Complex Functions Examples c-7

Applications of the Calculus of Residues

Leif Mejlbro

bookboon
The eBook company

Leif Mejlbro

Complex Functions Examples c-7

Applications of the Calculus of Residues

Complex Functions Examples c-7 – Applications of the Calculus of Residues

ISBN 978-87-7681-390-1

Contents

Introduction

This is the seventh book containing examples from the *Theory of Complex Functions*. In this volume we shall apply the calculations or residues in computing special types of trigonometric integrals, some types of improper integrals, including the computation of Cauchy's principal value of an integral, and the sum of some types of series. We shall of course assume some knowledge of the previous books and the corresponding theory.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
19th June 2014

1 Some practical formulæ in the applications of the calculation of residues

1.1 Trigonometric integrals

We have the following theorem:

Theorem 1.1 *Given a function $R(x,y)$ in two real variables in a domain of $\mathbb{R}^2$. If the formal extension, given by*

$$R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right),$$

is an analytic function in a domain $\Omega \subseteq \mathbb{C}$, which contains the unit circle $|z|=1$, then

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{z^1-1}{2iz}, \frac{z^2+1}{2z}\right)\frac{dz}{iz}.$$

In most applications, $R(\sin\theta, \cos\theta)$ is typically given as a "trigonometric rational function", on which the theorem can be applied, unless the denominator of the integrand is zero somewhere in the interval $[0, 2\pi]$.

1.2 Improper integrals in general

We shall now turn to the improper integrals over the real axis. The general result is the following extension of Cauchy's residue theorem:

Theorem 1.2 *Given an analytic function* $f : \Omega \to \mathbb{C}$ *on an open domain* Ω *which, apart from a finite number of points* $z_1, \dots, z_n$*, all satisfying* $\text{Im}\, z_j > 0$, $j = 1, \dots, n$*, contains the* closed *upper half plane, i.e.*

$$\Omega \cup \{z_1, \dots, z_n\} \supset \{z \in \mathbb{C} \mid \text{Im}\, z \geq 0\}.$$

If there exist constants $R > 0$, $c > 0$ *and* $a > 1$*, such that we have the estimate,*

$$|f(z)| \leq \frac{c}{|z|^a}, \qquad \text{when both } |z| \geq R \text{ and } \text{Im}\, z \geq 0,$$

then the improper integral of $f(x)$ *along the* X*-axis is convergent, and the value is given by the following residuum formula,*

$$\int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_{\text{Im}\, z_j > 0} \text{res}\,(f; z_j) = 2\pi i \sum_{j=1}^{n} \text{res}\,(f; z_j)\,.$$

1.3 Improper integrals, where the integrand is a rational function

We have the following important special case, where $f(z)$ is a rational function with no poles on the real axis. When this is the case, the theorem above is reduced to the following:

Theorem 1.3 *Given a rational function* $f(z) = \dfrac{P(z)}{Q(z)}$ *without poles on the real axis. If the degree of the denominator polynomial is at least 2 bigger than the degree of the numerator polynomial, then the improper integral of* $f(x)$ *along the real axis exists, and its value is given by a residuum formula,*

$$\int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_{\text{Im}\, z_j > 0} \text{res}\,(f; z_j)\,.$$

1.4 Improper integrals, where the integrand is a rational function time a trigonometric function

If the integrand is a rational function time a trigonometric function, we even obtain a better result, because the exponent of the denominator in the estimate can be chosen smaller:

Theorem 1.4 *Assume that* $f : \Omega \to \mathbb{C}$ *is an analytic function on an open domain* Ω*, which, apart from a finite number of points* $z_1, \dots, z_n$*, where all* $\text{Im}\, z_j > 0$, $j = 1, \dots, n$*, contains all of the* closed *upper half plane, i.e.*

$$\Omega \cup \{z_1, \dots, z_n\} \supset \{z \in \mathbb{C} \mid \text{Im}\, z \geq 0\}.$$

If there exist constants $R > 0$, $c > 0$ *and* $a > 0$*, such that we have the estimate*

$$|f(z)| \leq \frac{c}{|z|^a}, \qquad \text{if both } |z| \geq R \text{ and } \text{Im}\, z \geq 0,$$

then the improper integral of $f(x)\,e^{imx}$ along the X-axis exists for every $m>0$, and its value is given by the following residuum formula,

$$\int_{-\infty}^{+\infty} f(x)\,e^{imx}\,dx = 2\pi i \sum_{\text{Im } z_j>0} \text{res}\left(f(z)\,e^{imz}; z_j\right) = 2\pi i \sum_{j=1}^{n} \text{res}\left(f\,e^{imz}; z_j\right).$$

In the special case, where $f(z)$ is a rational function, we of course get a simpler result:

Theorem 1.5 *Given $f(z)=\dfrac{P(z)}{Q(z)}\cdot e^{imz}$, where $P(z)$ and $Q(z)$ are polynomials. Assume that*

1) the denominator $Q(z)$ does not have zeros on the real axis,

2) the degree of the denominator is at least 1 bigger than the degree of the numerator,

3) the constant m is a real positive *number.*

Then the corresponding improper integral along the real axis is convergent and its value is given by a residuum formula,

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\cdot e^{imx}\,dx = 2\pi i \sum_{\text{Im } z_j>0} \text{res}\left(\frac{P(z)}{Q(z)}\cdot e^{imz}; z_j\right).$$

The ungraceful assumption $m>0$ above can be repared by the following:

Theorem 1.6 *Assume that $f(z)$ is analytic in $\mathbb{C}\setminus\{z_1,\dots,z_n\}$, where none of the isolated singularities z_j lies on the real axis.*

If there exist positive constants R, a, $c>0$, such that

$$|f(z)| < \frac{c}{|z|^a}, \qquad \text{for } |z|\ge R,$$

then

$$\int_{-\infty}^{+\infty} f(x)\,e^{ixy}\,dx = \begin{cases} 2\pi i \sum_{\text{Im } z_j>0} \text{res}\left(f(z)\,e^{izy};\, z_j\right) & \text{for } y>0, \\ -2\pi i \sum_{\text{Im } z_j<0} \text{res}\left(f(z)\,e^{izy}; z_j\right) & \text{for } y<0. \end{cases}$$

In the final theorem of this section we give some formulæ for improper integrals, containing either $\cos mx$ or $\sin mx$ as a factor of the integrand. We may of course derive them from the theorem above, but it would be more helpful, if they are stated separately:

Theorem 1.7 *Given a function $f(z)$ which is analytic in an open domain Ω which – apart from a finite number of points $z_1, \dots, z_n$, where* $\mathrm{Im}\, z_j > 0$ *– contains the* closed *upper half plane* $\mathrm{Im}\, z_j > 0$. *Assume that $f(x) \in \mathbb{R}$ is real, if $x \in \mathbb{R}$ is real, and that there exist positive constants R, a, $c > 0$, such that we have the estimate,*

$$|f(z)| \le \frac{c}{|z|^a}, \qquad \text{for } \mathrm{Im}\, z \ge 0 \text{ and } |z| \ge R.$$

Then the improper integrals $\int_{-\infty}^{+\infty} f(x) \begin{array}{c}\cos(mx)\\ \sin(mx)\end{array} dx$ are convergent for every $m > 0$ with the values given by

$$\int_{-\infty}^{+\infty} f(x)\, \cos(mx)\, dx = \mathrm{Re}\left\{ 2\pi i \sum_{\mathrm{Im}\, z_j > 0} \mathrm{res}\left(f(z)\, e^{imz}; z_j\right) \right\},$$

and

$$\int_{-\infty}^{+\infty} f(x)\, \sin(mx)\, dx = \mathrm{Im}\left\{ 2\pi i \sum_{\mathrm{Im}\, z_j > 0} \mathrm{res}\left(f(z)\, e^{imz}; z_j\right) \right\},$$

respectively.

1.5 Cauchy's principal value

If the integrand has a *real* singularity $x_0 \in \mathbb{R}$, it is still possible in some cases with the right interpretation of the integral as a principal value, i.e.

$$\text{v.p.} \int_{-\infty}^{+\infty} f(x)\, dx := \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{x_0 - \varepsilon} + \int_{x_0+\varepsilon}^{+\infty} \right\} f(x)\, dx,$$

to find the value of this integral by some residuum formula.
Here *v.p. (= "valeur principal")* indicates that the integral is defined in the sense given above where one removes a symmetric interval around the singular point, and then go to the limit.

Using the definition above of the principal value of the integral we get

Theorem 1.8 *Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open domain Ω, where*

$$\Omega \supseteq \{z \in \mathbb{C} \mid \mathrm{Im}\, z \ge 0\} \setminus \{z_1, \dots, z_n\}.$$

Assume that the singularities z_j, which also lie on the real axis, all are simple *poles.*
If there exist constants $R > 0$, $c > 0$ and $a > 1$, such that we have the estimate

$$|f(z)| \le \frac{c}{|z|^a} \qquad \text{for } \mathrm{Im}\, z \ge 0 \text{ and } |z| \ge R,$$

then Cauchy's principal value v.p. $\int_{-\infty}^{+\infty} f(x)\, dx$ *exists, and its value is given by the following residuum formula,*

$$\text{v.p.} \int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum_{Im\ z_j > 0} \mathrm{res}\,(f; z_j) + \pi i \sum_{Im\ z_j = 0} \mathrm{res}\,(f; z_j).$$

This formula is easily remembered if one think of the real path of integration as “splitting” the residuum into two equal halves, of which one half is attached to the upper half plane, and the other half is attached to the lower half plane.

It is easy to extend the residuum formula for Cauchy’s principal value to the previous cases, in which we also include a trigonometric factor in the integrand.

1.6 Sum of some series

Finally, we mention a theorem with some residuum formulæ, which can be used to determine the sum of special types of series,

Theorem 1.9 *Let* $f:\Omega\to\mathbb{C}$ *be an analytic function in a domain of the type* $\Omega=\mathbb{C}\setminus\{z_1,\dots,z_n\}$, *where every* $z_j\notin\mathbb{Z}$.
If there exist constants R, $c>0$ *and* $a>1$, *such that*

$$|f(z)|\leq\frac{c}{|z|^a|}\qquad \text{for } |z|\geq R,$$

then the series $\sum_{n=-\infty}^{+\infty}f(n)$ *is convergent with the sum*

$$\sum_{n=-\infty}^{+\infty}f(n)=-\pi\sum_{j=1}^{n}\operatorname{res}\left(\cot(\pi z)\cdot f(z);z_j\right).$$

Furthermore, the alternating series $\sum_{n=-\infty}^{+\infty}(-1)^n f(n)$ *is also convergent. Its sum is given by*

$$\sum_{n=-\infty}^{+\infty}(-1)^n f(n)=-\pi\sum_{j=1}^{n}\operatorname{res}\left(\frac{f(z)}{\sin(\pi z)}\,;\,z_j\right).$$

2 Trigonometric integrals

Example 2.1 *Compute* $\int_0^{2\pi} e^{2\,\cos\theta}\,d\theta$.

Here, the auxiliary function is given $R(\xi,\eta) = e^{2\eta}$, in which ξ does not enter. The function

$$R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) = \exp\left(\frac{z^2+1}{z}\right)$$

is analytic in $\mathbb{C}\setminus\{0\}$, so

$$\int_0^{2\pi} e^{2\cos\theta}\,d\theta = \oint_{|z|=1} \exp\left(z+\frac{1}{z}\right)\frac{dz}{iz} = \frac{2\pi i}{i}\,\text{res}\left(\frac{1}{z}\exp\left(z+\frac{1}{z}\right)\,;\,0\right).$$

We note that both $z=0$ and $z=\infty$ are essential singularities, so we are forced to determine the Laurent series of the integrand in $0<|z|$. However, there is a shortcut here, because we shall only be interested in the coefficient a_{-1}. We see from

$$\frac{1}{z}\exp\left(z+\frac{1}{z}\right) = \frac{1}{z}\exp z\cdot\exp\frac{1}{z} = \frac{1}{z}\sum_{m=0}^{+\infty}\frac{1}{m!}\,z^m\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{z^n}, \qquad z\neq 0,$$

that a_{-1} is obtained by a *Cauchy multiplication* as the coefficient, which corresponds to $m=n$, thus

$$\int_0^{2\pi} e^{2\cos\theta}\,d\theta = 2\pi\sum_{n=0}^{+\infty}\frac{1}{(n!)^2},$$

which can be shown to be equal to $J_0(2i)$, where $J_0(z)$ is the zeroth Bessel function.

Example 2.2 *Compute* $\int_0^{2\pi}\dfrac{d\theta}{2+\cos\theta}$.

This integral can of course be computed in the traditional real way (change to $\tan\dfrac{\theta}{2}$, where one of course must be careful with the singularity at $\theta=\pi$). We have in fact,

$$\int_0^{2\pi}\frac{\delta\theta}{2+\cos\theta} = \int_0^{2\pi}\frac{d\theta}{3\cos^2\dfrac{\theta}{2}+\sin^2\dfrac{\theta}{2}} = 2{\cdot}2\int_0^{\frac{\pi}{2}}\frac{dt}{3\cos^2 t+\sin^2 t} = \frac{4}{3}\int_0^{+\infty}\frac{du}{1+\dfrac{1}{3}\,u^2} = \frac{4\sqrt{3}}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{\sqrt{3}}.$$

If we instead apply the Complex Function Theory, then we have the following computation

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} &= \oint_{|z|=1}\frac{1}{2+\dfrac{z^2+1}{2z}}\,\frac{dz}{iz} = \oint_{|z|=1}\frac{-2i}{z^2+4z+1}\,dz\\ &= (-2i)\cdot 2\pi i\,\text{res}\left(\frac{1}{z^2+4z+1}\,;\,-2+\sqrt{3}\right) = 4\pi\lim_{z\to-2+\sqrt{3}}\frac{1}{z+2+\sqrt{3}} = \frac{2\pi}{\sqrt{3}},\end{aligned}$$

where we have applied that z^2+4z+1 has the roots $-2\pm\sqrt{3}$, of which only $-2+\sqrt{3}$ lies inside $|z|=1$.

Example 2.3 *Prove that*

$$(a)\quad \int_0^{2\pi} \frac{\cos 2\theta}{5-3\cos\theta}\,d\theta = \frac{\pi}{18}, \qquad (b)\quad \int_0^{2\pi} \frac{\cos 3\theta}{5-3\cos\theta}\,d\theta = \frac{\pi}{54}.$$

(a) We shall use the substitution $z = e^{i\theta}$, where in particular,

$$\cos 2\theta = \frac{1}{2}\left\{e^{2i\theta}+e^{-2i\theta}\right\} = \frac{1}{2}\left\{z^2+\frac{1}{z^2}\right\}.$$

Then

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta}{5-3\cos\theta}\,d\theta &= \oint_{|z|=1} \frac{\frac{1}{2}\left\{z^2+\frac{1}{z^2}\right\}}{5-\frac{3}{2}\left\{z+\frac{1}{z}\right\}}\,\frac{dz}{iz} = \frac{1}{i}\oint_{|z|=1} \frac{z^2+\frac{1}{z^2}}{-3z^2+10z-3}\,dz \\
&= -\frac{1}{3i}\oint_{|z|=1} \frac{z^4+1}{z^2\left\{z^2-\frac{10}{3}\,z+1\right\}}\,dz = -\frac{1}{3i}\oint_{|z|=1} \frac{z^4+1}{z^2\left(z-\frac{1}{3}\right)(z-3)}\,dz \\
&= \frac{2\pi i}{-3i}\left\{\operatorname{res}\left(\frac{z^4+1}{z^2\left(z^2-\frac{10}{3}\,z+1\right)}\,;0\right)+\operatorname{res}\left(\frac{z^4+1}{z^2\left(z-\frac{1}{3}\right)(z-3)}\,;\frac{1}{3}\right)\right\}.
\end{aligned}$$

We obtain by RULE I,

$$\operatorname{res}\left(\frac{z^4+1}{z^2\left(z-\frac{1}{3}\right)(z-3)}\,;\frac{1}{3}\right) = \frac{\frac{1}{3^4}+1}{\frac{1}{3^2}\left(\frac{1}{3}-3\right)} = \frac{82}{3\cdot(-8)} = -\frac{41}{12},$$

and

$$\begin{aligned}
\operatorname{res}\left(\frac{z^4+1}{z^2\left(z^2-\frac{10}{3}\,z+1\right)}\,;0\right) &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left\{\frac{z^4+1}{z^2-\frac{10}{3}\,z+1}\right\} \\
&= \lim_{z\to 0}\left\{\frac{4z^3}{z^2-\frac{10}{3}\,z+1} - \frac{\left(z^4+1\right)\left(2z-\frac{10}{3}\right)}{\left(z^2-\frac{10}{3}\,z+1\right)^2}\right\} = \frac{10}{3}.
\end{aligned}$$

Finally, by insertion,

$$\int_0^{2\pi} \frac{\cos 2\theta}{5-3\cos\theta}\,d\theta = \frac{2\pi i}{-3i}\left\{-\frac{41}{12}+\frac{10}{3}\right\} = -\frac{2\pi}{3}\left\{\frac{-41+40}{12}\right\} = \frac{\pi}{18}.$$

(b) For the substitution $z = e^{i\theta}$, where we see that in particular,

$$\cos 3\theta = \frac{1}{2}\left\{e^{3i\theta}+e^{-3i\theta}\right\} = \frac{1}{2}\left\{z^3+\frac{1}{z^3}\right\},$$

we get

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-3\cos\theta}\,d\theta = \oint_{|z|=1} \frac{\frac{1}{2}\left\{z^3+\dfrac{1}{z^3}\right\}}{5-\frac{3}{2}\left\{z+\frac{1}{z}\right\}}\,\frac{dz}{iz} = -\frac{\frac{1}{2}}{\frac{3i}{2}}\oint_{|z|=1} \frac{z^3+\dfrac{1}{z^3}}{z^2-\frac{10}{3}\,z+1}\,dz$$

$$= \quad -\frac{2\pi}{3}\left\{\operatorname{res}\left(\frac{z^6+1}{z^2-\frac{10}{3}\,z+1}\cdot\frac{1}{z^3}\,;\,0\right)+\operatorname{res}\left(\frac{z^6+1}{z^3\left(z-\frac{1}{3}\right)(z-3)}\,;\,\frac{1}{3}\right)\right\}.$$

Here,

$$\operatorname{res}\left(\frac{z^3+\dfrac{1}{z^3}}{\left(z-\frac{1}{3}\right)(z-3)}\,;\,\frac{1}{3}\right) = \frac{\dfrac{1}{3^3}+3^3}{\dfrac{1}{3}-3} = -\frac{730}{9\cdot 8} = \frac{365}{36},$$

and

$$\begin{aligned}
\operatorname{res}\left(\frac{z^3+\dfrac{1}{z^3}}{z^2-\frac{10}{3}\,z+1}\,;\,0\right) &= \operatorname{res}\left(\frac{\dfrac{1}{z^3}}{z^2-\frac{10}{3}\,z+1}\,;\,0\right)\\
&= \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{1}{z^2-\frac{10}{3}\,z+1}\right\} = \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{-\frac{2z-\frac{10}{3}}{\left(z^2-\frac{10}{3}\,z+1\right)^2}\right\}\\
&= \frac{1}{2}\lim_{z\to 0}\left\{-\frac{2}{\left(z^2-\frac{10}{3}\,z+1\right)^2}+\frac{2\left(2z-\frac{10}{3}\right)^2}{\left(z^2-\frac{10}{3}\,z+1\right)^2}\right\} = -1+\frac{100}{9} = \frac{91}{9},
\end{aligned}$$

hence by insertion,

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-3\,\cos\theta}\,d\theta = -\frac{2\pi}{3}\left\{-\frac{365}{36}+\frac{91}{9}\right\} = \frac{2\pi}{3}\cdot\frac{364-364}{36} = \frac{\pi}{3\cdot 18} = \frac{\pi}{54}.$$

Example 2.4 *Prove that*

$$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\,\cos\theta} = \frac{2\pi}{1-a^2}, \qquad \textit{for } 0<a<1.$$

Find also the value, when $a>1$.

We get by the substitution $z=e^{i\theta}$ that

$$d\theta = \frac{dz}{iz} \qquad \text{and} \qquad \cos\theta = \frac{1}{2}\left\{z+\frac{1}{z}\right\},$$

thus

$$\begin{aligned}
\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\,\cos\theta} &= \oint_{|z|=1}\frac{1}{1+a^2-\left(z+\dfrac{1}{z}\right)a}\,\frac{dz}{iz} = -\frac{1}{i}\oint_{|z|=1}\frac{dz}{az^2-(1+a^2)\,z+a}\\
&= \frac{i}{a}\oint_{|z|=1}\frac{dz}{z^2-\left(a+\dfrac{1}{a}\right)z+1}.
\end{aligned}$$

The integrand has the poles $z=a$ and $z=\dfrac{1}{a}$. Of these, only $z=a$ lies inside the circle $|z|=1$, hence

$$\begin{aligned}
\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\,\cos\theta} &= \frac{i}{a}\cdot 2\pi i\cdot\operatorname{res}\left(\frac{1}{z^2-\left(a+\dfrac{1}{a}\right)z+1}\,;\,a\right)\\
&= -\frac{2\pi}{a}\lim_{z\to 1}\frac{1}{z-\dfrac{1}{a}} = -\frac{2\pi}{a}\cdot\frac{1}{a-\dfrac{1}{a}} = -\frac{2\pi}{a^2-1} = \frac{2\pi}{1-a^2}.
\end{aligned}$$

If $a > 1$, then $0 < \dfrac{1}{a} < 1$, and it follows from the above that

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\,\cos\theta} = \frac{1}{a^2}\int_0^{2\pi} \frac{d\theta}{1+\left(\dfrac{1}{a}\right)^2 - 2\cdot\dfrac{1}{a}\,\cos\theta} = \frac{1}{a^2}\cdot\frac{2\pi}{1-\dfrac{1}{a^2}} = \frac{2\pi}{a^2-1}.$$

Remark 2.1 We note that the case $a < 0$ gives the same values, only dependent on if $|a| < 1$ or $|a| > 1$. Finally, the case $a = 0$ is trivial. $\diamondsuit$

Summing up,

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\,\cos\theta} = \frac{2\pi}{|1-a^2|}, \qquad \text{for } a \in \mathbb{R}\setminus\{-1,1\}.$$

The integral is divergent, if $a = \pm 1$.

Example 2.5 *Prove that if $a > 1$, then*

$$\int_0^{2<pi} \frac{dt}{a+\sin t} = \frac{2\pi}{\sqrt{a^2-1}}.$$

It follows from

$$\int_0^{2\pi} R(\sin\theta,\cos\theta)\,d\theta = \oint_{|z|=1} R\left(\frac{z^2-1}{2iz},\,\frac{z^2+1}{2z}\right)\frac{dz}{iz},$$

that

$$\int_0^{2\pi}\frac{dt}{a+\sin t} = \oint_{|z|=1}\frac{1}{a+\dfrac{z^2-1}{2iz}}\,\frac{dz}{iz} = 2\oint_{|z|=1}\frac{1}{z^2+2i\,a\,z-1}\,dz.$$

The function $\dfrac{1}{z^2+2i\,a\,z-1}$ has the two simple poles

$$z = -i\,a \pm \sqrt{-a^2-1}.$$

Of these only $z = i\left\{\sqrt{a^2-1}-1\right\}$ lies inside the unit circle. Hence

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{a+\sin t} &= 2\cdot 2\pi i\cdot \operatorname{res}\left(\frac{1}{z^2+2i\,a\,z-1}\,;\,i\left\{\sqrt{a^2-1}-1\right\}\right)\\ &= 2\cdot 2\pi i \lim_{z\to i\left\{\sqrt{a^2-1}-a\right\}}\frac{1}{z+i\sqrt{a^2-1}+i\,a} = 2\cdot 2\pi i\cdot\frac{1}{2i\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.\end{aligned}$$

Example 2.6 *Compute*

$$\int_0^{2\pi} \cos(2\cos\theta)\,d\theta,$$

expressed as a sum $\sum_{n=0}^{+\infty} a_n$.

Applying the substitution $z = e^{i\theta}$ we get

$$\begin{aligned}\int_0^{2\pi} \cos(2\cos\theta)\,d\theta &= \oint_{|z|=1} \cos\left(z+\frac{1}{z}\right)\frac{dz}{iz} = \frac{2\pi i}{i}\,\text{res}\left(\frac{1}{z}\cos\left(z+\frac{1}{z}\right)\,;\,0\right)\\ &= 2\pi\cdot\text{res}\left(\frac{1}{z}\cos\left(z+\frac{1}{z}\right)\,;\,0\right).\end{aligned}$$

It follows from

$$\begin{aligned}&\frac{1}{z}\cos\left(z+\frac{1}{z}\right) = \frac{1}{z}\cdot\frac{1}{2}\left\{\exp\left(i\left\{z+\frac{1}{z}\right\}\right)+\exp\left(-i\left\{z+\frac{1}{z}\right\}\right)\right\}\\ &\quad= \frac{1}{z}\cdot\frac{1}{2}\left\{\exp(i\,z)\cdot\exp\left(\frac{i}{z}\right)+\exp(-i\,z)\cdot\exp\left(-\frac{i}{z}\right)\right\}\\ &\quad= \frac{1}{z}\cdot\frac{1}{2}\left\{\sum_{m=0}^{+\infty}\frac{1}{m!}\,i^m z^m\cdot\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{i^n}{z^n}+\sum_{m=0}^{+\infty}\frac{1}{m!}\,(-i)^m z^m\cdot\sum_{n=0}^{+\infty}\frac{1}{n!}\,(-i)^n\cdot\frac{1}{z^n}\right\},\end{aligned}$$

that the coefficient a_{-1} in the Laurent series expansion for

$$\frac{1}{z}\cos\left(z+\frac{1}{z}\right)$$

is determined by $m = n$, i.e.

$$a_{-1} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{i^n}{n!}\cdot\frac{i^n}{n!}+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-i)^n}{n!}\cdot\frac{(-i)^n}{n!} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2},$$

which can be shown to be equal to $J_0(2)$, where $J_0(z)$ is the zeroth Bessel function. Hence we conclude that at

$$\int_0^{2\pi}\cos(2\cos\theta)\,d\theta = 2\pi\,\text{res}\left(\frac{1}{z}\cos\left(z+\frac{1}{z}\right)\,;\,0\right) = 2\pi a_{-1} = 2\pi\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2} = 2\pi\,J_0(2).$$

Example 2.7 **(a)** *Determine the Taylor series from $z = 0$ of*

$$\frac{1}{z^2 - \left(a + \frac{1}{a}\right) z + 1}, \qquad \textit{where } 0 < a < 1,$$

in the form $\sum_{p=0}^{+\infty} a_p z^p$.
Find the radius of convergence r of the series.

(b) *Find the Laurent series of*

$$f(z) = \frac{z^n + z^{-n}}{z^n - \left(a + \frac{1}{a}\right) z + 1}, \qquad n \in \mathbb{N}, \quad 0 < a < 1,$$

in the domain given by $0 < |z| < r$, by using the result of **(a)**, *and then find the residuum of f at the point $z = 0$.*

(c) *Compute*

$$\int_0^{2\pi} \frac{\cos(n\, v)}{1 + a^2 - 2a\, \cos v}\, dv, \qquad n \in \mathbb{N}, \quad 0 < a < 1,$$

by transforming the integral into a line integral in the complex plane.

(a) First note that we have the factor expansion

$$z^2 - \left(a + \frac{1}{a}\right) z + 1 = (z - a)\left(z - \frac{1}{a}\right).$$

If $|z| < a \quad \left(< \frac{1}{a}\right)$, it follows by a decomposition and an application of the geometric series,

$$\begin{aligned}
\frac{1}{z^2 - \left(a + \frac{1}{a}\right) z + 1} &= \frac{1}{(z-a)\left(z - \frac{1}{a}\right)} = \frac{1}{a - \frac{1}{a}} \cdot \frac{1}{z - a} + \frac{1}{\frac{1}{a} - a} \cdot \frac{1}{z - \frac{1}{a}} \\
&= \frac{1}{a - \frac{1}{a}} \cdot \frac{-1}{a} \cdot \frac{1}{1 - \frac{z}{a}} + \frac{1}{\frac{1}{a} - a} \cdot \frac{1}{\left(-\frac{1}{a}\right)} \cdot \frac{1}{1 - az} \\
&= -\frac{1}{a^2 - 1} \sum_{p=0}^{+\infty} \left(\frac{z}{a}\right)^p - \frac{a^2}{1 - a^2} \sum_{p=0}^{+\infty} a^p z^p = \frac{1}{1 - a^2} \left\{ \sum_{p=0}^{+\infty} \frac{1}{a^p}\, z^p - \sum_{p=0}^{+\infty} a^{p+2} z^p \right\} \\
&= \sum_{p=0}^{+\infty} \frac{1}{a^p} \cdot \frac{1 - a^{2p+2}}{1 - a^2}\, z^p.
\end{aligned}$$

The radius of convergence is of course $r = a$, which e.g. follows from the fact that $z = a$ is the pole, which is closest to 0. We may also easily obtain this result by the criterion of roots.

(b) If $0 < |z| < a$ and $n \in \mathbb{N}$, then it follows from **(a)** that

$$\begin{aligned} f(z) &= \frac{z^n + z^{-n}}{z^2 - \left(a + \dfrac{1}{a}\right) z + 1} = \left(z^n + z^{-n}\right) \sum_{p=0}^{+\infty} \frac{1}{a^p} \cdot \frac{1 - a^{2p+2}}{1 - a^2}\, z^p \\ &= \sum_{p=0}^{+\infty} \frac{1}{a^p} \cdot \frac{1 - a^{2p+2}}{1 - a^2}\, z^{p+n} + \sum_{p=0}^{+\infty} \frac{1}{a^p} \cdot \frac{1 - a^{2p+2}}{1 - a^2}\, z^{p-n}, \end{aligned}$$

which we may reduce to the Laurent series

$$f(z) = \sum_{p=-n}^{n+1} \frac{1}{a^{p+n}} \cdot \frac{1 - a^{2p+2+2n}}{1 - a^2}\, z^p + \sum_{p=n}^{+\infty} \frac{1}{a^{p+n}} \cdot \frac{\left(a^{2n} + 1\right)\left(1 - a^{2p+2}\right)}{1 - a^2}\, z^p,$$

although this result is not much nicer.

We know that the residuum is given by $p = -1$, so

$$\operatorname{res}(f; 0) = a_{-1} = \frac{1}{a^{n-1}} \cdot \frac{1 - a^{2n}}{1 - a^2}.$$

(c) If we put $z = e^{iv}$, then

$$\cos nv = \frac{1}{2}\left\{e^{inv} + e^{-inv}\right\} \qquad \text{and} \qquad dv = \frac{dz}{iz}.$$

Then we get by insertion, reduction and an application of the residuum theorem (with the two poles $z = 0$ and $z = a$ inside the unit circle $|z| = 1$),

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos(nv)}{1+a^2-2a\,\cos v}\,dv &= \frac{1}{2}\oint_{|z|=1} \frac{z^n+z^{-n}}{1+a^2-a\left(z+z^{-1}\right)}\cdot\frac{dz}{iz} \\
&= \frac{1}{2i}\oint_{|z|=1}\frac{1}{-a}\cdot\frac{z^n+z^{-1}}{z^2-\left(a+\dfrac{1}{a}\right)z+1}\,dz = -\frac{1}{2ia}\cdot 2\pi i\left\{\operatorname{res}(f;0)+\operatorname{res}(f;a)\right\} \\
&= -\frac{\pi}{a}\left\{\frac{1}{a^{n-1}}\cdot\frac{1-a^{2n}}{1-a^2}+\lim_{z\to a}\frac{z^n+z^{-n}}{z-\dfrac{1}{a}}\right\} = -\pi\left\{\frac{1}{a^n}\cdot\frac{1-a^{2n}}{1-a^2}+\frac{1}{a}\cdot\frac{a^n+a^{-n}}{a-\dfrac{1}{a}}\right\} \\
&= -\pi\left\{\frac{1}{a^n}\cdot\frac{1-a^{2n}}{1-a^2}-\frac{1}{a^n}\cdot\frac{a^{2n}+1}{1-a^2}\right\} = -\frac{\pi}{a^n}\cdot\frac{1-a^{2n}-a^{2n}-1}{1-a^2} = 2\pi\cdot\frac{a^n}{1-a^2}.
\end{aligned}$$

Example 2.8 *Given the function*

$$f(z) = \frac{1 - e^{i2z}}{z^2}.$$

(1) *Prove that f has a simple pole at $z = 0$, so we have for $z \neq 0$,*

$$f(z) = \frac{b_1}{z} + g(z),$$

where g is an entire function, i.e. analytic in $\mathbb{C}$.
Find the residuum b_1.

Consider for $r > 0$ the half circle γ_r, given by the parametric description

$$\gamma_r(t) = r\,e^{it}, \qquad 0 \leq t \leq \pi.$$

(2) *Prove that*

$$\int_{\gamma_r} f(z)\,dz \to b_1 \pi\, i \qquad \textit{for } r \to 0.$$

(3) *Prove that*

$$\int_{\gamma_r} f(z)\,dz \to 0 \qquad \textit{for } r \to +\infty.$$

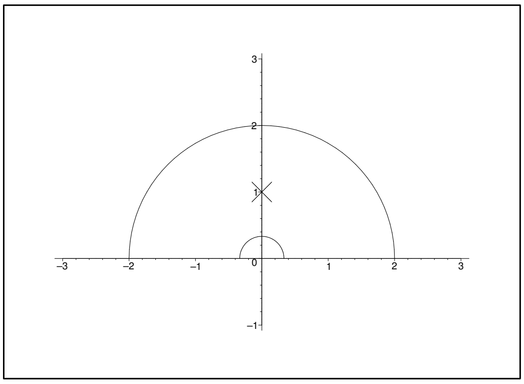


Figure 1: The curve $\Gamma_{\varepsilon,R}$.

Let $\Gamma_{\varepsilon,R} = I + II + III + IV$ denote the simple, closed curve on the figure, where $II = \gamma_R$ and $IV = -\gamma_\varepsilon$.

(4) *Compute*

$$\oint_{\Gamma_{\varepsilon,R}} f(z)\,dz,$$

and prove the formula

$$\int_0^{+\infty} \left\{ \frac{\sin x}{x} \right\}^2 dx = \frac{\pi}{2}.$$

1) It follows from the Laurent series expansion

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left\{1-e^{i2z}\right\} = \frac{1}{z^2}\left\{1-\sum_{n=0}^{+\infty}\frac{i^n}{n!}\cdot 2^n\, z^n\right\} \\ &= \frac{1}{z^2}\left\{1-1-2iz+\sum_{n=0}^{+\infty}\frac{i^n}{(n+2)!}\cdot 2^{n+2}\, z^{n+2}\right\} \\ &= -\frac{2i}{z}+4\sum_{n=0}^{+\infty}\frac{i^n}{(n+2)!}\cdot 2^n\cdot z^n, \qquad |z|>0, \end{aligned}$$

that f has a simple pole at 0 and that

$$g(z) = 4\sum_{n=0}^{+\infty}\frac{i^n}{(n+2)!}\cdot 2^n z^n, \qquad z\in\mathbb{C},$$

is an entire function, and that

$$\operatorname{res}(f;0) = -2i = b_1.$$

2) When we use the parametric description of the half circle, we get

$$\begin{aligned} \int_{\gamma_r} f(z)\,dz &= \int_{\gamma_r}\left\{-\frac{2i}{z}+g(z)\right\}dz = \int_0^{\pi}\left\{-\frac{2i}{r\,e^{it}}\cdot i\,r\,e^{it}\right\}dt + \int_{\gamma_r} g(z)\,dz \\ &= 2\int_0^{\pi} dt + \int_{\gamma_r} g(z)\,dz = 2\pi + \int_{\gamma_r} g(z)\,dz, \end{aligned}$$

where $2\pi = -2i\cdot i\pi = b_1\cdot i\pi$.

