Knots and Functional integration

Louis H. Kauffman

Preface. This book is a survey course in knot theory, starting with the basics of the combinatorial topology, and leading quickly to state summation models for knot invariants such as the Alexander polynomial, the Alexander-Conway polynomial, the Jones polynomial and the Homflypt and Kauffman polynomials. We briefly discuss relationships of these skein polynomials with surgery and three-manifolds. Then we embark on Vassiliev invariants and their relationship with the functional integral formulations of Edward Witten. We conceptualize the functional integral in terms of equivalence classes of functionals of gauge fields and we do not use measure theory. This approach makes it possible to discuss the mathematics intrinsic to the functional integral rigorously and without functional integration. Applications to loop quantum gravity are discussed.

This book originated in a course of lectures at the Institute Henri Poincare in 1997 and then courses given at the University of Illinois at Chicago in the late 1990s and the early part of the 21-st century. The point of the lectures was to develop basic knot theory from the point of view of state summations and the bracket model for the Jones polynomial, and then to move to Wittens functional integral, quantum field theoretic approach to link invariants and its relationship with Vassiliev invariants of knots and links. The Witten functional integral can be approached in a number of ways and we take what is probably the simplest route. We work with the formalism of the integral at the level of advanced calculus without measure theory. This means that the integral of a function or of a gauge field is not a number in our treatment, but an equivalence class of functions just as one would write $\int sin(x)dx = cos(x) + c$ in elementary calculus. In this case the integral "Is" an anti-derivative, and set-theoretically it is an equivalence class of functions where$f \sim g$ means that $f - g$ is the derivative of another function. In this way we can work mathematically with the formalism of the functional integral. Of course, at certain points we would like to say that "this means that the value of the integral is such an such". At such points we will take the leap and posit a value or a relative value, and in this way obtain a powerful heuristic for thinking about link invariants in a quantum field theoretic framework. There is no doubt that this is an interim measure in an evolution of ideas and methods that will eventually lead to a complete story satisfying to topologists and analysts.

This way to approach the Witten functional integral will serve us well in giving a unified picture of how Vassiliev invariants come to be represented by the Kontsevich integral via weight systems for chord diagrams. We find that the perturbative expansion of Wittens integral gives a series of elementary Feynman diagrams whose associated integrals are exactly the Kontsevich integrals. This is the simplest case of dealing with the perturbative expansion of the Witten integral, using the light-cone gauge in an analysis that was initiated by Froelich and King and that in fact goes back to the work of Khono in the 1980's. With the help of this, we see how the Kontsevich integrals are implicit in Wittens functional integral formulation. Mathematicians have been happy to see how to make the last vestiges of the Kontsevich integrals rigorous, and so found the Vassiliev invariants on them. We hope that in the future there will be an appropriate formulation of the functional integral so that one can start with such a central formulation and unfold the invariants from that point. For the present, we have an illuminating approach that should not be forgotten, as it contains the seeds for future mathematics. We will also discuss other topics that emanate from this central place, such as the possibility of finding a quantum field theoretic interpretation of Khovanov Homology that is as simple as the Witten functional integral interpretation for the Jones polynomial. And we will discuss relationships of these ideas with loop quantum gravity, quantum information theory and virtual knot theory.

CHAPTER 1

Topics in combinatorial knot theory

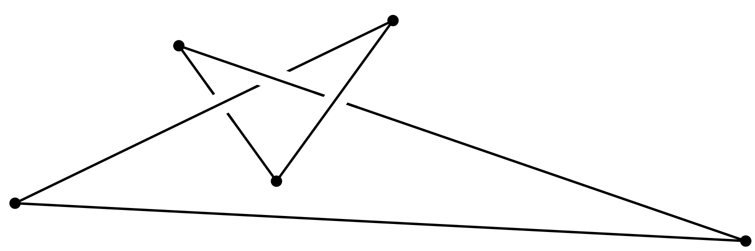
The purpose of this part of the book is to record concisely a number of different parts of basic knot theory. Much of this is related to physical models and to functional integration, as we shall see. Nevertheless, we develop the knot theory here as a chapter in pure combinatorial topology and keep this point of view as we proceed to more analytical ways of thinking. We begin this chapter with the patterns of Fox colorings for knots and links and how these colorings can be seen to yield topological information.

1. Reidemeister Moves

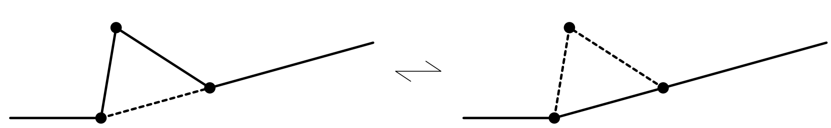
A *knot* is an embedding in three-dimensional space of a single circle. A *link* is an embedding of a collection of circles. Two knots (links) are *ambient isotopic* if there is a continuously varying family of embeddings connecting one to the other.

Unless otherwise specified I shall deal only with *tame* knots and links. In a tame knot every point on the knot has a neighborhood in 3-space that is equivalent to the standard (ball, arc-diameter) pair. Tame knots (links)can be represented up to ambient isotopy by *piecewise linear* knots and links. A link is piecewise linear if the embedding consists in straight line segments. Thus a piecewise linear link is an embedding of a collection of boundaries of n-gons (different n for different components and the n's are not fixed). A piecewise linear knot (link) is made from "straight sticks."

EXERCISE. Show that any "knot" made with ≤ 5 straight sticks is necessarily unknotted. (i. e. ambient isotopic to a circle in a plane.) What is misleading about this picture?



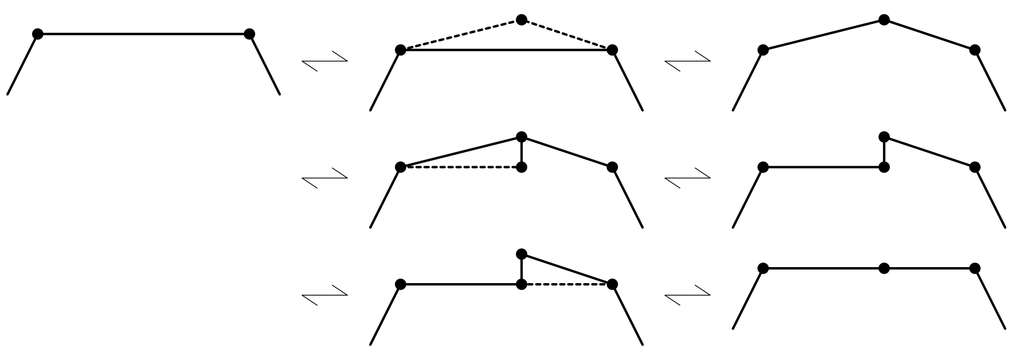
Reidemeister [**1**], in the 1920–1930's, studied knots and links from the PL viewpoint. He defined *PL-equivalence* of knots and links via the following 3-dimensional move:



In this move one can remove two sides of a triangle and put in the missing third side, or remove the third side and put in the other two sides. The interior of the triangle must *not* be pierced by any other arcs in the link.

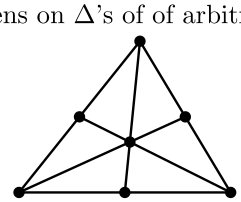
For tame links, PL-equivalence $\equiv$ ambient isotopy

We shall not prove this statement here. The reader may like to take it as an extended exercise. The main point is that subdivision is possible, as we show below, and any PL-equivalence can be approximated by a sequence of triangle moves.

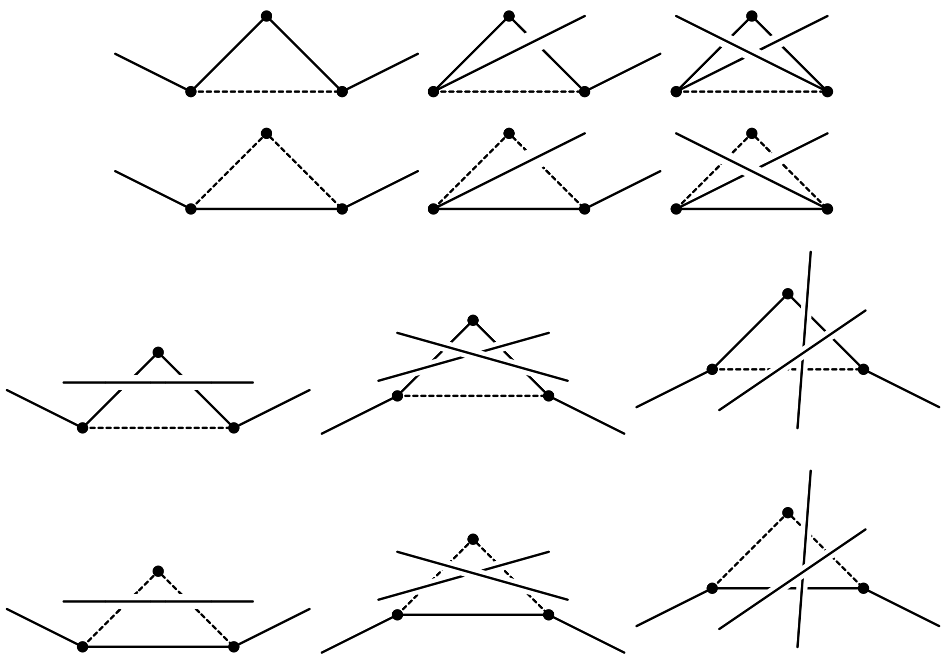


These diagrams demonstrate that the subdivision of an edge (or the removal of a vertex in collinear edges) can be accomplished via the Reidemeister Δ-move. (The triangles involved in the demonstration can be chosen sufficiently small to avoid piercing the rest of the link.)

Standard barycentric subdivision of a triangle can be used to assure that each Δ-move happens on Δ's of of arbitrarily small diameter. (Taking edge subdivision as axiomatic.)



Now consider planar projections of Δ-moves. We can assume that the point of projection gives an image that is in general position. Typical triangles will look like



By using barycentric subdivision we can assume that each Δ has a minimum number of edges projected over or under it. Since the edges that emanate from the triangle may be theses edges we must take such cases into account, plus the case of extra edges (one or two crossing each other) from elsewhere in the link. Cases (a) $\rightarrow$ (f) enumerate these local possibilities. Now lets translate (a) $\rightarrow$ (f) into knot diagrammatic language:

(a)

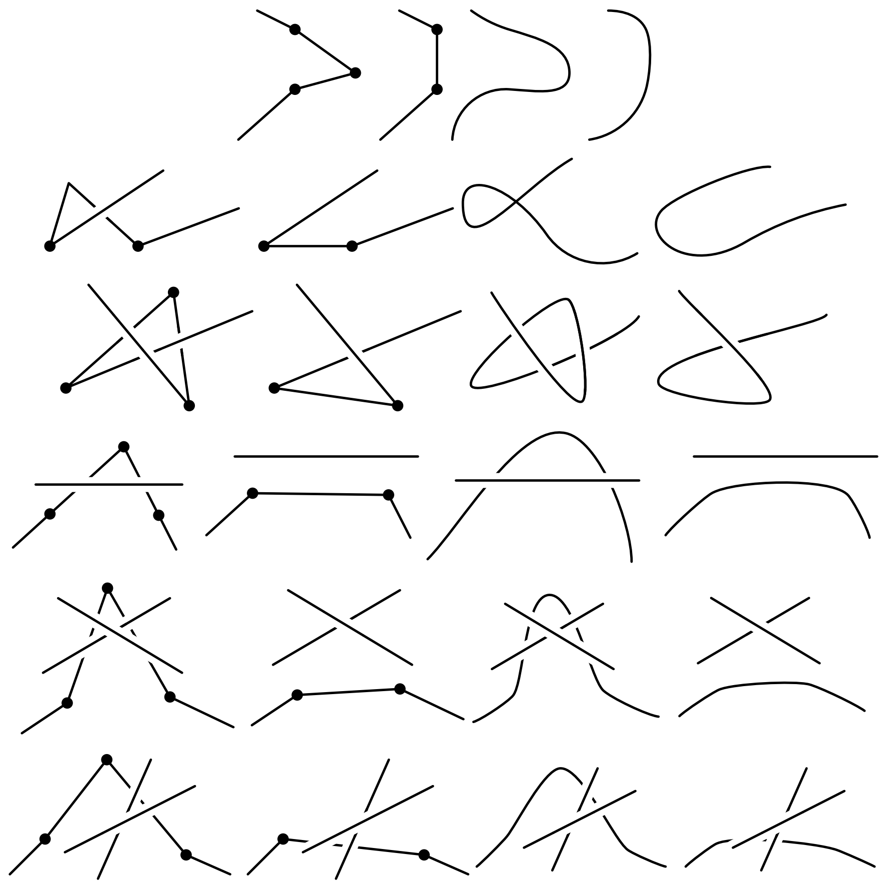
(b)

(c)

(d)

(e)

(f)



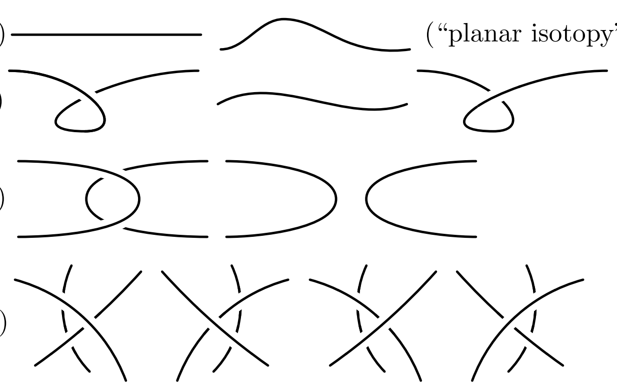
By our subdivision argument, local moves of this type on diagrams will generate ambient isotopy (PL isotopy) for tame links in 3-space. In fact, we can eliminate (c) and (e) from the list, generating them from the others. We take as fundamental

(a) $R(0)$ ("planar isotopy")

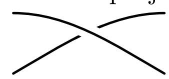
(b) $R(1)$

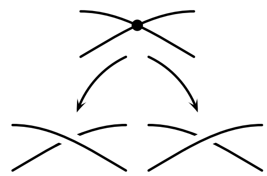
(d) $R(2)$

(f) $R(3)$

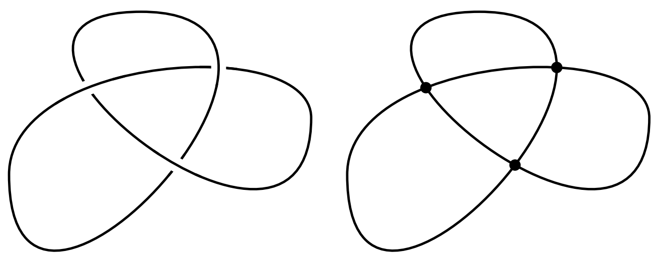


Exercise. Accomplish moves (c) and (e) from Reidemeister moves $R(0)$, $R(1)$, $R(2)$, $R(3)$.

Knot and link diagrams are smoothed-out pictures of projections of knots and links from 3-space with the crossing convention  that puts a break in the under-crossing line. You can regard the crossing as extra structure on a graphical vertex of degree 4:



In this sense a knot or link diagram is a 4-valent plane graph with extra structure:



$T \quad \pi(T) =$ underlying plane graph for the diagram

We have proved the

Theorem (Reidemeister). *Let $\mathbb{K}$ and $\mathbb{L}$ be two tame links in 3-space. Let K and L be diagrams for $\mathbb{K}$ and $\mathbb{L}$ obtained by projection $\mathbb{K}$ and $\mathbb{L}$ to a plane in general position with respect to $\mathbb{K} \cup \mathbb{L}$. Then $\mathbb{K}$ is ambient isotopic to $\mathbb{L}$ if and only if K and L are connected by a finite sequence of the Reidemeister moves* (0)*,* (1)*,* (2)*,* (3)*.*

This theorem reduces the topological classification of knots and links in three-dimensional space to a combinatorial question about the equivalence classes of diagrams under the Reidemeister moves. As we shall see, the diagrams provide a pivot for translating many different ideas into the topological domain.

Some examples are in order. First of all, consider the diagram in Figure 1. You can see that this diagram is unknotted because it always winds down in such a way that if you were to pull the rope upward it would all unravel. You can take, as an exercise to show the unknotting strictly by Reidemeister moves. Note that from the graphical point of view, when you apply a Reidemeister move, you must see either a 1-sided, 2-sided or 3-sided region in the plane. There are natural generalizations of the Reidemeister moves where a large-scale Reidemeister pattern occurs with other weaving above or below that pattern, but we are not using generalized moves here.

A second example is shown in Figure 2. Here the diagram is unknotted but there are no 3-moves and there are no moves that directly simplify the diagram. In order to unknot it by Reidemeister moves, you will have to make it more complicated!

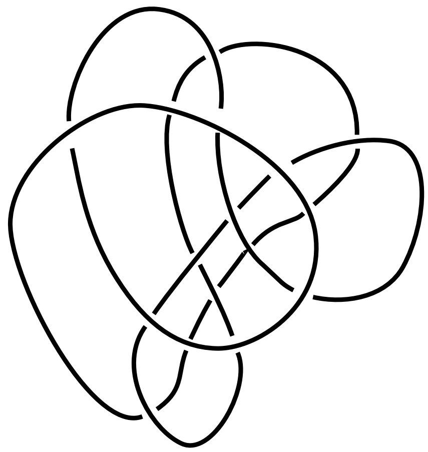


FIGURE 1. **Descending Unknot**

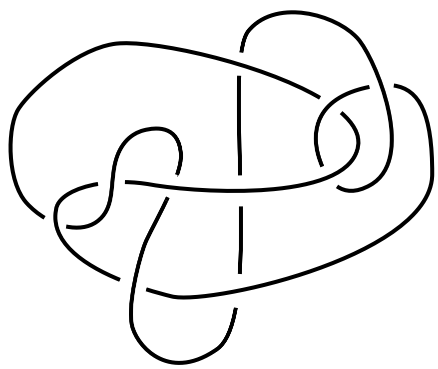


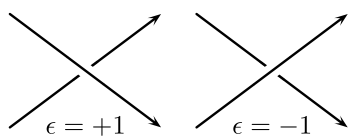
FIGURE 2. **Hard Unknot**

2. Linking number

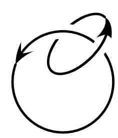
Given two oriented curves A and B in R^3, we can define a *linking number*, $Lk(A, B)$ that counts the (algebraic) number of times that one curve winds around the other. In this section we shall give a purely combinatorial definition of the linking number, by making a count involving the signs of the crossings between one curve and the other. The signs of crossings and the corresponding definition of linking number is shown in the diagrams below. The reader should note that one can think of a crossing as one arc going *half-way around* the other arc. Thus we assign $\pm 1/2$ to each crossing and define $Lk(A, B)$ to be the sum over all crossings of A with B of these half-integer contributions. The resulting summation is invariant under the Reidemeister moves.

$$Lk(A, B) = \sum_p \epsilon(p)/2$$

where p runs through crossings of A with B and $\epsilon(p)$ is the *sign* of the crossing as defined below. Note that any crossing has a sign, but we use only the crossings that occur between the two components A and B in computing the linking number. It is easy to see that the linking number is invariant under the action of the Reidemeister moves.



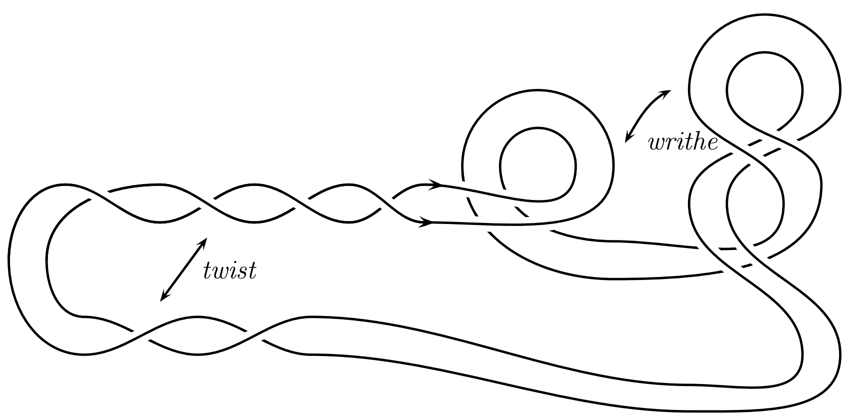
$$\mathrm{Lk}(A,B) \stackrel{\text{def}}{=} \sum_{\substack{p \\ p \text{ is a crossing of } A \text{ with } B}} \epsilon(p)/2$$



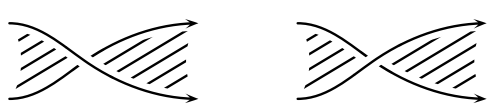
$: \ \mathrm{Lk}(L) = \frac{1}{2} + \frac{1}{2} = 1.$

$: \ \mathrm{Lk}(W) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0.$

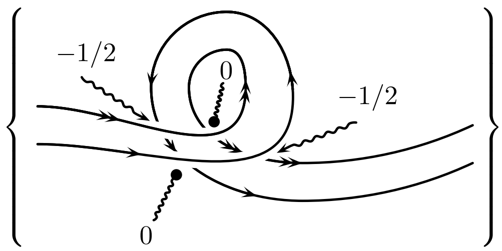
Exercise. Linking number is invariant under the Reidemeister moves.



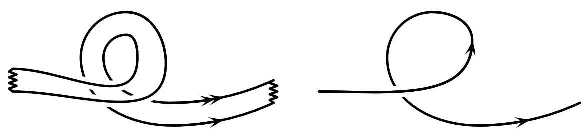
For a link consisting in the two edges of a twisted circular band, as illustrated above, we can divide contributions to the linking number into "twist" and "writhe".



twist



$$- \frac{1}{2} - \frac{1}{2} = -1$$



writhe

In counting *writhe* you look at the *self*-crossings of the core curve of the band and count them ±1. Thus writhe is defined for any oriented diagram D by the formula

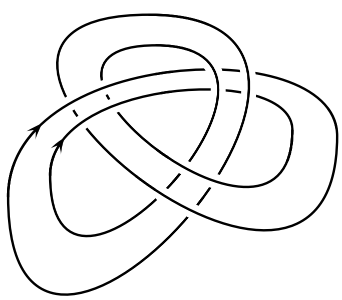
$$Writhe(D) = \sum_{p \in Cr(D)} \epsilon(p),$$

where $Cr(D)$ is the collection of all the crossings in the diagram D. In the case of the band B we let $Core(B)$ be the core curve of the band as illustrated above (it is a copy of either one of the two components of the boundary of the band) and define $Writhe(B) = Writhe(Core(B))$.

If L is the boundary of band B (with both components of L directed in the *same* direction) then

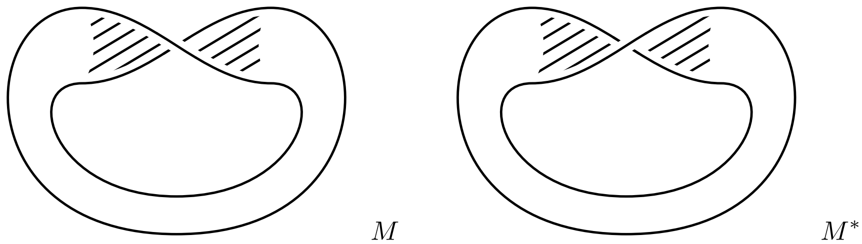
$$\mathrm{Lk}(L) = \mathrm{Twist}(B) + \mathrm{Writhe}(B).$$

This is diagrammatic version of the Calagareanu/Pohl/White formula [] (expressed by them in terms of 3-dimensional differential geometry) and used in studying topology of DNA.



$$\mathrm{Lk}(L) = \mathrm{Writhe}(B) = +3.$$

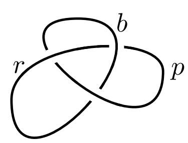
Exercise.



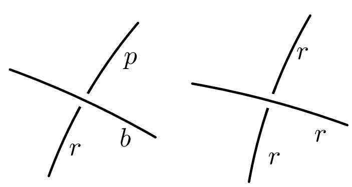
Two Möbius bands M, M^*. Mirror images of each other. Show that M and M^* are *not* ambient isotopic as surfaces in 3-space. Hint: Cut each band down the middle and consider the linking numbers of the two edges of each cut band (orienting the edges parallel to one another with respect to the band).

3. Coloring a trefoil knot

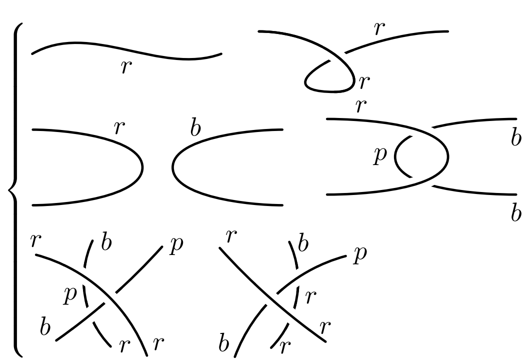
The linking number of a two-component link is a property of that link that remains the same when the link is transformed by Reidemeister moves. We shall now embark on the problem of finding properties of knots that are invariant under the Reidemeister moves. Surprisingly, this is harder to find than the linking number. We would like to somehow measure a 'self-linking number' for a knot. And indeed we see that it is possible to color the arcs in a trefoil diagram with three colors.



The arcs in the diagram are colored from the set $C = \{r, b, p\}$.



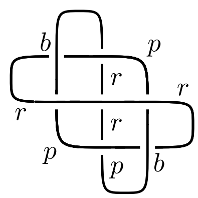
Any given crossing either has *three* distinct colors incident to it or *one*.



(Two other cases work similarly.)

These diagrams and a little extra case-work show that *any* diagram representing the trefoil knot can be colored with three colors (using all three colors) according to these rules. Since an unknotted diagram can receive only one color by the same procedure, *this proves that the trefoil knot is knotted*.

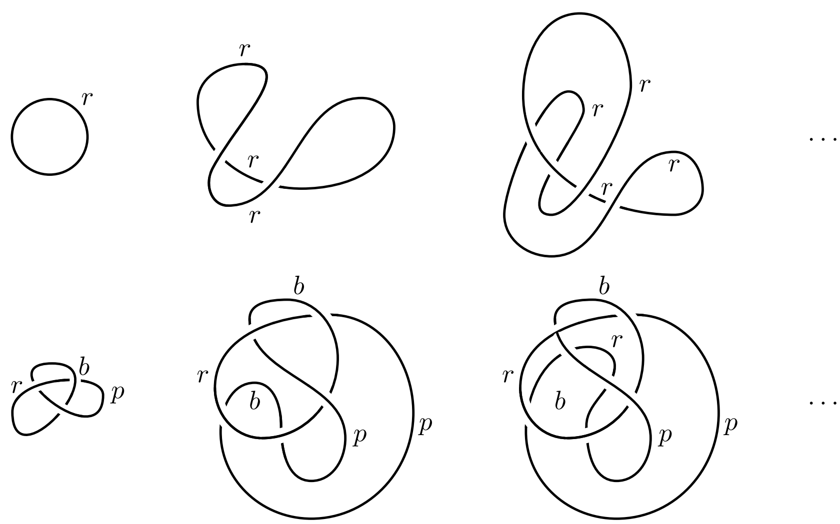
EXAMPLE.



This diagram also represents a trefoil knot.

It is an example of a diagram with a crossing that *requires* a single color.

Here are some examples of colorings induced by Reidemeister moves. Each time one does a Reidemeister move, the new diagram is uniquely colored in terms of its predecessor diagram.



Example. The connected sum of two knots is obtained by removing a small arc from each knot making it a knot in a "length of rope", connecting the two lengths of rope end to end, and closing the result, as shown in Figure 3. In this Figure we illustrate that fact that $T + K$ is 3-colorable for any knot K where $T + K$ denotes the connected sum of the trefoil and the knot K. We conclude that there does not exist a knot K such that $T + K$ is the trivial knot.

Example. We can prove that if K and K' are any two knots then $K+K'$ unknotted implies that both K and K' are themselves unknotted. This generalizes our result about the trefoil to all knots. You cannot cancel a knot with another knot by the operation of connected sum. We will give two proofs of this fact. The first proof uses a method of infinite repetition. We suppose that knots K and K' can cancel in a connected sum where all cancellation occurs in a $1 - 1$ tangle representation of $K + K'$. This means that we view $K + K'$ as a knot in a string with a left end and

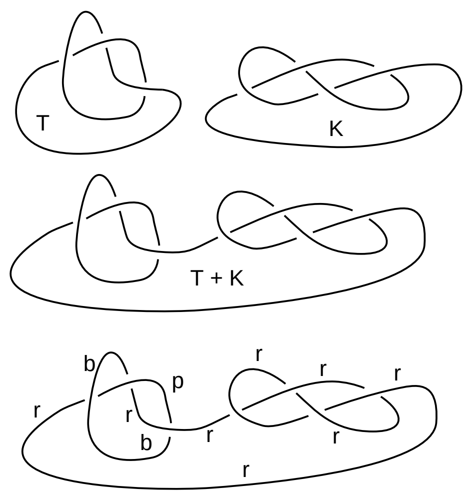


FIGURE 3. **Connected Sum of Trefoil T and Any Knot K is knotted.**

a right end. The left and right ends of the string are fixed during the cancellation process, and no part of the isotopy that cancels the two knots every crosses over the ends of the string. We can then say that the cancellation isotopy occurs in a box containing $K + K'$ with one end of the string emanating from the left end of the box and the other end of the string emanating from the right end of the box. With this we make a series of boxes, each one-half the size of the previous one and use the decreasing sequence of adjacent boxes to form $K_\infty = K + K' + K + K' + K + K' + ...$ where the knots get smaller and smaller as they approach the limit point of the stack of boxes. We then make a closed loop by connecting the limit point to the left most box string. Looked at this way K_∞ is a topologically embedded circle in three dimensional space. We now look at K_∞ in two ways. First of all we have

$$K_\infty = (K + K') + (K + K') + (K + K') + ...$$

The parentheses correspond to the original configuration of boxes, and each $K + K'$ is isotopic to an unknotted segment, and hence

$$K_\infty = U$$

where U is the unknot. On the other hand, we can reassociate the infinite connected sum to form

$$K_\infty = K + (K' + K) + (K' + K) + (K' + K)+) + ...,$$

and now we see that all the pairs $K' + K$ are isotopic to segments (since connected sum is commutative). Therefore we conclude that

$$K_\infty = K.$$

Hence

$$U = K_\infty = K.$$

We have shown that if $K + K'$ is unknotted, then each of K and K' is unknotted. You cannot cancel knots.

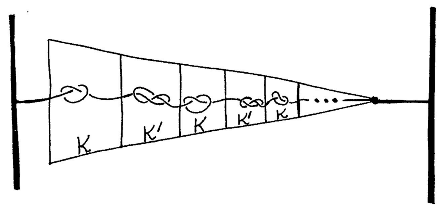


FIGURE 4. **An infinite connected sum.**

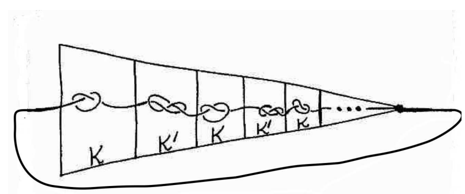


FIGURE 5. **A closed infinite connected sum.**

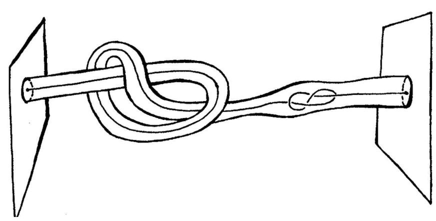


FIGURE 6. **Tube for Connected Sum.**

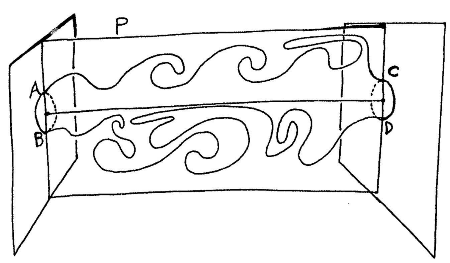
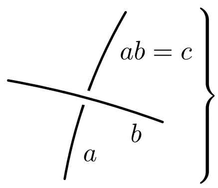


FIGURE 7. **Tube after Isotopy.**

Exercise. Prove that the Whitehead Link and the Borommean Rings are each non-trivial links by showing that neither link has a non-trivial three-coloring. (If either link were trivial, then it would have a three coloring induced by Reidemeister moves from a non-trivial coloring of the unlink).

4. First generalization of coloring

We generalize the three-coloring scheme by assuming that the arcs of the knot or link diagram are labeled with elements of a set A in such a way that when a labels an undercrossing arc at a given crossing and b labels the overcrossing arc, then $c = ab$ labels the remaining undercrossing arc. Here ab denotes a binary operation on the set A. This is illustrated below, and we shall see what rules this binary operation must satisfy to be compatible with the Reideister moves.



A binary operation on colors a, b produces a new color $c = ab$.

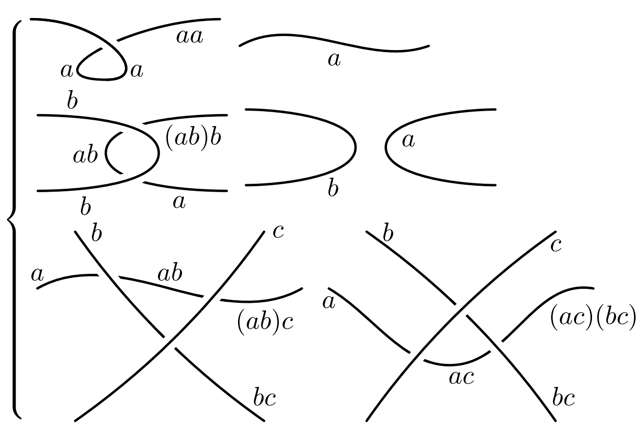
We will examine the behaviour of the coloring scheme under the Reidemeister moves, but first, here is a re-write of our 3-coloring rules in terms of this algebraic point of view.

In our 3-coloring example we had:

$$\left\{ \begin{array}{c} rr = r,\ bb = b,\ pp = p, \\ rb = br = p,\ rp = pr = b, bp = pb = r. \end{array} \right\}$$

Note. $\left. \begin{array}{l} (rb)p = (p)p = p \\ r(bp) = rr = r \end{array} \right\} \quad (xy)z \neq x(yz).$

We can think of this 3-color algebra as a very simple system of discriminations. Each color is seen to be different from the other because the product of any two of them is the third color. Any color is seen to be equal to itself because the product of the color with itself is itself. Here is a very simple algebra that is not associative!



$aa = a,$

$(ab)b = a,$

$(ab)c = (ac)(bc).$

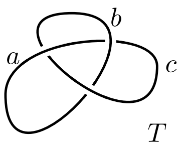
To see the general rules, we examine the binary operation in relation to the Reidemeister moves. Note that we do *not* assume that the product is associative. We find the results in the diagrams show above. We see that in order to compatibly induce colorings on diagrams obtained before and after a given Reidemeister move, we need that the moves satisfy the equations given in the axioms below.

Definition. An algebraic system with one binary operation $a, b \mapsto ab$ (not necessarily associative) and satisfying the axioms

1. $aa = a \; \forall a$,
2. $(ab)b = a \; \forall a, b$,
3. $(ab)c = (ab)(ac) \; \forall a, b, c$,

is called an *involutory quandle,*

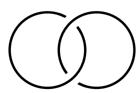
We associate an involutory quandle $\mathrm{IQ}(K)$ to any knot or link K by taking one generator for each arc in the diagram and one relation of the form $c = ab$ at each crossing. This ideas of associating a quandle to a knot or link diagram, and indeed the term quandle, is due to David Joyce [].

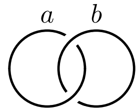


$$\left.\begin{array}{l} c = ab \\ b = ca \\ a = bc \end{array}\right\} \begin{array}{l} cb = (ab)b = a \\ ba = (ca)a = c \\ ac = (bc)c = b \end{array}$$

·	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

} $\mathrm{IQ}(T)$ reconstructs our 3-coloring.

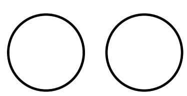
In general, the IQ of a knot or link can be quite complex. (It is not necessarily finite.) For example the Hopf link L  has a finite IQ:

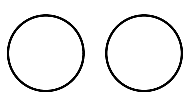


: $\begin{array}{l} ab = a \\ ba = b \end{array}$

·	a	b
a	a	a
b	b	b

} closed finite system

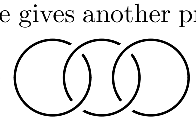
and the unlink  has:

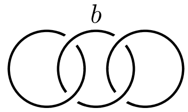


·	a	b	...
a	a	ab	
b	ba	a	
⋮			

an *infinite* IQ.

This distinction between finite and infinite gives another proof that L is linked.

On the other hand, you can check that $\widetilde{L}$  has an *infinite* IQ.



$$\begin{array}{c} bc = ba \\ c(ba) = c \end{array}$$

REMARK. If you are interested in working with this sort of algebra, the following Lemma is useful.

LEMMA (Winker). *In an* IQ, $a(bc) = ((ac)b)c$ *for any* a, b, c. *Hence any expression can be put in a canonical left-parenthesized form.* (*Notation.* Write $xyzw = ((xy)z)w$ etc.)

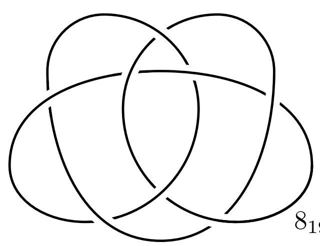
PROOF. $((ac)b)c = ((ac)c)(bc) = a(bc)$. □

Thus for $\mathrm{IQ}(\widetilde{L})$, above, we have:

$$\left\{\begin{array}{c} bc = ba \\ caba = c \end{array}\right\} \begin{array}{l} b = bac \\ c = caba \end{array}$$

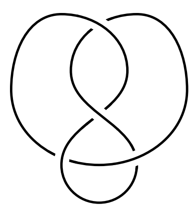
The collection $\{a, ab, bab, abab, \ldots\}$ is an infinite subset of $\mathrm{IQ}(\widetilde{L})$.

Winker verified that the first *knot* with infinite IQ is 8_{19} (numeration from Reidemeister's tables).



8_{19}

EXERCISE. Show that $\mathrm{IQ}(E)$ has five elements where

E:  (*figure eight knot*)

.

We could try to represent an IQ to a set C that is a module[1] over $\mathbb{Z}$ via $ab = ra + sb$ with $r, s \in \mathbb{Z}$:

$$a = aa = ra + sa = (r+s)a$$

therefore assume $r + s = 1$.

$$(ab)b = a\colon r(ra + sb) + sb = a$$
$$r^2 a + (r+1)sb = a$$

therefore assume $r = -1$. $\Rightarrow$ $s = 2$. Then

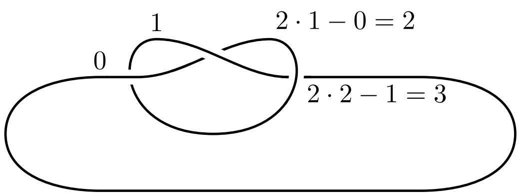
$$ab = 2b - a.$$

[1]Kakoj modul?

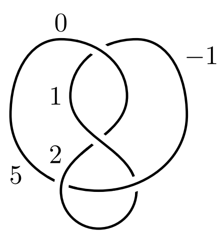
Check.

$$\begin{aligned}(ab)c &= 2c - (2b - a) = 2c - 2b + a;\\ (ac)(bc) &= -(2c - a) + 2(2c - b)\\ &= -2c + a + 4c - 2b\\ &= 2c - 2b + a.\end{aligned}$$

The rule $ab = 2b - a$ puts an IQ structure on any module over $\mathbb{Z}$.



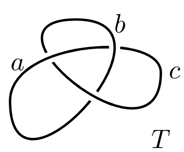
$\Rightarrow 3 = 0 \Rightarrow C = \mathbb{Z}/3\mathbb{Z}$.



$\Rightarrow C = \mathbb{Z}/5\mathbb{Z}$. } Note that not all 5 colors are used on *this* diagram.

In this scheme, each knot acquires a modulus[2] and the IQ structure occurs on a finite Abelian group. In order to define a unique modulus[3], one can take the determinant of a (reduced) relation matrix corresponding to the detining[4] for ths IQ.

e. g.



$$\begin{aligned}ab &= c \qquad & 2b - a &= c\\ ca &= b & 2a - c &= b\\ bc &= a & 2c - b &= a\end{aligned}$$

	a	b	c
	−1	2	−1
	2	−1	−1
	−1	−1	2

M = matrix obtained by striking out one row and one column from relation matrix.

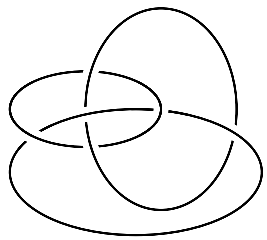
$$\mathrm{D}(T) \stackrel{\text{def}}{=} |\,\mathrm{Det}(M)|.$$

D(T) is called the *determinant of the knot* T. It can be used as a modulus[5] for coloring the knot diagram with $ab = 2b - a$, and D(T) *itself* is an invariant of the knot (link).

---

[2]????
[3]????
[4]????
[5]????

Sometime divisors of D(K) will suffice. For example,

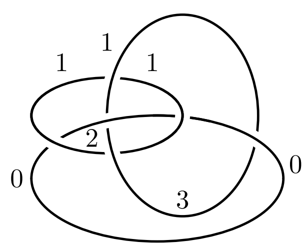


B=Borrommew rings

m. b. Borromean rings?

EXERCISE. $D(B) = 32$.

But[6]



$C = \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$.

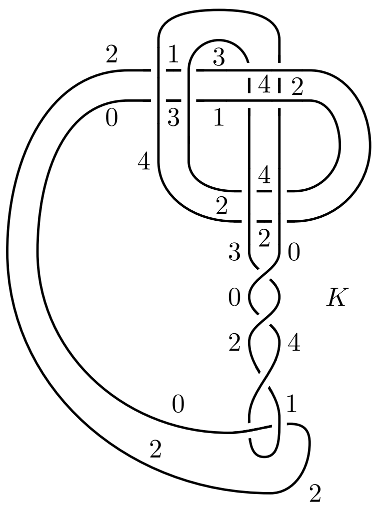
We can color in $\mathbb{Z}/4\mathbb{Z}$.
Note that this coloring actually uses colors $0, 1, 2, 3$.

CONJECTURE. No diagram for B can be non-trivially colored with less than 4 colors.

Harary[7] and Kauffman define the *coloring number* $C(K)$ of a knot or link K to be the *least number* of colors needed by a diagram to color K. It is not hard to see that $C(\text{fig eight}) = 4$. (Some diagrams of E require 5 colors, but there are diagrams that require 4 and no diagrams that can be colored in < 4.)

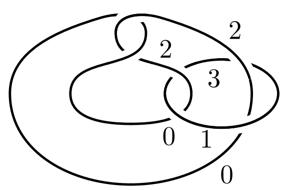
[6]K chemu eto?
[7]????

Every knot or link presents its own coloring problem in this context. Here is a $\mathbb{Z}/5\mathbb{Z}$ knot that I[8] conjecture has $C(K) = 5$:



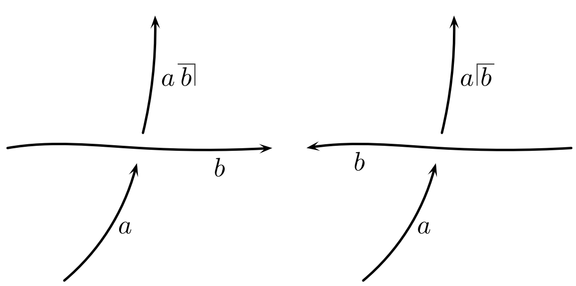
Conjecture. $C(K) = 5$.

Here is a $\mathbb{Z}/4\mathbb{Z}$ coloring of the Whitehead link.



Is $C(W) = 4$?

5. 2^{nd} generalization of coloring



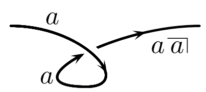
Now the operations keep track of orientation.

Notation. If $a * b = a\,\overline{b|}$ and $a\#b = a\overline{|b}$ then

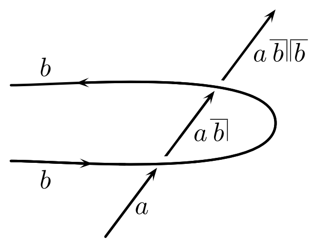
$$\left\{\begin{array}{c} (a*b)*c = a\,\overline{b|}\,\overline{c|} \\ a*(b*c) = a\,\overline{b\,\overline{c|}|} \\ (a*b)\#c = a\,\overline{b|}\overline{|c} \\ a*(b\#c) = a\,\overline{b\overline{|c}|} \end{array}\right\} \text{ etc.}$$

[8] ????

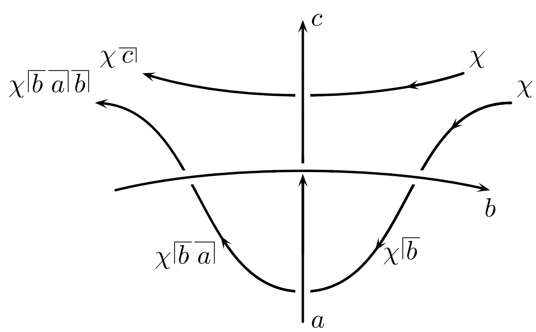
6. Translation of Reidemeister moves

1. 

$a\overline{a}| = a$

2. 

$a\overline{b}||\overline{b} = a \Leftrightarrow \overline{b}||\overline{b} = 1$

3. 

$c = a\overline{b}|$
$\overline{c}| = |\overline{b}\,\overline{a}|\overline{b}| \Leftrightarrow \overline{a\overline{b}|}| = |\overline{b}\,\overline{a}|\overline{b}|$

In this oriented context we are given a set C of "colors." Each element $a \in C$ is also an *automorphism* $\overline{a}|: C \to C$ (by property 2) with $|\overline{a} = \overline{a}|^{-1}$ and $\overline{a}|(x) = x\overline{a}|$, $|\overline{a}(x) = x|\overline{a}$. C is an *automorphic set* (the terminology is due to E. Brieskorn). The special rules corresponding to 1. and 3. give us the axioms for a *quandle*. If we just use 3. (not 1.) I call this a *crystal*. (Rourke and Fenn call the correspondence structure[9] with binary operations a *rack* (from "wrack" — a term used by J. H. Conway and Gavin[10] Wraith).)

Thus a *crystal* is a set (automorphic) C such that

$$\overline{a\overline{b}|}| = |\overline{b}\,\overline{a}|\overline{b}| \qquad |\overline{a\overline{b}|} = |\overline{b}|\overline{a}\,\overline{b}|$$
$$\overline{a|\overline{b}}| = \overline{b}|\overline{a}||\overline{b} \qquad |\overline{a|\overline{b}} = \overline{b}||\overline{a}|\overline{b}$$

for all $a, b \in C$.

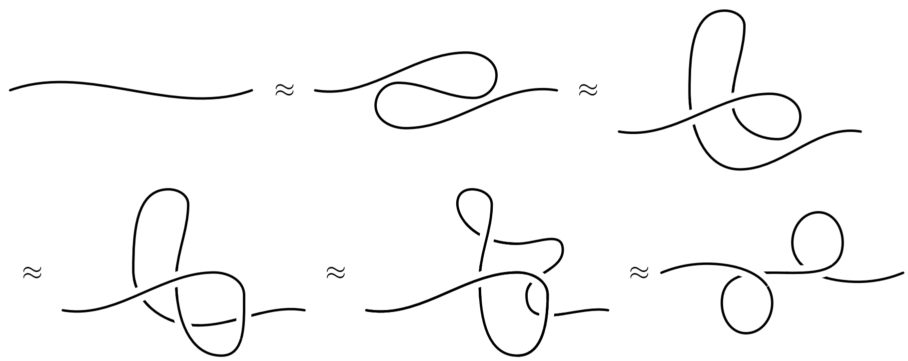
In terms of the binary operations $*$ and $\#$, these rules just say that they distribute over themselves:

$$\begin{aligned}(a*b)*c &= a\overline{b}|\overline{c}| \\ (a*c)*(b*c) &= a\overline{c}|\overline{b\overline{c}|}| \\ &= a\overline{c}||\overline{c}\,\overline{b}|\overline{c}| \\ &= a\overline{b}|\overline{c}| \\ &= (a*b)*c.\end{aligned}$$

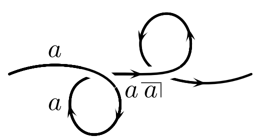
[9] V tekste "corres str"
[10] ili Cavin?

DEFINITION. *Regular Isotopy* ($\approx$) = the equivalence relation generated by Reidemeister moves 0,2,3.

EXERCISE. The Whitney trick.

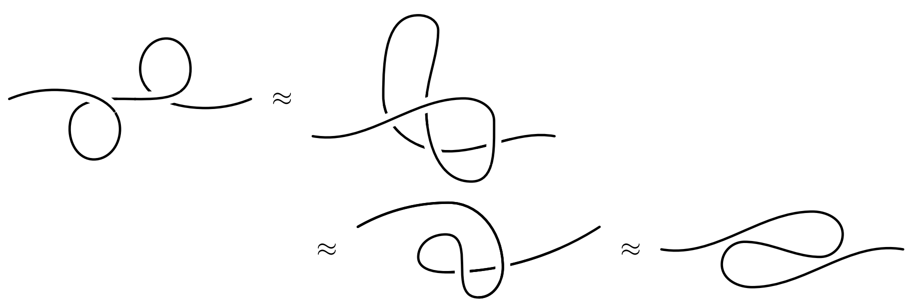


NOTE.

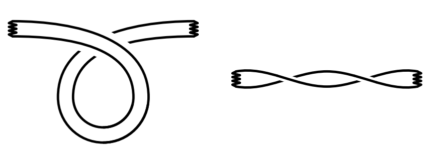


$$\begin{aligned} a\,\overline{a}\rceil\lceil\overline{a\,\overline{a}\rceil} &= a\,\overline{a}\rceil\lceil\overline{a}\,\lceil\overline{a}\,\overline{a}\rceil \\ &= a\lceil\overline{a}\,\overline{a}\rceil \\ &= a \end{aligned}$$

The crystal does the Whitney trick algebraically. Note that this algebra corresponds to the following topological moves:

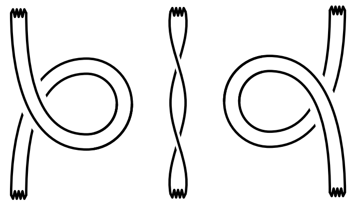


DEFINITION. A *framed link* is a link such that every component has a normal vector field. This is equivalent to using *bands* instead of circles. The twisting of the band catalogs the normal field.

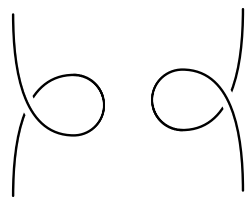


The type 1 Reidemeister move does not apply to framed links.

We can use link diagrams to classity framed links. But



Thus we must replace $R1$ by $R1'$:



This is called *ribbon equivalence*. (The equivalent relations are generated[11] by $R0, R1', R2, R3$.)

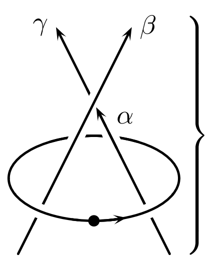
Ribbon equivalence is very important in studying 3-manifolds via surgery. More of this later on.

