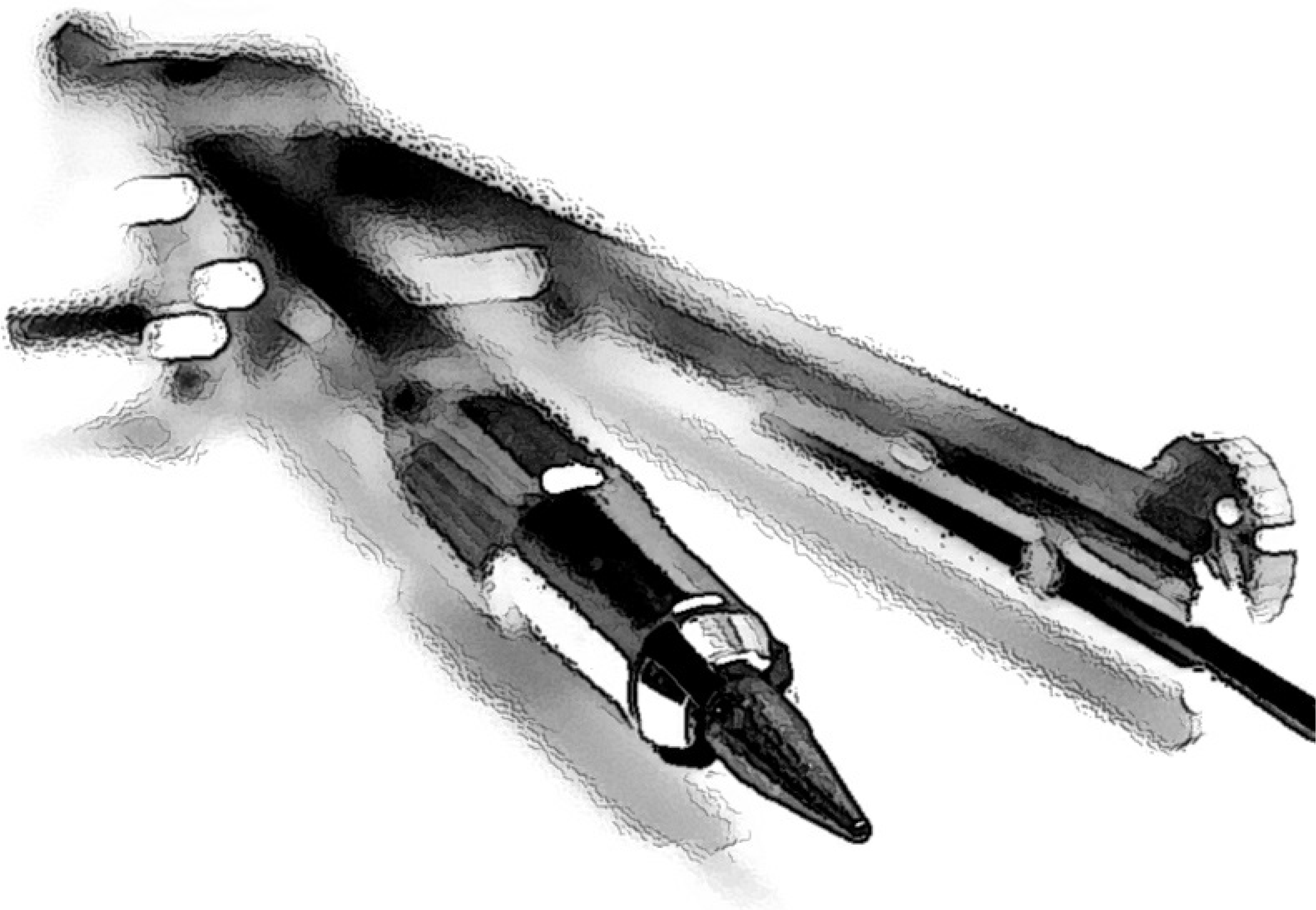
Linear algebra c-2

Geometrical Vectors, Vector Spaces and Linear Maps

Leif Mejlbro



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Leif Mejlbro

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Geometrical Vectors, Vector spaces and Linear Maps

Linear Algebra Examples c-2 – Geometrical Vectors, Vector Spaces and Linear Maps

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Introduction

Here we collect all tables of contents of all the books on mathematics I have written so far for the publisher. In the rst list the topics are grouped according to their headlines, so the reader quickly can get an idea of where to search for a given topic.In order not to make the titles too long I have in the numbering added

a for a compendium

b for practical solution procedures (standard methods etc.)

c for examples.

The ideal situation would of course be that all major topics were supplied with all three forms of books, but this would be too much for a single man to write within a limited time.

After the rst short review follows a more detailed review of the contents of each book. Only Linear Algebra has been supplied with a short index. The plan in the future is also to make indices of every other book as well, possibly supplied by an index of all books. This cannot be done for obvious reasons during the rst couple of years, because this work is very big, indeed.

It is my hope that the present list can help the reader to navigate through this rather big collection of books.

Finally, since this list from time to time will be updated, one should always check when this introduction has been signed. If a mathematical topic is not on this list, it still could be published, so the reader should also check for possible new books, which have not been included in this list yet.

Unfortunately errors cannot be avoided in a rst edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
5th October 2014

1 Geometrical vectors

Example 1.1 *Given $A_1A_2\cdots A_8$ a regular octogon of midpoint A_0. How many different vectors are there among the 81 vectors $\overrightarrow{A_iA_j}$, where i and j belong to the set $\{0,1,2,\dots,8\}$?*

Remark 1.1 There should have been a figure here, but neither LATEXnor MAPLE will produce it for me properly, so it is left to the reader. ◊

This problem is a typical combinatorial problem.

Clearly, the 9 possibilities $\overrightarrow{A_iA_i}$ all represent the $\mathbf{0}$ vector, so this will giver us 1 possibility.

From a geometrical point of view A_0 is not typical. We can form 16 vector where A_0 is the initial or final point. These can, however, be paired. For instance

$$\overrightarrow{A_1A_0} = \overrightarrow{A_0A_5}$$

and analogously. In this particular case we get 8 vectors.

Then we consider the indices modulo 8, i.e. if an index is larger than 8 or smaller than 1, we subtract or add some multiple of 8, such that the resulting index lies in the set $\{1,2,\dots,8\}$. Thus e.g. $9 = 1+8 \equiv 1(\mod 8)$.

Then we have 8 different vectors of the form $\overrightarrow{A_iA_{i+1}}$, and these can always be paired with a vector of the form $\overrightarrow{A_jA_{j-1}}$. Thus e.g. $\overrightarrow{A_1A_2} = \overrightarrow{A_6A_5}$. Hence the 16 possibilities of this type will only give os 8 different vectors.

The same is true for $\overrightarrow{A_iA_{i+2}}$ and $\overrightarrow{A_jA_{j-2}}$ (16 possibilities and only 8 vectors), and for $\overrightarrow{A_iA_{i+3}}$ and $\overrightarrow{A_jA_{j-3}}$ (again 16 possibilities and 8 vectors).

Finally, we see that we have for $\overrightarrow{A_iA_{i+4}}$ 8 possibilities, which all represent a diameter. None of these diameters can be paired with any other, so we obtain another 8 vectors.

Summing up,

	# possibilities	# vectors
$\mathbf{0}$ vector	9	1
A_0 is one of the points	16	8
$\overrightarrow{A_iA_{i\pm 1}}$	16	8
$\overrightarrow{A_iA_{i\pm 2}}$	16	8
$\overrightarrow{A_iA_{i\pm 3}}$	16	8
$\overrightarrow{A_1A_{i+4}}$	8	8
I alt	81	41

By counting we find 41 different vectors among the 81 possible combinations.

Example 1.2 *Given a point set G consisting of n points*

$$G = \{A_1, A_2, \dots, A_n\}.$$

Denoting by O the point which is chosen as origo of the vectors, prove that the point M given by the equation

$$\overrightarrow{OM} = \frac{1}{n}\left(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n}\right),$$

does not depend on the choice of the origo O.
The point M is called the midpoint *or the* geometrical barycenter *of the point set G.*
Prove that the point M satisfies the equation

$$\overrightarrow{MA_1} + \overrightarrow{MA_2} + \cdots + \overrightarrow{MA_n} = \vec{0},$$

and that M is the only point fulfilling this equation.

Let

$$\overrightarrow{OM} = \frac{1}{n}\left(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n}\right)$$

and

$$\overrightarrow{O_1M_1} = \frac{1}{n}\left(\overrightarrow{O_1A_1} + \overrightarrow{O_1A_2} + \cdots + \overrightarrow{O_1A_n}\right).$$

Then

$$\begin{aligned}\overrightarrow{O_1M} &= \overrightarrow{O_1O}+\overrightarrow{OM}=\overrightarrow{O_1O}+\frac{1}{n}\left(\overrightarrow{OA_1}+\overrightarrow{OA_2}+\cdots+\overrightarrow{OA_n}\right)\\ &= \frac{1}{n}\left\{\left(\overrightarrow{O_1O}+\overrightarrow{OA_1}\right)+\left(\overrightarrow{O_1O}+\overrightarrow{OA_2}\right)+\cdots+\left(\overrightarrow{O_1O}+\overrightarrow{OA_n}\right)\right\}\\ &= \frac{1}{n}\left(\overrightarrow{O_1A_1}+\overrightarrow{O_1A_2}+\cdots+\overrightarrow{O_1A_n}\right)=\overrightarrow{O_1M_1},\end{aligned}$$

from which we conclude that $M_1 = M$.

Now choose in particular $O = M$. Then

$$\overrightarrow{MM}=\vec{0}=\frac{1}{n}\left(\overrightarrow{MA_1}+\overrightarrow{MA_2}+\cdots+\overrightarrow{MA_n}\right),$$

thus

$$\overrightarrow{MA_1}+\overrightarrow{MA_2}+\cdots+\overrightarrow{MA_n}=\vec{0}.$$

On the other hand, the uniqueness proved above shows that M is the only point, for which this is true.

Example 1.3 *Prove that if a point set*

$$G = \{A_1, A_2, \dots, A_n\}$$

has a centrum of symmetry M, then the midpoint of the set (the geometrical barycenter) lie in M.

If A_i and A_j are symmetric with respect to M, then

$$\overrightarrow{MA_i}+\overrightarrow{MA_j}=\vec{0}.$$

Since every point is symmetric to precisely one other point with respect to M, we get

$$\overrightarrow{MA_1}+\overrightarrow{MA_2}+\cdots+\overrightarrow{MA_n}=\vec{0},$$

which according to Example 1.2 means that M is also the geometrical barycenter of the set.

Example 1.4 *Prove that if a point set $G = \{A_1, A_2, \dots, A_n\}$ has an axis of symmetry ℓ, then the midpoint of the set (the geometrical barycenter) lies on ℓ.*

Every point A_i can be paired with an A_j, such that $\overrightarrow{OA_i}+\overrightarrow{OA_j}$ lies on ℓ, and such that $G \setminus \{A_i, A_j\}$ still has the axis of symmetry ℓ.

Remark 1.2 The problem is here that A_j, contrary to Example 1.3 is not uniquely determined. $\Diamond$

Continue in this way by selecting pairs, until there are no more points left. Then the midpoints of all pairs will lie on ℓ. Since ℓ is a straight line, the midpoint of all points in G will also lie on ℓ.

Example 1.5 *Given a regular hexagon of the vertices A_1, A_2, ..., A_6. Denote the center of the hexagon by O. Find the vector $\overrightarrow{OM}$ from O to the midpoint (the geometrical barycenter) M of*

1. *the point set $\{A_1, A_2, A_3, A_4, A_5\}$,*
2. *the point set $\{A_1, A_2, A_3\}$.*

Remark 1.3 Again a figure would have been very useful and again neither LATEXnor MAPLE will produce it properly. The drawing is therefore left to the reader. ◊

1. It follows from

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4} + \overrightarrow{OA_5} + \overrightarrow{OA_6} = \vec{0},$$

by adding something and then subtracting it again that

$$\begin{aligned} \overrightarrow{OM} &= \frac{1}{5}\left\{\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4} + \overrightarrow{OA_5}\right\} \\ &= \frac{1}{5}\left\{\left(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4} + \overrightarrow{OA_5} + \overrightarrow{OA_6}\right) - \overrightarrow{OA_6}\right\} \\ &= -\frac{1}{5}\overrightarrow{OA_6} = \frac{1}{5}\overrightarrow{OA_3}. \end{aligned}$$

2. Since $\overrightarrow{OA_1} + \overrightarrow{OA_3} = \overrightarrow{OA_2}$ (follows from the missing figure, which the reader of course has drawn already), we get

$$\overrightarrow{OM} = \frac{1}{3}\left\{\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}\right\} = \frac{2}{3}\overrightarrow{OA_2}.$$

Example 1.6 *Prove by vector calculus that the medians of a triangle pass through the same point and that they cut each other in the proportion $1 : 2$.*

Remark 1.4 In this case there would be a theoretical possibility of sketching a figure in LATEX. It will, however, be very small, and the benefit of if will be too small for all the troubles in creating the figure. LATEXis not suited for figures. ◊

Let O denote the reference point. Let M_A denote the midpoint of BC and analogously of the others. Then the median from A is given by the line segment AM_A, and analogously.
It follows from the definition of M_A that

$$\overrightarrow{OM_A} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}),$$

$$\overrightarrow{OM_B} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}),$$

$$\overrightarrow{OM_C} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}).$$

Then we conclude that

$$\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{2}\overrightarrow{OA} + \overrightarrow{OM_A} = \frac{1}{2}\overrightarrow{OB} + \overrightarrow{OM_B} = \frac{1}{2}\overrightarrow{OM_C}.$$

Choose $O = M$, such that $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \vec{0}$, i.e. M is the geometrical barycenter. Then we get by multiplying by 2 that

$$\vec{0} = \overrightarrow{MA} + 2\overrightarrow{MM_A} = \overrightarrow{MB} + 2\overrightarrow{MM_B} = \overrightarrow{MC} + 2\overrightarrow{MM_C},$$

which proves that M lies on all three lines AM_A, BM_B and CM_C, and that M cuts each of these line segments in the proportion $2:1$.

Example 1.7 *We define the* median *from a vertex A of a tetrahedron $ABCD$ as the line segment from A to the point of intersection of the medians of the triangle BCD. Prove by vector calculus that the four medians of a tetrahedron all pass through the same point and cut each other in the proportion* $1:3$.
Furthermore, prove that the point mentioned above is the common midpoint of the line segments which connect the midpoints of opposite edges of the tetrahedron.

Remark 1.5 It is again left to the reader to sketch a figure of a tetrahedron. $\Diamond$

It follows from Example 1.6 that M_A is the geometrical barycentrum of $\triangle BCD$, i.e.

$$\overrightarrow{OM_A} = \frac{1}{3}\left(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right),$$

and analogously. Thus

$$\begin{aligned}\frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right) &= \frac{1}{3}\overrightarrow{OA} + \overrightarrow{OM_A} = \frac{1}{3}\overrightarrow{OB} + \overrightarrow{OM_B} = \frac{1}{3}\overrightarrow{OC} + \overrightarrow{OM_C} \\ &= \frac{1}{3}\overrightarrow{OD} + \overrightarrow{ONM_D}.\end{aligned}$$

By choosing $O = M$ as the geometrical barycenter of A, B, C and D, i.e.

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = \vec{0},$$

we get

$$\frac{1}{3}\overrightarrow{MA} + \overrightarrow{MM_A} = \frac{1}{3}\overrightarrow{MB} + \overrightarrow{MM_B} = \frac{1}{3}\overrightarrow{MC} + \overrightarrow{MM_C} = \frac{1}{3}\overrightarrow{MD} + \overrightarrow{MM_D},$$

so we conclude as in Example 1.6 that the four medians all pass through M, and that M divides each median in the proportion $3:1$.

Finally, by using M as reference point we get

$$\begin{aligned}
\vec{0} &= \frac{1}{4}\left\{\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}+\overrightarrow{MD}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MA}+\frac{1}{2}\overrightarrow{MB}\right\}+\frac{1}{2}\left\{\overrightarrow{MC}+\frac{1}{2}\overrightarrow{MD}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MA}+\frac{1}{2}\overrightarrow{MC}\right\}+\frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MB}+\frac{1}{2}\overrightarrow{MD}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MA}+\frac{1}{2}\overrightarrow{MD}\right\}+\frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MB}+\frac{1}{2}\overrightarrow{MC}\right\}.
\end{aligned}$$

Here e.g

$$\frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MA}+\frac{1}{2}\overrightarrow{MB}\right\}+\frac{1}{2}\left\{\frac{1}{2}\overrightarrow{MC}+\frac{1}{2}\overrightarrow{MD}\right\}=\vec{0}$$

represents M as well as the midpoint of the midpoints of the two opposite edges AB and CD. Analogously for in the other two cases.

Example 1.8 *In the tetrahedron $OABC$ we denote the sides of triangle ABC by a, b and c, while the edges OA, OB and OC are denoted by α, β and γ. Using vector calculus one shall find the length of the median of the tetrahedron from 0 expressed by the lengths of the six edges.*

Remark 1.6 It is again left to the reader to sketch a figure of the tetrahedron. $\Diamond$

It follows from Example 1.7 that

$$\overrightarrow{OM}=\frac{1}{4}\left(\overrightarrow{OO}+\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)=\frac{1}{4}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right),$$

hence

$$\begin{aligned}|\overrightarrow{OM}|^2 &= \frac{1}{16}\left\{|\overrightarrow{OA}|^2+|\overrightarrow{OB}|^2+|\overrightarrow{OC}|^2+2\overrightarrow{OA}\cdot\overrightarrow{OB}+2\overrightarrow{OA}\cdot\overrightarrow{OC}+2\overrightarrow{OB}\cdot\overrightarrow{OC}\right\}\\ &= \frac{1}{16}\left\{\alpha^2+\beta^2+\gamma^2+2\overrightarrow{OA}\cdot\overrightarrow{OB}+2\overrightarrow{OA}\cdot\overrightarrow{OB}+2\overrightarrow{OB}\cdot\overrightarrow{OC}\right\}.\end{aligned}$$

Then note that

$$\begin{aligned}\overrightarrow{OA}\cdot\overrightarrow{OB} &= \overrightarrow{OA}\cdot\left(\overrightarrow{OA}+\overrightarrow{AB}\right)=|\overrightarrow{OA}|^2+\overrightarrow{OA}\cdot\overrightarrow{AB}\\ &= \alpha^2+\overrightarrow{AB}\cdot\overrightarrow{OA}=\left(\overrightarrow{OB}+\overrightarrow{BA}\right)\cdot\overrightarrow{OB}\\ &= |\overrightarrow{OB}|^2+\overrightarrow{OB}\cdot\overrightarrow{BA}=\beta^2+\overrightarrow{AB}\cdot\overrightarrow{BO},\end{aligned}$$

thus

$$\begin{aligned}2\overrightarrow{OA}\cdot\overrightarrow{OB} &= \left\{\alpha^2+\overrightarrow{AB}\cdot\overrightarrow{OA}\right\}+\left\{\beta^2+\overrightarrow{AB}\cdot\overrightarrow{BO}\right\}\\ &= \alpha^2+\beta^2+\overrightarrow{AB}\cdot\left\{\overrightarrow{BO}+\overrightarrow{OA}\right\}=\alpha^2+\beta^2-\overrightarrow{AB}\cdot\overrightarrow{AB}\\ &= \alpha^2+\beta^2-c^2.\end{aligned}$$

Analogously,

$$2\overrightarrow{OA}\cdot\overrightarrow{OC}=\alpha^2+\gamma^2-b^2 \quad \text{og} \quad 2\overrightarrow{OB}\cdot\overrightarrow{OC}=\beta^2+\gamma^2-a^2.$$

It follows by insertion that

$$\begin{aligned}|\overrightarrow{OM}|^2 &= \frac{1}{16}\left\{\alpha^2+\beta^2+\gamma^2+\alpha^2+\beta^2-c^2+\alpha^2+\gamma^2-b^2+\beta^2+\gamma^2-a^2\right\}\\ &= \frac{1}{16}\left\{3\left(\alpha^2+\beta^2+\gamma^2\right)-\left(a^2+b^2+c^2\right)\right\},\end{aligned}$$

so

$$|\overrightarrow{OM}|=\frac{1}{4}\sqrt{3(\alpha^2+\beta^2+\gamma^2)-(a^2+b^2+c^2)}.$$

Example 1.9 *Prove for any tetrahedron that the sum of the squares of the edges is equal to four times the sum of the squares of the lengths of the line segments which connect the midpoints of opposite edges.*

Remark 1.7 It is left to the reader to sketch a tetrahedron for the argument below. $\Diamond$

Choose two opposite edges, e.g. OA and BC, where 0 is the top point, while ABC is the triangle at the bottom. If we use 0 as the reference point, then the initial point of OA is represented by the vector $\frac{1}{2}\overrightarrow{OA}$, and the end point is represented by

$$\overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}.$$

Hence, the vector, representing the connecting line segment between the midpoints of two opposite edges, is given by

$$\frac{1}{2}\left\{\overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OA}\right\} = \frac{1}{2}\left\{\overrightarrow{AB} + \overrightarrow{OC}\right\}.$$

Analogously we obtain the vectors of the other two pairs of opposite edges,

$$\frac{1}{2}\left\{\overrightarrow{BC} + \overrightarrow{OA}\right\} \qquad \text{og} \qquad \frac{1}{2}\left\{\overrightarrow{CA} + \overrightarrow{OB}\right\}.$$

Then four times the sum of the squares of these lengths is

$$\begin{aligned}&\left\{\overrightarrow{AB} + \overrightarrow{OC}\right\} \cdot \left\{\overrightarrow{AB} + \overrightarrow{OC}\right\} + \left\{\overrightarrow{BC} + \overrightarrow{OA}\right\} \cdot \left\{\overrightarrow{BC} + \overrightarrow{OA}\right\} + \left\{\overrightarrow{CA} + \overrightarrow{OB}\right\} \cdot \left\{\overrightarrow{CA} + \overrightarrow{OB}\right\} \\ &\qquad = |\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{OC} + |\overrightarrow{BC}|^2 + |\overrightarrow{OA}|^2 + 2\overrightarrow{BC} \cdot \overrightarrow{OA} + |\overrightarrow{CA}|^2 + |\overrightarrow{OB}|^2 + 2\overrightarrow{CA} \cdot \overrightarrow{OB}.\end{aligned}$$

The claim will be proved if we can prove that

$$\overrightarrow{AB} \cdot \overrightarrow{OC} + \overrightarrow{BC} \cdot \overrightarrow{OA} + \overrightarrow{CA} \cdot \overrightarrow{OB} = 0.$$

Now,

$$\begin{aligned}&\overrightarrow{AB} \cdot \overrightarrow{OC} + \overrightarrow{BC} \cdot \overrightarrow{OA} + \overrightarrow{CA} \cdot \overrightarrow{OB} \\ &\qquad = (\overrightarrow{OB} - \overrightarrow{OA} \cdot \overrightarrow{OC} + (\overrightarrow{OC} - \overrightarrow{OB}) \cdot \overrightarrow{OA} + (\overrightarrow{OA} - \overrightarrow{OC}) \cdot \overrightarrow{OB} \\ &\qquad = \overrightarrow{OB} \cdot \overrightarrow{OC} - \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{OA} - \overrightarrow{OB} \cdot \overrightarrow{OA} + \overrightarrow{OA} \cdot \overrightarrow{OB} - \overrightarrow{OC} \cdot \overrightarrow{OB} \\ &\qquad = 0,\end{aligned}$$

so we have proved that the sum of the squares of the edges is equal to four times the sum of the squares of the lengths of the line segments which combine the midpoints of opposite edges.

Example 1.10 *Prove by vector calculus that the midpoints of the six edges of a cube, which do not intersect a given diagonal, must lie in the same plane.*

Remark 1.8 It is left to the reader to sketch a cube where $ABCD$ is the upper square and $EFGH$ the lower square, such that A lies above E, B above F, C above G and D above H. $\Diamond$

Using the fixation of the corners in the remark above we choose the diagonal AG. Then the six edges in question are BC, CD, DH, HE, EF and FB.

Denote the midpoint of the cube by 0- Then it follows that the midpoint of BC is symmetric to the midpoint of HE with respect to 0. We have analogous results concerning the midpoints of the pairs (CD, EF) and (DH, BF).

The claim will follow if we can prove that the midpoints of BC, CD and DH all lie in the same plane as 0, because it follows by the symmetry that the latter three midpoints lie in the same plane.

Using 0 as reference point we get the representatives of the midpoints

$$\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC}), \quad \frac{1}{2}(\overrightarrow{OC}+\overrightarrow{OD}), \quad \frac{1}{2}(\overrightarrow{OD}+\overrightarrow{OH}) = \frac{1}{2}(\overrightarrow{OD}-\overrightarrow{OB}).$$

Now, these three vectors are linearly dependent, because

$$\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{OD}) - \frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC}) = \frac{1}{2}(\overrightarrow{OD}-\overrightarrow{OB}),$$

hence the three points all lie in the same plane as 0, and the claim is proved.

Example 1.11 *Find by using vector calculus the distance between a corner of a unit cube and a diagonal, which does not pass through this corner.*

Remark 1.9 It is left to the reader to sketch a unit cube of the corners $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$. $\Diamond$

Since we consider a unit cube, the distance is the same, no matter which corner we choose not lying on the chosen diagonal.

We choose in the given coordinate system the point $(0,0,0)$ and the diagonal from $(1,0,0)$ to $(0,1,1)$. The diagonal is represented by the vectorial parametric description

$$(1,0,0) - s(-1,1,1) = (1-s,s,s), \qquad s \in [0,1].$$

The task is to find $s \in [0,1]$, such that

$$|(1-s,s,s)| = \sqrt{(1-s)^2+s^2+s^2} = \sqrt{3s^2-2s+1},$$

becomes as small as possible, because $|(1-s,s,s)|$ is the distance from $(0,0,0)$ to the general point on the diagonal.

If we put $\varphi(s) = 3s^2 - 2s + 1$, then

$$\varphi'(s) = 6s - 2 = 0 \qquad \text{for } s = \frac{1}{3},$$

which necessarily must be a minimum. The point on the diagonal which is closest to $(0, 0, 0)$ is then $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and the distance is

$$\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}.$$

Example 1.12 *Formulate the geometrical theorems which can be derived from the vector identities*

1. $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = 2(\vec{a}^2 + \vec{b}^2)$.

2. $(\vec{a} + \vec{b} + \vec{c})^2 + (\vec{a} + \vec{b} - \vec{c})^2 + (\vec{a} - \vec{b} + \vec{c})^2 + (-\vec{a} + \vec{b} + \vec{c})^2 = 4(\vec{a}^2 + \vec{b}^2 + \vec{c}^2)$.

1. It follows from a figure that in a parallelogram the sum of the squares of the edges is equal to the sum of the squares of the diagonals, where we use that

$$2(\vec{a}^2 + \vec{b}^2) = \vec{a}^2 + \vec{b}^2 + \vec{a}^2 + \vec{b}^2.$$

Remark 1.10 I have tried without success to let LaTeX sketch a nice figure, so it is again left to the reader to sketch the parallelogram. Analogously in the second question. ♢.

2. This follows in a similar way. In a parallelepiped the sum of the squares of the edges, i.e. $4(\vec{a}^2+\vec{b}^2+\vec{c}^2)$, is equal to the sum of the squares of the diagonals.

Example 1.13 *Given three points P, Q and R, which define a plane π. Let P, Q and R be represented by the vectors $\vec{p}$, $\vec{q}$ and $\vec{r}$. Prove that the vector*

$$\vec{p}\times\vec{q}+\vec{q}\times\vec{r}+\vec{r}\times\vec{p}$$

is perpendicular to π.
Find an expression of the distance of the origo to $\vec{r}$.

Remark 1.11 Again it is left to the reader to sketch the figure. ♢

Since $\vec{q}-\vec{p}$ and $\vec{r}-\vec{q}$ are parallel to the plane π, the vectorial product

$$(\vec{q}-\vec{p})\times(\vec{r}-\vec{q})=\vec{q}\times\vec{r}-\vec{p}\times\vec{r}-\vec{q}\times\vec{q}+\vec{p}\times\vec{q}=\vec{p}\times\vec{q}+\vec{q}\times\vec{r}+\vec{r}\times\vec{p}$$

must be perpendicular to π.

Then

$$\vec{p}\cdot\{\vec{p}\times\vec{q}+\vec{q}\times\vec{r}+\vec{r}\times\vec{p}\}=\vec{p}\cdot(\vec{q}\times\vec{r}),$$

is the distance (with sign)

$$\frac{\vec{p}\cdot(\vec{q}\times\vec{r})}{|\vec{p}\times\vec{q}+\vec{q}\times\vec{r}+\vec{r}\times\vec{p}|}.$$

Example 1.14 *Let $\vec{a}=(\vec{b}\cdot\vec{e})\vec{b}+\vec{b}\times(\vec{b}\times\vec{e})$, where $\vec{a}$, $\vec{b}$ and $\vec{e}$ are vectors from the same point, and $\vec{e}$ is a unit vector. Prove that $\vec{b}$ is halving $\angle(\vec{e},\vec{a})$.*

The vector $\vec{b}\times(\vec{b}\times\vec{e})$ is perpendicular to $\vec{b}$, hence

$$\vec{a}=(\vec{b}\cdot\vec{e})\vec{b}+\vec{b}\times(\vec{b}\times\vec{e})$$

is an orthogonal splitting.