In particular, $g(z)$ is continuous, so $|g(z)|\le c$ for $|z|\le 1$. Therefore, if $0<r<1$, then we get the estimate

$$\left|\int_{\gamma_r} g(z)\,dz\right| \le c\cdot\pi\,r\to 0 \qquad \text{for } r\to 0+\,.$$

It follows that

$$\lim_{r\to 0+}\int_{\gamma_r} f(z)\,dz = b_1\pi i = 2\pi.$$

3) Assume that $r>0$ is large. It follows from

$$e^{i2z} = \exp(2ir\cdot(cost+i\,\sin t)) = \exp(-2r\,\sin t)\,\exp(2ir\,\cos t),$$

and $r>0$ and $0<t<\pi$ that $-2r\,\sin t<0$, hence

$$\left|e^{i2z}\right| \le \text{for } z\in\Gamma_r.$$

This implies the estimate

$$\left|\int_{\gamma_r} f(z)\,dz\right| \le \left|\int_0^{\pi}\frac{1+1}{r^2}\cdot r\,dt\right| = \frac{2\pi}{r}\to 0 \qquad \text{for } r\to+\infty.$$

4) Now, $f(z)$ is analytic everywhere inside $\Gamma_{\varepsilon,R}$, so it follows from Cauchy's integral theorem that

$$0 = \int_{\Gamma_{\varepsilon,R}} f(z)\,dz = -\int_{\gamma_\varepsilon} f(z)\,dz + \int_\varepsilon^R \frac{1-e^{2ix}}{x^2}\,dx + \int_{\gamma_R} f(z)\,dz + \int_{-R}^{-\varepsilon} \frac{1-e^{2ix}}{x^2}\,dx.$$

Since cosine is an even function and sine is an odd function, it follows by the symmetry that

$$\begin{aligned}
&\int_\varepsilon^R \frac{1-e^{2ix}}{x^2}\,dx + \int_{-R}^{-\varepsilon} \frac{1-e^{2ix}}{x^2}\,dx \\
&= \int_\varepsilon^R \frac{1-\cos 2x}{x^2}\,dx - \int_\varepsilon^R \frac{\sin 2x}{x^2}\,dx + \int_{-R}^{-\varepsilon} \frac{1-\cos 2x}{x^2}\,dx - \int_{-R}^{-\varepsilon} \frac{\sin 2x}{x^2}\,dx \\
&= 2\int_\varepsilon^R \frac{1-\cos 2x}{x^2}\,dx = 2\int_\varepsilon^R \left\{\frac{\sin x}{x}\right\}^2 dx.
\end{aligned}$$

Then by insertion and taking the limits $\varepsilon \to 0+$ and $R \to +\infty$,

$$\begin{aligned}
0 &= -\int_{\gamma_\varepsilon} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + 2\int_\varepsilon^R \left\{\frac{\sin x}{x}\right\}^2 dx \\
&\to -\pi + 0 + 2\int_0^{+\infty} \left\{\frac{\sin x}{x}\right\}^2 dx.
\end{aligned}$$

This limit is of course also equal to 0, so by a rearrangement,

$$\int_0^{+\infty} \left\{\frac{\sin x}{x}\right\}^2 dx = \frac{\pi}{2}.$$

3 Improper integrals in general

Example 3.1 *Compute the improper integrals*

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \exp\left(\frac{x}{x^2+1}\right) \cos\left(\frac{1}{x^2+1}\right) dx, \quad \textit{and} \quad \int_{-\infty}^{+\infty} \frac{1}{x^2+1} \exp\left(\frac{x}{x^2+1}\right) \sin\left(\frac{1}{x^2+1}\right) dx.$$

When we split into the real and the imaginary part, we get

$$\frac{1}{x-i} = \frac{x}{x^2+1} + i\,\frac{1}{x^2+1},$$

so it is quite natural to consider the analytic function

$$f(z) = \frac{1}{z^2+1} \exp\left(\frac{1}{z-i}\right), \qquad \text{for } z \in \mathbb{C} \setminus \{-i, i\}.$$

Since $\dfrac{1}{z-i} \to 0$ for $z \to \infty$, there clearly exists an $R > 1$, such that we have the estimate

$$|f(z)| \le \frac{2}{|z|^2} \qquad \text{for } |z| \ge R.$$

Then the assumptions of an application of the residuum formula are satisfies, so we conclude by the linear transform $w = z - i$ that

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{1}{x^2+1} \exp\left(\frac{x+i}{x^2+1}\right) dx &= 2\pi i \cdot \operatorname{res}\left(\frac{1}{z^2+1} \exp\left(\frac{1}{z-i}\right) ;\, i\right) \\ &= 2\pi i \cdot \operatorname{res}\left(\frac{1}{w^2+2iw} \exp\left(\frac{1}{w}\right) ;\, 0\right). \end{aligned}$$

Now, $w_0 = 0$ is an essential singularity of the function

$$\frac{1}{w^2+2iw} \exp\left(\frac{1}{w}\right),$$

so in order to find the residuum we shall expand into a Laurent series from $w_0 = 0$, then perform a Cauchy multiplication and finally determine a_{-1} by collecting all the coefficients of $\dfrac{1}{w}$. When $0 < |w| < 2$, we get

$$\frac{1}{w} \cdot \frac{1}{2i+w} \cdot \exp\frac{1}{w} = \frac{1}{2i} \cdot \frac{1}{w} \cdot \frac{1}{1+\dfrac{w}{2i}} \cdot \exp\frac{1}{w} = \frac{1}{2i} \cdot \frac{1}{w} \sum_{m=0}^{+\infty} \left\{\frac{-w}{2i}\right\}^m \sum_{n=0}^{+\infty} \frac{1}{n!}\,\frac{1}{w^n}.$$

Since we have separated the factor $\dfrac{1}{w}$, it follows that a_{-1} is equal to the constant term in the product of the two series, i.e. $m = n$. Thus

$$a_{-1} = \frac{1}{2i} \sum_{n=0}^{+\infty} \frac{1}{n!} \left\{\frac{i}{2}\right\}^n = \frac{1}{2i} \exp\left(\frac{i}{2}\right) = \frac{1}{2i}\left\{\cos\frac{1}{2} + i\,\sin\frac{1}{2}\right\},$$

and we conclude that

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\exp\left(\frac{x}{x^2+1}\right)\left\{\cos\left(\frac{1}{x^2+1}\right)+i\,\sin\left(\frac{1}{x^2+1}\right)\right\}dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,\exp\left(\frac{x+i}{x^2+1}\right)dx = 2\pi i a_{-1} = \pi\left\{\cos\frac{1}{2}+i\,\sin\frac{1}{2}\right\}.$$

When we separate the real and the imaginary parts, we get

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,\exp\left(\frac{x}{x^2+1}\right)\cos\left(\frac{1}{x^2+1}\right)dx = \pi\cdot\cos\frac{1}{2},$$

and

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,\exp\left(\frac{x}{x^2+1}\right)\sin\left(\frac{1}{x^2+1}\right)dx = \pi\cdot\sin\frac{1}{2}.$$

ALTERNATIVELY we may use that the function

$$g(w) = \frac{1}{w^2+2iw}\,\exp\left(\frac{1}{w}\right) = \frac{1}{w(w+2i)}\,\exp\left(\frac{1}{w}\right)$$

is analytic in $\mathbb{C}\setminus\{0,-2i\}$, so if we include the residuum at ∞, then the sum of the residues is zero. Hence

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,\exp\left(\frac{x+i}{x^2+1}\right)dx = 2\pi i\cdot\operatorname{res}(g(w)\,;\,0) = -2\pi i\{\operatorname{res}(g(w)\,;\,-2i)+\ \operatorname{res}(g(w)\,;\,\infty)\}.$$

Here, $-2i$ is a *simple pole*, so by Rule Ia,

$$-\ \operatorname{res}(g(w)\,;\,-2i) = -\lim_{z\to -2i}\frac{1}{w}\cdot\exp\left(\frac{1}{w}\right) = \frac{1}{2i}\,\exp\left(\frac{i}{2}\right).$$

Furthermore, $\lim_{w\to\infty}\exp\left(\frac{1}{w}\right) = \exp 0 = 1$, so $w=\infty$ is a *zero of order 2* of

$$g(w) = \frac{1}{w^2}\cdot\frac{1}{1+\dfrac{2i}{w}}\cdot\exp\left(\frac{1}{w}\right),$$

and it follows from Rule IV that

$$\operatorname{res}(g(w)\,;\,\infty) = 0.$$

Then by insertion,

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\cdot\exp\left(\frac{x+i}{x^2+1}\right)dx = 2\pi i\cdot\operatorname{res}(g(w)\,;\,,\infty) = \frac{2\pi i}{2i}\cdot\exp\left(\frac{i}{2}\right) = \pi\left\{\cos\frac{1}{2}+i\,\sin\frac{1}{2}\right\},$$

and the results follow as above by separating the real and the imaginary parts.

Example 3.2 *Assume that $x > 0$. Find the limit value*

$$\lim_{A \to +\infty} \int_{-A}^{A} \left(\frac{1}{t + ix} - \frac{1}{t - ix} \right) dt.$$

Here we get without using Complex Function Theory,

$$\lim_{A \to +\infty} \int_{-A}^{A} \left(\frac{1}{t + ix} - \frac{1}{t - ix} \right) dt = \lim_{A \to +\infty} \int_{-A}^{A} \frac{t - ix - t - ix}{t^2 + x^2}\, dt$$

$$= \lim_{A \to +\infty} \left\{ -2ix \int_{-A}^{A} \frac{dt}{t^2 + x^2} \right\} = -2i \lim_{A \to +\infty} \left[\text{Arctan} \left(\frac{t}{x} \right) \right]_{-A}^{A} = -2i\pi.$$

Example 3.3 *Let $a \in \mathbb{R}$ be a constant. Prove that the integral*

$$I(a) = \int_{-\infty}^{+\infty} e^{-(x+ia)^2} dx$$

is independent of a.
HINT: *We may assume that $a \in \mathbb{R}_+$. Denote by C the rectangle of the corners $-b$, b, $b+ia$ and $-b+ia$. Show that*

$$\oint_C \exp\left(-z^2\right)\, dz = 0.$$

Then prove that

$$\left|\int_0^a e^{-(b+iy)^2} dy\right| \le e^{-b^2} \int_0^a e^{y^2}\, dy.$$

By letting $b \to +\infty$, prove that $I(a) = I(0)$.

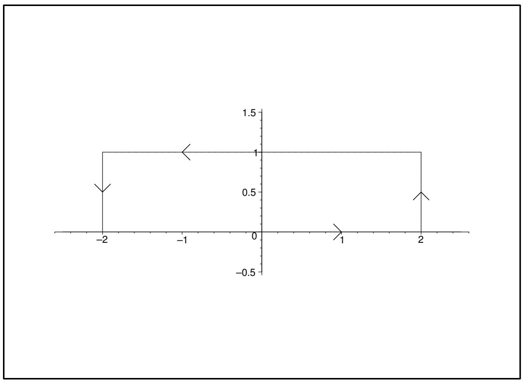


Figure 2: Example of one of the curves C. Here, $a = 1$ and $b = 2$.

Clearly, we may assume that $a > 0$, because we otherwise might consider an analogous curve in the lower half plane.
Now $\exp\left(-z^2\right)$ is analytic in $\mathbb{C}$, so

$$\oint_C \exp\left(-z^2\right)\, dz = 0.$$

We estimate the line integrals along the vertical lines by

$$\left|\int_0^a e^{-(\pm b+iy)^2} dy\right| = \left|\int_0^a e^{-b^2 \mp 2ibu + y^2} dy\right| \le e^{-b^2} \int_0^a e^{y^2} dy \to 0 \qquad \text{for } b \to +\infty.$$

Since

$$\oint_C e^{-z^2} dz = \int_{-b}^{b} e^{-x^2} dx + i \int_0^{-(b+iy)^2} dy - \int_{-b}^{b} e^{-(x+ia)^2} dx - i \int_0^a e^{-(-b+iy)^2} dy = 0,$$

and since $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ is an improper convergent integral, it follows by taking the limit $b \to +\infty$ that

$$I(a) = \int_{-\infty}^{+\infty} e^{-(x+ia)^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx = I(0).$$

Note that we also have

$$I(-a) = \overline{I(a)} = \overline{I(0)} = I(0).$$

Example 3.4 *Compute*

$$I(0) = \int_{+\infty}^{+\infty} e^{-x^2}\,dx.$$

HINT: *Use that*

$$\{I(0)\}^2 = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \int_{-\infty}^{+\infty} e^{-y^2}\,dy = \int\!\!\int_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)}\,dx\,dy.$$

Then use polar coordinates.

Since $e^{-\left(x^2+y^2\right)} > 0$ for every $(x,y) \in \mathbb{R}^2$, and since the function is continuous, all the transforms below are legal, if only the improper plane integral exists. (The only thing which may go wrong is that the value could be $+\infty$). Hence,

$$\begin{aligned} I(0)^2 &= \int_{-\infty}^{+\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{+\infty} e^{-y^2}\,dy = \int\!\!\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\theta = 2\pi \cdot \left[\frac{1}{2}\,e^{-r^2}\right]_0^{+\infty} = \pi, \end{aligned}$$

and thus

$$I(0) = \sqrt{\pi}. \qquad \Diamond$$

Example 3.5 *Integrate the function e^{iz^2} by using Cauchy's theorem along a triangle of corners 0, a and $a(1+i)$, where $a > 0$. Prove that the integral along the path from a to $a(1+i)$ tends to 0 for $a \to +\infty$, and then prove that*

$$\int_0^{+\infty} e^{ix^2}\,dx = \int_0^{+\infty} \cos\left(x^2\right)\,dx + i\int_0^{+\infty} \sin\left(x^2\right)\,dx = \frac{1+i}{\sqrt{2}}\int_0^{+\infty} e^{-x^2}\,dx.$$

The integrand is analytic, so it follows from Cauchy's theorem that

$$0 = \int_0^a e^{ix^2}\,dx + \int_0^a e^{i(a+it)^2}\,i\,dt - \int_0^a e^{i(1+i)^2t^2}(1+i)\,dt.$$

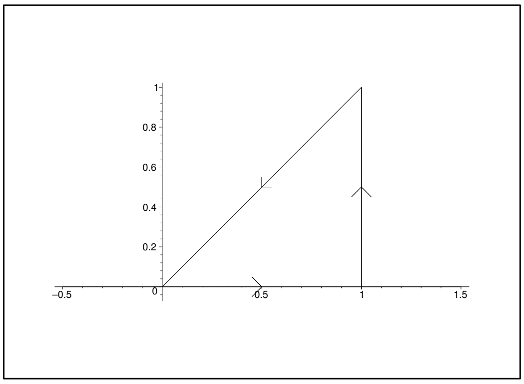


Figure 3: The curve C when $a = 1$.

We first consider

$$I_2 = \int_0^a e^{i(a+it)^2} i\, dt.$$

Here we get the estimate

$$|I_2| = \left| \int_0^a e^{i\left(a^2-t^2\right)} e^{-2at} i\, dt \right| \le \int_0^a e^{-2at}\, dt = \frac{1-e^{-2a^2}}{2a} < \frac{1}{2a}.$$

It follows immediately that

$$\int_0^{a(1+i)} e^{iz^2}\, dz \to 0 \qquad \text{for } a \to +\infty.$$

Then we introduce the substitution $u = t\sqrt{2}$ into the latter integral,

$$I_3 = \int_0^a e^{i(1+i)^2 t^2}(1+i)\, dt.$$

We get here

$$\begin{aligned} I_3 &= \int_0^a e^{i(1+i)^2 t^2}(1+i)\, dt = (1+i)\int_0^a e^{-2t^2} dt = \frac{1+i}{\sqrt{2}} \int_0^{a\sqrt{2}} e^{-u^2} du \\ &\to \frac{1+i}{\sqrt{2}} \int_0^{+\infty} e^{-x^2} dx \qquad \text{for } a \to +\infty. \end{aligned}$$

We finally conclude that the first integral I_1 is also convergent for $a \to +\infty$, and

$$\int_0^{+\infty} e^{ix^2} dx = \int_0^{+\infty} \cos\left(x^2\right)\, dx + i \int_0^{+\infty} \sin\left(x^2\right)\, dx = \frac{1+i}{\sqrt{2}} \int_0^{+\infty} e^{-x^2} dx.$$

Remark 3.1 If we use the result of Example 3.4, it follows by the symmetry that

$$\int_0^{+\infty} e^{ix^2}\,dx = \frac{1+i}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2},$$

hence

$$\int_0^{+\infty} \cos\left(x^2\right)\,dx = \int_0^{+\infty} \sin\left(x^2\right)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}. \qquad \Diamond$$

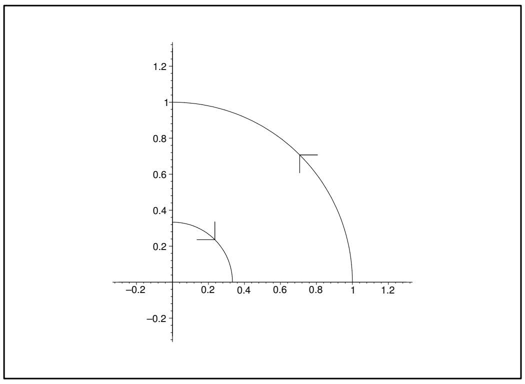
Example 3.6 *1) Find the domain of analyticity of the function*

$$f(z) = \frac{\text{Log } z}{z^2 - 1}.$$

Explain why f has a removable singularity at $z = 1$.

2) Let $C_{r,R}$ denote the simple, closed curve on the figure, where

$$0 < r < R < +\infty.$$



Compute the line integral

$$(1) \quad \oint_{C_{r,R}} f(z)\, dz.$$

3) Show that the improper integral

$$\int_0^{+\infty} \frac{\ln x}{x^2 - 1}\, dx$$

is convergent, and then find its value, e.g. by letting $r \to 0+$ and $R \to +\infty$ in (1).

1) Clearly, f is defined and analytic, when

$$z \in \mathbb{C} \setminus (\mathbb{R}_- \cup \{0, 1\}),$$

and the singularity at $z = 1$ is at most a simple pole,

$$f(z) = \frac{\text{Log } z}{z^2 - 1} = \frac{\text{res}(f; 1)}{z - 1} + \sum_{n=0}^{+\infty} a_n (z - 1)^n, \qquad 0 < |z - 1| < 1.$$

But since

$$\text{res}(f(f; 1) = \lim_{z \to 1} \frac{\text{Log } z}{z + 1} = \frac{\text{Log } 1}{2} = 0,$$

it follows that the Laurent series of f from $z = 1$ is a power series, so the singularity at $z = 1$ is removable.

ALTERNATIVELY, both the numerator and the denominator are 0 for $z = 1$, so we get by l'Hospital's rule that

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{\operatorname{Log} z}{z^2-1} = \lim_{z\to 1} \frac{\frac{1}{z}}{2z} = \frac{1}{2},$$

so the singularity is removable, and we may consider

$$f(z) = \begin{cases} \dfrac{\operatorname{Log} z}{z^2-1}, & x \in \mathbb{C} \setminus (\mathbb{R}_- \cup \{0,1\}), \\[2mm] \dfrac{1}{2}, & z = 1, \end{cases}$$

as an analytic function in $\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$.

ALTERNATIVELY it follows by a series expansion of

$$\operatorname{Log} z = \operatorname{Log}(1 + (z-1)) \qquad \text{for } 0 < |z-1| < 1,$$

that

$$f(z) = \frac{\operatorname{Log} z}{z^2-1} = \frac{1}{z+1} \cdot \frac{1}{z-1} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \cdot (z-1)^n = \frac{1}{z+1} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} (z-1)^n.$$

Here $\dfrac{1}{z+1}$ is continuous in all of the disc $|z-1| < 1$, so we conclude again that $z = 1$ is a removable singularity and that f can be analytically extended to $z = 1$ by putting

$$f(1) := \frac{1}{1+1} \left\{ \frac{1}{1+0} + 0 \right\} = \frac{1}{2}.$$

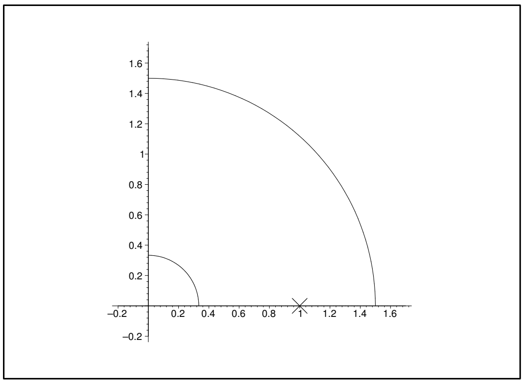


Figure 4: The path of integration $C_{r,R}$ with the removable singularity at $z = 1$.

2) Since we may consider f as an analytic function in $\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$, we conclude from *Cauchy's integral theorem* that

$$(2) \quad \oint_{C_{r,R}} f(z)\,dz = 0.$$

3) When we restrict the analytic function to $\mathbb{R}_+$, we get a continuous function $\dfrac{\ln x}{x^2-1}$, supplied by the value $\dfrac{1}{2}$ at $x = 1$. Since we only have $\ln x = 0$ for $x = 1$, we see that $x = 1$ is the only *possiple* zero. However, the value is here $\dfrac{1}{2} > 0$, so we conclude by the continuity that $\dfrac{\ln x}{x^2-1}$ is positive (and continuous) for $x \in \mathbb{R}_+$. Then we have the splitting

$$\int_0^{+\infty} \frac{\ln x}{x^2-1}\,dx = \int_0^{\frac{1}{2}} \frac{\ln x}{x^2-1}\,dx + \int_{\frac{1}{2}}^{2} \frac{\ln x}{x^2-1}\,dx + \int_2^{+\infty} \frac{\ln x}{x^2-1}\,dx.$$

The estimate

$$\begin{aligned} 0 \quad < \quad & \int_0^{\frac{1}{2}} \frac{\ln x}{x^2-1}\,dx \le \frac{1}{\left|\frac{1}{4}-1\right|} \int_0^{\frac{1}{2}} |\ln x|\,dx = \frac{4}{3}\int_0^{\frac{1}{2}} (-\ln x)\,dx \\ = \quad & \frac{4}{3}\left[-x\,\ln x + x\right]_{0+}^{\frac{1}{2}} = \frac{4}{3}\cdot\frac{1}{2}\,(1+\ln 2) = \frac{2}{3}\,(1+\ln 2) < +\infty, \end{aligned}$$

implies that the first integral exists.

It was mentioned above that we could consider $\dfrac{\ln x}{x^2-1}$ as a continuous function in the closed bounded interval $\left[\frac{1}{2}\,,\,2\right]$, from which we conclude that the second integral also is convergent.

Finally, it follows from the magnitudes of the functions, when $x \to +\infty$ that there exists a constant $C > 0$, such that

$$0 < \int_2^{+\infty} \frac{\ln x}{x^2-1}\,dx < C\int_2^{+\infty} \frac{1}{x^{\frac{3}{2}}}\,dx = C\sqrt{2} < +\infty,$$

and we conclude that the last integral also is convergent.

Summing up we have proved that the improper integral $\int_0^{+\infty} \dfrac{\ln x}{x^2-1}\,dx$ is convergent.

When we expand (2), then

$$\begin{aligned} 0 \quad = \quad & \int_r^R \frac{\ln x}{x^2-1}\,dx + \int_0^{\frac{\pi}{2}} \frac{\operatorname{Log}\left(R\,e^{i\theta}\right)}{R^2e^{2i\theta}-1}\cdot R\,i\,e^{i\theta}\,d\theta - \int_r^R \frac{\operatorname{Log}(it)}{(it)^2-1}\cdot i\,dt - \int_0^{\frac{\pi}{2}} \frac{\operatorname{Log}\left(r\,e^{i\theta}\right)}{r^2e^{2i\theta}-1}\,r\,i\,e^{i\theta}\,d\theta \\ = \quad & \int_r^R \frac{\ln x}{x^2-1}\,dx + i\int_r^R \frac{\ln t + i\,\frac{\pi}{2}}{1+t^2}\,dt + i\int_0^{\frac{\pi}{2}} \frac{\ln R + i\theta}{R^2e^{2i\theta}-1}\cdot R\,e^{i\theta}\,d\theta + i\int_0^{\frac{\pi}{2}} \frac{\ln r + i\theta}{1-r^2e^{2i\theta}}\cdot r\,e^{i\theta}\,d\theta, \end{aligned}$$

hence by a rearrangement,

$$\begin{aligned}&\int_r^R \frac{\ln x}{x^2-1}\,dx + i\int_r^R \frac{\ln t}{1+t^2}\,dt \\ &= \frac{\pi}{2}\int_r^R \frac{dt}{1+t^2} - i\int_0^{\frac{\pi}{2}} \frac{\ln R + i\theta}{R^2 e^{2i\theta}-1}\cdot R\,e^{i\theta}\,d\theta - i\int_0^{\frac{\pi}{2}} \frac{\ln r + i\theta}{1-r^2 e^{2i\theta}}\cdot r\,e^{i\theta}\,d\theta.\end{aligned}$$

By taking the limits $r \to 0+$ and $R \to +\infty$ on each of the terms on the right hand side we get

$$\lim_{r\to 0+}\lim_{R\to+\infty} \frac{\pi}{2}\int_r^R \frac{dt}{1+t^2} = \frac{\pi}{2}\int_0^{+\infty} \frac{dt}{1+t^2} = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4},$$

and

$$\begin{aligned}\left| i\int_0^{\frac{\pi}{2}} \frac{\ln R + i\theta}{R^2 e^{2i\theta}-1}\cdot R\,e^{i\theta}\,d\theta\right| &\leq \int_0^{\frac{\pi}{2}} \frac{\ln R + \frac{\pi}{2}}{R^2-1}\cdot R\,d\theta \\ &= \frac{\pi}{2}\cdot\frac{R\left(\ln R + \frac{\pi}{2}\right)}{R^2-1} \to 0 \qquad \text{for } R \to +\infty,\end{aligned}$$

and

$$\left| i \int_0^{\frac{\pi}{2}} \frac{\ln r + i\theta}{1-r^2} e^{2i\theta}\, d\theta \right| \le \frac{\pi}{2} \cdot \frac{r\left(|\ln r| + \frac{\pi}{2}\right)}{1-r^2} \to 0 \qquad \text{for } r \to 0+,$$

respectively. Hence, by summing up,

$$\int_0^{+\infty} \frac{\ln x}{x^2-1}\, dx + i \int_0^{+\infty} \frac{\ln t}{t^2+1}\, dt = \frac{\pi^2}{4}.$$

Finally, by separating the real and the imaginary parts,

$$\int_0^{+\infty} \frac{\ln x}{x^2-1}\, dx = \frac{x^2}{4} \qquad \text{og} \qquad \int_0^{+\infty} \frac{\ln x}{x^2+1}\, dx = 0.$$

Example 3.7 *Given the function*

$$f(z) = \frac{e^z}{1+e^{4z}}.$$

(1) *Find all the isolated singularities of f in $\mathbb{C}$.*
Determine the type of each of them and their residuum.

Given for each $r_1 > 0$ and $r_2 > 0$ the closed curve

$$\gamma_{r_1,r_2} = I_{r_1,r_2} + II_{r_2} + III_{r_1,r_2} + IV_{r_1}$$

(cf. the figure), which form the boundary of the domain

$$A_{r_1,r_2} = \{z \in \mathbb{C} \mid -r_1 < Re(z) < r_2 \text{ and } 0 < Im(z) < \pi\}.$$

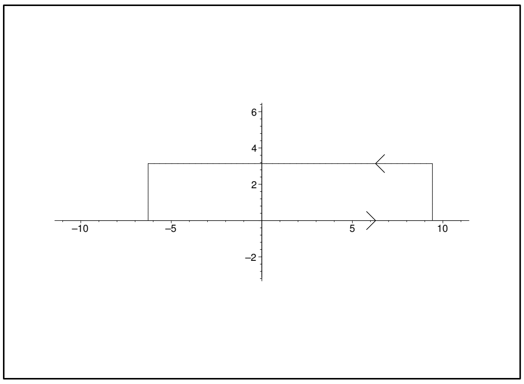


Figure 5: The curve γ_{r_1,r_2} with the direction given on $I_{r_1,r_2} = [-r_1, r_2]$ and III_{r_1,r_2}.

(2) *Prove that*

$$\oint_{\gamma_{r_1,r_2}} f(z)\,dz = \frac{\sqrt{2}}{2}\,\pi.$$

(3) *Prove that the line integrals along the vertical curves II_{r_2} and IV_{r_1} tend to 0 for r_2 and r_1 tending to $+\infty$.*

(4) *Find*

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{4x}}\,dx.$$

1) Since $e^z \neq 0$ for every $z \in \mathbb{C}$, the singularities are determined by

$$e^{4z} = -1 = e^{i(\pi+2p\pi)}, \qquad p \in \mathbb{Z},$$

so the isolated singularities are

$$z_p = i\left\{\frac{\pi}{4} + p\cdot\frac{\pi}{2}\right\}, \qquad p \in \mathbb{Z}.$$

We see from

$$\frac{d}{dz}\left\{e^{4z}+1\right\}_{|z=z_p} = 4\,e^{4z_p} = -4 \neq 0,$$

that these singularities are all simple poles with the residues

$$\operatorname{res}(f;z_p) = \frac{1}{-4}\,\exp\left(i\left\{\frac{\pi}{4}+p\,\frac{\pi}{2}\right\}\right) = -\frac{\sqrt{2}}{8}\,(1+i)\cdot\exp\left(i\,\frac{\pi}{2}\cdot p\right), \qquad p\in\mathbb{Z}.$$

2) We have inside the curve γ_{r_1,r_2} only the two poles z_0 and z_1, hence by *Cauchy's residuum theorem*,

$$\begin{aligned}\oint_{\gamma_{r_1,r_2}} \frac{e^z}{1+e^{4z}}\,dz &= 2\pi i\left\{\operatorname{res}(f;z_0)+\operatorname{res}(f;z_1)\right\} = 2\pi i\left\{-\frac{\sqrt{2}}{8}\,(1+i)(1+i)\right\}\\ &= 2\pi i\cdot\left(-\frac{\sqrt{2}}{8}\right)\cdot 2i = \frac{\sqrt{2}}{2}\,\pi = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

3) We may choose the parametric descriptions of the vertical paths of integration in the form $z(t) = r + it$, $t \in [0,\pi]$, where either $r = r_2$ or $r = -r_1$.
If $r = r_2 > 0$, then we get the estimate

$$\left|\int_{II_{r_2}} f(z)\,dz\right| \le \int_0^{\pi}\left|\frac{e^{r_2+it}}{1+e^{4r_2+4it}}\right|dt \le \pi\cdot\frac{e^{r_2}}{e^{r_2}-1} \to 0 \qquad \text{for } r_2\to+\infty.$$

If $r = -r_1 < 0$, then we get instead

$$\left|\int_{IV_{r_1}} f(z)\,dz\right| \le \int_0^{\pi}\left|\frac{e^{-r_1+it}}{1+e^{-4r_1+4it}}\right|dt \le \pi\cdot\frac{e^{-r_1}}{1-e^{-4r_1}} \to 0 \qquad \text{for } r_1\to+\infty.$$

4) Finally,

$$\int_{I_{r_1,r_2}} f(z)\,dz = \int_{-r_1}^{r_2}\frac{e^x}{1+e^{4x}}\,dx,$$

and

$$\int_{III_{r_1,r_2}} f(z)\,dz = \int_{r_2}^{-r_1}\frac{e^x\,e^{i\pi}}{1+e^{4x}e^{i4\pi}}\,dx = \int_{-r_1}^{r_2}\frac{e^x}{1+e^{4x}}\,dx.$$

It follows from **(2)** and **(3)** that

$$\frac{\sqrt{2}}{2}\,\pi = \lim_{r_1,\,r_2\to+\infty}\oint_{\gamma_{r_1,r_2}} f(z)\,dz = 2\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{4x}}\,dx,$$

hence

$$\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{4x}}\,dx = \frac{\sqrt{2}}{4}\,\pi.$$

Remark 3.2 It is possible to find the value of the improper integral (which clearly is convergent) without using the calculus of residues. First we get by the substitution $t = e^x$,

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{4x}}\,dx = \int_0^{+\infty} \frac{dt}{1+t^4}.$$

Then we decompose the integrand in the following way,

$$\begin{aligned}\frac{1}{1+t^4} &= \frac{1}{(t^4+2t^2+1)-2t^2} = \frac{1}{(t^2+1)^2-\left(\sqrt{2}\,t\right)^2} = \frac{1}{\left(t^2+\sqrt{2}\,t+1\right)\left(t^2-\sqrt{2}\,t+1\right)} \\ &= \frac{at+b}{t^2+\sqrt{2}\,t+1}+\frac{ct+d}{t^2-\sqrt{2}\,t+1},\end{aligned}$$

hence

$$\begin{aligned}1 &= (at+b)\left(t^2-\sqrt{2}t+1\right)+(ct+d)\left(t^2+\sqrt{2}t+1\right) \\ &= (a+c)t^3+(-\sqrt{2}a+b+\sqrt{2}c+d)t^2+(a-\sqrt{2}b+c+\sqrt{2}d)t+(b+d).\end{aligned}$$

We get $a+c=0$, i.e. $c=-a$, and $b+d=1$, so

$$-2\sqrt{2}\,a+1=0 \qquad \text{and} \qquad -\sqrt{2}\,b+\sqrt{2}\,d=0.$$

It follows that

$$a = \frac{1}{2\sqrt{2}} = -c \qquad \text{and} \qquad b=d=\frac{1}{2},$$

thus

$$\frac{1}{1+t^4} = \frac{1}{4\sqrt{2}}\cdot\frac{2t+\sqrt{2}}{t^2+\sqrt{2}\,t+1}+\frac{1}{4}\cdot\frac{1}{t^2+\sqrt{2}\,t+1}-\frac{1}{4\sqrt{2}}\cdot\frac{2t-\sqrt{2}}{t^2-\sqrt{2}\,t+1}+\frac{1}{4}\cdot\frac{1}{t^2-\sqrt{2}\,t+1}.$$

Finally, we get the primitive

$$\int\left\{\frac{2t+\sqrt{2}}{t^2+\sqrt{2}\,t+1}-\frac{2t+\sqrt{2}}{t^2-\sqrt{2}\,t+1}\right\}dt = \ln\left(\frac{t^2+\sqrt{2}\,t+1}{t^2-\sqrt{2}\,t+1}\right) \to 0,$$

for $t \to 0+$, and for $t \to +\infty$. We therefore conclude that

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{4x}}\,dx &= \int_0^{+\infty}\frac{dt}{1+t^4} = \frac{1}{4}\int_0^{+\infty}\left\{\frac{1}{t^2+\sqrt{2}\,t+1}+\frac{1}{t^2-\sqrt{2}\,t+1}\right\}dt \\ &= \frac{1}{4}\int_0^{+\infty}\left\{\frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(t-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right\}dt \\ &= \frac{\sqrt{2}}{4}\left[\text{Arctan}\left(\sqrt{2}\,t+1\right)+\ \text{Arctan}\left(\sqrt{2}\,t-1\right)\right]_0^{\infty} \\ &= \frac{\sqrt{2}}{4}\left\{\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}+\frac{\pi}{4}\right\} = \frac{\sqrt{2}\,\pi}{4}. \qquad \Diamond\end{aligned}$$

Example 3.8 *Denote by A the domain*

$$A = \mathbb{C} \setminus \{z \in \mathbb{C} \mid \text{Re}(z) = 0 \text{ and } \text{Im}(z) \leq 0\},$$

and denote by $\sqrt{z}$ the branch of the square root which is analytic in A, and which is equal to the usual real square root on the positive real axis $\mathbb{R}_+$.
Furthermore, let

$$\Gamma_{r,R} = I_{r,R} + II_R + III_{r,R} + IV_r \qquad \text{for } 0 < r < 1 < R,$$

denote the simple closed curve on the figure.

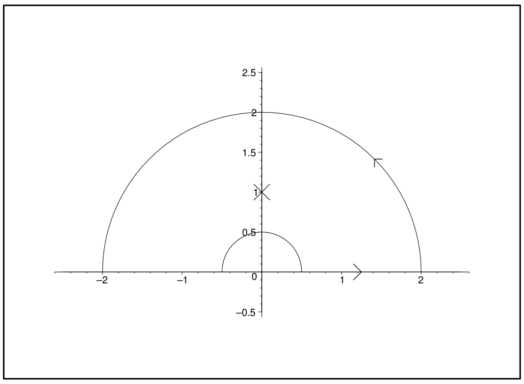


Figure 6: The closed curve $\Gamma_{r,R}$ med $I_{r,R} = [r, R]$ and the circular arc II_R with a direction, (and $III_{r,R}$ and IV_r follow in a natural way). The pole i of $f(z)$ is indicated inside $\Gamma_{r,R}$.

Put

$$f(z) = \frac{1}{\sqrt{z}\,(z^2+1)}.$$

1) *Prove that*

$$\oint_{\Gamma_{r,R}} f(z)\,dz = \frac{\pi}{\sqrt{2}}\,(1-i).$$

2) *Prove that the integrals of f along the half circles II_R and IV_r tend to 0 when R tends to ∞, and r tends to 0.*

3) *Prove that the integral*

$$\int_0^{+\infty} \frac{1}{\sqrt{x}\,(x^2+1)}\,dx$$

is convergent and find its value.

1) The only singularity of $f(z)$ inside $\Gamma_{r,R}$ is the simple pole $z = i$, so it follows by *Cauchy's residuum theorem* that

$$\begin{aligned}\oint_{\Gamma_{r,R}} f(z)\,dz &= 2\pi i \operatorname{res}\left(\frac{1}{\sqrt{z}\,(z^2+1)}\,,\,i\right) = 2\pi i \lim_{z\to i} \frac{1}{\sqrt{z}\cdot 2z} = 2\pi i \cdot \frac{1}{\dfrac{1+i}{\sqrt{2}}\cdot 2i} \\ &= \pi \cdot \frac{1-i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}\,(1-i).\end{aligned}$$

2) We estimate the integral along the curve IV_r of the parametric description $z(t) = r\,e^{i(\pi-t)}$, $t \in [0,\pi]$ and $0 < r < 1$, by

$$\left|\int_{IV_r} \frac{1}{\sqrt{z}\,(z^2+1)}\,dz\right| \le \int_0^\pi \frac{r\,dt}{\sqrt{r}\cdot(1-r^2)} = \frac{\pi\sqrt{r}}{1-r^2} \to 0 \qquad \text{for } r \to +\infty.$$

Along med II_R we choose the parametric description $z(t) = R\cdot e^{it}$, $t \in [0,\pi]$, $R > 1$, and then get the estimate

$$\left|\int_{II_R} \frac{1}{\sqrt{z}\,(z^2+1)}\,dz\right| \le \int_0^\pi \frac{R}{\sqrt{R}\cdot(R^2-1)}\,dt = \frac{\pi\sqrt{R}}{R^2-1} \to 0 \qquad \text{for } R \to +\infty.$$

3) We obtain along $III_{r,R}$,

$$\int_{III_{r,R}} \frac{dz}{\sqrt{z}\,(z^2+1)} = \frac{1}{i}\int_{-R}^{-r} \frac{dx}{\sqrt{|x|}\,(x^2+1)} = \frac{1}{i}\int_r^R \frac{dx}{\sqrt{x}\,(x^2+1)} = -i\int_r^R \frac{dx}{\sqrt{x}\,(x^2+1)}.$$

Taking the limits $r \to 0+$ and $R \to +\infty$ and applying the results of **(1)** and **(2)** we get

$$(1-i)\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}\,(1-i),$$

hence

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

Alternatively we may change the variable to $t = \sqrt{x}$, $x = t^2$, $t \in \mathbb{R}_+$,

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}\,(x^2+1)} = 2\int_0^{+\infty} \frac{dt}{t^4+1}.$$

Then we decompose in the following way,

$$\begin{aligned}\frac{1}{1+t^4} &= \frac{1}{(t^4+2t^2+1)-2t^2} = \frac{1}{(t^2+1)^2-\left(\sqrt{2}\,t\right)^2} = \frac{1}{\left(t^2+\sqrt{2}\,t+1\right)\left(t^2-\sqrt{2}\,t+1\right)}\\ &= \frac{at+b}{t^2+\sqrt{2}\,t+1}+\frac{ct+d}{t^2-\sqrt{2}\,t+1},\end{aligned}$$

hence

$$\begin{aligned}1 &= (at+b)\left(t^2-\sqrt{2}t+1\right)+(ct+d)\left(t^2+\sqrt{2}t+1\right)\\ &= (a+c)t^3+(-\sqrt{2}a+b+\sqrt{2}c+d)t^2+(a-\sqrt{2}b+c+\sqrt{2}d)t+(b+d).\end{aligned}$$

We get $a+c=0$, thus $c=-a$, and $b+d=1$, so

$$-2\sqrt{2}\,a+1=0 \qquad \text{and} \qquad -\sqrt{2}\,b+\sqrt{2}\,d=0.$$

Then

$$a = \frac{1}{2\sqrt{2}} = -c \qquad \text{and} \qquad b=d=\frac{1}{2},$$

and

$$\frac{1}{1+t^4} = \frac{1}{4\sqrt{2}}\cdot\frac{2t+\sqrt{2}}{t^2+\sqrt{2}\,t+1}+\frac{1}{4}\cdot\frac{1}{t^2+\sqrt{2}\,t+1}-\frac{1}{4\sqrt{2}}\cdot\frac{2t-\sqrt{2}}{t^2-\sqrt{2}\,t+1}+\frac{1}{4}\cdot\frac{1}{t^2-\sqrt{2}\,t+1}.$$

Finally, we see that the primitive is given by

$$\int\left\{\frac{2t+\sqrt{2}}{t^2+\sqrt{2}\,t+1}-\frac{2t+\sqrt{2}}{t^2-\sqrt{2}\,t+1}\right\}dt = \ln\left(\frac{t^2+\sqrt{2}\,t+1}{t^2-\sqrt{2}\,t+1}\right) \to 0,$$

for $t \to 0+$, and for $t \to +\infty$. We therefore conclude that

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{\sqrt{x}\,(x^2+1)}\,dx &= 2\int_0^{+\infty} \frac{dt}{1+t^4} = \frac{1}{2}\int_0^{+\infty} \left\{ \frac{1}{t^2+\sqrt{2}\,t+1} + \frac{1}{t^2-\sqrt{2}\,t+1} \right\} dt \\
&= \frac{1}{2}\int_0^{+\infty} \left\{ \frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \frac{1}{\left(t-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} \right\} dt \\
&= \frac{\sqrt{2}}{2}\left[\operatorname{Arctan}\left(\sqrt{2}\,t+1\right) + \operatorname{Arctan}\left(\sqrt{2}\,t-1\right)\right]_0^{\infty} \\
&= \frac{\sqrt{2}}{2}\left\{\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}+\frac{\pi}{4}\right\} = \frac{\sqrt{2}\,\pi}{2}.
\end{aligned}$$

4 Improper integral, where the integrand is a rational function

Example 4.1 *Find the value of the improper integral*

$$\int_{-\infty}^{+\infty} \frac{dx}{x^4+1}.$$

1) It is possible to compute the integral by a real decomposition; but this is not an easy method. We shall here shortly sketch it in order to demonstrate the difficulties connected with it: By "adding something and then subtracting it again, followed by factorizing the difference of two squares" we get

$$x^4 = x^4+2x^2+1-2x^2 = \left\{x^2+1\right\}^2 - \left\{\sqrt{2}\,x\right\}^2 = \left\{x^2+\sqrt{2}\,x+1\right\}\left\{x^2-\sqrt{2}\,x+1\right\}.$$

We conclude that there exist real constants A, B, C and $D \in \mathbb{R}$, such that

$$\begin{aligned}\frac{1}{x^4+1} &= \frac{1}{\{x^2+1\}^2 - \{\sqrt{2}\,x\}^2 = \{x^2+\sqrt{2}\,x+1\}\{x^2-\sqrt{2}\,x+1\}} \\ &= \frac{Ax+B}{x^2+\sqrt{2}\,x+1} + \frac{Cx+D}{x^2-\sqrt{2}\,x+1}.\end{aligned}$$

Then by the usual decomposition,

$$A = \frac{1}{2\sqrt{2}}, \qquad B = \frac{1}{2}, \qquad C = -\frac{1}{2\sqrt{2}}, \qquad D = \frac{1}{2},$$

and we find a primitive of $\dfrac{1}{x^4+1}$ in the usual way.