K an oriented link. $\widehat{C}(K)$ = the *crystal of* K obtained by taking one generator for each arc of the K-diagram and one relation $c = a\,\overline{b|}$ or $c = a\,\overline{|b}$ at each crossing.

Let $C(K)$ = *reduced crystal* ($\equiv$ quandle) where we include the axiom $a\,\overline{a|} = a\,\overline{|a} = a\ \forall a$.

Theorem. *Let $G(K)$ denote the group of automorphisms of $C(K)$ generated by the elements of $C(K)$. Then $G(K)$ is isomorphic to the fundamental group of the complement of K in S^3: $G(K) \cong \pi(K) = \pi_1(S^3 - K)$.*

Proof.



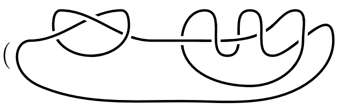
In $\pi(K)$ $\alpha\beta = \beta\gamma \ \Rightarrow\ \gamma = \beta^{-1}\alpha\beta$.

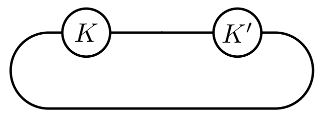
This corresponds to $\overline{c|} = \overline{|b}\,\overline{a|}\,\overline{|b}$ in $C(K)$.

We know from the Van Kampen Theorem that $\pi(K)$ is given by one generator per arc of the diagram and one relation per crossing in the form above. (Wirting[12] presentation.) The rest is a bit of universal algebra. □

[11] V originale eta fraza sokrashena. Tak pravilno?

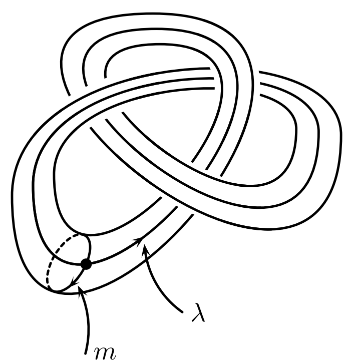
[12] m.b. Wirtinger?

Theorems of *Waldhausen* show that, for a (prime) knot K (

is composite. $K\#K'$: 

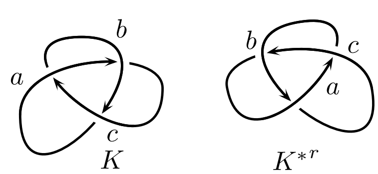
prime means $K \not\simeq K'\#K''$ with K' and K'' non-trivial.) the topological type of $K \subset S^3$ is determined by $(\pi(K), P(K))$ where $P(K)$ is the *peripheral subgroup* of $\pi(K)$. The peripheral subgroup is the subgroup generated by elements of $\pi_1(S^3 - K)$ that lie on the surface of a tabular neighborhood of K.

$P(K)$ is generated by one meridian m and one longitude λ.



Since $C(K)$ is generated by meridians, one can prove that: $C(K) \cong C(K') \Rightarrow$ either $K' \simeq K$ or $K' \simeq K^{*r}$ where

$$\begin{cases} K^* = \text{mirror of } K \text{ image (flip all crossings)}, \\ K^r = \text{reverse orientation of } K. \end{cases}$$

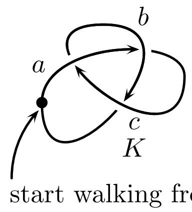


$$\left\{ \begin{array}{l} a\,\overline{b|} = c \\ c\,\overline{a|} = b \\ b\,\overline{c|} = a \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} a\overline{|c} = b \\ b\overline{|a} = c \\ c\overline{|b} = a \end{array} \right\}$$

In this case $K \cong K^r$ so $C(K) = C(K^*)$.

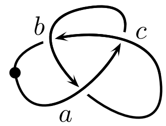
If we add the longitude $\lambda(K)$ to $C(K)$ then the resulting structure completely classifies K.

In $\pi(K)$, $\lambda(K)$ is the element obtained by taking the product of the elements that you *underpass* as you walking along the diagram.



$$\lambda(K) = \overline{b|}\,\overline{a|}\,\overline{c|}.$$

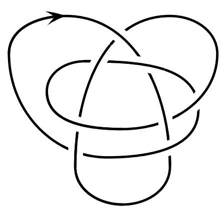
Note that in the example above



$$\lambda(K^{*r}) = \overline{|c}\,\overline{|a}\,\overline{|b};$$

the isomorphism $C(K) \cong C(K^{*r})$ does not carry longitude to longitude.

REMARK. There are exist knots K s.t. $K \not\simeq K^r$. The first example is 8_{17}:



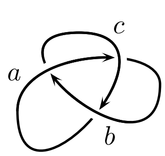
$$\left.\right\} \begin{array}{l} \Rightarrow\ 8_{17} \not\simeq 8_{17}^{*} \\ K \not\simeq K^{*}\text{: one says } K \text{ is chiral} \end{array}$$

FACT. $8_{17} \not\simeq 8_{17}^r$, $8_{17} \simeq 8_{17}^{*r}$

A knot K is said to be *reversible* if $K \simeq K^r$. A knot K is said to be *achiral*[13] if $K \simeq K^*$.

It would be nice to have a purely combinatorial proof that $(C(K), \lambda(K))$ classifies the knot K. The problem is in reconstructing the knot from the algebra after doing algebraic manipulations.

EXERCISE.



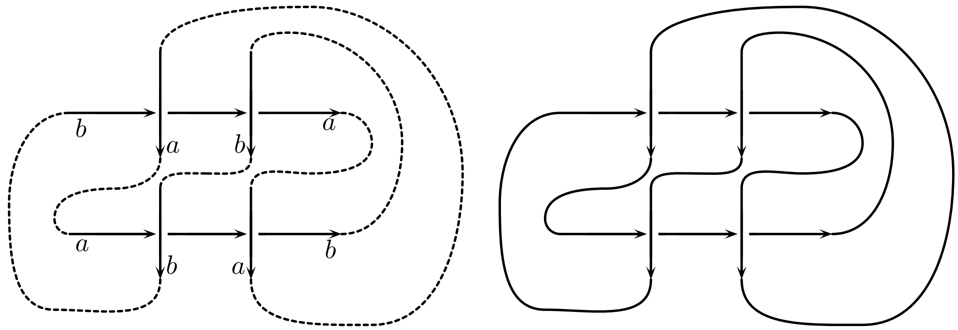
$$\left.\begin{array}{l} a = c\,\overline{b|} \\ b = a\,\overline{c|} \\ c = b\,\overline{a|} \end{array}\right\} \quad \begin{array}{l} a = b\,\overline{a|}\,\overline{b|} \\ b = a\,\overline{b\,\overline{a|}|} = a\,\overline{|a}\;\overline{b|}\,\overline{a|} = a\,\overline{b|}\,\overline{a|} \end{array}$$

So

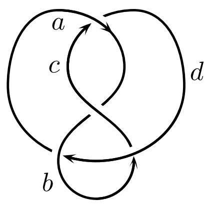
$$a = b\,\overline{a|}\,\overline{b|},$$
$$b = a\,\overline{b|}\,\overline{a|}.$$

[13] tak?

Try reconstructing from here:



Exercise.



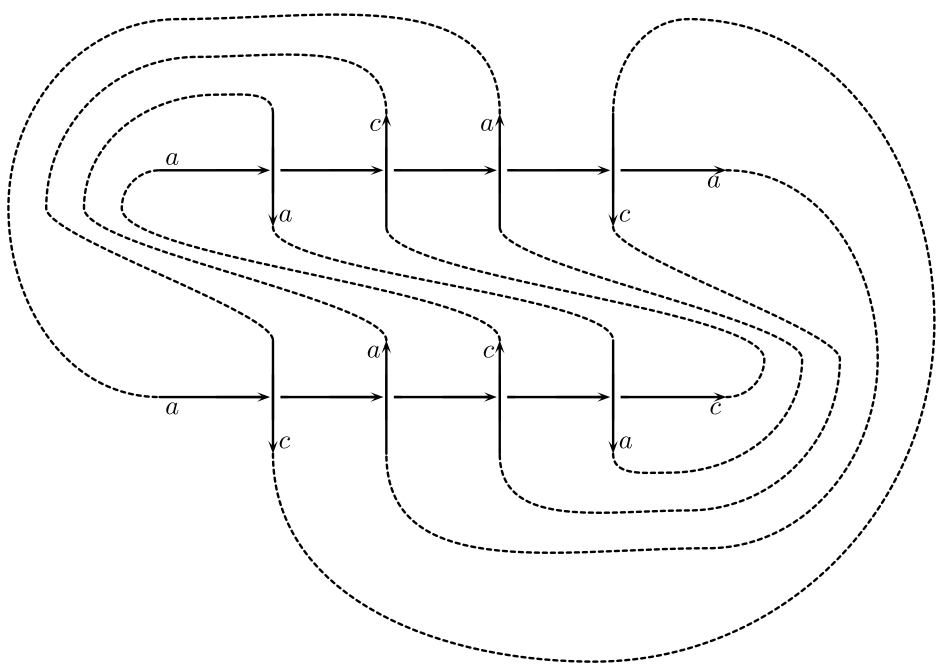
$$\left.\begin{aligned} a &= d\lceil\overline{b} \\ b &= a\,\overline{c}\rceil \\ c &= b\lceil\overline{d} \\ d &= c\,\overline{a}\rceil \end{aligned}\right\} \qquad \left.\begin{aligned} a &= c\,\overline{a}\rceil\lceil\overline{b} \\ b &= a\,\overline{c}\rceil \\ c &= b\lceil\overline{c\,\overline{a}\rceil} = b\lceil\overline{a}\lceil\overline{c}\,\overline{a}\rceil \end{aligned}\right\}$$

$$\begin{aligned} a &= c\,\overline{a}\rceil\lceil\overline{a\,\overline{c}\rceil} = c\,\overline{a}\rceil\lceil\overline{c}\lceil\overline{a}\,\overline{c}\rceil \\ c &= \overline{c}\rceil\lceil\overline{a}\lceil\overline{c}\,\overline{a}\rceil \end{aligned}$$

So

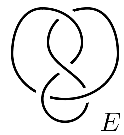
$$\begin{aligned} a &= c\,\overline{a}\rceil\lceil\overline{c}\lceil\overline{a}\,\overline{c}\rceil \\ c &= \overline{c}\rceil\lceil\overline{a}\lceil\overline{c}\,\overline{a}\rceil \end{aligned}$$

Try reconstructing from here:



Even in these cases of simple repeated substitutions to contracted self-referential equations, the reconstruction is not obvious!

Exercises. 1) Show



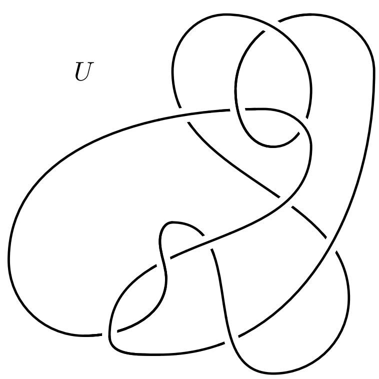
E

is achival, i. e. $E \simeq E^*$.

2) Show π () has presentation $(a, b \mid aba = bab)$. Show there is a non-trivial representation $\rho\colon \pi$ () $\xrightarrow{\text{onto}} S_3 =$ symmetric group on 3 letters. How is this related to our 3-coloring? m.b. ne $\xrightarrow{onto}$, a $\twoheadrightarrow$?

3) Generalize the coloring and representation part of 2) via the IQ operation $ab = 2b - a$.

4) U is unknotted, but no simplifying (making fewer crossings) R-moves are available (and no $R3$ either). Hence U must be made more complex before becoming simple (via R-moves).



7. Alexander module

This is a special sort of crystal representation.

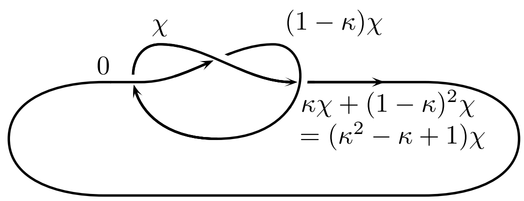
$$\left.\begin{aligned} a\,\overline{b|} &= ta + (1-t)b \\ a|\overline{b} &= t^{-1}a + (1-t^{-1})b \end{aligned}\right\} a, b \in C \quad C = \text{module over}\mathbb{Z}[t, t^{-1}].$$

Exercise. (a) Check that these operations satisfy the axioms for a reduced crystal.

(b) If

$$\begin{aligned} a\,\overline{b|} &= ra + sb \\ a|\overline{b} &= pa + qb \end{aligned}$$

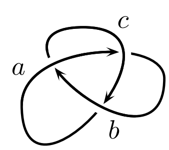
for fixed r, s, p, q, find out for what r, s, p, q these operations will satisfy the crystal axioms.



$\Rightarrow (t^2 - t + 1)x = \emptyset$

Coloring in the Alexander module, we generalize the modulus (determinant) to the determinant of the corresponding matrix of relations over $\mathbb{Z}[t, t^{-1}]$. This gives the *Alexander polynomial.* It is determined up to $\pm t^n$ $n \in \mathbb{Z}$. We write $X \doteq Y$ to mean $X = \pm t^n Y$.

Exercise.

$K :$ 

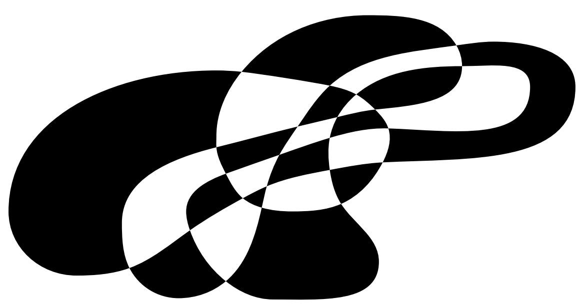
$$a = c\overline{b|} = tc + (1-t)b$$
$$b = a\overline{c|} = ta + (1-t)c$$
$$c = b\overline{a|} = tb + (1-t)a$$

	a	b	c
	-1	$(1-t)$	t
	t	-1	$(1-t)$
	$(1-t)$	t	-1

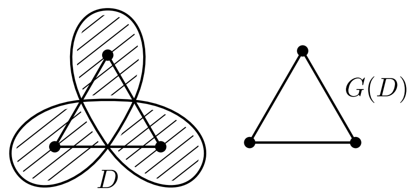
$\Delta_K(t) \doteq t^2 - t + 1$ *Alexander Polynomial.*

The *Alexander polynomial* $\Delta_K(t)$ has the property that $\Delta_K(t)x = 0$ for all x in the Alexander module (= module over $\mathbb{Z}[t, t^{-1}]$ with relations given as above). (This module is isomorphic with the homology module of the infinite cyclic covering of the link complement over $\mathbb{Z}[t, t^{-1}]$ where $t\colon X_\infty \to X_\infty$ is the covering transformation in the infinite cyclic cover.)

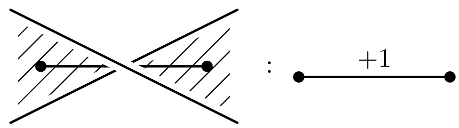
Exercise. Show that every planar 4-valent graph (link projection graph or "universe"), regarded as a map, can be colored with two colors so that each pair of regions sharing an edge are colored differently.

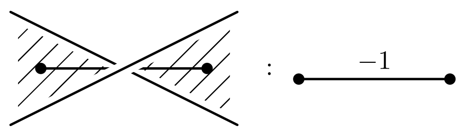


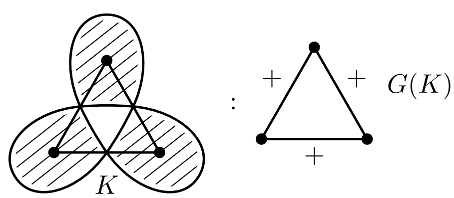
Call this the "checkerboard" shading of the diagram, with the unbounded region colored white. Colors white + black. Make a graph $G(D)$ associated with the universe D by taking one vertex of D for each shaded region and one edge for each crossing on the boundary of that region:



For a knot or link diagram, associate signs ± 1 to each edge of $G(D)$:

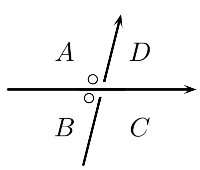




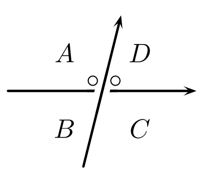


In this way, a *signed graph* $G(D)$ is associated to each link diagram D. Work out a complete translation of the Reidemeister moves to this category of signed graphs.

Exercise.

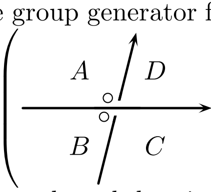


Associate a relation $AB^{-1}CD^{-1} = 1$



Associate a relation $DA^{-1}BC^{-1} = 1$

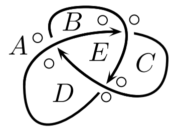
For a given link diagram K put down one group generator for each *region* and one relation for each crossing as shown above. (



$AB^{-1}CD^{-1} = 1$)

Set $A_0 = 1$ where $A =$ region label for the *unbounded* region in the plane.

Call $G(K)$ the resulting group.

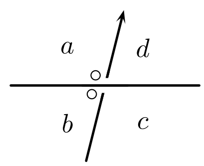


$$\left\{\begin{array}{l} A\overline{D}E\overline{B} = 1 \\ A\overline{B}E\overline{C} = 1 \\ D\overline{A}C\overline{E} = 1 \\ A = 1 \end{array}\right\} \leftrightarrow \begin{array}{l} \overline{D}E\overline{B} = 1 \\ \overline{B}E\overline{C} = 1 \\ DC\overline{E} = 1 \end{array}$$

(a) Show that $K \simeq K' \;\Rightarrow\; G(K) \cong G(K')$.

(b) $G(K)$ is the so-called *Dehn presentation* of $\pi_1(S^3 - K)$. Think about this by using a base point above the plane and letting A correspond to a loop that drops down through A and returns up trough the unbounded region.

(c) Show, using the Dehn presentation, that $Ab[G,G]$ (the abelianization[14] of the commutator subgroup of G) has an additive presentation:



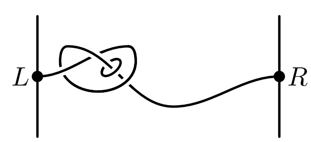
$$xa - xb + c - d = 0$$

a here $a \mapsto xa$ corresponds to the action $A \mapsto XAX^{-1}$ where X is a fixed element of linking number 1 with K.

(d) Use the relations in (c) to get a polynomial $\Delta_K(x)$ by taking an appropriate determinant. (See J.W. Alexander. Trans. Amer. Math. Soc. 20 (1923) pp. 275-306).

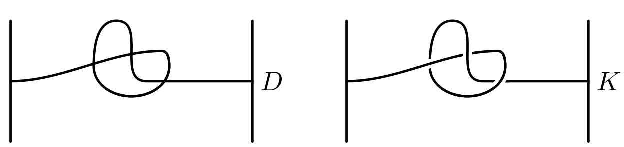
8. (A') A state summation model for the Alexander (Conway) polynomial

For simplicity, I will use $|-|$[15] tangles here.

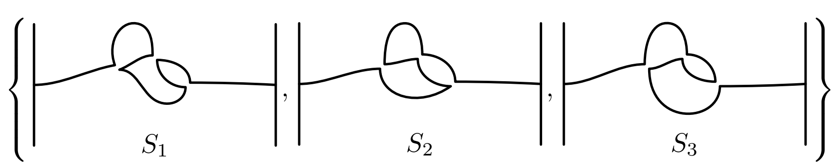


A $|-|$ tangle is a rope(s) tied between two walls. You can move it around, but the end-points are fixed. Extra loop components are possible.

$\{\text{States}\} = \mathcal{S}(K) =$ all "self-avoiding" walks from L to R.



$A(D) =$



---

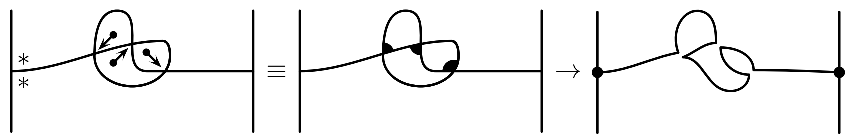
[14] abelianisation??

[15] 1-1?

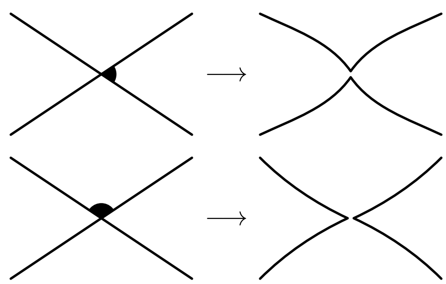
For each $S \in A(K)$, we will define $\langle K|S\rangle$ n such way that

$$\nabla_K = \sum_S \langle K|S\rangle$$

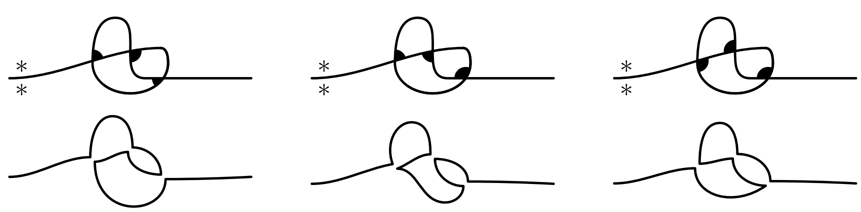
is an invariant of K. ∇_K will turn out to be the Conway version (normalization) of the Alexander polynomial. (See L. Kauffman. Formal Knot Theory, P. U. P. (1983) for all details.)



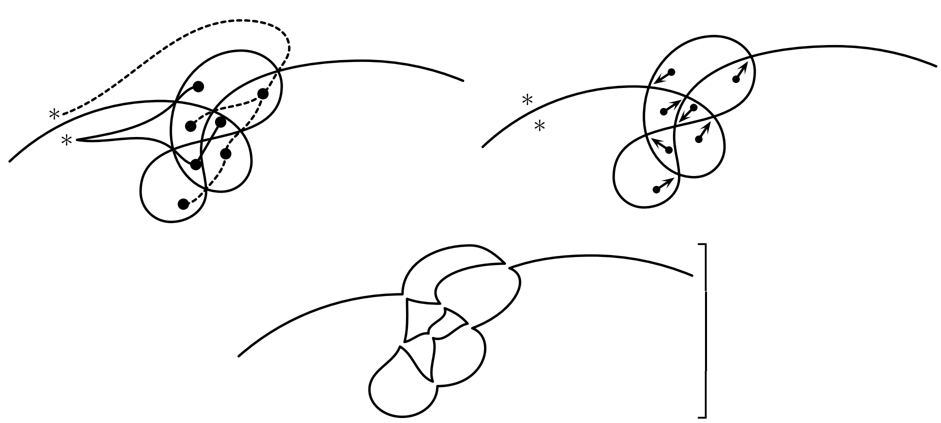
each non-starred region has a pointer that points to *one* vertex. No vertex receives more than one pointer.



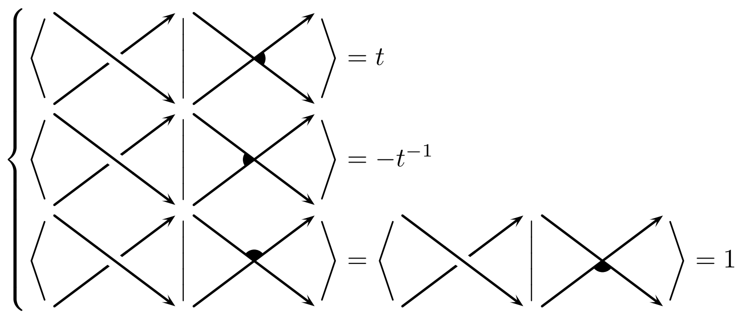
$A(K) \leftrightarrow P(K) =$ Pointer States as described above.

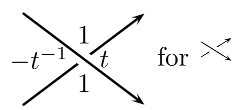


[There states are also in $|-|$ correspondence with maximal trees in the checkerboard graph.

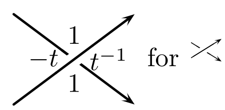


Now the Vertex Weights:

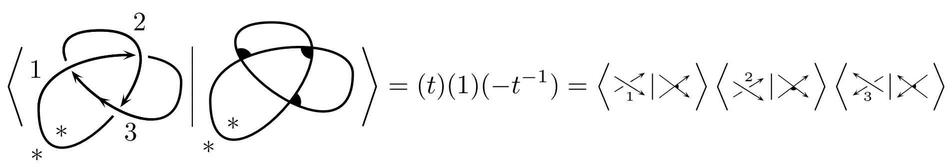




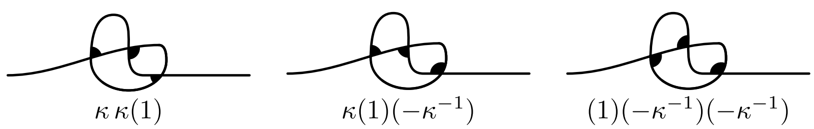
Similarly

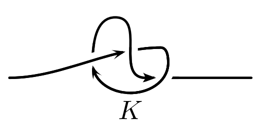


$$\langle K|S\rangle = \prod_{p\in K} \mathrm{wt}(p/s)$$

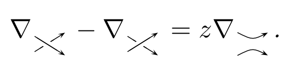


We shall see that ∇_K is a polynomial in $Z = t - t^{-1}$, and $\nabla_K(t) \doteq \Delta_K(t^2)$.

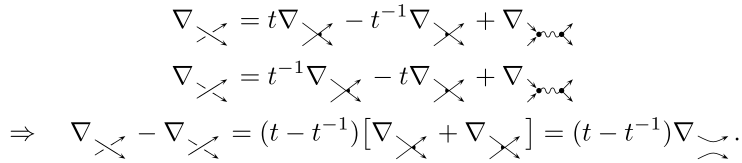




$$\nabla_K = t^2 - 1 + t^{-2} = (t - t^{-1})^2 + 1$$
$$\nabla_K = z^2 + 1$$
$$\nabla_K(\sqrt{t}) = t - 1 + t^{-1} \doteq t^2 - t + 1 = \Delta_K(t).$$

THEOREM. 

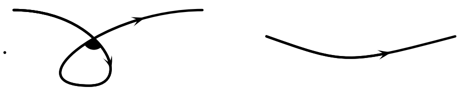
PROOF.



□

THEOREM. $K \sim K' \;\Rightarrow\; \nabla_K = \nabla_{K'}$.

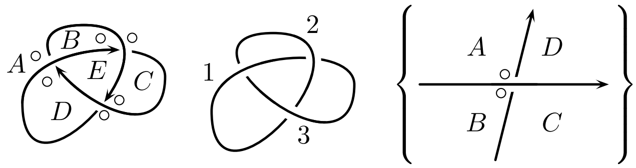
PROOF. Check invariance under Reidemeister moves.

e. g. N. B. 

□

I discovered this model by abstracting the combinatorics inherent in Alexander's original paper on his polynomial.

EXERCISE. This exercise sharpens part (d) of the last exercise on page 17.1. Alexander would compute $\Delta_K(t)$ for the trefoil as follows.



$tA - tB + C - D = 0$

$\mathcal{M} =$

	A	B	C	D	E
1	t	-1	0	$-t$	1
2	t	$-t$	-1	0	1
3	t	0	$-t$	-1	1

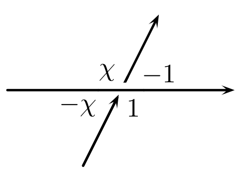
Let M be matrix obtained from $\mathcal{M}$ by striking out *any two adjacent columns.* $\Delta_K(t) \doteq |M|$. E.g.

$$M = \begin{pmatrix} 0 & -t & 1 \\ -1 & 0 & 1 \\ -t & -1 & 1 \end{pmatrix} \quad \text{(strike } A, B\text{)}$$

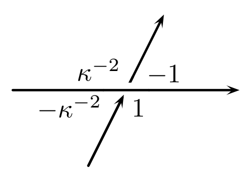
$\Rightarrow$

$$|M| = -t\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ -t & -1 \end{vmatrix} = -t(-1+t) + 1 \doteq t^2 - t + 1.$$

The state model we have discussed in this section is a result of abstracting the combinatorial structure of Alexander's original definition.

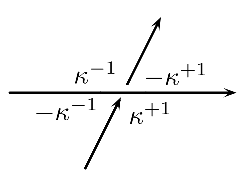


$\rightsquigarrow$



(use variable t^{-2})

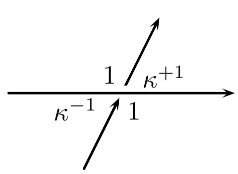
$\rightsquigarrow$



(multiply some rows by t^{+1})

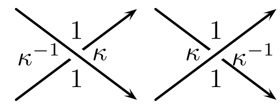
v formule na meste ...... ne poniatno

$\rightsquigarrow$



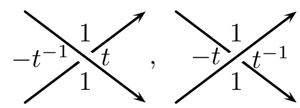
(throw out terms that have a ...... (Some work here))

Result is



for Determinant weights.

The state model sums over the same "states" as the product terms in the determinant and, by a combinatorial miracle, gets the sings

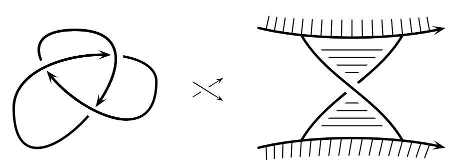


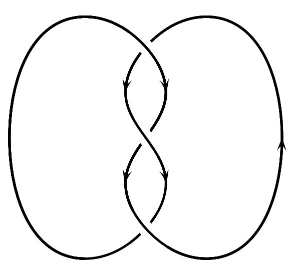
instead of from the determinant! *The above remarks constitute a proof that*

$$\nabla_K(\underbrace{\sqrt{t} - 1/\sqrt{t}}_{z}) \doteq \Delta_K(t).$$

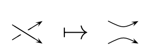
9. (B) Surfaces, S-equivalence and...

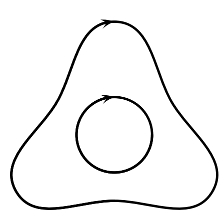
Given an oriented link $K \subset S^3$, $\exists$ an oriented surface F with $\partial F = K$ (∂F denotes the oriented boundary of F): proof via Seifert's algorithm $\mapsto$

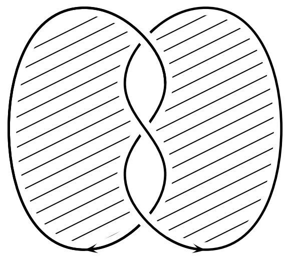




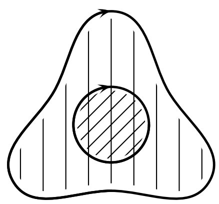
1) From Seifert Circuits via





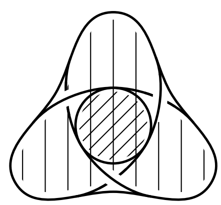


2) Add disk to each Seifert circuit.

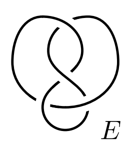


Keeping them disjoint.

3) Add twisted bands as shown above

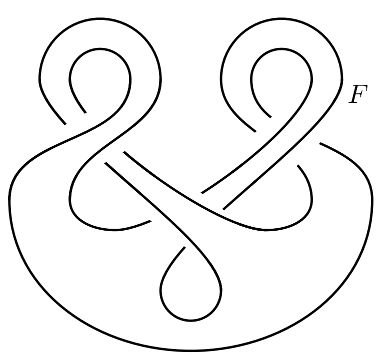


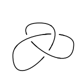
EXERCISE. 1) Apply Seifert's algorithm to obtain an oriented spanning surface for the figure eight knot



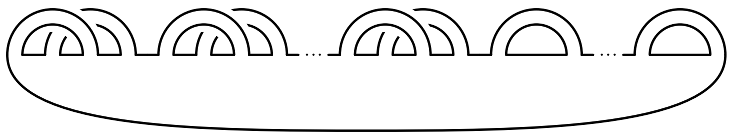
.

2)



This F is a disk with attached bands. Show $\partial F \cong$ .

3) Every surface (orientable) with boundary has the abstract form



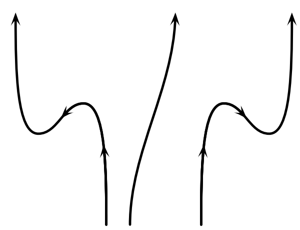
of a disk with attached bands.

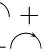
4) Every link $K \subset S^3$ bounds an embedded disk with attached (twisted, knotted, linked) bands.

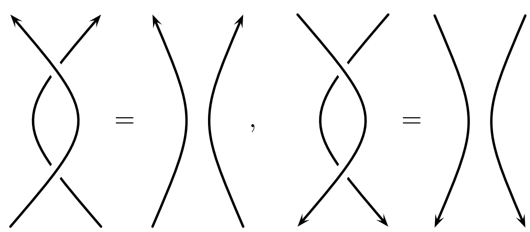
10. Supplement an quantum link invariants

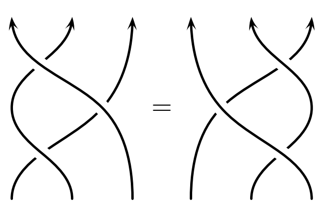
m.b. ne "0.", a "1."?

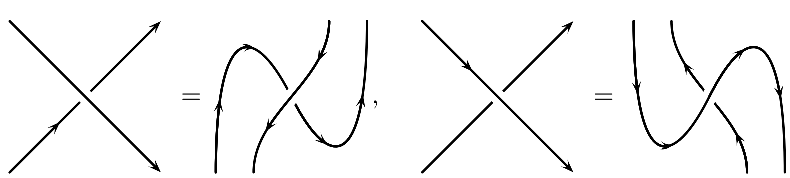
Use oriented tangle category.

0. 

So  +  inverses
+ +  inverses

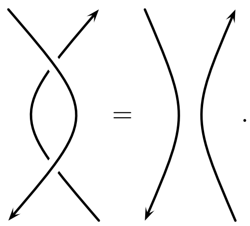
2.  given.

3.  given (up + down).

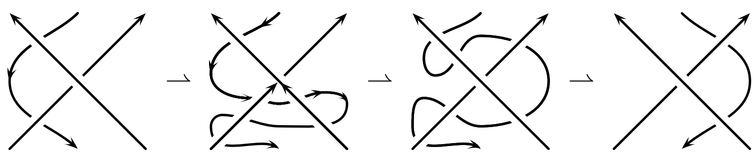
4. 

given in all its oriented forms.

Vo vtoroj kartimke punkta 4 ne narisovano kto iz strelok legit vyshe. Tak pravilno?

Need: 

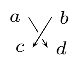
Then other variants of 3. follow.



Now re-write as tensor equations under assumption that

$$\overrightarrow{M}_{ab} = \lambda^{-a/2}\delta_{ab} \qquad \overleftarrow{M}_{ab} = \lambda^{+a/2}\delta_{ab}$$

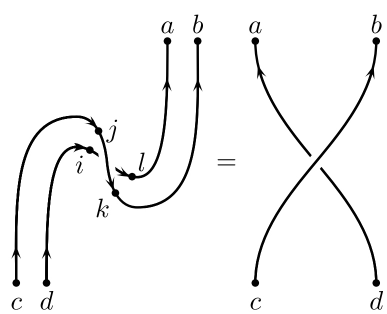
$$\underrightarrow{M}^{ab} = \lambda^{+a/2}\delta^{ab} \qquad \underleftarrow{M}^{ab} = \lambda^{-a/2}\delta^{ab}$$

 $\longleftrightarrow \uparrow R^{ab}_{cd}$ 

$= \downarrow R^{ab}_{cd}$.

LEMMA 1. $\uparrow R^{ab}_{cd} = \lambda^{\frac{-(c+d)+(a+b)}{2}} \downarrow R^{dc}_{ba}$

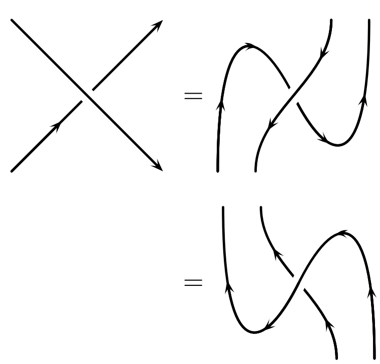
PROOF.



$$\overrightarrow{M}_{cj}\,\overrightarrow{M}_{di} \downarrow R^{ij}_{kl}\, \underrightarrow{M}^{kb}\, \underrightarrow{M}^{la} = \uparrow R^{ab}_{cd}$$
$$\lambda^{-c/2}\lambda^{-d/2} \downarrow R^{dc}_{ba} \lambda^{+b/2}\lambda^{+a/2} = \uparrow R^{ab}_{cd}$$

Thus $\uparrow R^{ab}_{cd} = \lambda^{\frac{-(c+d)+(a+b)}{2}} \downarrow R^{dc}_{ba}$.

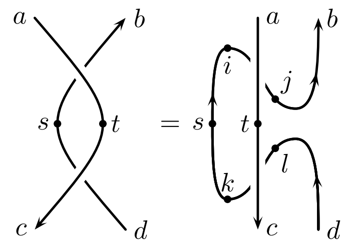
v 3-j kartinke neponiatno kak orientirovan vneshnij put (v originale — v obe! storoni)



yields same condition

$$\uparrow R^{ab}_{cd} = \lambda^{\frac{-(c+d)+(a+b)}{2}} \downarrow R^{dc}_{ba}.$$

□



$$\delta^a_c\delta^b_d = \overset{\frown}{M}_{si}\ \underset{\smile}{M}^{sk}\ \underset{\smile}{M}^{ib}\ \overset{\frown}{M}_{ld} \downarrow R^{ia}_{tj} \downarrow \overline{R}^{tl}_{kc}$$

$$\delta^a_c\delta^b_d = \lambda^{-s/2}\lambda^{-s/2}\lambda^{+b/2}\lambda^{+d/2} \downarrow R^{sa}_{tb} \downarrow \overline{R}^{td}_{sc}$$

$$\boxed{\delta^a_c\delta^b_d = \sum_{s,t} \lambda^{\frac{-s+b}{2}}\lambda^{\frac{-s+d}{2}} \downarrow R^{sa}_{tb} \downarrow \overline{R}^{td}_{sc}}$$

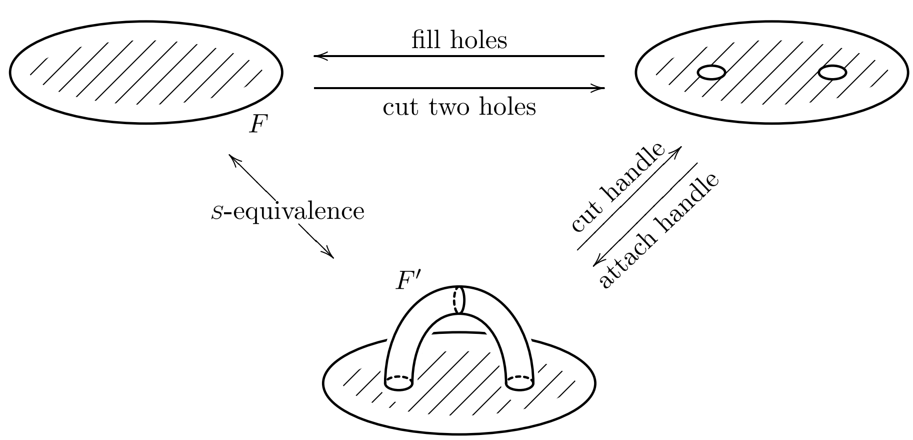
A similar equation holds for $\uparrow R$. We should compare these using the Lemma that relates $\uparrow R$ and $\downarrow R$.

Should also (later) check under what conditions[16] $\uparrow R$ and $\downarrow R$ can both be solutions to YBE.

Note that we would get corresponding equations for $\uparrow R$ but[17] that these will be changed by exactly the given relations between $\uparrow R$ and $\downarrow R$.

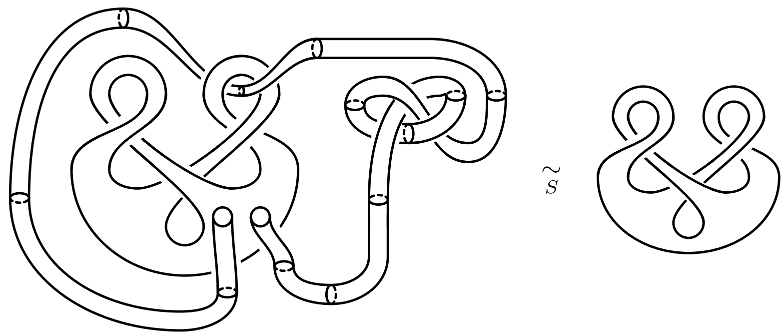
Note also that $\uparrow R$ satisfies[18] YBE $\Leftrightarrow$ $\downarrow R$ satisfies YBE.

Question. Do $\exists$ non-spin preserving solutions to YBE such that we get a link invariant that can detect differences between some knots and their reverses?



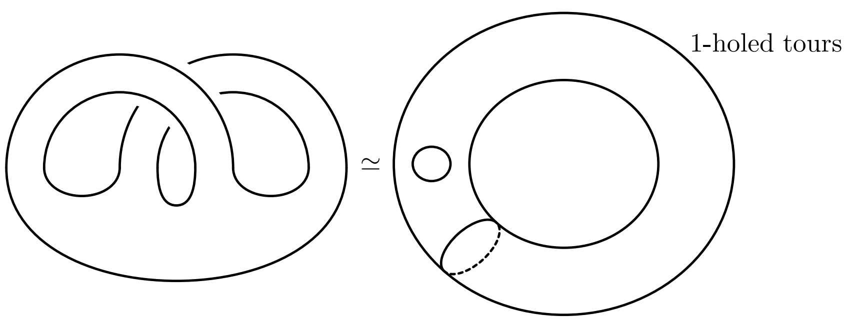
[16] ?
[17] ?
[18] ?

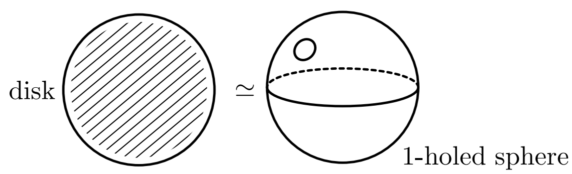
Exercise 1.



Note how the tube is knotted and linked with the surface.

Exercise 2.





Two surfaces are said to be *S-equivalent* if one can be obtained from the other by a combination of ambient isotopy and a finite number of handle additions and/or subtractions.

Note that $F \underset{S}{\sim} F' \Rightarrow \partial F \underset{\text{isotopy}}{\simeq} \partial F'$.

We will prove the

Theorem (S-theorem).[19] *Two surfaces are S-equivalent if and only if their boundaries are isotopic:*

$$F \underset{S}{\sim} F' \Leftrightarrow \partial F \simeq \partial F'.$$

Moral. Use S-equivalence of surfaces to study isotopy of knots and links.

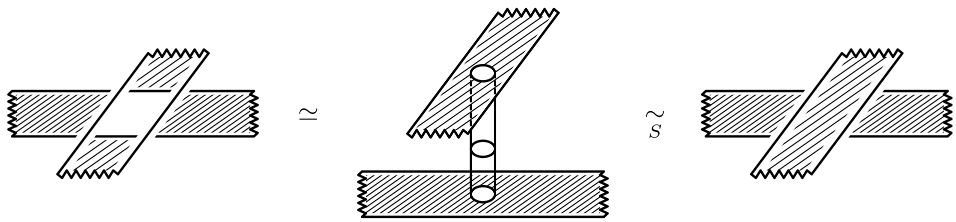
In order to prove the Theorem we need some Lemmas.

Lemma 1. *Any orientable spanning surface for a link K is S-equivalent to a surface obtained via Seifert's algorithm.*

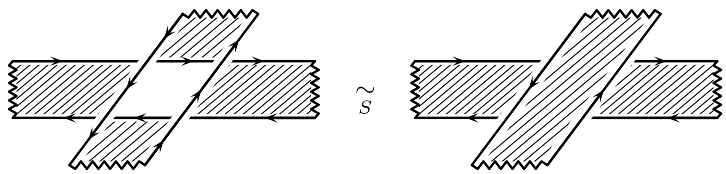
---

[19]The elementary proof given here is due to Lou Kauffman, Dror Bar-Natau, and Jason Fulman.

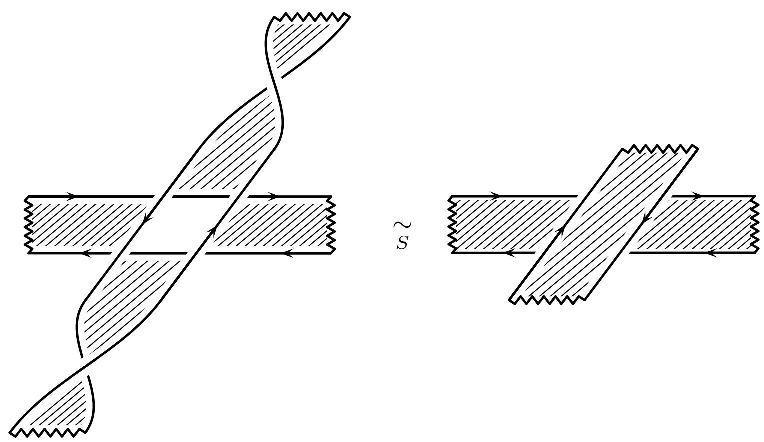
Proof. Note that



Thus



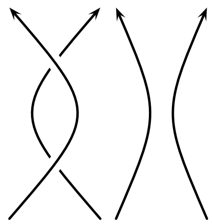
and



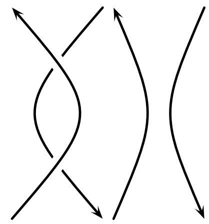
Any disk-with-bands surface can be represented by an embedding so that the bands pass over and under one another as illustrated locally above. The local situations shown above demonstrate that we can add tubes to get a surface obtained via Seifert's algorithm. □

Lemma 2. *If K' is obtained from K by a sequence of Reidemeister moves, then $F(K) \underset{S}{\sim} F(K')$ where $F(K)$ denotes the spanning surface for K obtained via Seifert's algorithm. (If K is a link diagram, we assume that $F(K)$ is connected and that each Reidemeister move preserves connectivity — so that $F(K')$ is also a connected surface.)*

Before proving Lemma 2 it is useful to point out that some of the oriented versions of the Reidemeister moves are consequences of the others. Lets assume that we have both versions of the type II move to work with:

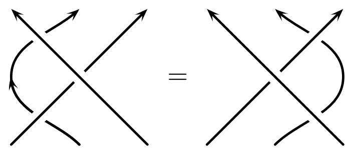


and

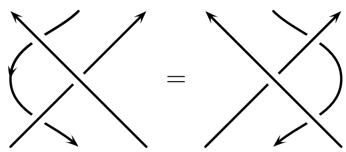


Then we can cut down the number of type III moves needed as shown in the following discussion.

There are two basic types of type III moves:

III$_0$  non-cyclic,

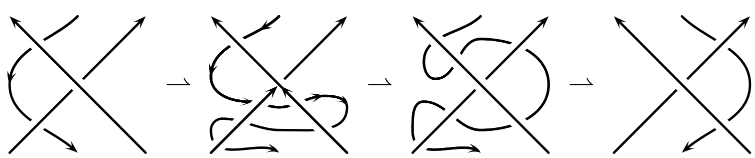
(the inner Δ does *not* make an oriented cycle) and

III$_1$  cyclic,

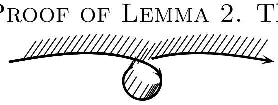
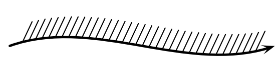
(the inner Δ does make an oriented cycle).

Useful Fact. Cyclic type III moves can be accomplished by combinations of type II moves and non-cyclic type III moves.

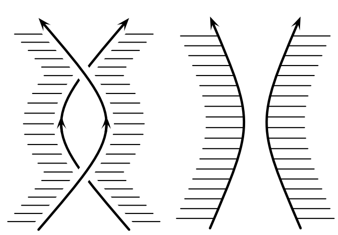
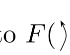
Proof.

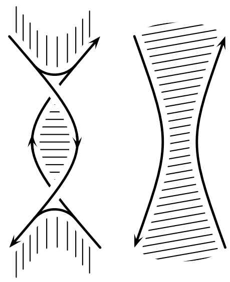


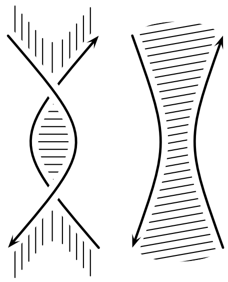
□

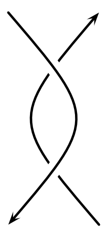
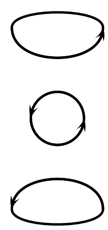
Proof of Lemma 2. There is nothing to check for type I Reidemeister moves, since  $\simeq$ .

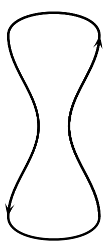
Now look at type II.

Case 1.  $\Rightarrow F($$) =$ tube added to $F($$)$.

Case 2.  These surfaces are isotopic, *but* this is not the only case. Continue.

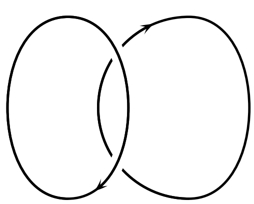
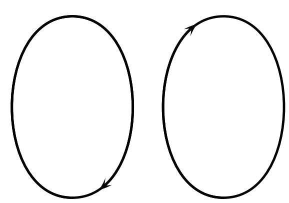
 will happen when the upper and lower Seifert circuits of

 are disjoint, schematically as in  .

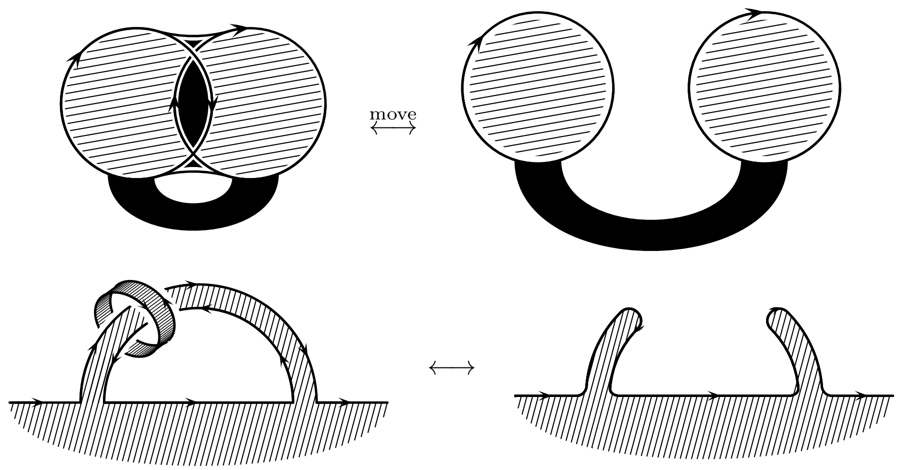
Then  is a single Seifert circuit, filled in as above.

Chto-to neponiatnoe s napravlenijami

If the upper and lower Seifert circuits are not disjoint, then we have

 and 

with no extra structure other than two Seifert circles on the right, this move would disconnect the surface and take us out of category. Hence, the situation has the form

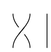
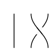
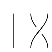


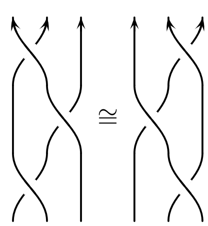
(Black boxes representing rest of the Seifert surface.)