Furthermore, $\vec{b}\times(\vec{b}\times\vec{e})$ is perpendicular to $\vec{b}\times\vec{e}$, and this vector lies in the half space which is given by the plane defined by $\vec{b}$ and $\vec{b}\times\vec{e}$, given that this half space does *not* contain $\vec{e}$. Then the claim will follow, if we can prove that $\varphi=\cos\psi$, where φ denotes the angle between $\vec{a}$ and $\vec{b}$, and ψ denotes the angle between $\vec{b}$ and $\vec{e}$.

Now,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos(\angle(\vec{a},\vec{b})) \quad \text{og} \quad \vec{b}\cdot\vec{e} = |\vec{e}|\cos(\angle(\vec{b},\vec{e})),$$

thus it suffices to prove that $\vec{a}\cdot\vec{b} = |\vec{a}|(\vec{b}\cdot\vec{e})$. We have

$$\vec{a}\cdot\vec{b} = (\vec{b}\cdot\vec{e})\vec{b}\cdot\vec{b} = |\vec{b}|^2(\vec{b}\cdot\vec{e})$$

and

$$\begin{aligned}|\vec{a}|^2 &= (\vec{b}\cdot\vec{e})|\vec{b}|^2 + \left\{|\vec{b}|\cdot|\vec{b}\times\vec{e}|\sin(\angle(\vec{b},\vec{b}\times\vec{e}))\right\}^2 = (\vec{b}\cdot\vec{e})^2\cdot|\vec{b}|^2 + |\vec{b}|^2\cdot|\vec{\times}\vec{e}|^2 \\ &= |\vec{b}|^2\left\{|\vec{b}|^2\cos^2(\angle(\vec{b},\vec{e})) + |\vec{b}|^2\sin^2(\angle(\vec{b},\vec{e}))\right\} = |\vec{b}|^4,\end{aligned}$$

so $|\vec{a}| = |\vec{b}|^2$, and we see that

$$\vec{a}\cdot\vec{b} = |\vec{b}|^2(\vec{b}\cdot\vec{e}) = |\vec{a}|(\vec{b}\cdot\vec{e})$$

as required and the claim is proved.

ALTERNATIVELY if follows from the rule of the double vectorial product that

$$\vec{b}\times(\vec{b}\times\vec{e}) = (\vec{b}\cdot\vec{e})\vec{b} - |\vec{b}|^2\vec{e},$$

thus $\vec{a} = 2(\vec{b}\cdot\vec{e})\vec{b} - |\vec{b}|^2\vec{e}$. Then

$$|\vec{a}|^2 = 4(\vec{b}\cdot\vec{e})^2|\vec{b}|^2 + |\vec{b}|^4 - 4(\vec{b}\cdot e^2)|\vec{b}|^2 = |\vec{b}|^4,$$

i.e. $|\vec{a}| = |\vec{b}|^2$, and we find again that

$$\vec{a}\cdot\vec{b} = |\vec{b}|^2(\vec{b}\cdot\vec{e}) = |\vec{a}|(\vec{b}\cdot\vec{e}).$$

Example 1.15 *Prove the formula*

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}.$$

We get by insertion into the formula of the double vectorial product

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c},$$

followed by pairing the vectors that

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) \\ &\quad = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b} = \vec{0} -\end{aligned}$$

Example 1.16 *Given three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, where we assume that*

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}.$$

What can be said about their positions?

Using that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

and

$$(\vec{a} \times \vec{b}) \times \vec{c} = -c \times (\vec{a} \times \vec{b}) = -(\vec{c} \cdot \vec{b})\vec{a} + (\vec{c} \cdot \vec{a})\vec{b},$$

it follows by identification that

$$(\vec{a} \cdot \vec{b})\vec{c} = (\vec{c} \cdot \vec{b})\vec{a}.$$

This holds if either $\vec{c} = \pm\vec{a}$, or if $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}$.

Example 1.17 *Explain the geometrical contents of the equations*

$$1) \quad (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{0}, \qquad 2) \quad (\vec{a} \times \vec{b}) \times (\vec{d} \times \vec{d}) = \vec{0}.$$

1. This condition means that $\vec{a} \times \vec{b}$ an $\vec{c} \times \vec{d}$ are perpendicular to each other. Since also $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a} \times \vec{b}$, we conclude that $\vec{a}$, $\vec{b}$ and $\vec{c} \times \vec{d}$ must be linearly dependent of each other.

   Analogously, $\vec{c}$, $\vec{d}$ and $\vec{a} \times \vec{b}$ are linearly dependent.

2. This condition means that $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are proportional, thus $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ all lie in the same plane.

Example 1.18 *Prove that*

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$$

and interpret this formula as a theorem on areas of parallelograms.

By a direct computation,

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b} = 2\vec{a} \times \vec{b}.$$

Then interpret $|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})|$ as the area of the parallelogram, which is defined by the vectors $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$. This area is twice the area of the parallelogram, which is defined by $\vec{a}$ and $\vec{b}$, where $2\vec{a}$ and $2\vec{b}$ are the diagonals of the previous mentioned parallelogram.

Example 1.19 *Compute the vectorial product*

$$\vec{e} \times (\vec{e} \times (\vec{e} \times (\vec{e} \times a))),$$

where $\vec{e}$ is a unit vector.

We shall only repeat the formula of the double vectorial product

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

a couple of times. Starting from the inside we get successively

$$\begin{aligned}\vec{e} \times (\vec{e} \times (\vec{e} \times (\vec{e} \times \vec{a}))) &= \vec{e} \times (\vec{e} \times \{(\vec{e} \cdot \vec{a})\vec{e} - (\vec{e} \cdot \vec{e})\vec{a}\}) \\ &= -\vec{e} \times (\vec{e} \times \vec{a}) = -(\vec{e} \cdot \vec{a})\vec{e} + (\vec{e} \cdot \vec{e})\vec{a} \\ &= \vec{a} - (\vec{e} \cdot \vec{a})\vec{e},\end{aligned}$$

which is that component of $\vec{a}$, which is perpendicular of $\vec{e}$, hence

$$\vec{a} = \vec{e} \times (\vec{e} \times (\vec{e} \times (\vec{e} \times a))) + (\vec{e} \cdot \vec{a})\vec{e}.$$

Example 1.20 *Consider an ordinary rectangular coordinate system in the space of positive orientation, in which there are given the vectors* $\vec{a}(1,-1,2)$ *and* $\vec{b}(-1,k,k)$*. Find all values of* k*, for which the equation*

$$\vec{r} \times \vec{a} = \vec{b}$$

has solutions and find in each case the solutions.

A necessary condition of solutions is that $\vec{a}$ and $\vec{b}$ are perpendicular to each other, i.e.

$$0 = \vec{a} \cdot \vec{b} = -1 - k + 2k = k - 1, \qquad \text{thus } k = 1.$$

The only possibility is therefore $\vec{b}(-1,1,1)$.

Then notice that

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{vmatrix} = (-3,-3,0) = -3(1,1,0),$$

and

$$(1,1,0,) \times \vec{a} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (2,-2,-2) = -2\,\vec{b},$$

hence

$$\left(-\frac{1}{2}, -\frac{1}{2}, 0\right) \times \vec{a} = \vec{b}.$$

Thus, one solution is given by $\vec{r}_0 = -\frac{1}{2}(1,1,0)$. Since all solutions of the homogeneous equation $\vec{r} \times \vec{a} = \vec{0}$ is given by $k\vec{a}$, $k \in \mathbb{R}$, the total solution of the inhomogeneous equation is

$$\vec{r} = -\frac{1}{2}(1,1,0) + k(1,-1,2), \qquad k \in \mathbb{R}.$$

Example 1.21 *Consider an ordinary rectangular coordinate system in the space of positive orientation, in which there are given the vectors* $\vec{a}(1,-1,2)$, $\vec{b}(-1,k,k)$, $\vec{c}(3,1,2)$. *Find all values of* k, *for which the equation*

$$\vec{r} \times \vec{a} + k\,\vec{b} = \vec{c}$$

has solutions and find these solutions.

Since

$$\vec{r} \times a = \vec{c} - k\,\vec{b}$$

is perpendicular to $\vec{a}$, we must have

$$\begin{aligned} 0 &= \vec{a}\cdot\vec{c} - k\,\vec{a}\cdot\vec{b} = (1,-1,2)\cdot(3,1,2) - k(1,-1,2)\cdot(-1,k,k) \\ &= 6 - k\{-1+k\} = -k^2 + k + 6 = -(k+2)(k-3), \end{aligned}$$

so the only possibilities are $k = -2$ and $k = 3$.

If $k = -2$, then

$$\vec{c} - k\,\vec{b} = (3,1,2) + 2(-1,-2,-2) = (1,-3,-2).$$

It follows from

$$\vec{a} \times (1,-3,-2) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 2 \\ 1 & -3 & -2 \end{vmatrix} = (8,4,-2) = 2(4,2,-1)$$

and

$$(4,2,-1) \times \vec{a} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 4 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = (3,-9,-6) = 3(1,-3,-2),$$

that a particular solution is $\vec{r}_0 = \dfrac{1}{3}(4,2,-1)$.

The complete solution is then obtained by adding a multiple of $\vec{a}$, thus

$$\vec{r} = \frac{1}{3}(4,2,-1) + (k-1)(1,-1,2) = (1,1,-1) + k(1,-1,2), \quad k \in \mathbb{R}.$$

If $k = 3$, then

$$\vec{c} - k\,\vec{b} = (3,1,2) - 3(-1,3,3) = (6,-8,-7).$$

It follows from

$$\vec{a} \times (6,-8,-7) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 2 \\ 6 & -8 & -7 \end{vmatrix} = (23,19,-2)$$

and

$$(23, 19, -2) \times \vec{a} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 23 & 19 & -2 \\ 1 & -1 & 2 \end{vmatrix} = (36, -48, -42) = 6(6, -8, -7),$$

that

$$\frac{1}{6}(23, 19, -2) \times \vec{a} = (6, -8, -7) = \vec{c} - k\,\vec{b},$$

so a particular solution is given by $\vec{r} = \dfrac{1}{6}(23, 19, -2)$.
Since $\vec{a} \times \vec{a} = \vec{0}$, the complete set of solutions is given by

$$\vec{r} = \frac{1}{6}(23, 19, -2) + k_1(1, -1, 2), \qquad k_1 \in \mathbb{R}.$$

A nicer expression if obtained if we choose $k_1 = k + \dfrac{1}{6}$, in which case

$$\vec{r} = (4, 3, 0) + k(1, -1, 2), \qquad k \in \mathbb{R}.$$

2 Vector spaces

Example 2.1 *Given the following subsets of the vector space $\mathbb{R}^n$:*

1. *The set of all vectors in $\mathbb{R}^n$, the first coordinate of which is an integer.*
2. *The set of all vectors in $\mathbb{R}^n$, the first coordinate of which is zero.*
3. *The set of all vectors in $\mathbb{R}^n$, $(n \geq 2)$, where at least one for the first two coordinates is zero.*
4. *The set of all vectors in $\mathbb{R}^n$ $(n \geq 2)$, for which the first two coordinates satisfy the equation $x_1 + 2x_2 = 0$.*
5. *The set of all vectors in $\mathbb{R}^n$ $(n \geq 2)$, for which the first two coordinates satisfy the equation $x_1 + 2x_2 = 1$.*

Which of these subsets above are also subspaces of $\mathbb{R}^n$?

1. This set is not a subspace. For example, $(1, \dots)$ belongs to the set, while $\frac{1}{2}(1, \dots) = (\frac{1}{2}, \dots)$ does not.
2. This set is a subspace. In face, every linear combination of elements from the set must have 0 as its first coordinate.
3. This set is not a subspace. Both $(1, 0, \dots)$ and $(0, 1, \dots)$ belong to the set, but their sum $(1, 1, \dots)$ does not.
4. This set is a subspace. The equation $x_1 + 2x_2 = 0$ describes geometrically an hyperplane through 0. Any linear combination of elements satisfying this condition will also fulfil this condition.
5. This set is not a subspace. In fact, $(0, \dots, 0)$ does not belong to the set- The equation $x_1 + 2x_2 = 1$ describes geometrically an hyperplane which is parallel to the subspace of 4).

Example 2.2 *Prove that the following vectors in $\mathbb{R}^4$ are linearly independent:*

1. $\mathbf{a}_1 = (0, -1, -1, -1), \quad \mathbf{a}_2 = (1, 0, -1, -1), \quad \mathbf{a}_3 = (1, 1, 0, -1), \quad \mathbf{a}_4 = (1, 1, 1, 0).$
2. $\mathbf{a}_1 = (1, 1, 0, 0), \quad \mathbf{a}_2 = (2, 1, 1, 0), \quad \mathbf{a}_3 = (3, 1, 1, 1).$

1. We setup the matrix with $\mathbf{a}_i$ as the i-th row and reduce,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_2 \\ R_2 := R_3 - R_2 \\ R_3 := R_4 - R_3 \\ R_4 := -R_1 \end{matrix} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} \sim \\ R_1 := R_1 + R_3 \\ R_2 := R_2 - R_3 \\ R_4 := R_4 - R_2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 + R_4 \\ R_3 := R_3 - R_4 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that the rank is 4. This means that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent.

2. Analogously,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - R_1 \\ R_3 := R_3 - R_2 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

which clearly is of rank 3, so $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent.

Example 2.3 *Check if the matrices*

$$\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 \\ 8 & 3 \end{pmatrix}, \qquad \begin{pmatrix} -5 & -8 \\ -16 & 4 \end{pmatrix}$$

are linearly dependent or linearly independent in the vector space $\mathbb{R}^{2\times 2}$.

Every matrix may be considered as a vector in $\mathbb{R}^4$, where the vector is organized such that we first take the first row and then the second row. Hence,

$$\begin{pmatrix} 2 & -1 & 4 & 6 \\ 3 & 2 & 8 & 3 \\ -5 & -8 & -16 & 4 \end{pmatrix} \begin{matrix} \sim \\ R_1 := 2R_2 - 3R_1 \\ R_3 := 5R_1 + 2R_2 \end{matrix} \begin{pmatrix} 2 & -1 & 4 & 6 \\ 0 & 7 & 4 & -12 \\ 0 & -42 & 72 & 32 \end{pmatrix}$$
$$\begin{matrix} \sim \\ R_3 := R_3 + 6R_2 \end{matrix} \begin{pmatrix} 2 & -1 & 4 & 6 \\ 0 & 7 & 4 & -12 \\ 0 & 0 & 98 & -40 \end{pmatrix}.$$

Since the rank is 3 for the three vector, the vectors are – and hence also the corresponding matrices – linearly independent.

Example 2.4 *Find* a*, such that the vectors* $(1,2,3)$*,* $(-1,0,2)$ *and* $(1,6,a)$ *in* $\mathbb{R}^3$ *are linearly dependent.*

We get by reduction,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 6 & a \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & a-3 \end{pmatrix}$$
$$\begin{matrix} \sim \\ R_3 := R_3 - 2R_2 \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & a-13 \end{pmatrix}.$$

The rank is 3, unless $a = 13$, so the vectors are only linearly dependent for $a = 13$.
We see that if $a = 13$, then

$$(1,6,13) = 3(1,2,3) + 2(-1,0,2),$$

so we have checked our result.

Example 2.5 *Check if the three polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$, below considered as vectors in the vector space $P_2(\mathbb{R})$, are linearly dependent or linearly independent:*

$$P_1(x) = 1 - x, \qquad P_2(x) = x(1-x), \qquad P_3(x) = 1 - x^2.$$

It follows immediately by inspection that

$$P_3(x) = 1 - x^2 = (1-x) + (x - x^2) = P_1(x) + x(1-x) = P_1(x) + P_2(x),$$

showing that the polynomials are linearly dependent.

Example 2.6 *Given in the vector space $P_2(\mathbb{R})$ the vectors*

$$P_1(x) = 1 + x - 3x^2, \quad P_2(x) = 1 + 2x - 3x^2, \quad P_3(x) = -x + x^2.$$

Prove that $(P_1(x), P_2(x), P_3(x))$ is a basis of $P_2(\mathbb{R})$, and write the vector

$$P(x) = 2 + 3x - 3x^2$$

as a linear combination of $P_1(x)$, $P_2(x)$ and $P_3(x)$.

We first note that $P_2(x) - P_1(x) = x$, thus

$$x^2 = x + (-x + x^2) = (P_2(x) - P_1(x)) + P_3(x).$$

Then

$$\begin{aligned} 1 &= P_1(x) - x + 3x^2 \\ &= P_1(x) - P_2(x) + P_1(x) + 3P_3(x) + 3P_2(x) - 3P_1(x) \\ &= 3P_3(x) + 2P_2(x) - P_1(x), \end{aligned}$$

so we have at least

$$\begin{array}{rcrcrcr} 1 &=& 3P_3(x) &+& 2P_2(x) &-& P_1(x), \\ x &=& & & P_2(x) &-& P_1(x), \\ x^2 &=& P_3(x) &+& P_2(x) &-& P_1(x), \end{array}$$

from which

$$P(x) = 2 + 3x - 3x^2 = 3P_3(x) + 4P_2(x) - 2P_1(x).$$

We shall now return to the uniqueness. This may be proved alone by the above. However, we shall here choose a more secure method. The uniqueness clearly follows, if we can prove that

$$\alpha P_1(x) + \beta P_2(x) + \gamma P_3(x) = 0$$

implies $\alpha = \beta = \gamma = 0$.
Putting $x = 0$ into the equation above we get $\alpha + \beta = 0$.
Putting $x = 1$ into the equation, we get $-\alpha = 0$, thus $\alpha = 0$, and hence also $\beta = 0$. Then it follows that $\gamma = 0$, and $P_1(x)$, $P_2(x)$, $P_3(x)$ form a basis of $P_2(\mathbb{R})$.

Example 2.7 *Consider the vector space $C^0(\mathbb{R})$ of real, continuous functions defined on $\mathbb{R}$ with the given vectors (functions) $f(t) = \sin^2 t$, $g(t) = \cos 2t$, and $h(t) = 2$. Find the dimension of* $\text{span}\{f, g, h\}$.

It follows from

$$f(t) = \sin^2 t = \frac{1}{2}\{1 - \cos 2t\} = \frac{1}{4}\,h(t) - \frac{1}{2}\,g(t),$$

that f, g and h are linearly dependent, i.e. of at most rank 2. Since g and h clearly are linearly independent, the rank is 2, hence

$$\dim\ \text{span}\{f, g, h\} = 2.$$

Example 2.8 *Find a basis of the space of solutions of the system of equations*

$$\begin{array}{rcrcrcrcrcl} & & x_2 & + & 3x_3 & - & x_4 & + & x_5 & = & 0, \\ & & & & x_3 & - & x_4 & - & 5x_5 & = & 0, \\ x_1 & + & x_2 & - & x_3 & + & 2x_4 & + & 6x_5 & = & 0. \end{array}$$

First we reduce the matrix of coefficients,

$$\begin{pmatrix} 0 & 1 & 3 & -1 & 1 \\ 0 & 0 & 1 & -1 & -5 \\ 1 & 1 & -1 & 2 & 6 \end{pmatrix} \begin{array}{c} \sim \\ R_1 := R_3 - R_1 \\ R_2 := R_1 - 3R_2 \\ R_3 := R_2 \end{array} \begin{pmatrix} 1 & 0 & -4 & 3 & 5 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & -5 \end{pmatrix}$$

$$\begin{array}{c} \sim \\ R_1 := R_1 + 4R_3 \end{array} \begin{pmatrix} 1 & 0 & 0 & -1 & -15 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & -5 \end{pmatrix},$$

corresponding to the reduced equations

$$\begin{array}{rcrcr} x_1 & = & x_4 & + & 15x_5, \\ x_2 & = & -2x_4 & - & 16x_5, \\ x_3 & = & x_4 & + & 5x_5. \end{array}$$

Choosing $x_4 = s$ and $x_5 = t$ as parameters we find the set of solutions

$$(s + 15t, -2s - 16t, s + 5t, s, t) = s(1, -2, 1, 1, 0) + t(15, -16, 5, 0, 1), \qquad s,\, t \in \mathbb{R}.$$

Hence, a basis of the space of solutions may therefore be consisting of the vectors

$$(1, -2, 1, 1, 0) \quad \text{and} \quad (15, -16, 5, 0, 1).$$

Example 2.9 *Given in the vector space $P_2(\mathbb{R})$ a basis $\{P_1(x), P_2(x), P_3(x)\}$.*
The polynomials $3 + 2x + 7x^2$, $2 + x + 4x^2$ and $5 + 2x^2$ have with respect to this basis the coordinates

$$(1, -2, 0), \qquad (1, -1, 0), \qquad (0, 1, 1).$$

Find the polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$ of the basis.

The conditions mean that

$$\begin{array}{rcrcrcrcrcr} P_1(x) & - & 2P_2(x) & & & = & 3 & + & 2x & + & 7x^2, \\ P_1(x) & - & P_2(x) & & & = & 2 & + & x & + & 4x^2, \\ & & P_2(x) & + & P_3(x) & = & 5 & & & + & 2x^2. \end{array}$$

This is a very simple system, and it follows immediately that

$$\begin{aligned} P_1(x) &= 2\{P_1(x) - P_2(x)\} - \{P_1(x) - 2P_2(x)\} \\ &= 2\left\{2 + x + 4x^2\right\} - \left\{3 + 2x + 7x^2\right\} = 1 + x^2, \\ P_2(x) &= \{P_1(x) - P_2(x)\} - \{P_1(x) - 2P_2(x)\} \\ &= \left\{2 + x + 4x^2\right\} - \left\{3 + 2x + 7x^2\right\} = -1 - x - 3x^2, \\ P_3(x) &= -P_2(x) + 5 + 2x^2 = 1 + x + 3x^2 + 5 + 2x^2 \\ &= 6 + x + 5x^2. \end{aligned}$$

Summing up we have

$$P_1(x) = 1 + x^2, \qquad P_2(x) = -1 - x - 3x^2, \qquad P_3(x) = 6 + x + 5x^2.$$

Example 2.10 *Prove that the two vectors*

$$\mathbf{a}_1 = (1,0,1,0,1,0) \qquad \textit{and} \qquad \mathbf{a}_1 = (0,1,1,1,1,-1)$$

span the same subspace of $\mathbb{R}^6$ *as the two vectors*

$$\mathbf{b}_1 = (4,-5,-1,-5,-1,5) \qquad \textit{and} \qquad \mathbf{b}_2 = (-3,2,-1,2,-1,-2).$$

Obviously, the pairs $\{\mathbf{a}_1, \mathbf{a}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ are separately linearly independent. The claim follows if we can prove that the system $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2\}$ is of rank 2. It follows by reduction that

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 4 & -5 & -1 & -5 & -1 & 5 \\ -3 & 2 & -1 & 2 & -1 & -2 \end{pmatrix} \underset{\substack{R_3 := R_3 - 4R_1 \\ R_4 := R_4 + 3R_1}}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & -5 & -5 & -5 & -5 & 5 \\ 0 & 2 & 2 & 2 & 2 & -2 \end{pmatrix},$$

which clearly is of rank 2, and the claim is proved.

ALTERNATIVELY we see that

$$\mathbf{b}_1 = (4,-5,-1,-5,-1,5) = (4,0,4,0,4,0) + (0,-5,-5,-5,-5,5) = 4\mathbf{a}_1 - 5\mathbf{a}_2,$$

and

$$\mathbf{b}_2 = (-3,2,-1,2,-1,-2) = (-3,0,-3,0,-3,0) + (0,2,2,2,2,-2) = -3\mathbf{a}_1 + 2\mathbf{a}_2,$$

thus

$$\begin{aligned} \mathbf{b}_1 &= 4\mathbf{a}_1 - 5\mathbf{a}_2, \qquad & \mathbf{a}_1 &= -\tfrac{2}{7}\,\mathbf{b}_1 - \tfrac{5}{7}\,\mathbf{b}_2, \\ \mathbf{b}_2 &= -3\mathbf{a}_1 + 2\mathbf{a}_2, \qquad & \mathbf{a}_2 &= -\tfrac{3}{7}\,\mathbf{b}_1 - \tfrac{4}{7}\,\mathbf{b}_2, \end{aligned}$$

and the claim follows.

Example 2.11 *Prove that the vectors*

$$\mathbf{b}_1 = (1,1,1,1), \quad \mathbf{b}_2 = (1,0,1,2), \quad \mathbf{b}_3 = (2,1,0,2), \quad \mathbf{b}_4 = (2,1,1,1),$$

form a basis of $\mathbb{R}^4$*, and find the coordinates of the vectors* $(2,1,1,2)$ *and* $(1,0,0,1)$ *with respect to this basis.*

We get by reducing the (4×4) matrix, which has the $\mathbf{b}_i$ as its rows:

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_2 \\ R_2 := R_1 - R_2 \\ R_3 := R_3 - R_1 - R_2 \\ R_4 := R_4 - R_3 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} \sim \\ R_1 := R_1 - R_4 \\ R_3 := R_4 \\ R_4 := R_3 + 2R_4 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is of rank 4, hence the four vectors $\mathbf{b}_1, \dots, \mathbf{b}_4$ form a basis of $\mathbb{R}_4$.

Then we shall find (x_1, x_2, x_3, x_4), such that

$$(2,1,1,2) = x_1(1,1,1,1) + x_2(1,0,1,2) + x_3(2,1,0,2) + x_4(2,1,1,1),$$

thus written as a system of equations,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

We reduce the total matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 \end{array}\right) \begin{matrix} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_1 - R_3 \\ R_4 := R_4 - R_1 \end{matrix} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right)$$

$$\begin{matrix} \sim \\ R_1 := R_1 - R_2 \\ R_2 := R_4 \\ R_4 := R_2 - R_4 \end{matrix} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right) \begin{matrix} \sim \\ R_1 := R_1 - R_4 \\ R_3 := 2R_3 - R_4 \end{matrix}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right).$$

It follows immediately that $x_1 = x_4 = x_2$, and $x_3 = \frac{1}{3}$. Now $x_3 + 2x_4 = 1$, so $x_4 = \frac{1}{3}$, thus

$$\mathbf{x} = \frac{1}{3}(1,1,1,1),$$

which is easy to check.

Finally,

$$(1,0,0,1) = (2,1,1,2) - (1,1,1,1),$$

so

$$\mathbf{x} = \frac{1}{3}(1,1,1,1) - \mathbf{b}_1 = \frac{1}{3}(-2,1,1,1).$$

Example 2.12 *Assume that $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d} \in V_g^3$ have the coordinates*

$$(3,1,2), \quad (2,-4,1), \quad (-1,2,1), \quad (-3,-1,1)$$

with respect to an ordinary rectangular coordinate system in the space.

1. *Prove that $\left(\vec{a},\vec{b},\vec{c}\right)$ form a basis for V_g^3.*
2. *Find the coordinates of the vector $\vec{d}$ with respect to the basis $\left(\vec{a},\vec{b},\vec{c}\right)$.*

1. Reducing

$$\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -4 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{array}{l} \sim \\ R_1 := -R_3 \\ R_2 := R_2 + 2R_3 \\ R_3 := R_1 + 3R_3 \end{array} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 7 & 5 \end{pmatrix},$$

it follows that this system is of rank 3, so $\left\{\vec{a},\vec{b},\vec{c}\right\}$ form a basis of V_g^3.

2. Then we shall find $\mathbf{x}$, such that

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -4 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

We get by a reduction of the total matrix,

$$\left(\begin{array}{rrr|r} 3 & 2 & -1 & -3 \\ 1 & -4 & 2 & -1 \\ 2 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_2 \\ R_2 := R_1 - 3R_2 \\ R_3 := R_3 - 2R_2 \end{array} \left(\begin{array}{rrr|r} 1 & -4 & 2 & -1 \\ 0 & 14 & -7 & 0 \\ 0 & 9 & -3 & 3 \end{array}\right)$$
$$\begin{array}{l} \sim \\ R_2 := R_2/14 \\ R_3 := R_3/9 \end{array} \left(\begin{array}{rrr|r} 1 & -4 & 2 & -1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - R_2 \\ R_1 := R_1 + 4R_2 \end{array}$$
$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 + 3R_3 \\ R_3 := 6R_3 \end{array} \left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

It follows that $\mathbf{x} = (-1,1,2)$.

Example 2.13 *Given the subsets M, N of a vector space V, we define $M+N$ as the subset*

$$M+N=\{u+v \mid u\in M,\, v\in N\}.$$

Prove that if M and N are subspaces of V, then $M+N$ is a subspace of V, and $M+N$ is the span of $M\cup N$, i.e. $M+N$ consists of all linear combinationes of vectors from the union $M\cup N$ af M and N.

We first prove that $M+N$ is a vector space.
Assume that u_1, $u_2\in M$ and v_1, $v_2\in N$ and $\lambda\in\mathbb{L}$. Then u_1+v_1, $u_2+v_2\in M+N$. We shall prove that this is also the case of $(u_1+v_1)+\lambda(u_2+v_2)$. Now,

$$(u_1+v_1)+\lambda(u_2+v_2)=(u_1+\lambda u_2)+(v_1+\lambda v_2).$$

Since M and N are subspaces, we have $u_1+\lambda u_2\in M$ and $v_1+\lambda v_2\in N$, and the sum belongs to $M+N$.
Putting $\lambda=1$ we get condition U1, and putting $u_2=0$ and $v_2=0$ we obtain U2, and we have proved that $M+N$ is a subspace.

