_0}\frac{1}{2^j}<\infty,\end{aligned}$$

provided we set j_0 larger than ℓ and such that $j_0-\frac{n}{2}-2k-1\ge1$, which is clearly possible. In this case $\epsilon_j^{j-\frac{n}{2}-2k-1}\le\epsilon_j$ for $j\ge j_0$, and we can use (3.4.31) to get the estimate. But this shows absolute convergence in the seminorm $\mathrm{p}_{K_\ell\times K_\ell,k}$ as the finitely many terms with $N\le j\le j_0-1$ do not matter. Since ℓ and k were arbitrary we get $\mathcal{C}^\infty$-convergence. Note that it is crucial that each term of Σ_2 is already smooth, quite differently from the ideas in Proposition 3.4.5. □

By a completely analogous argument one can estimate the terms in the sum Σ_1 and show that again finitely many conditions on each $\epsilon_j\in(0,1]$ yield $\mathcal{C}^\infty$-convergence also of Σ_1. We do not write down the explicit condition but leave this as an exercise. The result is the following:

Lemma 3.4.11 *There are choices of $\epsilon_j\in(0,1]$ analogous to* (3.4.31) *such that the sum Σ_1 converges absolutely in the $\mathcal{C}^\infty$-topology to some section $\Sigma_1\in\Gamma^\infty\left(E^*\boxtimes E\big|_{U'\times U'}\right)$.*

Finally, we consider the third sum Σ_3. Here the argument is slightly different leading nevertheless to the same consequences.

Lemma 3.4.12 *Let $\ell, k \in \mathbb{N}_0$ and let $j \geq N$ satisfy $j \geq 2k + \frac{n}{2}$. Then we have*

$$\mathrm{p}_{K_\ell \times K_\ell, k}\left(\left(\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\right) D(V^j) R^{\pm}_{U'}(2+2j, \cdot\,)\right) \leq (\epsilon_j + \epsilon_{j+1}) f(k, \ell, j), \tag{3.4.32}$$

with some constants $f(k, \ell, j)$ not depending on the choices of the ϵ_j.

Proof. We estimate

$$\begin{aligned}
&\mathrm{p}_{K_\ell \times K_\ell, k}\left(\left(\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\right) D(V^j) R^{\pm}_{U'}(2+2j, \cdot\,)\right) \\
&= \mathrm{p}_{K_\ell \times K_\ell, k}\left(\left(\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\right) \eta^{k+1} D(V^j) c(2+2j, n) \eta^{j-\frac{n}{2}-k}\right) \\
&\leq c_k c(2+2j, n) \\
&\qquad \left(\mathrm{p}_{K_\ell \times K_\ell, k}\left(\chi\left(\frac{\eta}{\epsilon_j}\right)\eta^{k+1}\right) + \mathrm{p}_{K_\ell \times K_\ell, k}\left(\chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\eta^{k+1}\right)\right) \mathrm{p}_{K_\ell \times K_\ell, k}\left(D(V^j)\eta^{j-\frac{n}{2}-k}\right) \\
&\leq c_k c(2+2j, n) \left(\epsilon_j e(k, \ell, j) + \epsilon_{j+1} e(k, \ell, j)\right) \mathrm{p}_{K_\ell \times K_\ell, k}\left(D(V^j)\eta^{j-\frac{n}{2}-k}\right),
\end{aligned}$$

with some constants $e(k, \ell, j)$ obtained from a Leibniz rule and arguments as in the proof of Lemma 3.4.3 and Lemma 3.4.1. Note that for $j \geq 2k - \frac{n}{2}$ the function $\eta^{j-k-\frac{n}{2}}$ is still $\mathcal{C}^k$ whence the last seminorm is still finite. Putting all the constants together, we get the desired estimate. □

Again, we can turn (3.4.32) into a condition on the ϵ_j in order to make the seminorm smaller than $\frac{1}{2^j}$.

Lemma 3.4.13 *Let the $\epsilon_j \in (0, 1]$ be chosen such that in addition to (3.4.17) we have*

$$\epsilon_j \cdot \max\left\{\max_{\substack{\ell \leq j \\ 2k+\frac{n}{2} \leq j}} f(k, \ell, j), \max_{\substack{\ell \leq j-1 \\ 2k+\frac{n}{2} \leq j-1}} f(k, \ell, j-1)\right\} \leq \frac{1}{2^j}. \tag{3.4.33}$$

Then the sum Σ_3 converges absolutely with respect to the $\mathcal{C}^\infty$-topology and yields a smooth section $\Sigma_3 \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U' \times U'}\right)$.

Proof. Note that (3.4.33) are again finitely many condition on each ϵ_j whence we indeed can find an $\epsilon_j \in (0, 1]$ satisfying (3.4.33). Now Lemma 3.4.12 yields the estimate

$$\begin{aligned}
&\mathrm{p}_{K_\ell \times K_\ell, k} \sum_{j \geq N}\left(\left(\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\right) D(V^j) R^{\pm}_{U'}(2+2j, \cdot\,)\right) \\
&\leq \mathrm{p}_{K_\ell \times K_\ell, k} \sum_{j=N}^{j_0-1}\left(\left(\chi\left(\frac{\eta}{\epsilon_j}\right) - \chi\left(\frac{\eta}{\epsilon_{j+1}}\right)\right) D(V^j) R^{\pm}_{U'}(2+2j, \cdot\,)\right) + \sum_{j=j_0}^{\infty} (\epsilon_j + \epsilon_{j+1}) f(k, \ell, j) \\
&\leq \text{const.} + 2\sum_{j=j_0}^{\infty} \frac{1}{2^j} < \infty,
\end{aligned}$$

if we take $j_0 \geq N$ such that $j_0 \geq 2k + \frac{n}{2}$ and $j_0 \geq \ell$. Indeed, in this case we have

$$\epsilon_j \cdot \max_{\substack{\ell \leq j \\ 2k+\frac{n}{2} \leq j}} f(k, \ell, j) \leq \frac{1}{2^j} \quad \text{and} \quad \epsilon_{j+1} \cdot \max_{\substack{\ell \leq j \\ 2k+\frac{n}{2} \leq j}} f(k, \ell, j) \leq \frac{1}{2^j},$$

both by (3.4.33). But then the absolute convergence of Σ_3 is clear as the finitely many terms $N \leq j \leq j_0 - 1$ do not change the convergence. ☐

Collecting the results of the previous lemmas we arrive at the following result:

Proposition 3.4.14 *There is a choice of $\epsilon_j \in (0, 1]$ such that the approximate solution $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ satisfies in addition to the properties described in Proposition 3.4.5 and Corollary 3.4.6*

$$D\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \delta_p + K^{\pm}_{U'}(p, \,\cdot\,) \tag{3.4.34}$$

with some smooth section $K^{\pm}_{U'} \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U'\times U'}\right)$ for $p \in U'$.

Proof. Indeed, the section $K^{\pm}_{U'}$ is obtained from the computation in (3.4.26) as

$$K^{\pm}_{U'} = \left(1 - \chi\left(\frac{\eta}{\epsilon_N}\right)\right) D(V^{N-1}) R^{\pm}_{U'}(2N, \,\cdot\,) + \Sigma_1 + \Sigma_2 + \Sigma_3.$$

The convergence results on the series Σ_1, Σ_2, and Σ_3 yield $K^{\pm}_{U'} \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U'\times U'}\right)$ as we wanted. Note that in total, we only have to impose finitely many conditions on each ϵ_j according to Proposition 3.4.5, *i.)*, Lemma 3.4.10, the analogue condition from Σ_1, and Lemma 3.4.13. ☐

Remark 3.4.15 (Parametrix) The proposition just says that we have constructed a parametrix $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ of D for every $p \in U'$, see also [31, Sect. 7.1] for more information on parametrices.

In Proposition 3.2.15 we had some estimates for $|R^{\pm}_{U'}(p)(\varphi)|$ *locally uniform* in p. Since $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ is build out of the $R^{\pm}_{U'}(\alpha, p)$ we can expect a similar feature also for $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$. Indeed, this is the case:

For a fixed $\varphi \in \Gamma^\infty_0(E^*\big|_{U'})$ we can view $U' \ni p \mapsto \widetilde{\mathcal{R}}^{\pm}_{U'}(p)(\varphi) \in E^*_p$ as a section of E^* defined on U'. This section has nice features, it will be smooth again. More precisely, we have the following statements:

Proposition 3.4.16 *Let $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ be the approximate fundamental solution. Moreover, let $k \in \mathbb{N}_0$ and $K, L \subset U'$ be compact. Then we have:*

i.) There is a constant $c_{K,L} > 0$ such that

$$\left|\widetilde{\mathcal{R}}^{\pm}_{U'}(p)(\varphi)\right| \leq c_{K,L}\, \mathrm{p}_{K,n+1}(\varphi) \tag{3.4.35}$$

for all $p \in L$ and $\varphi \in \Gamma^\infty_K\left(E^*\big|_{U'}\right)$. In particular, the distribution $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ is of global order $\leq n + 1$.

ii.) The section $\widetilde{\mathcal{R}}^{\pm}_{U'}(\,\cdot\,)(\varphi)$ of $E^*\big|_{U'}$ is smooth for all $\varphi \in \Gamma^\infty_0(E^*\big|_{U'})$.

iii.) There are constants $c_{K,L,k} > 0$ such that

$$\mathrm{p}_{L,k}(\widetilde{\mathcal{R}}^{\pm}_{U'}(\,\cdot\,)(\varphi)) \leq c_{K,L,k}\, \mathrm{p}_{K,k+n+1}(\varphi) \tag{3.4.36}$$

for all $\varphi \in \Gamma^\infty_K(E^*\big|_{U})$.

iv.) The operator

$$\mathcal{R}^{\pm}_{U'} : \Gamma^\infty_0(E^*\big|_{U'}) \ni \varphi \mapsto \big(p \mapsto \mathcal{R}^{\pm}_{U'}(p)(\varphi)\big) \in \Gamma^\infty(E^*\big|_{U'}) \tag{3.4.37}$$

is continuous in the $\mathcal{C}^\infty_0$*- and* $\mathcal{C}^\infty$*-topology.*

Proof. Clearly, the estimate (3.4.35) is a particular case of the more general situation in (3.4.36) for $k = 0$. Thus fix $k \in \mathbb{N}_0$. Then we have

$$\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \sum_{j=0}^{N+1} V_p^j R^{\pm}_{U'}(2+2j, p) + \sum_{j=N}^{N+k-1} \chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R^{\pm}_{U'}(2+2j, p) + \sum_{j=N+k}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right) V_p^j R^{\pm}_{U'}(2+2j, p), \tag{$*$}$$

and we know that the third contribution converges in the $\mathcal{C}^k$-topology to

$$f_k(p,q) = \sum_{j=N+k}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right)(q) V_p^j(q) R^{\pm}_{U'}(2+2j, p)(q),$$

which is a section $f_k \in \mathcal{C}^k(E^* \boxtimes E\big|_{U'\times U'})$. Now let $\varphi \in \Gamma^\infty_K(E^*\big|_{U'})$ then the pairing of f_k with φ is

$$p \mapsto f_k(p, \cdot\,)\varphi = \int_{U'} f(p,q)\cdot\varphi(q)\, \mu_g(q) = \int_K f_k(p,q)\cdot\varphi(q)\,\mu_g(q), \tag{$**$}$$

which still yields a $\mathcal{C}^k$-section. In fact, we immediately obtain an estimate of the form

$$\mathrm{p}_{L,k}(f_k(\,\cdot\,)\varphi) \leq \mathrm{vol}(K)\, \mathrm{p}_{L\times K,k}(f_k)\, \mathrm{p}_{K,0}(\varphi)$$

by differentiating into the integral ($**$), which is legal as the compactly supported integrand is $\mathcal{C}^k$ in p and all first derivatives in p-direction yield still a continuous integrand in p and q. The first and second contribution in ($*$) are slightly more complicated. First we note that the sums are all finite and each term is of the form $\Phi^k(p, \cdot\,)R^{\pm}_{U'}(2+2j,p)$ with a smooth section $\Phi^j \in \Gamma^\infty(E^* \boxtimes E\big|_{U'\times U'})$. Thus applying this to a fixed test section $\varphi \in \Gamma^\infty_K(E^*\big|_{U'})$ gives by the very definition of the Riesz distributions the map

$$\begin{aligned} p \mapsto \Phi^j(p,\cdot\,)R^{\pm}_{U'}(2+2j,p)(\varphi) &= R^{\pm}_{U'}(2+2j,p)\left(\Phi^j(p,\cdot\,)\varphi(\,\cdot\,)\right) \\ &= R^{\pm}(2+2j)\left(\widetilde{\varrho}_p(\,\cdot\,)\exp_p^*(\Phi^j(p,\cdot\,)\varphi(\,\cdot\,))\right). \end{aligned} \tag{$***$}$$

If we want now to estimate the p-dependence we can rely on Lemma 1.3.38: The function $(p,q) \mapsto \widetilde{\varrho}_p(q)\exp_p^*(\Phi^j(p,q)\varphi(q))$ is smooth in both variables and has support in $U' \times K$ thanks to the support condition on φ. Thus the lemma applies and yields a smooth function of p. Moreover, we can differentiate into the application of $R^{\pm}(2+2j)$ and have for the p-derivatives of ($***$)

$$\begin{aligned} &\frac{\partial^{|I|}}{\partial x^I}\left(p \mapsto \Phi^j(p,\cdot\,)R^{\pm}_{U'}(2+2j,p)(\varphi)\right) \\ &\quad = R^{\pm}(2+2j)\left(\frac{\partial^{|I|}}{\partial x^I}\left(p \mapsto \Phi^j(p,\cdot\,)R^{\pm}_{U'}(2+2j,p)(\varphi)\right)\right), \end{aligned} \tag{☺}$$

where x are some generic coordinates for the p-variable. Now we know that for $j \geq 0$ the Riesz distribution $R^{\pm}(2+2j)$ is of order $\leq n+1$. In fact, the order is much less for some j, see also the low dimensional discussion in Section 3.1.3, but the above estimate on the order will do the job. Thus for each term we get an estimate of the form

$$\mathrm{p}_{L,k}\left(\Phi^k R^{\pm}_U(\,\cdot\,)\varphi\right) \leq c^j_{K,L}\, \mathrm{p}_{K,k+n+1}(\varphi),$$

as we need the $n+1$ derivatives of φ for $R^{\pm}(2+2j)$ and up to k derivatives from the differentiation and the chain rule coming from (☺). In the constant $c^j_{K,L}$ we get contributions of the first k derivatives of Φ^j, $\exp_p$ and $\widetilde{\varrho}_p$ as well as from the continuity of $R^{\pm}(2+2j)$. Thus we arrive at finitely many estimates for the finitely many terms in $(*)$ which can be combined into (3.4.36). This shows the third part. But then the fourth part is clear as well. □

Remark 3.4.17 The estimate in (3.4.36) also shows that we can apply the operator $\widetilde{\mathcal{R}}^{\pm}_{U'}$ to less regular sections than smooth ones. In fact, $\widetilde{\mathcal{R}}^{\pm}_{U'}$ extends to a well-defined continuous linear operator

$$\widetilde{\mathcal{R}}^{\pm}_{U'} : \Gamma^{k+n+1}_0(E^*\big|_{U'}) \longrightarrow \Gamma^k(E^*\big|_{U'}) \tag{3.4.38}$$

for all $k \geq 0$ with respect to the $\mathcal{C}^{k+n+1}_0$- and $\mathcal{C}^k$-topology, respectively. This will sometimes be a useful extension.

The last features we will need are some support properties of the "defect" $K^{\pm}_{U'}$ of $\widetilde{\mathcal{R}}^{\pm}_{U'}$ being a fundamental solution.

Lemma 3.4.18 *The smooth section* $K^{\pm}_{U'} \in \Gamma^{\infty}\left(E^* \boxtimes E\big|_{U'\times U'}\right)$ *satisfies*

$$(p,q) \in \operatorname{supp} K^{\pm}_{U'} \subseteq U' \times U' \Longrightarrow q \in J^{\pm}_{U'}(p). \tag{3.4.39}$$

Proof. Assume that $K^{\pm}_{U'}(p,q)$ is non-zero. From

$$K^{\pm}_{U'}(p,q) = \left(1 - \chi\left(\frac{\eta_p(q)}{\epsilon_j}\right)\right) D\left(V^{N-1}_p R^{\pm}_{U'}(2N,p)\right)(q) + \Sigma_1(p,q) + \Sigma_2(p,q) + \Sigma_3(p,q)$$

and the fact that each series $\Sigma_1, \Sigma_2, \Sigma_3$ has only terms involving $R^{\pm}_{U'}(2+2j,p)(q)$, to have a non-zero contribution we necessarily need $q \in J^{\pm}_{U'}(p)$. Thus $K^{\pm}_{U'}(p,q) \neq 0$ implies $q \in J^{\pm}_{U'}(p)$. Since the support of $K^{\pm}_{U'}$ is the closure of all those point with $K^{\pm}_{U'}(p,q) \neq 0$ it is contained in the closure of those points $(p,q) \in U' \times U'$ with $q \in J^{\pm}_{U'}(p)$, all closures taken with respect to $U' \times U'$. Since U' is assumed to be geodesically convex, one can show that the causal relation

$$J^{\pm}_{U'} = \left\{(p,q) \in U' \times U' \;\middle|\; q \in J^{\pm}_{U'}(p)\right\} \subseteq U' \times U'$$

is actually closed. Note that this is a stronger statement than all $J^{\pm}_{U'}(p)$ being closed in U', see e.g. [45, Prop. 2.10] or [46, Lemma 2 in Chap. 14]. But then (3.4.39) follows at once. □

Remark 3.4.19 (Future and past stretched subsets) A subset $S \subseteq U' \times U'$ with the feature that $(p,q) \in S$ implies $q \in J^{\pm}_{U'}(p)$ is also called future or past stretched, respectively. Thus the support of $K^{\pm}_{U'}$ is future and past stretched with respect to U', respectively.

We are now in the position to collect all the features of the approximate fundamental solution $\widetilde{\mathcal{R}}^{\pm}_{U'}$ we shall need in the following:

Theorem 3.4.20 (Approximate fundamental solution) *Let* $U' \subseteq M$ *be geodesically convex and let* $V^j \in \Gamma^{\infty}\left(E^* \boxtimes E\big|_{U'\times U'}\right)$ *be the Hadamard coefficients with respect to the normally hyperbolic operator* $D \in \operatorname{DiffOp}^2(E)$. *Then there exists a sequence* $\epsilon_j \in (0,1]$ *for* $j \geq N > \frac{n}{2}$ *such that*

$$\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \sum_{j=0}^{N-1} V^j_p R^{\pm}_{U'}(2+2j,p) + \sum_{j=N}^{\infty} \chi\left(\frac{\eta_p}{\epsilon_j}\right) V^j_p R^{\pm}_{U'}(2+2j,p) \tag{3.4.40}$$

converges in the weak topology to a distribution* $\widetilde{\mathcal{R}}^{\pm}_{U'}(p) \in \Gamma^{-\infty}\left(E\big|_{U'}\right) \otimes E^*_p$ *with the following properties:*

i.) For the support and singular support we have

$$\operatorname{supp} \widetilde{\mathcal{R}}^{\pm}_{U'}(p) \subseteq J^{\pm}_{U'}(p), \tag{3.4.41}$$

$$\operatorname{sing\,supp} \widetilde{\mathcal{R}}^{\pm}_{U'}(p) \subseteq C^{\pm}_{U'}(p). \tag{3.4.42}$$

ii.) We have

$$D\widetilde{\mathcal{R}}^{\pm}_{U'}(p) = \delta_p + K^{\pm}_{U'}(p, \,\cdot\,) \tag{3.4.43}$$

with a smooth section $K^{\pm}_{U'} \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U'\times U'}\right)$.

iii.) The support of $K^{+}_{U'}$ *is future stretched and the support of* $K^{-}_{U'}$ *is past stretched.*

iv.) For a test section $\varphi \in \Gamma^\infty_0\left(E^*\big|_{U'}\right)$ *the section* $p \mapsto \widetilde{\mathcal{R}}^{\pm}_{U'}(p)(\varphi)$ *is smooth.*

v.) For compact subsets $K, L \subseteq U'$ *there exist constants* $c_{K,L} > 0$ *such that*

$$\left|\widetilde{\mathcal{R}}^{\pm}_{U'}(p)(\varphi)\right| \leq c_{K,L}\, \mathrm{p}_{K,n+1}(\varphi) \tag{3.4.44}$$

for all $p \in L$ *and* $\varphi \in \Gamma^\infty_K\left(E^*\big|_{U'}\right)$. *In particular, for the global order of* $\widetilde{\mathcal{R}}^{\pm}_{U'}(p)$ *we have*

$$\operatorname{ord}\left(\widetilde{\mathcal{R}}^{\pm}_{U'}(p)\right) \leq n + 1. \tag{3.4.45}$$

3.4.2 Construction of the Local Fundamental Solution

Having a (well-behaved) parametrix to a differential operator there is a more or less standard procedure of how one can obtain a fundamental solution from it. Roughly speaking, the defect in having a fundamental solution is so small that one can use a geometric series to resolve this problem.

We will choose now an open subset $U \subseteq U'$ such that

$$U^{\mathrm{cl}} \subseteq U' \tag{3.4.46}$$

is compact. Later on, we will need additional properties of U but for the time being the compactness of U^{cl} will suffice. Then we consider the following integral operator build out of $K^{\pm}_{U'} \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U'\times U'}\right)$. Let φ be a section of E^* defined at least on U^{cl} then we can naturally pair $K^{\pm}_{U'}(p, q) \cdot \varphi(q)$ and integrate. This gives

$$\left(\mathcal{K}^{\pm}_{U}\varphi\right)(p) = \int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p, q) \cdot \varphi(q)\, \mu_g(q). \tag{3.4.47}$$

Depending on the properties of φ the integral will be well-defined and yields a rather nice section of E^* defined on U'. One rather general scenario is the following:

Definition 3.4.21 *With respect to some auxiliary positive fiber metric on* E^* *we define*

$$\Gamma_b\left(E^*\big|_U\right) = \left\{\varphi : U \longrightarrow E^* \;\middle|\; \varphi(q) \in E^*_q \text{ and } \varphi \text{ is bounded and measurable}\right\}. \tag{3.4.48}$$

Here the fiber metric is used to define a norm on each fiber. With respect to these norms we want φ to be bounded over U. The following technical lemma is well-known and obtained in a completely standard way:

Lemma 3.4.22 (The Banach space $\Gamma_b\left(E^*\big|_U\right)$) *Let* $U \subseteq M$ *be open with compact closure.*

i.) The definition of $\Gamma_b\left(E^*\big|_U\right)$ *does not depend on the auxiliary smooth fiber metric.*

ii.) *The vector space* $\Gamma_b\left(E^*\big|_U\right)$ *becomes a Banach space via the norm*

$$\mathrm{p}_{U,0}(\varphi) = \sup_{q\in U} \|\varphi(q)\|_{E^*_q}. \tag{3.4.49}$$

iii.) *Different choices of positive fiber metrics on* E^* *yield equivalent Banach norms* (3.4.49).

iv.) *The restriction map*

$$\Gamma^k(E^*) \ni \varphi \mapsto \varphi\big|_U \in \Gamma_b(E^*\big|_U) \tag{3.4.50}$$

is continuous for all $k \in \mathbb{N}_0 \cup \{+\infty\}$.

Proof. The measurability of a section is intrinsically defined and refers only to the Borel σ-algebra of the topological space M. Clearly, the boundedness does not depend on the choice of the fiber metric. Only the numerical value of the bound depends on this choice. Obviously, (3.4.49) is a norm and different choices of the fiber metric yield equivalent norms in (3.4.49). This can entirely be copied from our considerations in Theorem 1.1.5. We have to show completeness of $\Gamma_b(E^*\big|_U)$. Thus let $\varphi_n \in \Gamma_b(E^*\big|_U)$ be a Cauchy sequence with respect to $\mathrm{p}_{U,0}$. Then we have uniform convergence of $\varphi_n(q) \longrightarrow \varphi(q)$ on U^{cl}. Since every φ_n is bounded the limit is bounded as well. Finally, already the pointwise limit of measurable functions (and hence by local triviality: of sections) is known to be measurable again, see e.g. [2, Satz X.1.11]. Thus $\varphi \in \Gamma_b\left(E^*\big|_U\right)$ is the desired limit of φ_n. Finally, if $\varphi \in \Gamma^k(E^*)$ then $\varphi\big|_U \in \Gamma_b\left(E^*\big|_U\right)$ since over a compactum U^{cl} any continuous section is bounded and measurable. Moreover, by elementary features of the supremum we have

$$\mathrm{p}_{U,0}\left(\varphi\big|_U\right) = \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi),$$

with our previous definition of the seminorm $\mathrm{p}_{K,\ell}$. This gives the continuity of (3.4.50). ☐

We claim that the operator $\mathcal{K}^\pm_U$ is well-defined on $\Gamma_b\left(E^*\big|_U\right)$ and maps into the smooth sections in a continuous manner.

Lemma 3.4.23 *Let* $k \in \mathbb{N}_0$ *and* $U \subseteq U'$ *open with compact closure* $U^{\mathrm{cl}} \subseteq U'$.

i.) *For* $\varphi \in \Gamma_b\left(E^*\big|_U\right)$ *we have* $\mathcal{K}^\pm_U\varphi \in \Gamma^\infty\left(E^*\big|_{U'}\right)$.

ii.) *We have an estimate of the form*

$$\mathrm{p}_{K,k}\left(\mathcal{K}^\pm_U\varphi\right) \le \mathrm{vol}(U^{\mathrm{cl}})\, \mathrm{p}_{K\times U^{\mathrm{cl}},k}(K^\pm_{U'})\, \mathrm{p}_{U,0}(\varphi) \tag{3.4.51}$$

for all $\varphi \in \Gamma_b\left(E^*\big|_U\right)$ *and compact* $K \subseteq U'$.

Proof. We first proof continuity. Thus let $p \in U'$ be fixed and consider $p_n \longrightarrow p$. Since the integrand $K^\pm_U(p_n, q)\cdot\varphi(q)$ is bounded by some integrable function, namely by the constant function $\mathrm{p}_{K\times U^{\mathrm{cl}},0}(K^\pm_{U'})\, \mathrm{p}_{U,0}(\varphi)$ where K is any compactum containing the convergent sequence p_n, we can apply Lebesgue's dominated convergence and find

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{K}^\pm_U\varphi)(p_n) &= \lim_{n\to\infty} \int_{U^{\mathrm{cl}}} K^\pm_{U'}(p_n, q)\cdot\varphi(q)\, \mu_g(q) \\
&\overset{\text{Lebesgue}}{=} \int_{U^{\mathrm{cl}}} \lim_{n\to\infty} K^\pm_{U'}(p_n, q)\cdot\varphi(q)\, \mu_g(q) \\
&= \int_{U^{\mathrm{cl}}} K^\pm_{U'}(p, q)\cdot\varphi(q)\, \mu_g(q) \\
&= (\mathcal{K}^\pm_U\varphi)(p),
\end{aligned}$$

which is the continuity of $\mathcal{K}^{\pm}_U\varphi$. By an analogous argument we can also exchange the partial differentiation with the integration yielding a continuous partial derivative

$$\frac{\partial}{\partial x^i}\mathcal{K}^{\pm}_U\varphi = \int_{U^{\mathrm{cl}}} \frac{\partial K^{\pm}_{U'}(p,q)}{\partial x^i} \cdot \varphi(q)\, \mu_g(q), \tag{$*$}$$

all with respect to some local trivialization of E^*. Thus $\mathcal{K}^{\pm}_U\varphi$ turns out to be $\mathcal{C}^1$ and by induction we get $\mathcal{K}^{\pm}_U\varphi \in \Gamma^\infty\left(E^*\big|_{U'}\right)$. This shows the first part. For the second, we use a local trivialization and $(*)$ to obtain

$$\frac{\partial^{|I|}}{\partial x^I}(\mathcal{K}^{\pm}_U\varphi)\Big|_p = \int_{U^{\mathrm{cl}}} \frac{\partial^{|I|}K^{\pm}_{U'}}{\partial x^I}(p,q)\cdot\varphi(q)\,\mu_g(q),$$

from which we get

$$\begin{aligned}
\mathrm{p}_{U^{\mathrm{cl}},k}(\mathcal{K}^{\pm}_U\varphi) &\le \sup_{\substack{p\in U^{\mathrm{cl}}\\ |I|\le k}} \int_{U^{\mathrm{cl}}} \left\| \frac{\partial^{|I|}K^{\pm}_{U'}}{\partial x^I}(p,q)\right\| \|\varphi(q)\|\ \mu_g(q)\\
&\le \operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{K\times U^{\mathrm{cl}},k}(K^{\pm}_{U'})\,\mathrm{p}_{U,0}(\varphi).
\end{aligned}$$

□

With other words, the integral operator behaves like a convolution integral: the result inherits the better properties concerning smoothness of both factors under the integral.

The problem is now that the operator $\mathcal{K}^{\pm}_U$ is far from being "local": it changes and typically enlarges the support strictly. Thus it is slightly tricky to define powers of $\mathcal{K}^{\pm}_U$. However, as we did not insist on φ being continuous at all we can proceed as follows: For $\varphi \in \Gamma_b\left(E^*\big|_U\right)$ the section $\mathcal{K}^{\pm}_U\varphi$ is smooth and defined on the *larger* open subset U'. Thus restricting $\mathcal{K}^{\pm}_U\varphi$ back to U^{cl} yields a section which is clearly measurable and bounded *and* still smooth on the interior U of U^{cl}. Thus we have

$$\Gamma_b\left(E^*\big|_U\right) \ni \varphi \ \mapsto\ \mathcal{K}^{\pm}_U\varphi\big|_U \in \Gamma_b\left(E^*\big|_U\right). \tag{3.4.52}$$

By some slight abuse of notation we denote the composition $\varphi \mapsto \mathcal{K}^{\pm}_U\varphi \mapsto \mathcal{K}^{\pm}_U\varphi\big|_U$ again simply by $\mathcal{K}^{\pm}_U$.

Lemma 3.4.24 *The linear operator*

$$\mathcal{K}^{\pm}_U : \Gamma_b\left(E^*\big|_U\right) \ni \varphi \ \mapsto\ \mathcal{K}^{\pm}_U\varphi\big|_U \in \Gamma_b\left(E^*\big|_U\right) \tag{3.4.53}$$

is continuous with operator norm

$$\left\|\mathcal{K}^{\pm}_U\right\| \le \operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(K^{\pm}_{U'}). \tag{3.4.54}$$

Proof. From Lemma 3.4.23 we know that for all $\varphi \in \Gamma_b\left(E^*\big|_U\right)$ we have

$$\mathrm{p}_{U,0}(\mathcal{K}^{\pm}_U\varphi) = \mathrm{p}_{U^{\mathrm{cl}},0}(\mathcal{K}^{\pm}_U\varphi) \le \operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(\mathcal{K}^{\pm}_U)\,\mathrm{p}_{U,0}(\varphi),$$

which gives the continuity as well as the estimate on the operator norm (3.4.54). □

Corollary 3.4.25 *If the open subset $U \subseteq U'$ is sufficiently small in the sense that*

$$\operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(\mathcal{K}^{\pm}_U) < 1, \tag{3.4.55}$$

then the operator

$$\mathsf{id} + \mathcal{K}^{\pm}_U : \Gamma_b\left(E^*\big|_U\right) \longrightarrow \Gamma_b\left(E^*\big|_U\right) \tag{3.4.56}$$

is invertible with continuous inverse given by the absolutely norm-convergent geometric series

$$\left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1} = \sum_{j=0}^{\infty}(-\mathcal{K}^{\pm}_U)^j. \tag{3.4.57}$$

Proof. Since the operator norm of $\mathcal{K}_U^{\pm}$ is smaller or equal to $\operatorname{vol}(U^{\mathrm{cl}})\, \mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(\mathcal{K}_U^{\pm})$ the statement follows from general arguments on the geometric series and the fact that bounded operators on a Banach space form a Banach space themselves with respect to the operator norm. □

Note that since $\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(\mathcal{K}_U^{\pm})$ is only getting smaller for smaller U^{cl}, there always exists a small enough $U \subseteq U'$ around a given point in U'.

The idea is now to use the inverse $(\mathsf{id} + \mathcal{K}_U^{\pm})^{-1}$ to correct the approximate solution $\widetilde{\mathcal{R}}_{U'}^{\pm}$ at least on some small enough $U \subseteq U'$. There are now two problems: the inverse a priori maps into $\Gamma_b\left(E^*\big|_U\right)$ but we want some smooth section instead of a bounded and measurable one. Moreover, we want to control the support of the result at least in so far that we get "causal behaviour".

The first problem is solved by a more careful investigation of the geometric series: indeed the operator $\mathcal{K}_U^{\pm}$ already maps into much nicer sections than just bounded and measurable ones. By Lemma 3.4.23 they are restrictions of smooth sections on U'.

The second problem will persist unless we make some additional assumptions on the subset U. It has to be causal, see Section 2.2.3. We will postpone this investigation to Section 3.4.3.

We start to discuss the smoothness properties. For continuous sections things are still very simple as there is a good and easy notion of a continuous section over a compact subset. In fact, the continuous sections over U^{cl} form a *closed subspace*

$$\Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \subseteq \Gamma_b\left(E^*\big|_U\right) \tag{3.4.58}$$

with respect to the norm $\mathrm{p}_{U,0} = \mathrm{p}_{U^{\mathrm{cl}},0}$. Clearly, restricting a continuous section $\varphi \in \Gamma^0\left(E^*\big|_{U'}\right)$ to U^{cl} yields $\varphi\big|_{U^{\mathrm{cl}}} \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$. From Lemma 3.4.23, *i.)* we obtain

$$\mathcal{K}_U^{\pm} : \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \tag{3.4.59}$$

in a continuous way. Moreover, the operator norm estimate (3.4.54) for the restriction (3.4.59) of $\mathcal{K}_U^{\pm}$ to continuous sections is still valid. Since $\Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ is a Banach space by its own, we get a continuous invertible operator

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} = \sum_{j=0}^{\infty} (-\mathcal{K}_U^{\pm})^j : \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \tag{3.4.60}$$

with absolutely norm-convergent geometric series analogously to Corollary 3.4.25.

In order to control the smoothness properties of the inverse of $\mathsf{id} + \mathcal{K}_U^{\pm}$ we introduce the following subspaces of $\Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$. The tricky point is to define smoothness on a *closed* subset U^{cl} instead of an open one in such a way that we still get a good functional space.

Definition 3.4.26 (The space $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$) *Let $k \in \mathbb{N}_0$, then a section $\varphi \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ is called $\mathcal{C}^k$ on U^{cl} if it can be approximated by sections $\varphi_n\big|_{U^{\mathrm{cl}}}$, with $\varphi_n \in \Gamma^k\left(E^*\big|_{U_n}\right)$ with respect to the norm $\mathrm{p}_{U^{\mathrm{cl}},k}$, where $U_n \supseteq U^{\mathrm{cl}}$ is open. The set of all such section is denoted by*

$$\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) = \left\{\varphi \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \;\middle|\; \varphi \text{ is } \mathcal{C}^k\right\}. \tag{3.4.61}$$

Remark 3.4.27 For sections in $\Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ which are $\mathcal{C}^k$ in U and have bounded derivatives the seminorm $\mathrm{p}_{U^{\mathrm{cl}},k}$ is actually a norm with $\mathrm{p}_{U^{\mathrm{cl}},0} \leq \mathrm{p}_{U^{\mathrm{cl}},k}$. We obtain a norm topology on the subset of sections $\varphi \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ which are restrictions of $\mathcal{C}^k$-sections defined on an (arbitrarily small) open neighborhood of U^{cl}. By definition, $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ is the Banach space completion of these sections. Note however that for $\varphi \in \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ it is *not clear* whether there is a section $\widetilde{\varphi} \in \Gamma^k\left(E^*\big|_{\widetilde{U}}\right)$ with

$$\varphi = \widetilde{\varphi}\big|_{U^{\mathrm{cl}}} \tag{3.4.62}$$

for some open $\tilde{U} \supseteq U^{\mathrm{cl}}$. In fact, the existence of such a $\mathcal{C}^k$-section $\widetilde{\varphi}$ depends very much of the form of the boundary ∂U^{cl} of U^{cl} which can be very "wild".

Though this is a difficult question in general, we shall not be bothered by it too much as in the end we are only interested in $\varphi\big|_U$ for $\varphi \in \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ which is $\mathcal{C}^k$ on U. In fact, we have that

$$\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \ni \varphi \;\mapsto\; \varphi\big|_U \in \Gamma^k\left(E^*\big|_U\right) \tag{3.4.63}$$

is a continuous injective linear map with

$$\mathrm{p}_{K,k}(\varphi) \le \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi) \tag{3.4.64}$$

for all compact $K \subseteq U^{\mathrm{cl}}$. This is obvious. Note however, that in general (3.4.63) is far from being *surjective.*

Remark 3.4.28 Let $D \in \mathrm{DiffOp}^k(E^*)$ be a differential operator of order k and $\ell \ge k$. Then there is a canonical extension of $D\big|_U$ to $\Gamma^\ell(E^*\big|_{U^{\mathrm{cl}}})$ such that for $\varphi \in \Gamma^\ell\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we have $D\varphi \in \Gamma^{\ell-k}\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ and

$$D : \Gamma^\ell\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^{\ell-k}\left(E^*\big|_{U^{\mathrm{cl}}}\right) \tag{3.4.65}$$

is continuous. Indeed, let $\widetilde{\varphi} \in \Gamma^\ell\left(E^*\big|_{\widetilde{U}}\right)$ then $\mathrm{p}_{U^{\mathrm{cl}},\ell-k}(D\widetilde{\varphi}) \le c\,\mathrm{p}_{U^{\mathrm{cl}},k}(\widetilde{\varphi})$ for some $c > 0$ depending on D by Theorem 1.2.8. Since the restrictions of such $\widetilde{\varphi}$ to U^{cl} form a dense set in the Banach space $\Gamma^\ell\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we obtain the result.

Lemma 3.4.29 *The operator* $\mathcal{K}_U^\pm : \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ *restricts to a continuous linear operator*

$$\mathcal{K}_U^\pm : \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \tag{3.4.66}$$

for all $k \in \mathbb{N}_0$ *whose image are restrictions of smooth sections of* E^* *defined on* U'. *The operator norm of* (3.4.66) *is bounded by*

$$\left\|\mathcal{K}_U^\pm\right\| \le \mathrm{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},k}(K_{U'}^\pm). \tag{3.4.67}$$

Proof. Since $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \subseteq \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \subseteq \Gamma_b\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we can use Lemma 3.4.23 to get the estimate

$$\mathrm{p}_{U^{\mathrm{cl}},k}(\mathcal{K}_U^\pm\varphi) \le \mathrm{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},k}(K_{U'}^\pm)\,\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi)$$

and $\mathcal{K}_U^\pm\varphi \in \Gamma^\infty\left(E^*\big|_{U'}\right)$. Since in general $\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) \le \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi)$ for $\varphi \in \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we have the continuity and also the estimate for the operator norm of $\mathcal{K}_U^\pm$ as in (3.4.67). □

If we want to repeat the argument of invertibility of $\mathcal{K}_U^\pm : \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we face the following problem: for a fixed k we can certainly shrink U in such a way that the operator norm (3.4.67) becomes less than one, but as we are interested in *all* $k \in \mathbb{N}$ the countable intersection of all shrinkings of U might be empty. Thus we have to proceed differently. The idea is that we influence the *numerical value* of the operator norm of $\mathcal{K}_U^\pm$ by passing to a different but equivalent Banach norm for $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$.

Lemma 3.4.30 *Let* $U \subseteq U'$ *be small enough such that*

$$\delta = \mathrm{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(\mathcal{K}_U^\pm) < 1, \tag{3.4.68}$$

and let $k \in \mathbb{N}_0$. *Then*

$$\widetilde{\mathrm{p}}_{U^{\mathrm{cl}},k}(\varphi) = \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) + \frac{1-\delta}{2\,\mathrm{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},k}(K_{U'}^\pm)+1}\,\mathrm{p}_{U^{\mathrm{cl}},k}(\varphi) \tag{3.4.69}$$

defines a norm on $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \subseteq \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ *which is equivalent to* $\mathrm{p}_{U^{\mathrm{cl}},k}$. *With respect to this Banach norm the operator* $\mathcal{K}_U^\pm$ *has operator norm*

$$\left\|\mathcal{K}_U^\pm\right\|^\sim \le \frac{1+\delta}{2} < 1. \tag{3.4.70}$$

Proof. We know that $1-\delta > 0$ by assumption. Thus it is an easy task to see that the two norms $\widetilde{\mathrm{p}}_{U^{\mathrm{cl}},k}$ and $\mathrm{p}_{U^{\mathrm{cl}},k}$ are equivalent, since $\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) \leq \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi)$ as well as $\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) < \widetilde{\mathrm{p}}_{U^{\mathrm{cl}},k}(\varphi)$. Moreover, by (3.4.67) we have for the operator norm of $\mathcal{K}^{\pm}_U$ the following estimate

$$
\begin{aligned}
\widetilde{\mathrm{p}}_{U^{\mathrm{cl}},k}(\mathcal{K}^{\pm}_U\varphi) &= \mathrm{p}_{U^{\mathrm{cl}},0}(\mathcal{K}^{\pm}_U\varphi) + \frac{1-\delta}{2\operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},k}(K^{\pm}_{U'})+1}\,\mathrm{p}_{U^{\mathrm{cl}},k}(\mathcal{K}^{\pm}_U\varphi) \\
&\leq \delta\,\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) + \frac{1-\delta}{2\operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},k}(K^{\pm}_{U'})+1}\operatorname{vol}(U^{\mathrm{cl}})\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}}}(K^{\pm}_{U'})\,\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) \\
&\leq \left(\delta + \frac{1-\delta}{2}\right)\mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) \\
&\leq \frac{1+\delta}{2}\widetilde{\mathrm{p}}_{U^{\mathrm{cl}},k}(\varphi)
\end{aligned}
$$

for $\varphi \in \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$. Since $0 \leq \delta < 1$ by assumption (3.4.68) we conclude $\frac{1+\delta}{2} < 1$ as desired. □

Corollary 3.4.31 *Let $k \in \mathbb{N}_0$. Then the operator*

$$
\mathsf{id} + \mathcal{K}^{\pm}_U : \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \tag{3.4.71}
$$

is linear, continuous, and bijective with continuous inverse given by the absolutely norm-convergent series

$$
\left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1} = \sum_{j=0}^{\infty}\left(-\mathcal{K}^{\pm}_U\right)^j. \tag{3.4.72}
$$

Proof. This is now obvious by the lemma. □

We shall now compute the inverse of $\mathsf{id} + \mathcal{K}^{\pm}_U$ slightly more explicit: in fact, it is again an integral operator with a nice kernel. The j-th power of $\mathcal{K}^{\pm}_U$ is explicitly given by

$$
\begin{aligned}
\left((\mathcal{K}^{\pm}_U)^j\varphi\right)(p) &= \int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p,z_1)\left((\mathcal{K}^{\pm}_U)^{j-1}\varphi\right)(z_1)\,\mu_g(z_1) \\
&= \int_{U^{\mathrm{cl}}}\cdots\int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p,z_1)\cdots K^{\pm}_{U'}(z_{j-1},z_j)\varphi(z_j)\,\mu_g(z_1)\cdots\mu_g(z_j) \\
&= \int_{U^{\mathrm{cl}}}\left(\int_{U^{\mathrm{cl}}}\cdots\int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p,z_1)\cdots K^{\pm}_{U'}(z_{j-1},q)\,\mu_g(z_1)\cdots\mu_g(z_{j-1})\right)\varphi(q)\,\mu_g(q)
\end{aligned} \tag{3.4.73}
$$

by Fubini's theorem. Thus $(\mathcal{K}^{\pm}_U)^j$ has again a nice kernel given by

$$
K^{\pm(j)}_U(p,q) = \int_{U^{\mathrm{cl}}}\cdots\int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p,z_1)\cdots K^{\pm}_{U'}(z_{j-1},q)\,\mu_g(z_1)\cdots\mu_g(z_{j-1}). \tag{3.4.74}
$$

For this kernel we have the following properties:

Lemma 3.4.32 *Let $j \in \mathbb{N}$. Then the j-th power of $\mathcal{K}^{\pm}_U$ has again a smooth integral kernel $K^{\pm(j)}_{U'} \in \Gamma^\infty\left(E^* \boxtimes E\big|_{U'\times U'}\right)$ explicitly given by*

$$
K^{\pm(j)}_U(p,q) = \int_{U^{\mathrm{cl}}}\cdots\int_{U^{\mathrm{cl}}} K^{\pm}_{U'}(p,z_1)\cdots K^{\pm}_{U'}(z_{j-1},q)\,\mu_g(z_1)\cdots\mu_g(z_{j-1}), \tag{3.4.75}
$$

satisfying the estimate

$$\mathrm{p}_{K\times K,k}\left(K_{U'}^{\pm(j)}\right) \leq \operatorname{vol}(U^{\mathrm{cl}})\, \mathrm{p}_{(K\cup U^{\mathrm{cl}})\times(K\cup U^{\mathrm{cl}}),k}(K_{U'}^{\pm})\delta^{j-2}, \tag{3.4.76}$$

with δ *as in* (3.4.68) *where* $K \subseteq U'$ *is compact.*

Proof. The above computation (3.4.73) shows that (3.4.75) is indeed the kernel of $(\mathcal{K}_U^{\pm})^j : \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$. From the explicit formula (3.4.75) and an argument analogous to the one in the proof of Lemma 3.4.23 we see that $K_U^{\pm(j)}$ has a continuation for all $(p,q) \in U' \times U'$ to a *smooth* section by the very same expression (3.4.75). Moreover, we can differentiate $K_U^{\pm(j)}$ by differentiating under the integral. This yields

$$\begin{aligned}
\mathrm{p}_{K\times K,k}\left(K_U^{\pm(j)}\right) &\leq \int_{U^{\mathrm{cl}}}\cdots\int_{U^{\mathrm{cl}}} \mathrm{p}_{K\times U^{\mathrm{cl}},k}(K_{U'}^{\pm})\underbrace{\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(K_{U'}^{\pm})\cdots \mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(K_{U'}^{\pm})}_{j-1 \text{ times}} \\
&\qquad \mathrm{p}_{U^{\mathrm{cl}}\times K,k}(K_{U'}^{\pm})\,\mu_g(z_1)\cdots\mu_g(z_{j-1}) \\
&\leq \operatorname{vol}(U^{\mathrm{cl}})^{j-1}\,\mathrm{p}_{U^{\mathrm{cl}}\times U^{\mathrm{cl}},0}(K_{U'}^{\pm})^{j-2}\,\mathrm{p}_{(K\cup U^{\mathrm{cl}})\times(K\cup U^{\mathrm{cl}}),l}(K_{U'}^{\pm})^2 \\
&= \operatorname{vol}(U^{\mathrm{cl}})\delta^{j-2}\,\mathrm{p}_{(K\cup U^{\mathrm{cl}})\times(K\cup U^{\mathrm{cl}}),l}(K_{U'}^{\pm})^2,
\end{aligned}$$

since only the first and last $K_{U'}^{\pm}$ in (3.4.75) depend on the points $p, q \in K \subseteq U'$ which are used for differentiation in $\mathrm{p}_{K\times K,k}$. Thanks to the factorization of the variables, we do not get extra (k-dependent) constants from the Leibniz rule. Thus (3.4.76) follows. □

Corollary 3.4.33 *The operator* $(\mathsf{id} + \mathcal{K}_U^{\pm})^{-1} \circ \mathcal{K}_U^{\pm}$ *has a smooth kernel explicitly given by the series* $\sum_{j=1}^{\infty}(-1)^{j-1}K_{U'}^{\pm(j)}$*, which converges in the* $\mathcal{C}^\infty$*-topology of* $\Gamma^\infty\left(E^*\boxtimes E\big|_{U'\times U'}\right)$.

Proof. By the lemma, each $K_{U'}^{\pm(j)}$ is smooth on $U' \times U'$. Moreover, with respect to a given seminorm $\mathrm{p}_{K\times K,k}$, the above series converges since $\delta < 1$ by assumption on U^{cl}. This shows that the series $\sum_{j=1}^{\infty}(-1)^{j-1}K_{U'}^{\pm(j)}$ converges (even absolutely) with respect to $\mathrm{p}_{K\times K,k}$. Since $K \subseteq U'$ and $k \in \mathbb{N}_0$ are arbitrary, we have $\mathcal{C}^\infty$-convergence. Clearly, when restricting to $U^{\mathrm{cl}} \times U^{\mathrm{cl}}$, the series is the kernel of $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}$. □

Lemma 3.4.34 *Let* $\varphi \in \Gamma^\infty\left(E^*\big|_{U'}\right)$ *be smooth. Then* $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\left(\varphi\big|_{U^{\mathrm{cl}}}\right)$ *is in* $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ *for all* $k \in \mathbb{N}_0$*. Moreover,*

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_U \in \Gamma^\infty\left(E^*\big|_U\right) \tag{3.4.77}$$

and the map

$$\Gamma^\infty\left(E^*\big|_{U'}\right) \ni \varphi \;\mapsto\; \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_U \in \Gamma^\infty\left(E^*\big|_U\right) \tag{3.4.78}$$

is continuous. The image is even in the subset of those smooth sections on U *which are restrictions of smooth sections of* E^* *on* U'.

Proof. First we note that $\varphi\big|_{U^{\mathrm{cl}}} \in \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ by the very definition as in Definition 3.4.26. Moreover, since

$$\mathrm{p}_{U^{\mathrm{cl}},k}\left(\varphi\big|_{U^{\mathrm{cl}}}\right) = \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi),$$

the restriction map is a continuous map

$$\Gamma^\infty(E^*\big|_{U'}) \longrightarrow \Gamma^k(E^*\big|_{U^{\mathrm{cl}}})$$

for any $k \in \mathbb{N}_0$. Now $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right) \in \Gamma^k(E^*\big|_{U^{\mathrm{cl}}})$ by Corollary 3.4.31 and applying $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$ is again continuous. Finally, restricting a section in $\Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ to U gives a $\mathcal{C}^k$-section in the usual sense by (3.4.63) in Remark 3.4.27. Moreover, this restriction is again continuous whence finally

$$\Gamma^\infty\left(E^*\big|_{U'}\right) \ni \varphi \;\mapsto\; \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_U \in \Gamma^k\left(E^*\big|_U\right)$$

is continuous for all $k \in \mathbb{N}_0$. In particular, it follows that $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_U \in \Gamma^\infty\left(E^*\big|_U\right)$. Since the inverse is given by the geometric series we see that

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right) = \varphi\big|_{U^{\mathrm{cl}}} - \left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right) \left(\varphi\big|_{U^{\mathrm{cl}}}\right).$$

Now $\varphi\big|_{U^{\mathrm{cl}}}$ is the restriction of the smooth section φ on U' to U^{cl}. Also the operator $(\mathsf{id} + \mathcal{K}_U^{\pm})^{-1} \circ \mathcal{K}_U^{\pm}$ has a smooth integral kernel defined even on $U' \times U'$ by Corollary 3.4.31. Hence the result $\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right) \left(\varphi\big|_{U^{\mathrm{cl}}}\right)$ can also be viewed as the smooth section

$$\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right) \left(\varphi\big|_{U^{\mathrm{cl}}}\right)(p) = \int_{U^{\mathrm{cl}}} \left(\sum_{j=1}^{\infty} (-1)^{j-1} K_U^{\pm(j)}(p,q)\right) \varphi(q)\, \mu_g(q), \tag{$*$}$$

defined even for $p \in U'$. Since the kernel of $(*)$ is smooth it follows easily that

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm} : \Gamma^k\left(E^*\big|_{U^{\mathrm{cl}}}\right) \longrightarrow \Gamma^\infty\left(E^*\big|_{U'}\right)$$

is a continuous linear map: this can be done analogously to the argument in Lemma 3.4.23 where we only have to replace $K_U^{\pm}$ by the smooth kernel of $(*)$ in (3.4.51). This shows that $\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right) \left(\varphi\big|_{U^{\mathrm{cl}}}\right) \in \Gamma^\infty\left(E^*\big|_{U'}\right)$ and hence (3.4.78). Moreover, the composition of all the involved maps including the last restriction to U are continuous. Thus (3.4.78) is continuous as well. □

Note that $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$ is defined even on U' via the integral formula. But here it is no longer the inverse of the operator $\mathsf{id} + \mathcal{K}_U^{\pm}$.

We can now use the inverse of $\mathsf{id} + \mathcal{K}_U^{\pm}$ to build a true fundamental solution as follows:

Definition 3.4.35 (Local fundamental solution) *Let $U' \subseteq M$ be geodesically convex and $U \subseteq U'$ be open with compact closure $U^{\mathrm{cl}} \subseteq U'$ such that the volume of U^{cl} is small enough. For $p \in U$ we define*

$$F_U^{\pm}(p) : \Gamma_0^\infty\left(E^*\big|_U\right) \ni \varphi \;\mapsto\; \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\widetilde{\mathcal{R}}_U^{\pm}(\,\cdot\,)(\varphi)\right)\Big|_p \in E_p^*. \tag{3.4.79}$$

Theorem 3.4.36 (Local fundamental solution) *Let $U' \subseteq M$ be geodesically convex and let $U \subseteq U'$ be open with compact closure $U^{\mathrm{cl}} \subseteq U'$ such that the volume of U^{cl} is small enough. Then for $p \in U$ the map*

$$F_U^{\pm}(p) : \Gamma_0^\infty\left(E^*\big|_U\right) \longrightarrow E_p^* \tag{3.4.80}$$

is a local fundamental solution of D at p such that for every $\varphi \in \Gamma_0^\infty\left(E^*\big|_U\right)$

$$F_U^{\pm}(\,\cdot\,)\varphi : p \;\mapsto\; F_U^{\pm}(p)(\varphi) \tag{3.4.81}$$

is a smooth section of E^* over U. In fact,

$$F_U^{\pm} : \Gamma_0^\infty\left(E^*\big|_U\right) \ni \varphi \;\mapsto\; F_U^{\pm}(\,\cdot\,)(\varphi) \in \Gamma^\infty\left(E^*\big|_U\right) \tag{3.4.82}$$

is a continuous linear map.

Proof. From Theorem 3.4.20, *iv.)* we know that $\widetilde{\mathcal{R}}^{\pm}_{U'}(\,\cdot\,)(\varphi)$ defines a smooth section of E^* over U'. By Proposition 3.4.16 we know that $\widetilde{\mathcal{R}}^{\pm}_{U'}: \Gamma^\infty_0(E^*\big|_{U'}) \ni \varphi \mapsto \widetilde{\mathcal{R}}^{\pm}_{U'}(\,\cdot\,)\varphi \in \Gamma^\infty(E^*\big|_{U'})$ is continuous in the $\mathcal{C}^\infty_0$- and $\mathcal{C}^\infty$-topology, respectively. By Lemma 3.4.34, also the map

$$\Gamma^\infty\left(E^*\big|_{U'}\right) \ni \varphi \;\mapsto\; \left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1}\left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_U \in \Gamma^\infty\left(E^*\big|_U\right)$$

is continuous, whence it follows that (3.4.82) is continuous and linear. This also implies (3.4.81). Thus it remains to shows that $F^{\pm}_U(p)$ is indeed a fundamental solution of D at p. We compute

$$\begin{aligned}
\left(DF^{\pm}_U(p)\right)(\varphi) &= F^{\pm}_U(p)(D^{\mathrm{T}}\varphi)\\
&= \left(\left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1}\left(\widetilde{\mathcal{R}}^{\pm}_U(\,\cdot\,)(D^{\mathrm{T}}\varphi)\right)\right)\Big|_p\\
&= \left(\left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1}\left(\left(D\widetilde{\mathcal{R}}^{\pm}_U(\,\cdot\,)\right)(\varphi)\right)\right)\Big|_p\\
&= \left(\left(\mathsf{id} + \mathcal{K}^{\pm}_U\right)^{-1}\left(\varphi + \mathcal{K}^{\pm}_U\varphi\right)\right)\Big|_p\\
&= \varphi(p)
\end{aligned}$$

by (3.4.34). But this is precisely the defining property of a fundamental solution. □

Corollary 3.4.37 *Let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator. Then every point in M has a small neighborhood $U \subseteq M$ such that on U we have a fundamental solution $F^{\pm}_U(p)$ for all $p \in U$, i.e.*

$$DF^{\pm}_U(p) = \delta_p, \tag{3.4.83}$$

and such that the linear map

$$F^{\pm}_U : \Gamma^\infty_0\left(E\big|_U\right) \ni \varphi \;\mapsto\; \left(p \mapsto F^{\pm}_U(p)(\varphi)\right) \in \Gamma^\infty\left(E\big|_U\right) \tag{3.4.84}$$

is continuous.

3.4.3 Causal Properties of $F^{\pm}_U$

The construction of the integral operator $\mathcal{K}^{\pm}_U$ and the invertibility of $\mathsf{id} + \mathcal{K}^{\pm}_U$ works for arbitrary small enough $U \subseteq U'$. However, since $\mathcal{K}^{\pm}_U$ is *non-local* the nice support properties of $\widetilde{\mathcal{R}}^{\pm}_{U'}$ are typically destroyed. To guarantee good causal behaviour we need to put some extra conditions on U.

Remark 3.4.38 Let $U \subseteq U'$ be causal, i.e. for $p, q \in U^{\mathrm{cl}} \subseteq U'$ we have $J^{\pm}_{U'}(p, q) \subseteq U^{\mathrm{cl}}$ and the diamond is compact. Then U^{cl} is causally compatible with U'. Indeed, if say $q \in J^{+}_{U'}(p)$ then we can join p and q by a unique future directed geodesic which is entirely in $J^{+}_{U'}(p, q)$. Thus this curve is also entirely in U^{cl} whence $q \in J^{+}_{U^{\mathrm{cl}}}(p)$ proving that U^{cl} is causally compatible with U'.

In the following, we assume that $U \subseteq U'$ is in addition a causal subset. As a first consequence we have

$$J^{\pm}_{U^{\mathrm{cl}}}(p) = J^{\pm}_{U'}(p) \cap U^{\mathrm{cl}} \tag{3.4.85}$$

for $p \in U^{\mathrm{cl}}$.

Lemma 3.4.39 *Let $U \subseteq U'$ be in addition causal. Then for $\varphi \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we have*

$$\operatorname{supp}(\mathcal{K}^{\pm}_U\varphi) \subseteq J^{\mp}_{U^{\mathrm{cl}}}(\operatorname{supp}\varphi). \tag{3.4.86}$$

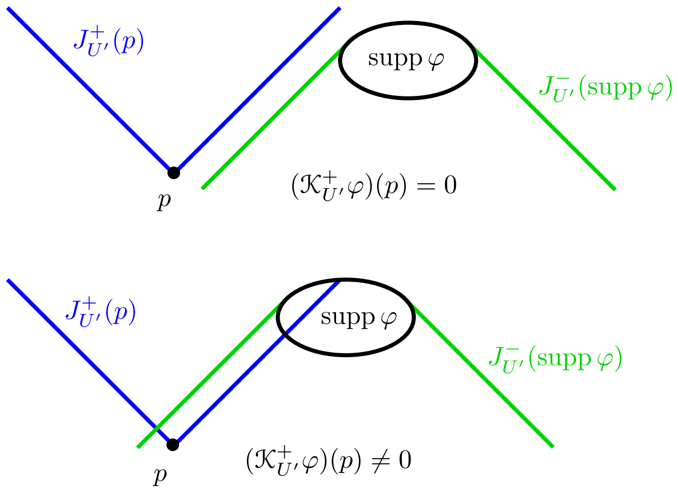

Figure 3.7: The relation between the supports in the proof of Lemma 3.4.39.

Proof. We know that $(p, q) \in \operatorname{supp} K^\pm_{U'}$ implies $q \in J^\pm_{U'}(p)$ by Lemma 3.4.18. Thus for $p \in U^{\mathrm{cl}}$ and

$$\left(\mathcal{K}^\pm_U \varphi\right)(p) = \int_{U^{\mathrm{cl}}} K^\pm_{U'}(p, q) \cdot \varphi(q)\, \mu_g(q)$$

we get $(\mathcal{K}^\pm_U \varphi)(p) = 0$ if the integrand vanishes identically. But if $K^\pm_{U'}(p, q) \cdot \varphi(q) \neq 0$ for some (p, q) then on one hand $q \in J^\pm_{U'}(p)$ by the support features of $K^\pm_{U'}(p, q)$ and $q \in \operatorname{supp}\varphi$ on the other hand. Thus $q \in J^\pm_{U'}(p) \cap \operatorname{supp}\varphi$ follows. We conclude that necessarily $(\mathcal{K}^\pm_U \varphi)(p) = 0$ if $J^\pm_{U'}(p) \cap \operatorname{supp}\varphi = \emptyset$. From this we conclude

$$\operatorname{supp}(\mathcal{K}^\pm_U \varphi) \subseteq J^\mp_{U'}(\operatorname{supp}\varphi) \cap U^{\mathrm{cl}} = J^\mp_{U^{\mathrm{cl}}}(\operatorname{supp}\varphi)$$

see also Figure 3.7. □

To compute the support of $\left(\mathsf{id} + \mathcal{K}^\pm_U\right)^{-1} \varphi$ one may have the idea that with (3.4.86) also the finite powers of $\mathcal{K}^\pm_U$ have the property (3.4.86). This is indeed correct as by induction and (3.4.86)

$$\operatorname{supp}\left((\mathcal{K}^\pm_U)^j \varphi\right) \subseteq J^\mp_U\left(\operatorname{supp}\left((\mathcal{K}^\pm_U)^{j-1}\varphi\right)\right) \subseteq J^\mp_U\left(J^\mp_U \cdots J^\mp_U(\operatorname{supp}\varphi)\right) = J^\mp_U(\operatorname{supp}\varphi), \tag{3.4.87}$$

since clearly $J^\mp_U(A) = J^\mp_U(J^\mp_U(A))$ for arbitrary $A \subseteq U$. However, taking the geometric series for $\left(\mathsf{id} + \mathcal{K}^\pm_U\right)^{-1}$ would require to take the closure of the union of countably many closed subsets of $J^\mp_U(\operatorname{supp}\varphi)$. Now $J^\mp_U(\operatorname{supp}\varphi)$ need not be closed at all, even though $\operatorname{supp}\varphi$ is closed. Thus we can *not* conclude by this argument that the support of $\left(\mathsf{id} + \mathcal{K}^\pm_U\right)^{-1} \varphi$ lies in $J^\mp_U(\operatorname{supp}\varphi)$. However, we can proceed as follows:

Lemma 3.4.40 *For all $j \in \mathbb{N}$ the supports of the integral kernels $K^{\pm(j)}_{U'}$ of $(\mathcal{K}^\pm_U)^j$ are future respectively past stretched, i.e.*

$$(p, q) \in \operatorname{supp} K^{\pm(j)}_{J'} \subseteq U' \times U' \Longrightarrow q \in J^\pm_{U'}(p). \tag{3.4.88}$$

Moreover, the support of the integral kernel of $(\mathsf{id} + \mathcal{K}^\pm_U)^{-1} \circ \mathcal{K}^\pm_U$ is also future respectively past stretched.