Thus the move between the two surfaces in this case consist in taking the boundary connected sum with a punctured torus (or removing it). This is an S-equivalence!

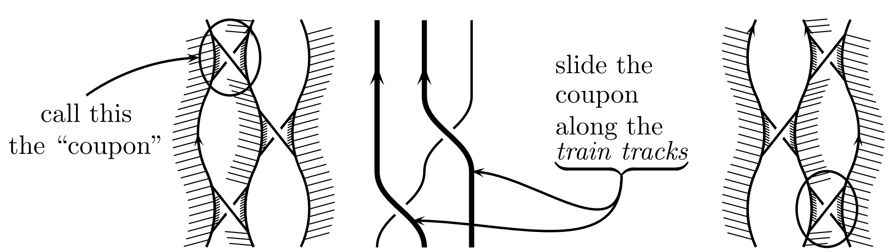
We have now shown that Reidemeister moves of type II (in both orientations) induce S-equivalence on the corresponding Seifert surfaces.

It remains to examine how the Seifert surfaces change under type III moves of non-cyclic type. For this purpose it is useful to use some algebraic terminology from the theory of braids:

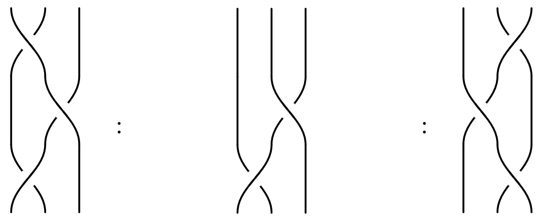
 $= \sigma_1,$  $= \sigma_1^{-1},$  $= \sigma_2,$  $= \sigma_2^{-1}.$

 $: \sigma_1\sigma_2\sigma_1 \cong \sigma_2\sigma_1\sigma_2.$

Look at the Seifert surface of $\sigma_1\sigma_2\sigma_1$:



This shows that $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ have *isotopic* Seifert surfaces. Other cases such as



$\sigma_1\sigma_2\sigma_1^{-1}$ ⟨ twisted train tracks ⟩ $\sigma_2^{-1}\sigma_1\sigma_2$

But

$$\begin{aligned}\sigma_1\sigma_2\sigma_1^{-1} &\cong \sigma_2^{-1}(\sigma_2\sigma_1\sigma_2)\sigma_1^{-1}\\ &\cong \sigma_2^{-1}(\sigma_1\sigma_2\sigma_1)\sigma_1^{-1}\\ &\cong \sigma_2^{-1}\sigma_1\sigma_2.\end{aligned}$$

Since $\sigma_i\sigma_i^{-1} = 1 \leftrightarrow$  $\cong$  this corresponds to isotopies of the corresponding Seifert surfaces. This completes the proof of Lemma 2. □

With Lemmas 1 and 2 now proved, we have completed the proof of the S-*Theorem. Two connected surfaces F and F' are S-equivalent if and only if their boundaries are isotopic.*

11. The Seifert Pairing and Invariants of S-equivalence

Given an oriented surface F embedded in $\mathbb{R}^3$, we define the *Seifert pairing*

$$\theta\colon H_1(F) \times H_1(F) \to \mathbb{Z}$$

where H_1 denotes the first homology group by the formula

$$\theta(a,b) = lk(a*, b)$$

where lk denotes linking number, and a^* is the result of pushing a representative for a off the surface in the direction of the positive normal to the surface. This pairing is an invariant of the ambient isotopy class of the embedding of F in $\mathbb{R}^3$. We can examine how it behaves under S-equivalence.

The following functions of θ are *invariants* of S-equivalence. (See L. Kauffman. On knots. for the proofs.)

(1) $D(K) = \mathrm{Det}(\theta + \theta^T)$ (θ^T denotes the transpose of θ).
This is the determinant of the knot. It is the same (!) as the determinant that we have defined via the quandle/crystal.

(2) $\sigma(K) = \mathrm{Signature}(\theta + \theta^T)$.
This is the *signature* of the knot. Since the matrix $\theta + theta^T$ is symmetric, it has a well-defined *signature* ($= p_+ - p_-$ where $p_+ =$ #positive entries, $p_- =$ #negative entries on diagonal of $P\theta P^T =$ diagonal matrix, P invertible over Q[20].)

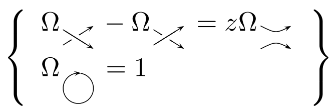
Exercise. Show that $\sigma(K^*) = -\sigma(K)$ when $K*$ is the mirror image of K.

---

[20] ???

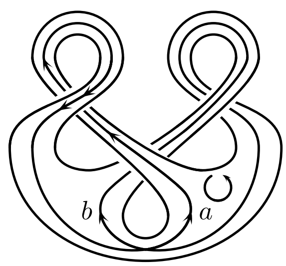
(3) $\Omega(t) = \mathrm{Det}(t^{-1}\theta - t\theta^T)$ (here t is a (Laurent) polynomial variable.)

It is not hard to show that Ω is an invariant of S-equivalence, hence that it is an invariant of $K = \partial F$. $\Omega_K(z) = \Omega(t)$ where $z = t - t^{-1}$. Ω_K is a polynomial in z satisfying



In other words, $\Omega_K(z)$ is the Conway normalization of the Alexander polynomial!

Example.

F: 

θ	a	b
a	-1	0
b	-1	-1

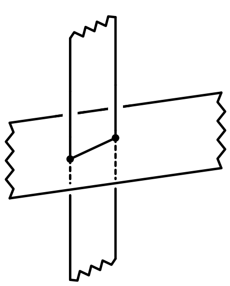
$$t^{-1}\theta - t\theta^T = \begin{bmatrix} -t^{-1} & 0 \\ -t^{-1} & -t^{-1} \end{bmatrix} + \begin{bmatrix} t & t \\ 0 & t \end{bmatrix} = \begin{bmatrix} -t^{-1}+t & t \\ -t^{-1} & t - t^{-1} \end{bmatrix}$$

$$\Omega_K = \mathrm{Det}(t^{-1}\theta - t\theta^T) = (t - t^{-1})^2 + 1 = z^2 + 1.$$

$$\theta + \theta^T = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \underset{r}{\rightharpoonup} \begin{bmatrix} -2 & -1 \\ 0 & -2+1/2 \end{bmatrix} \underset{c}{\rightharpoonup} \begin{bmatrix} -2 & 0 \\ 0 & -3/2 \end{bmatrix}$$

$$\sigma(K) = -2 \Rightarrow K \not\approx K^*.$$

Exercise. A knot is said to be a *ribbon knot* if it bounds a singular disk in $\mathbb{R}^3$ whose singularities are all of the form

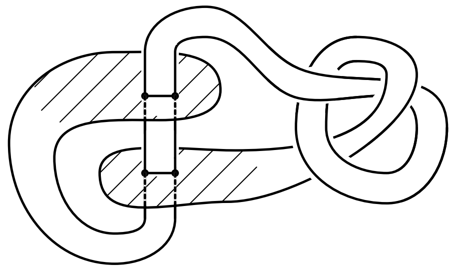


arc between two boundary points of the disk intersects transversely an arc on the interior of the disk.

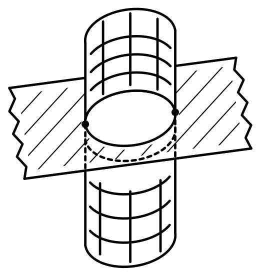
(Such a knot bounds a non-singular disk in *upper 4-space* $= \mathbb{R}^4_+ = \{(x,y,z,t)\colon x,y,z \in \mathbb{R},\ t \in \mathbb{R}$ and $t \geqslant 0\}$. It is a long-standing conjecture that every such knot (i. e.

every *slice knot* = knot bounding non-singular disk in upper 4-space) is ribbon.)

e. g.



You can make spanning surfaces for ribbon knots via

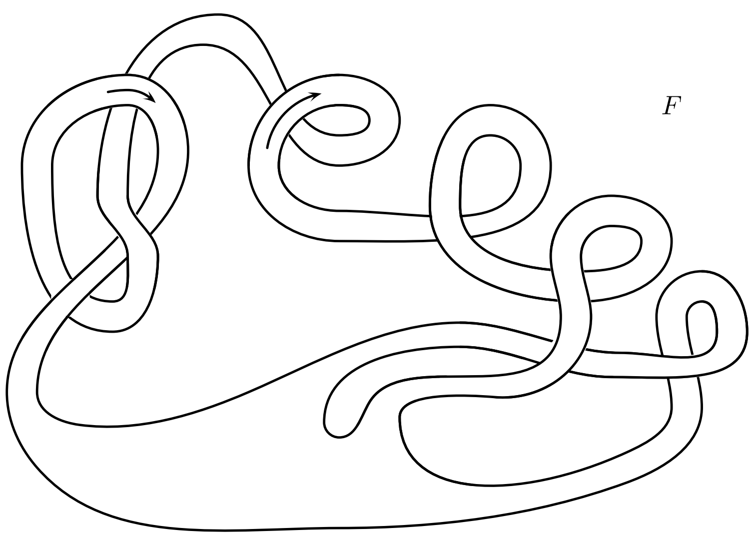


replacing each singular line by a hole and attaching one piece of ribbon to one-half the boundary of the hole and the other piece of ribbon to the other half of boundary of the hole.

Show that K ribbon $\Rightarrow \exists\ f(t)$ a polynomial in t such that

$$\Omega_K(t) = f(t)f(1/t).$$

EXERCISE. $K = \partial F$



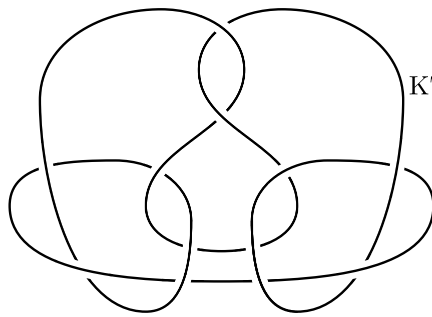
Show that θ has the form $\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$.

Conclude that $\Omega_K(z) = \nabla_K(z) = 1$.

This is an example of a non-trivial knot with Alexander polynomial equal to 1.

We shall need new methods (or more work with $\pi(K)$)) to prove it to be non-trivial.

Exercise. Use the Conway switching identities for ∇_K to show that the knot drawn below has $\nabla_{\mathrm{KT}} = 1$:



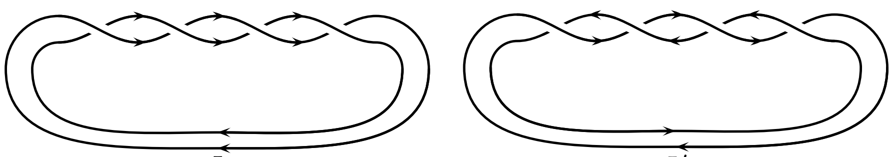
This is the *Kinoshita-Terasaka* knot.

Subexercise: Find a diagram for KT that only 11 crossings.

Exercise. Find a formula for the genus of the Seifert surface in terms of the number of Seifert circuits and the number of diagram regions in the link diagram. (Answer: $g = 1/2(-\mu + R - S)$, $\mu = \#\text{comps}$, $S = \#\text{Seifert circuits}$, $R = \#\text{regions}$.)

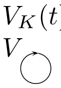
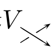
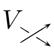
Exercise. Find other knots of Alexander polynomial 1.

Exercise.

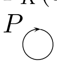
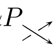
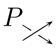


Compare the Alexander (Conway) polynomials of these two links. Generalize your result.

Exercise. After Vaughan Jones discovered (1984) a Laurent polynomial invariant of knots and links satis

- $V_K(t) = V_{K'}(t)$ if $K \simeq K'$
- V $= 1$
- tV $- t^{-1}V$ $= (\sqrt{t} - \frac{1}{\sqrt{t}})V$

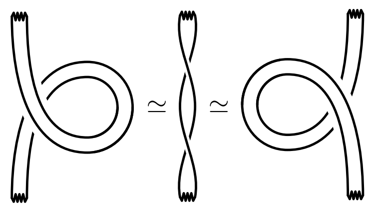
it quickly became apparent to many people (Homflypt) that there was a beautiful 2-variable invariant:

- $P_K(\alpha, z) = P_{K'}(\alpha, z)$ if $K \simeq K'$
- P $= 1$
- αP $- \alpha^{-1}P$ $= zP$

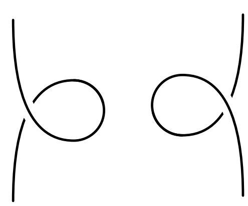
(a) *Believe* that the Jones and Homflypt polynomials are well-defined and do some computations.

(b) Find your own proof that Homflypt polynomial is a well-defined invariant.

EXERCISE. a)



b) a) leads to diagrammatic notion of *ribbon equivalence* obtained by replacing Reidemeister move #1 by 1′:

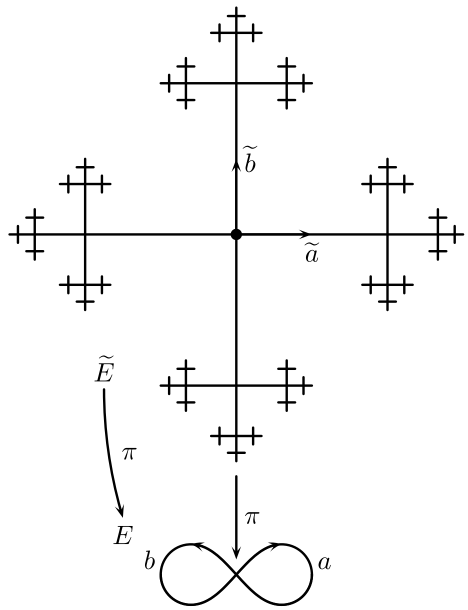


$$\boxed{\text{Ribbon Equivalence}} = \boxed{0., 1'., 2., 3.} \equiv \boxed{\text{Isotopy Classes of Framed Links}}$$

Show that the crystal defined by axioms $\overline{a\rceil}\,\overline{a} = 1$, $\overline{a\,\overline{b\rceil}\rceil} = \lceil\overline{b}\,\overline{a\rceil}\,\overline{b}\rceil$ etc. (but *not* $a\,\overline{a\rceil} = a$) is an invariant of framed links.

EXERCISE. Use the FKT model to show that the highest degree term in the normalized Alexander polynomial of a knot (i. e. $\nabla_K(\sqrt{t} - 1/\sqrt{t})$) is t^g where g =genus of the Seifert spanning surface for the diagram used in the calculation *when* the knot is an alternating knot. Conclude that an alternating diagram K has Seifert surface of minimal genus among Seifert surfaces spanning the diagrams equivalent to K under Reidemeister moves. (See *Formal knot theory* for this result and a generalization of it.)

Exercise (Fox Free Calculus and Fox's Group Theoretic Definition of the Alexander Polynomial). [21]



universal covering of figure eight

In any covering space

$$\boxed{\widetilde{\omega\tau} = \widetilde{\omega} + \omega\widetilde{\tau}}$$

where $\widetilde{\omega}$ denotes the lift of a loop in the base to a path emanating from base point in the total space.

$G = \pi_1(E) = (a, b \mid\ )$ free group on generators a,b.

$\omega \in \pi_1(E) \ \Rightarrow\ \widetilde{\omega} = A\widetilde{a} + B\widetilde{b}$, $A, B \in \mathbb{Z}[G]$, the group ring of G.

Definition. $\frac{\partial\omega}{\partial a} = A$, $\frac{\partial\omega}{\partial b} = B$.

$$D\colon \partial_a = \partial/\partial a, \quad \partial_b = \partial/\partial b$$

$$\boxed{D(\omega\tau) = D(\omega) + \omega D(\tau)}$$

Fox derivation.

If $D = \partial/\partial a$

$$\begin{aligned}
&\Rightarrow D(a) = 1\\
&\quad D(a^2) = D(a) + aD(a) = 1 + a\\
&\quad D(a^3) = D(a) + aD(a^2) = 1 + a(1+a) = 1 + a + a^2\\
&\qquad \dots\\
&\quad D(a^n) = 1 + a + \ldots + a^{n-1} = [n]_a
\end{aligned}$$

$$\begin{aligned}
a^{-n}a^n = 1 &\Rightarrow D(a^{-n}) + a^{-n}D(a^n) = 0\\
&\Rightarrow D(a^{-n}) = -a^{-n}(1 + a + \ldots + a^{n-1})\\
&\quad\ D(a^{-n}) = -(a^{-n} + a^{-n+1} + \ldots + a^{-1}).
\end{aligned}$$

[21] This exercise is written in cryptic style. See "A quick trip trough knot theory" by R. Fox in *Topology of 3-manifolds* edit by M. K. Fort Jr. Prentice—Hall (1962) pp. 120–167.

DEFINITION. $G = (g_1, \ldots, g_n \mid r_1, \ldots, r_m)$ is a group with n generators and m relations, define the Jacobian matrix

$$J = (\partial r_i / \partial g_j).$$

If $G = \pi_1(S^3 - K)$, K a knot then have

$$\phi\colon G \to (t :) \cong \mathbb{Z}, \qquad \phi(x) = t^{lk(x,K)}.$$

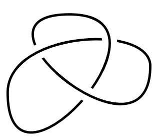
Let $J^\phi = (\phi(\partial r_i / \partial g_j))$. Then

$\Delta_K(t) \doteq$ generators of the ideal in $\mathbb{Z}[t, t^{-1}]$generated by $(m-1) \times (n-1)$ minors in J^ϕ.

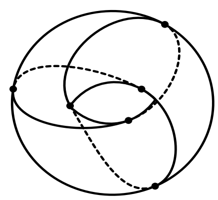
(Fox's definition of the Alexander polynomial)

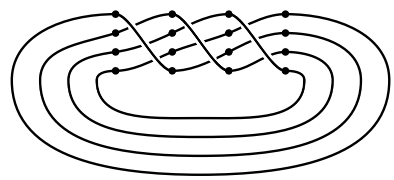
This definition ....[22] the polynomial from *any* representation of the fundamental group.

EXAMPLE (Torus Knots $K(p,q)$). (p,q) relatively prime.

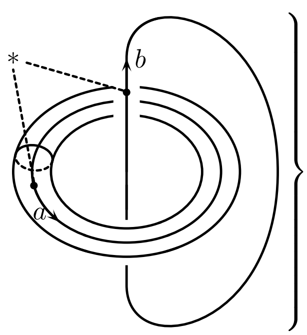


$K(2,3)$





$K(3,4)$



$\pi(K(p,q)) = (a, b\colon a^p = b^q), \quad r = a^p - b^q$

$$\psi(a) = t^q, \quad \psi(b) = t^p$$

$$J = (\partial_a r, \partial_b r) = (1 + a + \ldots + a^{p-1}, -(1 + b + \ldots + b^{q-1}))$$
$$J = \left(\frac{a^p - 1}{a - 1}, -\frac{b^p - 1}{b - 1} \right)$$
$$J^\phi = \left(\frac{t^{pq} - 1}{t^q - 1}, -\frac{t^{pq} - 1}{t^p - 1} \right)$$

$\Rightarrow$

$$\boxed{\Delta_{K(p,q)} \doteq \gcd\left(\frac{t^{pq} - 1}{t^q - 1}, \frac{t^{pq} - 1}{t^p - 1} \right)}$$

EXERCISE. Check it out for $p = 2$, $q = 3$.

[22]???

Now note: $c = bab^{-1}$, $r = c - bab^{-1}$

$$\frac{\partial r}{\partial c} = 1, \quad \frac{\partial r}{\partial b} = -1 - ba(-b^{-1}) = -1 + bab^{-1} \quad \frac{\partial r}{\partial a} = -b$$

If $a\,\overline{b|} = bab^{-1} = c$

$$\frac{\partial r}{\partial c}[c] + \frac{\partial r}{\partial b}[b] + \frac{\partial r}{\partial a}[c] = 0$$

expresses corresponding linear relation.

$$[c] + (-1 + bab^{-1})[b] - b[a] = 0$$
$$[c] = +b[a] + (1 - bab^{-1})[b]$$
$$\downarrow \psi\colon a, b \mapsto t$$
$$[c] = t[a] + (1 - t)[b].$$

Moral. In the Wirtinger presentation, or in using the crystal, Fox's definition coincides with our "modular" definition of $\Delta_K(t)$.

Exercise (Dehornoy and Laver's Magma). A *magma* is an algebraic system with *one* binary operation $*$ and *one* axiom:

$$a * (b * c) = (a * b) * (a * c).$$

(The operation is left distributive over itself.)

Consider the magma $\mathcal{M}(a)$: free magma on one generator a.

$$\begin{aligned}
&a\\
&a * a\\
&a * (a * a) = (a * a) * (a * a)\\
&\quad = ((a * a) * a) * ((a * a) * a)\\
&\quad = (((a * a) * a) * (a * a)) * (((a * a) * a) * a)
\end{aligned}$$

at this point it branches into two possible distributions

What is the structure of $\mathcal{M}(a)$?

Dehornoy and Laver conjectured and then proved that $\mathcal{M}(a)$ is totally ordered by *left truncation*.

$$X * Y : X \text{ is a left truncate of } Y.$$

In order to prove this, they first regarded $X * Y = X[Y]$ (operator notation $\sim X\,\overline{Y|}$) and wrote

$$X * (Y * Z) = (X * Y) * (X * Z)$$
$$\updownarrow$$
$$X[Y[Z]] = X[Y][X[Z]]$$

Interpret.: $X, Y, Z, \ldots \in \mathit{Set}\,\mathcal{A}$

$$X \in \mathcal{A} \mapsto X : \mathcal{A} \to \mathcal{A}$$

($X[Y]$ denotes application of X to Y) and

$$\text{each } \tau\colon \mathcal{A}^{\mathcal{A}} \to \mathcal{A}^{\mathcal{A}} \quad (\tau \in \mathcal{A})$$
$$\text{s. t. } \tau[X[Y]] = \tau[X][\tau[Y]].$$

Then they looked for set-theoretic models and seemed to need axioms asserting existence of inaccessible cardinals to make free models of $\mathcal{M}(a)$. But Dehornoy found a way to embed $\mathcal{M}(a) \subset B_\infty$, the Artin braid group!

$$\begin{aligned}
&\beta \colon \mathcal{M}(a) \hookrightarrow B_\infty \\
&\beta(a) = 1 \\
&\beta(X[Y]) = \beta(X) s(\beta(Y)) \sigma_1 s(\beta(X)^{-1})
\end{aligned}$$

(Dehornoy's inductive definition.)

$s(\text{braid})$ = braid resulting from shifting all of its strands to the right by one strand.

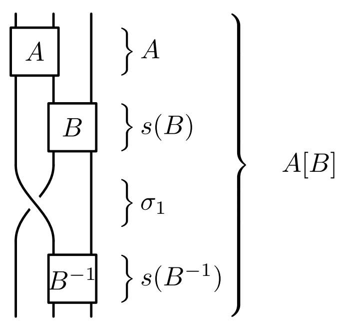
$(\text{braid})^{-1}$ = inverse braid (turn upside down and flip all crossings)

Exercise. Show that if we define the operation

$$A\boxed{B} = As(B)\sigma_1 s(B^{-1})$$

for $A, B \in B_\infty$. Then

$$A\boxed{B\boxed{C}} = A\boxed{B}\boxed{A\boxed{C}}$$

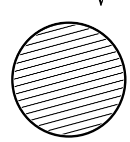
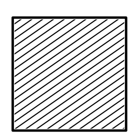


12. (C) Three Manifolds, Surgery and Kirby Calculus

∂ =boundary

$$S^3 = \text{3-dimensional sphere} = \partial(D^4)$$

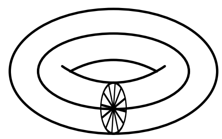
$$D^4 = \text{4-ball} = \{\overrightarrow{x} \in \mathbb{R}^4 \mid \|\overrightarrow{x}\| = \sqrt{x_1^2 + \ldots + x_4^2} \leqslant 1\}.$$

$D^4 \cong D^2 \times D^2$ just as  $\cong$ .

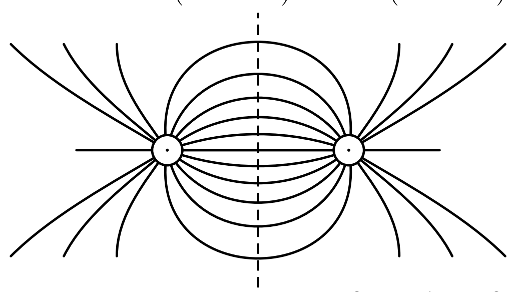
Therefore

$$\begin{aligned}
S^3 &= \partial D^4 \\
&\cong \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2 \\
&= (S^1 \times D^2) \cup (D^2 \times S^1). \\
S^3 &\cong (S^1 \times D^2) \cup (D^2 \times S^1)
\end{aligned}$$

The three-sphere is the union of two solid tori.

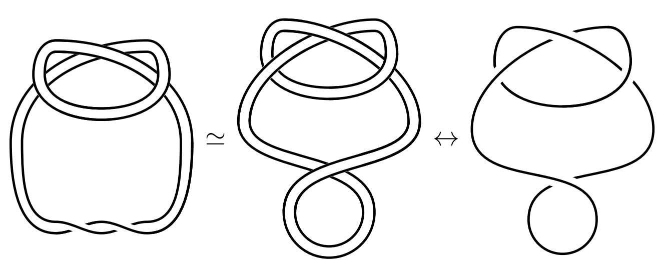
 , $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Look at $\mathbb{R}^3 - (S^1 \times D^2) \subset S^3 - (S^1 \times D^2)$



Spin it about the dotted axis to get $\{S^3 - (S^1 \times D^2)\} - \{\infty\}$,
and see that $S^3 - (S^1 \times D^2) \simeq \mathring{D}^2 \times S^1$.

Framed Links up to ambient isotopy	$\leftrightarrow$	Link Diagrams up to ribbon equivalence



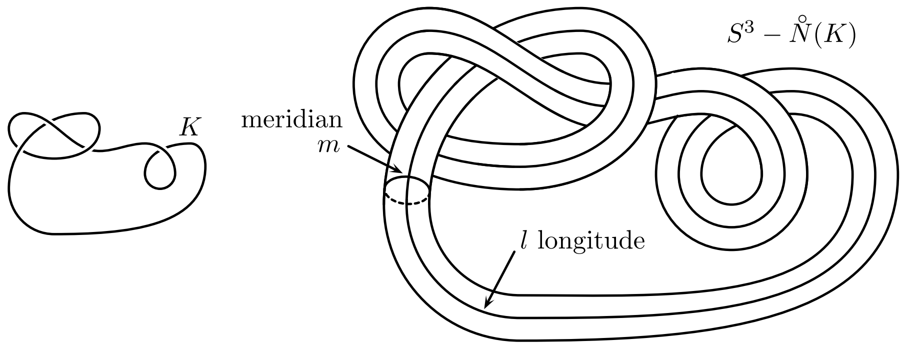
[For Kirby Calculus, see

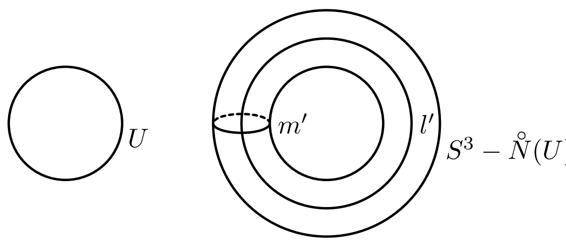
1. R. Kirby. A calculus for framed links in S^3. Invent. Math. 45, pp. 36–56 (1978).

2. R. A. Fenn and C. P. Rourke. On Kirby's Calculus of Links. Topology 18, pp. 1–15 (1979).

3. E. César de Sá. A Link Calculus for 4-Manifolds. (1977) Lecture Notes #722 Springer–Verlag. pp. 16–30.]

13. Surgery on a Blackboard Framed Link





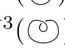
$$M^3(K) \overset{\text{def}}{=} [S^3 - \mathring{N}(K)] \cup_\partial [S^3 - \mathring{N}(U)]$$

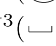
$$m \leftrightarrow m', \quad l \leftrightarrow l'.$$

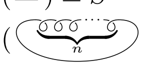
[Note: $S^3 - \mathring{N}(U) \cong S^1 \times D^2$]

EXAMPLES.

$M^3($$) \cong S^2 \times S^1$

$M^3($$) \cong M^3($$) \cong S^3$

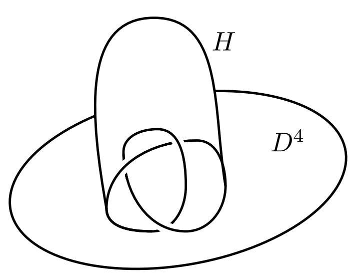
$M^3($$) \cong S^3$ no surgery at all

$M^3($$) \cong L(n,1)$ Lens space

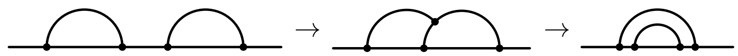
ALTERNATE DEFINITION. Framed link L gives $\alpha\colon S^1 \times D^2 \to S^3$.

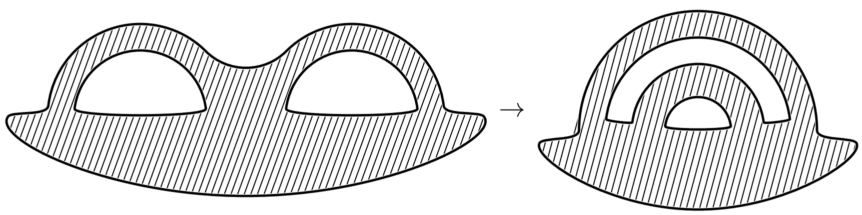
$$M^3(K) = \partial[D^4 \cup_\alpha (D^2 \times D^2)]\colon \partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1).$$

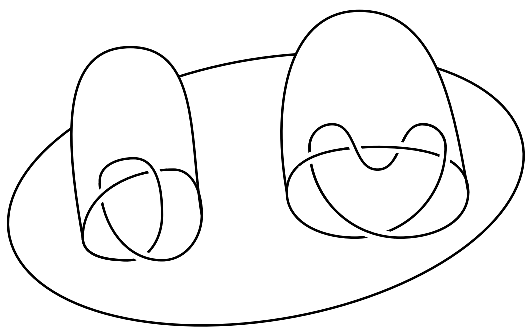
handle H attached to ∂D^4.



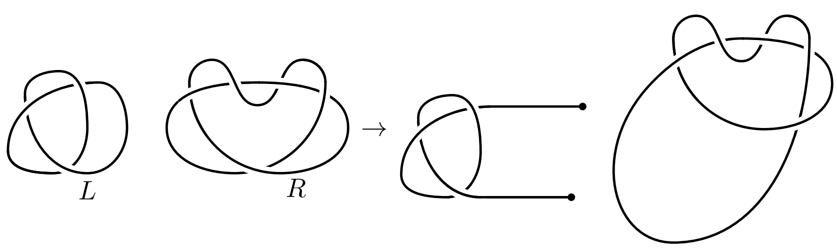
Handle Sliding: 4D analog of



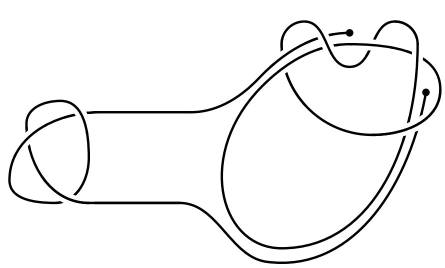




On the *base(s)* of the handles, the result is:

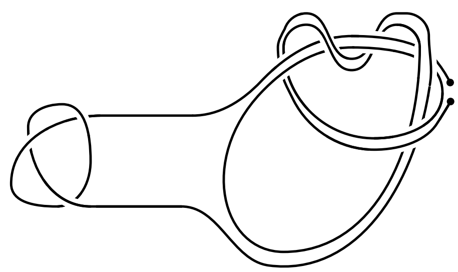


(part of left has slide up on R's handle.)

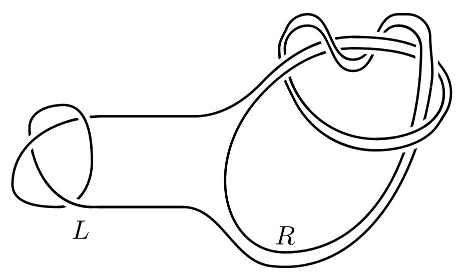


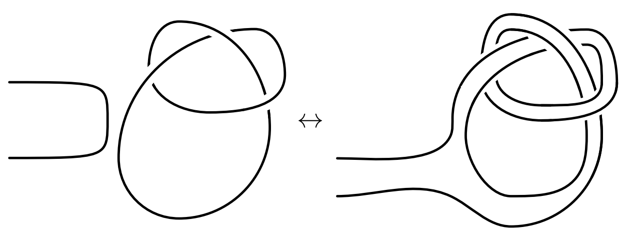
(")

The end result of the handle-slide is that L is replaced by $L' = L\#\widetilde{R}$ where $\widetilde{R}$ is a parallel copy of R (parallelism[23] defined by the framing).

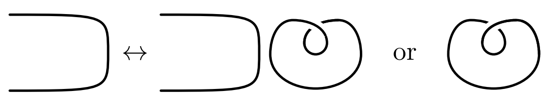


Handle sliding preserves the topological type of the boundary of the 4-manifold





$$K \leftrightarrow K' \Rightarrow M^3(K) \cong M^3(K').$$



$$K \leftrightarrow K' \Rightarrow M^3(K) \cong M^3(K').$$

Two link diagrams in the ribbon category (links up to ribbon equivalence) are said to be *Kirby equivalint* ($\underset{K}{\sim}$) if one can be obtained from the other by a finite combination of

- ribbon equivalence
- HS
- bu or bd

THEOREM (R. Kirby). $M^3(K) \cong M^3(K') \;\Leftrightarrow\; K \underset{K}{\sim} K'$.

PROOF. See Kirby's paper. □

[23]???

Working with link diagrams via ribbon equivalence and HS and bu, bd is called *Kirby Calculus*.

Note that it is corollary of the Kirby Theorem that

$$M^3(K) \cong S^3 \Leftrightarrow K \underset{K}{\sim} \varnothing.$$

Is it possible to recognize whether a link K is Kirby equivalent to nothing? The famous Poincaré Conjecture becomes a conjecture about exactly this:

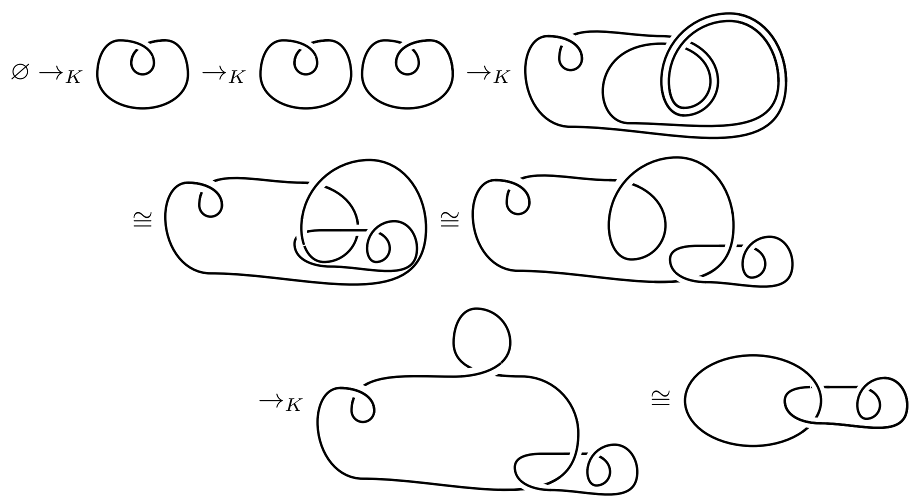
Poincaré conjecture. $M^3(K) \cong S^3 \Leftrightarrow \pi_1(M^3(K)) = \{1\}$.

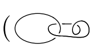
Exercise. $\pi_1(M^3(K)) \cong \pi_1(S^3 - K)/\langle\lambda(K)\rangle$ where $\langle\lambda(K)\rangle$ denotes the normal subgroup generated by the longitude of K.

Thus the *Kirby Calculus Version of the Poincaré Conjecture* is

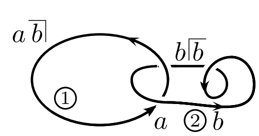
$$\pi_1(S^3 - K)/\langle\lambda(K)\rangle = \{1\} \Leftrightarrow K \underset{K}{\sim} \varnothing.$$

Example.



Thus $M($$) \cong S^3$.

Lets check $\pi_1(L) = \pi_1(S^3 - L)/\langle\lambda_1, \lambda_2\rangle$.



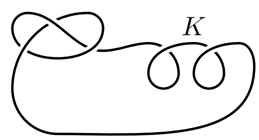
relations are

$$a = a\,\overline{b}|, \quad b = b|\overline{b}\,\overline{a}| = b\,\overline{a}| = \beta$$
$$\lambda_1 = \overline{b}|, \quad \lambda_2 = |\overline{b}\,\overline{a}| = \beta^{-1}\alpha$$

(Letting $\alpha = \overline{a}|$, $\beta = \overline{b}|$)

$$\frac{\pi_1(S^3 - L)}{\langle\lambda_1, \lambda_2\rangle} = \frac{(\alpha, \beta \mid \alpha = \beta^{-1}\alpha\beta,\ \beta = \alpha^{-1}\beta\alpha)}{\langle\beta, \beta^{-1}\alpha\rangle} = \{1\}.$$

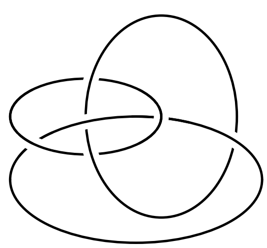
EXAMPLE.



CLAIM. $\pi_1(M^3(K)) = (a, b, c \mid a^5 = b^3 = c^2 = abc)$.

This 3-manifold is the Poincaré Dodecahedral Space, obtained from a dodecahedron by identifying opposite sides with a $2\pi/5$ twist.

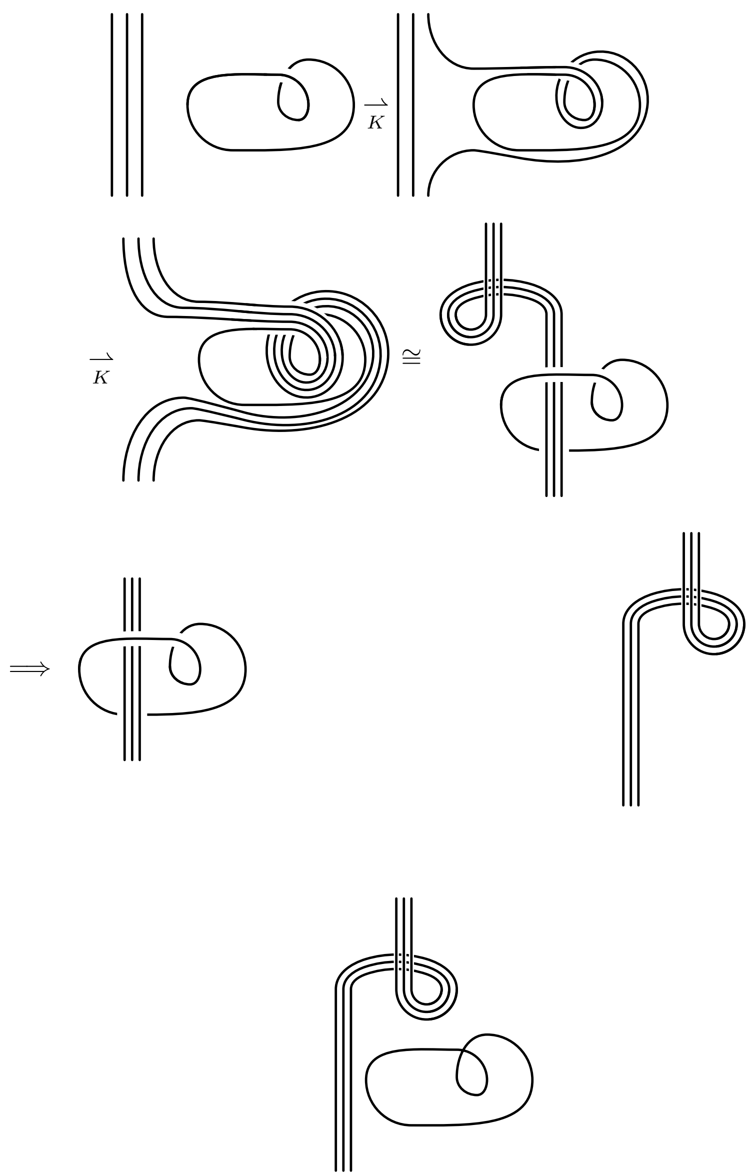
This π_1 is a finite group — with 120 elements — and $Abel(\pi_1) = H_1(M^3(K))$ is trivial.

EXAMPLE.  Borrommean Rings

EXERCISE. Show that $\pi_1(M^3(B)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by algebraic calculation.

EXERCISE. Prove that $M^3(B) \cong S^1 \oplus S^1 \oplus S^1$. calculation.

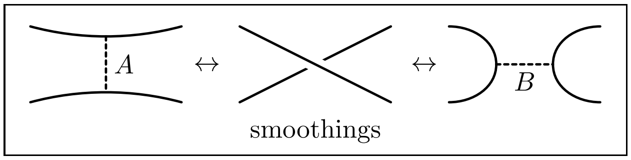
Example. Rourke and Fenn showed that candle sliding could be replaced by the “Rourke—Fenn” move in Kirby calculus axioms.

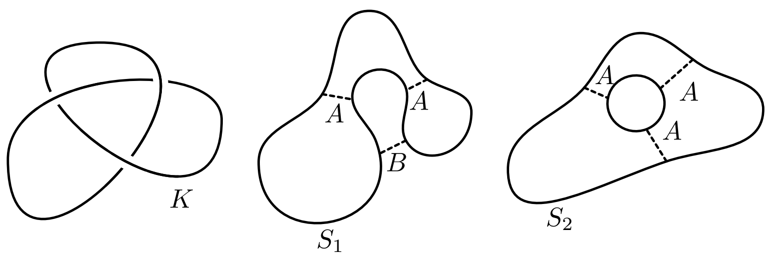


Rourke—Fenn move

14. D. Bracket Polynomial and Jones Polynomial

DEFINITION. $\langle K\rangle = \sum_{S} \langle K|S\rangle\, d^{\|S\|-1}$





2 of the 8 states of K

(A state S_1 is obtained by smoothing all the crossings of K.)

$\langle K|S\rangle$ = product of the weights A,B.

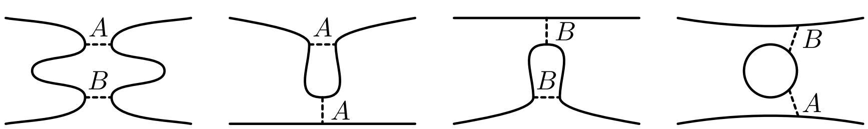
$$\begin{bmatrix} \text{e.g. } \langle K|S_1\rangle = A^2B \\ \langle K|S_2\rangle = A^3 \end{bmatrix}$$

$\|S\|$ = #of Jordan curves in S

[e.g. $\|S_1\| = 1$, $\|S_2\| = 2$.]

THEOREM. (*a*) $\langle \times \rangle = A\langle \asymp \rangle + B\langle \,\rangle\langle\, \rangle$
(*b*) $\langle \bowtie \rangle = AB\langle \,\rangle\langle\, \rangle + (ABd + A^2 + B^2)\langle \asymp \rangle$

PROOF. (a) obvious from definition.
(b) Use (a):

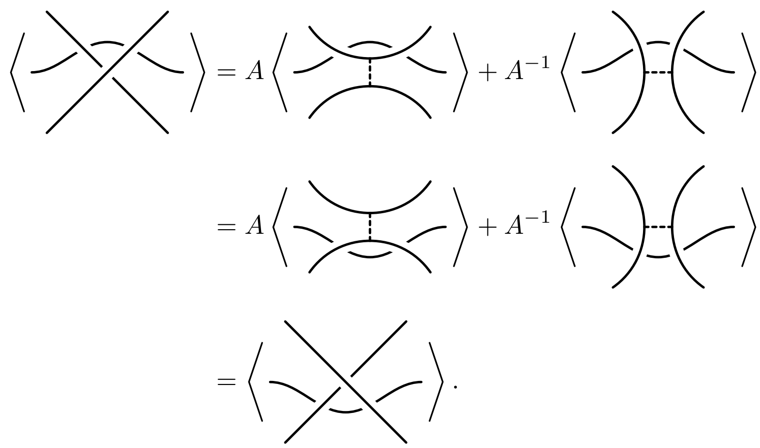


□

Let $B = A^{-1}$, $d = -A^2 - A^{-2}$. Then

$$\langle \bowtie \rangle = \langle \,\rangle\langle\, \rangle$$

and



Thus $\langle K \rangle$ with $B = A^{-1}$, $d = -A^2 - A^{-2}$ is an invariant of regular isotopy.

CHAPTER 2

State Models and State Summations

Let K be a *mathematical object* (by which I usually mean an object in some category, a space, a combinatorial decomposition of some space, a diagram, an expression in some folmal system...). Suppose that there is associated with K a set $\mathcal{S}$ called the "set of states S of K." Let us assume that there is a ring $\mathcal{R}$ and a mapping

$$\langle K|\ \rangle : \mathcal{S} \to \mathcal{R}$$

such that $\langle K|S\rangle \in \mathcal{R}$ for each $S \in \mathcal{S}$.

We define the *state summation* $\langle K\rangle$ by the formula

$$\langle K\rangle = \sum_S \langle K|S\rangle\, d^{\|S\|}$$

where $\|\cdot\| : \mathcal{S} \to \mathcal{R}$ is a function dependent only on the set of states itself. (d is an algebraic in determinate such that $d^{\|S\|} \in \mathcal{R}$ as well.)

In statistical mechanics one defines partition functions $Z = \sum\limits_{S \in \mathcal{S}} e^{\|S\|}$ where $\|S\| = -\frac{1}{kT}E(S)$ ($e = \lim\limits_{n\to\infty}(1+\frac{1}{n})^n$. $k \leftrightarrow$ Boltrmann's[1] constant. $T \leftrightarrow$ temperature. $E(S) =$ the "energy" of the state S of the system under study.)

In this context K may denote an observable for the system. For example $\langle K|S\rangle$ could be the energy for the state S. Then $\langle K\rangle$ represents the average energy for the system.

In general, we will write

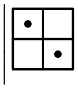
$$[K] = \sum_S [K \mid S]$$

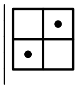
for a state summation with respect to a general mathematical object K and its corresponding set of states $\mathcal{S}$. If such sums are infinite, then questions of integration need to be considered. If finite, them matters of definition need to be looked into.

Many well-known mathematical gadgets are quite naturally seen as state sums.

[1] ???

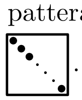
Example.

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \right.$$  $$\rangle = +ad, \qquad t($$  $$) = +0$$

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \right.$$  $$\rangle = +bc, \qquad t($$  $$) = +1$$

$$\langle M \rangle = \sum_S \langle M|S\rangle \, (-1)^{t(S)}$$

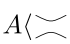
$$\Rightarrow \langle M \rangle = \mathrm{Det}(M).$$

$\mathcal{S}$ =rook patteras[2] on $n \times n$ board; $t\colon \mathcal{S} \to \mathbb{Z}$, $t(S) = \#$of transpositions needed to bring to .

In these notes we have seen state summations for the Alexander—Conway polynomial and for the bracket polynomial. In both of these cases, the states are "naturally" associated with the mathematical object K (the diagram) and the intent of the state sum was to produce something invariant under the Reidemeister moves. This strategy is quite different from the strategies we have seen up until now for producing topological invariants of spaces. We shall see, as the course proceeds, that one can define state sums to yield invariants of 3-manifolds as well as knots, and that there are many variations on this theme of state summation — some being suffused with algebra, others partaking of functional integration and geometry.

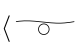
Lets begin again by looking at the bracket polynomial, first seeing what it does topologically:

1. A. Applications of Topological Bracket Polynomial

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$$\langle \bigcirc K \rangle = \delta \langle K \rangle, \quad \delta = -A^2 - A^{-2}$$

1°.

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= (A(-A^2 - A^{-2}) + A^{-1})\langle$ $\rangle$

$\langle$  $\rangle = (-A^3)\langle$  $\rangle$

Similarly,

$\langle$  $\rangle = (-A^{-3})\langle$  $\rangle$

1°. Define $w(K) = \sum_p \epsilon(p)$, sum over *all* crossings in oriented diagram K.

$\epsilon($  $) = +1, \quad \epsilon($  $) = -1.$

Define

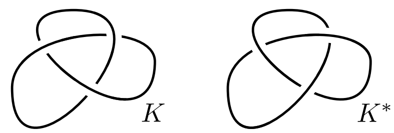
$$f_K = (-A^3)^{-w(K)} \langle K \rangle.$$

Proposition. (a) $\langle K \rangle$ is an invariant of regular isotopy.

(b) f_K is an invariant of ambient isotopy (R1, R2, R3 + R0) for oriented links.

[2] ???

Proof. (a) done already. (b) easy. □



Definition. The *mirror image* of K (denoted K^*) is obtained by reversing (switching) all the crossings of K.

Proposition. $\langle K^*\rangle(A) = \langle(\rangle A^{-1})$, $f_{K^*}(A) = f_K(A^{-1})$.

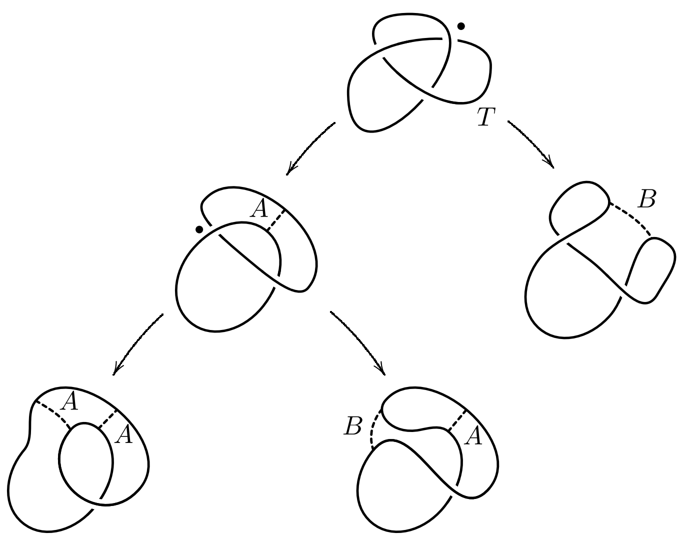
Proof. Immediate from definition. □

Thus

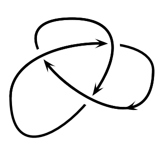
$$K \underset{\substack{\text{ambient}\\\text{isotopy}}}{\cong} K^* \Rightarrow f_K(A) = f_K(A^{-1}).$$

If $F_K(A) \neq f_K(A^{-1})$ then $K \ncong K^*$. In this way, f_K can (sometimes) detect *chirality*. (A knot K is *chiral* if $K \ncong K^*$ and *achiral* if $K \cong K^*$.)