Clearly, every element of $M+N$ can be written as a linear combination of vectors from $M\cup N$. Conversely, if $w_1,\,\dots,\,w_n\in M\cup N$, and $\lambda_1,\,\dots,\,\lambda_n\in\mathbb{L}$, then each w_i either belongs to M or to N. Therefore, we can write the linear combination $\lambda_1w_1+\cdots+\lambda_nw_n$ into a linear combination of vectors from M (a subspace, so this contribution lies in M) and an linear combination of vectors from N (which lies in N, because N is a subspace). Then

$$\lambda_1w_1+\cdots+\lambda_n\in M+N,$$

and the claim is proved.

Example 2.14 *Let V_1 and V_2 be two subspaces of a vector space V.*

1. *Prove that $V_1 \cap V_2$ is a subspace in V, while $V_1 \cup V_2$ in general is not a vector space.*
2. *Let $V_1 + V_2$ denote the vector space spanned by $V_1 \cup V_2$. Prove that*

$$\dim V_1 + \dim V_2 = \dim(V_1 \cap V_2) + \dim(V_1 + V_2).$$

*(*Grassmann's *formula of dimensions).*

1. Let $u, v \in V_1 \cap V_2$ and $\lambda \in \mathbb{L}$. Then V_1 is a subspace, so if $u, v \in V_1 \cap V_2 \subseteq V_1$, then $u + \lambda v \in V_1$. Analogously, $u + \lambda v \in V_2$, hence $u + \lambda v \in V_1 \cap V_2$, and we have proved that $V_1 \cap V_2$ is a subspace.

   Choosing $V = \mathbb{R}^2$ and $V_1 = \mathbb{R} \times \{0\}$, $V_2 = \{0\} \times \mathbb{R}$, thus V is represented by the plane, and V_1 by the x axis and V_2 by the y axis it is obvious that $V_1 \cup V_2$ is the union of the two axes, which is not a subspace.

2. First choose a basis $\mathbf{a}_1, \dots, \mathbf{a}_k$ of $V_1 \cap V_2$. Then supply this to either a basis of

$$V_1 : \qquad \mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}'_{k+1}, \dots, \mathbf{a}'_{k+p},$$

or to

$$V_2 : \qquad \mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}''_{k+1}, \dots, \mathbf{a}''_{k+q}.$$

The point is that no proper linear combination of $\mathbf{a}'_{k+1}, \dots, \mathbf{a}'_{k+p}$ can lie in V_2, because this would imply that

$$\lambda_1 \mathbf{a}'_{k+1} + \cdots + \lambda_p \mathbf{a}'_{k+p} \in V_1 \cap V_2$$

for some set of constants $(\lambda_1, \dots, \lambda_p) \neq \mathbf{0}$. This is in contradiction with the fact that already $\mathbf{a}_1, \dots, \mathbf{a}_k$ form a basis of $V_1 \cap V_2$.

Analogously, no proper linear combination of $\mathbf{a}''_{k+1}, \dots, \mathbf{a}''_{k+q}$ can lie in V_1.

It follows [cf. e.g. Example 2.13] that we can choose

$$\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{a}'_{k+1}, \dots, \mathbf{a}'_{k+p}, \mathbf{a}''_{k+1}, \dots, \mathbf{a}''_{k+q},$$

as a basis of $V_1 + V_2$, hence

$$\dim(V_1 + V_2) = k + p + q.$$

It follows from

$$\dim(V_2 \cap V_2) = k, \quad \dim V_1 = k + p, \quad \dim V_2 = k + q,$$

that

$$\begin{aligned} \dim V_1 + \dim V_2 &= (k+p) + (k+q) = k + (k+p+q) \\ &= \dim(V_1 \cap V_2) + \dim(V_1 + V_2), \end{aligned}$$

and the formula is proved.

Example 2.15 *Given in the vector space $P_2(\mathbb{R})$ the vectors*

$$P_1(x) = 1 + x^2 \qquad \text{and} \qquad P_2(x) = -1 + x + x^2$$

and the vectors

$$Q_1(x) = -1 + 3x + 5x^2 \qquad \text{and} \qquad Q_2(x) = -1 + 4x + 7x^2.$$

Furthermore, let $U = \text{span}\{P_1(x), P_2(x)\}$.

1. *Prove that $Q_1(x)$ and $Q_2(x)$ both belong to U.*
2. *Prove that $(P_1(x), P_2(x))$ and $(Q_1(x), Q_2(x))$ both form a basis of U.*
3. *Let $\mathbf{P}$ denote the basis $(P_1(x), P_2(x))$, and let $\mathbf{Q}$ denote the basis $(Q_1(x), Q_2(x))$. Find the matrix of the change of basis $\mathbf{M}_{PQ}$, which in U goes from the $\mathbf{Q}$ coordinates to the $\mathbf{P}$ coordinates.*

1. We shall prove that $Q_1(x)$ and $Q_2(x)$ can be expressed as linear combinations of $P_1(x)$ and $P_2(x)$. It follows from

$$Q_1(x) = -1 + 3x + 5x^2 = \alpha P_1(x) + \beta P_2(x) = (\alpha - \beta) + \beta x + (\alpha + \beta)x^2$$

that $\beta = 3$ and $\alpha + \beta = 5$, and thus $\alpha = 2$. Finally, a check shows that $\alpha - \beta = 2 - 3 = 1$, so

$$Q_1(x) = 2P_1(x) + 3P_2(x).$$

Analogously,

$$Q_2(x) = -1 + 4x + 7x^2 = \gamma P_1(x) + \delta P_2(x) = (\gamma - \delta) + \delta x + (\gamma + \delta)x^2.$$

Analogously, we see that the only possibility is $\delta = 4$ and $\gamma = 3$, and as another check we have $\gamma - \delta = 3 - 4 = -1$ (OK), hence

$$Q_2(x) = 3P_1(x) + 4P_2(x).$$

Thus, we have proved that $Q_1(x)$, $Q_2(x) \in U$.

2. We get according to 1),

$$\begin{array}{l} Q_1(x) = 2P_1(x) + 3P_2(x), \\ Q_2(x) = 3P_1(x) + 4P_2(x), \end{array} \qquad \text{dvs.} \qquad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}.$$

It follows from

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

[the simple computations are left to the reader] that

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \qquad \text{dvs.} \qquad \begin{array}{l} P_1(x) = -4Q_1(x) + 3Q_2(x), \\ P_2(x) = 3Q_1(x) - 2Q_2(x), \end{array}$$

thus every $P_i(x)$ is uniquely expressed by a linear combination of the Q_i. Thus we conclude that both $(P_1(x), P_2(x))$ and $(Q_1(x), Q_2(x))$ form a basis of U.

3. In the two bases,

$$(Q_1(x)\;\; Q_2(x)) \begin{pmatrix} x_{Q1} \\ x_{Q2} \end{pmatrix} = (P_1(x)\;\; P_2(x)) \begin{pmatrix} x_{P1} \\ x_{P2} \end{pmatrix},$$

where $\mathbf{x_Q}$ are the $\mathbf{Q}$ -coordinates and $\mathbf{x_P}$ are the $\mathbf{P}$ coordinates. By taking the transpose if follows from 2) that

$$(Q_1(x)\;\; Q_2(x)) = (P_1(x)\;\; P_2(x)) \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = (P_1(x)\;\; P_2(x))\, \mathbf{M}_{P\,Q},$$

hence

$$\mathbf{M}_{P\,Q} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix},$$

because we have in this case

$$(Q_1\;\; Q_2) \begin{pmatrix} x_{Q1} \\ x_{Q2} \end{pmatrix} = (P_1\;\; P_2) \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_{Q1} \\ x_{Q2} \end{pmatrix} = (P_1\;\; P_2) \begin{pmatrix} x_{P1} \\ x_{P2} \end{pmatrix}.$$

Example 2.16 *Given in* $\mathbb{R}^4$ *the vectors*

$$\mathbf{a}_1 = (1,-1,2,1), \quad \mathbf{a}_2 = (0,1,1,3), \quad \mathbf{a}_3 = (1,-2,2,-1),$$

$$\mathbf{a}_4 = (0,1,-1,3), \quad \mathbf{a}_5 = (1,-2,2,-3).$$

Prove that $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)$ *form a basis of* $\mathbb{R}^4$*, and find the coordinates of* $\mathbf{a}_5$ *in this basis.*

It follows that $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)$ form a basis of $\mathbb{R}^4$, if and only if

$$\mathbf{a}_5 = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4$$

has a unique solution $\mathbf{x}$. Writing all $\mathbf{a}_i$ as column vectors it follows that

$$\mathbf{a}_5 = (\mathbf{a}_1\;\; \mathbf{a}_2\;\; \mathbf{a}_3\;\; \mathbf{a}_4) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

thus

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 \\ 2 & 1 & 2 & -1 \\ 1 & 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -3 \end{pmatrix}.$$

We have earlier met this task, so we reduce

$$\left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & 1 & 2 & -1 & 2 \\ 1 & 3 & -1 & 3 & -3 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_3 - 2R_1 \\ R_4 := R_4 - R_1 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 3 & -2 & 3 & -4 \end{array}\right)$$

$$\begin{array}{l} \sim \\ R_2 := R_3 \\ R_3 := R_3 - R_2 \\ R_4 := R_4 - 3R_2 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & 1=1 & & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_4 \\ R_3 := R_4 \\ R_4 := R_4 - R_3 \end{array}$$

$$\left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 + R_4/2 \\ R_4 := R_4/2 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

It follows that the solution $\mathbf{x} = (2, -1, -1, -1)$ is unique, so

(1) $\quad \mathbf{a}_5 = 2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4,$

and $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ form a basis of $\mathbb{R}^4$.

Remark 2.1 It is easy to check (1). This is left to the reader.

Example 2.17 *Given in* $\mathbb{R}^3$ *the three vectors*

$$\mathbf{a}_1 = (1,0,-1), \qquad \mathbf{a}_2 = (1,1,1), \qquad \mathbf{a}_3 = (1,-1,1).$$

Prove that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ *form a basis of* $\mathbb{R}^3$*, and find the coordinates of the vectors* $\mathbf{e}_1$*,* $\mathbf{e}_2$*,* $\mathbf{e}_3$ *(the usual basis) with respect to the basis* $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$*.*

It suffices to prove that

$$(\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3)\begin{pmatrix} b_1 \\ b_2 \\ b_2 \end{pmatrix}$$

always has a unique solution for given $\mathbf{b}$. We reduce

$$\left(\begin{array}{rrr|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & -1 & b_2 \\ -1 & 1 & 1 & b_3 \end{array}\right) \begin{array}{c} \sim \\ R_3 := (R_1+R_3)/2 \end{array} \left(\begin{array}{rrr|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 1 & 1 & \frac{1}{2}(b_1+b_2) \end{array}\right)$$

$$\begin{array}{l} \sim \\ R_1 := R_1 - R_3 \\ R_2 := \frac{1}{2}(R_2+R_3) \\ R_3 := \frac{1}{2}(R_3-R_2) \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(b_1-b_3) \\ 0 & 1 & 0 & \frac{1}{4}(b_1+2b_2+b_3) \\ 0 & 0 & 1 & \frac{1}{4}(b_1-2b_2+b_3) \end{array}\right),$$

where it again is easy to check the solution.
Since we after the reductions have the unit matrix in the front, we conclude that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ form a basis of $\mathbb{R}^3$.

We get the coordinates of $\mathbf{e}_1$ by putting $b_1 = 1$ and $b_2 = b_3 = 0$, i.e.

$$\mathbf{e}_1 = \frac{1}{2}\,\mathbf{a}_1 + \frac{1}{4}\,\mathbf{a}_2 + \frac{1}{4}\,\mathbf{a}_3 \sim \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Analogously,

$$\mathbf{e}_2 = 0 \cdot \mathbf{a}_1 + \frac{1}{2}\,\mathbf{a}_2 - \frac{1}{2}\,\mathbf{a}_3 \sim \left(0, \frac{1}{2}, -\frac{1}{2}\right)$$

and

$$\mathbf{e}_3 = -\frac{1}{2}\,\mathbf{a}_1 + \frac{1}{4}\,\mathbf{a}_2 + \frac{1}{4}\,\mathbf{a}_3 \sim \left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Example 2.18 *Let $U \subseteq \mathbb{R}^{2\times 2}$ denote the set of symmetric matrices, i.e.* $\mathbf{A}$ *belongs to* U*, if and and only if* $\mathbf{A} = \mathbf{A}^T$.

1. *Prove that U is a subspace of $\mathbb{R}^{2\times 2}$.*
2. *Find a basis of U and find the dimension of U.*

1. Given $\mathbf{A}$, $\mathbf{B} \in U$ and $\lambda \in \mathbb{L}$. Then
$$(\mathbf{A} + \lambda\mathbf{B})^T = \mathbf{A}^T + \lambda\mathbf{B}^T = \mathbf{A} + \lambda\mathbf{B},$$
which is the condition of $\mathbf{A} + \lambda\mathbf{B} \in U$. This proves that U is a subspace.
2. A basis of U is e.g.
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
The diagonal elements are obvious, and we conclude by the symmetry that we can only have one further dimension. The dimension is 3.

Remark 2.2 The results are easily extended to $U \subseteq \mathbb{R}^{n\times n}$. The basis is determined of the elements of e.g. the upper triangular matrix, because the symmetry then fixes the elements of the lower triangular matrix. Since there are $\frac{1}{2}\,n(n+1)$ elements in an upper triangular matrix, the dimension is in general $\frac{1}{2}\,n(n+1)$. $\Diamond$

Example 2.19 *Given in $\mathbb{R}^4$ the vectors*
$$\mathbf{a}_1 = (1,1,-1,-1), \quad \mathbf{a}_2 = (1,2,-3,-1), \quad \mathbf{a}_3 = (2,1,0,-2), \quad \mathbf{a}_4 = (0,-4,3,0).$$

1. *Find the dimension of* $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$*, and find a basis of* $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.
Find the coordinates of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ with respect to this basis.
2. *Let $\mathbf{x} = (x_1, x_2, x_3)$ be any vector in $\mathbb{R}^4$. Prove that*
$$\mathbf{x} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} \text{ if and only if } x_1 + x_4 = 0.$$

1. The dimension of $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is equal to the rank of the matrix $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$, where the $\mathbf{a}_1$ are written as column vectors. We get by reduction,
$$\begin{aligned}(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4) \quad &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & -4 \\ -1 & -3 & 0 & 3 \\ -1 & -1 & -2 & 0 \end{pmatrix} \begin{array}{c} \sim \\ R_1 := R_2 - R_1 \\ R_3 := R_3 + R_1 \\ R_4 := R_4 + R_1 \end{array} \\ &\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & -4 \\ 0 & -2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \sim \\ R_3 := R_3 + 2R_2 \end{array} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

the rank of which is 3, hence $\dim\ \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = 3$.

Then notice that

$$\mathbf{a}_2 - \mathbf{a}_1 = (0, 1, -2, 0) \quad \text{and} \quad \mathbf{a}_3 - 2\mathbf{a}_1 = (0, -1, 2, 0),$$

so these two vector combinations are linearly dependent. Since the rank is 3, e.g. $(\mathbf{a}_1, \mathbf{a}_1 - \mathbf{a}_1, \mathbf{a}_4)$ must form a basis, possibly $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4)$ instead. It follows from

$$(\mathbf{a}_2 - \mathbf{a}_1) + (\mathbf{a}_3 - 2\mathbf{a}_1) = \mathbf{0},$$

that

$$\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_1 - \mathbf{a}_2 = 3\mathbf{a}_1 - \mathbf{a}_2.$$

The coordinates with respect to the basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4)$ are

$$\begin{array}{rcccc} \mathbf{a}_1 & = & 1 \cdot \mathbf{a}_1 & \sim & (1, 0, 0), \\ \mathbf{a}_2 & = & 1 \cdot \mathbf{a}_2 & \sim & (0, 1, 0), \\ \mathbf{a}_3 & = & 3\mathbf{a}_1 - \mathbf{a}_2 & \sim & (3, -1, 0), \\ \mathbf{a}_4 & = & 1 \cdot \mathbf{a}_4 & \sim & (0, 0, 1). \end{array}$$

2. The equation

$$\mathbf{x} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + y_3\mathbf{a}_3 + y_4\mathbf{a}_4$$

corresponds to the total matrix

$$\{\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{a}_4|\mathbf{x}\} = \left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & x_1 \\ 1 & 2 & 1 & -4 & x_2 \\ -1 & -3 & 0 & 3 & x_3 \\ -1 & -1 & -2 & 0 & x_4 \end{array}\right) \begin{array}{c} \sim \\ R_4 := R_4 + R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & x_1 \\ 1 & 2 & 1 & -4 & x_2 \\ -1 & -3 & 0 & 3 & x_3 \\ 0 & 0 & 0 & 0 & x_1 + x_4 \end{array}\right).$$

We saw in 1) that the matrix of coefficients is of rank 3. Hence, the equation has solutions $\mathbf{y}$, if and only if the total matrix is of rank 3, i.e. if and only if $x_1 + x_4 = 0$.

Example 2.20 *Given in the vector spacet* $\mathbb{R}^4$ *the vectors*

$$\mathbf{u}_1 = (1, -1, 2, 3), \quad \mathbf{u}_2 = (2, -3, 3, 5), \quad \mathbf{u}_3 = (-1, 4, 1, 0),$$

and

$$\mathbf{v}_1 = (3, -8, 1, 4), \quad \mathbf{v}_2 = (1, -7, -4, -3), \quad \mathbf{v}_3 = (-1, 8, 5, 4), \quad \mathbf{v}_4 = (1, 0, 3, 4).$$

1. *Prove that the subspace spanned by the vectors* $\mathbf{u}_1$, $\mathbf{u}_2$ *and* $\mathbf{u}_3$ *is the same as the subspace spanned by the vectors* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *and* $\mathbf{v}_4$.
2. *Find the dimension and a basis of the subspace.*

Here we start by 2).

2. It follows immediately that

$$5\mathbf{u}_1 - 3\mathbf{u}_2 = \mathbf{u}_3,$$

thus the dimension is at most 2. On the other hand, any two of the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are linearly independent, so the dimension is 2.

Since $\mathbf{u}_1 + \mathbf{u}_3 = (0, 3, 3, 3)$, an easy basis is

$$\left\{-\mathbf{u}_3, \frac{1}{3}(\mathbf{u}_1 + \mathbf{u}_3)\right\} = \{(1, -4, -1, 0), (0, 1, 1, 1)\},$$

where both vectors most conveniently have a 0 as one of its coordinates.

1. It follows from

$$\begin{aligned}\mathbf{v}_1 &= (3,-8,1,4) = 3(1,-4,-1,0)+4(0,1,1,1),\\ \mathbf{v}_2 &= (1,-7,-4,-3) = 1\cdot(1,-4,-1,0)-3(0,1,1,1),\\ \mathbf{v}_3 &= (-1,8,5,4) = -1\cdot(1,-4,-1,0)+4(0,1,1,1),\\ \mathbf{v}_4 &= (1,0,3,4) = 1\cdot(1,-4,-1,0)+4(0,1,1,1),\end{aligned}$$

that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ all lie in span$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, so

$$\dim\ \text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\} \le \dim\ \text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\} = 2.$$

On the other hand, e.g.. $\mathbf{v}_1$ and $\mathbf{v}_2$ are clearly linearly independent, hence

$$\dim\ \text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\} \ge 2.$$

We conclude that

$$\text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\} = \text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\},$$

and that the dimension is 2.

Example 2.21 *Given in the vector space* $\mathbb{R}^4$ *the vectors*

$$\mathbf{u}_1 = (1,-1,1,2),\quad \mathbf{u}_2 = (1,-1,2,1),\quad \mathbf{u}_3 = (1,-1,2,2).$$

1. *Find the dimension of the subspace* $U = \text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

2. *Given three linearly independent vectors*

$$\mathbf{v}_1 = (2,-1,3,0),\quad \mathbf{v}_2 = (1,-1,1,1),\quad \mathbf{v}_3 = (2,-1,4,0).$$

Prove that $\mathbf{v}_2$ *belongs to the subspace* U*, and describe this vector as a linear combination of* $\mathbf{u}_1$*,* $\mathbf{u}_2$*,* $\mathbf{u}_3$*. Prove that* $\mathbf{v}_1$ *and* $\mathbf{v}_3$ *do* not *belong to* U*.*

3. *Prove that there exists a proper linear combination of* $\mathbf{v}_1$ *and* $\mathbf{v}_3$*, which belongs to* U*, and find such a linear combination.*

4. *Find the dimension of the subspace* $U \cap V$*, where*

$$V = \text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}.$$

1. It follows immediately that

$$\mathbf{u}_3-\mathbf{u}_2 = (0,0,0,1) \quad\text{and}\quad \mathbf{u}_3-\mathbf{u}_1 = (0,0,1,0).$$

Then $\{\mathbf{u}_3,\mathbf{u}_3-\mathbf{u}_1,\mathbf{u}_3-\mathbf{u}_2\}$ is a basis, hence $\dim U = 3$. We may choose the basis

(2) $\{(1,-1,0,0),(0,0,1,0),(0,0,0,1)\}$,

which will be more convenient in the following. Note, however, that

$$\begin{aligned}(1,-1,0,0) &= \mathbf{u}_3-1(\mathbf{u}_3-\mathbf{u}_1)-2(\mathbf{u}_3-\mathbf{u}_2)\\ &= \mathbf{u}_3-2\mathbf{u}_3+2\mathbf{u}_1-2\mathbf{u}_3+2\mathbf{u}_2\\ &= 2\mathbf{u}_1+2\mathbf{u}_2-3\mathbf{u}_3.\end{aligned}$$

2. Applying the basis from (2) we get

$$\mathbf{v}_2 = (1,-1,1,1) = (1,-1,0,0) + (0,0,1,0) + (0,0,0,1),$$

hence $\mathbf{v}_2 \in U$.

Since the first two coordinates of $\mathbf{v}_1$ and $\mathbf{v}_3$ are $(2,-1)$, and since only the vector $(1,-1,0,0)$ in the basis have any of the two first coordinates different from zero, neither $\mathbf{v}_1$ nor $\mathbf{v}_3$ lie in U.

3. The only possibilities are $\alpha(\mathbf{v}_1 - \mathbf{v}_3)$, $\alpha \in \mathbb{L}$, e.g.

$$\mathbf{v}_3 - \mathbf{v}_1 = (0,0,1,0) = \mathbf{u}_3 - \mathbf{u}_1,$$

cf. the above.

Summing up we have

$$\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3 \quad \text{and} \quad \mathbf{v}_3 - \mathbf{v}_1 = \mathbf{u}_3 - \mathbf{u}_1,$$

thus

$$\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 \in U \cap V \quad \text{and} \quad \mathbf{u}_3 - \mathbf{u}_1 = \mathbf{v}_3 - \mathbf{v}_1 \in U \cap V.$$

Hence the dimension is at least 2. On the other hand, it cannot be larger than 2, because this would imply that $\dim U \cap V = 3$, thus e.g. $\mathbf{v}_1$ would belong to U. Since this is not the case, the dimension is at most 2.

Summing up we have found that

$$\dim(U \cap V) = 2.$$

Example 2.22 *Given in $\mathbb{R}^5$ the vectors*

$$\mathbf{a}_1 = (1,-1,1,1,2), \quad \mathbf{a}_2 = (0,1,0,-1,0), \quad \mathbf{a}_3 = (3,0,3,0,6),$$

$$\mathbf{a}_4 = (0,0,-1,1,1) \quad \textit{and} \quad \mathbf{a}_5 = (1,1,0,0,3).$$

1. *Define $U = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$. Find $\dim U$.*
2. *Find a basis of U among the five given vectors, and find the coordinates of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ and $\mathbf{a}_5$ with respect to this basis.*

1. We get by reduction,

$$\{\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5\} = \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 6 & 1 & 3 \end{pmatrix} \begin{array}{l} \sim \\ R_2 := R_1 + R_2 \\ R_3 := R_1 - R_3 \\ R_4 := R_4 - R_2 \\ R_5 := R_5 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{l} \sim \\ R_4 := (R_3 + R_4)/2 \\ R_5 := R_3 - R_5 \end{array} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

which has the rank 4, so $\dim U = 4$.

2. It follows by inspection that

$$\mathbf{a}_3 = 3\mathbf{a}_1 + 3\mathbf{a}_2,$$

hence a basis is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}$.

The coordinates are

$$\begin{array}{lcccl} \mathbf{a}_1 & = & 1 \cdot \mathbf{a}_1 & \sim & (1,0,0,0,0), \\ \mathbf{a}_2 & = & 1 \cdot \mathbf{a}_1 & \sim & (0,1,0,0,0), \\ \mathbf{a}_3 & = & 3\mathbf{a}_1 + 3\mathbf{a}_2 & \sim & (3,3,0,0,0), \\ \mathbf{a}_4 & = & 1 \cdot \mathbf{a}_4 & \sim & (0,0,0,1,0), \\ \mathbf{a}_5 & = & 1 \cdot \mathbf{a}_5 & \sim & (0,0,0,0,1). \end{array}$$

Example 2.23 *Given in* $\mathbb{R}^3$ *the vectors*

$$\mathbf{a}_1 = (1,1,1), \quad \mathbf{a}_2 = (0,1,1), \quad \mathbf{a}_3 = (0,0,1),$$

as well as the vectors

$$\mathbf{b}_1 = (1,0,1), \quad \mathbf{b}_2 = (1,2,1), \quad \mathbf{b}_3 = (1,2,2).$$

1. *Prove that* $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a})$ *and* $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ *both form a basis of* $\mathbb{R}^3$.
2. *Find the matrix of the change of basis* $\mathbf{M}_{\mathbf{a}\,\mathbf{b}}$, *going from* $\mathbf{b}$ *coordinates to* $\mathbf{a}$ *coordinates.*

1. It follows from

$$|\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0,$$

that $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ are linearly independent, hence they form a basis of $\mathbb{R}^3$.

From

$$|\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} = \\ S_2 := S_2 - S_1 \\ S_3 := S_3 - S_2 \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0,$$

follows that the same is true for $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$.

2. First compute

$$\begin{aligned} \mathbf{b}_1 &= (1,0,1) = \mathbf{a}_1 - (0,1,0) = \mathbf{a}_1 - \mathbf{a}_2 + (0,0,1) = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3, \\ \mathbf{b}_2 &= (1,2,1) = \mathbf{a}_1 + (0,1,0) = \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3, \\ \mathbf{b}_3 &= (1,2,2) = \mathbf{a}_1 + (0,1,1) = \mathbf{a}_1 + \mathbf{a}_2. \end{aligned}$$

Using the columns as the coordinates of the $\mathbf{b}_i$ with respect to the $\mathbf{a}_j$ we get

$$\mathbf{M}_{\mathbf{a}\,\mathbf{b}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Example 2.24 *Let U and W be subspaces of a vector space. Prove that the following are equivalent:*

1. $\forall \mathbf{u}, \mathbf{u}' \in U, \forall \mathbf{w}, \mathbf{w}' \in W : \mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}' \Rightarrow \mathbf{u} = \mathbf{u}' \wedge \mathbf{w} = \mathbf{w}'$.
2. $\forall \mathbf{u} \in U, \forall \mathbf{w} \in W : \mathbf{u} + \mathbf{w} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{w} = \mathbf{0}$.
3. $U \cap W = \{\mathbf{0}\}$.

If U and W have one (and hence all) of the properties 1., 2. and 3., the vector space $X = U + W$ is called the direct sum *of U and V (cf.* EXAMPLE 2.13*) and we write*

$$X = U \oplus W.$$

Remark 2.3 Here the symbol "$\forall$" is a shorthand for "for all". $\Diamond$

1. $\Rightarrow$ **2**.. Assume 1. and that $\mathbf{u} + \mathbf{w} = \mathbf{0}$ for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Since $\mathbf{0} \in U \cap W$, if follows by 1. that

$$\mathbf{u} + \mathbf{w} = \mathbf{0} + \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{0} \wedge \mathbf{w} = \mathbf{0},$$

and 2. follows.

2. $\Rightarrow$ **3**.. Assume 2., and assume that if $\mathbf{v} \in U \cap W$, then also $-\mathbf{v} \in U \cap W$, thus $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, where we consider $\mathbf{v} \in U$ as an element of U and $-\mathbf{v} \in W$ as an element of W. Then by 2. we get $\mathbf{v} = -\mathbf{v} = \mathbf{0}$, and we have proved that $\mathbf{0}$ is the only element of $U \cap W$, hence

$$U \cap W = \{\mathbf{0}\}.$$

3. $\Rightarrow$ **1**.. Assume that $U \cap W = \{\mathbf{0}\}$. If $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$, then $\mathbf{u} - \mathbf{u}' \in U$ and $\mathbf{w}' - \mathbf{w} \in W$, hence

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}\}.$$

It follows that $\mathbf{u} - \mathbf{u}' = \mathbf{0}$ and $\mathbf{w}' - \mathbf{w} = \mathbf{0}$, and we have proved that $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

Thus we have proved that the three conditions are equivalent.

Example 2.25 *Let U be a subspace of a vector space V. If for another subspace W of V we have that $U \oplus W = V$, we call W a* complementary *subspace of U.*

1. *Prove that every subspace of a (finite dimensional) vector space V has a complementary subspace.*
2. *Prove that if V is finite dimensional and $\{\mathbf{0}\} \neq U \neq V$, then U has several different complementary subspaces.*

Remark 2.4 This example assumes Example 2.24. $\Diamond$

1. If $U = V$, then $W = \{\mathbf{0}\}$, and if $U = \{\mathbf{0}\}$, then $W = V$.

   Assume that $\{\mathbf{0}\} \neq U \neq V$. Then choose a basis $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ of U. Continue by supplying it to a basis

   $$(\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_n)$$

   of V. Then $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ is a basis of some subspace W, which clearly satisfies $U \cap W = \{\mathbf{0}\}$, and $U + W = V$, hence

   $$V = U \oplus W.$$

2. Now let $\{\mathbf{0}\} \neq U \neq V$ and construct the basis

   $$(\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_n)$$

   as above. Then $k > 0$ and $n > 0$, and e.g.

   $$W = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \quad W' = \operatorname{span}\{\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_1 + \mathbf{b}_n\}$$

   are different complementary subspaces of U.