Proof. Assume that $K^{\pm(j)}_{U'}(p, q) \neq 0$. Then the integrand in (3.4.75) can not be identically zero whence there have to be $z_1, \ldots, z_{j-1} \in U^{\mathrm{cl}}$ with $z_1 \in J^\pm_{U'}(p), \ldots, z_{j-1} \in J^\pm_{U'}(z_{j-2}), q \in J^\pm_{U'}(z_{j-1})$. But this means $q \in J^\pm_{U'}(p)$ proving (3.4.88) with the same closure argument as in the proof of Lemma 3.4.40. Now we consider the $\mathcal{C}^\infty$-convergent sum of the $K^{\pm(j)}_{U'}$. If $\sum_{j=1}^\infty (-1)^{j-1} K^{\pm(j)}_{U'}(p, q) \neq 0$ for some (p, q) then at least for one j we have $K^{\pm(j)}_{U'}(p, q) \neq 0$. Thus $q \in J^\pm_{U'}(p)$ and we can proceed as before. □

Corollary 3.4.41 *For $\varphi \in \Gamma^0\left(E^*\big|_{U^{\mathrm{cl}}}\right)$ we have*

$$\operatorname{supp}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\varphi\right) \subseteq J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi). \tag{3.4.89}$$

Proof. Clearly, we have

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\varphi = \varphi - \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\varphi.$$

With $\operatorname{supp}\varphi \subseteq J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi)$ and the above lemma the statement follows at once as in Lemma 3.4.39. ☐

Using this property of $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$ for *causal* U we arrive at the following statement:

Theorem 3.4.42 (Local Green functions) *Let $U \subseteq U'$ be small enough and causal. Then the fundamental solutions $F_U^{\pm}(p)$ from Theorem 3.4.36 are advanced and retarded Green functions, i.e. we have*

$$\operatorname{supp} F_U^{\pm}(p) \subseteq J_U^{\pm}(p). \tag{3.4.90}$$

Proof. Let $\varphi \in \Gamma_0^{\infty}\left(E^*\big|_U\right)$ be a test section. Then

$$\begin{aligned}
\operatorname{supp}\left(F_U^{\pm}(\,\cdot\,)(\varphi)\right) &= \operatorname{supp}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \widetilde{\mathcal{R}}_{U'}^{\pm}(\,\cdot\,)(\varphi)\right) \\
&\subseteq J_{U^{\mathrm{cl}}}^{\mp}\left(\widetilde{\mathcal{R}}_{U'}^{\pm}(\,\cdot\,)(\varphi)\right) \\
&\subseteq J_{U^{\mathrm{cl}}}^{\mp}\left(J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi)\right) = J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi), \qquad (*)
\end{aligned}$$

since $\operatorname{supp}\widetilde{\mathcal{R}}_{U'}^{\pm}(p)\big|_U \subseteq J_{U^{\mathrm{cl}}}^{\pm}(p)$ whence for $\operatorname{supp}\varphi \cap J_{U^{\mathrm{cl}}}^{\pm}(p) = \emptyset$ we conclude $\widetilde{\mathcal{R}}_{U'}^{\pm}(p)(\varphi) = 0$. Thus $p \notin J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi)$ implies $\widetilde{\mathcal{R}}_{U'}^{\pm}(p)(\varphi) = 0$. Since for compactly supported φ we have a *closed* $J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi)$ by U being causal we conclude that $\operatorname{supp}\widetilde{\mathcal{R}}_{U'}^{\pm}(\,\cdot\,)(\varphi) \subseteq J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp}\varphi)$. This shows $(*)$. Thus if $\operatorname{supp}\varphi \cap J_{U^{\mathrm{cl}}}^{\pm}(p) = \emptyset$ for $p \in U^{\mathrm{cl}}$ then $p \notin J_U^{\mp}(\operatorname{supp}\varphi)$ and thus $p \notin \operatorname{supp}(F_U^{\pm}(\,\cdot\,)(\varphi))$ whence $F_U^{\pm}(p)(\varphi) = 0$ follows. But this implies (3.4.90) as $J_{U^{\mathrm{cl}}}^{\pm}(p)$ is closed thanks to U being causal. ☐

Since every point in a time-oriented Lorentz manifold has an arbitrarily small causal neighborhood we finally arrive at the following result:

Corollary 3.4.43 *Let $D \in \operatorname{DiffOp}^2(E)$ be normally hyperbolic. Then every point in M has a small enough causal neighborhood $U \subseteq M$ such that on U we have advanced and retarded Green functions $F_U^{\pm}(p)$ at $p \in U$, i.e.*

$$DF^{\pm}(p) = \delta_p \tag{3.4.91}$$

and

$$\operatorname{supp} F_U^{\pm}(p) \subseteq J_U^{\pm}(p), \tag{3.4.92}$$

such that in addition

$$F_U^{\pm} : \Gamma_0^{\infty}\left(E^*\big|_U\right) \ni \varphi \;\mapsto\; \left(p \;\mapsto\; F_U^{\pm}(p)(\varphi)\right) \in \Gamma^{\infty}\left(E^*\big|_U\right) \tag{3.4.93}$$

is a continuous linear map.

3.5 Solving the Wave Equation Locally

In this section we show how the Green functions $F_U^{\pm}(p)$ can be used to obtain solutions to the wave equation

$$Du = v \tag{3.5.1}$$

with a prescribed source term v. The main idea is that a suitable v can be written as a superposition of δ-functionals. Since $F_U^{\pm}(p)$ solves (3.5.1) for $v = \delta_p$ we get a solution to (3.5.1) for arbitrary v by taking the corresponding superposition of the fundamental solutions $F_U^{\pm}(p)$. Of course, at the moment we are restricted to v having compact support in U.

Then we are interested in two extreme cases: for a distributional v we can only expect to obtain distributions u as solutions. However, if v has good regularity then we can expect u to be regular as well.

3.5.1 Local Solutions for Distributional Inhomogeneity

Let $v \in \Gamma_0^{-\infty}(E\big|_U)$ be a generalized section of E with compact support in U. We want to solve

$$Du^{\pm} = v \tag{3.5.2}$$

with some $u^{\pm} \in \Gamma^{-\infty}(E\big|_U)$.

Remark 3.5.1 Since a normally hyperbolic differential operator D describes a wave equation we expect from physical considerations that a source term v causes *propagating* waves whence the support of $u^{\pm}$ is expected to be non-compact: In fact, the best we can hope for is that in spatial directions the support stays compact while in time directions we will have non-compact support at least in either the future or the past. Up to now we are dealing with the local situation $U \subseteq M$ where thanks to the simple geometry those questions are rather harmless. Later on this issue will become more subtle.

Lemma 3.5.2 *Let $U \subseteq M$ be a small enough open subset such that the construction of $F_U^{\pm}$ as in Section 3.4 applies.*

i.) The map $F_U^{\pm} : \Gamma_0^{\infty}(E^*\big|_U) \longrightarrow \Gamma^{\infty}(E^*\big|_U)$ induces a linear map

$$(F_U^{\pm})' : \Gamma_0^{-\infty}(E\big|_U) \longrightarrow \Gamma^{-\infty}(E\big|_U) \tag{3.5.3}$$

by dualizing, i.e. for $v \in \Gamma_0^{-\infty}(E\big|_u)$ and $\varphi \in \Gamma_0^{\infty}(E^*\big|_U)$ one defines

$$\left((F_U^{\pm})'(v)\right)(\varphi) = v\left(F_U^{\pm}(\varphi)\right). \tag{3.5.4}$$

ii.) The map $(F_U^{\pm})'$ is weak continuous.*

iii.) We have

$$D(F_U^{\pm})'(v) = v \tag{3.5.5}$$

for all $v \in \Gamma_0^{-\infty}(E\big|_U)$.

Proof. For the first part we recall that we have the identification

$$\Gamma_0^{\infty}(E^*\big|_U) \ni \varphi \;\mapsto\; \varphi \otimes \mu_g \in \Gamma_0^{\infty}(E^*\big|_U \otimes |\Lambda^{\mathrm{top}}|T^*M)$$

from which we obtain the identification

$$\Gamma^{-\infty}(E\big|_U) \ni u \;\mapsto\; (\varphi \mapsto u(\varphi \otimes \mu_g)) \in \Gamma_0^{\infty}(E^*\big|_U)'. \tag{$*$}$$

Since tensoring with $\mu_g > 0$ does not change the supports we can dualize the continuous map

$$F_U^{\pm} : \Gamma_0^{\infty}(E^*\big|_U) \longrightarrow \Gamma^{\infty}(E^*\big|_U)$$

to a map

$$(F_U^{\pm})' : \Gamma^{\infty}(E^*\big|_U)' \longrightarrow \Gamma_0^{\infty}(E^*\big|_U)'. \tag{$**$}$$

Using $(*)$ and the fact that the dual space of all test sections are the compactly supported generalized sections, see Theorem 1.3.18, we get

$$\Gamma_0^{-\infty}(E\big|_U) \stackrel{(*)}{\longrightarrow} \Gamma^{\infty}(E^*\big|_U)' \stackrel{(F_U^{\pm})'}{\longrightarrow} \Gamma_0^{\infty}(E^*\big|_U)' \stackrel{(*)}{\longrightarrow} \Gamma^{-\infty}(E\big|_U),$$

whose composition we denote by $(F_U^{\pm})'$ as well. This is the map (3.5.3). Dualizing yields a weak* continuous map in $(**)$. Finally, the identifications $(*)$ are weak* continuous as well, hence it results in a weak* continuous map (3.5.3). Note that in (3.5.4) we have hidden the aspect of the reference density μ_g in the pairing of v and $F_U^{\pm}(\varphi)$. This shows the first and second part. For the third part we unwind the definition of $DF_U^{\pm\prime}$. Let $\varphi \in \Gamma_0^{\infty}(E^*\big|_U)$ be a test section and compute

$$\begin{aligned}
\left(D\left((F_U^{\pm})'(v)\right)\right)(\varphi) &= \left((F_U^{\pm})'(v)\right)(D^{\mathrm{T}}\varphi)\\
&= v\left(p \mapsto F_U^{\pm}(D^{\mathrm{T}}\varphi)\big|_p\right)\\
&= v\left(p \mapsto (F_U^{\pm}(p))(D^{\mathrm{T}}\varphi)\right)\\
&= v\left(p \mapsto \varphi(p)\right)\\
&= v(\varphi),
\end{aligned}$$

using the definition of the dualized map and the feature $DF_U^{\pm}(p) = \delta_p$. But this means (3.5.5). □

Remark 3.5.3 (Fundamental solutions) We note that in the above proof we have not used any details of the properties of D or $F_U^{\pm}$. The only thing we needed was the property that

$$F_U^{\pm} : \Gamma_0^{\infty}(E^*\big|_U) \ni \varphi \mapsto \left(p \mapsto F_U^{\pm}(p)(\varphi)\right) \in \Gamma^{\infty}(E\big|_U) \tag{3.5.6}$$

is continuous in the $\mathcal{C}_0^{\infty}$- and $\mathcal{C}^{\infty}$-topology in order to dualize (3.5.6) to a map (3.5.3) *and* the fundamental solution property

$$DF_U^{\pm}(p) = \delta_p \tag{3.5.7}$$

in order to compute $D(F_U^{\pm})'(v)$ as in (3.5.5). Thus the above argument shows one principle usage of fundamental solutions: they allow to solve the inhomogeneous equations in a distributional sense. Of course, up to now we have just found on particular solution for each inhomogeneity v but no uniqueness. In fact, for our wave equations we expect to have many solutions as we expect traveling waves for trivial inhomogeneity $v = 0$. Thus we have to specify boundary conditions in order to get more specific solutions. In order to control the "boundary conditions" in our case, we use the fundamental solutions $F_U^{\pm}(p)$ as in Theorem 3.4.42, i.e. on a causal $U \subseteq M$.

Lemma 3.5.4 *Let $U \subseteq M$ be small enough and causal and let $F_U^{\pm}(p)$ be the corresponding fundamental solutions as in Theorem 3.4.42. For $v \in \Gamma_0^{-\infty}(E\big|_U)$ we have*

$$\operatorname{supp}(F_U^{\pm})'(v) \subset J_U^{\pm}(\operatorname{supp} v). \tag{3.5.8}$$

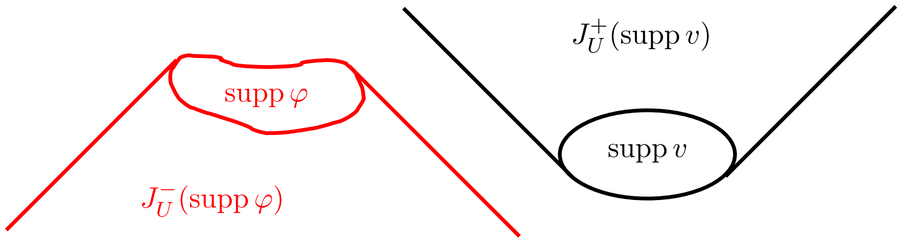

Figure 3.8: The supports of φ and v in Lemma 3.5.4.

Proof. We use the causality property $\operatorname{supp} F_U^{\pm}(p) \subseteq J_U^{\pm}(p)$ for all $p \in U$ of the fundamental solution. Thus let $\varphi \in \Gamma_0^\infty(E^*\big|_U)$ be a test section with $\operatorname{supp} \varphi \cap J_U^{\pm}(\operatorname{supp} v) = \emptyset$. We have to show $((F_U^{\pm})'(v))(\varphi) = 0$ for all such φ. We compute

$$(F_U^{\pm\prime}(v))(\varphi) = v\left(F_U^{\pm}(\varphi)\right) = v\left(p \ \mapsto \ F_U^{\pm}(p)(\varphi)\right).$$

From the proof of Theorem 3.4.42 we know that $\operatorname{supp}(p \mapsto F_U^{\pm}(p)(\varphi)) \subset J_{U^{\mathrm{cl}}}^{\mp}(\operatorname{supp} \varphi)$. But $\operatorname{supp} u \cap J_U^{\pm}(\operatorname{supp} \varphi) = \emptyset$ by assumption whence $v(p \mapsto F_U^{\pm}(p)(\varphi)) = 0$ follows, see also Figure 3.8. □

Remark 3.5.5 Even though we do not yet have the uniqueness properties, already at this stage we see some very nice features familiar from our physically motivated expectations:

i.) Using the solution $u^+ = (F_U^+)'(v)$ of the inhomogeneous wave equation we see that the influence of the source term v is only in the future of v. This is a physically reasonable behaviour. The interpretation is that at some time one switches on a source term, e.g. an oscillating dipole, and observes emitted waves u^+ in the future of v. In particular, the signals emitted by v can not propagate faster than with light speed. The solution u^- is the other extreme which for physical reasons is not acceptable.

ii.) In the flat situation of the Minkowski spacetime $(\mathbb{R}^n, \eta)$ we can take $U = \mathbb{R}^n$ and obtain $F_{\mathbb{R}^n}^{\pm}(0) = R^{\pm}(2)$ and $F_{\mathbb{R}^n}^{\pm}(p)$ is the translated Riesz distribution for arbitrary $p \in \mathbb{R}^n$. Then the construction of the solutions $(F_{\mathbb{R}^n}^{\pm})'(v)$ for a given v is the well-known solution procedure as known e.g. from electrodynamics [32, 53].

iii.) Of particular interest is the following situation: a charged pointlike particle with charge e moves along a trajectory $t \mapsto \vec{x}(t)$ in Minkowski spacetime with velocity $|\vec{v}(t)| = |\dot{\vec{x}}(t)| < 1$. As usual, we set the speed of light $c = 1$ by choosing an appropriate unit system. Then the charge density is $\varrho(t, \vec{x}) = e\delta_{\vec{x}(t)}$ while the current density is $\vec{j}(t, \vec{x}) = e\vec{v}(t)\delta_{x(t)}$, viewed both as distributions on the spatial $\mathbb{R}^{n-1}$ inside Minkowski spacetime. They combine into an $\mathbb{R}^n$-valued distribution on $\mathbb{R}^n$ denoted by j. The corresponding solution $A = (F_U^{\pm})'(j)$ of $\square A = j$ is then known as the *Lienhard-Wiechert potential*. It describes the electromagnetic potential of the radiation emitted by the moving charge, see e.g. [53, Sect. 3.6] or [32, Sect. 14.1].

iv.) From our construction, $(F_U^{\pm})'$ is only defined on the distributional sections with *compact* support. However, the example of the moving charge gives an inhomogeneity with non-compact support, at least in timelike directions: Here only the support in spatial directions is compact for all times. Thus for physical applications it will be necessary to extend the domain of $(F_U^{\pm})'$ to more general distributions.

3.5.2 Local Solution for Smooth Inhomogeneity

In a next step we want to discuss the additional properties of the solutions $(F_U^{\pm})'(v)$ of the inhomogeneous wave equation $Du = v$ for distributional v having some kind of regularity. Of particular interest is the case where v is actually *smooth* and hence a test section $v \in \Gamma_0^\infty(E\big|_U)$.

To this end we first collect some more specific properties of the operator $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$. It will be advantageous to consider integral operators with smooth kernel in general. Thus we consider the following situation: Let $U \subset M$ be open with U^{cl} compact and let $U^{\mathrm{cl}} \subseteq U'$ with U' open. Moreover, let $K \in \Gamma^\infty(E^* \boxtimes E\big|_{U' \times U'})$ be a smooth kernel on the larger open subset $U' \times U'$. For sections $\varphi \in \Gamma_b(E^*\big|_U)$ we consider the integral operator

$$(\mathcal{K}\varphi)(p) = \int_{U^{\mathrm{cl}}} K(p, q) \cdot \varphi(q)\, \mu_g(q) \tag{3.5.9}$$

analogously to (3.4.47), where $p \in U'$. Repeating the arguments from Lemma 3.4.23 and Lemma 3.4.29 we obtain the following general result:

Lemma 3.5.6 *Let $U \subseteq U^{\mathrm{cl}} \subseteq U'$ with U, U' open and U^{cl} compact. For the integral operator $\mathcal{K}$ corresponding to a smooth kernel $K \in \Gamma^\infty(E^* \boxtimes E\big|_{U' \times U'})$ as in* (3.5.9) *the following statements are true:*

i.) For $\varphi \in \Gamma_b(E^*\big|_U)$ one has $\mathcal{K}\varphi \in \Gamma^k(E^*\big|_{U^{\mathrm{cl}}})$ for all $k \in \mathbb{N}_0$ and $\mathcal{K}\varphi\big|_U \in \Gamma^\infty(E^*\big|_U)$.

ii.) The maps (all denoted by $\mathcal{K}$)

$$\mathcal{K} : \Gamma_b(E^*\big|_U) \ni \varphi \;\mapsto\; \mathcal{K}\varphi \in \Gamma^k(E^*\big|_{U^{\mathrm{cl}}}) \tag{3.5.10}$$

and

$$\mathcal{K} : \Gamma_b(E^*\big|_U) \ni \varphi \;\mapsto\; \mathcal{K}\varphi\big|_U \in \Gamma^\infty(E^*\big|_U) \tag{3.5.11}$$

are continuous. In fact, for $k \in \mathbb{N}_0$ one even has

$$\mathrm{p}_{U^{\mathrm{cl}},k}(\mathcal{K}\varphi) \le c\, \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi) \tag{3.5.12}$$

for some $c > 0$ depending on k.

Proof. For the first part we can copy the proof of Lemma 3.4.23, *i.)* and show that (3.5.9) yields a smooth section $\mathcal{K}\varphi \in \Gamma^\infty(E^*\big|_{U'})$. Its restriction to U^{cl} is then in $\Gamma^k(E^*\big|_{U^{\mathrm{cl}}})$ by the very definition, see Definition 3.4.26. Moreover, the restriction to the open U is of course still smooth. For the second part it suffices to show (3.5.11). But clearly

$$\mathrm{p}_{K,k}(\mathcal{K}\varphi) \le \mathrm{vol}(U^{\mathrm{cl}})\, \mathrm{p}_{K \times U^{\mathrm{cl}},k}(K)\, \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi)$$

as in Lemma 3.4.23, *ii.)*. But then the continuity is clear by the definition of the locally convex and Banach topologies of $\Gamma_b(E^*\big|_U)$, $\Gamma^k(E^*\big|_{U^{\mathrm{cl}}})$ and $\Gamma^\infty(E^*\big|_U)$, respectively. □

We apply this lemma now to the Green functions $F_U^{\pm}(p)$ in the following way.

Lemma 3.5.7 *Let $U \subseteq U^{\mathrm{cl}} \subseteq U' \subseteq M$ be as in Section 3.4 with U small enough and let $\mathcal{K}_U^{\pm}$ be the integral operator from* (3.4.47).

i.) For every $k \in \mathbb{N}_0$ there is a $c > 0$ such that for $\varphi \in \Gamma_b(E^*\big|_{U'})$ we have

$$\mathrm{p}_{U^{\mathrm{cl}},k}\left(\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right)(\varphi)\right) \le c\, \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi). \tag{3.5.13}$$

ii.) For $\varphi \in \Gamma^k(E^*\big|_{U'})$ there is a $\widetilde{c} > 0$ such that

$$\mathrm{p}_{U^{\mathrm{cl}},k}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\left(\varphi\big|_{U^{\mathrm{cl}}}\right)\right) \le \widetilde{c}\, \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi). \tag{3.5.14}$$

Proof. From Corollary 3.4.33 we know that the operator $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}$ has a smooth kernel in $\Gamma^\infty(E^* \boxtimes E\big|_{U' \times U'})$. Thus the previous Lemma 3.5.6, *ii.)* applies and (3.5.12) gives (3.5.13). For the second part we note that

$$\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right)\Big|_{U^{\mathrm{cl}}} = \varphi\big|_{U^{\mathrm{cl}}} - \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}(\varphi)\big|_{U^{\mathrm{cl}}},$$

as we already argued in the proof of Lemma 3.4.23. But then

$$\mathrm{p}_{U^{\mathrm{cl}},k}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \left(\varphi\big|_{U^{\mathrm{cl}}}\right)\right) = \mathrm{p}_{U^{\mathrm{cl}},k}\left(\varphi - \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}(\varphi)\right) \leq \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi) + c\, \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi)$$

with $c > 0$ from (3.5.13). Since $\mathrm{p}_{U^{\mathrm{cl}},k}(\varphi) \geq \mathrm{p}_{U^{\mathrm{cl}},0}(\varphi)$ we take $\widetilde{c} = 1 + c$ to obtain (3.5.14). □

The importance in the above estimates is that we can control the "loss of derivatives": the operator $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$ is not loosing orders of differentiation while $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}$ is even gaining smoothness in (3.5.13). We combine this now with the properties of $\widetilde{\mathcal{R}}_{U'}^{\pm}$ from Proposition 3.4.16 to obtain the following property of the operator $F_U^{\pm}$:

Proposition 3.5.8 *Let $U \subseteq U^{\mathrm{cl}} \subseteq U^{\mathrm{cl}}$ be as before and let $F_U^{\pm} = \left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \widetilde{\mathcal{R}}_{U'}^{\pm}(\,\cdot\,)$ be the operator as in Definition 3.4.35. Then for all compacta $K \subseteq U$ and all $k \in \mathbb{N}_0$ we have a $c_{K,k} > 0$ such that*

$$\mathrm{p}_{U^{\mathrm{cl}},k}(F_U^{\pm}(\varphi)) \leq c_{K,k}\, \mathrm{p}_{K,k+n+1}(\varphi) \tag{3.5.15}$$

for all $\varphi \in \Gamma_K^\infty(E^*\big|_U)$.

Proof. We know already from the proof of Theorem 3.4.36 that the operator $F_U^{\pm}$ is continuous but (3.5.15) gives a more precise statement of this. We have by (3.5.14) and (3.4.36)

$$\begin{aligned}
\mathrm{p}_{U^{\mathrm{cl}},k}(F_U^{\pm}(\varphi)) &= \mathrm{p}_{U^{\mathrm{cl}},k}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}\left(\widetilde{\mathcal{R}}_{U'}^{\pm}\right)(\varphi)\right) \\
&\leq \widetilde{c}\, \mathrm{p}_{U^{\mathrm{cl}},k}\left(\widetilde{\mathcal{R}}_{U'}^{\pm}(\varphi)\right) \\
&\leq \widetilde{c}\, c_{K,U^{\mathrm{cl}},k+n+1}\, \mathrm{p}_{K,k+n+1}(\varphi),
\end{aligned}$$

which is (3.5.15). □

Corollary 3.5.9 *The operator $F_U^{\pm}$ has a continuous extension to an operator*

$$F_U^{\pm} : \Gamma_0^{k+n+1}(E^*\big|_U) \longrightarrow \Gamma^k(E^*\big|_U) \tag{3.5.16}$$

for all $k \geq 0$, and the estimate (3.5.15) *also holds for $\varphi \in \Gamma_K^{k+n+1}(E^*\big|_U)$.*

Proof. The estimate (3.5.15) for all compact subsets $K \subseteq U$ is just the continuity of $F_U^{\pm}$ in the $\mathcal{C}_0^{k+n+1}$- and $\mathcal{C}^k$-topology. Thus by the usual density argument we have a unique continuous extension (3.5.16) still obeying the estimate (3.5.15). □

As usual we can also dualize (3.5.16) and get a weak* continuous map

$$(F_U^{\pm})' : \Gamma_0^{-k}(E\big|_U) \longrightarrow \Gamma^{-k-n-1}(E\big|_U), \tag{3.5.17}$$

again for all $k \geq 0$. Recall that by Remark 1.3.8 the topological dual spaces of $\Gamma^k(E^*\big|_U)$ and $\Gamma_0^k(E^*\big|_U)$ can be identified with $\Gamma_0^{-k}(E\big|_U)$ and $\Gamma^{-k}(E\big|_U)$, respectively. Note again, that $\Gamma^{-0}(E\big|_U)$ are *not* just the continuous sections $\Gamma^0(E\big|_U)$. The importance of Proposition 3.5.8 and Corollary 3.5.9 is that we only loose a fixed amount of derivatives under $F_U^{\pm}$. In this sense the order of the map $F_U^{\pm}$ is globally bounded by $n + 1$.

In general, a continuous operator $A : \Gamma_0^\infty(E^*) \longrightarrow \Gamma^\infty(E^*)$ gives a dual operator $A' : \Gamma_0^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(E)$ as we did this above for $A = F_U^\pm$. Now this operator A' does not necessarily map $\Gamma_0^\infty(E) \subseteq \Gamma_0^{-\infty}(E)$ into $\Gamma^\infty(E) \subseteq \Gamma^{-\infty}(E)$. For this additional property, A needs to be a "symmetric" operator for the natural pairing. We will now show this feature for $F_U^\pm$. We consider the following situation. Let v be a distributional section of E with compact support in U as before but we assume that v is actually a $\mathcal{C}^\ell$-section with $\ell \in \mathbb{N}_0$. Then for a test section $\varphi \in \Gamma_0^\infty(E^*\big|_U)$ we have

$$(F_U^\pm)'(v)(\varphi) = v(F_U^\pm(\varphi)) = \int_U v(p) \cdot F_U^\pm(\varphi)\big|_p \, \mu_g(p) = \int_U v(p) \cdot F_U^\pm(p)(\varphi) \, \mu_g(p), \tag{3.5.18}$$

according to our convention for the pairing of $\Gamma_0^{-\infty}(E\big|_U)$ and $\Gamma^\infty(E^*\big|_U)$. For the Riesz distributions we already had some symmetry properties as explained in Proposition 3.2.16. Thus the question is whether we can extend this to $F_U^\pm$ as well and move $F_U^\pm$ to the other side in the natural pairing (3.5.18). We start with the corresponding symmetry property of $\widetilde{\mathcal{R}}_{U'}^\pm$.

Lemma 3.5.10 *Let $\widetilde{\mathcal{R}}_{U'}^\pm$ be as before and let $k \in \mathbb{N}_0$. The for all $u \in \Gamma_0^{k+n+1}(E\big|_{U'})$ we have*

i.) $\widetilde{\mathcal{R}}_{U'}^\pm$ dualizes to a weak continuous linear map*

$$\left(\widetilde{\mathcal{R}}_{U'}^\pm\right)' : \Gamma_0^{-k}(E\big|_{U'}) \longrightarrow \Gamma^{-k-n-1}(E\big|_{U'}). \tag{3.5.19}$$

ii.) We have $(\widetilde{\mathcal{R}}_{U'}^\pm)'(u) \in \Gamma^k(E\big|_{U'})$ explicitly given by

$$\left(\left(\widetilde{\mathcal{R}}_{U'}^\pm\right)'(u)\right)(q) = \sum_{j=0}^\infty \left(\widetilde{V}_q^j\right)^{\mathrm{T}} R_{U'}^\mp(2+2j, q)(u), \tag{3.5.20}$$

where $\widetilde{V}^j = V^j$ for $j \le N-1$ and $\widetilde{V}^j = V^j \chi(\frac{\eta}{\epsilon_j})$ for $j \ge N$ for abbreviation and

$$^{\mathrm{T}} : \Gamma^\infty\left(E^* \boxtimes E\big|_{U' \times U'}\right) \longrightarrow \Gamma^\infty\left(E \boxtimes E^*\big|_{U' \times U'}\right) \tag{3.5.21}$$

is the canonical transposition also flipping the arguments.

Proof. The first part is clear since $\widetilde{\mathcal{R}}_{U'}^\pm$ is a continuous linear map

$$\widetilde{\mathcal{R}}_{U'}^\pm : \Gamma_0^{k+n+1}(E^*\big|_U) \longrightarrow \Gamma^k(E^*\big|_U)$$

by Remark 3.4.17 and the duals are just given by $\Gamma^{-k-n-1}(E\big|_{U'})$ and $\Gamma_0^{-k}(E\big|_{U'})$ respectively. Thus it remains to evaluate $(\widetilde{\mathcal{R}}_{U'}^\pm)'(u)$. Since we can interpret u as distributional section of any order we want, it is sufficient to evaluate the result on smooth test sections $\varphi \in \Gamma_0^\infty(E^*\big|_{U'})$ since they will by dense in every other test section space $\Gamma_0^\ell(E^*\big|_{U'})$. Thus we compute

$$\begin{aligned}
(\widetilde{\mathcal{R}}_{U'}^\pm)'(u)(\varphi) &= u\left(\widetilde{\mathcal{R}}_{U'}^\pm(\varphi)\right) \\
&= \int_{U'} u(p) \cdot \widetilde{\mathcal{R}}_{U'}^\pm(\varphi)\big|_p \, \mu_g(p) \\
&= \int_{U'} u(p) \cdot \widetilde{\mathcal{R}}_{U'}^\pm(p)(\varphi) \, \mu_g(p) \\
&= \int_{U'} u(p) \cdot \sum_{j=0}^{N-1} V_p^j R_{U'}^\pm(2+2j, p)(\varphi) \, \mu_g(p)
\end{aligned}$$

$$+ \int_{U'} u(p) \cdot \left(\sum_{j=N}^{\infty} V_p^j \chi\left(\frac{\eta_p}{\epsilon_j}\right) R_{U'}^{\pm}(2+2j,p)(\varphi) \right) \mu_g(p).$$

We set $\widetilde{V}_p^j = V_p^j$ for $j \le N-1$ and $\widetilde{V}_p^j = V_p^j \chi(\frac{\eta_p}{\epsilon_j})$ for $j \ge N$ to abbreviate the single terms. Then we have

$$(\widetilde{\mathcal{R}}_{U'}^{\pm})'(u)(\varphi) = \sum_{j=0}^{N+k-1} \int_{U'} u(p) \cdot \widetilde{V}_p^j R_{U'}^{\pm}(2+2j,p)(\varphi)\, \mu_g(p)$$
$$+ \sum_{j=N+k}^{\infty} \int_{U'} u(p) \cdot \int_{U'} \widetilde{V}_p^j(q) R_{U'}^{\pm}(2+2j,p)(q) \cdot \varphi(q)\, \mu_g(q)\mu_g(p), \qquad (☺)$$

since in the second series we have $\mathcal{C}^k$-convergence by Proposition 3.4.5, *ii.)* and compact support. Thus the series can indeed be taken outside the integrals. For the first $N+k$ terms we use Proposition 3.2.16 in a slightly more general setting: the function

$$(p,q) \mapsto u(p) \cdot \widetilde{V}_p^j(p,q) \cdot \varphi(q)$$

is compactly supported in $U' \times U'$ but only $\mathcal{C}^{k+n+1}$ instead of $\mathcal{C}^\infty$. However, the involved Riesz distributions are all of order $\le n+1$ whence we still can apply Proposition 3.2.16, *ii.)*, e.g. by arguing with the usual density trick. This gives

$$\sum_{j=0}^{N+k-1} \int_{U'} R_{U'}^{\pm}(2+2j,p)\left(q \mapsto u(p) \cdot \widetilde{V}^j(p,q) \cdot \varphi(q)\right) \mu_g(p)$$
$$= \sum_{j=0}^{N+k-1} \int_{U'} R_{U'}^{\mp}(2+2j,q)\left(p \mapsto u(p) \cdot \widetilde{V}^j(p,q)\right) \cdot \varphi(q))\, \mu_g(q)$$
$$= \sum_{j=0}^{N+k-1} \int_{U'} R_{U'}^{\mp}(2+2j,q)\left(p \mapsto u(p) \cdot \widetilde{V}^j(p,q)\right) \cdot \varphi(q)\, \mu_g(q).$$

Now it is useful to consider the transposition map

$$^{\mathrm{T}} : \Gamma^\infty\left(E * \boxtimes E\big|_{U' \times U'}\right) \longrightarrow \Gamma^\infty\left(E \boxtimes E^*\big|_{U' \times U'}\right),$$

defined in the usual way by exchanging the order of arguments $(p,q) \leftrightarrow (q,p)$ and the E- and $E*$-parts, respectively. Thus we have

$$\sum_{j=0}^{N+k-1} \int_{U'} R_{U'}^{\mp}(2+2j,q)\left(p \mapsto u(p) \cdot \widetilde{V}^j(p,q)\right) \cdot \varphi(q)\, \mu_g(q)$$
$$= \sum_{j=0}^{N+k-1} \int_{U'} R_{U'}^{\mp}(2+2j,q)\left(p \mapsto \widetilde{V}^{j\,\mathrm{T}}(p,q) \cdot u(p)\right) \cdot \varphi(q)\, \mu_g(q)$$
$$= \sum_{j=0}^{N+k-1} \int_{U'} \widetilde{V}^{j\,\mathrm{T}} R_{U'}^{\mp}(2+2j,\,\cdot\,)(u)\big|_q \cdot \varphi(q)\, \mu_g(q).$$

By the smoothness of $\widetilde{V}^{j\,\mathrm{T}}$ and Proposition 3.2.15 we conclude that the section

$$q \mapsto \sum_{j=0}^{\infty} \left((\widetilde{V}^j R^{\mp}_{U'}(2+2j,\,\cdot\,))^{\mathrm{T}}(u) \right)(q)$$

is actually a $\mathcal{C}^k$-section of E on U' since u is $\mathcal{C}^{k+n+1}$. It remains to consider the second part of (☺). First we again use Proposition 3.2.16, *i.)* to move $R^{\pm}_{U'}(2+2j,p)$ to the other side. Afterwards we exchange the order of integration and summation back by the same $\mathcal{C}^k$-convergence yielding eventually

$$\begin{aligned}
&\sum_{j=N+k}^{\infty} \int_{U'}\int_{U'} u(p)\cdot \widetilde{V}^j_p(q) R^{\pm}_{U'}(2+2j,p)(q)\cdot \varphi(q)\,\mu_g(p)\mu_g(q) \\
&\quad= \sum_{j=N+k}^{\infty} \int_{U'}\int_{U'} R^{\mp}_{U'}(2+2j,q)(p)u(p)\cdot \widetilde{V}^j(p,q)\cdot\varphi(q)\,\mu_g(p)\mu_g(q) \\
&\quad= \int_{U'} \varphi(q)\cdot \int_{U'} \left(\sum_{j=N+k}^{\infty} (\widetilde{V}^k_q)^{\mathrm{T}} R^{\mp}_{U'}(2+2j,q) \right)(p)\cdot u(p)\,\mu_g(p)\mu_g(q).
\end{aligned}$$

The series still converges *in the* $\mathcal{C}^k$*-topology* as we only switched the labels. Thus the inner integrand is a $\mathcal{C}^k$-section on $U'\times U'$ being paired with a compactly supported $\mathcal{C}^{k+n+1}$-section u. This gives still a $\mathcal{C}^k$-section on U' which is then paired with the remaining φ. We conclude that

$$\left((\widetilde{\mathcal{R}}^{\pm}_{U'})'(u) \right)(\varphi) = \int_{U'} \left(\sum_{j=0}^{\infty} (\widetilde{V}^j)^{\mathrm{T}} R^{\mp}_{U'}(2+2j,\,\cdot\,)(u) \right)(q)\cdot\varphi(q)\,\mu_g(q)$$

with a $\mathcal{C}^k$-section

$$\left((\widetilde{\mathcal{R}}^{\pm}_{U'})'(u) \right)(q) = \sum_{j=0}^{\infty} (\widetilde{V}^j_q)^{\mathrm{T}} R^{\mp}_{U'}(2+2j,q)(u)$$

as claimed. □

Remark 3.5.11 The Riesz distributions $R^{\pm}_{U'}(\alpha,p)$ enjoy the symmetry property $R^{\pm}_{U'}(\alpha,p)(q) = R^{\mp}_{U'}(\alpha,q)(p)$ as soon as $\mathsf{Re}(\alpha) > n$. For all $\alpha\in\mathbb{C}$, the correct analog of this symmetry was obtained in Proposition 3.2.16, *ii.)*. Thus extending the transposition $^{\mathrm{T}}$ from smooth to continuous or even distributional sections we have

$$\left(R^{\pm}_{U'}\right)^{\mathrm{T}} = R^{\mp}_{U'} \tag{3.5.22}$$

in the sense of Proposition 3.2.16, *ii.)*. Moreover, since in the series (3.5.20) we have the "same" coefficients as for the original series defining $\widetilde{\mathcal{R}}^{\pm}_{U'}$ only at flipped points, we get the same sort of estimates and convergence results. In particular we have

$$\left(\widetilde{\mathcal{R}}^{\pm}_{U'}\right)' = \left(\widetilde{\mathcal{R}}^{\mp}_{U'}\right)^{\mathrm{T}} \tag{3.5.23}$$

on distributional sections which are at least $\mathcal{C}^{n+1}$. This allows to efficiently compute $\left(\widetilde{\mathcal{R}}^{\pm}_{U'}\right)'(u)$ for $u\in\Gamma^{n+1}_0(E\big|_{U'})$ by means of the nicely convergent series (3.5.20) or (3.5.23).

Corollary 3.5.12 *Let* $u\in\Gamma^{\infty}_0(E\big|_{U'})$ *then* $\left(\widetilde{\mathcal{R}}^{\pm}_{U'}\right)'(u)\in\Gamma^{\infty}(E\big|_{U'})$.

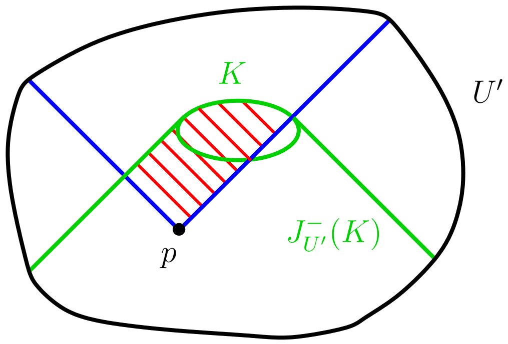

Figure 3.9: The intersection of the future of a point p with the past of a compactum K, all in a geodesically convex U'.

Corollary 3.5.13 *Let $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $u \in \Gamma_0^{k+n+1}(E\big|_{U'})$. Then the series* (3.5.20) *converges in the $\mathcal{C}^k$-topology.*

Proof. This follows analogously to the statements for $\widetilde{\mathcal{R}}^{\pm}_{U'}$ as in Proposition 3.4.5: the finitely many terms with $j \leq N + k - 1$ are already $\mathcal{C}^k$ by themselves and the remaining sum converges in $\mathcal{C}^k$ *before* applying to u on $U' \times U'$. Then the integration over p together with the compactly supported u can be exchanged with the summation by the usual arguments. It gives then the $\mathcal{C}^k$-convergence on U'. □

We can use the lemma also to extend $\widetilde{\mathcal{R}}^{\pm}_{U'}$ as well as its dual $\left(\widetilde{\mathcal{R}}^{\pm}_{U'}\right)'$ and $\left(\widetilde{\mathcal{R}}^{\mp}_{U'}\right)^{\mathrm{T}}$ to some more general test sections and distributions with not necessarily compact support. We consider the following situation: Let $K \subseteq U'$ be compact, then the intersection $J^+_{U'}(p) \cap J^-_{U'}(K)$ is still compact since U' is geodesically convex, see Figure 3.9. In fact, also the intersection $J^+_{U'}(L) \cap J^-_{U'}(K)$ is compact for another compactum $L \subseteq U'$. Suppose $\operatorname{supp}\varphi \subseteq J^-_{U'}(K)$ for a test section $\varphi \in \Gamma^k(E^*\big|_{U'})$ with not necessarily compact support. Then for every j and every $p \in U'$ the overlap

$$\begin{aligned}
\operatorname{supp}\left(\widetilde{V}^j_p R^+(2+2j,p)\right) \cap \operatorname{supp}\varphi &\subseteq \operatorname{supp} R^+_{U'}(2+2j,p) \cap \operatorname{supp}\varphi \\
&\subseteq J^+_{U'}(p) \cap \operatorname{supp}\varphi \\
&\subseteq J^+_{U'}(p) \cap J^-_{U'}(K)
\end{aligned}$$

is compact. Thus $\widetilde{V}^j_p R^+_{U'}(2+2j,p)(\varphi)$ is defined by Proposition 1.3.20 in a non-ambiguous way. By the same argument, also $\widetilde{\mathcal{R}}^+_{U'}(\varphi)$ is well-defined. Moreover, since for $p \in L$ the support of $\widetilde{V}^j_p R^+(2+2j,p)$ has still compact overlap with $\operatorname{supp}\varphi$ we can replace φ by some $\chi\varphi$ as in the proof of Proposition 1.3.20 and get the same convergence results of the series

$$\widetilde{\mathcal{R}}^+_{U'}(p)(\varphi) = \sum_{j=0}^{\infty} \widetilde{V}^j_p R^+(2+2j,p)(\varphi) \tag{3.5.24}$$

as for compactly supported φ. In conclusion, this gives a $\mathcal{C}^k$-convergence if φ is of class $\mathcal{C}^{k+n+1}$ for all $k \in \mathbb{N}_0 \cup \{+\infty\}$. With the same argument, also the series $\left(\widetilde{\mathcal{R}}^-_{U'}\right)^{\mathrm{T}}$ converges. Here of course we need $u \in \Gamma^{k+n+1}(E\big|_{U'})$ with $\operatorname{supp} u \subseteq J^+_{U'}(K)$ to make the series

$$(\widetilde{\mathcal{R}}^-_{U'})^{\mathrm{T}}(u) = \sum_{j=0}^{\infty} (\widetilde{V}^j_{\cdot})^{\mathrm{T}} R^-_{U'}(2+2j,\,\cdot\,)(u) \tag{3.5.25}$$

converge in the $\mathcal{C}^k$-topology. We collect these results in the following lemma:

Lemma 3.5.14 *Let $K \subseteq U'$ be compact and $k \in \mathbb{N}_0 \cup \{+\infty\}$.*

i.) Assume $u \in \Gamma^{k+n+1}(E\big|_{U'})$ has support in $J^{\mp}_{U'}(K)$. Then

$$(\widetilde{\mathcal{R}}^{\mp}_{U'})^{\mathrm{T}}(u) = \sum_{j=0}^{\infty} (\widetilde{V}^{j}_{\cdot})^{\mathrm{T}} R^{\mp}_{U'}(2+2j, \,\cdot\,)(u) \tag{3.5.26}$$

converges in the $\mathcal{C}^k$-topology.

ii.) Assume $\varphi \in \Gamma^{k+n+1}(E^*\big|_{U'})$ has support in $J^{\mp}_{U'}(K)$. Then

$$\widetilde{\mathcal{R}}^{\pm}_{U'}(p)(\varphi) = \sum_{j=0}^{\infty} \widetilde{V}^{j}_{p} R^{\pm}(2+2j, p)(\varphi) \tag{3.5.27}$$

converges in the $\mathcal{C}^k$-topology.

We can now study the dual of $F^{\pm}_{U}$ under the assumption that $U \subset U^{\mathrm{cl}} \subset U'$ is *causal* in order to have good support properties of the integral operator $\mathcal{K}^{\pm}_{U}$.

Lemma 3.5.15 *Let $u \in \Gamma^{\infty}_{0}(E\big|_{U})$. Then*

$$(F^{\pm}_{U})'(u) = \left(\widetilde{\mathcal{R}}^{\mp}_{U}\right)^{\mathrm{T}} \left(q \;\mapsto\; u(q) - \int_{U} u(p) \cdot L^{\pm}_{U}(p,q)\, \mu_g(p)\right) \tag{3.5.28}$$

with $L^{\pm}_{U}$ being the smooth integral kernel of $\left(\mathsf{id} + \mathcal{K}^{\pm}_{U}\right)^{-1} \circ \mathcal{K}^{\pm}_{U}$. Thus $(F^{\pm}_{U})'(u) \in \Gamma^{\infty}(E\big|_{U})$.

Proof. For $\varphi \in \Gamma^{\infty}_{0}(E^*\big|_{U})$ we have to evaluate the pairing

$$\begin{aligned}
(F^{\pm}_{U})'(u)(\varphi) &= u\left(F^{\pm}_{U}(\varphi)\right) \\
&= \int_{U} u(p) \cdot F^{\pm}_{U}(\varphi)\big|_{p}\, \mu_g(p) \\
&= \int_{U} u(p) \cdot \left(\mathsf{id} + \mathcal{K}^{\pm}_{U}\right)^{-1} \left(\widetilde{\mathcal{R}}^{\pm}_{U}(\,\cdot\,)(\varphi)\right)\Big|_{p}\, \mu_g(p) \\
&= \int_{U} u(p) \left(\widetilde{\mathcal{R}}^{\pm}_{U}(p)(\varphi) - \left(\mathsf{id} + \mathcal{K}^{\pm}_{U}\right)^{-1} \circ \mathcal{K}^{\pm}_{U} \left(\widetilde{\mathcal{R}}^{\pm}_{U}(\,\cdot\,)(\varphi)\right)\Big|_{p}\right) \mu_g(p).
\end{aligned}$$

Now $\left(\mathsf{id} + \mathcal{K}^{\pm}_{U}\right)^{-1} \circ \mathcal{K}^{\pm}_{U}$ is again an integral operator whose kernel is smooth and given by the truncated geometric series as in Corollary 3.4.33. Thus denote its kernel by $L^{\pm}_{U} \in \Gamma^{\infty}(E^* \boxtimes E\big|_{U' \times U'})$, noting that even though we only integrate over U^{cl} the kernel has a smooth continuation to $U' \times U'$. Since we integrate at least continuous functions and sections over compact sets U^{cl} and $U^{\mathrm{cl}} \times U^{\mathrm{cl}}$, respectively, we can exchange the orders of integration and obtain

$$\begin{aligned}
(F^{\pm}_{U})'(u)(\varphi) &= \int_{U} u(p) \cdot \widetilde{\mathcal{R}}^{\pm}_{U}(\varphi)(p)\, \mu_g(p) - \int_{U'} \int_{U'} u(p) \cdot L^{\pm}_{U}(p,q) \cdot \widetilde{\mathcal{R}}^{\pm}_{U}(\varphi)(q)\, \mu_g(q) \mu_g(p) \\
&= \int_{U} \left(u(q) - \int_{U} u(p) \cdot L^{\pm}_{U}(p,q)\, \mu_g(p)\right) \cdot \widetilde{\mathcal{R}}^{\pm}_{U}(\varphi)(q)\, \mu_g(q) \\
&= \int_{U} v(q) \cdot \widetilde{\mathcal{R}}^{\pm}_{U}(\varphi)(q)\, \mu_g(q), \qquad (*)
\end{aligned}$$

with

$$v(q) = u(q) - \int_U u(p) \cdot L_U^{\pm}(p,q) \mu_g(p).$$

Now the second term in v is smooth and has a smooth extension to U'. The first contribution u is compactly supported in U and smooth whence it also has a smooth extension to U': we conclude $v \in \Gamma^\infty(E\big|_U)$. We claim that in $(*)$ we are allowed to move $\widetilde{\mathcal{R}}_U^{\pm}$ from φ to v on the other side of the pairing. Indeed, by the causal properties of $L_U^{\pm}$ according to Lemma 3.4.40 we know

$$\operatorname{supp} L_U^{\pm} \subseteq \left\{(p,q) \;\middle|\; q \in J_{U'}^{+}(p)\right\} \subseteq U' \times U'.$$

Thus when restricting to U^{cl} and using that U is causal we see that the integrand $u(p) \cdot L_U^{\pm}(p,q)$ is possibly non-trivial only for $p \in \operatorname{supp} v$ and $q \in J_{U'}^{\pm}(p) \cap U^{\mathrm{cl}} = J_{U^{\mathrm{cl}}}^{\pm}(p)$. But this is equivalent to $p \in J_{U^{\mathrm{cl}}}^{\mp}(q)$ and hence the integrand is possibly non-trivial only for $\operatorname{supp} u \cap J_{U^{\mathrm{cl}}}^{\mp}(q) \neq \emptyset$. In other words, $\operatorname{supp}\left(\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1} \circ \mathcal{K}_U^{\pm}\right)(u) \subseteq J_{U^{\mathrm{cl}}}^{\pm}(\operatorname{supp} u)$. Hence $\operatorname{supp} v \subseteq J_{U^{\mathrm{cl}}}^{\pm}(\operatorname{supp} u)$. Note that due to the transposed integration this differs from the considerations for $L_U^{\pm}$ acting on $\varphi \in \Gamma_0^\infty(E^*\big|_U)$. But then expanding the series over j in $\widetilde{\mathcal{R}}_U^{\pm}(\varphi)$ we get

$$\begin{aligned}
\int_U v(q) \cdot \widetilde{\mathcal{R}}_U^{\pm}(\varphi)(q)\, \mu_g(q) &= \int_U v(q) \cdot \sum_{j=0}^{\infty} \widetilde{V}_q^j R_U^{\pm}(2+2j, q)(\varphi)\, \mu_g(q) \\
&= \sum_{j=0}^{\infty} \int_U v(q) \cdot \widetilde{V}_q^j R_U^{\pm}(2+2j, p)(\varphi)\, \mu_g(q) \\
&= \sum_{j=0}^{\infty} \int_U \left(\widetilde{V}_p^{j\,\mathrm{T}} R_U^{\mp}(2+2j, p)(v)\right) \cdot \varphi(p)\, \mu_g(p) \\
&= \int_U \sum_{j=0}^{\infty} \widetilde{V}_p^{j\,\mathrm{T}} R_U^{\mp}(2+2j, p)(v) \cdot \varphi(p)\, \mu_g(p) \\
&= \int_U (\widetilde{\mathcal{R}}_U^{\mp})^{\mathrm{T}}(v)(p) \cdot \varphi(p)\, \mu_g(p).
\end{aligned}$$

Here we used that $\mathcal{C}^0$-convergent series can be exchanges with integration over compacta and $R_U^{\pm}$ can be transposed as in Proposition 3.2.16, *ii.)* even though v has non-compact support: The main point is that the overlap of the supports is compact even though $\operatorname{supp} v \subseteq J_{U^{\mathrm{cl}}}^{\pm}(\operatorname{supp} u)$ typically is non-compact. But then we know that the series still converges in the $\mathcal{C}^0$-topology and can be moved inside the integral by Lemma 3.5.14. □

Remark 3.5.16 A careful counting of derivatives shows that the operator $\left(\mathsf{id} + \mathcal{K}_U^{\pm}\right)^{-1}$ does not eat orders of differentiation and $(\widetilde{\mathcal{R}}_U^{\mp})^{\mathrm{T}}$ needs at most $n+1$. Thus we also obtain the statement that

$$(F_U^{\pm})' : \Gamma_0^{k+n+1}(E\big|_U) \longrightarrow \Gamma^k(E\big|_U) \tag{3.5.29}$$

holds for all $k \in \mathbb{N}_0 \cup \{+\infty\}$.

We summarize the result of this section in the following theorem:

Theorem 3.5.17 *Let* $k \in \mathbb{N}_0 \cup \{+\infty\}$ *and* $u \in \Gamma_0^{k+n+1}(E\big|_U$. *Then* $(F_U^{\pm})'(u)$, *explicitly given by* (3.5.27), *is a* $\mathcal{C}^k$*-section of* $E\big|_U$ *with*

$$\operatorname{supp}(F_U^{\pm})'(u) \subseteq J_U^{\pm}(\operatorname{supp} u) \quad \textit{and} \quad D(F_U^{\pm})'(u) = u. \tag{3.5.30}$$

In particular, we have a smooth local solution of the wave equation for a smooth and compactly supported inhomogeneity.

Chapter 4

The Global Theory of Geometric Wave Equations

Since in a time-oriented Lorentz manifold every point has a causal neighborhood we see from the results in the last chapter that locally we have advanced and retarded fundamental solutions, i.e. Green functions, for a given normally hyperbolic differential operator. Moreover, we have seen how these fundamental solutions can be used to construct solutions to the inhomogeneous wave equations for different kinds of inhomogeneities.

The topic in this chapter is now to globalize these results from the (small) neighborhoods to the whole Lorentz manifold. Here the global causal structure yields obstructions of various kinds: in general we will not be able to find global Green functions. Instead, we will need some assumptions on the global geometry. Here the best situation will be obtained for globally hyperbolic Lorentz manifolds. On such spacetimes we can then also formulate and solve the Cauchy problem for the wave equation. This nice solutions theory allows to treat the wave equation essentially as an (infinite-dimensional) Hamiltonian dynamical system. We will illustrate this point of view by determining the relevant Poisson algebra of observables.

4.1 Uniqueness Properties of Fundamental Solutions

It will be easier to show uniqueness of fundamental solutions than their actual existence. In the following we will provide criteria under which there is at most one advanced and one retarded fundamental solution. In order to treat a rather general situation we first recall some more refined techniques for the description of the causal structure.

4.1.1 Time Separation

The time separation function τ on M will be the Lorentz analogue of the Riemannian distance d. However, in various aspects it behaves quite differently. It will help us to formulate appropriate conditions on M to ensure uniqueness properties for the fundamental solutions. We recall here its definition and some of the basic properties.

Definition 4.1.1 (Arc length) *Let $\gamma : [a, b] \longrightarrow M$ be a (piecewise) $\mathcal{C}^1$ curve in a semi-Riemannian manifold (M, g). Then its arc length is defined by*

$$L(\gamma) = \int_a^b \sqrt{\left|g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\right|} \, \mathrm{d}\, t. \tag{4.1.1}$$

Clearly, the definition makes sense for piecewise $\mathcal{C}^1$-curves as well. The following is obvious:

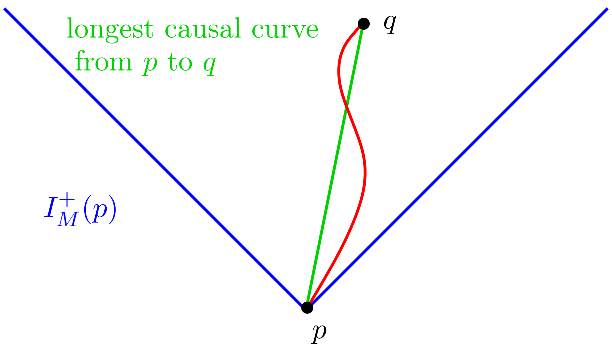

Figure 4.1: The twin paradoxon

Lemma 4.1.2 *The arc length of a piecewise $\mathcal{C}^1$ curve γ is invariant under monotonous piecewise $\mathcal{C}^1$ reparametrization.*

Unlike in Riemannian geometry, for different points p and q there may still be curves γ joining p and q which have arc length 0, namely if $\dot{\gamma}$ is timelike. This makes the concept of a "distance" more complicated. One has the following definition:

Definition 4.1.3 (Time separation) *The time separation function $\tau : M \times M \longrightarrow \mathbb{R} \cup \{+\infty\}$ in a time-oriented Lorentz manifold (M, g) is defined by*

$$\tau(p, q) = \sup \left\{ L(\gamma) \;\middle|\; \gamma \textit{ is a future directed causal curve from } p \textit{ to } q \right\} \tag{4.1.2}$$

if $q \in J^+_M(p)$ and $\tau(p, q) = 0$ if $q \notin J^+_M(p)$.

In contrast to the Riemannian situation where one uses the infimum over all arc lengths of curves joining p and q to define the Riemannian distance, the time separation τ has some new features: first it is clear that $\tau(p, q) = 0$ may happen even for $p \neq q$; this is possible already in Minkowski spacetime. Moreover, in general $\tau(p, q)$ is *not* a symmetric function as it involves the choice of the time-orientation. Again, this can easily be seen for Minkowski spacetime and points $p \neq q$ with $q \in I^+_M(p)$. In this case $\tau(p, q)$ is the Minkowski length of the vector $\vec{pq} = q - p$. The fact that all other future directed causal curves from p to q are shorter is the mathematical fact underlying the so-called *twin paradoxon*. In the more weird examples of Lorentz manifolds it may happen that $\tau(p, q) = +\infty$ for some or even all pairs of points: the Lorentz cylinder from Figure 2.18 is an example. By spiralling around the cylinder we find a future directed timelike geodesic γ from p to q of arbitrarily big length $L(\gamma)$. This already indicates that the points p and q with $\tau(p, q) = +\infty$ will be responsible for bad behaviour of the causal structure.

Recall that a lightlike curve γ from p to q is called *maximizing* if there is no timelike curve from p to q. Then we have the following useful Lemma:

Lemma 4.1.4 *If there is a causal curve γ from p to q which is not a maximizing lightlike curve then there also exists a timelike curve from p to q.*

The proof can be found e.g. in [46, Thm. 10.51], see also the discussion in [45, Thm. 2.30]. The geometric meaning of this is illustrated in Figure 4.2. In fact, it can be shown that a maximizing lightlike curve is, up to reparametrization, a lightlike geodesic without conjugate points between the endpoints. Moreover, one can show that the timelike curve in the lemma can be chosen arbitrarily close to the original causal curve γ. Using this lemma one arrives at the following properties of the time separation:

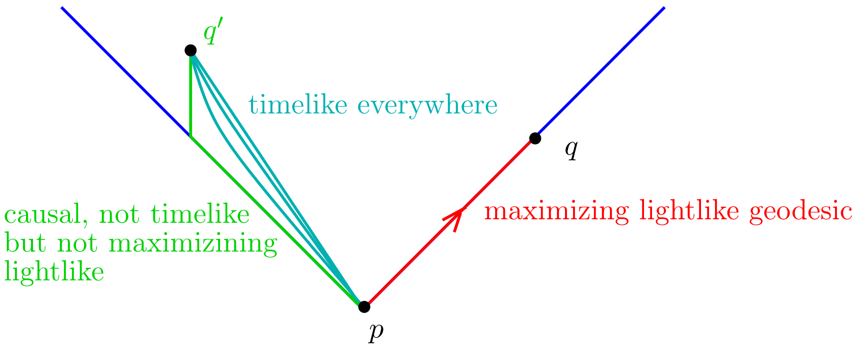

Figure 4.2: Illustration for Lemma 4.1.4.

Theorem 4.1.5 (Time separation) *Let (M, g) be a time-oriented Lorentz manifold and $p, q, r \in M$.*

i.) One has $\tau(p, q) > 0$ iff $p \ll q$.

ii.) If there exists a timelike closed curve through p then we have $\tau(p, p) = +\infty$. Otherwise one has $\tau(p, p) = 0$.

iii.) If $0 < \tau(p, q) < +\infty$ then $\tau(q, p) = 0$.

iv.) For $p \leq q \leq r$ one has a reverse triangle inequality, i.e.

$$\tau(p, q) + \tau(q, r) \leq \tau(p, r). \tag{4.1.3}$$

v.) Suppose $p, q \in U \subseteq M$ with an open geodesically convex U. If $q \in I_U^+(p)$ then the geodesic $\gamma(t) = \exp_p(t \exp_p^{-1}(q))$ maximizes the arc length of all causal curves from p to q which are entirely in U and $\tau_U(p, q) = \sqrt{g_p(\exp_p^{-1}(q), \exp_p^{-1}(q))}$.

vi.) The time separation function τ is lower semi continuous, i.e. for convergent sequence $p_n \longrightarrow p$ and $q_n \longrightarrow q$ one has

$$\liminf_{n\to\infty} \tau(p_n, q_n) = \tau(p, q). \tag{4.1.4}$$

Proof. We only sketch the arguments and refer to [46, Chapter 14] or [45, Sect. 2.5] for details. If $p \ll q$ then there is a timelike future directed curve γ from p to q. Thus $L(\gamma) > 0$ and $\tau(p, q) \geq L(\gamma)$. Conversely, suppose $\tau(p, q) > 0$ then there is a causal future directed curve γ from p to q which cannot be a lightlike curve as for lightlike curves we have arc length 0. By Lemma 4.1.4 we can deform γ into a timelike curve whence $p \ll q$ follows. This gives the first part. If we have a timelike closed loop γ through p then clearly $L(\gamma) > 0$. Thus winding around more and more often produces $L(\gamma^n) = nL(\gamma) \longrightarrow +\infty$, showing $\tau(p, p) = +\infty$. Otherwise, there can be at most a maximizing lightlike loop through p or $p \notin J_M^+(p)$ at all, by Lemma 4.1.4. In both cases $L(\gamma) = 0$ for all (possibly none at all) curves whence $\tau(p, p) = 0$. The third part is clear since $0 < \tau(p, q)$ shows that there is a timelike curve from p to q and hence $p \ll q$. If also $\tau(q, p) > 0$ then also $q \ll p$ whence we would obtain a closed timelike loop from p to p with non-trivial length $L(\gamma)$. Running around this loop n times and then to q gives a timelike curve from p to q with arc length at least $nL(\gamma) \longrightarrow +\infty$. This contradicts $\tau(p, q) < \infty$, see also Figure 4.3. For the fourth part, let $p \leq q \leq r$ be given and let $\epsilon > 0$. We find future directed causal curves γ_1 from p to q and γ_2 from q to p with

$$\tau(p, q) < L(\gamma_1) + \epsilon \quad \text{and} \quad \tau(q, r) < L(\gamma_2) + \epsilon$$

by definition of τ as supremum. Since $\tau(p, r)$ is clearly not less than $L(\gamma_1) + L(\gamma_2)$ as γ_2 after γ_1 is joining p to r, we find

$$\tau(p, r) \geq L(\gamma_1) + L(\gamma_2) > \tau(p, q) - \epsilon + \tau(q, r) - \epsilon,$$

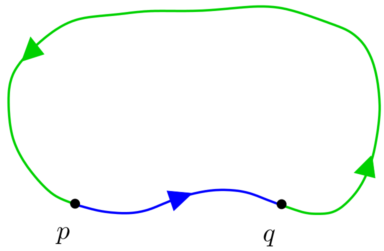

Figure 4.3: A timelike loop from p to q.

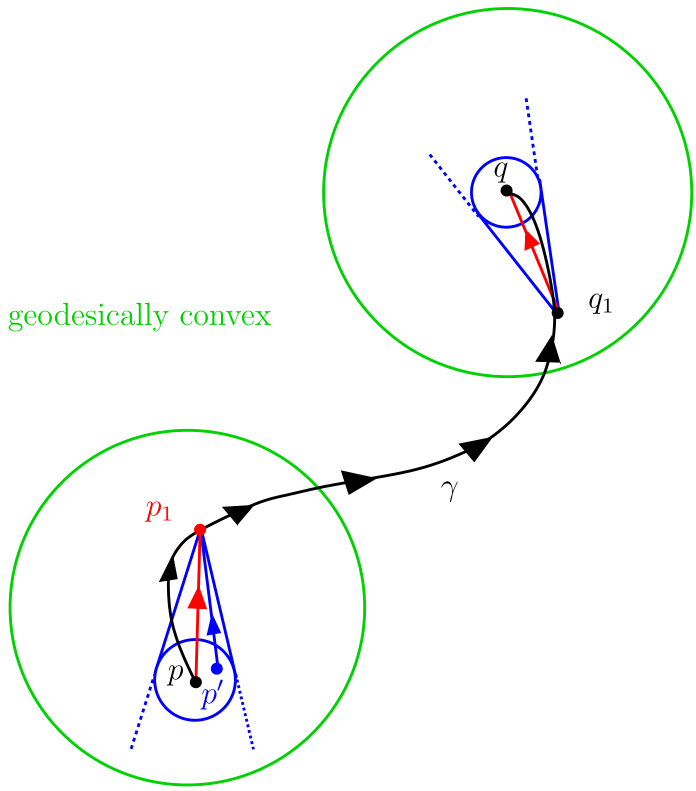

Figure 4.4: Illustration for proof of Theorem 4.1.5, *vi.)*.

whence $\tau(p,r) \geq \tau(p,r) + \tau(q,r) - 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we get the reverse triangle inequality. For the fifth part we refer to e.g. [46, Lem. 5.33 and Prop. 5.34]. Using this we can prove the last part as follows: for $\tau(p,q) = 0$ nothing is to be shown. Thus consider $0 < \tau(p,q) < +\infty$. Now we fix $\epsilon > 0$. Then we have to find a neighborhood U of p and a neighborhood V of q such that for $p' \in U$ and $q' \in V$ we have $\tau(p',q') > \tau(p,q) - \epsilon$. Since $0 < \tau(p,q) < +\infty$ we find a timelike curve γ from p to q with $\tau(p,q) < L(\gamma) + \frac{\epsilon}{3}$ by the first part. Now we choose a geodesically convex neighborhood V' of q and fix a point $q_1 \in V'$ on the curve γ such that the curve γ from q_1 to q stays inside V', see Figure 4.4. Since the curve γ from q_1 to q is inside V' and timelike, we know from the fifth part that the geodesic segment from q_1 to q in V' maximizes the arc length and hence it is longer (or equal) as the curve γ from q_1 to q. Now we fix a smaller neighborhood V of q by the condition that $q' \in V$ is in the causal future of q_1 and the geodesic $c_{q_1,q'}(t) = \exp_{q_1}(t \exp_{q_1}^{-1}(q'))$ from q_1 to q' has arc length

$$L(c_{q_1,q'}) > L(c_{q_1,q}) - \frac{\epsilon}{3}.$$

This is clearly possible as the arc length depends continuously on the endpoint. From the two conditions we see that the curve from p to q' first along γ and then along $c_{q_1,q'}$ has arc length

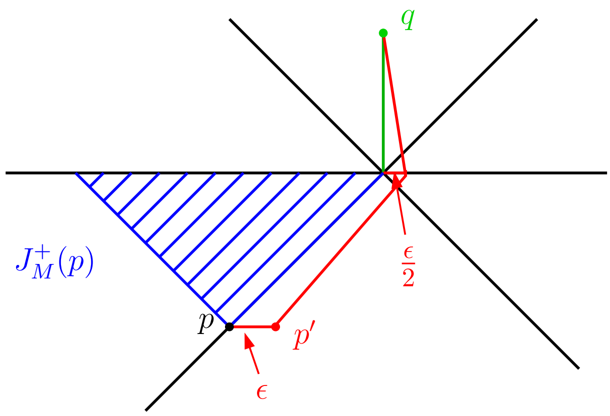

Figure 4.5: A discontinuous time separation.