Example.



$$\langle T\rangle = A^2(-A^3) + (A^{-1}A)(-A^{-3}) + A^{-1}(-A^{-3})^2$$
$$\langle T\rangle = -A^5 - A^{-3} + A^{-7}$$



$$w(T) = 1 + 1 + 1 = 3$$

$$f_T = (-A^3)^{-3}(-A^5 - A^{-3} + A^{-7})$$
$$f_T = A^{-4} + A^{-12} - A^{-16}$$
$$f_T(A) \neq f_T(A^{-1}) \;\Rightarrow\; T \ncong T^*.$$

This is the simplest proof I know that the trefoil knot is chiral.

Notice also that the difference between the highest and lowest terms (i. e. degrees) of $\langle K\rangle$ is an ambient isotopy invariant. i. e.

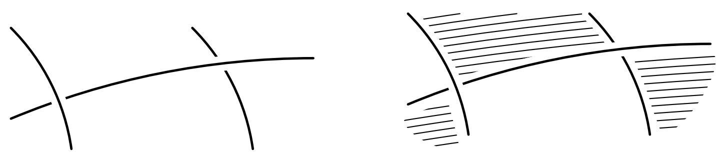
$$\text{Let}\quad \begin{cases} M_K = \text{highest degree in } \langle K\rangle & (\text{e. g. } M_T = 5) \\ m_K = \text{lowest degree in } \langle K\rangle & (\text{e. g. } m_T = -12) \end{cases}$$

Let $Span(\langle K\rangle) = M_K - m_K$.

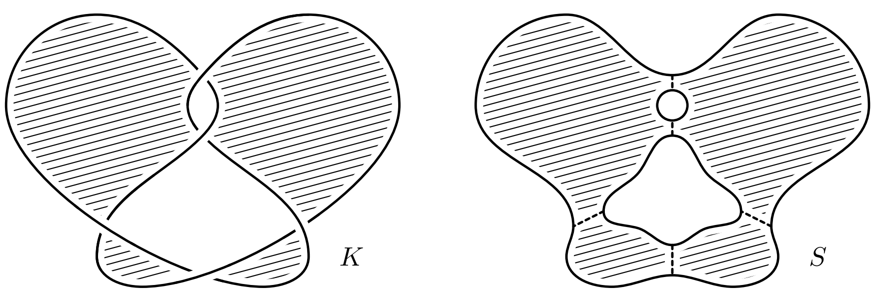
PROPOSITION. $Span(\langle K\rangle)$ is an ambient isotopy invariant of K.

PROOF. $Span(\langle K\rangle) = Span(f_T)$. ☐

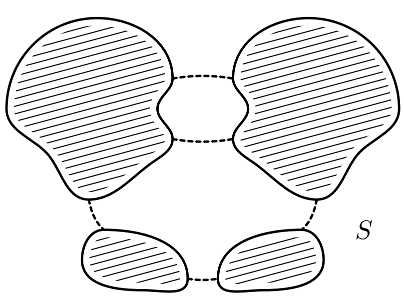
We begin with *alternating* knots and links:



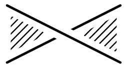
In the checkerboard shading of an alternating link all the crossings have same shaded type. This gives a way to locate states S with highest degree $\langle K|S\rangle$.



$\langle K|S'\rangle = A^{-5} = A^{-v}$ $\qquad \|S'\| = 3 = W$

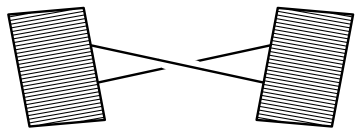


$\langle K|S\rangle = A^{5} = A^{v} \quad \|S\| = 4 = B$

We see that, quite generally, for any alternating diagram (with all shaded crossings of the type ) we have two special states S and S' with

$$\begin{cases} \langle K|S\rangle = A^v, \ v = \#\text{crossings in } K, \\ \quad \|S\| = B = \#\text{shaded regions in } K, \\ \langle K|S'\rangle = A^{-v}, \\ \quad \|S'\| = \#\text{unshaded regions in } K. \end{cases}$$

DEFINITION. A connected diagram K is *reduced* if it does not contain an *isthmus*:


K

An isthmus is a crossing that can be smoothed to cause the diagram to fall into disjoint pieces in the plane.

PROPOSITION. If K is reduced and alternating then the states S and S' described above contribute the highest and lowest degrees of $\langle K\rangle$. More precisely,

$$M_K = v + 2(B-1)$$
$$m_K = -v - 2(W-1)$$

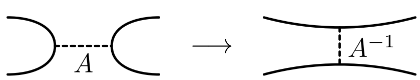
and $Span\langle K\rangle = 4v$.

Thus the number of crossings of a reduced alternating diagram is an ambient isotopy invariant of the corresponding knot.

PROOF. $\langle K\rangle = \sum_\sigma \langle K|\sigma\rangle\, d^{\|\sigma\|-1}$ where $\|\sigma\| = \#\text{loops in } \sigma$ and $d = (-A^2 - A^{-2})$.

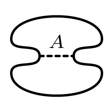
Thus S contributes $\langle K|S\rangle\, d^{\|S\|-1} = A^v(-A^2 - A^{-2})^{B-1}$ and the highest degree contribution from this degree can not be cancelled from any other state.

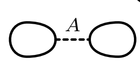
Examine what happens to the degree when we switch a smoothing:



$$\begin{array}{rcl} A^l(-A^2 - A^{-2})^k & & A^{l-2}(-A^2 - A^{-2})^{k\pm 1} \\ l + 2k & \longmapsto & l - 2 + 2(k\pm 1) \\ & & = l + 2k - 4 \text{ or } l + 2k. \end{array}$$

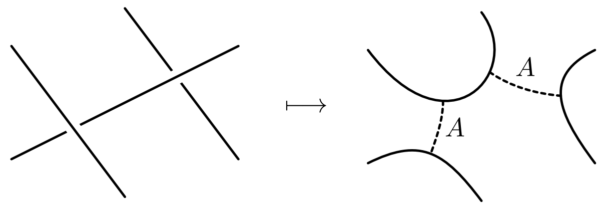
where

$k \pm 1 = \begin{cases} k+1 & \text{if the interaction (left side) is } \end{cases}$ ,

$\qquad k-1$ if the interaction is .

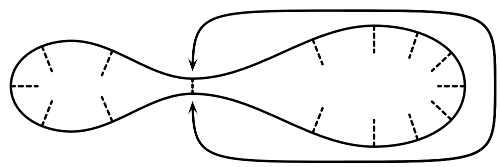
IMPORTANT FACT. In a reduced alternating diagram, the A-state (all smoothings of type A) S has no self-interacting sites.

PROOF. (1)



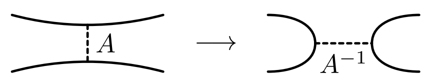
$\Rightarrow$ For any Jordan curve in the A-state, all interactions occur on the *same side* of that curve.

(2) If we have a self-interaction site, then (1) $\Rightarrow$ it is an isthums! Contradiction.



□

Therefore



$$\begin{array}{rcl} A^{l}(-A^{2}-A^{-2})^{k} & & A^{l-2}(-A^{2}-A^{-2})^{k-1} \\ l+2k & \longmapsto & l-2+2(k-1) \\ & & = l+2k-4. \end{array}$$

In switching *any single* smoothing from the A-state the degree (largest possible degree) *drops by 4*. As we saw on previous page, the best that any further switch can do is to leave the degree the same. Therefore the A-state contributes highest degree and the A^{-1} state contributes lowest degree.

It is easy to see that $\|S\| = B$ (# of shaded regions) and $\|S'\| = W$ (# of unshaded regions). Thus

$$M_K = v + 2(B-1),$$
$$m_K = -v - 2(W-1).$$
$$\begin{aligned} Span\,\langle K\rangle &= M_K - m_K \\ &= 2v + 2(B+W-2) \\ &= 2v + 2(\#\text{regions} - 2) \quad [\#\text{regions}= v+2] \\ &= 2v + 2(v) \\ &= 4v \end{aligned}$$

□

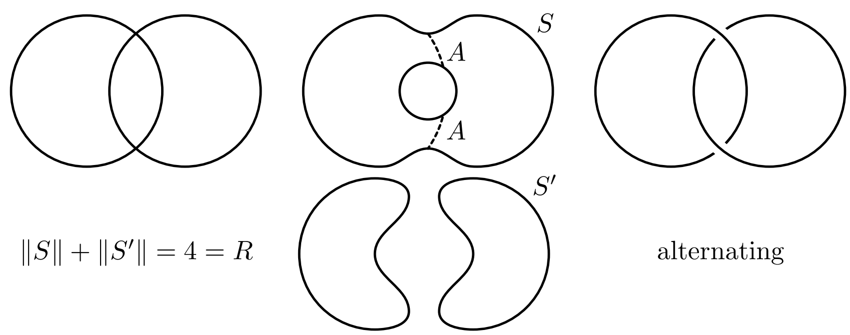
Lemma. *Let S be a state with connected universe U (U is the 4-valent plane graph giving rise to the states). Let S' denote the state obtained by resmoothing all sites of S. Let $R = \#$regions in U. Then*

$$\|S\| + \|S'\| \leqslant R$$

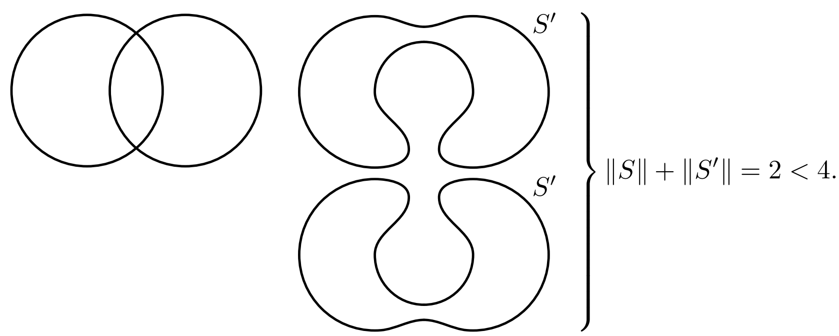
and the inequality is strict if S and S' are not *the A (resp A^{-1}) states for the alternating diagram that overlies U.*

Example.

(a)



(b)



Proof of Lemma. Proof by induction on # regions in diagram. Bottom of induction is $\bigcirc$: $R = 2$, $S = S'$, $\|S\| = \|S'\| = 1$: $1 + 1 = 2$.

Now consider a crossing in a connected universe U.

Let $\times$ denote both this crossing and rest of U by induction. Let $\asymp$, $)($ denote universes obtained by smoothing crossing.

Since $\times$ is connected, one of $\asymp$, $)($ is connected. We may assume that $\asymp$ is connected for sake of argument. Then

$$R(\times) = R(\asymp) + 1,$$

and Lemma is true for $\asymp$ by induction hypothesis.

Let S, S' be dual states for U. By symmetry we can assume $S = \asymp\!\!\!\!\mid$. Thus $S \leftrightarrow \widetilde{S}$ — a state of $\asymp$. By assumption, $S' =)\!\!-\!\!($. Thus $\widetilde{S}'$: $\asymp\!\!\!\!\mid$ is a state of $\asymp$ dual to $\widetilde{S}$ obtained by switching *one site* of S'. So we have

$$\|\widetilde{S}'\| = \|S'\| = \pm 1, \qquad \|\widetilde{S}\| = \|S\|.$$

Therefore

$$\|S\| + \|S'\| \leqslant \|\widetilde{S}\| + \|\widetilde{S}'\| \leqslant R(\asymp) + 1 = R(\times).$$

This completes the proof. □

THEOREM. *Let K be a reduced, non-alternating diagram. Then*

$$Span\langle K\rangle < 4v(K).$$

PROOF. Repeat the argument. That is, let S =A-state for K, S' its dual. Then

$$\begin{aligned} M_K &\leqslant v(K) + 2(\|S\| - 1), \\ m_K &\geqslant -v(K) - 2(\|S'\| - 1). \end{aligned}$$

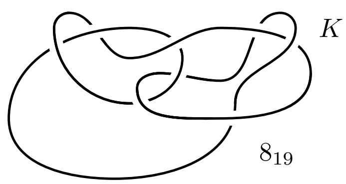
Therefore

$$\begin{aligned} Span\langle K\rangle &\leqslant 2v(K) + 2(\|S\| + \|S'\| - 2) \\ &< 2v(K) + 2(R - 2) = 4v(K). \end{aligned}$$

□

COROLLARY. *If K is a reduced alternating diagram then K is a minimal diagram in the sense that no equivalent diagram (R0, R1, R2, R3) has fewer crossings.*

EXAMPLE.



It can be verified that this is a minimal diagram for K.

Consequence: K is not an alternating knot.

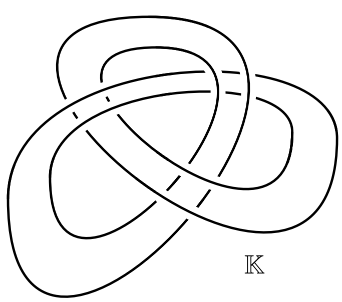
DEFINITION. K is said to be an *adequate* diagram if

1) K reduced
2) Neither the A-state S nor the A^{-1} state S' have any self-interacting sites.

Then our arguments show:

THEOREM. *K adequate* $\Rightarrow$ $Span\langle K\rangle = 2v(K) + 2(\|S\| + \|S'\| - 2)$.

EXAMPLE. r-fold parallel cables are adequate.



EXERCISE. $\|S\| = 4$, $\|S'\| = 6$, $Span\langle\mathbb{K}\rangle = 40$.

EXERCISE. K reduced and adequate. Assume K oriented with writhe $w(K)$. Let

$$\left\{ \begin{array}{l} S \;= A\text{-state of } K \\ S' = A^{-1}\text{-state of } K \end{array} \right\}$$

Show that $K \cong K^*$ (i. e. K achiral)

$$\Rightarrow \ \|S\| - \|S'\| = 3w(K).$$

(Conclude e. g. that $\mathbb{K}$ above (r =2 cable of trefoil) must be chiral.)

SOLUTION TO ABOVE EXERCISE. Suppose K is reduced and adequate. Then

$$\begin{aligned} \langle K\rangle &= A^{v(K)+2(\|S\|-1)} + \ldots \pm A^{-v(K)-2(\|S'\|-1)} \\ f_K &= (-A^3)^{-w(K)}\langle K\rangle. \\ \max\deg f_K &= v(K) + 2(\|S\|-1) - 3w(K) \\ \min\deg f_K &= -v(K) - 2(\|S'\|-1) - 3w(K) \\ K \simeq K^* &\Rightarrow f_K(A) = f_K(A^{-1}) \\ &\Rightarrow \begin{cases} \quad v(K) + 2(\|S\|-1) - 3w(K) \\ = v(K) + 2(\|S'\|-1) + 3w(K) \end{cases} \\ &\Rightarrow \|S\| - \|S'\| = 3w(K). \end{aligned}$$

□

Note that in the case of alternating knots,

$$\|S\| = W = \#\text{White regions},$$
$$\|S'\| = B = \#\text{Black regions}.$$

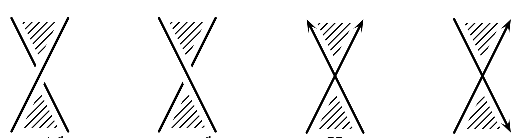
Thus K alternating reduced and *achiral* $\Rightarrow 3w(K) = W - B$.

In this case of alternating knots we can do better. A general theorem of Gordon and Litherland (C. Mc A Gordon and R. A. Litherland. On the Signature of a Links, Invent. Math. 47, pp. 53–69 (1978)) states that for a knot k with diagram D, the signature of k is given by the formula

$$\sigma(k) = \text{Signature}(\mathcal{Z}) - \eta(D)$$

where $\mathcal{Z}$ = the Goeritz matrix for the white regions and

$$\eta(D) = \sum_{p \text{ of type II}} \eta(p) :$$



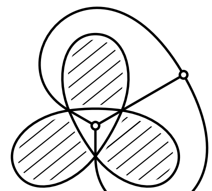
$\eta{=}{+}1$ $\eta{=}{-}1$ II

Here we define Goeritz matrix $\mathcal{Z}$ via the graph on white regions with

$$\left\{ \begin{array}{l} \widetilde{\mathcal{Z}}_{ij} = -\displaystyle\sum_{\substack{\text{crossings } p \\ \text{touching regions } i+j \\ (i \neq j)}} \eta(p) \\ \widetilde{\mathcal{Z}}_{ii} = -\displaystyle\sum_{i \neq j} \widetilde{\mathcal{Z}}_{ij} \end{array} \right\}$$

$\mathcal{Z}_{ij}$ = any $(n-1)\times(n-1)$ matrix obtained from $\widetilde{\mathcal{Z}}$ by striking out $k+$ ......[3] row and column.

Example.

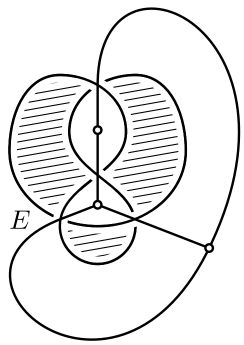


k

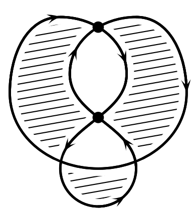
$$\widetilde{\mathcal{Z}} = \begin{pmatrix} 2 & -1 \\ -1 & \boxed{2} \end{pmatrix}$$

$$\eta(k) = +3, \quad \sigma(k) = \sigma(\mathcal{Z}) - 3 = 1 - 3 = -2.$$

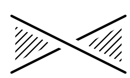
Example.



$$\widetilde{\mathcal{Z}} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



$$\eta = 2, \quad \sigma(E) = 2 - 2 = 0.$$

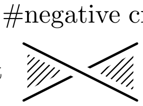
In general, if k *is reduced alternating* of type  with respect to shadings (i. e. all crossings have $\eta = +1$), then $\sigma(\mathcal{Z}) = W - 1$ where W = #white regions.

Therefore

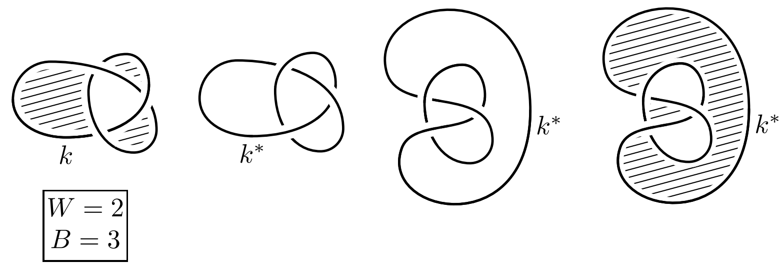
$$\sigma(k) = W - 1 - P_+$$

(P_+ = #positive crossings in k's diagram). Hence

$$\sigma(k^*) = B - 1 - P_-$$

(P_- = #negative crossings in k's diagram, B refers to k's diagram. We rewrote k^* so that  comes to shading.)

[3] k+skolko?



Thus for k *reduced and altarnating*, we have that

$$\sigma(k) - \sigma(k^*) = W - B - w(k)$$

where W and B refer to the number of white and black regions in the diagram for k.

Thus

(a) $(W - B) - w(k)$ is an invariant of k.

(b) $-3w(k) + 2(W - 1)$, $-3w(k) - 2(B - 1)$ are each invariants of k via our bracket calculation. Hence $-3w(k) + (W - B)$ is an invariant of k.

(c) Therefore $[-3w(k) + (W - B)] - [-w(k) + (W - B)]$ is an invariant of k.

Therefore $w(k)$ *is an invariant of* k.

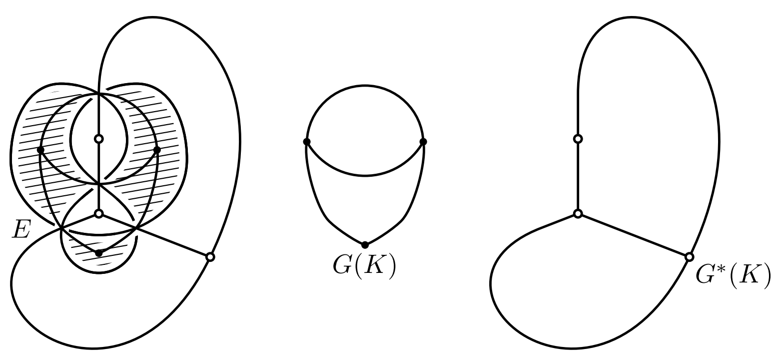
We have proved (by an argument due to Kunio Marasugi) the

THEOREM. *For k reduced and alternating, the writhe $w(k)$ is an ambient isotopy invariant of k.*

COROLLARY. *A reduced alternating k such that $k \simeq k^*$ must have zero writhe and also must have $W = B$.*

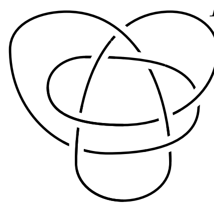
REMARK. The problem of classifying achiral reduced alternating knots is not completely solved. One class of such knots is those whose graphs $G(K)$ are isomorphic to their planar .....[4] i. e. white and black checkerboard graphs are isomorphic. (Show $G(K) \cong G^*(K) \Rightarrow K \cong K^*$ (K alternating).)

EXERCISE.



[4] ??

Exercise.



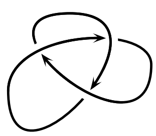
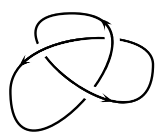
K

(This is 8_{17} from Reidemeister tables.) Show

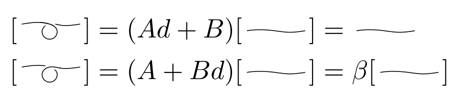
1) $G(K) \cong G^*(K)$

2) $\overrightarrow{K} \cong \overleftarrow{K}^*$ where $\rightarrow$ means oriented and $\leftarrow$ K oriented is equivalent to K^* reverse oriented.

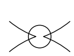
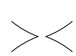
Deeper Fact. Deeper Fact: $\overrightarrow{8_{17}} \not\cong \overleftarrow{8_{17}}$.

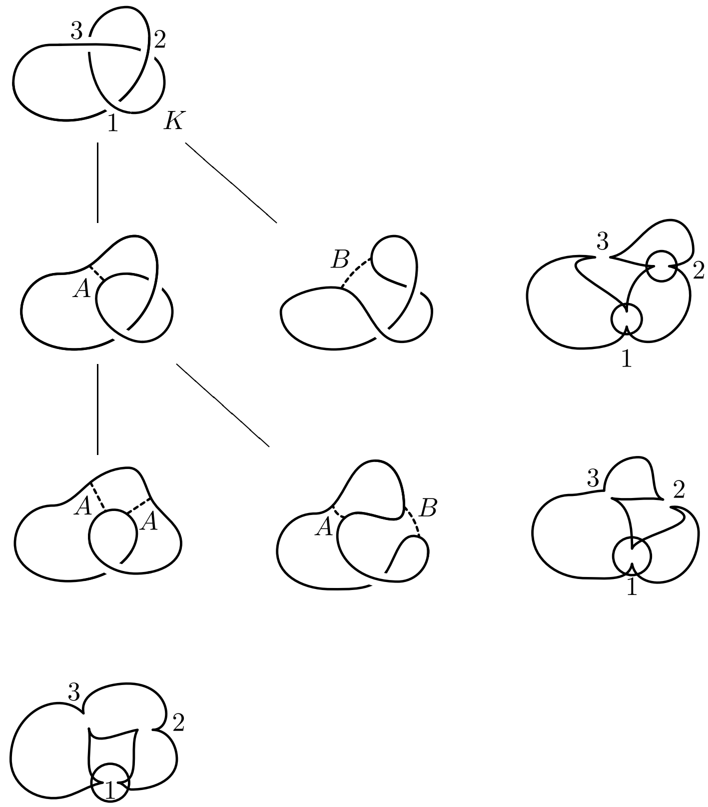
 $\overrightarrow{T}$  $\overleftarrow{T}$ Here $\overrightarrow{T} \cong \overleftarrow{T}$

2. Spanning Tree Expansion

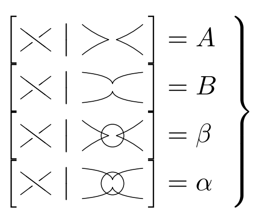


(Topological case: $\alpha = -A^3$, $\beta = -A^{-3}$.)

 active,  inactive



$$[K] = A^2\alpha + AB\beta + B\beta^2 \xrightarrow{\text{Topological}} A^2(-A^3) + AA^{-1}(-A^{-3}) + A^{-1}(-A^{-3})^2$$
$$= -A^5 - A^{-3} + A^{-7}.$$



FACT. A site i is *active* if in i $i < j$ for all j interactions between the *two loops* in i.

$$[K] = \sum_{S \in \mathcal{T}} [K \mid S]$$

$\mathcal{T}$ = set of single component states with marked activity (N. B. $\mathcal{T} \leftrightarrow$ Maximal Trees in Checkerboard Graph $G(K)$.)

EXERCISE. Use the spanning tree expansion to show that

1) There is *no cancellation* in the spanning free sum for a reduced alternating knot, and the polynomial f_K (topological bracket) has sign $(-1)^{n+C}$ for terms of degree $4 - n$.

2) The sum of the absolute valnes of the coefficients of $\langle K \rangle$ (for K reduced alternating) equals the number of maximal trees in $G(K)$.

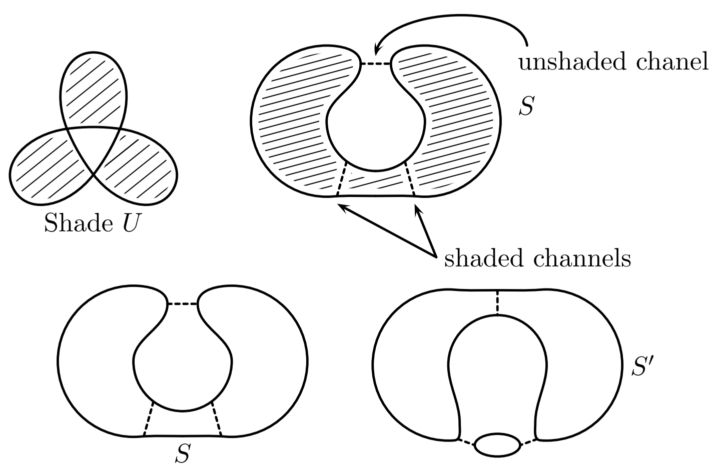
3) Give examples of term cancellations in spanning tree expansion of $\langle K \rangle$ when K is not alternating.

Open Problem. K any knot, $f_K(A) = 1$. Does this imply that K is ambient isotopic to the unknotted circle?

Exercise. Given a state S of a connected universe (planar projection diagram) U in the plane, there is an *orientable surface* $F(S)$ of genus $g(S)$ such that

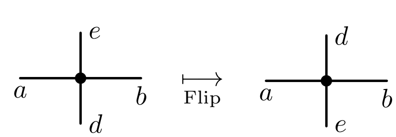
$$\|S\| + \|S'\| = v(U) - (2g(S) - 2).$$

$F(S)$:

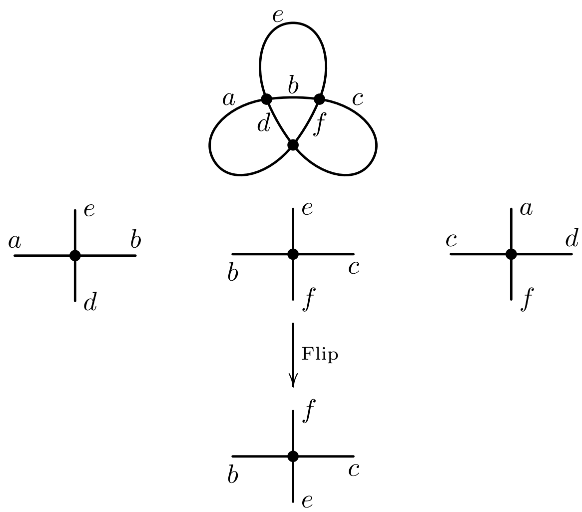


$\|S\| + \|S'\| = 3, \quad v = 3, \quad g = 1.$

Flip the crossings for unshaded channels:



e. g.

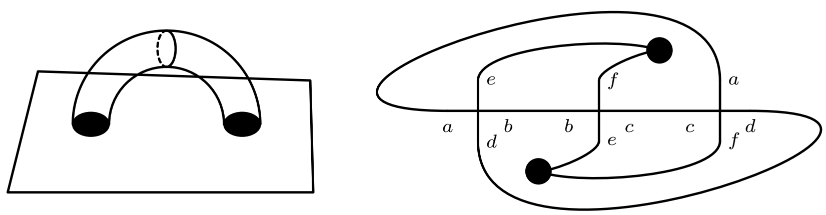


Show: K adequate $\Rightarrow$

$$Span\langle K\rangle = 4(v(K) - g(S))$$

where $S = A$-state of K.

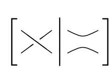
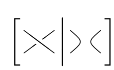
New crossing list requires surface of genus $g(S)$:

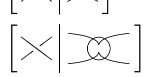
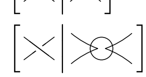


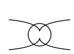
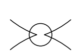
Here $g(S) = 1$.

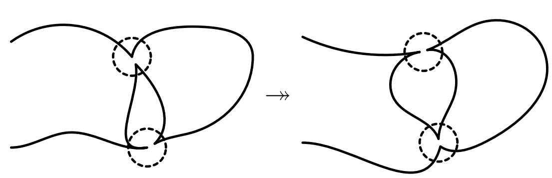
3. Trees, Single component States and $\langle K\rangle(\sqrt{i})$

We have seen $\langle K\rangle = \sum_{\|S\|=1} [K \mid S]$ where

 $= A,$  $= A^{-1}$

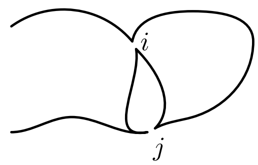
 $= -A^{-3},$  $= -A^{3}$

Here  and  refer to active sites with respect to a choice of vertex ordering. For the states S with $\|S\| = 1$, you can go from any state to any other state by a sequence of *double flips* where a double flip has the form:

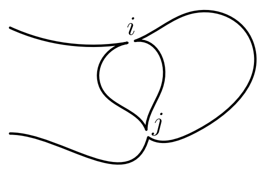


In other words, you change two sites such that one change disconnects and other change reconnects.

In the case of alternating links we can see what the effect of a double flips is on the state evaluation. Note that in general the two sites involved in the double flip are of opposite type (one interior and one exterior). Thus if inactive, one is A and the other is A^{-1}. If active, they have opposite types of activity.



Here i can be active and j cannot be active.



Here i can be active and j cannot be active.

General case includes: $AA \leftrightarrow A^{-1}A^{-1}$, $-A^3A^{-1} \leftrightarrow A(-A^{-3})$.

Thus the cases are

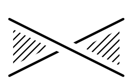
0) $i+j$ inactive $\longrightarrow i+j$ inactive
1) $i+j$ inactive $\longrightarrow i$ active, j inactive
2) i active, j inactive $\longrightarrow i$ active, j inactive
3) i active, j inactive $\longrightarrow i$ inactive, j inactive
0) $(A)(A^{-1}) \longrightarrow (A^{-1})(A)$: no sign change, deg 0 change
1) $(A)(A^{-1}) \longrightarrow (-A^{-3})(A)$: sign change, deg 4 change
2) $(-A^{-3})(A^{-1}) \longrightarrow (-A^{-3})(A)$: no sign change, deg 8 change
3) $(-A^{-3})(A^{-1}) \longrightarrow (A^{-1})(A)$: sign change, deg 4 change

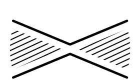
Thus we see that for alternating knots and links, there can be *no cancellation* in the sum

$$\langle K \rangle = \sum_{\|S\|=1} [K \mid S].$$

All terms of f_K are of the form $\pm A^{4n}$ some $n \in \mathbb{Z}$. The sign is $(-1)^{n+c}A^{4n}$ for some constant c.

To see this, lets first look again the maximal degree term for $\langle K \rangle$. Lets suppose that with respect to shading K is of the type ...[5]:

 .

Then the A-state is locally  with W circuits (W = # of unshaded regions).

[5] ??

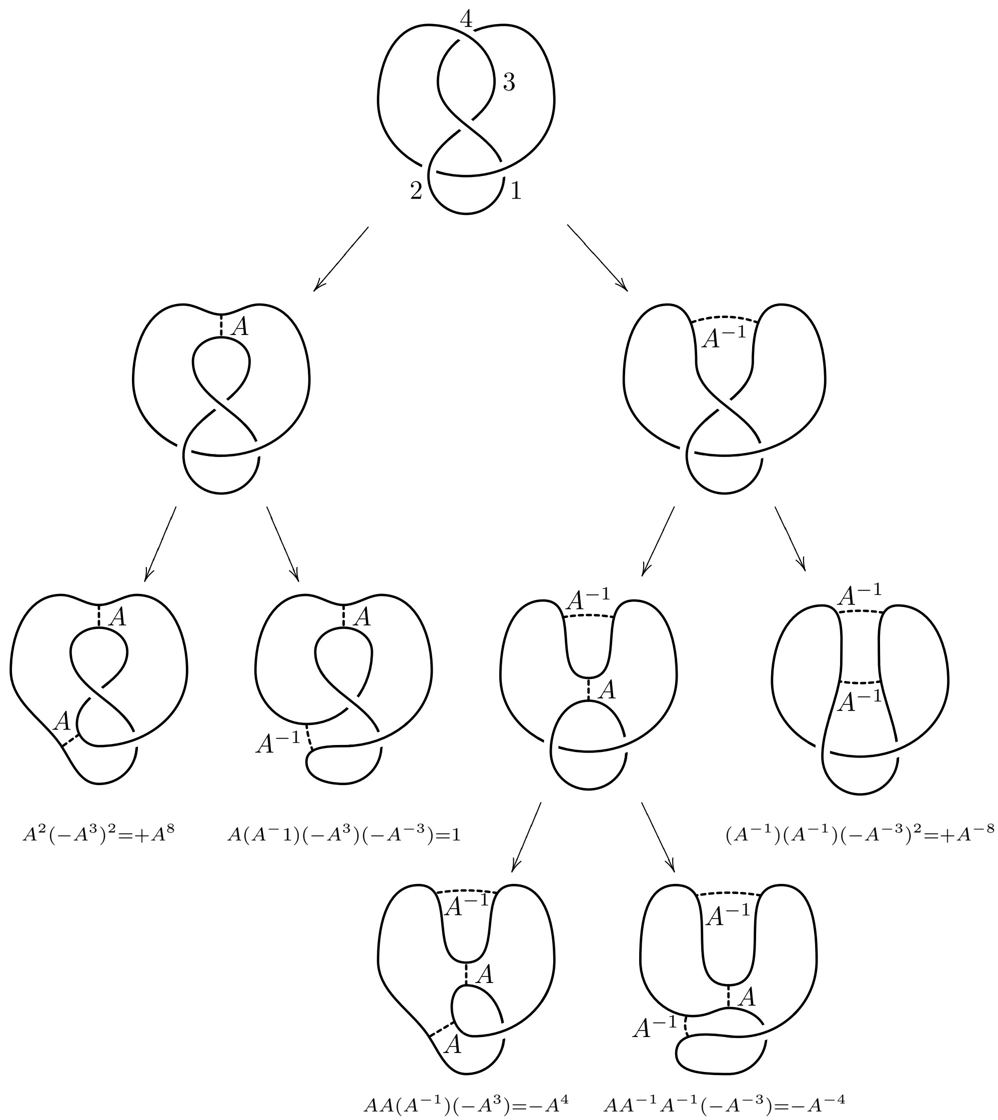
We can obtain a corresponding monocyclic state ($\|S\| = 1$) by reassembling $(W-1)$ of these sites. With appropriate labelling of the vertices these $(W-1)$ sites are *active*. Thus we get a state S, $\|S\| = 1$ with

$$\begin{aligned}[K \mid S] &= A^{v-(W-1)}(-A^3)^{(W-1)} \\ &= A^{v-2(W-1)}(-1)^{(W-1)}.\end{aligned}$$

This is exactly the highest degree as predicted by our previous theory.

In $f_K = (-A^3)^{-w(K)}\langle K\rangle$ this term receives a writhe compensation *and* we know from the structure of the polynomial f_K that f_K is a polynomial in (A^4).

EXAMPLE.



$$\langle E\rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8}$$

There is one term for each tree in $G(K)$. Note how the terms alternate in sign.

Now consider $\langle K\rangle(\sqrt{i})$ where $i^2 = -1$.

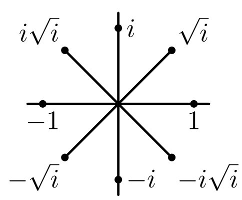
$$A = \sqrt{i}, \quad A^{-1} = 1/\sqrt{i} = \sqrt{i}/i = -i\sqrt{i}, \quad A^4 = i^2 = -1.$$
$$-A^2 - A^{-2} = -i - 1/i = -i + i = \emptyset.$$

This means that the usual bracket sum for $\langle K\rangle(\sqrt{i})$ is a sum over single component states.

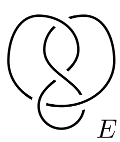
Also,

$$-A^3 = -i\sqrt{i} = A^{-1}, \quad -A^{-3} = A.$$

Thus we see that here the spanning free expansion and the standard bracket expansion *coincide* (and one dose not need to look at activity for $\langle K\rangle(\sqrt{i})$). Furthermore, we see from our previous argument about sign changes in the expansion $\langle K\rangle = \sum_{\|S\|=1} [K \mid S]$ that for an alternating knot or link K *there are no sign changes* in the terms expansion of $\langle K\rangle(\sqrt{i})$. This means that for K alternating $\langle K\rangle(\sqrt{i}) = w\#\mathcal{T}(G(K))$ where $w \in \{\pm i\sqrt{i}, \pm\sqrt{i}, \pm i, \pm 1\} = \Omega_8$ and $\#\mathcal{T}(G(K))$ denotes the number of maximal trees in $G(K)$).



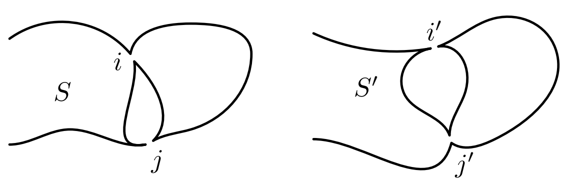
Ω_8

e. g. for ,

E

$$\langle K\rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8}$$
$$\langle K\rangle(\sqrt{i}) = 1 + 1 + 1 + 1 + 1 = 5.$$

Now look at *any* knot K and consider $\langle K\rangle(\sqrt{i})$.



What happens in a double-flip?

Say

i contributes A^{ϵ}	$\epsilon = \pm 1$
j contributes A^{μ}	$\mu = \pm 1$

then

$$i' \text{ contributes } A^{-\epsilon}$$
$$j' \text{ contributes } A^{-\mu}$$

So

$$\left.\begin{array}{l}\langle K|S\rangle = xA^{\epsilon+\mu} \\ \langle K|S'\rangle = xA^{-(\epsilon+\mu)}\end{array}\right\downarrow \text{ degree change } = 2(\epsilon+\mu)$$

Now $A = \sqrt{i}$, $A^{-1} = i\sqrt{i}$, $A^2 = i$, $A^4 = -1$ etc.

The change is either 0 ($\epsilon + \mu = 0$) or $2(\pm 2) = \pm 4$.

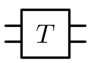
Thus

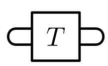
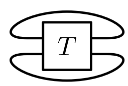
$$\langle K|S'\rangle = \pm \langle K|S\rangle .$$

This means that $\langle K\rangle(\sqrt{i}) = wN$ where $N \in \mathbb{Z}$ and $w \in \Omega_8$.

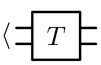
Thus $|N| = \#\mathcal{T}(G(K))$ when K is alternating and more generally $|N|$ is an integer invariant of the knot or link K.

In fact, we shall now see that $\mathcal{N}(K) = |N|$ (N as above) is equal to $\mathrm{Det}(K)$ as defined in earlier sections. Thus $\langle K\rangle(\sqrt{i})$ is a version of $\mathrm{Det}(K)$.

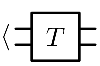
Let  denote a *tangle* with 2 inputs and 2 outputs.

 $= \mathcal{D}(T)$,  $= \mathcal{N}(T)$.

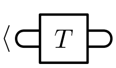
Then

$\langle$  $\rangle(\sqrt{i}) = \langle \mathcal{D}(T)\rangle(\sqrt{i})\langle \asymp \rangle + \langle \mathcal{N}(T)\rangle(\sqrt{i})\langle) (\rangle.$

Proof. Certainly

$\langle$  $\rangle(\sqrt{i}) = X\langle \asymp \rangle + Y\langle) (\rangle.$

Therefore

$\langle$  $\rangle(\sqrt{i}) = X\langle \bigcirc \rangle(\sqrt{i}) + Y\langle \bigcirc \bigcirc \rangle(\sqrt{i}) = X + \emptyset.$

Therefore

$$X = \langle \mathcal{D}(T)\rangle(\sqrt{i}).$$

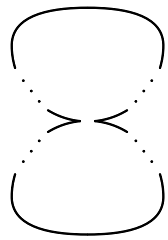
Similarly,

$$Y = \langle \mathcal{N}(T)\rangle(\sqrt{i}).$$

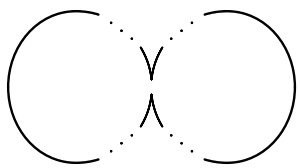
$\square$

Claim. $F(T) = \frac{1}{i}\frac{\langle \mathcal{N}(T)\rangle(\sqrt{i})}{\langle \mathcal{D}(T)\rangle(\sqrt{i})}$ is a rational number.

Proof. Suffices to show that states of $\langle \mathcal{N}(T)\rangle(\sqrt{i})$ and $\langle \mathcal{D}(T)\rangle(\sqrt{i})$ differ by a factor of i. (Recall previous discussion.) Now if



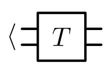
is a state of $\mathcal{N}(T)$ then



is a state of $\mathcal{D}(T)$.

These states differ at just *one cite*: $A \to A^{-1}$ with $A = \sqrt{i}$, the factor is i or i^{-1}. □

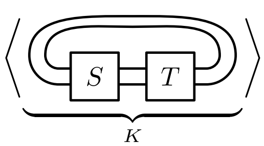
Thus we have

(*)  $(\sqrt{i}) = w \operatorname{Den}(T)\langle \asymp \rangle + wi \operatorname{Num}(T)\langle \rangle \langle \rangle$

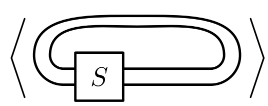
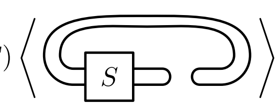
where $\operatorname{Den}(T)$ and $\operatorname{Num}(T)$ are integer valued invariants of the tangle T ($w \in \Omega_8$). They are the determinants (we will see) of $\mathcal{N}(T) + \mathcal{D}(T)$.

$F(T) = \frac{1}{i}\frac{\langle \mathcal{N}(T)\rangle(\sqrt{i})}{\langle \mathcal{D}(T)\rangle(\sqrt{i})} = \frac{\operatorname{Num}(T)}{\operatorname{Den}(T)}$ is called the *fraction* of the tangle T.

Here is nice application of formula $*$, due to David Krebes. Consider



$(\sqrt{i})$ using formula $*$

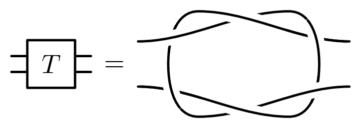
$= w \operatorname{Den}(T)$  $+ iw \operatorname{Num}(T)$ 

$= w \operatorname{Den}(T) i\lambda \operatorname{Num}(S) + iw \operatorname{Num}(T)\lambda \operatorname{Den}(S)$

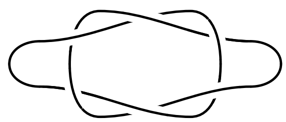
$= iw\lambda[\operatorname{Den}(T)\operatorname{Num}(S) + \operatorname{Num}(T)\operatorname{Den}(S)], \quad (w, \lambda \in \Omega_8)$.

ne sovsem poniatno Conclusion. If $K = \operatorname{Num}(S * T)$ as above and $m / \operatorname{Num}(T) + m / \operatorname{Den}(T)$ then $m / \operatorname{Num}(S * T) = N(K)$. Thus if $m \neq 0$ then K is necessarily non-trivial.

EXAMPLE (Kerbes).

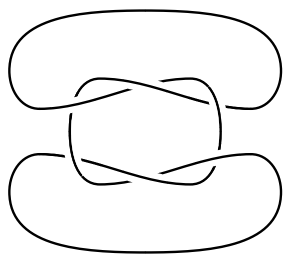


$\mathcal{D}(T)$:



Check $N(\mathcal{D}(T)) = 6$

$\mathcal{N}(T)$:

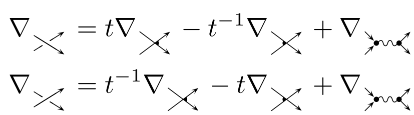


Check $N(\mathcal{N}(T)) = 3^2 = 9$

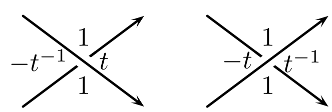
Since $\gcd(\mathcal{D}(T), \mathcal{N}(T)) = 3$, any knot in which T occurs is *necessarily knoted.*
(Does this have applications to DNA?)

Fractions of tangles is an old object. See e. g. for a new proof of Conway's theorem using rational tangles and classifying them their fractions.

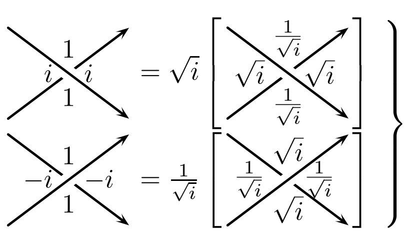
Now go back to our state sum for Alexander polynomial.



(sum over states S, $\|S\| = 1$)



Let $t = i$, $z = t - t^{-1} = 2i \Rightarrow$



$$\boxed{\nabla_K(2i) = \sqrt{i}^{w(K)}\langle K\rangle(\sqrt{i})}$$

Now the rest of the story of the relationship with classical knot determinant you'll have to get via e. g. [*On knots* by L. Kauffman. Princeton Univ. Press 1987] where we show:

1) $\Delta_K(t) = \nabla_K(\sqrt{t} - 1/\sqrt{t})$
2) $\Delta_K(-1) = \mathrm{Det}(K)$ where $\mathrm{Det}(K)$ is defined as we did earlier.

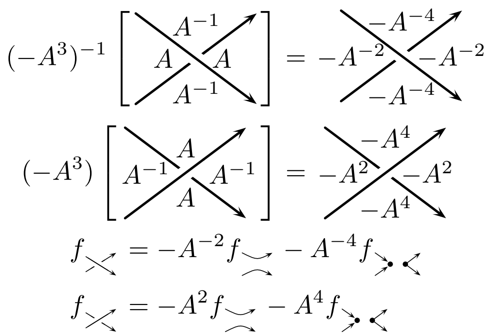
Thus $\mathrm{Det}(K) = \nabla_K(2i) = \sqrt{i}^{w(K)}\langle K\rangle(\sqrt{i})$.

Note that we have located the special element in Ω_8 that will multiply $\langle K\rangle(\sqrt{i})$ to make an integer.

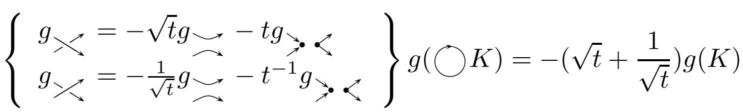
---

4. B. Vassiliev Invariants and the Jones Polynomial

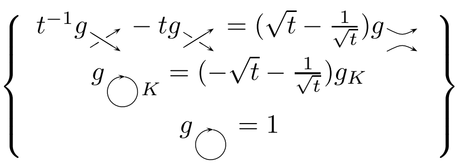
$f_K(A) = (-A^3)^{-w(K)}$



Let $A = t^{-1/4}$, $g_K(t) = f)K(t^{-1/4})$. Then

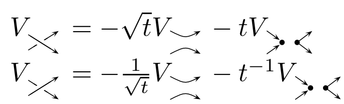


So g_K is an ambient isotopy invariant satisfying

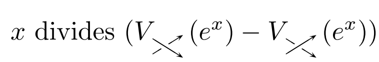


This suffices to identify $g_K = V_K(t)$, the original Jones polynomial.

Thus



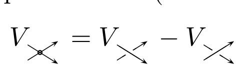
Conclusion. Let $t = e^x$. Then



If

$$V_K(e^x) = \sum_{n=0}^{\infty} \mathcal{V}_n(K)x^n,$$

then *if* we extend V_K to a graph invariant (discussion below) by



then

Hence, if G has K nodes,

$$x \text{ divides } V_G.$$

Therefore $\mathcal{V}_n(G) = 0$ if $n < \#\,\text{nodes}(G) = \#(G)$ of, for a given n,

$$\mathcal{V}_n(G) = 0 \quad \text{for all } G \text{ with } \#(G) > n.$$

We say the $\mathcal{V}_n(G)$ are of *finite type*.

- An invariant of (rigid vertex isotopy) graphs satisfying

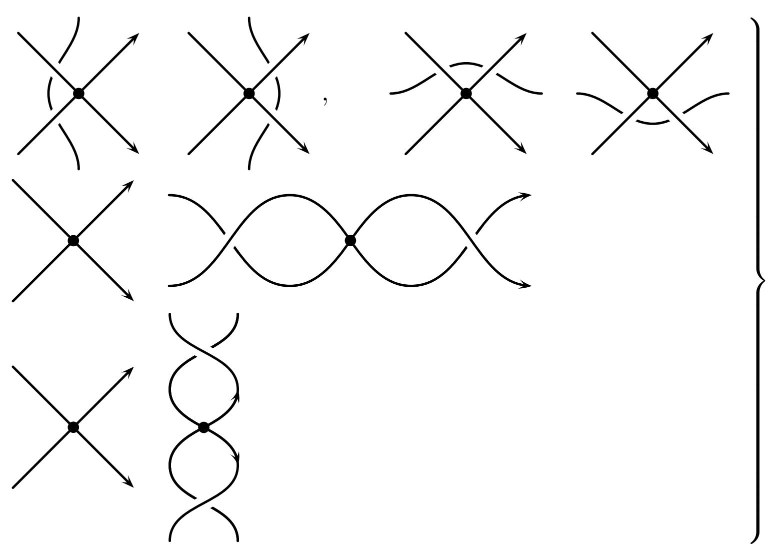
$$V_{\times\!\!\!\!\bullet} = V_{\times} - V_{\times},$$

  is said to be a *Vassiliev invariant*.
- A Vassiliev invariant $\mathcal{V}_n$ is of *finite type* n if

$$\mathcal{V}_n(G) = 0 \quad \text{for all } G \text{ with } \#(G) > n.$$

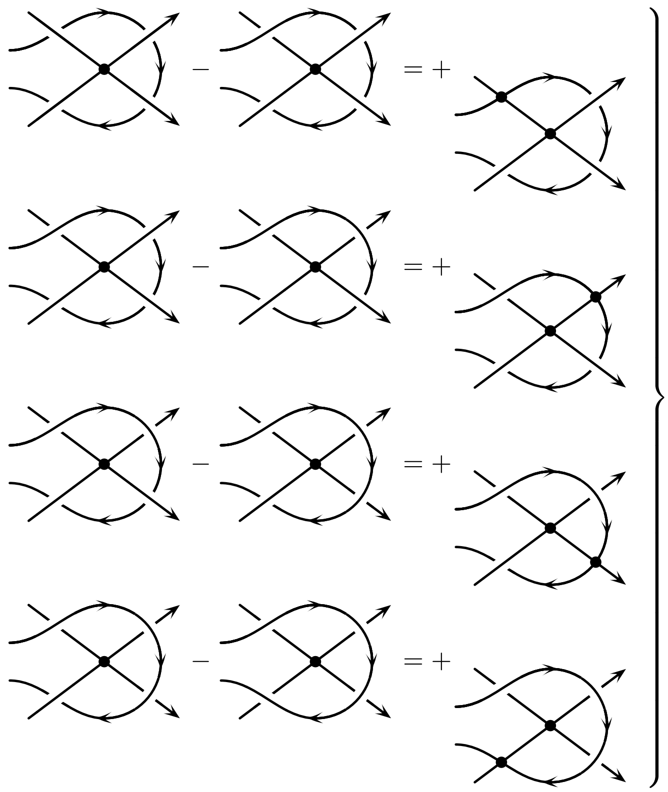
We have shown that *the Jones polynomial is built from an infinite set of Vassiliev invariants, each of finite type.*

5. Rigid Vertex Isotopy



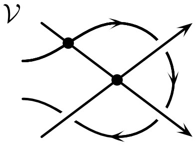
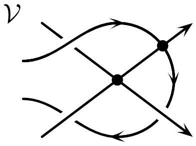
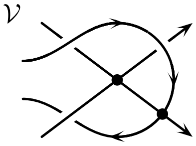
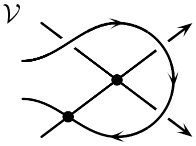
Plus usual Reidemeister moves.