3 Linear maps

Example 3.1 *Find the matrix with respect to the ordinary basis of* $\mathbb{R}^3$ *for the linear map* f *of* $\mathbb{R}^3$ *into* $\mathbb{R}^3$*, where* f *is mapping the vectors* $(2,1,0)$*,* $(0,0,2)$ *and* $(1,1,0)$ *into* $(1,4,1)$*,* $(4,2,2)$ *and* $(1,2,1)$*, respectively.*
Find the range of the subspace which is spanned by the vectors $(1,2,3)$ *and* $(-1,2,0)$*.*

The formulation above invites to the following,

$$\begin{array}{llll} \mathbf{a}_1=(2,1,0), & \mathbf{a}_2=(0,0,2) & \text{and} & \mathbf{a}_3=(1,1,0),\\ \mathbf{b}_1=(1,0,0), & \mathbf{b}_2=(0,1,0) & \text{and} & \mathbf{b}_3=(0,0,1),\\ \mathbf{c}_1=(1,4,1), & \mathbf{c_2}=(4.2.2) & \text{and} & \mathbf{c}_3=(1,2,1),\\ \mathbf{d}_1=(1,0,0), & \mathbf{d}_2=(0,1,0) & \text{and} & \mathbf{d}_3=(0,0,1), \end{array}$$

where

$$\mathbf{b}_1=\mathbf{a}_1-\mathbf{a}_3,\quad \mathbf{b}_2=-\mathbf{a}_1+2\mathbf{a}_3,\quad \mathbf{b}_3=\frac{1}{2}\,\mathbf{a}_2,$$

hence

$$\mathbf{M}_{\mathbf{a}\,\mathbf{b}} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \\ -1 & 2 & 0 \end{pmatrix}$$

and

$$\mathbf{F}_{\mathbf{d}\,\mathbf{b}} = \begin{pmatrix} 1 & 4 & 1 \\ 4 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check the result.
It follows by the linearity from

$$f(1,2,3) = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+6 \\ 2+3 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 5 \end{pmatrix}$$

and

$$f(-1,2,0) = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

that the range is spanned by the vectors $(8,5,5)$ and $(2,-1,2)$, thus

$$\begin{aligned} f(U) &= \{x(8,5,5) + y(2,-1,2) \mid x,\, y \in \mathbb{L}\} \\ &= \{(8x+2y, 5x-y, 5x+2y) \mid x,\, y \in \mathbb{L}\}. \end{aligned}$$

Example 3.2 *Given a map* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ *by*

$$f(\mathbf{X}) = \mathbf{AX} - \mathbf{XA}, \qquad \textit{where } \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

1. *Prove that f is linear.*

2. *Find the kernel of f.*

1. It follows from

$$\begin{aligned} f(\mathbf{X}+\lambda\mathbf{Y}) &= \mathbf{A}(\mathbf{X}+\lambda\mathbf{Y}) - (\mathbf{X}+\lambda\mathbf{Y})\mathbf{A} \\ &= \{\mathbf{AX}-\mathbf{XA}\} + \lambda\{\mathbf{AY}-\mathbf{YA}\} = f(\mathbf{X}) + \lambda f(\mathbf{Y}), \end{aligned}$$

that f is linear.

2. Assume that $\mathbf{X} \in \ker(f)$. Then

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ 4_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} x_{11}+2x_{21} & x_{12}+x_{22} \\ -x_{21} & -x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & 2x_{11}-x_{12} \\ x_{21} & 2x_{21}-x_{22} \end{pmatrix} \\ &= \begin{pmatrix} 2x_{21} & -2x_{11}+2x_{12}+x_{22} \\ -2x_{21} & -2x_{21} \end{pmatrix}, \end{aligned}$$

hence $x_{21} = 0$ and $-2x_{11} + 2x_{12} + x_{22} = 0$. Choosing $x_{11} = s$ and $x_{12} = t$ as parameters we get

$$\ker(f) = \left\{ \begin{pmatrix} s & t \\ 0 & 2(s-t) \end{pmatrix} \;\middle|\; s, t \in \mathbb{L} \right\}, \qquad \dim \ker(f) = 2.$$

Example 3.3 *Let U and W be subspaces of a vector space and define $V = U \oplus W$ (cf.* EXAMPLE 2.24*). Assume that the vector $\mathbf{v} \in V$ is given by*

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \qquad \textit{where } \mathbf{u} \in U \textit{ and } \mathbf{w} \in W.$$

Prove that the map $f : \mathbf{v} \to \mathbf{u}$ is linear and that the composite map $f \circ f = f^2 = f$.
Prove that $U = f(V)$ and $W = \ker f$.
The map f is called the projection *onto U in the direction W.*

Consider $\mathbf{v}_1, \mathbf{v}_2 \in V$ of the unique splitting

$$\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{w}_1, \quad \mathbf{v}_2 = \mathbf{u}_2 + \mathbf{w}_2, \qquad \mathbf{u}_1, \mathbf{u}_2 \in U, \quad \mathbf{w}_1, \mathbf{w}_2 \in W.$$

If $\lambda \in \mathbb{L}$, then

$$\begin{aligned} f(\mathbf{v}_1 + \lambda \mathbf{v}_2) &= f(\mathbf{u}_1 + \lambda \mathbf{u}_2 + (\mathbf{w}_1 + \lambda \mathbf{w}_2)) \\ &= \mathbf{u}_1 + \lambda \mathbf{u}_2 = f(\mathbf{v}_1) + \lambda f(\mathbf{v}_2), \end{aligned}$$

proving that the map is linear.

Then

$$f(\mathbf{v}) = f(\mathbf{u} + \mathbf{v}) = \mathbf{u}, \qquad \text{thus } f \circ f(\mathbf{v}) = f(\mathbf{u}) = \mathbf{u}.$$

In particular, $f(U) = U$, hence $U \subseteq f(V) \subseteq U$, and we conclude that $f(V) = U$.
Finally, if $\mathbf{w} \in W$, then $f(\mathbf{w}) = \mathbf{0}$, hence $W \subseteq \ker(f)$.
Conversely, if $\mathbf{u} + \mathbf{v} \in W$, er $f(\mathbf{u}) = \mathbf{u} = \mathbf{0}$, then $\ker(f) = W$.

Example 3.4 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map which corresponds to the following matrix in the ordinary basis of $\mathbb{R}^3$,*

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

1. *Find a basis of the range $f(\mathbb{R}^3)$.*
2. *Prove that the vector $\mathbf{b} = (6, 2, -2)$ belongs to both the kernel of f and the range of f.*

1. Since $f(\mathbf{e}_1) = (1, 0, -1)$, $f(\mathbf{e}_2) = (1, 1, 1)$ and $f(\mathbf{e}_3) = (4, 1, -2)$, the range $f(\mathbb{R}^3)$ is spanned by these three vectors. Since

   $$f(\mathbf{e}_3) - f(\mathbf{e}_2) = 3f(\mathbf{e}_1), \qquad \text{dvs. } f(3\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3) = \mathbf{0},$$

   the range is only of dimension 2. A basis is e.g.

   $$\{f(\mathbf{e}_1), f(\mathbf{e}_2)\} = \{(1, 0, -1), (1, 1, 1)\}.$$

2. Since $\mathbf{b} = (6, 2, -2) = 2(3\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$, we get $f(\mathbf{b}) = \mathbf{0}$, so $\mathbf{b} \in \ker(f)$.
   It then follows by inspection that

   $$f(1, 1, 1) = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} = \mathbf{b} \in f(\mathbb{R}^3),$$

   so $\mathbf{b}$ does also belong to the range.

Example 3.5 *Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map, which is given with respect to the ordinary bases of $\mathbb{R}^5$ and $\mathbb{R}^3$ by the matrix*

$$\mathbf{F} = \begin{pmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 2 & 4 & 1 \\ 3 & 4 & 5 & 1 & 1 \end{pmatrix}.$$

1. *Find $\{\mathbf{x} \in \mathbb{R}^5 \mid f(\mathbf{x}) = (4, 3, 6)\}$, and $\ker f$.*
2. *Find a basis of range $f(\mathbb{R}^5)$.*

1. The equation $f(\mathbf{x}) = (4, 3, 6)$ corresponds to the system

   $$\begin{pmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 2 & 4 & 1 \\ 3 & 4 & 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}.$$

We reduce the total matrix,

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 1 & 4 \\ 0 & 1 & 2 & 4 & 1 & 3 \\ 3 & 4 & 5 & 1 & 1 & 6 \end{array}\right) \underset{R_3 := R_3 - 3R_1 + 2R_2}{\sim} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 1 & 4 \\ 0 & 1 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\underset{R_1 := R_1 - 2R_2}{\sim} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -5 & -1 & -2 \\ 0 & 1 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The rank is 2, so by choosing the parameters $c_3 = s$, $x_4 = t$, $x_5 = u$, we obtain the solution

$$\{(-2 + s + 5t + u, 3 - 2s - 4t - u, s, t, u) s,\ t,\ u \in \mathbb{R}\},$$

and the kernel is

$$\begin{aligned} \ker f &= \{(s + 5t + u, -2s - 4t - u, s, t, u) \mid s,\ t,\ u \in \mathbb{R}\} \\ &= \{s(1, -2, 1, 0, 0) + t(5, -4, 0, 1, 0) + u(1, -1, 0, 0, 1) \mid s,\ t,\ u \in \mathbb{R}\}. \end{aligned}$$

The kernel is therefore spanned by the vectors

$$\{(1, -2, 1, 0, 0).(5, -4, 0, 1, 0), (1, -1, 0, 0, 1)\}.$$

2. It follows from the reduction of the total matrix that the range – hence also the matrix of coefficients – is of dimension 2. Since

$$f(\mathbb{R}^5) = \operatorname{span}\{(1,0,3),(2,1,4),(3,2,5),(3,4,1),(1,1,1)\},$$

we obtain a basis by choosing two linearly independent vectors from this set, e.g.

$$f(\mathbb{R}^5) = \operatorname{span}\{(1,0,3),(1,1,1)\} = \operatorname{span}\{(1,0,3),(0,1,-2)\},$$

etc.

Example 3.6 *A linear map $f : \mathbb{C}^4 \to \mathbb{C}^4$ is in the usual coordinates given by the matrix*

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & -i & 0 \\ 1 & -i & i & 1 \\ -1 & 0 & -1 & 0 \\ i & -1 & -1 & -i \end{pmatrix}.$$

Find the kernel and the range of this map.
Find the intersection of the kernel and the range.
Find the set $\{\mathbf{x} \in \mathbb{C}^4 \mid f(\mathbf{x}) = (1,-i,-i,-1+2i)\}$.

We get by reduction,

$$\left(\begin{array}{rrrr|r} 1 & 0 & -i & 0 & 0 \\ 1 & -i & i & 1 & 0 \\ -i & 0 & -1 & 0 & 0 \\ i & -1 & -1 & -i & 0 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 + R_4 \\ R_4 := R_4 - iR_1 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & -i & 0 & 0 \\ 0 & i & -2i & -1 & 0 \\ 0 & -1 & -2 & -i & 0 \\ 0 & -1 & -2 & -i & 0 \end{array}\right)$$

$$\begin{array}{l} \sim \\ R_2 := iR_2 \\ R_4 := R_3 - R_4 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & -i & 0 & 0 \\ 0 & -1 & 2 & -i & 0 \\ 0 & -1 & -2 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_2 - R_3 \\ R_2 := -R_2 \end{array}$$

$$\left(\begin{array}{rrrr|r} 1 & 0 & -i & 0 & 0 \\ 0 & 1 & -2 & i & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - iR_3/4 \\ R_2 := R_2 + R_3/2 \\ R_4 := R_4/4 \end{array} \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Then the equations of the kernel are $x_1 = 0$, $x_2 + ix_4 = 0$, $x_3 = 0$, thus

$$\ker(f) = \{s(0,-i,0,1) \mid s \in \mathbb{C}\}$$

The kernel has dimension 1, so the range is of dimension 3. Since the second and the fourth column of the matrix are linearly dependent, the range is

$$f(\mathbb{C}^4) = \operatorname{span}\{(1,1,-i,i),(-i,i,-1,-1),(0,1,0,i)\},$$

because we can exclude the second column.

We have only two possibilities for $f(\mathbb{C}^4)\cap\ker(f)$. Either this intersection is $\ker(f)$, or it is $\{\mathbf{0}\}$. If the intersection is $\ker(f)$, then the four vectors $(1,1,-i,i)$, $(-i,i,-1,-1)$, $(0,1,0,-i)$ [from $f(\mathbb{C}^4)$] and $(0,-i,0,1)$ [from $\ker(f)$] must be linearly dependent. We get by reduction,

$$\left(\begin{array}{rrr|r} 1 & -i & 0 & 0 \\ 1 & i & 1 & -i \\ -i & -1 & 0 & 0 \\ i & -1 & -i & 1 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 + R_4 \\ R_4 := R_4 - iR_1 \end{array} \left(\begin{array}{rrr|r} 1 & -i & 0 & 0 \\ 0 & -2i & -1 & i \\ 0 & -2 & -i & 1 \\ 0 & -2 & -i & 0 \end{array}\right)$$

$$\begin{array}{c} \sim \\ R_4 := R_3 - R_4 \end{array} \left(\begin{array}{rrr|r} 1 & -i & 0 & 0 \\ 0 & -2i & -1 & i \\ 0 & -2 & -i & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_2 := -R_3/2 \\ R_3 := R_2 - iR_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -i & 0 & 0 \\ 0 & 1 & \frac{i}{2} & -\frac{1}{2} \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_2 + iR_3/4 \\ R_3 := -R_3/2 \end{array} \left(\begin{array}{rrr|r} 1 & -i & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} \sim \\ R_1 := R_1 + iR_2 \end{array} \left(\begin{array}{rrr|r} 1 & 0 & 0 & -\frac{i}{2} \\ 0 & 1 & 0 & -frac12 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The rank is 3, so the vectors are linearly dependent, and

$$f(\mathbb{C}^4)\cap\ker f = \ker f.$$

It follows further from the reduction above that

$$(0,-i,0,1) = -\frac{i}{2}\,(1,1,-i,i) - \frac{1}{2}\,(-i,i,-1,-1).$$

Finally, we shall describe the set

$$U = \{\mathbf{x}\in\mathbb{C}^4 \mid f(\mathbf{x}) = (1,-i,-i,-1+2i)\}.$$

The corresponding total matrix is reduced to

$$\left(\begin{array}{rrrr|c} 1 & 0 & -i & 0 & 1 \\ 1 & -i & i & 1 & -i \\ -i & 0 & -1 & 0 & -i \\ i & -1 & -1 & -i & -1+2i \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 + R_4 \\ R_4 := R_4 - iR_1 \end{array} \left(\begin{array}{rrrr|c} 1 & 0 & -i & 0 & 1 \\ 0 & i & -2i & -1 & 1+i \\ 0 & -1 & -2 & -i & -1+i \\ 0 & -1 & -2 & -i & -1+i \end{array}\right)$$

$$\begin{array}{l} \sim \\ R_2 := iR_2 \\ R_4 := R_3 - R_4 \end{array} \left(\begin{array}{rrrr|c} 1 & 0 & -i & 0 & 1 \\ 0 & -1 & 2 & -i & -1+i \\ 0 & -1 & -2 & -i & -1+i \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_2 - R_3 \\ R_2 := -R_2 \end{array}$$

$$\left(\begin{array}{rrrr|c} 1 & 0 & -i & 0 & 1 \\ 0 & 1 & -2 & i & 1-i \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & i & 1-i \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

hence

$$U = \{(1,1-i,0,0) + s(0,-i,0,1) \mid s\in\mathbb{R}\}.$$

CHECK. The computations here have been so complicated that one ought to check the result:

$$\begin{pmatrix} 1 & 0 & -i & 0 \\ 1 & -i & i & 1 \\ -i & 0 & -1 & 0 \\ i & -1 & -1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 1-i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i-1 \\ -i \\ i-1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -i \\ -i \\ -1+2i \end{pmatrix}.$$

We see that the result is correct.

Example 3.7 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

Denote by $f : \mathbb{R}^3 \to \mathbb{R}^4$ *the linear map which in the usual bases of* $\mathbb{R}^3$ *and* $\mathbb{R}^4$ *is given by the matrix* $\mathbf{A}$.

1. *Prove that* $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,0)$ *and* $\mathbf{v}_3 = (1,-2,1)$ *forms a basis of* $\mathbb{R}^3$.
   Find the coordinates of $f(\mathbf{v}_1)$, $f(\mathbf{v}_2)$ *and* $f(\mathbf{v}_3)$ *with respect to the usual basis of* $\mathbb{R}^4$.
2. *Prove that* $\mathbf{D}$ *is regular and compute* $\mathbf{D}^{-1}$.
   Prove that $\mathbf{d}_1 = (1,-1,1,1)$, $\mathbf{d}_2 = (1,0,2,-1)$, $\mathbf{d}_3 = (0,0,1,0)$ *and* $\mathbf{d}_4 = (0,0,0,1)$ *form a basis of* $\mathbb{R}^4$. *Find the coordinates of* $(1,1,3,-3)$ *with respect to the basis* $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$.
3. *Find the coordinates of* $f(\mathbf{v}_1)$, $f(\mathbf{v}_2)$ *and* $f(\mathbf{v}_3)$ *with respect to the basis* $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$.
   Find the matrix of f *with respect to the basis* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *i* $\mathbb{R}^3$ *and the basis* $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$ *i* $\mathbb{R}^4$.

1. It follows from

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

that the three vectors are linearly independent. Since the dimension of $\mathbb{R}^3$ is 3, we conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.
Then we find

$$f(\mathbf{v}_1) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \qquad f(\mathbf{v}_2) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix},$$

and

$$f(\mathbf{v}_3) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2+1 \\ -1+0+1 \\ 1-4+3 \\ 1+2-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. We conclude from

$$\det \mathbf{D} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} \underset{R_2}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

that $\mathbf{D}$ is regular. We can now find the inverse in various ways of which we demonstrate two of them:

(a) By the well-known reduction,

$$(\mathbf{D}\,|\,\mathbf{I}) \quad = \quad \left(\begin{array}{rrrr|rrrr} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_3 := R_3 - 2R_1 - R_2 \\ R_4 := R_4 + R_1 + 2R_2 \end{array}$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := -R_2 \\ R_2 := R_1 + R_2 \end{array}$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right),$$

from which we conclude that

$$\mathbf{D}^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

(b) ALTERNATIVELY we shall try to find $\mathbf{K_D}$ in order to compare the two methods. We compute all the subdeterminants of the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

where $\det \mathbf{D} = 1$, cf. the above. We get

$$A_{11} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0, \qquad A_{12} = -\begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1,$$

$$A_{13} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -2, \qquad A_{14} = -\begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1,$$

$$A_{21} = -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1, \quad A_{22} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1,$$

$$A_{23} = -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1,$$

$$A_{24} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2,$$

$$A_{31} = 0, \qquad A_{32} = -\begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0,$$

$$A_{33} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1, \qquad A_{34} = -\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 0,$$

$$A_{41} = -0, \qquad A_{42} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0,$$

$$A_{43} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 0, \qquad A_{44} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1.$$

We conclude that

$$\mathbf{K_D} = \begin{pmatrix} 0 & 1 & -2 & 1 \\ -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}^{-1} = \frac{\mathbf{K_D}^T}{\det \mathbf{D}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

We see by comparison that we get the same result by the two methods. In order to be absolutely certain, we also CHECK the result:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows from

$$|\mathbf{d}_1 \;\; \mathbf{d}_2 \;\; \mathbf{d}_3 \;\; \mathbf{d}_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1,$$

that $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$ are linearly independent, so they form a basis of $\mathbb{R}^4$.

Then we reduce the total matrix,

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} \sim \\ R_1 := -R_2 \\ R_2 := R_1 + R_2 \\ R_3 := R_3 + R_2 \\ R_4 := R_4 + R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 2 = 1 & 0 & & 4 \\ 0 & -1 & 0 & 1 & -2 \end{array}\right)$$

$$\begin{array}{l} \sim \\ R_3 := R_3 - 2R_2 \\ R_4 := R_4 + R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

so the coordinates are $(-1, 2, 0, 0)$.
A CHECK gives

$$-1 \cdot (1, -1, 1, 1) + 2(1, 0, 2, -1) = (1, 1, 3, -3),$$

which can also be written

$$(1, 1, 3, -3) = -\mathbf{d}_1 + 2\mathbf{d}_2.$$

3. We have found earlier that

$$f(\mathbf{v}_1) = (1, -1, 1, 1), \qquad f(\mathbf{v}_2) = (1, 0, 2, -1), \qquad f(\mathbf{v}_3) = (0, 0, 0, 0),$$

which interpreted to the given vectors very conveniently also can be written

$$f(\mathbf{v}_1) = \mathbf{d}_1, \qquad f(\mathbf{v}_2) = \mathbf{d}_2, \qquad f(\mathbf{v}_3) = \mathbf{0}.$$

The matrix is represented by the columns $f(\mathbf{v}_1)$, $f(\mathbf{v}_2)$, $f(\mathbf{v}_3)$, i.e.

$$\mathbf{F}_{\mathbf{d}\,\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 3.8 *A linear map $f : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by*

$$f(\mathbf{v}_1) = \mathbf{v}_1 + 2\mathbf{v}_2, \qquad f(\mathbf{v}_2) = i\mathbf{v}_1 + \mathbf{v}_2,$$

given the basis $(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{C}^2$,

1. *Find the matrix equation of f with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2)$.*
2. *Prove that $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$ form a basis of $\mathbb{C}^2$.*
3. *Find the matrix equation of f with respect to the basis $(\mathbf{w}_1, \mathbf{w}_2)$.*

1. The matrix equation is ${}_{\mathbf{v}}\mathbf{y} = \mathbf{F}_{\mathbf{v}\,\mathbf{v}}({}_{\mathbf{v}}\mathbf{x})$, where

$$\mathbf{F}_{\mathbf{v}\,\mathbf{v}} = \begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix}.$$

2. If $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$, then

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{w}_1 + \mathbf{w}_2) \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{2}(\mathbf{w}_1 - \mathbf{w}_2).$$

   The elements of the basis $\mathbf{v}_1$, $\mathbf{v}_2$ can uniquely be expressed by $\mathbf{w}_1$, $\mathbf{w}_2$, hence $(\mathbf{w}_1, \mathbf{w}_2)$ is also basis of $\mathbb{C}^2$.

3. It suffices to indicate the matrix of the map,

$$\begin{aligned} \mathbf{F}_{\mathbf{w}\,\mathbf{w}} &= \mathbf{M}_{\mathbf{w}\,\mathbf{v}}\mathbf{F}_{\mathbf{v}\,\mathbf{v}}\mathbf{M}_{\mathbf{v}\,\mathbf{w}} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 3 & 1+i \\ -1 & -1+i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4+i & 4+i \\ -2+i & -i \end{pmatrix} \\ &= \begin{pmatrix} 2+\frac{i}{2} & 2+\frac{i}{2} \\ -1+\frac{i}{2} & -\frac{i}{2} \end{pmatrix}. \end{aligned}$$

Example 3.9 *Given in* $\mathbb{R}^4$ *the vectors*

$$\mathbf{b}_1 = (1,2,2,0), \quad \mathbf{b}_2 = (0,1,1,1), \quad \mathbf{b}_3 = (0,0,1,1), \quad \mathbf{b}_4 = (1,1,1,1).$$

1. *Prove that* $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ *and* $\mathbf{b}_4$ *form a basis of* $\mathbb{R}^4$.
2. *Let a linear map* $f : \mathbb{R}^4 \to \mathbb{R}^3$ *be given, such that*

$$f(\mathbf{b}_1) = (1,1,2), \quad f(\mathbf{b}_2) = (3,-1,1), \quad f(\mathbf{b}_3) = (4,0,3), \quad f(\mathbf{b}_4) = (-5,3,0).$$

   Find the matrix of f*, when we use the usual basis in* $\mathbb{R}^3$ *and the basis* $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ *in* $\mathbb{R}^4$. *Find the dimension of the range.*
3. *Given the vectors* $\mathbf{v}_1 = \mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3$ *and* $\mathbf{v}_2 = -\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_4$. *Prove that* $\mathbf{v}_1$, $\mathbf{v}_2$ *span the kernel* $\ker f$.
4. *Find all vectors* $\mathbf{x} \in \mathbb{R}^4$*, which satisfy the equation* $f(\mathbf{x}) = f(\mathbf{b}_1)$*, expressed by the vectors* $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$.

1. It follows from

$$|\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4| = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{S_1 := S_1 - S_4}{=} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix}$$
$$\underset{R_1}{=} -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \underset{\substack{R_2 := R_2 - R_1 \\ R_3 := R_3 + R_1}}{=} -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$
$$\underset{S_1}{=} -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 2 \neq 0,$$

   that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ are linearly independent in $\mathbb{R}^4$, hence they form a basis of $\mathbb{R}^4$.

2. The matrix corresponding to the map is

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}.$$

3. A simple check gives

$$f(\mathbf{v}_1) = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \mathbf{0}$$

   and

$$f(\mathbf{v}_2) = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0},$$

hence $\mathbf{v}_1$, $\mathbf{v}_1 \in \ker f$. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independens, thus $\dim \ker f \geq 2$.
On the other hand, $\operatorname{rg} \mathbf{F} \geq 2$, hence $\dim \ker f \leq 2$.
Summing up we see that $\dim \ker f = 2$, so $\mathbf{v}_1$, $\mathbf{v}_2$ span $\ker f$.

4. If $f(\mathbf{x}) = f(\mathbf{b}_1)$, then it follows by the linearity that

$$\mathbf{0} = f(\mathbf{x}) - f(\mathbf{b}_1) = f(\mathbf{x} - \mathbf{b}_1),$$

thus $\mathbf{x} - \mathbf{b}_1 \in \ker f = \{s\mathbf{v}_1 + t\mathbf{v}_2 \mid s,\, t \in \mathbb{R}\}$. This gives us the solutions

$$\begin{aligned} \mathbf{x} &= \mathbf{b}_1 + s\mathbf{v}_1 + t\mathbf{v}_2 \\ &= \mathbf{b}_1 + s(\mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3) + t(-\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_4) \\ &= (1+s-t)\mathbf{b}_1 + (s+2t)\mathbf{b}_2 - s\mathbf{b}_3 + t\mathbf{b}_4, \qquad s,\, t \in \mathbb{R}. \end{aligned}$$

Example 3.10 *Consider in a 2-dimensional vector space V over $\mathbb{R}$ a basis $(\mathbf{a}_1, \mathbf{a}_2)$ and a linear map f of V into V, which in the basis $(\mathbf{a}_1, \mathbf{a}_2)$ has the corresponding matrix*

$$\mathbf{F} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Find the matrix of f with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2)$, where $\mathbf{b}_1 = \mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{b}_2 = \mathbf{a}_1 - \mathbf{a}_2$.

Now,

$$\mathbf{F}_{\mathbf{b}\,\mathbf{b}} = (\mathbf{M}_{\mathbf{a}\,\mathbf{b}})^{-1}\,\mathbf{F}_{\mathbf{a}\,\mathbf{a}}\mathbf{M}_{\mathbf{a}\,\mathbf{b}},$$

where

$$\mathbf{M}_{\mathbf{a}\,\mathbf{b}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad (\mathbf{M}_{\mathbf{a}\,\mathbf{b}})^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

hence

$$\begin{aligned} \mathbf{F}_{\mathbf{b}\,\mathbf{b}} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+c & a-c \\ b+d & b-d \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} a+b+c+d & a+b-c-d \\ a-b+c-d & a-b-c+d \end{pmatrix}. \end{aligned}$$

Example 3.11 *Let $f : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be a linear map satisfying*

$$f(1+4x) = 1-2x \quad \textit{and} \quad f(-2-9x) = 2+4x.$$

1. *Find the matrix of f med with respect to the basis of monomials $(1, x)$.*
2. *Find the polynomial $f(1+3x)$.*

1. Since f is linear, we get by inspection,

   $$9f(1+4x) + 4f(-2-9x) = f(1) = 9\{1-2x\} + 4\{2+4x\} = 17-2x,$$

   hence

   $$4f(x) = f(1+4x) - f(1) = \{1-2x\} - \{17-2x\} = -16,$$

   and whence

   $$f(1) = 17-2x \quad \text{and} \quad f(x) = -4,$$

   so the matrix is

   $$\begin{pmatrix} 17 & -4 \\ -2 & 0 \end{pmatrix}.$$

2. Then by the linearity,

$$f(1+3x)=f(1)+3f(x)=\{17-2x\}-12=5-2x.$$

Example 3.12 *A linear map $f:\mathbb{R}^3\to\mathbb{R}^3$ is in the usual basis of $\mathbb{R}^3$ given by the matrix equation*

$$\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix} 1 & -3 & 1\\ -1 & -3 & 2\\ -1 & -3 & 2\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}.$$

1. *Prove that the vectors*

$$\mathbf{v}_1=(1,0,1),\quad \mathbf{v}_2=(0,1,2),\quad \mathbf{v}_3=(1,1,2)$$

form a basis of $\mathbb{R}^3$, and find the image vectors $f(\mathbf{v}_1)$, $f(\mathbf{v}_2)$, $f(\mathbf{v}_3)$.