2) A *variant* of the method of decomposition above is to note that all four poles z_j are simple, so

$$\frac{1}{z^4+1} = \frac{\operatorname{res}(f\,;\,z_1)}{z-z_1} + \frac{\operatorname{res}(f\,;\,z_2)}{z-z_2} + \frac{\operatorname{res}(f\,;\,z_3)}{z-z_3} + \frac{\operatorname{res}(f\,;\,z_4)}{z-z_4},$$

where

$$\operatorname{res}(f\,;\,z_j) = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{1}{4}\,z_j,$$

by Rule II. We see that the z_j are complex (they occur in complex conjugated pairs), so the terms shall afterwards be paired together in the same way, before we find the real primitives.

3) Finally, we show that it is much easier to use the residuum formula. We shall first check the assumptions. The integrand is a rational function with a zero of order 4 at ∞ and no pole on the real axis. This implies that the improper integral is convergent and we can find it value by computing the residues of the poles in the upper half plane. The four simple poles are $\exp\left(\dfrac{ip\pi}{4}\right)$, $p = 1,\,3,\,5,\,7$, of which only

$$\exp\left(\frac{i\pi}{4}\right) = \frac{1+i}{\sqrt{2}} \qquad \text{og} \qquad \exp\left(\frac{3i\pi}{4}\right) = \frac{-1+i}{\sqrt{2}}$$

lies in the upper half plane.

If we as above use Rule II with $A(z) = 1$ and $B(z) = z^4 + 1$, where $B'(z) = 4z^3$, then we get for any of the poles z_0 that

$$\text{res}\,(f\,;\,z_0) = \frac{A\,(z_0)}{B'\,(z_0)} = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = -\frac{1}{4}\,z_0,$$

because $z_0^4 = -1$ for all of them.

Then by the residuum formula,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{x^4+1} &= 2\pi i\left\{\text{res}\left(\frac{1}{z^4+1}\,;\,\frac{1+i}{\sqrt{2}}\right) + \text{res}\left(\frac{1}{z^4+1}\,;\,\frac{-1+i}{\sqrt{2}}\right)\right\} \\ &= 2\pi i\cdot\left\{-\frac{1}{4}\right\}\left\{\frac{1+i}{\sqrt{2}}+\frac{-1+i}{\sqrt{2}}\right\} = -\frac{2\pi i}{4}\cdot\frac{2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Example 4.2 *The improper integral* $\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\,dx$ *is not convergent. Discuss what happens if one nevertheless use the residuum formula.*

The only singularity of the analytic extension of the integrand in the upper half plane is the simple pole at $z_0 = i$. Here we have

$$\text{res}\left(\frac{z}{z^2+1}\,;\,i\right) = \lim_{z\to i}\frac{z}{2z} = \frac{1}{2},$$

so if we unconsciously put this into the residuum formula, then

$$\text{“}\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\,dx = 2\pi i\cdot\text{res}\left(\frac{z}{z^2+1}\,;\,i\right) = \pi i\text{”}.$$

This is of course not true, because *if* we could attach the improper integral a value (it is not convergent, so one should at least use *“Cauchy's principal value”* in order just to get a little sense into this expression), and then it is obvious that a possible value should be reel and not at all imaginary. The example shows that residuum formulæ formally often can be applied in cases, in which their assumptions are not fulfilled. If so, they will usually give a wrong result.

Example 4.3 *Compute*

$$(a)\ \int_{-\infty}^{+\infty}\frac{dx}{\left(1+x^2\right)^2},\qquad (b)\ \int_{-\infty}^{+\infty}\frac{x^2}{\left(1+x^2\right)^2}\,dx,\qquad (c)\ \int_{-\infty}^{+\infty}\frac{dx}{\left(1+x^2\right)^3}.$$

(a) The integrand has a zero of fourth order at ∞, and since $\left(1+x^2\right)^2 \neq 0$ for every $x \in \mathbb{R}$, the integral is convergent. The integrand has the two double poles $\pm i$, of which only $+i$ lies in the upper half plane, so

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dx}{\left(1+x^2\right)^2} &= 2\pi i\cdot\text{res}\left(\frac{1}{\left(1+z^2\right)^2}\,;\,i\right) = 2\pi i\cdot\frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left\{\frac{1}{(z+i)^2}\right\} \\ &= 2\pi i\lim_{z\to i}\frac{-2}{(z+i)^3} = \frac{-4\pi i}{(2i)^3} = \frac{\pi}{2}.\end{aligned}$$

(b) The difference of degrees is 2 where the denominator is dominating, and the integrand has only the singularities $\pm i$ (double poles, which do not lie on $\mathbb{R}$). Hence, the integral exists and

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)^2}\,dx &= 2\pi i \cdot \text{res}\left(\frac{z^2}{(z^2+1)^2}\,;\,i\right) = 2\pi i \cdot \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left\{\frac{z^2}{(z+i)^2}\right\} \\ &= 2\pi i \lim_{z\to i}\left\{\frac{2z}{(z+i)^2} - \frac{2z^2}{(z+i)^3}\right\} = 2\pi i\left\{\frac{2i}{(2i)^3} - \frac{2i^2}{(2i)^3}\right\} \\ &= -\pi\left\{\frac{(2i)^2}{(2i)^2} - \frac{1}{2}\,\frac{(2i)^3}{(2i)^3}\right\} = \frac{\pi}{2}.\end{aligned}$$

Alternatively, we of course also have

$$\frac{x^2}{(1+x^2)^2} = \frac{x^2+1-1}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2},$$

and then it follows from **(a)** that

$$\int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)^2}\,dx = \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx - \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2}\,dx = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

(c) The integrand has a zero of order 6 at ∞ and no singularity on the x-axis, and poles of order 3 at $z = \pm i$. Hence,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} &= 2\pi i \cdot \text{res}\left(\frac{1}{(z^2+1)^3}\,;\,i\right) = 2\pi i \cdot \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left\{\frac{1}{(z+i)^3}\right\} \\ &= \pi i \lim_{z\to i}\frac{d}{dz}\left\{\frac{-3}{(z+i)^4}\right\} = \pi i \lim_{z\to i}\frac{(-3)(-4)}{(z+i)^5} = \frac{12\pi i}{(2i)^5} = \frac{12\pi i}{32 i} = \frac{3\pi}{8}.\end{aligned}$$

Example 4.4 *Prove that*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi\sqrt{2}}{2}, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{x-1}{x^5-1}\,dx = \frac{4\pi}{5}\,\sin\frac{2\pi}{5}.$$

(a) The integrand $\dfrac{x^2}{x^4+1}$ has a zero of second order at ∞, and no singularity on the x-axis. The poles in the upper half plane,

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad \text{and} \qquad z_2 = \frac{-1+i}{\sqrt{2}},$$

are both simple with $z_1 \cdot z_2 = -1$, so

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx &= 2\pi i\left\{\text{res}\left(\frac{z^2}{z^4+1}\,;\,z_1\right) + \text{res}\left(\frac{z^2}{z^4+1}\,;\,z_2\right)\right\} \\ &= 2\pi i\left\{\lim_{z\to z_1}\frac{z^2}{4z^3} + \lim_{z\to z_2}\frac{z^2}{4z^3}\right) = \frac{2\pi i}{4}\left(\frac{1}{z_1} + \frac{1}{z_2}\right) = -\frac{\pi i}{2}\,(z_1+z_2) \\ &= -\frac{\pi}{2}\left\{\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}\right\} = -\frac{\pi i}{2}\cdot\frac{2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}} = \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

(b) First we note that we have a removable singularity at $x = 1$, because

$$\lim_{x \to 1} \frac{x-1}{x^5-1} = \lim_{x \to 1} \frac{1}{5x^4} = \frac{1}{5}$$

either by l'Hospital's rule, or by a simple division,

$$\frac{x-1}{x^5-1} = \frac{1}{x^4+x^3+x^2+x+1} \to \frac{1}{5} \qquad \text{for } x \to 1.$$

There is no other singularity on $\mathbb{R}$ than the removable singularity at $z = 1$, and the integrand has a zero of order 4 at ∞. We therefore conclude that the improper integral is convergent. The poles in the upper half plane,

$$z_1 = \exp\left(\frac{2i\pi}{5}\right) \qquad \text{and} \qquad z_2 = \exp\left(\frac{4i\pi}{5}\right),$$

are both simple, so the value of the improper integral is

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x-1}{x^5-1}\,dx &= 2\pi i\left\{\operatorname{res}\left(\frac{z-1}{z^5-1}\,;\,z_1\right)+\operatorname{res}\left(\frac{z-1}{z^5-1}\,;\,z_2\right)\right\}\\
&= 2\pi i\left\{\lim_{z\to z_1}\frac{z-1}{5z^4}+\lim_{z\to z_2}\frac{z-1}{5z^4}\right\}\\
&= \frac{2\pi i}{5}\left\{\lim_{z\to z_1}\frac{z^2-z}{z^5}+\lim_{z\to z_2}\frac{z^2-z}{z^5}\right\}=\frac{2\pi i}{5}\left\{z_1^2-z_1+z_2^2-z_2\right\}\\
&= \frac{2\pi i}{5}\left\{\exp\left(\frac{4i\pi}{5}\right)-\exp\left(\frac{2i\pi}{5}\right)+\exp\left(\frac{8i\pi}{5}\right)-\exp\left(\frac{4i\pi}{5}\right)\right\}\\
&= \frac{2\pi i}{5}\left\{\exp\left(-\frac{2\pi i}{5}\right)-\exp\left(\frac{2\pi i}{5}\right)\right\}=\frac{2\pi i}{5}\cdot\left\{-2i\,\sin\frac{2\pi}{5}\right\}=\frac{4\pi}{5}\,\sin\frac{2\pi}{5}.
\end{aligned}$$

Remark 4.1 Since

$$\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}\qquad\text{and}\qquad\sin\frac{\pi}{5}=\frac{\sqrt{10-2\sqrt{5}}}{4},$$

it follows by insertion that

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{x-1}{x^5-1}\,dx &= \frac{4\pi}{5}\cdot 2\sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5}=\frac{\pi}{10}\cdot\left(1+\sqrt{5}\right)\cdot\sqrt{10-2\sqrt{5}}\\
&= \frac{\pi}{10}\sqrt{10+8\sqrt{5}}=\frac{\pi}{5}\sqrt{10+2\sqrt{5}}.
\end{aligned}$$

Example 4.5 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)^n},\qquad n\in\mathbb{N},\qquad\qquad (b)\quad \int_0^{+\infty}\frac{x^2+1}{x^2+1}\,dx.$$

In both cases we see that the improper integrals are convergent (the denominator is dominating, and the difference of the degrees is at least 2, and we have no singularity on the real axis $\mathbb{R}$), so the values can be found by means of the residues in the upper half plane.

(a) Since $z=i$ is an n-tuple pole, we find

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)^n} &= 2\pi i\cdot\operatorname{res}\left(\frac{1}{\left(z^2+1\right)^n}\,;\,i\right)=2\pi i\cdot\frac{1}{(n-1)!}\lim_{z\to i}\frac{d^{n-1}}{dz^{n-1}}\left\{(z+i)^{-n}\right\}\\
&= \frac{2\pi i}{(n-1)!}(-n)(-n-1)\cdots(-2n+2)\lim_{z\to i}\frac{1}{(z+i)^{2n-1}}\\
&= 2\pi i\cdot\frac{(-1)^{n-1}\cdot(2n-2)!}{(n-1)!(n-1)!}\cdot\frac{1}{2^{2n-1}}\cdot\frac{1}{i^{2n}}=\frac{\pi}{2^{2n-2}}\begin{pmatrix}2n-2\\ n-1\end{pmatrix}.
\end{aligned}$$

(b) If z_0 is one of the simple poles, then $z_0^4=-1$, so the residuum is given by

$$\operatorname{res}\left(\frac{z^2+1}{z^4+1}\,;\,z_0\right)=\frac{z_0^2+1}{4z_0^3}=-\frac{z_0}{4}\left(z_0^2+1\right).$$

Then by using the symmetry, (the integrand is an even function),

$$\begin{aligned}
\int_0^{+\infty} \frac{x^2}{x^4+1}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,dx \\
&= \pi i\left\{\operatorname{res}\left(\frac{z^2+1}{z^4+1}\,;\,\exp\left(i\,\frac{\pi}{4}\right)\right)+\operatorname{res}\left(\frac{z^2+1}{z^4+1}\,;\,\exp\left(i\,\frac{3\pi}{4}\right)\right)\right\} \\
&= -\frac{\pi i}{4}\left\{\exp\left(i\,\frac{\pi}{4}\right)\left(\exp\left(i\,\frac{\pi}{2}\right)+1\right)+\exp\left(i\,\frac{3\pi}{4}\right)\left(\exp\left(i\,\frac{3\pi}{2}\right)+1\right)\right\} \\
&= -\frac{\pi i}{4\sqrt{2}}\{(1+i)(1+i)+(-1+i)(1-i)\} = -\frac{\pi i}{4\sqrt{2}}\{2i-(-2i)\} = -\frac{4i\cdot\pi i}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Remark 4.2 It is possible, though far from easy to compute the value of the integral by only using the known real methods of integration from Calculus, i.e. by a decomposition. We shall here only sketch the method.
If we only can factorize the denominator, the rest is standard, though still difficult. The trick is here the usual one: Add something and then subtract it again,

$$x^4+1 = x^4+2x^2+1-2x^2 = \left(x^2+1\right)^2-(\sqrt{2}\,x)^2 = \left(x^2+\sqrt{2}\,x+1\right)\left(x^2-\sqrt{2}\,x+1\right).$$

Now we have written the denominator as a product of polynomials of degree 2, so we can *in principle* decompose and then compute the integral. However, the factorization of the denominator shows that this will be fairly difficult to carry through in practice. ◊

Example 4.6 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^6}, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{x^2}{1+x^6}\,dx.$$

The denominator is dominating with at least 4 degrees in the exponent, and there are no poles on the x-axis. Therefore, both improper integrals are convergent, and we can compute them by a residuum formula.

(a) The integrand $\frac{1}{1+z^6}$ has in the upper half plane the three simple poles

$$\exp\left(i\,\frac{\pi}{6}\right), \qquad \exp\left(i\,\frac{\pi}{2}\right)=i, \qquad \exp\left(i\,\frac{5\pi}{6}\right).$$

Let z_0 be anyone of these poles. Then in particular $z_0^6=-1$, and it follows that

$$\operatorname{res}\left(\frac{1}{1+z^6}\,;\,z_0\right) = \frac{1}{6z_0^5} = \frac{z_0}{6z_0^6} = -\frac{1}{6}\,z_0.$$

By insertion;

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dx}{1+x^6} &= 2\pi i\left\{\operatorname{res}\left(\frac{1}{1+z^6}\,;\,\exp\left(i\,\frac{\pi}{6}\right)\right)+\operatorname{res}\left(\frac{1}{1+z^6}\,;\,i\right)+\operatorname{res}\left(\frac{1}{1+z^6}\quad\exp\left(i\,\frac{5\pi}{6}\right)\right)\right\} \\
&= -\frac{2\pi i}{6}\left\{\exp\left(i\,\frac{\pi}{6}\right)+i+\exp\left(i\,\frac{5\pi}{6}\right)\right\} = -\frac{\pi i}{3}\left\{2i\,\sin\frac{\pi}{6}+i\right\} = \frac{2\pi}{3}.
\end{aligned}$$

Remark 4.3 The denominator can be factorized in the following way

$$\begin{aligned} 1+x^6 &= \left(1+x^2\right)\left(x^4-x^2+1\right) = \left(1+x^2\right)\left(x^4+2x^3+1-3x^2\right) \\ &= \left(x^2+1\right)\left(\left(x^2+1\right)^2-(\sqrt{3}\,x)^2\right) = \left(x^2+1\right)\left(x^2+\sqrt{3}\,x+1\right)\left(x^2-\sqrt{3}\,x+1\right), \end{aligned}$$

so we can *in principle* decompose the integrand and then integrate in the usual way known from Calculus. However, the coefficients clearly show that this will be very difficult to carry through. ♢

(b) Here we must not forget what we learned in the "kindergarten":

$$\int_{-\infty}^{+\infty} \frac{x^2}{1+x^6}\,dx = \frac{1}{3}\int_{-\infty}^{+\infty} \frac{d\left(x^3\right)}{1+\left(x^3\right)^2} = \frac{1}{3}\left[\operatorname{Arctan}\left(x^3\right)\right]_{-\infty}^{+\infty} = \frac{\pi}{3}.$$

Alternatively (and this time far more difficult) we see that we have the same simple poles as in **(a)**, and then we get by the general expression of the residuum at the pole z_0, where $z_0^6=-1$,

$$\operatorname{res}\left(\frac{z^2}{1+z^6}\,;\,z_0\right) = \frac{z_0^2}{6z_0^5} = \frac{z_0^3}{6z_0^6} = -\frac{1}{6}\,z_0^3,$$

hence

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^6}\,dx &= 2\pi i\left\{\text{res}\left(\frac{z^2}{1+z^6}\,;\,\exp\left(i\,\frac{\pi}{6}\right)\right)+\text{res}\left(\frac{z^2}{1+z^6}\,;\,i\right)+\text{res}\left(\frac{z^2}{1+z^6}\,;\,\exp\left(i\,\frac{5\pi}{6}\right)\right)\right\}\\ &= -\frac{2\pi i}{6}\left\{\exp\left(i\,\frac{\pi}{2}\right)+i^3+\exp\left(i\,\frac{5\pi}{2}\right)\right\} = -\frac{\pi i}{3}\{i-i+i\} = \frac{\pi}{3}.\end{aligned}$$

Example 4.7 *Compute* $\int_{-\infty}^{+\infty} \frac{dx}{1+x^8}$.

The function $\frac{1}{1+z^8}$ has a zero of order 8 at ∞ and no singularity on the real axis. Hence the improper integral is convergent, and its value can be found by the residues at the poles in the upper half plane. All poles are simple, and we have in the upper plane the four poles

$$z_1 = \exp\left(i\,\frac{\pi}{8}\right), \quad z_2 = \exp\left(i\,\frac{3\pi}{8}\right), \quad z_3 = \exp\left(i\,\frac{5\pi}{8}\right), \quad z_4 = \exp\left(i\,\frac{7\pi}{8}\right).$$

We have for every pole z_k that $z_k^9 = -1$, so

$$\text{res}\left(\frac{1}{1+z^8}\,;\,z_k\right) = \frac{1}{8z_k^7} = -\frac{1}{8}\,z_7.$$

Then by the residuum formula,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{1+x^8} &= 2\pi i\left(-\frac{1}{8}\right)\cdot\{z_1+z_2+z_3+z_4\} = -\frac{2\pi i}{8}\cdot 2i\left\{\sin\frac{\pi}{8}+\sin\frac{3\pi}{8}\right\} = \frac{\pi}{2}\cdot 2\,\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\\ &= \frac{\pi}{2}\sqrt{2}\cdot\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \frac{\pi}{2}\sqrt{1+\frac{1}{\sqrt{2}}}.\end{aligned}$$

Alternatively, it is possible to decompose. Here, we shall only show how one factorizes the denominator $1+x^8$:

$$\begin{aligned}1+x^8 &= \left(x^8+2x^4+1\right)-2x^4 = \left(x^4+1\right)^2-\left(\sqrt{2}\cdot x^2\right)^2 = \left(x^4+\sqrt{2}\,x^2+1\right)\left(x^4-\sqrt{2}\,x^2+1\right)\\ &= \left\{\left(x^4+2x^2+1\right)-(2-\sqrt{2})x^2\right\}\left\{\left(x^4+2x^2+1\right)-\left(2+\sqrt{2}\right)x^2\right\}\\ &= \left\{\left(x^2+1\right)^2-\left(\sqrt{2-\sqrt{2}}\cdot x\right)^2\right\}\left\{\left(x^2+1\right)^2-\left(\sqrt{2+\sqrt{2}}\,x\right)^2\right\}\\ &= \left(x^2+\sqrt{2-\sqrt{2}}\,x+1\right)\left(x^2-\sqrt{2-\sqrt{2}}\,x+1\right)\left(x^2+\sqrt{2+\sqrt{2}}\,x+1\right)\left(x^2-\sqrt{2+\sqrt{2}}\,x+1\right).\end{aligned}$$

Obviously, the following decomposition becomes very difficult, although it can in principle be carried through.

Example 4.8 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x\,dx}{\left(x^2+4x+13\right)^2}, \qquad (b)\quad \int_0^{+\infty} \frac{x^2 dx}{\left(x^2+a^2\right)^2}, \quad a \in \mathbb{R}_+.$$

(a) It follows from

$$x^2+4x+13 = (x+2)^2+3^2,$$

that the integrand has the double poles $-2 \pm 3i$, which do not lie on the real axis. The difference of the degrees is 3 with the denominator dominating, so the integral exists, and the value can be expressed by the residuum at $z = -2+3i$:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{x\,dx}{\left(x^2+4x+13\right)^2} = 2\pi i \cdot \text{res}\left(\frac{z}{\left(z^2+4z+13\right)^2}\,;\,-2+3i\right)\\
&= \frac{2\pi}{1!} \lim_{z\to -2+3i} \frac{d}{dz}\left\{\frac{z}{(z+2+3i)^2}\right\} = 2\pi i \lim_{z\to -2+3i}\left\{\frac{1}{(z+2+2i)^2} - \frac{2z}{(z+2+3i)^3}\right\}\\
&= 2\pi i\left\{\frac{1}{(6i)^2} - \frac{2(-2+2i)}{(6i)^3}\right\} = \frac{2\pi i}{(6i)^3}\{6i+4-6i\} = -\frac{\pi}{27}.
\end{aligned}$$

(b) Here we have the double poles $\pm i\,a \notin \mathbb{R}$, and since the difference in degrees is 2 with the denominator dominating, it follows by the symmetry that

$$\begin{aligned}
&\int_0^{+\infty} \frac{x^2}{\left(x^2+a^2\right)^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{\left(x^2+a^2\right)^2}\,dx = \pi i \cdot \text{res}\left(\frac{z^2}{\left(z^2+a^2\right)^2}\,;\,i\,a\right)\\
&= \pi i \lim_{z\to i\,a} \frac{d}{dz}\left\{\frac{z^2}{(z+i\,a)^2}\right\} = \pi i \lim_{z\to i\,a}\left\{\frac{2z}{(z+i\,a)^2} - \frac{2z^2}{(z+i\,a)^3}\right\}\\
&= \pi i\left\{\frac{2i\,a}{(2i\,a)^2} - \frac{2(i\,a)^2}{(2i\,a)^3}\right\} = \frac{\pi}{2a}\left\{\frac{(2i\,a)^2}{(2i\,a)^2} - \frac{1}{2}\,\frac{(2i\,a)^3}{(2i\,a)^3}\right\} = \frac{\pi}{4a}.
\end{aligned}$$

Example 4.9 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx, \qquad (b)\quad \int_0^{+\infty} \frac{x^2-1}{x^4+5x^2+4}\,dx.$$

(a) The integrand $\dfrac{z^2-z+2}{x^4+10x^2+9}$ has a zero of second order at ∞ and its poles are given by

$$z^2 = -5 \pm \sqrt{25-9} = -5 \pm 4,$$

i.e. the simple poles are

$$3i, \qquad -3i, \qquad i \quad \text{and} \quad -i.$$

None of these lies on the x-axis. Since $f(z)$ is analytic outside the poles and since we have the estimate

$$|f(z)| \leq \frac{C}{|z|^2} \qquad \text{for } |z| \geq 4 \quad \text{og} \quad \text{Im}(z) \geq 0,$$

we conclude that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx &= 2\pi i\,\{\text{res}(f;i)+\text{res}(f;3i)\} \\
&= 2\pi\left\{\lim_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)}+\lim_{z\to 3i}\frac{z^2-z+2}{(z^2+1)(z+3i)}\right\} = 2\pi i\left\{\frac{-1-i+2}{2i\cdot 8}+\frac{-9-3i+2}{-8\cdot 6i}\right\} \\
&= \frac{\pi}{24}\{3-3i+9+31-2\} = \frac{5\pi}{12}.
\end{aligned}$$

Alternatively, one may apply the traditional real method of integration, by using that we have proved that the integral exists. In particular,

$$\int_{-\infty}^{+\infty} \frac{-x\,dx}{x^4+10x^2+9} = 0,$$

because the integrand is an odd function. Then

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx &= \int_{-\infty}^{+\infty}\frac{x^2+2}{(x^2+9)(x^2+1)}\,dx = \frac{1}{8}\int_{-\infty}^{+\infty}\frac{dx}{x^2+1}+\frac{7}{8}\int_{-\infty}^{+\infty}\frac{dx}{x^2+9} \\
&= \pi\left\{\frac{1}{8}+\frac{7}{3\cdot 8}\right\} = \frac{10\pi}{24} = \frac{5\pi}{12}.
\end{aligned}$$

(b) The difference of the degrees is 2 where the denominator is dominating, and the denominator is furthermore positive for every real x. Hence, the improper integral is convergent. Since the integrand is an even function, it follows by the symmetry, followed by an application of the residuum formula that

$$\begin{aligned}
\int_0^{+\infty}\frac{x^2-1}{x^4+5x^2+4}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2-1}{(x^2+4)(x^2+1)}\,dx = \pi i\,\{\text{res}(f;i)+\text{res}(f;2i)\} \\
&= \pi i\left\{\lim_{z\to i}\frac{z^2-1}{(z^2+4)(z+i)}+\lim_{z\to 2i}\frac{z^2-1}{(z^2+1)(z+2i)}\right\} = \pi i\left\{\frac{-2}{3\cdot 2i}+\frac{-5}{(-3)\cdot 4i}\right\} \\
&= \pi\left\{\frac{5}{12}-\frac{1}{3}\right\} = \frac{\pi}{12}.
\end{aligned}$$

Alternatively we decompose:

$$\begin{aligned}
\int_0^{+\infty}\frac{x^2-1}{x^4+5x^2+4}\,dx &= \int_0^{+\infty}\frac{x^2-1}{(x^2+4)(x^2+1)}\,dx = -\frac{2}{3}\int_0^{+\infty}\frac{dx}{x^2+1}+\frac{5}{3}\int_0^{+\infty}\frac{dx}{x^2+4} \\
&= \left[-\frac{2}{3}\,\text{Arctan}\,x+\frac{1}{2}\cdot\frac{5}{3}\,\text{Arctan}\left(\frac{x}{2}\right)\right]_0^{+\infty} = \left(\frac{5}{6}-\frac{2}{3}\right)\frac{\pi}{2} = \frac{\pi}{12}.
\end{aligned}$$

Example 4.10 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1}, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)\left(x^2+4\right)}.$$

(a) The integrand $\dfrac{1}{z^2+z+1}$ has a zero of second order at ∞ and the poles

$$z = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2} \notin \mathbb{R}.$$

Hence, the improper integral exists, and we may find its value by means of the residuum at $-\dfrac{1}{2}+i\,\dfrac{\sqrt{3}}{2}$, i.e. at the pole in the upper half plane:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} &= 2\pi i \cdot \operatorname{res}\left(\frac{1}{z^2+z+1}\,;\,-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right) = 2\pi i \lim_{z\to -\frac{1}{2}+i\,\frac{\sqrt{3}}{2}} \frac{1}{2z+1} \\ &= 2\pi i \cdot \frac{1}{-1+i\sqrt{3}+1} = \frac{2\pi i}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.\end{aligned}$$

ALTERNATIVELY, the traditional computation gives

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} = \int_{-\infty}^{+\infty} \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{\sqrt{\frac{3}{4}}}\left[\operatorname{Arctan}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)\right]_{-\infty}^{+\infty} = \frac{\pi}{\sqrt{\frac{3}{4}}} = \frac{2\pi}{\sqrt{3}}.$$

(b) The integrand $\dfrac{1}{(z^2+1)(z^2+4)}$ has a zero of order 4 at ∞ and the simple poles $\pm i$ and $\pm 2i \notin \mathbb{R}$. Hence, the improper integral exists, and its value can be found by means of the residues at the poles in the upper half plane. We get

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i\left\{\operatorname{res}\left(\frac{1}{(z^2+1)(z^2+4)}\,;\,i\right)+\operatorname{res}\left(\frac{1}{(z^2+1)(z^2+4)}\,;\,2i\right)\right\}$$
$$= 2\pi i\left\{\frac{1}{2i\cdot 3}+\frac{1}{(-3)4i}\right\} = \pi\left\{\frac{1}{3}-\frac{1}{6}\right\} = \frac{\pi}{6}.$$

ALTERNATIVELY,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+4)} &= \frac{1}{3}\int_{-\infty}^{+\infty}\frac{dx}{x^2+1} - \frac{1}{3}\int_{-\infty}^{+\infty}\frac{dx}{x^2+4} = \frac{1}{3}\left[\operatorname{Arctan} x - \frac{1}{2}\operatorname{Arctan}\frac{x}{2}\right]_{-\infty}^{+\infty} \\ &= \frac{1}{3}\left\{\pi-\frac{\pi}{2}\right\} = \frac{\pi}{6}.\end{aligned}$$

Example 4.11 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}, \qquad (b)\quad \int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)(x^2+2x+2)}.$$

(a) Here $\dfrac{1}{z^2+2z+2}$ has a zero of second order at ∞ and simple poles at $z=-1\pm i\notin\mathbb{R}$. Hence, the improper integral is convergent, and its value can be found by a residuum formula. However, the easiest method here is actually the *traditional one*,

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2} = \int_{-\infty}^{+\infty}\frac{dx}{(x+1)^2+1} = [\operatorname{Arctan}(x+1)]_{-\infty}^{+\infty} = \pi.$$

For comparison we get by the *calculus of residues*,

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2} &= 2\pi i\cdot\operatorname{res}\left(\frac{1}{z^2+2z+2}\,;\,-1+i\right) = 2\pi i\lim_{z\to -1+i}\frac{1}{z+1+i} \\ &= \frac{2\pi i}{-1+i+1+i} = \frac{2\pi i}{2i} = \pi.\end{aligned}$$

(b) The integrand has a zero of order 4 at ∞, and the simple poles $\pm i$, $-1 \pm i \notin \mathbb{R}$, so we conclude that the improper integral is convergent, and its value is given by

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+2x+2)} \\
&= 2\pi i \left\{ \text{res}\left(\frac{1}{(z^2+1)(z^2+2z+2)} \,;\, i \right) + \text{res}\left(\frac{1}{(z^2+1)(z^2+2z+2)} \,;\, -1i \right) \right\} \\
&= 2\pi i \left\{ \frac{1}{2i} \cdot \frac{1}{-1+2i+2} + \frac{1}{\{(-1+i)^2+1\} \cdot 2i} \right\} = \pi \left\{ \frac{1}{1+2i} + \frac{1}{1-2i} \right\} \\
&= \pi \cdot \frac{1-2i+1+2i}{1+4} = \frac{2\pi}{5}.
\end{aligned}$$

Example 4.12 *1) Explain why the improper integral*

$$\int_{-\infty}^{+\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$$

is convergent, and find its value.

2) Compute the complex line integral

$$\oint_{|z|=3} \frac{z^2 dz}{(z^2+1)^2 (z^2+4)}.$$

1) The integrand is a rational function with a zero of order 4 at ∞ and with no poles on the real axis. The poles are $z = \pm i$ (double poles) and $z = \pm 2i$ (simple poles), so the integral is convergent, and its value can be found by a residuum formula,

$$\int_{-\infty}^{+\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = 2\pi i \left\{ \text{res}\left(\frac{z^2}{(z^2+1)^2(z^2+4)} \,;\, i \right) + \text{res}\left(\frac{z^2}{(z^2+1)^2(z^2+4)} \,;\, 2i \right) \right\}.$$

Here we get straight away,

$$\begin{aligned}
&\text{res}\left(\frac{z^2}{(z^2+1)^2(z^2+4)} \,;\, i \right) = \frac{1}{1!} \lim_{z\to i} \frac{d}{dz} \left\{ \frac{z^2}{(z+i)^2(z^2+4)} \right\} \\
&\lim_{z\to i} \left\{ \frac{2z}{(z+i)^2(z^2+4)} - \frac{2z^2}{(z+i)^3(z^2+4)} - \frac{z^2 \cdot 2z}{(z+i)^2(z^2+4)^2} \right\} \\
&= \frac{2i}{(2i)^3 \cdot 3} - \frac{-2}{(2i)^3 \cdot 3} - \frac{(-1)\cdot 2i}{(2i)^2 3^2} = -\frac{2i}{4\cdot 3} + \frac{2i}{8\cdot 3} - \frac{2i}{9\cdot 4} \\
&= \frac{i}{36}(-6+3-2) = -\frac{5i}{36},
\end{aligned}$$

and

$$\text{res}\left(\frac{z^2}{(z^2+1)^2(z^2+4)} \,;\, 2i \right) = \lim_{z\to 2i} \frac{z^2}{(z^2+1)^2(z+2i)} = \frac{-4}{(-3)^2 \cdot 4i} = \frac{4i}{36},$$

so by insertion,

$$\int_{-\infty}^{+\infty} \frac{x^2 dx}{\left(x^2+1\right)^2 \left(x^2+4\right)} = 2\pi i \left\{ -\frac{5i}{36} + \frac{4i}{36} \right\} = 2\pi \cdot \frac{1}{36} = \frac{\pi}{18}.$$

ALTERNATIVELY, we may first decompose to get

$$\frac{u}{(u+1)^2(u+4)} = \frac{A}{u+4} + \frac{B}{n+1} + \frac{C}{(u+1)^2}.$$

Here we immediately get

$$A = \frac{-4}{(-3)^2} = -\frac{4}{9} \qquad \text{and} \qquad C = \frac{-1}{3} = -\frac{1}{3}.$$

Then by insertion, rearrangement and reduction,

$$\begin{aligned}
\frac{B}{u+1} &= \frac{u}{(u+1)^2(u+4)} + \frac{4}{9}\,\frac{1}{u+4} + \frac{1}{3}\,\frac{1}{(u+1)^2} \\
&= \frac{1}{9} \cdot \frac{1}{(u+1)^2(u+4)} \left\{9u + 4(u+1)^2 + 3(u+4)\right\} \\
&= \frac{1}{9} \cdot \frac{1}{(u+1)^2(u+4)} \left\{4(u+1)^2 + 12(u+1)\right\} \\
&= \frac{4}{9} \cdot \frac{u+1+3}{(u+1)(u+4)} = \frac{4}{9} \cdot \frac{1}{u+1}.
\end{aligned}$$

Then put $u = x^2$ to get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{x^2}{\left(x^2+1\right)^2 \left(x^2+4\right)}\,dx \\
&= -\frac{4}{9} \int_{-\infty}^{+\infty} \frac{dx}{x^2+4} + \frac{4}{9} \int_{-\infty}^{+\infty} \frac{dx}{x^2+1} - \frac{1}{3} \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} \\
&= -\frac{4}{9} \cdot \frac{1}{2} \left[\operatorname{Arctan}\left(\frac{x}{2}\right)\right]_{-\infty}^{+\infty} + \frac{4}{9} \left[\operatorname{Arctan} x\right]_{-\infty}^{+\infty} - \frac{1}{3} \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} \\
&= \frac{2\pi}{9} - \frac{1}{3} \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2}.
\end{aligned}$$

We can now compute the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2}$$

in a number of ways:

a) We get by the *calculus of residues*,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} &= 2\pi i \cdot \operatorname{res}\left(\frac{1}{\left(z^2+1\right)^2}\,;\, i \right) = 2\pi i \cdot \frac{1}{1!} \lim_{z \to i} \frac{d}{dz} \left\{ \frac{1}{(z+i)^2} \right\} \\
&= 2\pi i \lim_{z \to i} \left\{ -\frac{2}{(z+i)^3} \right\} = 2\pi i \cdot \left(-\frac{2}{(2i)^3} \right) = \frac{2\pi i \cdot (-2i)}{8} = \frac{\pi}{2}.
\end{aligned}$$

b) ALTERNATIVELY, we get by a *partial integration*

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} &= \left[\frac{x}{x^2+1}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{2x\cdot x}{\left(x^2+1\right)^2}\,dx = \int_{-\infty}^{+\infty} \frac{2\left(x^2+1\right)-2}{\left(x^2+1\right)^2}\,dx \\ &= 2\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} - 2\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2},\end{aligned}$$

and then by a rearrangement,

$$\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

Finally, by insertion,

$$\int_{-\infty}^{+\infty} \frac{x^2}{\left(x^2+1\right)^2\left(x^2+4\right)}\,dx = \frac{2\pi}{9} - \frac{\pi}{6} = \frac{\pi}{18}\,(4-3) = \frac{\pi}{18}.$$

2) Since all pole lie inside $|z|=3$, and since we have a zero of order 4 at ∞, we get by changing the direction on the path of integration,

$$\oint_{|z|=3} \frac{z^2dz}{\left(z^2+1\right)^2\left(z^2+4\right)} = -\oint_{|z|=3}^{\star} \frac{z^2dz}{\left(z^2+1\right)^2\left(z^2+4\right)} = -2\pi\, i\cdot \text{res}\left(\frac{z^2}{\left(z^2+1\right)\left(z^2+4\right)}\,;\,\infty\right) = 0.$$

Example 4.13 *1) Find all complex solutions of the equation*

$$z^4+5z^2+4=0.$$

2) Prove that the improper integral

$$\int_0^{+\infty}\frac{2x^2-1}{x^4+5x^2+4}\,dx$$

is convergent, and find its value

1) We get by the factorization

$$0=z^4+5z^2+4=\left(z^2+1\right)\left(z^2+4\right),$$

the four roots

$$i,\qquad -i,\qquad 2i,\qquad -2i.$$

2) The integrand is a rational function with no pole on the x-axis and with a zero of second order at ∞. Hence, the improper integral is convergent. Since the integrand is even it follows by a reflection and the residues at the singularities i and $2i$ in the upper half plane that

$$\begin{aligned}\int_0^{+\infty}\frac{2x^2-1}{x^4+5x^2+4}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{2x^2-1}{x^4+5x^2+4}\,dx\\ &= \frac{2\pi i}{2}\left\{\operatorname{res}\left(\frac{2z^2-1}{z^4+5z^2+4}\,;\,i\right)+\ \operatorname{res}\left(\frac{2z^2-1}{z^4+5z^2+4}\,;\,2i\right)\right\}\\ &= \pi i\left\{\lim_{z\to i}\frac{2z^2-1}{4z^3+10z}+\lim_{z\to 2i}\frac{2z^2-1}{4z^3+10z}\right\}\\ &= \pi i\left\{\frac{1}{i}\cdot\frac{-2-1}{-4+10}+\frac{1}{2i}\cdot\frac{-8-1}{-16+10}\right\}=\pi\left\{-\frac{1}{2}+\frac{9}{16}\right\}=\frac{\pi}{4}.\end{aligned}$$

Alternatively, the traditional method of decomposition gives that

$$\frac{2x^2-1}{x^4+5x^2+4}=\frac{2x^2-1}{(x^2+1)(x^2+4)}=-\frac{1}{x^2+1}+\frac{3}{x^2+4},$$

hence

$$\int_0^{+\infty}\frac{2z^2-1}{x^4+5x^2+4}\,dx=-\int_0^{+\infty}\frac{1}{x^2+1}\,dx+\int_0^{+\infty}\frac{3}{x^2+4}\,dx=-\frac{\pi}{2}+\frac{3}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}.$$

Example 4.14 *Prove that the improper integral*

$$\int_{-\infty}^{+\infty} \frac{1+x^2}{1+x^4}\,dx$$

is convergent, and find its value.