Exercise. If R_K is an invariant of ambient isotopy for knots and links, then $R_{\times\!\!\!\!\bullet} = R_{\times} - R_{\times}$ makes a well-defined extension to rigid vertex graph isotopy invariant.



(Diagrams mean eval by a Vassiliev invariant.)

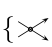
CONCLUSION. For any Vassiliev invariant $\mathcal{V}$

$\mathcal{V}$  $-\,\mathcal{V}$  $-\,\mathcal{V}$  $+\,\mathcal{V}$  $= \varnothing$

This is called the *four term relation.*

LEMMA. *$\mathcal{V}_n$ a Vassiliev invariant of type n. G a graph with $\#G = n$. Then $\mathcal{V}_n(G)$ is independent of the embedding of G in $\mathbb{R}^3$.*

PROOF. If  is a crossing in $G \underset{\alpha}{\subset} \mathbb{R}^3$ then  gives $G \underset{\alpha'}{\subset} \mathbb{R}^3$ and

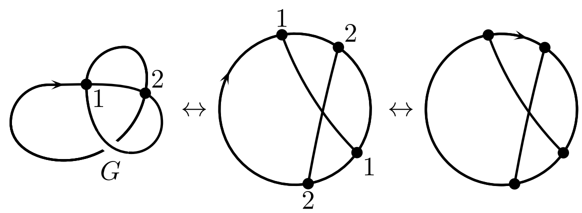
$$\mathcal{V}_{G \underset{\alpha}{\subset} \mathbb{R}^3} - \mathcal{V}_{G \underset{\alpha'}{\subset} \mathbb{R}^3} = \mathcal{V}_n G' = \{$$  $$\} \subset \mathbb{R}^3 = \varnothing$$

$(\#G' = n+1)$. Therefore

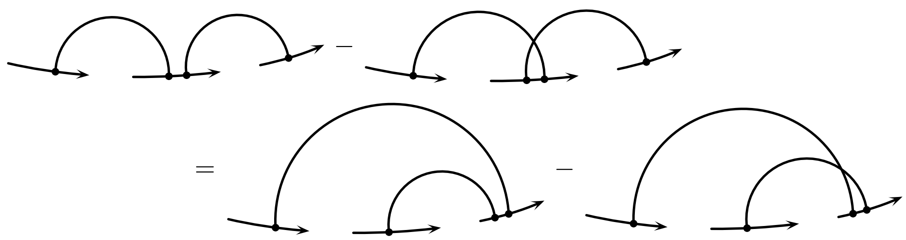
$$\mathcal{V}_{G \underset{\alpha}{\subset} \mathbb{R}^3} = \mathcal{V}_{G \underset{\alpha'}{\subset} \mathbb{R}^3}.$$

This suffices. □

We now look at the four term relation for $\mathcal{V}_n$ and $\#G = n$. These are called *top row evaluations of* $\mathcal{V}_n$. We encode G as a circle with arcs (chord diagram).

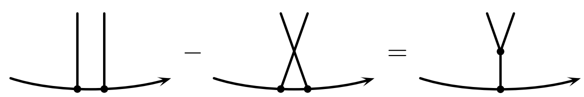


Exercise (!). Show that on the top row the four term relation becomes

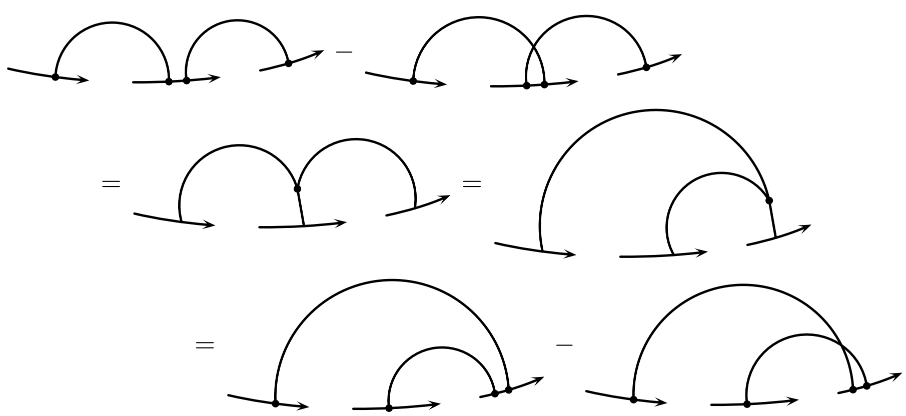


Now *define*

(∗∗)



with the assumption that the trivalent vertex can be moved around in the plane, legs cyclically permated etc. Then (magic):



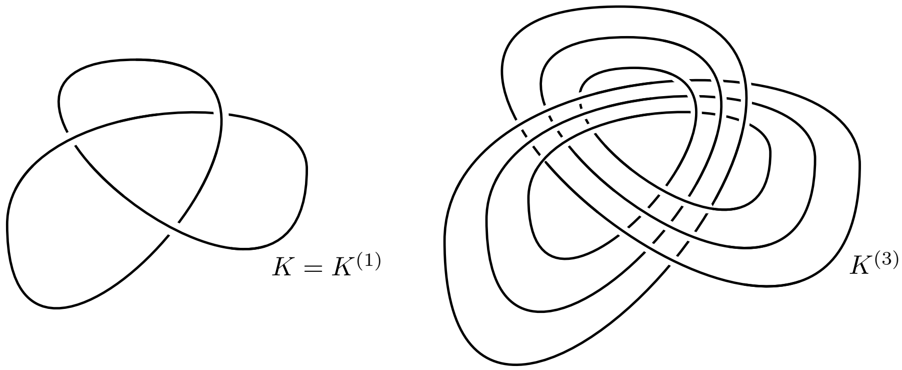
The category of diagrams satisfying (∗∗) is a generalization[6] of the category of Lie algebras.

6. An Adequate Digression

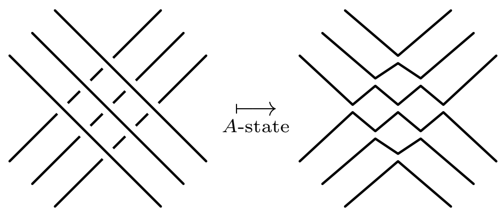
This digression is devoted to a further remark about using adequate links. It is easy to see that if K is *alternating reduced*, then $K^{(r)}$ is adequate, where $K^{(r)}$ is

[6] ???

the r-fold parallel cable of K. e. g.



Now look at structure of A-state of $K^{(r)}$:



$$\begin{aligned} S^{(r)} &= A\text{-state of } K^{(r)} \\ S'^{(r)} &= A^{-1}\text{-state of } K^{(r)} \\ W &= \#\text{white regions for } K \\ B &= \#\text{black regions for } K \\ \mathcal{V} &= \#\text{of crossings for } K \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \|S^{(r)}\| &= r\|S\| = rW \\ \|S'^{(r)}\| &= r\|S'\| = rB \end{aligned} \quad \text{and} \quad w(K^{(r)}) = r^2 w(K)$$

$$\begin{aligned} \max\deg f_{K^{(r)}} &= r^2\mathcal{V} + 2(\|S^{(r)}\| - 1) - 3w(K^{(r)}) \\ &= r^2\mathcal{V} + 2(rW - 1) - 3r^2 w(K) \end{aligned}$$

and

$$\begin{aligned} \max\deg f_{K^{(r)}} &= r^2(\mathcal{V}(K) - 3w(K)) + r(2W(K)) - 2 \\ \min\deg f_{K^{(r)}} &= -r^2(\mathcal{V}(K) + 3w(K)) - r(2B(K)) + 2 \end{aligned}$$

An invariant of $K^{(r)}$ is also an invariant of K. These invariant numbers hold for infinitely many r. Therefore the coefficients in the quadratics are themselves invariant. Hence

$$\left\{ \begin{array}{l} W(K) \\ B(K) \\ v(K) - 3w(K) \\ v(K) + 3w(K) \end{array} \right\}$$

all invariants of $K \Rightarrow$

$$\{v(K), W(K), B(K), w(K)\}$$

all invariants of K alternating and reduced. (This argument is due to[7])

EXERCISE. Generalize discussion on this page to show

THEOREM. *K adequate $\Rightarrow v(K)$, $w(K)$, $\|S(K)\|$, $\|S'(K)\|$ are all invariants of K.*

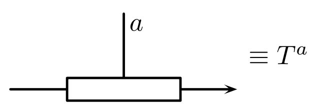
Call a diagram A-good (A^{-1}-good) if its A state S (A^{-1}-state S') has no self-touching loops.

K A-good $\Rightarrow v(K) - 3w(K)$ and $\|S\|$ are invariants of K.

K A^{-1}-good $\Rightarrow v(K) + 3w(K)$ and $\|S'\|$ are invariants of K.

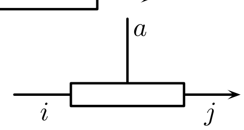
---

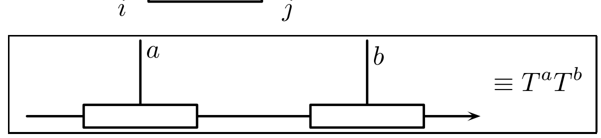
7. [Back to] Chord Diagrams and Lie Algebras



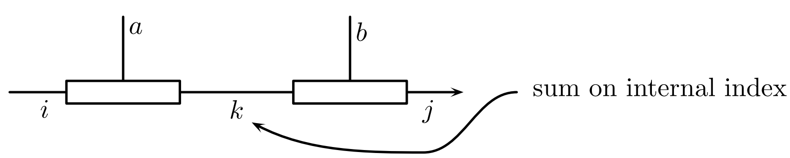
a matrix with a label a.

$(T^a)_{ij} =$





$$(T^aT^b)_{ij} = \sum_k (T^a)_{ik}(T^b)_{kj}$$



In a (matrix representation of a) Lie Algebra there is a basis $\{T^1, T^2, \ldots, T^n\}$ and

$$T^aT^b - T^bT^a = f_{abc}T^c \quad \text{sum on } c.$$

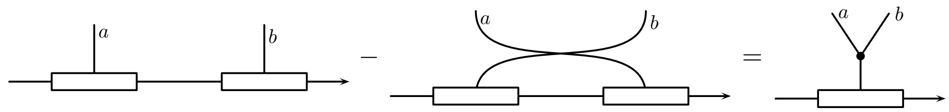
Let *assume* that the structure constants f_{abc} are totally antisymmetric: i. e. if $\langle \pi a, \pi b, \pi c\rangle$ is a permutation of $\langle a, b, c\rangle$ then $f_{\pi a\, \pi b\, \pi c} = \text{sign}(\pi) f_{abc}$, and $f_{abc} = 0$ if any two indexes are equal. There are many classical examples with these properties.

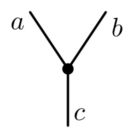
---

[7] Komu on prinadlegit?

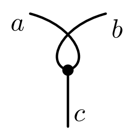
e. g. a standard representation of the SU(N). Lie algebra — which we will use shortly.

$$[T^a, T^b] = T^aT^b - T^bT^a = f_{abc}T^c$$





$\equiv f_{abc}$ depends on cyclic order abc



$\equiv -f_{abc}$

---

e. g. SU(2)

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(Adjoint Representation $(T^a)_{bc} = -i\varepsilon_{abc}$)

$$[T^a, T^b] = i\varepsilon_{abc}T_c$$

$e^{\mathcal{L}} = G$, G group, $\mathcal{L}$ = Lie algebra of G

e. g. SU(N) = $G = \{U \mid U \; N \times N$ complex matrix, $U^* = U^{-1}$, $|U| = 1\}$

$(e^M)^* = e^{M^*}$. So $e^{M^*} = e^{-M}$ or $M^* = -M$.

Write $M = iH$. Then

$$M^* = -M \;\Leftrightarrow\; -iH^* = -iH \;\Leftrightarrow\; H^* = H.$$

$|e^M| = e^{\mathrm{tr}(M)}$. Hence $\mathrm{tr}(H) = 1$. We conclude that the Lie algebra $\mathcal{L}$ of SU(N) consist in $N \times N$ Hermitian matrices of trace zero. Then $e^{iH} \in$ SU(N). For SU(2):

$$H = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\sigma_1\sigma_2 = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{i}{2}\sigma_3$$

$$\sigma_2\sigma_1 = \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\frac{i}{2}\sigma_3$$

$$[\sigma_1, \sigma_2] = i\sigma_3.$$

And generally, $[\sigma_i, \sigma_j] = i\varepsilon_{ijk}\sigma_k$ gives the Lie algebra of Hermitian matrices for SU(2).

In general in a *Lie algebra*, have $[a,b]$ linear in each variable, satisfies Jacobi identity

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0.$$

and

$$[a,b]=-[b,a].$$

In a basis $\{T^1,\dots,T^n\}$

$$[T^a,T^b]=C_k^{ab}T^k$$

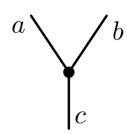
$\{C_k^{ab}\}$ structure constants.

Adjoint representation: $\mathrm{ad}(X)(Y)=[X,Y]$.

N. B. $$\begin{aligned}\mathrm{ad}(X)\,\mathrm{ad}(Y)Z-\mathrm{ad}(Y)\,\mathrm{ad}(X)Z &= \mathrm{ad}(X)[Y,Z]-\mathrm{ad}(Y)[X,Z]\\ &=[X,[Y,Z]]-[Y,[X,Z]]\\ &=[X,[Y,Z]]+[Y,[Z,X]]\\ &=-[Z,[X,Y]]=[[X,Y],Z]=\mathrm{ad}([X,Y])Z.\end{aligned}$$

8. Jacobi Identity $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]$

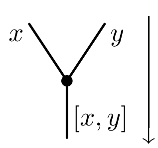
We want to see a diagrammatic version of the Jacobi identity. Look again at the formalism



$$\equiv C_c^{ab} \qquad [T^a,T^b]=C_c^{ab}T^c.$$

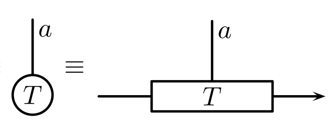
We have interpreted it in terms of indices and the structure constants C_c^{ab}.

It can *also* be interpreted as an "input/output" diagram for multiplication:



where x and y are any elements in the Lie algebra generated by $\{T^1,T^2,\dots,T^n\}$.

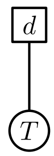
Identities about the algebra then translate directly into diagrammatic identities about the structure constants. To see the structure of this translation process, let

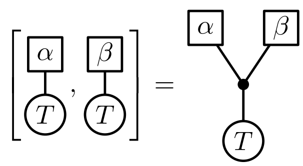
$T^a \equiv$ 

$\sum_a d_a T^a \equiv$ 

$\longleftarrow$ implicit summation on a closed segment

$\{d_a\}$ are scalars, $\{T^a\}$ elements of the algebra.

Then $\widehat{d} =$  is a general element of the algebra. Then

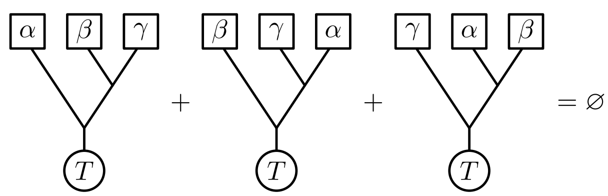
$[\widehat{\alpha}, \widehat{\beta}] =$ 

corresponds directly to the calculation

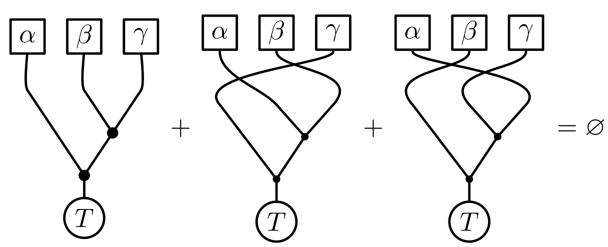
$$[\widehat{\alpha}, \widehat{\beta}] = [\alpha_i T^i, \beta_j T^j] = \alpha_i \beta_j C_k^{ij} T^k.$$

Now lets translate the *Jacobi identity*:

$$[\widehat{\alpha}, [\widehat{\beta}, \widehat{\gamma}]] + [\widehat{\beta}, [\widehat{\gamma}, \widehat{\alpha}]] + [\widehat{\gamma}, [\widehat{\alpha}, \widehat{\beta}]] = \varnothing$$

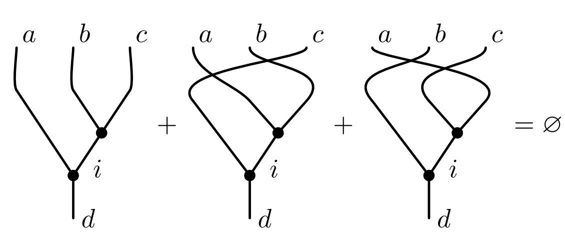


This is equivalent to



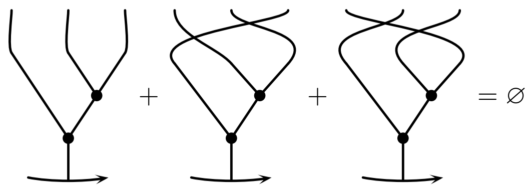
(The last two terms are in opposite order on previous page.)

By linear independence, the Jacobi identity therefore implies the following identity about the structure constants:

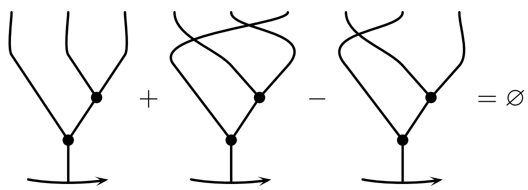


$$\boxed{C_d^{ai} C_i^{bc} + C_d^{ci} C_i^{ab} + C_d^{bi} C_i^{ca} = \varnothing}$$

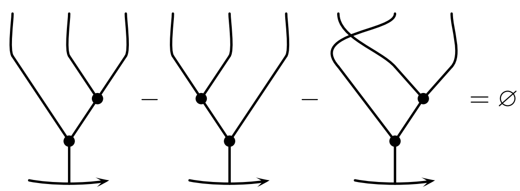
We now take this diagrammatic form of Jacobi identity and transform it into a move convenient pattern.



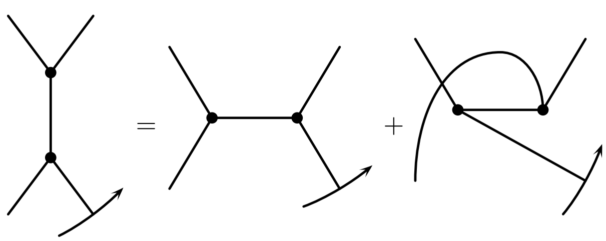
$\Rightarrow$

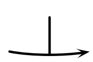


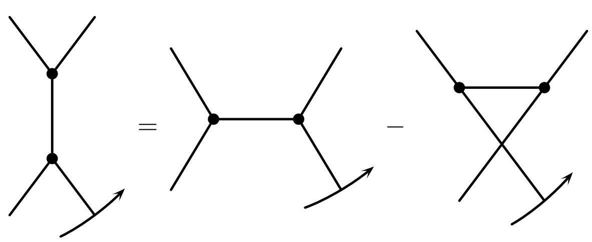
$\Rightarrow$



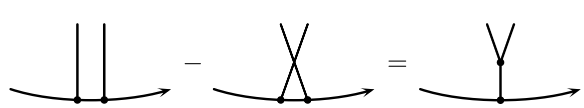
$\Rightarrow$



(Note that the arrow  lets you decode which are the upper and lower indices for the structure constants.) $\Rightarrow$



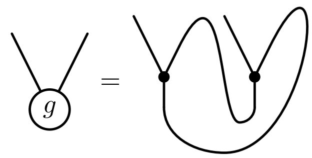
This way of writing the Jacobi identity is called "IHX" (for obvious reasons). The identity



is called "STU." And we have used antisymmetry of C_c^{ab} to derive IHX from STU.

Actually, *we would like* to *have* structure constants such that C^{ab}_c *is unchanged by cyclic permutation of abc.* In the general case this can be accomplished by using the *Killing form*,

$$g^{\sigma\lambda} = C^{\sigma\rho}_{\tau} C^{\lambda\tau}_{\rho} \qquad (\textit{Note}:\ g^{\sigma\lambda} = g^{\lambda\sigma}.)$$



This defines an inner product on the Lie algebra via $\langle T^\sigma, T^\lambda \rangle \stackrel{\text{def}}{=} g^{\sigma\lambda}$.

Theorem (Cartan). *A Lie algebra is semi-simple (no non-trivial abelian subalgebra) iff* $\mathrm{Det}(g^{\sigma\lambda}) \neq 0$. *Thus for semi-simple algebras the Killing form is non-degenerate.*

Proof. Strictly speaking, we want to show that if $\mathfrak{g}$ is the given Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ is a non-trivial abelian *ideal* then $\mathrm{Det}(g^{\sigma\lambda}) = 0$. By an ideal $\mathfrak{a}$, we mean that $[\alpha, \beta] \in \mathfrak{a}$ for any $\alpha \in \mathfrak{a}$, $\beta \in \mathfrak{g}$.

In terms of structure constants this means that we can choose a basis $\{T^\sigma\}$ s. t. $\sigma \in \widetilde{\mathfrak{a}} \leftrightarrow$ basis for $\mathfrak{a}$, and fall set of indices runs over $\widetilde{\mathfrak{g}} \supset \widetilde{\mathfrak{a}}$.

Thus

$$[T^\sigma, \mathcal{T}^\tau] = C^{\sigma\tau}_{\rho} T^\rho, \qquad \sigma \in \widetilde{\mathfrak{a}}, \quad \tau \in \mathfrak{g} \Rightarrow \rho \in \widetilde{\mathfrak{a}}.$$

Now lets use $'$ ("primes") to indicate indices for an abelian ideal $\mathfrak{a}$.

$$\begin{aligned} g^{\sigma\lambda'} &= C^{\sigma\rho}_{\tau} C^{\lambda'\tau}_{\rho} \\ &= C^{\sigma\rho'}_{\tau} C^{\lambda'\tau}_{\rho'} \qquad (C^{\lambda'\tau}_{\rho} = 0 \text{ if } \rho \notin \widetilde{\mathfrak{a}}) \\ &= -C^{\rho'\sigma}_{\tau} C^{\lambda'\tau}_{\rho'} \\ &= -C^{\rho'\sigma}_{\tau'} C^{\lambda'\tau'}_{\rho'} \qquad (C^{\rho'\sigma}_{\tau} = 0 \text{ if } \tau \notin \widetilde{\mathfrak{a}}) \\ &= -C^{\rho'\sigma}_{\tau'} \cdot \varnothing \qquad (\mathfrak{a} \text{ is abelian}) \\ g^{\sigma\lambda'} &= \varnothing. \end{aligned}$$

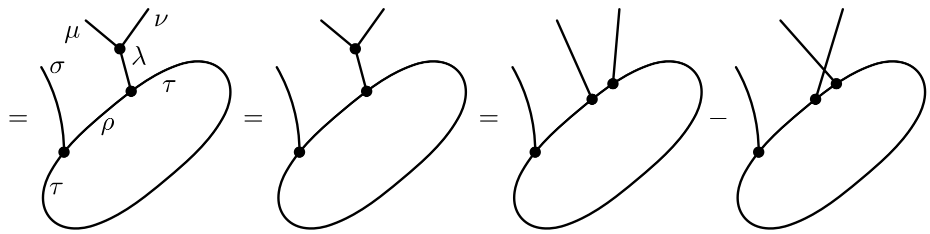
Thus the λ' row of the matrix $(g^{\sigma\lambda})$ is zero and hence $\mathrm{Det}(g^{\sigma\lambda}) = 0$. □

Although we can define weight systems from Lie algebras more generally, we shall here assume semi-simplicity and thus can assume that the basis $\{T^\sigma\}$ is orthogonal with respect to the Killing form. The reason that this is good for our purposes is that *in general*

Lemma. *Let* $C^{\sigma\mu\nu} = g^{\sigma\lambda} C^{\mu\nu}_{\lambda}$. *Then* $C^{\sigma\mu\nu}$ *is invariant under cyclic permutation of* $\sigma\mu\nu$.

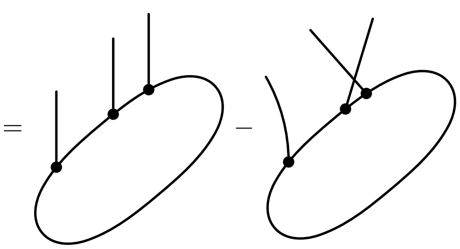
Proof.

$$C^{\sigma\mu\nu} = g^{\sigma\lambda}C^{\mu\nu}_{\lambda} = C^{\sigma\rho}_{\tau}C^{\lambda\tau}_{\rho}C^{\mu\nu}_{\lambda}$$



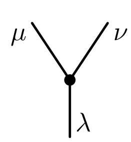
(via Jacobi identity IHX)

$C^{\sigma\mu\nu} =$



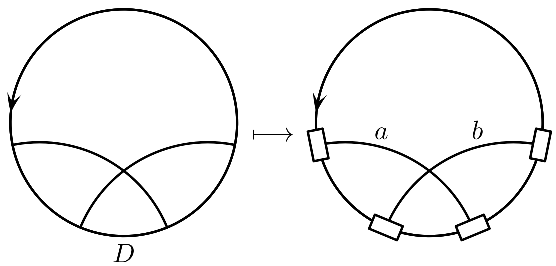
Each term is individually invariant under cyclic permutation. □

Thus if we choose an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form, then $C^{\mu\nu}_{\lambda}$ is already invariant under cyclic permutation, and the assignment

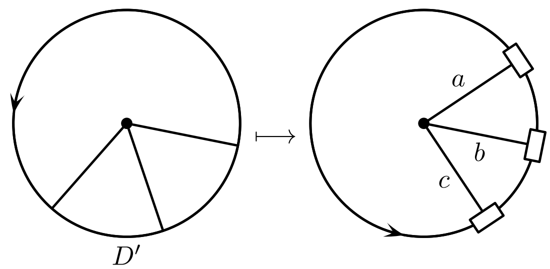


$$\leftrightarrow C^{\mu\nu}_{\lambda} = f_{\mu\nu\lambda}$$

will produce a weight system for Vassiliev invariants. Specifically, we mean the following:



$$\mathrm{wt}(D) = \sum_{a,b} \mathrm{tr}(T^a T^b T^a T^b)$$

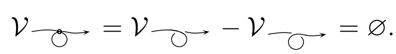


$$\mathrm{wt}(D') = \sum_{a,b,c} \mathrm{tr}(T_c T_b T_a) f_{abc}$$

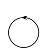
The proof of the Lemma on the page 88 then shows that these weight systems satisfy the 4-term relation. They do not in general satisfy the *isolated chord condition*

 $= \varnothing :$

i. e.

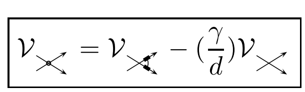


Of course we know that a *framed version* of Vassiliev invariants would not demand the isolated chord condition. On the other hand,

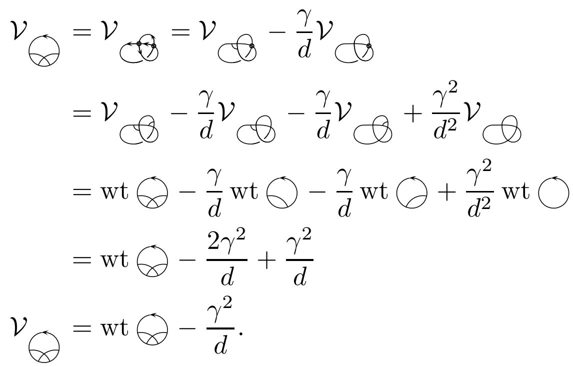
wt  $= \mathrm{tr}(\mathbf{1}) = \dim(\mathfrak{g}) = d,$

wt  $= \sum_a \mathrm{tr}(T^a T^a) = \gamma(\mathfrak{g}) = \gamma.$

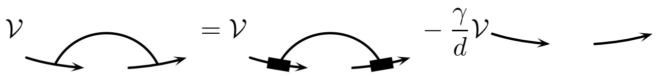
Therefore, define recursively: *state sum for weight system*



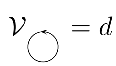
to obtain a weight system corresponding to the given Lie algebra and satisfying both 4-term relation and *isolated chord condition.* e. g.

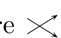


Alternate notation:



and



Here we have written the recursion in knot diagrammatic form where  denotes a "virtual" crossing of the lines in the diagram. We will make use of this formalism in discussing the Witten functional integral.

Example (SU(N) weight system). There is a nice basis for $\mathfrak{su}(N)$ = Lie algebra of SU(N) is fundamental representation given by Hermitian matrices of trace = zero. $H : H = H^*$ = conjugate transpose of H.

Note: $\mu = e^{iH} \implies \mu^* = e^{-iH^*} = e^{-iH} = \mu^{-1} \implies \mu \in \mathrm{SU}(N)$.

In this basis $T^a = \lambda^a/2$ where $\{\lambda^a\}$ is a basis for $N \times N$ Hermitian matrices,

$$[T^a, T^b] = i f_{abc} T^c, \qquad \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

with f_{abc} totally antisymmetric. And

$$\mathrm{tr}\left(\sum_a T^a T^a\right) = (N^2-1)/2.$$

For example, the $\{\lambda^a\}$ for SU(3) is shown below.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

EXERCISE. In the general case, define the *Casimir element* C via the Killing form by

$$C = \sum g_{\rho\sigma} T^\rho T^\sigma \qquad (g_{\rho\sigma}) = (g_{\rho\sigma})^{-1}.$$

Prove that $[C, T^a] = \varnothing \ \forall a$. Thus C is in the center of $\mathfrak{g}$. In a matrix representation this implies that C is a multiple of the identity matrix.

(*Note*: $\operatorname{tr}((\lambda^i)^2) = 2$ for $\forall i$.

$$\begin{aligned} \operatorname{tr}\left(\sum_a T^a T^a\right) &= \frac{1}{2} \operatorname{tr}\left(\sum_a (\lambda^a)^2\right) \\ &= \frac{1}{4}(8 \cdot 2) = \frac{1}{2}(3^2 - 1) \ ) \end{aligned}$$

EXERCISE. Show that the dimension of the space of trace zero $N \times N$ Hermitian matrices (complex entries, dim over $\mathbb{R}$ = reals) is $(N^2 - 1)$.

For SU(2) we have $\lambda^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\lambda^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\begin{pmatrix} a & b - ci \\ b + ci & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In terms of the $\{\lambda^a\}$ basis, *any* Hermitian $(N \times N)$ matrix can be written in the form

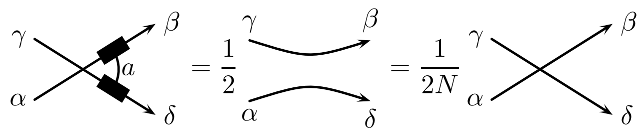
$$H = m_0 \mathbf{1}_N + \sum_{a=1}^{N^2-1} m_a \lambda^a$$

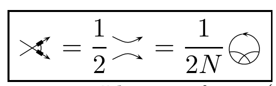
for real numbers $(m_0, m_1, \ldots, m_{N^2-1})$.

$$\begin{aligned} \Longrightarrow \ m_0 &= \frac{1}{N} \operatorname{tr}(H) \\ m_a &= \frac{1}{2} \operatorname{tr}(H\lambda^a) \qquad (\operatorname{tr}(\lambda^a \lambda^b) - 2\delta^{ab}) \end{aligned}$$

Thus

$$
\begin{aligned}
H &= \frac{1}{N}\,\mathrm{tr}(H)\mathbf{1}_N + \sum_{a=1}^{N^2-1} \frac{\mathrm{tr}}{(}H\lambda^a)\lambda^a \\
\Longrightarrow H_{\alpha\beta} &= \frac{1}{N}\left(\sum_{\gamma,\delta} H_{\delta\gamma}\delta_{\gamma\delta}\right)\delta_{\alpha\beta} + \frac{1}{2}\sum_{\alpha}\sum_{\gamma,\delta}(\lambda^a)_{\alpha\beta}(\lambda^a)_{\gamma\delta}H_{\delta\gamma} \\
\Longrightarrow &\boxed{\delta_{\alpha\delta}\delta_{\beta\gamma} = \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\sum_a(\lambda^a)_{\alpha\beta}(\lambda^a)_{\gamma\delta}} \\
\Longrightarrow &\sum_a (T^a)_{\alpha\beta}(T^a)_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma} = \frac{1}{2N}\delta_{\alpha\beta}\delta_{\gamma\delta}
\end{aligned}
$$

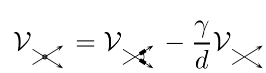




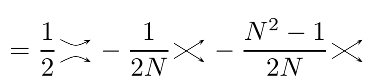
The Fierz Identity for $\mathfrak{su}(N)$

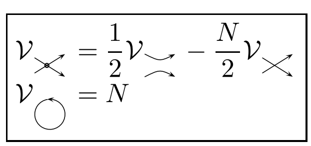
Note that this will yield a completely diagrammatic methid for computing the $\mathfrak{su}(N)$ weight systems.

Now lets use the Fiers identity to write out the recursion for the weight system for $\mathfrak{su}(N)$ satisfying 4-term relation and isolated chord condition:



($\gamma = (N^2-1)/2$, $d = N$ (See page 99))





This then is a state summation for doing top row evaluations for the $\mathfrak{su}(N)$ weight system.

For example

$$\mathcal{V} = \frac{1}{2}\mathcal{V} - \frac{N}{2}\mathcal{V}$$

(top row so we don't care if crossing is under or over.)

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{2}\mathcal{V} - \frac{N}{2}\mathcal{V}\right) - \frac{N}{2}\left(\frac{1}{2}\mathcal{V} - \frac{N}{2}\mathcal{V}\right)\\
&= \frac{1}{2}(\frac{1}{2}N - \frac{N}{2}N^2) - \frac{N}{2}(\frac{1}{2}N^2 - \frac{N}{2}N)\\
&= \frac{1}{4}N - \frac{1}{4}N^2 - \frac{1}{4}N^3 + \frac{1}{4}N^3\\
&= \frac{1}{4}N(1 - N^2)
\end{aligned}$$

NOTABENE. This method for compating the weights is *not* the evaluation of the invariant itself. The invariant gives ∅ for an unknotted circle, while this algorithm gives the trace of the identity matrix to a bare chord diagram.

9. The Homfly Polynomial and Au(N)

The framed version of the Homfly polynomial rends

$$\left\{\begin{array}{l}
H - H = zH\\
H = \alpha H\\
H = \alpha^{-1} H\\
H = 1\\
H \text{ invariant of regular isotopy.}
\end{array}\right\}$$

Letting $P_K = \alpha^{-w(K)} H_K$, we get an ambient isotopy invariant polynomial P_K satisfying

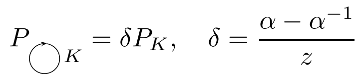
$$\left\{\begin{array}{l}
\alpha P - \alpha^{-1} P = zP\\
P = 1\\
P \text{ an invariant of ambient isotopy.}
\end{array}\right\}$$

Just as with the Jones polynomial, we want to see that letting $\alpha = e^{Nx}$, $z = e^{x} - e^{-x}$ will yield a series $P_K(x) = \sum_{n=0}^{\infty} P_n(K)x^n$ where $P_n(K)$ is a Vassiliev invariant of type n. But note

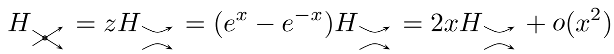
$$\begin{aligned}
P &= P - P = e^{Nx}P - e^{-Nx}P + o(x)\\
&= (e^{x} - e^{-x})P + o(x).
\end{aligned}$$

Hence $x \mid P$, proving this assertion as before.

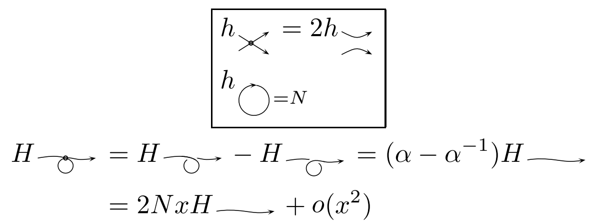
To understand the weight systems note:



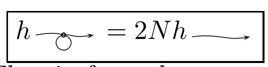
$$\alpha = e^{Nx} = 1 + Nx + o(x^2)$$
$$\alpha^{-1} = e^{-Nx} = 1 - Nx + o(x^2)$$
$$\alpha - \alpha^{-1} = 2Nx + o(x^2)$$
$$z = e^x - e^{-x} = 2x + o(x^2)$$
$$\delta = (\alpha - \alpha - 1)/z = N + o(x^2)$$



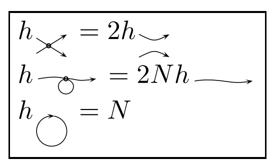
$\Longrightarrow$ *framed weight system*:



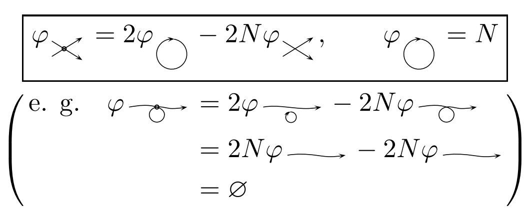
$\Longrightarrow$



Thus in framed system we have



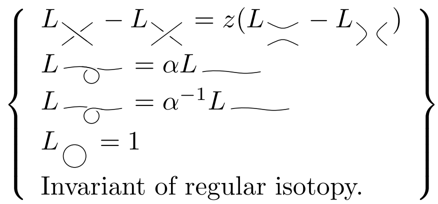
From this we get the recursion for the *unframed* system φ:



Up to a global factor of $2^{\#\text{nodes}}$, this weight system is identical to the $\mathfrak{su}(N)$ weight system. We conclude that *the Homfly polynomial* $P_K(\alpha, z)$ for $\alpha = e^{Nx}$, $z = e^x - e^{-x}$ *is a generator for Vassiliev invariants based in the* $\mathfrak{su}(N)$ *weight system*. In particular *the Jones polynomial corresponds to* SU(2).(*)

(*)We are using $tV_+ - t^{-1}V_- = (\sqrt{t} - 1/\sqrt{t})V_0$ for Jones polynomial. This corresponds to having $A = i\alpha$ in $\langle K\rangle$ so that loop value is $\alpha^2 + \alpha^{-2} + \alpha \implies$ positive loop values in Vassiliev evaluations.

A similar story holds for $O(N)$ in relation to the Kauffman polynomial:

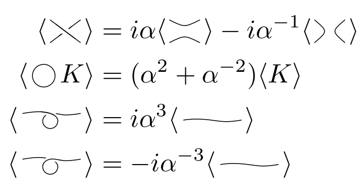


More generally, one can start with a quantum group (Hopf algebra) and/or a solution to the Yang—Baxter equation and get streams of associated Vassiliev invariants. If the quantum group is associated with (a representation of) a classical Lie algebra, then the weight systems of the associated invariants will be those coming from this (representation of the) Lie algebra.

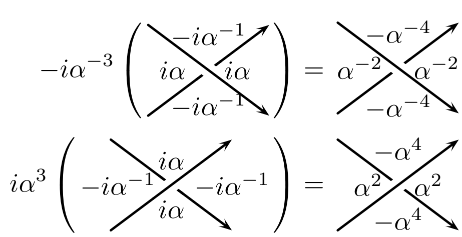
We will omit discussion of these *quantum link invariants* from these notes, but see e. g. L. Kauffman, *Knots and Diagrams*, Proceedings of Lectures at Knots 1996, Tokyo, World Sci. Pub (1997) for a discussion of this aspect.

10. Remarks on the Jones Polynomial

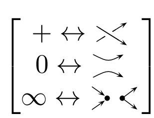
We have advertised $t^{-1}V_+ - tV_- = (\sqrt{t} - 1/\sqrt{t})V_0$ as the original Jones polynomial. It is often convenient to regard the other Homfly specialization $t\widetilde{V}_+ - t^{-1}\widetilde{V}_- = (\sqrt{t} - 1/\sqrt{t})\widetilde{V}_0$ as the Jones polynomial. In particular, the loop value for the latter is $\delta = \sqrt{t} + 1/\sqrt{t}$ and this will make some things in life a bit easier! These two polynomials are really just versions of each other. For example, if we start with $\langle K \rangle$ and write $A = i\alpha$, then



Going to f_K as before:



$$\boxed{\begin{array}{l} f_+ = \alpha^{-2} f_0 - \alpha^{-4} f_\infty \\ f_- = \alpha^{2} f_0 - \alpha^{4} f_\infty \end{array}}$$



Let $\widetilde{V}_K$ be the polynomial obtained via

$$\boxed{t = \alpha^4}$$

$$\boxed{\begin{array}{l}\widetilde{V}_+ = 1/\sqrt{t}\widetilde{V}_0 - 1/t\widetilde{V}_\infty \\ \widetilde{V}_- = \sqrt{t}\widetilde{V}_0 - t\widetilde{V}_\infty\end{array}}$$

$$\delta = \sqrt{t} + 1/\sqrt{t} \quad (\sqrt{t} = \alpha^2)$$

$\Longrightarrow$

$$t\widetilde{V}_+ - t^{-1}\widetilde{V}_- = (\sqrt{t} - 1/\sqrt{t})\widetilde{V}_\infty$$

and

For Vassiliev:

$$\widetilde{V}_{\times} = (1/\sqrt{t} - \sqrt{t})\widetilde{V}_0 + (t - 1/t)\widetilde{V}_\infty$$

Let $t = e^x$:

$$\left\{\begin{array}{r} e^{-x/2} - e^{x/2} = -x + o(x^2) \\ e^{x} - e^{-x} = 2x + o(x^2) \\ e^{x/2} + e^{-x/2} = 2 + o(x^2) \end{array}\right\}$$

Thus for $\widetilde{V}$, we have the *weight system recursion*:

$$\widetilde{\mathcal{V}}_{\times} = -\widetilde{\mathcal{V}}_{\asymp} + 2\widetilde{\mathcal{V}}_{\succ\bullet\prec}$$

$$\widetilde{\mathcal{V}}_{\bigcirc} = 2$$

How is this related to our $\mathfrak{su}(2)$ calculation?

The answer is that as far as this algorithm is concerned, we can replace $\succ\bullet\prec$ by a linear combination of $\asymp$ and $\times$. Then we need:

There is a mistake here that is cleared up on pages 81–82. EXERCISE. Find out what is wrong here and fix it.

$$\succ\bullet\prec = a\asymp + b\times$$

$$\Longrightarrow \quad \infty\!\!\circ = a\bigcirc + b\infty \quad \Longrightarrow \quad 2 = a + b$$

$$8 = a\,{}^{\circ}_{\circ} + b\,8 \quad \Longrightarrow \quad 1 = 2a + b$$

$$\Longrightarrow \quad a = -1, \; b = 3$$

$$\boxed{\succ\bullet\prec = -\asymp + 3\times}$$

whence

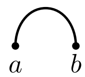
$$\widetilde{\mathcal{V}} = -\widetilde{\mathcal{V}} + 2(-\widetilde{\mathcal{V}} + 3\widetilde{\mathcal{V}})$$

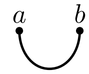
$$\widetilde{\mathcal{V}} = -3\widetilde{\mathcal{V}} + 6\widetilde{\mathcal{V}}$$

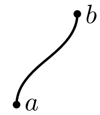
giving a multiple of the $\mathfrak{su}(2)$ weight system.

11. Quantum Link Invariants (Briefly)

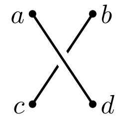
Actually, what is going on here is very interesting in relation to the so-called *quantum link invariant* models for $\langle K\rangle$ and its generalizations. Lets look at this. I will do the notes in cryptic form.

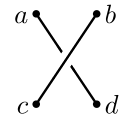


$\leftrightarrow M_{ab}$


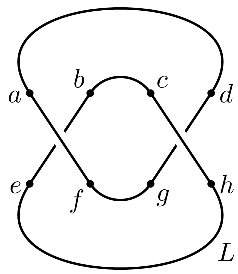
$\leftrightarrow M^{ab}$


$\leftrightarrow \delta_a^b$



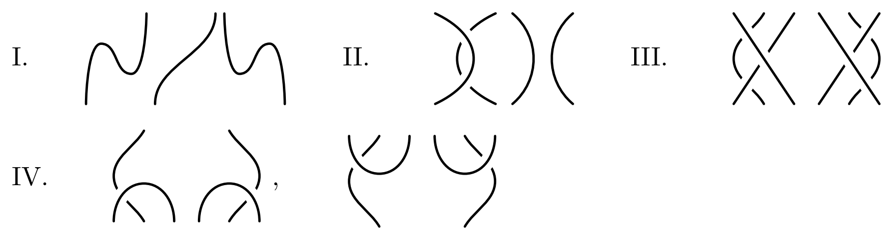
$\leftrightarrow R^{ab}_{cd}$


$\leftrightarrow \overline{R}^{ab}_{cd}$



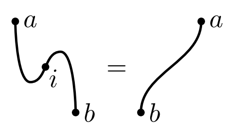
$$\longmapsto \mathcal{T}(L) = \sum_{a,\dots,h} M_{ad}M_{bc}R^{ab}_{ef}R^{cd}_{gh}M^{fg}M^{eh}$$

Need:

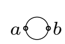


for regular isotopy with respect to a ↑ direction.

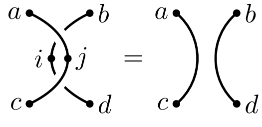
Thus



$$\Leftrightarrow \sum_i M^{ai}M_{ib} = \delta^a_b.$$

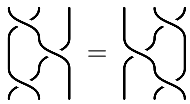
We can solve this equation with $M = \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix}$, $i^2 = -1$, and $M^{ab} = M_{ab}$. Note that $\tau(\bigcirc) = \sum_{a,b}$  $= \sum_{a,b} M_{ab}M^{ab} = \sum_{a,b}(M_{ab})^2 = (iA)^2 + (-iA^{-1})^2 = -A^2 - A^{-2}$, the correct loop value for the bracket.

Equation II is

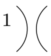
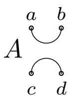
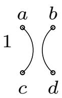
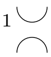
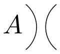


$$\Leftrightarrow \sum_{i,j} R^{ab}_{ij}\overline{R}^{ij}_{cd} = \delta^a_c\delta^b_d \Leftrightarrow R\overline{R} = I \otimes I.$$

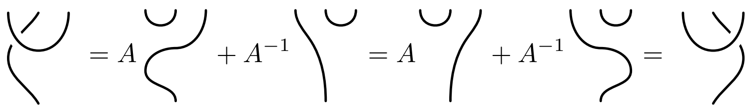
Equation III is the (so-called) *Quantum Yang-Baxter Equation* (QYBE)



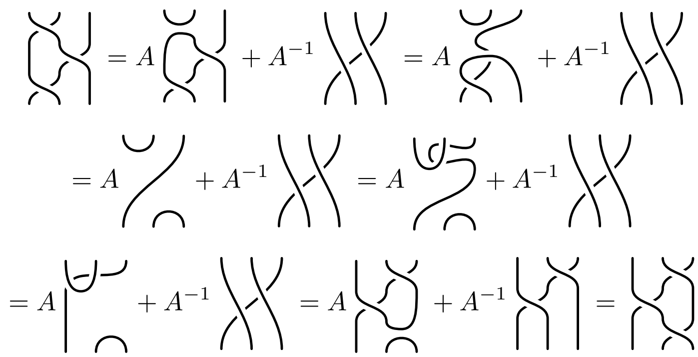
$$: (R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

In our case, we want  $= A$  $+ A^{-1}$  and so we *define* $R^{ab}_{cd} = A$  $+ A^{-1}$  $= AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d$ and similarly for $\overline{R} = A^{-1}$  $+ A$ .

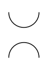
Equation IV follows from the definition



Knowing this, we deduce III:

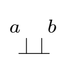


Note that this diagrammatic derivation translates directly into an algebraic proof that R (and $\overline{R}$) satisfies the QYBE.

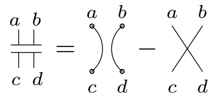
EXERCISE. a) $U =$ , $U^{ab}_{cd} = M^{ab}M_{cd}$. Show that with M as above, $U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -A^2 & 1 & 0 \\ 0 & 1 & -A^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) Use a) to show that $R = \begin{pmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & (-A^3 + A^{-1}) & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^{-1} \end{pmatrix}$. Find $\overline{R}$.

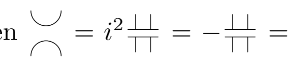
Now note that $M = i\widetilde{\varepsilon}$, $\widetilde{\varepsilon} = \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}$ and for $A = +1$ we have $\widetilde{\varepsilon} = \varepsilon \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$. The ε is algebraically significant because $P\varepsilon P^T = \mathrm{Det}(P)\varepsilon$ for any 2×2 matrix (with commuting entries). Thus the group $SL(2, \mathbb{C})$ can be defined by $SL(2, \mathbb{C}) = \{P \mid P \ 2 \times 2 \text{ matrix over } \mathbb{C},\ P\varepsilon P^T = \varepsilon\}$. Now let

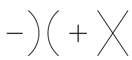
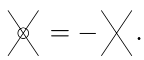
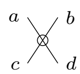
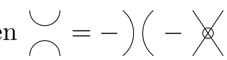
 $= \varepsilon^{ab}$,  $= \varepsilon_{ab}$

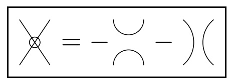
where $\varepsilon^{ab} = \varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab}$.

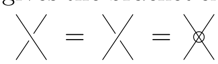
CLAIM.  i. e. $\boxed{\varepsilon^{ab}\varepsilon_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c}$

EXERCISE. Prove this claim.