2. *Find the kernel of f. Explain why the range $f(\mathbb{R}^3)$ is a 2-dimensional subspace of $\mathbb{R}^3$, and that the vectors*

$$\mathbf{w}_1=(2,1,1),\qquad \mathbf{w}_2=(-1,1,1)$$

form a basis of $f(\mathbb{R}^3)$.

3. *Find the matrix of f with respect to the basis $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$.*

4. *A linear map $g:f(\mathbb{R}^3)\to\mathbb{R}^3$ is given by*

$$g(\mathbf{w}_1)=\mathbf{v}_1,\qquad g(\mathbf{w}_2)=\mathbf{v}_2.$$

Find the matrix of the composite map $g\circ f:\mathbb{R}^3\to\mathbb{R}^3$ with respect to the basis $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$, and prove that

$$f\circ g\circ f=f.$$

1. It follows from

$$|\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3|=\begin{vmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 2 & 2\end{vmatrix}=\begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 2 & -2\end{vmatrix}=-1\neq 0,$$

that $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$ forms a basis of $\mathbb{R}^3$.

Then by a computation,

$$f(\mathbf{v}_1)=\begin{pmatrix} -1 & -3 & 1\\ -1 & -3 & 2\\ -1 & -3 & 2\end{pmatrix}\begin{pmatrix} 1\\ 0\\ 1\end{pmatrix}=\begin{pmatrix} 2\\ 1\\ 1\end{pmatrix},$$

$$f(\mathbf{v}_2)=\begin{pmatrix} 1 & -3 & 1\\ -1 & -3 & 2\\ -1 & -3 & 2\end{pmatrix}\begin{pmatrix} 0\\ 1\\ 2\end{pmatrix}=\begin{pmatrix} -1\\ 1\\ 1\end{pmatrix},$$

$$f(\mathbf{v}_3) = \begin{pmatrix} 1 & -3 & 1 \\ -1 & -3 & 2 \\ -1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0},$$

thus

$$f(\mathbf{v}_1) = (2,1,1), \quad f(\mathbf{v}_2) = (-1,1,1), \quad f(\mathbf{v}_3) = \mathbf{0}.$$

2. Obviously, $f(\mathbf{v}_1)$, $f(\mathbf{v}_1) \in f(\mathbb{R}^3)$, and $\mathbf{v}_3 \in \ker f$. Since $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ are linearly independent, we must have

$$\dim f(\mathbb{R}^3) = 2 \quad \text{and} \quad \dim \ker f = 1.$$

We get from $\mathbf{v}_3 \in \ker f$ that

$$\ker f = \{s\mathbf{v}_3 \mid s \in \mathbb{R}\} = \{s(1,1,2) \mid s \in \mathbb{R}\}.$$

Now, $\mathbf{w}_1 = (2,1,1) = f(\mathbf{v}_1)$ and $\mathbf{w}_2 = (-1,1,1) = f(\mathbf{v}_2)$, so it follows from the above that $(\mathbf{w}_1, \mathbf{w}_2)$ form a basis of $f(\mathbb{R}^3)$.

3. Then by reduction,

$$(\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \mid \mathbf{w}_1) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array}\right) \underset{R_3 := R_1 + 2R_2 - R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right),$$

from which we conclude that

$$\mathbf{w}_1 = -\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3.$$

Analogously,

$$(\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \mid \mathbf{w}_2) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array}\right) \underset{R_3 := R_1 + 2R_2 - R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

from which

$$\mathbf{w}_2 = -\mathbf{v}_1 + \mathbf{v}_2.$$

Since $f(\mathbf{v}_3) = \mathbf{0}$, the matrix of f with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is given by

$$\mathbf{F}_{\mathbf{v}\,\mathbf{v}} = (f(\mathbf{v}_1) \;\; f(\mathbf{v}_2) \;\; f(\mathbf{v}_3)) = (\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \mathbf{0}) = \begin{pmatrix} -1 & -1 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

4. Note that since $\dim f(\mathbb{R}^3) = 2$, the map g is uniquely determined. It follows that

$$\mathbf{v}_1 = g(\mathbf{w}_1) = g(f(\mathbf{v}_1)) = (g \circ f)(\mathbf{v}_1),$$

$$\mathbf{v}_2 = g(\mathbf{w}_2) = g(f(\mathbf{v}_2)) = (g \circ f)(\mathbf{v}_2),$$

hence the matrix of the composite map with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally,

$$(f \circ g \circ f)(\mathbf{v}_1) = f(\mathbf{v}_1) = \mathbf{w}_1,$$

$$(f \circ g \circ f)(\mathbf{v}_2) = f(\mathbf{v}_2) = \mathbf{w}_2.$$

The maps are linear, and $(\mathbf{w}_1, \mathbf{w}_2)$ is a basis of $f(\mathbb{R}^3)$, and

$$(f \circ g \circ f)(\mathbf{v}_3) = f(\mathbf{v}_3) = \mathbf{0}.$$

Hence we conclude that

$$f \circ g \circ g = f.$$

Example 3.13 *Let V denote a vector space of dimension 2, and let $(\mathbf{a}_1, \mathbf{a}_2)$ be a basis of V. Furthermore, let two linear maps be given, f and g, of V into V. It is assumed that*

$$g(\mathbf{a}_1) = 3\mathbf{a}_1 - \mathbf{a}_2, \quad g(\mathbf{a}_2) = \mathbf{a}_1, \quad f(\mathbf{a}_1) = \mathbf{a}_1 - \mathbf{a}_2, \quad f(3\mathbf{a}_1 - \mathbf{a}_2) = 2\mathbf{a}_1 - \mathbf{a}_2.$$

1. *Find $f(\mathbf{a}_2)$.*
2. *Find the matrices of f and g with respect to the basis $(\mathbf{a}_1, \mathbf{a}_2)$.*
3. *Check if $f \circ g = g \circ f$.*

1. Due to the linearity,

$$f(\mathbf{a}_1) = -f(3\mathbf{a}_1 - \mathbf{a}_2) + 3f(\mathbf{a}_1) = -\{2\mathbf{a}_1 - \mathbf{a}_2\} + 3\{\mathbf{a}_1 - \mathbf{a}_2\} = \mathbf{a}_1 - 2\mathbf{a}_2.$$

2. The matrix of f with respect to the basis $(\mathbf{a}_1, \mathbf{a}_2)$ is

$$\{f(\mathbf{a}_1) \;\; f(\mathbf{a}_2)\} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

The matrix of g with respect to the basis $(\mathbf{a}_1, \mathbf{a}_2)$ is

$$\{g(\mathbf{a}_1) \;\; g(\mathbf{a}_2)\} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. Since

$$f \circ g \sim \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix},$$

and

$$g \circ f \sim \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix},$$

the two matrices are identical, hence

$$f \circ g = g \circ f.$$

Alternatively we compute

$$(f \circ g)(\mathbf{a}_1) = f(3\mathbf{a}_1 - \mathbf{a}_2) = 3(\mathbf{a}_1 - \mathbf{a}_2) - (\mathbf{a}_1 - 2\mathbf{a}_2) = 2\mathbf{a}_1 - \mathbf{a}_2,$$

$$(g \circ f)(\mathbf{a}_1) = g(\mathbf{a}_1 - \mathbf{a}_2) = (3\mathbf{a}_1 - \mathbf{a}_2) = (3\mathbf{a}_1 - \mathbf{a}_2) - \mathbf{a}_1 = 2\mathbf{a}_1 - \mathbf{a}_2,$$

and

$$(f \circ g)(\mathbf{a}_2) = f(\mathbf{a}_1) = \mathbf{a}_1 - \mathbf{a}_2,$$

$$(g \circ f)(\mathbf{a}_2) = g(\mathbf{a}_1 - 2\mathbf{a}_2) = (3\mathbf{a}_1 - \mathbf{a}_2) - 2\mathbf{a}_1 = \mathbf{a}_1 - \mathbf{a}_2.$$

It follows that $f \circ g = g \circ f$ on all vectors of the basis, hence by the linearity everywhere.

Example 3.14 *Let* $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ *be a basis of* $\mathbb{R}^4$*, and let* $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ *be a basis of* $\mathbb{R}^3$*. Given a linear map* $f : \mathbb{R}^4 \to \mathbb{R}^3$ *by*

$$f(\mathbf{a}_1) = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3, \qquad f(\mathbf{a}_2) = \mathbf{c}_1 + \mathbf{c}_2,$$

$$f(\mathbf{a}_3) = f(\mathbf{a}_1) - f(\mathbf{a}_2), \qquad f(\mathbf{a}_4) = f(\mathbf{a}_1) + 2f(\mathbf{a}_3).$$

1. *Find the matrix of* f *with respect to the bases above of* $\mathbb{R}^4$ *and* $\mathbb{R}^3$.
2. *Find a basis of the range* $f(\mathbb{R}^4)$.
3. *Find a basis of the kernel* $\ker f$.

1. We first compute

$$\begin{aligned} f(\mathbf{a}_3) &= f(\mathbf{a}_1) - f(\mathbf{a}_2) = \mathbf{c}_3, \\ f(\mathbf{a}_4) &= f(\mathbf{a}_1) + 2f(\mathbf{a}_3) = \mathbf{c}_1 + \mathbf{c}_2 + 3\mathbf{c}_3. \end{aligned}$$

This gives us the matrix

$$\{f(\mathbf{a}_1)\ \ f(\mathbf{a}_2)\ \ f(\mathbf{a}_3)\ \ f(\mathbf{a}_4)\} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

2. Obviously, $\dim f(\mathbb{R}^3) = 2$, and

$$f(\mathbf{a}_2) = \mathbf{c}_1 + \mathbf{c}_2, \qquad f(\mathbf{a}_3) = \mathbf{c}_3$$

form a basis of the range $f(\mathbb{R}^4)$.

3. We get by reduction,

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 \end{array}\right)$$
$$\begin{array}{c} \sim \\ R_1 := R_1 - R_3 \\ R_2 := R_3 \\ R_3 := R_2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Choosing $x_3 = s$ and $x_4 = t$ as parameters it follows that

$$x_1 = -s - 3t, \qquad x_2 = s + 2t,$$

and all elements of kernel are given by

$$(-s-3t, s+2t, s, t) = s(-1,1,1,0) + t(-3,2,1), \qquad s,\, t \in \mathbb{R}.$$

It follows in particular that a basis of $\ker f$ is e.g.

$$(-1,1,1,0) \quad \text{and} \quad (-3,2,1).$$

Example 3.15 *Given a linear map* $f : \mathbb{R}^4 \to \mathbb{R}^3$ *with the following matrix (with respect to the usual basis of* $\mathbb{R}^4$ *and the usual basis of* $\mathbb{R}^3$*)*

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{pmatrix}.$$

1. *Explain why the vectors* $\mathbf{u}_1 = (-1, 0, 0, 1)$, $\mathbf{u}_2 = (-1, -2, 2, -1)$ *and* $\mathbf{u}_3 = (2, -2, 2, -4)$ *belong to the kernel of* f.
2. *Find the dimensions of the kernel* $\ker f$ *and the range* $f(\mathbb{R}^4)$.
3. *Find a basis of* $\ker f$.

1. It follows from

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}, \qquad \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix} = \mathbf{0},$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 2 \\ -4 \end{pmatrix} = \mathbf{0},$$

that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ all belong to the kernel n of f.

Then we note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent. On the other hand, since $\mathbf{u}_3 = \mathbf{u}_2 - 3\mathbf{u}_1$, we can so far only conclude that $\dim \ker f \geq 2$.

We reduce the matrix,

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 0 & 3 & 3 \\ -1 & 2 & 1 & -1 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2/3 \\ R_3 := R_1 + R_3 \end{matrix} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 3 & 0 \end{pmatrix}$$
$$\begin{matrix} \sim \\ R_1 := R_2 \\ R_2 := R_1 - R_2 \\ R_3 := R_3/3 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

which clearly is of rank 2, thus $\dim f(\mathbb{R}^4) = 2$.

It follows from the theorem of dimensions that

$$\dim \mathbb{R}^4 = 4 = \dim f(\mathbb{R}^4) + \dim \ker f = 2 + \dim \ker f,$$

and we conclude that $\dim \ker f = 2$.

2. We have proved above that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent in $\ker f$, and since $\dim \ker f = 2$, we conclude that $(\mathbf{u}_1, \mathbf{u}_2)$ is a basis of $\ker f$.

Example 3.16 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map which i the usual basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{R}^3$ is given by the matrix*

$$\mathbf{F} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Given the vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ by

$$\mathbf{b}_1 = (1, -1, 1), \quad \mathbf{b}_2 = (-1, 1, 0), \quad \mathbf{b}_3 = (1, 0, 0).$$

Prove that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is a basis of $\mathbb{R}^3$.
Find the matrix of f with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ i $\mathbb{R}^3$.

It follows from

$$|\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3| = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

that $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are linearly independent, hence they form a basis of $\mathbb{R}^3$.

Then we use that

$$\mathbf{F}_{\mathbf{b}\,\mathbf{b}} = (\mathbf{M}_{\mathbf{e}\,\mathbf{b}})^{-1} \mathbf{F}_{\mathbf{e}\,\mathbf{e}} \mathbf{M}_{\mathbf{e}\,\mathbf{b}},$$

where

$$\mathbf{M}_{\mathbf{e}\,\mathbf{b}} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We conclude from

$$\left(\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \sim \\ R_1 := R_3 \\ R_2 := R_2 + R_3 \\ R_3 := R_1 - R_3 \end{array} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\underset{R_3 := R_2 + R_3}{\sim} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right),$$

that

$$(\mathbf{M}_{\mathbf{e}\,\mathbf{a}})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

hence

$$\begin{aligned} \mathbf{F}_{\mathbf{b}\,\mathbf{b}} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 2 & -2 & 2 \end{pmatrix}. \end{aligned}$$

Example 3.17 *Given two bases in* $\mathbb{R}^2$*, namely* $(\mathbf{a}_1, \mathbf{a}_2)$ *and* $(\mathbf{b}_1, \mathbf{b}_2)$*, where* $\mathbf{b}_1 = 2\mathbf{a}_1 + 5\mathbf{a}_2$ *and* $\mathbf{b}_2 = \mathbf{a}_1 + 4\mathbf{a}_2$*.*
Let a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$f(\mathbf{a}_1) = \mathbf{b}_1 \quad \textit{and} \quad f(\mathbf{b}_2) = -\mathbf{1}_1 + 2\mathbf{a}_2.$$

1. *Find the matrix of* f *with respect to the basis* $(\mathbf{a}_1, \mathbf{a}_2)$*.*
2. *Find the matrix of* f *with respect to the basis* $(\mathbf{b}_1, \mathbf{b}_2)$*.*

1. It follows from $f(\mathbf{a}_1) = \mathbf{b}_1 = 2\mathbf{a}_1 + 5\mathbf{a}_2$ and

$$f(\mathbf{a}_2) = \frac{1}{3}\{f(\mathbf{b}_2) - f(\mathbf{a}_1)\} = \frac{1}{3}\{-\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_1 - 5\mathbf{a}_2\} = -\mathbf{a}_1 - \mathbf{a}_2,$$

that

$$\mathbf{F}_{\mathbf{a}\,\mathbf{a}} = \begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}.$$

2. Since

$$\mathbf{M}_{\mathbf{a}\,\mathbf{b}} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \quad \text{and} \quad (\mathbf{M}_{\mathbf{a}\,\mathbf{b}})^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix},$$

we get

$$\begin{aligned}\mathbf{F}_{\mathbf{b}\,\mathbf{b}} &= (\mathbf{M}_{\mathbf{1}\,\mathbf{b}})^{-1}\mathbf{F}_{\mathbf{a}\,\mathbf{a}}\mathbf{M}_{\mathbf{a}\,\mathbf{b}} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} -7 & -5 \\ 15 & 9 \end{pmatrix}.\end{aligned}$$

Example 3.18 *Given in* $\mathbb{R}^3$ *the vectors*

$$\mathbf{v}_1 = (1,0,1), \quad \mathbf{v}_2 = (1,1,0) \quad \textit{and} \quad \mathbf{v}_3 = (0,1,1).$$

1. *Prove that* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *form a basis of* $\mathbb{R}^3$.
2. *Given a linear map* $f : \mathbb{R}^3 \to \mathbb{R}^4$ *by*

$$f(\mathbf{v}_1) = (3,9,1,0), \quad f(\mathbf{v}_2) = (4,5,-1,1) \quad \textit{and} \quad f(\mathbf{v}_3) = (5,6,0,-1).$$

Find the matrix of f *with respect to the usual bases of* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

1. It follows from

$$|\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0,$$

that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, so they form a basis of $\mathbb{R}^3$.

2. We shall first express $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Since

$$\begin{aligned}(\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3 \mid \mathbf{I}) &= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c}\sim \\ R_3 := R_3 - R_1\end{array} \\ &\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l}\sim \\ R_1 := R_1 - R_2 \\ R_3 := (R_2 + R_3)/2\end{array} \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \quad \begin{array}{l}\sim \\ R_1 := R_1 + R_3 \\ R_2 := R_2 - R_3\end{array} \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right),\end{aligned}$$

we get

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Then the matrix expressed in the usual bases is given by

$$\begin{pmatrix} 3 & 4 & 5 \\ 9 & 5 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 5 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Example 3.19 *1. Explain why there is precisely one linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$, which fulfils*

$$f(1,1,1) = (4,0,0,6), \quad f(1,1,0) = (2,0,0,3), \quad f(1.0.-1) = (-1,-1,1,-1).$$

2. Find the matrix of f with respect to the usual bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

3. Find the dimension and a basis of the range.

4. Give a parametric description of the kernel.

1. The vectors $(1,1,1)$, $(1,1,0)$ and $(1,0,-1)$ form a basis of $\mathbb{R}^3$. In fact, it follows from

$$\alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,-1) = (0,0,0)$$

that $\alpha+\beta+\gamma = 0$, $\alpha+\beta = 0$ and $\alpha = \gamma$, hence $\gamma = \alpha = \beta = 0$, and the vectors are independent. Hence, there is precisely one linear map, which satisfies the given conditions.

2. We conclude from

$$\left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 + R_3 \\ R_2 := 2R_2 - R_1 - R_3 \\ R_3 := R_1 - R_2 \end{array} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right),$$

that

$$\mathbf{M}_{\mathbf{v}\,\mathbf{e}} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix},$$

hence

$$\mathbf{F}_{\mathbf{e}\mathbf{e}} = \mathbf{F}_{\mathbf{e}\mathbf{v}}\mathbf{M}_{\mathbf{v}\,\mathbf{e}} = \begin{pmatrix} 4 & 2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 6 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{pmatrix}.$$

3. Clearly, $\mathbf{F}_{\mathbf{e}\mathbf{v}}$, and thus $\mathbf{F}_{\mathbf{e}\mathbf{e}}$, has rank 2, so the range is of dimension 2.
A basis is composed of two of the three columns of $\mathbf{F}_{\mathbf{e}\mathbf{e}}$, e.g.

$$(1,1,-1,1) \quad \text{and} \quad (2,0,0,3).$$

4. It follows from

$$x_1(1,-1,1,2)+x_2(1,1,-1,1)+x_3(2,0,0,3)=(0,0,0,0)$$

that

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & 0, \\ -x_1 & + & x_2 & & & = & 0, \\ x_1 & - & x_2 & & & = & 0, \\ 2x_1 & + & x_2 & + & 3x_3 & = & 0, \end{array}$$

hence $x_2 = x_1$, and whence $x_3 = -x_1$. We conclude that

$$\ker f = \{s(1,1,-1) \mid s \in \mathbb{R}\}.$$

Example 3.20 *The linear map* $f : \mathbb{R}^3 \to \mathbb{R}^4$ *is with respect to the usual bases of* $\mathbb{R}^3$ *and* $\mathbb{R}^4$ *given by the matrix equation*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & -1 \\ -3 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1. *Find the dimension of the kernel* $\ker f$ *and the dimension of the range* $f(\mathbb{R}^3)$.
2. *Find a basis of the range* $f(\mathbb{R}^3)$.

1. We reduce the matrix of coefficients

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & -1 \\ -3 & -1 & 5 \end{pmatrix} \begin{matrix} \sim \\ R_1 := R_2 - R_3 - R_1 \\ R_4 := R_4 + R_3 - R_1 \end{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 2, so $\dim f(\mathbb{R}^3) = 2$, and it follows from

$$\dim \mathbb{R}^3 = 3 = \dim f(\mathbb{R}^3) + \ker f,$$

that $\dim \ker f = 1$.

2. A basis of the range is given by any two of the columns of the matrix, e.g.

$$(1, 2, 1, -3) \quad \text{and} \quad (1, 0, -1, 5).$$

Example 3.21 *Given in the vector space* $\mathbb{R}^2$ *the vectors*

$$\mathbf{a}_1 = (-8, 3) \quad \textit{and} \quad \mathbf{a}_2 = (-5, 2).$$

1. *Explain why* $(\mathbf{a}_1, \mathbf{a}_2)$ *is a basis of* $\mathbb{R}^2$.
2. *A linear map* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is given by*

$$f(\mathbf{a}_1) = 2\mathbf{a}_1 - 3\mathbf{a}_2 \quad \textit{and} \quad f(\mathbf{a}_2) = -\mathbf{a}_1 + 2\mathbf{a}_2.$$

Find the matrix of f *with respect to the basis* $(\mathbf{a}_1, \mathbf{a}_2)$ *of* $\mathbb{R}^2$.

3. *Find the matrix of* f *with respect to the usual basis of* $\mathbb{R}^2$.

1. It follows from

$$|\mathbf{a}_1 \;\; \mathbf{a}_2| = \begin{vmatrix} -8 & -5 \\ 3 & 2 \end{vmatrix} = -1 \neq 0,$$

that $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent. The dimension is 2, so $(\mathbf{a}_1, \mathbf{a}_2)$ is a basis of $\mathbb{R}^2$.

2. The matrix is given by the columns $f(\mathbf{a}_1)$, $f(\mathbf{a}_2)$,

$$\mathbf{F}_{\mathbf{a}\,\mathbf{a}} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

3. Since

$$\mathbf{F}_{\mathbf{e}\,\mathbf{e}} = \mathbf{M}_{\mathbf{e}\,\mathbf{a}}\mathbf{F}_{\mathbf{a}\,\mathbf{a}}\mathbf{M}_{\mathbf{a}\,\mathbf{a}},$$

where

$$\mathbf{M}_{\mathbf{e}\,\mathbf{a}} = \begin{pmatrix} -8 & -5 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{\mathbf{a}\,\mathbf{e}} = (\mathbf{M}_{\mathbf{e}\,\mathbf{a}})^{-1} = \begin{pmatrix} -2 & -5 \\ 3 & 8 \end{pmatrix},$$

we get

$$\begin{aligned}\mathbf{F}_{\mathbf{e}\,\mathbf{e}} &= \begin{pmatrix} -8 & -5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -2 & -5 \\ 3 & 8 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -5 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} -4 & -11 \\ 3 & 8 \end{pmatrix}.\end{aligned}$$

Example 3.22 *Given in the vector space* $\mathbb{R}^3$ *the vectors*

$$\mathbf{v}_1 = (1,2,0), \quad \mathbf{v}_2 = (0,1,4) \quad \textit{and} \quad \mathbf{v}_3 = (0,0,1),$$

and in $\mathbb{R}^4$ *the vectors*

$$\mathbf{w}_1 = (1,0,0,0), \quad \mathbf{w}_2 = (1,1,0,0), \mathbf{w}_3 = (1,1,1,0), \quad \mathbf{w}_4 = (1,1,1,1).$$

1. *Prove that* $(\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3)$ *form a basis of* $\mathbb{R}^4$.
2. *A linear map* $f : \mathbb{R}^3 \to \mathbb{R}^4$ *is given by*

$$f(\mathbf{v}_1) = \mathbf{w}_1 + \mathbf{w}_2, \quad f(\mathbf{v}_2) = \mathbf{w}_2 + \mathbf{w}_3, \quad f(\mathbf{v}_3) = \mathbf{w}_3 + \mathbf{w}_4.$$

   Find the matrix of f *with respect to the basis* $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *i* $\mathbb{R}^3$ *and* $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$ *i* $\mathbb{R}$.
3. *Find the matrix of* f *with respect to the usual bases in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

1. We just have to check the linear independency. It follows from

$$|\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 1,$$

and

$$|\mathbf{w}_1\ \ \mathbf{w}_2\ \ \mathbf{w}_3\ \ \mathbf{w}_4| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1,$$

that the vectors are linearly independent, so they are bases in the two spaces.

2. We just the columns in coordinates,

$$\mathbf{F}_{\mathbf{w}\,\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. We shall find

$$\mathbf{F}_{\mathbf{e}_4\,\mathbf{e}_3} = \mathbf{M}_{\mathbf{e}_4\,\mathbf{w}}\mathbf{F}_{\mathbf{w}\,\mathbf{v}}\mathbf{M}_{\mathbf{v}\,\mathbf{e}_3}.$$

Here,

$$\mathbf{M}_{\mathbf{e}_4\,\mathbf{w}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{\mathbf{e}_3\,\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}.$$

It follows from

$$(\mathbf{M}_{\mathbf{e}_3\,\mathbf{v}} \mid \mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \sim \\ R_2 := R_2 - 2R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \sim \\ R_3 := R_3 - 4R_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 8 & -4 & 1 \end{array}\right),$$

that

$$\mathbf{M}_{\mathbf{v}\,\mathbf{e}_3} = (\mathbf{M}_{\mathbf{e}_3\,\mathbf{v}})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{pmatrix}.$$

Finally, we get by insertion,

$$\begin{aligned} \mathbf{M} - \mathbf{e}_4\,\mathbf{e}_3 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 6 & -3 & 1 \\ 8 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -6 & 2 \\ 13 & -6 & 2 \\ 14 & -7 & 2 \\ 8 & -4 & 1 \end{pmatrix}. \end{aligned}$$

Example 3.23 *Given a map* $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ *by*

$$f((x_1, x_2, x_3)) = (x_1 + 3x_2 + 2x_3, x_1 - x_2 + 3x_3, 3x_1 + x_2 + 8x_3).$$

1. *Prove that* f *is linear.*
2. *Find the kernel of* f*, and find all* $a \in \mathbb{R}$*, for which the vector* $(8, 4, 8a)$ *belongs to the range* $f(\mathbb{R}^3)$*.*

1. Since

$$f((x_1, x_2, x_3)) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & 3 \\ 3 & 1 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

the map is clearly linear.

2. We reduce the matrix of coefficients

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & 3 \\ 3 & 1 & 8 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 - 3R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ 0 & 4 & -1 \end{pmatrix} \underset{R_3 := R_3 - R_2}{\sim} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank is 2, so $\dim\ker f = 3 - 2 = 1$, and the elements of the kernel satisfy

$$x_1 + 3x_2 = -2x_3, \qquad 4x_2 = x_3.$$

Using the parametric description $x_3 = 4s$, we get $x_2 = s$ and

$$x_1 = -3x_2 - 2x_3 = -3s - 8s = -11s,$$

thus

$$\ker f = \{s(-11, 1, 4) \mid s \in \mathbb{R}\}.$$

It follows from

$$\left(\begin{array}{rrr|r} 1 & 3 & 2 & 8 \\ 1 & -1 & 3 & 4 \\ 3 & 1 & 8 & 8a \end{array}\right) \begin{array}{l} \sim \\ R_2 := R_1 - R_2 \\ R_3 := R_3 - 3R_2 \end{array} \left(\begin{array}{rrr|c} 1 & 3 & 2 & 8 \\ 0 & 4 & -1 & 4 \\ 0 & 4 & -1 & 8a-12 \end{array}\right)$$
$$\begin{array}{c} \sim \\ R_3 := R_3 - R_2 \end{array} \left(\begin{array}{rrr|c} 1 & 3 & 2 & 8 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & 0 & 8a-16 \end{array}\right),$$

that $(8, 4, 8a) \in f(\mathbb{R}^3)$, if and only if the rank of the total matrix is 2, i.e. if and only if $8a - 16 = 0$, from which $a = 2$.

Example 3.24 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear map, which in the usual basis of $\mathbb{R}^3$ is given by the matrix*

$$\mathbf{F} = \begin{pmatrix} 4 & -11 & -3 \\ 1 & -2 & 0 \\ 1 & -4 & -1 \end{pmatrix}.$$

Furthermore, let

$$\mathbf{b}_1 = (1, 0, 1), \quad \mathbf{b}_2 = (1, 1, {}_1\,), \quad \mathbf{b}_3 = (-3, -1, 0)$$

be given vectors of $\mathbb{R}$.

1. *Prove that*

$$f(\mathbf{b}_1) = \mathbf{b}_2, \quad f(\mathbf{b}_2) = -\mathbf{b}_1 + \mathbf{b}_3, \quad f(\mathbf{b}_3) = -\mathbf{b}_2.$$

2. *Prove that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is a basis of $\mathbb{R}^3$.*
   Find the matrix of f with respect to this basis, and find the dimension of the range.