$L(\gamma) - \frac{\epsilon}{3}$. An analogous construction around p specifies a p_1 and the neighborhood U. Then for $p' \in U$ and $q' \in V$ we have a timelike curve by first taking the geodesic from p' to p_1 then via γ from p_1 to q_1 and finally along the geodesic from q_1 to q'. Its arc length is at least $L(\gamma) - 2\frac{\epsilon}{3}$. Since γ was chosen such that $L(\gamma) + \frac{\epsilon}{3} > \tau(p, q)$ we see that the arc length of the curve from p' to q' is at least $\tau(p, q) - \frac{\epsilon}{3} - 2\frac{\epsilon}{3} = \tau(p, q) - \epsilon$. It follows that for all p', q' in these neighborhoods we have $\tau(p', q') \geq \tau(p, q) - \epsilon$. This shows the lower semi continuity of τ for the case $\tau(p, q) < \infty$. The construction for $\tau(p, q) = \infty$ proceeds analogously by choosing large $L(\gamma)$ and neighborhoods as before. □

The following example shows that τ is *not* continuous in general:

Example 4.1.6 (Discontinuous time separation) Consider the Minkowski plane with a half axis removed, i.e. $M = \mathbb{R}^2 \setminus (-\infty, 0]$, see Figure 4.5. Let $p = (-1, -1)$ then the causal future $J_M^+(p)$ is the triangle under the removed axis. In particular, $q = (1, 0)$ is *not* in the future of p whence $\tau(p, q) = 0$. However, for $p' = (-1, -1 + \epsilon)$ with $0 < \epsilon < 1$ the point q is in $J_M^+(p')$. The broken geodesic from p' to $(0, \frac{\epsilon}{2})$ and then from $(0, \frac{\epsilon}{2})$ to q are both timelike and the length of the first is

$$L(\gamma_1) = \sqrt{1 - (1 - \frac{\epsilon}{2})^2} = \sqrt{1 - 1 + \epsilon - \frac{\epsilon^2}{4}} = \sqrt{\epsilon - \frac{\epsilon^2}{4}}$$

while the length of the second curve is

$$L(\gamma_2) = \sqrt{1 - \frac{\epsilon^2}{4}}.$$

It follows that $\tau(p', q)$ is at least $\sqrt{\epsilon - \frac{\epsilon^2}{4}} + \sqrt{1 - \frac{\epsilon^2}{4}}$, whence

$$\limsup_{\epsilon \to 0} \tau(p', q) \geq 1 \tag{4.1.5}$$

follows at once (in fact equality holds). But since $p' \longrightarrow p$ for $\epsilon \longrightarrow 0$ we see that τ is *not* upper semi continuous and hence not continuous. In fact, moving q further upwards we can make the jump arbitrarily high.

The question is now whether we have spacetimes where τ is continuous (and finite). Clearly, Minkowski spacetime is an example where τ is continuous and finite. More generally, convex spacetimes have this feature:

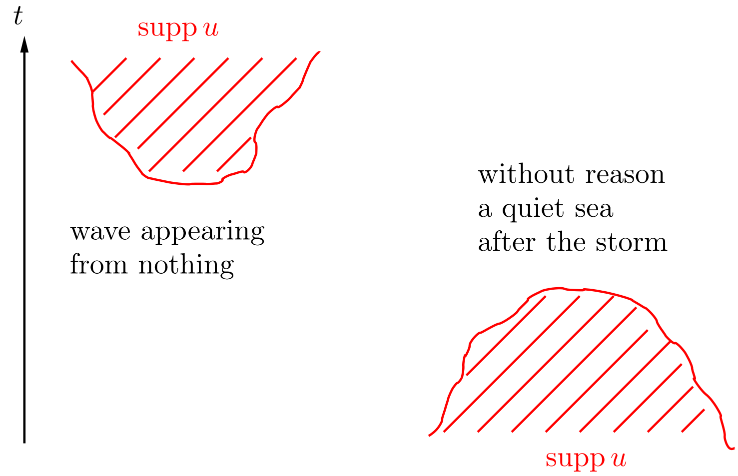

Figure 4.6: Waves with either past or future compact support should not exist.

Example 4.1.7 (Time separation for convex spacetimes) Suppose that M is geodesically convex, or $U \subseteq M$ is a geodesically convex neighborhood. Then the time separation τ_U on U is finite and continuous. Indeed, this follows from Theorem 4.1.5, *v.)* at once.

Slightly less obvious is the following situation of a globally hyperbolic spacetime: In fact, this statement can be seen as an additional motivation for the definition of globally hyperbolic spacetimes as in Definition 2.2.28. However, it was noted that Definition 2.2.28 implies strong causality as well. Using this observation, we can quote the following result [46, Prop. 21 in Chap. 14]:

Example 4.1.8 (Time separation for globally hyperbolic spacetimes) Suppose that (M, g) is globally hyperbolic. Then the time separation τ is finite and continuous, see also [45, Thm. 3.83].

With these two fundamental examples in mind we conclude this short subsection on time separation and refer to [46, Chap. 14] for additional information.

4.1.2 Uniqueness of Solutions to the Wave Equation

In general, the wave equation

$$Du = 0 \tag{4.1.6}$$

has many solutions $u \in \Gamma^{-\infty}(E)$: physically such solutions correspond to propagating waves without sources. However, also from our physical intuition we expect that a propagating wave without any possibility to interact with source terms has to "travel forever". Thus a non-trivial solution of (4.1.6) with either future or past compact support should not exist, see Figure 4.6. Assuming some (technical) conditions about the causality structure of the spacetime this is indeed true.

To formulate these conditions first recall that the causal relation $\leq$ is called *closed* if for any sequence $p_n \longrightarrow p$ and $q_n \longrightarrow q$ with $p_n \leq q_n$ we have $p \leq q$ as well. Equivalently, this means that

$$J_M^+ = \left\{(p, q) \in M \times M \;\middle|\; p \leq q\right\} \subseteq M \times M \tag{4.1.7}$$

is a closed subset of $M \times M$.

We consider now the following three properties which will turn out to be sufficient to guarantee the uniqueness of the solutions to (4.1.6) with future or past compact support.

i.) (M, g) is causal, i.e. there are no causal loops.

ii.) J_M^+ is closed.

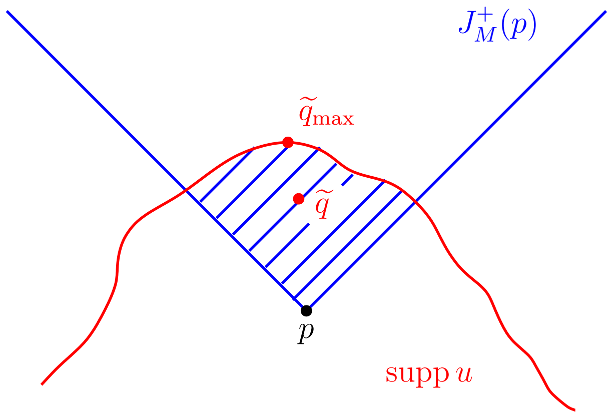

Figure 4.7: Finding the "top" of the support of u.

iii.) The time separation τ is finite and continuous.

Concerning the relation among these three properties some remarks are in due:

Remark 4.1.9 (Causally simple spacetimes) A time-oriented Lorentz manifold (M, g) which satisfies the causality condition *i.)* is called *causally simple* if in addition $J_M^{\pm}(p)$ are closed for all $p \in M$, see e.g. [45, Sect. 3.10]. One can show that this is equivalent to being causal and J_M^+ being closed which is equivalent to being causal and $J_M^{\pm}(K)$ being closed for all compact subsets $K \subseteq M$. Thus *i.)* and *ii.)* just say that (M, g) is causally simple.

Remark 4.1.10

i.) The finiteness of τ clearly implies that there are no timelike loops.

ii.) There are examples of causally simple spacetimes which do not satisfy *iii.)*. So this is indeed an additional requirement.

iii.) Convex spacetimes satisfy all three requirements, see Example 4.1.7.

iv.) Also globally hyperbolic spacetimes satisfy all three conditions, see e.g. the discussion in [45, Thm. 3.83].

With these conditions we can now prove the following theorem:

Theorem 4.1.11 *Assume that a time-oriented Lorentz manifold (M, g) satisfies the three conditions i.), ii.), iii.). Let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator on some vector bundle $E \longrightarrow M$ and let $u \in \Gamma^{-\infty}(E)$ be a distributional section. If u has either past or future compact support and satisfies the homogeneous wave equation*

$$Du = 0, \tag{4.1.8}$$

then $u = 0$.

Proof. We follow [4, Thm. 3.1.1] and consider the case of a future compact support $\operatorname{supp} u$. We have to show $\operatorname{supp} u = \emptyset$. We assume the converse and choose a point $\widetilde{q} \in \operatorname{supp} u$. The future compactness of $\operatorname{supp} u$ means that for all $p \in M$ the subset $\operatorname{supp} u \cap J_M^+(p) \subseteq M$ is compact. Choosing $p \in I_M^-(\widetilde{q})$ we obtain a non-empty intersection $\operatorname{supp} u \cap J_M^+(p)$, see Figure 4.7. We now want to find the "top" of the intersection $\operatorname{supp} u \cap J_M^+(p)$: since the time separation τ is continuous the map $q \mapsto \tau(p, q)$ for $q \in \operatorname{supp} u \cap J_M^+(p)$ takes its maximal value $\tau(p, \widetilde{q}_{\max})$ at some (not necessarily unique) $\widetilde{q}_{\max} \in \operatorname{supp} u \cap J_M^+(p)$ by compactness. We consider now the intersection $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$ which is still compact and non-empty since $\widetilde{q}_{\max} \in \operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Figure 4.7 suggests that this subset

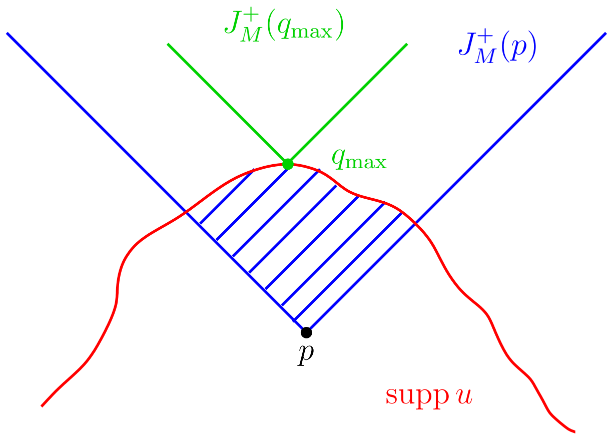

Figure 4.8: The point $q_{\max}$ on top of $\operatorname{supp} u \cap J_M^+(p)$.

is actually rather small. In fact, for $q \in \operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$ we have on one hand $\tau(p,q) \geq \tau(p, \widetilde{q}_{\max})$ since $q \geq \widetilde{q}_{\max}$ and $\tau(p,q) \leq \tau(p, \widetilde{q}_{\max})$ by the maximality of $\widetilde{q}_{\max}$. Thus

$$\tau(p,q) = \tau(p, \widetilde{q}_{\max})$$

for all $q \in \operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Among all the $q \in \operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$ we want to find a particular $q_{\max}$ such that the intersection $\operatorname{supp} u \cap J_M^+(q_{\max})$ contains *only* $q_{\max}$ and no other points. In order to find such an optimal point we proceed as follows. The compact subset $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$ is *partially ordered* via $\leq$. Indeed, $p \leq p$ as well as transitivity, $p \leq p'$ and $p' \leq p''$ implies $p \leq p''$, are always true. Since we do not have causal loops also $p \leq p'$ and $p' \leq p$ implies $p = p'$. Now assume that we have an increasing chain of elements $\{q_i\}_{i \in I}$, i.e. a subset of points of which any two are in relation "$\leq$". Our manifold being second countable we can find a countable dense subset $\{q_n\}_{n \in \mathbb{N}} \subset \{q_i\}_{i \in I}$ which is ordered again since it is the subset of an ordered set. We define Q_n to be the maximum of $\{q_1, \ldots, q_n\}$ for all $n \in \mathbb{N}$. This gives a sequence (Q_n) of elements in $\{q_n\}_{n \in \mathbb{N}}$ such that for every q_k there is an n_0 with $q_k \leq Q_n$ for all $n \geq n_0$. Now the Q_n have accumulation points in the compact subset $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Thus fixing a suitable subsequence Q_{n_m} this converges to some Q_∞ which is still in $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Since the relation $\leq$ is closed we see that Q_∞ is an upper bound for all the q_n, i.e. we have $q_n \leq Q_\infty$ for all $n \in \mathbb{N}_0$. Since the $\{q_n\}_{n \in \mathbb{N}_0} \subseteq \{q_i\}_{i \in I}$ are dense and "$\leq$" is a closed relation, we also have

$$q_i \leq Q_\infty$$

for all indexes $i \in I$. This shows that inside $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$ every increasing chain has an upper bound. Thus we are in the position to use Zorn's Lemma and conclude that there are maximal elements for all of $\operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Thus we pick one of these not necessarily unique ones and obtain a $q_{\max} \in \operatorname{supp} u \cap J_M^+(p)$ such that on one hand $q \mapsto \tau(p,q)$ attains its maximum at $q_{\max}$ *and* we have $q \leq q_{\max}$ for all $q \in \operatorname{supp} u \cap J_M^+(\widetilde{q}_{\max})$. Thus it follows that

$$\operatorname{supp} u \cap J_M^+(q_{\max}) = \{q_{\max}\}$$

by the maximality property with respect to "$\leq$". Thus we arrive at the following picture, see Figure 4.8, where $q_{\max}$ is now on the top of $\operatorname{supp} u \cap J_M^+(p)$ and $J_M^+(q_{\max})$ does not intersect $\operatorname{supp} u \cap J_M^+(p)$ except in $q_{\max}$. Now we consider a causal neighborhood $U \subseteq U'$ of $q_{\max}$ in some convex $U' \subseteq M$ with U^{cl} compact in U', such that the volume of U^{cl} is sufficiently small. Consider a sequence of points $p_i \in U$ which converge to $q_{\max}$ and are contained in $I_M^-(q_{\max}) \cap I^+(p)$. Then for large enough i the intersection $J_M^+(p_i) \cap \operatorname{supp} u$ is entirely contained in U. Indeed, assume this is not true. Then for each $i \in \mathbb{N}$ we can find a $q_i \in J_M^+(p_i) \cap \operatorname{supp} u$ which is *not* in U. By the compactness of $J_M^+(p) \cap \operatorname{supp} u$ we

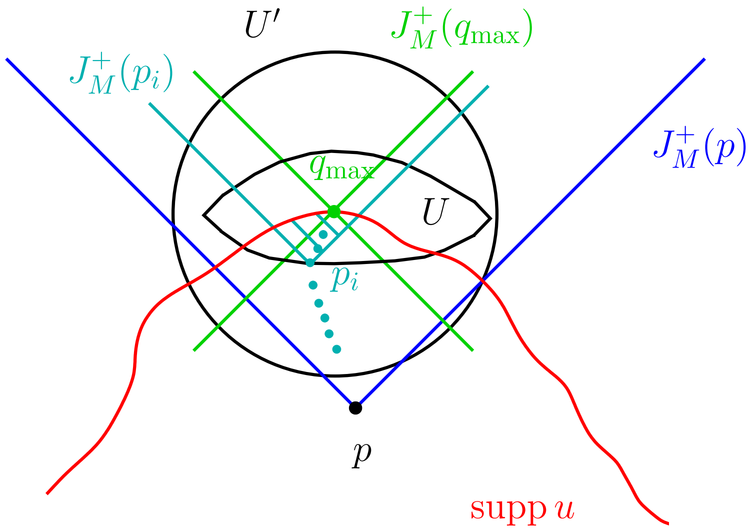

Figure 4.9: The sequence p_i approaching $q_{\max}$.

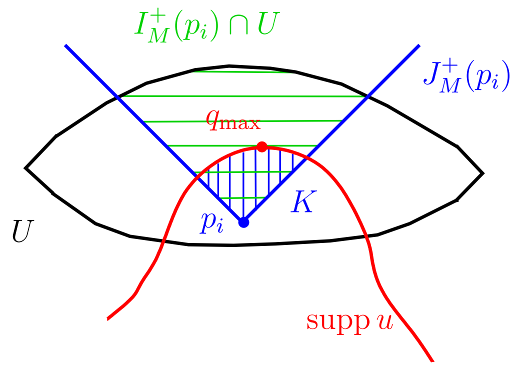

Figure 4.10: The neighborhood of $q_{\max}$.

can assume that $q_i \longrightarrow q$ converges inside $J_M^+(p) \cap \operatorname{supp} u$, probably we have to pass to a suitable subsequence. Since $q_i \in J_M^+(p_i)$ and $q_i \longrightarrow q$ as well as $p_i \longrightarrow q_{\max}$ we conclude by the closedness of the relation "$\leq$" that $q \geq q_{\max}$. Thus $q \in J_M^+(q_{\max}) \cap \operatorname{supp} u = \{q_{\max}\}$ and hence $q = q_{\max}$. On the other hand, $q_i \notin U$ implies $q \notin U$ as U is open which gives a contradiction to $q_{\max} \in U$. Thus we arrive indeed at the situation as in Figure 4.9. We choose such a point p_i and consider the compact subset $K = J_M^+(p_i) \cap \operatorname{supp} u \subseteq U$. The open subset $\widetilde{U} = I_M^+(p_i) \cap U$ contains $q_{\max}$ and is therefor an open neighborhood of $q_{\max}$, see Figure 4.10. Now we want to show that $u(\varphi) = 0$ for all test sections $\varphi \in \Gamma_0^\infty(E^*\big|_{\widetilde{U}})$. Since with $D \in \operatorname{DiffOp}^2(E)$ also the transposed operator $D^{\mathrm{T}} \in \operatorname{DiffOp}^2(E^*)$ is normally hyperbolic we can solve the inhomogeneous wave equation

$$D^{\mathrm{T}}\psi = \varphi$$

with some $\psi \in \Gamma^\infty(E^*\big|_U)$ by Theorem 3.5.17. In particular, we know that with φ being smooth also ψ is smooth. Moreover, this theorem also provides us information on the support: we can take the advanced solution for which we have $\operatorname{supp}\psi \subseteq J_U^+(\operatorname{supp}\varphi) \subseteq J_M^+(p_i) \cap U$, see Figure 4.11. Thus we get

$$\operatorname{supp} u \cap \operatorname{supp}\psi \subseteq \operatorname{supp} u \cap J_M^+(p_i) \cap U \subseteq \operatorname{supp} u \cap J_M^*(p_i) = K.$$

This is now the compactness criterion we need for applying u to the section ψ according to Proposi-

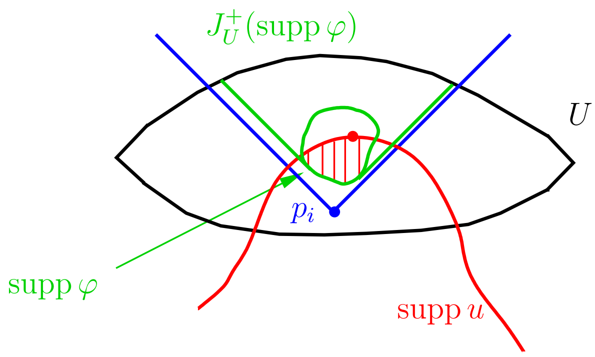

Figure 4.11: The support of φ and its future.

tion 1.3.20. Note that both have non-compact support in general. But then we have

$$u(\varphi) = u(D^{\mathrm{T}}\psi) = Du(\psi) = 0$$

by $Du = 0$. This shows that u vanishes on all test sections $\varphi \in \Gamma_0^\infty(E^*\big|_{\widetilde{U}})$. Thus the support of u is disjoint from $\widetilde{U}$. Now we arrived at the desired contradiction as $q_{\max} \in \operatorname{supp} u$ but $\widetilde{U}$ is an open neighborhood of $q_{\max}$. Hence $\operatorname{supp} u = \emptyset$ follows and thus $u = 0$. The case of past compact support is analogous. □

From this theorem we immediately obtain several statements about the solutions of the wave equations. Under the same assumptions on the global structure of M, i.e. we require a causally simple spacetime with finite and continuous time separation, one obtains the following statement:

Corollary 4.1.12 *Let (M, g) be a causally simple Lorentz manifold with finite and continuous time separation. Then for every normally hyperbolic differential operator $D \in \operatorname{DiffOp}^2(E)$ there exists at most one fundamental solution at $p \in M$ with past compact support and at most one with future compact support.*

Proof. Indeed if $DF = \delta_p = D\widetilde{F}$ then $F - \widetilde{F}$ solves the homogeneous wave equation *and* has still past (or future) compact support. Thus $F - \widetilde{F} = 0$ by the preceding theorem. □

Now we pass to a globally hyperbolic spacetime (M, g). On one hand we know from Remark 4.1.10 that (M, g) satisfies the hypothesis of Theorem 4.1.11. On the other hand on a globally hyperbolic spacetime the subset $J_M^\pm(p)$ are always past/future compact: indeed, by the very definition of global hyperbolicity, $J_M^+(p) \cap J_M^-(q) = J_M(p, q)$ is a *compact* diamond for all $p, q \in M$. This is just the statement that $J_M^+(p)$ is past compact and $J_M^-(q)$ is future compact. This gives immediately the following result:

Corollary 4.1.13 *Let (M, g) be a globally hyperbolic Lorentz manifold. Then for every normally hyperbolic differential operator $D \in \operatorname{DiffOp}^2(E)$ there exists at most one advanced and at most one retarded Green function at $p \in M$.*

Example 4.1.14 (Uniqueness of Green functions) Let $(\mathbb{R}^n, \eta)$ be the flat Minkowski spacetime as before. Since this is a globally hyperbolic spacetime we have the following global and unique Green functions:

i.) The Riesz distributions $R^\pm(2)$ are the unique advanced and retarded Green functions for $\square$ at 0. Their translates to arbitrary $p \in \mathbb{R}^n$ are the unique advanced and retarded Green functions for $\square$ at p.

ii.) The distributions $\widetilde{\mathcal{R}}^\pm(p) = \sum_{k=0}^\infty (-m^2)^k R^\pm(2 + 2k, p)$ are the unique advanced and retarded Green functions at $p \in \mathbb{R}^m$ of the Klein-Gordon operator $\square + m^2$ on Minkowski spacetime.

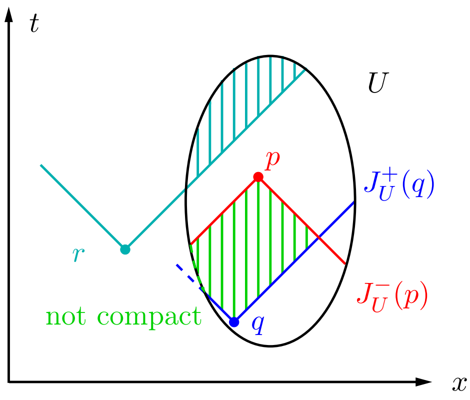

Figure 4.12: Convex domain in Minkowski spacetime with non-unique Green functions.

Finally, we mention that on convex domains we can not conclude the uniqueness of advanced and retarded Green functions without further assumptions. Even though geodesically convex domains satisfy the hypothesis of Theorem 4.1.11 it may *not* be true that $J_U^{\mp}(p)$ is past or future compact, respectively. This is clear from the example in Figure 4.12. Indeed, if in this situation we take the Green function $R^{\pm}(2)(p)$ of $\Box$ on $(\mathbb{R}^n, \eta)$ and restrict them to U we obtain advanced and retarded Green functions $\{R^{\pm}(2)(p)\big|_U\}_{p\in U}$ for all points $p \in U$. Taking now a point $r \in \mathbb{R}^n$ as in Figure 4.12 and adding $R^+(2)(r)\big|_U$ to $R^+(2)(q)\big|_U$ we still have an advanced Green function since $\Box R^+(2)(r) = 0$ on U. However, as $\operatorname{sing\,supp} R^+(2)(r) = C^+(r)$ by Proposition 3.1.12 for n even, we see that this new advanced Green function differs from $R^+(2)(q)\big|_U$ on the intersection $C^+(r) \cap U$, even in an essential way. Thus we cannot hope for uniqueness of advanced and retarded Green functions in general.

4.2 The Cauchy Problem

In order to pose the Cauchy problem we have to assume that we have a Cauchy hypersurface on which we can specify the initial values. Thus in this section we assume that (M, g) is a globally hyperbolic spacetime and $\iota : \Sigma \hookrightarrow M$ is a smooth spacelike Cauchy hypersurface in M whose existence is guaranteed by Theorem 2.2.31. Furthermore, the future directed timelike normal vector field of Σ will be denoted by $\mathfrak{n} \in \Gamma^\infty(TM\big|_\Sigma)$ as in Section 2.3.

Remark 4.2.1 When solving the wave equation $Du = v$ in a distributional sense for $u, v \in \Gamma^{-\infty}(E)$ one might be tempted to ask for the *initial conditions* of u on Σ. However, since $\iota : \Sigma \hookrightarrow M$ is far from being a submersion the restriction $\iota^* u$ is *not at all* well-defined. To see the problem one should try to define $\iota^*\delta$ for the δ distribution on $\mathbb{R}$ and $\iota : \{0\} \hookrightarrow \mathbb{R}$. Thus for the Cauchy problem to make sense we either have to specify conditions on u and v which ultimately allow to define $\iota^* u$ etc., or we restrict ourselves directly to regular initial conditions and solutions of some $\mathcal{C}^k$-regularity. As usual, the most convenient situation will be the $\mathcal{C}^\infty$-case.

In view of the above remark we will therefore focus on regular and smooth solutions and initial conditions. Thus the Cauchy problem consists in the following task: Given an inhomogeneity $v \in \Gamma^\infty(E)$ we want to find a solution $u \in \Gamma^\infty(E)$ of

$$Du = v \tag{4.2.1}$$

for given initial conditions $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$, i.e.

$$\iota^\# u = u_0, \tag{4.2.2}$$

$$\iota^{\#}\nabla^{E}_{\mathfrak{n}}u = \dot{u}_0. \tag{4.2.3}$$

Here ∇^E will always be the covariant derivative on E determined by D as usual. Note that the left hand side of (4.2.2) is indeed well-defined as for $p \in \Sigma$ the value $\nabla^E_{\mathfrak{n}(p)}u \in E_p$ is defined as ∇^E is function linear in the tangent vector field argument. Thus we can interpret $p \mapsto \nabla^E_{\mathfrak{n}(p)}u$ indeed as a section of $\iota^{\#}E$.

4.2.1 Uniqueness of the Solution to the Cauchy Problem

As for the solutions of the homogeneous wave equation also for the Cauchy problem the uniqueness will be easier to show than the existence. We start with some preparatory material on the adjoint D^{T} of D. Recall from Theorem 1.2.15 that $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$ is determined by

$$\int_M \varphi(Du)\,\mu_g = \int_M (D^{\mathrm{T}}\varphi)u\,\mu_g \tag{4.2.4}$$

for $\varphi \in \Gamma^\infty(E^*)$ and $u \in \Gamma^\infty(E)$ with at least one of them having compact support. We want to compute now D^{T} explicitly.

Lemma 4.2.2 *Let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator written as $D = \Box^\nabla + B$ with $B \in \Gamma^\infty(\mathsf{End}(E))$ and the connection d'Alembertian $\Box^\nabla$ build out of the connection ∇^E defined by D.*

i.) The transposed operator $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$ is given by

$$D^{\mathrm{T}} = \Box^\nabla + B^{\mathrm{T}} \tag{4.2.5}$$

where $\Box^\nabla$ is the connection d'Alembertian with respect to the induced connection ∇^{E^*} for E^* coming from ∇^E.

ii.) For $s \in \Gamma^\infty(E)$ and $\psi \in \Gamma^\infty(E^*)$ we have

$$\Box(\psi(s)) = (\Box^\nabla\psi)(s) + \psi(\Box s) + \left\langle g^{-1}, (\mathsf{D}^{E^*}\psi) \vee (\mathsf{D}^E s)\right\rangle. \tag{4.2.6}$$

iii.) For $s \in \Gamma^\infty(E)$ and $\psi \in \Gamma^\infty(E^*)$ we have

$$(D^{\mathrm{T}}\psi)(s) - \psi(Ds) = \mathrm{div}\left(\left((\mathsf{D}^{E*}\psi)(s) - \psi(\mathsf{D}^E s)\right)^{\#}\right). \tag{4.2.7}$$

Proof. For the first part we use Theorem 1.2.21 as well as the result from Example 2.1.24. In this example we found that $\Box^\nabla = (\frac{\mathrm{i}}{\hbar})^2\,\varrho_{\mathrm{Std}}(2g^{-1} \otimes \mathsf{id}_E)$. Since the remaining part B is $\mathcal{C}^\infty(M)$-linear it is clear that $B = \varrho_{\mathrm{Std}}(B)$ in the sense that the tensor field B acts pointwise as endomorphism on sections of E. By Theorem 1.2.21 we have $\varrho_{\mathrm{Std}}(B)^{\mathrm{T}} = \varrho_{\mathrm{Std}}(B^{\mathrm{T}})$ as there are no degrees to be lowered by the divergence operator $\mathrm{div}^{\mathsf{End}(E)}_{\mu_g}$. In fact, we have $\varphi(Bs) = (B^{\mathrm{T}}\varphi)(s)$ by definition of the pointwise transposition from which $\varrho_{\mathrm{Std}}(B)^{\mathrm{T}} = \varrho_{\mathrm{Std}}(B^{\mathrm{T}})$ is immediate. The transpose of $\Box^\nabla$ is more involved: here we need to compute the divergence of $2g^{-1} \otimes \mathsf{id}_E$. First we note that the one-form α measuring the non-parallelness of the integration density μ_g is vanishing thanks to Proposition 2.1.15, *iii.)*. Thus $\mathrm{div}^{\mathsf{End}(E)}_{\mu}$ coincides with the connection divergence $\mathrm{div}^{\mathsf{End}(E)}_{\nabla}$ where we have to use the induced connection on $\mathsf{End}(E)$ coming from ∇^E. Thus we have to compute

$$\begin{aligned}\mathrm{div}^{\mathsf{End}(E)}_{\nabla}(g^{-1} \otimes \mathsf{id}_E) &= \mathrm{i_s}(\mathrm{d}x^i)\nabla^{\mathsf{End}(E)}_{\frac{\partial}{\partial x^i}}(g^{-1} \otimes \mathsf{id}_E)\\ &= \mathrm{i_s}(\mathrm{d}x^i)\left(\nabla_{\frac{\partial}{\partial x^i}} g^{-1} \otimes \mathsf{id}_E + g^{-1} \otimes \nabla^{\mathsf{End}(E)}_{\frac{\partial}{\partial x^i}}\mathsf{id}_E\right)\end{aligned}$$

$$= 0 + 0,$$

since on one hand g^{-1} is parallel for the Levi-Civita connection and on the other hand id_E is a parallel section with respect to $\nabla^{\mathsf{End}(E)}$. In fact, the latter result is just the definition of $\nabla^{\mathsf{End}(E)}$: for $A \in \Gamma^\infty(\mathsf{End}(E))$ and $s \in \Gamma^\infty(E)$ the induced connection $\nabla^{\mathsf{End}(E)}$ is determined by

$$(\nabla^{\mathsf{End}(E)} A)(s) = \nabla^E(As) - A(\nabla^E s).$$

Thus id_E is covariantly constant since the right hand side will be zero for $A = \mathsf{id}_E$. We conclude that

$$\begin{aligned} D^{\mathrm{T}} &= \left(\frac{\mathrm{i}}{\hbar}\right)^2 \varrho_{\mathrm{Std}}(2g^{-1} \otimes \mathsf{id}_E)^{\mathrm{T}} + \varrho_{\mathrm{Std}}(B)^{\mathrm{T}} \\ &= \left(\frac{\mathrm{i}}{\hbar}\right)^2 \varrho_{\mathrm{Std}}(2g^{-1} \otimes \mathsf{id}_{E^*}) + B^{\mathrm{T}} \\ &= \Box^\nabla + B^{\mathrm{T}}, \end{aligned}$$

where now $\Box^\nabla$ is the connection d'Alembertian on E^* with respect to the induced connection ∇^{E^*}. For the second part we first show the following Leibniz rule of $\Box$ with respect to natural pairings, see also Lemma 2.1.25. We compute

$$\begin{aligned} \Box(\psi(s)) &= \frac{1}{2}\left\langle g^{-1}, \mathsf{D}^2(\psi(s))\right\rangle \\ &= \frac{1}{2}\left\langle g^{-1}, \mathsf{D}((\mathsf{D}^{E^*}\psi)(s) + \psi(\mathsf{D}^E s))\right\rangle \\ &= \frac{1}{2}\left\langle g^{-1}, ((\mathsf{D}^{E^*})^2\psi)(s) + 2(\mathsf{D}^{E^*}\psi) \vee (\mathsf{D}^E s) + \psi((\mathsf{D}^E)^2 s)\right\rangle \\ &= (\Box^\nabla\psi)(s) + \left\langle g^{-1}, (\mathsf{D}^{E^*}\psi) \vee (\mathsf{D}^E s)\right\rangle + \psi(\Box^\nabla s), \end{aligned}$$

where we have used the compatibility of the symmetrized covariant derivative operators D, D^E and D^{E^*} with natural pairings. This compatibility is immediate from the definition of these operators, see Proposition 1.1.3, *iii.)*. This shows the second part. For the last part we know from Theorem 1.2.21 that $(D^{\mathrm{T}}\psi)(s) - \psi(Ds)$ vanishes after integrating over M with respect to μ_g. Thus it has to be a divergence of some vector field with respect to μ_g. However, this vector field is only unique up to a divergence free vector field. Thus (4.2.6) gives an explicit representative. First we notice that the contribution of B cancels as $(B^{\mathrm{T}}\psi)(s) - \psi(Bs) = 0$ holds pointwise. Thus we only have to consider $(\Box^\nabla\psi)(s) - \psi(\Box^\nabla s)$. We compute using the compatibility with natural pairing again

$$\begin{aligned} &\mathsf{D}\left(\left(\mathsf{D}^{E^*}\psi\right)(s)\right) - \mathsf{D}\left(\psi\left(\mathsf{D}^E s\right)\right) \\ &= \left(\left(\mathsf{D}^{E^*}\right)^2\psi\right)(s) + \left(\mathsf{D}^{E^*}\psi\right)\left(\mathsf{D}^E s\right) - \left(\mathsf{D}^{E^*}\psi\right)\left(\mathsf{D}^E s\right) - \psi\left(\left(\mathsf{D}^E\right)^2 s\right) \\ &= \left(\left(\mathsf{D}^{E^*}\right)^2\psi\right)(s) - \psi\left(\left(\mathsf{D}^E\right)^2 s\right). \end{aligned}$$

Hence we obtain for the left hand side of (4.2.6)

$$(D^{\mathrm{T}}\psi)(s) - \psi(Ds) = \frac{1}{2}\left\langle g^{-1}, \left(\mathsf{D}^{E^*}\right)^2\psi\right\rangle(s) - \psi\left(\frac{1}{2}\left\langle g^{-1}, \left(\mathsf{D}^E\right)^2 s\right\rangle\right)$$

$$= \frac{1}{2} \left\langle g^{-1}, \mathsf{D} \left(\left(\mathsf{D}^{E^*} \psi \right)(s) - \psi \left(\mathsf{D}^{E} s \right) \right) \right\rangle,$$

since natural pairings commute. Now the one-form in this pairing is determined by

$$\left(\left(\mathsf{D}^{E^*} \psi \right)(s) - \psi \left(\mathsf{D}^{E} s \right) \right)(\chi) = \left(\nabla^{E^*}_{\chi} \psi \right)(s) - \psi \left(\nabla^{E}_{\chi} s \right)$$

for $\chi \in \Gamma^\infty(TM)$. Since g^{-1} is covariantly constant for the Levi-Civita connection, we have in general

$$\frac{1}{2} \left\langle g^{-1}, \mathsf{D}\, \alpha \right\rangle = \operatorname{div}(\alpha^\#)$$

for arbitrary one-forms $\alpha \in \Gamma^\infty(T^*M)$. This completes the proof. □

Now we consider again a small convex open subset $U' \subseteq M$ and a causal open subset $U \subseteq U^{\mathrm{cl}} \subseteq U'$ of sufficiently small volume so that we can use our local fundamental solutions from Chapter 3. The subset U being causal includes the diamonds $J_U(p, q)$ being compact and since it is inside the convex U' there are no causal loops in U. Thus U is globally hyperbolic and by Theorem 2.2.31 we have a smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow U$ in U. In fact, we recall from [45, Thm. 2.14] that every point in M has a neighborhood basis of globally hyperbolic open subsets. Thus we can safely assume the existence of a smooth Cauchy hypersurface in U. Since Σ is spacelike the pull-back of g to Σ gives a *negative* definite metric (beware of our signature convention) which includes a corresponding volume density. We denote this by $\mu_\Sigma \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*\Sigma)$ and use it for integration on Σ. Denote the fundamental solutions of $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$ on U as constructed analogously to the ones of D by $G^\pm_U(p) \in \Gamma^{-\infty}(E^*\big|_U) \otimes E_p$ where $p \in U$. Then we have operators

$$G^\pm_U : \Gamma^\infty_0(E\big|_U) \longrightarrow \Gamma^\infty(E\big|_U), \tag{4.2.8}$$

enjoying properties analogously to the $F^\pm_U$. In particular, we have a dual map

$$(G^\pm_U)' : \Gamma^{-\infty}_0(E^*\big|_U) \longrightarrow \Gamma^{-\infty}(E^*\big|_U), \tag{4.2.9}$$

which restricts to a map

$$(G^\pm_U)' : \Gamma^\infty_0(E^*\big|_U) \longrightarrow \Gamma^\infty(E^*\big|_U) \tag{4.2.10}$$

by Theorem 3.5.17. We will need the difference between the advanced and retarded fundamental solutions. We define the map

$$G_U = G^+_U - G^-_U : \Gamma^\infty_0(E\big|_U) \longrightarrow \Gamma^\infty(E\big|_U), \tag{4.2.11}$$

which gives a dual map

$$G'_U = (G^+_U)' - (G^-_U)' : \Gamma^{-\infty}(E^*\big|_U) \longrightarrow \Gamma^{-\infty}(E^*\big|_U). \tag{4.2.12}$$

On smooth sections $\varphi \in \Gamma^\infty_0(E^*\big|_U)$, viewed as distributional sections, the map G'_U is determined by

$$(G'_U\varphi)(u) = \varphi(G_U(u)) = \int_U \varphi(p) \cdot \left(G^+_U(p)u - G^-_U(p)u \right) \, \mu_g(p), \tag{4.2.13}$$

where $u \in \Gamma^\infty_0(E\big|_U)$ is a test section of $E\big|_U$. Since we know by Theorem 3.5.17 that $G'_U(\varphi)$ is actually a smooth section of $E^*\big|_U$, it makes sense to restrict this section to Σ. Then we obtain the following lemma:

Lemma 4.2.3 *Assume $u \in \Gamma^\infty(E\big|_U)$ is a solution to the homogeneous wave equation $Du = 0$ and let $\varphi \in \Gamma^\infty_0(E^*\big|_U)$. Then we have*

$$\int_U \varphi(p) \cdot u(p) \, \mu_g(p) = \int_\Sigma \left((\nabla^{E^*}_{\mathfrak{n}} G'_U(\varphi)) \cdot u_0(\sigma) - G'_U(\varphi)(\sigma) \cdot \dot{u}_0(\sigma) \right) \mu_\Sigma(\sigma), \tag{4.2.14}$$

where $u_0 = \iota^\# u$, $\dot{u}_0 = \iota^\# \nabla^E_{\mathfrak{n}} u \in \Gamma^\infty(i^\# E)$ are the initial values of u on Σ.

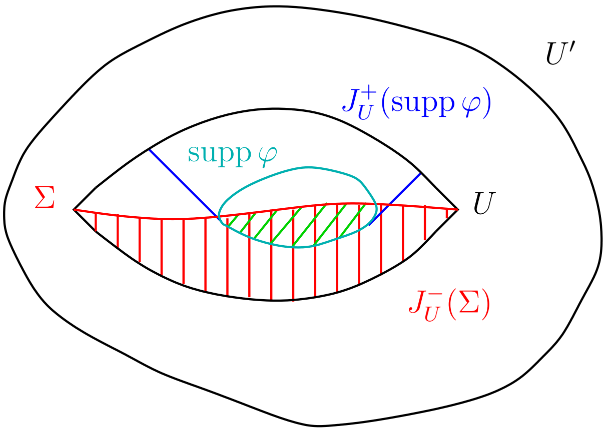

Figure 4.13: Sketch of the situation of the proof for Lemma 4.2.3.

Proof. Let $\varphi \in \Gamma_0^\infty(E^*\big|_U)$ be a test section and let $\psi^\pm = (G_U^\pm)(\varphi) \in \Gamma^{-\infty}(E^*\big|_u)$ which is in $\Gamma^\infty(E^*\big|_U)$ by Theorem 3.5.17. We know from this theorem that $D^\mathrm{T}\psi^\pm = \varphi$ and $\operatorname{supp}\psi^\pm \subseteq J_U^\pm(\operatorname{supp}\varphi)$. For a Cauchy surface Σ and an arbitrary compact subset $K \subseteq U$ one knows that $J_U^\pm(K) \cap J_U^\mp(\Sigma)$ is again compact, see Figure 4.13. For a proof of this fact we refer to [4, Cor. A.5.4] or [45, p. 44]. We know that the (globally hyperbolic) spacetime U decomposes into the disjoint unions

$$U = I_U^-(\Sigma)\dot\cup\Sigma\dot\cup I_U^+(\Sigma),$$

where $I_U^\pm(\Sigma)$ are open and Σ is the common boundary of these open subsets, see Remark 2.2.18. Since we have chosen even a smooth Cauchy hypersurface, we can apply Gauss' Theorem in the form of Theorem B.11 to the vector field

$$X^\pm = \left(\left(\mathsf{D}^{E^*}\psi^\pm\right)(u) - \psi^\pm\left(\mathsf{D}^E u\right)\right)^\# \in \Gamma^\infty(TU).$$

Indeed, this vector field has support in $J_U^\pm(\operatorname{supp}\varphi)$. Thus the integrations over $I_U^\mp(\Sigma)$ and $J_U^\mp(\Sigma)$ as well as over Σ itself are well defined because the integrands all have compact support. We consider first the case of $I_U^-(\Sigma)$. Then the future directed normal vector $\mathfrak{n}$ on Σ points *outwards* whence

$$\int_{I_U^-(\Sigma)} \operatorname{div}(X^+)\,\mu_g = \int_{\partial I_U^-(\Sigma)=\Sigma} g(X^+,\mathfrak{n})\,\mu_\Sigma \tag{$*$}$$

by Theorem B.11. We evaluate both sides explicitly. First we have

$$\int_{I_U^-(\Sigma)} \operatorname{div}(X^+)\,\mu_g = \int_{I_U^-(\Sigma)} \left((D^\mathrm{T}\psi^+)(u) - \psi^+(Du)\right)\mu_g = \int_{I_U^-(\Sigma)} \varphi(u)\,\mu_g,$$

by Lemma 4.2.2 and $Du = 0$ as well as $D^\mathrm{T}\psi^+ = \varphi$. For the right hand side of $(*)$ we get

$$\begin{aligned}
\int_\Sigma g(X^+,\mathfrak{n})\,\mu_\Sigma &= \int_\Sigma \left(g\left(\left(\mathsf{D}^{E*}\psi^+(u)\right)^\#,\mathfrak{n}\right) - g\left(\psi^+\left(\mathsf{D}^E u\right)^\#,\mathfrak{n}\right)\right)\mu_\Sigma \\
&= \int_\Sigma \left(\left(\mathsf{D}^{E^*}\psi^+(u)\right)(\mathfrak{n}) - \left(\psi^+(\mathsf{D}\,u)\right)(\mathfrak{n})\right)\mu_\Sigma \\
&= \int_\Sigma \left(\left(\nabla_\mathfrak{n}^{E^*}\psi^+\right)(u) - \psi^+\left(\nabla_\mathfrak{n}^E u\right)\right)\mu_\Sigma \\
&= \int_\Sigma \left(\left(\nabla_\mathfrak{n}^{E^*}\psi^+\right)(u_0) - \psi^+(\dot u_0)\right)\mu_\Sigma,
\end{aligned}$$

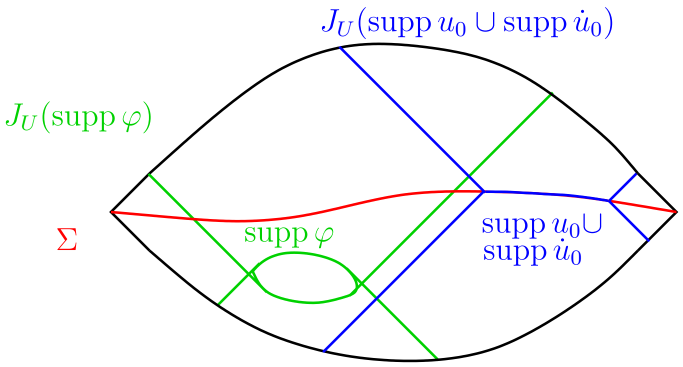

Figure 4.14: The support of the initial data.

where we have omitted the restriction $\iota^\#$ in our notation for the sake of simplicity. Analogously, we obtain for $I_U^+(\Sigma)$ the result

$$\int_{I_U^+(\Sigma)} \operatorname{div}(X^-)\, \mu_g = -\int_\Sigma g(X^-, \mathfrak{n})\, \mu_\Sigma, \tag{$**$}$$

since now $\mathfrak{n}$ is pointing *inwards*. Evaluating both sides gives

$$\int_{I_U^+(\Sigma)} \operatorname{div}(X^-)\, \mu_g = \int_{I_U^+(\Sigma)} \varphi(u)\, \mu_g$$

and

$$-\int_\Sigma g(X^-, \mathfrak{n})\mu_\Sigma = -\int_\Sigma \left(\left(\nabla_{\mathfrak{n}}^{E^*} \psi^- \right)(u_0) - \psi^-(\dot{u}_0) \right) \mu_\Sigma.$$

Thus taking the sum of $(*)$ and $(**)$ gives the equality

$$\int_U \varphi(u)\, \mu_g = \int_\Sigma \left(\nabla_{\mathfrak{n}}^{E^*}(\psi^+ - \psi^-)(u_0) - (\psi^+ - \psi^-)(\dot{u}_0) \right) \mu_\Sigma,$$

which is (4.2.14) by the definition of ψ^+ and ψ^-. □

Lemma 4.2.4 *Assume $u \in \Gamma^\infty(E\big|_U)$ is a solution to the homogeneous wave equation $Du = 0$ and let $u_0, \dot{u}_0 \in \Gamma^\infty(\iota^\# E)$ denote the initial values of u on Σ. Then*

$$\operatorname{supp} u \subseteq J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0). \tag{4.2.15}$$

Proof. We determine the support of u viewed as distributional section. This will coincide with the true support thanks to Remark 1.3.15, *i.)*. Thus let $\varphi \in \Gamma_0^\infty(E^*\big|_U)$ be a test section. Then we know that $\operatorname{supp}(G_U^\pm)'(\varphi) \subseteq J_U^\pm(\operatorname{supp}\varphi)$ by Lemma 3.5.4. It follows that $G_U'(\varphi)$ has its support in $J_U(\operatorname{supp}\varphi)$. Suppose that $\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0$ will not intersect $J_U(\operatorname{supp}\varphi)$, see Figure 4.14. Then this is equivalent to say that $\operatorname{supp}\varphi$ does not intersect $J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0)$. But by (4.2.14) the integral over Σ is clearly 0 whence $\int_U \varphi(u)\mu_g = 0$ follows. Thus u, viewed as distribution, vanishes on all these φ where $\operatorname{supp}\varphi \cap J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0) = \emptyset$. But this means $\operatorname{supp} u \subseteq J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0)^{\mathrm{cl}}$. It remains to show that $J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0)$ is closed. In fact, this is true in general as we shall sketch now: Let $A \subseteq \Sigma$ be closed and consider $J_M^+(A)$ for simplicity. Let $p_n \in J_M^+(A)$ be a sequence of points with $p_n \longrightarrow p \in M$. Choose a point q in the chronological future of p, i.e. we have $p \in I_M^-(q)$. Since $I_M^-(q)$ is open, all but finitely many p_n are in $I_M^-(q)$ whence q is in the chronological future of these p_n. Thus in particular $q \in J_M^+(A)$ as we can join the curves from A to p_n and then from p_n to q.

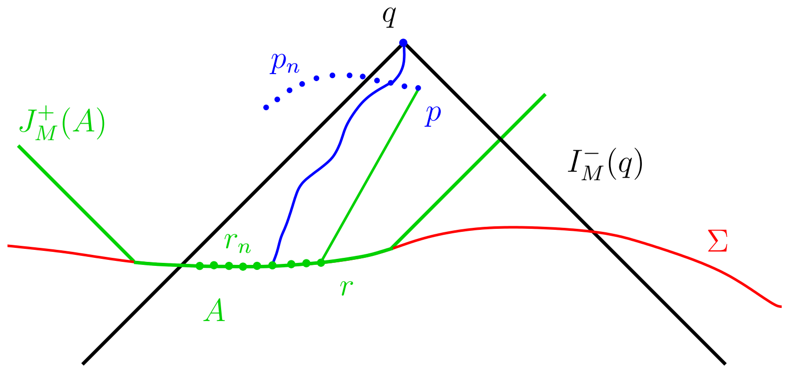

Figure 4.15: The causal influence of a closed set $A \subseteq \Sigma$ in a Cauchy hypersurface is closed again.

Now we find causal curves γ_n from p_n through Σ entirely inside $J_M^+(A)$ giving us a point $r_n \in \Sigma \cap A$. Since these curves are in the cone $J_M^-(q)$ we have $r_n \in \Sigma \cap J_M^-(q)$. For a Cauchy hypersurface one knows that $\Sigma \cap J_M^-(q)$ is always compact. Thus also $A \cap \Sigma \cap J_M^-(q)$ is compact and hence the r_n converge to some $r \in A$ after passing to a suitable subsequence. But then the curves γ_n converge to some limiting curve γ joining r with p, see [46, Lemma 14.14] for details on the notion of limiting curves. By continuity γ is still causal and thus $p \in J_M^+(A)$, see Figure 4.15. The argument for $J_M^-(A)$ is analogous. □

Later on we will be interested in those $u \in \Gamma^\infty(E\big|_U)$ where the initial values $u_0, \dot{u}_0 \in \Gamma^\infty(\iota^\# E)$ have *compact support* in Σ.

Let us now prove the uniqueness property of the Cauchy problem. Lemma 4.2.3 states that *locally* on U the solution u of the wave equation is determined by its initial values u_0 and $\dot{u}_0$ on Σ, since the left hand side of (4.2.14) determines u as a distribution and hence by the injective embedding according to Remark 1.3.5 also as a section. Thus we need to globalize this uniqueness statement.

Theorem 4.2.5 *Let (M, g) be globally hyperbolic and let $\iota : \Sigma \hookrightarrow M$ be a smooth spacelike Cauchy hypersurface with future directed normal vector field $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$. Assume that $u \in \Gamma^\infty(E)$ is a solution to the wave equation $Du = 0$ with initial conditions*

$$u_0 = 0 = \dot{u}_0. \tag{4.2.16}$$

Then

$$u = 0. \tag{4.2.17}$$

Proof. First we note that by Theorem 2.2.31 there is a Cauchy temporal function $\mathfrak{t}$ on M such that the level surface for $\mathfrak{t} = 0$ coincides with Σ. We set

$$\iota_t : \Sigma_t = \left\{ p \in M \;\middle|\; \mathfrak{t}(p) = t \right\} \hookrightarrow M$$

for all times $t \in \mathbb{R}$. The gradient of $\mathfrak{t}$ is by definition future directed and timelike and for a tangent vector $v_p \in T_p\Sigma_t$ we have $\mathrm{d}\,\mathfrak{t}\big|_p(v_p) = 0$ whence the gradient of $\mathfrak{t}$ is orthogonal to $T_p\Sigma_t$ at $p \in \Sigma_t$. Normalizing the gradient will give a globally defined vector field $\mathfrak{n} \in \Gamma^\infty(TM)$ such that for every $t \in \mathbb{R}$ the restriction $\mathfrak{n}_t = \iota_t^\# \mathfrak{n} \in \Gamma^\infty(\iota_t^\# TM)$ is the future directed normal vector field of Σ_t. Now let $p \in M$ be given and let $t_0 = \mathfrak{t}(p)$ be its time value, i.e. $p \in \Sigma_{t_0}$. Assume $t_0 > 0$ (the case $t_0 < 0$ is treated analogously). Then we define

$$t_{\max} = \sup \left\{ t \in [0, t_0] \;\middle|\; u \text{ vanishes on } J_M^-(p) \cap \cup_{0 \le \tau \le t} \Sigma_\tau \right\}.$$

Since u vanishes on Σ_0 this is well-defined and we have $0 \le t_{\max} \le t_0$, see also Figure 4.16. The

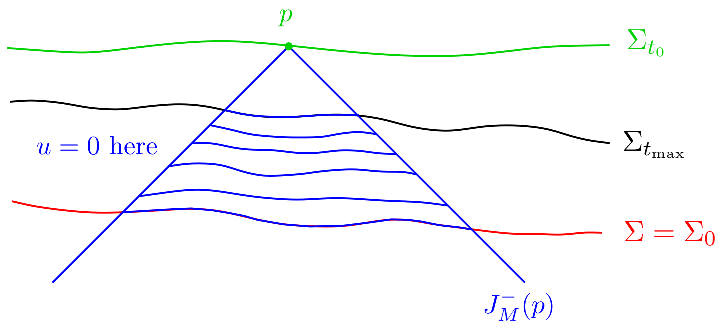

Figure 4.16: The definition of $t_{\max}$

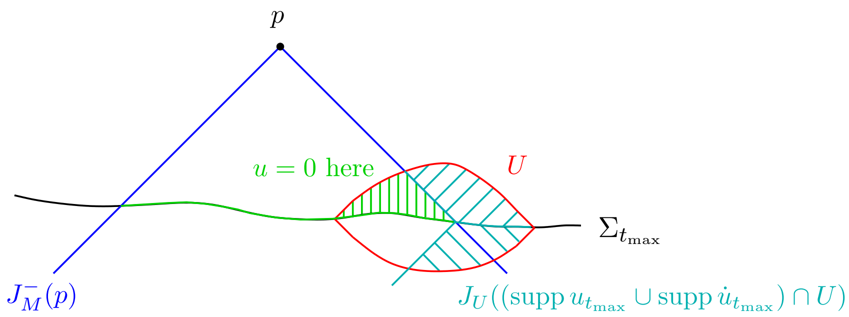

Figure 4.17: Showing that u is zero locally above $\Sigma_{\max}$.

idea is now to show $t_{\max} = t_0$ whence by continuity u vanishes also at p. As p was arbitrary this will imply $u = 0$ everywhere for positive times. Then the analogous argument would give $u = 0$ also for negative times. Thus let us assume the controversy, i.e. $t_{\max} < t_0$. Let $q \in J_M^-(p) \cap \Sigma_{t_{\max}}$, then we can find a small open causal neighborhood $U \subseteq U^{\mathrm{cl}} \subseteq U'$ of q such that on one hand we have our local fundamental solutions and on the other hand $U \cap \Sigma_{t_{\max}}$ is still a Cauchy hypersurface. Note that this additional requirement can still be achieved, see e.g. [4, Lem. A.5.6]. In fact, the Cauchy development $D(V)$ of a small enough open neighborhood $q \in V \subseteq \Sigma_{t_{\max}}$ of q in $\Sigma_{t_{\max}}$ will do the job, see also Remark 2.2.22. We consider the initial values of u on this Cauchy hypersurface and denote them by $u_{t_{\max}} = \iota^{\#}_{t_{\max}} u$ and $\dot{u}_{t_{\max}} = \iota^{\#}_{t_{\max}} \nabla^E_{\mathfrak{n}(p)} u$ as usual. From Lemma 4.2.4 we know that u restricted to the small open subset U has the following property

$$\operatorname{supp} u \subseteq J_U\left(\operatorname{supp} u_{t_{\max}} \cup \operatorname{supp} \dot{u}_{t_{\max}} \cap U\right).$$

Now by continuity and the choice of $t_{\max}$ we know that $u_{t_{\max}} = 0 = \dot{u}_{t_{\max}}$ on $\Sigma_{t_{\max}} \cap J_M^-(p)$. In particular, $u_{t_{\max}} = 0 = \dot{u}_{t_{\max}}$ in the open subset $U \cap \Sigma_{t_{\max}} \cap J_M^-(p)$ of $\Sigma_{t_{\max}}$, see Figure 4.17. But then Lemma 4.2.4 shows that u still vanishes on $J_M^-(p) \cap J_M^+(\Sigma_{t_{\max}} \cap U)$, i.e. in this part of U which is above $\Sigma_{t_{\max}}$ and in the past of p. Since $J_M^-(p) \cap \Sigma_{t_{\max}}$ is compact we can cover this part of the Cauchy hypersurface $\Sigma_{t_{\max}}$ with finitely many $U_1, \ldots, U_N$ for which the above argument applies. Now the union $U_1 \cup \ldots \cup U_N$ is an open neighborhood of $J_M^-(p) \cap \Sigma_{t_{\max}}$ and hence u vanishes on this open subset $(U_1 \cup \ldots \cup U_N) \cap J_M^-(p) \cap J_M^+(\Sigma_{t_{\max}})$ in the future of $J_M^+(\Sigma_{t_{\max}})$. But this means that there is an $\epsilon > 0$ such that on $\Sigma_t \cap J_M^-(p)$ the section u still vanishes for all $t \in [t_{\max}, t_{\max} + \epsilon)$. This is in contradiction to the maximality of $t_{\max}$ and hence $t_{\max} = t_0$ whence $u(p) = 0$ by continuity. This shows that $u = 0$ on $J_M^+(\Sigma)$ and an analogous argument gives $u = 0$ on $J_M^-(\Sigma)$. □

As this is one of the central theorems we give an alternative proof of the uniqueness statement. In particular, it will give some new insight and an additional technique which turns out to be useful also at other places.

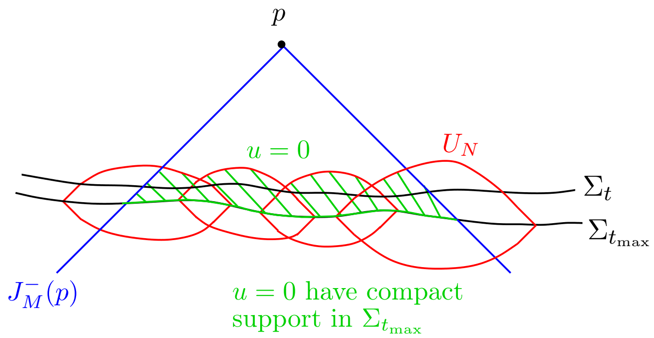

Figure 4.18: Showing that u is zero in a small neighborhood of $\Sigma_{t_{\max}}$.

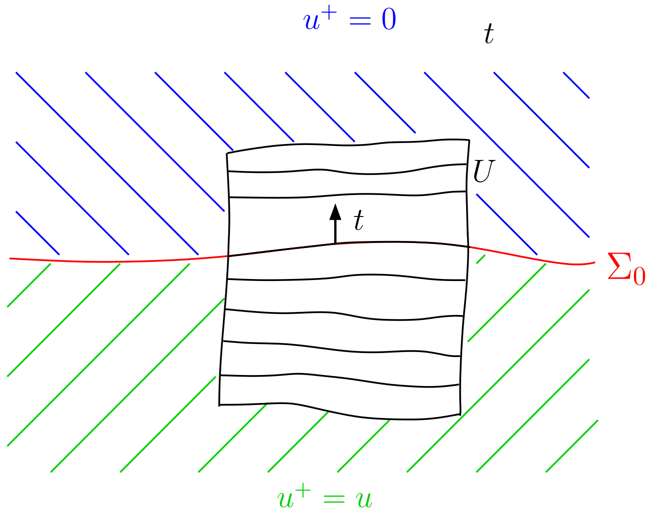

Figure 4.19: The neighborhood U.

Alternative Proof of Theorem 4.2.5. Again we use a foliation of M by smooth spacelike Cauchy hypersurfaces Σ_t where for each $t \in \mathbb{R}$ the set Σ_t is the level hypersurface of a Cauchy temporal function as before. We define now

$$u^+(p) = \begin{cases} u(p) & \text{for } t(p) \leq 0 \\ 0 & \text{for } t(p) > 0 \end{cases},$$

and claim that this is a $\mathcal{C}^2$-section still satisfying the wave equation $Du^+ = 0$. Since $M = I_M^+(\Sigma_0) \cup \Sigma_0 \cup I_M^-(\Sigma_0)$ with open $I_M^\pm(\Sigma_0)$ and Σ_0 the common boundary of $I_M^\pm(\Sigma_0)$ we can check the regularity of u^+ on each piece. Clearly on $I_M^\pm(\Sigma_0)$ we have $u^+\big|_{I_M^\pm(\Sigma_0)} \in \Gamma^\infty(E\big|_{I_M^\pm(\Sigma_0)})$ and $Du^+\big|_{I_M^\pm(\Sigma_0)} = 0$. Thus we only have to check that u^+ is $\mathcal{C}^2$ at Σ_0, then by continuity $Du^+ = 0$ will follow everywhere. Thus let $p \in \Sigma_0$ and choose a small open neighborhood $V \subseteq \Sigma_0$ of p allowing for local coordinates $x^1, \ldots, x^{n-1}$ and a trivialization of the bundled $E\big|_{\Sigma_0}$. By the splitting theorem we have an open neighborhood $U \subseteq M$ of p such that the time function t gives a diffeomorphism $U \simeq (-\epsilon, \epsilon) \times V$ and the metric $g\big|_U$ is given by

$$g\big|_U = \beta \, \mathrm{d}t^2 - g_t$$

with $\beta \in \mathcal{C}^\infty(U)$ positive and g_t a smooth time-dependent metric on Σ_0, see Theorem 2.2.31. In fact, we have this block diagonal structure even globally, see also Figure 4.19. Now $u_0 = 0$ implies that u^+ is continuous at Σ_0. Moreover, all partial derivatives of u in $x^1, \ldots, x^{n-1}$ direction vanish on Σ_0 and hence the partial derivative of u^+ in $x^1, \ldots, x^{n-1}$ directions are continuous as well. The block

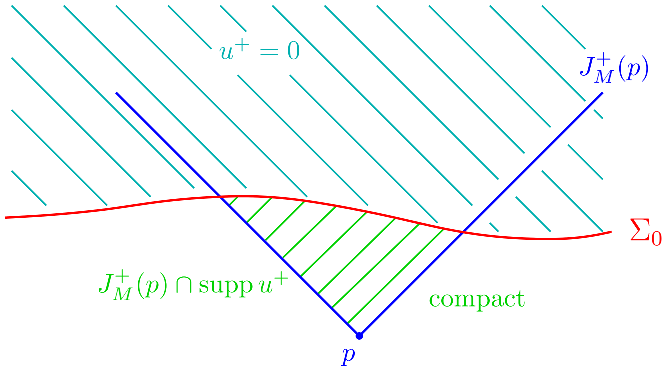

Figure 4.20: The section u^+ has future compact support.

diagonal form of the metric shows that $\frac{\partial}{\partial t}$ is parallel to $\mathfrak{n}$ at Σ_0 whence the condition $\dot{u}_0 = 0$ means that the partial $\frac{\partial}{\partial t}$-derivative of u vanishes at Σ_0. Indeed this differs (in our trivialization) from the covariant derivative by $\mathcal{C}^\infty(M)$-linear combinations of the components of u_0, which vanish by $u_0 = 0$. We conclude that u^+ is $\mathcal{C}^1$. For the second derivative we first observe that the contributions $\frac{\partial^2}{\partial x^i \partial x^j} u$ all vanish on Σ_0 since $u_0 = 0$ is constant. Moreover, since u is $\mathcal{C}^2$, the contributions $\frac{\partial}{\partial t}\frac{\partial}{\partial x^i} u = \frac{\partial}{\partial x^i}\frac{\partial}{\partial t} u$ vanish on Σ_0 since $\frac{\partial}{\partial t} u = 0$ identically on Σ_0. For the last combination $\frac{\partial^2}{\partial t^2} u$ we have to use the wave equation. Locally the wave equation reads

$$\left(\frac{1}{\beta} \frac{\partial^2}{\partial t^2} - g_t^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right) u + a \frac{\partial u}{\partial t} + b^i \frac{\partial u}{\partial x^i} + Bu = 0,$$

where g_t^{ij} is the inverse metric to the metric g_t on Σ_t, and a, b^i, B are coefficient functions. Evaluating this on Σ_0 using the previous results gives $\frac{\partial^2 u}{\partial t^2} = 0$ on Σ_0. Thus the second partial derivatives are also continuous in this local chart. It follows that u^+ is $\mathcal{C}^2$. By continuity it follows that $Du^+ = 0$ everywhere. But then Theorem 4.1.11 gives immediately $u^+ = 0$ since clearly u^+ has future compact support, see Figure 4.20, and M being globally hyperbolic fulfills the conditions of Theorem 4.1.11. But this implies $u\big|_{I^-_M(\Sigma_0)} = 0$. An analogous argument for

$$u^-(p) = \begin{cases} 0 & p \in I^-_M(\Sigma_0) \\ u(p) & p \in I^+_M(\Sigma_0) \end{cases}$$

shows that $u\big|_{I^+_M(\Sigma_0)} = 0$ as well. □

Remark 4.2.6 The alternative proof gives yet another interpretation of Cauchy hypersurfaces. They are the hypersurfaces Σ along which solutions of the wave equation can be sewed together if they match on Σ. The argument in this approach will be used at several instances again.