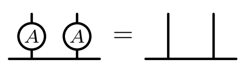
Now let $\cup = i\perp\perp$, $\cap = i\top\top$ (case of $A = 1$). Then 

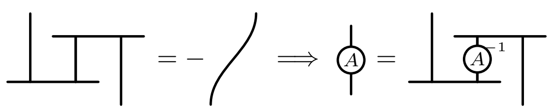
 Define . i. e.  $= -\delta^a_d\delta^b_c$. Then  or

(⋆) 

This gives the bracket expansion at the special value $A = -1$ and loop value $\delta = -2$. Here  and *each term has an algebraic interpretation so that* (⋆) *is an algebraic identity*. This shows how $\langle K\rangle$ has an underlying structure related to $SL(2)$. Following this line of thought further leads to the subject of *quantum groups*.

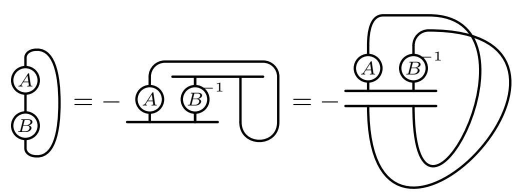
EXAMPLE. $SL(2) = \{A \mid A\varepsilon A^T = \varepsilon\}$:

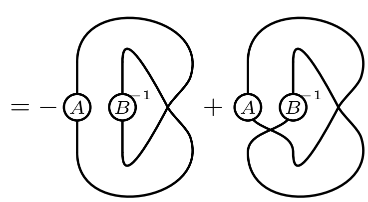
 $\boxed{A^a_i\varepsilon^{ij}A^b_j = \varepsilon^{ab}}$

 for $A \in SL(2)$.

THEOREM. $A, B \in SL(2) \implies \mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$.

PROOF.

$\mathrm{tr}(AB) =$ 



$= -\,\mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)\,\mathrm{tr}(B^{-1})$

Therefore

$$\mathrm{tr}(A)\,\mathrm{tr}(B^{-1}) = \mathrm{tr}(AB^{-1}) + \mathrm{tr}(AB).$$

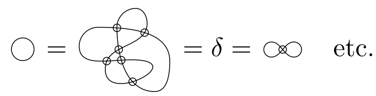
Therefore

$$\mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}).$$

□

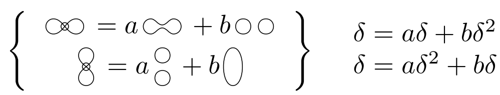
NOTABENE. The minima and maxima are identity lines in this proof.

Note that if we had an algorithm that assigned δ to each loop value, including

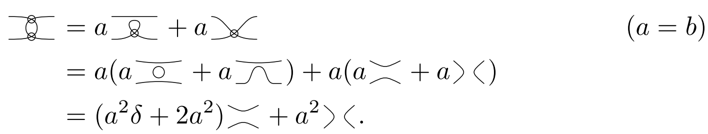


then we can look for a linear combination

that is compatible with the loop evaluation. Assuming $\delta \neq 0$ this means:

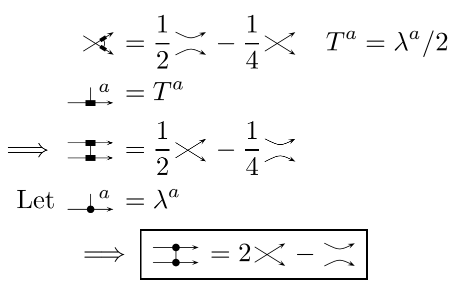


whence $\left\{ \begin{array}{l} 1 = a + b\delta \\ 1 = a\delta + b \end{array} \right\} \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$. This has infinitely many solutions for $\delta = 1$. The solution $a = b = 1/(\delta + 1)$ for any δ. $\delta = -2$ is the *only* solution that gives a topological calculus in the sense that [diagram] $= \rangle\langle$; for by the above we repeat an old argument:



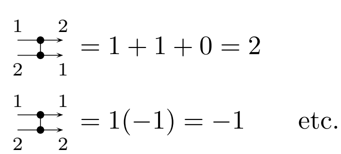
Thus need $a^2 = 1$ $a^2\delta + 2a^2 = 0 \implies \delta = -2$. *This calculation shows that it is only at the special value corresponding to $SL(2)$ that this linear combinations trick can work topologically.*

12. Fierz Identity for $\mathfrak{su}(2)$



$$\lambda^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You can check this directly. For example,



From the Fierz identity for $\mathfrak{su}(2)$ we conclude the weight system recursion formula $\mathcal{V}_{\times} = \frac{1}{2}\mathcal{V}_{\asymp} - \frac{1}{4}\mathcal{V}_{\times} - \frac{2^2-1}{2\cdot 2}\mathcal{V}_{\times} = \frac{1}{2}\mathcal{V}_{\asymp} - \mathcal{V}_{\times}$, $\mathcal{V}_{\bigcirc} = 2$. Lets compare this directly with the two recursions we have for the standard Jones polynomial: $t^{-1}V_+ - tV_- = (\sqrt{t} - 1/\sqrt{t})V_0$. Using the expansion

$$V_{\times} = -\sqrt{t}V_{\asymp} - tV_{\succ\cdot\prec}$$
$$V_{\times} = -1/\sqrt{t}V_{\asymp} - t^{-1}V_{\succ\cdot\prec}$$
$$\delta - \sqrt{t} - 1/\sqrt{t}$$

and $t = e^x$, we obtained

(A) $$\boxed{\begin{array}{l} \mathcal{V}_{\dot{\times}} = -\mathcal{V}_{\asymp} - 2\mathcal{V}_{\succ\cdot\prec} \\ \mathcal{V}_{\bigcirc} = -2 \end{array}}$$

Using framing compensation, we have $H_+ - H_- = zH_0$, $H_{\text{curl}} = t^{-1}H_{2.246}$, $z = \sqrt{t} - 1/\sqrt{t}$, $\delta = -\sqrt{t} + 1/\sqrt{t}$ $(V_K = t^{w(K)}H_K)$.

Letting $t = e^x$, we get framed $h_{\dot{\times}} = h_{\asymp}$ compensating the framing

(B) $$\boxed{\begin{array}{l} \mathcal{V}_{\dot{\times}} = \mathcal{V}_{\asymp} + 2\mathcal{V}_{\times} \\ \mathcal{V}_{\bigcirc} = -2 \end{array}}$$

Both (A) and (B) should give the *same* results, being two ways to express the weight system for V_K. This means: $-\mathcal{V}_{\asymp} - 2\mathcal{V}_{\succ\cdot\prec} = \mathcal{V}_{\asymp} + 2\mathcal{V}_{\times} \Leftrightarrow \asymp + \times + \succ\cdot\prec = 0$ in loop value $= -2$. But this is exactly the "binor identity"

$$\asymp + \times +)(= 0, \qquad \bigcirc = -2$$

that we have shown to be compatible with loop counts. So indeed (A) and (B) will produce the same counts and we see that *the binor calculus for $SL(2)$ underlies this weight system.*

To compare with loop value $= +2$, note

$$\mathcal{V}_{\dot{\times}} = \mathcal{V}_{\asymp} + 2\mathcal{V}_{\times}, \qquad \mathcal{V}_{\bigcirc} = -2$$
$$\Longrightarrow \mathcal{V}_G = \sum_{\sigma} 2^{\#\times}(-1)^{\#\asymp}2^{\|\sigma\|}$$

where a state σ is obtained from G by replacing each node $\dot{\times}$ by either $\times$ or $\asymp$. We only count crossings and smoothings arising from the nodes of G. $N = \#$nodes, $\|\sigma\| = \#$loops in resulting diagram. Then $\#\times + \#\asymp = N$. So

$$\mathcal{V}_G = \sum_{\sigma} 2^{\#\times}(-1)^N(-1)^{\#\times}2^{\|\sigma\|}$$
$$\mathcal{V}_G = (-1)^N \sum_{\sigma} (-2)^{\#\times}2^{\|\sigma\|}$$

$\Longrightarrow$ If $\widetilde{V}_G = (-1)^N\mathcal{V}_G$ then $\widetilde{V}_{\dot{\times}} = \widetilde{V}_{\asymp} - 2\widetilde{V}_{\times}$, $\widetilde{V}_{\bigcirc} = 2$. Thus, up to a constant factor, $\mathcal{V}_G$ is exactly the $\mathfrak{su}(2)$ Lie algebra weight system. This documents the relationship between $\mathfrak{su}(2)$ and the Jones polynomial.

CHAPTER 3

IV. Vassiliev Invariants and Witten's Functional Integral

Witten's integral

$$\boxed{Z_K = \int dA\, e^{\frac{ik}{4\pi}\mathcal{L}(A)}\mathcal{W}(A)}$$

$$A(x) = \sum_{a=1}^{d}\sum_{k=1}^{3} A_k^a(x) T_a\, dx^k, \quad x \in \mathbb{R}^3$$

$\{T_a\}$ is a Lie algebra basis such that

$$[T_a, T_b] = f_{abc}T_c \qquad \text{(sum on } c\text{)}$$

f_{abc} totally antisymmetric. *We will assume that* $\mathrm{tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$ *as in case* $\mathfrak{su}(N)$.

$A(x)$:
- called a *gauge connection*.
- $A_k^a(x)$ smooth function on $\mathbb{R}^3 \subset S^3$.
- Thus A is a Lie algebra valued 1-form.

$$\mathcal{L}(A) = \int_{S^3} \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \qquad \text{"Chern-Simous Lagrangian"}$$

$$\mathcal{W}_K(A) = \mathrm{tr}\big(\mathbb{P}e^{\oint_K A}\big) = \mathrm{tr}\prod_{x\in K}(\mathbf{1} + A(x)). \qquad \text{"Wilson Loop"}$$

The product definition of the Wilson loop explains directly what the $\mathbb{P}$ (for "path ordering") means. In $\prod_{x\in K}(\mathbf{1} + A(x))$ we take the (limit of) the product of the matrices in the given order around the loop K.

We shall see that the functional integral formalism indicates that Z_K is an invariant of regular isotopy of K. This comes about from a reciprocal relationship in varying the connection A and varying the loop K. Variation of A generates curvature $(dA + A \wedge A)$ via $\mathcal{L}$. Variation of K generates curvature via $\mathcal{W}_K$. The result will take the form

$$Z_{\times} = Z_{\times_+} - Z_{\times_-} = \frac{c}{k} Z_{\times_\bullet} + o(1/k^2).$$

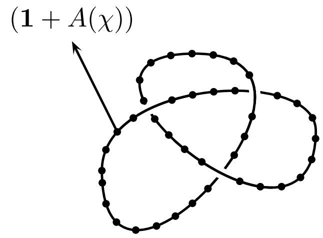
In other words,

$$Z_K = \sum_{n=0}^{\infty} \mathcal{V}_n(K)(\frac{1}{k^n})$$

where $\mathcal{V}_n(K)$ is a Vassiliev invariant of type n. The weight systems for these (framed) invariants are exactly the weight systems we studied abstractly in the last section. Later we will extract more analytically rigorous constructions for those invariants by following the lesson of the functional integral more deeply.

1. Calculus on the Wilson Line

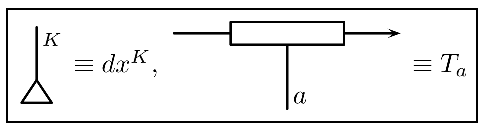
$$\mathcal{W}_K(A) = \mathrm{tr} \prod_{x \in K} (\mathbf{1} + A_K^a(x) T_a \, dx^K)$$

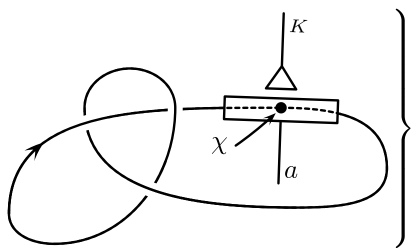


$$\frac{\delta \mathcal{W}_K}{\delta A_K^a(\chi)} = T_a \, d\chi^K \quad \mathcal{W}_K$$

insert at χ in the loop

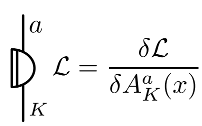
Notation.

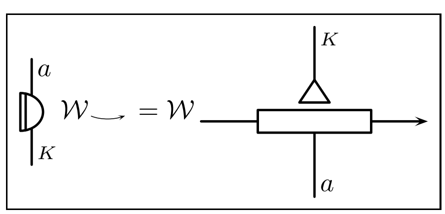




Picture of $T_a \, dx^K$ inserted into the Wilson loop at point x.

Here we we functionally differentiate purely formally and omit mention of delta functions. Let





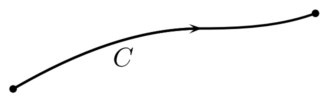
These results really refer to varying in the field only in a very small neighborhood of the point x.

Curvature of A: $F = dA + A \wedge A$

For an arbitrary gauge field $A(x) = A_i^a(x) T_a \, dx^i$ we regard

$$R(C; A) = e^{\oint_C A}$$

as a "parallel transport" from one endpoint of the curve C to the other.

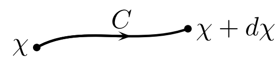


Definition. We say that A and A' are *gauge equivalent* if there is a smooth map $U\colon \mathbb{R}^3 \subset S^3 \to \mathcal{L}$ where $\mathcal{L}$ is the Lie group whose Lie algebra $\mathfrak{g}$ is represented by $\{T_a\}$, and $A' = UAU^{-1} + (dU)U^{-1}$.

Specifically, for $A = A_\mu\, dx^\mu$ ($A_\mu = A^a_\mu(x)T_a \in \mathfrak{g}$) we have

$$A'_\mu = UA_\mu U^{-1} + (\partial_\mu U)U^{-1}.$$

This definition is (partly) explained by the following Lemma.

Lemma. *For an infinitesimal path* C: 

the condition that A' *is gauge equivalent to* A *via* U *is equivalent to the statement*

$$R(C; A') = U(x + dx)R(C; A)U^{-1}(x).$$

Proof.

$$\begin{aligned}(1 + A'_\mu\, dx^\mu) &= U(x+dx)(1 + A_\mu\, dx^\mu)U^{-1}(x)\\ &= U(x+dx)U^{-1}(x) + U(x+dx)A_\mu U^{-1}(x)dx^\mu\\ &= (U + \partial_\mu U\, dx^\mu)U^{-1} + (U + \partial_\mu U\, dx^\mu)A_\mu U^{-1}dx^\mu\\ &= 1 + (\partial_\mu U)U^{-1}dx^\mu + UA_\mu U^{-1}dx^\mu\\ &= 1 + (UA_\mu U^{-1} + (\partial_\mu U)U^{-1})dx^\mu.\end{aligned}$$

□

Fact (not proved here).[1] Take $\mathcal{L} = SU(2)$. $U\colon S^3 \to SU(2)$ a gauge transformation. $n =$ degree of U. Then if $A^U = UAU^{-1} + (dU)U^{-1}$ then with

$$CS(A) = \frac{1}{4\pi}\int_{S^3} \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A),$$

$CS(A^U) = CS(A) + 2\pi n$. Hence $e^{ikCS(A)}$ *is an invariant under gauge transformations of* A *when* k *is an integer.*

The previous Lemma shows that the Wilson Loop $\mathcal{W}_K(A)$ is invariant under gauge transformations. Hence the integrand in

$$Z_K = \int dA\, e^{ikCS(A)}\mathcal{W}_K(A)$$

is gauge invariant. Thus this integral is taken over $\widetilde{\mathcal{A}}/\widetilde{\mathcal{L}}$ where $\widetilde{\mathcal{A}} =$ all gauge connections and $\widetilde{\mathcal{L}}$ denotes the group of gauge transformations.

Getting on to curvature, we now look at $R(C; A)$ for an *infinitesimal loop* C and see $dA + A \wedge A$ pop out:

[1]See: Jackiw, Annals of Physics 194, 197–223 (1989).

Lemma. *A, B matrices, λ a scalar. Then*

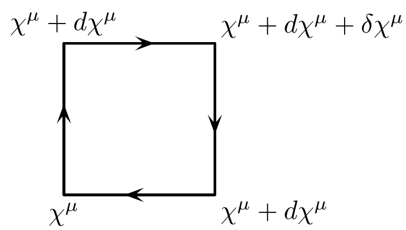
$$e^{\lambda A}e^{\lambda B} = e^{\lambda(A+B)+(\lambda^2/2)[A,B]} + o(\lambda^3).$$

Proof.

$$\begin{aligned}
e^{\lambda A}e^{\lambda B} &= (1 + \lambda A + \lambda^2 A^2/2! + \dots)(1 + \lambda B + \lambda^2 B^2/2! + \dots) \\
&= 1 + \lambda(A+B) + \lambda^2 AB + \lambda^2 A^2/2! + \lambda^2 B^2/2! + o(\lambda^3) \\
&= 1 + \lambda(A+B) + \lambda^2(A^2 + 2AB + B^2)/2 + o(\lambda^3) \\
&= 1 + \lambda(A+B) + \lambda^2(A^2 + AB + BA + B^2)/2 + \lambda^2(AB - BA)/2 + o(\lambda^3) \\
&= 1 + \lambda(A+B) + \lambda^2(A+B)^2/2 + (\lambda^2/2)[A,B] + o(\lambda^3) \\
&= e^{\lambda(A+B)+(\lambda^2/2)[A,B]} + o(\lambda^3).
\end{aligned}$$

□

$\square \equiv \circlearrowright :$



$$R_{\circlearrowright} = \mathbb{P}e^{\oint_\square A} = R_{\lrcorner} R_{\ulcorner}$$

$$R_{\lrcorner} = R(x, x+dx)R(x+dx, x+dx+\delta x)$$

$$R_{\ulcorner} = R(x+dx+\delta x, x+\delta x)R(x+\delta x, x)$$

$$R_{\lrcorner} = e^{A_\mu(x)dx^\mu} e^{A_\nu(x+dx)\delta x^\nu}$$

$$\equiv e^{(A_\mu dx^\mu + A_\nu \delta x^\nu + \partial_\mu A_\nu\, dx^\mu \delta x^\nu + \frac{1}{2}[A_\mu, A_\nu]\, dx^\mu \delta x^\nu)}$$

$$R_{\ulcorner} \equiv e^{[-(A_\mu\, dx^\mu + \partial_\nu A_\mu\, dx^\mu dx^\nu + A_\nu \delta x^\nu) + \frac{1}{2}[A_\mu, A_\nu]\, dx^\mu \delta x^\nu]}$$

$$\implies R_{\circlearrowright} \equiv e^{(\partial_\mu A\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu])dx^\mu dx^\nu}$$

$$R_{\circlearrowright} = e^{F_{\mu\nu}\, dx^\mu dx^\nu}$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]}$$

(Here $\equiv$ means equal up to 2^{nd} order differentials.)

Note. $A = A_\mu\, dx^\mu$

$$\begin{aligned}
dA + A \wedge A &= \partial_\mu A_\nu\, dx^\mu \wedge dx^\nu + A_\mu A_\nu dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) dx^\mu \wedge dx^\nu
\end{aligned}$$

$$\boxed{dA + A \wedge A = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu}$$

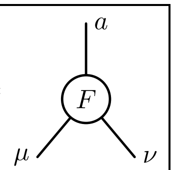
Holonomy around an infinitesimal loop measures the curvature of the gauge connection.

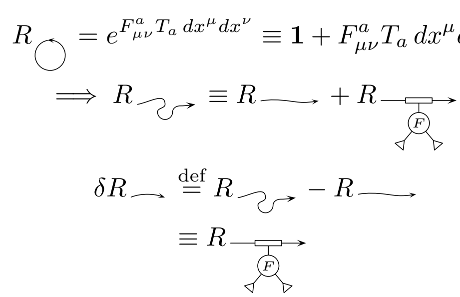
Now put in the Lie algebra: $[T_a, T_b] = f_{abc}T_c$.

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\
&= \partial_\mu A^a_\nu T_a - \partial_\nu A^b_\mu T_b + [A^a_\mu T_a, A^b_\nu T_b] \\
&= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T_a + A^a_\mu A^b_\nu [T_a, T_b] \\
&= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T_a + A^a_\mu A^b_\nu f_{abc} T_c \\
&= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T_a + A^b_\mu A^c_\nu f_{bca} T_a \qquad (f_{bca} = f_{abc}) \\
&= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f_{abc})T_a
\end{aligned}$$

$$\boxed{\begin{aligned} F_{\mu\nu} &= F^a_{\mu\nu} T_a \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f_{abc} \end{aligned}}$$

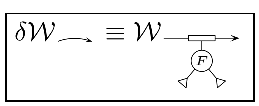
Don't forget this formula for $F^a_{\mu\nu}$! We will use it shortly.

NOTATION. $F^a_{\mu\nu} =$ 



Wilson Loop is measured by curvature insertion.

Since $\mathcal{W}_K = \mathrm{tr}(R_K)$,



2. Varying the Chern-Simous Lagrangian

Recall our definitions:

$$\mathcal{L}(A) = \int_{M^3} \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \qquad (CS(A) = \frac{1}{4\pi}\mathcal{L}(A)).$$

Here $M^3 = S^3$, but M^3 can be any can be any compact 3-manifold.

$$A(x) = A^a_i(x) T_a \, dx^i, \qquad [T_a, T_b] = f_{abc} T_c$$

f_{abc} totally antisymmetric in abc. $\mathrm{tr}(T_a T_b) = \delta_{ab}/2$.

Let $\mathcal{T} = \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$. Then

$$\begin{aligned}\mathcal{T} &= \mathrm{tr}(A_j^a \partial_k A_l^b T_a T_b + \frac{2}{3} A_j^a A_k^b A_l^c) dx^j \wedge dx^k \wedge dx^l \\ &= \varepsilon^{jkl} [A_j^a \partial_k A_l^b \,\mathrm{tr}(T_a T_b) + \frac{2}{3} A_j^a A_k^b A_l^c \,\mathrm{tr}(T_a T_b T_c)] dx^1 \wedge dx^2 \wedge dx^3\end{aligned}$$

$$\boxed{\mathcal{T} = \varepsilon^{jkl} [A_j^a \partial_k A_l^b \,\mathrm{tr}(T_a T_b) + \frac{2}{3} A_j^a A_k^b A_l^c \,\mathrm{tr}(T_a T_b T_c)] d\,vol}$$

NOTE. For a given choice of abc

$$\begin{aligned}\varepsilon^{jkl} A_j^a A_k^b A_l^c T_a T_b T_c &= \sum_{\pi \in Perm\{a,b,c\}} \mathrm{sign}(\pi) A_1^a A_2^b A_3^c T_{\pi a} T_{\pi b} T_{\pi c} \\ &= A_1^a A_2^b A_3^c [[T_a, T_b] T_c - [T_a, T_c] T_b + [T_b, T_c] T_a]\end{aligned}$$

Hence, going back to summations,

$$\begin{aligned}\varepsilon^{jkl} A_j^a A_k^b A_l^c \,\mathrm{tr}(T_a T_b T_c) &= A_1^a A_2^b A_3^c [f_{abi} \,\mathrm{tr}(T_i T_c) - f_{aci} \,\mathrm{tr}(T_i T_b) + f_{bci} \,\mathrm{tr}(T_i T_a)] \\ &= A_1^a A_2^b A_3^c [f_{abc}/2 - f_{acb}/2 + f_{bca}/2] \\ \big(&= \frac{2}{3} A_1^a A_2^b A_3^c f_{abc} \big) \\ &= \frac{1}{4} A_1^a A_2^b A_3^c [f_{abc} - f_{acb} + f_{bca} - f_{bac} + f_{cab} - f_{cba}] \\ &= \frac{1}{4} \varepsilon^{ijk} A_i^a A_j^b A_k^c f_{abc}\end{aligned}$$

$$\boxed{\mathcal{T} = \frac{\varepsilon^{jkl}}{2} \Big[\sum_a A_j^a \partial_k A_l^a + \frac{1}{3} \sum_{a,\,b,\,c} A_j^a A_k^b A_l^c f_{abc} \Big] d\,vol}$$

$$\mathcal{L}(A) = \int_{M^3} \mathcal{T}$$

$$\begin{aligned}\frac{\delta}{\delta A_l^r} (\varepsilon^{ijk} A_i^a A_j^b A_k^c f_{abc}) &= \varepsilon^{ljk} A_j^b A_k^c f_{rbc} + \varepsilon^{ilk} A_i^a A_k^c f_{arc} + \varepsilon^{ijl} A_i^a A_j^b f_{abr} \\ &= 3 \varepsilon^{ljk} A_j^b A_k^c f_{rbc} \qquad \text{(using antisymmetry of } \varepsilon \textit{ and } f).\end{aligned}$$

$$\int\limits_{M^3} d\,vol\,\frac{\delta}{\delta A_l^r}\Big(\varepsilon^{ijk}\sum_a A_i^a\partial_j A_k^a\Big)$$

$$= \int\limits_{M^3} d\,vol\,\varepsilon^{ijk}\sum_a\Big(\frac{\delta A_i^a}{\delta A_l^r} + A_i^a\,\frac{\delta[\partial_j A_k^a]}{\delta A_l^r}\Big)$$

$$= \int\limits_{M^3} d\,vol\Big(\varepsilon^{ljk}\partial_j A_k^r + \varepsilon^{ijk}A_i^a\,\frac{\partial_j[\delta A_k^a]}{\delta A_l^r}\Big) \quad [\delta\partial_j = \partial_j\delta]$$

integrating by parts

$$= \int\limits_{M^3} d\,vol\,\varepsilon^{ljk}\partial_j A_k^r - \int\limits_{M^3} d\,vol\,\varepsilon^{ijk}\partial_j A_i^a\,\frac{\delta A_k^a}{\delta A_l^r}$$

$$= \int\limits_{M^3} d\,vol\,\varepsilon^{ljk}\partial_j A_k^r - \int\limits_{M^3} d\,vol\,\varepsilon^{ijl}\partial_j A_i^r$$

$$= \int\limits_{M^3} d\,vol\,\varepsilon^{ljk}\partial_j A_k^r - \int\limits_{M^3} d\,vol\,\varepsilon^{kjl}\partial_j A_k^r$$

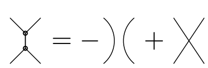
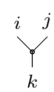
$$= 2\int\limits_{M^3} d\,vol\,\varepsilon^{ljk}\partial_j A_k^r$$

Hence

$$\boxed{\frac{\delta\mathcal{L}}{\delta A_i^a} = \int\limits_{M^3} d\,vol[\varepsilon^{ijk}\partial_j A_k^a + \frac{1}{2}\varepsilon^{ijk}A_j^b A_k^c f_{abc}}$$

Now we use:

$$\varepsilon_{rsi}\varepsilon^{ijk} = -\delta_s^j\delta_r^k + \delta_r^j\delta_s^k$$

 for 

$= \varepsilon^{ijk}$.

$$\varepsilon_{rsi}\frac{\delta\mathcal{L}}{\delta A_i^a} = \int\limits_{M^3} d\,vol[\varepsilon_{rsi}\varepsilon^{ijk}\partial_j A_k^a + \frac{1}{2}\varepsilon_{rsi}\varepsilon_{ijk}A_j^b A_k^c f_{abc}]$$

$$= \int\limits_{M^3} d\,vol[(-\partial_s A_r^a + \partial_r A_s^a) + \frac{1}{2}(-A_s^b A_r^c + A_r^b A_s^c)f_{abc}]$$

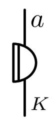
$$= \int\limits_{M^3} d\,vol[(\partial_r A_s^a - \partial_s A_r^a) + A_r^b A_s^c f_{abc}]$$

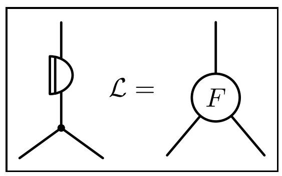
$$\boxed{\varepsilon_{rsi}\frac{\delta\mathcal{L}}{\delta A_i^a} = \int\limits_{M^3} d\,vol(F_{rs}^a)}$$

where F_{rs}^a is the curvature tensor for A.

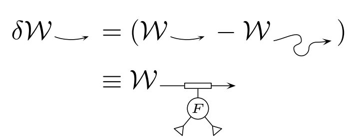
Another way to put this result (in our cryptic notation that does not write out the Dirac delta function):

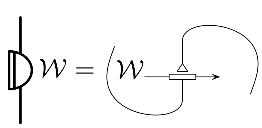
$$\boxed{\varepsilon_{rsi}\frac{\delta \mathcal{L}}{\delta A_i^a(x)} = F_{rs}^a(x)}$$

Recall  $\mathcal{L} = \dfrac{\delta \mathcal{L}}{\delta A_K^a(x)}$



Summary



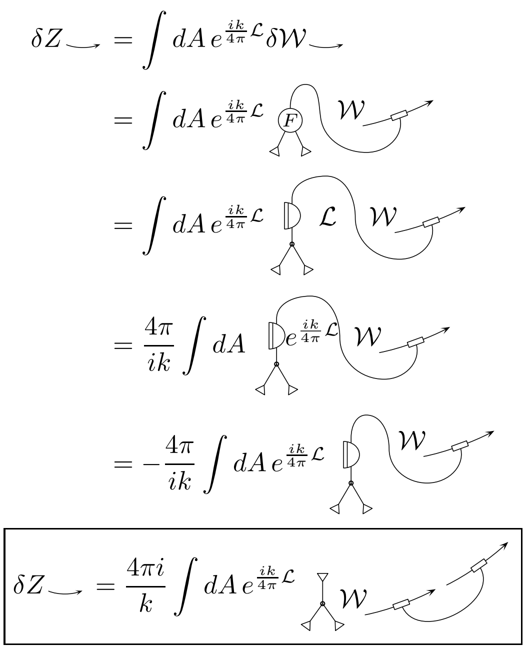


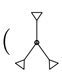
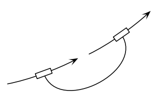
We are now ready to make a variational analysis of the functional integral formalism. We make the following two assumptions about the integration:

1) $\int dA\, X(\mathbb{D}Y) + \int dA\, (\mathbb{D}X)Y = \varnothing$ (Integration by parts)

2) $Z_K = \sum_{n=0}^{\infty} \left(\frac{1}{k}\right)^n Z_n(K)$ ($\frac{1}{k}$ expansion)

Now we compute δZ_K where $\delta Z = Z - Z$, the change Z_K under a small variation of the loop K:



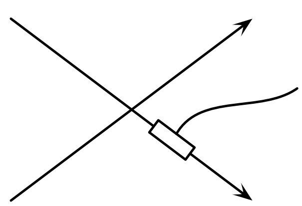
( = volume form,  = double Lie algebra insertion into Wilson line.)

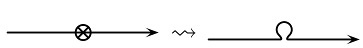
Conclusion. • If the loop is moved so as not to cross itself and not to trace out any volume in 3-space, then $\delta Z_K = \varnothing$. Hence Z_K *is an invariant of regular isotopy.*

• We need to look at

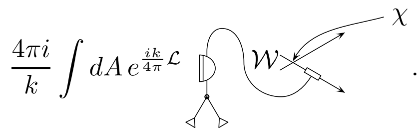
Lets begin by discussing $Z - Z$. Here denotes a Wilson loop that touches itself. (We postpone the case of multiple Wilson loops for the moment.) Note that we are making a sharp notational distinction between and .

$Z = Z - Z$ by *definition.* Let $\Delta = Z - Z$. Apply the variational argument to this with as the initial state and as the variational argument the first Lie algebra insertion occurs along :

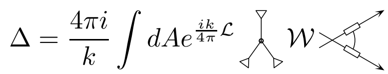


due to curvature:  .

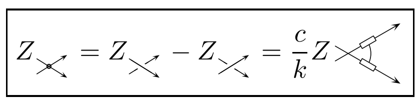
The next possibility for insertion occurs when we differentiate the Wilson line:



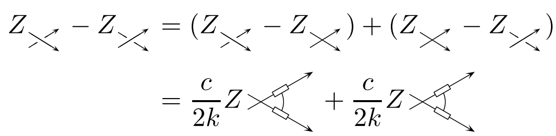
Since the upward movement of the line does not generate any volume the derivate will not contribute in the ↘ direction. However, we *are* differentiating a product and the other relevant term $(\mathbb{D}_a + x)$ is when we return trough x in the ↗ direction. This direction makes a non-zero angle with the area generated by the moving line. Thus we *will* receive a second insertion in the ↗ direction, giving



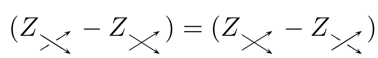
This means that up to normalizing the volume element we can state that there is a constant c such that



Here we write

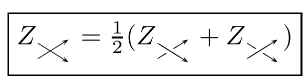


Note that we *assume* that $|\Delta_+| = |\Delta_-|$:



whence

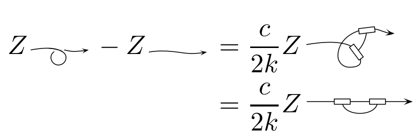
(∗)

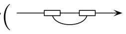
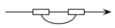
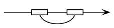


We also assume that Z (curl) $= Z$ (line).

In this context, (∗) actually only makes sense up to $\equiv (o(1/k))$. [For example, consider the Homfly polynomial H (curl) $= \alpha H$ (line), H (curl) $= \alpha^{-1} H$ (line). Then H (line) $\equiv H$ (curl) $\equiv \frac{1}{2}(H$ (curl) $+ H$ (curl)$) \equiv \frac{1}{2}(\alpha + \alpha^{-1})H$ (line) is OK if interpreted: $\alpha = e^{1/k}$, H (line) $= \frac{1}{2}(\alpha + \alpha^{-1})H$ (line) $+ o(1/k)$.] But the *differences* Z (crossing) $- Z$ (crossing) and Z (crossing) $- Z$ (crossing) are good to order $o(1/k^2)$ as desired. [e. g. H (curl) $- H$ (line) $= (\alpha - 1)H$ (line) $= (1 - \alpha^{-1})H$ (line) $+ o(1/k^2) =$ H (line) $- H$ (curl) $+ o(1/k^2)$.]

Now



The Casimir element (diagram) in the Lie algebra is central and, we can assume, diagonal in the representation. Thus in fact (by Schur's Lemma) (diagram) is a multiple of the identity. If $N = \dim$ of representative space and $\gamma = \mathrm{tr}($$)$ then  $= (\gamma/N)\mathbf{1}$. Anyway, since Wilson line takes trace, this means Z $= \frac{\gamma}{N} Z$ whence Z $- Z$ $= \frac{c}{2k}\frac{\gamma}{N} Z$

Z $= (1 + \dfrac{c}{2k}\dfrac{\gamma}{N}) Z$

Z $= e^{\frac{c\gamma}{2Nk}} Z$ $+ o(1/k^2)$.

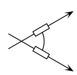
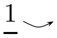
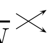
Similarly,

Z $= e^{-\frac{c\gamma}{2Nk}} Z$ $+ o(1/k^2)$.

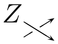
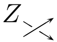
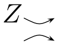
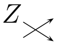
We have shown that the Lie algebra insertion and framing compensations familiar from Vassiliev invariants are implicit in the structure of Witten's functional integral!

3. $SU(N)$ Again

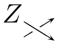
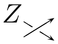
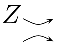
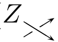
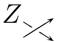
Recall the Fierz identity:

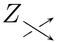
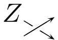
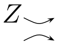
 $= \dfrac{1}{2}$ $- \dfrac{1}{2N}$.

Thus

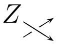
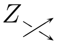
Z $- Z$ $\equiv \dfrac{c}{k}\left(\dfrac{1}{2} Z\right.$ $\left. - \dfrac{1}{2N} Z\right.$$\left.\right)$

$(x \equiv y \Leftrightarrow x = y + o(1/k^2))$

$\implies Z$ $- Z$ $\equiv \dfrac{c}{k}\left(\dfrac{1}{2} Z\right.$ $- \dfrac{1}{4N}(Z$ $+ Z$$)\left.\right)$

$\implies \left(1 + \dfrac{c}{4kN}\right) Z$ $- \left(1 - \dfrac{c}{4kN}\right) Z$ $\equiv \dfrac{c}{2k} Z$.

Let $q = 1 + \frac{c}{4k} + \dots$ such that

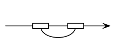
$q^{1/N} Z$ $- q^{-1/N} Z$ $= (q - q^{-1}) Z$.

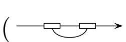
(This is *heuristic*. We would have to work harder to obtain the existence of such a q.)

Now

Z $- Z$ $\equiv \dfrac{c}{2k}\dfrac{\gamma}{N} Z$

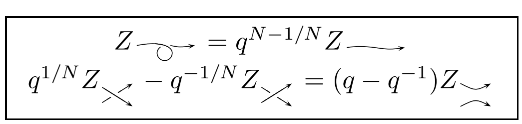
and here

 $= \mathrm{diag}((N^2-1)/2N, \dots, (N^2-1)/2N)$ ($N \times N$ matrix)

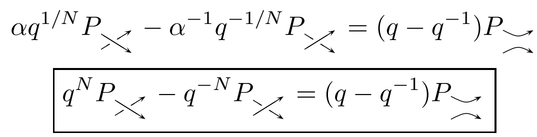
$\gamma = \mathrm{tr}($$) = (N^2-1)/2$

$\implies Z$ $\equiv \left(1 + \dfrac{c}{4k}\left(\dfrac{N^2-1}{N}\right)\right) Z$

So



Normalizing $P_K = \alpha^{-w(K)} Z_K$, $\alpha = q^{N-1/N}$, we have



This is the $\mathfrak{su}(N)$ specialization of the Homfly polynomial.

4. Links and Wilson Loops

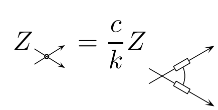
If $K = K_1 \sqcup K_2 \sqcup \ldots \sqcup K_s$ is a link with s components then Z_K is defined via

$$Z_K = \int dA\, e^{\frac{ik}{4\pi}\mathcal{L}(A)} \mathcal{W}_{K_1}(A)\mathcal{W}_{K_2}(A)\ldots\mathcal{W}_{K_s}(A).$$

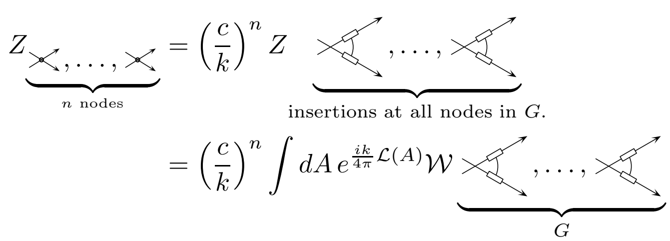
How does this affect the switching formula when $\searrow$ and $\nearrow$ are on different components? The answer is the same as before. The $\mathbb{D}$ operator is applied to the product $\prod\limits_i \mathcal{W}_{K_i}(A)$. $\searrow_x$ and $\nearrow_x$ occur in two factors of that product. If the Lie algebra element is originally inserted in $\searrow$, then to obtain $\neq 0$ volume , the result of $\mathbb{D}$ will insert in $\nearrow$ and we get the same formula as before.

5. The Vassiliev Invariants

We assumed $Z_K = \sum\limits_{n=0}^{\infty} Z_n(K)\left(\frac{1}{k}\right)^n$.



If G is a graph with n nodes, then, as in the usual argument, $Z_n(G)$ is independent of the embedding of $G \subset \mathbb{R}^3$.



We wish to articulate that part of this integral that has no $(1/k)$ dependence. To get a $(1/k)$ expansion of Z_K, *replace A by $A/\sqrt{k}$*:

$$\begin{aligned} Z_K &= \int dA e^{\frac{i}{4\pi}\int_{M^3} \mathrm{tr}(A\wedge dA)} e^{\frac{i}{4\pi\sqrt{k}}\int_{M^3}\mathrm{tr}(\frac{2}{3}A\wedge A\wedge A)} \mathcal{W}_K(A/\sqrt{k}) \\ &= \mathrm{tr}\Big(\prod_{x\in K}(\mathbf{1} + A(x)/\sqrt{k})\Big). \end{aligned}$$

This shows that, up to a scale factor, $\{Z_n(G)\}$ is precisely our (framed) Lie algebra weight system (for G with $\#G = n$).

6. More About the $\left(\frac{1}{k}\right)$-Expansion

$$Z_K = \int dA\, e^{\frac{i}{4\pi}\int_{M^3}\mathrm{tr}(A\wedge dA)} e^{\frac{i}{6\pi}\int_{M^3}\mathrm{tr}(\frac{1}{\sqrt{k}}A\wedge A\wedge A)}\, \mathrm{tr}\Big(\prod_{x\in K}(\mathbf{1}+A(x)/\sqrt{k})\Big).$$

$$\mathrm{tr}\Big(\prod_{x\in K}(\mathbf{1}+\frac{1}{\sqrt{k}}A(x))\Big) = \mathrm{tr}\Big(\mathbf{1}+\frac{1}{\sqrt{k}}\int_K A + \frac{1}{k}\iint_{K_1<K_2} A(x_1)A(x_2)+\ldots\Big)$$

where

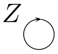
$$\int_{K_1<\ldots<K_n}\cdots\int A(x_1)A(x_2)\ldots A(x_n) = \int_{\underbrace{K\times\ldots\times K}_{n}=K^n}\cdots\int A(x_1)A(x_2)\ldots A(x_n)$$

$$\overrightarrow{x} = (x_1,x_2,\ldots,x_n)\in K^n \text{ with } x_1<x_2<\ldots<x_n.$$

This is an iterated integrals expression for the Wilson loop.

Remark. Strictly speaking, we should have defined

$$Z_K = \frac{\int dA\, e^{\frac{ik}{4\pi}\mathcal{L}(A)}\mathcal{W}(A)}{\int dA\, e^{\frac{ik}{4\pi}\mathcal{L}(A)}}$$

so that Z  $=1$

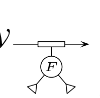
Remark. Suppose $\psi(A)$ is a (complex-valued) function on gauge fields. Define the *loop transform* of ψ by the formula

$$\widehat{\psi}(K) = \int \mathcal{L}A\psi(A)\mathcal{W}_K(A)$$

for $K\subset S^3$ any knot. The loop transform of a function on gauge fields is due to Lee Smolin, Ted Jacobson, and Carlo Rovelli and is used (as a short of giant Fourier transform) in their theory of quantum gravity. They use the loop transform to transfer the action of certain differential operators on the $\psi(A)$ to corresponding operators on the loops.

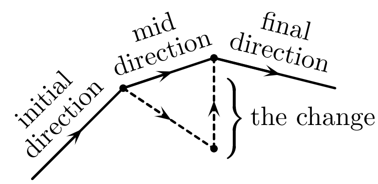
7. Remarks on Framing and Planarity

In our variational argument we wrote

$\delta\mathcal{W}$ ⌣ $=\mathcal{W}$  $\Big\}\quad F^a_{\nu\mu}T_a\, ax^\nu dx^\mu$

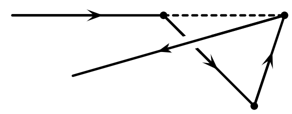
It is at this point in the derivation that the summation for the curvature insertion may actually be restricted to a proper subspace of $\mathbb{R}^3$, affording the possibility that the volume form appearing later may be equal to zero. In fact if the deformation of the line does not result in any self-touching, then we would be restricted to the proper subspace troughout the rest of the derivation.

Now consider what sorts of deformation will result in the proper subspace.



If the initial and final directions are in the same plane as the change, then we will be in a proper subspace, otherwise not. This is why an arbitrary Reidemeister 3-dimensional move can change the value of the formal integral.

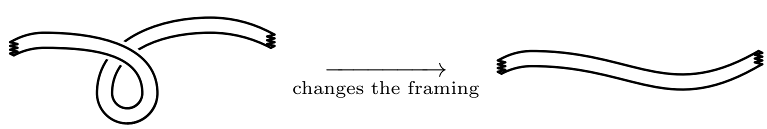
We shall say that a 3-dimensional Reidemeister move is *regular* if initial direction, mid-direction, final direction and the change all occur in a plane. Typically,



is *not* planar (hence not regular). We have shown that Z_K is invariant under regular 3-dimensional move.

In order to study Z_K as an invariant of knots and links, we can, in effect, choose a framing by representing K as a diagram with respect to a chosen plane. Then Z_K is an invariant of regular isotopy with respect to this plane and it can be normalized to give an invariant of ambient isotopy.

A framing on an arbitrary 3-space embedding of K should give us the same control with respect to Z_K. When we specify a diagrammatic projection plane, we are actually specifying a framing (the blackboard framing) as discussed earlier in these notes.



Problem. Think about the 3-dimensional normalization for Z_K with respect to a framing for K.

8. A Preview of Things to Come

In order to go move deeply into this functional integrals approach to knot and link invariants we need to take the gauge theory more seriously. In particular Z_K is an integral over gauge connections modulo gauge equivalence. One way to make better formal sense of this is to use "gauge fixing": A choice of restriction on the gauge so that each orbit under the action of the gauge group is represented exactly once. Depending on the particular method of gauge fixing, different features of the functional integral come forth.

A particular nice example of gauge fixing is the *axial gauge* used by Frölich and King. They write ($G = \mathfrak{su}(N)$) $M = S^3$, $x \in M$, $x = (x^+, x^-, t)$

$$A(x) = a_+(x)dx^+ + a_-(x)dx^- + a_0(x)dt$$

$$\boxed{a_-(x) \equiv 0} \text{ gauge choice.}$$

For $\mathcal{L}(A) = ikS(A)$, $S(A) = \frac{1}{4\pi}\int \operatorname{tr}(a_+\partial_- a_0)dx^+ \wedge dx^- \wedge dt$ and, after complexification, $x^+ = z \in \mathbb{C}$, $x^- = \overline{z} \in \mathbb{C}$ they find

$$\langle a_+^j(z,t)a_0^k(w,s)\rangle = -2\delta^{jk}\frac{\delta(t-s)}{z-w}$$
$$\langle a - a_j\rangle = \langle a_+a_+\rangle = \langle a_0a_0\rangle = 0.$$

This gives rise to a transport equation of the form

$$\frac{dQ}{dt} = \lambda \sum_{1\leqslant i<j\leqslant n} \frac{z_i' - z_j'}{z_i - z_j}\Omega_{ij}\phi_n, \qquad \Omega_{ij} = \sum_a I \otimes \ldots \otimes T_a \otimes \ldots \otimes T_a \otimes \ldots \otimes I$$

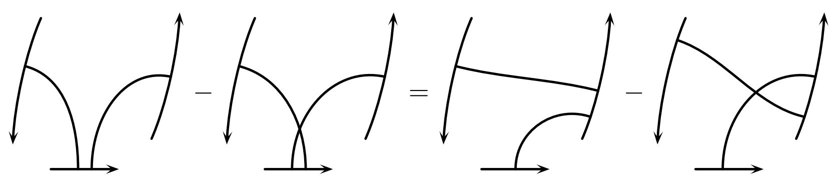
and is the underlying mechanism for *Kontsevich Integral*:

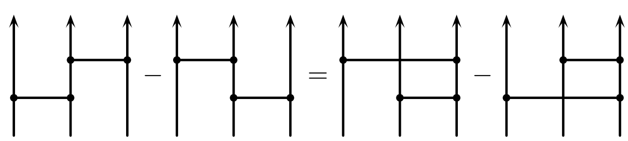
$$\sum_{m=0}^{\infty}(2\pi i)^{-m} \int_{t_1<\ldots<t_m} \sum_{\substack{\text{pairings}\\ P=\{(z_i,z_i')\}}} (-1)^{\#P\downarrow} D_P \bigwedge_{i=1}^{m}\Big(\frac{dz_i - dz_i'}{z_i - z_i'}\Big)$$

$$\Omega_n = \sum_{1\leqslant i\leqslant j\leqslant n} \Omega_{ij}\omega_{ij} \text{ flat } \Leftrightarrow \Omega_n \wedge \Omega = 0$$

$\Leftrightarrow \Omega_{ij}$ satisfies inf??????? braid relations

$\Leftrightarrow$ 4TR ((4-term relation??))





$$[t^{12}, t^{23}] = [t^{23}, t^{13}]$$

9. 3-Manifold Invariants — Preview

We can form

$$Z_{M^3} = \int DA\, e^{\frac{ik}{4\pi}\int_{M^3}\operatorname{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)}$$

for a given compact manifold and choice of Lie algebra $\mathfrak{g}$. Up to "framing" and blowing up and blowing down (see our section on Kirby calculus) this should be an invariant of 3-manifolds. Witten argued via field theory that for each compact surface S there should be a Hilbert space $\mathcal{H}(S)$ s.t. $S = \partial M_1$ (M_1 3-manifold), then $\exists\, \langle M_1| \in \mathcal{H}(S)$ *and* if $M = M_1 \cup_\partial M_2$ then $Z_{M^3} = \langle M_1|M_2\rangle$ for an appropriate product on $\mathcal{H}(S)$. This linearicas[2] the problem. He further argued that $D^2 \times S^1 \supset \circ \times S^1 = G$ can be regarded as $\mathcal{W}(G,\rho)$ (Wilson line for a given representation ρ of $\mathfrak{g}$). Then $\langle D^2 \times S^1; G, \rho|$ gives vector in $\mathcal{H}(S^1 \times S^1)$ for each representation ρ. Witten claimed *these span* $\mathcal{H}(S^1 \times S^1)$.

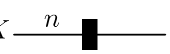
[2] ???

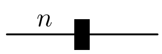
Conclusion. If $M^3 = M^3(K)$ — surgery on a link $K \subset S^3$, then $\exists$ constant c_ρ s.t.

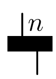
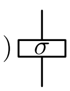
$$Z_{M^3} = \sum_\rho c_\rho Z_{S^3,K,\rho}$$

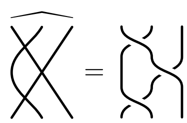
where $K = K_1 \cup \ldots \cup K_n$ and K, ρ really means various ρ's to different composers of K. The 3-manifold invariant is expressed as a sum over invariants of links with different representations of $\mathfrak{g}$ attached to their components.

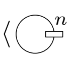
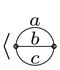
Rigorous combinatorial models exist! For $\mathfrak{su}(2)$ one can take

$\langle K \rangle_{[n]} = \langle K$  $\rangle$

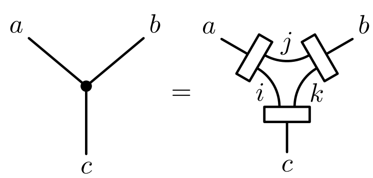
where  is an n-symmetrizer (n-parallel lines)

 $= \dfrac{1}{\{n\}!} \sum_{\sigma \in S_n} (\text{coeff}^{+(\sigma)})$ ,

e. g.  braid lift of permutation

$\Delta_n = \langle$  $\rangle$, $\theta(a,b,c) = \langle$  $\rangle$

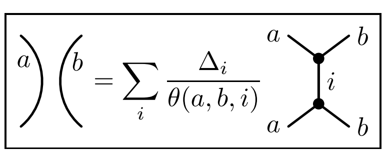
where



$$\left.\begin{array}{r} i + j = a \\ j + k = b \\ i + k = c \end{array}\right\}$$

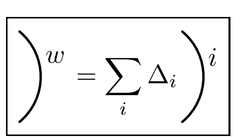
extends $\langle\ \rangle_{[n]}$ to graphs with 3-vertices.

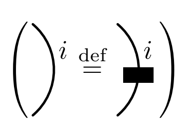
One can show:



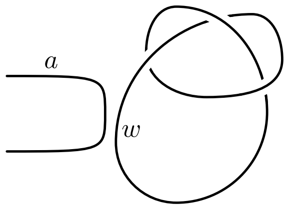
(Finite sum by letting $A = e^{i\pi/r}$ in $\langle K \rangle$.)