We estimate the integrand for $|x| > 1$ in the following way,

$$0 < g(x) := \frac{1+x^2}{1+x^4} = \frac{1}{x^2}\,\frac{1+\left(\dfrac{1}{x}\right)^2}{1+\left(\dfrac{1}{x}\right)^4} < \frac{2}{x^2}.$$

Since the improper integral $\int_1^{+\infty} \dfrac{1}{x^2}\,dx$ is convergent, the given integral is also convergent.

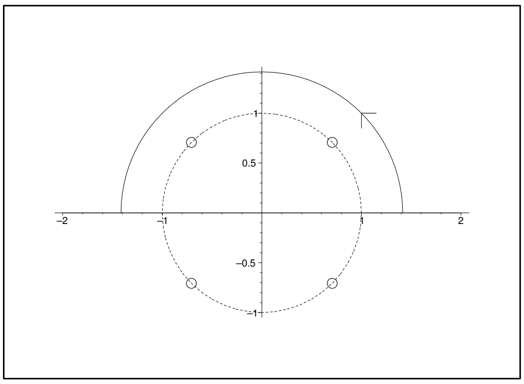


Figure 7: The curve C_R for $R > 1$ and the singularities $\pm\dfrac{1}{\sqrt{2}} \pm \dfrac{i}{\sqrt{2}}$.

We can now find the value of the improper integral as a Cauchy principal value via the residuum theorem.
The denominator has the *simple* poles at the points

$$z_p = \exp\left(i\,\frac{\pi}{4} + p\,\frac{\pi}{2}\right), \qquad p \in \{0,\,1,\,2,\,3\},$$

where the former two lie inside the circle of integration C_R. We get by a small computation

$$\operatorname{res}\left(g(z);z_p\right) = \frac{1+z_p^2}{4z_p^3} = -\frac{1}{4}\,z_p\left(1+z_p^2\right) = \begin{cases} -\dfrac{1}{4}\cdot\dfrac{1+i}{\sqrt{2}}\,(1+i), & p=0,\\[2ex] -\dfrac{1}{4}\cdot\dfrac{-1+i}{\sqrt{2}}\,(1-i), & p=1.\end{cases}$$

Both residues are $-\dfrac{i}{2\sqrt{2}}$, so

$$2\pi i\cdot\frac{-i}{\sqrt{2}} = \oint_{C_R} g(z)\,dz = \int_{-R}^{R} g(z)\,dz + \int_0^{R} g\left(R\,e^{it}\right)\cdot i\,R\,e^{i\theta}\,d\theta, \quad R>1.$$

If $R > 2$, then we have the following estimate on C_R,

$$\left|g\left(R\,e^{i\theta}\right)\right| = \frac{1}{R^2}\cdot\frac{\left|1+\dfrac{1}{R^2}\,e^{-2it}\right|}{\left|1+\dfrac{1}{R^4}\,e^{-4it}\right|} \le \frac{1}{R^2}\cdot\frac{1+\dfrac{1}{R^2}}{1-\dfrac{1}{R^4}} < \frac{1}{R^2}\cdot\frac{\frac{5}{4}}{\frac{15}{16}}.$$

It follows easily that the line integral along the circular arc tends to zero, when $R \to +\infty$, so we finally get by taking this limit,

$$\int_{-\infty}^{+\infty} \frac{1+x^2}{1+x^4}\,dx = \pi\sqrt{2}.$$

Example 4.15 *Given the function*

$$f(z) = \frac{1}{z^3+1},$$

and for every $R > 1$ *the closed curve* $\gamma_R = I + II + III$ *(see the figure), enclosing the domain*

$$U_R = \left\{ z = r\, e^{it} \;\middle|\; 0 < r < R \text{ and } 0 < t < \frac{2\pi}{3} \right\}.$$

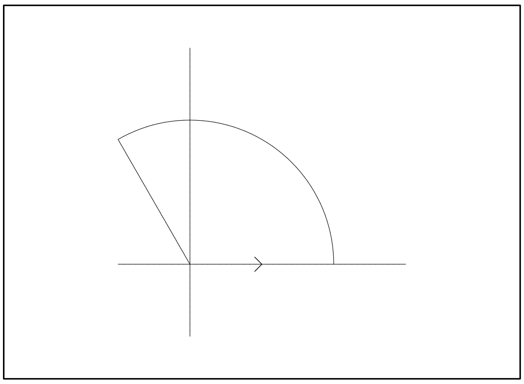


Figure 8: The curve $\gamma_R = I + II + III$, enclosing U_R. Here, $I = [0, R]$ is an interval on the x-axis, II is the circular arc, and III is the oblique line.

1) Find

$$\int_{\gamma_R} f\, dz.$$

2) Prove that the line integral along the circular arc II *tends towards 0, when* R *tends towards* $+\infty$.

3) Prove that

$$\int_0^{+\infty} \frac{1}{x^3+1}\, dx = \frac{2\pi}{3\sqrt{3}}.$$

1) The function $f(z) = \dfrac{1}{z^3+1}$ has the simple poles

$$z_1 = -1, \qquad z_2 = \exp\left(-i\,\frac{\pi}{3}\right), \qquad z_3 = \exp\left(i\,\frac{\pi}{3}\right).$$

If $R > 1$, then only $z_3 = \exp\left(i\,\dfrac{\pi}{3}\right)$ lies inside γ_R, so it follows by the residuum theorem that

$$\begin{aligned} \oint_{\gamma_R} f(z)\, dz &= 2\pi i \cdot \operatorname{res}(f; z_3) = 2\pi i \cdot \frac{1}{3z_3^2} = \frac{2\pi i}{3} \cdot \frac{z_3}{z_3^3} \\ &= -\frac{2\pi i}{3} \exp\left(i\,\frac{\pi}{3}\right) = -\frac{2\pi i}{3} \left\{ \frac{1}{2} + i\,\frac{\sqrt{3}}{2} \right\} = \frac{\pi}{3} \left\{ \sqrt{3} - i \right\}. \end{aligned}$$

2) We get along II the estimate

$$\left|\int_{II} \frac{dz}{z^3+1}\right| \le \frac{1}{R^3-1}\cdot\frac{2\pi}{3}\,R \to 0 \qquad \text{for } R\to+\infty.$$

3) Along III we choose the parametric description

$$(R-r)\,\exp\left(i\,\frac{2\pi}{3}\right), \qquad r\in[0,R],$$

so

$$\int_{III} f(z)\,dz = \int_0^R \frac{-\exp\left(i\,\frac{2\pi}{3}\right)}{1+(R-r)^3\exp(2\pi i)}\,dr = -\exp\left(i\,\frac{2\pi}{3}\right)\int_0^R \frac{dx}{x^3+1}.$$

Then by insertion and the limit $R\to+\infty$,

$$\begin{aligned}\frac{\pi}{3}\,(\sqrt{3}-i) &= \lim_{R\to+\infty}\oint_{\gamma_R} f(z)\,dz = 0+\left\{1-\exp\left(i\,\frac{2\pi}{3}\right)\right\}\int_0^{+\infty}\frac{dx}{x^3+1}\\ &= \left\{\frac{3}{2}-i\frac{\sqrt{3}}{2}\right\}\int_0^{+\infty}\frac{dx}{x^3+1} = \frac{\sqrt{3}}{2}\,(\sqrt{3}-i)\int_0^{+\infty}\frac{dx}{x^3+1},\end{aligned}$$

so by a rearrangement,

$$\int_0^{+\infty}\frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

Remark 4.4 The integral can in fact also be computed by more elementary methods. We get by a decomposition,

$$\begin{aligned}\frac{1}{x^3+1} &= \frac{1}{(x+1)\,(x^2-x+1)} = \frac{1}{3}\,\frac{1}{x+1}+\frac{1}{3}\cdot\frac{3-x^2+x-1}{(x^2-x+1)\,(x+1)}\\ &= \frac{1}{3}\,\frac{1}{x+1}-\frac{1}{3}\cdot\frac{x-2}{x^2-x+1} = \frac{1}{3}\cdot\frac{1}{x+1}-\frac{1}{3}\cdot\frac{\left(x-\frac{1}{2}\right)-\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}},\end{aligned}$$

hence

$$\begin{aligned}\int_0^{+\infty}\frac{dx}{x^3+1} &= \frac{1}{3}\int_0^{+\infty}\frac{dx}{x+1}-\frac{1}{6}\int_0^{+\infty}\frac{2x-1}{x^2-x+1}\,dx+\frac{1}{2}\int_0^{+\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\\ &= \frac{1}{6}\left[\ln\left(\frac{x^2+2x+1}{x^2-x+1}\right)\right]_0^{+\infty}+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\left[\text{Arctan}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^{+\infty}\\ &= 0+\frac{1}{\sqrt{3}}\left\{\frac{\pi}{2}+\ \text{Arctan}\left(\frac{1}{\sqrt{3}}\right)\right\} = \frac{1}{\sqrt{3}}\left\{\frac{\pi}{2}+\frac{\pi}{6}\right\} = \frac{2\pi}{3\sqrt{3}}. \qquad \Diamond.\end{aligned}$$

Example 4.16 *Given the function*

$$f(z)=\frac{z^2}{z^4+z^2+1}.$$

1) Find all isolated singularities of f in $\mathbb{C}$, and specify their types.

2) Prove by using the calculus of residues that the improper integral

$$\int_0^{+\infty}\frac{x^2}{x^4+x^2+1}\,dx$$

is convergent of the value $\dfrac{\pi}{2\sqrt{3}}$.
One may use that $f(x)$ is an even function.

1) First note that

$$\left(z^2-1\right)\left(z^4+z^2+1\right)=z^6-1=0$$

for

$$z=\exp\left(i\,\frac{p\pi}{3}\right),\qquad p\in\mathbb{Z}.$$

When we again remove the roots $z=\pm 1$ of the auxiliary factor z^2-1, we see that the simple poles are

$$\exp\left(i\,\frac{\pi}{3}\right)=\frac{1}{2}+i\,\frac{\sqrt{3}}{2},\qquad \exp\left(i\,\frac{2\pi}{3}\right)=-\frac{1}{2}+i\,\frac{\sqrt{3}}{2},$$

$$\exp\left(-i\,\frac{\pi}{3}\right)=\frac{1}{2}-i\,\frac{\sqrt{3}}{2},\qquad \exp\left(-i\,\frac{2\pi}{3}\right)=-\frac{1}{2}-i\,\frac{\sqrt{3}}{2}.$$

2) Since we have a zero of second degree at ∞, and since we do not have any pole on the x-axis, we conclude that the improper integral is convergent. The integrand is even, so we get by an extended residuum theorem that

$$\begin{aligned}\int_0^{+\infty}\frac{x^2}{x^4+x^2+1}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2}{x^4+x^2+1}\,dx\\ &= \pi i\left\{\text{res}\left(f(z)\,;\,\exp\left(i\,\frac{\pi}{3}\right)\right)+\text{res}\left(f(z)\,;\,\exp\left(i\,\frac{2\pi}{3}\right)\right)\right\},\end{aligned}$$

because $\exp\left(i\,\dfrac{\pi}{3}\right)$ and $\exp\left(i\,\dfrac{2\pi}{3}\right)$ are the only singularities in the upper half plane.

Using the rearrangement

$$f(z)=\frac{z^2}{z^4+z^2+1}=\frac{z^2\left(z^2-1\right)}{z^6-1},$$

we get

$$\begin{aligned}\operatorname{res}\left(f(z)\,;\,\exp\left(i\,\frac{\pi}{3}\right)\right) &= \left[\frac{z^2\left(z^2-1\right)}{6z^5}\right]_{z=\exp\left(i\,\frac{\pi}{3}\right)} = \frac{1}{6}\cdot\exp(i\pi)\cdot\left\{\exp\left(\frac{2i\pi}{3}\right)-1\right\} \\ &= -\frac{1}{6}\left\{-\frac{1}{2}+i\,\frac{3}{2}-1\right\} = \frac{1}{12}\,\{3-i\sqrt{3}\},\end{aligned}$$

and

$$\begin{aligned}\operatorname{res}\left(f(z)\,;\,\exp\left(i\,\frac{2\pi}{3}\right)\right) &= \left[\frac{z^2\left(z^2-1\right)}{6z^5}\right]_{z=\exp\left(i\,\frac{2\pi}{3}\right)} = \frac{1}{6}\cdot 1\cdot\left\{\exp\left(\frac{4i\pi}{3}\right)-1\right\} \\ &= -\frac{1}{6}\left\{-\frac{1}{2}-i\,\frac{3}{2}-1\right\} = \frac{1}{12}\,\{-3-i\sqrt{3}\},\end{aligned}$$

hence by insertion,

$$\int_0^{+\infty}\frac{x^2}{x^4+x^2+1}\,dx = \frac{\pi i}{12}\cdot(-2i\sqrt{3}) = \frac{\pi\sqrt{3}}{6} = \frac{\pi}{2\sqrt{3}}.$$

Remark 4.5 Since

$$z^4+z^2+1=z^4+2z^2+1-z^2=\left(z^2+1\right)^2-z^2=\left(z^2+z+1\right)\left(z^2-z+1\right),$$

it is of course also possible – though not quite easy – to use the method of decomposition. This variant is left to the reader as an exercise. ◊

Example 4.17 *Given the function*

$$f(z)=\frac{z^2}{z^4+1},$$

and for every $R>1$ *a positively oriented curve*

$$\Gamma_R=I_R+II_R+III_R,$$

(cf. the figure), which surrounds the domain

$$U_R=\left\{z=r\,e^{it}\;\middle|\;0<r<R \text{ og } 0<t<\frac{\pi}{2}\right\}.$$

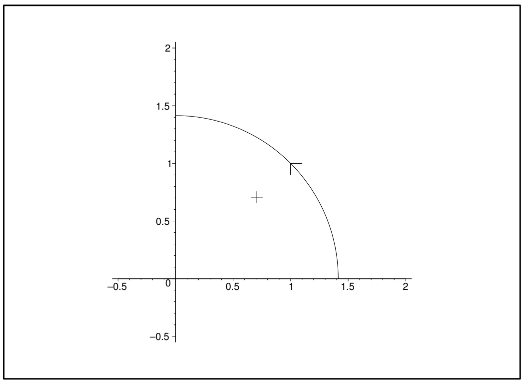


Figure 9: The curve Γ_R, starting with $I_R=[0,R]$ on the x-axis and with the singularity $\exp\left(i\,\frac{\pi}{4}\right)$ inside the curve.

1) Prove that

$$\oint_{\Gamma_R} f(z)\,dz=\frac{\pi}{2\sqrt{2}}\,(1+i).$$

2) Show that the line integral along the circular arc II_R *tends towards 0 for* R *tending towards* $+\infty$*, and find the value of*

$$\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx.$$

1) The function $f(z)$ has the four simple poles

$$z_p = \exp\left(i\left\{\frac{\pi}{4}+p\,\frac{\pi}{2}\right\}\right), \qquad p \in \{0,\,1,\,2,\,3\}.$$

Of these only

$$z_0 = \exp\left(i\,\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\,(1+i)$$

lies inside Γ_R, when $R>1$. Then by *Cauchy's residuum theorem*,

$$\begin{aligned}\oint_{\Gamma_R} f(z)\,dz &= 2\pi i\,\operatorname{res}\left(\frac{z^2}{z^4+1}\,;\,z_0\right) = 2\pi i\cdot\frac{z_0^2}{4z_0^3} = \frac{\pi i}{2}\cdot\frac{1}{z_0}\\ &= i\cdot\frac{\pi}{2}\cdot\frac{\sqrt{2}}{1+i} = \frac{\pi}{\sqrt{2}}\cdot\frac{i(1-i)}{2} = \frac{\pi}{2\sqrt{2}}\,(1+i).\end{aligned}$$

2) We use along the circular arc II_R the parametric description $z(t)=R\,e^{it}$, $t\in\left[0\,,\,\frac{\pi}{2}\right]$, so we get the estimate for $R>1$,

$$\left|\int_{II_R}\frac{z^2}{z^4+1}\,dz\right| \le \int_0^{\frac{\pi}{2}}\frac{R^2}{R^4-1}\cdot R\,dt = \frac{\pi}{2}\cdot\frac{1}{R-\dfrac{1}{R^3}} \to 0$$

when $R\to+\infty$.

Finally, we use along III_R on the imaginary axis the parametric description $z(t)=(R-t)i$, $t\in[0,R]$, giving

$$\int_{III_R}\frac{z^2}{z^4+1}\,dz = \int_0^R\frac{(R-t)^2i^2}{(R-t)^4i^4+1}\cdot(-i)dt = i\int_0^R\frac{t^2}{t^4+1}\,dt.$$

Then by **(1)** we get by insertion and taking the limit $R\to+\infty$,

$$\frac{\pi}{2\sqrt{2}}\,(1+i) = (1+i)\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx,$$

hence

$$\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

ALTERNATIVELY, we get by a decomposition,

$$\begin{aligned}\frac{x^2}{x^4+1} &= \frac{x^2}{x^4+2x^2+1-2x^2} = \frac{x^2}{\left(x^2+1\right)^2-\left(\sqrt{2}\,x\right)^2} = \frac{x^2}{\left(x^2+\sqrt{2}\,x+1\right)\left(x^2-\sqrt{2}\,x+1\right)}\\ &= \frac{ax+b}{\left(x^2+\sqrt{2}\,x+1\right)}+\frac{cx+d}{\left(x^2-\sqrt{2}\,x+1\right)},\end{aligned}$$

thus

$$\begin{aligned}x^2 &= (ax+b)\left(x^2-\sqrt{2}\,x+1\right)+(cx+d)\left(x^2+\sqrt{2}\,x+1\right)\\ &= (a+c)x^3+(b-\sqrt{2}a+d+\sqrt{2}c)x^2+(a-\sqrt{2}b+c+\sqrt{2}d)x+b+d.\end{aligned}$$

By identifying the coefficients we clearly obtain that

$$a+c=0 \qquad \text{and} \qquad b+d=0,$$

so

$$\sqrt{2}\,(-a+c)=1, \qquad \text{thus } c=-a=\frac{1}{\sqrt{2}},$$

and $b=d=0$. Hence

$$\begin{aligned}
\frac{x^2}{x^4+1} &= \frac{1}{2\sqrt{2}}\left\{\frac{x}{x^2-\sqrt{2}\,x+1}-\frac{x}{x^2+\sqrt{2}\,x+1}\right\}\\
&= \frac{1}{4\sqrt{2}}\left\{\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}+\frac{\sqrt{2}}{x^2-\sqrt{2}\,x+1}-\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}+\frac{\sqrt{2}}{x^2+\sqrt{2}\,x+1}\right\}\\
&= \frac{1}{4\sqrt{2}}\left\{\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}-\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}\right\}+\frac{1}{4}\left\{\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right\}.
\end{aligned}$$

Clearly, the improper integral $\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx$ is convergent, and

$$\begin{aligned}
&\int_0^{+\infty}\frac{x^2}{x^4+1}\,dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2}{x^4+1}\,dx\\
&= \frac{1}{2}\lim_{R\to+\infty}\frac{1}{4\sqrt{2}}\int_{-R}^{R}\left\{\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}-\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}\right\}dx\\
&\qquad +\frac{1}{2}\cdot\frac{1}{4}\left[\sqrt{2}\,\mathrm{Arctan}\left(\sqrt{2}\,x-1\right)+\sqrt{2}\,\mathrm{Arctan}\left(\sqrt{2}\,x+1\right)\right]_{-\infty}^{+\infty}\\
&= \frac{1}{8\sqrt{2}}\lim_{R\to+\infty}\left[\ln\left(\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right)\right]_{-R}^{R}+\frac{1}{4\sqrt{2}}\cdot(\pi+\pi)=0+\frac{\pi}{2\sqrt{2}}=\frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Example 4.18 *Given the rational function*

$$f(z) = \frac{z^2+z+1}{z^4+z^2+1}.$$

1) Find all the isolated singularities of f in $\mathbb{C}$, and specify their types.

2) Prove by calculus of residues that

$$p.v.\int_{-\infty}^{+\infty} \frac{x^2+x+1}{x^4+x^2+1}\,dx = \frac{2\pi}{\sqrt{3}}.$$

1) **First variant.** If $z \neq \pm 1$, then

$$\frac{z^2+z+1}{z^4+z^2+1} = \frac{(z+1)(z-1)\left(z^2+z+1\right)}{\left(z^2-1\right)\left(z^4+z^2+1\right)} = (z+1)\cdot\frac{z^3-1}{z^6-1} = \frac{z+1}{z^3+1} = \frac{1}{z^2-z+1},$$

and the simple poles are

$$z_1 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2} \qquad \text{and} \qquad z_2 = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

Second variant. Obviously, $z = \pm 1$ are not poles. Now,

$$\left(z^2-1\right)\left(z^4+z^2+1\right) = z^6 - 1 = 0$$

for

$$z = \exp\left(i\,\frac{p\pi}{3}\right), \qquad \text{for } p \in \{0,\,1,\,2,\,3,\,4,\,5\},$$

and since we shall remove $p = 0$ and $p = 3$, because they stem from the auxiliary factor $z^2 - 1$, the singularities are

$$\begin{aligned}
\tilde{z}_1 &= \exp\left(i\,\frac{\pi}{3}\right) = \frac{1}{2} + i\,\frac{\sqrt{3}}{2},\\
\tilde{z}_2 &= \exp\left(i\,\frac{2\pi}{3}\right) = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},\\
\tilde{z}_4 &= \exp\left(i\,\frac{4\pi}{3}\right) = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2},\\
\tilde{z}_5 &= \exp\left(i\,\frac{5\pi}{3}\right) = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}.
\end{aligned}$$

Each one of these is at most a simple pole, and they could even be removable singularities.

Analogously,

$$(z-1)\left(z^2+z+1\right) = z^3 - 1,$$

so since $z = 1$ is a "false" singularity coming from the auxiliary factor $z - 1$, the numerator has the roots

$$\tilde{z}_2 = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \text{and} \quad \tilde{z}_4 = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},$$

which will cancel the same zeros in the denominator. Thus

$$\tilde{z}_2 = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \text{and} \quad \tilde{z}_4 = -\frac{1}{2} + i\,\frac{sqrt3}{2},$$

are removable singularities, while

$$\tilde{z}_1 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \text{and} \quad \tilde{z}_5 = \frac{1}{2} - i\,\frac{\sqrt{3}}{2},$$

are simple poles.

2) The integrand is defined on $\mathbb{R}$, and since the integrand has a zero of order 2 at ∞, the improper integral is convergent, and we do not need the notation "p.v." (= "principal value"). The improper integral can be computed in a number of ways.

First method. By a *simple integration* (without using the calculus of residues) it follows from the **first solution** above that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x^2+x+1}{x^4+x^2+1}\,dx &= \int_{-\infty}^{+\infty} \frac{dx}{x^2-x+1} = \int_{-\infty}^{+\infty} \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{2}{3}}\\
&= \frac{2}{\sqrt{3}]}\left[\operatorname{Arctan}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)\right]_{-\infty}^{+\infty} = \frac{2\pi}{\sqrt{3}}.
\end{aligned}$$

Second method. We shall in the *calculus of residues* use that only $z_1 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2} = \exp\left(i\,\frac{\pi}{3}\right)$ lies in the upper half plane and that $z_1^3 = -1$. Then

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^2+x+1}{x^4+x^2+1}\,dx &= 2\pi i \operatorname{res}\left(\frac{z+1}{z^3+1}\,;\,z_1\right) = 2\pi i\left\{\frac{z_1+1}{3z_1^2}\right\} = \frac{2\pi i}{3}\cdot\frac{z_1^2+z_1}{z_1^3}\\ &= -\frac{2\pi i}{3}\,\{\tilde{z}_2+\tilde{z}_1\} = -\frac{2\pi i}{3}\cdot\left(2\cdot i\,\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{\sqrt{3}}.\end{aligned}$$

Third method. *Calculus of residues without a reformulation* gives the following difficult computations,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^2+x+1}{x^4+x^2+1}\,dx &= 2\pi i \operatorname{res}\left(\frac{z^2+z+1}{z^4+z^2+1}\,;\,z_1\right)\\ &= 2\pi i\left\{\frac{z_1^2+z_1+1}{4z_1^3+2z_1}\right\} = 2\pi i\left\{\frac{\tilde{z}_2+\tilde{z}_1+1}{4\tilde{z}_3+2\tilde{z}_1}\right\}\\ &= 2\pi i\left\{\frac{\left(-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)+1}{4(-1)+2\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)}\right\}\\ &= 2\pi i\cdot\frac{1+i\sqrt{3}}{-4+1+i\sqrt{3}} = \frac{2\pi i}{\sqrt{3}}\cdot\frac{1+i\sqrt{3}}{-\sqrt{3}+i}\\ &= \frac{2\pi}{\sqrt{3}}\cdot\frac{i-\sqrt{3}}{-\sqrt{3}+i} = \frac{2\pi}{\sqrt{3}}.\end{aligned}$$

5 Improper integrals, where the integrand is a rational function times a trigonometric function

Example 5.1 *The transfer function of a RC-filter is given by*

$$f(z) = \frac{1}{1 + 2\pi\, i\, RC\, z}.$$

Find the corresponding answer.

The corresponding answer is given by the improper integral

$$h(t) = \int_{-\infty}^{+\infty} f(x)\, e^{2\pi i x t}\, dt = \frac{1}{2\pi i RC} \int_{-\infty}^{+\infty} \frac{1}{x - \dfrac{i}{2\pi RC}}\, e^{i2\pi x t}\, dx.$$

Here, $z_1 = \dfrac{i}{2\pi RC}$ is the only pole of the corresponding analytic function

$$f(z) = \frac{1}{2\pi i RC} \cdot \frac{1}{z - \dfrac{i}{2\pi RC}},$$

and it is obvious that there exist constants k, and $r > \dfrac{1}{2\pi RC}$, such that we have the estimate

$$|f(z)| < \frac{k}{|z|} \qquad \text{for } |z| > r.$$

Since $f(z)$ does not have any singularity in the lower half plane, we conclude from the corresponding residuum formula, which here is empty that

$$h(t) = \int_{-\infty}^{+\infty} f(x)\, e^{2\pi i x t}\, dx = \frac{1}{2\pi i RC} \int_{-\infty}^{+\infty} \frac{1}{x - \dfrac{i}{2\pi RC}}\, e^{i2\pi x t}\, dx = 0 \qquad \text{for } t < 0.$$

If instead $t > 0$, then, since we have already checked the assumptions of the validity of the residuum formula,

$$h(t) = \int_{-\infty}^{+\infty} f(x)\, e^{2\pi i x t} = \frac{2\pi i}{2\pi i RC} \operatorname{res}\left(\frac{e^{i2\pi z t}}{z - \dfrac{i}{2\pi RC}}\, ;\, \frac{i}{2\pi RC} \right) = \frac{1}{RC} \cdot \exp\left(-\frac{t}{RC} \right).$$

Example 5.2 *Compute the improper integrals*

$$\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+1}\,dx \qquad og \qquad \int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1}\,dx.$$

Here we must consider the analytic function

$$\frac{z\,e^{iz}}{z^2+1}, \qquad \text{for } z \neq \pm i,$$

instead of $\dfrac{z\cos z}{z^2+1}$ and $\dfrac{z\sin z}{z^2+1}$. Clearly, the rational function $\dfrac{z}{z^2+1}$ has a zero of first order at ∞ and no pole on the X-axis, so the assumptions of the residuum formula are fulfilled. Since $m = 1 > 0$, the pole $z = i$ in the upper half plane is the only relevant singularity. Hence by the residuum formula,

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{x^2+1}\,dx = 2\pi i \cdot \operatorname{res}\left(\frac{z\,e^{iz}}{z^2+1}\,;\,i\right) = 2\pi i \cdot \frac{i\,e^{i\cdot i}}{i+i} = \frac{\pi i}{e}.$$

Then by separating into the real and the imaginary parts,

$$\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2+1}\,dx = 0 \qquad \text{og} \qquad \int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{e}.$$

Example 5.3 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2+9}\,dx, \qquad (b)\quad \int_{0}^{+\infty} \frac{\cos \pi\, x}{x^4+4}\,dx.$$

(a) Here $\dfrac{x}{x^2+9}$ is a rational function of real coefficients and with a zero of first order at ∞. The denominator does not have real zeros and

$$\left|\frac{z}{z^2+9}\right| \le \frac{C}{|z|} \qquad \text{for } |z| \ge 4,$$

so we conclude that the improper integral is convergent. Using that

$$\sin x = \operatorname{Im}\left(e^{i\cdot 1\cdot x}\right),$$

where $1 > 0$, it follows by the residuum formula that

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2+9}\,dx &= \operatorname{Im}\left\{2\pi i \cdot \operatorname{res}\left(\frac{z\,e^{i\,z}}{z^2+9}\,;\,3i\right)\right\} = 2\pi\operatorname{Re}\left\{\lim_{z\to 3i} \frac{z\,e^{iz}}{z+3i}\right\} = 2\pi \operatorname{Re}\left\{\frac{3i\,e^{-3}}{3i+3i}\right\}\\ &= 2\pi \cdot \frac{3\,e^{-3}}{6} = \frac{\pi}{e^3}.\end{aligned}$$

(b) Here $\dfrac{1}{z^4+1}$ has a zero of fourth order at ∞ and no poles on the x-axis. Hence, the integral is convergent. Since $\dfrac{\cos \pi x}{x^4+4}$ is an even function, it follows by the symmetry and a residuum formula that

$$\begin{aligned}\int_0^{+\infty} \frac{\cos \pi x}{x^4+4}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos \pi x}{x^4+4}\,dx = \frac{1}{2}\,(\mathrm{Re})\left\{\int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x^4+4}\,dx\right\} \\ &= \pi i\left\{\mathrm{res}\left(\frac{e^{i\pi z}}{z^4+4}\,;\,1+i\right) + \mathrm{res}\left(\frac{e^{i\pi z}}{z^4+4}\,;\,-1+i\right)\right\} \\ &= \pi i\left\{\left[\frac{e^{i\pi z}}{4z^3}\right]_{z=1+i} + \left[\frac{e^{i\pi z}}{4z^3}\right]_{z=-1+i}\right\} = \pi i\left\{\left[\frac{z\,e^{i\pi z}}{4z^4}\right]_{z=1+i} + \left[\frac{z\,e^{i\pi\, z}}{4z^4}\right]_{z?-1+i}\right\} \\ &= \frac{\pi i}{4\cdot(-4)}\cdot\left\{(1+i)e^{\pi(-1+i)} + (-1+i)e^{\pi(-1-i)}\right\} = \frac{\pi i}{16}\,e^{-\pi}\cdot 2i = -\frac{1}{8}\,\pi\,e^{-\pi}.\end{aligned}$$

Example 5.4 *Compute*

$$(a)\quad \int_0^{+\infty} \frac{\cos x}{(1+x^2)^3}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx.$$

(a) We see that $\dfrac{1}{(1+x^2)^3}$ has a zero or order 6 at ∞ and no real pole. Hence the improper integral exists. The integrand is an even function, so by the symmetry, followed by an application of a residuum formula,

$$\begin{aligned}&\int_0^{+\infty} \frac{\cos x}{(1+x^2)^3}\,dx = \frac{1}{2}\,(\mathrm{Re})\int_{-\infty}^{+\infty} \frac{e^{ix}}{(1+x^2)^3}\,dx = \pi i\cdot \mathrm{res}\left(\frac{e^{iz}}{(1+z^2)^3}\,;\,i\right) \\ &= \pi i\cdot\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left\{\frac{e^{iz}}{(z+i)^3}\right\} = \frac{\pi i}{2}\lim_{z\to i}\frac{d}{dz}\left\{\frac{i\,e^{iz}}{(z+i)^3} - \frac{3\,e^{iz}}{(z+i)^4}\right\} \\ &= \frac{\pi i}{2}\lim_{z\to i}\left\{\frac{-e^{iz}}{(z+i)^3} - \frac{6i\,e^{iz}}{(z+i)^4} + \frac{12\,e^{iz}}{(z+i)^5}\right\} = \frac{\pi i}{2}\left\{-\frac{e^{-1}}{(2i)^3} - \frac{6i\,e^{-1}}{(2i)^4} + 12\cdot\frac{e^{-1}}{(2i)^5}\right\} \\ &= \frac{2\pi i}{4(2i)^5}\cdot\frac{1}{e}\,\left\{-(2i)^2 - 6i\cdot(2i) + 12\right\} = \frac{\pi}{4\cdot 2^4\,e}\,\{4+12+12\} = \frac{7\pi}{16\,e}.\end{aligned}$$

(b) We get by a decomposition that

$$\frac{1}{(1+x^2)(4+x^2)} = \frac{1}{3}\cdot\frac{1}{x^2+1} - \frac{1}{3}\cdot\frac{1}{x^2+4},$$

so it follows immediately that the integral is convergent. Then by the residuum formula,

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx = \frac{1}{3}\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,dx - \frac{1}{3}\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+4}\,dx \\ &= \frac{1}{3}\,\mathrm{Re}\left\{2\pi i\cdot \mathrm{res}\left(\frac{e^{iz}}{z^2+1}\,;\,i\right)\right\} - \frac{1}{3}\,\mathrm{Re}\left\{2\pi i\cdot \mathrm{res}\left(\frac{e^{iz}}{z^2+4}\,;\,2i\right)\right\} \\ &= \frac{1}{3}\,\mathrm{Re}\left\{2\pi i\cdot\frac{e^{-1}}{2i}\right\} - \frac{1}{3}\,\mathrm{Re}\left\{2\pi i\cdot\frac{e^{-2}}{4i}\right\} = \frac{1}{3}\cdot\frac{\pi}{e} - \frac{1}{3}\cdot\frac{1}{2e^2} = \frac{\pi}{6e^2}\,(2e-1).\end{aligned}$$

Example 5.5 *Prove that*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\cos x}{\left(1+x^2\right)^2}\,dx = \frac{\pi}{e}.$$

Clearly, both integrals are convergent, and we can apply the residuum formula. Thus

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \operatorname{Re}\left\{\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}\,dx\right\} = \operatorname{Re}\left\{2\pi i \cdot \operatorname{res}\left(\frac{e^{iz}}{z^2+1}\,;\, i\right)\right\} = 2\pi \operatorname{Re}\left\{i \cdot \frac{e^{-1}}{2i}\right\} = \frac{\pi}{e},$$

and

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\cos x}{\left(1+x^2\right)^2}\,dx &= \operatorname{Re}\left\{\int_{-\infty}^{+\infty} \frac{e^{ix}}{\left(x^2+1\right)^2}\,dx\right\} = \operatorname{Re}\left\{2\pi i \cdot \operatorname{res}\left(\frac{e^{iz}}{\left(z^2+1\right)^2}\,;\, i\right)\right\} \\
&= 2\pi \operatorname{Re}\left\{i \cdot \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\right\} = 2\pi \operatorname{Re}\left\{i\left[\frac{i\,e^{iz}}{(z+i)^2} - \frac{2\,e^{iz}}{(z+i)^3}\right]_{z=i}\right\} \\
&= 2\pi \operatorname{Re}\left\{i\left(\frac{i\,e^{-1}}{(2i)^2} - \frac{2e^{-1}}{(2i)^3}\right)\right\} = 2\pi \operatorname{Re}\left\{\frac{e^{-1}}{4} - \frac{2\,e^{-1}}{8i^2}\right\} = \frac{2\pi}{e}\cdot\left(\frac{1}{4}+\frac{1}{4}\right) = \frac{\pi}{e}.
\end{aligned}$$

Example 5.6 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x\,\cos x}{x^2-2x+10}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2-2x+10}\,dx.$$

It follows from

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ox}}{x^2-2x+10}\,dx = \int_{-\infty}^{+\infty} \frac{x\,\cos x}{x^2-2x+10}\,dx + i\int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2-2x+10}\,dx,$$

that it suffices to prove that

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{x^2-2x+10}\,dx$$

exists and to find the value of this integral.