In view of the alternative proof we see that the uniqueness of the solution to the Cauchy Problem is a direct consequence of Theorem 4.1.11 alone. The considerations in Section 4.2.1 before are not needed. Moreover, since Theorem 4.1.11 works even for distributional sections $u \in \Gamma^{-\infty}(E)$ the regularity needed for the uniqueness is actually much smaller than $\mathcal{C}^\infty$:

Theorem 4.2.7 *Let (M, g) be globally hyperbolic and let $\iota : \Sigma \hookrightarrow M$ be a smooth spacelike Cauchy hypersurface with future directed normal vector field $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$. Let $v \in \Gamma^0(E)$ be a continuous section and $u \in \Gamma^2(E)$ a $\mathcal{C}^2$-section satisfying the inhomogeneous wave equation*

$$Du = v. \tag{4.2.18}$$

Then u is uniquely determined by its initial conditions $u_0 = \iota^\# u$ and $\dot{u}_0 = \iota^\# \nabla^E_{\mathfrak{n}} u$ on Σ.

Proof. Requiring $u \in \Gamma^2(E)$ is the minimal requirement to view (4.2.18) as a *pointwise* equation. In fact, since continuous sections still embed into $\Gamma^{-\infty}(E)$ we also have $Du = v$ in the sense of distributional sections. Suppose $\widetilde{u} \in \Gamma^2(E)$ is an alternative solution with the same initial conditions. Then $u - \widetilde{u}$ is a $\mathcal{C}^2$-solution of the homogeneous wave equation. For this we can repeat the argument from the alternative proof of Theorem 4.2.5 since we only needed $\mathcal{C}^2$ there. Thus $u - \widetilde{u} = 0$ as distributions by Theorem 4.2.5 and hence $u - \widetilde{u} = 0$ as $\mathcal{C}^2$-sections as well. □

4.2.2 Existence of Local Solutions to the Cauchy Problem

After the uniqueness we pass to the existence of solutions to the Cauchy problem. We will assume that the Cauchy data as well as the inhomogeneity of the wave equation have compact support.

The first statement is still a local result to the Cauchy problem:

Proposition 4.2.8 *Let (M, g) be a time-oriented Lorentz manifold with a smooth spacelike hypersurface $\iota : \Sigma \hookrightarrow M$ with future directed normal vector field $\mathfrak{n}$. Moreover, let $U \subseteq U^{\mathrm{cl}} \subseteq U'$ be a sufficiently small causal open subset of M such that $\Sigma \cap U \hookrightarrow U$ is a Cauchy hypersurface for U. Then there exists a unique solution $u \in \Gamma^\infty(E\big|_U)$ for given initial values $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E\big|_U)$ and given inhomogeneity $v \in \Gamma_0^\infty(E\big|_U)$ of the inhomogeneous wave equation*

$$Du = v \tag{4.2.19}$$

with $\iota^\# u = u_0$ and $\iota^\# \nabla^E_{\mathfrak{n}} u = \dot{u}_0$. In addition we have

$$\operatorname{supp} u \subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0 \cup \operatorname{supp} v). \tag{4.2.20}$$

Proof. As usual, sufficiently small means that we have our local fundamental solutions and therefor the result of Chapter 3. The uniqueness of u follows directly from Theorem 4.2.5. We can apply the splitting theorem for globally hyperbolic manifolds in the form of Theorem 2.2.31 to U, see also [45, Thm. 2.78]. Thus we find a Cauchy temporal function t on U inducing an isometry of U to $\mathbb{R} \times (\Sigma \cap U)$ such that the metric becomes $\beta\, \mathrm{d}t^2 - g_t$ with $\beta \in \mathcal{C}^\infty(U)$ positive and g_t a time dependent Riemannian metric on $\Sigma \cap U$. Every t-level surface is Cauchy and we have the normal vector field

$$\mathfrak{n} = \frac{1}{\sqrt{\beta}} \frac{\partial}{\partial t} \in \Gamma^\infty(TU),$$

which is normal to every level surface. Moreover, since by definition $U \subseteq U'$ is contained in a *convex* domain U' the vector bundle E is trivializable over U' and hence over U. Therefore we can choose a frame $\{e_\alpha\}$ over U of $E\big|_U$ and write $u = u^\alpha e_\alpha$ with smooth functions $u^\alpha \in \mathcal{C}^\infty(U)$ for every $u \in \Gamma^\infty(E\big|_U)$. This allows to identify a section u with a collection of scalar function u^α. The normally hyperbolic operator D is now of the form

$$D = \frac{1}{\beta} \frac{\partial^2}{\partial t^2} + \widetilde{D}, \tag{$*$}$$

where $\widetilde{D}$ contains at most first t-derivatives, still up to second derivatives in Σ-directions, and it has matrix-valued coefficient functions with respect to our trivialization induced by the e_α. We claim now that the initial conditions together with the wave equation determine all t-derivatives of a solution along Σ. The argument is similar to the proof of Theorem 4.2.5. Suppose u is a smooth solution of $Du = v$ with initial conditions u_0 and $\dot{u}_0$. We already know that $\dot{u}_0$ is determined by u_0 and $\frac{\partial u}{\partial t}\big|_\Sigma$ and conversely $\frac{\partial u}{\partial t}\big|_\Sigma$ is determined by $\dot{u}_0$ and u_0. Using $(*)$ we see that

$$\frac{\partial^2 u}{\partial t^2} = \beta(Du - \widetilde{D}u) = \beta(v - \widetilde{D}u). \tag{$**$}$$

This shows that $\frac{\partial^2 u}{\partial t^2}\big|_\Sigma$ is determined by u_0 and $\frac{\partial u}{\partial t}\big|_\Sigma$, namely we have

$$\frac{\partial^2 u}{\partial t^2}\Big|_\Sigma = (\beta v)\Big|_\Sigma - (\beta \widetilde{D} u)\Big|_\Sigma,$$

where the right hand side uses only u_0 and $\frac{\partial u}{\partial t}\big|_\Sigma$ since $\widetilde{D}$ is at most of first order in the t-variable. Moreover, differentiating $(**)$ j-times we get

$$\frac{\partial^{j+2} u}{\partial t^{j+2}} = \frac{\partial^j (\beta v)}{\partial t^j} - \frac{\partial^j}{\partial t^j}(\beta \widetilde{D} u).$$

Hence on Σ we have

$$\frac{\partial^{j+2} u}{\partial t^{j+2}}\Big|_\Sigma = \frac{\partial^j (\beta v)}{\partial t^j}\Big|_\Sigma - \frac{\partial^j}{\partial t^j}(\beta \widetilde{D} u)\Big|_\Sigma. \tag{$***$}$$

We see that the right hand side is a $\mathcal{C}^\infty(\Sigma)$-linear combination of the $u_0, \frac{\partial u}{\partial t}\big|_\Sigma, \ldots, \frac{\partial^{j+1} u}{\partial t^{j+1}}\big|_\Sigma$ plus an affine term $\frac{\partial^j(\beta v)}{\partial t^j}\big|_\Sigma$. Thus by induction we conclude that all t-derivatives of u on Σ are determined by u_0 and $\frac{\partial u}{\partial t}\big|_\Sigma$, and of course by the choice of the inhomogeneity v. Moreover, since we have a $\mathcal{C}^\infty(\Sigma)$-affine linear combination we conclude that

$$\operatorname{supp}\left(\frac{\partial^j u}{\partial t^j}\Big|_\Sigma\right) \subseteq \Bigg(\underbrace{\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0 \cup \operatorname{supp} v}_{K}\Bigg) \cap \Sigma = K \cap \Sigma$$

is contained in a compact subset $K' = K \cap \Sigma$ of Σ for all j. Now we use these recursion formulas to *define* sections $u_j \in \Gamma^\infty(E\big|_\Sigma)$ by $(***)$ for all $j \geq 2$. First we note that we indeed can find a global section $\widetilde{u} \in \Gamma^\infty(E)$ whose t-derivatives on Σ are given by the u_j: this is essentially a consequence of the Borel Lemma for Fréchet spaces, see e.g. [60, Satz 5.3.33]. For convenience we repeat the argument here: We choose a cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subseteq [-1,1]$ and $\chi\big|_{[-\frac{1}{2},\frac{1}{2}]} = 1$. As we did frequently in Section 3.4 we consider as Ansatz a series

$$\widetilde{u}(t,p) = \sum_{j=0}^\infty \chi\left(\frac{t}{\epsilon_j}\right)\frac{t^j}{j!}\, u_j(p) \tag{$\star$}$$

with numbers $0 < \epsilon_j \leq 1$ yet to be chosen. We want to choose them in such a way that the series converges in the $\mathcal{C}^\infty$-topology of $\Gamma^\infty(E\big|_U)$. Clearly, each term has support in $[-1,1] \times K'$ whence we only have to consider the seminorms of $\Gamma^\infty(E\big|_U)$ estimating derivatives on this compactum. It is clear from the Ansatz and the properties of χ that if we have $\mathcal{C}^\infty$-convergence then $\frac{\partial^j \widetilde{u}}{\partial t^j}\big|_{t=0} = u_j$ for all j. Thus let us estimate the k-th seminorm $\mathrm{p}_{[-1,1]\times K',k}$ of each term of $(\star)$. With the usual Leibniz rule and the fact that the seminorms factorize on factorizing functions we get from Lemma 3.4.1

$$\mathrm{p}_{[-1,1]\times K',k}\left(\chi\left(\frac{t}{\epsilon_j}\right)\frac{t^j}{j!}\, u_j\right) \leq \frac{\epsilon_j}{j!}\,\mathrm{p}_{[-1,1],k}(\chi)\,\mathrm{p}_{K',k}(u_j).$$

This allows to choose the ϵ_j such that

$$\epsilon_j \max_{k\leq j} \mathrm{p}_{[-1,1],k}(\chi)\,\mathrm{p}_{K',k}(u_j) < 1.$$

Then the series $(\star)$ converges in the $\mathcal{C}^k$-norm $\mathrm{p}_{[-1,1]\times K',k}$ absolutely as the first terms do not spoil the convergence. Thus we have absolute $\mathcal{C}^\infty$-convergence in total. This shows the existence of a $\widetilde{u} \in \Gamma^\infty(E\big|_U)$ with

$$\frac{\partial^j \widetilde{u}}{\partial t^j}\Big|_\Sigma = u_j$$

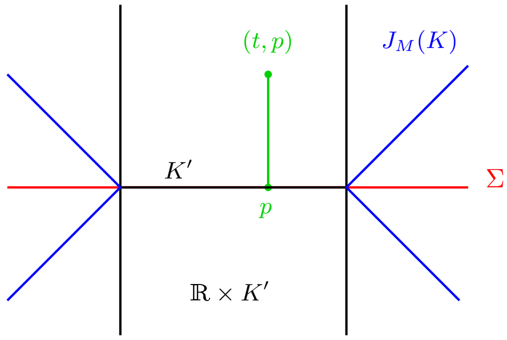

Figure 4.21: The splitting yields simple timelike curves.

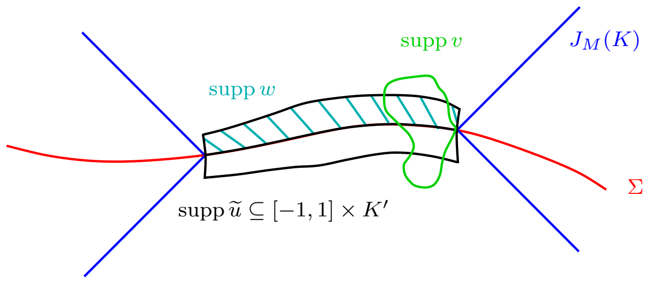

Figure 4.22: The supports of the several sections in the proof of Proposition 4.2.8.

and

$$\operatorname{supp} \widetilde{u} \subseteq J_M(K).$$

Indeed, the last claim follows from the fact that $\operatorname{supp} \widetilde{u} \subseteq [-1, 1] \times K'$ and $\mathbb{R} \times K' \subseteq J_M(K)$ since for every $(t, p) \in \mathbb{R} \times K'$ the curve $\tau \mapsto (\tau, p)$ connects $(0, p)$ to (t, p) and the curve is clearly timelike. This follows from the splitting of the metric, see also Figure 4.21. From the construction of $\widetilde{u}$ we see that $D\widetilde{u}$ coincides with v including *all* time derivatives on Σ. In other words, $D\widetilde{u} - v$ vanishes on Σ up to infinite order. Thus we can consider the definition

$$w_\pm = \begin{cases} D\widetilde{u} - v & \text{on } I_M^\pm(\Sigma) \\ 0 & \text{on } J_M^\mp(\Sigma), \end{cases}$$

which gives a *smooth* section $w_\pm \in \Gamma^\infty(E\big|_U)$. Since both $\widetilde{u}$ and v have compact support, also $w_\pm$ is compactly supported. Thus we can solve the inhomogeneous wave equation

$$D\widetilde{\widetilde{u}}_\pm = w_\pm$$

on the open subset U according to Theorem 3.5.17 with a smooth solution $\widetilde{\widetilde{u}}_\pm \in \Gamma^\infty(E\big|_U)$ such that $\operatorname{supp} \widetilde{\widetilde{u}}_\pm \subseteq J_U^\pm(\operatorname{supp} w_\pm)$. Since $\operatorname{supp} w \subseteq (\operatorname{supp} D\widetilde{u} \cup \operatorname{supp} v) \cap J^+(\Sigma) \subseteq J_M^+(K)$ we conclude $J_M^+(\operatorname{supp} w) \subseteq J_M^+(K)$. This shows that $\operatorname{supp} \widetilde{\widetilde{u}}_\pm \subseteq J_M^\pm(K) \cap U = J_U^\pm(K)$. In particular, $\widetilde{\widetilde{u}}\big|_{J_M^\mp(\Sigma)} = 0$. Now we consider the smooth section $u_\pm \in \Gamma^\infty(E\big|_U)$ defined by

$$u_\pm = \widetilde{u} - \widetilde{\widetilde{u}}_\pm.$$

Since $\widetilde{\widetilde{u}}_\pm$ vanishes on $J_M^\mp(\Sigma)$ we have $u_\pm = \widetilde{u}$ on $J_M^\mp(\Sigma)$. In particular, $u_\mp$ coincides with $\widetilde{u}$ up to all orders on Σ by continuity of the t-derivatives. Thus $u_\pm$ satisfies the correct initial conditions. Moreover, on $I_U^\pm(\Sigma)$ we have

$$Du_\pm\big|_{I_U^\pm(\Sigma)} = D\widetilde{u}\big|_{I_U^\pm(\Sigma)} - D\widetilde{\widetilde{u}}_\pm\big|_{U_U^\pm(\Sigma)} = (w + v)\big|_{I_U^\pm(\Sigma)} - w\big|_{I_U^\pm(\Sigma)} = v\big|_{I_U^\pm}(\Sigma),$$

whence on this open part of U the section $u_\pm$ solves the inhomogeneous wave equation. Since both u_+ and u_- agree on Σ up to infinite orders, as they agree with $\widetilde{u}$, we can glue them together and set

$$u = \begin{cases} u_+ & \text{on } I_U^+(\Sigma) \\ u_- & \text{on } I_U^-(\Sigma). \end{cases}$$

On one hand, this yields a smooth section $u \in \Gamma^\infty(E\big|_U)$ on all of U. Moreover, u solves the inhomogeneous wave equation on both open parts $I_U^\pm(\Sigma)$ and hence on all of U by continuity. Finally, we know that

$$\operatorname{supp}(u_\pm) \subseteq \operatorname{supp}\widetilde{u} \cup \operatorname{supp}\widetilde{\widetilde{u}}_\pm \subseteq J_U(K) \cup J_U^\pm(K) = J_U(K),$$

whence also $\operatorname{supp} u \subseteq J_U(K)$. This completes the proof. □

We can refine the above argument for finite order of differentiability. Here on one hand the Borel-Lemma is not needed as we can simply take a polynomial in t multiplied by the cut-off function in order to have compact support. On the other hand, we have to count orders of differentiation carefully:

Proposition 4.2.9 *Let $k \geq 2$. Under the same general assumptions as in Proposition 4.2.8 we assume to have initial values $u_0 \in \Gamma_0^{2(k+n+1)+2}(\iota^\# E\big|_U)$, $\dot{u}_0 \in \Gamma_0^{2(k+n+1)+1}(\iota^\# E\big|_U)$ and an inhomogeneity $v \in \Gamma_0^{2(k+n+1)}(E\big|_U)$. Then there exists a unique solution $u \in \Gamma^k(E\big|_U)$ of the inhomogeneous wave equation*

$$Du = v \tag{4.2.21}$$

with initial conditions $\iota^\# v = u_0$ and $\iota^\# \nabla_\mathfrak{n} u = \dot{u}_0$. For the support we still have

$$\operatorname{supp} u \subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0 \cup \operatorname{supp} v). \tag{4.2.22}$$

Proof. As in the proof of Proposition 4.2.8 we define the sections u_j recursively by

$$\frac{\partial^{j+1} u}{\partial t^{j+1}}\Big|_\Sigma = \frac{\partial^j(\beta v)}{\partial t^j}\Big|_\Sigma - \frac{\partial^j}{\partial t^j}(\beta \widetilde{D} u)\Big|_\Sigma \tag{$*$}$$

with $u_j = \frac{\partial^j u}{\partial t^j}\big|_\Sigma$. Since for the right hand side we only have up to $j+1$ time derivatives we need $u_0, u_1, \ldots, u_{j+1}$ in order to determine u_{j+2}. In the local coordinates on U we split the operator $\widetilde{D}$ into $\widetilde{D} = D_2 + D_1 \frac{\partial}{\partial t}$ where D_2, D_1 are operators differentiating only in spacial directions. The coefficients of D_2, D_1 depend on all variables and D_2 is of order two while D_1 is of order one. Then the recursion $(*)$ for $u = u_0 + t u_1 + \frac{t^2}{2} u_2 + \ldots$ can be written as

$$\begin{aligned} u_{j+1} &= \frac{\partial^j}{\partial t^j}(\beta v)\Big|_{t=0} - \frac{\partial^j}{\partial t^j} \beta D_2 \sum_{k=0}^j \frac{t^k}{k!} u_k\Big|_{t=0} - \frac{\partial^j}{\partial t^j} \beta D_1 \frac{\partial}{\partial t} \sum_{k=0}^j \frac{t^{k+1}}{(k+1)!} u_{k+1}\Big|_{t=0} \\ &= \frac{\partial^j}{\partial t^j}(\beta v)\Big|_{t=0} - \sum_{k=0}^j \binom{j}{k} \frac{\partial^{j-k}}{\partial t^{j-k}}(\beta D_2)\Big|_{t=0} u_k - \sum_{k=0}^j \binom{j}{k} \frac{\partial^{j-k}}{\partial t^{j-k}}(\beta D_1)\Big|_{t=0} u_{k+1}. \end{aligned} \tag{$**$}$$

Note that $\frac{\partial^{j-k}}{\partial t^{j-k}}(\beta D_2)\big|_{t=0}$ is again a differential operator of order two while $\frac{\partial^{j-k}}{\partial t^{j-k}}(\beta D_1)\big|_{t=0}$ is of order one. This determines u_{j+2} recursively in terms of spacial derivatives of $u_0, \ldots, u_{j+1}$. We claim that u_{j+2} contains at most $j+2$ derivatives of u_0, at most $j+1$ derivatives of u_1 and at most j derivatives of v. Indeed, for $j = 0$ we have

$$u_2 = \beta v\big|_\Sigma - \beta D_2\big|_\Sigma u_0 - \beta D_1\big|_\Sigma u_1,$$

which shows the claim for this j. By inductions we see from (∗∗) that $\frac{\partial^{j-k}}{\partial t^{j-k}}(\beta D_2)\big|_{t=0} u_k$ contains at most $k+2$ derivatives of u_0 and hence at most $j+2$ derivatives since $k = 0, \ldots, j$. Moreover, it contains at most $k+1$ derivatives of u_1 and hence at most $k+1 \leq j+1$. Finally it contains at most $k-1$ derivatives of v and thus also here things match. For the second sum one proceeds analogously. Finally, the first term gives j derivatives of v, which also matches our claim. Now assume we are give u_0 and u_1 of class $\mathcal{C}^{2(k+n+1)+2}$ and $\mathcal{C}^{2(k+n+1)+1}$, respectively. Moreover, suppose $v \in \Gamma_0^{2(k+n+1)}(E\big|_U)$. Then the u_j defined by the recursion (∗) are of class $\mathcal{C}^{2(k+n+1)+2-j}$. Thus the finite sum

$$\widetilde{u}(t,p) = \chi(t) \sum_{j=0}^{k+n+1} \frac{t^j}{j!} u_j(p)$$

gives a section of class at least $\mathcal{C}^{k+n+1+2}$. Moreover, the recursion shows that $D\widetilde{u} - v$ vanishes up to order t^{k+n+1}. Thus gluing this with zero gives a section

$$w_\pm = \begin{cases} D\widetilde{u} - v & \text{on } I_U^\pm(\Sigma) \\ 0 & \text{else,} \end{cases}$$

which is still of class $\mathcal{C}^{k+n+1}$ everywhere. Then $\widetilde{\widetilde{u}}_\pm$ is of class $\mathcal{C}^k$ by Theorem 3.5.17 and thus $u_\pm$ are both of class $\mathcal{C}^k$. Since $\widetilde{\widetilde{u}}_\pm$ is $\mathcal{C}^k$ and vanishes on the open subset $I_U^\mp(\Sigma)$, the $u_\pm = \widetilde{u} - \widetilde{\widetilde{u}}_\pm$ agree with $\widetilde{u}$ on Σ up to order t^k. Thus also the glued solution u is of class $\mathcal{C}^k$ as claimed. The statement about the support is analogous to the smooth case. □

Remark 4.2.10 Having Lemma 4.2.3 in mind, it is tempting to define the solution of the Cauchy problem (at least in the homogeneous case $v = 0$) by the formula (4.2.14): Using instead of a test section φ a δ-functional at p would directly give

$$u(p) = \int_\Sigma \left(\nabla^E_{\mathfrak{n}} G'_U(\delta_p)\big|_\sigma \cdot u_0(\sigma) - G'_U(\delta_p)\big|_\sigma \cdot \dot{u}_0(\sigma) \right) \mu_\Sigma(\sigma). \tag{4.2.23}$$

However, here we face two problems. First one has to shows that u is indeed a solution of $Du = 0$ with the correct initial conditions. Second, and more severe, one has to justify the restriction of the *distributions* $\nabla^E_{\mathfrak{n}} G'_U(\delta_p)$ and $G'_U(\delta_p)$ to the hypersurface, which is indeed a nontrivial task. Thus we leave (4.2.23) as a heuristic formula and stay with Proposition 4.2.8 and Proposition 4.2.9.

4.2.3 Existence of Global Solutions to the Cauchy Problem

To approach the global existence of solutions we assume as before that M is globally hyperbolic with a smooth spacelike Cauchy hypersurface Σ. Now we again use the splitting theorem $M \cong \mathbb{R} \times \Sigma$ with the first coordinate being the Cauchy temporal function and Σ_t the Cauchy hypersurface of constant time t where we shift the origin to $\Sigma_0 = \Sigma$. For every $p \in M$ we have a unique time t with $p \in \Sigma_t$. On each Σ_t we have a Riemannian metric g_t such that $g = \beta\, dt^2 - g_t$. This allows to speak of the open balls around $p \in \Sigma_t$ of radius $r > 0$ with respect to this metric g_t. We denote these by $B_r(p)$ without explicit reference to t. Note that $B_r(p) \subseteq \Sigma_t$ is open in Σ_t but not in M, see also Figure 4.23. Here we use the Riemannian distance d_{g_t} in Σ_t with respect to g_t for defining the ball, i.e.

$$d_{g_t}(p,q) = \inf \left\{ \int_a^b g_t(\dot{\gamma}(\tau), \dot{\gamma}(\tau))\, d\tau \;\middle|\; \gamma(a) = p, \gamma(b) = q, \gamma(\tau) \in \Sigma_t \right\}, \tag{4.2.24}$$

where γ is an at least piecewise $\mathcal{C}^1$ curve joining $p, q \in \Sigma_t$ *inside* Σ_t. Having such a ball we consider its Cauchy development $D_M(B_r(p)) = D^+_M(B_r(P)) \cup D^-_M(B_r(p))$ in M according to Definition 2.2.19, see again Figure 4.23. We now want to find r small enough that $D_M(B_r(p))$ is a nice open neighborhood

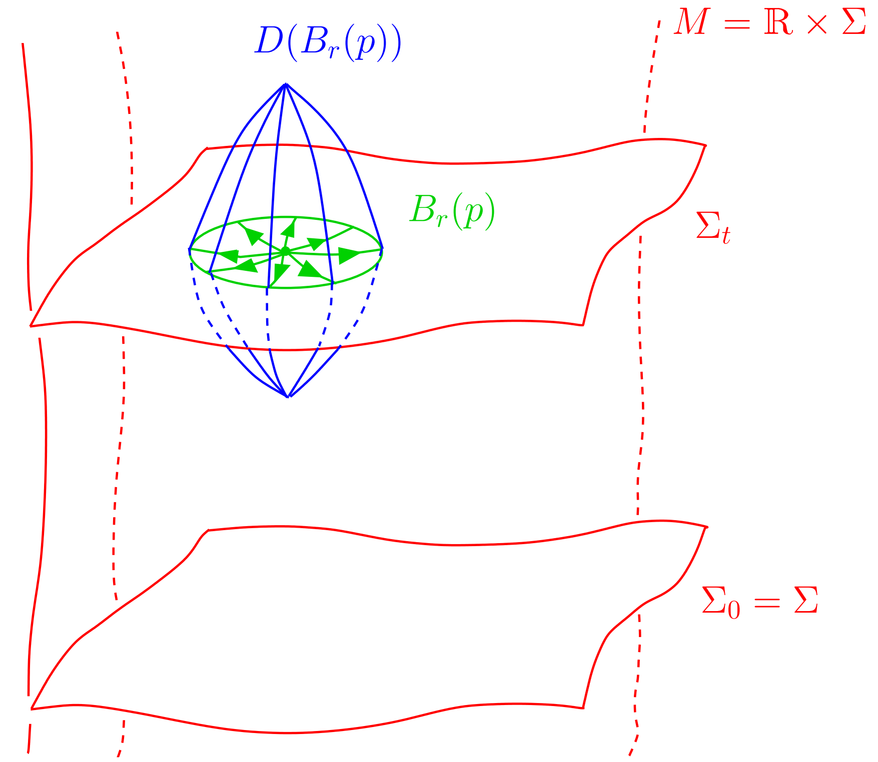

Figure 4.23: An open ball $B_r(p)$ in a Cauchy hypersurface Σ_t and its Cauchy development $D(B_r(p))$.

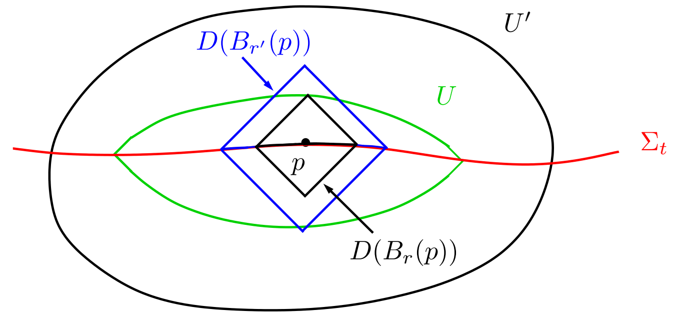

Figure 4.24: Illustration for the proof of Lemma 4.2.11.

of p allowing a local fundamental solution: in this case we call an open neighborhood a *relatively compact causal open neighborhood of small volume* or short *RCCSV* for abbreviation. We start with a couple of technical lemmas, following [4]:

Lemma 4.2.11 *The function* $\rho : M \longrightarrow (0, +\infty]$ *defined by*

$$\rho(p) = \sup \left\{ r > 0 \;\middle|\; D(B_r(p)) \textit{ is RCCSV} \right\} \tag{4.2.25}$$

is well-defined and lower semi-continuous.

Proof. We have to show first that the set of $r > 0$ with $D(B_r(p))$ RCCSV is non-empty. To this end we choose an RCCSV neighborhood $U \subseteq U^{\mathrm{cl}} \subseteq U'$ as before. Then $U \cap \Sigma_t$ will be an open neighborhood of p in Σ_t hence it contains a $B_r(p) \subseteq \Sigma_t$. The problem might be that the Cauchy development of $B_r(p)$ may reach too far outside of U or even U' such that it is not RCCSV for free, see Figure 4.24. In fact, we have to choose a small enough r such that $D(B_r(p)) \subseteq U$. In this case

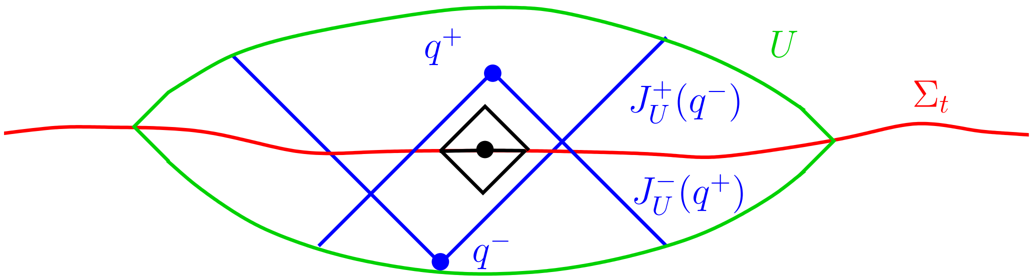

Figure 4.25: Constructing a small enough open ball around p.

it is causal in U' and has small enough volume. We choose points $q^\pm \in U$ with $p \in J_U^\pm(q^\mp)$. Then we consider the open subset $J_U^+(q-) \cap J_U^-(q^+)$ which is an neighborhood of p, see Figure 4.25. The intersection of this neighborhood of p (in M) with Σ_t gives an open neighborhood of p in Σ_t. Now we choose a $B_r(p)$ contained in this neighborhood. We claim that $D_M(B_r(p))$ is in U. First we note that $J_U^\pm(q^\mp) = U \cap J_M^\pm(q^\mp)$ since U is causally compatible with M. Now if $q \in D_M^+(B_r(p))$ then every past-inextensible causal curve meets $B_r(p)$. We claim that $q \in I^-(q^+)$. Assume that this is not the case. Then we have a past-inextensible curve from p to q which has to pass through the backward light cone of q^+. Denote this intersection point by q_0. Since we are inside a geodesically convex neighborhood U', we can take the unique lightlike geodesic from q^+ to this q_0 which is past directed. Since this geodesic is on the light cone, it hits the Cauchy hypersurface Σ_t *not* in the open subset $I_U^-(q^+)$ but on its boundary, say in the point q_1. Thus it will not intersect the even smaller open ball $B_r(p)$. Thus the combined curve from q back to q_0 and then back to q_1 will never hit $B_r(p)$, no matter how we extend it further in past directions. This contradicts $q \in D_M^+(p)$ whence we conclude that $q \in I_U^-(q^+)$. A simpler argument shows that q is also in the chronological future of q^- and hence in the intersection of the two open subsets $I_U^+(q^-)$ and $I_U^-(q^+)$. An analogous argument shows that a point in $D_M^-(p)$ is also in this intersection. We finally arrived at the desired statement that $D_M(B_r(p))$ is in U.

Now let $p \in M$ and $r > 0$ with $\rho(p) > r$ be given. In particular $D_M(B_r(p))$ will be RCCSV. Then we have to show that for a given $\epsilon > 0$ we have

$$\rho(p') > r - \epsilon$$

for all p' in an appropriate open neighborhood of p. We consider the following function defined for $p' \in D_M(B_r(p))$ by

$$\lambda(p') = \sup \left\{ r' > 0 \;\middle|\; B_{r'}(p') \subset D_M(B_r(p)) \right\},$$

i.e. we ask for the balls around p' to be contained in the Cauchy development of $B_r(p)$. Note that p' may correspond to a different time $t' \neq t$ which has to be taken into account in the definition of the radius r', i.e. we use $g_{t'}$. We claim that there is an open neighborhood V of p such that for all $p' \in V$ we have

$$\lambda(p') > r - \epsilon.$$

Assume that this is not true. Then we can find a sequence $p_n \longrightarrow p$ of points in $D_M(B_r(p))$ with $\lambda(p_n) \leq r - \epsilon$ for all n. Then it follows that for $r' = r - \frac{\epsilon}{2}$ the ball $B_{r'}(p_n)$ is *not* entirely contained in $D_M(B_r(p))$ for all n. This allows to find a point $q_n \in B_{r'}(p_n) \setminus D_M(B_r(p))$. Since $D_M(B_r(p))$ is RCCSV the closure $D_M(B_r(p))^{\mathrm{cl}}$ is compact and thus also $B_r(p)^{\mathrm{cl}} \subseteq D_M(B_r(p))^{\mathrm{cl}}$. Since the metric g_t and hence the distance function d_{g_t} depend (at least) continuous on t we conclude that with the convergence of $p_n \longrightarrow p$ and $r' < r$ we have $B_{r'}(p_n) \subseteq [-1, 1] \times B_r(p)^{\mathrm{cl}}$ for all $n \geq n_0$. But then also the points $q_n \in B_{r'}(p_n) \subseteq [-1, 1] \times B_r(p)^{\mathrm{cl}}$ are in this compact "box", see Figure 4.26. Therefore we find a convergent subsequence which we denote by $q_n \longrightarrow q$ as well. Now $p_n \longrightarrow p$ and $q_n \in B_{r'}(p_n)^{\mathrm{cl}}$ whence

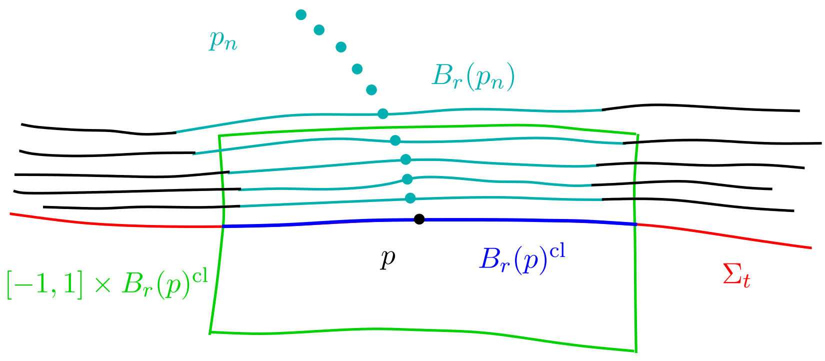

Figure 4.26: Balls around the p_n with radius r' are finally inside the box $[-1, 1] \times B_r(p)^{\mathrm{cl}}$.

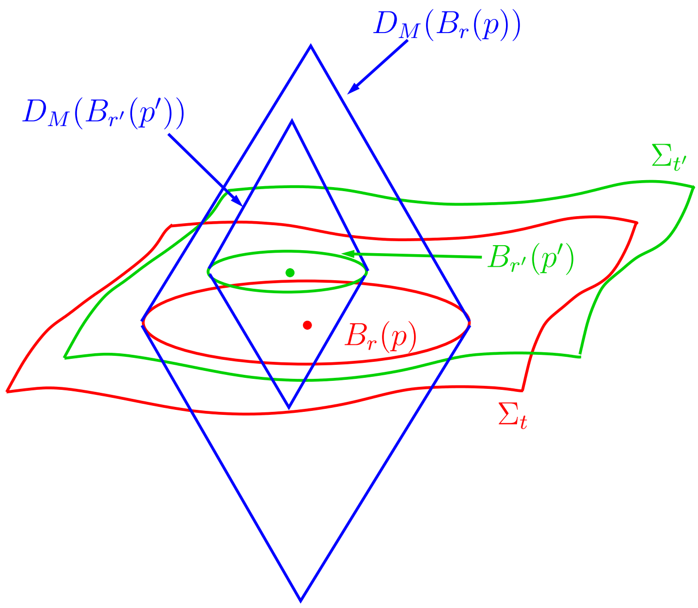

Figure 4.27: For points $p' \in D_M(B_r(p))$ the Cauchy development of a smaller ball is included in that of $B_r(p)$.

$q \in B_{r'}(p)^{\mathrm{cl}}$ follows. Since $B_{r'}(p)^{\mathrm{cl}} \subseteq B_r(p)^{\mathrm{cl}}$ we conclude $q \in B_r(p)$. But $D_M(B_r(p))$ is open and hence eventually all sequence elements q_n are contained in $D_M(B_r(p))$ which is a contradiction. Thus our original claim was in fact true. Thus let $p' \in V$ be in this neighborhood and let $r - \epsilon < r' < \lambda(p')$. Then by definition we have $B_{r'}(p') \subseteq D_M(B_r(p))$ and hence by Remark 2.2.21 we have

$$D(B_{r'}(p')) \subseteq D(B_r(p)).$$

Since the larger Cauchy development $D_M(B_r(p))$ is RCCSV this is also true for the smaller $D_M(B_{r'}(p'))$. Indeed, $D_M(B_{r'}(p'))$ is causal in the surrounding convex U' and has smaller volume than $D_M(B_r(p))$ Since $D_M(B_{r'}(p'))^{\mathrm{cl}} \subseteq D_M(B_r(p))^{\mathrm{cl}}$ it is also pre-compact as wanted. But this shows $\rho(p') \geq r' > r - \epsilon$, which is the lower semi-continuity. □

Geometrically, this semi-continuity means that for a given $B_r(p)$ around p we can find a ball $B_{r'}(p')$ around p' with only slightly smaller $r' < r$ such that the Cauchy development of $B_{r'}(p')$ is still entirely in the one of $B_r(p)$, see also Figure 4.27. The next auxiliary function we shall need is the following. We define for $r > 0$ and $p \in M$ (always with respect to the chosen Cauchy temporal function)

$$\theta_r(p) = \sup \left\{ \tau > 0 \;\middle|\; J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \cap ([t-\tau, t+\tau] \times \Sigma) \subseteq D_M(B_r(p)) \right\}, \tag{4.2.26}$$

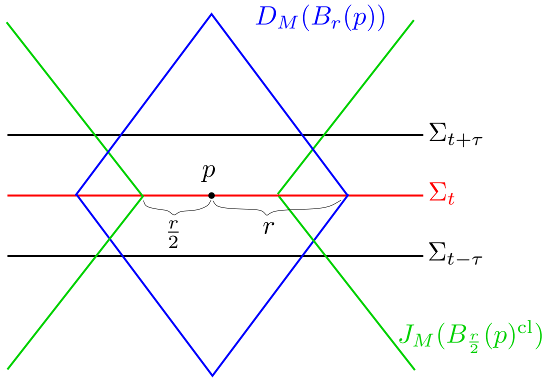

Figure 4.28: Illustration for the function $\theta_r(p)$.

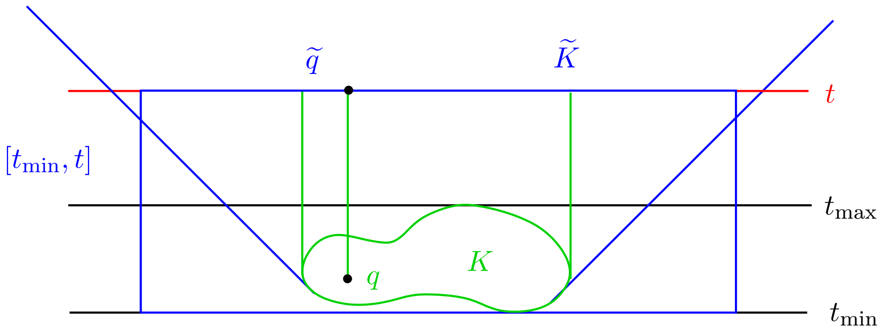

Figure 4.29: The compact subset K is in $[t_{\min}, t] \times \widetilde{K}$.

where t is the time corresponding to the point p, i.e. $p \in \{t\} \times \Sigma \subseteq M$. The picture to have in mind is sketched in Figure 4.28. Again, we first show that this is well-defined, i.e. the subset of $\tau > 0$ with $J_M(B_{\frac{r}{2}}\mathrm{cl}) \cap ([t-\tau, t+\tau] \times \Sigma) \subseteq D_M(B_r(p))$ is non-empty:

Lemma 4.2.12 *For every $p \in M$ and $r > 0$ there exists a $\tau > 0$ such that*

$$J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \cap ([t-\tau, t+\tau] \times \Sigma) \subseteq D_M(B_r(p)), \tag{4.2.27}$$

where $t \in \mathbb{R}$ is the unique time with $p \in \Sigma_t$.

Proof. First we note the following statement: for a compact subset $K \subseteq M = \mathbb{R} \times \Sigma$ let $t_{\min}$ and $t_{\max}$ be the minimum and maximum of the time function on K, respectively. Then consider an arbitrary time $t \geq t_{\max}$ and let $\widetilde{K} = J_M^+(K) \cap \Sigma_t$ which we can identify with a subset of Σ again since $\Sigma_t \simeq \Sigma$. Guided by Figure 4.29 we claim that K is contained in $[t_{\min}, t] \times \widetilde{K}$: indeed, let $p \in K$ be given, then there is a timelike curve from p to $q \in \widetilde{K}$ which is just $\tau \mapsto (\tau, p)$ where τ ranges from the time $t(p) \geq t_{\min}$ of p to t. Thus in the trivialization p corresponds to $(t(p), q) \in [t_{\min}, t] \times \widetilde{K}$. Since K is compact, one knows that $J_M^+(K) \cap \Sigma = \widetilde{K}$ is compact as well, see e.g. [45, p. 44]. This shows that $J_M^+(K) \cap ([t_{\min}, t] \times \Sigma) \subseteq [t_{\min}, t] \times \widetilde{K}$ is compact, too. As we can argue analogously for $J_M^-(K)$ we see that for any compact subset $K \subseteq M$ the subset $J_M(K) \cap ([t_1, t_2] \times \Sigma)$ is compact for $t_1 \leq t_{\min}$ and $t_2 \geq t_{\max}$. In particular, $J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}}) \cap ([t - \frac{1}{2}, t + \frac{1}{2}] \times \Sigma)$ is compact for all $n \geq 1$ since here $t_{\min} = t = t_{\max}$.

Now assume such a τ with (4.2.27) does not exist. Then we find $q_n \in ([t - \frac{1}{n}, t + \frac{1}{n}] \times \Sigma) \cap J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}})$ which are not in $D_M(B_r(p))$. Since the subset $J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}}) \cap ([t - \frac{1}{n}, t + \frac{1}{n}] \times \Sigma)$ is *compact* we can pass to a convergent subsequence, which we also denote by q_n converging to some q. Clearly, the point q has time value t. But this means $q \in J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}}) \cap \{t_0\} \times \Sigma = B_{\frac{r}{2}}(p)^{\mathrm{cl}}$. Now

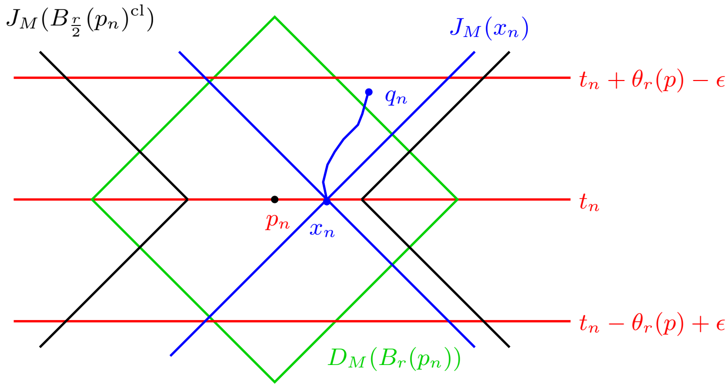

Figure 4.30: Construction of the points q_n in the proof of Lemma 4.2.13.

$D_M(B_r(q))$ is an open neighborhood of $B_{\frac{r}{2}}(p)^{\mathrm{cl}}$, thus we have necessarily $q_n \in D_M(B_r(q))$ for almost all n. This a contradiction and hence we have a $\tau > 0$ as wanted. □

Lemma 4.2.13 *The function $\theta_r : M \longrightarrow (0, \infty]$ is well-defined and lower semi-continuous.*

Proof. By the last lemma, the function is well-defined. We consider $p \in M$ and $\epsilon > 0$. Then we have to show that for all p' in a suitable neighborhood of p we still have $\theta_r(p') \geq \theta(p) - \epsilon$. Let $t \in \mathbb{R}$ be the time of p. We assume that there is *no* such open neighborhood of p. Thus we find a sequence $p_n \longrightarrow p$ of points with $\theta_r(p_n) < \theta_r(p) - \epsilon$ for all n. Since $I_M(B_r(p))$ as well as $(t - \tau, t + \tau) \times \Sigma$ are open neighborhoods of p we have

$$p_n \in J_M\left(B_r(p)^{\mathrm{cl}}\right) \cap ([-T, T] \times \Sigma) \tag{$*$}$$

for T large enough and $n \geq n_0$. As already argued in the proof of Lemma 4.2.12, this subset is compact. For the times t_n of $p_n \in \{t_n\} \times \Sigma$ we know $t_n \longrightarrow t$ as $p_n \longrightarrow p$. Since $\theta_r(p_n) < \theta_r(p) - \epsilon$ we have

$$J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \cap ([t_n - \theta_r(p) + \epsilon, t_n + \theta_r(p) - \epsilon] \times \Sigma) \nsubseteq D_M(B_r(p_n)).$$

Hence we can choose points $q_n \in J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \cap ([t_n - \theta_r(p) + \epsilon, t_n + \theta_r(p) - \epsilon] \times \Sigma)$ which are *not* in $D_M(B_r(p_n))$, see Figure 4.30. By definition we find $x_n \in B_{\frac{r}{2}}(q_n)^{\mathrm{cl}}$ with $q_n \in J_M(x_n)$. From $p_n \longrightarrow p$ we also conclude that for sufficiently large n we have

$$J_M\left(B_{\frac{r}{2}}(p_n)^{\mathrm{cl}}\right) \subseteq J_M\left(B_r(p)^{\mathrm{cl}}\right),$$

see also Figure 4.31. This shows that $q_n \in J_M\left(B_{\frac{r}{2}}(p_n)^{\mathrm{cl}}\right) \subseteq J_M\left(B_r(p)^{\mathrm{cl}}\right)$ whence together with $q_n \in [t_n - \theta_r(p) + \epsilon, t_n + \theta_r(p) - \epsilon] \times \Sigma$ we see that all the q_n are in the compact subset $(*)$. For the x_n this is also true as we have $x_n \in B_{\frac{r}{2}}(p_n)^{\mathrm{cl}} \subseteq J_M\left(B_{\frac{r}{2}}(p_n)^{\mathrm{cl}}\right)$. We may pass to convergent subsequences $q_n \longrightarrow q$ and $x_n \longrightarrow x$. Since $x_n \in B_{\frac{r}{2}}(p_n)^{\mathrm{cl}}$ with $p_n \longrightarrow p$ we conclude by continuity of the Riemannian distance function that $x \in B_{\frac{r}{2}}(p)^{\mathrm{cl}}$. Moreover, since the causal relation "$\leq$" is closed on a globally hyperbolic spacetime, see Remark 4.1.10, we conclude from $q_n \in J_M(x_n)$ and the convergence of the sequences that $q \in J_M(x)$ and hence $q \in J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}})$. In addition, since $q_n \in [t_n - \theta_r(p) + \epsilon, t_n + \theta_r(p) - \epsilon] \times \Sigma$ and $t_n \longrightarrow t$ we conclude that $q \in [t - \theta_r(p) + \epsilon, t + \theta_r(p) - \epsilon] \times \Sigma$. Thus we can use the definition of the function $\theta_r(p)$ at p and conclude from

$$q \in J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \cap ([t - \theta_r(p) + \epsilon, t + \theta_r(p) - \epsilon] \times \Sigma)$$

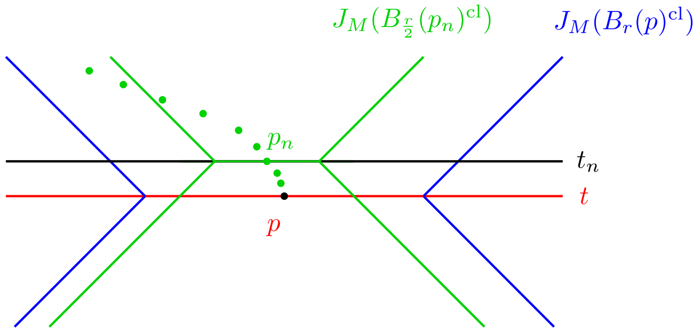

Figure 4.31: The double cones of the half radius balls around p_n are included in the double cone of the full radius ball around p for large n.

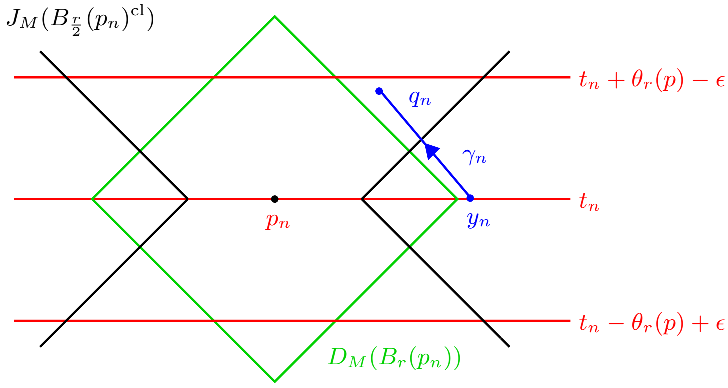

Figure 4.32: The causal curve γ_n from the proof of Lemma 4.2.13 which does not meet $B_r(p_n)$.

that $q \in D_M(B_r(p))$. Indeed, $\theta_r(p) > \theta_r(p) - \epsilon$ whence we can apply (4.2.26) for $\tau = \theta_r(p) - \epsilon$. Since the q_n are not in $D_M(B_r(p_n))$ we have an inextensible causal curve γ_n through q_n which does not meet $B_r(p_n)$, see Figure 4.32. However, since Σ_{t_n} is a Cauchy hypersurface, it meets γ_n in exactly one point, say y_n, see also Remark 2.2.18. Now we claim that the y_n are also in a compact subset. To this end we consider again a large enough T such that all times occurring are in $[-T, T]$. First note that it may well happen that none of the y_n are in the compact subset $(*)$ if $p_n \longrightarrow p$ but all the p_n have the same time and come, say from the "right". In this case, already Minkowski spacetime gives us y_n not in $(*)$. However, the intersection $J_M(B_{\frac{r}{2}}(p)^{\mathrm{cl}}) \cap \Sigma_T = L$ is compact and hence the past of L intersected with the time interval $[-T, T] \times \Sigma$ is again compact, as we argued in the proof of Lemma 4.2.12, see Figure 4.33. But now $q_n \in J_M\left(B_{\frac{r}{2}}(p)^{\mathrm{cl}}\right) \subseteq J_M\left(B_r(p)^{\mathrm{cl}}\right)$ shows that $q_n \in J_M^-(L)$. But then also the past $J_M^-(q_n)$ is in the past of $J_M^-(L)$ and thus $y_n \in J_M^-(q_n) \subseteq J_M^-(L)$. Since the time of y_n is $t_n \in [-T, T]$ we conclude that $y_n \in J_M^-(L) \cap ([-T, T] \times \Sigma)$ for all n. Clearly, if y_n is in the future of q_n, i.e. the Figure 4.33 is reversed, the same holds for $J_M^+(L) \cap ([-T, T] \times \Sigma)$. Taking the union $J_M(L) \cap ([-T, T] \times \Sigma)$ will therefore give a compactum for which all y_n are inside. Thus we can also here pass to a convergent subsequence $y_n \longrightarrow y$. Necessarily $y \in \Sigma_t$ as $t_n \longrightarrow t$. Since $y_n \notin B_r(p_n)$ we conclude $y \in B_r(p)$ by continuity of the distance function d_{g_t} with respect to t. Since all the curves γ_n are causal, we have $q_n \in J_M(y_n)$ and by the closedness of the causal relation "$\leq$" on a globally hyperbolic spacetime we conclude $q \in J_M(y)$. Hence there are inextensible causal curves through y and q. But since every such curve meets Σ_t in only one point, namely in y, it can not meet $B_r(p)$. However, $q \in D_M(B_r(p))$, which is a contradiction. □

The importance of the two lower semi-continuous functions ρ and θ_r is that they are bounded from below on every compact subset: this is an adaption of the statement that a continuous function

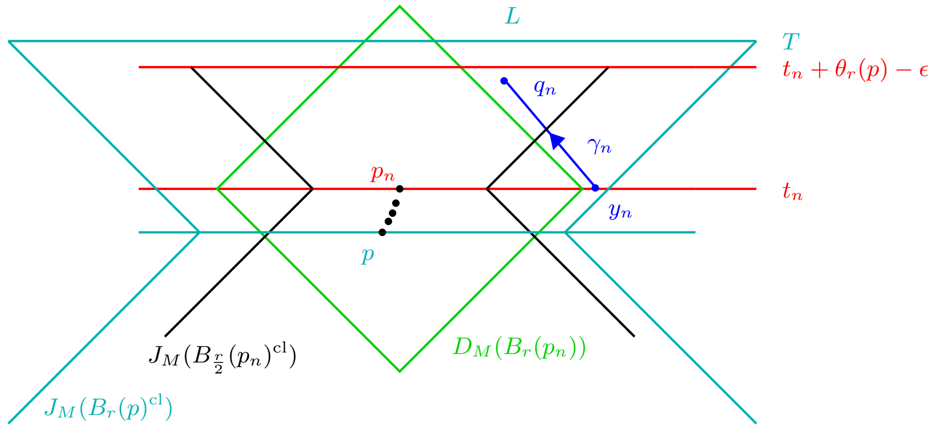

Figure 4.33: The compactum L.

is bounded (it takes maximum and minimum) on a compact subset. Indeed, let $f : K \longrightarrow \mathbb{R}$ be lower semi-continuous and K compact. Then for all $p \in K$ and $\epsilon > 0$ we find an open neighborhood $U(p)$ of p such that $f(p') \geq f(p) - \epsilon$ for all $p' \in U(p)$. Covering K with finitely many such neighborhoods $U(p_1), \ldots, U(p_n)$ we see that $f(p') \geq \min_i f(p_i) - \epsilon$ whence f is *bounded* from below. Let $c = \inf_{p \in K} f(p)$ the infimum of f. Then we have a sequence $p_n \in K$ with $f(p_n) \longrightarrow c$. Now K is compact whence p_n has a convergent subsequence which we denote also by $p_n \longrightarrow p$. Thus let $\epsilon > 0$ and choose $U \subseteq K$ such that $f(p') \geq f(p) - \epsilon$ for all $p' \in U$. Now all but finitely p_n are in U whence $f(p_n) \geq f(p) - \epsilon$ for all but finitely many n. It follows that also the limit $\lim_n f(p_n)$ satisfies $c = \lim_n f(p_n) \geq f(p) - \epsilon$. Thus $c \geq f(p) - \epsilon$ for all $\epsilon > 0$. But by construction of c we know $c \leq f(p)$ whence $f(p) = c$ follows.

It follows that on a compact subset $K \subseteq M$ the functions ρ and θ_r are bounded from zero. We use this in the following lemma:

Lemma 4.2.14 *Let $K \subseteq M$ by compact. Then there is a $\delta > 0$ such that for all times $t \in \mathbb{R}$ and all $u_t, \dot{u}_t \in \Gamma^\infty(\iota_t^\# E)$ on Σ_t with support $\operatorname{supp} u_t, \operatorname{supp} \dot{u}_t \subseteq K$ we have a smooth solution u of the homogeneous wave equation $Du = 0$ on the time slice $(t - \delta, t + \delta) \times \Sigma$ with the initial conditions $u\big|_{\Sigma_t} = u_t$ and $\nabla^E_{\mathfrak{n}} u\big|_{\Sigma_t} = \dot{u}_t$. Moreover, for the support one has*

$$\operatorname{supp} u \subseteq J_M\left(\operatorname{supp} u_t \cup \operatorname{supp} \dot{u}_t\right). \tag{4.2.28}$$

Proof. Since ρ is lower semi-continuous according to Lemma 4.2.11 and positive, it admits a minimum on the compact subset K. Thus we find an $r_0 > 0$ with $\rho(p) > 2r_0$ for all $p \in K$. For this radius, the function θ_{2r_0} is lower semi-continuous according to Lemma 4.2.13 and positive. Hence we find a $\delta > 0$ with $\theta_{2r_0} > \delta$ on K. We claim that this δ will do the job. Thus let $t \in \mathbb{R}$ be given. Since $\Sigma_t \cap K$ is again compact, we can cover $\Sigma_t \cap K$ with finitely many open balls $B_{r_0}(p_1), \ldots, B_{r_0}(p_N)$ of radius r_0, where as usual the notion of "ball" refers to the Riemannian manifold (Σ_t, g_t). We can find a smooth partition of unity $\chi_1, \ldots, \chi_N$ subordinate to the cover $B_{r_0}(p_1) \cup \ldots \cup B_{r_0}(p_N)$, i.e. on this open cover of $\Sigma_t \cap K$ we have $\chi_1 + \cdots + \chi_N = 1$ and $\operatorname{supp} \chi_\alpha \subseteq B_{r_0}(p_\alpha)$ for all $\alpha = 1, \ldots, N$. It follows that we can decompose the initial conditions u_t and $\dot{u}_t$ into smooth pieces having compact support in $B_{r_0}(p_\alpha)$ by considering $\chi_\alpha u_t$ and $\chi_\alpha \dot{u}_t$, respectively. Clearly, we still have $\chi_\alpha u_t, \chi_\alpha \dot{u}_t \in \Gamma_0^\infty(\iota_t^\# E)$ and $\chi_1 u_t + \cdots + \chi_N u_t = u_t$ as well as $\chi_1 \dot{u}_t + \cdots + \chi_N \dot{u}_t = \dot{u}_t$. By definition of ρ the Cauchy development $D_M(B_{2r_0}(p_\alpha))$ of the balls with twice the radius is still RCCSV, see Figure 4.34. Thus we can apply Proposition 4.2.8 to these open subsets and obtain a smooth solution $u_\alpha \in \Gamma^\infty\left(E\big|_{D_M(B_{2r_0}(p_\alpha))}\right)$ of the homogeneous wave equation $Du_\alpha = 0$ on $D_M(B_{2r_0}(p_\alpha))$ for the initial conditions

$$u_\alpha\big|_{\Sigma_t} = \chi_\alpha u_t \quad \text{and} \quad \nabla^E_{\mathfrak{n}} u_\alpha\big|_{\Sigma_t} = \chi_\alpha \dot{u}_t.$$

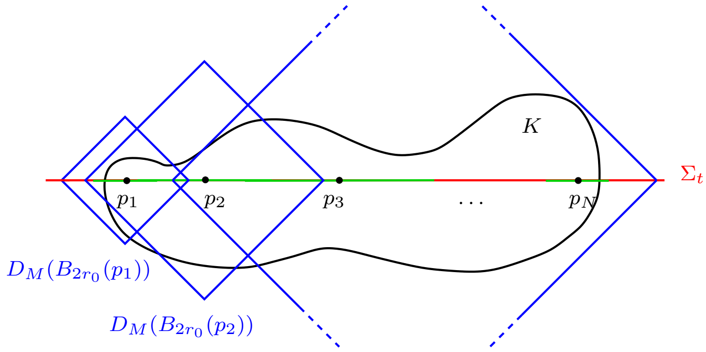

Figure 4.34: The covering of the compact subset $K \cap \Sigma_t$ and the Cauchy development of the balls.

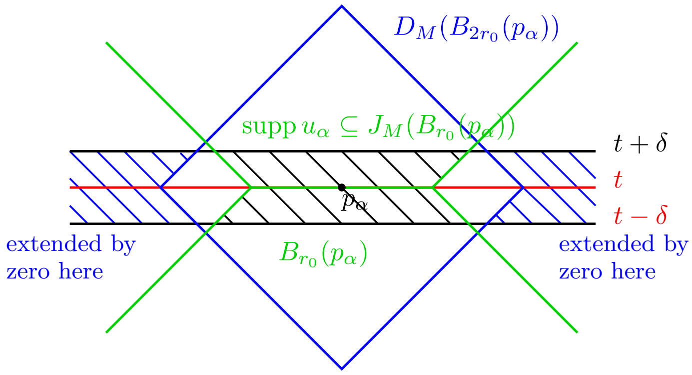

Figure 4.35: The local solutions u_α constructed in the proof of Lemma 4.2.14 and their support.

Moreover, since we consider the homogeneous wave equation, the supports satisfy

$$\operatorname{supp} u_\alpha \subseteq J_M \left(\operatorname{supp} \chi_\alpha u_t \cup \operatorname{supp} \chi_\alpha \dot{u}_t\right). \tag{$*$}$$

By definition of the function θ_{2r_0} and the choice of δ we see that

$$J_M\left(B_{r_0}(p_\alpha)^{\mathrm{cl}}\right) \cap ([t-\delta, t+\delta] \times \Sigma) \subseteq D_M(B_{2r_0}(p_\alpha)).$$

Hence the solution u_α is defined on the subset $J_M\left(B_{r_0}(p_\alpha)^{\mathrm{cl}}\right) \cap ([t-\delta, t+\delta] \times \Sigma)$. Moreover, since $\operatorname{supp} \chi_\alpha u_t, \operatorname{supp} \chi_\alpha \dot{u}_t \subseteq B_{r_0}(p_\alpha)$ we conclude from $(*)$ that

$$\operatorname{supp} u_\alpha \subseteq J_M\left(B_{r_0}(p_\alpha)^{\mathrm{cl}}\right).$$

Since u_α is smooth on $D_M(B_{2r_0}(p_\alpha))$ we can safely extend u_α by zero to $(t-\delta, t+\delta) \times \Sigma$, see Figure 4.35, and have a section $u_\alpha \in \Gamma^\infty\left(E\big|_{(t-\delta,t+\delta)\times\Sigma}\right)$ satisfying $\operatorname{supp} u_\alpha \subseteq J_M\left(B_{r_0}(p_\alpha)^{\mathrm{cl}}\right) \cap ([t-\delta, t+\delta] \times \Sigma)$ and $Du_\alpha = 0$ as well as

$$u_\alpha\big|_{\Sigma_t} = \chi_\alpha u_t \quad \text{and} \quad \nabla^E_{\mathfrak{n}} u_\alpha\big|_{\Sigma_t} = \chi_\alpha \dot{u}_t.$$

Then their sum $u = u_1 + \ldots + u_N$ will still satisfy $Du = 0$ on $(t-\delta, t+\delta) \times \Sigma$ and

$$u\big|_{\Sigma_t} = u_t \quad \text{as well as} \quad \nabla^E_{\mathfrak{n}} u\big|_{\Sigma_t} = \dot{u}_t,$$

since the χ_α are a partition of unity. Finally,

$$\operatorname{supp} u \subseteq \operatorname{supp} u_1 \cup \ldots \cup \operatorname{supp} u_N$$

$$\subseteq J_M(\operatorname{supp}\chi_1 u_t \cup \operatorname{supp}\chi_1\dot{u}_t) \cup \ldots \cup J_M(\operatorname{supp}\chi_N u_t \cup \operatorname{supp}\chi_N\dot{u}_t)$$

$$\subseteq J_M(\operatorname{supp}\chi_1 u_t \cup \operatorname{supp}\chi_1\dot{u}_t \cup \operatorname{supp}\chi_N u_t \cup \operatorname{supp}\chi_N\dot{u}_t)$$

$$\subseteq J_M(\operatorname{supp} u_t \cup \operatorname{supp}\dot{u}_t),$$

since on one hand $J_M(A) \cup J_M(B) \subseteq J_M(A \cup B)$ and on the other hand $\operatorname{supp}\chi_\alpha u_t \subseteq \operatorname{supp} u_t$ and $\operatorname{supp}\chi_\alpha \dot{u}_t \subseteq \operatorname{supp}\dot{u}_t$ for all α. This completes the proof. □

Remark 4.2.15 We see from the proof that we do not loose any differentiability by the globalization process. Only for the local solvability of the Cauchy problem we need to count orders of differentiation carefully. The reason is that the partition of unity can be chosen smooth and hence we do not spoil regularity by decomposing everything into small pieces. Thus we get from Proposition 4.2.9 the analogous statement: for initial conditions $u_t \in \Gamma^{2(k+n+1)+2}(\iota_t^\# E)$ and $\dot{u}_t \in \Gamma^{2(k+n+2)+1}(\iota_t^\# E)$ with the same support conditions we get a solution $u \in \Gamma^k\left(E\big|_{(t-\delta,t+\delta)\times\Sigma}\right)$, where of course $k \geq 2$. The statement on the support is also still valid.

Now we come to the existence of global solutions to the Cauchy problem. As before, $M = \mathbb{R} \times \Sigma$ is globally hyperbolic with a smooth spacelike Cauchy hypersurface.