Let

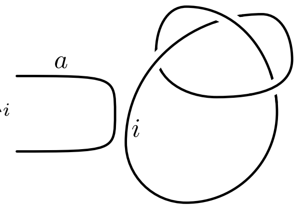




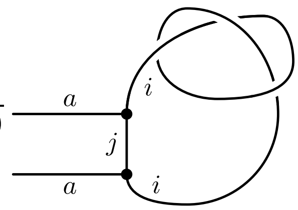
Then



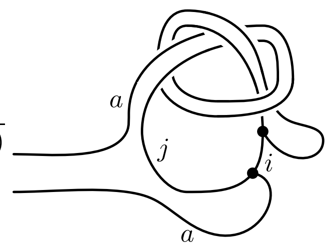
$= \sum_i \Delta_i$



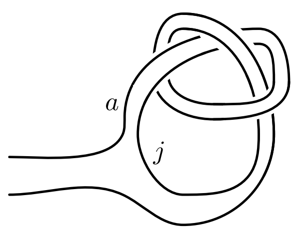
$= \sum_i \Delta_i \sum_j \frac{\Delta_j}{\theta(a,i,j)}$



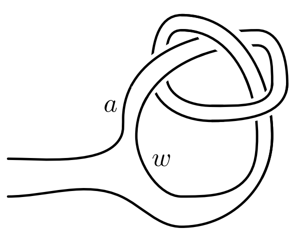
$= \sum_j \Delta_j \sum_i \frac{\Delta_i}{\theta(a,i,j)}$



$= \sum_j \Delta_j$



$=$



(We mean compute bracket polynomial all the time.)

MORAL. $d(K) = \sum_i \Delta_i \langle K \rangle_{[i]}$ is handle-sliding invariant. An appropriate normalization of $d(K)$ yields the $\mathfrak{su}(2)$ 3-manifold invariant and *it matches with* $Z_{M^3} = \int DA\, e^{\frac{ik}{4\pi}\mathcal{L}_{M^3}(A)}$!

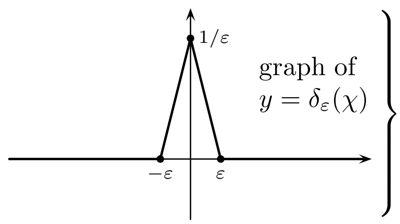
Much more work is being done and remains to be done in this domain. One of the most interesting problems is to prove in the combinatorial model the conjectures about $Z_{M^3}(k)$ as $k \to \infty$. This, by the functional integral, is dominated by the flat (curvature zero) gauge connections on M^3. Numerical evidence for $\mathcal{L}(K)$ (e. g. by finite Fourier transforms of the values of $\mathcal{L}(K)$) shows that it works. Many exact calculations in special cases show this as well.

End of the IHP/CEB Course
December, 1977

Section. Linking Numbers, Gauge Fixing and Pertarbative Expansion
This will be section VI. of the notes, but first we shall .. ..
..

10. V. Various remarks

(A) Lets begin by recalling the Dirac delta function: $\delta(x)$.
$\delta(x)$ is not a function really, the idea is:



$$\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = 1$$

$$\delta(x) = \text{“}\lim_{\varepsilon\to 0}\text{”}\, \delta_\varepsilon(x)\colon \begin{cases} \delta(0) = \infty \\ \delta(x) = 0 \qquad x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

This is justified in the theory of distributions.

NOTE.

$$1 = \int d(f(x))\delta(f(x)) = \int dx\, f'(x)\delta(f(x)) = \int\limits_{x\colon f(x)=0} dx(f'|_{f=0}\delta(fx))$$

---

11. Functional Derivatives

$$\frac{\delta F[f(x)]}{\delta(fx)} = \lim_{\varepsilon\to 0} \frac{F(f(x)+\varepsilon\delta(x-y)) - F(x)}{\varepsilon}$$

$$\text{e. g.} \quad F[f(x)] = f(x)^n$$

$$\frac{\delta F[f(y)]}{\delta fx} = n(f(y))^{n-1}\delta(y-x)$$

$$\text{and} \quad \frac{\delta}{\delta fx}\int F[f(y)]\, dy = \int n(f(y))^{n-1}\delta(y-x)dy = n(f(x))^{n-1}$$

Apply these definitions to our previous formal work differentiating the Wilson loop.

Note that

$$\frac{\delta f(y)}{\delta f(x)} = \frac{f(y)+\varepsilon\delta(y-x) - f(y)}{\varepsilon} = \delta(y-x)$$

and

$$\frac{\delta \int_a^b f(y)dy}{\delta f(x)} = \int\limits_a^b \delta(y-x)dx = 1.$$

We can also write $\frac{\delta F[f]}{\delta f}$ where there is no x dependence in the denominator. Thus

$$\frac{\delta f^n}{\delta f} = nf^{n-1}$$

and

$$\frac{\delta}{\delta f}\int f^n(y)dy = \int\limits_a^b nf^{n-1}(y)dy.$$

Putting in the x inserts a delta function in the integral.

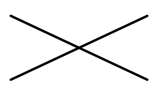
What about different functions, as in $\int\limits_a^b f^2(y)g(y)dy$

$$\boxed{\frac{\delta g}{\delta f} = 0 \qquad \text{if } g \text{ is not a function of } f.}$$

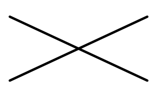
$$\Longrightarrow \frac{\delta \int_a^b f^2 g\, dy}{\delta f} = \int\limits_a^b 2f(y)g(y)dy$$

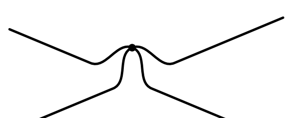
$$\text{and} \quad \frac{\delta \int_a^b f^2 g\, dy}{\delta f(x)} = \int_a^b 2f(y)g(y)\delta(x-y)dy$$
$$= 2f(x)g(x).$$

It is in this sense that we did variational calculus in the last section.

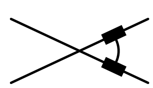
(B) Consider *just-touching* Wilson lines:  .

What do our variational arguments tell us about these, assuming we move such a point as a ring vertex.





In such a deformation we are moving two parts of the Wilson loop simultaneously. This should be a multiple of the standard insertion

 .

(C) Smolin and Rovelli* use the functional integral as a "Fourier transform": Given $\psi(A)$ a function on gauge connections A, they define the *loop transform* $\widehat{\psi}(K)$

---

*See e. g. Lee Smolin. Quantum Gravity in the Self-Dual Representation. In Contemp. Math. Vol. 71, 1988, pp. 55–97.

defined on loops $K \subset \mathbb{R}^3$ (or $K \subset M^3$ a given 3-manifold) via:

$$\widehat{\psi}(K) = \int DA\psi(A)\mathcal{W}_K(A).$$

One can then transfer operators to loops formally as shown below:

Let $\Delta\psi$ be a differential operator applied to ψ. Then

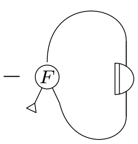
$$\widehat{\Delta\psi}(K) = \int DA(\Delta\psi)\mathcal{W}_K = -\int DA\psi(A)(\Delta\mathcal{W}_K(A))$$

(via integration by parts.)

Since $\Delta\mathcal{W}_K$ can often be reformulated in terms of insertion operations on the loops, this provides a new language!

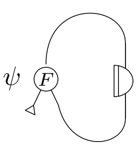
Here are some examples.

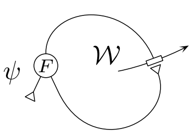
1°. Let

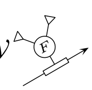
$$\Delta = -$$

$$= -\sum_{r,a,s} F^a_{r,s}\, dx^r \frac{\delta}{\delta A^a_s}.$$

Then

$$\widehat{\Delta\psi} = +\int DA(\Delta\psi)\mathcal{W}_{\smile}$$

$$= -\int DA\,\psi\Delta\mathcal{W}_{\smile} = \int DA\,\psi$$

$$\mathcal{W}_{\smile}$$

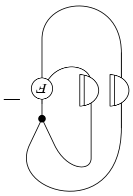
$$= \int DA\,\psi$$


$$= \int DA\,\psi\mathcal{W}$$


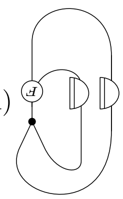
$$= \int DA\,\psi(A)[\mathcal{W}_{\smile} - \mathcal{W}_{\sim}].$$

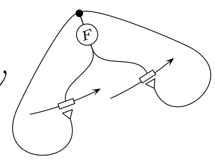
Thus the operator Δ translates into a loop deformation operator, and $\Delta\psi = 0$ corresponds to $\widehat{\psi}(K)$ a link invariant! In the Ashtekar Theory $\Delta\psi = 0$ is called the *diffeomorphism constraint.* It is supposed to be globally satisfied by the functions $\psi(A)$ in this theory. As we know, even in the case of the chern simous functional $\boxed{\psi(A) = e^{\frac{ik}{4\pi}CS(A)}}$ we at best get $\widehat{\Delta\psi}(\longrightarrow) = 0$ for flat deformations of the loop. It would be interesting to know what sort of analytic constraint this would means. Perhaps the theory can be used with such a restraint. Then framed knot invariants would be directly related to the Ashtekar theory.

2°. Let

$\mathcal{H} = -$  .

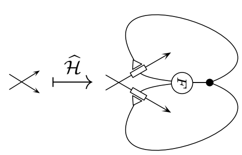
Then

$\widehat{\mathcal{H}\psi} = \int DA\,\psi(A)$  $\mathcal{W}$

$= \int DA\,\psi\mathcal{W}$  .

$\boxed{\mathcal{H}\psi = 0}$ is called the *Hamiltonian Constraint* in the Ashtekar theory.

In order for $\mathcal{H}\psi$ to be non-zero, $\mathcal{H}$ must act at an intersection of loops.

 .

(Otherwise the epsilon  causes $\widehat{\mathcal{H}\psi}$ to vanish.)

This apparently means that (framed) link invariants are fundamental for quantum gravity!

(D) *Gaussian Integrals and their Infinite Dimensional Generalizations*

Let

$$\Delta = \int_{-\infty}^{\infty} e^{-\lambda x^2/2}\,dx.$$

Then

$$\begin{aligned}
\Delta^2 &= \iint_{\mathbb{R}^2} e^{-\frac{\lambda(x^2+y^2)}{2}} dxdy = \iint_{\mathbb{R}^2} e^{-\frac{\lambda(x^2+y^2)}{2}} r\, dr d\theta \\
&= 2\pi \int_0^\infty e^{-\frac{\lambda r^2}{2}} r\, dr = \frac{+2\pi}{\lambda} \int_0^\infty e^{-\frac{\lambda r^2}{2}} d(\lambda r^2) \\
&= \frac{2\pi}{\lambda} \int_0^\infty e^{-t}\, dt = \frac{2\pi}{\lambda} (-e^{-t})\big|_0^\infty = 2\pi/\lambda
\end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\lambda x^2/2} dx = \sqrt{2\pi/\lambda}} \qquad (\lambda > 0)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}+Jx} &= \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2}(x^2-2/\lambda Jx+J^2/\lambda^2)} + \frac{\lambda}{2}\frac{J^2}{\lambda^2} \\
&= \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2}(x-J/\lambda)^2} e^{J^2/2\lambda}
\end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}+Jx} = \sqrt{\frac{2\pi}{\lambda}} e^{J^2/2\lambda}}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}} x^N = \left(\frac{\delta^N}{\delta J^N}\right) \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}+Jx}\Big|_{J=0} = \sqrt{\frac{2\pi}{\lambda}} \left(\frac{\delta}{\delta J}\right)^N e^{J^2/2\lambda}\Big|_{J=0}$$

$$g(J) = e^{J^2/2\lambda}\sqrt{\frac{2\pi}{\lambda}}$$

$$\begin{aligned}
\frac{\partial g}{\partial J}\Big|_{J=0} &= \frac{J}{\lambda} e^{J^2/2\lambda}\sqrt{\frac{2\pi}{\lambda}}\Big|_{J=0} \\
\frac{\partial^2 g}{\partial J}\Big|_{J=0} &= \left(\frac{1}{\lambda} e^{J^2/2\lambda}\sqrt{\frac{2\pi}{\lambda}} + \left(\frac{J}{\lambda}\right)^2 e^{J^2/2\lambda}\sqrt{\frac{2\pi}{\lambda}}\right)\Big|_{J=0} \\
&= \frac{1}{\lambda}\sqrt{\frac{2\pi}{\lambda}}
\end{aligned}$$

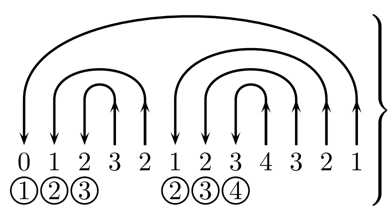
$h(J) = e^{J^2/2}$. We want to evaluate $\frac{\partial^N h}{\partial J^N}\Big|_{J=0}$.

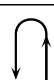
$$\begin{cases} \dfrac{\partial h}{\partial J} = Je^{J^2/2} = Jh \\ \dfrac{\partial^2 h}{\partial J^2} = h + J^2 h \\ \dfrac{\partial^3 h}{\partial J^3} = Jh + 2Jh + J^3 h = 3Jh + J^3 h \\ \dfrac{\partial^4 h}{\partial J^4} = 3h + 3J^2 h + 3J^2 h + J^4 h \\ \qquad = 3h + 6J^2 h + J^4 h \\ \qquad \text{Let } D = \frac{\partial}{\partial J}: \\ D^5 h = 3Jh + 12Jh + 6J^3 h + 4J^3 h + J^5 h \\ \qquad = 15Jh + 10J^3 h + J^5 h \\ D^6 h = 15h + 30J^2 h + 10J^4 h + 5J^4 h + J^6 h \\ \qquad = 15h + 45J^2 h + 15J^4 h + J^6 h \end{cases}$$

So $D^N h\big|_{J=0} \neq 0$ only if N is even.
Then think of $\uparrow$ as bringing down a J and $\downarrow$ as eliminating a J.

$$D^N(J^K h) = (\downarrow J^K)h + J^K(\uparrow h).$$

The terms that contribute to $|_{J=0}$ are sequences $\uparrow \ldots \downarrow\uparrow\downarrow\uparrow$ that sum to zero and never go below zero.

e. g. 

Pair off …  … numerical factor: $4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = 4!3!$ and $\left(\frac{1}{\lambda}\right)^6 = \left(\frac{1}{\lambda}\right)^{N/2}$

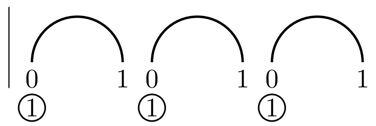
Note that:
$\downarrow J^K = KJ^{K-1}$ so down arrows contribute K.
$\uparrow g = \frac{J}{\lambda} g$ so up arrows contribute $(1/\lambda)$.
Let

$Cap(N) =$ 
$(n = 12)$

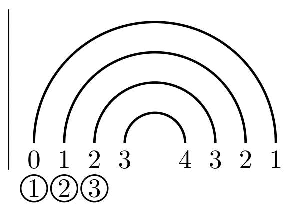
$= \{$well-formed parenthesisstructures with $N/2$parentheses pairs.$\}$

For each $\alpha \in Cap(N)$ define $|\alpha|$ = numerical contribution obtained from corresponding differentiation rules.

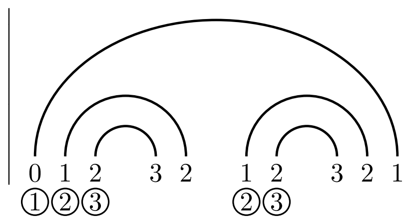
e. g.



$= 1 \cdot 1 \cdot 1 = 1$



$= 3!$



$= (3 \cdot 2)(3 \cdot 2 \cdot 1) = (3!)^2$

$$\Longrightarrow \boxed{\left(\sum_{\alpha \in Cap(N)} |\alpha|\right)\left(\frac{1}{\lambda}\right)^{N/2} = D^N(e^{-J^2/2\lambda})\Big|_{J=0}}$$

But[4]

$$e^{J^2/2\lambda} = \sum_{n=0}^{\infty} \frac{J^{2n}}{(n)!}\left(\frac{1}{\lambda}\right)\frac{1}{2^n}$$

$$\begin{aligned}
\sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}+Jx} &= \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} \frac{J^n}{n!} \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda x^2}{2}} x^n \\
&= \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} \frac{J^n}{n!} \sqrt{\frac{2\pi}{\lambda}} \left[D^n e^{J^2/2\lambda}\Big|_{J=0}\right] \\
&= \sum_{n=0}^{\infty} \frac{J^n}{n!} \Bigg(\sum_{\alpha \in Cap(2n)}\Bigg)\left(\frac{1}{\lambda}\right)^n
\end{aligned}$$

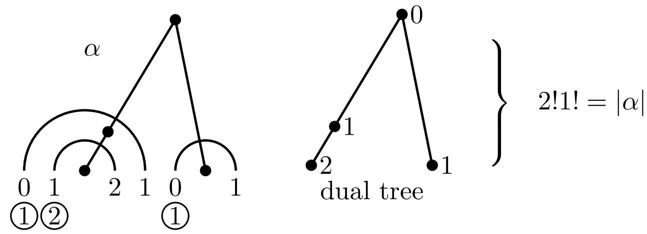
$$\Longrightarrow \frac{1}{n!2^n} = \frac{1}{(2n)!} \sum_{\alpha \in Cap(2n)} |\alpha|$$

$$\boxed{\sum_{\alpha \in Cap(2n)} |\alpha| = \frac{(2n)!}{n!2^n}}$$

We have proved this using integrals. Try to prove it by pure combinatorics!

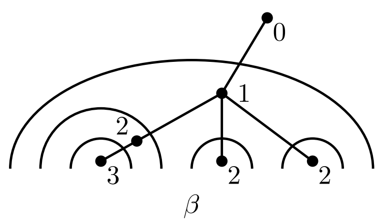
[4] kak oformit?

12. The Combinatorics of Parentheses



From the point of view of the tree dual to the parenthesis structure, the norm $|\alpha|$ of the parenthesis structure α is a product of depth counts.

e. g.

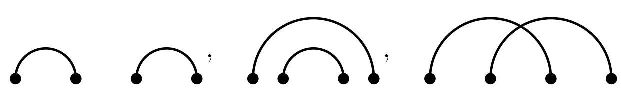


$|\beta| = 2^3 \cdot 3$

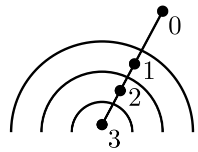
$$\begin{aligned}\frac{(2n)!}{2^n n!} &= \frac{(2n)(2n-1)(2n-2)\dots(1)}{2^n n!}\\ &= \left(\frac{(2n)(2n-2)(2n-4)\dots(2)}{2^n n!}\right)(2n-1)(2n-3)\dots(1)\\ &= (2n-1)(2n-3)(2n-5)\dots(1) = (2n-1)!!\end{aligned}$$

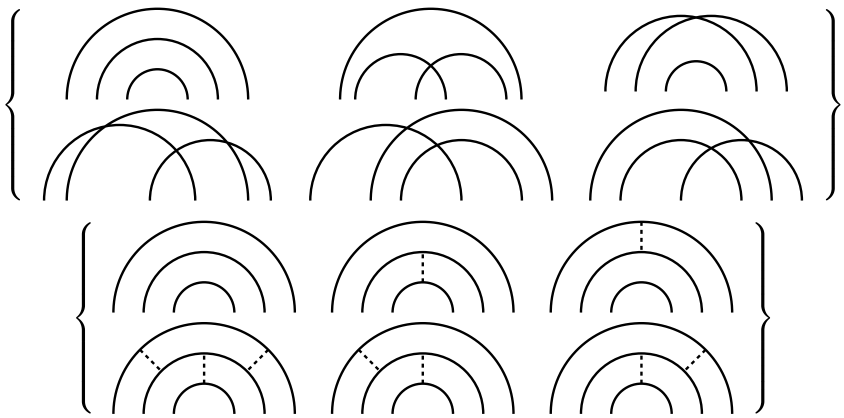
Now note that $(2n-1)(2n-3)\dots(1)$ is equal to the number of *unrestricted pairings* of a row of $2n$ points.

e. g. $n = 2$: $(4-1)(4-3) = 3 \cdot 1 = 3$

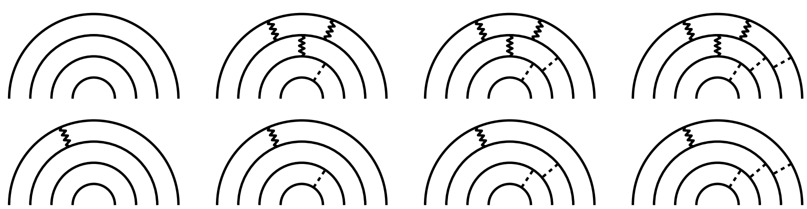


Now look:





Each time you add one more depth (n) you get n choices to add on: e. g.



systematically always go to the right for the new "paths".

This gives a combinatorial proof that

$$\sum_{\alpha \in Cap(2n)} |\alpha| = (2n-1)(2n-3)\dots(1).$$

Note also that

$$\frac{(2n)!}{2^n n!} = \frac{(n+1)!}{2^n}\left[\frac{1}{n+1}\binom{2n}{n}\right]$$

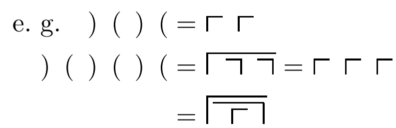
It terms out that $Cap(2n)$ has cardinality $C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} *Catalan Number*.

It is quite interesting to see why $\frac{1}{n+1}\binom{2n}{n}$ counts the number of parenthesis structures on $2n$ points. We will digress with two proofs of this fact. There may be more proofs lurking in the background!

(i) Here is a purely combinatorial way to count $|Cap(2n)|$.

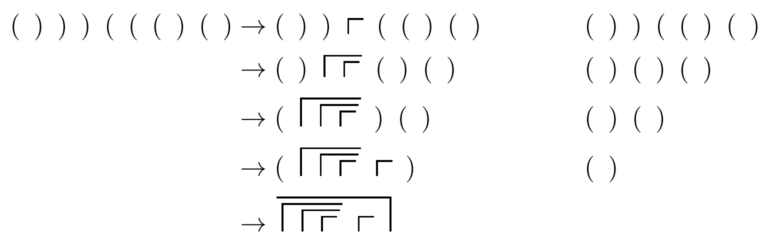
Start by looking at all sequences of (and) with n)'s and n ('s — not necessarily we formed. e. g.) ( ) (is such a sequence for $n = 2$.

FACT. Any such sequence can be constructed as an expression in *parentheses* $(\times) = \overline{\times}\rceil$ and *anti-parentheses* $)\times(= \lceil\overline{\times}$



As the second example shows, such decompositions are not unique.

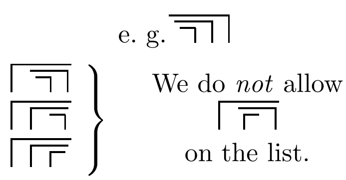
We will obtain a decomposition into parentheses and anti-parentheses by inductively taking each) (occurrence as an ¬ and containing as though the ¬'s are not present. The residue is written in ¬ forms. Next page for examples:



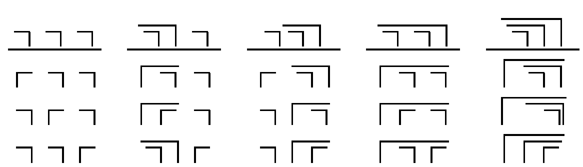
The decomposition is not unique.

However, list all parenthesis structures and associate to each one a structure collection with anti-parenthesis via the following rule: The original structure is obtained from any mixed *s* structure by flipping *one* anti-parenthesis and all the (anti-)parentheses containing it.

A mixed structure is said to be in *standard form* if it has this "one-flip" property.

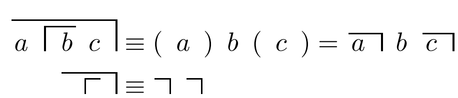


For $n = 3$, the complete list is:



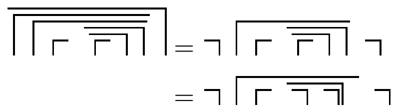
This process generates $(n+1)\#\,Cap(2n)$ distinct mixed structures. In fact it creates *all* mixed structures of) and (because the procedure we outlined for conversion can be construed to arise from flipping one) (and all) ('s containing it (non-uniquely). For example in the procedure above, we take it to the stage (⌐⌐) ( ) but stop here and convert to ⌐⌐ ¬ since the deleted sequence is already well-formed.

NOTE.

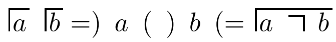


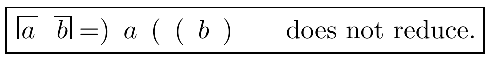
Equivalent expressions are obtained by *idemposition* (mod 2 cancellation on the horizontal boundary).

Thus:

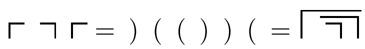


Apply ⌐¬ = ¬ ¬ repeatedly to put the mixed expression into standard form*. Note that you may also need to use

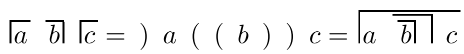




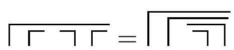
But



and



e. g.



Each expression can be converted into standard form, and hence arranged in the $(n+1)\#\,Cap(2n)$ list.

But there are *exactly* $\binom{2n}{n}$ expressions since these are strings of), (with n)'s and n ('s. Therefore

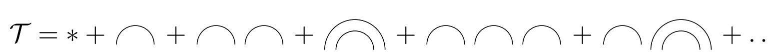
$$(n+1)\#\,Cap(2n) = \binom{2n}{n}.$$

Combing this with our previous work, we have shown

$$\sum_{\alpha\in Cap(2n)} |\alpha| = \frac{(n+1)!}{2^n}|\,Cap(2n)|$$

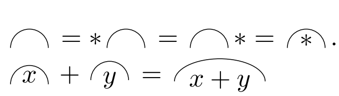
where $|\,Cap(2n)|$ denotes the cardinality of the set $Cap(2n)$.

There are other ways to count $|\,Cap(2n)| = \frac{1}{n+1}\binom{2n}{n}$. Here is a quick sketch of another method I find particularly amusing:



Formal sum of all cap structures.

NOTATION.



---

*You can prove that the *standard form is unique* by obtaining it by a canonical procedure:
1°. Do $a\,b\,c \Rightarrow a\,b\,c$ from deepest spaces upward.
2°. Do $a\,b\,c = a\,b\,c$ from top down.

Then

$$\mathcal{T} = \overset{\frown}{\mathcal{T}}\,\mathcal{T} + *.$$

Let F = result of replacing $* \mapsto 1$, $\frown \mapsto x$.

$$F = \sum_{n=0}^{\infty} d_n x^n, \qquad d_n = |\,Cap(2n)|.$$

Above equation $\implies$

$$\begin{gathered} F = xF^2 + 1 \\ xF^2 - F + 1 = 0 \\ F^2 - x^{-1}F + x^{-1} = 0 \\ F = (x^{-1} \pm \sqrt{x^{-2} - 4x^{-1}})/2 \end{gathered}$$

Newton $\Longrightarrow$

$$\begin{aligned}
\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \dots \\
&= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n
\end{aligned}$$

$$\begin{aligned}
\binom{1/2}{n} &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-n+1\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{-2n+3}{2}\right)}{n!} \\
&= (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}{2^n n!} \\
&= (-1)^{n-1}\frac{(2n-2)!}{2^{2n-1}n!(n-1)!}
\end{aligned}$$

$$\binom{1/2}{n} = \frac{(-1)^{n-1}}{2^{2n-1}n}\binom{2(n-1)}{n-1}$$

$$F = \frac{x^{-1} \pm x^{-1/2}\sqrt{x^{-1}-4}}{2}$$

$$\begin{aligned}
\sqrt{x^{-1}-4} &= \sum_{n=0}^{\infty}\binom{1/2}{n}(-4)^n (x^{-1})^{\frac{1}{2}-n} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2^{2n-1}n}\binom{2n-2}{n-1}(-1)^n 2^{2n} x^{-1/2} x^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)x^{-1/2}}{2^{-1}n}\binom{2n-2}{n-1}x^n \\
&= x^{-1/2} + \sum_{n=1}^{\infty}\frac{(-1)x^{-1/2}}{2^{-1}n}\binom{2n-2}{n-1}x^n
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\ F &= \frac{x^{-1} - x^{-1/2}\sqrt{x^{-1}-4}}{2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1} = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n.
\end{aligned}$$

This gives an alternative proof that

$$\frac{1}{n+1}\binom{2n}{n} = |\,Cap(2n)|.$$

---

13. Generalizing the Gaussian Integral

We worked with

$$\Delta = \int_{-\infty}^{\infty} e^{-\lambda x^2/2}dx.$$

Now let $\overrightarrow{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and A be an $n \times n$ symmetric matrix. Consider

$$\Delta = \int\limits_{\mathbb{R}^n} e^{-\overrightarrow{x} A \overrightarrow{x}^T/2} d\overrightarrow{x} \qquad \text{where} \quad d\overrightarrow{x} = dx_1 dx_2 \ldots dx_n.$$

A real symmetric matrix can be diagonalized by a rotation R:

$$RAR^{-1} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Volume does not change under a rotation. Hence we can replace A by the diagonal matrix of its eigenvalues.

Then

$$\begin{aligned} \Delta &= \int\limits_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^{n} \lambda_i x_i^2/2} dx_1 dx_2 \ldots dx_n \\ &= \prod_{i=1}^{n} \int\limits_{-\infty}^{\infty} e^{-\lambda_i x^2/2} dx_i = \prod_{i=1}^{n} \sqrt{\frac{2\pi}{\lambda_i}} \\ \Delta &= (2\pi)^{n/2} / \sqrt{\mathrm{Det}(A)} = (2\pi)^{n/2} \sqrt{\mathrm{Det}(A^{-1})} \end{aligned}$$

All this works nicely so long as we assume that A is invertible (or as the physicists say "no zero models").

$$\boxed{\int\limits_{\mathbb{R}^n} e^{-\overrightarrow{x} A \overrightarrow{x}^T/2} d\overrightarrow{x} = (2\pi)^{n/2} / \sqrt{\mathrm{Det}(A)}}$$

Now look at

$$Z(J) = \int\limits_{\mathbb{R}^n} e^{-\overrightarrow{x} A \overrightarrow{x}/2 + J^T \overrightarrow{x}} d\overrightarrow{x}$$

A an $n \times n$ symmetric and invertible matrix; $J \in \mathbb{R}^n$; $\overrightarrow{x}$ = column; $\overrightarrow{x}^T$ = row.
We assume A has positive eigenvalues.

Let $\langle x, y \rangle = x^T A y$ Then

$$\begin{aligned} \langle x + A^{-1}J, x + A^{-1}J \rangle &= \langle x, x \rangle + \langle A^{-1}J, x \rangle + \langle x, A^{-1}J \rangle + \langle A^{-1}J, A^{-1}J \rangle \\ &= \langle x, x \rangle + 2\langle A^{-1}J, x \rangle + \langle A^{-1}J, A^{-1}J \rangle \\ &= \langle x, x \rangle + 2(A^{-1}J)^T Ax + (A^{-1}J)^T A A^{-1} J \\ &= \langle x, x \rangle + 2J^T A^{-1} Ax + J^T A^{-1} A A^{-1} J \qquad (A = A^T) \\ &= \langle x, x \rangle + 2J^T x + J^T A^{-1} J. \end{aligned}$$

Therefore,

$$\begin{aligned}
Z(J) &= \int\limits_{\mathbb{R}^n} dx\, e^{-\langle x,x\rangle/2 + J^T x} \\
&= \int\limits_{\mathbb{R}^n} dx\, e^{-\frac{1}{2}\langle x + A^{-1}J, x + A^{-1}J\rangle + \frac{1}{2}J^T A^{-1} J} \\
&= \int\limits_{\mathbb{R}^n} dx\, e^{-\frac{1}{2}\langle x,x\rangle} e^{+\frac{1}{2}J^T A^{-1} J} \\
Z(J) &= Z(0) e^{+\frac{1}{2}J^T A^{-1} J}
\end{aligned}$$

Now $Z(0)$ can be computed by diagonalizing A. This can be accomplished by a rotation R of $\mathbb{R}^n$:

$$RAR^{-1} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Hence

$$\begin{aligned}
Z(0) &= \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}(\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2)} = \prod_{i=1}^{n} \int_{\mathbb{R}} dx\, e^{-\lambda_i x^2/2} \\
Z(0) &= (2\pi)^{n/2} / \sqrt{\operatorname{Det}(A)} \\
\int\limits_{\mathbb{R}^n} dx\, e^{-\langle x,x\rangle/2 + J_1 x_1 + \ldots + J_n x_n} &= \frac{(2\pi)^{n/2}}{\sqrt{\operatorname{Det}(A)}} e^{\frac{1}{2}J^T A^{-1} J}
\end{aligned}$$

whence

$$\frac{1}{\sqrt{2\pi}^n} \int\limits_{\mathbb{R}^n} dx\, e^{-\langle x,x\rangle/2} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} = \frac{1}{\sqrt{\operatorname{Det}(A)}} \frac{\partial^{|\alpha|}}{\partial J_1^{\alpha_1} \ldots J_n^{\alpha_n}} e^{\frac{1}{2}J^T A^{-1} J}\Big|_{J=0}.$$

This generalized the 1-variable calculation. e. g.

$$\begin{aligned}
\frac{\partial^2}{\partial J_i \partial J_j} e^{\frac{1}{2}J^T A^{-1} J}\Big|_{J=0} &= \frac{\partial^2}{\partial J_i \partial J_j} e^{\frac{1}{2}\sum\limits_{k,l} J_k A^{-1}_{kl} J_l}\Big|_{J=0} \\
&= A^{-1}_{kl} \overset{\text{def}}{=} \overbracket{x_i x_j} \overset{\text{def}}{=} \langle x_i x_j \rangle_0.
\end{aligned}$$

This tells us that *all the moments* (*of* $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$) *can be expressed in terms of moments of order two.*

The derivative $\partial^{|\alpha|}/\partial J_1^{\alpha_1} \ldots J_n^{\alpha_n}$ gives rise to $(2n)!/2^n n! = (2n-1)!!$ terms just as in the one dimensional case. This is the number of ways to form the pairs

$$\overbracket{x_{i_1} x_{i_2}} \overbracket{x_{i_3} x_{i_4}} \ldots \overbracket{x_{i_{2n-1}} x_{i_{2n}}}$$

where $i_1 i_2 \ldots i_{2n}$ is the list α_1 1's, α_2 2's, ..., α_n n's,. Thus

$$\begin{aligned}
\langle x_1 x_2 x_3 x_4 \rangle_0 &= \overbracket{x_1 \overbracket{x_2 x_3} x_4} + \overbracket{x_1 x_2 x_3} x_4 + \overbracket{x_1 x_2} \overbracket{x_3 x_4} \\
&= \overbracket{x_1 x_4} \overbracket{x_2 x_3} + \overbracket{x_1 x_3} \overbracket{x_2 x_4} + \overbracket{x_1 x_2} \overbracket{x_3 x_4}
\end{aligned}$$

NOTE.

$$\begin{aligned}
\langle x_1 x_2 x_3 x_4 \rangle &= \overbracket{x_1 x_3} \overbracket{x_2 x_2} + \overbracket{x_1 x_2} \overbracket{x_2 x_3} + \overbracket{x_1 x_2} \overbracket{x_2 x_3} \\
&= \overbracket{x_1 x_3} \overbracket{x_2 x_2} + 2 \overbracket{x_1 x_2} \overbracket{x_2 x_3}.
\end{aligned}$$

This is called the *Wick Expansion.*

Now, in analogy to the infinite dimensional Chern—Simous integral, consider

$$Z_k = \int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{ik(\frac{1}{2}\lambda_{ijk}x^i x^j + \lambda_{ij}k x^i x^j x^k)}.$$

(Using $A_{ij} = \lambda_{ij}$ to avoid confusion with gauge connection A.)

Let $x \mapsto \overrightarrow{x}' = \sqrt{k}\,\overrightarrow{x}$

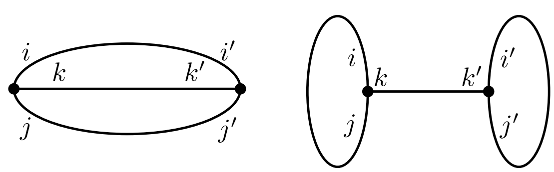
$$\begin{aligned} Z_k &= -k^{-N/2} \int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{\frac{i}{2}\lambda_{ij}x^i x^j} e^{\frac{i}{\sqrt{k}}\lambda_{ijk}x^i x^j x^k} \\ &= -k^{-N/2} \int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{\frac{i}{2}\lambda_{ij}x^i x^j} \sum_{m=0}^{\infty} \frac{i^m}{m! k^{m/2}} (\lambda_{ijk}x^i x^j x^k)^m \end{aligned}$$

m^{th} term in asymptotic expansion given by

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{\frac{i}{2}\lambda_{ij}(x^i x^j)} (\lambda_{ijk}x^i x^j x^k)^m \\ &= \left(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k}\right)^m \int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{\frac{i}{2}\lambda_{ij}x^i x^j + iJ_i x^i} \Big|_{J=0} \\ &= \frac{1}{(2\pi i)^{-N/2}\sqrt{\operatorname{Det}\lambda}} \left[\left(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k}\right)^m e^{+\frac{i}{2}\lambda^{ij}J_i J_j}\right]_{J=0} \qquad ((\lambda^{ij}) = \lambda^{-1}) \end{aligned}$$

e. g. $m = 2$:

$$\begin{aligned} &\left[\left(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k}\right)^2 \int\limits_{\mathbb{R}^n} d\overrightarrow{x}\, e^{\frac{i}{2}\lambda_{ij}x^i x^j + iJ_i x^i}\right]_{J=0} \\ &\qquad = \frac{1}{(2\pi i)^{N/2}\sqrt{\operatorname{Det}\lambda}} \Big(6\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ii'}\lambda^{jj'}\lambda^{kk'} + 9\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ii}\lambda^{kk'}\lambda^{i'j'}\Big) \end{aligned}$$



$n = \#$ of independent loops in the diagram. Thus we refer to the m-loop term.

For a closer analogy with our Chern—Simous integral, look at variables as matrix entries — say $(X^I_J)_{I,J=1,\dots,N}$ symmetric (or Hermitian) matrices. $(X_a)^I_J$, $a = 1, \dots, d$ and look at

$$\int d(X^I_{aJ}) \exp\Big(-\frac{1}{2}\sum \lambda_{ab}\operatorname{tr}(X_a X_b) + \sum g_{abc}\operatorname{tr}(X_a X_b X_c)\Big),$$
$$(d(X^I_{aJ}) = dx_1 \dots dx_{N^2})$$

Temporarily put $d = 1$ so have a single matrix in the integral with

$$\operatorname{tr}(X^2) = \sum_{i=1}^{N^2} x_i^2 \qquad\qquad (\operatorname{tr}(X^2) = \sum_{I,J} X^I_J X^J_I + X^I_J = X^J_I)$$

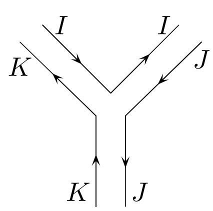
$$\operatorname{tr}(X^3) = \sum_{I,J,K} X^I_J X^J_K X^K_I$$

($O(N)$ or $U(N)$ acts by conjugacy on the integral.)

NOTE.

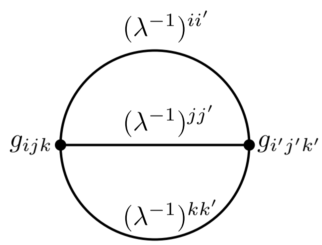
$$\left\{\begin{array}{c} \lambda_{ab}\operatorname{tr}(X_a X_b) \;=\; \lambda_{ab}\sum X^I_{aJ} X^J_{bI} \\ \qquad\qquad\qquad \stackrel{\text{def}}{=} \sum [\widetilde{\lambda}_{ab}]^{IK}_{JL} X^I_{aJ} X^K_{bL} \\ [\widetilde{\lambda}_{ab}]^{IK}_{JL} = \lambda_{ab}\delta^I_L \delta^J_K \\ \text{for matching to quadratic form language.} \end{array}\right\}$$

$X \underset{J}{\overset{I}{\rightleftarrows}} X$ corresponds to $X^I_J X^J_I$

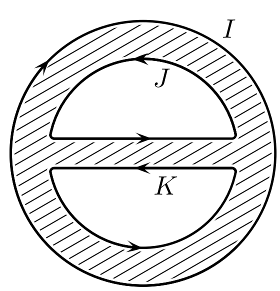


corresponds to $X^I_K X^K_J X^J_I$

Thus the graph



is replaced by the "fat graph"



.

$$\int d(X^I_J)\exp\left(-\frac{1}{2}\operatorname{tr}(X^2) + ig\operatorname{tr}(X^3)\right) \quad = \quad \frac{1}{\sqrt{2\pi}^{N^2}}\sum_{l=0}^{\infty}\frac{(ig)^l}{l!}N^{\#\partial\text{comps}}W_{d,S}$$

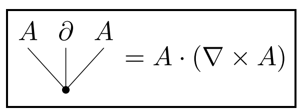
($W_{d,S}$ = of flat graphs will l v[6] and S boundary component.) Put $l = l(g, s)$

Now lets shift to an infinite dimensional case. Take Chern—Simous integral in Abelian gauge. In fact, we can just take formalism where there is no Lie algebra so that $A_i(x)dx^i = A(x)$. Then we have

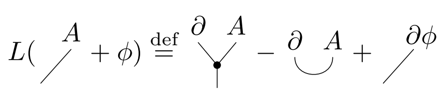
$$Z_K = \int DA\, e^{\frac{ik}{4\pi}\int(A\,dA)}\mathcal{W}_K(A)$$

to consider.

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{tr}(A\,dA) &= \int A_i\,dx^i \wedge \partial_j A_k\,dx^j \wedge dx^k \\ &= \int_{\mathbb{R}^3} A_i \partial_j A_k\,dx^i \wedge dx^j \wedge dx^k \\ &= \int_{\mathbb{R}^3} (\varepsilon^{ijk} A_i \partial_j A_k)d\,vol \\ &= \int_{\mathbb{R}^3} A\cdot(\nabla\times A)d\,vol\end{aligned}$$

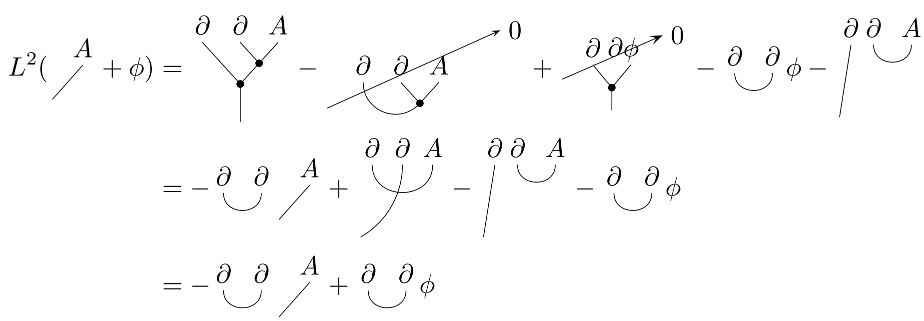


Define the following operator L:



$$[L(A+\phi) = \nabla\times A - \nabla\cdot A + \nabla\phi]$$

Then



Thus

$$\boxed{L^2(A+\phi) = -\nabla^2 A - \nabla^2\phi}$$

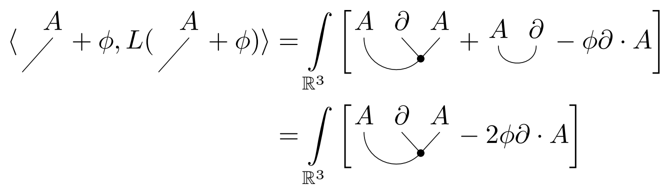
Thus the square of L gives the Laplacian.

Define

$$\langle X, Y\rangle = \int_{\mathbb{R}^3} X\cdot Y\,d\,vol\,.$$

[6]???

Then

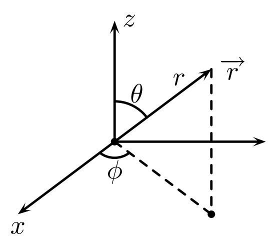


(integration by parts)

Thus

$$\langle (A,\phi), L(A,\phi)\rangle = \int\limits_{\mathbb{R}^3} (A\cdot(\nabla\times A) - 2\phi\nabla\cdot A)d\,vol$$

Note now how we can obtain an "inverse" for the Laplacian:



In polar coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Let $R = |\overrightarrow{r} - \overrightarrow{r}'|$

$$\overrightarrow{r}' \neq \overrightarrow{r} \implies \nabla^2\left(\frac{1}{4\pi R}\right) = \frac{1}{R^2}\left(\frac{\partial}{\partial R}\left(R^2\frac{\partial\left(\frac{1}{4\pi R}\right)}{\partial R}\right)\right)$$

Of course, if $\overrightarrow{r}' = \overrightarrow{r} \implies \frac{1}{4\pi R} \to \infty$, but note that if $\partial\mathbb{B} = S_\varepsilon$, a sphere of radius ε, then

$$\int\limits_{\mathbb{B}} \nabla^2\left(\frac{1}{4\pi R}\right) = \int\limits_{S_\varepsilon} \nabla\left(\frac{1}{4\pi R}\right)\cdot d\overrightarrow{S} = -\int\limits_{S_\varepsilon} \frac{1}{4\pi R^2}\,dS_\varepsilon$$
$$= \frac{1}{4\pi\varepsilon^2}\int dS_\varepsilon = -1.$$

Thus

$$\boxed{\nabla^2\left(\frac{1}{4\pi R}\right) = \delta^{(3)}(\overrightarrow{r} - \overrightarrow{r}')}$$

$$\frac{-1}{4\pi}\int\limits_{\mathbb{R}^3} \nabla^2\left(\frac{1}{|x-y|}\right)dy = +\int\limits_{\mathbb{R}^3} \delta^{(3)}(x-y)dy.$$

So

$$\frac{-1}{4\pi}\int\limits_{\mathbb{R}^3} \nabla^2\left(\frac{1}{|x-y|}\right)J(y)dy = J(x).$$

If

$$G(x-y) = -\frac{1}{4\pi}\left(\frac{1}{|x-y|}\right)$$

and

$$\boxed{G * J = \int_{\mathbb{R}^3} dy\, G(x-y) J(y)}$$

then

$$\nabla^2 G * J = J.$$

We have correspondingly constructed an inverse operator for $L^2 = -\nabla^2$. Thus $(G \mapsto -G)$

$$\begin{aligned} L^2 G * J &= J \\ L(LG * J) &= J \\ LG * J &= ((\nabla \times G) * J, -\nabla \cdot G * J) \\ \langle J, LG * J \rangle &= \int J \cdot (\nabla \times G) * J \\ &= \int J(x) \cdot \frac{(x-y) \times J(y)}{|x-y|^3}. \end{aligned}$$

Now we want to consider

$$\begin{aligned} SL(K, K') &= \int DA\, e^{-\frac{1}{2}\langle A, LA \rangle} \iint_{K \times K'} A(x) A(y) \quad \text{(1}^{\text{st}} \text{ non trivial Wilson Loop Contribution)} \\ &= \iint_{K \times K'} \int DA\, e^{-\frac{1}{2}\langle A, LA \rangle} A(x) A(y) \\ \langle A(x) A(y) \rangle &\stackrel{\text{def}}{=} \int DA\, e^{-\frac{1}{2}\langle A, LA \rangle} A(x) A(y) \\ &= \left(\frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \right) \Big|_{J=0} e^{\frac{1}{2}\langle J, L^{-1} J \rangle} \\ &\quad \text{(Ignoring some constants and following analogy with finite dimensional case.)} \\ &= \left(\frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \right) \Big|_{J=0} e^{\frac{1}{2}\langle J, LG * J \rangle} \\ &= \frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \Big|_{J=0} e^{\frac{1}{2}\int J(x) \cdot \frac{(x-y) \times J(y)}{|x-y|^3}} \\ &= ???????? \end{aligned}$$

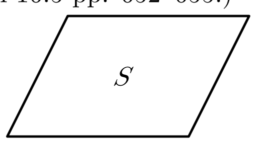
(...... in terms of $A_i(x)dx^i$.)

$$\begin{aligned} SL(K,K') &= \iint\limits_{K\times K'} \langle A(x)A(y)\rangle \\ &= c\iint\limits_{K\times K'} dx\cdot\frac{(x-y)\times dy}{|x-y|^3} \\ &= -c\iint\limits_{K\times K'} dx\cdot dy\times\frac{(x-y)}{|x-y|^3} \\ &= -c\iint\limits_{K\times K'} (dx\times dy)\cdot\frac{(x-y)}{|x-y|^3} \\ SL(K,K') &= -4\pi c\,\mathrm{Link}(K,K'). \end{aligned}$$

This shows how C.S. Functional integral computes linking nos.

14. Digression on Linking Number

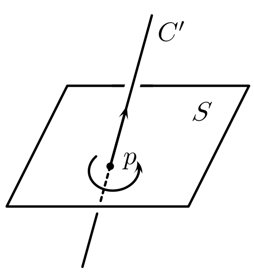
(The derivation below is due to Hagen Kleinert from his book "Path Integrals" W. S. (1995) Section 16.5 pp. 652–655.)