We see that $\dfrac{z}{z^2-2z+10}$ has a first order zero at ∞ and simple poles at $z=1\pm 3i\notin\mathbb{R}$, hence the improper integral exists. Since $m=1>0$, we can compute the integral by a residuum formula,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{x^2-2x+10}\,dx &= 2\pi i\cdot \operatorname{res}\left(\frac{z\,e^{iz}}{z^2-2z+10}\,;\,1+3i\right) = 2\pi i \lim_{z\to 1+3i} \frac{z\,e^{iz}}{z-1+3i}\\ &= 2\pi i\cdot\frac{(1+3i)e^{i(1+3i)}}{6i} = \frac{\pi}{3}\,(1+3i)e^{-3}\{\cos 1 + i\,\sin 1\}.\end{aligned}$$

Then by a separation into the real and the imaginary part,

(a)

$$\int_{-\infty}^{+\infty} \frac{x\,\cos x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\,(\cos 1 - 3\,\sin 1),$$

(b)

$$\int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\,(3\,\cos 1 + \sin 1).$$

Example 5.7 *Compute*

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2+4x+20}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}.$$

(a) The function $\dfrac{z}{z^2+4z+20}$ has a zero of first order at ∞. The poles are

$$-2\pm 4i\notin\mathbb{R},$$

so by a residuum formula,

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{x\,\sin x}{x^2+4x+20}\,dx = \operatorname{Im}\left\{\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{x^2+4x+20}\,dx\right\}\\ &\quad = \operatorname{Im}\left\{2\pi i\cdot\operatorname{res}\left(\frac{z\,e^{iz}}{z^2+4z+20}\,;\,-2+4i\right)\right\} = 2\pi\cdot\operatorname{Im}\left\{i\cdot\lim_{z\to -2+4i}\frac{z\,e^{iz}}{z+2+4i}\right\}\\ &\quad = 2\pi\cdot\operatorname{Im}\left\{i\cdot\frac{(-2+4i)e^{i(-2+4i)}}{8i}\right\} = \frac{\pi}{2\,e^4}\,(2\,\cos 2+\sin 2).\end{aligned}$$

(b) We have of course,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = [\text{Arctan }]_{-\infty}^{+\infty} = \pi.$$

ALTERNATIVELY, it follows by a residuum formula that

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 2\pi i \cdot \text{res}\left(\frac{1}{1+z^2}\,;\,i\right) = 2\pi i \lim_{z\to i} \frac{1}{z+i} = \frac{2\pi i}{2i} = \pi.$$

Example 5.8 *Prove that*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\,dx = \frac{\pi}{\cosh \frac{\pi}{2}}.$$

HINT: *Integrate the function* $\dfrac{\cos z}{\cosh z}$ *along a rectangle with the corners* $-R$, R, $R+\pi i$ *and* $-R+\pi i$, *and let* $R \to +\infty$.

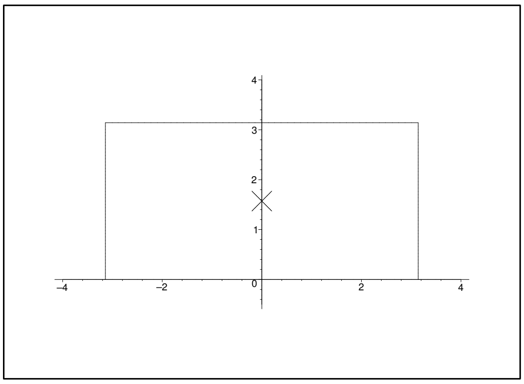


Figure 10: The curve C_π with the singularity $z_0 = i\,\dfrac{\pi}{2}$ inside C_π.

We shall use the hint, so we call the curve C_R. It follows from

$$\cosh z = 0 \qquad \text{for } z = i\,\frac{\pi}{2} + i\,p\pi, \quad p \in \mathbb{Z},$$

that $z = i\,\dfrac{\pi}{2}$ is the only singularity (a simple pole) lying inside C_R for every $R > 0$. Hence by Cauchy's integral formula

$$\oint_{C_R} \frac{\cos z}{\cosh z}\,dz = 2\pi i \cdot \text{res}\left(\frac{\cos z}{\cosh z}\,;\,i\,\frac{\pi}{2}\right) = 2\pi i \cdot \frac{\cos\left(i\,\frac{\pi}{2}\right)}{\sinh\left(i\,\frac{\pi}{2}\right)} = 2\pi i \cdot \frac{\cosh \frac{\pi}{2}}{i\,sin\frac{\pi}{2}} = 2\pi\,\cosh\frac{\pi}{2}.$$

On the other hand,

$$\oint_{C_R} \frac{\cos z}{\cosh z}\, dz = \int_{-R}^{R} \frac{\cos x}{\cosh x}\, dx - \int_{-R}^{R} \frac{\cos(x+i\pi)}{\cosh(x+i\pi)}\, dx + i\int_0^{\pi} \frac{\cos(R+iy)}{\cosh(R+iy)}\, dy - i\int_0^{\pi} \frac{\cos(-R+iy)}{\cosh(-R+iy)}\, dy.$$

We first note that

$$\begin{aligned} -\int_{-R}^{R} \frac{\cos(x+i\pi)}{\cosh(x+i\pi)}\, dx &= -\int_{-R}^{R} \frac{\cos x\cdot\cosh\pi - i\,\sin x\cdot\sinh\pi}{\cosh x\cdot\cos\pi + i\,\sinh x\cdot\sin\pi}\, dx \\ &= \cosh\pi \int_{-R}^{R} \frac{\cos x}{\cosh x}\, dx - i\,\sinh\pi \int_{-R}^{R} \frac{\sin x}{\cosh x}\, dx \\ &= \cosh\pi\cdot\int_{-R}^{R} \frac{\cos x}{\cosh x}\, dx + 0, \end{aligned}$$

because the latter integral has an odd integrand. Summing up we get for the first two terms,

$$\int_{-R}^{R} \frac{\cos x}{\cosh x}\, dx - \int_{-R}^{R} \frac{\cos(x+i\pi)}{\cosh(x+i\pi)}\, dx = (1+\cosh\pi)\int_{-R}^{R} \frac{\cos x}{\cosh x}\, dx.$$

Clearly, this integral is convergent for $R \to +\infty$, because the numerator of the integrand is bounded, and its denominator tends exponentially towards 0 by the limits $x \to \pm\infty$. We only have to show that the contributions from the vertical axes tend to zero for $R \to +\infty$. It follows from

$$\frac{\cos(R+iy)}{\cosh(R+iy)} = \frac{\cos R\cdot\cosh y - i\,\sin R\cdot\sinh y}{\cosh R\cdot\cos y + i\,\sinh R\cdot\sin y},$$

when $0 \le y \le \pi$ that

$$\begin{aligned} \left|\frac{\cos(R+iy)}{\cosh(R+iy)}\right|^2 &= \frac{\cos^2 R\cdot\cosh^2 y + \sin^2 R\cdot\sinh^2 y}{\cosh^2 R\cdot\cos^2 y + \sinh^2 R\cdot\sin^2 y} = \frac{\cos^2 R + \sinh^2 y}{\sinh^2 R + \cos^2 y} \\ &\le \frac{1+\sinh^2\pi}{\sinh^2 R} = \frac{\cosh^2\pi}{\sinh^2 R}. \end{aligned}$$

The length of the path of integration is π, so we conclude that

$$\left|\int_0^{\pi} \frac{\cos(R+iy)}{\cosh(R+iy)}\, dy\right| \le \pi\cdot\frac{\cosh\pi}{\sinh R} \to 0 \qquad \text{for } R \to +\infty.$$

Since also

$$\left|\frac{\cos(-R+iy)}{\cosh(-R+iy)}\right| \le \frac{\cosh\pi}{\sinh R},$$

it follows in the same way that the latter integral tends to 0 for $R \to +\infty$. Summing up we get by this limit,

$$(1+\cosh\pi)\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\, dx = 2\pi\cosh\frac{\pi}{2},$$

and since

$$1+\cosh\pi = 1 + \left\{2\cosh^2\frac{\pi}{2} - 1\right\} = 2\cosh^2\frac{\pi}{2},$$

we finally get that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\, dx = \frac{2\pi \cosh \frac{\pi}{2}}{2 \cosh^2 \frac{\pi}{2}} = \frac{\pi}{\cosh \frac{\pi}{2}}.$$

Example 5.9 *Compute*

$$(a) \quad \int_{-\infty}^{+\infty} \frac{\cos x}{x^2+4}, \qquad (b) \quad \int_{-\infty}^{+\infty} \frac{\sin 2x}{x^2+x+1}\, dx.$$

The denominator is in both cases a polynomial of degree grad 2 without zeros on the x-axis. The numerators are purely trigonometric, so we get by a residuum formula,

(a)

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+4}\, dx = (\mathrm{Re}) \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+4}\, dx = 2\pi i \cdot \mathrm{res}\left(\frac{e^{iz}}{z^2+4}\, ;\, 2i\right) = \frac{e^{i\cdot 2i}}{4i} \cdot 2\pi i = \frac{\pi}{2e^2}.$$

(b)

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\sin 2x}{x^2+x+1}\,dx &= \operatorname{Im}\left\{\int_{-\infty}^{+\infty} \frac{e^{2ix}}{x^2+x+1}\,dx\right\}\\ &= \operatorname{Im}\left\{2\pi i\cdot \operatorname{res}\left(\frac{e^{2iz}}{z^2+z+1}\,;\,-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\right\}\\ &= \operatorname{Re}\left\{2\pi \lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{e^{2iz}}{2z+1}\right\} = \operatorname{Re}\left\{2\pi\cdot\frac{e^{i(-1+i\sqrt{3})}}{-1+i\sqrt{3}+1}\right\}\\ &= \operatorname{Re}\left\{\frac{2\pi}{\sqrt{3}}(-i)e^{-\sqrt{3}-i}\right\} = -\frac{2\pi}{\sqrt{3}}\,e^{-\sqrt{3}}\sin 1.\end{aligned}$$

Example 5.10 *Compute*

$$(a)\quad \int_0^{+\infty} \frac{x^3\sin x}{x^4+1}\,dx, \qquad (b)\quad \int_0^{+\infty}\frac{x^2\cos 3x}{\left(x^2+1\right)^2}\,dx.$$

The integrands are in both cases even functions, so they may be extended by symmetry to all of $\mathbb{R}$. Furthermore, the difference of degrees of the numerator and the denominator of the rational function of the integrands is at least 1, where the denominators are dominating, so the integrals are convergent, and we can find their values by a residuum formula.

(a) The zeros of the denominator are determined by $z^4+1=0$, so

$$z=\pm\frac{1}{\sqrt{2}}\pm i\,\frac{1}{\sqrt{2}},$$

and we get

$$\begin{aligned}&\int_0^{+\infty}\frac{x^3\sin x}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^3\sin x}{x^4+1}\,dx = \frac{1}{2}\operatorname{Im}\left\{\int_{-\infty}^{+\infty}\frac{x^3e^{ix}}{x^4+1}\,dx\right\}\\ &= \frac{1}{2}\operatorname{Im}\left\{2\pi i\left[\operatorname{res}\left(\frac{z^3e^{iz}}{z^4+1}\,;\,\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+\operatorname{res}\left(\frac{z^3e^{iz}}{z^4+1}\,;\,-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]\right\}.\end{aligned}$$

Let z_0 be any pole. Then $z_0^4=-1$, and

$$\operatorname{res}\left(\frac{z^3e^{iz}}{z^4+1}\,;\,z_0\right)=\frac{z_0^3e^{iz_0}}{4z_0^3}=\frac{1}{4}\,e^{iz_0},$$

hence by insertion,

$$\begin{aligned}&\int_0^{+\infty}\frac{x^3\sin x}{x^4+1}\,dx = \pi\operatorname{Im}\left\{i\cdot\frac{1}{4}\exp\left(i\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right)+i\cdot\frac{1}{4}\exp\left(i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right)\right\}\\ &= \frac{\pi}{2}\operatorname{Im}\left\{i\cdot\exp\left(-\frac{1}{\sqrt{2}}\right)\cdot\frac{1}{2}\cdot\left\{\exp\left(\frac{i}{\sqrt{2}}\right)\exp\left(-\frac{i}{\sqrt{2}}\right)\right\}\right\} = \frac{\pi}{2}\exp\left(-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right).\end{aligned}$$

(b) Here $z = i$ is a double pole. It lies in the upper half plane, so we start by computing its residuum:

$$\begin{aligned}\operatorname{res}\left(\frac{z^3e^{3iz}}{\left(z^2+1\right)^2}\,;\,i\right) &= \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left\{\frac{z^2e^{3iz}}{(z+i)^2}\right\} = \lim_{z\to i}\left\{\frac{2ze^{3iz}}{(z+i)^2}+\frac{3iz^2e^{3iz}}{(z+i)^2}-\frac{2z^2e^{3iz}}{(z+i)^3}\right\} \\ &= \frac{2i\,e^{-3}}{(2i)^2}+\frac{3i(-1)e^{-3}}{(2i)^2}-\frac{2(-1)e^{-3}}{(2i)^3} = \frac{e^{-3}}{8}\left\{-4i+6i-\frac{2(-1)}{-i}\right\} = i\cdot\frac{1}{2e^3}.\end{aligned}$$

Then by the symmetry and the residuum formula,

$$\begin{aligned}\int_0^{+\infty}\frac{x^2\cos 3x}{\left(x^2+1\right)^2}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2\cos 3x}{\left(x^2+1\right)^2}\,dx = \frac{1}{2}\operatorname{Re}\left\{\int_{-\infty}^{+\infty}\frac{x^2e^{3ix}}{\left(x^2+1\right)^2}\,dx\right\} \\ &= \frac{1}{2}\operatorname{Re}\left\{2\pi i\cdot\operatorname{res}\left(\frac{z^2e^{3iz}}{\left(z^2+1\right)^2}\,;\,i\right)\right\} = \frac{1}{2}\operatorname{Re}\left\{2\pi i\cdot i\cdot\frac{1}{2e^3}\right\} = -\frac{\pi}{2e^3}.\end{aligned}$$

Example 5.11 *Compute*

$$(a)\quad \int_0^{+\infty}\frac{x\,\sin x}{\left(x^2+1\right)\left(x^2+4\right)}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty}\frac{\sin x}{x^2+4x+5}\,dx.$$

In both cases the integrand satisfies the assumptions for the application of the residuum formula.

(a) First we get by a decomposition,

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{3}\,\frac{1}{x^2+1}-\frac{1}{3}\,\frac{1}{x^2+4}.$$

The integrand is even, so by the symmetry, followed by an application of the residuum formula,

$$\int_0^{+\infty}\frac{x\,\sin x}{\left(x^2+1\right)\left(x^2+4\right)}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x\,\sin x}{\left(x^2+1\right)\left(x^2+4\right)}\,dx$$

$$\begin{aligned} &= \frac{1}{6}\int_{-\infty}^{+\infty}\frac{x\,\sin x}{x^2+1}\,dx-\frac{1}{6}\int_{-\infty}^{+\infty}\frac{x\,\sin x}{x^2+4}\,dx \\ &= \frac{1}{6}\operatorname{Im}\left\{2\pi i\cdot\operatorname{res}\left(\frac{z\,e^{iz}}{z^2+1}\,;\,i\right)\right\}-\frac{1}{6}\operatorname{Im}\left\{2\pi i\cdot\operatorname{res}\left(\frac{z\,e^{iz}}{z^2+4}\,;\,2i\right)\right\} \\ &= \frac{\pi}{3}\operatorname{Re}\left\{\operatorname{res}\left(\frac{z\,e^{iz}}{z^2+1}\,;\,i\right)-\operatorname{res}\left(\frac{z\,e^{iz}}{z^2+4}\,;\,2i\right)\right\} \\ &= \frac{\pi}{3}\operatorname{Re}\left\{\frac{i\,e^{i\cdot i}}{i+i}-i\cdot\frac{2i\,e^{i\cdot 2i}}{2i+2i}\right\} = \frac{\pi}{3}\operatorname{Re}\left\{\frac{e^{-1}}{2}-\frac{e^{-2}}{2}\right\} \\ &= \frac{\pi}{6}\left(\frac{1}{e}-\frac{1}{e^2}\right) = \frac{\pi}{6e^2}\,(e-1).\end{aligned}$$

(b) The poles are $z = -2 \pm i$, of which only $z_0 = -2+i$ lies in the upper half plane. Then by the residuum formula,

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\sin x}{x^2+4x+5}\,dx &= \operatorname{Im}\left\{2\pi i\operatorname{res}\left(\frac{e^{iz}}{(z+2+i)(z+2-i)}\,;\,-2+i\right)\right\} \\ &= \operatorname{Im}\left\{2\pi i\cdot\frac{e^{i(-2+i)}}{-2+i+2+i}\right\}(\operatorname{Im}\left\{\pi\,e^{-2i-1}\right\} = -\frac{\pi}{e}\,\sin 2.\end{aligned}$$

Example 5.12 *Prove that*

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi\,e^{-a} \qquad \textit{for } a \geq 0.$$

The claim is trivial for $a = 0$, because

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = [\text{Arctan } x]_{-\infty}^{+\infty} = \pi.$$

If $a > 0$, then the assumptions of using the residuum formula are satisfied, so

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx = 2\pi i \cdot \text{res}\left(\frac{e^{iaz}}{z^2+1}\,;\,i\right) = 2\pi i \cdot \frac{e^{i\,a\,i}}{i+i} = \pi\,e^{-a}.$$

Remark 5.1 If instead $a < 0$, then we get by a complex conjugation and an application of the first result that

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx = \overline{\int_{-\infty}^{+\infty} \frac{e^{-iax}}{x^2+1}\,dx} = \overline{\pi\,e^{-(-a)}} = \pi\,e^{a} = \pi\,e^{-|a|},$$

so we have in general that

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi\,e^{-|a|}, \qquad a \in \mathbb{R}. \qquad \Diamond$$

Example 5.13 *Prove that*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi\,e^{-a}}{a} \qquad \textit{for } a > 0.$$

The conditions of convergence of the improper integrals and the legality of the application of the residuum formula are fulfilled. Then by the symmetry,

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x^2+a^2}\,dx = 0,$$

so

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\,dx = (\text{Re})\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = 2\pi i \cdot \text{res}\left(\frac{e^{iz}}{z^2+a^2}\,;\,i\,a\right) = 2\pi i \cdot \frac{e^{i\cdot ia}}{2ia} = \frac{\pi\,e^{-a}}{a}.$$

Example 5.14 *Compute for* $a,\, b \in \mathbb{R}_+$,

$$(a)\quad \int_{-\infty}^{+\infty} \frac{x\,\sin ax}{x^2+b^2}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{\cos ax}{x^2+b^2}\,dx.$$

In both cases the conditions of convergence of the improper integrals and the application of a residuum formula are fulfilled. Hence, because $a,\, b > 0$,

(a)

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x\,\sin ax}{x^2+b^2}\,dx &= \operatorname{Im}\left\{\int_{-\infty}^{+\infty} \frac{x\,e^{iax}}{x^2+b^2}\,dx\right\} = \operatorname{Im}\left\{2\pi i \lim_{x\to ib} \frac{z\,e^{iaz}}{z+ib}\right\} \\ &= \operatorname{Im}\left\{2\pi i \cdot \frac{ib\,e^{-ab}}{2ib}\right\} = \pi\,e^{-ab}.\end{aligned}$$

(b)

$$\int_{-\infty}^{+\infty} \frac{\cos ax}{x^2+b^2}\,dx = (\mathrm{Re}) \int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+b^2}\,dx = 2\pi i \lim_{z\to ib} \frac{e^{iaz}}{z+ib} = 2\pi\,i \cdot \frac{e^{-ab}}{2ib} = \frac{\pi}{b}\,e^{-ab}.$$

Example 5.15 *Prove that the integral*

$$\int_{+\infty}^{+\infty} \frac{x\,\sin x}{1+x^4}\,dx$$

is convergent, and find its value.

We have an improper integral, where the integrand is a product of $\sin x$ and a *real* rational function without poles on the x-axis and with a zero of third order at ∞. From this we conclude that the integral is convergent, and its value is given by the residues at the poles in the upper half plane of the function $\frac{z\,e^{iz}}{1+z^4}$. We have more precisely,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x\,\sin x}{1+x^4}\,dx &= \operatorname{Im}\left\{2\pi i\left[\operatorname{res}\left(\frac{z\,e^{iz}}{z^4+1}\,;\,\frac{1+i}{\sqrt{2}}\right)+\operatorname{res}\left(\frac{z\,e^{iz}}{z^4+1}\,;\,\frac{-1+i}{\sqrt{2}}\right)\right]\right\}\\
&= 2\pi\operatorname{Re}\left\{\left[\operatorname{res}\left(\frac{z\,e^{iz}}{z^4+1}\,;\,\frac{1+i}{\sqrt{2}}\right)+\operatorname{res}\left(\frac{z\,e^{iz}}{z^4+1}\,;\,\frac{-1+i}{\sqrt{2}}\right)\right]\right\} = 2\pi\operatorname{Re}\left\{\left[\frac{z\,e^{iz}}{4z^3}\right]_{\frac{1+i}{\sqrt{2}}}+\left[\frac{z\,e^{iz}}{4z^3}\right]_{\frac{-1+i}{\sqrt{2}}}\right\}\\
&= 2\pi\operatorname{Re}\left\{\left[-\frac{z^2}{4}\,e^{iz}\right]_{\frac{1+i}{\sqrt{2}}}+\left[-\frac{z^2}{4}\,e^{iz}\right]_{\frac{-1+i}{\sqrt{2}}}\right\} = -\frac{\pi}{2}\operatorname{Re}\left\{i\,\exp\left(i\left(\frac{1+i}{\sqrt{2}}\right)\right)-i\,\exp\left(i\left(\frac{-1+i}{\sqrt{2}}\right)\right)\right\}\\
&= -\frac{\pi}{2}\operatorname{Re}\left(i\left\{\exp\left(-\frac{1}{\sqrt{2}}\right)\cdot\exp\left(\frac{i}{\sqrt{2}}\right)-\exp\left(-\frac{1}{\sqrt{2}}\right)\cdot\exp\left(-\frac{i}{\sqrt{2}}\right)\right\}\right)\\
&= -\frac{\pi}{2}\operatorname{Re}\left(i\,\exp\left(-\frac{1}{\sqrt{2}}\right)\cdot 2i\,\sin\frac{1}{\sqrt{2}}\right) = \pi\,\exp\left(-\frac{1}{\sqrt{2}}\right)\,\sin\left(\frac{1}{\sqrt{2}}\right).
\end{aligned}$$

Example 5.16 *Prove that*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4}\,dx = \pi\,\exp\left(-\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right).$$

We first note that the integrand $f(x)=\cos x\cdot\frac{1}{1+x^4}$ does not have poles on the x-axis and that the factor $\frac{1}{1+x^4}$ has a zero of order 4 at ∞. Since $\frac{1}{1+x^4}$ is a real rational function, we can obtain the value of the integral by a residuum formula.

Now $1+z^4=0$ for

$$z=\exp\left(i\left\{\frac{\pi}{4}+p\,\frac{\pi}{2}\right\}\right),\qquad p\in\mathbb{Z},$$

so we get by the residuum formula,

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{\cos x}{1+x^4}\,dx &= \operatorname{Re}\left\{\int_{-\infty}^{+\infty}\frac{e^{ix}}{1+x^4}\,dx\right\}\\
&= \operatorname{Re}\left(2\pi i\left\{\operatorname{res}\left(\frac{e^{iz}}{1+z^4}\,;\,\exp\left(i\,\frac{\pi}{4}\right)\right)+\operatorname{res}\left(\frac{e^{iz}}{1+z^4}\,;\,\exp\left(i\,\frac{3\pi}{4}\right)\right)\right\}\right).
\end{aligned}$$

All poles z_0 with $z_0^4 = -1$ are simple, so by RULE II,

$$\text{res}\left(\frac{e^{iz}}{1+z^4}\,;\,z_0\right) = \frac{e^{iz_0}}{4z_0^3} = \frac{z_0 e^{iz_0}}{4z_0^4} = -\frac{1}{4}\,z_0\,e^{iz_0}.$$

Finally,

$$\exp\left(i\,\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\,(1+i) \qquad \text{og} \qquad \exp\left(i\,\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}\,(-1+i),$$

hence by insertion,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4}\,dx &= \text{Re}\left[2\pi i\left(-\frac{1}{4}\right)\left\{\left(\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right)e^{i\,\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}} + \left(-\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right)e^{-i\,\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}\right\}\right] \\
&= \text{Re}\left[-\frac{\pi i}{2}\cdot e^{-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}\cdot\frac{1}{\sqrt{2}}\left\{(1+i)e^{i\,\frac{1}{\sqrt{2}}} + (-1+i)e^{-i\,\frac{1}{\sqrt{2}}}\right\}\right] \\
&= \text{Re}\left[-\frac{\pi i}{2\sqrt{2}}\,e^{-\frac{1}{\sqrt{2}}}\left\{\left(e^{\frac{i}{\sqrt{2}}} - e^{-\frac{i}{\sqrt{2}}}\right) + i\left(e^{\frac{i}{\sqrt{2}}} + e^{-\frac{i}{\sqrt{2}}}\right)\right\}\right] \\
&= \text{Re}\left[-\frac{\pi i}{2\sqrt{2}}\cdot e^{-\frac{1}{\sqrt{2}}}\cdot\left\{2i\cdot\sin\frac{1}{\sqrt{2}} + i\cdot 2\,\cos\frac{1}{\sqrt{2}}\right\}\right] = \text{Re}\left[\pi\,e^{-\frac{1}{\sqrt{2}}}\cdot\left\{\frac{1}{\sqrt{2}}\,\sin\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\cos\frac{1}{\sqrt{2}}\right\}\right] \\
&= \pi\,e^{-\frac{1}{\sqrt{2}}}\left\{\sin\frac{1}{\sqrt{2}}\cdot\cos\frac{\pi}{4} + \cos\frac{1}{\sqrt{2}}\cdot\sin\frac{\pi}{4}\right\} = \pi\,e^{-\frac{1}{\sqrt{2}}}\,\sin\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right).
\end{aligned}$$

ALTERNATIVELY and slightly shorter,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4}\,dx &= \text{Re}\left[2\pi i\left(-\frac{1}{4}\right)\,\left\{e^{i\,\frac{\pi}{4}}\cdot e^{i\,\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}} + e^{i\,\frac{3\pi}{4}}\,e^{-i\,\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}\right\}\right] \\
&= \text{Re}\left[\frac{\pi}{2i}\cdot e^{-\frac{1}{\sqrt{2}}}\,\left\{e^{i\,\frac{\pi}{4}}\,e^{i\,\frac{1}{\sqrt{2}}} - e^{-i\,\frac{\pi}{4}}\,e^{-i\,\frac{1}{\sqrt{2}}}\right\}\right] = \text{Re}\left[\pi\,e^{-\frac{1}{\sqrt{2}}}\cdot\frac{1}{2i}\left\{e^{i\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)} - e^{-i\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)}\right\}\right] \\
&= \pi\,e^{-\frac{1}{\sqrt{2}}}\,\sin\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right).
\end{aligned}$$

Example 5.17 *Prove that the improper integral*

$$\int_{-\infty}^{+\infty} \frac{\sin\left(x+\frac{\pi}{4}\right)}{(x^2+1)(x^2+4)}\,dx$$

is convergent. Then find the value of the integral.

Since

$$\sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin x+\cos x),$$

and since $\dfrac{1}{(x^2+1)(x^2+4)}$ is a real and even rational function with a zero of order 4 at ∞ and with no pole on the x-axis, the improper integral is convergent, and we can find its value by a residuum formula, where we use that the integral of an odd function over a symmetric interval is 0,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\sin\left(x+\frac{\pi}{4}\right)}{(x^2+1)(x^2+4)}\,dx &= \frac{1}{\sqrt{2}}\left\{\int_{-\infty}^{+\infty}\frac{\sin x}{(x^2+1)(x^2+4)}\,dx+\int_{-\infty}^{+\infty}\frac{\cos x}{(x^2+1)(x^2+4)}\,dx\right\}\\
&= 0+\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}\frac{\cos x}{(x^2+1)(x^2+4)}\,dx = \frac{1}{\sqrt{2}}(\mathrm{Re})\int_{-\infty}^{+\infty} e^{ix}\left\{\frac{1}{3}\cdot\frac{1}{x^2+1}-\frac{1}{3}\cdot\frac{1}{x^2+4}\right\}dx\\
&= \frac{2\pi i}{3\sqrt{2}}\left\{\mathrm{res}\left(\frac{e^{iz}}{z^2+1};\,i\right)-\mathrm{res}\left(\frac{e^{iz}}{z^2+4};\,2i\right)\right\} = \frac{2\pi i}{3\sqrt{2}}\cdot\left\{\frac{e^{-1}}{2i}-\frac{e^{-2}}{4i}\right\} = \frac{(2e-1)\pi}{6\sqrt{2}\cdot e^2}.
\end{aligned}$$

ALTERNATIVELY we may carry through the following computations,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\sin\left(x+\frac{\pi}{4}\right)}{(x^2+1)(x^2+4)}\,dx &= \mathrm{Im}\int_{-\infty}^{+\infty}\frac{\exp\left(i\left(x+\frac{\pi}{4}\right)\right)}{(x^2+1)(x^2+4)}\,dx\\
&= \mathrm{Im}\left\{2\pi i\left[\mathrm{res}\left(\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z^2+1)(z^2+4)};\,i\right)+\mathrm{res}\left(\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z^2+1)(z^2+4)};\,2i\right)\right]\right\}.
\end{aligned}$$

It follows from

$$\mathrm{res}\left(\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z^2+1)(z^2+4)};\,i\right) = \lim_{z\to i}\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z+i)(z^2+4)} = \frac{e^{i\left(i+\frac{\pi}{4}\right)}}{2i\cdot 3} = \frac{e^{-1}}{6i}\cdot e^{i\,\frac{\pi}{4}},$$

and

$$\mathrm{res}\left(\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z^2+1)(z^2+4)};\,2i\right) = \lim_{z\to 2i}\frac{e^{i\left(z+\frac{\pi}{4}\right)}}{(z^2+1)(z+2i)} = \frac{e^{i\left(2i+\frac{\pi}{4}\right)}}{-3\cdot 4i} = \frac{-e^{-2}}{12i}\cdot e^{i\,\frac{\pi}{4}},$$

that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\sin\left(x+\frac{\pi}{4}\right)}{(x^2+1)(x^2+4)}\,dx &= \mathrm{Im}\left\{2\pi i\left(\frac{e^{-1}}{6i}-\frac{e^{-2}}{12i}\right)\cdot\frac{1+i}{\sqrt{2}}\right\}\\
&= \mathrm{Im}\left\{\frac{\pi\cdot(2e-1)}{6e^2}\cdot\frac{1+i}{\sqrt{2}}\right\} = \frac{(2e-1)\pi}{6\sqrt{2}\cdot e^2}.
\end{aligned}$$

Example 5.18 *Given the function*

$$f(z) = \frac{z^2}{z^4+4}.$$

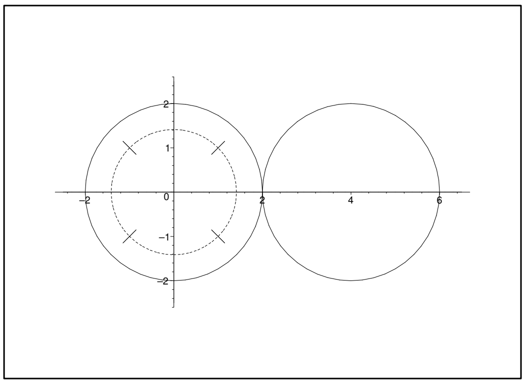
1) Find the singular points and their types in $\mathbb{C} \cup \{\infty\}$ *for* $f(z)$.

2) Find the value of the following two complex line integrals,

$$(a) \quad \oint_{|z-4|=2} f(z)\,dz, \qquad (b) \quad \oint_{|z|=2} f(z)\,dz.$$

3) Prove for every $\omega > 0$ *that*

$$\int_{-\infty}^{+\infty} \frac{t^2}{t^4+4}\, e^{i\omega t}\,dt = \frac{\pi}{2}\, e^{-\omega}\,(\cos\omega - \sin\omega).$$



1) Clearly, $z = \infty$ is a removable singularity (a zero of second order).
The denominator $z^4 + 4$ has the zeros

$$1+i, \qquad -1+i, \qquad -1-i, \qquad 1-i.$$

These are all simple pole of $f(z)$.

2) a) Since there is no pole of $f(z)$ inside the circle $|z-4| = 2$ (cf. the figure), it follows from Cauchy's integral theorem that

$$\oint_{|z-4|=2} f(z)\,dz = 0.$$

b) All singularities of $f(z)$ lie inside the circle $|z| = 2$, and $z = \infty$ is a zero of second order. Hence, by reversing the direction of the curve,

$$\oint_{|z|=2} f(z)\,dz = -\oint_{|z|=2}^{\star} f(z)\,dz = -2\pi i \cdot \operatorname{res}(f;\infty) = 0.$$

Alternatively, the residuum in a general pole z_0, for which $z_0^4 = -4$, is given by

$$\text{res}\left(\frac{z^2}{z^4+4}\,;\,z_0\right) = \frac{z_0^2}{4z_0^3} = \frac{1}{4z_0},$$

so

$$\begin{aligned}\oint_{|z|=2} f(z)\,dz &= 2\pi i\sum_{n=1}^{4}\text{res}\left(\frac{z^2}{z^4+4}\,;\,z_n\right) = 2\pi i\left\{\frac{1}{1+i}+\frac{1}{-1+i}+\frac{1}{-1-i}+\frac{1}{1-i}\right\}\\ &= 2\pi i\left\{\frac{1}{1+i}-\frac{1}{1-i}-\frac{1}{1+i}+\frac{1}{1-i}\right\} = 0.\end{aligned}$$

3) Since the integrand has a zero of order 2 at ∞, and since there are no real singularities, the improper integral exists, and when $\omega > 0$ its value can be found by the residues in the upper half plane,

$$\begin{aligned}&\int_{+\infty}^{+\infty}\frac{t^2}{t^4+4}\,e^{i\omega t}dt = 2\pi i\left\{\text{res}\left(\frac{z^2}{z^4+4}\,e^{i\omega z}\,;\,1+i\right)+\text{res}\left(\frac{z^2}{z^4+4}\,e^{i\omega z}\,;\,-1+i\right)\right\}\\ &= 2\pi i\left\{\lim_{z\to 1+i}\frac{z^2e^{i\omega z}}{4z^3}+\lim_{z\to -1+i}\frac{z^2e^{i\omega z}}{4z^3}\right\} = \frac{2\pi i}{4}\left\{\frac{e^{i\omega(1+i)}}{1+i}+\frac{e^{i\omega(-1+i)}}{-1+i}\right\}\\ &= \frac{\pi}{2}\cdot\left\{\frac{1-i}{2}\cdot e^{-\omega}e^{i\omega}-\frac{1+i}{2}\cdot e^{-\omega}e^{-i\omega}\right\} = \frac{\pi i}{2}\,e^{-\omega}\cdot\frac{1}{2}\left\{e^{i\omega}-i\,e^{i\omega}-e^{-i\omega}-i\,e^{-i\omega}\right\}\\ &= \frac{\pi}{2}\,e^{-\omega}\cdot i\left\{\frac{e^{i\omega}-e^{-i\omega}}{2i}\cdot i-i\cdot\frac{e^{i\omega}+e^{-i\omega}}{2}\right\} = \frac{\pi}{2}\,e^{-\omega}\cdot(\cos\omega-\sin\omega).\end{aligned}$$

Example 5.19 **(a)** *Given* $m > 0$*. Prove that the improper integral*

$$(3)\quad \int_{-\infty}^{+\infty} \frac{x^2 e^{imx}}{x^4+6x^2+25}\,dx$$

is convergent, and find its value.