Theorem 4.2.16 *Let (M, g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow M$.*

i.) For $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$ and $v \in \Gamma_0^\infty(E)$ there exists a unique global solution $u \in \Gamma^\infty(E)$ of the inhomogeneous wave equation $Du = v$ with initial conditions $\iota^\# u = u_0$ and $\iota^\# \nabla_{\mathfrak{n}}^E u = \dot{u}_0$. We have

$$\operatorname{supp} u \subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp}\dot{u}_0 \cup \operatorname{supp} v). \tag{4.2.29}$$

ii.) For $k \geq 2$ and $u_0 \in \Gamma_0^{2(k+n+1)+2}(\iota^\# E)$, $\dot{u}_0 \in \Gamma_0^{2(k+n+1)+1}(\iota^\# E)$ and $v \in \Gamma_0^{2(k+n+1)}(E)$ there exists a unique global solution $u \in \Gamma^k(E)$ of the inhomogeneous wave equation $Du = v$ with initial conditions $\iota^\# u = u_0$ and $\iota^\# \nabla_{\mathfrak{n}}^E u = \dot{u}_0$. It also satisfies (4.2.29).

Proof. Uniqueness follows in both cases from Theorem 4.2.5. We consider the first case with smooth initial conditions. Since all the supports are compact so is their union. Therefore, we can cover this compact subset with finitely many RCCSV subsets for which we can apply the local existence according to Proposition 4.2.8. Again, choosing an appropriate partition of unity subordinate to this cover, we can decompose the initial conditions and the inhomogeneity into pieces having their compact supports inside of the RCCSV subsets. If we succeed to show the existence of a global solution for such initial conditions and inhomogeneity with support in the RCCSV subset, we can afterwards sum up this finite number of solutions to get a solution for the arbitrary $u_0, \dot{u}_0$ and v. This shows that without restriction, we can assume that $\operatorname{supp} u_0, \operatorname{supp}\dot{u}_0$ and $\operatorname{supp} v$ are contained in a single RCCSV subset $U \subseteq U^{\mathrm{cl}} \subseteq U'$ as required by Proposition 4.2.8. We set $K = \operatorname{supp} u_0 \cup \operatorname{supp}\dot{u}_0 \cup \operatorname{supp} v \subseteq U$, which is still compact. By using a second partition of unity argument, we can cut K into even smaller pieces such that we have, for the fixed U, the properties

$$K \subseteq (-\epsilon, \epsilon) \times \Sigma$$

and

$$J_M(K) \cap ((-\epsilon, \epsilon) \times \Sigma) \subseteq U,$$

for an appropriate small $\epsilon > 0$, see Figure 4.36. Now let $u \in \Gamma^\infty(E\big|_U)$ be the solution according to Proposition 4.2.8. Since $\operatorname{supp} u \subseteq J_M(K)$ we see that we can smoothly extend u to the whole time slice $(-\epsilon, \epsilon) \times \Sigma$ by 0. We have to argue that we can extend this solution even further on arbitrarily

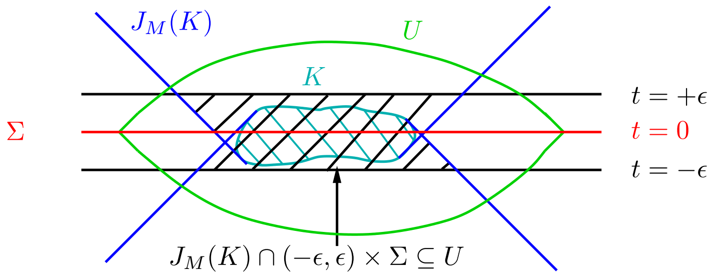

Figure 4.36: The compact set $K = \operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0 \cup \operatorname{supp} v$ of the proof of Theorem 4.2.16.

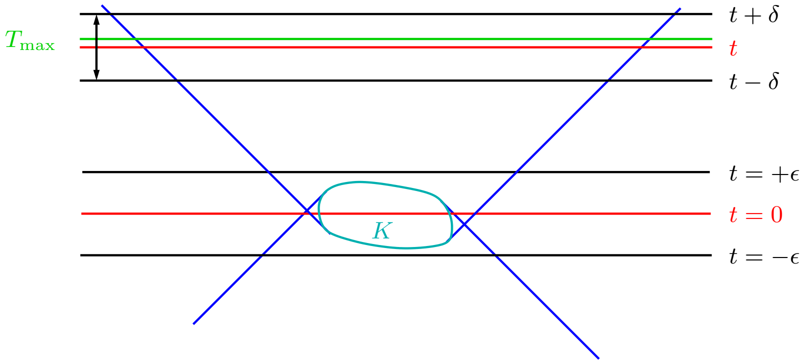

Figure 4.37: The choice of t for given $\delta > 0$ in the proof of Theorem 4.2.16.

large time slices $(-T, T) \times \Sigma$. Thus we set $T_{\max}$ to be the supremum of all those times T for which there exists a smooth extension of u to the slice $(-\epsilon, T) \times \Sigma$, still obeying the causality condition $\operatorname{supp} u \subseteq J_M(K)$. Since we have at least $T \geq \epsilon$ the supremum $T_{\max}$ is positive. Since K is in the slice $(-\epsilon, \epsilon) \times \Sigma$ we have $Du = 0$ on $[+\epsilon, T_{\max}) \times \Sigma$ since the inhomogeneity has $\operatorname{supp} v \subseteq K$. If we have two extensions, u until T and $\widetilde{u}$ until $\widetilde{T}$ with $T < \widetilde{T}$, then $\widetilde{u}\big|_{(-\epsilon,T)\times\Sigma} = u$ since the open piece $(-\epsilon, T) \times \Sigma$ is globally hyperbolic itself. Hence the uniqueness statement from Theorem 4.2.5 applies to $\widetilde{u}\big|_{(-\epsilon,T)\times\Sigma}$ and u. Thanks to this uniqueness we only have to show the existence of a solution for arbitrary, but fixed finite T, i.e. $T_{\max} = +\infty$. This will automatically give a solution defined for all times $t \in \mathbb{R}^+$ and hence a solution on $(-\epsilon, \infty) \times \Sigma$.

We assume the converse, i.e. $T_{\max} < \infty$. We consider $\widetilde{K} = ([-\epsilon, T_{\max}] \times \Sigma) \cup J_M(K)$ which is compact as we have already argued at the beginning of the proof of Lemma 4.2.12 in greater generality. We can therefore apply Lemma 4.2.14 to this compact subset $\widetilde{K}$ yielding a $\delta > 0$ as described there. Now we take a $t < T_{\max}$ with $T_{\max} - t < \delta$ but $K \subseteq (-\epsilon, t) \times \Sigma$. Note that since $K \subseteq (-\epsilon, \epsilon) \times \Sigma$ and $T_{\max} \geq \epsilon$, this is clearly possible no matter what $\delta > 0$ is, see Figure 4.37. On the whole slice $(t - \delta, t + \delta) \times \Sigma$ we solve the homogeneous wave equation $Dw = 0$ for the initial conditions

$$w\big|_{\Sigma_t} = u\big|_{\Sigma_t} \quad \text{and} \quad \nabla^E_{\mathfrak{n}} w\big|_{\Sigma_t} = \nabla^E_{\mathfrak{n}} u\big|_{\Sigma_t},$$

which is possible thanks to Lemma 4.2.14. On a smaller slice $(t - \eta, t + \eta) \times \Sigma$ the inhomogeneity v already vanishes by $\operatorname{supp} v \subseteq K$ since K is contained in the *open* slice $(-\epsilon, t) \times \Sigma$. Thus on this slice w and u both solve the homogeneous wave equation with the same initial conditions on Σ_t. Therefore $w = u$ on $(-\epsilon, t) \times \Sigma$, again by the uniqueness theorem. But this shows that w extends u to the slice $(-\epsilon, t + \delta) \times \Sigma$ in a smooth way. For the support we see that the initial conditions for w are contained in $J_M(K) \cap \Sigma_t$. For the future of t this means that $\operatorname{supp} w$ is still contained in $J_M(K)$, for the past of t we already know that $w = u$ whence in total $\operatorname{supp} w \subseteq J_M(K)$, see Figure 4.38. But $T_{\max} < t + \delta$ whence we get a contradiction since w is a valid extension of u with all desired properties. Thus $T_{\max} = +\infty$. An analogous argument shows that also in the past directions we can extend the

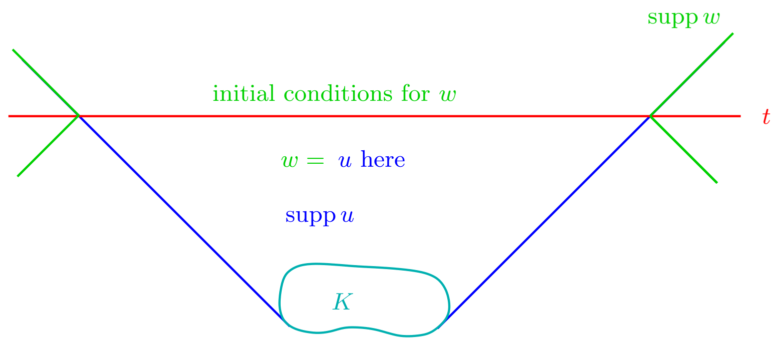

Figure 4.38: The extension w of u in the proof of Theorem 4.2.16.

solution to $t = -\infty$. This gives the first part. The second part proceeds completely analogous, using only Proposition 4.2.9 and Remark 4.2.15 instead. □

4.2.4 Well-Posedness of the Cauchy Problem

We have seen that the Cauchy problem for the inhomogeneous wave equation with smooth initial data and smooth compactly supported inhomogeneity admits a unique smooth solution. Also in the context of sufficiently large but finite differentiability we have a unique solution to the Cauchy problem. A Cauchy problem is called *well-posed* if for given initial data one has a unique solution which depends *continuously* on the initial data. Of course, this requires to specify the relevant topologies in detail. In typical situations, the relevant topologies should be clear from the context. Note also that for physical applications a continuous dependence on the initial data is certainly necessary in order to have a physically reasonable theory: initial data are always subject to (arbitrarily small but non-zero) uncertainties when measured. Thus a discontinuous dependence would lead to a physical theory without predictive power. But even if one has continuous dependence it may well happen for Cauchy problems that the discrepancy at finite times between solutions corresponding to very close initial conditions grows very fast in time, typically in an exponential way when quantified correctly. Thus it might be of interest to have the continuity even sharpened by some more quantitative description.

Back to our situation we want to show the well-posedness of the Cauchy problem with respect to the usual locally convex topologies of smooth or $\mathcal{C}^k$-sections. The main tool will be the following general statement from locally convex analysis:

Theorem 4.2.17 (Open mapping theorem) *Let $\mathcal{E}, \widetilde{\mathcal{E}}$ be Fréchet spaces and let $\phi : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$ be a continuous linear map. If ϕ is surjective then ϕ is an open map.*

As usual, a map ϕ is called *open* if the images of open subsets are again open. The proof of the open mapping theorem can e.g. be found in [51, Thm. 2.11]. We will need the following corollary of it:

Corollary 4.2.18 *Let $\phi : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$ be a continuous linear bijection between Fréchet spaces. Then ϕ^{-1} is continuous as well.*

Indeed, let $U \subseteq \mathcal{E}$ be open. Then the set-theoretic $(\phi^{-1})^{-1}(U)$, i.e. the pre-image of U under ϕ^{-1}, coincides simply with $\phi(U)$ which is open by the theorem. Thus ϕ^{-1} is continuous. Note that for general maps between topological spaces a continuous bijective map needs not have a continuous inverse at all.

We are now interested in the following situation: the result of Theorem 4.2.16 can be viewed as a map

$$\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(E), \tag{4.2.30}$$

sending $(u_0, \dot{u}_0, v)$ to the unique solution u of the wave equation $Du = v$ with initial conditions u_0 and $\dot{u}_0$. Clearly, the map (4.2.30) is linear which easily follows from the uniqueness statement of Theorem 4.2.16. Thus continuous dependence on the initial conditions will refer to the continuity of the map (4.2.30). Note that this even includes the continuous dependence on the inhomogeneity v. The relevant topologies are then the $\mathcal{C}^\infty$-topology on the target side and the canonical topology of the direct sum of the $\mathcal{C}^\infty_0$-topologies. Since the direct sum is finite, this is not problematic and essentially boils down to show $\mathcal{C}^\infty_0$-continuity for each summand. This way, we arrive at the following theorem:

Theorem 4.2.19 (Well-posed Cauchy problem I) *Let (M, g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow M$. Then the linear map* (4.2.30) *sending the initial conditions and the inhomogeneity to the corresponding solution of the Cauchy problem is continuous.*

Proof. First we note that the "inverse" map which evaluates an arbitrary section $u \in \Gamma^\infty(E)$ on the Cauchy hypersurface and applies D to it is continuous, i.e.

$$\mathcal{P} : \Gamma^\infty(E) \ni u \mapsto (\iota^\# u, \iota^\# \nabla^E_{\mathfrak{n}} u, Du) \in \Gamma^\infty(\iota^\# E) \oplus \Gamma^\infty(\iota^\# E) \oplus \Gamma^\infty(E) \tag{$*$}$$

is continuous in the $\mathcal{C}^\infty$-topologies. This is clear as all three components of $\mathcal{P}$ are continuous. Indeed, the restriction is continuous by a slight variation of the results from Proposition 1.1.20. The application of either $\nabla^E_{\mathfrak{n}}$ or D is continuous as well whence the continuity of each of the three components of $\mathcal{P}$ follows. However, for a general $u \in \Gamma^\infty(E)$ neither the restrictions $\iota^\# u$ and $\iota^\# \nabla^E_{\mathfrak{n}} u$ nor Du will have compact support. Thus we enforce this by considering a fixed compact subset $K \subseteq M$ and the subspaces $\Gamma^\infty_{K\cap\Sigma}(\iota^\# E)$ as well as $\Gamma^\infty_K(E)$ of $\Gamma^\infty(\iota^\# E)$ and $\Gamma^\infty(E)$ of those sections with compact support in the compact subsets $K \cap \Sigma$ and K, respectively. By Lemma 1.1.10 we know that both spaces are Fréchet spaces as they are $\mathcal{C}^\infty$-closed subspaces of the Fréchet spaces $\Gamma^\infty(\iota^\# E)$ and $\Gamma^\infty(E)$, respectively. Hence their direct sum is a closed subspace of the target in $(*)$ whence the pre-image

$$\mathcal{V}_K = \mathcal{P}^{-1}(\Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty_K(E)) \subseteq \Gamma^\infty(E)$$

is again closed. This way, it becomes a Fréchet subspace itself. Restricted to $\mathcal{V}_K$, the map $\mathcal{P}_K = \mathcal{P}\big|_{\mathcal{V}_K}$ becomes bijective, this is precisely the statement of Theorem 4.2.16. Indeed, $\mathcal{P}_K$ is surjective since every point in $\Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty_K(E)$ has a pre-image. This is just the existence of the solutions to the Cauchy problem. However, as the solution is unique, we have precisely one pre-image under $\mathcal{P}_K$. Since now all involved spaces are Fréchet themselves and $\mathcal{P}_K$ is obviously continuous, we can apply Corollary 4.2.18 to conclude that $\mathcal{P}_K$ has continuous inverse

$$\mathcal{P}_K^{-1} : \Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty_{K\cap\Sigma}(\iota^\# E) \oplus \Gamma^\infty(E) \longrightarrow \mathcal{V}_K \subseteq \Gamma^\infty(E)$$

for all $K \subseteq M$ compact. By the definition of the inductive limit topology this gives us immediately the continuity of the map (4.2.30) as claimed. In fact, this is again a general feature of LF topologies and this trick can be transferred to the general situation, see e.g. [34]. □

With an analogous argument we also obtain the well-posedness of the Cauchy problem in the following situation of finite differentiability:

Theorem 4.2.20 (Well-posed Cauchy problem II) *Let (M, g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow M$ and let $k \geq 2$. Then the linear map*

$$\Gamma_0^{2(k+n+1)+2}(\iota^\# E) \oplus \Gamma_0^{2(k+n+1)+1}(\iota^\# E) \oplus \Gamma_0^{2(k+n+1)}(E) \longrightarrow \Gamma^k(E) \tag{4.2.31}$$

sending $(u_0, \dot{u}_0, v)$ to the unique solution u of the inhomogeneous wave equation $Du = v$ with initial conditions $\iota^\# u = u_0$ and $\iota^\# \nabla^E_{\mathfrak{n}} u = \dot{u}_0$ is continuous.

Thus we have in both cases a well-posed Cauchy problem. There are, however, some small drawbacks of the above theorems: First, as already mentioned, we are limited to inhomogeneities v with compact support in M. Physically more appealing would be an inhomogeneity with compact support only in spacelike direction, i.e. the "eternally moving electron". Note that this is clearly an intrinsic concept on a globally hyperbolic spacetime. Moreover, the control of derivatives in Theorem 4.2.16 and hence in Theorem 4.2.20 seems not to be optimal. In particular, it would be nice to show that the map (4.2.31) has some fixed order *independent* of k.

4.3 Global Fundamental Solutions and Green Operators

While in Chapter 3 we have discussed the local existence of fundamental solutions as well as their properties we shall now pass to the global picture. From the uniqueness statements in Corollary 4.1.13 we see that the local advanced and retarded fundamental solutions necessarily agree with the restrictions of the corresponding global ones if the latter exist at all. Here we have to restrict to such an RCCSV neighborhood which is globally hyperbolic itself, i.e. a Cauchy development of a small enough ball in Σ. Then the question of existence of global fundamental solutions can be viewed as the question whether the given local fundamental solutions can be extended to the whole spacetime.

Actually, we shall proceed differently and construct the global fundamental solutions directly using the global statements on the Cauchy problem. As before, we assume throughout this section that (M, g) is globally hyperbolic.

4.3.1 Global Green Functions

We first consider the smooth version. Here we start with the following theorem:

Theorem 4.3.1 *Let (M, g) be a globally hyperbolic spacetime and $D \in \mathrm{DiffOp}^2(E)$ a normally hyperbolic differential operator. For every point $p \in M$ there is a unique advanced and retarded fundamental solution $F^{\pm}_M(p)$ of D at p. Moreover, for every test section $\varphi \in \Gamma^\infty_0(E^*)$ the section*

$$M \ni p \mapsto F^{\pm}_M(p)\varphi \in E^*_p \tag{4.3.1}$$

is a smooth section of E^* which satisfies the equation

$$D^{\mathrm{T}} F^{\pm}_M(\,\cdot\,)\varphi = \varphi. \tag{4.3.2}$$

Finally, the linear map

$$F^{\pm}_M : \Gamma^\infty_0(E^*) \ni \varphi \mapsto F^{\pm}_M(\,\cdot\,)\varphi \in \Gamma^\infty(E^*) \tag{4.3.3}$$

is continuous.

Proof. The uniqueness was already shown in Corollary 4.1.13. For the existence we consider the following construction: we first choose a splitting $M \simeq \mathbb{R} \times \Sigma$ with a Cauchy temporal function being the first coordinate of the product and Σ being a smooth spacelike Cauchy hypersurface. We denote as usual by Σ_t the level set of fixed time t, i.e. $\Sigma_t = \{t\} \times \Sigma \overset{\iota_t}{\hookrightarrow} M$, which is again a Cauchy hypersurface. Normalizing the gradient of t appropriately we obtain the smooth future-directed unit normal vector field $\mathfrak{n} \in \Gamma^\infty(TM)$ which, at Σ_t, is normal to Σ_t for all times t. Now let $\varphi \in \Gamma^\infty_0(E^*)$ be a test section of E^*. Since φ has compact support we find a time t such that $\operatorname{supp} \varphi$ is in the past of t. More precisely, we have $\operatorname{supp} \varphi \subseteq I^-_M(\Sigma_t)$, see Figure 4.39. We now apply Theorem 4.2.16, *i.)* to the transposed operator $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$ which we know to be normally hyperbolic as well. Thus we obtain a unique global and smooth solution $\psi^+ \in \Gamma^\infty(E^*)$ of the inhomogeneous wave equation $D^{\mathrm{T}}\psi^+ = \varphi$ for the initial conditions $\iota^\#_t \psi^+ = 0 = \iota^\#_t \nabla^{E^*}_{\mathfrak{n}} \psi^+$. First we note that ψ^+ does not depend

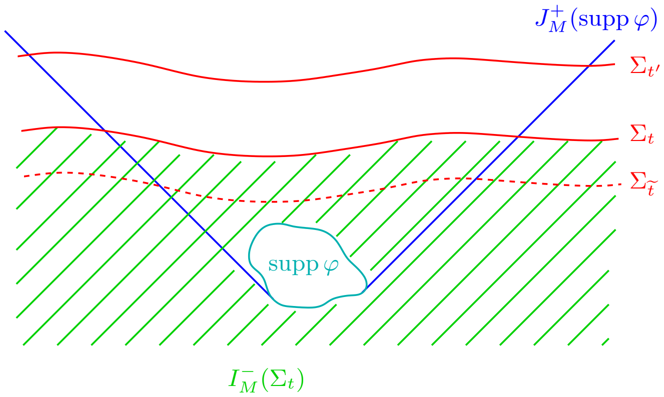

Figure 4.39: The various hypersurfaces chosen in the future of $\operatorname{supp}\varphi$ in the proof of Theorem 4.3.1.

on the precise choice of t. Indeed, let t' be another time with $\operatorname{supp}\varphi \subseteq I_M^-(\Sigma_{t'})$ and assume e.g. $t < t'$. Denote by $\psi^{+\prime} \in \Gamma^\infty(E^*)$ the corresponding solution of the Cauchy problem $D^{\mathrm{T}}\psi^{+\prime} = \varphi$ and $\iota_{t'}^{\#}\psi^{+\prime} = 0 = \iota_{t'}^{\#}\nabla_{\mathfrak{n}}^{E'}\psi^{+\prime}$. Then we find a $\widetilde{t} < t$ such that $\operatorname{supp}\varphi \subseteq I_M^-(\Sigma_{\widetilde{t}})$ since $I_M^-(\Sigma_t)$ is *open* while $\operatorname{supp}\varphi$ is closed, see again Figure 4.39. The open piece $(\widetilde{t}, \infty) \times \Sigma = \bigcup_{t > \widetilde{t}} \Sigma_t \subseteq M$ is still a globally hyperbolic spacetime on its own. Here $\psi^{+\prime}$ satisfies $D^{\mathrm{T}}\psi^{+\prime} = 0$ since $\operatorname{supp}\varphi$ is not in this part of M. Since $\psi^{+\prime}$ has vanishing initial conditions on $\Sigma_{t'}$ we conclude by the uniqueness properties of solutions that $\psi^{+\prime} = 0$ on $(\widetilde{t}, \infty) \times \Sigma$. This implies in particular the feature that $\iota_t^{\#}\psi^{+\prime} = 0 = \iota_t^{\#}\nabla_{\mathfrak{n}}^{E^*}\psi^{+\prime}$ whence both ψ^+ and $\psi^{+\prime}$ have vanishing initial conditions on Σ_t and satisfy the wave equation $D^{\mathrm{T}}\psi^{+\prime} = \varphi = D^{\mathrm{T}}\psi^+$ on all of M. Thus by the uniqueness according to Theorem 4.2.7 we conclude $\psi^+ = \psi^{+\prime}$. Hence the section ψ^+ does not depend on the choice of t as long as t is large enough. According to Theorem 4.2.19 the map which assigns φ to ψ^+ is a continuous linear map with respect to the $\mathcal{C}_0^\infty$- and $\mathcal{C}^\infty$-topology. Moreover, evaluating ψ^+ at a given point $p \in M$ is a E_p^*-valued continuous linear functional, namely the δ_p-functional. Thus the map $\varphi \mapsto \psi^+(p)$ is a continuous linear functional for every point $p \in M$. This defines the generalized section $F_M^+(p) \in \Gamma^{-\infty}(E) \otimes E_p^*$, i.e.

$$F_M^+(p) : \varphi \;\mapsto\; \psi^+(p)$$

with ψ^+ as above. By definition of $F_M^+(p)$ the map (4.3.1) is just the map $\varphi \mapsto \psi^+$ which is continuous according to Theorem 4.2.19 and yields a smooth section ψ^+. This shows (4.3.1) and (4.3.3) for the case of "+". We prove that $F_M^+(p)$ is a fundamental solution at p. For the two test sections $\varphi, D^{\mathrm{T}}\varphi \in \Gamma_0^\infty(E^*)$ we have resulting solutions $\psi^+, \widetilde{\psi}^+$ as above, i.e. $D^{\mathrm{T}}\psi^+ = \varphi$ and $D^{\mathrm{T}}\widetilde{\psi}^+ = D^{\mathrm{T}}\varphi$. Thus $D^{\mathrm{T}}(\widetilde{\psi}^+ - \varphi) = 0$ and both $\widetilde{\psi}^+$ and φ have vanishing initial conditions on Σ_t: the section φ even vanishes in an open neighborhood of Σ_t while $\widetilde{\psi}^+$ has vanishing initial conditions on Σ_t by construction. Thus by uniqueness we have $\widetilde{\psi}^+ - \varphi = 0$. Unwinding this gives

$$\left(DF_M^+(p)\right)(\varphi) = F_M^+(p)(D^{\mathrm{T}}\varphi) = \widetilde{\psi}^+(p) = \varphi(p),$$

hence $DF_M^+(p) = \delta_p$ follows as φ is an arbitrary test section. This gives us a fundamental solution $F_M^+(p)$ at every point $p \in M$. It remains to show that $F_M^+(p)$ is actually an advanced fundamental solution, i.e. $\operatorname{supp} F_M^+(p) \subseteq J_M^+(p)$. Since $J_M^+(p)$ is closed by global hyperbolicity of M we have to find an open neighborhood of $q \in M \setminus J_M^+(p)$ on which $F_M^+(p)$ vanishes. Thus let $q \notin J_M^+(p)$ be such a point. By closedness of $J_M^+(p)$ there is an open neighborhood of q such that $q' \notin J_M^+(p)$ for all q' in this neighborhood. We distinguish two cases. If $p \notin J_M^+(q)$ then we also have $p \notin J_M^+(q')$ for all q' in a small neighborhood of q. Thus we can choose q', q'' close to q with $q \in I_M^+(q') \cap I_M^+(q'')$ but $p \notin J_M^+(q'')$ and $p \notin J_M^-(q')$, see Figure 4.40. In this case $p \notin J_M(J_M(q'', q'))$. Since $I_M(q'', q')$ is an

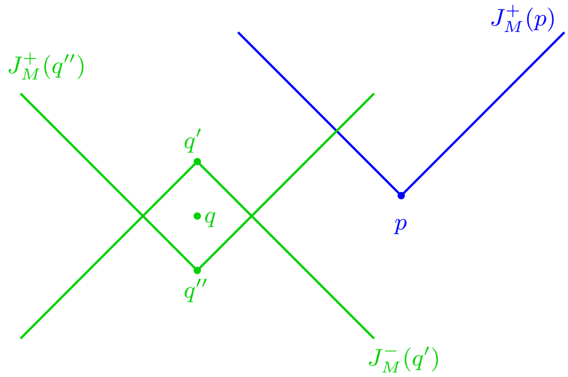

Figure 4.40: Choosing the points q' and q'' with $q \in J_M(q'', q')$ for q and p spacelike.

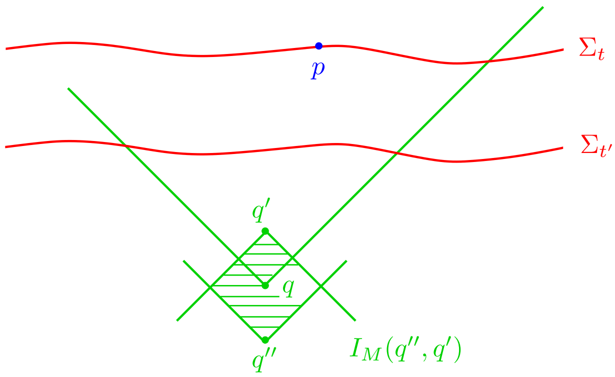

Figure 4.41: The points p and q do not lie spacelike to each other.

open neighborhood of q we have for all $\varphi \in \Gamma_0^\infty(E^*)$ with $\operatorname{supp} \varphi \subseteq I_M(q'', q')$ by Theorem 4.2.16, *i.)* the property $\operatorname{supp} \psi^+ \subseteq J_M(\operatorname{supp} \varphi) \subseteq J_M^+(q'') \cup J_M^-(q')$, where ψ^+ is the section with vanishing initial conditions for large times and $D^{\mathrm{T}} \psi^+ = \varphi$. Since $p \notin J_M^+(q'') \cup J_M^-(q')$ we have $0 = \psi^+(p) = F_M^+(p)\varphi$. However, this simple argument only works for p and q spacelike. Thus the other case is where p and q are not spacelike, but p is in $J_M^+(q)$. But then necessarily the time t of p is strictly larger than the one of q as $q \neq p$. We fix a time t' between t and the time of q and choose a point q' on Σ_t in the future $I_M^+(q)$ of q, see Figure 4.41. Moreover, let $q'' \in I_M^-(q)$ be arbitrary. This gives us an open diamond $I_M(q'', q')$ which is an open neighborhood of q. Let $\varphi \in \Gamma_0^\infty(E^*)$ have support in $I_M(q'', q')$ and let ψ^+ be the solution of $D^{\mathrm{T}} \psi^+ = \varphi$ with vanishing initial values for large times as before. Since $t > t'$ is clearly later than $\operatorname{supp} \varphi$ we have $\psi^+\big|_{\Sigma_t} = 0$. But this gives $\psi^+(p) = 0$ also in this case and hence $F_M^+(p)\varphi = 0$ for all such φ. This finally shows that $\operatorname{supp} F_M^+(p) \subseteq J_M^+(p)$ as wanted. The retarded case is analogous as usual. $\square$

We can strengthen the above result in the following way. As we have at least some rough counting of needed derivatives in Theorem 4.2.16, *ii.)* for the Cauchy problem we can use this to estimate the order of the Green functions $F_M^\pm(p)$:

Theorem 4.3.2 *Let (M, g) be a globally hyperbolic spacetime and $D \in \mathrm{DiffOp}^2(E)$ a normally hyperbolic differential operator. Then the unique advanced and retarded Green functions $F_M^\pm(p)$ of D at p are of global order*

$$\operatorname{ord} F_M^\pm(p) \leq 2n + 6. \tag{4.3.4}$$

More precisely, the linear map (4.3.3) *extends to a continuous linear map*

$$F_M^\pm : \Gamma_0^{2(k+n+1)}(E^*) \ni \varphi \mapsto F_M^\pm(\cdot)\varphi \in \Gamma^k(E^*) \tag{4.3.5}$$

for all $k \geq 2$ such that we still have

$$D^{\mathrm{T}} F_M^{\pm}(\,\cdot\,)\varphi = \varphi. \tag{4.3.6}$$

Proof. By Theorem 4.2.16 *ii.)* we can repeat the whole construction in the proof of Theorem 4.3.1 for a test section $\varphi \in \Gamma_0^{2(k+n+1)}(E^*)$. Indeed, the initial conditions for ψ^+ being zero for large times clearly satisfy the differentiability conditions of Theorem 4.2.16, *ii.)*. Thus we obtain a solution $\psi^+ \in \Gamma^k(E)$ of $D^{\mathrm{T}}\psi^+ = \varphi$. With the definition $F_M^+(p)\varphi = \psi^+(p)$ and hence $\psi^+ = F^+(\,\cdot\,)\varphi$ we get by Theorem 4.2.20 the continuity of (4.3.3). By construction, (4.3.6) still holds. Now let $p \in M$ be given and choose $k = 2$ which is the minimal one allowed by Theorem 4.2.16 and Theorem 4.2.20. Then the continuity of (4.3.3) implies that for all compact $K \subseteq M$ we find a constant $c > 0$ with

$$\left|F_M^+(p)\varphi\right| = \mathrm{p}_{\{p\},0}\left(F_M^+(\,\cdot\,)\varphi\right) \leq c\, \mathrm{p}_{K,2(k+n+1)}(\varphi).$$

But this shows that the local order of $F_M^+(\,\cdot\,)$ on the compactum K is less or equal than $2n+6$, independently on K. It is clear by the usual density argument that the map $F_M^+(p)$ defined here is indeed the unique extension of the advanced Green function defined in the previous Theorem. The retarded case is analogous. □

Remark 4.3.3 Again, the estimate on the order is usually very rough and even worse than the estimate we found in the local case. Nevertheless, the important point is that the order is *globally finite* and independent of p. Since in the construction of the solution ψ^+ we only needed the very special initial conditions $\iota^\#\psi^+ = 0 = \iota^\#\nabla_{\mathfrak{n}}^E\psi^+$ the proof of the local solution to the Cauchy problem as in Proposition 4.2.9 with finite differentiability simplifies drastically yielding a simplified recursion only involving the inhomogeneity. We leave it as an open task to improve the estimate (4.3.4) on the global order.

4.3.2 Green Operators

The fundamental solutions $F_M^{\pm}(p)$ were constructed as the map $\varphi \mapsto (p \mapsto F_M^{\pm}(p)\varphi)$ being a map $\Gamma_0^\infty(E^*) \longrightarrow \Gamma^\infty(E)$, i.e. the solution map from the Cauchy problem. We shall now investigate this map more closely as it provides almost an inverse to D. In general, one defines the following operators.

Definition 4.3.4 (Green Operators) *Let (M,g) be a time-oriented Lorentz manifold and $D \in \mathrm{DiffOp}^2(E)$ a normally hyperbolic differential operator. Then a continuous linear map*

$$G_U^{\pm} : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(E) \tag{4.3.7}$$

with

i.) $DG_M^{\pm} = \mathsf{id}_{\Gamma_0^\infty(E)}$,

ii.) $G_M^{\pm}D\big|_{\Gamma_0^\infty(E)} = \mathsf{id}_{\Gamma_0^\infty(E)}$,

iii.) $\mathrm{supp}(G_M^{\pm}u) \subseteq J_M^{\pm}(\mathrm{supp}\, u)^{\mathrm{cl}}$ *for all* $u \in \Gamma_0^\infty(E)$

is called an advanced and retarded Green operator for D, respectively.

Note that if the causal relation is not closed we have to put a closure in part *iii.)* by hand. In view of the local result in (4.2.8) one can imagine that a Green operator for D is linked to the fundamental solutions $G_M^{\pm}(p)$ of the dual differential operator $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$. In fact, we have the following proposition for general spacetimes, where we require $\mathrm{supp}\, G_M^{\pm}(p) \subseteq (J_M^{\pm}(p))^{\mathrm{cl}}$ in the case when the causal relation is not closed.

Proposition 4.3.5 (Green operators and fundamental solutions) *Let (M,g) be a time-oriented Lorentz manifold and $D \in \mathrm{DiffOp}^2(E)$ a normally hyperbolic differential operator.*

i.) *Assume $\{G^{\pm}_M(p)\}$ is a family of global advanced or retarded fundamental solutions of D^{T} at every point $p \in M$ with the following property: for every test section $u \in \Gamma^\infty_0(E)$ the section $p \mapsto G^{\pm}_M(p)u$ is a smooth section of E depending continuously on u and satisfying $DG^{\pm}_M(\,\cdot\,)u = u$. Then*

$$(G^{\pm}_M u)(p) = G^{\mp}_M(p)u \tag{4.3.8}$$

yield advanced and retarded Green operators for D, respectively.

ii.) *Assume $G^{\pm}_M$ are advanced or retarded Green operator for D, respectively. Then $G^{\pm}_M(p) : \Gamma^\infty_0(E) \longrightarrow \mathbb{C}$ defined by*

$$G^{\pm}_M(p)u = (G^{\mp}_M u)(p) \tag{4.3.9}$$

defines a family of advanced and retarded fundamental solutions of D^{T} at every point $p \in M$ with the properties described in i.), respectively.

Proof. For the first part we assume to have a family $\{G^{\pm}_M(p)\}_{p\in M}$ of advanced or retarded fundamental solutions of D^{T} with the above properties. By assumption, the resulting linear map (4.3.8) is continuous. It satisfies $DG^{\pm}_M = \mathsf{id}_{\Gamma^\infty_0(E)}$ also by assumption. Since the $G^{\pm}_M(p)$ are fundamental solutions of D^{T} we have

$$(G^{\pm}_M Du)(p) = G^{\mp}_M(p)(Du) = (D^{\mathrm{T}} G^{\mp}_M(p))(u) = \delta_p(u) = u(p)$$

for all $p \in M$ and $u \in \Gamma^\infty_0(E)$. Thus $G^{\pm}_M D = \mathsf{id}_{\Gamma^\infty_0(E)}$ as well. Finally, we have to check the support properties thereby explaining the flip from $\pm$ to $\mp$ in (4.3.8). Thus let $p \in M$ be given such that $0 \neq (G^{\pm}_M u)(p) = G^{\mp}(p)u$. Since the support of the distributions $G^{\mp}_M(p)$ is in $J^{\mp}_M(p)^{\mathrm{cl}}$ this implies that $\operatorname{supp} u$ has to intersect $J^{\mp}_M(p)^{\mathrm{cl}}$. Since $J^{\mp}_M(p)^{\mathrm{cl}} = I^{\mp}_M(p)^{\mathrm{cl}}$, see [45, Prop. 2.17], and since $\operatorname{supp} u$ has an open interior which is non-empty, we see that $\operatorname{supp} u$ also has to intersect $I^{\mp}_M(p)$. But then $p \in I^{\mp}_M(\operatorname{supp} u)$ whence $\operatorname{supp}(G^{\pm}_M u) \subseteq I^{\pm}_M(\operatorname{supp} u)^{\mathrm{cl}} = J^{\pm}_M(\operatorname{supp} u)^{\mathrm{cl}}$ follows, proving the first part. For the second part assume $G^{\pm}_M$ is given and define $G^{\pm}_M(p) = \delta_p \circ G^{\mp}_M$, according to (4.3.9). This is clearly a distribution since δ_p is continuous and $G^{\mp}_M$ is continuous by assumption. By construction, the section $p \mapsto G^{\pm}_M(p)u = (G^{\mp}_M u)(p)$ is smooth and depends continuously on u. We have

$$DG^{\mp}_M(\,\cdot\,)u = D\left(p \mapsto G^{\mp}_M(p)u\right) = DG^{\pm}_M u = u$$

as well as

$$\left(D^{\mathrm{T}} G^{\mp}_M(p)\right)(u) = G^{\mp}_M(p)(Du) = \left(G^{\pm}_M(Du)\right)(p) = u(p),$$

whence $G^{\mp}_M(p)$ is a fundamental solution satisfying also $DG^{\pm}_M(\,\cdot\,)u = u$. Finally, for the support we can argue as before in part *i.)*. ☐

Remark 4.3.6 (Green operators)

i.) If the causal relation "$\leq$" is closed then the definition of a Green operator simplifies and also the above proof simplifies. This will be the case for globally hyperbolic spacetimes.

ii.) At first glance, a Green operator of D looks like an inverse on the space of compactly supported sections. However, this is not quite correct as $G^{\pm}_M$ maps into $\Gamma^\infty(E)$ and not into $\Gamma^\infty_0(E)$. Nevertheless, the Green operator behaves very much like an inverse of $D\big|_{\Gamma^\infty_0(E)}$.

iii.) In general, Green operators do not exist: if e.g. M is a compact Lorentz manifold and $D = \square$ is the scalar d'Alembertian then the constant function 1 has compact support but satisfied $\square 1 = 0$. Thus $G\square 1 = 1$ is impossible for a linear map G.

In the case of a globally hyperbolic spacetime our construction of advanced and retarded fundamental solutions in Theorem 4.3.1 gives immediately advanced and retarded Green operators:

Corollary 4.3.7 *On a globally hyperbolic spacetime any normally hyperbolic differential operator has unique advanced and retarded Green operators.*

Proof. Indeed, the fundamental solutions were precisely constructed as in the proposition with the operator coming from the solvability of the Cauchy problem in Theorem 4.3.1. □

Having related the Green operators of D to the fundamental solutions of D^{T} we can also relate the Green operators of D and D^{T} directly. First we notice that, as we already did locally in Section 3.5, the Green operators allow for dualizing:

Proposition 4.3.8 *Let (M, g) be globally hyperbolic and let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator with advanced and retarded Green operators $G_M^{\pm} : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(E)$.*

i.) The dual map $(G_M^{\pm})' : \Gamma_0^{-\infty}(E^*) \longrightarrow \Gamma^{-\infty}(E^*)$ is weak continuous and satisfies*

$$D^{\mathrm{T}}(G_M^{\pm})'(\varphi) = \varphi = (G_M^{\pm})' D^{\mathrm{T}} \varphi \tag{4.3.10}$$

for all generalized sections $\varphi \in \Gamma_0^{-\infty}(E^*)$ with compact support.

ii.) For a generalized section $\varphi \in \Gamma_0^{-\infty}(E^*)$ with compact support we have

$$\operatorname{supp}(G_M^{\pm})'(\varphi) \subseteq J_M^{\mp}(\operatorname{supp} \varphi). \tag{4.3.11}$$

Proof. Since $G_M^{\pm} : \Gamma_0^\infty(E) \longrightarrow \Gamma^\infty(E)$ is linear and continuous we have an induced dual map $(G_M^{\pm})' : \Gamma^\infty(E)' = \Gamma_0^{-\infty}(E^*) \longrightarrow \Gamma_0^\infty(E)' = \Gamma^{-\infty}(E^*)$ where we identify the dual spaces as usual by means of the canonical volume density μ_g. Then $(G_M^{\pm})'$ is automatically weak* continuous. To prove (4.3.10) we take a test section $u \in \Gamma_0^\infty(E)$ and compute

$$\left(D^{\mathrm{T}}(G_M^{\pm})'(\varphi)\right)(u) = (G_M^{\pm})'(\varphi)(Du) = \varphi\left(G_M^{\pm} Du\right) = \varphi(u)$$

by the very definitions. Since $\Gamma_0^\infty(E) \subseteq \Gamma^\infty(E)$ is dense this is sufficient to show the first part of (4.3.10), which is understood as an identity between generalized sections with compact support. For the other part we compute

$$\left((G_M^{\pm})' D^{\mathrm{T}} \varphi\right)(u) = (D^{\mathrm{T}} \varphi)(G_M^{\pm} u) = \varphi\left(D G_M^{\pm} u\right) = \varphi(u).$$

Note that $D^{\mathrm{T}}\varphi$ has again compact support whence the above computation is indeed justified. This proves (4.3.10). For the second statement let $u \in \Gamma_0^\infty(E)$ be a test section. Then $(G_M^{\pm})'(\varphi)u = \varphi(G_M^{\pm} u)$. Since $\operatorname{supp}(G_M^{\pm} u) \subseteq J_M^{\pm}(\operatorname{supp} u)$ we see that $\varphi(G_M^{\pm} u)$ vanishes if $\operatorname{supp}\varphi \cap J_M^{\pm}(\operatorname{supp} u) = \emptyset$. But this means $J_M^{\mp}(\operatorname{supp}\varphi) \cap \operatorname{supp} u = \emptyset$. Thus for $\operatorname{supp} u \subseteq M \setminus J_M^{\mp}(\operatorname{supp}\varphi)$ we have $(G_M^{\pm})'(\varphi)u = 0$ which implies (4.3.11), since $J_M^{\pm}(\operatorname{supp}\varphi)$ is already closed. □

As in the local situation we can now apply $(G_M^{\pm})'$ to generalized sections φ which are actually smooth, i.e. $\varphi \in \Gamma_0^\infty(E^*)$. We expect that we obtain the Green operators of D^{T}. Here we need the following simple Lemma:

Lemma 4.3.9 *Let (M, g) be a globally hyperbolic spacetime and let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator with advanced and retarded Green operators $G_M^{\pm}$. Moreover, denote the corresponding Green operators of $D^{\mathrm{T}} \in \mathrm{DiffOp}^2(E^*)$ by $F_M^{\pm}$. Then we have for $\varphi \in \Gamma_0^\infty(E^*)$ and $u \in \Gamma_0^\infty(E)$*

$$\int_M (F_M^{\pm}\varphi) \cdot u \, \mu_g = \int_M \varphi \cdot (G_M^{\mp} u) \, \mu_g. \tag{4.3.12}$$

Proof. The lemma is a simple integrations by parts argument. First we note that $F_M^{\pm}\varphi$ has (non-compact) support in $J_M^{\pm}(\operatorname{supp}\varphi)$ while $G_M^{\mp} u$ has (non-compact) support in $J_M^{\mp}(\operatorname{supp} u)$ by the very definition of Green operators. It follows from the global hyperbolicity that the overlap $J_M^{\pm}(\operatorname{supp}\varphi) \cap J_M^{\mp}(\operatorname{supp}\varphi)$ is compact, see Figure 4.42. Thus writing $u = DG_M^{\mp} u$ we get

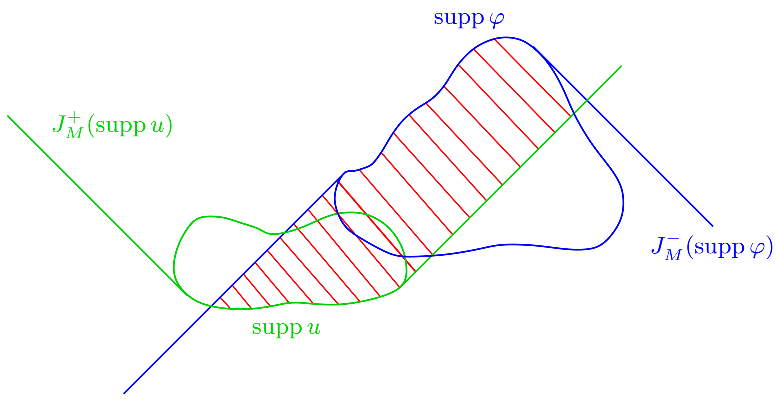

Figure 4.42: The compact overlap of $J_M^+(\operatorname{supp} u)$ and $J_M^-(\operatorname{supp} \varphi)$.

$$
\begin{aligned}
\int_M (F^{\pm}\varphi) \cdot u \, \mu_g &= \int_M (F_M^{\pm}\varphi) \cdot (DG_M^{\mp}u) \, \mu_g \\
&\stackrel{(*)}{=} \int_M (D^{\mathrm{T}} F_M^{\pm}\varphi) \cdot (G_M^{\mp}u) \, \mu_g \\
&= \int_M \varphi \cdot (G_M^{\mp}u) \, \mu_g,
\end{aligned}
$$

where we have used $D^{\mathrm{T}} F_M^{\pm}\varphi = \varphi$ and the compactness of the overlap to justify the integration by parts in $(*)$. □

From this lemma we immediately see that the dual operator $(G_M^{\mp})' : \Gamma_0^{-\infty}(E^*) \longrightarrow \Gamma^{-\infty}(E^*)$ applied to a distributional section which is actually smooth, i.e. to $\varphi \in \Gamma_0^{\infty}(E^*) \subseteq \Gamma_0^{-\infty}(E^*)$ is given by

$$
(G_M^{\mp})'\varphi = F_M^{\pm}\varphi. \tag{4.3.13}
$$

Indeed, this is just the content of (4.3.12) where we interpret the right hand side as the distributional section $\varphi \in \Gamma_0^{-\infty}(E^*)$ evaluated on $G_M^{\mp}(u)$ as usual. In particular, the dual map $(G_M^{\mp})'$ yields a smooth section and not just a distributional one when applied to $\varphi \in \Gamma_0^{\infty}(E^*)$. Moreover, since $F_M^{\pm}$ is continuous with respect to the $\mathcal{C}_0^{\infty}$- and $\mathcal{C}^{\infty}$-topology according to Theorem 4.3.1 we have also continuity of the dual operators $(G_M^{\mp})'$ on $\Gamma_0^{\infty}(E^*)$ with respect to the $\mathcal{C}_0^{\infty}$- and $\mathcal{C}^{\infty}$-topology. This way, we obtain the global analogues of the local results obtained in Section 3.4.2. We summarize the discussion the in the following theorem:

Theorem 4.3.10 *Let (M, g) be globally hyperbolic and let $D \in \operatorname{DiffOp}^2(E)$ be a normally hyperbolic differential operator. Denote the global advanced and retarded Green operators of D by $G_M^{\pm}$ and those of D^{T} by $F_M^{\pm}$, respectively.*

i.) For the dual operators we have

$$
(G_M^{\pm})'\big|_{\Gamma_0^{\infty}(E^*)} = F_M^{\mp} \tag{4.3.14}
$$

$$
(F_M^{\pm})'\big|_{\Gamma_0^{\infty}(E)} = G_M^{\mp}. \tag{4.3.15}
$$

ii.) The duals of the Green operators restrict to maps

$$
(G_M^{\pm})' : \Gamma_0^{\infty}(E^*) \longrightarrow \Gamma^{\infty}(E^*), \tag{4.3.16}
$$

$$
(F_M^{\pm})' : \Gamma_0^{\infty}(E) \longrightarrow \Gamma^{\infty}(E), \tag{4.3.17}
$$

which are continuous with respect to the $\mathcal{C}_0^{\infty}$- and $\mathcal{C}^{\infty}$-topologies, respectively.

iii.) The Green operators have unique weak continuous extensions to operators*

$$G_M^{\pm} : \Gamma_0^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(E) \tag{4.3.18}$$

$$F_M^{\pm} : \Gamma_0^{-\infty}(E^*) \longrightarrow \Gamma^{-\infty}(E^*) \tag{4.3.19}$$

satisfying

$$\operatorname{supp}(G_M^{\pm} u) \subseteq J_M^{\pm}(\operatorname{supp} u) \tag{4.3.20}$$

$$\operatorname{supp}(F_M^{\pm} \varphi) \subseteq J_M^{\pm}(\operatorname{supp} \varphi), \tag{4.3.21}$$

respectively. For these extensions one has

$$G_M^{\pm} = \left(F_M^{\mp}\big|_{\Gamma_0^\infty(E^*)}\right)' \tag{4.3.22}$$

$$F_M^{\pm} = \left(G_M^{\mp}\big|_{\Gamma_0^\infty(E)}\right)'. \tag{4.3.23}$$

Proof. Indeed, part *i.)* was already discussed and part *ii.)* is clear by part *i.)* and the continuity of Green operators. The last part is also clear since the corresponding dual operators provide us with an extension of the Green operators according to *i.)*. The uniqueness of the extension is clear as the smooth sections with compact support are (sequentially) dense in the distributional sections with compact support: this follows analogously to the density statement in Theorem 1.3.18, *v.)* for the case of arbitrary distributional sections. Then (4.3.20) and (4.3.21) are obtained from Proposition 4.3.8, *ii.)* applied to D^{T} and D, respectively. Finally (4.3.22) and (4.3.23) are clear. □

Remark 4.3.11 With some slight abuse of notation we do not distinguish between the Green operators and their canonical extension to generalized sections. This gives the short hand version

$$G_M^{\pm} = \left(F_M^{\mp}\right)' \tag{4.3.24}$$

of (4.3.22) and (4.3.23). In particular, the Green operators of D^{T} are completely determined by those of D and vice versa.

As a first application of the extended Green operators we obtain a solution of the wave equation for arbitrary compactly supported inhomogeneity with good causal behaviour:

Theorem 4.3.12 *Let (M, g) be a globally hyperbolic spacetime and $D \in \mathrm{DiffOp}^2(E)$ normally hyperbolic with advanced and retarded Green operators $G_M^{\pm}$.*

i.) The Green operators $G_M^{\pm} : \Gamma_0^{-\infty}(E) \longrightarrow \Gamma^{-\infty}(E)$ satisfy

$$DG_M^{\pm} = \mathsf{id}_{\Gamma_0^{-\infty}(E)} = G_M^{\pm} D\big|_{\Gamma_0^{-\infty}(E)}. \tag{4.3.25}$$

ii.) For every $v \in \Gamma_0^{-\infty}(E)$, every smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow M$ with

$$\operatorname{supp} v \subseteq I_M^{+}(\Sigma), \tag{4.3.26}$$

and all $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$ there exists a unique generalized section $u_+ \in \Gamma^{-\infty}(E)$ with

$$Du_+ = v, \tag{4.3.27}$$

$$\operatorname{supp} u_+ \subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0) \cup J_M^{+}(\operatorname{supp} v), \tag{4.3.28}$$

$$\operatorname{sing\,supp} u_+ \subseteq J_M^{+}(\operatorname{supp} v), \tag{4.3.29}$$

$$\iota^\# u_+ = u_0 \quad \text{and} \quad \iota^\# \nabla_{\mathfrak{n}}^E u = \dot{u}_0. \tag{4.3.30}$$

The section u_+ depends weak continuously on v and continuously on $u_0, \dot{u}_0$.*

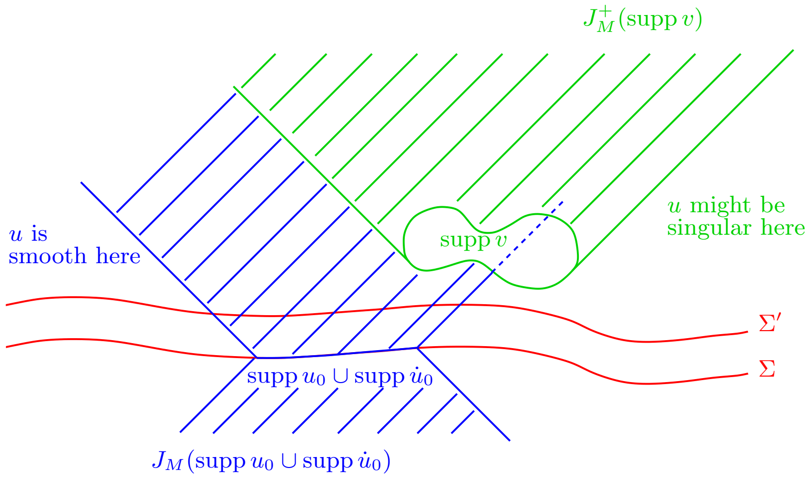

Figure 4.43: The figure shows the supports of the inhomogeneity v, the initial conditions $u_0, \dot{u}_0$ and where the solution to the inhomogeneous wave equation might be singular.

iii.) An analogous statement holds for the case $\operatorname{supp} v \subseteq I_M^-(\Sigma)$.

Proof. For the first part we can use the fact that all involved maps are weak* continuous and $\Gamma_0^\infty(E)$ is weak* dense in $\Gamma_0^{-\infty}(E)$. Then (4.3.25) is just a consequence of the defining properties of a Green operator on $\Gamma_0^\infty(E)$. For the second part we first notice that $G_M^+ v \in \Gamma^{-\infty}(E)$ is a generalized section with support in $J_M^+(\operatorname{supp} v)$ according to (4.3.20) and $DG_M^+ v = v$ according to the first part. Let $w \in \Gamma^{+\infty}(E)$ be the unique solution to the Cauchy problem $Dw = 0$ and $\iota^\# w = u_0$ and $\iota^\# \nabla_{\mathfrak{n}}^E w = \dot{u}_0$ whose existence and uniqueness is guaranteed by Theorem 4.2.16, *i.)*. We set $u = w + G_M^+ v$. This is a generalized section with $Du = v$ as w solves the homogeneous wave equation. Moreover, we have

$$
\begin{aligned}
\operatorname{supp} u &= \operatorname{supp}(w + G_M^+ v) \\
&\subseteq \operatorname{supp} w \cup \operatorname{supp} G_M^+ v \\
&\subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0) \cup J_M^+(\operatorname{supp} v),
\end{aligned}
$$

according to (4.2.29) and (4.3.20). Since w is smooth we also have $\operatorname{sing\,supp} u = \operatorname{sing\,supp} G_M^+ v \subseteq J_M^+(\operatorname{supp} v)$. Now $\operatorname{supp} v \subseteq J_M^+(\Sigma)$ implies that $M \setminus J_M^+(\operatorname{supp} v)$ is an open neighborhood of Σ, see Figure 4.43. Thus u is smooth on an open neighborhood of Σ whence the restriction of u is well-defined. Note that for a general element of $\Gamma^{-\infty}(E)$ this would not be possible. Thus (4.3.30) is meaningful and we have $\iota^\# u = \iota^\# w = u_0$ as well as $\iota^\# \nabla_{\mathfrak{n}}^E u = \iota^\# \nabla_{\mathfrak{n}}^E w = \dot{u}_0$. Hence u has all required properties. Note that u depends weak* continuously on v as G_M^+ is weak* continuous. Moreover, w depends continuously on u_0, $\dot{u}_0$ with respect to the $\mathcal{C}^\infty$- and $\mathcal{C}_0^\infty$-topologies. Finally, suppose that $\widetilde{u}$ is another generalized section satisfying the four properties (4.3.27) - (4.3.30). Then $u - \widetilde{u}$ solves the homogeneous wave equation and has singular support away from Σ, too. Thus we can speak of initial conditions of $u - \widetilde{u}$ on Σ which are now identically zero. Let Σ' be another Cauchy hypersurface separating Σ and $J_M^+(\operatorname{supp} v)$ as in Figure 4.43, which we clearly can find. Then in the globally hyperbolic spacetime $I_M^-(\Sigma')$ we have a smooth solution $u - \widetilde{u}$ of the homogeneous wave equation with vanishing initial conditions. Hence $(u - \widetilde{u})\big|_{I_M^-(\Sigma')} = 0$ by the uniqueness Theorem 4.2.5. But this implies that the generalized section $u - \widetilde{u}$ meets the conditions of Theorem 4.1.11, which gives $u - \widetilde{u} = 0$ everywhere. □

Remark 4.3.13 With other words, we have again a well-posed Cauchy problem in this more general context of generalized sections as inhomogeneities. Note that due to $u \in \Gamma^{-\infty}(E)$ the weak* continuity is the best we can hope for. Analogously to Theorem 4.2.16, *ii.)* we can also solve the analogous Cauchy problem with finite differentiability of the initial conditions. In this case we can have singular support outside of $J^+_M(\operatorname{supp} v)$ but only a rather mild one: on $M \setminus J^+_M(\operatorname{supp} v)$ the solution u is $\mathcal{C}^k$ whence the restrictions to Σ still make sense.

4.3.3 The Image of the Green Operators

In this section we want to characterize the image of the Green operators $G^\pm_M$ in $\Gamma^\infty(E)$ in some more detail. Since $\operatorname{supp}(G^\pm_M u) \subseteq J_M(\operatorname{supp} u)$ for $u \in \Gamma^\infty_0(E)$ we see already here that in general, the maps $G^\pm_M$ can not be surjective. In general, M can not be written as $J_M(K)$ for a compact subset. This would require a *compact* Cauchy hypersurface Σ. These considerations motivate the following definition:

Definition 4.3.14 (The space $\Gamma^k_{\mathrm{sc}}(E)$) *Let $k \in \mathbb{N} \cup \{+\infty\}$. For a time-oriented Lorentz manifold we denote by $\Gamma^k_{\mathrm{sc}}(E) \subseteq \Gamma^k(E)$ those section u for which there exists a compact subset $K \subseteq M$ with* $\operatorname{supp} u \subseteq J_M(K)$.

Of course, we are mainly interested in the globally hyperbolic case. The notion "sc" refers to *spacelike compact support*. We want to endow the subspace $\Gamma^k_{\mathrm{sc}}(E) \subseteq \Gamma^k(E)$ with a suitable topology analogous to the one of $\Gamma^k_0(E)$. Indeed, $\Gamma^k_{\mathrm{sc}}(E)$ is dense in $\Gamma^k(E)$ for the $\mathcal{C}^\infty$-topology as $\Gamma^k_0(E) \subseteq \Gamma^k_{\mathrm{sc}}(E) \subseteq \Gamma^k(E)$ is already dense. Thus we need a finer topology for $\Gamma^k_{\mathrm{sc}}(E)$ to have good completeness properties. Since $J_M(K)$ is closed in M on a globally hyperbolic spacetime we can use Lemma 1.1.10 to construct a LF topology for $\Gamma^k_{\mathrm{sc}}(E)$ as follows: For $K \subseteq K'$ we have $J_M(K) \subseteq J_M(K')$ whence

$$\Gamma^k_{J_M(K)}(E) \hookrightarrow \Gamma^k_{J_M(K')}(E) \tag{4.3.31}$$

is continuous in the $\mathcal{C}^k_{J_M(K)}$- and $\mathcal{C}^k_{J_M(K')}$-topology and we have a closed image. Since the induced topology from the $\mathcal{C}^k_{J(K')}$-topology on the image of (4.3.31) is again the $\mathcal{C}^k_{J(K)}$-topology we indeed have a nice embedding. Finally, for an exhausting sequence $K_n \subseteq M$ of compacta we have eventually $J_M(K) \subseteq J_M(K_n)$. Thus a countable sequence of subsets exhausts all $J_M(K)$'s. These are the prerequisites for the strict inductive limit topology analogously to the case of $\Gamma^\infty_0(E)$ as formulated in Theorem 1.1.11. We call the resulting topology the $\mathcal{C}^k_{\mathrm{sc}}$*-topology*. Without going into further details we state the consequences literally translating from Theorem 1.1.11.

Theorem 4.3.15 (LF topology for $\Gamma^k_{\mathrm{sc}}(E)$) *Let (M, g) be a time-oriented Lorentz manifold with closed causal relation and let $k \in \mathbb{N}_0 \cup \{+\infty\}$. Endow $\Gamma^k_{\mathrm{sc}}(E)$ with the inductive limit topology coming from* (4.3.31).

i.) *$\Gamma^k_{\mathrm{sc}}(E)$ is a Hausdorff locally convex complete and sequentially complete topological vector space.*

ii.) *All inclusions $\Gamma^k_{J_M(K)}(E) \hookrightarrow \Gamma^k_{\mathrm{sc}}(E)$ are continuous and the $\mathcal{C}^k_{\mathrm{sc}}$-topology is the finest locally convex topology on $\Gamma^k_{\mathrm{sc}}(E)$ with this property. Every $\Gamma^k_{J_M(K)}(E)$ is closed in $\Gamma^k_{\mathrm{sc}}(E)$ and the induced topology from the $\mathcal{C}^k_{\mathrm{sc}}$-topology is again the $\mathcal{C}^k_{J_M(K)}$-topology.*

iii.) *A sequence $u_n \in \Gamma^k_{\mathrm{sc}}(E)$ is a $\mathcal{C}^k_{\mathrm{sc}}$-Cauchy sequence iff there is a compact subset $K \subseteq M$ with $u_n \in \Gamma^k_{J_M(K)}(E)$ and u_n is a $\mathcal{C}^k_{J_M(K)}$-Cauchy sequence. An analogous statement holds for convergent sequences.*

iv.) *If V is a locally convex vector space then a linear map $\Phi : \Gamma^k_{\mathrm{sc}}(E) \longrightarrow V$ is $\mathcal{C}^k_{\mathrm{sc}}$-continuous iff all restrictions $\Phi\big|_{\Gamma^k_{J_M(K)}} : \Gamma^k_{J_M(K)}(E) \longrightarrow V$ are $\mathcal{C}^k_{J_M(K)}$-continuous. It suffices to check this for an exhausting sequence of compacta.*

v.) If in addition M is globally hyperbolic with a smooth spacelike Cauchy hypersurface Σ then $\Gamma^k_{\mathrm{sc}}(E) = \Gamma^k(E)$ iff Σ is compact in which case the $\mathcal{C}^k_{\mathrm{sc}}$- and the $\mathcal{C}^k$-topologies coincide. Otherwise the Γ^k_{sc}-topology is strictly finer. In fact,

$$\iota^\# : \Gamma^k_{\mathrm{sc}}(E) \longrightarrow \Gamma^k_0(\iota^\# E) \tag{4.3.32}$$

is a surjective linear map which is continuous in the $\mathcal{C}^k_{\mathrm{sc}}$- and $\mathcal{C}^k_0$-topology. It furthermore has continuous right inverses.