1. We get by direct computation,

$$\begin{aligned}
f(\mathbf{b}_1) &= \begin{pmatrix} 4 & -11 & -3 \\ 1 & -2 & 0 \\ 1 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \mathbf{b}_2, \\
f(\mathbf{b}_2) &= \begin{pmatrix} 4 & -11 & -3 \\ 1 & -2 & 0 \\ 1 & -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} \\
&= -\mathbf{b}_1 + \mathbf{b}_3, \\
f(\mathbf{b}_3) &= \begin{pmatrix} 4 & -11 & -3 \\ 1 & -2 & 0 \\ 1 & -4 & -2 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -\mathbf{b}_2.
\end{aligned}$$

2. It follows from

$$|\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{b}_3| = \begin{vmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = 1 \neq 0,$$

that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is a basis of $\mathbb{R}^3$.

According to 1) the matrix of the map is

$$\mathbf{F}_{\mathbf{b}\,\mathbf{b}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly, this matrix has rank 2, hence the dimension of the range is 2.

Example 3.25 *Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map, where the corresponding matrix with respect to the usual bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ is given by*

$$\mathbf{F}_{\mathbf{e}\,\mathbf{e}} = \begin{pmatrix} 1 & -2 & 0 & a \\ 3 & -6 & 1 & b \\ -2 & 4 & 1 & c \end{pmatrix}, \qquad \text{where } a,\ b,\ c \in \mathbb{R},$$

and where $f(1, -1, -2, 1) = (2, 8, -2)$.

1. *Find a, b and c.*
2. *Find a basis of the range $f(\mathbb{R}^4)$, and find the coordinates of the image vector $(2, 8, -2)$ with respect to this basis.*

1. It follows from

$$\begin{pmatrix} 1 & -2 & 0 & a \\ 3 & -6 & 1 & b \\ -2 & 4 & 1 & c \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} a+3 \\ b+7 \\ c-8 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix},$$

thats $a = -1$, $b = 1$ and $c = 6$.

2. Then by reduction,

$$\begin{pmatrix} 1 & -2 & 0 & -1 \\ 3 & -6 & 1 & 1 \\ -2 & 4 & 1 & 6 \end{pmatrix} \begin{matrix} \sim \\ R_2 := R_2 - R_1 + R_3 \\ R_3 := R_3 + 2R_1 \end{matrix} \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 1 & 4 \end{pmatrix},$$

which clearly is of rank 2, so $\dim f(\mathbb{R}^4) = 2$.

Since already $(2, 8, -2) \in f(\mathbb{R}^4)$, we shall only choose any other column of the matrix in order to obtain a basis, e.g.

$$\mathbf{a}_1 = (2, 8, -2) \quad \text{and} \quad \mathbf{a}_2 = (0, 1, 1).$$

Then the coordinates of $(2, 8, -2)$ with respect to $(\mathbf{a}_1, \mathbf{a}_2)$ are of course $(1, 0)$.

Example 3.26 *A linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$ is given by*

$$f((1,0,0)) = (2,1,0,1), \quad f((1,1,0)) = (3,2,1,1), \quad f((0,1,2)) = (3,-1,-5,4).$$

1. *Find the matrix of f with respect to the usual bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.*
2. *Find the dimension and a basis of the kernel $\ker f$.*
3. *Find the dimension and a basis of the range $f(\mathbb{R}^3)$.*

1. Let $\mathbf{a}_1 = (1,0,0)$, $\mathbf{a}_2 = (1,1,0)$ and $\mathbf{a}_3 = (0,1,2)$. Then

$$|\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0,$$

thus $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is a basis. Clearly,

$$\mathbf{F}_{\mathbf{e}\,\mathbf{a}} = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -5 \\ 1 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{\mathbf{e}\,\mathbf{a}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

where

$$\begin{aligned}
(\mathbf{M}_{\mathbf{e}\,\mathbf{a}} \mid \mathbf{I}) \quad &= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 - R_2 \\ R_3 := R_3/2 \end{array} \\
&\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + R_3 \\ R_2 := R_2 - R_3 \end{array} \\
&\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right),
\end{aligned}$$

thus

$$\mathbf{M}_{\mathbf{a}\,\mathbf{e}} = (\mathbf{M}_{\mathbf{e}\,\mathbf{a}})^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We get by insertion

$$\mathbf{F}_{\mathbf{e}\,\mathbf{e}} = \mathbf{F}_{\mathbf{e}\,\mathbf{a}} \mathbf{M}_{\mathbf{a}\,\mathbf{e}} = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -5 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}.$$

2. (Actually point 3.) We get by reduction,

$$\mathbf{F}_{\mathbf{e}\,\mathbf{e}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \begin{array}{l} \sim \\ R_1 := R_4 \\ R_2 := R_2 - R_4 \\ R_3 := R_2 - R_3 - R_4 \\ R_4 := R_1 - R_2 - R_4 \end{array} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which follows that the rank is 2, thus $\dim f(\mathbb{R}^3) = 2$, and a basis is e.g.

$$\{(2,1,0,1),(1,1,1,0)\}.$$

3. (Actually point 2.) It follows from

$$\dim V = \dim \mathbb{R}^3 = 3 = \dim \ker f + \dim f(\mathbb{R}^3) = 2 + \dim \ker f,$$

that

$$\dim \ker f = 1.$$

Then by the reduction above, choosing $x_3 = s$ as parameter we get $x_1 = -2x_3 = -2s$ and $x_2 = 3s$ for $\mathbf{x} \in \ker f$, i.e.

$$\ker f = \{s(-2,3,1) \mid s \in \mathbb{R}\},$$

and a basis vector is e.g. $(-2,3,1)$.

Example 3.27 *Given in the vector space* $\mathbb{P}_2(\mathbb{R})$ *the vectors*

$$P_1(x) = 1 + x - x^2, \quad P_2(x) = 2 + x - x^2, \quad P_3(x) = 1 - x^2.$$

Furhtermore, let $f : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ *be the linear map, which is given in the monomial basis* $(1, x, x^2)$ *of* $\mathbb{P}_2(\mathbb{R})$ *by the matrix*

$$\mathbf{F}_{\mathbf{m}\,\mathbf{m}} = \begin{pmatrix} 1 & 6 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -3 \end{pmatrix}.$$

1. *Prove that* $(P_1(x), P_2(x), P_3(x))$ *is a basis of* $\mathbb{P}_2(\mathbb{R})$.
2. *Write* $f(6 - x - 2x^2)$ *partly as a linear combination of 1,* x *and* x^2*, and partly as a linear combination of* $P_1(x)$*,* $P_2(x)$ *and* $P_3(x)$*.*

1. The coordinates are in the monomial basis

$$\begin{array}{rcccl} P_1(x) & = & 1 + x - x^2 & \sim & (1, 1, -1), \\ P_2(x) & = & 2 + x - x^2 & \sim & (2, 1, -1), \\ P_3(x) & = & 1 - x^2 & \sim & (1, 0, -1). \end{array}$$

It follows from

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

that $\{P_1(x), P_2(x), P_3(x)\}$ is a basis of $\mathbb{P}_2(\mathbb{R})$.

2. Since $6 - x - 2x^2 \sim (6, -1, -2)$, we find in the monomial basis

$$\begin{pmatrix} 1 & 6 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -3 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ 4 \end{pmatrix},$$

thus

$$f(6 - x - 2x^2) = -8 - 3x + 4x^2.$$

Then it immediately follows that

$$\begin{array}{rcl} 1 & = & -P_1(x) + P_2(x), \\ x & = & P_1(x) - P_3(x), \\ x^2 & = & 1 - P_3(x) = -P_1(x) + P_2(x = -P_3(x), \end{array}$$

hence

$$\begin{array}{rcrcrcr} f(6 - x - 2x^2) & = & -8 & - & 3x & + & 4x^2 \\ & = & 8P_1(x) & - & 8P_2(x) & & \\ & & -3P_1(x) & & & + & 3P_3(x) \\ & & -4P_1(x) & + & 4P_2(x) & - & 4P_3(x) \\ & = & P_1(x) & - & 4P_2(x) & - & P_3(x), \end{array}$$

and the coordinates are $(1, -4, -1)$ with respect to the basis $\{P_1(x), P_2(x), P_3(x)\}$.

Example 3.28 *Let f be a linear map of $\mathbb{R}^3$ into itself. The vectors $\mathbf{b}_1 = (-1,1,1)$, $\mathbf{b}_2 = (1,0,-1)$ and $\mathbf{b}_3 = (0,1,1)$ form a basis of $\mathbb{R}^3$, and the matrix of f with respect to this basis is*

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}.$$

Find the matrix of f with respect to the usual basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

It follows from the given conditions above that

$$\mathbf{M}_{\mathbf{e}\,\mathbf{b}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Then by a reduction,

$$\begin{aligned}
(\mathbf{M}_{\mathbf{e}\,\mathbf{b}} \mid \mathbf{I}) \quad = \quad & \left(\begin{array}{rrr|rrr} -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := -R_1 \\ R_2 := R_1 + R_2 \\ R_3 := R_1 + R_3 \end{array} \\
& \left(\begin{array}{rrr|rrr} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} \sim \\ R_1 := R_1 + R_2 - R_3 \\ R_2 := R_2 - R_3 \end{array} \\
& \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right),
\end{aligned}$$

hence

$$\mathbf{M}_{\mathbf{b}\,\mathbf{e}} = (\mathbf{M}_{\mathbf{e}\,\mathbf{b}})^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathbf{F}_{\mathbf{e}\,\mathbf{e}} &= \mathbf{M}_{\mathbf{e}\,\mathbf{b}}\mathbf{F}_{\mathbf{b}\,\mathbf{b}}\mathbf{M}_{\mathbf{b}\,\mathbf{e}} \\
&= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 2 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}.
\end{aligned}$$

Example 3.29 *Given in the vector space* $\mathbb{P}_2(\mathbb{R})$ *the vectors*

$$P_0(x) = 1, \quad P_1(x) = 1 - x, \quad P_2(x) = 1 - 2x + \frac{1}{2}x^2.$$

Let a map $f : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ *be given by*

$$f(P) = P' + 2P, \qquad P \in \mathbb{P}_2(\mathbb{R}),$$

where P' *is the derivative of* P.

1. *Prove that* $(P_0(x), P_1(x), P_2(x))$ *is a basis of* $\mathbb{P}_2(\mathbb{R})$.
2. *Prove that* f *is linear.*
3. *Find the matrix of* f *with respect to the basis* $(P_0(x), P_1(x), P_2(x))$.

1. It follows from

$$\begin{aligned} 1 &= P_0(x), \\ x &= 1 - P_1(x) = P_0(x) - P_1(x), \\ x^2 &= 2P_2(x) - 2 + 4x \\ &= 2P_2(x) - 2P_0(x) + 4P_0(x) - 4P_1(x) \\ &= 2P_0(x) - 4P_1(x) + 2P_2(x), \end{aligned}$$

that the monomial basis can be expressed by $P_0(x)$, $P_1(x)$, $P_2(x)$, hence the set

$$\{P_0(x), P_1(x), P_2(x)\}$$

also forms a basis of $\mathbb{P}_2(\mathbb{R})$.

Alternatively the coordinates are

$$P_0(x) \sim (1, 0, 0), \quad P_1(x) \sim (1, -1, 0), \quad P_2(x) \sim \left(1, -2, \frac{1}{2}\right),$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2} \neq 0,$$

which also shows that $\{P_0(x), P_1(x), P_2(x)\}$ is a basis.

2. If P, $Q \in \mathbb{P}_2(\mathbb{R})$, and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} f(P + \lambda Q) &= (P + \lambda Q)' + 2(P + \lambda Q) \\ &= \{P' + 2R\} + \lambda\{Q' + 2Q\} = f(P) + \lambda f(Q), \end{aligned}$$

proving that f is linear.

3. Since

$$\begin{aligned} f(P_0) &= P_0' + 2P_0 = 2P_0, \\ f(P_1) &= P_1' + 2P_1 = -1 + 2P_1 = -P_0 + 2P_1, \\ f(P_2) &= P_2' + 2P_2 = -2 + x + 2P_2(x) = -1 - (1-x) + 2P_2(x) = -P_0 - P_1 + 2P_2, \end{aligned}$$

we get the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

with respect to the basis (P_0, P_1, P_2).

Example 3.30 *Let a map* $f : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ *be given by*

$$f(P(x)) = (x-1)P'(x) - xP(1).$$

1. *Prove that* f *is linear.*
2. *Find the matrix of* f *with respect to the monomial basis* $(1, x, x^2)$.

1. If P, $Q \in \mathbb{P}_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} f(P(x) + \lambda Q(x)) &= (x-1)\{P(x) + \lambda Q(x)\}' - x\{P(1) + \lambda Q(1)\} \\ &= \{(x-1)P'(x) - xP(1)\} + \lambda\{(x-1)Q'(x) - xQ(1)\} \\ &= f(P(x)) + \lambda f(Q(x)), \end{aligned}$$

and f is linear.

2. Since

$$\begin{aligned} f(1) &= (x-1)\cdot 0 - x \cdot 1 = -x, \\ f(x) &= (x-1)\cdot 1 - x \cdot 1 = -1, \\ f(x^2) &= (x-1)\cdot 2x - x \cdot 1 = -3x + 2x^2, \end{aligned}$$

the corresponding matrix is

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Example 3.31 *Given the matrix*

$$\mathbf{A}=\begin{pmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1+a & 0 & 1 \end{pmatrix}.$$

1. *Find* $\det \mathbf{A}$ *for every* a.

2. *Solve for all real* a *and* b *the equation*

$$\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ b \\ 0 \\ b \end{pmatrix}.$$

3. *In the matrix* $\mathbf{A}$ *we put* $a=1$. *Then we get another matrix* $\mathbf{A}_1$. *We consider in the following the linear map* $f:\mathbb{R}^4\to\mathbb{R}^4$, *which is given in the usual basis* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ *by the matrix*

$$\mathbf{y}=\mathbf{A}_1\mathbf{x}.$$

The subspace V *of* $\mathbb{R}^4$, *which is spanned by* $\mathbf{e}_1$ *and* $\mathbf{e}_3$, *is by* f *into a subspace* $f(V)$ *of* $\mathbb{R}^4$. *The subspace* W *of* $\mathbb{R}^4$, *which is spanned by* $\mathbf{e}_2$ *and* $\mathbf{e}_4$, *is mapped by* f *into some subspace* $f(W)$ *of* $\mathbb{R}^4$.

Prove that $f(V)\subset V$ *and that* $f(W)=W$.

4. *Find the eigenvalues and the corresponding eigenvectors of the map* f.

5. *Find a regular matrix* $\mathbf{V}$ *and an diagonal matrix* $\mathbf{\Lambda}$, *such that*

$$\mathbf{\Lambda}=\mathbf{V}^{-1}\mathbf{A}_1\mathbf{V}.$$

1. We get by some reductions,

$$\begin{aligned}\det\mathbf{A} &= \begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1+a & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1-a & 0 \\ 0 & 1+a & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1-a & 0 \\ 1+a & 0 & 1 \end{vmatrix} \\ &= -(a-1)\begin{vmatrix} 1 & 2 \\ 1+a & 1 \end{vmatrix} = -(a-1)\{1-2-2a\}=(a-1)(2a+1).\end{aligned}$$

It follows that $\det\mathbf{A}=0$, if and only if either $a=1$ or $a=-\dfrac{1}{2}$.

2. If $a\neq 1$ and $a\neq -\dfrac{1}{2}$, then the solution is unique, and we get the reductions

$$(\mathbf{A}\mid\mathbf{b})=\left(\begin{array}{cccc|c} 1 & 0 & -a & 0 & 0 \\ 0 & 1 & 0 & 2 & b \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1+a & 0 & 1 & b \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 0 & -a & 0 & 0 \\ 0 & 1 & 0 & 2 & b \\ 0 & 0 & 1-a & 0 & 0 \\ 0 & 0 & 0 & -1-2a & -ab \end{array}\right),$$

hence

$$x_1 = x_3 = 0 \quad \text{and} \quad x_4 = \frac{ab}{1+2a} \quad \text{and} x_2 = b - \frac{2ab}{1+2a} = \frac{b}{1+2a}.$$

The unique solution is

$$\mathbf{x} = \left(0, \frac{b}{1+2a}, 0, \frac{ab}{1+2a}\right).$$

If $a = 1$, then we get the reductions

$$(\mathbf{A} \mid \mathbf{b}) = \left(\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & b \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & b \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -b \\ 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In this case we have infinitely many solutions,

$$\mathbf{x} = (0, -b, 0, b) + (s, 0, s, 0), \qquad s \in \mathbb{R}.$$

If $a = -\dfrac{1}{2}$, then we have the reductions

$$(\mathbf{A} \mid \mathbf{b}) = \left(\begin{array}{rrrr|r} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & b \\ -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 & b \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & b \\ 0 & 1 & 0 & 2 & 2b \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

If $b \neq 0$, then there are no solutions.
If $b = 0$, we get infinitely many solutions,

$$\mathbf{x} = (0, 2s, 0, -s) = s(0, 2, 0, -1), \qquad s \in \mathbb{R}.$$

3. The matrix $\mathbf{A}_1$ is

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

It follows that

$$\mathbf{A}_1\mathbf{e}_1 = \mathbf{e}_1 - \mathbf{e}_3 \quad \text{and} \quad \mathbf{A}_1\mathbf{e}_3 = -\mathbf{e}_1 + \mathbf{e}_3 = -\mathbf{A}_1\mathbf{e}_1,$$

thus

$$f(V) = \{s(\mathbf{e}_1 - \mathbf{e}_3) \mid s \in \mathbb{R}\} \subset V.$$

Furthermore,

$$\mathbf{A}_1\mathbf{e}_2 = \mathbf{e}_2 \quad \text{and} \quad \mathbf{A}_1\mathbf{e}_4 = 2\mathbf{e}_2 + \mathbf{e}_4,$$

hence

$$f(W) = \text{span}\{\mathbf{e}_2, 2\mathbf{e}_2 + \mathbf{e}_4\} = \text{span}\{\mathbf{e}_2, \mathbf{e}_4\} = W.$$

4. We compute the characteristic polynomials,

$$\begin{aligned}
\det(\mathbf{A}_1 - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & 2 \\ -1 & 0 & 1-\lambda & 0 \\ 0 & 2 & 0 & 1-\lambda \end{vmatrix} \\
&= \begin{vmatrix} -\lambda & 0 & -\lambda & 0 \\ 0 & 3-\lambda & 0 & 3-\lambda \\ -1 & 0 & 1-\lambda & 0 \\ 0 & 2 & 0 & 1-\lambda \end{vmatrix} \\
&= \lambda(\lambda-3) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1-\lambda & 0 \\ 0 & 2 & 0 & 1-\lambda \end{vmatrix} \\
&= \lambda(\lambda-3) \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix} \\
&= \lambda(\lambda-3)(\lambda-2)(\lambda+1).
\end{aligned}$$

We see that the four eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = -1, \quad \lambda_4 = 3.$$

For $\lambda_1 = 0$ we get the reduction

$$\mathbf{A}_1 - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_1 = (1,0,1,0)$, where $\|\mathbf{v}_1\| = \sqrt{2}$.

For $\lambda_2 = 2$ we get

$$\mathbf{A}_1 - \lambda_2\mathbf{I} = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_2 = (1,0,-1,0)$, where $\|\mathbf{v}_2\| = \sqrt{2}$.

For $\lambda_3 = -1$ we get

$$\mathbf{A}_1 - \lambda_3\mathbf{I} = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence an eigenvector is e.g. $\mathbf{v}_3 = (0,1,0,-1)$, where $\|\mathbf{v}_3\| = \sqrt{2}$.

For $\lambda_4 = 3$ we get

$$\mathbf{A}_1 - \lambda_4\mathbf{I} = \begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 2 \\ -1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_4 = (0,1,0,1)$ where $\|\mathbf{v}_4\| = \sqrt{2}$.

5. It follows that

$$\mathbf{V}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0&0\\0&0&1&1\\1&-1&0&0\\0&0&-1&1\end{pmatrix}\quad\text{med}\quad\mathbf{\Lambda}=\begin{pmatrix}0&0&0&0\\0&2&0&0\\0&0&-1&0\\0&0&0&3\end{pmatrix}.$$

Example 3.32 . *A linear map $f:\mathbb{R}^4\to\mathbb{R}^4$ is in the usual basis of $\mathbb{R}^4$ given by the matrix equation*

$$\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}a&a&2-a&a^2-a\\0&a&0&2-a\\2-a&a^2-a&a&2a^2-3a\\0&2-a&0&a\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

where a is a real number.

1. *Find the characteristic polynomial of f, and prove that $\lambda=2$ is an eigenvalue of f.*
2. *Find for every a the dimension of the eigenspace corresponding to the eigenvalue $\lambda=2$.*
3. *Find all a, for which one can find a basis of $\mathbb{R}^4$ consisting of eigenvectors of f.*
4. *Prove for $a=0$ that there exists an orthonormal basis of $\mathbb{R}^4$ (with the usual scalar product) consisting of eigenvectors eigenvectors of f. Find such basis, and also the matrix equation of f with respect to this basis.*

1. The characteristic polynomial is

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}a-\lambda&a&2-a&a^2-a\\0&a-\lambda&0&2-a\\2-a&a^2-a&a-\lambda&2a^2-3a\\0&2-a&0&a-\lambda\end{vmatrix}\\
&= \begin{vmatrix}2-\lambda&a^2&2-\lambda&3a^2-4a\\0&a-\lambda&0&2-a\\2-a&a^2-a&a-\lambda&2a^2-3a\\0&2-a&0&a-\lambda\end{vmatrix}\\
&= (a-\lambda)\begin{vmatrix}2-\lambda&2-\lambda&3a^2-4a\\2-a&a-\lambda&2a^2-3a\\0&0&a-\lambda\end{vmatrix}\\
&\qquad +(2-a)\begin{vmatrix}2-\lambda&a^2&2-\lambda\\2-a&a^2-a&a-\lambda\\0&2-a&0\end{vmatrix}\\
&= (\lambda-a)^2\begin{vmatrix}2-\lambda&2-\lambda\\2-a&a-\lambda\end{vmatrix}-(a-2)^2\begin{vmatrix}2-\lambda&2-\lambda\\2-a&a-\lambda\end{vmatrix}\\
&= \left\{(\lambda-a)^2-(a-2)^2\right\}(2-\lambda)\begin{vmatrix}1&1\\2-a&a-\lambda\end{vmatrix}\\
&= (\lambda-2)(\lambda-2a+2)(2-\lambda)(a-\lambda-2+a)\\
&= (\lambda-2)^2(\lambda-\{2a-2\})^2.\end{aligned}$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 2a - 2$, both of algebraic multiplicity 2, if $a \neq 2$.

If $a = 2$, then $\lambda_1 = 2$ is of algebraic multiplicity 4.

2. If $\lambda = 2$ and $a \neq 2$, then we have the reductions

$$\begin{pmatrix} a-2 & a & 2-a & a^2-a \\ 0 & a-2 & 0 & 2-a \\ 2-a & a^2-a & a-2 & 2a^2-3a \\ 0 & 2-a & 0 & a-2 \end{pmatrix} \sim \begin{pmatrix} a-2 & a & 2-a & a^2-a \\ 0 & 1 & 0 & -1 \\ 0 & a^2 & 0 & 3a^2-4a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} a-2 & a & 2-a & a^2-a \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 4(a^2-a) \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} a-2 & a & 2-a & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & a(a-1) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $a \neq 2$ and $a \neq 0$, $a \neq 1$, then the rank is 3, hence the dimension of the eigenspace is $4-3=1$ with the eigenvector $(1,0,1,0)$.

If $a = 0$ or $a = 1$, then the rank is 2, and then dimension of the eigenspace is $4-2=2$.

If $a = 2$, then we get instead,

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is of rank 2, so the eigenspace is of dimension $4-2=2$.

3. According to 1) and 2) the algebraic and the geometric multiplicity do not agree for $\lambda = 2$, if $a \neq 0$ and $a \neq 1$.

   The only possibility of such a basis, is therefore when either $a = 0$ or $a = 1$. The case $a = 0$ is treated in 4), so here we consider $a = 1$. Then it follows from 2) that the eigenspace corresponding to $\lambda = 2$ is of dimension 2.

   Then we check the other eigenvalue $\lambda_2 = 2\cdot 1 - 2 = 0$. Its algebraic multiplicity is 2. Furthermore, we have the reduction

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   The eigenspace is of dimension $4-2=2$, thus for $a = 1$ there exists a basis consisting of eigenvectors.

4. Finally, we check $a = 0$. The two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, both of algebraic multiplicity 2. Since

$$\mathbf{A}_0 - 2\mathbf{I} = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

are two orthonormal eigenvectors corresponding to $\lambda_1 = 2$,

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1,0,1,0) \quad \text{and} \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}(0,1,0,1).$$

For $\lambda_2 = -2$ we instead obtain

$$\mathbf{A}_0 + 2\mathbf{I} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the two orthonormal eigenvectors corresponding to $\lambda_2 = -2$ are

$$\mathbf{q}_3 = \frac{1}{\sqrt{2}}(1,0,-1,0) \quad \text{and} \quad \mathbf{q}_4 = \frac{1}{\sqrt{2}}(0,1,0,-1).$$

The matrix equation of f is now with respect to the basis $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Example 3.33 *Let the map $f : V_g^3 \to V_g^3$ be given by*

$$f(\vec{x}) = \vec{x} \times \vec{\imath} + (\vec{x} \cdot \vec{\jmath})\vec{k} + \vec{x},$$

where the three geometrical vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ form an orthonormal basis of positive orientation.

1. *Prove that f is a linear map.*
2. *Express $f(\vec{\imath})$, $f(\vec{\jmath})$ and $f(\vec{k})$ as linear combinations of $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$, and find the matrix $\mathbf{F}$ of f with respect to the basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$.*
3. *Check if $\mathbf{F}$ can be diagonalized.*

1. We infer from

$$\begin{aligned} f(\vec{x} + \lambda\vec{y}) &= (\vec{x} + \lambda\vec{y}) \times \vec{\imath} + ((\vec{x} + \lambda\vec{y}) \cdot \vec{\jmath})\vec{k} + (\vec{x} + \lambda\vec{y}) \\ &= \{\vec{x} \times \vec{\imath} + (\vec{x} \cdot \vec{\jmath})\vec{k} + \vec{x}\} + \lambda\{\vec{y} \times \vec{\imath} + (\vec{y} \cdot \vec{\jmath})\vec{k} + \vec{y}\} \\ &= f(\vec{x}) + \lambda f(\vec{y}), \end{aligned}$$

that f is a linear map.

2. Then by a computation,

$$\begin{aligned} f(\vec{i}) &= \vec{i}\times\vec{i}+(\vec{i}\cdot\vec{j})\vec{k}+\vec{i}=\vec{i}, \\ f(\vec{j}) &= \vec{j}\times i+(\vec{j}\cdot\vec{j})\vec{k}+\vec{j}=-\vec{k}+\vec{k}+\vec{j}=\vec{j}, \\ f(\vec{k}) &= \vec{k}\times\vec{i}+(\vec{k}\cdot\vec{j})\vec{k}+\vec{k}=\vec{j}+\vec{k}. \end{aligned}$$

The corresponding matrix is

$$\mathbf{F}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. It is not possible to diagonalize $\mathbf{F}$, because $\lambda = 1$ is of geometric multiplicity 2 and of algebraic multiplicity 3. In fact,

$$\mathbf{F}-\mathbf{I}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is of rank 1, hence the eigenspace is only of dimension $3-1=2$.

ALTERNATIVELY we have a 1 just above the diagonal (Jordan's form of matrices).

Example 3.34 *Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map which with respect to the usual basis of $\mathbb{R}^4$ is given by the matrix*

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

and let $g : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map, which with respect to the usual basis of $\mathbb{R}^4$ is given by the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Consider also the composite map $h = f \circ g$.

1. *Find the vectors $\mathbf{x}$ and $\mathbf{x}$, such that*

$$f(\mathbf{y}) = \mathbf{b} \quad \text{and} \quad h(\mathbf{x}) = \mathbf{b},$$

where $\mathbf{b} = (1, 5, 4, -9)$.

2. *Prove that*

$$\mathbf{U} = \mathbf{D}\mathbf{L}^T,$$

where $\mathbf{D}$ is a diagonal matrix, and apply this result to prove that the matrix of h with respect to the usual basis of $\mathbb{R}^4$ is symmetric and positive definit.

1. It follows from

$$f(\mathbf{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_1 + y_2 \\ 2y_1 + y_3 \\ -y_2 + y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 4 \\ -9 \end{pmatrix}$$

that $\mathbf{y} = (1, 6, 2, -3)$.

From $\mathbf{b} = h(\mathbf{x} = f \circ g(\mathbf{x}) = f(\mathbf{y})$ we get the equation $g(\mathbf{x}) = \mathbf{y}$, thus

$$g(\mathbf{x}) = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + 2x_3 \\ 2x_2 - 2x_4 \\ 2x_3 \\ 3x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \\ -3 \end{pmatrix},$$

hence $x_4 = -1$ and $x_3 = 1$, and whence $x_2 = 3 + x_4 = 2$ and

$$x_1 = 1 + x_2 - 2x_3 = 1 + 2 - 2 = 1.$$

We infer that

$$\mathbf{x} = (1, 2, 1, -1).$$

2. The only possibility of $\mathbf{D}$ is a diagonal matrix, which has the same diagonal elements as $\mathbf{U}$. Then

$$\begin{aligned}\mathbf{DL}^T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \mathbf{U},\end{aligned}$$

and $\mathbf{U} = \mathbf{DL}^T$.