(b) *What is the value of the improper integral (3), when* $m < 0$ *instead?*

(a) Clearly, $\dfrac{x^2}{x^2+6x^2+25}$ has a zero of order 2 at ∞, and the denominator is ≥ 25 for every $x \in \mathbb{R}$. Hence, the improper integral is convergent, even for every $m \in \mathbb{R}$, and when $m \geq 0$ we can find the value by a residuum formula. When the denominator is put equal to zero,

$$z^4+6z^2+25 = \left(z^2+5\right)^2 - (2z)^2 = 0$$

we get

$$z^2 = -3 \pm \sqrt{9-25} = -3 \pm 4i = (\pm 1 + 2i)^2,$$

so we have four simple poles,

$$1+2i, \qquad -1+2i, \qquad 1-2i, \qquad -1-2i,$$

of which only the former two lie in the upper half plane. Hence, for $m \geq 0$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x^2\, e^{imx}}{x^4+6x^2+25}\,dx &= 2\pi i\left\{\text{res}\left(\frac{z^2e^{imz}}{z^4+6z^2+25}\,,\,1+2i\right) + \text{res}\left(\frac{z^2e^{imz}}{z^4+6z^2+25}\,,\,-1+2i\right)\right\}\\
&= 2\pi i\left\{\lim_{z\to 1+2i}\frac{z^2e^{imz}}{4z^3+12z} + \lim_{z\to -1+2i}\frac{z^2e^{imz}}{4z^3+12z}\right\} = \frac{2\pi i}{4}\left\{\lim_{z\to 1+2i}\frac{z\,e^{imz}}{z^2+3} + \lim_{z\to -1+2i}\frac{z\,e^{imz}}{z^2+3}\right\}\\
&= \frac{\pi i}{2}\left\{\frac{(1+2i)e^{im(1+2i)}}{1-4+4i+3} + \frac{(-1+2i)e^{im(-1+2i)}}{1-4-4i+3}\right\}\\
&= \frac{\pi i}{8}\left\{(1+2i)e^{im}\cdot e^{-2m} - (-1+2i)e^{-im}\cdot e^{-2m}\right\} = \frac{\pi}{8}\,e^{-2m}\left\{\left(e^{im}+e^{-im}\right) + 2i\left(e^{im}-e^{-im}\right)\right\}\\
&= \frac{\pi}{8}\,e^{-2m}\{2\cos m + 2i\cdot 2i\sin m\} = \frac{\pi}{4}\,e^{-2m}\{\cos m - 2\sin m\},
\end{aligned}$$

which is also true for $m = 0$, where

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4+6x^2+25}\,dx = \frac{\pi}{4}.$$

(b) If $m < 0$, then we get by complex conjugation,

$$\int_{-\infty}^{+\infty} \frac{x^2 e^{imx}}{x^4+6x^2+25}\,dx = \overline{\int_{-\infty}^{+\infty} \frac{x^2 e^{i|m|x}}{x^4+6x^2+25}\,dx} = \frac{\pi}{4}\cdot e^{-2|m|}\{\cos|m| - 2\sin|m|\},$$

where we have used the result from **(a)** with $|m|$ instead of m.

Summing up we have for every $m \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} \frac{x^2 e^{imx}}{x^4+6x^2+25}\,dx = \frac{\pi}{4}\cdot e^{-2|m|}\{\cos|m| - 2\sin|m|\}.$$

Example 5.20 *Find the Fourier transform of the function*

$$f(x)=\frac{x+1}{x^2+2x+2},$$

i.e. compute

$$\hat{f}(\xi)=\int_{-\infty}^{+\infty}\frac{x+1}{x^2+2x+2}\,e^{-\xi x}\,dx,$$

first for $\xi<0$, and then for $\xi>0$.

We see that

$$f(z)=\frac{P(z)}{Q(z)}=\frac{z+1}{z^2+1+2z+2}=\frac{z+1}{(z+1)^2+1}$$

is a rational function, where

1) the polynomial $Q(z)=(z+1)^2+1$ of the denominator has the simple zeros $z=-1\pm i$, where none of these is lying on the real axis;

2) the polynomial of the denominator is of 1 degree bigger than the polynomial of the numerator;

3) if $\xi<0$, then $m=-\xi>0$.

Hence, the conditions of convergence of the improper integral are satisfied for $\xi<0$, and since $-1+i$ is the only (simple) pole in the upper half plane, the value of the improper integral is given by a residuum formula,

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{+\infty}\frac{x+1}{x^2+2x+2}\,e^{-i\xi x}\,dx=2\pi i\cdot\operatorname{res}\left(\frac{z+1}{(z+1)^2+1}\cdot e^{-i\xi z}\,;\,-1+i\right)\\ &= 2\pi i\lim_{z\to -1+i}\frac{z+1}{2(z+1)}\cdot e^{-i\xi z}=\pi i\cdot e^{-i\xi(-1+i)}=\pi i\cdot e^{\xi(1+i)},\qquad \xi<0,\end{aligned}$$

where we have applied Rule II.

Now $P(z)$ and $Q(z)$ have *real* coefficients, so if $\xi>0$, then we get by complex conjugation,

$$\hat{f}(\xi)=\int_{-\infty}^{+\infty}\frac{x+1}{x^2+2x+2}\,e^{-i\xi x}\,dx=\overline{\int_{-\infty}^{+\infty}\frac{x+1}{x^2+2x+2}\,e^{i\xi x}\,dx}=\overline{\pi i\cdot e^{-\xi(1+i)}}=-\pi i\cdot e^{\xi(-1+i)}.$$

Summing up,

$$\hat{f}(\xi)=\begin{cases}\pi i\cdot e^{\xi(1+i)}=\pi i\cdot e^{-|\xi|(1+i)} & \text{for } \xi<0,\\[2ex] -\pi i\cdot e^{\xi(-1+i)}=-\pi i\cdot e^{-|\xi|(1-i)} & \text{for } \xi>0.\end{cases}$$

When $\xi=0$, the integral does not converge.

Remark 5.2 For $\xi < 0$ we have

$$\hat{f}(\xi) = \pi i\, e^{-i|\xi|(1+i)} = \pi\, e^{-|\xi|} \cdot i\,\{\cos|\xi| - i\,\sin|\xi|\} = \pi\, e^{-|\xi|}\,\{\sin|\xi| + i\,\cos|\xi|\},$$

so by a complex conjugation when $\xi > 0$ we get all things considered,

$$\hat{f}(\xi) = \begin{cases} \pi\, e^{-|\xi|}\,\{\sin|\xi| + i\,\cos|\xi|\}, & \text{for } \xi < 0, \\[2mm] \pi\, e^{-|\xi|}\,\{\sin|\xi| - i\,\cos|\xi|\}, & \text{for } \xi > 0. \end{cases} \qquad \Diamond$$

In a VARIANT we may use the change of variable $t = x + 1$. Then we have the following calculation for $\xi < 0$:

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{+\infty} \frac{x+1}{x^2+2x+2}\, e^{-i\xi x}\, dx = \int_{-\infty}^{+\infty} \frac{t}{t^2+1}\, e^{-i\xi(t-1)}\, dt = e^{i\xi} \int_{-\infty}^{+\infty} \frac{t}{t^2+1}\, e^{-i\xi t}\, dt \\ &= 2\pi i \cdot e^{i\xi} \cdot \operatorname{res}\left(\frac{z}{z^2+1}\, e^{-i\xi z}\,;\, i \right) = 2\pi i \cdot e^{i\xi} \left[\frac{z}{2z} \cdot e^{-i\xi z} \right]_{z=i} = \pi i \cdot e^{i\xi} \cdot e^{\xi} = \pi i\, e^{(1+i)\xi}. \end{aligned}$$

Alternatively we may use for $\xi > 0$ another residuum formula, because the conditions of its use are still valid. We get

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{+\infty} \frac{x+1}{x^2+2x+1}\, e^{-i\xi x}\, dx = -2\pi i \cdot \operatorname{res}\left(\frac{z+1}{z^2+2z+2}\, e^{-i\xi z}\,;\, -1-i\right) \\ &= -2\pi i \lim_{z\to -1-i} \left\{ \frac{z+1}{2z+2}\, e^{-i\xi z}\right\} = -2\pi i \lim_{z\to -1-i}\left\{ \frac{1}{2}\, e^{-i\xi z}\right\} \\ &= -\pi i\, e^{-i\xi(-1-i)} = -\pi i\, e^{\xi(-1+i)}.\end{aligned}$$

Example 5.21 *Given the function f by*

$$f(z) = \frac{z\, e^{iz}}{\left(z^2+1\right)^2}.$$

1) Find the singularities and their type of f in $\mathbb{C} \cup \{\infty\}$.

2) Compute the complex line integral

$$\oint_{C_R} f(z)\, dz,$$

where C_R denotes the simple closed curve, which consists of

the half circle $z = R\, e^{i\theta}, \quad 0 \le \theta \le \pi, \quad R > 1,$

and

the interval $[-R, R]$ on the real axis.

3) Prove that the improper integral

$$\int_0^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\, dx$$

is convergent, and compute its value.

1) Clearly, $z = \pm i$ are double poles. Furthermore, ∞ is an essential singularity. In fact, we have

$$f(-iy) \to +\infty \qquad \text{for } y \to +\infty,$$

and also

$$f(x) \to 0 \qquad \text{for } x \to +\infty,$$

so we can obtain at least two different limit values for $z \to \infty$.

2) We have only the singularity $z = i$ lying inside C_R, so we get by a residuum formula,

$$\begin{aligned}\oint_{C_R} f(z)\,dz &= 2\pi i \cdot \operatorname{res}\left(\frac{z\,e^{iz}}{\left(z^2+1\right)^2}\,;\,i\right) = 2\pi i \lim_{z\to i}\frac{d}{dz}\left\{\frac{z\,e^{iz}}{(z+i)^2}\right\}\\ &= 2\pi i \lim_{z\to i}\left\{\frac{e^{iz}}{(z+i)^2} + \frac{i\,z\,e^{iz}}{(z+i)^2} - \frac{2z\,e^{iz}}{(z+i)^3}\right\}\\ &= 2\pi i\left\{\frac{e^{-1}}{(2i)^2} - \frac{e^{-1}}{(2i)^2} - \frac{2i\,e^{-1}}{(2i)^3}\right\} = \frac{\pi i}{2e}.\end{aligned}$$

3) Since we have a zero of order 3 at $infinity$, we get by taking the limit $R \to +\infty$ that

$$\int_{-\infty}^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx = \operatorname{Im}\left\{\lim_{R\to+\infty}\oint_{C_R} f(z)\,dz\right\} = \frac{\pi}{2e}.$$

Since the integrand is even, we finally get

$$\int_0^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx = \frac{\pi}{4e}.$$

Example 5.22 *Given the function*

$$f(z)=\frac{z\,e^{iz}}{\left(z^2+1\right)^2}.$$

1) Find the singular points and their types of f in $\mathbb{C}$.

2) Let x_1, x_2, y_1 denote any positive real numbers where $y_1>1$, and let $\gamma=\gamma_{x_1,x_2,y_1}$ denote the closed curve (run through in the positive sense), which surrounds the domain

$$A_{x_1,x_2,y_1}=\{z\in\mathbb{C}\mid -x_1<Re(z)<x_2 \text{ and } 0<Im(z)<y_1\}.$$

Prove that

$$\oint_\gamma f(z)\,dz=i\,\frac{\pi}{2e}.$$

3) Prove that the improper integral

$$\int_0^{+\infty}\frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx$$

is convergent and find its value.

1) The denominator has the two double zeros πi, and since the numerator is $\neq 0$ in these points, we conclude that $\pm i$ are double poles.

2) We see that $+i$ is the only singularity inside γ, hence it follows by the residuum theorem that

$$\begin{aligned}\oint_\gamma f(z)\,dz &= 2\pi i\,\text{res}\left(\frac{z\,e^{iz}}{\left(z^2+1\right)^2}\,;\,i\right)=2\pi i\lim_{z\to i}\frac{d}{dz}\left(\frac{z\,e^{iz}}{(z+i)^2}\right)\\ &= 2\pi i\lim_{z\to i}\left\{\frac{e^{iz}}{(z+i)^2}+\frac{i\,z\,e^{iz}}{(z+i)^2}-\frac{2z\,e^{iz}}{(z+i)^3}\right\}\\ &= 2\pi\,i\,e^{-1}\left\{\frac{1}{(2i)^2}+\frac{i^2}{(2i)^2}-\frac{2i}{(2i)^3}\right\}=\frac{2\pi i}{e}\cdot\frac{1}{4}=i\,\frac{\pi}{2e}.\end{aligned}$$

3) It follows from

$$\frac{x}{\left(x^2+1\right)^2}\sim\frac{1}{|x|^3}\qquad\text{for } |x| \text{ large,}$$

that the improper integral is convergent.

When we apply the parametric description $z(t)=-x_1+it$, $0<t<y_1$, for one part of γ we here get the estimate of the integrand,

$$|f(z)|=\left|\frac{z\,e^{iz}}{\left(z^2+1\right)^2}\right|\le\left|\frac{z}{z^2+1}\right|\cdot\frac{e^{-t}}{|z^2+|}\le\frac{|x_1|}{\left(|x_1|^2-1\right)^2}\,e^{-t},$$

and the line integral along this part of γ fulfils the estimate

$$\left|\int_0^{y_1} f\left(-x_1+it\right)\, i\,dt\right| \le \frac{|x_1|}{\left(|x_1|^2-1\right)^2} \to 0 \qquad \text{for } x_1 \to +\infty.$$

Analogously we get

$$\left|\int_0^{y_1} f\left(x_2+it\right)\, i\,dt\right| \le \frac{|x_2|}{\left(|x_2|^2-1\right)^2} \to 0 \qquad \text{for } x_2 \to +\infty.$$

Finally, we get for the curvilinear part by choosing the parametric description $z(t) = t + iy_1$, $t \in [-x_1, x_2]$ [that

$$|f(z)| = \left|\frac{z(t)|}{\left(z(t)^2+1\right)^2}\right| e^{-y_1},$$

so the corresponding line integral is estimated by

$$\left|\int_{-x_1}^{x_2} f(z)\,dz\right| \le \text{ constant} \cdot e^{-y_1} \to 0 \qquad \text{for } y_1 \to +\infty.$$

Then by taking the limits $x_1 \to +\infty$ and $x_2 \to +\infty$ and $y_1 \to +\infty$,

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{\left(x^2+1\right)^2}\,dx = i\,\frac{\pi}{2e}.$$

We conclude from

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{\left(x^2+1\right)^2}\,dx = \int_{-\infty}^{+\infty} \frac{x\,\cos x}{\left(x^2+1\right)^2}\,dx + i\int_{-\infty}^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx = 2i\int_0^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx,$$

that

$$\int_0^{+\infty} \frac{x\,\sin x}{\left(x^2+1\right)^2}\,dx = \frac{\pi}{4e}.$$

Example 5.23 *Given the function*

$$f(z) = \frac{z\,e^{iz}}{\left(z^2+4\right)^2}.$$

Denote by $\Gamma_\varrho = \gamma_\varrho + C_\varrho^+$ *the simple closed curve run through in the positive direction, consisting of* γ_ϱ*, the line segment* $[-\varrho, +\varrho]$ *on the real axis and the half circle* C_ϱ^+ *in the upper half plane of centrum 0 and radius* ϱ*.*

1) Find the isolated singularities and their types of f *in* $\mathbb{C}$*.*

2) Prove for $\varrho > 2$ *that*

$$\oint_{\Gamma_\varrho} f(z)\,dz = i\,\frac{\pi}{4e^2}.$$

3) Prove that

$$\int_{C_\varrho^+} f(z)\,dz \to 0 \qquad \text{as } \varrho \to +\infty.$$

4) Compute the improper integrals

$$p.v.\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{\left(x^2+4\right)^2}\,dx \qquad og \qquad \int_0^{+\infty} \frac{x\,\sin x}{\left(x^2+4\right)^2}\,dx.$$

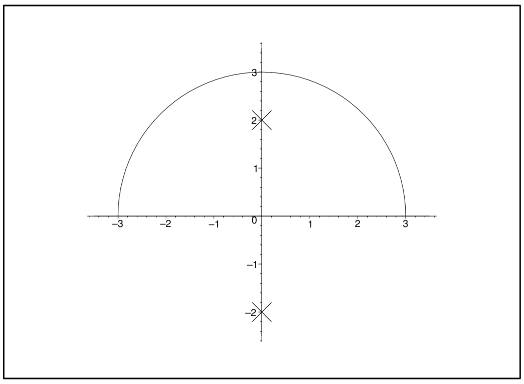


Figure 11: The closed path of integration C_ϱ and the two singularities $\pm 2i$.

1) The function $f(z)$ has the two double poles $\pm 2i$.

2) When $\varrho > 2$, only the double pole $2i$ lies inside Γ_ϱ. Hence by *Cauchy's residuum theorem*,

$$\begin{aligned}
\oint_{\Gamma_\varrho} f(z)\,dz &= 2\pi i \,\text{res}\left(\frac{z\,e^{iz}}{\left(z^2+4\right)^2}\,;\,2i\right) = 2\pi i \cdot \frac{1}{1!}\lim_{z\to 2i}\frac{d}{dz}\left\{\frac{z\,e^{iz}}{(z+2i)^2}\right\} \\
&= 2\pi i \lim_{z\to 2i}\left\{\frac{e^{iz}}{(z+2i)^2} + \frac{i\,z\,e^{iz}}{(z+2i)^2} - \frac{2z\,e^{iz}}{(z+2i)^3}\right\} \\
&= 2\pi i\left\{\frac{e^{-2}}{(4i)^2} + \frac{i\cdot 2i\cdot e^{-2}}{(4i)^2} - \frac{4i\cdot e^{-2}}{(4i)^3}\right\} = \frac{2\pi i}{e^2}\left\{-\frac{1}{16}+\frac{1}{8}+\frac{1}{16}\right\} = i\,\frac{\pi}{4e^2}.
\end{aligned}$$

3) A parametric description of C_ϱ^+ may be chosen as $z(t)\varrho\, e^{it}$, $t \in [0,\pi]$, so we get the following estimate when $\varrho > 2$,

$$\begin{aligned}\left|\int_{C_\varrho^+} f(z)\,dz\right| &\le \int_0^\pi \frac{\varrho\,|\exp(i\,\varrho\{\cos t + i\,\sin t\})|}{(varrho^2-4)^2}\cdot\varrho\,dt = \frac{\varrho^2}{(\varrho^2-4)^2}\int_0^\pi \exp(-\varrho\cdot\sin t)\,dt \\ &\le \frac{\pi\varrho^2}{(\varrho^2-4)^2} \to 0 \qquad \text{for } \varrho \to +\infty.\end{aligned}$$

4) Both the improper integrals are trivially absolutely convergent, so it is not necessary to write "p.v." (= "principal value") here.
It follows by a residuum formula, where we use the limits above,

$$\int_{-\infty}^{+\infty} \frac{x\,e^{ix}}{(x^2+4)^2}\,dx = \lim_{\varrho\to+\infty}\oint_{\Gamma_\varrho} \frac{z\,e^{iz}}{(z^2+4)^2}\,dz = i\cdot\frac{\pi}{4e^2},$$

and then by a reflection argument,

$$\begin{aligned}\int_0^{+\infty} \frac{x\,\sin x}{(x^2+4)^2}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x\,\sin x}{(x^2+4)^2}\,dx = \frac{1}{2}\,\mathrm{Im}\left\{\int_{-\infty}^{+\infty}\frac{x\,e^{ix}}{(x^2+4)^2}\,dx\right\} \\ &= \frac{1}{2}\,\mathrm{Im}\left\{i\cdot\frac{\pi}{4e^2}\right\} = \frac{\pi}{8e^2}.\end{aligned}$$

6 Improper integrals, where the integrand is a rational function times an exponential function

Example 6.1 *Given $a \in]0,1[$, prove that*

$$(a) \quad \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx = \frac{\pi}{\sin \pi a}, \qquad (b) \quad \int_{-\infty}^{+\infty} \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{\cos \frac{\pi a}{2}}.$$

HINT: *Integrate the function $\dfrac{e^{az}}{e^z+1}$ along a rectangle with the corners $-R$, R, $R+2\pi i$ and $-R+2\pi i$, and then let $R \to +\infty$. The integral of* **(b)** *is found analogously, but it can also be derived from* **(a)**.

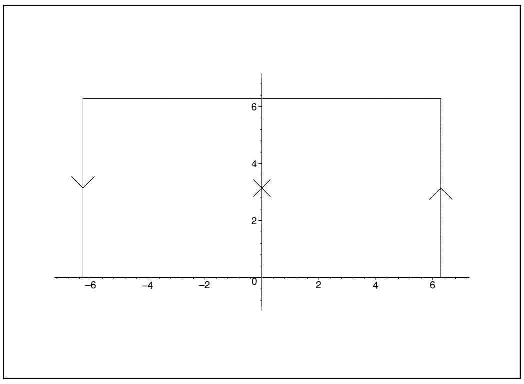


Figure 12: The curve $C_{2\pi}$ and the simple pole πi inside $C_{2\pi}$.

(a) Since $e^z+1=0$ for $z=\pi i+2\pi i$, $p \in \mathbb{Z}$, it follows that $z_0 = \pi i$ is the only singularity inside C_R for $R>0$, and this singularity is clearly a simple pole. Then we get by the residuum theorem,

$$\oint_{C_R} \frac{e^{az}}{e^z+1}\,dz = 2\pi i \cdot \text{res}\left(\frac{e^{az}}{e^z+1}\,;\,\pi i\right) = 2\pi i \lim_{z\to\pi i} \frac{e^{az}}{e^z} = 2\pi i \cdot \frac{1}{-1} \cdot e^{a\pi i} = -2\pi i\, e^{a\pi i}.$$

On the other hand,

$$\oint_{C_R} \frac{e^{az}}{e^z+1}\,dz = \int_{-R}^{R} \frac{e^{ax}}{e^x+1}\,dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}\,i\,dy + i\int_R^{-R} \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}\,dx + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\,i\,dy.$$

Using that $0<a<1$, it follows by some trivial estimates (though with a different argument) that the second and the fourth integral tend to 0 for $R \to +\infty$. Furthermore, by some trivial estimates, each of the two remaining integrals converges for $R \to +\infty$, and we have

$$\begin{aligned} -2\pi i\, e^{a\pi i} &= \lim_{R\to+\infty} \oint_{C_R} \frac{e^{az}}{e^z+1}\,dz = \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx - e^{a\cdot 2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx \\ &= \left(1-e^{2a\pi i}\right) \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx. \end{aligned}$$

Finally, by a rearrangement,

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx = \frac{2\pi i\, e^{a\pi i}}{e^{2a\pi i}-1} = \frac{\pi}{\frac{1}{2i}\left(e^{a\pi i}-e^{-a\pi i}\right)} = \frac{\pi}{\sin \pi a}.$$

(b) It follows from

$$\frac{\cosh ax}{\cosh x} = \frac{e^{ax}+e^{-ax}}{e^x+e^{-x}} = \frac{e^{(a+1)x}+e^{(1-a)x}}{e^{2x}+1} = \frac{\exp\left(\frac{a+1}{2}\,2x\right)}{e^{2x}+1} + \frac{\exp\left(\frac{1-a}{2}\,2x\right)}{e^{2x}+1},$$

and

$$0 < \frac{1+a}{2} < 1 \qquad \text{and} \qquad 0 < \frac{1-a}{2} < 1,$$

and **(a)** that

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\cosh ax}{\cosh x}\,dx &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{2}(a+1)t}}{e^t+1}\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{2}(1-a)t}}{e^t+1}\,dt = \frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{a+1}{2}\,\pi\right)} + \frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{1-a}{2}\,\pi\right)} \\ &= \frac{\pi}{2}\left\{\frac{1}{\sin\left(\frac{\pi}{2}+\frac{a\pi}{2}\right)} + \frac{1}{\sin\left(\frac{\pi}{2}-\frac{a\pi}{2}\right)}\right\} = \frac{\pi}{2}\left\{\frac{1}{\cos\frac{a\pi}{2}} + \frac{1}{\cos\left(-\frac{a\pi}{2}\right)}\right\} = \frac{\pi}{\cos\frac{\pi a}{2}}.\end{aligned}$$

Example 6.2 *Prove that*

$$\int_0^{+\infty} \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}$$

by using the path of integration sketched on the figure ant then let $R \to +\infty$ *and* $\delta \to 0+$.

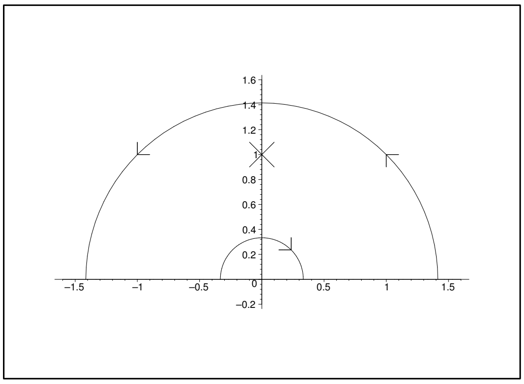


Figure 13: The curve $C_{\sqrt{2},\frac{1}{3}}$ and the simple pole i.

Let $\text{Log}^\star z$ denote the branch of the logarithm, which is given by

$$\text{Im}\{\text{Log}^\star z\} \in \left]-\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

i.e. we choose the branch of the logarithm, for which the branch cut lies along the *negative imaginary axis*. Then

$$f(z) = \frac{(\text{Log}^\star z)^2}{1+z^2}$$

is analytic in the open upper half plane with the exception of the simple pole $z = i$. Therefore, if $R > 1$ and $\delta < 1$, and we denote the curve by $C_{R,\delta}$, then

$$\oint_{C_{R,\delta}} \frac{(\text{Log}^\star z)^2}{1+z^2}\,dz = 2\pi i \cdot \text{res}\left(\frac{(\text{Log}^\star z)^2}{1+z^2}\,;\,i\right) = 2\pi i \cdot \frac{(\text{Log}^\star i)}{i+i} = \pi \cdot \left(i\,\frac{\pi}{2}\right)^2 = -\frac{\pi^3}{4},$$

which in particular shows that the value of the line integral is independent of $R > 1$ and $\delta < 1$.

The curve $C_{R,\delta}$ is composed of the interval $[\delta, R]$, the circular arc C_R, the interval $[-R, -\delta]$ and the circular arc C_δ (with obvious notations). If we put $t = -x$, then we get on the interval $[-\delta, -R]$,

$$\begin{aligned}\int_{-R}^{-\delta} \frac{(\text{Log}^\star x)^2}{1+x^2}\,dx &= \int_{-R}^{-\delta} \frac{\ln|x| + i\pi)^2}{1+x^2}\,dx = \int_\delta^R \frac{(\ln t + i\pi)^2}{1+t^2}\,dt \\ &= \int_\delta^R \frac{(\ln t)^2}{1+t^2}\,dt + 2i\pi \int_\delta^R \frac{\ln t}{1+t^2}\,dt - \pi^2 \int_\delta^R \frac{1}{1+t^2}\,dt.\end{aligned}$$

On the circular arc C_R we put $z = R\,e^{i\theta}$, $\theta \in [0,\pi]$, and then

$$|\text{Log}^{\star} z|^2 = |\ln R + i\theta|^2 = (\ln R)^2 + \theta^2.$$

We get the following estimate

$$\left|\int_{C_R} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz\right| \le \frac{(\ln R)^2+\pi^2}{R^2-1}\cdot \pi R \to 0 \qquad \text{for } R \to +\infty.$$

Analogously we get the following estimate of the circular arc C_δ,

$$\left|\int_{C_\delta} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz\right| \le \frac{(\ln \delta)^2+\pi^2}{1-\delta^2}\cdot \pi \delta \to 0 \qquad \text{for } \delta \to 0+,$$

because

$$(\ln\delta)^2\cdot\delta = \frac{\left(\ln\dfrac{1}{\delta}\right)^2}{\dfrac{1}{\delta}} \to 0 \qquad \text{for } \frac{1}{\delta}\to +\infty, \quad \text{i.e. for} \quad \delta \to 0+\,.$$

Summing up we have for $R>1$ and $0<\delta<1$,

$$\begin{aligned}
-\frac{\pi^3}{4} &= \oint_{C_{R,\delta}} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz \\
&= \int_\delta^R \frac{(\ln x)^2}{1+x^2}\,dx + \int_\delta^R \frac{(\ln t)^2}{1+t^2}\,dt + 2\pi i\int_\delta^R \frac{\ln t}{1+t^2}\,dt \\
&\qquad -\pi^2\int_\delta^R \frac{1}{1+t^2}\,dt + \int_{C_R} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz + \int_{C_\delta} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz \\
&= 2\int_\delta^R \frac{(\ln x)^2}{1+x^2}\,dx - \pi^2\int_\delta^R \frac{dt}{1+t^2} + \int_{C_R} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz \\
&\qquad + \int_{C_\delta} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz + 2i\pi\int_\delta^R \frac{\ln t}{1+t^2}\,dt.
\end{aligned}$$

Then by a rearrangement,

$$2\int_\delta^R \frac{(\ln x)^2}{1+x^2}\,dx = 2i\pi\int_\delta^R \frac{\ln t}{1+t^2}\,dt = \pi^2\int_\delta^R \frac{dt}{1+t^2} - \frac{\pi^2}{4} - \int_{C_R} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz - \int_{C_\delta} \frac{(\text{Log}^{\star} z)^2}{1+z^2}\,dz.$$

Here the left hand side is separated in its real and imaginary part.
This equation now holds for every $R>1$ and $\delta \in]0,1[$. The right hand side has a limit value for $R\to+\infty$ and $\delta\to 0+$, independent of each other,

$$\pi^2\cdot\frac{\pi}{2} - \frac{\pi^3}{4} - 0 - 0 = \frac{\pi^3}{4},$$

hence the limit value of the left hand side must also exist, and it is equal to $\dfrac{\pi^3}{4}$. Hence by separating into the real and the imaginary part we get

$$\int_0^{+\infty} \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8} \qquad \text{og} \qquad \int_0^{+\infty} \frac{\ln x}{1+x^2}\,dx = 0.$$

Example 6.3 **(a)** *Given the function*

$$F(z)=\frac{1}{z}\cdot\frac{\tanh\sqrt{z}}{\sqrt{z}}=\frac{1}{z}\cdot\frac{\sinh\sqrt{z}}{\sqrt{z}}\cdot\frac{1}{\cosh\sqrt{z}}.$$

Prove that $F(z)$ is an analytic function in a domain

$$\Omega=\mathbb{C}\setminus\{z_n\mid n\in\mathbb{N}_0\},$$

independent of the choice of the branch of the square root.

(b) *Find the poles $\{z_n\mid n\in\mathbb{N}_0\}$ of $F(z)$, as well as their orders.*

(c) *Let C_p, $p\in\mathbb{N}$, denote the simple, closed curve in the z-plane, which is composed of the line segment*

$$z=1+it,\qquad |t|\le\sqrt{p^4\pi^4-1},$$

and the circular arc

$$\Gamma_p:\quad |z|=p^2\pi^2,\qquad Re(z)\le 1.$$

Find for every fixed $t\ge 0$ the value of the line integral

$$\frac{1}{2\pi i}\oint_{C_p}e^{zt}\,F(z)\,dz=\frac{1}{2\pi i}\oint_{C_p}\frac{e^{zt}}{z}\,\frac{\tanh\sqrt{z}}{\sqrt{z}}\,dz.$$

(d) *Given that*

$$|\tanh w|\le 2\qquad \text{for } w=p\pi\,e^{i\theta},\quad \theta\in\mathbb{R},\quad p\in\mathbb{N},$$

prove that for every fixed $t\ge 0$,

$$\lim_{p\to+\infty}\int_{\Gamma_p}\frac{e^{zt}}{z\sqrt{z}}\,\tanh\sqrt{z}\,dz=0.$$

(e) *Using that $F(z)$ has an inverse Laplace transform given by*

$$f(t)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}e^{zt}\,F(z)\,dz=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{(1+is)t}\,F(1+is)\,ds,\qquad t\ge 0,$$

where the integral is convergent, find $f(t)$ expressed by a series and prove that this series is convergent for every $t\ge 0$.

(a) We use that $\left(\sqrt{z}\right)^2=z$, no matter the choice of the branch of the square root. Then by some series expansions,

$$\cosh\sqrt{z}=\sum_{n=0}^{+\infty}\frac{1}{(2n)!}\left(\sqrt{z}\right)^{2n}=\sum_{n=0}^{+\infty}\frac{1}{(2n)!}\,z^n$$

and

$$\frac{\sinh\sqrt{z}}{\sqrt{z}}=\frac{1}{\sqrt{z}}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)!}\left(\sqrt{z}\right)^{2n+1}=\frac{1}{\sqrt{z}}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)!}\,z^n\sqrt{z}=\sum_{n=0}^{+\infty}\frac{1}{(2n+1)!}\,z^n$$

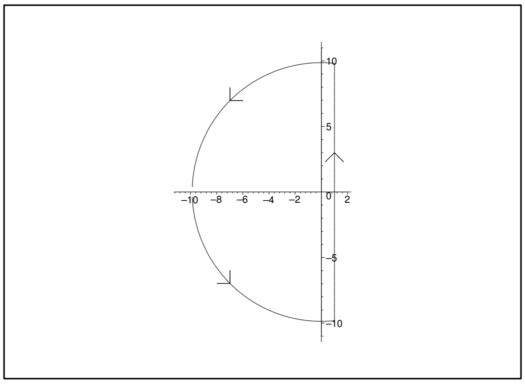


Figure 14: The path of integration C_p for $p = 1$.

so we have indeed defined a analytic function, which is independent of the choice of the branch of the square root. Notice in particular that

(4) $\lim_{z\to 0} \cosh\sqrt{z} = 1$ and $\lim_{z\to 0} \frac{\sinh\sqrt{z}}{\sqrt{z}} = 1.$

We therefore conclude that

$$F(z) = \frac{1}{z}\,\frac{\tanh\sqrt{z}}{\sqrt{z}} = \frac{1}{z}\,\frac{\sinh\sqrt{z}}{\sqrt{z}}\cdot\frac{1}{\cosh\sqrt{z}}$$

is analytic in a domain Ω, which does not contain $z = 0$ or the zeros of $\cosh\sqrt{z}$.

(b) The zeros of $\cosh\sqrt{z}$ are found in the following way,

$$\sqrt{z} = i\left(\frac{\pi}{2} + p\pi\right), \qquad p \in \mathbb{Z},$$

thus

$$z = -\left(\frac{\pi}{2} + p\pi\right)^2, \qquad p \in \mathbb{Z}.$$

Then note that p and $-p-1$, $p \in \mathbb{N}_0$ give the same z, so we can now replace $\mathbb{Z}$ by N_0

When p is replaced by $p-1$, then the singularities become

$$z_0 = 0 \qquad \text{and} \qquad z_p = -(2p-1)^2\,\frac{\pi^2}{4}, \quad p \in \mathbb{N}.$$

Then we determine the order of z_p, $p \in \mathbb{N}_0$. Since

$$F(z) = \frac{1}{z}\cdot\frac{1}{\cosh\sqrt{z}}\cdot\frac{\sinh\sqrt{z}}{\sqrt{z}},$$

we conclude from (4) that $z_0 = 0$ is a simple pole.

When

$$z_p = -(2p-1)^2 \, \frac{\pi^2}{4}, \qquad p \in \mathbb{N},$$

we get

$$\frac{\sinh \sqrt{z_p}}{z_p \sqrt{z_p}} \neq 0 \qquad \text{og} \qquad \cosh \sqrt{z_p} = 0,$$

and since

$$\lim_{z \to z_p} \frac{d}{dz} \cosh \sqrt{z} = \lim_{z \to z_p} \sinh \sqrt{z} \cdot \frac{1}{2\sqrt{z}} = \frac{\sinh \sqrt{z_p}}{2\sqrt{z_p}} \neq 0,$$

we conclude that every z_p is a simple pole.

(c) Using that $z_p = -\left(p - \frac{1}{2}\right)^2 \pi^2$, it follows from Cauchy's residuum theorem that

$$\frac{1}{2\pi i} \oint_{C_p} e^{zt} \, F(z) \, dz = \frac{1}{2\pi i} \oint_{C_p} \frac{e^{zt}}{z} \cdot \frac{\tanh \sqrt{z}}{\sqrt{z}} \, dz = \sum_{n=0}^{p} \text{res} \left(e^{zt} \, F(z) \, ; \, z_p \right),$$

because only z_0, z_1, $\dots$, z_p lie inside C_p.

Then by Rule Ia,

$$\operatorname{res}\left(e^{zt}\,F(z)\,;\,z_0\right)=\lim_{z\to 0} e^{zt}\cdot\frac{\sinh\sqrt{z}}{\sqrt{z}}\cdot\frac{1}{\cosh\sqrt{z}}=1\cdot 1\cdot\frac{1}{1}=1,$$

where we again have used (4).

In the computation of

$$\operatorname{res}\left(e^{zt}\,F(z)\,;\,z_n\right),\qquad n\in\mathbb{N},$$

we shall use Rule II, because z_n is a simple pole. We put

$$A(z)=\frac{e^{zt}}{z}\cdot\frac{\sinh\sqrt{z}}{\sqrt{z}}\qquad\text{and}\qquad B(z)=\cosh\sqrt{z},$$

and get by Rule II,

$$\operatorname{res}\left(e^{zt}\,F(z)\,;\,z_n\right)=\frac{A\left(z_n\right)}{B'\left(z_n\right)}=\lim_{z\to z_n}\left\{\frac{e^{zt}}{z}\cdot\frac{\sinh\sqrt{z}}{\sqrt{z}}\right\}\cdot\frac{1}{\sinh\sqrt{z}\cdot\dfrac{1}{2\sqrt{z}}}=\lim_{z\to z_n}\frac{2\,e^{zt}}{z}=\frac{2}{z_n}\,e^{z_n t},$$

hence by insertion,

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_p} e^{zt}\,F(z)\,dz &= \sum_{n=0}^{p}\operatorname{res}\left(e^{zt}\,F(z)\,;\,z_n\right)=1+\sum_{n=1}^{p}\frac{2}{z_n}\,e^{z_n t}\\ \text{(5)}\qquad &= 1-\frac{8}{\pi^2}\sum_{n=1}^{p}\frac{1}{(2n-1)^2}\,\exp\left(-\left\{n-\frac{1}{2}\right\}^2\pi^2 t\right).\end{aligned}$$

(d) If $z\in\Gamma_p$, then $|z|=p^2\pi^2$ and $|\sqrt{z}|=p\pi$. According to the given formula,

(6) $\left|\tanh\sqrt{z}\right|\le 2\qquad$ for $|z|=p^2\pi^2$.