Proof. First we note that for an exhausting sequence K_n of compacta we have $K \subseteq K_n$ for all compacta and n suitably large. Thus countably many K_n will suffice to specify the inductive limit topology of $\Gamma^k_{\mathrm{sc}}(E)$. Since we have the continuous embedding with closed image (4.3.31) and the correct induced topology on the image, we are indeed in the situation of a countable strict inductive limit of Fréchet spaces, see again e.g. [34, Sect. 4.6] for details. In particular, the parts *i.) - iv.)* are consequences of the general properties of LF topologies. For the last part it is clear that if Σ is a compact Cauchy hypersurface then $J_M(\Sigma) = M$ whence the $\mathcal{C}^k_{\mathrm{sc}}$-topology simply coincides with the $\mathcal{C}^k$-topology as $\Gamma^k_{J_M(\Sigma)}(E)$ *is* already the inductive limit. Thus assume that Σ is not compact. Moreover, let $K \subseteq \Sigma$ be a compact subset in Σ. Then the restriction of a section $u \in \Gamma^k_{J_M(K)}(E)$ to Σ yields a section $\iota^\# u \in \Gamma^k_K(\iota^\# E)$. Moreover, we clearly have that the linear map

$$\iota^\# : \Gamma^k_{J_M(K)} \ni u \;\mapsto\; \iota^\# u \in \Gamma^k_K(i^\# E) \tag{$*$}$$

is continuous. This is clear from the concrete form of the seminorms defining the $\mathcal{C}^k$-topology on M and Σ, respectively. Here we see that in general

$$\iota^\# : \Gamma^\infty_{\mathrm{sc}} \longrightarrow \Gamma^\infty_0(\iota^\# E),$$

hence $\Gamma^\infty_{\mathrm{sc}}(E) \subsetneq \Gamma^\infty(E)$ follows from $\Gamma^\infty_0(i^\# E) \subsetneq \Gamma^\infty(i^\# E)$ at once. Moreover, since $(*)$ is continuous for all such $K \subseteq M$ we see that also

$$\Gamma^\infty_{J_M(K)}(E) \longrightarrow \Gamma^\infty_K(\iota^\# E) \hookrightarrow \Gamma^\infty_0(\iota^\# E) \tag{$**$}$$

is continuous. Now we use that an exhausting sequence $K_n \subseteq \Sigma$ *inside of* Σ still provides an exhausting sequence $J_M(K_n) \subseteq M$ of M. Thus we can use $(**)$ to conclude the continuity of (4.3.32) by part *iv.)*. Conversely, using the fact that M is diffeomorphic to $\mathbb{R} \times \Sigma$ we can extend a section $u_0 \in \Gamma^\infty_0(\iota^\# E)$ to M by using the *prolongation map*

$$\mathrm{prol}(u_0)\big|_{(t,\sigma)} = u_0(\sigma), \tag{$\star$}$$

i.e. $\mathrm{prol}(u_0) = \mathrm{pr}_2^\# u_0$. Note that the vector bundle E on M can be identified with the pull-back bundle $\mathrm{pr}_2^\# \iota^\# E \longrightarrow M$ since the time axis is topologically trivial. Here $\mathrm{pr}_2 : M = \mathbb{R} \times \Sigma \longrightarrow \Sigma$ is the projection onto Σ as usual. Note that $(\star)$ makes use of the diffeomorphism $M \simeq \mathbb{R} \times \Sigma$ and is *not* canonical. If $u_0 \in \Gamma^k_K(\iota^\# E)$ then $\mathrm{prol}(u_0) \in \Gamma^k_{\mathrm{pr}_2^{-1}(K)}(E) \subseteq \Gamma^k_{J_M(K)}(E)$ since clearly $\mathrm{pr}_2^{-1}(K)$ is inside $J_M(K)$, see Figure 4.44, as the curve $t \mapsto (t, \sigma)$ is clearly timelike, see also the proof of Proposition 4.2.8. Since $\mathrm{prol}(u_0)$ is "constant" in time it is easy to see that $\mathrm{prol} : \Gamma^k_K(\iota^\# E) \longrightarrow \Gamma^k_{\mathrm{pr}_2^{-1}(K)}(E) \subseteq \Gamma^k_{J_M(K)}(E)$ is continuous. Then also

$$\mathrm{prol} : \Gamma^k_K(\iota^\# E) \longrightarrow \Gamma^k_{\mathrm{sc}}(E)$$

is continuous by part *ii.)*. Now the characterization of the $\mathcal{C}^\infty_0$-topology asserts that $\mathrm{prol} : \Gamma^k_0(\iota^\# E) \longrightarrow \Gamma^k_{\mathrm{sc}}(E)$ is continuous as well since $K \subseteq \Sigma$ was an arbitrary compact subset, see again Theorem 1.1.11, *iv.)*. Since by construction $\iota^\# \mathrm{prol} = \mathsf{id}$ we finally showed the last part. Note that the $\mathcal{C}^k_{\mathrm{sc}}$-topology is clearly strictly finer because $\Gamma^k_{\mathrm{sc}}(E) \subseteq \Gamma^k(E)$ is dense in the Γ^k-topology but $\Gamma^k_{\mathrm{sc}}(E)$ is complete in the Γ^k_{sc}-topology. □

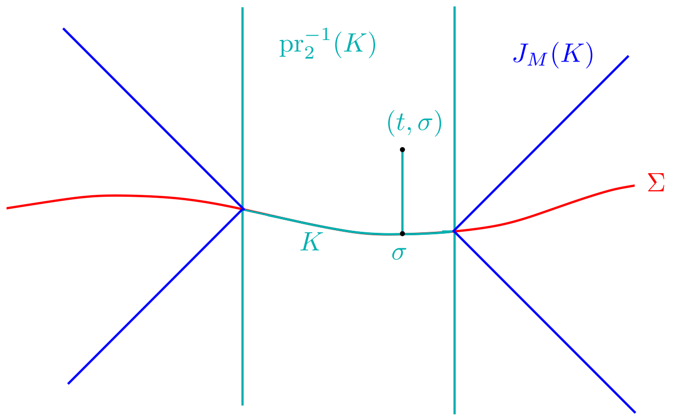

Figure 4.44: The pre-image of a compactum $K \subseteq \Sigma$ under the projection pr_2 is inside the causal future of K in a globally hyperbolic manifold.

Remark 4.3.16 (The $\mathcal{C}^k_{\mathrm{sc}}$-topology) We can repeat the discussion of continuous maps also for the $\mathcal{C}^k_{\mathrm{sc}}$-topology in complete analogy to the case of the $\mathcal{C}^k_0$-topology as in Subsection 1.1.2 and Subsection 1.2.3. In particular, any differential operator $D \in \mathrm{DiffOp}^k(E; F)$ of order k gives a continuous linear map

$$D : \Gamma^{k+\ell}_{\mathrm{sc}}(E) \longrightarrow \Gamma^{\ell}_{\mathrm{sc}}(F) \tag{4.3.33}$$

with respect to the $\mathcal{C}^{k+\ell}_{\mathrm{sc}}$- and the $\mathcal{C}^{\ell}_{\mathrm{sc}}$-topology for all $\ell \in \mathbb{N}_0 \cup \{+\infty\}$. We also have approximation theorems resulting from the ones in Subsection 1.1.3.

The space $\Gamma^\infty_{\mathrm{sc}}(E) \subseteq \Gamma^\infty(E)$ is the natural target space for the Green operators $G^\pm_M$ since the causality requirement

$$\mathrm{supp}(G^\pm_M(u)) \subseteq J_M(\mathrm{supp}\, u) \tag{4.3.34}$$

immediately implies $G^\pm_M(u) \in \Gamma^k_{\mathrm{sc}}(E)$. The continuity of $G^\pm_M$ with respect to the $\mathcal{C}^\infty$-topology on $\Gamma^\infty(E)$ implies also the continuity with respect to the in general strictly finer $\mathcal{C}^\infty_{\mathrm{sc}}$-topology:

Proposition 4.3.17 *Let (M, g) be a time-oriented Lorentz manifold with closed causal relation. Assume that $G^\pm_M$ are advanced or retarded Green operators for a normally hyperbolic differential operator $D \in \mathrm{DiffOp}^2(E)$. Then*

$$G^\pm_M : \Gamma^\infty_0(E) \longrightarrow \Gamma^\infty_{\mathrm{sc}}(E) \tag{4.3.35}$$

is continuous with respect to the $\mathcal{C}^\infty_{\mathrm{sc}}$- and $\mathcal{C}^\infty_0$-topology.

Proof. We know that $G^\pm_M : \Gamma^\infty_0(E) \longrightarrow \Gamma^\infty(E)$ is continuous by definition. Thus let $K \subseteq M$ be compact then $G^\pm_M : \Gamma^\infty_K(E) \longrightarrow \Gamma^\infty(E)$ is continuous in the $\mathcal{C}^\infty_K$- and $\mathcal{C}^\infty$-topology be Theorem 1.1.11, *iv.)*. Since the image is in $\Gamma^\infty_{J_M(K)}(E)$ and the $\mathcal{C}^\infty_{J_M(K)}$-topology of $\Gamma^\infty_{J_M(K)}$ is the subspace topology inherited from $\Gamma^\infty(E)$ we have continuity of

$$G^\pm_M : \Gamma^\infty_K(E) \longrightarrow \Gamma^\infty_{J_M(K)}(E)$$

for all compact subsets $K \subseteq M$. By Theorem 4.3.15, *ii.)* we conclude that also

$$G^\pm_M : \Gamma^\infty_K(E) \longrightarrow \Gamma^\infty_{\mathrm{sc}}(E)$$

is continuous. Since K was arbitrary, by Theorem 1.1.11, *iv.)* we have the continuity of (4.3.35). □

Now we come to the main result of this section which describes the image of the *difference* of the advanced and the retarded Green operator: as already in the local case we consider the *propagator*

$$G_M = G^+_M - G^-_M : \Gamma^\infty_0(E) \longrightarrow \Gamma^\infty_{\mathrm{sc}}(E), \tag{4.3.36}$$

if $G_M^{\pm}$ are advanced and retarded Green operators for a normally hyperbolic differential operator D. Here we have the following statement:

Theorem 4.3.18 *Let (M, g) be a time-oriented Lorentz manifold with closed causal relation. Assume that a normally hyperbolic differential operator $D \in \mathrm{DiffOp}^2(E)$ has advanced and retarded Green operators $G_M^{\pm}$.*

i.) The sequence of linear maps

$$0 \longrightarrow \Gamma_0^\infty(E) \stackrel{D}{\longrightarrow} \Gamma_0^\infty(E) \stackrel{G_M}{\longrightarrow} \Gamma_{\mathrm{sc}}^\infty(E) \stackrel{D}{\longrightarrow} \Gamma_{\mathrm{sc}}^\infty(E) \tag{4.3.37}$$

is a complex of continuous linear maps.

ii.) The complex (4.3.37) is exact at the first $\Gamma_0^\infty(E)$.

iii.) If (M, g) is globally hyperbolic then (4.3.37) is exact everywhere.

Proof. The continuity refers to the natural topologies of $\Gamma_0^\infty(E)$ and $\Gamma_{\mathrm{sc}}^\infty(E)$, respectively, and follows from Remark 4.3.16 and Proposition 4.3.17. From the very definition of Green operators it follows that $G_M \circ D = 0 = D \circ G_M$ on $\Gamma_0^\infty(E)$. This shows that (4.3.37) is a complex. To show exactness at the first $\Gamma_0^\infty(E)$ we have to show that D is injective on $\Gamma_0^\infty(E)$. Thus let $u \in \Gamma_0^\infty(E)$ with $Du = 0$ be given. Then $0 = G_M^+ Du = u$ shows the injectivity of D. For the last part assume that (M, g) is globally hyperbolic. To show exactness at the second $\Gamma_0^\infty(E)$ we have to show $\operatorname{im} D\big|_{\Gamma_0^\infty(E)} = \ker G_M$. We already know "$\subseteq$" hence we consider $u \in \Gamma_0^\infty(E)$ with $G_M u = 0$. We know that $v = G_M^+ u = G_M^- u$ has support in $J_M^+(\operatorname{supp} u)$ as well as in $J_M^-(\operatorname{supp} u)$ as $G_M^{\pm}$ are advanced and retarded Green operators. This shows $\operatorname{supp} v \subseteq J_M^+(\operatorname{supp} u) \cap J_M^-(\operatorname{supp} u)$ which is compact. Indeed, the intersection of J_M^+ and J_M^- of compact subsets like $\operatorname{supp} u$ is again compact on a globally hyperbolic spacetime. This implies $v \in \Gamma_0^\infty(E)$. Since in general $DG_M^+ u = u$ we see $u = Dv$ with u compactly supported. This shows exactness at the second place. To show exactness at the third place we have to show that $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ with $Du = 0$ is actually of the form $u = G_M v$ with $v \in \Gamma_0^\infty(E)$. Thus let $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ be such a section. For $u \in \Gamma_{\mathrm{sc}}^\infty(E)$, the support of u is contained in some $J_M(K')$ with $K' \subseteq M$ compact. Choosing an open neighborhood of K' with compact closure K, i.e. $K' \subseteq \mathring{K} \subseteq K$, we see that $\operatorname{supp} u \subseteq I_M^+(K) \cup I_M^-(K)$. The two subsets $I_M^{\pm}(K)$ provide an open cover of the open subset $I_M(K) \subseteq M$. Thus we can find a subordinate partition of unity $\chi^+, \chi^- \in \mathcal{C}^\infty(I_M(K))$ with $\operatorname{supp} \chi^{\pm} \subseteq I_M^{\pm}(K)$ and $\chi^+ + \chi^- = 1$ on $I_M(K)$. Setting $u^{\pm} = \chi^{\pm} u$ we have $u = u^+ + u^-$ with $\operatorname{supp} u^{\pm} \subseteq I_M^{\pm}(K) \subseteq J_M^{\pm}(K)$. From $Du = 0$ we see $Du^+ = -Du^-$ which we denote by v. Since $\operatorname{supp} Du^{\pm} \subseteq \operatorname{supp} u^{\pm}$ we conclude $\operatorname{supp} v \subseteq \operatorname{supp} u^+ \cap \operatorname{supp} u^- \subseteq J_M^+(K) \cap J_M^-(K)$ which is compact, i.e. $v \in \Gamma_0^\infty(E)$. In particular we can apply $G_M^{\pm}$ to v. We want to show $G_M^+ Du^+ = u^+$: Even though $Du^+ = v$ has compact support we can not directly apply the defining property of G_M^+ since u^+ does not have compact support. However, we can interpret u^+ in a distributional sense and compute for a test section $\varphi \in \Gamma_0^\infty(E^*)$

$$\begin{aligned}
\int_M \varphi(p) \cdot (G_M^+ Du^+)(p)\, \mu_g &\stackrel{(*)}{=} \int_M (F_M^- \varphi) \cdot (Du^+)(p)\, \mu_g \\
&\stackrel{(**)}{=} \int_M (D^{\mathrm{T}} F_M^- \varphi)(p) \cdot u^+(p)\, \mu_g \\
&= \int_M \varphi(p) \cdot u^+(p)\, \mu_g,
\end{aligned}$$

where we have used Lemma 4.3.9 in $(*)$ and integration by parts in $(**)$ which is possible since $F_M^- \varphi$ has support in $J_M^-(\operatorname{supp} \varphi)$ while u^+ has support in $J_M^+(K)$. Hence the overlap of their supports is compact even though their supports are not. Then the above computation shows $G_M^+ Du^+ = u^+$.

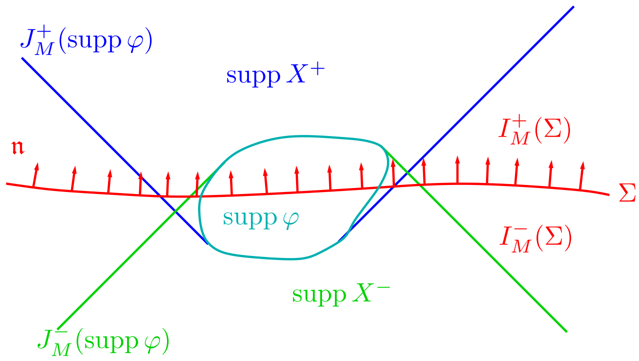

Figure 4.45: The supports of the functions φ and $X^{\pm}$.

Analogously we find $G_M^- D u^- = u^-$. Putting these results together gives $G_M v = G_M^+ v - G_M^- v = G_M^+ D u^+ + G_M^- D u^- = u^+ + u^- = u$. Therefore, u is in the image of G_M with a pre-image in $\Gamma_0^\infty(E)$ as wanted. □

Remark 4.3.19 (Propagator) The simple description of the image and kernel of the operator $G_M = G_M^+ - G_M^-$ has many important consequences. In physics in (quantum) field theory this operator is called the *propagator* which is one of the most crucial ingredients in any perturbative (quantum) field theory. It also appears as the kernel of the Poisson bracket in classical field theory which we will discuss in Section 4.4.

As an application of the operator G_M we obtain a global version of Lemma 4.2.3 expressing the solution of the homogeneous Cauchy problem in terms of the initial data:

Theorem 4.3.20 *Let (M, g) be a globally hyperbolic spacetime and let $\iota : \Sigma \hookrightarrow M$ be a smooth spacelike Cauchy hypersurface. Let $D \in \operatorname{DiffOp}^2(E)$ be normally hyperbolic and let $F_M^{\pm}$ be the advanced and retarded Green operators of D^{T}. Then the solution $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ of the homogeneous wave equation $Du = 0$ with initial values $\iota^\# u = u_0$ and $\iota^\# \nabla_{\mathfrak{n}}^E u = \dot{u}_0$ on Σ is determined by*

$$\int_M \varphi(p) \cdot u(p) \mu_g(p) = \int_\Sigma \left((\nabla_{\mathfrak{n}}^E F_M(\varphi))(\sigma) \cdot u_0(\sigma) - F_M(\varphi)(\sigma) \cdot \dot{u}_0(\sigma) \right) \mu_\Sigma \tag{4.3.38}$$

for $\varphi \in \Gamma_0^\infty(E^*)$.

Proof. The proof is literally the same as for Lemma 4.2.3. Therefore it will be enough to sketch the arguments. We consider the sections $F_M^{\pm}(\varphi) \in \Gamma_{\mathrm{sc}}^\infty(E^*)$ which have supports $\operatorname{supp} F_M^{\pm}(\varphi) \subseteq J_M^{\pm}(\operatorname{supp}\varphi)$. Taking covariant derivatives and pairing with u gives the vector field

$$X^{\pm} = \left((\mathsf{D}^{E^*} F_M^{\pm}(\varphi)) \cdot u - F_M^{\pm}(\varphi) \cdot (\mathsf{D}^E u) \right)^\# \in \Gamma^\infty(TM)$$

which has again support in $J_M^{\pm}(\operatorname{supp}\varphi)$. In particular, $\operatorname{supp} X^{\pm} \cap I_M^{\mp}(\Sigma)$ as well as $\operatorname{supp} X^{\pm} \cap \Sigma$ are (pre-) compact and hence the following integrations will be well-defined. Integrating over $I_M^-(\Sigma)$ the unit normal field $\mathfrak{n}$ is pointing outwards as it is future-directed. Conversely, integrating over $I_M^+(\Sigma)$ the vector field $-\mathfrak{n}$ is pointing outwards. Thus by Theorem B.11 we get

$$\int_{I_M^{\pm}(\Sigma)} \operatorname{div}(X^{\mp}) \, \mu_g = \mp \int_\Sigma g(X^{\mp} \mathfrak{n}) \, \mu_\Sigma,$$

where we of course have restricted $X^{\pm}$ to Σ on the right hand side. For the left hand side we obtain

$$\int_{I_M^{\pm}(\Sigma)} \operatorname{div}(X^{\mp}) \, \mu_g = \int_{I_M^{\pm}(\Sigma)} \operatorname{div} \left((\mathsf{D}^{E^*} F_M^{\pm}(\varphi)) \cdot u - F_M^{\pm}(\varphi) \cdot (\mathsf{D}^E u) \right)^\# \mu_g$$

$$\overset{(4.2.7)}{=} \int_{I_M^{\pm}(\Sigma)} \left((D^{\mathrm{T}} D_M^{\mp}(\varphi)) \cdot u - F^{\pm}(\varphi)(Du)\right) \mu_g$$

$$= \int_{I_M^{\pm}(\Sigma)} \varphi \cdot u \, \mu_g,$$

since $F_M^{\pm}$ are the Green operators of D^{T} and $Du = 0$. For the right hand side we have

$$\mp \int_{\Sigma} g(X^{\mp}, \mathfrak{n}) \, \mu_\Sigma = \mp \int_{\Sigma} \Big((\mathsf{D}^{E^*} F_M^{\mp}(\varphi)) \cdot u - F_M^{\mp}(\varphi) \cdot \mathsf{D}^E \, u \Big) \cdot \mathfrak{n} \, \mu_\Sigma$$

$$= \mp \int_{\Sigma} \Big((\nabla_{\mathfrak{n}}^{E^*} F_M^{\mp}(\varphi)) \cdot u - F_M^{\mp}(\varphi) \cdot \nabla_{\mathfrak{n}}^E u \Big) \, \mu_g$$

$$= \mp \int_{\Sigma} \Big((\nabla_{\mathfrak{n}}^{E^*} F_M^{\mp}(\varphi)) \cdot u_0 - F_M^{\mp}(\varphi) \cdot \dot{u}_0 \Big) \, \mu_\Sigma,$$

with an analogous computation as in Lemma 4.2.3. Putting things together gives (4.3.38). □

Remark 4.3.21 From this formula we see that the homogeneous Cauchy problem can again be encoded completely in terms of the Green operators. Since also the inhomogeneous Cauchy problem with vanishing initial conditions can be solved by means of the Green operators thanks to Theorem 4.3.12 we see that the Cauchy problem and the construction of the Green operators are ultimately the same problem.

4.4 A Poisson Algebra

In this section we describe a first attempt to establish a Hamiltonian picture for the wave equation based on a certain Poisson algebra of observables coming from the canonical symplectic structure on the space of initial conditions. Throughout this section, (M, g) will be globally hyperbolic. For the vector bundle $E \longrightarrow M$ we have to be slightly more specific: We choose E to be a *real* vector bundle. The reason will be to get the correct linearity properties of the Poisson bracket later. From a physical point of view, many of the complex vector bundles actually arise as complexifications of real ones. Then the wave operators in question have the additional property to commute with the complex conjugation of the sections of the complexified bundles. This will be important in applications in physics later on, in particular for CPT-like theorems in quantum field theories, see e.g [28, 57]. For an overview on the geometrical aspects of (finite-dimensional) classical mechanics we refer to [1, 43, 60]

4.4.1 Symmetric Differential Operators

Now we equip the vector bundle E with an additional structure, namely a fiber metric h. In most applications this fibre metric will be positive definite, a fact which we shall not use though. In any case, the fibre metric induces a musical isomorphism $\flat : E \longrightarrow E^*$ with inverse $\# : E^* \longrightarrow E$ as usual. On sections we have

$$\flat : \Gamma^\infty(E) \ni u \;\mapsto\; u^\flat = h(u, \cdot\,) \in \Gamma^\infty(E^*). \tag{4.4.1}$$

There should be no confusion with the sharp and flat map coming from the Lorentz metric g. Using this additional structure one can define symmetric differential operators as usual:

Definition 4.4.1 (Symmetric differential operators) *Let (E, h) be a real vector bundle with fibre metric and $D \in \mathrm{DiffOp}^\bullet(E)$. Then the adjoint of D with respect to h is defined to be the unique $D^* \in \mathrm{DiffOp}^\bullet(E)$ with*

$$\int_M h(D^* u, v) \mu_g = \int_M h(u, Dv) \mu_g \tag{4.4.2}$$

for all $u, v \in \Gamma_0^\infty(E)$. *The operator* D *is called symmetric if*

$$D = D^*. \tag{4.4.3}$$

Remark 4.4.2 (Symmetric differential operators)

i.) The definition of the adjoint D^* with respect to h is well-defined indeed. Namely, if $D \in \mathrm{DiffOp}^k(E)$ then one has

$$D^* u = (D^{\mathrm{T}} u^\flat)^\# \tag{4.4.4}$$

with the adjoint operator $D^{\mathrm{T}} \in \mathrm{DiffOp}^k(E^*)$ as we discussed it before in Theorem 1.2.15. This follows from the simple computation

$$\int_M h((D^{\mathrm{T}} u^\flat)^\#, v)\, \mu_g = \int_M (D^{\mathrm{T}} u^\flat) \cdot v\, \mu_g = \int_M u^\flat \cdot Dv\, \mu_g = \int_M h(u, Dv) \mu_g, \tag{4.4.5}$$

which shows that (4.4.4) solves the condition (4.4.2). It is clear that D^* is again a differential operator of the same order as D and it is necessarily unique since the inner product is non-degenerate.

ii.) The adjoint D^* depends on h but also on the density μ_g in the integration (4.4.2). The map $D \mapsto D^*$ is a linear involutive anti-automorphism, i.e. we have

$$(D^*)^* = D \quad \text{and} \quad (D\widetilde{D})^* = \widetilde{D}^* D^* \tag{4.4.6}$$

for $D, \widetilde{D} \in \mathrm{DiffOp}^\bullet(E)$.

iii.) In the case of a complex vector bundle one proceeds similarly: for a given (pseudo-) Hermitian fibre metric one defines the adjoint D^* by the same condition (4.4.2). Now $D \mapsto D^*$ is antilinear in addition to (4.4.6) and $\mathrm{DiffOp}^\bullet(E)$ becomes a *-algebra over $\mathbb{C}$ by this choice. Differential operators with $D = D^*$ are now called *Hermitian*. A particular case is obtained for a complexified vector bundle $E_{\mathbb{C}} = E \otimes \mathbb{C}$. If h is a fibre metric on E then it induces a Hermitian fibre metric on $E_{\mathbb{C}}$ by setting

$$h_{\mathbb{C}}(u \otimes z, v \otimes w) = h(u, v)\overline{z}w \tag{4.4.7}$$

for $u, v \in E_p$ and $z, w \in \mathbb{C}$. Then a symmetric operator $D \in \mathrm{DiffOp}^\bullet(E)$ yields a Hermitian operator $D_{\mathbb{C}} \in \mathrm{DiffOp}^\bullet(E_{\mathbb{C}})$ which commutes in addition with the *complex conjugation* of sections.

In most physically interesting situations the wave operator D will be symmetric. As a motivation we consider the following example:

Example 4.4.3 (Symmetric connection d'Alembertian) Let (E, h) be a real vector bundle with fibre metric h. Moreover, let ∇^E be a covariant derivative which is *metric* with respect to h, i.e.

$$\mathscr{L}_X\, h(u, v) = h(\nabla^E_X u, v) + h(u, \nabla^E_X v) \tag{4.4.8}$$

for all $X \in \Gamma^\infty(TM)$ and $u, v \in \Gamma^\infty(E)$. We claim that in this case the connection d'Alembertian is symmetric. Indeed, (4.4.8) immediately implies that the symmetric covariant derivative operators D^E and D^{E^*} with respect to ∇^E and ∇^{E^*} are intertwined by $\#$ and $\flat$ as follows

$$\left(\mathsf{D}^E\, u\right)^\flat = \mathsf{D}^{E^*}\, u^\flat \tag{4.4.9}$$

for $u \in \Gamma^\infty(\mathrm{S}^k T^*M \otimes E)$ and $\flat$ applied to the E-part only. This is a simple verification. But then we have for $\Box^\nabla$ by Lemma 4.2.2, *i.)*

$$\left(\Box^\nabla\right)^{\mathrm{T}} u^\flat = \left\langle g^{-1}, \left(\mathsf{D}^{E^*}\right)^2 u^\flat \right\rangle = \left\langle g^{-1}, \left(\left(\mathsf{D}^E\right)^2 u\right)^\flat \right\rangle = \left\langle g, \left(\mathsf{D}^E\right)^2 u\right\rangle^\flat = \left(\Box^\nabla u\right)^\flat, \tag{4.4.10}$$

since the natural pairing of the S^2T^*M component with g^{-1} obviously commutes with the musical isomorphism $\flat$ acting only on the E-component. But this implies

$$\Box^\nabla = \left(\Box^\nabla\right)^* \tag{4.4.11}$$

as claimed. More generally, if $B \in \Gamma^\infty(\mathsf{End}(E))$ is also symmetric with respect to h, which is now a pointwise criterion, then $D = \Box^\nabla + B$ is symmetric as well.

This construction is also compatible with complexification: if h is extended to $E_\mathbb{C} = E \otimes \mathbb{C}$ as in Remark 4.4.2, *iii.)* then the connection ∇^E also extends to $E_\mathbb{C}$ yielding a metric connection $\nabla^{E_\mathbb{C}}$ with respect to $h_\mathbb{C}$. The condition (4.4.8) is then satisfied for real tangent vector fields $X \in \Gamma^\infty(TM)$ while we have to replace X by $\overline{X}$ in the first term of the right hand side of (4.4.8) in general. With this (pseudo) Hermitian fibre metric $h_\mathbb{C}$ and the covariant derivative $\nabla^{E_\mathbb{C}}$ the property (4.4.9) still holds, resulting in (4.4.11) for the connection d'Alembertian $\Box^\nabla$ on $E_\mathbb{C}$. Again $\Box^\nabla$ is not only Hermitian but also commutes with the complex conjugation of sections of $E_\mathbb{C}$. Note that for general complex vector bundles there is no notion of complex conjugation of sections.

From now on we shall focus on a symmetric and normally hyperbolic differential operator D. In fact, we shall also assume that ∇^E is metric. Then $D = D^*$ means $B = B^*$ for $D = \Box^\nabla + B$. Since $D = D^*$ essentially means that we can identify D with D^T via $\flat$ and $\#$ we expect a similar relation between the Green operators, extending the already found relations between $F_M^\pm$ and $G_M^\pm$ as in Theorem 4.3.10. In fact, one has the following characterization:

Proposition 4.4.4 (Symmetry of Green operators) *Let (M, g) be globally hyperbolic and let $D \in \mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator on the real vector bundle E. Assume that D is symmetric with respect to a fibre metric h on E.*

i.) For the Green operators of D and D^T and $u \in \Gamma_0^\infty(E)$ we have

$$\left(G_M^\pm u\right)^\flat = F_M^\pm u^\flat. \tag{4.4.12}$$

ii.) For $u, v \in \Gamma_0^\infty(E)$ we have

$$\int_M h\left(u, G_M^\pm v\right)\, \mu_g = \int_M h\left(G_M^\mp u, v\right)\, \mu_g. \tag{4.4.13}$$

iii.) The Green operators of the canonical $\mathbb{C}$-linear extension of D to $E_\mathbb{C} = E \otimes \mathbb{C}$ are the canonical $\mathbb{C}$-linear extension of the Green operators $G_M^\pm$ of D. They still satisfy (4.4.12),

$$\int_M h_\mathbb{C}\left(u, G_M^\pm v\right)\, \mu_g = \int_M h_\mathbb{C}\left(G_M^\mp u, v\right)\, \mu_g \tag{4.4.14}$$

for $u, v \in \Gamma_0^\infty(E_\mathbb{C})$ and additionally the reality condition

$$\overline{G_M^\pm u} = G_M^\pm \overline{u}. \tag{4.4.15}$$

Proof. Clearly, $u \in \Gamma_0^\infty(E)$ has compact support iff $u^\flat$ has compact support, making (4.4.12) meaningful. We compute for $\varphi \in \Gamma_0^\infty(E^*)$

$$D^\mathrm{T}\left(G_M^\pm \varphi^\#\right)^\flat \stackrel{(4.4.4)}{=} \left(D G_M^\pm \varphi^\#\right)^\flat = \left(\varphi^\#\right)^\flat = \varphi,$$

since $G_M^\pm$ is a Green operator for D. Analogously,

$$\left(G_M^\pm \left(D^\mathrm{T}\varphi\right)^\#\right)^\flat \stackrel{(4.4.4)}{=} \left(G_M^\pm D \varphi^\#\right)^\flat = \left(\varphi^\#\right)^\flat = \varphi.$$

Now $\varphi \mapsto \left(G_M^\pm \varphi^\#\right)^\flat$ is clear linear and continuous since $\#, \flat$ as well as $G_M^\pm$ are continuous. Finally, since $\#$ and $\flat$ preserve supports we have $\operatorname{supp}\left(G_M^\pm \varphi^\#\right)^\flat \subseteq J_M^\pm(\operatorname{supp}\varphi)$. This shows that the map $\varphi \mapsto \left(G_M^\pm \varphi^\#\right)^\flat$ is indeed an advanced and retarded Green operator for D^{T}, respectively. By uniqueness according to Corollary 4.3.7 we get (4.4.13). Using this, we compute

$$
\begin{aligned}
\int_M h\left(u, G_M^\pm v\right)\ \mu_g &= \int_M u^\flat \cdot \left(G_M^\pm v\right)\ \mu_g \\
&\stackrel{(4.3.12)}{=} \int_M \left(F_M^\mp u^\flat\right) \cdot v\ \mu_g \\
&\stackrel{(4.4.13)}{=} \int_M \left(G_M^\mp u\right)^\flat \cdot v\ \mu_g \\
&= \int_M h\left(G_M^\mp u, v\right)\ \mu_g
\end{aligned}
$$

for $u, v \in \Gamma_0^\infty(E)$. Now consider $u, v \in \Gamma_0^\infty(E_{\mathbb{C}})$. Then $\overline{Du} = D\overline{u}$ yields the hermiticity $D = D^*$ with respect to $h_{\mathbb{C}}$. With the same kind of uniqueness argument we see that the Green operators $G_M^\pm$ of D, canonically extended to $G_M^\pm : \Gamma_0^\infty(E_{\mathbb{C}}) \longrightarrow \Gamma^\infty(E_{\mathbb{C}})$, yield the Green operators of the extension $D \in \operatorname{DiffOp}^2(E_{\mathbb{C}})$. Moreover, we clearly have (4.4.15) by construction. But then (4.4.14) follows from (4.4.15) and (4.4.13) at once. □

Remark 4.4.5 Extending our notation of the adjoint to more general operators we can rephrase the result of (4.4.13) or (4.4.14) by saying

$$\left(G_M^\pm\right)^* = G_M^\mp. \tag{4.4.16}$$

Note that Proposition 4.4.4, *iii.)* still holds for arbitrary Hermitian $D = D^*$ on arbitrary complex vector bundles except for (4.4.15). In both cases, it follows that the propagator $G_M = G_M^+ - G_M^-$ is *antisymmetric*

$$G_M^* = -G_M \tag{4.4.17}$$

or *anti-Hermitian* in the complex case, respectively. In the complex case we can rescale G_M by i to obtain a *Hermitian* operator

$$(\mathrm{i}G_M)^* = \mathrm{i}G_M. \tag{4.4.18}$$

4.4.2 Interlude: The Lagrangian and the Hamiltonian Picture

To put the following construction in the right perspective we briefly remind on the Lagrangian and Hamiltonian approach to field equations as it can be found in various textbooks on classical and quantum field theory. Most of our present considerations should be taken as heuristic as it would require a lot more effort to justify them on a mathematically rigorous basis. They serve as a motivation for our definition of certain Poisson brackets.

Many field equations in physics arise from an action principle where an action functional is defined on the space of all field configurations on the whole spacetime by means of a Lagrangian density. Such a Lagrangian density $\mathcal{L}$ can be viewed as a function on the (first) jet bundle J^1E of E which takes values in the densities $|\Lambda^{\mathrm{top}}|T^*M$ on M. Roughly speaking, the k-th jet bundle J^kE of E is a fibre bundle over M whose fibre at $p \in M$ consists of equivalence classes of Taylor expansions of sections of E around p up to order k. Two sections are called equivalent if they have the same Taylor expansion at p up to order k. This is a coordinate independent statement whence the jet bundles serve the following purpose: we can make geometrically sense of the statement that a map $\mathcal{L} : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ depends at $p \in M$ only on the first k derivatives of $u \in \Gamma^\infty(E)$ at

p. In our case one typically has $k = 1$ and symbolically writes $\mathcal{L}(u, \partial u)$ to emphasize that $\mathcal{L}(u, \partial u)\big|_p$ depends only on $u(p)$ and $\frac{\partial u}{\partial x^i}(p)$. Having specified such a Lagrangian density $\mathcal{L}$ the action $S(u)$ is defined by the (hopefully existing) integral of $\mathcal{L}(u, \partial u)$ over M. Then the stationary points of the action functional are supposed to be those sections which satisfy the wave equation. With other words one wants the Euler-Lagrange equations for $\mathcal{L}$ to be the wave equation under consideration. Note that the precise formulation of an action principle is far from being trivial: on one hand, one has to require certain integrability conditions on the sections in order to have a well-defined action. On the other hand, in deriving the Euler-Lagrange equations one usually neglects certain boundary terms or considers only variations with compact support. Thus it is not evident that the Euler-Lagrange equations really describe the stationary points of S. Even worse, in typical situations the solutions of the Euler-Lagrange equations yield sections u with *no* good integrability properties at all. Our wave equation is a good example as here the non-trivial solutions have to have non-compact support in timelike directions. This way, it may well happen that none of the solutions of the Euler-Lagrange equation is in the domain of definition of the action S at all, except for some trivial solutions like $u = 0$. To handle these difficulties a more sophisticated variational calculus is required which is not within the reach for us at this stage. Therefore, we take a more pragmatic point of view and take the Lagrangian density $\mathcal{L}$ and the corresponding Euler-Lagrange equations as the starting point instead of the action S itself. These equations and hence the wave equation are the ultimate goal anyway.

The idea is now to treat the Euler-Lagrange equations for the Lagrangian density as Euler-Lagrange equations of a suitably defined Lagrangian function defined on the space of initial conditions: this way we can interpret the field theoretic wave equations as a classical mechanical system, though of course with infinitely many degrees of freedom. The idea is roughly as follows: the initial conditions of the wave equation are specified on a fixed smooth spacelike Cauchy hypersurface $\iota : \Sigma \hookrightarrow M$. There we have to specify the value of the section $u_0 \in \Gamma^\infty_0(\iota^\# E)$ and the normal derivative $\dot{u}_0 \in \Gamma^\infty_0(\iota^\# E)$. Mechanically speaking, this corresponds to the initial position and the initial velocity. Thus the (velocity-) phase space of the Lagrangian approach is the tangent bundle of the space of initial positions in complete analogy to Lagrangian mechanics for finite-dimensional systems. Since the initial positions are described by the vector space $\Gamma^\infty_0(\iota^\# E)$ the notion of tangent bundle is simple: we just have to take $\Gamma^\infty_0(\iota^\# E) \times \Gamma^\infty_0(\iota^\# E)$, i.e. two copies of the configuration space. The Lagrange function now consists in evaluating the Lagrange density on u_0 and $\dot{u}_0$ on Σ and integrating over Σ: this indeed makes sense as the Lagrange density $\mathcal{L}$ can be written relative to the density μ_g as $\mathcal{L}(u, \partial u) = \widetilde{\mathcal{L}}(u, \partial u)\mu_g$ with a function $\widetilde{\mathcal{L}}(u, \partial u)$ on the first jet bundle. Then we can take this function and evaluate it on u_0 and $\dot{u}_0$ instead of u and ∂u and consider the density $\widetilde{\mathcal{L}}(u_0, \dot{u}_0)\mu_\Sigma$ on Σ. Again, we ignore the technical details which are less severe as for the action since we are interested in u_0 and $\dot{u}_0$ with compact support anyway. The integration over Σ is thus easily defined.

Having a Lagrangian mechanical point of view for our wave equation we can try to pass to a Hamiltonian description by the usual Legendre transform. This amounts to the passage from the tangent bundle to the cotangent bundle of the configuration space $\Gamma^\infty_0(\iota^\# E)$. While the tangent bundle of a vector space is conceptually easy, the cotangent bundle is more subtle: here the fact that $\Gamma^\infty_0(\iota^\# E)$ is infinite-dimensional becomes crucial. Thus we have to decide which dual we want to take. Of course, the algebraic dual seems inappropriate whence we take the topological dual which we identify with $\Gamma^{-\infty}(\iota^\# E^*)$ as usual by means of μ_Σ. Then the cotangent bundle of $\Gamma^\infty_0(\iota^\# E)$ is $\Gamma^\infty_0(\iota^\# E) \times \Gamma^{-\infty}(\iota^\# E^*)$. The following proposition shows that this is indeed a symplectic vector space in a very good sense. We formulate it for a general vector bundle $F \longrightarrow M$ over an arbitrary manifold.

Proposition 4.4.6 (Symplectic vector space)

i.) Let W be a Hausdorff locally convex topological vector space with topological dual and consider

$V = W \oplus W'$. Then on V the two-form

$$\omega_{\mathrm{can}}\left((w,\varphi),(w',\varphi')\right) = \varphi'(w) - \varphi(w') \tag{4.4.19}$$

is antisymmetric and non-degenerate.

ii.) Let $F \longrightarrow M$ be a real vector bundle. Then on $\Gamma_0^\infty(F^* \otimes |\Lambda^{\mathrm{top}}|T^*M) \oplus \Gamma^{-\infty}(F)$ the two-form

$$\omega_{\mathrm{can}}\left((\varphi,u),(\varphi',u')\right) = u'(\varphi) - u(\varphi') \tag{4.4.20}$$

is antisymmetric and non-degenerate.

iii.) Let $F \longrightarrow M$ be a real vector bundle and let μ be a positive density on M. Then on $\Gamma_0^\infty(F^*) \oplus \Gamma^{-\infty}(F)$ the two form

$$\omega_{\mathrm{can}}\left((\varphi,u),(\varphi',u')\right) = u'(\varphi\otimes\mu) - u(\varphi'\otimes\mu) \tag{4.4.21}$$

is antisymmetric and non-degenerate.

Proof. Clearly, ω_{can} is bilinear in all three cases and antisymmetric on the nose. Assume that $(w,\varphi) \in W \oplus W'$ is such that $\omega_{\mathrm{can}}((w,\varphi),\,\cdot\,) = 0$. Then it follows that $\varphi'(\omega) = 0$ for all $\varphi' \in W'$ and $\varphi(w') = 0$ for all $w' \in W$. This clearly implies $\varphi = 0$. Since W is Hausdorff, by some Hahn-Banach-like statements it follows that W' is large enough to separate points, see e.g. [34, Sect. 7.2]. Thus also $w = 0$ follows which proves that (4.4.19) is non-degenerate. The second and third part are only special cases. □

Since in our situation we have a canonical positive density on Σ, namely μ_Σ, we can apply the third part and conclude that $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma^{-\infty}(\iota^\# E^*)$ is indeed a symplectic vector space.

Without going into the details we can now use the Lagrange function to define a Legendre transform by which we can pull back the canonical symplectic form of the cotangent bundle to the tangent bundle. This constructions boils down to the following simple map, at least in all cases relevant for us. By means of the fibre metric h_Σ on $\iota^\# E$ coming from h on E we can map a tangent vector $\dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$ to a cotangent vector in $\Gamma^{-\infty}(\iota^\# E^*)$ by taking $\dot{u}^\flat \in \Gamma_0^\infty(\iota^\# E^*)$ and interpret this smooth section of $\iota^\# E^*$ as a distributional section $\dot{u}_0^\flat \in \Gamma^{-\infty}(\iota^\# E^*)$. Clearly, this yields an injective linear map

$$\Gamma_0^\infty(\iota^\# E) \ni \dot{u}_0 \;\mapsto\; \dot{u}_0^\flat \in \Gamma_0^\infty(\iota^\# E^*) \subseteq \Gamma^{-\infty}(\iota^\# E^*), \tag{4.4.22}$$

which allows to pull back ω_{can} to the tangent bundle. This results in the following, still non-degenerate two-form:

Lemma 4.4.7 *The pull-back of the symplectic form ω_{can} from the cotangent bundle of $\Gamma_0^\infty(\iota^\# E)$ to its tangent bundle $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ via (4.4.22) is explicitly given by*

$$\omega_h\left((u_0,\dot{u}_0),(v_0,\dot{v}_0)\right) = \int_\Sigma \left(h_\Sigma(u_0,\dot{v}_0) - h_\Sigma(\dot{u}_0,v_0)\right)\mu_\Sigma \tag{4.4.23}$$

for $u_0, \dot{u}_0, v_0, \dot{v}_0 \in \Gamma_0^\infty(\iota^\# E)$. The two-form ω_h turns $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ also into a symplectic vector space.

Proof. Evaluating (4.4.21) for the distributional sections $\dot{u}_0^\flat, \dot{v}_0^\flat$ gives immediately (4.4.23). We have to check the non-degeneracy: but since h_Σ is non-degenerate we can always find *smooth* $(v_0, \dot{v}_0)$ for a given $(u_0, \dot{u}_0) \neq 0$, resulting in a non-trivial pairing via ω_h. □

Remark 4.4.8 (Weak vs. strong symplectic) The symplectic structure ω_{can} on the cotangent bundle is even a *strong* symplectic form if one defines the topological dual of $\Gamma^{-\infty}(\iota^\# E^*)$ in an appropriate way: ω_{can} induces an isomorphism from $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma^{-\infty}(\iota^\# E^*)$ to its topological dual.

For ω_h this is clearly not the case as the topological dual of $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ are two copies of $\Gamma^{-\infty}(\iota^\# E^*)$ but with $\omega_h((u_0, \dot{u}_0), \cdot\,)$ we only obtain the (very small) part of smooth sections of $\iota^\# E^*$ and not the generalized ones. This is an effect of infinite dimension as in finite dimensions an injective linear map from a vector space to its dual is necessarily bijective. This indicates that for a Hamiltonian description one has to expect some (bad!) surprises.

In any case, we only want to use the symplectic form to define the Poisson algebra of observables of our "mechanical" system. In the most general approach this algebra consists of smooth functions on the (co-) tangent bundles. However, we do not want to enter the quite nontrivial discussion on the appropriate definition of smooth functions on the LF space $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$. There are several competing options which we do not discuss here. To get a flavour of the complications one should consult e.g. [38]. Instead, we focus only on a very small class of functions, the polynomials on the tangent bundle.

4.4.3 The Poisson Algebra of Polynomials

If V is a Hausdorff locally convex topological vector space over $\mathbb{R}$, what should the polynomials on V be? Clearly, a homogeneous polynomial of degree 1 is just a *linear* functional $V \longrightarrow \mathbb{R}$ and hence an element of the dual space of V. Having a topological vector space V we require *continuity* for the homogeneous polynomials of degree 1 whence we end up with an element of V'.

Passing to homogeneous quadratic polynomials we certainly like to have expression as

$$p(v) = \sum_{i=1}^N \varphi_i(v)\psi_i(v)$$

with $\varphi_i, \psi_i \in V'$ to be part of our observables. Indeed, if we insist on an *algebra* this is even forced by the algebraic features: such a $p : V \longrightarrow \mathbb{R}$ is the sum of products of elements in V'. Since we can multiply further we also have to include functions of the form

$$p(v) = \sum_{i=1}^N \varphi_1^{(i)}(v) \cdots \varphi_k^{(i)}(v) \tag{4.4.24}$$

with $\varphi_1^{(i)}, \ldots, \varphi_k^{(i)} \in V'$ and $v \in V$. Such a function certainly deserves the name "homogeneous polynomial of degree k". Taking also linear combinations of such polynomials of different homogeneity, which is again required if we want an *algebra* of observables, we end up with functions $p : V \longrightarrow \mathbb{C}$ of the form

$$p(v) = c + \sum_{k=1}^{\ell} \sum_{i=1}^{N_k} \varphi_{1,k}^{(i)}(v) \cdots \varphi_{k,k}^{(i)}(v) \tag{4.4.25}$$

with $\varphi_{\chi,k}^{(i)} \in V'$ and $v \in V$ and a constant $c \in \mathbb{C}$.

Definition 4.4.9 (Polynomial functions) *Let V be a Hausdorff locally convex topological vector space. Then the polynomial functions generated by the constants and the linear functions $\varphi \in V'$ are denoted by* $\operatorname{Pol}^\bullet(V)$.

These functions can be identified with the symmetric algebra over V'.

Proposition 4.4.10 *Let V be a Hausdorff locally convex topological vector space. Then the polynomial functions $p : V \longrightarrow \mathbb{R}$ of the form* (4.4.25) *are in canonical bijection with the symmetric algebra $\mathrm{S}^\bullet V'$ over V'. The isomorphism is explicitly given by*

$$\mathcal{J} : \mathrm{S}^\bullet V' \ni \varphi_1 \vee \cdots \vee \varphi_k \;\mapsto\; \mathcal{J}(\varphi_1 \vee \cdots \vee \varphi_k) = \mathcal{J}(\varphi_1) \cdots \mathcal{J}(\varphi_k) \in \operatorname{Pol}^\bullet(V), \tag{4.4.26}$$

where for degree 0 *and* 1 *we have explicitly*

$$\mathcal{J}(\varphi)(v) = \varphi(v) \quad \textit{and} \quad \mathcal{J}(\mathbb{1})(v) = 1. \tag{4.4.27}$$

On arbitrary homogeneous elements $\Phi \in \mathrm{S}^k V'$ *we have*

$$\mathcal{J}(\Phi)(v) = \frac{1}{k!}\Phi(v, \ldots, v). \tag{4.4.28}$$

Proof. This is abstract nonsense on the symmetric algebra. First we recall that $\mathrm{S}^k V'$ consists of linear combinations of totally symmetrized tensor products of k elements $\varphi_1, \ldots, \varphi_k \in V'$. We adopt the convection

$$\varphi_1 \vee \cdots \vee \varphi_k = \sum_{\sigma \in S_k} \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(k)}$$

without prefactors. Then it is well-known that $\mathrm{S}^\bullet V'$ with $\vee$ is *the* (up to canonical isomorphisms) free commutative algebra generated by $\mathbb{1}$ and V'. Since the polynomials (4.4.25) are, by construction, also generated by V' and the constants, we get a unique algebra homomorphism $\mathcal{J}$ by specifying it on the generators by (4.4.27). Evaluating this on higher tensor products gives immediately (4.4.28) *with* prefactor. It remains to show that $\mathcal{J}$ is injective, since the surjectivity is clearly the definition of the polynomials. Thus assume that $\Phi = \sum_{k=1}^{\ell} \Phi_k \in \mathrm{S}^\bullet V'$ with homogeneous components $\Phi_k \in \mathrm{S}^k V'$ satisfies $\mathcal{J}(\Phi) = 0$. Then for all $v \in V$ we have $\sum_{k=1}^{\ell} \frac{1}{k!}\Phi_k(v, \ldots, v) = 0$. Rescaling v to tv with $t \in \mathbb{R}$ we see that the polynomial

$$p(t) = \sum_{k=1}^{\ell} \frac{1}{k!}\Phi_k(v, \ldots, v)t^k = 0$$

vanishes identically. Hence $\Phi_k(v, \ldots, v) = 0$ for all k separately. Now the polarization identities allow to express $\Phi_k(v_1, \ldots, v_k)$ in terms of linear combinations of terms $\Phi_k(w, \ldots, w)$ with w being certain linear combinations of the $v_1, \ldots, v_k$. E.g. for quadratic ones we have

$$\Phi_2(v_1, v_2) = \frac{1}{2}\left(\Phi_2(v_1 + v_2, v_1 + v_2) - \Phi_2(v_1, v_1) + \Phi_2(v_2, v_2)\right)$$

and so on. But then $\Phi_k(v, \ldots, v) = 0$ for all $v \in V$ implies $\Phi_k = 0$ in $\mathrm{S}^k V'$. Thus $\mathcal{J}$ is injective. □

Remark 4.4.11 (Polynomial functions) Let again V be a Hausdorff locally convex vector space.

i.) From Proposition 4.4.10 we have that

$$\mathcal{J} : \mathrm{S}^\bullet V' \longrightarrow \mathrm{Pol}^\bullet(V) \tag{4.4.29}$$

is an isomorphism of commutative, unital, and graded algebras.

ii.) More generally, one could define a polynomial function $p : V \longrightarrow \mathbb{R}$ on V of degree k to be a function with the property

$$p(tv) = t^k p(v) \tag{4.4.30}$$

for all $v \in V$ and $t \in \mathbb{R}$ *plus* some suitable continuity at the origin. This continuity is already needed in finite dimensions to exclude functions like

$$p(v) = \begin{cases} 0 & \text{for } v = 0 \\ \frac{v^i v^j v^k}{\sum_{\ell=1}^{\dim V}(v^\ell)^2} & \text{for } v \neq 0 \end{cases} \tag{4.4.31}$$

to be a "linear polynomial" in v. Here $v = v^i e_i$ with a basis $e_i \in V$.

iii.) Since V' carries a natural Hausdorff locally convex topology, the weak* topology, one can endow $\mathrm{S}^k V'$ with a locally convex topology as well: in fact, there are several and typically inequivalent possibilities. The usage of such topologies can be two-fold: on one hand we can complete each $\mathrm{S}^k V'$ which amounts to obtaining polynomial functions of homogeneous degree k of the form

$$p(v) = \sum_{i=1}^{\infty} \varphi_1^{(i)}(v) \cdots \varphi_k^{(i)}(v), \tag{4.4.32}$$

where the topology on $\mathrm{S}^k V'$ is now used to make sense out of the limit. But we can also complete into another direction: the direct sum $\mathrm{S}^\bullet V' = \bigoplus_{k=0}^{\infty} \mathrm{S}^k V'$ can be completed to include also "transcendental" functions and not just polynomials. In particular, one would be interested in functions as $f(v) = \mathrm{e}^{\varphi(v)}$ with $\varphi \in V'$. This leads to notions of holomorphic or real analytic functions on V. While the first completion does not give anything new in finite dimension the second is already interesting in finite dimensions. If V is infinite-dimensional, both types of completions are typically non-trivial and depend on the precise choices of the topologies on the (symmetric) tensor products.

After these general considerations we come back to our original task: on the symplectic vector space $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ we want to establish a polynomial algebra with a *Poisson bracket*.

So the first guess is to use the symmetric algebra over $\Gamma^{-\infty}(\iota^\# E^*) \oplus \Gamma^{-\infty}(\iota^\# E^*)$, which is the topological dual of $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ via the usual identification, and endow this symmetric algebra with a Poisson bracket. The problem is here the following: Since ω_h is only a weak symplectic form, not every linear functional has a Hamiltonian vector field. Thus the Poisson bracket can not be defined that easily on all linear functionals and hence on all polynomials. This forces us to proceed differently: we take as a beginning the subspace

$$\Gamma_0^\infty(\iota^\# E^*) \oplus \Gamma_0^\infty(\iota^\# E^*) \subseteq \Gamma^{-\infty}(\iota^\# E^*) \oplus \Gamma^{-\infty}(\iota^\# E^*) \tag{4.4.33}$$

as dual space of $\Gamma_0^\infty(\iota^\# E) \oplus \Gamma_0^\infty(\iota^\# E)$ and consider the symmetric algebra over this much smaller space. Here the following result is easy to obtain:

Proposition 4.4.12 *On the symmetric algebra over $\Gamma_0^\infty(i^\# E^*) \oplus \Gamma_0^\infty(i^\# E^*)$ exists a unique Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_h$ induced by ω_h with the property*

$$\Big\{\mathcal{J}(\varphi_0, \dot{\varphi}_0), \mathcal{J}(\psi_0, \dot{\psi}_0)\Big\}_h = \int_\Sigma \Big(h_\Sigma^{-1}(\varphi_0, \dot{\psi}_0) - h_\Sigma^{-1}(\dot{\varphi}_0, \psi_0)\Big)\, \mu_\Sigma \tag{4.4.34}$$

for $(\varphi_0, \dot{\varphi}_0), (\psi_0, \dot{\psi}_0) \in \Gamma_0^\infty(\iota^\# E^*) \oplus \Gamma_0^\infty(\iota^\# E^*)$. The Hamiltonian vector field of the linear functional $\mathcal{J}(\varphi_0, \dot{\varphi}_0)$ with respect to ω_h is the constant vector field

$$X_{\mathcal{J}(\varphi_0, \dot{\varphi}_0)} = (-\dot{\varphi}_0^\#, \varphi_0^\#). \tag{4.4.35}$$

Proof. First we note that any Poisson bracket on a symmetric algebra $\mathrm{S}^\bullet W$ of any vector space W is uniquely determined by its values on W alone: since a Poisson bracket satisfies by definition a Leibniz rule in both arguments it is determined by its values on a set of generators of the algebra. Since necessarily $\{\mathbb{1}, \cdot\,\} = 0 = \{\,\cdot\,, \mathbb{1}\}$ for any Poisson bracket it is therefore sufficient to specify it on the generators $W \subseteq \mathrm{S}^\bullet W$. Thus $\{\,\cdot\,,\,\cdot\,\}_h$ will be uniquely determined by (4.4.34). To motivate the formula (4.4.34) we first prove (4.4.35). Thus let $(\varphi_0, \dot{\varphi}_0)$ be given. Since this is viewed as a linear function the *differential* is constant and given by $(\varphi_0, \dot{\varphi}_0)$ at every point, i.e.

$$\mathrm{d}\,\mathcal{J}(\varphi_0, \dot{\varphi}_0)\big|_{(u_0, \dot{u}_0)} = (\varphi_0, \dot{\varphi}_0). \tag{$*$}$$

Thus the Hamiltonian vector field, defined by $\omega_h(X_f, \cdot) = \mathrm{d} f(\cdot)$ in general, is determined by

$$\begin{aligned}\int_\Sigma (\varphi_0(v_0) + \dot\varphi_0(\dot v_0))\, \mu_\Sigma &= \mathrm{d}\mathcal{J}(\varphi_0, \dot\varphi_0)\big|_{(u_0,\dot u_0)}(v_0, \dot v_0) \\ &= \omega_h\left(X_{\mathcal{J}(\varphi_0,\dot\varphi_0)}\big|_{(u_0,\dot u_0)}, (v_0, \dot v_0)\right) \\ &= \int_\Sigma \left(h_\Sigma\left(X_{\mathcal{J}(\varphi_0,\dot\varphi_0)}\big|_{(u_0,\dot u_0)}, \dot v_0\right) - h_\Sigma\left(\dot X_{\mathcal{J}(\varphi_0,\dot\varphi_0)}\big|_{(u_0,\dot u_0)}, v_0\right)\right) \mu_\Sigma.\end{aligned}$$

This shows that $X_{\mathcal{J}(\varphi_0,\dot\varphi_0)}$ is the constant vector field with the two components

$$X_{\mathcal{J}(\varphi_0,\dot\varphi_0)}\big|_{(u_0,\dot u_0)} = (-\dot\varphi_0^\#, \varphi_0^\#)$$

at every point $(u_0, \dot u_0)$, i.e. (4.4.35). Now the Poisson bracket is, by definition $\{f, g\} = X_g(f) = \mathrm{d}f(X_g)$. Hence we get the *constant* function

$$\begin{aligned}\left\{\mathcal{J}(\varphi_0, \dot\varphi_0), \mathcal{J}(\psi_0, \dot\psi_0)\right\}\Big|_{(u_0,\dot u_0)} &= (\varphi_0, \dot\varphi_0)(\dot\psi_0^\#, -\psi_0^\#) \\ &= \int_\Sigma \left(\varphi_0(\dot\psi_0^\#) - \dot\varphi_0(\psi_0^\#)\right) \mu_\Sigma \\ &= \int_\Sigma \left(h_\Sigma^{-1}(\varphi_0, \dot\psi_0) - h_\Sigma^{-1}(\dot\varphi_0, \psi_0)\right) \mu_\Sigma,\end{aligned}$$

using the dual fibre metric h_Σ^{-1} on $\iota^\# E^*$. This explains the statement (4.4.34). For finite dimensional vector spaces (or manifolds) we could now argue with the usual calculus of smooth functions that, thanks to the closedness of ω_h, the Poisson bracket is indeed a Poisson bracket. In infinite dimensions we can not just rely on the analogy, in particular since ω_h is only a weak symplectic structure. Instead of establishing an appropriate calculus also in this situation, which in principle can be done, we prove the *existence* of a Poisson bracket on the polynomials by hand. In fact, this follows from the next proposition at once. □

Proposition 4.4.13 *Let W be a real vector space and let*

$$\pi : W \times W \longrightarrow \mathbb{R} \tag{4.4.36}$$

be an antisymmetric bilinear form. Then on $\mathrm{S}^\bullet W$ there is a unique Poisson bracket $\{\cdot, \cdot\}_\pi$ with

$$\{\cdot, \cdot\}_\pi : \mathrm{S}^k W \times \mathrm{S}^\ell W \longrightarrow \mathrm{S}^{k+\ell-2} W, \tag{4.4.37}$$

such that for $v, w \in W = \mathrm{S}^1 W \subseteq \mathrm{S}^\bullet W$ one has

$$\{v, w\}_\pi = \pi(v, w)\mathbb{1}. \tag{4.4.38}$$

Proof. Again, the uniqueness is clear since by the Leibniz rule, a Poisson bracket is determined by its values on the generators. Enforcing the Leibniz rule gives us the explicit expression

$$\{v_1 \vee \cdots \vee v_k, w_1 \vee \cdots \vee w_\ell\}_\pi = \sum_{i,j} \pi(v_i, w_j) v_1 \vee \cdots \overset{i}{\wedge} \cdots \vee v_k \vee w_1 \vee \cdots \overset{j}{\wedge} \cdots \vee w_\ell$$

as the unique extension of π to $\mathrm{S}^\bullet W$ which satisfies the Leibniz rule in both arguments. Since π is antisymmetric, $\{\cdot, \cdot\}_\pi$ is antisymmetric as well. It remains to check the Jacobi identity. Thus let

$$\mathrm{Jac}_\pi(f, g, h) = \{f, \{g, h\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi$$

be the Jacobiator of $\{\,\cdot\,,\,\cdot\,\}_\pi$ for arbitrary $f, g, h \in \mathrm{S}^\bullet W$. We have to show that $\mathrm{Jac}_\pi(f, g, h) = 0$. Now it is a simple algebraic fact that Jac_π is a derivation in each argument. Thus $\mathrm{Jac}_\pi(f, g, h) = 0$ iff the Jacobiator vanishes on *generators* already. In our case $\mathrm{Jac}_\pi(v, w, u) = 0$ is clear, since $\{v, w\}_\pi$ is already constant. The grading statement (4.4.37) is clear. □

This way we obtain a Poisson algebra of polynomials modeled by the symmetric algebra over $\Gamma^\infty_0(\iota^\# E^*) \oplus \Gamma^\infty_0(\iota^\# E^*)$. Without going into the details we note that this Poisson bracket has reasonable continuity properties with respect to the usual LF topology of $\Gamma^\infty_0(\iota^\# E^*) \oplus \Gamma^\infty_0(\iota^\# E^*)$. To explain these properties we first rewrite

$$\Gamma^\infty_0(\iota^\# E^*) \oplus \Gamma^\infty_0(\iota^\# E^*) = \Gamma^\infty_0(\iota^\# (E^* \oplus E^*)) \tag{4.4.39}$$

as usual. Then we have the following lemma:

Lemma 4.4.14 *There is a canonical injection*

$$\mathrm{S}^k \Gamma^\infty_0(\iota^\# (E^* \oplus E^*)) \hookrightarrow \Gamma^\infty_0(\iota^\# (E^* \oplus E^*) \underbrace{\boxtimes \cdots \boxtimes}_{k\text{-times}} \iota^\# (E^* \oplus E^*))^{\mathrm{S}_k} \tag{4.4.40}$$

of the symmetric power of $\Gamma^\infty_0(\iota^\# (E^* \oplus E^*))$ of degree k into the sections of the k-th external tensor product of $\iota^\# (E^* \oplus E^*)$ with itself which are totally symmetric under the internal action of the permutations of the fibres. Explicitly, we have

$$(\varphi_1 \vee \cdots \vee \varphi_k)(p_1, \ldots, p_k) = \sum_{\sigma \in \mathrm{S}_k} \varphi_{\sigma(1)}(p_1) \boxtimes \cdots \boxtimes \varphi_{\varphi(k)}(p_k) \tag{4.4.41}$$

for $p_1, \ldots, p_k \in \Sigma$ and $\Gamma^\infty_0(\iota^\# (E^* \oplus E^*))$.

Proof. Clearly, (4.4.41) is injective and well-defined, yielding a totally symmetric section with compact support. This follows analogously to Theorem 1.3.35. □

Remark 4.4.15 As in Theorem 1.3.35 this map is continuous in a very precise way: we have estimates analogously to the ones in (1.3.69). Without introducing this notion, we note that (4.4.40) is continuous with respect to the *projective* tensor product topology of $\mathrm{S}^k \Gamma^\infty_0(\iota^\# (E^* \oplus E^*))$, see e.g. [34, Chap. 15] for more details on this π-topology. Moreover, we note that the image of (4.4.40) is sequentially dense in the totally symmetric sections. This can also be shown analogously to Theorem 1.3.35. In fact, this even allows to extend the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_h$ to the direct sum over the right hand side of (4.4.40) for $k \in \mathbb{N}_0$ by a continuity argument. However, we shall not enter this discussion here.

From now on, we shall omit the explicit usage of the symbol $\mathcal{J}$ in (4.4.29) to simplify our notation and identify elements in $\mathrm{S}^\bullet V$ with the polynomials in $\mathrm{Pol}^\bullet(V)$ directly.

4.4.4 The Covariant Poisson Algebra

Up to now the Poisson algebra of observables has certain deficits from a physical point of view: its definition depends on the *choice* of a Cauchy hypersurface. In particular, it is not quite clear whether we get different Poisson algebras for different choices and, if not, how they are related in detail. In fact, since on a globally hyperbolic spacetime M all smooth spacelike Cauchy hypersurfaces are diffeomorphic and since any two positive definite fibre metrics are isometric, one can cook up an isomorphism of the Poisson algebras corresponding to (Σ_1, h_{Σ_1}) and (Σ_2, h_{Σ_2}), respectively. However, this does not seem to be a very conceptual statement as the isomorphism is just there by "pure luck".

More severe than these aesthetic arguments is the conceptual disadvantage that all nice symmetries between time- and spacelike directions will be "broken" by the choice of Σ. As example, one considers again Minkowski spacetime $(\mathbb{R}^n, \eta)$ with its Poincare symmetry $O(1, n-1) \ltimes \mathbb{R}^n$. Choosing an arbitrary smooth spacelike Cauchy hypersurface Σ results in destroying the symmetry: the Poincare group action will *not* respect the splitting $\mathbb{R}^n \simeq \mathbb{R} \times \Sigma$, even if Σ is a spacelike linear subspace. Thus the true symmetry of the situation might be hidden after choosing a splitting $\mathbb{R} \times \Sigma$.

Thus we look for a Poisson algebra isomorphic to the one constructed in Proposition 4.4.12 which is intrinsically defined without reference to Σ. This will be accomplished by the following construction, essentially going back to Peierls [47], see also [19–21, 41, 42] for a more modern treatment and applications to the (deformation) quantization of classical field theories as well as the thesis [33]. Note however, that we are only dealing with rather simple polynomial functions here instead of more general smooth functions.