The matrix of h is $\mathbf{A} = \mathbf{LU} = \mathbf{LDL}^T$, where clearly

$$\mathbf{A}^T = \left(\mathbf{LDL}^T\right)^T = \mathbf{LDL}^T = \mathbf{A},$$

hence $\mathbf{A}$ is symmetric.

The eigenvalues are the diagonal elements of $\mathbf{D}$, i.e. 1, 2, 2, 3. These are all positive, hence $\mathbf{A}$ is positive definite.

Example 3.35 *Given the matrices*

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{I}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{A} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

1. *Prove that*

$$\det(\mathbf{M} - \lambda \mathbf{I}_{2\times 2}) = \det(\mathbf{A} - (\lambda - 1)\mathbf{I}_{2\times 2})\det(\mathbf{A} - (\lambda + 1)\mathbf{I}_{2\times 2}).$$

2. *Then denote by $f : \mathbb{R}^4 \to \mathbb{R}^4$ the linear map, which with respect to the usual basis of $\mathbb{R}^4$ has $\mathbf{M}$ as matrix.*

   Find the eigenvalues and the corresponding eigenvectors of f.

3. *Find the dimension of the range of f and a parametric description of the range.*

4. *Find a vector $\neq \mathbf{0}$, which is orthogonal to the range (with respect to the usual scalar product of $\mathbb{R}^4$), and setup an equation of the range.*

1. By insertion

$$\begin{aligned}
\det(\mathbf{M} - \lambda \mathbf{I}_{4\times 4}) &= \det \begin{pmatrix} \mathbf{A} - \lambda \mathbf{I}_{2\times} & \mathbf{I}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{A} - \lambda \mathbf{I}_{2\times 2} \end{pmatrix} \\
&= \det \begin{pmatrix} \mathbf{A} - (\lambda - 1)\mathbf{I}_{2\times 2} & \\ \mathbf{I}_{2\times 2} & \mathbf{A} - \lambda \mathbf{I}_{2\times 2} \end{pmatrix} \\
&= \det \left\{ \begin{pmatrix} \mathbf{A} - (\lambda - 1)\mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{I}_{2\times 2} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{2\times 2} & \mathbf{I}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{A} - \lambda \mathbf{I}_{2\times 2} \end{pmatrix} \right\} \\
&= \det\left(\mathbf{A} -)\lambda - 1)\mathbf{I}_{2\times 2}\right) \cdot \det \begin{pmatrix} \mathbf{I}_{2\times 2} & \mathbf{I}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{A} - (\lambda - 1)\mathbf{I}_{2\times 2} \end{pmatrix} \\
&= \det\left(\mathbf{A} - (\lambda - 1)\mathbf{I}_{2\times 2}\right) \cdot \det\left(\mathbf{A} - (\lambda + 1)\mathbf{I}_{2\times 2}\right).
\end{aligned}$$

2. The roots of

$$\det(\mathbf{A} - \mu \mathbf{I}_{2\times}) = \begin{vmatrix} 2-\mu & 1 \\ 1 & 2-\mu \end{vmatrix} = (\mu - 2)^2 - 1 = (\mu - 1)(\mu - 3)$$

are $\mu_1 = 1$ and $\mu_2 = 3$, hence $\mathbf{M}$ has the four eigenvalues

$$\begin{aligned}
&\lambda_1 + 1 = \mu_1 = 1, \quad \text{dvs.} \quad \lambda_1 = 0, \\
&\lambda_2 + 1 = \mu_3 = 3, \quad \text{dvs.} \quad \lambda_2 = 2, \\
&\lambda_3 - 1 = \mu_1 = 1, \quad \text{dvs.} \quad \lambda_3 = 2, \\
&\lambda_4 - 1 = \mu_2 = 3, \quad \text{dvs.} \quad \lambda_4 = 4.
\end{aligned}$$

For $\lambda_1 = 0$ we reduce,

$$\mathbf{M} - \lambda_1 \mathbf{I} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $\mathbf{v}_1 = (1, -1, -1, 1)$.

For $\lambda_2 = \lambda_3 = 2$ we reduce,

$$\mathbf{M} - \lambda_2 \mathbf{I} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Two linearly independent eigenvectors, which span the eigenspace, are e.g.

$$\mathbf{v}_2 = (1, 0, 0, -1) \quad \text{and} \quad \mathbf{v}_3 = (0, 1, -1, 0).$$

For $\lambda_4 = 4$ we reduce,

$$\begin{aligned} \mathbf{M} - \lambda_4 \mathbf{I} &= \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

An eigenvector is e-g. $\mathbf{v}_4 = (1, 1, 1, 1)$.

3. If we apply

$$\mathbf{q}_1 = \frac{1}{2}(1, -1, -1, 1), \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}(1, 0, 0, -1),$$

$$\mathbf{q}_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0), \quad \mathbf{q}_4 = \frac{1}{2}(1, 1, 1, 1)$$

as an orthonormal basis, the map is written in the form

$$f(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Clearly, $\dim f(\mathbb{R}^3) = 3$, and

$$\begin{aligned} f(\mathbb{R}^3) &= \text{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \\ &= \{s(1, 0, 0, -1) + t(0, 1, -1, 0) + u(1, 1, 1, 1) \mid s,\ t,\ u \in \mathbb{R}\} \\ &= \{(s + u, t + u, -t + u, -s + u) \mid s,\ t,\ u \in \mathbb{R}\}. \end{aligned}$$

4. It follows from the above that $\mathbf{v}_1 = (1, -1, -1, 1)$ is orthogonal on the range, hence an equation of the range is

$$\mathbf{v}_1 \cdot \mathbf{x} = x_1 - x_2 - x_3 + x_4 = 0.$$

Example 3.36 *Concerning a linear map* $f : \mathbb{C}^3 \to \mathbb{C}^3$ *it is given that its eigenvalues are* $\lambda_1 = 1$, $\lambda_2 = 1+i$ *and* $\lambda_3 = 1-i$. *The corresponding eigenvectors are* $\mathbf{v}_1 = (1,1,0)$, $\mathbf{v}_2 = (0,1,i)$ *and* $\mathbf{v}_3 = (0,1,-i)$.

1. *Find the image vector* $f(\mathbf{w})$, *where* $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, *and find a vector* $\mathbf{v}$ *with the image vector* $f(\mathbf{v}) = (0, 2i, 2i)$.
2. *Find the kernel of the map, the dimension of the range, as well as the characteristic polynomial.*

   (Hint: Apply e.g. the matrix of f *with respect to the basis* $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$*).*
3. *Find the matrix of* f *with respect to the usual basis of* $\mathbb{C}^3$.

1. Given that

$$f(\mathbf{v}_1) = \mathbf{v}_1, \quad f(\mathbf{v}_2) = (1+i)\mathbf{v}_2, \quad f(\mathbf{v}_3) = (1-i)\mathbf{v}_3,$$

   such that

$$\begin{aligned} f(\mathbf{w}) &= f(\mathbf{v}_1) + f(\mathbf{v}_2) + f(\mathbf{v}_3) = \mathbf{v}_1 + (1+i)\mathbf{v}_2 + (1-i)\mathbf{v}_3 \\ &= (1,1,0) = \{(1+i)(0,1,i) + (1-i)(0,1,-i)\} \\ &= (1,1,0) = 2\mathrm{Re}\{(1+i)(0,1,i)\} \\ &= (1,1,0) + 2\mathrm{Re}\{(0,1+i,i-1)\} = (1,1,0) + 2(0,1,-1) \\ &= (1,3,2). \end{aligned}$$

   We infer from

$$\begin{aligned} (0,2i,2i) &= i(\mathbf{v}_2 + \mathbf{v}_3) + (\mathbf{v}_2 - \mathbf{v}_3) = (1+i)\mathbf{v}_2 - (1-i)\mathbf{v}_3 \\ &= f(\mathbf{v}_2) - f(\mathbf{v}_3) = f(\mathbf{v}_2 - \mathbf{v}_3), \end{aligned}$$

   that $\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_3 = (0,0,2i)$.

2. The range is of dimension 3, because all three eigenvalues are simple. Thus, the kernel must be $\{\mathbf{0}\}$.

   The characteristic polynomial has the eigenvalues as roots, so it is given by

$$\begin{aligned} &(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = (\lambda - 1)(\lambda - 1 - i)(\lambda - 1 + i) \\ &\qquad = (\lambda - 1)(\lambda^2 - 2\lambda + 2) = \lambda^3 - 3\lambda^2 + 4\lambda - 2, \end{aligned}$$

   where we in practice should keep the factorization.

3. It follows from

$$\begin{aligned} (1,0,0) &= \mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_3, \\ (0,1,0) &= \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3, \\ (0,0,1) &= -\frac{i}{2}\mathbf{v}_2 + \frac{i}{2}\mathbf{v}_3, \end{aligned}$$

that

$$\begin{aligned}
f(\mathbf{e}_1) &= \mathbf{v}_1 - \frac{1}{2}(1+i)\mathbf{v}_2 - \frac{1}{2}(1-i)\mathbf{v}_3 = \mathbf{v}_1 - \operatorname{Re}(1+i)\mathbf{v}_2 \\
&= (1,1,0) - \operatorname{Re}(0,1+i,i-1) = (1,1,0)-(0,1,-1) = (1,0,1), \\
f(\mathbf{e}_2) &= \frac{1}{2}(1+i)\mathbf{v}_2 + \frac{1}{2}(1-i)\mathbf{v}_3 = \operatorname{Re}(1+i)\mathbf{v}_2 = (0,1,-1), \\
f(\mathbf{e}_3) &= -\frac{i}{2}(1+i)\mathbf{v}_2 + \frac{i}{2}(1-i)\mathbf{v}_3 = -\operatorname{Re}(i(1+i)\mathbf{v}_2) \\
&= -\operatorname{Re}(0,i-1,-1-i) = (0,1,1).
\end{aligned}$$

The columns of the matrix are $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, $f(\mathbf{e}_3)$, hence

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Example 3.37 *Consider the vector space $\mathbb{R}^4$ with the usual scalar product, and the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$, which with respect to the usual basis of $\mathbb{R}^4$ is given by the matrix equation*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 3 & -3 & -1 & 1 \\ 1 & 3 & -3 & -1 \\ -1 & 1 & 3 & -3 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

1. *Find the kernel of f and the dimension of the range $f(\mathbb{R}^4)$.*
   Prove that every vector of $\ker f$ is orthogonal on every vector from $f(\mathbb{R}^4)$, and then infer that
   $$f(\mathbb{R}^4) = \left\{ \mathbf{y} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for alle } \mathbf{x} \in \ker f \right\}.$$
2. *Prove that the vectors*
   $$\mathbf{q}_1 = \frac{1}{2}(-1,1,-1,1), \quad \mathbf{q}_2 = \frac{1}{2}(-1,-1,1,1), \quad \mathbf{q}_3 = \frac{1}{2}(-1,1,1,-1),$$
   form an orthonormal basis of the range $f(\mathbb{R}^4)$.
   Find a vector $\mathbf{q}_4$, such that $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ is an orthonormal basis of $\mathbb{R}^4$.
3. *Express $f(\mathbf{q}_1)$, $f(\mathbf{q}_2)$, $f(\mathbf{q}_3)$, $f(\mathbf{q}_4)$ as linear combinations of $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$.*
   Find the matrix of f with respect to the basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$.
4. *Find all the eigenvalues and the corresponding eigenvectors of f.*
   (Hint: One may apply the result of 3)).

1. The sum of all columns is $\mathbf{0}$, hence $(1,1,1,1)$ belongs to $\ker f$.

   Then we get by reduction

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 3 & -3 & -1 & 1 \\ 1 & 3 & -3 & -1 \\ -1 & 1 & 3 & -3 \\ -3 & -1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -3 & -1 \\ 0 & -12 & 8 & 4 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 3 & -3 & -1 \\ 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

   which is of rank 3, so $\dim f(\mathbb{R}^4) = 3$, and

$$\ker f = \{ s(1,1,1,1) \mid s \in \mathbb{R} \}$$

   is of dimension 1.

The range is spanned by the columns of $\mathbf{A}$. The sum of the rows is $\mathbf{0}$, hence every column is orthogonal to $(1,1,1,1) \in \ker f$, whence

$$f(\mathbb{R}^4) = \left\{\mathbf{y} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{y}\rangle = 0 \text{ for alle } \mathbf{x} \in \ker f\right\}.$$

2. It follows immediately by choosing $\mathbf{q}_4 = \dfrac{1}{2}(1,1,1,1) \in \ker f$ that

$$\langle \mathbf{q}_i, \mathbf{q}_4\rangle = 0 \qquad \text{for } i = 1,\, 2,\, 3,$$

because the sum of the coordinates of each $\mathbf{q}_i$, $i = 1, 2, 3$, is 0. This implies that $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ all lie in the range. Clearly, they are all normed, and since

$$\begin{aligned}
\langle \mathbf{q}_1, \mathbf{q}_2\rangle &= \frac{1}{4}(1-1-1+1) = 0,\\
\langle \mathbf{q}_1, \mathbf{q}_3\rangle &= \frac{1}{4}(1+1-1-1) = 0,\\
\langle \mathbf{q}_2, \mathbf{q}_3\rangle &= \frac{1}{4}(1-1+1-1) = 0,
\end{aligned}$$

they are even orthonormal. It follows by choosing $\mathbf{q}_4$ that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ is an orthonormal basis of $\mathbb{R}^4$.

3. Now,

$$\begin{aligned}
f(\mathbf{q}_1) &= \frac{1}{2}\begin{pmatrix} 3 & -3 & -1 & 1\\ 1 & 3 & -3 & -1\\ -1 & 1 & 3 & -3\\ -3 & -1 & 1 & 3\end{pmatrix}\begin{pmatrix} -1\\ 1\\ -1\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4\\ 4\\ -4\\ 4\end{pmatrix} = 4\mathbf{q}_1,\\
f(\mathbf{q}_2) &= \frac{1}{2}\begin{pmatrix} 3 & -3 & -1 & 1\\ 1 & 3 & -3 & -1\\ -1 & 1 & 3 & -3\\ -3 & -1 & 1 & 3\end{pmatrix}\begin{pmatrix} -1\\ -1\\ 1\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0\\ -8\\ 0\\ 8\end{pmatrix} = 4\mathbf{q}_2 - 4\mathbf{q}_3,\\
f(\mathbf{q}_3) &= \frac{1}{2}\begin{pmatrix} 3 & -3 & -1 & 1\\ 1 & 3 & -3 & -1\\ -1 & 1 & 3 & -3\\ -3 & -1 & 1 & 3\end{pmatrix}\begin{pmatrix} -1\\ 1\\ 1\\ -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix} -8\\ 0\\ 8\\ 0\end{pmatrix} = \begin{pmatrix} -4\\ 0\\ 4\\ 0\end{pmatrix} = 4\mathbf{q}_2 + 4\mathbf{q}_3,\\
f(\mathbf{q}_4) &= \mathbf{0}.
\end{aligned}$$

The matrix with respect to this basis is

$$\mathbf{M} = \begin{pmatrix} 4 & 0 & 0 & 0\\ 0 & 4 & 4 & 0\\ 0 & -4 & 4 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

4. We have that $\lambda_1 = 4$ with the eigenvector $\mathbf{q}_1$, and $\lambda_4 = 0$ with the eigenvector $\mathbf{q}_4$.

Any other possible eigenvector must be of the form $\mathbf{q} = \mathbf{q}_2 + \alpha\mathbf{q}_3$. We infer from 3),

$$\begin{aligned}
f(\mathbf{q} &= f(\mathbf{q}_2) + \alpha f(\mathbf{q}_3) = 4\mathbf{q}_2 - 4\mathbf{q}_3 + 4\alpha\mathbf{q}_2 + 4\alpha\mathbf{q}_3\\
&= 4(\alpha+1)\mathbf{q}_2 + 4(\alpha-1)\mathbf{q}_3.
\end{aligned}$$

The eigenvalue is $\lambda = 4(\alpha+1)$, and the requirement is here that

$$\alpha \cdot 4(\alpha+1) = 4(\alpha-1),$$

thus $\alpha^2 + \alpha = \alpha - 1$, hence $\alpha^2 = -1$, and whence $\alpha = \pm i$.
Thus we have two complex eigenvalues. We shall, however, only work in $\mathbb{R}$ in this example, so we find that

$$\mathbf{q}_1 \text{ where } \lambda_1 = 4 \quad \text{and} \quad \mathbf{q}_4 \text{ where } \lambda_4 = 0$$

are the only (real) eigenvectors with ncorresponding real eigenvalues.

Remark 3.1 For $\lambda_2 = i$ we get the complex eigenvector

$$\mathbf{q}_2 + i\mathbf{q}_3 = \frac{1}{2}(-1-i, -1+i, 1+i, 1-i).$$

For $\lambda_3 = -i$ we get the complex eigenvector

$$\mathbf{q}_3 - i\mathbf{q}_3 = \frac{1}{2}(-1+i, -1-i, 1-i, 1+i).$$

They are of course complex conjugated. $\Diamond$

Example 3.38 *Given in* $\mathbb{R}^4$ *the vectors*

$$\mathbf{v}_1 = (1,2,4,-2), \quad \mathbf{v}_2 = (1,0,3,-2), \quad \mathbf{v}_3 = (-1,1,-3,5),$$

$$\mathbf{v}_4 = (-1,0,-3,1), \quad \textit{and} \quad \mathbf{v}_5 = (-1,4,-2,7).$$

1. *Prove that* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ *is a basis of* $\mathbb{R}^4$, *and find the coordinates of* $\mathbf{v}_5$ *with respect to this basis.*

2. *A linear map* $f : \mathbb{R}^4 \to \mathbb{R}^4$ *is given by*

$$\begin{aligned} f(\mathbf{v}_1) &= \mathbf{v}_1 + \mathbf{v}_2, & f(\mathbf{v}_2) &= -\mathbf{v}_1 + \mathbf{v}_2, \\ f(\mathbf{v}_3) &= \mathbf{v}_3 + \mathbf{v}_4, & f(\mathbf{v}_4) &= -\mathbf{v}_3 + \mathbf{v}_4. \end{aligned}$$

   Find the matrix of f *with respect to the basis* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, *and find the coordinates of* $f(\mathbf{v}_5)$ *with respect to basis* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.

3. *Prove that* f *does not have eigenvectors.*

4. *Prove that* f *maps the subspace* U, *spanned by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *onto* U.

1. Let us check if we can solve the equation

$$x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 + t\mathbf{v}_4 = \mathbf{v}_5,$$

   i.e. in matrix formulation

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 0 & 1 & 0 \\ 4 & 3 & -3 & -3 \\ -2 & -2 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 7 \end{pmatrix}.$$

   We reduce,

$$\begin{aligned} (\mathbf{A} \mid \mathbf{b}) &= \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 2 & 0 & 1 & 0 & 4 \\ 4 & 3 & -3 & -3 & -2 \\ -2 & -2 & 5 & 1 & 7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 2 & 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 5 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right). \end{aligned}$$

   From this we infer two things:

   (a) Since the matrix of coefficients has rank 4, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly independent, thus they form a basis of $\mathbb{R}^4$.

(b) In this basis the coordinates of $\mathbf{v}_5$ are $(1,1,2,1)$.

2. The matrix is

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and the coordinates of $f(\mathbf{v}_5)$ are

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix},$$

thus

$$f(\mathbf{v}_5) \sim (0,2,1,3).$$

3. The characteristic polynomial is

$$\left\{(\lambda-1)^2+1\right\}^2$$

with the two complex double roots $\lambda = 1 \pm i$. There are no real eigenvalues, hence f does not have eigenvectors.

4. This is obvious, because the image vectors $\mathbf{v}_3 + \mathbf{v}_4$ and $-\mathbf{v}_3 + \mathbf{v}_4$ lie in U and they are linearly independent. Now, U has dimension 2, so $f(\mathbf{v}_3)$ and $f(\mathbf{v}_4)$ also span U.

Example 3.39 *Let $\vec{a}$ and $\vec{b}$ be given vectors of V_g^3, for which*

$$|\vec{a}| = |\vec{b}| = \sqrt{2} \quad \textit{and} \quad \vec{a}\cdot\vec{b} = 1.$$

We define a map $f : V_g^3 \to V_g^3$ by

$$f(\vec{x}) = \vec{a}\times\vec{x} + (\vec{a}\cdot\vec{x})\vec{b} \qquad \textit{for } \vec{x} \in V_g^3.$$

1. Prove that f is a linear map.

2. Now, put $\vec{c} = \vec{a}\times\vec{b}$.
Explain why $\vec{a}$, $\vec{b}$, $\vec{c}$ form a basis of the vector space V_g^3, and find the matrix of f with respect to this basis.

3. Find all eigenvectors of f, expressed by the vectors $\vec{a}$ and $\vec{b}$.

4. Find the range $f(V_g^3)$.

1. It is obvious that f is linear:

$$\begin{aligned} f(\vec{x}+\lambda\vec{y}) &= \vec{a}\times(\vec{x}+\lambda\vec{y}) + (\vec{a}\cdot\{\vec{x}+\lambda\vec{y}\})\vec{b} \\ &= \vec{a}\times\vec{x} + (\vec{a}\cdot\vec{x})\vec{b} + \lambda\{\vec{a}\times\vec{y} + (\vec{a}\cdot\vec{y})\vec{b}\} \\ &= f(\vec{x}) + \lambda(\vec{y}). \end{aligned}$$

2. It follows from $\vec{a}\cdot\vec{b} = 1 \neq 2 = |\vec{a}|^2 = |\vec{b}|^2$, that $\vec{a}$ and $\vec{b}$ are linearly independent, hence $\vec{c} \neq \vec{0}$, and $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent, so they form a basis of V_3^g.

Using $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = 2$ we compute

$$\begin{aligned} f(\vec{a}) &= \vec{a}\times\vec{a} + (\vec{a}\cdot\vec{a})\vec{b} = 2\vec{b}, \\ f(\vec{b}) &= \vec{a}\times\vec{b} + (\vec{a}\cdot\vec{b})\vec{b} = \vec{b}+\vec{c}, \\ f(\vec{c}) &= \vec{a}\times(\vec{a}\times\vec{b}) + (\vec{a}\cdot(\vec{a}\times\vec{b}))\vec{b} \\ &= (\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b} + \vec{0} = \vec{a} - 2\vec{b}, \end{aligned}$$

hence the matrix with respect to the basis $\vec{a}$, $\vec{b}$, $\vec{c}$ is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}.$$

3. The characteristic polynomial is

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & 0 & 1 \\ 2 & 1-\lambda & -2 \\ 0 & 1 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 1-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1-\lambda \\ 0 & 1 \end{vmatrix} \\ &= -\lambda\{\lambda(\lambda-1)+2\}+2 = -\lambda^3+\lambda^2-2\lambda+2 \\ &= -(\lambda-1)\{\lambda^2+2\}.\end{aligned}$$

It follows that $\lambda = 1$ is the only real eigenvalue. It follows from the reduction

$$\mathbf{A}-\mathbf{I} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

that the coordinates of the eigenvector is $(1,1,1)$, hence

$$\vec{a}+\vec{b}+\vec{c}$$

is an eigenvector.

4. Clearly, $\mathbf{A}$ is of rank 3, so the range is all of V_g^3,

$$f(V_g^3) = V_g^3.$$

Example 3.40 *A linear map* $f : \mathbb{R}^4 \to \mathbb{R}^4$ *is with respect to the usual basis of* $\mathbb{R}^4$ *given by the matrix*

$$\mathbf{F} = \begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix}.$$

1. *Prove that the kernel* $\ker f$ *has dimension 2, and that the vectors*

$$\mathbf{q}_1 = \frac{1}{3}(0,2,2,1) \quad \text{and} \quad \mathbf{q}_2 = \frac{1}{3}(2,0,-1,2)$$

form an orthonormal basis of $\ker f$ *(where we use the usual scalar product of* $\mathbb{R}^4$*).*

2. *Prove that* $\mathbf{q}_3 = \dfrac{1}{3}(2,1,0,-2)$ *is orthogonal on every vector of* $\ker f$.
3. *Find the vector* $\mathbf{q}_4$*, such that* $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ *is an orthonormal basis of* $\mathbb{R}^4$.
4. *Find* $f(\mathbf{q}_3)$ *and* $f(\mathbf{q}_4)$*, and the matrix of* f *with respect to the basis* $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$.

1. First we reduce,

$$\begin{aligned}\mathbf{F} &= \begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & -2 \\ -5 & 0 & -2 & 4 \\ -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 15 & -12 & -6 \\ 0 & 5 & -4 & -2 \\ 0 & -10 & 8 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & -2 \\ 0 & 5 & -4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

which is clearly of rank 2, so the kernel is of dimension $4 - 2 = 2$.

CHECK:

$$4\mathbf{F}\mathbf{q}_1 = \begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10-8-2 \\ 0-4+4 \\ 4+0-4 \\ -8+8+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$3\mathbf{F}\mathbf{q}_2 = \begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0+0+4-4 \\ -10+0+2+8 \\ 8+0+0-8 \\ 4+0-4+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

hence both $\mathbf{q}_1$ and $\mathbf{q}_2$ belong to $\ker f$. Since

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = \frac{1}{9}(0+0-2+2) = 0,$$

they are orthogonal and in particular linearly independent, so they span $\ker f$. Since

$$\|\mathbf{q}_1\| = \frac{1}{3}\sqrt{4+4+1} = \|\mathbf{q}_2\| = 1,$$

the vectors $\mathbf{q}_1$, $\mathbf{q}_2$ form an orthonormal basis of $\ker f$.

2. Obviously, $\|\mathbf{q}\| = 1$. Since

$$\mathbf{q}_3 \cdot \mathbf{q}_1 = \frac{1}{9}(0+2+0-2) = 0 \text{ and } \mathbf{q}_3 \cdot \mathbf{q}_2 = \frac{1}{9}(4+0+0-4) = 0,$$

the vector $\mathbf{q}_3$ is orthogonal to both $\mathbf{q}_1$ and $\mathbf{q}_2$, hence to all of $\ker f$.

3. If we choose $\mathbf{v} = \mathbf{e}_1$, then clearly $\mathbf{e}_1$ is linearly independent of $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$. Then we get by using the Gram-Schmidt method,

$$\begin{aligned}
&\mathbf{e}_1 - (\mathbf{e}_1 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{e}_1 \cdot \mathbf{q}_2)\mathbf{q}_2 - (\mathbf{e}_1 \cdot \mathbf{q}_3)\mathbf{q}_3 \\
&\quad = (1,0,0,0) - \frac{2}{9}(2,0,-1,2) - \frac{2}{9}(2,1,0,-2) \\
&\quad = (1,0,0,0) - \frac{2}{9}(4,1,-1,0) = \frac{2}{9}(1,-2,2,0).
\end{aligned}$$

This vector is orthogonal to $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$. We get by norming $\mathbf{q}_4 = \frac{1}{3}(1,-2,2,0)$.

4. Here,

$$f(\mathbf{q}_3) = \mathbf{M}\mathbf{q}_3 = \frac{1}{3}\begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ -18 \\ 18 \\ 0 \end{pmatrix} = 9\mathbf{q}_4$$

and

$$f(\mathbf{q}_4) = \mathbf{M}\mathbf{q}_4 = \frac{1}{3}\begin{pmatrix} 0 & 5 & -4 & -2 \\ -5 & 0 & -2 & 4 \\ 4 & 2 & 0 & -4 \\ 2 & -4 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -18 \\ -9 \\ 0 \\ 18 \end{pmatrix} = -9\mathbf{q}_3.$$

Since $f(\mathbf{q}_1) = f(\mathbf{q}) = \mathbf{0}$, the matrix of f with respect to the basis $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 9 & 0 \end{pmatrix}.$$

Example 3.41 *A linear map* $f:\mathbb{R}^4\to\mathbb{R}^4$ *is with respect to the usual basis of* $\mathbb{R}^4$ *given by the matrix*

$$\mathbf{F}=\begin{pmatrix}1&0&0&-3\\2&3&0&3\\-2&-1&2&-3\\0&0&0&4\end{pmatrix}.$$

1. *Prove that the kernel* $\ker f$ *is of dimension 0.*
2. *Find the eigenvalues of* f*, and show that there are two of the eigenvectors which form an angle of* $\frac{\pi}{6}$*, another two which form an angle of* $\frac{\pi}{4}$*, and two which form an angle of* $\frac{\pi}{3}$*. We assume here that the vector space* $\mathbb{R}^4$ *has the usual scalar product.*
3. *Prove that it is possible to choose a basis of* $\mathbb{R}^4$ *from the set of eigenvectors and find the matrix of* f *with respect to this basis.*
4. *Find a regular matrix* $\mathbf{V}$ *and a diagonal matrix* $\mathbf{\Lambda}$*, such that* $\mathbf{V}^{-1}\mathbf{F}\mathbf{V}=\mathbf{\Lambda}$*.*

1. The characteristic polynomial of $\mathbf{F}$ is

$$\begin{aligned}\det(\mathbf{F}-\lambda\mathbf{I}) &= \begin{vmatrix}1-\lambda&0&0&-3\\2&3-\lambda&0&3\\-2&-1&2-\lambda&-3\\0&0&0&4-\lambda\end{vmatrix}=(4-\lambda)\begin{vmatrix}1-\lambda&0&0\\2&3-\lambda&0\\-2&-1&2-\lambda\end{vmatrix}\\ &= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).\end{aligned}$$

Since $\lambda=0$ is not a root of this polynomial, the kernel $\ker f$ has dimension 0.