We have on Γ_p that $\operatorname{Re}(z)\le 1$ and $|z|=p^2\pi^2$, so we get by (6) for every fixed $t\ge 0$ the following estimate,

$$\begin{aligned}\left|\int_{\Gamma_p}\frac{e^{zt}}{z\sqrt{z}}\,\tanh\sqrt{z}\,dz\right| &\le \max_{z\in\Gamma_p}\left|\frac{e^{zt}}{z\sqrt{z}}\cdot\tanh\sqrt{z}\right|\cdot\ell\left(\Gamma_p\right)\le\frac{e^{t\cdot 1}}{p^3\pi^3}\cdot 2\cdot 2\pi p^2\pi^2\\ &= \frac{4\,e^t}{p}\to 0\qquad\text{for } p\to+\infty,\end{aligned}$$

thus

(7) $$\lim_{\substack{p\in\mathbb{N}\\ p\to+\infty}}\int_{\Gamma_p}\frac{e^{zt}}{z\sqrt{z}}\,\tanh\sqrt{z}\,dz=0.$$

(e) We conclude from (5) that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_p} e^{zt}\,F(z)\,dz &= \frac{1}{2\pi i}\int_{1-i\sqrt{p^4\pi^4-1}}^{1+i\sqrt{p^4\pi^4-1}} e^{zt}\,F(z)\,dz+\frac{1}{2\pi i}\int_{\Gamma_p} e^{zt}\,F(z)\,dz\\ &=1-\frac{8}{\pi^2}\sum_{n=1}^{p}\frac{1}{(2n-1)^2}\,\exp\left(-\left\{n-\frac{1}{2}\right\}^2\pi^2 t\right),\end{aligned}$$

hence by a rearrangement,

$$\frac{1}{2\pi i}\int_{1-i\sqrt{p^4\pi^4-1}}^{1+i\sqrt{p^4\pi^4-1}} e^{zt}\,F(z)\,dz = 1-\frac{8}{\pi^2}\sum_{n=1}^{p}\frac{1}{(2n-1)^2}\,\exp\left(-\left\{p-\frac{1}{2}\right\}^2\pi^2 t\right)-\frac{1}{2\pi i}\int_{\Gamma_p} e^{zt}\,F(z)\,dz.$$

Then by (7) by taking the limit $p \to +\infty$, $p \in \mathbb{N}$,

$$(8)\;\; f(t) := \frac{1}{2\pi i}\int_{1-i\,\infty}^{1+i\,\infty} e^{zt}\,F(z)\,dz = 1-\frac{8}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\,\exp\left(-\left\{n-\frac{1}{2}\right\}^2\pi^2 t\right).$$

Clearly,

$$\exp\left(-\left\{n-\frac{1}{2}\right\}^2\pi^2 t\right) \le 1 \qquad \text{for } t \ge 0 \text{ and } n \in \mathbb{N},$$

so we have the estimate

$$\left|\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\,\exp\left(-\left\{n-\frac{1}{2}\right\}^2\pi^2 t\right)\right| \le \sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

and the series is absolutely and uniformly convergent for $t \ge 0$.

Remark 6.1 This example is a simplified version of a problem connected with oil drilling in the North Sea. One wanted to find the inverse Laplace transform of

$$F(z;\lambda,\omega) = \frac{1}{z}\,\frac{\tanh\sqrt{\varphi(z)}}{\sqrt{\varphi(z)}},$$

where

$$\varphi(z) = \varphi(z;\lambda,\omega) = z - \frac{\omega}{\lambda(z+\omega)},$$

and where λ and ω are two positive parameters, which are fixed by some practical measurements. The principles for solving this original problem are the same as the simplified example presented here, but one must admit that the computations are far more difficult that in this special case, where $\varphi(z) = z$. ◊

Remark 6.2 All though it is not required we shall here also prove (5), i.e.

$$(9)\;\; |\tanh w| \le 2 \qquad \text{for } w = p\pi\cdot e^{i\theta},\quad \theta\in\mathbb{R},\quad p\in\mathbb{N}.$$

We first introduce for $p \in \mathbb{N}$ a real auxiliary function ψ_p by

$$(10)\;\; \psi_p(\theta) = \cosh(2p\pi\cdot\cos\theta) + \cos(2p\pi\cdot\sin\theta), \qquad \theta\in\mathbb{R}.$$

Then we prove that

$$(11)\;\; \cos(2p\pi\cdot\sin\theta) \ge 0 \qquad \text{for Arcsin}\left(1-\frac{1}{4p}\right) \le |\theta| \le \frac{\pi}{2}.$$

Since $\cos(-u) = \cos u$, we may assume in (11) that

$$\theta \in \left[\text{Arcsin} \left(1 - \frac{1}{4p} \right), \frac{\pi}{2} \right].$$

Since $\sin\theta$ is increasing in this interval, we get

$$2p\pi \, \sin\theta \in \left[2p\pi \left(1 - \frac{1}{4p} \right), 2p\pi \right] = \left[2p\pi - \frac{\pi}{2} \, , \, 2p\pi \right],$$

and since $\cos u \geq 0$, when $u = 2p\pi \, \sin\theta \in \left[2p\pi - \frac{\pi}{2} \, , \, 2p\pi \right]$, we have proved (11).

Then we prove that

$$\text{(12)} \quad \cosh(2p\pi \, \cos\theta) \geq \cosh\left(\frac{\pi}{2} \sqrt{8p-1} \right) \text{ for } |\theta| \in \left[0 \, , \, \text{Arcsin} \left(1 - \frac{1}{4p} \right) \right].$$

We may again assume that $\theta \in \left[0 \, , \, \text{Arcsin} \left(1 - \frac{1}{4p} \right) \right]$, and using that $\cos\theta$ is decreasing in this interval, it follows that

$$\begin{aligned} \cos\theta \quad &\geq \quad \cos\left(\text{Arcsin} \left(1 - \frac{1}{4p} \right) \right) = +\sqrt{1 - \sin^2\left(\text{Arcsin} \left(1 - \frac{1}{4p} \right) \right)} \\ &= \quad \sqrt{1 - \left(1 - \frac{1}{4p} \right)^2} = \sqrt{1 - \left\{ 1 - \frac{1}{2p} + \frac{1}{16p^2} \right\}} = \frac{\sqrt{8p-1}}{4p}, \end{aligned}$$

and since cosh is increasing in $\mathbb{R}_+$, we get

$$\cos(2\pi \, \cos\theta) \geq \cosh\left(2p\pi \cdot \frac{\sqrt{8p-1}}{4p} \right) = \cosh\left(\frac{\pi}{2} \sqrt{8p-1} \right).$$

Now,

$$\psi_p(\theta + \pi) = \cosh(-2p\pi \, \cos\theta) + \cos(-2p\pi \, \sin\theta) = \psi_p(\theta),$$

and

$$\cosh\left(\frac{\pi}{2} \sqrt{8p-1} \right) \geq 2 \qquad \text{for alle } p \in \mathbb{N},$$

so we conclude in general by (11) and (12) that

$$\psi_p(\theta) = \cosh(2p\pi \, \cos\theta) + \cos(2p\pi \, \sin\theta) \geq \begin{cases} \cosh 0 + 0 = 1, \\ \\ \cosh\left(\frac{\pi}{2} \sqrt{8p-1} \right) - 1 \geq 1, \end{cases}$$

where at least one of the two estimates holds for any θ.
Summing up we have proved that

$$\text{(13)} \quad \psi_p(\theta) = \cosh(2p\pi \, \cos\theta) + \cos(2p\pi \, \sin\theta) \geq 1.$$

Then we use the definitions of the hyperbolic function of a complex variable,

$$\begin{aligned} |\tanh w|^2 &= \frac{|\sinh w|^2}{|\cosh w|^2} = \frac{\cosh^2 u - \cos^2 v}{\cosh^2 u - \sin^2 v} = 1 - \frac{\cos^2 v - \sin^2 v}{\cosh^2 u - \frac{1}{2} + \frac{1}{2} - \sin^2 v} \\ &= 1 - \frac{2\,\cos 2v}{\left(2\,\cosh^2 - 1\right) + \left(1 - 2\,\sin^2 v\right)} = 1 - \frac{2\,\cos 2v}{\cosh 2u + \cos 2v}. \end{aligned}$$

Then put

$$w = p\pi\, e^{i\theta} = p\pi\,\cos\theta + i\,p\pi\,\sin\theta = u + iv,$$

and apply (13) to get

$$\left|\tanh\left(p\pi\, e^{i\theta}\right)\right|^2 = 1 - \frac{2\,\cos(2p\pi\,\sin\theta)}{\cosh(2p\pi\,\cos\theta) + \cos(2p\pi\,\sin\theta)} \le 1 + \frac{2}{\psi_p(\theta)} \le 3,$$

thus

$$\left|\tanh\left(p\pi\, e^{i\theta}\right)\right| \le \sqrt{3} \quad (< 2),$$

and we have proved (9) with the even smaller constant $\sqrt{3}$. $\Diamond$

Example 6.4 *Compute*

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+9} \exp\left(\frac{x+3i}{x^2+9}\right) dx.$$

It follows from $\dfrac{z+3i}{z^2+9} = \dfrac{1}{z-3i}$, that this function can be extended analytically to $-3i$, so we get the estimate

$$\left|\frac{z+3i}{z^2+9}\right| = \frac{1}{|z-3i|} \le \frac{4}{|z|} \qquad \text{for } |z| \ge 4,$$

hence

$$\left|\exp\left(\frac{z+3i}{z^2+9}\right)\right| \le \exp\left(\frac{1}{|z-3i|}\right) \le \exp\left(\frac{4}{|z|}\right) \le e \qquad \text{for } |z| \ge 4.$$

Then we estimate the integrand by

$$\left|\frac{1}{z^2+9} \exp\left(\frac{z+3i}{z^2+9}\right)\right| \le \frac{k}{|z|^2} \qquad \text{for } |z| \ge 4.$$

The singularities are $z = \pm 3i$, where none of them lies on the real axis. We conclude that the improper integral is convergent and that its value can be found by a residuum formula,

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+9} \exp\left(\frac{x+3i}{x^2+9}\right) dx = 2\pi i \operatorname{res}\left(\frac{1}{z^2+9} \exp\left(\frac{1}{z-3i}\right); 3i\right).$$

The idea here is that the sum of the residues is 0. Since ∞ is a zero of second order, we have

$$\operatorname{res}\left(\frac{1}{z^2+9} \exp\left(\frac{1}{z-3i}\right); \infty\right) = 0.$$

Now $z = -3i$ is a simple pole, so

$$\operatorname{res}\left(\frac{1}{z^2+9} \exp\left(\frac{1}{z-3i}\right); -3i\right) = \frac{1}{-6i} \exp\left(-\frac{1}{6i}\right) = \frac{i}{6} \exp\left(\frac{i}{6}\right).$$

The sum of the residues is zero, so it follows from the above that

$$\operatorname{res}\left(\frac{1}{z^2+9} \exp\left(\frac{1}{z-3i}\right); 3i\right) = -\frac{i}{6} \exp\left(\frac{i}{6}\right).$$

Finally, by insertion

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+9} \exp\left(\frac{x+3i}{x^2+9}\right) dx = 2\pi i \left\{-\frac{i}{6} \exp\left(\frac{i}{6}\right)\right\} = \frac{\pi}{3} \exp\left(\frac{i}{6}\right).$$

Remark 6.3 We notice by separating the real and the imaginary part that it follows from this that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+9} \exp\left(\frac{x}{x^2+9}\right) \cos\left(\frac{3}{x^2+9}\right) dx = \frac{\pi}{3} \cos\frac{1}{6},$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+9} \exp\left(\frac{x}{x^2+9}\right) \sin\left(\frac{3}{x^2+9}\right) dx = \frac{\pi}{3} \sin\frac{1}{6}. \qquad \Diamond$$

ALTERNATIVELY one may compute the residuum at $z = 3i$ directly.
We get by the change of variable $w = z - 3i$ that

$$\operatorname{res}\left(\frac{1}{z^2+9}\,\exp\left(\frac{1}{z-3i}\right);3i\right) = \operatorname{res}\left(\frac{1}{w(w+6i)}\,\exp\left(\frac{1}{w}\right);0\right).$$

Here $w_0 = 0$ is an essential singularity, so we must find the Laurent series expansion and find the coefficient a_{-1} of $\frac{1}{w}$. When $0 < |w| < 6$, then

$$\frac{1}{w(w+6i)}\,\exp\left(\frac{1}{w}\right) = \frac{1}{w}\cdot\frac{1}{6i}\cdot\frac{1}{1+\frac{w}{6i}}\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{w^n} = \frac{1}{w}\left\{\frac{1}{6i}\sum_{p=0}^{+\infty}(-1)^p\left(\frac{w}{6i}\right)^p\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{w^n}\right\}.$$

It follows immediately that a_{-1} is the constant term inside the parenthesis, so a_{-1} is found by putting $p = n$, thus

$$\operatorname{res}\left(\frac{1}{z^2+9}\,\exp\left(\frac{1}{z-3i}\right);3i\right) = \frac{1}{6i}\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(6i)^n}\cdot\frac{1}{n!} = \frac{1}{6i}\,\exp\left(-\frac{1}{6i}\right) = -\frac{i}{6}\,\exp\left(\frac{i}{6}\right),$$

and we get as previously that

$$\int_{-\infty}^{+\infty}\frac{1}{x^2+9}\,\exp\left(\frac{x+3i}{x^2+9}\right)dx = \frac{\pi}{3}\,\exp\left(\frac{i}{6}\right) = \frac{\pi}{3}\left\{\cos\frac{1}{6}+i\,\sin\frac{1}{6}\right\}.$$

Example 6.5 *Given the function*

$$f(z) = \frac{e^{iz}}{\cosh z}, \qquad \textit{where } \cosh z = \frac{e^z + e^{-z}}{2}.$$

Define for every $R > 0$ the den simple closed curve

$$\Gamma_R = \Gamma_R^1 + \Gamma_R^2 + \Gamma_R^3 + \Gamma_R^4$$

which is the sides of the rectangle shown on the figure.

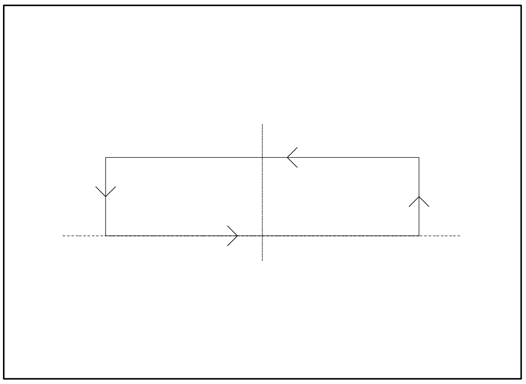


Figure 15: The curve Γ_R is composed of the four straight line segments: $\Gamma_R^1 = [-R, R]$ on the x-axis, $\Gamma_R^2 = R + i[0, \pi]$, parallel with the y-axis, $\Gamma_R^3 = [-R, R] + i\pi$ parallel with the x-axis, and $\Gamma_R^4 = -R + i[0, \pi]$ parallel with the y-axis, and with the given sense of direction.

1) *Find all isolated singularities of f in $\mathbb{C}$.*
   Determine for each of them its type and its residuum.

2) *Prove that*

$$\int_{\Gamma_R} f(z)\, dz = 2\pi \exp\left(-\frac{\pi}{2}\right).$$

3) *Prove that the line integrals along Γ_R^2 and Γ_R^4 tend to 0 for $R \to +\infty$.*
   HINT: *One may use that*

$$|\cosh(x + iy)| = \sqrt{\sinh^2 x + \cos^2 y}.$$

4) *Prove that the improper integral*

$$\int_0^{+\infty} \frac{\cos x}{\cosh x}\, dx$$

   is convergent, and find its value.

1) The numerator and the denominator are both analytic in all of $\mathbb{C}$, and the numerator is $\neq 0$ everywhere, so the singularities are given by the zeros of the denominator $\cosh z$, i.e.

$$z_p = i\left\{\frac{\pi}{2}+p\pi\right\}, \qquad p \in \mathbb{Z}.$$

It follows from

$$\frac{d}{dz}\cosh z_p = \sinh z_p \neq 0, \qquad \text{for alle } p \in \mathbb{Z},$$

that they are all simple pole of $f(z)$.

Finally,

$$\begin{aligned}\operatorname{res}\left(f\,,\, i\left\{\frac{\pi}{2}+p\pi\right\}\right) &= \left[\frac{e^{iz}}{\sinh z}\right]_{z=z_p} = \frac{\exp\left(-\frac{\pi}{2}-p\pi\right)}{i(-1)^p} \\ &= (-1)^{p-1} i\,\exp\left(-\left\{\frac{\pi}{2}+p\pi\right\}\right), \qquad p \in \mathbb{Z}.\end{aligned}$$

2) For every $R > 0$ the curve Γ_R surrounds only the singularity $z_0 = i\,\dfrac{\pi}{2}$.
Then we use the residuum theorem,

$$\oint_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \operatorname{res}\left(f\,,\, i\,\frac{\pi}{2}\right) = 2\pi \cdot \exp\left(-\frac{\pi}{2}\right).$$

3) The vertical line segment Γ_R^2 (possibly Γ_R^4) has e.g. the parametric description

$$z(t) = R + it, \qquad t \in [0,\pi],$$

so we obtain the estimate

$$\left|\int_{\Gamma_R^2} f(z)\,dz\right| = \left|\int_0^\pi \frac{e^{i(R+it)}}{\cosh(R+it)}\cdot i\,dt\right| \leq \int_0^\pi \frac{e^{-t}}{|\cosh(R+it)|}\,dt.$$

From

$$|\cosh(R+it)| = \sqrt{\sinh^2 R + \cos^2 t} \geq |\sinh R|,$$

we get the estimate

$$\left|\int_{\Gamma_R^2} f(z)\,dz\right| \leq \int_0^\pi \frac{1}{|\sinh R|}\,dt = \frac{\pi}{|\sinh R|} \to 0 \qquad \text{for } R \to +\infty.$$

We have only assumed in the argument above that $R \in \mathbb{R}$, so we also have

$$\left|\int_{\Gamma_R^4} f(z)\,dz\right| \leq \frac{\pi}{|\sinh R|} \to 0 \qquad \text{for } R \to +\infty.$$

4) Now $\cosh x \geq 1 + \frac{1}{2}\,x^2$, so it follows from the estimate

$$\left|\int_{-\infty}^{+\infty} \frac{a\,\cos x + b\,\sin x}{\cosh x}\,dx\right| \leq \int_{-\infty}^{+\infty} \frac{C}{1+\frac{1}{2}\,x^2}\,dx < +\infty,$$

that the improper integral is convergent.

When we return to the complex problem, then we get by the symmetry that

$$\int_{\Gamma_R^1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{\cosh x}\,dx = \int_{-R}^{R} \frac{\cos x + i\,\sin x}{\cosh x}\,dx = \int_{-R}^{R} \frac{\cos x}{\cosh x}\,dx,$$

and analogously,

$$\int_{\Gamma_R^3} f(z)\,dz = \int_{+R}^{-R} \frac{e^{i(x+i\pi)}}{\cosh(x+i\pi)}\,dx = +e^{-\pi}\int_{-R}^{R} \frac{\cos x + i\,\sin x}{\cosh x}\,dx = e^{-\pi}\int_{-R}^{R} \frac{\cos x}{\cosh x}\,dx,$$

because an integration of an odd function $\left(\text{her } \dfrac{\sin x}{\cosh x}\right)$ over a symmetric interval $[-R,R]$ is always 0.

Then we get by taking the limit $R \to +\infty$ in **(2)**,

$$2\pi\,\exp\left(-\frac{\pi}{2}\right) = \lim_{R\to+\infty}\int_{-R}^{R} \frac{\cos x}{\cosh x}\,dx = \left(1+e^{-\pi}\right)\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\,dx,$$

so $\dfrac{\cos x}{\cosh x}$ being even, we get by a reflection argument that

$$\int_{0}^{+\infty} \frac{\cos x}{\cosh x}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\,dx = \frac{2\pi\,\exp\left(-\frac{\pi}{2}\right)}{2\left(1+e^{-\pi}\right)} = \frac{\pi}{2}\cdot\frac{1}{\cosh\left(\frac{\pi}{2}\right)} = \frac{\pi\,\exp\left(-\frac{\pi}{2}\right)}{1+e^{-\pi}}.$$

7 Cauchy's principal value

Example 7.1 *Compute the (important) improper integral*

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx.$$

It is not possible directly to apply the analytic function $\dfrac{\sin z}{z}$ in the various solution formulæ, because it does not fulfil any of the inequalities required for the legality of some relevant residuum formula. Another problem is that we here only shall integrate along the positive real axis, i.e. not a "closed curve" in $\mathbb{C}^*$, and we cannot talk of a domain which is surrounded by the path of integration.

Instead we shall rewrite the integrand by means of Euler's formulæ. In order to avoid the singularity at the point 0 we shall integrate over an interval of the form $[\varepsilon, R]$. Then we get

$$\int_\varepsilon^R \frac{\sin x}{x} dx = \frac{1}{2i} \int_\varepsilon^R \left\{ \frac{e^{ix}}{x} - \frac{e^{-ix}}{x} \right\} dx = \frac{1}{2i} \left\{ \int_\varepsilon^R - \int_{-\varepsilon}^{-R} \right\} \frac{e^{ix}}{x}\, dx = \frac{1}{2i} \left\{ \int_{-R}^{-\varepsilon} + \int_\varepsilon^R \right\} \frac{e^{ix}}{x}\, dx.$$

If the right hand side has a limit value for $\varepsilon \to 0+$ and $R \to +\infty$, then the limit of the left hand side does also exist, and we have

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{1}{2i}\, \text{vp.} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\, dx.$$

The analytic function $\frac{1}{z}\,e^{iz}$ has only the simple pole at $z=0$, and this lies on the real axis, so it will contribute to *Cauchy's principal value* with the amount

$$\pi i \operatorname{res}\left(\frac{e^{iz}}{z}\,;\,0\right)=\pi i.$$

Since we have the structure $\frac{1}{z}\,e^{iz}=\frac{1}{z}\,e^{imz}$ of the integrand, where $m=1>0$ and $\frac{1}{z}$ has a zero of first order at ∞, taking the limit $R\to+\infty$ will not cause any problem, and we have checked the conditions for the application of the residuum formula for Cauchy's principal value. Finally, the integrand does not have any other singularity that $z=0$, so we conclude that

$$\int_0^{+\infty}\frac{\sin x}{x}\,dx=\frac{1}{2i}\,\text{vp.}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,dx=\frac{1}{2i}\cdot\pi i=\frac{\pi}{2}.$$

Example 7.2 *Compute*

$$\text{vp.}\oint_{|z|=2}\frac{dz}{2z^2+3z-2}.$$

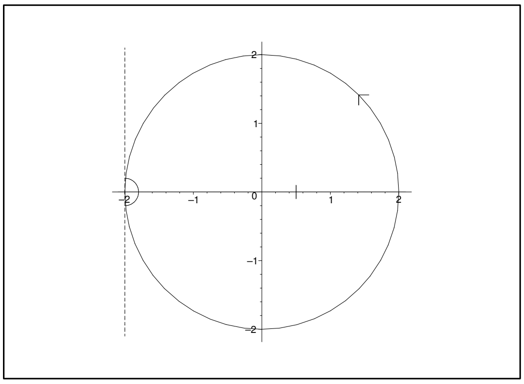


Figure 16: The circle $|z|=2$ with the evasive circular arc Γ_ε around the point -2, and the singularity $\frac{1}{2}$ inside the curve.

We first note that the denominator $2z^2+3z-2$ is 0 for

$$z=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm5}{4}=\begin{cases}-2,\\[2mm] \dfrac{1}{2}.\end{cases}$$

We see that the pole $z=-2$ lies on the path of integration, while the pole $z=\frac{1}{2}$ lies inside the curve. It follows by a decomposition that

$$\frac{1}{2z^2+3z-2}=\frac{1}{2}\cdot\frac{1}{(z+2)\left(z-\frac{1}{2}\right)}=\frac{1}{2}\cdot\frac{1}{\frac{5}{2}}\cdot\frac{1}{z-\frac{1}{2}}+\frac{1}{2}\cdot\frac{1}{\left(-\frac{5}{2}\right)}\cdot\frac{1}{z+2}=\frac{1}{5}\cdot\frac{1}{z-\frac{1}{2}}-\frac{1}{5}\cdot\frac{1}{z+2},$$

and hence

$$\begin{aligned}\text{vp.}\oint_{|z|=2}\frac{dz}{2z^2+3z-2} &= \text{vp.}\oint_{|z|=2}\frac{1}{5}\left\{\frac{1}{z-\frac{1}{2}}-\frac{1}{z+2}\right\}\\ &= \frac{1}{5}\oint_{|z|=2}\frac{dz}{z-\frac{1}{2}}-\frac{1}{5}\lim_{\varepsilon\to 0}\left\{\int_{C_\varepsilon+\Gamma_\varepsilon}-\int_{\Gamma_\varepsilon}\frac{1}{z+2}\,dz\right\},\end{aligned}$$

where

$$C_\varepsilon=\{z\in\mathbb{C}\mid |z|=2,\;|z+2|\geq\varepsilon\}$$

and

$$\Gamma_\varepsilon=\{z\in\mathbb{C}\mid |z+2|=\varepsilon,\;|z|\leq 2\}.$$

It follows from Cauchy's integral theorem that

$$\int_{C_\varepsilon+\Gamma_\varepsilon}\frac{dz}{z+2}=0,$$

so we get the following reduced expression

$$\text{vp.}\oint_{|z|=2}\frac{dz}{2z^2+3z-2}=\frac{2\pi i}{5}+\lim_{\varepsilon\to 0+}\frac{1}{5}\int_{\Gamma_\varepsilon}\frac{1}{z+2}\,dz.$$

We choose for Γ_ε the following parametric description,

$$z=-2+\varepsilon\cdot e^{i\theta}\qquad\text{for }\theta\in[\Theta_0(\varepsilon),\Theta_1(\varepsilon)],$$

where

$$\Theta_0(\varepsilon)\to-\frac{\pi}{2}\quad\text{and}\quad\Theta_1(\varepsilon)\to+\frac{\pi}{2}\qquad\text{for }\varepsilon\to 0,$$

and where the interval of the path of integration is run through in the opposite direction of the direction of the plane. Then we get by insertion and taking the limit,

$$\begin{aligned}\text{vp.}\oint_{|z|=2}\frac{dz}{2z^2+3z-2} &= \frac{2\pi i}{5}+\frac{1}{5}\lim_{\varepsilon\to 0+}\int_{\Theta_1(\varepsilon)}^{\Theta_2(\varepsilon)}\frac{i\,\varepsilon\,e^{i\theta}\,d\theta}{\varepsilon\,e^{i\theta}}=\frac{2\pi i}{5}+\frac{i}{5}\lim_{\varepsilon\to 0+}\{\Theta_0(\varepsilon)-\Theta_1(\varepsilon)\}\\ &= \frac{2\pi i}{5}+\frac{i}{5}\left\{-\frac{\pi}{5}-\frac{\pi}{2}\right\}=\frac{2\pi i}{5}-\frac{\pi i}{5}=\frac{\pi i}{5}.\end{aligned}$$

Example 7.3 *Let C denote the square with the corners 1, i, -1, $-i$. Compute*

$$\text{vp.}\oint_C\frac{dz}{z^4-1}.$$

It is obvious that the corners of C are the poles of the integrand, so for given $\varepsilon>0$ we define the auxiliary curves

$$C_\varepsilon=\{z\in\mathbb{C}\mid z\in C,\;|z-a|\geq\varepsilon,\;a=1,\,i,\,-1,\,-i\}$$

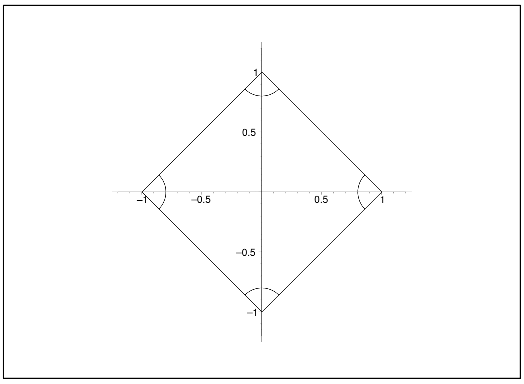


Figure 17: The curves C and C_ε with the arcs of evasion.

and

$$\Gamma_{a,\varepsilon} = \{z \in \mathbb{C} \mid |z-a| = \varepsilon,\, z \text{ inside } C\}, \qquad a = 1,\, i,\, -1,\, -i,$$

of positive direction. Then

$$\begin{aligned}
\text{vp.} \oint_C \frac{dz}{z^4-1} &= \lim_{\varepsilon} \frac{dz}{z^4-1} = \lim_{\varepsilon} \left\{ \int_{C_\varepsilon - \Gamma_{1,\varepsilon} - \Gamma_{i,\varepsilon} - \Gamma_{-1,\varepsilon} - \Gamma_{-i,\varepsilon}} \frac{dz}{z^4-1} + \sum_{a=1,i,-1,-i} \int_{\Gamma_{a,\varepsilon}} \frac{dz}{z^4-1} \right\} \\
&= \lim_{\varepsilon \to 0} \sum_{a=1,i,-1,-i} \int_{\Gamma_{a,\varepsilon}} \frac{dz}{z^4-1}.
\end{aligned}$$

By a decomposition,

$$\frac{1}{z^4-1} = \sum_{a=1,i,-1,-i} \text{res}\left(\frac{1}{z^4-1}\,;\, a \right) \cdot \frac{1}{z-a} = \frac{1}{4} \sum_{a=1,i,-1,-i} \frac{a}{z-a}.$$

Then we use the parametric descriptions

$$\Gamma_{a,\varepsilon}: \quad z = a + \varepsilon\, e^{i\theta}, \quad \theta \in \left[\Theta(a), \Theta(a) + \frac{\pi}{2} \right], \quad a = 1,\, i,\, -1,\, -i,$$

in order to get

$$\begin{aligned}
\text{vp.} \oint_C \frac{dz}{z^4-1} &= \frac{1}{4} \sum_{a=1,i,-1,-i} a \int_{\Gamma_{a,\varepsilon}} \frac{dz}{z-a} = \frac{1}{4} \sum_{a=1,i,-1,-i} a \int_{\Theta(a)}^{\Theta(a)+\frac{\pi}{2}} \frac{\varepsilon\, i\, e^{i\theta}}{\varepsilon\, e^{i\theta}}\, d\theta \\
&= \frac{1}{4} \sum_{a=1,i,-1,-i} a\, i \cdot \frac{\pi}{2} = \frac{\pi i}{8} \sum_{a=1,i,-1,-i} a = 0.
\end{aligned}$$

Example 7.4 *Compute*

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\,(x^2+1)}.$$

The integrand

$$f(z) = \frac{1}{z\,(z^2+1)}$$

is a rational function, and we have the three simple poles 0, i and $-i$. Of these, only 0 lies on the real axis, i.e. on the path of integration. Since

$$z^3 f(z) = \frac{z^3}{z\,(z^2+1)} \to 1 \qquad \text{for } z \to \infty,$$

there exists an $R > 1$, such that

$$\left| \frac{z^3}{z\,(z^2+1)} \right| \le 2 \qquad \text{for } |z| \ge R,$$

thus

$$(14) \quad \left| \frac{1}{z\,(z^2+1)} \right| \le \frac{2}{|z|^3} \qquad \text{for } |z| \ge R.$$

It follows that vp.$\int_{-\infty}^{+\infty} \dfrac{dx}{x\left(x^2+1\right)}$ exists and that it can be computed by a residuum formula,

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\left(x^2+1\right)} = 2\pi i \, \text{res}\left(\frac{1}{z\left(z^2+1\right)}\,;\, i\right) + \pi i \, \text{res}\left(\frac{1}{z\left(z^2+1\right)}\,;\, 0\right),$$

because $z = i$ lies to the left of the path of integration seen in its direction, and because its weight is $2\pi i$, while the residuum at the pole $z = 0$ on the x-axis roughly speaking is halved with only part going to the upper half plane and the other half to the lower half plane.

In the present case it suffices to convince oneself that the integral is convergent, because the integrand is an odd function, so the only possible value is 0, i.e.

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\left(x^2+1\right)} = 0.$$

For COMPLETENESS we compute

$$\text{res}\left(\frac{1}{z\left(z^2+1\right)}\,;\, i\right) = \lim_{z \to i} \frac{\frac{1}{z}}{2z} = \frac{1}{2i^2} = -\frac{1}{2}, \qquad (\text{REGEL II}),$$

where $P(z) = \dfrac{1}{z}$ and $Q(z) = z^2 + 1$, and

$$\text{res}\left(\frac{1}{z\left(z^2+1\right)}\,;\, 0\right) = \lim_{z \to 0} \frac{1}{z^2+1} = 1, \qquad (\text{REGEL IA}),$$

and we have (CONTROL),

$$\begin{aligned} \text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\left(x^2+1\right)} &= 2\pi i \, \text{res}\left(\frac{1}{z\left(z^2+1\right)}\,;\, i\right) + \pi i \, \text{res}\left(\frac{1}{z^2\left(z^2+1\right)}\,;\, 0\right) \\ &= 2\pi i \cdot \left(-\frac{1}{2}\right) + \pi i = 0. \end{aligned}$$

Example 7.5 *Compute*

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\left(x^3+1\right)}.$$

The integrand

$$f(z) = \frac{1}{z\left(z^3+1\right)}$$

is a rational function with a zero of fourth order at ∞. It is analytic in all of the complex plane except for the *simple* poles

$$0, \quad -1, \quad \frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \frac{1}{2} + i\,\frac{\sqrt{3}}{2},$$

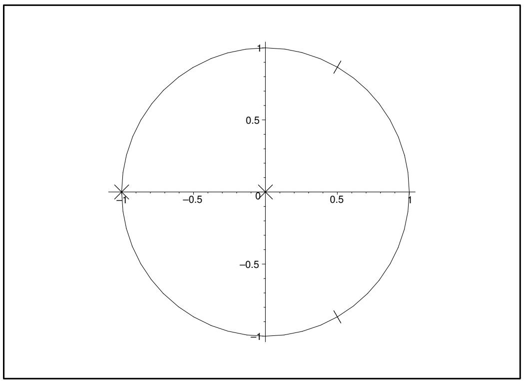


Figure 18: The four poles, of which two are lying on the path of integration, i.e. the real axis.

cf. the figure. We conclude from

$$z^4 f(z) = \frac{z^4}{z\left(z^3+1\right)} \to 1 \qquad \text{for } z \to \infty,$$

that there exists an $R > 0$, such that

$$\left|z^4 f(z)\right| \le 2, \quad \text{dvs.} \quad |f(z)| \le \frac{2}{|z|^4} \quad \text{for } |z| \ge R.$$

Now, $a = 4 > 1$, so Cauchy's principal value exists. In fact, we have *simple* poles on the x-axis, and $a > 1$ in (15). The value is given by the residuum formula

$$\begin{aligned} \text{vp.} \int_{-\infty}^{+\infty} \frac{dx}{x\left(x^3+1\right)} &= 2\pi i \,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\, \frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) \\ &\quad +\pi i \,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\, 0\right) + \pi i \,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\, -1\right). \end{aligned}$$

Then by Rule Ia,

$$\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\,0\right) = \lim_{z\to 0} \frac{1}{z^3+1} = 1.$$

The other two poles satisfy the equation $z_0^3 = -1$. Putting

$$P(z) = \frac{1}{z} \qquad \text{and} \qquad Q(z) = z^3 + 1,$$

then $P(z)$ and $Q(z)$ are analytic in a neighbourhood of z_0, and since

$$\frac{1}{z\left(z^3+1\right)} = \frac{P(z)}{Q(z)},$$

it follows from Rule II that

$$\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\,z_0\right) = \text{res}\left(\frac{P(z)}{Q(z)}\,;\,z_0\right) = \frac{P\left(z_0\right)}{Q'\left(z_0\right)} = \frac{1}{z_0}\cdot\frac{1}{3z_0^2} = \frac{1}{3}\cdot\frac{1}{z_0^3} = -\frac{1}{3}.$$

This is true for both $z_0 = -1$ and for $z_0 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$, so it follows that at

$$\begin{aligned}\text{vp.}\int_{-\infty}^{+\infty}\frac{dx}{x\left(x^3+1\right)} &= 2\pi i\,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\,\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\\ &\qquad +\pi i\,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\,0\right)+\pi i\,\text{res}\left(\frac{1}{z\left(z^3+1\right)}\,;\,-1\right)\\ &= 2\pi i\left(-\frac{1}{3}\right)+\pi i+\pi i\left(-\frac{1}{3}\right)=\pi i\left(-\frac{2}{3}+1-\frac{1}{3}\right)=0.\end{aligned}$$

Note also that since the integrand is real, the result shall also be real.

Example 7.6 *Compute*

$$\text{vp.}\int_{-\infty}^{+\infty}\frac{dx}{x^6-1}.$$

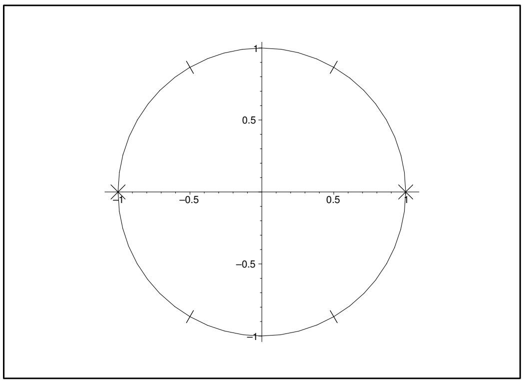


Figure 19: The poles of the integrand. Two of these, ± 1 lie on the path of integration.

The integrand $f(z) = \frac{1}{z^6-1}$ is a rational function with a zero of order 6 at ∞, i.e.

$$z^6 f(z) = \frac{z^6}{z^6-1} \to 1 \qquad \text{for } z \to \infty.$$

Hence there exists an $R > 0$, such that

$$f(z)| = \left|\frac{1}{z^6-1}\right| \le \frac{2}{|z|^6} \qquad \text{for } |z| \ge R.$$

Since $f(z)$ has only simple poler and da $a = 6 > 1$, it follows that Cauchy's principal value exists and is given by the following residuum formula,

$$\begin{aligned}\text{vp.}\int_{-\infty}^{+\infty}\frac{dx}{x^6-1} &= 2\pi i\left\{\text{res}\left(\frac{1}{z^6-1}\,;\,\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)+\text{res}\left(\frac{1}{z^6-1}\,;\,-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\right\}\\ &\qquad +\pi i\left\{\text{res}\left(\frac{1}{z^6-1}\,;\,1\right)+\text{res}\left(\frac{1}{z^6-1}\,;\,-1\right)\right\}.\end{aligned}$$

If $z_0^6 = 1$, then we have by RULE II,

$$\text{res}\left(\frac{1}{z^6-1}\,;\,z_0\right) = \frac{1}{6z_0^5} = \frac{1}{6}\,\frac{z_0}{z_0^6} = \frac{1}{6}\,z_0.$$

We can compute all four residues by this rule, so

$$\begin{aligned}\text{vp.}\int_{-\infty}^{+\infty}\frac{dx}{x^6-1} &= 2\pi i\frac{1}{6}\left\{\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)+\left(-\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\right\}+\pi i\cdot\frac{1}{6}\,\{1+(-1)\}\\ &= 2\pi i\cdot\frac{1}{6}\cdot i\sqrt{3}+0=-\frac{2\pi\sqrt{3}}{6}=-\frac{\pi}{\sqrt{3}}.\end{aligned}$$

As a very weak control we see that since the integrand is real, the result is also real.

Example 7.7 *1) Prove that the Cauchy principal value*

$$K(a) = \text{ vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,a\,x}-1}{x^2}\,dx$$

exists for every real number a, and show that

$$K(a) = -\pi|a|.$$

2) Compute the integral

$$\int_{-\infty}^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx,$$

expressed by $K(a)$ and $K(b)$.