We consider $\Gamma^\infty_0(E^*)$ which we can use to evaluate arbitrary sections $u \in \Gamma^\infty_{\mathrm{sc}}(E)$ on the whole spacetime M. Again, the symmetric algebra $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ serves as polynomial algebra on *all fields* $\Gamma^\infty_{\mathrm{sc}}(E)$, whether they are solutions to $Du = 0$ or not. The evaluation is the normal one, i.e. for $\varphi \in \Gamma^\infty_0(E^*)$ we set

$$\varphi(u) = \int_M \varphi(p) \cdot u(p)\, \mu_g(p) \tag{4.4.42}$$

and extend this to $\mathrm{S}^\bullet\Gamma^\infty_0(E)$ as before. Then these symmetric tensors become again an observable algebra. However, it should be emphasized clearly that we are dealing with polynomials on a much too large space $\Gamma^\infty_{\mathrm{sc}}(E)$ at the moment. Surprisingly, we will even have a Poisson bracket on this too large algebra:

Proposition 4.4.16 *Let (M, g) be a globally hyperbolic spacetime and $D \in \mathrm{DiffOp}^2(E)$ a normally hyperbolic differential operator that is symmetric with respect to a fibre metric h on E. Then on the symmetric algebra $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ there is a unique Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ determined by*

$$\{\varphi, \psi\} = \int_M h^{-1}\left(F_M\varphi, \psi\right) \mu_g \tag{4.4.43}$$

for $\varphi, \psi \in \Gamma^\infty_0(E^*)$, where $F_M = F^+_M - F^-_M$ as before. It satisfies

$$\left\{\mathrm{S}^k\Gamma^\infty_0(E^*), \mathrm{S}^\ell\Gamma^\infty_0(E^*)\right\} \subseteq \mathrm{S}^{k+\ell-2}\Gamma^\infty_0(E^*). \tag{4.4.44}$$

Proof. Since F_M is an antisymmetric operator with respect to the integration and h according to Remark 4.4.5, see also (4.4.17), the right hand side of (4.4.43) defines an antisymmetric bilinear form on $\Gamma^\infty_0(E^*)$. Thus, Proposition 4.4.13 can be applied. □

Definition 4.4.17 (Covariant Poisson bracket) *The Poisson bracket on $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ resulting from* (4.4.43) *is called the covariant Poisson bracket corresponding to D.*

Even though $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ is enough to separate points on the too large space of all fields $\Gamma^\infty_{\mathrm{sc}}(E)$, the covariant Poisson bracket becomes trivial for elements not sensitive to solutions of the wave equation. More precisely, we have the following result:

Lemma 4.4.18 *Let $\varphi \in \Gamma^\infty_0(E^*)$. Then the following statements are equivalent:*

i.) φ is a Casimir element of the covariant Poisson algebra $(\mathrm{S}^\bullet\Gamma^\infty_0(E^*), \{\,\cdot\,,\,\cdot\,\})$, i.e. we have

$$\{\varphi, \,\cdot\,\} = 0. \tag{4.4.45}$$

ii.) φ vanishes on solutions $u \in \Gamma^\infty_{\mathrm{sc}}(E)$ of the wave equation $Du = 0$, i.e.

$$\int_M \varphi \cdot u\, \mu_g = 0. \tag{4.4.46}$$

iii.) φ *is in the kernel of* F_M, *i.e.*

$$F_M \varphi = 0. \tag{4.4.47}$$

Proof. We show *i.)* $\Rightarrow$ *iii.)* $\Rightarrow$ *ii.)* $\Rightarrow$ *i.)*. Assume $\{\varphi, \cdot\} = 0$, then $0 = \{\varphi, \psi\} = \int_M h^{-1}(F_M\varphi, \psi)\mu_g$ for all $\psi \in \Gamma_0^\infty(E^*)$ which implies $F_M\varphi = 0$ since the pairing is non-degenerate. Now, if $F_M\varphi = 0$ then by Theorem 4.3.18, *iii.)* applied to D^{T} we know $\varphi = D^{\mathrm{T}}\chi$ for some $\chi \in \Gamma_0^\infty(E^*)$. Thus

$$\int_M \varphi \cdot u \,\mu_g = \int_M D^{\mathrm{T}}\chi \cdot u \,\mu_g = \int_M \varphi \cdot Du \,\mu_g = 0$$

for any solution $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ of the wave equation. Finally, assume that *ii.)* holds and let $\psi \in \Gamma_0^\infty(E^*)$ be arbitrary. Then $(F_M\psi)^\# = G_M\psi^\#$ solves the homogeneous wave equation. Thus

$$0 = \int_M \varphi \cdot (F_M\psi)^\# \mu_g = \int_M h^{-1}(\varphi, F_M\psi)\mu_g = -\{\varphi, \psi\}$$

for all $\psi \in \Gamma_0^\infty(E^*)$. By the Leibniz rule this implies $\{\varphi, \cdot\} = 0$ in general, since these ψ generate the whole algebra. ☐

We can rephrase the result of the lemma as follows: the kernel of F_M is a subspace $\ker F_M \subseteq \Gamma_0^\infty(E^*)$ which generates an ideal inside $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$. The generators of this ideal are Casimir elements whence the ideal is in fact even a *Poisson ideal*. Thus the quotient algebra of $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ by this ideal becomes a Poisson algebra itself. Now we want to relate this quotient to the canonical Poisson algebra defined on a Cauchy hypersurface Σ as constructed in the previous subsection. We want to establish a Poisson isomorphism which is compatible with the evaluation on solutions of the wave equation. To make these things more precise we again consider the result from Theorem 4.3.20. If $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ is the unique solution of $Du = 0$ with initial conditions $u_0, \dot{u}_0$ on Σ then the evaluation of $\varphi \in \Gamma_0^\infty(E^*)$ on u can be expressed by

$$\int_M \varphi \cdot u \,\mu_g = \int_\Sigma \Big((i^\# \nabla_{\mathfrak{n}}^{E^*} F_M\varphi) \cdot u_0 - (i^\# F_M\varphi) \cdot \dot{u}_0 \Big) \mu_\Sigma, \tag{4.4.48}$$

according to Theorem 4.3.20. Comparing this with the evaluation of a section $(\varphi_0, \dot{\varphi}_0) \in \Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$ on initial conditions according to (4.4.34), i.e.

$$(\varphi_0, \dot{\varphi}_0)\big|_{(u_0, \dot{u}_0)} = \int_\Sigma (\varphi_0 \cdot u_0 + \dot{\varphi}_0 \cdot \dot{u}_0)\,\mu_\Sigma, \tag{4.4.49}$$

suggests to map $\varphi \in \Gamma_0^\infty(E^*)$ to the section $(\varphi_0, \dot{\varphi}_0) \in \Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$ given by

$$\varphi_0 = \iota^\# \nabla_{\mathfrak{n}}^{E^*} F_M\varphi \quad \text{and} \quad \dot{\varphi}_0 = -\iota^\# F_M\varphi. \tag{4.4.50}$$

We denote this "restriction map" by

$$\varrho_\Sigma : \Gamma_0^\infty(E^*) \ni \varphi \;\mapsto\; \Big(\iota^\# \nabla_{\mathfrak{n}}^{E^*} F_M\varphi, -\iota^\# F_M\varphi \Big) \in \Gamma_0^\infty(\iota^\#(E^* \oplus E^*)). \tag{4.4.51}$$

Since $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ is freely generated by $\Gamma_0^\infty(E^*)$ we can extend ϱ_Σ in a unique way to a unital algebra homomorphism to $\mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$ which we still denote by

$$\varrho_\Sigma : \mathrm{S}^\bullet\Gamma_0^\infty(E^*) \longrightarrow \mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*)). \tag{4.4.52}$$

Then the above discussion results in the following lemma:

Lemma 4.4.19 *Let* $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ *be a solution of the homogeneous wave equation with initial conditions* $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$ *on* Σ. *Then for every* $\Phi \in \mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ *we have*

$$\Phi(u) = \varrho_\Sigma(\Phi)(u_0, \dot{u}_0). \tag{4.4.53}$$

Proof. We know (4.4.53) for $\Phi = \varphi \in \Gamma_0^\infty(E^*)$ by construction. For the constants we have by definition $\varrho_\Sigma(\mathbb{1}) = \mathbb{1}$ whence (4.4.53) is also true here. For higher symmetric tensors $\Phi \in \mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ the evaluation on u was defined to be compatible with the $\vee$-product, i.e.

$$(\varphi_1 \vee \cdots \vee \varphi_k)(u) = \varphi_1(u) \cdots \varphi_k(u).$$

Since we used the same sort of evaluation also for the symmetric tensors in $\mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$ the statement follows from the algebra homomorphism property of ϱ_Σ. □

Since the initial conditions determine the solution uniquely and vice versa it is tempting to use the algebra homomorphism ϱ_Σ to relate the Poisson algebras on M and on Σ. Indeed, we have the following result:

Lemma 4.4.20 *The algebra homomorphism ϱ_Σ is a homomorphism of Poisson algebras*

$$\varrho_\Sigma : (\mathrm{S}^\bullet\Gamma_0^\infty(E^*), \{\,\cdot\,,\,\cdot\,\}) \longrightarrow (\mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*)), \{\,\cdot\,,\,\cdot\,\}_h). \tag{4.4.54}$$

Proof. Since the Poisson brackets satisfy a Leibniz rule by definition and since ϱ_Σ is a unital algebra homomorphism it suffices to check the claim on generators. Thus let $\varphi, \psi \in \Gamma_0^\infty(E^*)$ be given and let $(\varphi_0, \dot\varphi_0) = \varrho_\Sigma(\varphi)$ and $(\psi_0, \dot\psi_0) = \varrho_\Sigma(\psi)$ be the corresponding sections in $\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$. Moreover, both Poisson brackets $\{(\varphi_0, \dot\varphi_0), (\psi_0, \dot\psi_0)\}_h$ and $\{\varphi, \psi\}$ are constants, i.e. multiples of the unit elements, respectively. Thus we only have to compute these number as $\varrho_\Sigma(\mathbb{1}) = \mathbb{1}$ by definition. We have

$$\begin{aligned}
\Big\{(\varphi_0, \dot\varphi_0), (\psi_0, \dot\psi_0)\Big\} &= \int_\Sigma \Big(h_\Sigma^{-1}(\varphi_0, \dot\psi_0) - h_\Sigma^{-1}(\dot\varphi_0, \psi_0) \Big) \mu_\Sigma \\
&= - \int_\Sigma \Big((\iota^\# \nabla_{\mathfrak{n}}^{E^*} F_M\varphi) \cdot (\iota^\# F_M\psi)^\# - (\iota^\# F_M\varphi) \cdot (\iota^\# \nabla_{\mathfrak{n}}^{E^*} F_M\psi)^\# \Big) \mu_\Sigma. \qquad (*)
\end{aligned}$$

Now $F_M\psi$ is a solution of the wave equation, $D^{\mathrm{T}} F_M\psi = 0$. Since D is symmetric, $u = (F_M\psi)^\# = G_M\psi^\#$ is a solution of $Du = 0$. The initial conditions for u on Σ are given by

$$u_0 = \iota^\# u = \iota^\#(F_M\psi)^\# \quad \text{and} \quad \dot u_0 = \iota^\# \nabla_{\mathfrak{n}}^E u = \iota^\#(\nabla_{\mathfrak{n}}^{E^*} F_M\psi)^\#$$

since the connections ∇^E and ∇^{E^*} are compatible with the musical isomorphisms as ∇^E is assumed to be metric with respect to h. By Theorem 4.3.20 we conclude

$$\begin{aligned}
\Big\{(\varphi_0, \dot\varphi_0), (\psi_0, \dot\psi_0)\Big\} &\overset{(*)}{=} - \int_M \varphi \cdot u \, \mu_g \\
&= - \int_M \varphi \cdot (F_M\psi)^\# \mu_g \\
&= - \int_M h^{-1}(\varphi, F_M\psi) \, \mu_g \\
&= \int_M h^{-1}(F_M\varphi, \psi) \, \mu_g \\
&= \{\varphi, \psi\}.
\end{aligned}$$

This shows that the constants coincide and thus the claim follows. □

Lemma 4.4.21 *The Poisson homomorphism ϱ_Σ is surjective and its kernel coincides with the ideal generated by the Casimir elements in $\Gamma_0^\infty(E^*)$, which coincides with all those $\Phi \in \mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ which vanish on all solutions $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ of the homogeneous wave equation $Du = 0$.*

Proof. By definition we have $\varrho_\Sigma(\mathbb{1}) = \mathbb{1}$. Now let $(\varphi_0, \dot\varphi_0) \in \Gamma^\infty_0(\iota^\#(E^* \oplus E^*))$ be given. Then there is a unique solution $\Phi \in \Gamma^\infty_{\mathrm{sc}}(E^*)$ of the homogeneous wave equation $D^{\mathrm{T}}\Phi = 0$ with initial conditions

$$\iota^\#\Phi = -\dot\varphi_0 \quad \text{and} \quad \iota^\#\nabla^{E^*}_{\mathfrak{n}}\Phi = \varphi_0 \tag{$*$}$$

by Theorem 4.2.5, *i.)* applied to D^{T}. By Theorem 4.3.18, *iii.)* we know that $\Phi = F_M\varphi$ for some $\varphi \in \Gamma^\infty_0(E^*)$. But then $\varrho_\Sigma(\varphi) = (\varphi_0, \dot\varphi_0)$ follows directly from $(*)$. Since the sections $(\varphi_0, \dot\varphi_0) \in \Gamma^\infty_0(\iota^\#(E^* \oplus E^*))$ generate the whole symmetric algebra and ϱ_Σ is an algebra homomorphism, the surjectivity follows. Now let $\Phi \in \mathrm{S}^\bullet\Gamma^\infty_0(E^*)$. Then $\varrho_\Sigma(\Phi) = 0$ iff for all $(u_0, \dot u_0)$ we have $\varrho_\Sigma(\Phi)(u_0, \dot u_0) = 0$. But this is equivalent to $\Phi(u) = 0$ for all solutions $u \in \Gamma^\infty_{\mathrm{sc}}(E)$ of the homogeneous wave equation $Du = 0$ by Lemma 4.4.19. Thus the kernel of ϱ_Σ consists precisely of those $\Phi \in \mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ which vanish on solutions. Since the kernel is clearly a (Poisson) ideal as ϱ_Σ is a (Poisson) algebra homomorphism and since the Casimir elements $\varphi \in \Gamma^\infty_0(E^*)$ vanish on solutions by Lemma 4.4.18 it follows that the ideal generated by the Casimir elements is part of the kernel. Now in symmetric degree one the converse is true: $\varphi \in \Gamma^\infty_0(E^*)$ is a Casimir element iff it is in the kernel. Thus we see that the induced map

$$\varrho_\Sigma : \Gamma^\infty_0(E^*) \big/ \big\{\varphi \in \Gamma^\infty_0(E^*) \;\big|\; \{\varphi, \cdot\,\} = 0\big\} \longrightarrow \Gamma^\infty_0(\iota^\#(E^* \oplus E^*))$$

is already a linear isomorphism. Thus we have an algebra isomorphism

$$\varrho_\Sigma : \mathrm{S}^\bullet\left(\Gamma^\infty_0(E^*) \big/ \big\{\varphi \in \Gamma^\infty_0(E^*) \;\big|\; \{\varphi, \cdot\,\} = 0\big\}\right) \longrightarrow \mathrm{S}^\bullet\Gamma^\infty_0(\iota^\#(E^* \oplus E^*)).$$

By a general argument, one has canonically $\mathrm{S}^\bullet(V/W) = \mathrm{S}^\bullet V/\mathcal{J}(W)$ for every linear subspace $W \subseteq V$, where $\mathcal{J}(W) \subseteq \mathrm{S}^\bullet V$ is the ideal generated by the elements in W. Hence we can conclude that ϱ_Σ is already injective on $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ modulo the ideal generated by the Casimir elements in $\Gamma^\infty_0(E^*)$. Thus the two ideals coincide. □

The covariant Poisson bracket gives us automatically the correct quotient procedure: the *vanishing ideal* of the subspace of solutions to the wave equation is a Poisson ideal, which can now be characterized in many equivalent ways: it is the ideal generated by the Casimir elements (and hence easily seen to be a Poisson ideal), or, equivalently, the ideal generated by the kernel of F_M, or, equivalently, the kernel of any of the Poisson homomorphisms ϱ_Σ for any Cauchy hypersurface ϱ_Σ. However, the physically important interpretation is the first: two $\Phi, \Psi \in \mathrm{S}^\bullet\Gamma^\infty_0(E)$ should be considered to be the same observables if they yield the same "expectation values"

$$\Phi(u) = \Psi(u) \tag{4.4.55}$$

for all physically relevant $u \in \Gamma^\infty_{\mathrm{sc}}(E)$, i.e. for all *solutions* of the wave equation. Note that a priori it is not clear whether this vanishing ideal of the subspace of solutions is a Poisson ideal at all. We can now summarize the results so far.

Theorem 4.4.22 (Covariant Poisson algebra) *Let (M, g) be a globally hyperbolic spacetime and let E be a real valued vector bundle with fibre metric and metric connections ∇^E. Let $D = \square^\nabla + B$ be a symmetric, normally hyperbolic differential operator on E. Moreover, let $\{\,\cdot\,,\,\cdot\,\}$ be the covariant Poisson bracket for $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ and let $\iota : \Sigma \hookrightarrow M$ be a smooth spacelike Cauchy hypersurface.*

i.) The following subspaces of $\mathrm{S}^\bullet\Gamma^\infty_0(E^*)$ coincide:

- *The vanishing ideal of the solutions of the wave equation $Du = 0$, i.e.*

$$\big\{\Phi \in \mathrm{S}^\bullet\Gamma^\infty_0 \;\big|\; \Phi(u) = 0 \textit{ for all } u \in \Gamma^\infty_{\mathrm{sc}}(E) \textit{ with } Du = 0\big\}. \tag{4.4.56}$$

- *The ideal generated by the Casimir elements $\varphi \in \Gamma^\infty_0(E^*)$.*
- *The ideal generated by the kernel of $F_M : \Gamma^\infty_0(E^*) \longrightarrow \Gamma^\infty_{\mathrm{sc}}(E^*)$.*

- *The kernel of the Poisson homomorphism*

$$\varrho_\Sigma : \mathrm{S}^\bullet\Gamma_0^\infty(E^*) \longrightarrow \mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*)). \tag{4.4.57}$$

ii.) The subspace in i.) is a Poisson ideal.

iii.) The quotient Poisson algebra $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)\big/\ker\varrho_\Sigma$ *is canonically isomorphic to the Poisson algebra* $\mathrm{S}^\bullet(\Gamma_0^\infty(E^*)\big/\ker F_M)$ *endowed with the induced bracket coming from* (4.4.43) *and*

$$\varrho_\Sigma : \mathrm{S}^\bullet\Gamma_0^\infty(E^*)\big/\ker\varrho_\Sigma \longrightarrow \mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*)) \tag{4.4.58}$$

is an isomorphism of Poisson algebra. It is compatible with evaluation on solutions and initial data, respectively, in the sense of (4.4.53)*.*

Proof. All the statements are clear from the preceding lemmas. □

Remark 4.4.23 (Covariant Poisson bracket) The remarkable feature of the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ as well as on the quotient $\mathrm{S}^\bullet(\Gamma_0^\infty(E^*)\big/\ker F_M)$ is that it does not refer to a splitting $\mathbb{R}\times\Sigma$ of M. Instead it is "*fully covariant*", i.e. defined in global and canonical terms only. Nevertheless, via ϱ_Σ it is isomorphic to the Poisson algebra on the Cauchy hypersurface Σ. The price is that for the construction of $\{\,\cdot\,,\,\cdot\,\}$ we have to use the dynamics already. This is a new feature as in geometrical mechanics the Poisson structure is understood as a *purely kinematical* ingredient of the theory. The dynamics comes only after specifying a Hamiltonian as an element of the a priori given Poisson algebra. Thus the above "covariant" Poisson bracket may also deserve the name "*dynamical Poisson bracket*".

Remark 4.4.24 (Time evolution) Using the Poisson isomorphisms ϱ_Σ for different Cauchy hypersurfaces we get a time evolution from one Cauchy hypersurface to another one. For smooth Cauchy hypersurfaces Σ, Σ' we have

$$\varrho_{\Sigma'} \circ \varrho_\Sigma^{-1} : \mathrm{S}^\bullet\Gamma_0^\infty(\iota^\#(E^* \oplus E^*)) \longrightarrow \mathrm{S}^\bullet\Gamma_0^\infty(\iota'^\#(E^* \oplus E^*)) \tag{4.4.59}$$

with $\varrho_\Sigma, \varrho_{\Sigma'}$ as in (4.4.58). This is an isomorphism of Poisson algebras. In this sense, the time evolution of the wave equation is "symplectic".

The next observation would be indeed very complicated and almost impossible to detect inside the canonical Poisson algebras of polynomials on the initial data. Here the global point of view indeed tuns out to be superior: Since we interpret the $\Phi \in \mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ as polynomial observables we can speak of a support of them. Indeed, we define for

$$\Phi = \sum_{k=0}^{N} \varphi_1^{(k)} \vee \cdots \vee \varphi_k^{(k)} \tag{4.4.60}$$

the *support* of Φ to be the (finite) union of the supports of the $\varphi_1^{(k)}, \ldots, \varphi_k^{(k)}$. In this sense we can speak of an observable being located in a certain region of the spacetime. The physical interpretation is that Φ corresponds to an observation (measurement) performed on the solution u in the spacetime region determined by $\operatorname{supp}\Phi$. Since we consider only those Φ coming from compactly supported $\varphi \in \Gamma_0^\infty(E^*)$ the support of Φ is also compact. Causality now means that two measurements Φ and Φ' should not influence each other in any way if they are performed in spacelike regions of M. The next proposition says that this is indeed the case:

Proposition 4.4.25 (Locality) *Let* $U, U' \subseteq M$ *be open subsets such that* U *is spacelike to* U'*. Then for all* $\Phi, \Phi' \in \mathrm{S}^\bullet\Gamma_0^\infty(E^*)$ *with* $\operatorname{supp}\Phi \subseteq U$ *and* $\operatorname{supp}\Phi' \subseteq U'$ *we have*

$$\big\{\Phi, \Phi'\big\} = 0. \tag{4.4.61}$$

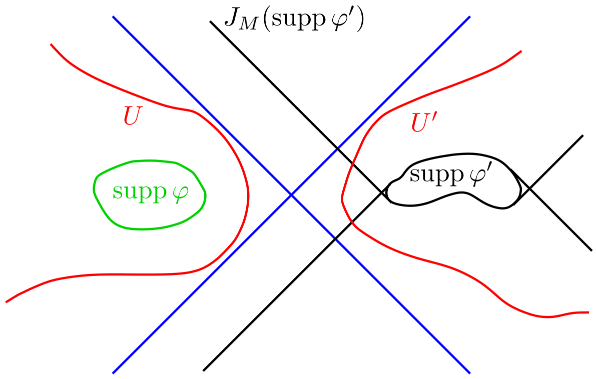

Figure 4.46: Illustration of the locality concept.

Proof. By the Leibniz rule it is again sufficient to consider $\varphi, \varphi' \in \Gamma_0^\infty(E^*)$ with $\operatorname{supp}\varphi \subseteq U$ and $\operatorname{supp}\varphi' \subseteq U'$ only. But here (4.4.61) is obvious since $\operatorname{supp}\varphi \subseteq U$ and $\operatorname{supp} F_M\varphi' \subset J_M(\operatorname{supp}\varphi') \subseteq J_M(U')$ have no overlap. Thus the integral (4.4.43) vanishes, see also Figure 4.46. □

For a later yet to be found transition to a quantum field theory, i.e. a quantization of the classical observable algebra, it is useful to consider also the complexification of $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$. This is the ultimate definition of the classical observable algebra.

Definition 4.4.26 (Classical observable algebra) *The classical observable algebra of the classical field theory determined by the wave equation is the unital Poisson* **-algebra*

$$\mathcal{A}(M) = \mathrm{S}^\bullet\left(\Gamma_0^\infty(E^*)\big/\ker F_M\right) \otimes \mathbb{C} \tag{4.4.62}$$

endowed with the complex conjugation as **-involution, the symmetric tensor product as associative and commutative product, and the covariant Poisson bracket induced from* $\mathrm{S}^\bullet\Gamma_0^\infty(E^*)$*.*

Here a Poisson *-algebra means that the *-involution is compatible with the Poisson bracket in the sense that $\{\,\cdot\,,\,\cdot\,\}$ is real, i.e. for $\Phi, \Psi \in \mathrm{S}^\bullet(\Gamma_0^\infty(E^*)\big/\ker F_M) \otimes \mathbb{C}$ we have

$$\overline{\{\Phi, \Psi\}} = \left\{\overline{\Phi}, \overline{\Psi}\right\}, \tag{4.4.63}$$

which is obvious as we complexified a Poisson algebra over $\mathbb{R}$. This is the ultimate reason that we insisted on a real vector bundle from the beginning. Alternatively, we can write the complexification as

$$\mathrm{S}^\bullet\Gamma_0^\infty(E^*) \otimes \mathbb{C} = \mathrm{S}^\bullet_{\mathbb{C}}\Gamma_0^\infty(E^* \otimes \mathbb{C}), \tag{4.4.64}$$

where we take the symmetric algebra over the complex numbers of the section of the complexified bundle. Again, this is compatible with the quotient procedure since F_M behaves well under complexification, according to Proposition 4.4.4, *iii.)*.

The locality property clearly passes to the quotient in the following sense: for an open subset $U \subseteq M$ we define analogously to (4.4.62)

$$\mathcal{A}_M(U) = \mathrm{S}^\bullet\left(\Gamma_0^\infty(E\big|_U)\big/\ker F_M\big|_{\Gamma_0^\infty(E|_U)}\right) \otimes \mathbb{C}, \tag{4.4.65}$$

and call this the subalgebra of observables located in U. Clearly, we have natural embeddings

$$\mathcal{A}_M(U) \hookrightarrow \mathcal{A}_M(U') \hookrightarrow \mathcal{A}_M(M) \tag{4.4.66}$$

for all $U \subseteq U' \subseteq M$ and each $\mathcal{A}_M(U)$ is a Poisson *-algebra itself. In this sense, $\mathcal{A}_M(M)$ becomes the inductive limit (int the category of Poisson *-algebras) of the collection of the $\mathcal{A}_M(U)$. The important consequence of Proposition 4.4.25 says that we have a local net of observable algebras:

Theorem 4.4.27 (Local net of observables) *The collection of Poisson* **-algebras* $\{\mathcal{A}_M(U) \mid U \subseteq M \text{ is open}\}$ *forms a net of local observables with inductive limit* $\mathcal{A}(M)$*, satisfying the causality condition*

$$\{\mathcal{A}_M(U), \mathcal{A}_M(U')\} = 0 \tag{4.4.67}$$

for $U, U' \subseteq M$ *spacelike to each other.*

Remark 4.4.28 This property is the classical analogy of one of the Haag-Kastler axioms for an (algebraic or axiomatic) quantum field theory: observables in spacelike regions should commute. We refer to [28] for further information on algebraic quantum field theory. Note that it would be extremely complicated to encode this net structure in the canonical Poisson algebra over Σ: here the covariant approach turns out to be the better choice.

We also have the following version of the time slice axiom:

Theorem 4.4.29 (Time slice axiom) *Let* $\iota : \Sigma \hookrightarrow M$ *be a smooth spacelike Cauchy hypersurface coming from a splitting* $M \simeq \mathbb{R} \times \Sigma$*. Let* $\epsilon > 0$*, then we have*

$$\mathcal{A}_M((-\epsilon, \epsilon) \times \Sigma) = \mathcal{A}_M(M). \tag{4.4.68}$$

Proof. This equality is of course not true on the level of the polynomial algebra $\mathrm{S}^\bullet\Gamma_0^\infty(E)$ itself since there are certainly elements with support *outside* $(-\epsilon, \epsilon) \times \Sigma$. The point is that they are equivalent to elements in $\mathrm{S}^\bullet\Gamma_0^\infty\left(E^*\big|_{(-\epsilon,\epsilon)\times\Sigma}\right)$ modulo the kernel of F_M. First we note that $(-\epsilon, \epsilon) \times \Sigma$ is again a globally hyperbolic spacetime by its own. Moreover, the embedding of $(-\epsilon, \epsilon) \times \Sigma$ into $M = \mathbb{R} \times \Sigma$ is causally compatible. We can now apply our theory of Green operators to D and D^{T} restricted to $(-\epsilon, \epsilon) \times \Sigma$ and obtain unique Green operators $F^\pm_{(-\epsilon,\epsilon)\times\Sigma}$ for D^{T} on $(-\epsilon, \epsilon) \times \Sigma$, too. Since $(-\epsilon, \epsilon) \times \Sigma$ is causally compatible in M, the support properties of $F^\pm_{(-\epsilon,\epsilon)\times\Sigma}$ match those of $F^\pm_M$ "restricted" to $(-\epsilon, \epsilon) \times \Sigma$. Thus by uniqueness we conclude that for $\varphi \in \Gamma_0^\infty\left(E^*\big|_{(-\epsilon,\epsilon)\times\Sigma}\right)$ we have

$$F^\pm_{(-\epsilon,\epsilon)\times\Sigma}\varphi = F^\pm_M\varphi\big|_{(-\epsilon,\epsilon)\times\Sigma}. \tag{$*$}$$

This implies that on $\mathrm{S}^\bullet\Gamma_0^\infty\left(E^*\big|_{(-\epsilon,\epsilon)\times\Sigma}\right)$ the covariant Poisson bracket coming from $F_{(-\epsilon,\epsilon)\times\Sigma}$ coincides with the restriction of the covariant Poisson bracket coming from F_M. Moreover, if $\varphi \in \Gamma_0^\infty(E^*)$ with support in $(-\epsilon, \epsilon) \times \Sigma$ vanishes on $u \in \Gamma_{\mathrm{sc}}^\infty(E)$ satisfying $Du = 0$ on M it also vanishes on $u \in \Gamma_{\mathrm{sc}}^\infty(E\big|_{(-\epsilon,\epsilon)\times\Sigma})$ satisfying $Du = 0$ on $(-\epsilon, \epsilon) \times \Sigma$. Indeed, in the condition $\varphi(u) = 0$ only $u\big|_{(-\epsilon,\epsilon)\times\Sigma}$ enters. This shows that

$$\ker F_{(-\epsilon,\epsilon)\times\Sigma} = \ker\left(F_M\big|_{\Gamma_0^\infty\left(E\big|_{(-\epsilon,\epsilon)\times\Sigma}\right)}\right).$$

Therefore the Poisson *-algebra $\mathcal{A}_M((-\epsilon, \epsilon)\times\Sigma)$ built using F_M and the Poisson *-algebra $\mathcal{A}_{(-\epsilon,\epsilon)\times\Sigma}((-\epsilon, \epsilon)\times\Sigma)$ coincide. Now we have the Poisson *-isomorphisms

$$\varrho_\Sigma : \mathcal{A}_{(-\epsilon,\epsilon)\times\Sigma} \longrightarrow \mathrm{S}^\bullet(\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))) \otimes \mathbb{C},$$

according to Theorem 4.4.22 applied to the spacetime $(-\epsilon, \epsilon) \times \Sigma$ as well as

$$\varrho_\Sigma^{-1} : \mathrm{S}^\bullet(\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))) \otimes \mathbb{C} \longrightarrow \mathcal{A}_M(M),$$

also using Theorem 4.4.22, now for the spacetime M. But this shows the equality (4.4.68). □

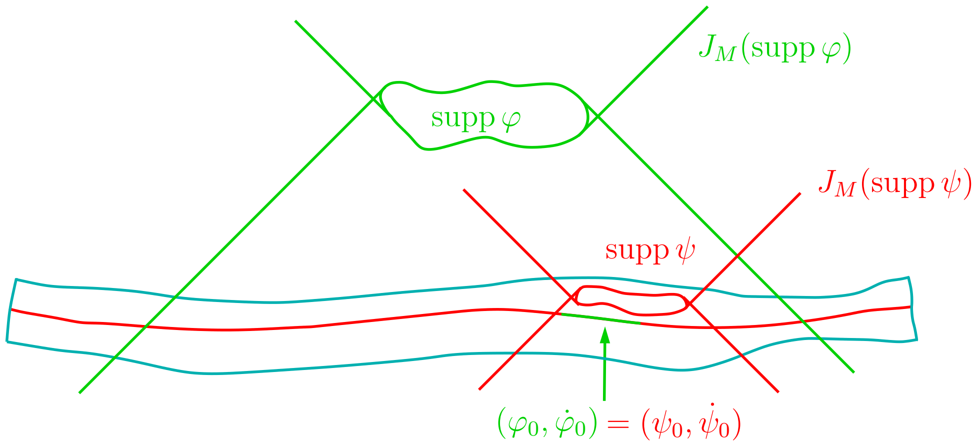

Figure 4.47: The time slice axiom

Remark 4.4.30 (Time slice axiom) We can rephrase this statement by saying that for every $\varphi \in \Gamma_0^\infty(E^*)$ with arbitrary compact support there is also a $\psi \in \Gamma_0^\infty\left(E^*\big|_{(-\epsilon,\epsilon)\times\Sigma}\right)$ having compact support very close to Σ such that their images under ϱ_Σ in $\Gamma_0^\infty(\iota^\#(E^* \oplus E^*))$ coincide, see also Figure 4.47. Since we know that ϱ_Σ is injective up to elements in $\ker F_M$ which is the image of D^{T} by (4.3.37) according to Theorem 4.3.18, we see that for every $\varphi \in \Gamma_0^\infty(E^*)$ there is a $\psi \in \Gamma_0^\infty(E^*)$ with support in $(-\epsilon, \epsilon) \times \Sigma$ such that

$$\varphi - \psi \in \ker F_M = \operatorname{im} D^{\mathrm{T}}, \tag{4.4.69}$$

see also [4, Lem. 4.5.6] for another approach to this question. Physically speaking, the time slice feature (4.4.68) says that on the level of observables a Cauchy hypersurface already determines everything. In view of our previous results this is of course not very surprising.

Appendix A

Parallel Transport, Jacobi Vector Fields, and all that

In this appendix we collect some facts on parallel transports, Taylor expansions and Jacobi vector fields needed in the computation of the derivatives of densities.

A.1 Taylor Expansion of Parallel Transports

Let ∇ be a torsion-free covariant derivative for M and let ∇^E be a covariant derivative for a vector bundle $E \longrightarrow M$. The aim is to compute the Taylor expansion of the parallel transport with respect to E along curves in M. Of particular interest will be the geodesics with respect to ∇.

Out of ∇ and ∇^E we can build covariant derivatives for all kind of bundles constructed from TM and E via dualizing and taking tensor products. We will denote them all by ∇ or ∇^E if E is involved. If $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is a smooth curve defined on some open interval then the pull-back connection of ∇ or ∇^E will be denoted by $\nabla^\#$. The canonical vector field on $\mathbb{R}$ is $\frac{\partial}{\partial t}$.

Lemma A.1.1 *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve in M and let $s \in \Gamma^\infty(\gamma^\# E)$ be a section of E along γ. For $t, t_0 \in I$ and all $k \in \mathbb{N}_0$ we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(P_{\gamma,t_0\to t}\right)^{-1} s(t) = \left(P_{\gamma,t_0\to t}\right)^{-1} \left(\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}} s(t)\right), \tag{A.1.1}$$

where $P_{\gamma,t_0\to t} : E_{\gamma(t_0)} \to E_{\gamma(t)}$ denotes the parallel transport along γ with respect to ∇^E.

Proof. We choose a vector space basis $e_\alpha(t_0) \in E_{\gamma(t_0)}$ and define smooth sections $e_\alpha \in \Gamma^\infty(\gamma^\# E)$ along γ by

$$e_\alpha(t) = P_{\gamma,t_0\to t} e_\alpha(t_0),$$

i.e. by parallel transporting $e_\alpha(t_0)$ to every point $\gamma(t)$ for $t \in I$. Since the parallel transport is a linear isomorphism, for every t the $e_\alpha(t)$ still form a basis of $E_{\gamma(t)}$. By the very definition, the $e_\alpha(t)$ solve the differential equation

$$\nabla^\#_{\frac{\partial}{\partial t}} e_\alpha(t) = 0 \tag{$*$}$$

with initial conditions $e_\alpha(t_0) \in \mathrm{e}_{\gamma(t_0)}$. Thus they are covariantly constant along γ. Now let $s \in \Gamma^\infty(\gamma^\# E)$ be arbitrary. Then there are unique smooth functions $s^\alpha \in \mathcal{C}^\infty(I)$ with

$$s(t) = s^\alpha(t) e_\alpha(t).$$

By linearity of $P_{\gamma,t_0\to t}$ we have

$$\left(P_{\gamma,t_0\to t}\right)^{-1}(s(t)) = s^\alpha(t) \left(P_{\gamma,t_0\to t}\right)^{-1}(e_\alpha(t)) = s^\alpha(t) \left(P_{\gamma,t_0\to t}\right)^{-1} P_{\gamma,t_0\to t}(e_\alpha(t_0)) = s^\alpha(t) e_\alpha(t_0).$$

This shows that we can express the left hand side of (A.1.1), being a curve in the vector space $E_{\gamma(t_0)}$, with respect to the *fixed* basis $e_\alpha(t_0)$. Thus the t-derivatives are easily computed giving

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\gamma,t_0\to t}\right)^{-1} s(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(s^\alpha(t) e_\alpha(t_0)) = \dot{s}^\alpha(t) e_\alpha(t_0) = \dot{s}^\alpha(t)\left(P_{\gamma,t_0\to t}\right)^{-1}\left(P_{\gamma,t_0\to t}(e_\alpha(t_0))\right) \\
&= \left(P_{\gamma,t_0\to t}\right)^{-1}\left(\dot{s}^\alpha(t) e_\alpha(t)\right) \stackrel{(*)}{=} \left(P_{\gamma,t_0\to t}\right)^{-1}\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\left(s^\alpha(t) e_\alpha(t)\right)\right) \\
&= \left(P_{\gamma,t_0\to t}\right)^{-1}\left(\nabla^{\#}_{\frac{\partial}{\partial t}} s(t)\right),
\end{aligned}$$

by the covariant constancy of the $e_\alpha(t)$. This shows (A.1.1) for $k = 1$ and from here we can proceed by induction. □

The next lemma will be useful to compute the Taylor coefficients of a function of several variables in an efficient way. The proof is a simple computation.

Lemma A.1.2 *Let $F \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $k \in \mathbb{N}_0$. Then one has*

$$\left.\frac{\partial^k F}{\partial v^{i_1} \cdots \partial v^{i_k}}\right|_{v=0} = \frac{1}{k!} \frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}} \left(\left.\frac{\mathrm{d}^k}{\mathrm{d}\, t^k} F(tv)\right|_{t=0} \right). \tag{A.1.2}$$

The following technical lemma will allow us to compute iterated covariant derivatives in terms of the symmetrized covariant derivative.

Lemma A.1.3 *For $s \in \Gamma^\infty(E)$ one inductively defines*

$$\begin{aligned}
\nabla^0 s &= s, \\
\left(\nabla^1 s\right)(X) &= \nabla^E_X s, \\
\left(\nabla^k s\right)(X_1, \ldots, X_k) &= \left(\nabla^E_{X_1} \nabla^{k-1} s\right)(X_2, \ldots, X_k)
\end{aligned} \tag{A.1.3}$$

for $X_1, \ldots, X_k \in \Gamma^\infty(TM)$. Then $\nabla^k s \in \Gamma^\infty(\otimes^k T^*M \otimes E)$ is a well-defined tensor field and we have

$$\sum_{\sigma \in \mathrm{S}_k} \left(\nabla^k s\right)(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = \left(\left(\mathsf{D}^E\right)^k s\right)(X_1, \ldots, X_k) \tag{A.1.4}$$

for the totally symmetric part of $\nabla^k s$.

Proof. By induction it is clear that $\nabla^k s$ is $\mathcal{C}^\infty(M)$-linear in each argument. Thus it defines a tensor field of the above type. To prove (A.1.4) we first note that for $k = 0, 1$ we have $\nabla^0 s = s = (\mathsf{D}^E)^0 s$ and $\nabla^1 s = \mathsf{D}^E s$ as wanted. We proceed by induction and have

$$\begin{aligned}
\sum_{\sigma \in \mathrm{S}_k} \left(\nabla^k s\right)(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) &= \sum_{\sigma \in \mathrm{S}_k} \left(\nabla^E_{X_{\sigma(1)}} \nabla^{k-1} s\right)(X_{\sigma(2)}, \ldots, X_{\sigma(k)}) \\
&= \sum_{\ell=1}^k \sum_{\substack{\sigma \in \mathrm{S}_k \\ \sigma(1)=\ell}} \left(\nabla^E_{X_\ell} \nabla^{k-1} s\right)(X_{\sigma(2)}, \ldots, X_{\sigma(k)}) \\
&= \sum_{\ell=1}^k \left(\nabla^E_{X_\ell} (\mathsf{D}^E)^{k-1} s\right)(X_2, \ldots, \stackrel{\ell}{\wedge}, \ldots, X_k)
\end{aligned}$$

$$= \left((D^E)^k s\right)(X_1, \ldots, X_k),$$

since the permutations $\sigma \in \mathrm{S}_k$ with $\sigma(1) = \ell$ are precisely the permutations of the remaining $1, \ldots, \overset{\ell}{\wedge}, \ldots, k$ entries. □

Since covariant derivatives are extended to tensor bundles and dual bundles in such a way that we have Leibniz rules with respect to tensor products and natural pairings, the parallel transport enjoys homomorphism properties in the following sense:

Lemma A.1.4 *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve in M and $f \in \mathcal{C}^\infty(M)$, $s_{t_0}, \widetilde{s}_{t_0} \in E_{\gamma(t_0)}$ and $\alpha_{t_0} \in E^*_{\gamma(t_0)}$.*

i.) Viewing f as a sections of $\otimes^0 E$, $\gamma^\# f = \gamma^* f$ is parallel if and only if f is constant along γ.

ii.) For all $t \in I$ we have

$$P_{\gamma,t_0\to t}(s_{t_0} \otimes \widetilde{s}_{t_0}) = P_{\gamma,t_0\to t}(s_{t_0}) \otimes P_{\gamma,t_0\to t}(\widetilde{s}_{t_0}). \tag{A.1.5}$$

iii.) For all $t \in I$ we have

$$\alpha_{t_0}(s_{t_0}) = P_{\gamma,t_0\to t}(\alpha_{t_0}(s_{t_0})) = P_{\gamma,t_0\to t}(\alpha_{t_0})\left(P_{\gamma,t_0\to t}(s_{t_0})\right). \tag{A.1.6}$$

Proof. For the first part we observe that by definition $\nabla^E_X f = X(f)$ when viewing a function as a tensor field. Moreover, $\gamma^\# f = \gamma^* f$ and hence $\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# f = \frac{\partial}{\partial t}\gamma^* f\big|_t = \dot{\gamma}(t) f\big|_{\gamma(t)}$ which is zero iff $f \circ \gamma$ is constant. It follows that for a *number* $z \in \otimes^0 E_{\gamma(t_0)} = \mathbb{C}$ we simply have $P_{\gamma,t_0\to t}(z) = z$ for all times. This shows the first part. For the second part we note that the left hand side is the unique solution of

$$\nabla^\#_{\frac{\partial}{\partial t}} P_{\gamma,t_0\to t}(s_{t_0} \otimes \widetilde{s}_{t_0}) = 0$$

with initial condition $s_{t_0} \otimes \widetilde{s}_{t_0}$ for $t = t_0$. For the right hand side we compute

$$\begin{aligned}
&\nabla^\#_{\frac{\partial}{\partial t}} \left(P_{\gamma,t_0\to t}(s_{t_0}) \otimes P_{\gamma,t_0\to t}(\widetilde{s}_{t_0})\right) \\
&\quad = \nabla^\#_{\frac{\partial}{\partial t}} \left(P_{\gamma,t_0\to t}(s_{t_0})\right) \otimes P_{\gamma,t_0\to t}(\widetilde{s}_{t_0}) + P_{\gamma,t_0\to t}(s_{t_0}) \otimes \nabla^\#_{\frac{\partial}{\partial t}} P_{\gamma,t_0\to t}(\widetilde{s}_{t_0}) = 0,
\end{aligned}$$

by the Leibniz rule of $\nabla^\#$ for sections $\Gamma^\infty(\gamma^\#(E \otimes E))$ with respect to $\otimes$. Since the right hand side of (A.1.5) is $s_{t_0} \otimes \widetilde{s}_{t_0}$ for $t = t_0$ we have (A.1.5) by uniqueness. Analogously, one shows *iii.)*. □

By combination of *ii.)* and *iii.)* we obtain the compatibility of parallel transport with the usual tensor product constructions and multilinear pairings. We shall use this frequently in the following.

Since a covariant derivative ∇^E also induces a covariant derivative for the density bundles we consider the compatibility of the parallel transport with the evaluation of a density on a basis. To this end we first recall the definition of the covariant derivative of a density. If $A_\alpha^\beta \in \Gamma^\infty(T^*U)$ denote the local connection one-forms of ∇^E with respect to a local frame $e_\alpha \in \Gamma^\infty\left(E\big|_U\right)$ then the covariant derivative of a z-density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|^z E^*)$ is defined locally by

$$(\nabla_X \mu)(e_1, \ldots, e_N) = X(\mu(e_1, \ldots, e_N)) - z \sum_{\alpha=1}^N A_\alpha^\alpha(X)\mu(e_1, \ldots, e_N), \tag{A.1.7}$$

where $N = \operatorname{rank} E$ and $z \in \mathbb{C}$, see e.g. [60, Sect. 2.2] for this approach and the proof that (A.1.7) indeed gives a globally defined $\nabla_X \mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|^z E^*)$. We shall interpret (A.1.7) in a more global way. Since μ is *not* multilinear in the arguments $e_1, \ldots, e_N$ we can *not* expect a simple Leibniz rule (and hence an alternative global definition of $\nabla_X \mu$) for the covariant derivative of a z-density. Instead we shall solve the differential equation

$$\nabla^{\#}_{\frac{\partial}{\partial t}} \mu = 0 \tag{A.1.8}$$

for a $\mu \in \Gamma^\infty(\gamma^{\#}|\Lambda^{\mathrm{top}}|^z E^*)$ explicitly. To this end, we note that $\gamma^{\#}(|\Lambda^{\mathrm{top}}|^z E^*) = |\Lambda^{\mathrm{top}}|^z (\gamma^{\#} E)^*$. Thus we can evaluate (A.1.8) in a local frame of $\gamma^{\#} E$ giving the equivalent local condition

$$0 = (\nabla^{\#}_{\frac{\partial}{\partial t}} \mu)(e_1, \ldots, e_N) = \frac{\partial}{\partial t}(\mu(e_1, \ldots, e_N)) - z \sum_{\alpha=1}^{N} A^{\#}{}^{\alpha}_{\alpha}\left(\frac{\partial}{\partial t}\right) \mu(e_1, \ldots, e_N) \tag{A.1.9}$$

where $A^{\#}{}^{\alpha}_{\beta} \in \Gamma^\infty(T^*I)$ are the local connection one-forms of $\nabla^{\#}$ with respect to the local frame $e_\alpha \in \Gamma^\infty(\gamma^{\#} E)$. Note that (A.1.9) is valid not only for frames of the form $\gamma^{\#} e_\alpha$ but for all frames. In particular, we can choose a covariantly constant frame as in the proof of Lemma A.1.1. This simply means that $A^{\#}{}^{\beta}_{\alpha} = 0$ for such a frame. Thus we arrive at the statement that for a covariantly constant frame we have (A.1.9) iff

$$\frac{\partial}{\partial t}(\mu(e_1, \ldots, e_N)) = 0. \tag{A.1.10}$$

This means that for a covariantly constant frame the function $\mu(e_1, \ldots, e_N)$ is constant. Conversely, if $\mu(e_1, \ldots, e_N)$ is constant for a covariantly constant frame then $A^{\#}{}^{\beta}_{\alpha} = 0$. Hence by (A.1.7) we conclude that μ is covariantly constant. From this we obtain the following statement:

Lemma A.1.5 (Parallel transport of densities) *Let $z \in \mathbb{C}$ and $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ a smooth curve. For a z-density $\mu \in |\Lambda^{\mathrm{top}}|^z E^*_{\gamma(t_0)}$ and a basis $e_1, \ldots, e_N \in E_{\gamma(t_0)}$ we have*

$$\mu(e_1, \ldots, e_N) = (P_{\gamma, t_0 \to t}(\mu)) \left(P_{\gamma, t_0 \to t}(e_1), \ldots, P_{\gamma, t_0 \to t}(e_N)\right). \tag{A.1.11}$$

Proof. Let $\mu(t) = P_{\gamma, t_0 \to t}(\mu) \in |\Lambda^{\mathrm{top}}|^z E^*_{\gamma(t)}$ and let $e_\alpha \in \Gamma^\infty(\gamma^{\#} E)$ be a covariantly constant frame, i.e. $e_\alpha(t) = P_{\gamma, t_0 \to t}(e_\alpha(t_0))$. Then we know that

$$\mu(e_1(t_0), \ldots, e_N(t_0)) = \mu(t)(e_1(t), \ldots, e_N(t))$$

by our previous considerations. But this is (A.1.11). □

Thus also here the parallel transport has "homomorphism properties". Note however that the covariant derivative does not obey a simple Leibniz rule with respect to the "pairing" of a z-density and a frame.

Now we consider geodesics $\gamma(t) = \exp_p(tv)$ with respect to ∇ instead of arbitrary curves. Since in this case $\dot{\gamma}$ is covariantly constant along γ we obtain the following lemma:

Lemma A.1.6 *Let $s \in \Gamma^\infty(E)$ and let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ by a geodesic. Then we have for all $k \in \mathbb{N}_0$ and $t \in I$*

$$\Big(\underbrace{\nabla^{\#}_{\frac{\partial}{\partial t}} \cdots \nabla^{\#}_{\frac{\partial}{\partial t}}}_{k \text{ times}} \gamma^{\#} s\Big)(t) = \left(\nabla^k s\big|_{\gamma(t)}\right)(\dot{\gamma}(t), \ldots, \dot{\gamma}(t)). \tag{A.1.12}$$

Proof. For $k = 0$ the statement is clearly correct. For $k = 1$ we have

$$\left(\nabla^{\#}_{\frac{\partial}{\partial t}} \gamma^{\#} s\right)(t) = \left(\nabla_{\dot{\gamma}(t)} s\right)(\gamma(t)) = \nabla s\big|_{\gamma(t)}(\dot{\gamma}(t))$$

by definition of $\nabla^{\#}$. Thus (A.1.12) holds for $k = 1$ as well. The general case follows by induction since

$$\nabla^k s\big|_{\gamma(t)}(\dot{\gamma}(t), \ldots, \dot{\gamma}(t)) = \left(\nabla^{\#}_{\frac{\partial}{\partial t}} \nabla^{k-1} s\right)\Big|_{\gamma(t)}(\dot{\gamma}(t), \ldots, \dot{\gamma}(t))$$

$$= \left(\nabla^{\#}_{\frac{\partial}{\partial t}}\left(\gamma^{\#}\nabla^{k-1}s\right)\right)\Big|_t(\dot\gamma(t),\ldots,\dot\gamma(t))$$

$$= \nabla^{\#}_{\frac{\partial}{\partial t}}\left((\gamma^{\#}\nabla^{k-1}s)\right)\Big|_t(\dot\gamma(t),\ldots,\dot\gamma(t))$$

$$-\sum_{\ell=1}^{k-1}\gamma^{\#}\nabla^{k-1}s\big|_t(\dot\gamma(t),\ldots,\nabla^{\#}_{\frac{\partial}{\partial t}}\dot\gamma\big|_t,\ldots,\dot\gamma(t))$$

$$= \nabla^{\#}_{\frac{\partial}{\partial t}}\cdots\nabla^{\#}_{\frac{\partial}{\partial t}}s\big|_t - 0,$$

using that $\dot\gamma$ is covariantly constant. □

Using Lemma A.1.3 we can rephrase the statement (A.1.12) using the symmetrized covariant derivative since we only evaluate $\nabla^k s\big|_{\gamma(t)}$ on k times the same vector $\dot\gamma(t)$. Thus we have

$$\nabla^{\#}_{\frac{\partial}{\partial t}}\cdots\nabla^{\#}_{\frac{\partial}{\partial t}}s = \frac{1}{k!}\left(\gamma^{\#}\left(\mathsf{D}^E\right)^k s\right)(\dot\gamma,\ldots,\dot\gamma), \tag{A.1.13}$$

taking into account the correct combinatorics. We can use this now to compute the Taylor coefficients of the parallel transport along geodesics in general:

Proposition A.1.7 (Taylor coefficients of the parallel transport) *Let $k \in \mathbb{N}_0$ and $s \in \Gamma^\infty(E)$ be given. Denote by $\gamma_v(t) = \exp_p(tv)$ the geodesic starting at p with velocity $v \in T_pM$. Then the Taylor coefficients of the parallel transport in radial directions are given by*

$$\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\left(P_{\gamma_v,0\to1}\right)^{-1}s(\gamma_v(1))\Big|_{v=0} = \mathrm{i_s}(e_{i_1})\cdots\mathrm{i_s}(e_{i_k})\frac{1}{k!}\left(\mathsf{D}^E\right)^k s\big|_p, \tag{A.1.14}$$

where $v^1,\ldots,v^k$ are the linear coordinates on T_pM with respect to a vector space basis $e_1,\ldots,e_n \in T_pM$.

Proof. First note that $s(\gamma_v(1)) \in E_{\gamma_v(1)}$ whence $\left(P_{\gamma_v,0\to1}\right)^{-1}s(\gamma_v(1)) \in E_{\gamma_v(0)} = E_p$ is indeed a vector in E_p for all $v \in T_pM$. Thus the map

$$v \;\mapsto\; \left(P_{\gamma_v,0\to1}\right)^{-1}s(\gamma_v(1))$$

is a smooth E_p-valued function on T_pM defined on an open neighborhood of 0. Thus we can apply Lemma A.1.2 to compute its Taylor coefficients. We obtain

$$\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\left(P_{\gamma_v,0\to1}\right)^{-1}s(\gamma_v(1))\Big|_{v=0}$$

$$\overset{\text{Lem. A.1.2}}{=} \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big|_{t=0}\left(P_{\gamma_{tv},0\to1}\right)^{-1}s(\gamma_{tv}(1))$$

$$= \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big|_{t=0}\left(P_{\gamma_v,0\to t}\right)^{-1}s(\gamma_v(t))$$

$$\overset{\text{Lem. A.1.1}}{=} \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big|_{t=0}\left(P_{\gamma_v,0\to t}\right)^{-1}\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\cdots\nabla^{\#}_{\frac{\partial}{\partial t}}s(\gamma_v(t))\right)$$

$$\overset{\text{(A.1.13)}}{=} \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k!}\frac{\partial^k}{\partial t^k}\Big|_{t=0}\left(P_{\gamma_v,0\to t}\right)^{-1}\left(\frac{1}{k!}\left(\gamma^{\#}\left(\mathsf{D}^E\right)^k s\right)(\dot\gamma,\ldots,\dot\gamma)\right)\Big|_{t=0}$$

$$= \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k!}\frac{1}{k!}\left(\mathsf{D}^E\right)^k s\big|_p(v,\ldots,v)$$

$$= \frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}} \frac{1}{k!} \frac{1}{k!} v^{j_1} \cdots v^{j_k} \left(\mathsf{D}^E\right)^k s\big|_p(e_{j_1}, \ldots, e_{j_k})$$

$$= \frac{1}{k!} \left(\mathsf{D}^E\right)^k s\big|_p(e_{i_1}, \ldots, e_{i_k}),$$

using $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. □

Of course, the Taylor expansion of $\left(P_{\gamma_v, 0\to 1}\right)^{-1} s(\gamma_v(1))$ around 0 needs not to converge at all. In fact, the Borel Lemma, see e.g. [60, Remark 5.3.34], shows that *all* possible numerical values appear as Taylor coefficients of smooth functions. Nevertheless, we can use this proposition to obtain the *formal* Taylor series in a very nice way:

Corollary A.1.8 *The formal Taylor series of the function $T_pM \ni v \mapsto \left(P_{\gamma_v, 0\to 1}\right)^{-1} s(\gamma_v(1)) \in E_p$ is given by*

$$\left(P_{\gamma_v, 0\to 1}\right)^{-1} s(\gamma_v(1)) \sim \mathcal{J}\left(\mathrm{e}^{\mathsf{D}^E} s\right)(v), \tag{A.1.15}$$

where $\mathcal{J} : \bigoplus_{k=0}^{\infty} \mathrm{S}^k T_p^* M \otimes E_p \longrightarrow \mathrm{Pol}^\bullet(T_pM) \otimes E_p$ is the canonical isomorphism, extended to formal series in the symmetric and polynomial degree, respectively.

Proof. This is now just a matter of computation. By Proposition A.1.7 we have in the sense of a formal series in v

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}} \left(P_{\gamma_v, 0\to 1}\right)^{-1} s(\gamma_v(1))\Big|_{v=0} v^{i_1} \cdots v^{i_k} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{k!} \left(\mathsf{D}^E\right)^k s\Big|_p(e_{i_1}, \ldots, e_{i_k}) v^{i_1} \cdots v^{i_k} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{k!} \left(\mathsf{D}^E\right)^k s(v, \ldots, v) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{J}\left(\left(\mathsf{D}^E\right)^k s\right)(v) \\
&= \mathcal{J}\left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathsf{D}^E\right)^k s\right) \\
&= \mathcal{J}\left(\mathrm{e}^{\mathsf{D}^E} s\right)(v).
\end{aligned}$$

□

In a more informal way one can say that the Taylor expansion of the parallel transport along geodesics around initial velocity 0 is given by the exponential of the symmetrized covariant derivative.

We can specialize this statement to functions instead of general sections. Here we simply have for $f \in \mathcal{C}^\infty(M)$

$$\left(P_{\gamma_v, 0\to 1}\right)^{-1} f(\gamma_v(1)) = f(\gamma_v(1)) = f(\exp_p(v)) = (\exp_p^* f)(v),$$

since by Lemma A.1.4 the parallel transport of numbers is trivial. Thus we obtain the Taylor expansion of $\exp_p^*$ around 0:

Corollary A.1.9 (Taylor expansion of $\exp_p^*$) *Let $V \subseteq T_pM$ be an open neighborhood of 0 such that $\exp_p\big|_V$ is a diffeomorphism onto $U = \exp_p(V) \subseteq M$. Moreover, let $f \in \mathcal{C}^\infty(U)$. Then the formal Taylor series of $\exp_p^* f \in \mathcal{C}^\infty(V)$ around 0 is given by*

$$\exp_p^* f \sim \mathcal{J}\left(\mathrm{e}^{\mathsf{D}} f\right). \tag{A.1.16}$$

With other words, the Taylor expansion in normal coordinates around p coincides with the Taylor expansion using D.

A.2 Jacobi Vector Fields and the Tangent Map of $\exp_p$

In this section we consider not a single curve γ in M but families of curves which are smoothly parametrized by an additional variable. With other words, we consider smooth *surfaces*

$$\sigma : \Sigma \longrightarrow M \tag{A.2.1}$$

in M where $\Sigma \subseteq \mathbb{R}^2$ is open. For convenience, we mainly restrict to $\Sigma = I \times I'$ where $I, I' \subseteq \mathbb{R}$ are open intervals. Hence Σ is an open rectangle. The two variables will be denoted by $(t, s) \in \Sigma$. The canonical vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ on Σ give now rise to vector fields

$$\dot{\sigma} = T\sigma \left(\frac{\partial}{\partial t}\right) \quad \text{and} \quad \sigma' = T\sigma \left(\frac{\partial}{\partial s}\right), \tag{A.2.2}$$

which we can view as vector fields along σ, i.e. sections

$$\dot{\sigma}, \sigma' \in \Gamma^\infty(\sigma^\# TM) \tag{A.2.3}$$

of the pulled back tangent bundle. The first lemma gives a geometric interpretation of the torsion of a covariant derivative. Note that for $\nabla^\#$ there is no intrinsic definition of torsion possible.

Lemma A.2.1 *Let ∇ be a covariant derivative for M and $\sigma : \Sigma \longrightarrow M$ a smooth surface. Then*

$$\nabla^\#_{\frac{\partial}{\partial t}} \sigma' - \nabla^\#_{\frac{\partial}{\partial s}} \dot{\sigma} = \sigma^\# \operatorname{Tor}(\dot{\sigma}, \sigma'). \tag{A.2.4}$$

In particular, if ∇ is torsion-free we have

$$\nabla^\#_{\frac{\partial}{\partial t}} \sigma' - \nabla^\#_{\frac{\partial}{\partial s}} \dot{\sigma} = 0. \tag{A.2.5}$$

Proof. This is just a simple consequence of the definition of the pull-back connection $\nabla^\#$. If (U, x) is a local chart we have

$$\dot{\sigma}(t, s) = \frac{\partial \sigma^i}{\partial t}(t, s) \frac{\partial}{\partial x^i}\Big|_{\sigma(t,s)} \quad \text{and} \quad \sigma'(t, s) = \frac{\partial \sigma^i}{\partial s}(t, s) \frac{\partial}{\partial x^i}\Big|_{\sigma(t,s)},$$

where $\sigma^i = x^i \circ \sigma$. Then

$$\nabla^\#_{\frac{\partial}{\partial t}} \sigma'\Big|_{t,s} = \frac{\partial}{\partial t}\frac{\partial \sigma^i}{\partial s}(t, s) \frac{\partial}{\partial x^i}\Big|_{\sigma(t,s)} + \frac{\partial \sigma^i}{\partial s}(t, s) \Gamma^k_{ji}(\sigma(t, s)) \frac{\partial \sigma^j}{\partial s}(t, s) \frac{\partial}{\partial x^k}\Big|_{\sigma(t,s)},$$

and analogously for $\nabla^\#_{\frac{\partial}{\partial s}} \dot{\sigma}$. From this the claim (A.2.4) follows since $\operatorname{Tor}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. But then (A.2.5) is clear. □

Lemma A.2.2 *Let ∇^E be a covariant derivative for $E \longrightarrow M$ and $\sigma : \Sigma \longrightarrow M$ a smooth surface in M. Then for $e \in \Gamma^\infty(\sigma^\# E)$ we have*

$$\nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial s}} e - \nabla^\#_{\frac{\partial}{\partial s}} \nabla^\#_{\frac{\partial}{\partial t}} e = R^E\big|_\sigma(\dot{\sigma}, \sigma')e. \tag{A.2.6}$$

Proof. This is just a particular case of the statement that the local curvature two-forms of ∇^E are the pull-backs of the local curvature two-forms of ∇ together with $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$. □

We can now turn to Jacobi vector fields: they will turn out to be the infinitesimal version of a family of geodesics. One defines for a yet arbitrary curve a Jacobi vector field as follows:

Definition A.2.3 (Jacobi vector field) *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve in M. Then a vector field $J \in \Gamma^\infty(\gamma^\# TM)$ is called Jacobi vector field along γ if it satisfies the differential equation*

$$\nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} J(t) = R_{\gamma(t)}(\dot{\gamma}(t), J(t))\dot{\gamma}(t) \tag{A.2.7}$$

for all $t \in I$.

Up to now it is not necessary for γ to be a geodesic, though later on in most applications γ will be a geodesic. We investigate (A.2.7) in a local chart (U, x). As usual we set $\gamma^i = x^i \circ \gamma$. Then we have for

$$J(t) = J^i(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \tag{A.2.8}$$

the first covariant derivative

$$\nabla^\#_{\frac{\partial}{\partial t}} J = \frac{\mathrm{d}J^i}{\mathrm{d}t} \frac{\partial}{\partial x^i} + \Gamma^k_{ij} J^i \dot{\gamma}^j \frac{\partial}{\partial x^k}. \tag{A.2.9}$$

Analogously, one computes the second covariant derivative

$$\begin{aligned} \nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} J = {} & \frac{\mathrm{d}^2 J^i}{\mathrm{d}t^2} \frac{\partial}{\partial x^i} + \Gamma^k_{ij} \frac{\mathrm{d}J^i}{\mathrm{d}t} \dot{\gamma}^j \frac{\partial}{\partial x^k} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\Gamma^k_{ij} \dot{\gamma}^j \right) J^i \frac{\partial}{\partial x^k} \\ & + \Gamma^k_{ij} \frac{\mathrm{d}J^i}{\mathrm{d}t} \dot{\gamma}^j \frac{\partial}{\partial x^k} + \Gamma^k_{ij} J^i \dot{\gamma}^j \Gamma^m_{kl} \frac{\partial}{\partial x^m}, \end{aligned} \tag{A.2.10}$$

where always the data on M has to be evaluated at $\gamma(t)$. On the other hand we have for the right hand side of (A.2.7)

$$R(\dot{\gamma}, J)\dot{\gamma} = R^\ell{}_{kij} \dot{\gamma}^i J^j \dot{\gamma}^k \frac{\partial}{\partial x^\ell}. \tag{A.2.11}$$

It follows that (A.2.7) is locally a system of linear second order differential equations for the coefficient functions J^i on $I \subseteq \mathbb{R}$ having the identity as leading symbol and time-dependent coefficients for the first and zeroth order terms. Thus we can apply the well-known theorems on existence and uniqueness of solutions for such ordinary differential equations:

Proposition A.2.4 *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve and $a \in I$. Then for every $v, w \in T_{\gamma(t)}M$ there exists a unique Jacobi vector field $J_{v,w}$ along γ with*

$$J_{v,w}(a) = v \quad \textit{and} \quad \nabla^\#_{\frac{\partial}{\partial t}} J_{v,w}(a) = w. \tag{A.2.12}$$

Moreover, the map

$$T_{\gamma(t)}M \oplus T_{\gamma(t)}M \ni (v, w) \ \mapsto \ J_{v,w} \in \Gamma^\infty(\gamma^\# TM) \tag{A.2.13}$$

is a linear injection.