2. The eigenvalues are $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$ and $\lambda_4=4$, are all simple.

For $\lambda_1=1$ we reduce

$$\begin{aligned}\mathbf{F}-\lambda_1\mathbf{I} &= \begin{pmatrix}0&0&0&3\\2&2&0&3\\-2&-1&1&-3\\0&0&0&3\end{pmatrix}\sim\begin{pmatrix}0&0&0&1\\2&2&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}\\ &\sim \begin{pmatrix}0&0&0&1\\1&0&-1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}.\end{aligned}$$

An eigenvector is $\mathbf{v}_1=(1,-1,1,0)$ and its length is $\sqrt{3}$.

For $\lambda_2=2$ we get

$$\mathbf{F}-\lambda_2\mathbf{I}=\begin{pmatrix}-1&0&0&-3\\2&1&0&3\\-2&-1&0&-3\\0&0&0&2\end{pmatrix}$$

with the obvious eigenvector $\mathbf{v}_2 = (0,0,1,0)$.

For $\lambda_3 = 3$ we get

$$\mathbf{F} - \lambda_3\mathbf{I} = \begin{pmatrix} -2 & 0 & 0 & -3 \\ 2 & 0 & 0 & 3 \\ -2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the obvious eigenvector $\mathbf{v}_3 = (0,-1,1,0)$ of length $\sqrt{2}$.

For $\lambda_4 = 4$ we reduce

$$\begin{aligned} \mathbf{F} - \lambda_4\mathbf{I} &= \begin{pmatrix} -3 & 0 & 0 & -3 \\ 2 & -1 & 0 & 3 \\ -2 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

where the eigenvector is $\mathbf{v}_4 = (1,-1,1,-1)$ of length 2.

Thus

$$\begin{aligned}
\lambda = 1, &\qquad \mathbf{q}_1 = \frac{1}{\sqrt{3}}(1,-1,1,0),\\
\lambda_2 = 2, &\qquad \mathbf{q}_2 = (0,0,1,0),\\
\lambda_3 = 3, &\qquad \mathbf{q}_3 = \frac{1}{\sqrt{2}}(0,-1,1,0),\\
\lambda_4 = 4, &\qquad \mathbf{q}_4 = \frac{1}{2}(1,-1,1,-1).
\end{aligned}$$

Then

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = \frac{1}{\sqrt{3}}, \qquad \mathbf{q}_1 \cdot \mathbf{q}_3 = \frac{1}{\sqrt{6}} \cdot (-2) = -\sqrt{\frac{2}{3}},$$

$$\mathbf{q}_1 \cdot \mathbf{q}_4 = \frac{1}{2\sqrt{3}}(1+1+1) = \frac{\sqrt{3}}{2}, \qquad \mathbf{q}_2 \cdot \mathbf{q}_3 = \frac{1}{\sqrt{2}},$$

$$\mathbf{q}_2 \cdot \mathbf{q}_4 = \frac{1}{2}, \qquad \mathbf{q}_3 \cdot \mathbf{q}_4 = \frac{1}{2\sqrt{2}} \cdot (+2) = \frac{1}{\sqrt{2}}.$$

Since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, the angle between $\mathbf{q}_1$ and $\mathbf{q}_4$ and $\frac{\pi}{6}$.

Since $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, the angle between $\mathbf{q}_3$ and $\mathbf{q}_4$, and between $\mathbf{q}_2$ and $\mathbf{q}_3$ is $\frac{\pi}{4}$.

Since $\cos\frac{\pi}{3} = \frac{1}{2}$, the angle between $\mathbf{q}_2$ and $\mathbf{q}_4$ is $\frac{\pi}{4}$.

3. The claim follows from that $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$ span all of $\mathbb{R}^r$.

   The matrix is

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

4. We still have to find $\mathbf{V}$. The columns of $\mathbf{V}$ are $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, hence

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Example 3.42 *A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix (with respect to the usual basis of $\mathbb{R}^3$):*

$$\mathbf{A} = \begin{pmatrix} 4 & -8 & 12 \\ -1 & 2 & -3 \\ -2 & 4 & -6 \end{pmatrix}.$$

1. *Find parametric descriptions of the kernel* $\ker f$ *and the range* $f(\mathbb{R}^3)$.

2. *Find all eigenvalues and corresponding eigenvector of* f.

3. *Explain why* $\mathbf{A}$ *cannot be diagonalized.*

1. We get by reduction,

$$\mathbf{A} = \begin{pmatrix} 4 & -8 & 12 \\ -1 & 2 & -3 \\ -2 & 4 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is of rank 1, so $\ker f$ has the dimension $3 - 1 = 2$. A parametric description is

$$0 = (1, -2, 3) \cdot (x, y, z) = x - 2y + 3z.$$

Putting $\mathbf{v} = (4, -1, -2)$, it follows that $\mathbf{A} = (\mathbf{v} \quad -2\mathbf{v} \;\; 3\mathbf{v})$, thus the range is

$$\begin{aligned} f(\mathbb{R}^3) &= \{x\mathbf{v} - 2y\mathbf{v} + 3z\mathbf{v} \mid x,\, y,\, z \in \mathbb{R}\} \\ &= \{(x - 2y + 3z)\mathbf{v} \mid x,\, y,\, z \in \mathbb{R}\} \\ &= \{s\mathbf{v} \mid s \in \mathbb{R}\}. \end{aligned}$$

2. The characteristic polynomial is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -8 & 12 \\ -1 & 2-\lambda & -3 \\ -2 & 4 & -6-\lambda \end{vmatrix} = -\begin{vmatrix} \lambda-4 & 8 & -12 \\ 1 & \lambda-2 & 3 \\ 2 & -4 & \lambda+6 \end{vmatrix}$$

$$= \{(\lambda-4)(\lambda-2)(\lambda+6)+48+48+24\lambda-48+12\lambda-48-8\lambda-48\}$$
$$= -\left\{\left(\lambda^2 - 6\lambda + 8\right)(\lambda + 6) + 28\lambda - 48\right\}$$
$$= -\left\{\lambda^3 - 36\lambda + 8\lambda + 48 + 28\lambda - 48\right\} = -\lambda^3,$$

hence $\lambda = 0$ is a root of algebraic multiplicity 3, and only of geometric multiplicity 2.

The kernel $\ker f$ is equal to the complete set of eigenvectors.

3. Since the algebraic and the geometric multiplicities are not equal, $\mathbf{A}$ cannot be diagonalized.

Example 3.43 *Let a be a real number. A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is assumed to satisfy*

$$f(\mathbf{v}_1) = \mathbf{v}_2, \quad f(\mathbf{v}_1 - \mathbf{v}_2) = a(\mathbf{v}_1 - \mathbf{v}_)), \quad f(\mathbf{v}_3) = \mathbf{v}_3,$$

where

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (1,0,0)$$

are vectors in $\mathbb{R}^3$.

Furthermore, given the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & -a & 0 \\ 1 & a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. *Prove that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$.*
2. *Explain why $\mathbf{B}$ is the matrix of f with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.*
3. *Find the eigenvalues of $\mathbf{B}$.*
4. *Show that $\mathbf{B}$ is similar to a diagonal matrix when $a \neq 1$, while $\mathbf{B}$ cannot be diagonalized for $a = 1$.*
5. *Find the matrix of f with respect to the usual basis of $\mathbb{R}^3$.*

1. Since

$$\det(\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0,$$

the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, hence they form a basis of $\mathbb{R}^3$.

2. We infer from

$$f(\mathbf{v}_1 - \mathbf{v}_2) = a\mathbf{v}_1 - a\mathbf{v}_2 = f(\mathbf{v}_1) - f(\mathbf{v}_2) = \mathbf{v}_2 - f(\mathbf{v}_2)$$

that

$$f(\mathbf{v}_2) = \mathbf{v}_2 - a\mathbf{v}_1 + a\mathbf{v}_2 = -a\mathbf{v}_1 + (1+1)\mathbf{v}_2.$$

The matrix of f is

$$(\mathbf{f}(\mathbf{v}_1) \;\; \mathbf{f}(\mathbf{v}_2) \;\; \mathbf{f}(\mathbf{v}_3)) = \begin{pmatrix} 0 & -a & 0 \\ 1 & a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{B}.$$

3. The characteristic polynomial is

$$\begin{aligned} \det(\mathbf{B} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & -a & 0 \\ 1 & a+1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)\begin{vmatrix} \lambda & a \\ -1 & \lambda-a-1 \end{vmatrix} \\ &= -(\lambda-1)\begin{vmatrix} \lambda & a \\ \lambda-1 & \lambda-1 \end{vmatrix} = -(\lambda-1)^2(\lambda-a), \end{aligned}$$

hence the three eigenvalues are 1, 1, a.

4. If $a \neq 1$, we get the reduction

$$\mathbf{B} - 1 \cdot \mathbf{I} = \begin{pmatrix} -1 & -a & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is of rank 1. Two linearly independent eigenvectors are e.g. $(a, -1, 0)$ and $(0, 0, 1)$.

Since $\lambda = a$ is a simple eigenvalue, there exists an eigenvector, hence $\mathbf{B}$ can be diagonalized for $a \neq 1$.

Remark 3.2 For the sake of completeness we here add the necessary reduction

$$\mathbf{B} - a\mathbf{I} = \begin{pmatrix} -a & -a & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

An eigenvector is e.g. $(1, -1, 0)$. $\Diamond$

If $a = 1$, then $\lambda = 1$ is a triple root, and

$$\mathbf{B} - 1 \cdot \mathbf{I} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The geometric multiplicity of $\lambda = 1$ for $a = 1$ is again $2 \neq 3$, so $\mathbf{B}$ cannot be diagonalized for $a = 1$.

5. The matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

is transforming the $\mathbf{v}$ coordinates to the usual coordinates, where

$$\mathbf{M}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathbf{M}^{-1}\mathbf{B}\mathbf{M} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -a & 0 \\ 1 & a+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -a & -a & 0 \\ a+2 & a+2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ a+1 & a+2 & 1 \\ -2a-1 & -2a-2 & -1 \end{pmatrix}. \end{aligned}$$

Example 3.44 *Given in* $\mathbb{R}^5$ *the four vectors*

$$\mathbf{a}_1 = (1,0,3,-2,-1), \qquad \mathbf{a}_2 = (0,1,1,-3,2),$$
$$\mathbf{a}_3 = (-1,-1,-2,-1,1), \qquad \mathbf{a}_4 = (1,-2,3,-2,-3).$$

1. *Prove that the four vectors span a three-dimensional subspace* U *of* $\mathbb{R}^5$ *and that* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ *is a basis of* U. *Find* $\mathbf{a}_4$ *as a linear combination of* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$.

2. *Let* $f : U \to U$ *be a linear map given by*

$$\begin{aligned} f(\mathbf{a}_1+\mathbf{a}_2) &= 2\mathbf{a}_3+2\mathbf{a}_4,\\ f(\mathbf{a}_2+\mathbf{a}_3) &= 2\mathbf{a}_1+2\mathbf{a}_4,\\ f(\mathbf{a}_3+\mathbf{a}_1) &= 2\mathbf{a}_2+2\mathbf{a}_4. \end{aligned}$$

   Find the matrix $\mathbf{A}$ *of* f *with respect to the basis* $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$.

3. *Prove that* $\mathbf{A}$ *is similar to a diagonal matrix.*

1. Let $\mathbf{B} = (\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \mid \mathbf{a}_4)$, all as columns. Then $\mathbf{B}$ is equivalent to

$$\mathbf{B} = \left(\begin{array}{rrr|r} 1 & 0 & -1 & 1\\ 0 & 1 & -1 & -2\\ 3 & 1 & -2 & 3\\ -2 & -3 & -1 & -2\\ -1 & 2 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & 1\\ 0 & 1 & -1 & -2\\ 0 & 1 & 1 & 0\\ 0 & -3 & -3 & 0\\ 0 & 2 & 0 & -2 \end{array}\right)$$
$$\sim \left(\begin{array}{rrr|r} 1 & 0 & -1 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right).$$

   We infer that $\text{span}(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4) = \text{span}(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$ is a three-dimensional subspace U and that

$$\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3.$$

   CHECK:

$$\begin{aligned} 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 &= (2,0,6,-4,-2) - (0,1,1,-3,2) + (-1,-1,-2,-1,1)\\ &= (2-0-1, 0-1-1, 6-1-2, -4+3-1, -2-2+1)\\ &= (1,-2,3,-2,-3)\\ &= \mathbf{a}_4. \qquad \lozenge \end{aligned}$$

2. Since f is linear, we get

$$\begin{array}{ccccccccc} f(\mathbf{a}_1) & + & f(\mathbf{a}_2) & & & = & 2\mathbf{a}_3 & + & 2\mathbf{a}_4,\\ & & f(\mathbf{a}_2) & + & f(\mathbf{a}_3) & = & 2\mathbf{a}_1 & + & 2\mathbf{a}_4,\\ f(\mathbf{a}_1) & & & + & f(\mathbf{a}_3) & = & 2\mathbf{a}_2 & + & 2\mathbf{a}_4, \end{array}$$

and

$$\begin{array}{rcrcrcrcr} f(\mathbf{a}_1) & & & - & f(\mathbf{a}_3) & = & -2\mathbf{a}_1 & + & 2\mathbf{a}_3, \\ f(\mathbf{a}_1) & - & f(\mathbf{a}_2) & & & = & -2\mathbf{a}_1 & + & 2\mathbf{a}_2, \\ & & f(\mathbf{a}_2) & - & f(\mathbf{a}_3) & = & -2\mathbf{a}_2 & + & 2\mathbf{a}_3, \end{array}$$

hence

$$\begin{aligned} f(\mathbf{a}_1) &= -\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{a}_1 + 2\mathbf{a}_3, \\ f(\mathbf{a}_2) &= \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = 3\mathbf{a}_1 - 2\mathbf{a}_2 + 2\mathbf{a}_3, \\ f(\mathbf{a}_3) &= \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 = 3\mathbf{a}_1. \end{aligned}$$

The matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}.$$

3. Then by reduction,

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

and $\mathbf{A}$ is similar to a diagonal matrix.

ALTERNATIVELY,

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & -2-\lambda & 0 \\ 2 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 0 & -2-\lambda \end{vmatrix} \\ &= -(\lambda-3)(\lambda-1)(\lambda+2).\end{aligned}$$

The characteristic polynomial has 3 simple real roots, hence $\mathbf{A}$ is similar to a diagonal matrix.

Example 3.45 *Let $f:\mathbb{R}^2\to\mathbb{R}^2$ denote the linear, which in the usual basis $(\mathbf{e}_1,\mathbf{e}_2)$ of $\mathbb{R}^2$ is given by the matrix description*

$$_e\mathbf{y}=\begin{pmatrix} 1 & -1 \\ 3 & -7 \end{pmatrix} {}_e\mathbf{x}.$$

Furthermore, let $\mathbf{b}_1=(1,1)$ and $\mathbf{b}_2=(2,1)$.

1. *Prove that $(\mathbf{b}_1,\mathbf{b}_2)$ is a basis of $\mathbb{R}^2$ and find the matrix description of f with respect to this basis.*
2. *Let $g:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be the bilinear function, which in the usual basis $(\mathbf{e}_1,\mathbf{e}_2)$ of $\mathbb{R}^2$ is given by*

$$g(\mathbf{x},\mathbf{y})={}_e\mathbf{x}^T\begin{pmatrix} 1 & -1 \\ 3 & -7 \end{pmatrix} {}_e\mathbf{y}.$$

Find the matrix of g with respect to the basis $(\mathbf{b}_1,\mathbf{b}_2)$.

1. We infer from

$$|\mathbf{b}_1\ \ \mathbf{b}_2|=\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}=-1\neq 0,$$

that $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent, hence $(\mathbf{b}_1,\mathbf{b}_2)$ is a basis of $\mathbb{R}^2$.

We have of course

$$_e\mathbf{x}={}_e\mathbf{M}_b\,{}_b\mathbf{x}=\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} {}_b\mathbf{x},$$

where

$$_b\mathbf{M}_e=\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

hence

$$_b\mathbf{y}={}_b\mathbf{M}_e\,{}_e\mathbf{y}=\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} {}_b\mathbf{x}.$$

Summing up we get

$$_b\mathbf{y}=\begin{pmatrix} -8 & -3 \\ 4 & 2 \end{pmatrix} {}_b\mathbf{x}.$$

2. Here,

$$\begin{aligned} g(\mathbf{x},\mathbf{y}) &= {}_e\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ 3 & -7 \end{pmatrix} {}_b\mathbf{x}^T \\ &= {}_b\mathbf{x}^T \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} {}_b\mathbf{y} \\ &= {}_b\mathbf{x}^T \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix} {}_b\mathbf{y} = {}_b\mathbf{x}^T \begin{pmatrix} -4 & 0 \\ -4 & 1 \end{pmatrix} {}_b\mathbf{y}. \end{aligned}$$

Example 3.46 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear map, which in the usual basis of $\mathbb{R}^3$ is given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 2 & 3 & -1 \end{pmatrix}.$$

1. *Check if $\mathbf{x} = (1,2,2)$ an eigenvector of f.*
2. *Check if $\lambda = 1$ is an eigenvalue of f.*
3. *Now, given that $\lambda = 0$ is an eigenvalue of f.*
   Find the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. *Does $\mathbf{y} = (0,3,1)$ belong to the range of f?*

1. By a mechanical insertion,

$$\mathbf{Ax} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+4-2 \\ 2+6-2 \\ 2+6-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

   We see that $\mathbf{x} = (1,2,2)$ is an eigenvector of the eigenvalue $\lambda = 3$.

2. By reduction,

$$\mathbf{A} - 1 \cdot \mathbf{I} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 2 & -1 \\ 2 & 3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   so $\lambda = 1$ is not an eigenvalue.

   (Alternatively one could here start by finding the characteristic polynomial and then show that $\lambda = 1$ is not a root. $\lozenge$)

3. We get by reduction,

$$\mathbf{A} - 0 \cdot \mathbf{I} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   The rank here is 2, hence the geometric multiplicity is $3 - 2 = 1$

4. By reduction,

$$(\mathbf{A} \mid \mathbf{y}) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & -1 & 3 \\ 2 & 3 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right).$$

The matrix of coefficients is of rank 2, and the total matrix is of rank 3, hence the equation $\mathbf{Ax} = \mathbf{y}$ does not have solutions, and $\mathbf{y}$ does not belong to the range.

Example 3.47 *A map* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is given by*

$$f(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y},$$

where $\mathbf{y} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, *and* $\langle \mathbf{x}, \mathbf{y} \rangle$ *is the usual scalar product of* $\mathbf{x}$ *and* $\mathbf{y}$ *i* $\mathbb{R}^2$.

1. *Prove that* f *is linear.*
2. *Find the matrix* ${}_e\mathbf{F}_e$ *of* f *with respect to the usual basis of* $\mathbb{R}^2$.
3. *Find a basis of* $\ker f$.
4. *Find a basis of the range* $f(\mathbb{R}^2)$.

1. The linearity is obvious,

$$\begin{aligned} f(\mathbf{x} + \lambda \mathbf{z}) &= (\mathbf{x} + \lambda \mathbf{z}) - \langle \mathbf{x} + \lambda \mathbf{z}, \mathbf{y} \rangle \mathbf{y} \\ &= (\mathbf{x} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}) + \lambda(\mathbf{z} - \langle \mathbf{z}, \mathbf{y} \rangle \mathbf{y}) = f(\mathbf{x}) + \lambda f(\mathbf{z}). \end{aligned}$$

2. It follows from

$$f(\mathbf{e}_2) = \mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{y} \rangle \mathbf{y} = (0,1) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}(-1,1),$$

that

$${}_e\mathbf{F}_e = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

3. Since rank ${}_e\mathbf{F}_e = 1$, we see that $\dim \ker f = 2 - 1 = 1$. Since

$$f(\mathbf{y} = \mathbf{y} - \langle \mathbf{y}, \mathbf{y} \rangle \mathbf{y} = \mathbf{y} - \mathbf{y} = \mathbf{0},$$

the vector $\mathbf{y}$ lies in the kernel, hence $\{\mathbf{y}\}$ is a basis of $\ker f$.

4. A basis of $f(\mathbb{R}^2)$ is $\left\{\frac{1}{\sqrt{2}}(1,-1)\right\}$.

Remark 3.3 Notice that

$$\det({}_e\mathbf{F}_e - \lambda\mathbf{I}) = \begin{vmatrix} \frac{1}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} = \left(\lambda - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \lambda(\lambda - 1),$$

thus $\lambda = 0$ and $\lambda = 1$ are the two eigenvalues.

Corresponding to the eigenvalue $\lambda = 0$ we have the eigenvector $\mathbf{y} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and corresponding to the eigenvalue $\lambda = 1$ we have the orthogonal eigenvector $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. $\Diamond$

Example 3.48 *A linear map* $f : \mathbb{R}^4 \to \mathbb{R}^4$ *is with respect to the usual basis described by the matrix*

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 5 \\ 1 & 3 & 1 & 5 \\ 4 & 0 & -2 & 3 \end{pmatrix}.$$

1. *Find the LU factorization of* $\mathbf{F}$ *and indicate* $\dim f(\mathbb{R}^4)$.
2. *Prove that the four vectors*

$$\mathbf{v}_1 = (1,2,1,4),\ \mathbf{v}_2 = (0,1,3,0),\ \mathbf{v}_3 = (0,0,1,-1),\ \mathbf{v}_4 = (0,0,0,1)$$

form a basis of $\mathbb{R}^4$.

3. *Find the matrix* ${}_v\mathbf{F}_e$ *(i.e. with respect to the usual basis in the domain and with respect to the basis* $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$*).*

1. We get by a simple Gauß reduction

$$\begin{aligned} \mathbf{F} &= \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 5 \\ 1 & 3 & 1 & 5 \\ 4 & 0 & -2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -6 & -5 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -6 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix} = \mathbf{U}. \end{aligned}$$

It follows from $\mathbf{F} = \mathbf{LU}$ that

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 5 \\ 1 & 3 & 1 & 5 \\ 4 & 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 4 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix} = \mathbf{LU}.$$

Now, $\det \mathbf{F} = \det \mathbf{U} = -30 \neq 0$, so $\dim f(\mathbb{R}^4) = 4$.

2. The columns of $\mathbf{F}$ are $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and $\det \mathbf{F} \neq 0$, hence $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly independent, hence they form a basis of $\mathbb{R}^4$.
3. The image of $\mathbf{e}_i$ is $\mathbf{v}_i$, hence the matrix is ${}_v\mathbf{F}_e = \mathbf{I}$.

Example 3.49 *A linear map f of the vector space $P_2\mathbb{R}_+)$ into $P_3(\mathbb{R}_+)$ is given by*

$$f(P(x)) = \int_0^x P(t)\,dt,$$

where $P_n(\mathbb{R}_+) = (P_n(\mathbb{R}_+), +, \mathbb{R})$ denotes the vector space of real polynomials $P_n(x)$, $x \in \mathbb{R}_+$ of degree $\leq n$.

1. *Compute $f(1+x+x^2)$.*
2. *Find ${}_m\mathbf{F}_m$ of f with respect to the monomial basis in both $P_2(\mathbb{R}_+)$ and $P_3(\mathbb{R}_+)$.*
3. *Find the kernel $\ker f$ and the dimension of the range $V = f(P_2(\mathbb{R}_+))$.*
4. *We define a linear map g of V into $P_2(\mathbb{R}_+)$ by*

$$g(Q(x)) = \frac{1}{x}\,Q(x), \qquad Q(x) \in V.$$

Find the matrix ${}_m\mathbf{H}_m$ with respect to the monomial basis of the composite map $g \circ f$ of $P_2(\mathbb{R}_+)$ into $P_2(\mathbb{R}_+)$.

5. *Find the eigenvalues and the eigenvectors of the map $g \circ f$.*

1. We get by a direct computation

$$f(1+x+x^2) = \int_0^x (1+t+t^2)\,dt = x + \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3.$$

2. The matrix is (cf. 1))

$${}_m\mathbf{F}_m = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

3. Clearly, $\ker f = \{0\}$, and

$$\dim V = \dim f(P_2(\mathbb{R}_+)) = 3.$$

4. It follows immediately from 2) that

$${}_m\mathbf{H}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

5. We infer from 4) that $\lambda_1 = 1$ is an eigenvalue corresponding to $P_1(x) = 1$, that $\lambda_2 = \dfrac{1}{2}$ is an eigenvalue corresponding to $P_2(x) = x$, and that $\lambda_3 = \dfrac{1}{3}$ is an eigenvalue corresponding to $P_3(x) = x^2$.

Example 3.50 *Let $f:\mathbb{R}^4\to\mathbb{R}^2$ be the linear map, which in the usual bases of $\mathbb{R}^4$ and $\mathbb{R}^2$ is given by the matrix*

$$\mathbf{F}=\begin{pmatrix}1&0&1&-1\\1&1&-1&1\end{pmatrix}.$$

1. *Find the kernel of f.*
2. *Consider $\mathbb{R}^4$ with the usual scalar product.*
   Find an orthonormal basis of $\ker f$.

1. We get by a reduction,

$$\mathbf{F}=\begin{pmatrix}1&0&1&-1\\1&1&-1&1\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1\\0&1&-2&2\end{pmatrix}.$$

Choosing $x_3=s$ and $x_4=t$ as parameters we get for every element of $\ker f$ that

$$x_1=-s+t \qquad \text{and} \qquad x_2=2s-2t,$$

thus

$$\begin{aligned}\mathbf{x}&=(-s+t,2s-2t,s,t)=s(-1,2,1,0)+t(1,-2,0,1)\\&=-s(1,-2,-1,0)+t(1,-2,+,1).\end{aligned}$$

By changing sign of s we get

$$\ker f=\{s(1,-2,-1,0)+t(1,-2,0,1)\mid s,\,t\in\mathbb{R}\},$$

hence $\ker f$ is spanned by the vectors $(1,-2,-1,0)$ and $(1,-2,0,1)$.

2. Since $\mathbf{v}_1=\dfrac{1}{\sqrt{6}}(1,-2,-1,0)$ is normed, and (Gram-Schmidt's method)

$$\begin{aligned}&(1,-2,0,1)-\frac{1}{6}\langle(1,-2,0,1),(1,-2,-1,0)\rangle(1,-2,-1,0)\\&\quad=(1,-2,0,1)-\frac{1}{6}(1+4)(1,-2,-1,0)\\&\quad=\frac{1}{6}(6-5,-12+10,0+5,6+0)\\&\quad=\frac{1}{6}(1,-2,5,6)\end{aligned}$$

is orthogonal to $\mathbf{v}_1$ and

$$\|(1,-2,5,6)\|=\sqrt{1+4+25+36}=\sqrt{66},$$

we have

$$\mathbf{v}_2=\frac{1}{\sqrt{66}}(1,-2,5,6).$$

An orthonormal basis of $\ker f$ is e.g. given by

$$\mathbf{v}_1=\frac{1}{\sqrt{6}}(1,-2,-1,0) \quad \text{and} \quad \mathbf{v}_2=\frac{1}{\sqrt{66}}(1,-2,5,6).$$

Example 3.51 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear map, which in the usual basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is given by the matrix*

$$_e\mathbf{F}_e = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}.$$

1. *Find the kernel* $\ker f$.
2. *Prove that* $\mathbf{u}_1 = (-4, 2, 2)$ *and* $\mathbf{u}_2 = (2, -4, 2)$ *form a basis of the range* $f(\mathbb{R}^3)$.
3. *Consider* $\mathbb{R}^3$ *with the usual scalar product.*
   Prove that any vector of the kernel of f is orthogonal to every vector in the range of f.
4. *Given a basis* $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, *where*

   $$\mathbf{b}_1 = (1, 2, 0), \quad \mathbf{b}_2 = (2, 3, 0), \quad \mathbf{b}_3 = (0, 0, 1).$$

   Find the matrices $_e\mathbf{M}_b$ and $_b\mathbf{M}_e$ of the change of coordinates.
5. *Prove that*

   $$\mathbf{B} = \begin{pmatrix} -12 & -10 & -2 \\ 6 & 4 & 2 \\ 6 & 10 & -4 \end{pmatrix}$$

   is the matrix of f with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$.

1. We get by some reductions,

   $$\begin{aligned} _e\mathbf{F}_e &= \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

   which has rank 2, hence $\dim \ker f = 3 - 2 = 1$. A generating vector is $(1, 1, 1)$, so

   $$\ker f = \{s(1, 1, 1) \mid s \in \mathbb{R}\}.$$

2. Clearly, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent and since they are columns of $_e\mathbf{F}_e$ they lie in the range. Now, the range has dimension 2, hence $\mathbf{u}_1$ and $\mathbf{u}_2$ form a basis of $f(\mathbb{R}^3)$.
3. Since $\langle(1, 1, 1), (-4, 2, 2)\rangle = 0$ and $\langle(1, 1, 1), (2, -4, 2)\rangle = 0$, any vector of $\ker f$ must be orthogonal to every vector of $f(\mathbb{R}^3)$.
4. Since

   $$_e\mathbf{M}_b = (\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we infer that

$$ {}_b\mathbf{M}_e = \left({}_e\mathbf{M}_b\right)^{-1} = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

5. Finally,

$$ \begin{aligned} \mathbf{B} &= {}_b\mathbf{M}_e \, {}_e\mathbf{F}_e \, {}_e\mathbf{M}_b \\ &= \begin{pmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 16 & -14 & -2 \\ -10 & 8 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -12 & -10 & -2 \\ 6 & 4 & 2 \\ 6 & 10 & -4 \end{pmatrix}, \end{aligned} $$

which should be proved.

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