Given a surface 

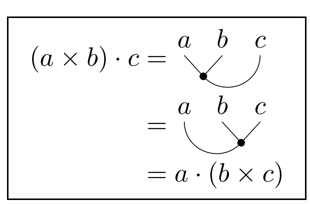
let

$$\delta_i(x',s) = \int\limits_S dS\,\delta^{(3)}(x'-x).$$

If $C'\cap S = \{p\}$



$$\Longrightarrow \int\limits_{C'} dx_i'\delta_i(x',S) = 1$$

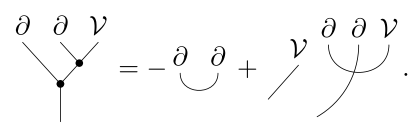


Say $\partial S = C$.

$$
\begin{aligned}
4\pi\mathcal{L}(C,C') &= \int\limits_C dx \cdot \int\limits_{C'} \frac{dx'\, x(x-x')}{|x-x'|^3} \\
&= \int\limits_C dx \cdot \Big(\nabla \times \int\limits_{C'} \frac{dx'}{|x-x'|}\Big) \\
&= \int\limits_S dS \cdot \Big(\nabla \times \Big(\nabla \times \int\limits_{C'} \frac{dx'}{|x-x'|}\Big)\Big)
\end{aligned}
$$

Now

$$\nabla \times (\nabla \times \mathcal{V}) = \nabla(\nabla \cdot \mathcal{V}) - \nabla^2 \mathcal{V}$$

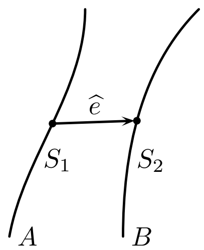


$$
\begin{aligned}
\nabla \cdot \int\limits_{C'} \frac{dx'}{|x-x'|} &= \int\limits_{C'} dx' \cdot \frac{(x-x')}{|x-x'|^3} = 0 \\
\nabla^2 \frac{1}{|x-x'|} &= -4\pi\delta^{(3)}(x-x')
\end{aligned}
$$

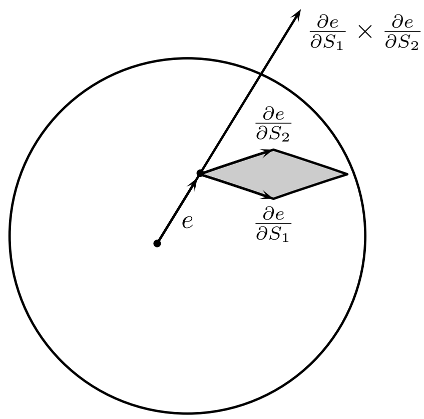
Therefore

$$
\begin{aligned}
\mathcal{L}(C,C') &= \int\limits_S dS \cdot \int\limits_{C'} dx'\, \delta^{(3)}(x-x') = \int\limits_{C'} dx'_i\, \delta_i(x' \mid s) \\
&= \mathrm{Link}(C,C').
\end{aligned}
$$

15. Interpretation of Linking Number as Mapping Degree



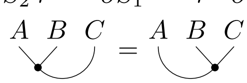
$e = \widehat{e}/|\widehat{e}|, \quad r = |\widehat{e}|$
$e\colon A \times B \to S^2$
(L............. map for two curves.)



area $= \left(\frac{\partial e}{\partial S_1} \times \frac{\partial e}{\partial S_2}\right) \cdot e$

$$\frac{\partial e}{\partial S_i} = \frac{\partial \widehat{e}}{\partial S_i}\frac{1}{r} + \widehat{e}\frac{\partial(1/r)}{\partial S_i}$$

$$\frac{\partial e}{\partial S_1} \times \frac{\partial e}{\partial S_2} = \frac{\partial \widehat{e}}{\partial S_1} \times \frac{\partial \widehat{e}}{\partial S_2}\frac{1}{r^2} + \frac{\partial \widehat{e}}{\partial S_1} \times \widehat{e}\frac{1}{r}\frac{\partial(1/r)}{\partial S_2} + \widehat{e} \times \frac{\partial \widehat{e}}{\partial S_2}\frac{1}{r}\frac{\partial(1/r)}{\partial S_1}$$



$$(A \times B) \cdot C = A \cdot (B \times C)$$

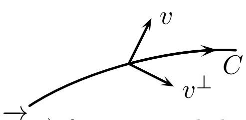
But
$$\left(\frac{\partial \widehat{e}}{\partial S_1} \times \frac{\partial(1/r)}{\partial S_2}\right) \cdot e = \frac{\partial \widehat{e}}{\partial S_1} \cdot (e \times e)\frac{\partial(1/r)}{\partial S_2} = 0.$$

Thus
$$\left(\frac{\partial e}{\partial S_1} \times \frac{\partial e}{\partial S_2}\right) \cdot e = \left(\frac{\partial \widehat{e}}{\partial S_1} \times \frac{\partial \widehat{e}}{\partial S_2}\right) \cdot \widehat{e}/r^3.$$

Therefore
$$\mathrm{Link}(A, B) = \frac{1}{4\pi}\iint\limits_{A \times B} \frac{\left(\frac{\partial \widehat{e}}{\partial S_1} \times \frac{\partial \widehat{e}}{\partial S_2}\right) \cdot \widehat{e}}{r^3} dS_1 dS_2$$

$$\Longrightarrow \ \mathrm{Link}(A, B) = \frac{1}{4\pi}\iint\limits_{A \times B} e^* d\Sigma, \qquad d\Sigma = \text{area form on } S^2.$$

For a single curve C with framing v:



$$(v, v^{\perp}, \overrightarrow{T}_C) \text{ forming righthand frame}$$

$$T_w(C, v) \overset{\text{def}}{=} \frac{1}{2\pi} \int v^{\perp} \cdot dv$$

twist

$c\colon C \times C \to S^2$

$$\mathrm{Wr}(C) \overset{\text{def}}{=} \int\limits_{C \times C} e^* \, d\Sigma$$

writhe,

$$\mathrm{Lk}(C, v) \overset{\text{def}}{=} \mathrm{Link}(C, C^v), \qquad C^v = C \text{ translated along } v$$

Theorem (Calagareann/White/Pohl).

$$\mathrm{Lk}(C, v) = \mathrm{Tw}(C, v) + \mathrm{Wr}(C)$$
$$\text{“Link = Twist + Writhe”}$$

Proof. Omitted*. □

This theorem has interesting applications to the study of DNA molecules in molecular biology.

The proof of the theorem (above) by W. F. Pohl. $M =$ ribbon of vectors v. $\partial M = C \cup C^v$. $\Delta = \{(m, m) \mid m \in M\} \subset M \times M$.

Blow up $\Delta \subset M \times M$ by replacing Δ by bundle of oriented normal directions to Δ in $M \times M$. Call $\widetilde{M \times M}$ the blow up $M \times M$. $S(C, M) =$ closure $(C \times M - \Delta)$ in $\widetilde{M \times M}$.

$$\partial S(C, M) = [C \times C] \cup [C \times C^v] \cup [T(M, C)],$$

$T(M, C) =$ unit tangent vectors to M based at points of C and pointing “into” M.

For $(x, y) \in C \times M - \Delta$, let $e(x, y) = (y - x)/|y - x|$. This extends to $T(M, C)^*$.

$$e\colon S(C, M) \to S^2$$

$$\Longrightarrow \int\limits_{C \times C} e^* \, d\Sigma + \int\limits_{C \times C^v} e^* \, d\Sigma + \int\limits_{T(M,C)} e^* \, d\Sigma = 0$$

$$\Longrightarrow \quad \boxed{\mathrm{Lk}(C, C^v) = \frac{1}{4\pi} \int\limits_{T(M,C)} + e^* \, d\Sigma + \mathrm{Wr}(C)}$$

(with orientation induced from $C \times M$).

*See “DNA and Differential Geometry” by W. F. Pohl. Math. Intell. 3, pp. 20–27.

*See W. F. Pohl, Some Integral Formulas for Space Curves and Their Generalization. Am. J. Math. 90, 1321–1345 (1968)

Now let $e^\perp$ be $\perp e$, $e^\perp \in Span(\{)e, t\}$, t unit tangent $t \in T(M, C)$ orientation $ee^\perp$ agrees with orientation vt of tangent planes to M along C.

$$\left.\begin{aligned} e &= \cos\phi\, v + \sin\phi\, t \\ e^\perp &= -\sin\phi\, v + \cos\phi\, t \end{aligned}\right\} \qquad \begin{aligned} de \cdot e^\perp &= d\phi + dv \cdot t \\ de \cdot v^\perp &= \cos\phi\, dv \cdot v^\perp + \sin\phi\, dt \cdot v^\perp \end{aligned}$$

Since $v, v^\perp, t$ are functions of curve parameter on C only, $(dv \cdot t) \wedge (dv \cdot v^\perp) = (dv \cdot t) \wedge (dt \cdot v^\perp) = 0$.

Therefore $e^* \, d\Sigma = (de \cdot e^\perp) \wedge (de \cdot v^\perp) = \cos\phi\, d\phi \wedge (dv \cdot v^\perp) + \sin\phi\, d\phi \wedge (dt \cdot v^\perp)$ on $T(M, C)$ and

$$\begin{aligned} \frac{1}{4\pi} \int\limits_{T(M,C)} e^* \, d\Sigma &= \frac{1}{4\pi} \int\limits_C \int\limits_{-\pi}^{\pi} \cos\phi\, d\phi (dv \cdot v^\perp) + \sin\phi\, d\phi (dt \cdot v^\perp) \\ &= \frac{1}{2\pi} \int\limits_C v^\perp \cdot dv = \mathrm{Tw}(C, C^v). \end{aligned}$$

□

16. [Nazvanije za stranicej]

$\mathcal{L} = \frac{1}{2}\lambda_{ij} x^i x^j + \lambda_{ijk} x^i x^j x^k$ finite dimensional[9].

Ne ochenn poniatno

Suppose $\mathcal{L}$ invariant under action of l-dimensional Lie group G. Visit[10] each orbit of G *once* by .... $F :: \mathbb{R}^n \to \mathbb{R}$ with an unique .... on each G-orbit and insert $\delta^l(F(\overrightarrow{x}))$ into integral. Multiply by volume of a G-orbit.

Note. $\int dx \delta(x) = 1 \implies \int df(x)\delta(f(x)) = 1 \implies \int f'(x) dx \delta(f(x)) = 1$.

So we must look at

$$Z = \int\limits_{\mathbb{R}^N} d^N x e^{ik(\frac{1}{2}\lambda_{ij} x^i x^j + \lambda_{ijk} x^i x^j x^k)} \delta^l(F(\overrightarrow{x}))\, \mathrm{Det} \left(\frac{\partial F^a}{\partial T_b} \right) (\overrightarrow{x})$$

$\{T_b \mid b = 1, \ldots, l\}$ generators for $\mathfrak{g} = Lie\ Alg(G)$.

$$\delta^l(F(\overrightarrow{x})) = \int\limits_{\mathbb{R}^l} d^l \phi\, e^{iF^a(\overrightarrow{x})\phi_a}$$

Ne ochenn poniatno (opiat)

And we want to ..... the Det term into the Lagrangian. We will use "non-commutative" (Berezin) integration:

Berezin Integral.

θ = single Grassmann variable.

Definition.

$$\boxed{\begin{aligned} \int d\theta\, \mathbf{1} &= 0 \\ \int d\theta\, \theta &= 1 \end{aligned}} \qquad \begin{aligned} \theta d\theta &= -d\theta\, \theta \\ \theta^2 &= 0 \end{aligned}$$

$$F(\theta) = a + b\theta \implies \int d\theta\, F(\theta) = b dF/d\theta.$$

---

[9] ???

[10] ???

Let $c^1, \dots, c^l; \overline{c}^1, \dots, \overline{c}^l$ be distinct anticommuting Grassmann variables. Thus

$$(c^i)^2 = (\overline{c}^i)^2 = 0$$

$$c^i c^j = -c^j c^i, \quad \overline{c}^i \overline{c}^j = -\overline{c}^j \overline{c}^i, \quad c^i \overline{c}^j = -\overline{c}^j c^i,$$

$M = l \times l$ matrix with entries in a commutative ring.

$$c^t = (c^1, \dots, c^l)$$

$$\overline{c}^t = (\overline{c}^1, \dots, \overline{c}^l)$$

Then:

$$\mathrm{Det}(M) = (-1)^{l(l+1)/2} \int d\overrightarrow{c}\, d\overrightarrow{\overline{c}}\, e^{c^t M \overline{c}}$$

EXERCISE.

$$\int dc_1\, dc_2\, d\overline{c}_1\, d\overline{c}_2\, e^{\begin{pmatrix} c_1 & c_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{c}_1 \\ \overline{c}_1 \end{pmatrix}}$$

$$\int dc_1\, dc_2\, d\overline{c}_1\, d\overline{c}_2\, \left[ac_1\overline{c}_1 + bc_1\overline{c}_2 + cc_2\overline{c}_1 + dc_2\overline{c}_2\right]^2 \Big/ 2$$

$$\int (adc_1\overline{c}_1 c_2\overline{c}_2 + bcc_1\overline{c}_2 c_2\overline{c}_1 + dac_2\overline{c}_2 c_1\overline{c}_1 + cbc_2\overline{c}_1 c_1\overline{c}_2)/2$$

$$= ad - bc = \mathrm{Det}(M).$$

Thus Z can be written as $\propto$? ? ? ? ?

$$Z \propto \int_{\mathbb{R}^N} d^N x \int_{\mathbb{R}^l} d^l \phi \int d^l \overline{c} d^l c\, e^{i[k\mathcal{L} + F^a(\overrightarrow{x})\phi_a + \overline{c}_a \left(\frac{\partial F^a}{\partial T_b}\right) c^b]} = \int e^{i\mathcal{L} t_0 t}$$

REMARK. Another way to work with Det: $A\colon V \to V$, $\widehat{A}\colon \Lambda^* V \to \Lambda^* V$ ($\Lambda^* V =$ Exterior algebra on V)

$$S\colon \Lambda^* V \to \Lambda^*, \qquad S|_{\Lambda^k V}(x) = (-\lambda)^k x$$

$\Longrightarrow$ $\mathrm{Det}(A - \lambda I) = \mathrm{tr}(S\widehat{A})$.

---

17. Gauge Fixing Chern—Simous Action

A) Fröhlich and King — Axial Gauge (Light—Cone Gauge)
Let (x^0, x^1, x^2) denote point in 3-space.
Think of $x^2 = t$ as "time".
Light-cone coordinates

$$x^+ = x^1 + x^2 = x^1 + t$$

$$x^- = x^1 - x^2 = x^1 - t.$$

For gauge connection A, $A_\pm = A_1 \pm A_2$.

$$A_i = \sum_{a=1}^{r} A_i^a T_a / \sqrt{2}$$

where $\{T_a\}_{a=1}^r$ orthonormal set of generators of Lie algebra of gauge group $G = SU(N)$, s. t. $\mathrm{tr}(T_a T_b) = -\delta_{ab}$. (These are Fröhlich and King conventions.)

Light-cone gauge: $\boxed{A^a_- = 0 \text{ for } a = 1, \ldots, r}$*

$$A(x)a_=(x)dx^+ + a_-(x)dx^- + a_0(x)dt$$

$$CS(A) = \frac{1}{4\pi}\int \operatorname{tr}(a \wedge da + \frac{2}{3}a \wedge a \wedge a) \qquad \text{with } \boxed{a_- = 0}.$$

$$= \frac{1}{4\pi}\int \operatorname{tr}(a \wedge da).$$

$$a \wedge da = (a_+\,dx^+ + a_0\,dt)$$
$$\wedge\,(\partial_- a_+\,dx^- \wedge dx^+ \partial_0 a_+\,dt \wedge dx^+ \partial_+ a_0\,dx^+ \wedge dt \partial_- a_0\,dx^- \wedge dt)$$
$$= (a_+\,dx^+ + a_0\,dt)$$
$$\wedge\,(\partial_- a_+\,dx^- \wedge dx^+ (\partial_0 a_+ - \partial_+ a_0) dt \wedge dx^+ + \partial_- a_0\,dx^- \wedge dt)$$
$$= a_+ \partial_- a_0\,dx^+ \wedge dx^- \wedge dt + a_0 \partial_- a_+\,dt \wedge dx^- \wedge dx^+$$
$$a \wedge da = (a_+\partial_- a_0 - a_0 \partial_- a_+) dx^+ \wedge dx^- \wedge dt$$

$$CS(A) = \frac{1}{4\pi}\int \operatorname{tr}(a_+\partial_- a_0 - a_0\partial_- a_+)dx^+ \wedge dx^- \wedge dt$$

$$CS(A) = \frac{1}{2\pi}\int \operatorname{tr}(a_+\partial_- a_0)dx^+ \wedge dx^- \wedge dt$$

$$a_+\partial_- a_0 - a_0\partial_- a_+ = \begin{pmatrix} -a_0\partial_- & a_+\partial_- \end{pmatrix}\begin{pmatrix} a_+ \\ a_0 \end{pmatrix}$$
$$= \begin{pmatrix} a_+ & a_0 \end{pmatrix}\begin{pmatrix} 0 & \partial_- \\ -\partial_- & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_0 \end{pmatrix}.$$

Let

$$L_- = \begin{pmatrix} 0 & \partial_- \\ -\partial_- & 0 \end{pmatrix}, \qquad L_+ = \begin{pmatrix} 0 & \partial_+ \\ -\partial_+ & 0 \end{pmatrix}$$

$$l_- L_+ = \begin{pmatrix} 0 & \partial_- \\ -\partial_- & 0 \end{pmatrix}\begin{pmatrix} 0 & \partial_+ \\ -\partial_+ & 0 \end{pmatrix}\begin{pmatrix} 0 & -\partial_-\partial_+ \\ -\partial_-\partial_+ & 0 \end{pmatrix}$$

$$\partial_-\partial_+ = \Box = \frac{\partial^2}{\partial x_- \partial x_+} = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

for $x^- = (x^1 - x^2)/sqrt2$, $x^+ = (x^1 + x^2)/\sqrt{2}$

We want

$$\partial - D(x - y) = \delta(x - y)$$

So

$$D = \partial_+(\partial_-\partial_+)^{-1} = \partial_+(\Box^{-1}).$$

Claim.

$$\boxed{\begin{aligned} \Box^{-1}(x) &= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^3}\int \frac{e^{i(k_0x^0 + k_1x^1 + k_2x^2)}}{k_2^2 - k_1^2 + i\varepsilon} d^3k \\ &= \delta(x^0) \text{ a distribution in } (x^1, x^2). \end{aligned}}$$

---

*$\mathcal{F} = \sum a^a_- T_a$

$$\frac{\partial \mathcal{F}^a}{\partial T_b} = a^a_- \delta_{ab}$$

So Ghost determinant is diagonal. Absorb into the measure.

We return to the claim in a moment. Note that

$$CS(A) = \frac{1}{2\pi} \int \operatorname{tr}(a_+ \partial_- a_0) dx^+ \wedge dx^- \wedge dt.$$

This means that we only need to consider the operator ∂_- and its inverse. Furthermore, the only non-zero 2-pt function is $\langle a_+ a_0 \rangle$. Fröhlich and King claim the following result:

$$\begin{aligned}\langle a_+^j(x) a_0^k(y) \rangle &= 2\lambda \delta^{jk} \operatorname{sign}(x^- - y^-) \delta(x^+ - y^+) \delta(x^0 - y^0) \\ &+ \frac{2\lambda}{i\pi} \delta^{jk} P \left(\frac{1}{x^+ - y^+} \right) \delta(x^0 - y^0) \qquad (\lambda \text{ a constant})\end{aligned}$$

We will get around this complexity by using $x^1 + ix^2$.

.........:

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Fourier transform of f. $\implies f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx.$

$$\delta(t - t^{-1}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk) e^{-ik(t-t')}$$

Fourier representation of delta function.

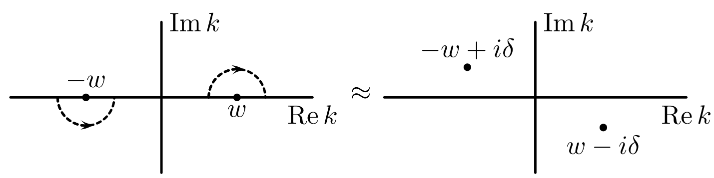
Example. Solve

$$\left(\frac{d^2}{dt^2} + w^2 \right) G(t - t') = -\delta(t - t').$$

Solution. Write $\delta(t - t')$ as above and

$$\begin{aligned} G(t - t') &= \int \frac{dk}{\sqrt{2\pi}} e^{-ik(t-t')} G(k) \\ \implies \frac{1}{\sqrt{2\pi}} (-k^2 + w^2) G(k) &= -\frac{1}{2\pi} \\ \implies G(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 - w^2} \\ G(t - t') &= \frac{1}{2\pi} \int dk \, \frac{e^{-ik(t-t')}}{k^2 - w^2} \end{aligned}$$

Can rewrite contour



$$\begin{aligned} G_F(k) &= \lim_{\varepsilon \to +0} \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + w^2 + i\varepsilon} \\ &= \lim_{\delta \to +0} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{k + w - i\delta} \right) \left(\frac{1}{k - w + i\delta} \right) \end{aligned}$$

And one can use this mode of analysis to penetrate King and Fröhlich.

We take a different tack:

$$\partial_+\partial_- = \partial_1^2 - \partial_2^2.$$

Replace x^2 by $ix^2 \implies$

$$\partial_+\partial_- \mapsto \partial_1^2 + \partial_2^2 = \nabla^2.$$
$$z = x^1 + ix^2$$

CLAIM. $\boxed{\nabla^2 \ln(z) = 2\pi\delta(z)}$

$$z = re^{i\theta}$$
$$\ln(z) = \ln(r) + i\theta$$

$$\boxed{\nabla^2 = \frac{1}{r^1}\frac{\partial}{\partial r}\left(r^1\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}} \qquad \text{2-dimensional Laplasian}$$

$$\boxed{\begin{array}{l} \nabla\ln(r) = \vec{r}/r \\ \nabla\ln(r)\cdot\frac{\vec{r}}{r} = \frac{\vec{r}\,\vec{r}}{r^2} = 1 \end{array}}$$

$$\frac{\partial\ln(z)}{\partial r} = 1/r \implies \nabla^2\ln(z) = 0 \qquad \text{for } z \neq 0.$$

$$\int\limits_{B_\varepsilon} \nabla^2\ln(z)dz = \int\limits_{S_\varepsilon} \nabla\ln(z)dz = 2\pi.$$

For the record:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$Jac = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|\,Jac\,| = r$$

$$\overset{-1}{Jac} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\implies \boxed{\tfrac{\partial}{\partial x} = \cos(\theta)\tfrac{\partial}{\partial r} - \tfrac{\sin(\theta)}{r}\tfrac{\partial}{\partial\theta}\,\tfrac{\partial}{\partial y} = \sin(\theta)\tfrac{\partial}{\partial r} + \tfrac{\cos(\theta)}{r}\tfrac{\partial}{\partial\theta}}$$

$$\implies \boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}$$

$$A_\pm = A_1 \pm A_2$$
$$x^+ = (x^1 + x^2)/2, \qquad x^- = (x^1 - x^2)/2,$$
$$\boxed{A(x) = A_+\,dx^+ + A_-\,dx^- + A_0\,dx^0}$$

Check: $(A_1 + A_2)(dx^1 + dx^2)/2 + (A_1 - A_2)(dx^1 - dx^2)/2 + A_0\,dx^0 = A_1\,dx^1 + A_2\,dx^2 + A_0\,dx^0$

Replace x^2 by ix^2.

$$\begin{aligned}\partial_+ &= \frac{1}{2}(\partial_1 + \partial_2)\\ \partial_- &= \frac{1}{2}(\partial_1 - \partial_2)\end{aligned}$$

$$\rightsquigarrow \boxed{\begin{aligned}\partial_+ &= \frac{1}{2}(\partial_1 - i\partial_2)\\ \partial_- &= \frac{1}{2}(\partial_1 + i\partial_2)\end{aligned}}$$

$$\left.\begin{aligned} z &= x_1 + ix_2\\ \frac{\partial}{\partial z} &= \frac{\partial x_1}{\partial z}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial z}\frac{\partial}{\partial x_2}\\ \left.\begin{aligned} z &= x_1 + ix_2\\ \overline{z} &= x_1 - ix_2\end{aligned}\right\} & \quad \begin{aligned} x_1 &= (z+\overline{z})/2\\ x_2 &= (z-\overline{z})/2\end{aligned}\\ \frac{\partial x_1}{\partial z} = \frac{1}{2}, & \qquad \frac{\partial x_2}{\partial z} = \frac{1}{2i}\end{aligned}\right\} \Longrightarrow \boxed{\partial_+ = \frac{\partial}{\partial z}}$$

In the axial gauge,

$$CS(A) = \frac{1}{2\pi}\int \mathrm{tr}(A_+\partial_- A_0)dx^+ \wedge dx^- \wedge dx^0$$
$$\partial_+(\partial_-\partial_+)^{-1} = \partial_-^{-1} = \partial_+\frac{1}{2\pi}\ln(z) = \frac{1}{2\pi z}.$$

The upshot of this is (Letting $\alpha = A_+(z,t)$, $\overline{\alpha} = A_-(z,t) \equiv 0$, $t = x^0$):

$$\begin{aligned}\langle \alpha^a(t,z)\alpha^b(s,w)\rangle &= 0\\ \langle A_0^a(t,z)A_0^b(s,w)\rangle &= 0\\ \langle \alpha^a(t,z)A_0^b(s,w)\rangle &= 4\lambda\frac{\delta^{ab}\delta(t-s)}{z-w} \quad \lambda \text{ a constant}\\ G * J(z) &= \int dw\, G(z-w)J(w)\\ \langle J(z), G * J(z)\rangle &= \int \mathrm{tr}\Big(J(z)\int \frac{dw\, J(w)}{z-w}\Big)dz\\ &= \iint \mathrm{tr}\left(\frac{J(z)J(w)}{z-w}\right)dzdw\end{aligned}$$

We want to apply these correlation functions to the Wilson loop calculations and get corresponding information about the knot and link invariants.

$$\begin{aligned}
Z_K &= \int DA\, e^{-\frac{1}{2}\langle A, LA\rangle} \mathcal{W}_K(A) = \int DA\, e^{-\frac{1}{2}\langle A, LA\rangle} \prod_{x\in K} (\mathbf{1} + \frac{1}{\sqrt{k}} A(x)) \\
&= \sum_n \frac{1}{k^{n/2}} \oint\!\!\oint\!\!\oint_{K_1<\ldots<K_n} \int DA\, A(x_1)\ldots A(x_n) e^{-\frac{1}{2}\langle A, LA\rangle} \\
&= \sum_n \frac{1}{k^{n/2}} \oint\!\!\oint\!\!\oint_{K_1<\ldots<K_n} \langle A(x_1)\ldots A(x_n)\rangle \\
&= \text{(Wick Expression)} \sum_n \frac{1}{k^{n/2}} \sum_{\text{pairs}} \oint\!\!\oint\!\!\oint_{K_1<\ldots<K_n} \langle \overline{A(x_1)A(x_2)}\rangle \ldots \langle \overline{A(x_{n-1})A(x_n)}\rangle
\end{aligned}$$

Now think about structure of the pairs and the structure of the Wilson loops.

In axial gauge, equation for parallel transport along a curve $\sigma(t) = (t, z(t))$ is (letting $0 \leqslant t \leqslant 1$, $u(0) = 1_{\mathbb{C}^N}$)

$$du(t) = \left(\frac{1}{2}\alpha\, dz + A_0\, dt\right) u(t)$$

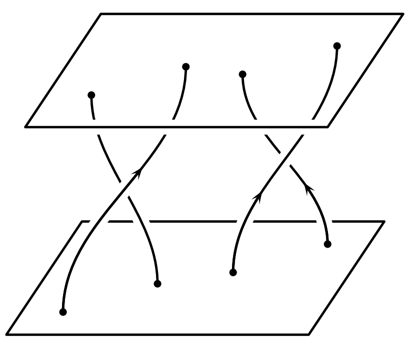
where α depends only on flds

New variables:

$$l(t) = \int_0^t \alpha(s, z(s)) dz(s)$$

$$m(t) = \int_0^t A_0(s, z(s)) dz(s)$$

$$du(t) = \left(\frac{1}{2} dl(t) + dm(t)\right) u(t).$$



Curves
$\sigma_i(t) = (t, z_i(t)), \qquad i = 1, \dots, n$
$\phi_n(t) = \langle u_1 \otimes \dots \otimes u_n(t) \rangle$

$$\begin{aligned}
d\phi_n(t) = \sum_{i=1}^{n} &\langle u_1(t) \otimes \dots \otimes du_i(t) \otimes \dots \otimes u_n(t) \rangle \\
&+ \sum_{1 \leqslant i \leqslant j \leqslant n} \langle u_1(t) \otimes \dots \otimes du_i(t) \otimes \dots \otimes du_j(t) \otimes \dots \otimes u_n(t) \rangle \\
= \sum_{i=1}^{n} &\langle I \otimes \dots \otimes \left(\frac{1}{2} dl_i(t) + dm_i(t)\right) \otimes \dots \otimes I \rangle \phi_n(t) \\
&+ \sum_{1 \leqslant i \leqslant j \leqslant n} \langle I \otimes \dots \otimes \left(\frac{1}{2} dl_i(t) + dm_i(t)\right) \otimes \dots \\
&\otimes \left(\frac{1}{2} dl_j(t) + dm_j(t)\right) \otimes \dots \otimes I \rangle \phi_n(t)
\end{aligned}$$

$$\langle l_i^a(t) m_j^b(t) \rangle 4\lambda \delta^{ab} \int_0^t \int_0^t \frac{dz_i(s) ds' \, \delta(s - s')}{z_i(s) - z_i(s')}$$

$$4\lambda \delta^{ab} \int_0^t \frac{z_i'(s)}{z_i(s) - z_j(s)}$$

$$\langle dl_i^a(t) dm_j^b(t) \rangle = 4\lambda \delta^{ab} \frac{z_i'(t)}{z_i(t) - z_j(t)} dt$$

$$\langle dl_i^a(t) \otimes dm_j^b(t) \rangle = 2\lambda \frac{z_i'}{z_i - z_j} \sum_{a=0}^{r} T_a \otimes T_a dt$$

$$\Omega_{ij} = \sum_{a=1}^{r} I \otimes \ldots \otimes T_a \otimes \ldots \otimes T_a \otimes \ldots \otimes I$$

$$d\phi_n = \lambda \sum_{1 \leqslant j \leqslant j \leqslant n} \frac{z_i' - z_j'}{z_i - z_j} \Omega_{ij} \phi_n \, dt$$

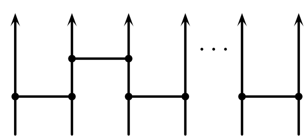
$$\boxed{\frac{d\phi_n}{dt} = \lambda \sum_{1 \leqslant j \leqslant j \leqslant n} \frac{z_i' - z_j'}{z_i - z_j} \Omega_{ij} \phi_n}$$

Now lets step back and look at the structure of this evolution.

$$X = X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$$

Path $B\colon [0,1] \to X_n$ is exactly an n-stranded braid.

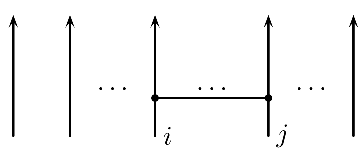
$\mathcal{D}_n =$ diagrams of form



$\mathcal{A}_n = \mathcal{D}_n / (4 - termrelations)$

$\Omega_n = \mathcal{A}_n$ *valued connection on* X via

$\Omega_{ij} =$



$$\omega_{ij} = d \ln(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}$$

$$\Omega_n = \sum_{1 \leqslant i \leqslant j \leqslant n} \Omega_{ij} \omega_{ij}$$

Flatness $\implies$ transport is independent of homotopy type of the path $B \implies$ get representation of $\pi_1(X_n) \cong B_n =$ Artin Braid Group

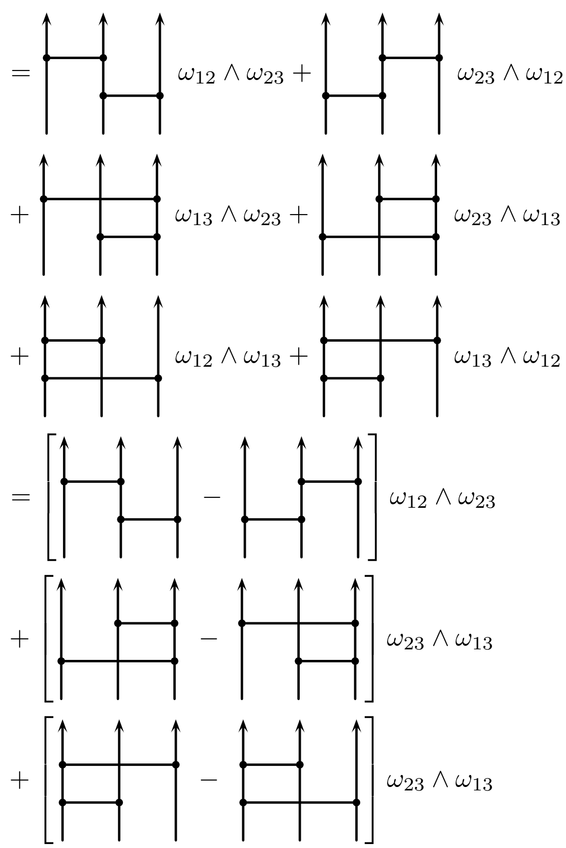
Lemma. Ω_n *is flat.*

Proof. $d\Omega = \sum \Omega_{ij} d^2 \ln(z_i - z_j) = 0$. So we need to show $\omega_n \wedge \Omega_n = 0$.

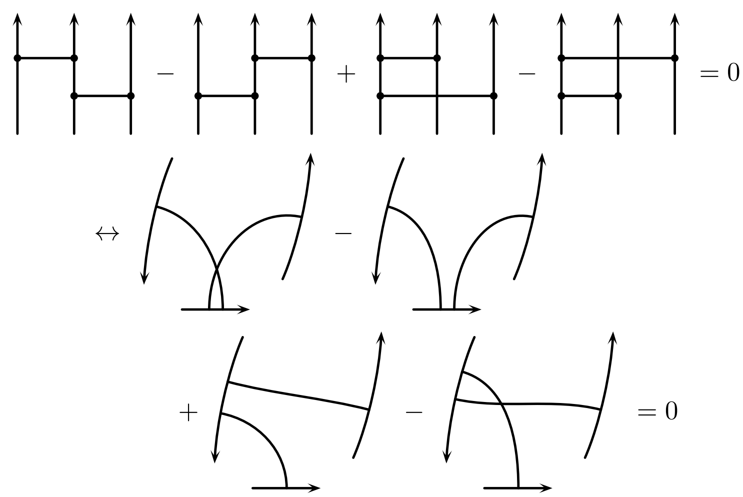
$$\Omega_n \wedge \Omega_n = \sum_{\substack{1 \leqslant i \leqslant j \leqslant n \\ 1 \leqslant k \leqslant l \leqslant n}} \Omega_{ij} \Omega_{kl} \omega_{ij} \wedge \omega_{kl}.$$

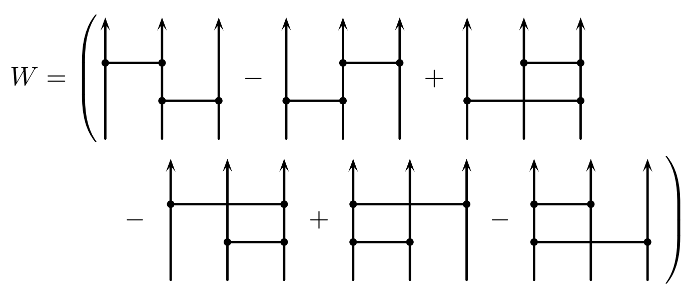
Look at

$$W = \sum_{\substack{i<j,\, k<l \\ \{i,j,k,l\}=\{1,2,3\}}} \Omega_{ij}\Omega_{kl}\omega_{ij}\wedge\omega_{kl} \qquad (\boxed{1\,2}, \boxed{1\,3}, \boxed{2\,3})$$



But four term relation in $[\Omega_{12}, \Omega_{23}] + [\Omega_{12}, \Omega_{13}] = 0$

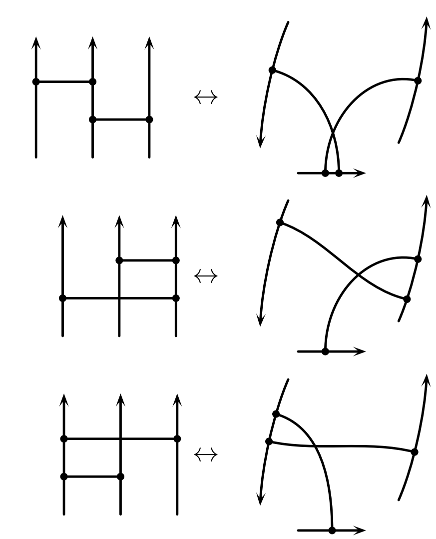




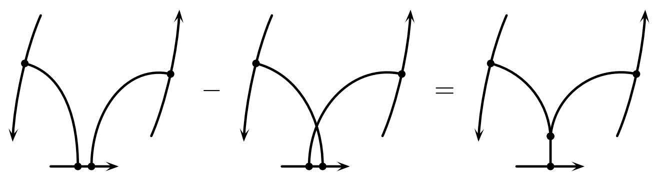
$$\times \left(\omega_{12} \wedge \omega_{23}\omega_{23} \wedge \omega_{13}\omega_{13} \wedge \omega_{12}\right) = 0$$

□

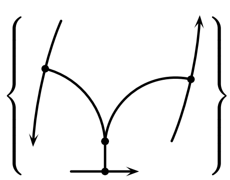
To see this more clearly:



and

(STU) 

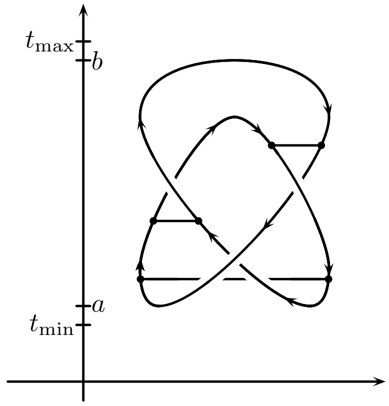
Therefore

$W =$  $(\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{13} + \omega_{13} \wedge \omega_{12})$

$$\begin{aligned}
\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{13} + \omega_{13} \wedge \omega_{12} &= \left(\frac{dz_1 - dz_2}{z_1 - z2}\right)\left(\frac{dz_2 - dz_3}{z_2 - z3}\right)\left(\frac{z_1 - z_3}{z_1 - z3}\right) \\
&+ \left(\frac{dz_2 - dz_3}{z_2 - z3}\right)\left(\frac{dz_1 - dz_3}{z_1 - z3}\right)\left(\frac{z_1 - z_2}{z_1 - z2}\right) \\
&+ \left(\frac{dz_1 - dz_3}{z_1 - z3}\right)\left(\frac{dz_1 - dz_2}{z_1 - z2}\right)\left(\frac{z_2 - z_3}{z_2 - z3}\right) \\
&= ((z_1 - z3) - (z_1 - z2) - (z_2 - z3))dz_1 \wedge dz_2 \\
&\quad + \ldots = 0
\end{aligned}$$

18. The Kontsevich Integral

(See e. g. "The Fundamental Theorem of Vassiliev Invariants" by Bar-Natan and Stoimorow[12])



Here $m = 3$

$$Z_m^{[a,b]}(K) = \frac{1}{(2\pi i)^m} \int\limits_{a<t_1<\ldots<t_m<b} \sum_{P=\{(z_i,z_i')|i=1,\ldots,m\}} (-1)^{\#P\downarrow} D_p^{[a,b]} \bigwedge_{i=1}^{m} \frac{d(z_i - z_i')}{z_i - z_i'}$$

$$\left\{ \begin{array}{l} [a,b] \subset [t_{\min}, t_{\max}] \\ P = \text{ a set of } m \text{ admissible pairings.} \\ D_P^{[a,b]} = \text{ Chord (algebra) diagram evaluation for pairings } P. \\ P\downarrow = \# \text{ of points } (z_i, t_i) \text{ or } (z_i', t_i) \text{ were } K \text{ is decreasing out.} \end{array} \right.$$

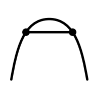
For suffiently large interval $[a, b]$, we write $Z_m(K)$ since this will capture the whole knot or link. Note that $Z_m(K)$ is a *generalization* of the $\frac{1}{(\sqrt{k})^{m\cdot 2}} = \frac{1}{k^m}$ term from the Wilson loop in the gauge theory case. i. e.

$$\int DAe^{\frac{ik}{4\pi}\mathcal{L}_{CS}} \mathcal{W}_K(A) \propto \sum_m \frac{1}{k^m} Z_m(K).$$

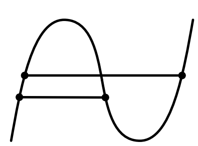
We will return to this point after discussing the Kontsevich Integral. The factor $(-1)^{\#P\downarrow}$ seems to make a difference in th two methods.

The $Z_m^{[a,b]}(K)$ are called *Kontsevich Integrals.*

Of course, we now want to worry about how well-defined is $Z_m^{[a,b]}(K)$. In particular, what about

 or 

as these chords have $z_i - z_i'$ small? *If we use chord diagram evaluations that are zero on isolated chords there is no problem.* Otherwise we need to look. Similarly for



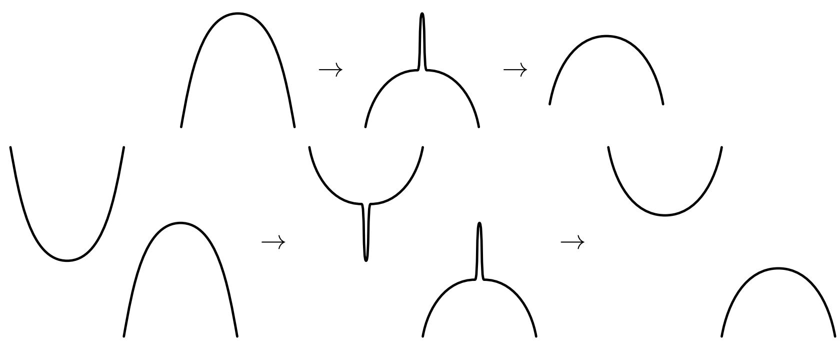
as these chords approach one another.

We will return to these issues.

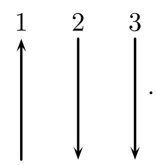
[12]????

Certainly, by flatness, $Z_m(K)$ is invariant under horizontal deformations of K in horizontal slices with no critical points*.

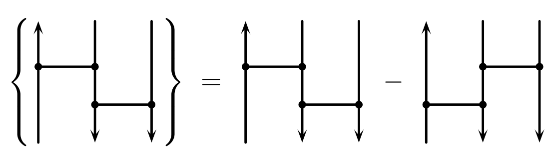
Needle deformations:



The verification can be reduced to 3 lines, as before: e. g.



Let



etc.

*However, this last is based on using a generalization of the connections Ω_n to connections $\Omega_{n,n}$ with underlying algebra gen by diagrams with $2n$ arrows, first n up, second n down. Then

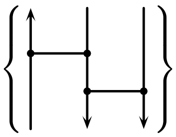
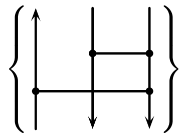
$$\Omega_{n,n} = \sum_{1\leqslant i\leqslant j\leqslant 2n} S_i S_j \Omega_{ij}\omega_{ij}$$

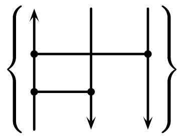
where

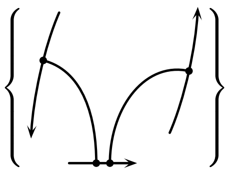
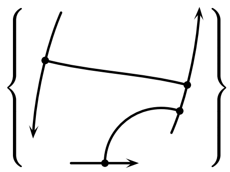
$$S_i = \begin{cases} +1 & i \leqslant n \\ -1 & i > n \end{cases}.$$

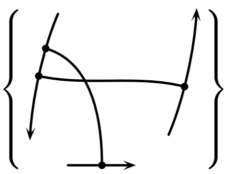
The claim is that *the connection $\Omega_{n,n}$ is flat.*

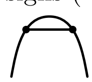
Then

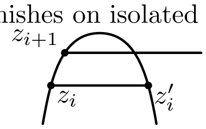
$W' =$  $\omega_{12} \wedge \omega_{23}(-1)^3 +$  $\omega_{23} \wedge \omega_{13}(-1)^3$

$+$  $\omega_{13} \wedge \omega_{12}(-1)^2$

$=$  $\omega_{12} \wedge \omega_{23}(-1) +$  $\omega_{23} \wedge \omega_{13}(-1)^3$

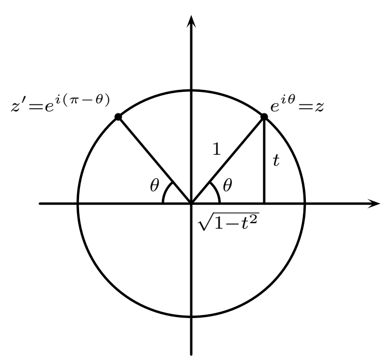
$+$  $\omega_{13} \wedge \omega_{12}(-1)^2$

This Lemma for $\Omega_{n,n}$ explains the use of the signs $(-1)^{\#P\downarrow}$ in the Kontsevich integral. It remains to discuss singularities. For  we get out by using D_P that vanishes on isolated chords.

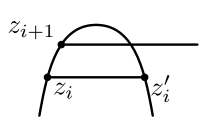
For 

it is claimed that the integration domain for $z + i + 1$ is as small as $z_i - z_i'$ and this smallness cancels the singularity coming from the denominator for the (z_i, z_i') chord.

Remark.



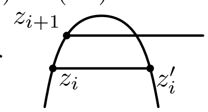
$$W = z - z' = e^{i\theta} + e^{-i\theta}$$
$$W = 2\cos(\theta) = 2\sqrt{1-t^2}$$
$$dW = 2(-2t)(1-t^2)^{-1/2}$$
$$\frac{dW}{W} = \frac{-4t\,dt}{1-t^2} = \frac{2d(1-t^2)}{1-t^2} = 2\frac{dS}{S}.$$

Thus the singularity for 

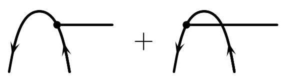
is $\int \frac{dS}{S}$ logarithmic.

LEMMA. *Suppose K and K' differ only in that K has a needle of width ε, then for some fixed norm on the algebra of chord diagrams $\mathcal{A}$, $\|Z_m(K) - z_m(K')\| \sim \varepsilon$.*

PROOF. Consider vertical needle .

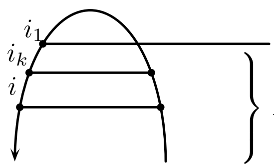
Contributions to $Z(K) - Z(K')$ come from pairings that pair at least one pair of strands on the needle. For  we get $\varnothing$ via isolated chord conditions.

Otherwise we have

 $\sim \varepsilon$

since these contribute opposite signs via $(-1)^{\#P\downarrow}$. (This is when the strands are not paired together.)

If the strands are paired together, then it must be in the round part of the needle at the top, otherwise $dz_i - dz_i' = 0$. Thus we have



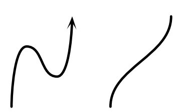
K pairings of the two strands in the rounded part.

$\delta_a = |z_{ja} - z'_{ja}| \implies$ the integral in this case is bounded by a constant multiple of

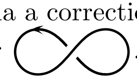
$$\int_0^\varepsilon \frac{d\delta_1}{\delta_1} \int_0^{\delta_1} \frac{d\delta_2}{\delta_2} \cdots \int_0^{\delta_{k-1}} \frac{d\delta_k}{\delta_k} \int_{z_{ik}}^{z'_{ik}} \frac{dz_i - dz'_i}{z_i - z'_i} \sim \varepsilon$$

□

However, we have nothing that lets us change the number of critical points. i. e. Chto eto za znak?

.

This is handled via a correction factor.

Let ∞ stand for .

$$Z(\infty) = \bigcirc + \text{(higher order terms)}$$

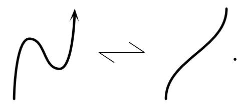
($\bigcirc$ — unknot diagram.) So series for Z(∞) is invertible.

DEFINITION. $K \subset \mathbb{C} \times \mathbb{R}$ with c critical points. ($c \equiv 0 \pmod 2$ in all cases.) Then define

$$\widetilde{Z}(K) = \frac{Z(K)}{(Z(\infty))^{c/2}}.$$

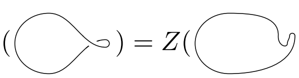
THEOREM. *$\widetilde{Z}(K)$ is invariant under arbitrary deformations of the knot K.*

PROOF. Want to show that



does not alter value of $\widetilde{Z}$. Let K^h, K^s be identical knots apart from small region where K^h has hump and K^s is straight. Want to show $Z(K^s)Z(\infty) = Z(K^h)$. Move hump in K^h so it is very far away from rest of knot (so can ignore pairings from hump to rest of knot). (N. B. you need see that  is allowed. More "needle" arg.) Thus $Z(K^h)$ factors via pairings which pair things on "main" part of knot and pairings that pair strands of hump. But

$$Z(\infty) = Z(\infty) = Z(\ \ ) = Z(\ \ ).$$



□

THEOREM. $\widetilde{Z}(K) \in \mathcal{A}_{\mathbb{R}}$ (i. e. diags with real coefficients)

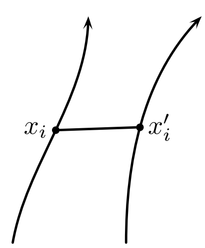
PROOF. Rotate K by 180° around real axis to jet $K' \implies Z(K') = \overline{Z(K)}$ and $Z(\infty') = Z(\infty)$. But $K' \approx K$ so $\widetilde{Z}(K) = \overline{\widetilde{Z}(K)}$. □

REMARK. Le and Murakami have shown that $\widetilde{Z}(K)$ has rational coefficients.

19. Return to Wilson Loop

$$Z_K = \int DA\, e^{-\frac{1}{2}\langle A, LA\rangle} \sum_m \frac{1}{k^m} \int\limits_{K_1 < \ldots < K_m} A(x_1)\ldots A(x_{2m})$$

$$= \sum_m \frac{1}{k^m} \int\limits_{K_1 < \ldots < K_{2m}} \sum_{P=\{(x_i, x_i')|i=1,\ldots,m\}} \qquad (x_i < x_i' \text{ in ordering})$$



$$x_i = (z_i, t)$$
$$x_i' = (z_i', t')$$

$$\langle A(x_i)A(x_i')\rangle = \langle A_k^a(x_i)A_l^b(x_i')\rangle T_a T_b\, dx^k dx^l.$$

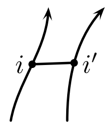
We re-write in complexified axial gauge coordinates. Then only contribution is

$$\langle A_+^a(z,t)A_0^b(z',t')\rangle = \left(\frac{4\lambda\delta^{ab}\delta(t-t')}{z-z'}\right)$$

So

$$\begin{aligned}\langle A(x_i)A(x_i')\rangle &= \langle A_+^a(x_i)A_0^a(x_i')\rangle T_a T_a dx^+ \wedge dt \\ &\quad + \langle A_0^a(x_i)A_+^a(x_i')\rangle T_a T_a dt \wedge dx^+\end{aligned} \quad dx^+ \equiv dx^1 + i dx^2 = dz, \qquad dt\delta(t) = 1$$

So

$$\langle A(x_i)A(x_i')\rangle = \frac{dz - dz'}{z - z'}$$ 

We get

$$Z_K \propto \sum_m \frac{1}{k^m} \int\limits_{K_1 < \ldots < K_m} \sum_P D_P \bigwedge_{i=1}^{m} \left(\frac{dz_i - dz_i'}{z_i - z_i'}\right)$$

This is a *Wilson Loop ordering* version of the Kontsevich integral.

QUESTION. Is the Wilson loop ordered integral

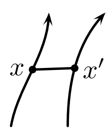
$$\int\limits_{K_1 < \ldots < K_n} \sum_P D_P \bigwedge_{i=1}^{m} \left(\frac{dz - dz_i'}{z_i - z_i'}\right)$$

the "same" as the Kontsevich integral?

We shall look at this!

The answer is: They are the same!

Just note



$$\Bigg\} : \frac{dz\delta(t - t')dt'}{z}, \frac{\delta(t - t')dtdz'}{z'}$$

If you switch one line, both of these terms change sign!

Thus our Wilson Loop Integrals become the Kontcevich Integrals with respect to a global time direction.

Bibliography

[1] K. Reidemeister, "KnotenTheorie", Chelesea Pub. (1933).
[2] E. Witten, Quantum Field Theory and the Jones Polynomial, *Comm. in Math. Phys.*, **121**, (1989), 351-399.
[3] E. Witten, Knot Invariants from Four-Dimensional Gauge Theory. Davide Gaiotto, Edward Witten. physics.hep-th. arXiv:1106.4789
[4] E. Witten, Fivebranes and Knots. Edward Witten. physics.hep-th. arXiv:1101.3216