Proof. We cover the image of γ by local charts. Then locally we have existence and uniqueness by the local form of (A.2.7). The uniqueness then guarantees that the local solutions patch together nicely on the overlaps of the charts. Then the linearity of (A.2.13) is a consequence of the linearity of (A.2.7). □

Now we consider the particular case of a geodesic $\gamma(t) = \exp_p(tv)$. In this case we can describe the Jacobi vector fields with initial values $J(a) = 0$ explicitly as follows:

Theorem A.2.5 *Let $v, w \in T_pM$ and let $I \times I' \subseteq \mathbb{R}^2$ be a small enough open rectangle around $(0, 0)$ such that*

$$\sigma : I \times I' \ni (t, s) \ \mapsto \ \sigma(t, s) = \exp_p(t(v + sw)) \tag{A.2.14}$$

is well-defined. Moreover, let $\gamma(t) = \sigma(t, 0)$ be the geodesic with initial velocity v at $p \in M$. Then

$$J(t) = \sigma'(t, 0) \in \Gamma^\infty(\gamma^\# TM) \tag{A.2.15}$$

is the Jacobi vector field along γ with initial values

$$J(0) = 0 \quad \text{and} \quad \nabla^\#_{\frac{\partial}{\partial t}} J(0) = w. \tag{A.2.16}$$

Proof. First we notice that for small enough I, I' around 0 the map σ is well-defined and hence J is a smooth vector field along the geodesic γ. We compute by the chain rule

$$\begin{aligned}\sigma'(t, s) &= \frac{\partial}{\partial s} \exp_p(t(v + sw)) = \left(T_{t(v+sw)} \exp_p\right) \left(\frac{\mathrm{d}}{\mathrm{d}s'}\Big|_{s'=0} (s' \mapsto t(v + (s + s')w))\right) \\ &= \left(T_{t(v+sw)} \exp_p\right)(tw) = t\left(T_{t(v+sw)} \exp_p\right)(w),\end{aligned}$$

where we have used the linearity of the tangent map and the canonical identification $T_{t(v+sw)} T_p M \simeq T_p M$ as usual. It follows that $J(0) = \sigma'(0, 0) = 0$ is satisfied indeed. Moreover, we compute

$$\nabla^\#_{\frac{\partial}{\partial t}} \sigma'(t, s) = \nabla^\#_{\frac{\partial}{\partial t}} \left(t(T_{t(v+sw)} \exp_p)(w)\right) = \left(T_{t(v+sw)} \exp_p\right)(w) + t\nabla^\#_{\frac{\partial}{\partial t}} \left((T_{t(v+sw)} \exp_p)(w)\right),$$

by the Leibniz rule for a covariant derivative. It follows that

$$\nabla^\#_{\frac{\partial}{\partial t}} J(0) = \nabla^\#_{\frac{\partial}{\partial t}} \sigma'(t, 0)\Big|_{t=0} = (T_{t(v+sw)} \exp_p)(w)\Big|_{t=s=0} + 0 = T_0 \exp_p(w) = w,$$

since $T_0 \exp_p = \mathsf{id}$. This shows that J has the correct initial conditions (A.2.16). Finally we compute

$$\begin{aligned}\nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} J(t) &= \nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} \sigma'(t, s)\Big|_{s=0} = \nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial s}} \dot\sigma(t, s)\Big|_{s=0} \\ &= \nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} \dot\sigma(t, s)\Big|_{s=0} + R\Big|_{\sigma(t,0)} (\dot\sigma(t, 0), \sigma'(t, 0)\dot\sigma(t, 0),\end{aligned}$$

by Lemma A.2.1 and the torsion-freeness of ∇ as well as by Lemma A.2.2. Now for all s the curve $t \mapsto \sigma(t, s) = \exp_p(t(v + sw))$ is a geodesic whence $\nabla^\#_{\frac{\partial}{\partial t}} \dot\sigma(t, s) = 0$ identically in s. This finally shows that the Jacobi equation, i.e. (A.2.7), is satisfied. □

Corollary A.2.6 *Let $v, w \in T_p M$. Then*

$$J(t) = t\left(T_{tv} \exp_p\right)(w) \tag{A.2.17}$$

is the unique Jacobi vector field along $\gamma(t) = \exp_p(tv)$ with $J(0) = 0$ and $\nabla^\#_{\frac{\partial}{\partial t}} J(0) = w$.

By covariant differentiation of the Jacobi differential equation we obtain the covariant derivatives of the Jacobi vector field up to all orders, at least recursively. To this end, we first notice that the right hand side of (A.2.7) can be viewed as a natural pairing of $\gamma^\# R \in \Gamma^\infty(\gamma^\# \mathsf{End}\, TM \otimes \Lambda^2 TM)$ with $\dot\gamma, J \in \Gamma^\infty(\gamma^\# TM)$. Thus using the covariant derivative $\nabla^\#$ on all the involved bundles gives

$$\nabla^\#_{\frac{\partial}{\partial t}} \left(\nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} J\right) = \nabla^\#_{\frac{\partial}{\partial t}} \left(\gamma^\# R(\dot\gamma, J)\dot\gamma\right) = \left(\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\right)(\dot\gamma, J)\dot\gamma + (\gamma^\# R)(\dot\gamma, \nabla^\#_{\frac{\partial}{\partial t}} J)\dot\gamma, \tag{A.2.18}$$

since $\nabla^\#_{\frac{\partial}{\partial t}} \dot\gamma = 0$ for a geodesic. Moreover,

$$\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\Big|_t = \left(\nabla_{\dot\gamma(t)} R\right)\Big|_{\gamma(t)} \tag{A.2.19}$$

allows to compute the covariant derivatives of $\gamma^\# R$ in terms of the covariant derivatives of R on M. By iteration, the successive use of the Leibniz rule of $\nabla^\#$ with respect to natural pairings yields the following statement:

Lemma A.2.7 *Let $J \in \Gamma^\infty(\gamma^\# TM)$ be a Jacobi vector field along a geodesic γ. Then*

$$\exp\left(\lambda \nabla^\#_{\frac{\partial}{\partial t}}\right) \nabla^\#_{\frac{\partial}{\partial t}} \nabla^\#_{\frac{\partial}{\partial t}} J = \left(\exp\left(\lambda \nabla^\#_{\frac{\partial}{\partial t}}\right) \gamma^\# R\right)(\dot\gamma, J)\dot\gamma + (\gamma^\# R)\left(\dot\gamma, \exp\left(\lambda \nabla^\#_{\frac{\partial}{\partial t}}\right)\right)\dot\gamma \tag{A.2.20}$$

in the sense of a formal power series in the formal parameter λ.

Proof. Either this is shown by differentiating both sides with respect to λ and observing that the resulting differential equations coincide thanks to (A.2.18), or by induction in the summation parameter of the exponential series. □

Remark A.2.8 The lemma can be used to efficiently compute $(\nabla^\#_{\frac{\partial}{\partial t}})^k J$ at $t = 0$ for $k \in \mathbb{N}_0$. Indeed, it provides a recursion scheme giving

$$\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 J(0) = R(v, J(0)) = 0, \tag{A.2.21}$$

$$\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^3 J(0) = \left(\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\right)(v, J(0))v + R\left(v, \nabla^\#_{\frac{\partial}{\partial t}} J(0)\right) v = 0 + R(v, w)v, \tag{A.2.22}$$

since $\dot\gamma(0) = v$ and $J(0) = 0$ as well as $\nabla^\#_{\frac{\partial}{\partial t}} J(0) = w$. The next terms are

$$\begin{aligned}\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^4 J(0) &= \left(\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 \gamma^\# R\right)(v, 0)v + 2\left(\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\right)(v, w)v + R\left(v, \left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 J(0)\right) v \\ &= 2\left(\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\right)(v, w)v \end{aligned} \tag{A.2.23}$$

and

$$\begin{aligned}\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^5 J(0) &= \left(\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^3 \gamma^\# R\right)(v, 0)v + 3\left(\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 \gamma^\# R\right)\left(v, \nabla^\#_{\frac{\partial}{\partial t}} J(0)\right) v \\ &\quad + 3\left(\nabla^\#_{\frac{\partial}{\partial t}} \gamma^\# R\right)\left(v, \left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 J(0)\right) v + R\left(v, \left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^3 J(0)\right) v \\ &= 3\left(\left(\nabla^\#_{\frac{\partial}{\partial t}}\right)^2 \gamma^\# R\right)(v, w)v + 3R(v, R(v, w)v)v, \end{aligned} \tag{A.2.24}$$

using successively those computations done in lower orders. Moreover, an easy induction shows that $(\nabla^\#_{\frac{\partial}{\partial t}})^k J_w(0)$ is a homogeneous polynomial in v of order $k-1$ and linear in w. Here one uses that $\nabla^\#_{\frac{\partial}{\partial t}}$ of an arbitrary tensor field $\gamma^\# T$ is linear in v.

We can use this to compute the Taylor expansion of the tangent map of the exponential map $\exp_p$. For any $v \in T_pM$ the tangent map $T_v \exp_p$ is a linear map $T_v \exp_p : T_pM \longrightarrow T_{\exp_p(v)}M$. In order to compute its Taylor expansion around $v = 0$ we first have to identify $T_{\exp_p(v)}M$ with T_pM again by using the parallel transport $P_{\gamma_v, 0\to 1} : T_pM \longrightarrow T_{\exp_p(v)}M$ along the geodesic $t \mapsto \gamma_v(t) = \exp_p(tv)$. This way we obtain a linear map

$$(P_{\gamma_v, 0\to 1})^{-1} \circ T_v \exp_p : T_pM \longrightarrow T_pM \tag{A.2.25}$$

for every $v \in T_pM$ small enough. We want to compute now the Taylor coefficients of

$$T_pM \ni v \;\mapsto\; \left(P_{\gamma_v,0\to 1}\right)^{-1} \circ T_v \exp_p \in \mathsf{End}(T_pM) \tag{A.2.26}$$

around $v = 0$. To do so we evaluate the endomorphism on a fixed vector $w \in T_pM$ and consider the map

$$v \;\mapsto\; \left(P_{\gamma_v,0\to 1}\right)^{-1} \circ T_v \exp_p(w). \tag{A.2.27}$$

In order to compute the partial derivatives of (A.2.27) in the v-variable it suffices to consider the derivatives of the map

$$t \;\mapsto\; \left(P_{\gamma_v,0\to 1}\right)^{-1} \circ T_{tv} \exp_p(w) \tag{A.2.28}$$

around $t = 0$ instead and use Lemma A.1.2 afterwards. Since $T_{tv}\exp_p(w) = \frac{1}{t} J_w(t)$ is a multiple of the unique Jacobi vector field $J_w \in \Gamma^\infty(\gamma_v^\# TM)$ along γ_v with $J_w(0) = 0$ and $\nabla^\#_{\frac{\partial}{\partial t}} J_w(0) = w$ we can compute its covariant derivatives by means of Lemma A.2.7 and Remark A.2.8 recursively. Finally, we note that

$$P_{\gamma_{tv},0\to 1} = P_{\gamma_v,0\to t}, \tag{A.2.29}$$

whence we have to consider the map

$$t \mapsto \left(P_{\gamma_v,0\to t}\right)^{-1} \left(\frac{1}{t} J_w(t)\right), \tag{A.2.30}$$

of which we want to compute the Taylor coefficients around $t = 0$. Collecting things we obtain the following result:

Theorem A.2.9 (Taylor coefficients of $T\exp_p$) *Let $p \in M$ and $v, w \in T_pM$. Then for all $k \in \mathbb{N}_0$ we have*

$$\frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}}\bigg|_{v=0} \left(P_{\gamma_v,0\to 1}\right)^{-1} \circ T_v \exp_p(w) = \frac{1}{(k+1)!} \frac{\partial^k}{\partial v^{i_1} \cdots \partial v^{i_k}} \underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k+1 \text{ times}} J_w(t)\bigg|_{t=0}. \tag{A.2.31}$$

The first terms of the (formal) Taylor expansion around $v = 0$ are therefore given by

$$\left(P_{\gamma_v,0\to 1}\right)^{-1} \circ T_v \exp_p(w) = w + \frac{1}{6} R_p(v,w)v + \frac{1}{12} (\nabla_v R)_p(v,w)v + \cdots. \tag{A.2.32}$$

Proof. By Corollary A.2.6 we have $tT_{tv}\exp_p(w) = J_w(t)$ whence we can compute the $\nabla^\#_{\frac{\partial}{\partial t}}$-derivatives of the vector field $t \mapsto T_{tv}\exp_p(w)$ at $t = 0$ as follows. By the Leibniz rule we have

$$\begin{aligned}
\underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k \text{ times}} J_w(t)\bigg|_{t=0} &= \underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k \text{ times}} (tT_{tv}\exp_p(w))\bigg|_{t=0} \\
&= tT_{tv}\exp_p(w)\bigg|_{t=0} + k\, \underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k-1 \text{ times}} T_{tv}\exp_p(w)\bigg|_{t=0} + 0,
\end{aligned}$$

whence for $k \geq 1$ we get

$$\underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k-1 \text{ times}} (tT_{tv}\exp_p(w))\bigg|_{t=0} = \frac{1}{k} \underbrace{\nabla^\#_{\frac{\partial}{\partial t}} \cdots \nabla^\#_{\frac{\partial}{\partial t}}}_{k \text{ times}} J_w(t)\bigg|_{t=0}. \tag{$*$}$$

Now the right hand side is recursively computable by Lemma A.2.7, see Remark A.2.8 for the first terms. We can collect the results and obtain

$$
\begin{aligned}
&\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\Big|_{v=0}\left(\left(P_{\gamma_v,0\to 1}\right)^{-1}\circ T_v\exp_p\right)(w)\\
&\stackrel{(A.1.2)}{=}\frac{1}{k!}\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0}\left(\left(P_{\gamma_{tv},0\to 1}\right)^{-1}\circ T_{tv}\exp_p\right)(w)\\
&\stackrel{(A.2.29)}{=}\frac{1}{k!}\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0}\left(P_{\gamma_v,0\to t}\right)^{-1}\left(T_{tv}\exp_p(w)\right)\\
&\stackrel{(A.1.1)}{=}\frac{1}{k!}\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\left(P_{\gamma_v,0\to t}\right)^{-1}\underbrace{\nabla^{\#}_{\frac{\partial}{\partial t}}\cdots\nabla^{\#}_{\frac{\partial}{\partial t}}}_{k\text{ times}}T_{tv}\exp_p(w)\Big|_{t=0}\\
&\stackrel{(*)}{=}\frac{1}{k!}\frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}}\frac{1}{k+1}\underbrace{\nabla^{\#}_{\frac{\partial}{\partial t}}\cdots\nabla^{\#}_{\frac{\partial}{\partial t}}}_{k+1\text{ times}}J_w(t)\Big|_{t=0},
\end{aligned}
$$

which shows (A.2.31). As we know from Remark A.2.8, the $(k+1)$-st covariant derivative of J_w at 0 is a homogeneous polynomial in v of order k. This is also clear from the proof of Lemma A.1.2. Now we compute the first orders of the Taylor expansion explicitly. Since we already know $T_0\exp_p=\mathsf{id}$ the zeroth order is given as in (A.2.32). In fact, this was used to show $\nabla^{\#}_{\frac{\partial}{\partial t}}J_w(0)=w$. For the first order $k=1$ we get

$$
\frac{\partial}{\partial v^i}\left(P_{\gamma_v,0\to 1}\right)^{-1}T_v\exp_p(w)\Big|_{v=0}=\frac{\partial}{\partial v^i}\frac{1}{2}\nabla^{\#}_{\frac{\partial}{\partial t}}\nabla^{\#}_{\frac{\partial}{\partial t}}J_w(0)=0
$$

by (A.2.21). The next order gives

$$
\begin{aligned}
\frac{\partial^2}{\partial v^i\partial v^j}\left(P_{\gamma_v,0\to 1}\right)^{-1}\circ T_v\exp_p(w)\Big|_{v=0}&=\frac{\partial^2}{\partial v^i\partial v^j}\frac{1}{6}\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\right)^3 J_w(0)\\
&=\frac{1}{6}\frac{\partial^2}{\partial v^i\partial v^j}R_p(v,w)v\\
&=\frac{1}{6}\left(R_p\Big(\frac{\partial}{\partial x^j},w\Big)\frac{\partial}{\partial x^i}+R_p\Big(\frac{\partial}{\partial x^i},w\Big)\frac{\partial}{\partial x^j}\right).
\end{aligned}
$$

Thus

$$
\frac{1}{2!}\frac{\partial^2}{\partial v^i\partial v^j}\left(P_{\gamma_v,0\to 1}\right)^{-1}\circ T_v\exp_p(w)v^iv^j=\frac{1}{6}R(v,w)v,
$$

explaining the quadratic term in (A.2.32). The cubic term is obtained from (A.2.23)

$$
\begin{aligned}
\frac{\partial^3}{\partial v^i\partial v^j\partial v^k}\left(P_{\gamma_v,0\to 1}\right)^{-1}\circ T_v\exp_p(w)\Big|_{v=0}&=\frac{1}{4!}\frac{\partial^3}{\partial v^i\partial v^j\partial v^k}2\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\gamma^{\#}R\right)(v,w)v\\
&=\frac{1}{4!}\frac{\partial^3}{\partial v^i\partial v^j\partial v^k}2\left(\nabla_v R\right)_p(v,w)v,
\end{aligned}
$$

from which we get

$$
\frac{1}{3!}\frac{\partial^3}{\partial v^i\partial v^j\partial v^k}\left(P_{\gamma_v,0\to 1}\right)^{-1}\circ T_v\exp_p(w)\Big|_{v=0}v^iv^jv^k=\frac{1}{12}\left(\nabla_v R\right)_p(v,w)v,
$$

as claimed in (A.2.32). □

Remark A.2.10 More symbolically we can write the (formal) Taylor expansion of the tangent map of $\exp_p$ as

$$
\begin{aligned}
(P_{\gamma_v,0\to 1})^{-1} \circ T_v \exp_p(w) &\sim_{v\to 0} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}} \left((P_{\gamma_v,0\to 1})^{-1} \circ T_v \exp_p \right)(w)\Big|_{v=0} v^{i_1}\cdots v^{i_k} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial v^{i_1}\cdots\partial v^{i_k}} \left(\frac{1}{(k+1)!} \underbrace{\nabla^{\#}_{\frac{\partial}{\partial t}} \cdots \nabla^{\#}_{\frac{\partial}{\partial t}}}_{k+1 \text{ times}} J_w(t)\Big|_{t=0} \right) v^{i_1}\cdots v^{i_k} \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{\nabla^{\#}_{\frac{\partial}{\partial t}} \cdots \nabla^{\#}_{\frac{\partial}{\partial t}}}_{k+1 \text{ times}} J_w(t)\Big|_{t=0} \\
&= \exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}} \right) J_w(t)\Big|_{t=0},
\end{aligned}
$$

since on one hand $\nabla^{\#}_{\frac{\partial}{\partial t}} \cdots \nabla^{\#}_{\frac{\partial}{\partial t}} J_w(t)$ is a homogeneous polynomial in v of degree k for $k+1$ derivatives and since the zeroth term of the exponential series does not contribute due to $J_w(0) = 0$. Of course the formula

$$
(P_{\gamma_v,0\to 1})^{-1} \circ T_v \exp_p(w) \sim_{v\to 0} \exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}} \right) J_w(t)\Big|_{t=0} \tag{A.2.33}
$$

is only the formal Taylor expansion: in general, the right hand side will not converge in any reasonable sense. Note however that the combinatorics to compute the covariant derivatives of J_w at $t = 0$ is fairly simple and given by universal polynomials in the curvature and its covariant derivatives at p.

As a last application of our investigations of Jacobi vector fields we specialize to the case of a semi-Riemannian manifold (M, g) and the Levi-Civita connection ∇. Then one has the following result, known as the *Gauss Lemma*:

Proposition A.2.11 (Gauss Lemma) *Let (M, g) be a semi-Riemannian manifold and $p \in M$. Then for $v, w \in T_pM$ we have*

$$
g_{\exp_p(v)}\left(T_v \exp_p(v), T_v \exp_p(w) \right) = g_p(v, w), \tag{A.2.34}
$$

whenever v is still in the domain of $\exp_p$.

Proof. We consider the surface $\sigma(t, s) = \exp_p(t(v + sw))$ which is defined for $t \in [0, 1]$ and s small enough. Then we have

$$
\dot\sigma(t, s) = T_{\exp_p(t(v+sw))} \exp_p(v + sw) \quad \text{and} \quad \sigma'(t, s) = T_{\exp_p(t(v+sw))} \exp_p(tw)
$$

by the chain rule as we computed already in the proof of Theorem A.2.5. Thus we have to compute $g_{\exp_p(v)}(\dot\sigma(1, 0), \sigma'(1, 0))$. We consider the geodesic $t \mapsto \exp_p(t(v + sw)) = \gamma_s(t)$ with initial velocity vector $v + sw$. First we note by $\nabla^{\#}_{\frac{\partial}{\partial t}} \dot\gamma_s = 0$ that

$$
\frac{\partial}{\partial t} g_{\gamma_s(t)}(\dot\gamma_s(t), \dot\gamma_s(t)) = 2 g_{\gamma_s(t)}\left(\nabla^{\#}_{\frac{\partial}{\partial t}} \dot\gamma_s(t), \dot\gamma_s(t) \right) = 0, \tag{$*$}
$$

by the fact that g is covariantly constant. It follows that $g_{\gamma_s(t)}(\dot{\gamma}_s(t), \dot{\gamma}_s(t)) = g_p(v + sw, v + sw)$. In fact, this is the Gauss Lemma for $w = v$. To proceed we compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{\sigma(t,s)}(\dot{\sigma}(t,s), \sigma'(t,s)) &= g_{\sigma(t,s)}\left(\nabla^{\#}_{\frac{\partial}{\partial t}} \dot{\sigma}(t,s), \sigma'(t,s)\right) + g_{\sigma(t,s)}\left(\dot{\sigma}(t,s), \nabla^{\#}_{\frac{\partial}{\partial t}} \sigma'(t,s)\right) \\
&= 0 + g_{\sigma(t,s)}\left(\dot{\sigma}(t,s), \nabla^{\#}_{\frac{\partial}{\partial t}} \dot{\sigma}(t,s)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial s}\left(g_{\sigma(t,s)}\left(\dot{\sigma}(t,s), \dot{\sigma}(t,s)\right)\right),
\end{aligned}$$

using the fact that ∇ is torsion-free, see Lemma A.2.1. Since all curves $t \mapsto \sigma(t,s) = \gamma_s(t)$ are geodesics we know that

$$g_{\sigma(t,s)}(\dot{\sigma}(t,s), \dot{\sigma}(t,s)) = g_{\sigma(0,s)}(\dot{\sigma}(0,s), \dot{\sigma}(0,s)) + g_p(v + sw, v + sw),$$

whence

$$\frac{1}{2}\frac{\partial}{\partial s}\left(g_{\sigma(t,s)}\left(\dot{\sigma}(t,s), \dot{\sigma}(t,s)\right)\right) = g_p(v, w) + s g_p(w, w).$$

Putting things together we have for $s = 0$ and all t

$$\frac{\partial}{\partial t} g_{\sigma(t,0)}(\dot{\sigma}(t,0), \sigma'(t,0)) = g_p(w, w)$$

independent of t. Hence we conclude $g_{\sigma(t,0)}(\dot{\sigma}(t,0), \sigma'(t,0)) = t g_p(v, w)$ and setting $t = 1$ gives the desired result (A.2.34). □

Remark A.2.12 (Gauss Lemma) The geometric interpretation of the Gauss Lemma is two-fold. For $v = w$ we see that the length-square of the tangent vector of a geodesic is constant. In the Riemannian setting this simply means that the length itself stays constant whence geodesics are curves with "constant velocity". In the Hamiltonian picture, this part of the Gauss Lemma can be interpreted as energy conservation under the Hamiltonian time evolution, see e.g. [60, Aufgabe 3.10, vii.)] for this point of view. The case with arbitrary w means that along a geodesic at least the "angles" with respect to the tangent vector of the geodesic are preserved.

A.3 Jacobi Determinants of the Exponential Map

Now we will use the formal Taylor expansion of $T_v \exp_p$ around $v = 0$ to consider the following problem. Given a positive density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ on M, i.e. $\mu > 0$ everywhere, we can compare the constant density μ_p on T_pM with μ via the exponential map $\exp_p$ of ∇. More precisely, we consider an open neighborhood of the zero section such that

$$\pi \times \exp : V \subseteq TM \longrightarrow U \subseteq M \times M \tag{A.3.1}$$

is a diffeomorphism onto its image, denoted by U. In fact, U is an open neighborhood of the diagonal since $(\pi \times \exp)(0_p) = (p, p)$ for $0_p \in T_pM$.

Definition A.3.1 *Let $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ be a positive density on M. Then the function $\rho : U \longrightarrow \mathbb{R}$ is defined by*

$$\rho(p, q)(\exp_{p\,*}\mu_p)_q = \mu_q \tag{A.3.2}$$

for $(p, q) \in U$.

Lemma A.3.2 *Let $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ by a positive density. Then $\rho \in \mathcal{C}^\infty(U)$ and $\rho > 0$.*

Proof. We have

$$\rho(p, q) = \frac{\mu_q}{(\exp_{p\,*}\mu_p)\big|_q} > 0.$$

Moreover, the map $(p, q) \mapsto \exp_{p*}\mu_p\big|_q$ is a smooth map on U with values in $|\Lambda^{\mathrm{top}}|T_q^*M$. Since at every point (p, q) the value is a *positive* density the quotient is well-defined and smooth. □

Remark A.3.3 Geometrically speaking, the function ρ measures how much the density μ at q differs from the density μ at p when the latter is moved to q by means of the exponential map. Thus ρ encodes the change of volume as one moves around in M. Note that ρ is *not* symmetric.

Sometimes we fix a reference point $p \in M$ and consider the function $\rho_p : U_p \longrightarrow \mathbb{R}$ defined by

$$\rho_p(q) = \rho(p, q) \tag{A.3.3}$$

for $q \in U_p \subseteq M$ where U_p is an open neighborhood on which we have normal coordinates, i.e. $U_p = \exp_p(V_p)$ with $V_p = V \cap T_pM$. Thus we have

$$\rho_p \exp_{p*}\mu_p = \mu \tag{A.3.4}$$

on U_p. Moreover, it will also be convenient to compare the densities on the tangent space of p and not on M. Thus one defines the function $\widetilde{\rho} : V \longrightarrow \mathbb{R}$ by

$$\widetilde{\rho}(v_p)\mu_p = (\exp_p^*\mu)(v_p). \tag{A.3.5}$$

Thus $\widetilde{\rho}$ is the prefactor of the *constant density* μ_p on T_pM such that we obtain the pull-back of μ. Clearly, we have

$$\widetilde{\rho}(v_p) = \rho(p, \exp_p(v_p)), \tag{A.3.6}$$

whence also $\widetilde{\rho} = \rho \circ (\pi \times \exp) \in \mathcal{C}^\infty(V)$ is smooth and positive. Again, we write $\widetilde{\rho}_p \in \mathcal{C}^\infty(V_p)$ for the restriction of $\widetilde{\rho}$ to a particular tangent space of a fixed reference point $p \in M$.

The aim is now to compute the (formal) Taylor expansion of $\widetilde{\rho}_p$ around $v = 0$ which is equivalent to the (formal) Taylor expansion of ρ_p in normal coordinates around p. To this end, we first give another interpretation of ρ and $\widetilde{\rho}$. In fact, we have *two* aspects of comparing the volumes. On one hand, the density μ is not "constant" along M since there is simply no intrinsic way to formulate such a statement. On the other hand, the exponential map needs not to be volume preserving. We try to separate these two effects as follows: Using the unique geodesic $t \mapsto \exp_p(tv)$ from p to $q = \exp_p(v)$ we can parallel transport μ_q back to p using the parallel transport induced by ∇ on the density bundle. This gives a *constant* density $(P_{\gamma_v,0\to1})^{-1}\mu_{\exp_p(v)} \in |\Lambda^{\mathrm{top}}|T_p^*M$ on T_pM for every $v \in V_p$. Thus this will be a *constant* multiple of μ_p depending parametrically on v. This v-dependence measures how much μ is *not parallel* with respect to ∇. Secondly, we consider the tangent map

$$T_v \exp_p : T_pM \longrightarrow T_{\exp_p(v)}M, \tag{A.3.7}$$

and want to determine its change of volume features. Since source and target are different vector spaces there is no way to define a "determinant" of this linear map, we first have to take care that we get a map from a tangent space into the *same* tangent space. Thus we consider

$$(P_{\gamma_v,0\to1})^{-1} \circ T_v \exp_p : T_pM \longrightarrow T_pM \tag{A.3.8}$$

instead.

Combining the effects we use the density $(P_{\gamma_v,0\to 1})^{-1}\mu_{\exp_p(v)}$ and evaluate on a basis $e_1,\ldots,e_n \in T_pM$ after applying $(P_{\gamma_v,0\to 1})^{-1}\circ T_v\exp_p$ to it. In order to get a result which is independent of the chosen basis we normalize it by $\mu_p(e_1,\ldots,e_n)$, i.e. we consider the quantity

$$\frac{1}{\mu_p(e_1,\ldots,e_n)}\left((P_{\gamma_v,0\to 1})^{-1}(\mu_{\exp_p(v)})\right)\left(P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p(e_1),\ldots,P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p(e_n)\right) \tag{A.3.9}$$

for $v \in V_p \subseteq T_pM$. Then we have the following statement:

Lemma A.3.4 *For $p \in M$ and $v \in V_p \subseteq T_pM$ we have*

$$\widetilde{\rho}(p) = \frac{1}{\mu_p(e_1,\ldots,e_n)}\left((P_{\gamma_v,0\to 1})^{-1}(\mu_{\exp_p(v)})\right)\left(P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p(e_1),\ldots,P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p(e_n)\right), \tag{A.3.10}$$

where $e_1,\ldots,e_n \in T_pM$ is a basis.

Proof. Using Lemma A.1.5 we compute

$$\begin{aligned}
&\left(P^{-1}_{\gamma_v,0\to 1}(\mu_{\exp_p(v)})\right)\left(P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p e_1,\ldots,P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p e_n\right)\\
&= \mu_{\exp_p(v)}(T_v\exp_p(e_1),\ldots,T_v\exp_p(e_n))\\
&= (\exp_p^*\mu)\big|_v(e_1,\ldots,e_n)
\end{aligned}$$

by the definition of the pull-back of a density. But then the right hand side of (A.3.10) is

$$\frac{\exp_p^*\mu\big|_v(e_1,\ldots,e_n)}{\mu_p(e_1,\ldots,e_n)} = \widetilde{\rho}_p(v)$$

by (A.3.5) proving the lemma. □

Since we have a good understanding of the Taylor expansion of $P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p$ as well as of the parallel transport $P_{\gamma_v,0\to 1}$ itself, we can use these results to obtain the complete Taylor expansion of the function $\widetilde{\rho}_p$ around $v = 0$, at least up to the usual recursive computation of the derivatives of the Jacobi vector fields.

Theorem A.3.5 *Let $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ be a positive density on M and let $p \in M$. Then the function $\widetilde{\rho}_p$ from* (A.3.10) *has the following formal Taylor expansion around $v = 0$*

$$\widetilde{\rho}_p(v) \sim_{v\to 0} \mathcal{J}(\mathrm{e}^{\mathsf{D}}\mu)(v)\cdot\det\left(P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p\right), \tag{A.3.11}$$

the first orders of which are explicitly given by

$$\widetilde{\rho}_p(v) = 1 + \alpha_p(v) + \frac{1}{2}(\nabla_v\alpha\big|_p)(v) + \frac{1}{2}\alpha(v)^2 - \frac{1}{6}\operatorname{Ric}_p(v,v) + \cdots \tag{A.3.12}$$

up to terms of order higher than 2. Here $\alpha \in \Gamma^\infty(T^*M)$ is the one-form with $\nabla_X\mu = \alpha(X)\mu$.

Proof. We fix a basis $e_1,\ldots,e_n \in T_pM$, then we first have

$$\left((P_{\gamma_v,0\to 1})^{-1}\mu_{\exp_p(v)}\right)(Ae_1,\ldots,Ae_n) = |\det A|\left((P_{\gamma_v,0\to 1})^{-1}\mu_{\exp_p(v)}\right)(e_1,\ldots,e_n)$$

for any linear map $A : T_pM \longrightarrow T_pM$. Since in our case $A = P^{-1}_{\gamma_v,0\to 1}\circ T_v\exp_p$ is continuously connected to id_{T_pM} via $v \longrightarrow 0$, we see that the determinant is always positive. Thus we can evaluate the determinant of $P^{-1}_{\gamma_v,0\to1} \circ T_v \exp_p$ in the usual multilinear way. Since in general $Ae_1 \wedge \cdots \wedge Ae_n = \det(A) e_1 \wedge \cdots \wedge e_n$, we have to compute

$$\begin{aligned} &P^{-1}_{\gamma_v,0\to1}\left(T_v \exp_p(e_1)\right) \wedge \cdots \wedge P^{-1}_{\gamma_v,0\to1}\left(T_v \exp_p(e_n)\right) \\ &\qquad \sim_{v\to0} \left(\exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\right) J_{e_1}(t)\Big|_{t=0}\right) \wedge \cdots \wedge \left(\exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\right) J_{e_n}(t)\Big|_{t=0}\right), \end{aligned}$$

and compare it to $e_1 \wedge \cdots \wedge e_n$. From here we get the first orders explicitly by (A.2.32).

$$\begin{aligned} &\left(\exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\right) J_{e_1}(t)\Big|_{t=0}\right) \wedge \cdots \wedge \left(\exp\left(\nabla^{\#}_{\frac{\partial}{\partial t}}\right) J_{e_n}(t)\Big|_{t=0}\right) \\ &\quad = \left(e_1 + \frac{1}{6} R_p(v, e_1)v + \cdots\right) \wedge \cdots \wedge \left(e_n + \frac{1}{6} R_p(v, e_n)v + \cdots\right) \\ &\quad = e_1 \wedge \cdots \wedge e_n + \frac{1}{6} \sum_{\ell=1}^{n} e_1 \wedge \cdots \wedge R_p(v, e_\ell)v \wedge \cdots \wedge e_n + \cdots \\ &\quad = e_1 \wedge \cdots e_n + \frac{1}{6} \sum_{\ell=1}^{n} e_1 \wedge \cdots \wedge e^k(R_p(v, e_\ell)v) e_k \wedge \cdots \wedge e_n + \cdots \\ &\quad = e_1 \wedge \cdots e_n + \frac{1}{6} e^\ell(R_p(v, e_\ell)v) e_1 \wedge \cdots e_n + \cdots \\ &\quad = \left(1 - \frac{1}{6} \operatorname{Ric}_p(v, v) + \cdots\right) e_1 \wedge \cdots \wedge e_n, \end{aligned}$$

whence up to second order we get

$$\det\left(P^{-1}_{\gamma_v,0\to1} \circ T_v \exp_p\right) = 1 - \frac{1}{6} \operatorname{Ric}_p(v, v) + \cdots.$$

The second step consists in Taylor expanding the parallel transport of μ. Here we have by Corollary A.1.8 the formal Taylor expansion

$$P^{-1}_{\gamma_v,0\to1} \mu_{\exp_p(v)} \sim_{v\to0} \mathcal{J}(\mathrm{e}^{\mathsf{D}} \mu)(v), \tag{$*$}$$

where $\mathsf{D} : \Gamma^\infty(\mathrm{S}^\bullet T^*M \otimes |\Lambda^{\mathrm{top}}|T^*M) \longrightarrow \Gamma^\infty(\mathrm{S}^{\bullet+1} T^*M \otimes |\Lambda^{\mathrm{top}}|T^*M)$ is the symmetrized covariant derivative on the density bundle. This shows (A.3.11). The first orders of $(*)$ are given by

$$\mathcal{J}(\mathrm{e}^{\mathsf{D}} \mu)(v) = \mu + (\mathsf{D}\,\mu)(v) + \frac{1}{2}\frac{1}{2}(\mathsf{D}^2\,\mu)(v, v) + \cdots.$$

Using $\nabla_X \mu = \alpha(X)\mu$ we get

$$\begin{aligned} \mathsf{D}\,\mu &= \alpha \otimes \mu, \\ \mathsf{D}^2\,\mu &= \mathsf{D}\,\alpha \otimes \mu + \alpha \vee \mathsf{D}\,\mu \\ &= \mathsf{D}\,\alpha \otimes \mu + \alpha \vee \alpha \otimes \mu, \\ \mathsf{D}^3\,\mu &= \mathsf{D}^2\,\alpha \otimes \mu + (\mathsf{D}\,\alpha \vee \alpha + \alpha \vee \mathsf{D}\,\alpha) \otimes \mu + \alpha \vee \alpha \vee \alpha \otimes \mu \\ &= \left(\mathsf{D}^2\,\alpha + 2\,\mathsf{D}\,\alpha \vee \alpha + \alpha \vee \alpha \vee \alpha\right) \otimes \mu, \end{aligned}$$

$$
\begin{aligned}
\mathsf{D}^4 \mu &= \big(\mathsf{D}^3 \alpha + 2 \mathsf{D}^2 \alpha \vee \alpha + 2 \mathsf{D} \alpha \vee \mathsf{D} \alpha + 3 \mathsf{D} \alpha \vee \alpha \vee \alpha\big) \otimes \mu \\
&\qquad + \big(\mathsf{D}^2 \alpha + 2 \mathsf{D} \alpha \vee \alpha + \alpha \vee \alpha \vee \alpha\big) \vee \alpha \otimes \mu \\
&= \big(\mathsf{D}^3 \alpha + 3 \mathsf{D}^2 \alpha \vee \alpha + 2 \mathsf{D} \alpha \vee \mathsf{D} \alpha + 5 \mathsf{D} \alpha \vee \alpha \vee \alpha + \alpha \vee \alpha \vee \alpha \vee \alpha\big) \otimes \mu,
\end{aligned}
$$

and so on by the Leibniz rule. Thus in particular

$$
(\mathsf{D}\,\mu)(v) = \alpha(v)\mu \quad \text{and} \quad (\mathsf{D}^2\,\mu)(v, v) = (\mathsf{D}\,\alpha)(v, v)\mu + 2\alpha(v)\alpha(v)\mu.
$$

Note that $(\mathsf{D}\,\alpha)(v, v) = 2(\nabla_v \alpha)(v)$ by the definition of D acting on a one-form. Collecting all terms gives

$$
\begin{aligned}
\mathcal{J}(\mathrm{e}^{\mathsf{D}}\mu)(v) &= \mu + \alpha(v)\mu + \frac{1}{4}(2(\nabla_v\alpha)(v) + 2\alpha(v)\alpha(v))\mu + \cdots \\
&= \left(1 + \alpha(v) + \frac{1}{2}\left((\nabla_v\alpha)(v) + \alpha(v)^2\right) + \cdots \right)\mu.
\end{aligned}
$$

Putting things together we have up to second order in v

$$
\begin{aligned}
\widetilde{\rho}_p(v) &= \left(1 + \alpha(v) + \frac{1}{2}(\alpha_v(\alpha))(v) + \frac{1}{2}\alpha(v)^2 + \cdots \right) \cdot \left(1 - \frac{1}{6}\operatorname{Ric}_p(v, v) + \cdots \right) \\
&= 1 + \alpha_p(v) + \frac{1}{2}\left((\nabla_v\alpha)\big|_p(v) + \alpha_p(v)^2\right) - \frac{1}{6}\operatorname{Ric}_p(v, v) + \cdots .
\end{aligned}
$$

□

Remark A.3.6 Again, we note that the evaluation of arbitrarily high orders of the Taylor expansion of $\widetilde{\rho}_p$ is reduced to the fairly easy computation of $\mathsf{D}^k\,\mu$ for arbitrary k as well as to the slightly more involved Taylor expansion of the determinant of the tangent map of exp. However, for the tangent map itself we have a fairly easy and completely algebraic procedure via the Jacobi fields. Since also $\mathsf{D}^k\,\mu$ can be computed in terms of covariant derivatives of the one-form α by a simple recursion, we can consider the problem of finding higher orders in $\widetilde{\rho}_p$ to be algebraic and simple.

Appendix B

A Brief Reminder on Stokes Theorem

In this appendix we collect a few basic facts on Stokes' Theorem and its applications in semi-Riemannian geometry.

We start with the following situation: let $U \subseteq M$ be an open subset and assume that its topological boundary $\iota : \partial U \hookrightarrow M$ is an embedded submanifold of codimension one. In this situation we say that U has a *smooth boundary*.

Lemma B.1 (Transverse vector field) Let $U \subseteq M$ be a non-empty open subset with smooth boundary. Then there exists a transverse vector field $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ on ∂U, i.e. for all $p \in \partial U$ the vector $\mathfrak{n}(p)$ is transverse to $T_p(\partial U) \subseteq T_p M$.

Proof. By assumption we have an atlas of submanifold charts. Since the codimension is one, we can label the coordinates $(x^1, \ldots, x^n)$ in such a chart in a way that $x^n = 0$ corresponds to the boundary ∂U and $x^n > 0$ yields points inside U, see Figure B.1. Clearly, we can find an atlas with this feature. Now $-\frac{\partial}{\partial x^n}$ is pointing outwards of U in such a chart. It is now easy to check that the property of pointing outwards is *convex*, i.e. convex combinations of (locally defined) vector fields which point outwards point outwards again. Thus a partition of unity argument gives a smooth vector field $\mathfrak{n}$ on ∂U which points outwards at every point. In particular $\mathfrak{n}(p)$ is transverse to $T_p(\partial U)$ at every $p \in \partial U$. □

On a connected component of ∂U a transverse vector field is either pointing outwards or pointing inwards. We can use transverse vector field to induce orientations:

Lemma B.2 Assume M is orientable and $\omega \in \Gamma^\infty(\Lambda^{\mathrm{top}} T^* M)$ is a nowhere vanishing n-form. If $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ is a transverse vector field to ∂U then $\mathrm{i}_{\mathfrak{n}} \omega \in \Gamma^\infty(\Lambda^{\mathrm{top}} T^* \partial U)$ is a nowhere vanishing $(n-1)$-form on ∂U.

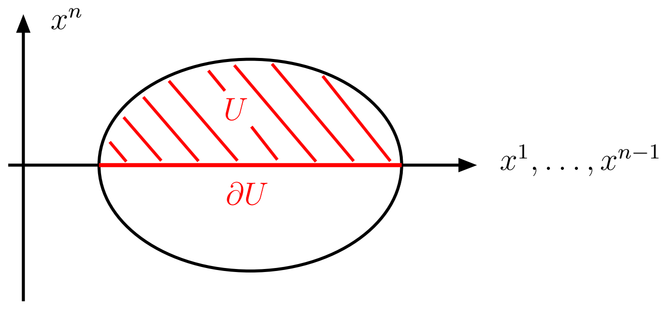

Figure B.1: A chart for the boundary ∂U.

Proof. Of course, here we view $\mathrm{i}_{\mathfrak{n}}\,\omega$ as a $(n-1)$-form defined on ∂U only. If $e_2, \ldots, e_n$ form a basis in $T_p\partial U$ then $\mathfrak{n}(p), e_2, \ldots, e_n \in T_pM$ form a basis by transversality. Thus, ω evaluated on this basis is non-zero, hence $\mathrm{i}_{\mathfrak{n}}\,\omega$ is nowhere vanishing. □

Definition B.3 (Induced orientation) Let $U \subseteq M$ be open with smooth boundary ∂U. If M is oriented then the induced orientation of ∂U is defined by the $(n-1)$-form $\mathrm{i}_{\mathfrak{n}}\,\omega$ where $\omega \in \Gamma^\infty(\Lambda^{\mathrm{top}}T^*M)$ is a positively oriented n-form and $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ is a transverse vector field pointing outwards.

Remark B.4 It is an easy check that this is indeed well-defined, i.e. the induced orientation of ∂U only depends on the orientation of M but not on the choices of ω and $\mathfrak{n}$.

With respect to these orientations we can integrate top degree forms. The fundamental feature of such integrations is then formulated in Stokes' Theorem:

Theorem B.5 (Stokes) Let M be oriented and let $U \subseteq M$ be a non-empty open subset with smooth boundary $\iota : \partial U \hookrightarrow M$, equipped with the induced orientation. Then for all $\omega \in \Gamma_0^\infty(\Lambda^{n-1}T^*M)$ we have

$$\int_U \mathrm{d}\,\omega = \int_{\partial U} \iota^*\omega. \tag{B.1}$$

For a proof of this well-known theorem one may consult any textbook on differential geometry, see. e.g. [44, Thm. 8.11] or [40, Thm. 14.9].

Remark B.6 There are many generalizations of (B.1) for forms and boundaries of less regularity than $\mathcal{C}^\infty$: this is reasonable to expect since ultimately (B.1) is an equation between integrals whence only measure-theoretic properties should be relevant. In particular, the theorem still holds for boundaries with corners, see [40, Thm. 14.20].

We shall now use this theorem to obtain similar results for the non-oriented situation: this is still plausible to be possible as changing the orientation from ω to $-\omega$ should produce the same sign on both sides of (B.1). We shall now see how this can be made precise.

Lemma B.7 Let $U \subseteq M$ be open with smooth boundary and let $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ be a transverse vector field.

i.) For $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$, $p \in \partial U$ and $e_2, \ldots e_n \in T_p(\partial U)$ the definition

$$(\mathrm{i}_{\mathfrak{n}}\,\mu)\big|_p(e_2, \ldots, e_n) = \mu_p(\mathfrak{n}(p), e_2, \ldots, e_n) \tag{B.2}$$

defines a smooth density $\mathrm{i}_{\mathfrak{n}}\,\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*(\partial U))$.

ii.) The map

$$\Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M) \ni \mu \;\mapsto\; \mathrm{i}_{\mathfrak{n}}\,\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*(\partial U)) \tag{B.3}$$

is continuous and $\mathcal{C}^\infty(M)$-linear in the sense that for $f \in \mathcal{C}^\infty(M)$ we have

$$\mathrm{i}_{\mathfrak{n}}(f\mu) = \iota^* f\,\mathrm{i}_{\mathfrak{n}}\,\mu. \tag{B.4}$$

iii.) For a positive density μ also $\mathrm{i}_{\mathfrak{n}}\,\mu$ is positive.

Proof. We choose a submanifold chart (V, x) of M such that $x^n = 0$ corresponds to ∂U in this chart. Then any transverse vector field $\mathfrak{n}$ has a nontrivial $\frac{\partial}{\partial x^n}$-component along $x^n = 0$, i.e. writing

$$\mathfrak{n}\big|_V = \mathfrak{n}^i \frac{\partial}{\partial x^i}$$

with $\mathfrak{n}^i \in \Gamma^\infty(\partial U \cap V)$ we have $\mathfrak{n}^n(x^1, \ldots, x^{n-1}) \neq 0$. If $e_2, \ldots e_n$ are a frame at $p \in \partial U$ the it is easy to check that $\mathrm{i}_\mathfrak{n}\, \mu$ transforms correctly under the change of frames. Thus (B.2) defines a density indeed. Moreover, if $\mu\big|_V = \mu_V |\, \mathrm{d}x^1 \wedge \cdots \wedge x^n|$ where $\mu_V \in \mathcal{C}^\infty(V)$ is the local form of μ in this chart then $\mathrm{i}_\mathfrak{n}\, \mu\big|_{\partial U \cap V} = \iota^*(\mu_V)|\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}||\mathfrak{n}^n|$ whence the local function representing $\mathrm{i}_\mathfrak{n}\, \mu$ is $\iota^*\mu_V|\mathfrak{n}^n|$. Since $\mathfrak{n}^n$ is everywhere different from zero, this is smooth again, showing that $\mathrm{i}_\mathfrak{n}\, \mu$ is indeed smooth. Moreover, if μ is positive we see that $\mathrm{i}_\mathfrak{n}\, \mu$ is positive as well. The continuity is again a consequence of the above local expression as we can use these submanifolds charts to characterize the Fréchet topologies of $\Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M)$ and $\Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*(\partial U))$, respectively. Finally, (B.4) is clear from the definition. □

Thus having specified a transverse vector field $\mathfrak{n}$ of ∂U we can speak of the *induced density* $\mathrm{i}_\mathfrak{n}\, \mu$ coming from a density μ on M. From the above definition it is clear that

$$\mathrm{i}_{f\mathfrak{n}}\, \mu = |f|\, \mathrm{i}_\mathfrak{n}\, \mu \tag{B.5}$$

for any nowhere vanishing function $f \in \mathcal{C}^\infty(\partial U)$.

We now specialize to the following situation: assume that M is in addition a semi-Riemannian manifold with metric g. Moreover, we assume that ∂U allows for a transverse vector field $\mathfrak{n}$ which is nowhere lightlike, where we shall use the notions of timelike, spacelike and lightlike vectors as in the Lorentzian situation. Then on each connected component $g(\mathfrak{n}, \mathfrak{n})$ is either positive or negative whence $\mathfrak{n}$ is either timelike or spacelike everywhere on this connected component. We can now achieve two things: first we can arrange $\mathfrak{n}$ in such a way that $\mathfrak{n}(p)$ is not only transverse to $T_p(\partial U)$ but orthogonal. Moreover, we can normalize $\mathfrak{n}(p)$ at every $p \in \partial U$. Finally, we choose $\mathfrak{n}(p)$ to point outwards: his determines $\mathfrak{n}(p)$ uniquely. Indeed, since $T_p(\partial U) \subseteq T_pM$ has codimension one the annihilator space $T_p(\partial U)^{\mathrm{ann}} \subseteq T_p^*M$ of one-forms annihilating $T_p(\partial U)$ is one-dimensional. Then $\mathfrak{n}(p) \in (T_p(\partial U)^{\mathrm{ann}})^\#$ is orthogonal to all of $T_p\partial U$ and uniquely determined as $\mathfrak{n}(p)^\flat \in T_p(\partial U)^{\mathrm{ann}}$ by definition. Then normalizing and orienting it gives a unique vector.

Definition B.8 (Normal vector field) Let (M, g) be semi-Riemannian and let $U \subseteq M$ be open with smooth boundary. Assume that the annihilator spaces $T_p(\partial U)^{\mathrm{ann}} \subseteq T_p^*M$ of ∂U are never lightlike (with respect to g^{-1}). Then the unique normalized transverse vector field $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ which is orthogonal to ∂U and pointing outward is called the normal vector field of ∂U.

This allows us to obtain a uniquely determined metric and density on ∂U as follows:

Definition B.9 Let M be semi-Riemannian and let $U \subseteq M$ be open with connected smooth boundary such that $T_p(\partial U)^{\mathrm{ann}} \subseteq T_p^*M$ is never lightlike. Then the induced metric on ∂U is $\iota^*g \in \Gamma^\infty(\mathrm{S}^2 T^*\partial U)$.

Lemma B.10 Under the above assumptions, ι^*g is a semi-Riemannian metric on ∂U. Moreover,

$$\mu_{\iota^*g} = \mathrm{i}_\mathfrak{n}\, \mu_g, \tag{B.6}$$

where $\mathfrak{n} \in \Gamma^\infty(\iota^\# TM)$ is the normal vector field of ∂U.

Proof. Let $p \in \partial U$. Then we have to show that $\iota^*g\big|_p$ is indeed non-degenerate (the Riemannian case is trivial). We find in T_pM a semi-Riemannian frame $e_1, \ldots, e_n$ such that $e_1 = \mathfrak{n}(p)$. Then $e_2, \ldots e_n$ are a basis of $T_p\partial U$ with

$$\iota^*g\big|_p(e_i, e_j) = g_p(e_i, e_j) = \pm\delta_{ij}$$

for $i, j = 2, \ldots, n$. Thus ι^*g is non-degenerate. Its signature can be obtained from knowing whether $\mathfrak{n}$ is time- or spacelike and from the signature of g. In particular, $e_2, \ldots e_n$ is a semi-Riemannian frame for ι^*g. From this we see that by definition $\mu_{\iota^*g}\big|_p(e_2, \ldots, e_n) = 1$. On the other hand $(\mathrm{i}_\mathfrak{n}\, \mu_g)(e_2, \ldots, e_n) =$

$\mu_g\big|_p(\mathfrak{n}(p), e_2, \ldots, e_n) = 1$ as $\mathfrak{n}(p), e_2, \ldots, e_n$ is a semi-Riemannian frame for g. Thus the two densities coincide as they coincide on one frame. □

We can now use the normal vector field $\mathfrak{n}$ to formulate Gauss' Theorem as a consequence of Stokes' Theorem:

Theorem B.11 (Gauss) Let (M, g) be a semi-Riemannian manifold and $U \subseteq M$ open with smooth connected boundary $\iota : \partial U \hookrightarrow M$. Assume that $T(\partial U)^{\mathrm{ann}}$ is never lightlike. Then for all vector fields $X \in \Gamma_0^\infty(TM)$ we have

$$\int_U \operatorname{div}(X)\mu_g = \epsilon \int_{\partial U} g(\iota^\# X, \mathfrak{n})\mu_{\iota^* g}, \tag{B.7}$$

where $\epsilon = \langle \mathfrak{n}, \mathfrak{n} \rangle \in \{1, -1\}$.

Proof. First we consider the oriented case. Thus the left hand side is

$$\int_U \operatorname{div}(X)\mu_g = \int_U \operatorname{div}(X)\Omega_g,$$

with the positively oriented volume form Ω_g yielding μ_g under the canonical map from forms to densities, see [60, Prop. 2.2.42]. Note that $\operatorname{div}(X)$ can alternatively be computed via $\operatorname{div}(X)\Omega_g = \mathscr{L}_X \Omega_g = \mathrm{d}(\mathrm{i}_X \Omega_g)$. Thus we can apply Stokes' Theorem and get

$$\int_U \operatorname{div}(X)\Omega_g = \int_U \mathrm{d}(\mathrm{i}_X \Omega_g) = \int_{\partial U} \iota^*(\mathrm{i}_X \Omega_g). \tag{$*$}$$

Now along ∂U we can decompose X into its $\mathfrak{n}$-component and parallel components. We have

$$X(p) = \epsilon g_p(X(p), \mathfrak{n}(p))\mathfrak{n}(p) + X_\parallel(p)$$

where $X_\parallel(p)$ is orthogonal to $\mathfrak{n}(p)$ and hence in $T_p\partial U$. Note that we need the constant ϵ here since $g_p(\mathfrak{n}(p), \mathfrak{n}(p)) = \epsilon$ may be -1 instead of 1. However, ϵ is constant on ∂U. Now we note that

$$\iota^* \mathrm{i}_{X_\parallel(p)} \Omega_g\big|_p = 0,$$

since evaluating $\mathrm{i}_{X_\parallel(p)} \Omega_g\big|_p$ on $n-1$ tangent vectors in $T_p\partial U$ means evaluating $\Omega_g\big|_p$ on n tangent vectors in $T_p\partial U$. Thus they are necessarily linear dependent. This shows that $\iota^* \mathrm{i}_{X(p)} \Omega_g\big|_p = \epsilon g_p(X(p), \mathfrak{n}(p))\, \mathrm{i}_{\mathfrak{n}(p)} \Omega_g\big|_p$. Finally, it is easy to see that $\mathrm{i}_{\mathfrak{n}(p)} \Omega_g\big|_p$ is the (by definition positively oriented) semi-Riemannian volume form of $\iota^* g$. This is clear be the same argument as for $\mu_{\iota^* g}$ in Lemma B.10. This finally shows

$$\int_U \operatorname{div}(X)\mu_g = \int_{\partial U} \operatorname{div}(X)\Omega_g = \epsilon \int_{\partial U} g(i^\# X, \mathfrak{n})\Omega_{i^* g} = \epsilon \int_{\partial U} g(i^\# X, \mathfrak{n})\mu_{i^* g}, \tag{$**$}$$

and hence (B.7). If we change the orientation from Ω_g to $-\Omega_g$ then the induced orientation $\Omega_{\iota^* g}$ changes to $-\Omega_{\iota^* g}$ since the normal vector field $\mathfrak{n}$ remains *unchanged*: "pointing outwards" does not depend on any choice of orientation. Thus we see that the left and right side of ($**$) both change their sign. From this we conclude that (B.7) also holds in the non-oriented case: indeed, by a partition of unity argument we can chop down X into small pieces having support in a chart. There we can choose an orientation and use ($**$). Summing up again is allowed as the validity of ($**$) does not depend on the local choices. □

A particular case of interest is the following. Assume (M, g) is a Lorentzian manifold and the boundary ∂U is spacelike. Then the normal vector field $\mathfrak{n}$ is timelike and we have

$$\int_U \operatorname{div}(X)\mu_g = \int_{\partial U} g(\iota^\# X, \mathfrak{n})\mu_{\iota^* g} \tag{B.8}$$

for all $X \in \Gamma_0^\infty(TM)$.

Bibliography

[1] Abraham, R., Marsden, J. E.: *Foundations of Mechanics.* Addison Wesley Publishing Company, Reading, Mass., 2. edition, 1985.

[2] Amann, H., Escher, J.: *Analysis III. Grundstudium Mathematik.* Birkhäuser Verlag, Basel, 2001.

[3] Bär, C., Fredenhagen, K. (eds.): *Quantum Field Theory on Curved Spacetimes*, vol. 786 in *Lecture Notes in Physics.* Springer-Verlag, Heidelberg, Berlin, New York, 2009. Concepts and Mathematical Foundations.

[4] Bär, C., Ginoux, N., Pfäffle, F.: *Wave equations on Lorentzian manifolds and quantization. ESI Lectures in Mathematics and Physics.* European Mathematical Society (EMS), Zürich, 2007.

[5] Baum, H., Kath, I.: *Normally hyperbolic operators, the Huygens property and conformal geometry.* Ann. Global Anal. Geom. **14**.4 (1996), 315–371.

[6] Beem, J. K., Ehrlich, P. E., Easley, K. L.: *Global Lorentzian Geometry*, vol. 202 in *Pure and Applied Mathematics.* Marcel Dekker, New York, Basel, Hong Kong, 2. edition, 1996.

[7] Bernal, A. N., Sánchez, M.: *On smooth Cauchy hypersurfaces and Geroch's splitting theorem.* Comm. Math. Phys. **243** (2003), 461–470.

[8] Bernal, A. N., Sánchez, M.: *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes.* Comm. Math. Phys. **257** (2005), 43–50.

[9] Bernal, A. N., Sánchez, M.: *Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions.* Lett. Math. Phys. **77**.2 (2006), 183–197.

[10] Bernal, A. N., Sánchez, M.: *Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal'.* Classical Quantum Gravity **24**.3 (2007), 745–749.

[11] Bordemann, M., Neumaier, N., Pflaum, M. J., Waldmann, S.: *On representations of star product algebras over cotangent spaces on Hermitian line bundles.* J. Funct. Anal. **199** (2003), 1–47.

[12] Bröcker, T., Jänich, K.: *Einführung in die Differentialtopologie*, vol. 143 in *Heidelberger Taschenbücher.* Springer-Verlag, Berlin, Heidelberg, New York, 1990. Korrigierter Nachdruck.

[13] Bronstein, I. N., Semendjajew, K. A.: *Taschenbuch der Mathematik.* Verlag Harri Deutsch, Thun, Frankfurt am Main, 24. edition, 1989.

[14] Brunetti, R., Fredenhagen, K.: *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds.* Commun. Math. Phys. **208** (2000), 623–661.

[15] Brunetti, R., Fredenhagen, K., Verch, R.: *The generally covariant locality principle—a new paradigm for local quantum field theory.* Commun. Math. Phys. **237** (2003), 31–68.

[16] Choquet-Bruhat, Y., Geroch, R.: *Global aspects of the Cauchy problem in general relativity.* Commun. Math. Phys. **14** (1969), 329–335.

[17] Choquet-Bruhat, Y., York, Jr. , J. W.: *The Cauchy problem.* In: Held, A. (eds.): *General relativity and gravitation*, 99–172. Plenum Press, New York, 1980. One hundred years after the birth of Albert Einstein.

[18] Dütsch, M., Fredenhagen, K.: *A Local (Perturbative) Construction of Observables in Gauge Theories: The Example of QED.* Commun. Math. Phys. **203** (1999), 71–105.

[19] Dütsch, M., Fredenhagen, K.: *Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion.* Commun. Math. Phys. **219** (2001), 5–30.

[20] Dütsch, M., Fredenhagen, K.: *Perturbative Algebraic Field Theory, and Deformation Quantization.* Field Inst. Commun. **30** (2001), 151–160.

[21] Dütsch, M., Fredenhagen, K.: *The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical Field Theory.* Commun. Math. Phys. **243** (2003), 275–314.

[22] Fischer, A. W., Marsden, J. E.: *The initial value problem and the dynamical formulation of general relativity.* In: Hawking, S. W., Israel, W. (eds.): *General relativity. An Einstein centenary survey*, 138–211. Cambridge University Press, Cambridge, 1979.

[23] Friedlander, F. G.: *The Wave Equation on a Curved Space-Time.* Cambridge University Press, Cambridge, 1975.

[24] Gallo, S., Hulin, D., Lafontaine, J.: *Riemannian Geometry.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1990.

[25] Geroch, R.: *Domain of dependence.* J. Math. Phys. **11** (1970), 437–449.

[26] Grothendieck, A.: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV.* Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361.

[27] Guillemin, V., Sternberg, S.: *Geometric Asymptotics*, vol. 14 in *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, Rhode Island, revised. edition, 1990.

[28] Haag, R.: *Local Quantum Physics.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1993.

[29] Hawking, S. W., Ellis, G. F. R.: *The large scale structure of space-time.* Cambridge University Press, London, 1973.

[30] Hollands, S., Wald, R. M.: *Axiomatic quantum field theory in curved spacetime.* Commun. Math. Phys. **293**.1 (2010), 85–125.

[31] Hörmander, L.: *The Analysis of Linear Partial Differential Operators I. Classics in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 2003. Reprint of the 2nd Edition 1990.

[32] Jackson, J. D.: *Classical Electrodynamics.* John Wiley & Sons, New York, 2. edition, 1975.

[33] Jakobs, S.: *Eichbrücken in der klassischen Feldtheorie (Gauge Bridges in Classical Field Theory).* master thesis, II. Institut für theoretische Physik, Department Physik, Universität Hamburg, Hamburg, February 2009.

[34] Jarchow, H.: *Locally Convex Spaces.* B. G. Teubner, Stuttdart, 1981.

[35] Kobayashi, S., Nomizu, K.: *Foundations of Differential Geometry I. Interscience Tracts in Pure and Applied Mathematics* no. **15**. John Wiley & Sons, New York, London, Sydney, 1963.

[36] Köthe, G.: *Topological Vector Spaces I. Grundlehren der mathematischen Wissenschaft* no. **159**. Springer-Verlag, Heidelberg, Berlin, New York, 1969.

[37] Köthe, G.: *Topological Vector Spaces II. Grundlehren der mathematischen Wissenschaft* no. **237**. Springer-Verlag, Heidelberg, Berlin, New York, 1979.

[38] Kriegl, A., Michor, P. W.: *The Convenient Setting of Global Analysis*, vol. 53 in *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[39] Lang, S.: *Fundamentals of differential geometry*, vol. 191 in *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1999.

[40] Lee, J. M.: *Introduction to smooth manifolds*, vol. 218 in *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 2003.

[41] Marolf, D.: *The generalized Peierls bracket.* Ann. Physics **236**.2 (1994), 392–412.

[42] Marolf, D.: *Poisson brackets on the space of histories.* Ann. Physics **236**.2 (1994), 374–391.

[43] Marsden, J. E., Ratiu, T. S.: *Introduction to Mechanics and Symmetry. Texts in applied mathematics* no. **17**. Springer-Verlag, New York, Heidelberg, 1999.

[44] Michor, P.: *Topics in Differential Geometry.* Schrödinger Institute, Wien, 2001. Lecture notes, available at `www.mat.univie.ac.at/~michor/`.

[45] Minguzzi, E., Sanchez, M.: *The causal hierarchy of spacetimes.* Preprint **gr-qc/0609119** (2006), 63 pages.

[46] O'Neill, B.: *Semi-Riemannian geometry*, vol. 103 in *Pure and Applied Mathematics.* Academic Press, New York, 1983. With applications to relativity.

[47] Peierls, R. E.: *The commutation laws of relativistic field theory.* Proc. Royal Soc. A **214** (1952), 143–157.

[48] Remmert, R.: *Funktionentheorie II.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1995.

[49] Riesz, M.: *L'intégrale de Riemann-Liouville et le problème de Cauchy.* Acta Math. **81** (1949), 1–223.

[50] Römer, H., Forger, M.: *Elementare Feldtheorie.* VCH Verlagsgesellschaft, Weinheim, 1993.

[51] Rudin, W.: *Functional Analysis.* McGraw-Hill Book Company, New York, 2. edition, 1991.

[52] Scharf, G.: *Finite Quantum Electrodynamics.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1995.

[53] Scheck, F.: *Theoretische Physik 3. Klassische Feldtheorie*. Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 2005.

[54] Sexl, R., Urbantke, H. K.: *Gravitation und Kosmologie*. Bibliographisches Institut, Mannheim, Wien, Zrich, 3. edition, 1987.

[55] Steinmann, O.: *Perturbative Quantum Electrodynamics and Axiomatic Field Theory*. Springer-Verlag, Berlin, Heidelberg, New York, 2000.

[56] Straumann, N.: *Allgemeine Relativitätstheorie und relativistische Astrophysik*, vol. 150 in *Lecture Notes in Physics*. Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1988.

[57] Streater, R. F., Wightman, A. S.: *PCT, spin and statistics, and all that*. W. A. Benjamin, New York, Amsterdam, 1964.

[58] Treves, F.: *Topological Vector Spaces, Distributions and Kernels*. Academic Press, New York, London, 1967.

[59] Wald, R. M.: *General relativity*. University of Chicago Press, Chicago, 1984.

[60] Waldmann, S.: *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung*. Springer-Verlag, Heidelberg, Berlin, New York, 2007.

Index