1) First note that the integrand

$$\frac{e^{i\,a\,z}-1}{z^2}$$

has only the pole $z=0$ (and ∞ as an essential singularity). The numerator has a zero of first order at $z=0$, hence $z=0$ is a pole of *first order*. From $\left|e^{i\,a\,z}\right| \le 1$ for $a \ge 0$ and $\text{Im}(z) \ge 0$, follows the estimate

$$\left|\frac{e^{i\,a\,z}-1}{z^2}\right| \le \frac{2}{R^2} \qquad \text{for } |z| \ge R \text{ and } \text{Im}(z) \ge 0,$$

where we also have assumed that $a \ge 0$. Thus the conditions of the existence of Cauchy's principal value are fulfilled for $a \ge 0$, and it is given by a residuum formula,

$$\begin{aligned} K(a) &= \text{ vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,a\,x}-1}{x^2}\,dx = \pi i \cdot \text{res}\left(\frac{e^{i\,i\,a\,z}-1}{z^2}\,;\,0\right) = \pi\,i \cdot \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(e^{i\,a\,z}-1\right) \\ &= \pi\,i\,\lim_{z\to 0} i\,a\,e^{i\,a\,z} = -\pi\,a = -\pi|a|. \end{aligned}$$

If $a<0$, i.e. $a = -|a|$, we get by a complex conjugation and the result above that

$$K(a) = \text{ vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,a\,z}-1}{x^2}\,dx = \overline{\text{vp.} \int_{-\infty}^{+\infty} \frac{e^{i|a|}-1}{x^2}\,dx} = \overline{K(|a|)} = -\pi|a|.$$

Summing up,

$$K(a) = K(|a|) = \text{ vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,a\,x}-1}{x^2}\,dx = -\pi|a|, \qquad a \in \mathbb{R}.$$

The result is *real*, so

$$\begin{aligned} K(a) &= \text{ vp. Re}\left\{\int_{-\infty}^{+\infty} \frac{e^{i\,a\,x}-1}{x^2}\,dx\right\} = \text{ vp.} \int_{-\infty}^{+\infty} \frac{\cos(ax)-1}{x^2}\,dx \\ &= \int_{-\infty}^{+\infty} \frac{\cos(ax)-1}{x^2}\,dx = -\pi|a|. \end{aligned}$$

The numerator $\cos(ax) - 1$ has a zero of at least second order at 0, so the integrand $\dfrac{\cos(ax)-1}{x^2}$ has a removable singularity at $x = 0$, and we can remove "vp." in front of the integral, and we have the estimate

$$\frac{\cos(ax)-1}{x^2} \le \frac{2}{x^2} \qquad \text{for } |x| \ge 1.$$

2) Clearly, the zero of the numerator at $x = 0$ has order 2, so the singularity at 0 is removable. Since

$$\left|\frac{\cos(ax)-\cos(bx)}{x^2}\right| \le \frac{2}{x^2} \qquad \text{for } x \ne 0,$$

and the integrand is continuous with a continuous extension to 0, we conclude that the improper integral exists and that it is given by

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\cos(ax)-\cos(bx)}{x^2}\,dx &= \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \frac{\cos(ax)-\cos(bx)}{x^2}\,dx \\
&= \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \frac{\cos(ax)-1}{x^2}\,dx - \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \frac{\cos(bx)-1}{x^2}\,dx \\
&= \text{vp.} \int_{-\infty}^{+\infty} \frac{\cos(ax)-1}{x^2}\,dx - \text{vp.} \int_{-\infty}^{+\infty} \frac{\cos(bx)-1}{x^2}\,dx \\
&= \int_{-\infty}^{+\infty} \frac{\cos(ax)-1}{x^2}\,dx - \int_{-\infty}^{+\infty} \frac{\cos(bx)-1}{x^2}\,dx \\
&= K(a) - K(b) = -\pi(|a|-|b|) = \pi(|b|-|a|).
\end{aligned}$$

Example 7.8 *1) Find the poles, their order and their residuum for the function*

$$f(z) = \frac{\operatorname{Log} z}{(z-1)^2(z-2)(z-3)}.$$

2) Use the calculus of residues to find Cauchy's principal value of the integral

$$\text{vp.} \int_{-\infty}^{+\infty} f(x)\,dx,$$

and then compute the integral

$$\int_{-\infty}^{0} \frac{dx}{(x-1)^2(x-2)(x-3)}.$$

1) We have a branch cut along $\mathbb{R}_- \cup \{0\}$, so it only makes sense to find the poles of the function outside this half line. It follows immediately that $z = 2$ and $z = 3$ are simple poles. Furthermore, $(z-1)^2$ has a zero of order 2, while $\operatorname{Log} z$ has a zero of order 1. Hence, $z = 1$ is also a simple pole.

The residuum at $z = 1$ is computed by considering $z = 1$ as a pole of at most order 2,

$$\begin{aligned}\operatorname{res}(f;1) &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left(\frac{\operatorname{Log} z}{(z-2)(z-3)}\right) = \lim_{z\to 1}\left\{\operatorname{Log} z\cdot\frac{d}{dz}\left(\frac{1}{(z-2)(z-3)}\right)+\frac{1}{z(z-2)(z-3)}\right\}\\ &= 0+\frac{1}{1\cdot(1-2)\cdot(1-3)} = \frac{1}{2}.\end{aligned}$$

Then we get

$$\operatorname{res}(f;2) = \frac{\operatorname{Log} 2}{(2-1)^2\cdot(2-3)} = -\ln 2,$$

and

$$\operatorname{res}(f;3) = \frac{\operatorname{Log} 3}{(3-1)^2\cdot(3-2)} = \frac{1}{4}\,\ln 3.$$

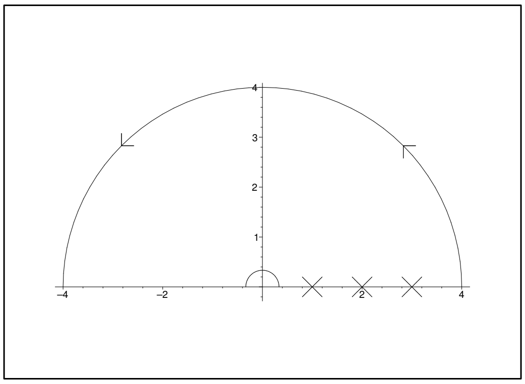


Figure 20: The path of integration $C_{R,\varepsilon} = C_{4,\frac{1}{3}}$.

2) Choose the path of integration $C_{R,\varepsilon}$ as the one given on the figure, where $0 < \varepsilon < 1 < 3 < R$. We shall allow the path of integration to pass through the simple poles at $z = 1$, 2 and 3, and they contribute to the integral with πi times their residues. We shall further assume that the part of the path of integration which runs along the negative and real axis, actually lies in the upper half plane above the branch cut. It follows from these assumptions that

$$\oint_{C_{R,\varepsilon}} f(z)\,dz = \pi\left\{\operatorname{res}(f;1)+\operatorname{res}(f;2)+\operatorname{res}(f;3)\right\} = \pi i\left\{\frac{1}{2}-\ln 2+\frac{1}{4}\,\ln 3\right\}.$$

We get the following estimate along the circular arc $|z| = R$, $z = R\,e^{i\theta}$, for $R\to\infty$,

$$\left|\int_{\theta=0}^{\pi}\frac{\operatorname{Log} z}{(z-1)^2(z-2)(z-3)}\,dz\right| \le \frac{\ln R+\pi}{(R-1)^2(R-2)(R-3)}\cdot\pi\,R\to 0.$$

Along the circular arc $|z| = \varepsilon$, i.e. $z = \varepsilon\,e^{i\,\theta}$, we get the following estimate for $\varepsilon\to 0+$,

$$\begin{aligned}\left|\int_{\theta=0}^{\pi}\frac{\operatorname{Log} z}{(z-1)^2(z-2)(z-3)}\,dz\right| &\le \frac{|\ln\varepsilon|+\pi}{1-\varepsilon)^2(2-\varepsilon)(3-\varepsilon)}\cdot\pi\varepsilon\\ &= \pi\cdot\frac{\varepsilon\,|\ln\varepsilon|+\pi\varepsilon}{(1-\varepsilon)^2}(2-\varepsilon)(3-\varepsilon)\to 0,\end{aligned}$$

where we have used that $\varepsilon|\ln\varepsilon| \to 0$ for $\varepsilon \to 0+$ due to the rules of magnitudes.

Hence we conclude by taking the limits $\varepsilon \to 0+$ and $R \to +\infty$ that

$$\text{vp.} \int_{-\infty}^{+\infty} f(x)\,dx = \pi\, i \left\{\frac{1}{2} - \ln 2 + \frac{1}{4}\,\ln 3\right\}.$$

This implies that

$$\begin{aligned}
&\pi\, i \left\{\frac{1}{2} - \ln 2 + \frac{1}{4}\,\ln 3\right\} \\
&= \lim_{\varepsilon\to 0+}\lim_{R\to+\infty}\left\{\int_{-R}^{-\varepsilon} \frac{\text{Log}\, x}{(x-1)^2(x-2)(x-3)}\,dx + \int_{\varepsilon}^{R} \frac{\text{Log}\, x}{(x-1)^2(x-2)(x-3)}\,dx\right\} \\
&= \lim_{\varepsilon\to 0+}\lim_{R\to+\infty}\left\{\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \frac{\ln|x|}{(x-1)^2(x-2)(x-3)}\,dx + i\pi\int_{-R}^{-\varepsilon} \frac{dx}{(x-1)^2(x-2)(x-3)}\right\}.
\end{aligned}$$

Then by taking the imaginary part and then the limits,

$$\int_{-\infty}^{0} \frac{dx}{(x-1)^2(x+2)(x-3)} = \frac{1}{2} - \ln 2 + \frac{1}{4}\,\ln 3.$$

As a CHECK we see that the latter result can also be derived by a decomposition and a simple integration. It follows from

$$\begin{aligned}
&\frac{1}{(x-1)^2(x-2)(x-3)}\\
&= \frac{1}{(1-2)(1-3)}\cdot\frac{1}{(x-1)^2}+\frac{A}{x-1}\\
&\qquad +\frac{1}{(2-1)^2(2-3)}\cdot\frac{1}{x-2}+\frac{1}{(3-1)^2(3-2)}\cdot\frac{1}{x-3}\\
&= \frac{1}{2}\,\frac{1}{(x-1)^2}+\frac{A}{x-1}-\frac{1}{x-2}+\frac{1}{4}\,\frac{1}{x-3},
\end{aligned}$$

that

$$\begin{aligned}
&\frac{A}{x-1}-\frac{1}{x-2}+\frac{1}{4}\,\frac{1}{x-3}\\
&= \frac{1}{(x-1)^2(x-2)(x-3)}-\frac{1}{2}\cdot\frac{1}{(x-1)^2}=\frac{2-(x-2)(x-3)}{2(x-1)^2(x-2)(x-3)}\\
&= \frac{-(x-1)(x-3)+x-3+2}{2(x-1)^2(x-2)(x-3)}=\frac{-x+3+1}{2(x-1)(x-2)(x-3)}\\
&= \frac{-x+4}{2(x-1)(x-2)(x-3)}\\
&= \frac{-1+4}{2(1-2)(1-3)}\cdot\frac{1}{x-1}+\frac{-2+4}{2(2-1)(2-3)}\cdot\frac{1}{x-2}+\frac{-3+4}{2(3-1)(3-2)}\cdot\frac{1}{x-3}\\
&= \frac{3}{4}\cdot\frac{1}{x-1}-\frac{1}{x-2}+\frac{1}{4}\cdot\frac{1}{x-3},
\end{aligned}$$

hence $A=\dfrac{3}{4}$, and then by insertion and a usual integration,

$$\begin{aligned}
&\int_{-\infty}^{0}\frac{dx}{(x-1)^2(x-2)(x-3)}=\int_{-\infty}^{0}\left\{\frac{1}{2}\,\frac{1}{(x-1)^2}+\frac{3}{4}\,\frac{1}{x-1}-\frac{1}{x-2}+\frac{1}{4}\,\frac{1}{x-3}\right\}dx\\
&= \lim_{R\to-\infty}\left[-\frac{1}{2}\,\frac{1}{x-1}+\frac{3}{4}\,\ln|x-1|-\ln|x-2|+\frac{1}{4}\,\ln|x-3|\right]_R^0\\
&= -\frac{1}{2}\cdot\frac{1}{(-1)}+0-\ln 2+\frac{1}{4}\,\ln 3-\lim_{x\to-\infty}\left\{-\frac{1}{2}\,\frac{1}{x-1}+\frac{1}{4}\,\ln\left|\frac{(x-1)^3(x-3)}{(x-2)^4}\right|\right\}\\
&= \frac{1}{2}-\ln 2+\frac{1}{4}\,\ln 3,
\end{aligned}$$

in accordance with the previous result.

Example 7.9 *Compute*

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,x\sqrt{\pi}}}{x\,(x^2+\pi)}\,dx.$$

Then prove that the improper integral

$$\int_{-\infty}^{+\infty} \frac{\sin(\sqrt{\pi}\,x)}{x\,(x^2+\pi)}\,dx$$

exists and find its value.

The integrand

$$f(z) = \frac{e^{i\,z\,\sqrt{\pi}}}{z\,(z^2+\pi)}$$

has the three singularities (they are all simple poles),

$$0, \qquad i\sqrt{\pi}, \qquad -i\sqrt{\pi},$$

and it is analytic in any other point of $\mathbb{C}$. If $\text{Im}(z) \geq 0$, then $\left|e^{i\,z\,\sqrt{\pi}}\right| \leq 1$, and since

$$\left|z^3 f(z)\right| = \left|e^{i\,z\,\sqrt{\pi}}\right| \cdot \left|\frac{z^2}{z^2+\pi}\right| \leq C \qquad \text{for } |z| \geq R \text{ and } \text{Im}(z) \geq 0,$$

where $R > \sqrt{\pi}$ is fixed, we conclude that

$$(15)\quad |f(z)| \leq \frac{C}{|z|^3} \qquad \text{for } |z| \geq R \text{ and } \text{Im}(z) \geq 0,$$

(note that we *cannot* here allow $\text{Im}(z) < 0$).

Remark 7.1 By a more careful analysis, which shall not be given here, one can show that one can choose

$$C = \frac{R^2}{R^2-\pi},$$

because

$$\left|\frac{z^2}{z^2+\pi}\right| = \left|\frac{z^2+\pi-\pi}{z^2+\pi}\right| = \left|1-\frac{\pi}{z^2+\pi}\right| = \left|\frac{\pi}{z^2+\pi}-1\right|$$

is maximum for $z = \pm i\,R$. $\Diamond$

The pole 0 on the real axis is *simple*, and we have $a = 3 > 1$ in (15). This implies that Cauchy's principal value exists and that it can be computed by a residuum formula,

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,x\,\sqrt{\pi}}}{x\,(x^2+\pi)}\,dx = 2\pi\,i\,\text{res}\left(\frac{e^{i\,z\,\sqrt{\pi}}}{z\,(z^2+\pi)}\,;\,i\,\sqrt{\pi}\right) + \pi\,i\,\text{res}\left(\frac{e^{i\,z\,\sqrt{\pi}}}{z\,(z^2+\pi)}\,;\,0\right)$$

$$= 2\pi\,i \lim_{z\to i\sqrt{\pi}} \frac{e^{i\,z\,\sqrt{\pi}}}{z\,(z+i\,\sqrt{\pi})} + \pi\,i \lim_{z\to 0} \frac{e^{i\,z\,\sqrt{\pi}}}{z^2+\pi} = \frac{2\pi\,e\cdot e^{-\pi}}{-2\pi} + \pi\,i\cdot\frac{1}{\pi} = i\cdot\left\{1-e^{-\pi}\right\}.$$

Now

$$\lim_{x \to 0} \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)} = \frac{1}{\sqrt{\pi}},$$

so we conclude that the integrand $\dfrac{\sin\left(\sqrt{\pi}\cdot x\right)}{x\left(x^2+\pi\right)}$ is continuous, so

$$\int_{-\infty}^{+\infty} \left| \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)} \right| dx \le \int_{-1}^{1} \left| \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)} \right| dx + \int_{|x|\ge 1} \frac{1}{x^2+\pi}\, dx,$$

because we have for $|x| \ge 1$,

$$\left| \frac{\sin\left(\sqrt{\pi}\, x\right)}{x} \right| \le 1, \qquad x \in \mathbb{R}.$$

It follows from the continuity that the former integral exists, and the latter integral is of Arctan type, i.e. in particular convergent. Hence we conclude that the improper integral

$$\int_{-\infty}^{+\infty} \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)}\, dx$$

is convergent and that its value is given by

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)}\, dx = \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \frac{\sin\left(\sqrt{\pi}\, x\right)}{x\left(x^2+\pi\right)}\, dx \right\}.$$

We have

$$
\begin{aligned}
(16)\;\; i\left\{1 - e^{-\pi}\right\} &= \text{vp.} \int_{-\infty}^{+\infty} \frac{e^{i\,x\,\sqrt{\pi}}}{x\left(x^2+\pi\right)}\, dx = \text{vp.} \int_{-\infty}^{+\infty} \frac{\cos\left(x\sqrt{\pi}\right) + i\,\sin\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx \\
&= \text{vp.} \int_{-\infty}^{+\infty} \frac{\cos\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx + i \int_{-\infty}^{+\infty} \frac{\sin\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx = 0 + i \int_{-\infty}^{+\infty} \frac{\sin\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx,
\end{aligned}
$$

because "vp" is superfluous on the sine integral according to the above. If one wants to be particular careful, then notice that we have by the definition,

$$\text{vp.} \int_{-\infty}^{+\infty} \frac{\cos\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx = \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{\cos\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx = 0,$$

because the integrand is an odd function in x, and because the improper integrals $\int_{-\infty}^{-\varepsilon} \cdots dx$ and $\int_{\varepsilon}^{+\infty} \cdots dx$ clearly exist.

Then we conclude from (16) that

$$\int_{-\infty}^{+\infty} \frac{\sin\left(x\sqrt{\pi}\right)}{x\left(x^2+\pi\right)}\, dx = 1 - e^{-\pi}.$$

8 Sum of special types of series

Example 8.1 *Find the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n-\frac{1}{2}\right)^2},$$

and then derive the value of the important sum

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

Putting $f(z) = \left(z-\frac{1}{2}\right)^{-2}$, it is obvious that $f(z)$ satisfies an estimate of the type $|f(z)| \leq \frac{c}{|z|^2}$ for $|z| \geq 1$. Since $z_0 = \frac{1}{2} \notin \mathbb{Z}$ is the only pole, the conditions for the application of some residuum formula are satisfied. The auxiliary function

$$g(z) := \frac{\cot(\pi z)}{\left(z-\frac{1}{2}\right)^2}, \qquad z \neq \frac{1}{2},$$

has *at most* a double pole at $z_0 = \frac{1}{2}$, so we may apply Rule I with $q = 2$. This gives

$$\operatorname{res}\left(\frac{\cot(\pi z)}{\left(z-\frac{1}{2}\right)^2};\frac{1}{2}\right) = \frac{1}{(2-1)!}\lim_{z\to\frac{1}{2}}\frac{d}{dz}\left\{\left(z-\frac{1}{2}\right)^2 g(z)\right\} = \lim_{z\to\frac{1}{2}}\left(-\left\{1+\cot^2(\pi z)\right\}\pi\right) = -\pi.$$

Finally, by insertion into the residuum formula for the sum of a series of this type,

$$\sum_{-\infty}^{+\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2} = -\pi\cdot\operatorname{res}\left(\frac{\cot(\pi z)}{\left(z-\frac{1}{2}\right)^2};\frac{1}{2}\right) = \pi^2.$$

It follows by a small rearrangement that

$$\pi^2 = \sum_{n=-\infty}^{+\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2} = \sum_{n=1}^{+\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2} + \sum_{n=0}^{+\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2} = 2\sum_{n=0}^{+\infty}\frac{4}{(2n+1)^2},$$

and then finally

$$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Remark 8.1 Since any number $m \in \mathbb{Z}$ can be written as

$$m = 2^r(2n+1), \qquad \text{for uniquely determined } r \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_0,$$

and since the series $\sum_{n=1}^{+\infty}\frac{1}{n^2}$ is absolutely convergent, it is easy to derive that the sum is given by

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{n^2} &= \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} + \frac{1}{2^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} + \frac{1}{(2^2)^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} + \cdots \\ &= \left\{1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right\}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = \frac{1}{1-\frac{1}{4}}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6},\end{aligned}$$

which is a very important result in the applications. $\Diamond$

Example 8.2 *Find for every given constant $a > 0$ the sum of the infinite series*

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+a^2}.$$

We can obviously for e.g. $|z| \geq 1$ find a constant $c > 0$, such that we have the estimate

$$|f(z)| = \left|\frac{1}{z^2+a^2}\right| \leq \frac{c}{|z|^2},$$

which proves one of the assumptions. Since $f(z)$ has only the two simple poles $z = \pm ia \notin \mathbb{Z}$, the other assumption for the application of the residuum formula is also fulfilled. Since $\cot(\pm i\pi a) \neq 0$, it suffices to compute the residues

$$\text{res}\left(\frac{1}{z^2+a^2}\,;\,ia\right) = \frac{1}{2ia} \qquad \text{and} \qquad \text{res}\left(\frac{1}{z^2+a^2}\,;\,-ia\right) = -\frac{1}{2ia}.$$

Then we get by the residuum formula,

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} &= -\pi\left\{\frac{1}{2ia}\cot(i\pi a) - \frac{1}{2ia}\cot(-i\pi a)\right\} \\ &= -\frac{\pi}{ia}\cdot\frac{\cos(i\pi a)}{\sin(i\pi a)} = -\frac{\pi}{ia}\cdot\frac{\cosh(\pi a)}{i\cdot\sinh(\pi a)} = \frac{\pi}{a}\cdot\coth(\pi a).\end{aligned}$$

Thus

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\coth(\pi a).$$

If $a = 1$, then we get in particular

$$\sum_{n=0}^{+\infty} \frac{1}{n^2} = \frac{1}{2} + \frac{\pi}{2}\coth \pi.$$

Example 8.3 *Let $a > 0$ denote a constant. Find the sum of the alternating series*

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+a^2}.$$

The underlying function $f(z) = \dfrac{1}{z^2+a^2}$ is the same as in Example 8.2, so we have already checked the assumptions of the relevant residuum formula in Example 8.2. The only difference is that the auxiliary factor $\cot(\pi z)$ has been replaced by $1/\sin(\pi z)$, so it follows immediately by insertion into the residuum formula that

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} = -\pi\left\{\frac{1}{2ai}\cdot\frac{1}{\sin(i\pi a)} - \frac{1}{2ai}\cdot\frac{1}{\sin(-i\pi a)}\right\} = -\frac{\pi}{ia}\cdot\frac{1}{\sin(i\pi a)} = \frac{\pi}{a}\cdot\frac{1}{\sinh(\pi a)}.$$

Since

$$\frac{(-1)^{-n}}{(-n)^2+a^2} = \frac{(-1)^n}{n^2+a^2},$$

it easily follows that

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2a^2} + \frac{1}{2}\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \cdot \frac{1}{\sinh(\pi a)}.$$

If in particular $a = 1$, then

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2} + \frac{\pi}{2\sinh\pi}.$$

Remark 8.2 Even if one may use the theory to find the sum of many convergent series, where the term has the structure of a rational function in n, one should not be misled to believe that this is true for every series of this type. The simplest example is

$$\sum_{n=1}^{+\infty} \frac{1}{n^3} \qquad (\approx 1,202),$$

the exact value of which is still unknown. $\Diamond$

Example 8.4 *Let $a \in \mathbb{R}_+ \setminus \mathbb{N}$. Find the sum of the series*

$$\sum_{n=0}^{+\infty} \frac{1}{n^2-a^2}.$$

The degree of the denominator is precisely 2 larger than the degree of the numerator, so the series

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2-a^2}$$

is convergent when $2a \notin \mathbb{N}$, and the value is given by

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \frac{1}{n^2-a^2} &= 2\sum_{n=0}^{+\infty} \frac{1}{n^2-a^2} + \frac{1}{a^2} = -\pi \sum_{j=1}^{k} \cot\left(a_j\pi\right) \operatorname{res}\left(f\,;\,a_j\right) \\
&= -\pi\left\{\cot(a\pi)\operatorname{res}\left(\frac{1}{2}(z+a)(z-a)\,;\,a\right) + \cot(-a\pi)\operatorname{res}\left(\frac{1}{(z+a)(z\ -a)}\,;\,-a\right)\right\} \\
&= -\pi\left\{\frac{1}{2a}\,\cos(a\pi) + \frac{1}{-2a}\,\cos(-a\pi)\right\} = -\frac{\pi}{a}\,\cot(a\pi),
\end{aligned}$$

hence by a rearrangement,

$$\sum_{n=0}^{+\infty} \frac{1}{n^2-a^2} = -\frac{1}{2a^2} - \frac{\pi}{2a}\,\cot(a\pi).$$

The expression (the series)

$$\sum_{n=0}^{+\infty} \frac{1}{n^2 - a^2}$$

is continuous in $a \in \mathbb{R}_+ \setminus \mathbb{N}$, so this formula also holds for $a = p + \frac{1}{2}$, $p \in \mathbb{N}_0$. Since $\cot\left(\left(p+\frac{1}{2}\right)\pi\right) = 0$, $p \in \mathbb{N}_0$, we obtain in this case

$$\sum_{n=0}^{+\infty} \frac{1}{n^2 - \left(p+\frac{1}{2}\right)^2} = -\frac{1}{2\left(p+\frac{1}{2}\right)^2} = -\frac{1}{2\left(p+\frac{1}{2}\right)^2} = -\frac{2}{(2p+1)^2}, \qquad p \in \mathbb{N}_0.$$

Alternatively,

$$\text{res}\left(\frac{\cot(az)}{z^2-a^2}\,;\,\pm a\right) = 0 \qquad \text{for } a = p + \frac{1}{2}, \quad p \in \mathbb{N}_0,$$

because the singularity is then removable.

Example 8.5 *Find the sum of the series*

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+4n+5}.$$

First we note that

$$n^2+4n+5=(n+2)^2+1.$$

We use that

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+1} = \frac{1}{2}+\frac{\pi}{2}\,\coth\pi$$

is the sum of a known series, so by a small rearrangement,

$$\sum_{n=0}^{+\infty} \frac{1}{n^2+4n+5} = \sum_{n=0}^{+\infty} \frac{1}{(n+2)^2+1} = \sum_{n=2}^{+\infty} \frac{1}{n^2+1} = -1-\frac{1}{2}+\sum_{n=0}^{+\infty} \frac{1}{n^2+1} = -1+\frac{\pi}{2}\,\coth\pi.$$

Example 8.6 *Find the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(n^2+1)\,(2n+1)}.$$

The polynomial of the denominator is of degree 3, and it does not have any zero in i $\mathbb{Z}$. We therefore conclude that the series is convergent, and its sum can be found by a residuum formula. The poles are

$$i, \qquad -i, \qquad -\frac{1}{2},$$

thus

$$\begin{aligned}
&\sum_{n=-\infty}^{+\infty} \frac{1}{(n^2+1)\,(2n+1)} \\
&\quad = -\pi\left\{\cot(i\,\pi)\cdot\operatorname{res}\left(\frac{1}{(z^2+1)\,(2z+1)}\,;\,i\right)\right. \\
&\qquad\qquad +\cot(-i\,\pi)\cdot\operatorname{res}\left(\frac{1}{(z^2+1)\,(2z+1)}\,;\,-i\right) \\
&\qquad\qquad \left.+\operatorname{res}\left(\frac{\cot(\pi\,z)}{(z^2+1)\,(2z+1)}\,;\,-\frac{1}{2}\right)\right\} \\
&\quad = -\pi\left\{\frac{\cos(i\pi)}{\sin(i\pi)}\,\frac{1}{(i+i)(2i+1)}+\frac{\cos(-i\pi)}{\sin(-i\pi)}\,\frac{1}{(-i-i)(-2i+1)}+0\right\} \\
&\quad = -\pi\left\{\frac{\cosh\pi}{\sinh\pi}\cdot\frac{1}{i}\cdot\frac{1}{2i(1+2i)}+\frac{\cosh\pi}{\sinh\pi}\cdot\frac{1}{(-i)}\cdot\frac{1}{(-2i)(1-2i)}\right\} \\
&\quad = \frac{\pi}{2}\,\coth\pi\cdot\left\{\frac{1}{1+2i}+\frac{1}{1-2i}\right\} = \frac{\pi}{2}\,\coth\pi\cdot\frac{1+2i+1-2i}{5} = \frac{\pi}{5}\,\coth\pi.
\end{aligned}$$

Remark 8.3 The function $\dfrac{1}{(z^2+1)(2z+1)}$ has a simple pole at $z=-\dfrac{1}{2}$, so the auxiliary function $\dfrac{\cot(\pi z)}{(z^2+1)(2z+1)}$ has a removable singularity for $z=-\dfrac{1}{2}$. This is in agreement with that the residuum of the auxiliary function is 0 at $z=-\dfrac{1}{2}$. ♢

Example 8.7 *Find the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{2n+1}{(n^2+1)(3n+1)}$$

The corresponding analytic function is

$$f(z)=\frac{1}{3}\cdot\frac{2z+1}{(z^2+1)\left(z+\frac{1}{3}\right)},$$

which has a zero of second order at ∞, and the simple poles

$$a_1=-\frac{1}{3}, \qquad a_2=i, \qquad a_3=-i,$$

where $2a_j \notin \mathbb{Z}$. It follows that the series is convergent with the sum

$$\sum_{n=-\infty}^{+\infty} \frac{2n+1}{(n^2+1)(3n+1)} = -\pi\left\{\cos\left(-\frac{\pi}{3}\right)\text{res}\left(f\,;\,-\frac{1}{3}\right)+\cot(i\pi)\text{res}(f;i)+\cot(-i\pi)\text{res}(f;-i)\right\}.$$

Here,

$$\text{res}\left(f\,;\,-\frac{1}{3}\right)=\frac{1}{3}\left[\frac{2z+1}{z^2+1}\right]_{z=-\frac{1}{3}}=\frac{1}{3}\cdot\frac{-\frac{2}{3}+1}{\frac{1}{9}+1}=\frac{1}{10},$$

$$\text{res}(f;i)=\left[\frac{2z+1}{(z+i)(3z+1)}\right]_{z=i}=\frac{2i+1}{2i(3i+1)}=\frac{-1-7i}{20},$$

$$\text{res}(f;-i)=\left[\frac{2z+1}{(z-i)(3z+1)}\right]_{z=-i}=\frac{-2i+1}{-2i(-3i+1)}=\frac{-1+7i}{20}.$$

Finally,

$$\cot(i\pi)=\frac{\cos(i\pi)}{\sin(i\pi)}=-i\,\frac{\cosh\pi}{\sinh\pi}=-i\,\coth\pi,$$

and

$$\cot(-i\pi)=i\,\coth\pi,$$

so the sum is

$$\sum_{n=-\infty}^{+\infty} \frac{2n+1}{(n^2+1)(2n+1)} = -\pi \left\{ -\frac{1}{\sqrt{3}} \cdot \frac{1}{10} - i \coth \pi \cdot \left(\frac{-1-7i}{20} \right) + i \coth \pi \cdot \left(\frac{-1+7i}{20} \right) \right\}$$

$$= \pi \left\{ \frac{1}{10\sqrt{3}} + i \coth \pi \cdot \left(-\frac{7i}{10} \right) \right\} = \frac{\pi\sqrt{3}}{30} + \frac{7\pi}{10} \cdot \coth \pi.$$

Example 8.8 *Prove that*

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{2} \sum_{-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

When we split the sum and change the variable $n = -m-1$, i.e. $m = -n-1$, then

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^3} &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} + \sum_{n=-\infty}^{-1} \frac{(-1)^n}{(2n+1)^3} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} + \sum_{m=0}^{+\infty} \frac{(-1)^{-m-1}}{(-2m-2+1)^3} \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} + \sum_{m=0}^{+\infty} \frac{(-1)^{m-4}}{(2m+1)^3} = 2\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3}, \end{aligned}$$

and the first equality follows.

We have a triple pole at $z = -\dfrac{1}{2}$, and the series is clearly convergent, so we obtain by a residuum formula,

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = -\pi \, \text{res} \left(\frac{1}{(2z+1)^3 \sin(\pi z)} \, ; \, -\frac{1}{2} \right).$$

A small rearrangement gives

$$\frac{1}{(2z+1)^3 \sin \pi z} = \frac{1}{8} \cdot \frac{1}{\left(z + \frac{1}{2} \right)^3} \cdot \frac{1}{\sin \pi z},$$

so

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^3} &= -\frac{\pi}{8} \cdot \frac{1}{2!} \lim_{z \to -\frac{1}{2}} \frac{d^2}{dz^2} \left(\frac{1}{\sin \pi z} \right) = \frac{\pi}{16} \lim_{z \to -\frac{1}{2}} \pi \frac{d}{dz} \left(\frac{\cos \pi z}{\sin^2 \pi z} \right) \\ &= \frac{\pi^2}{16} \lim_{z \to -\frac{1}{2}} \left\{ -\pi \frac{\sin \pi z}{\sin^2 \pi z} - 2\pi \cdot \frac{\cos^2 \pi z}{\sin^3 \pi z} \right\} = \frac{\pi^2}{16} \cdot \left\{ -\frac{\pi}{(-1)} - 2\pi \cdot 0 \right\} = \frac{\pi^3}{16}. \end{aligned}$$

Summing up,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Example 8.9 *Find*

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^4}.$$

We see that $z = -\frac{1}{2}$ is a four-tuple pole of

$$f(z) = \frac{1}{(2z+1)^4} = \frac{1}{16} \cdot \frac{1}{\left(z+\frac{1}{2}\right)^4},$$

so the sum is computed by a residuum formula,

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^4} &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{(2n+1)^4} = -\frac{\pi}{2}\,\mathrm{res}\left(\frac{\cot(\pi z)}{16\cdot\left(z+\frac{1}{2}\right)^4}\,;\,-\frac{1}{2}\right)\\
&= -\frac{\pi}{32}\,\mathrm{res}\left(\frac{\cot(\pi z)}{\left(z+\frac{1}{2}\right)^4}\,;\,-\frac{1}{2}\right) = -\frac{\pi}{32}\cdot\frac{1}{3!}\lim_{z\to-\frac{1}{2}}\frac{d^3}{dz^3}\,\cot(\pi z)\\
&= -\frac{\pi}{6\cdot 32}\lim_{z\to-\frac{1}{2}}\frac{d^2}{dz^2}\left\{-\pi\cdot\frac{1}{\sin^2(\pi z)}\right\} = \frac{\pi^2}{6\cdot 32}\lim_{z\to-\frac{1}{2}}\left\{-2\pi\,\frac{d}{dz}\left(\frac{\cos(\pi z)}{\sin^3(\pi z)}\right)\right\}\\
&= -\frac{\pi^3}{96}\lim_{z\to-\frac{1}{2}}\left\{-\frac{\pi\,\sin(\pi z)}{\sin^3(\pi z)}-3\pi\cdot\frac{\cos^2(\pi z)}{\sin^4(\pi z)}\right\} = \frac{\pi^4}{96}.\end{aligned}$$

Remark 8.4 It follows that

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{n^4} &= \left\{1+\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{8^4}+\frac{1}{16^4}+\cdots\right\}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^4} = \sum_{j=0}^{+\infty}\left(\frac{1}{2^4}\right)^j\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^4}\\
&= \frac{1}{1-\frac{1}{2^4}}\cdot\frac{\pi^4}{96} = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}. \qquad\Diamond\end{aligned}$$

Example 8.10 *Find the sum S of the series*

$$\sum_{n=0}^{+\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2}.$$

It is well-known that

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

hence

$$\sum_{n=0}^{+\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2} = 4\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = 4\cdot\frac{\pi^2}{8} = \frac{\pi^2}{2}.$$

Example 8.11 *Prove that the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n-\frac{1}{4}\right)\left(n-\frac{3}{4}\right)}$$

is convergent.
Then find the sum of the series.
Finally, find

$$\sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)},$$

where we only sum over the positive integers.

Here, $\left(n-\frac{1}{4}\right)\left(n-\frac{3}{4}\right)$ is a polynomial of second degree, which is not 0 for any $n \in \mathbb{Z}$. Hence, the series is convergent and the sum is given by a residuum formula,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n-\frac{1}{4}\right)\left(n-\frac{3}{4}\right)} = -\pi\left\{\operatorname{res}\left(\frac{\cot(\pi z)}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)};\frac{1}{4}\right)+\operatorname{res}\left(\frac{\cot(\pi z)}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)};\frac{3}{4}\right)\right\}$$

$$= -\pi\left\{\lim_{z\to\frac{1}{4}}\frac{\cot(\pi z)}{z-\frac{3}{4}}+\lim_{z\to\frac{3}{4}}\frac{\cot(\pi z)}{z-\frac{1}{4}}\right\} = -\pi\left\{\frac{\cot\frac{\pi}{4}}{-\frac{1}{2}}+\frac{\cot\frac{3\pi}{4}}{\frac{1}{2}}\right\} = -\pi(-2-2) = 4\pi.$$

Then we note that

$$\begin{aligned}\sum_{n=-\infty}^{0} \frac{1}{(4n-1)(4n-3)} &= \sum_{n=0}^{+\infty} \frac{1}{(-4n-1)(-4n-3)} = \sum_{n=0}^{+\infty} \frac{1}{(4n+3)(4n+1)} \\ &= \sum_{n=0}^{+\infty} \frac{1}{(4\{n+1\}-1)(4\{n+1\}-3)} = \sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)},\end{aligned}$$

hence

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)} &= \frac{1}{2}\sum_{-\infty}^{+\infty} \frac{1}{(4n-1)(4n-3)} \\ &= \frac{1}{2\cdot 4\cdot 4}\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n-\frac{1}{4}\right)\left(n-\frac{1}{4}\right)\left(n-\frac{3}{4}\right)} = \frac{4\pi}{32} = \frac{\pi}{8}.\end{aligned}$$

Alternatively, use *Leibniz's series*,

$$\frac{\pi}{4} = \text{ Arctan } 1 = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1},$$

where we have *added* parentheses in a convergent series, which is always possible without destroying the convergence or the limit value. Then

$$\begin{aligned}\frac{\pi}{4} &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)+\cdots \\ &= \sum_{n=1}^{+\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right) = \sum_{n=1}^{+\infty} \frac{(4n-1)-(4n-3)}{(4n-1)\cdot(4n-3)} = 2\sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)},\end{aligned}$$

hence

$$\sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)} = \frac{\pi}{8}.$$

Finally

$$\sum_{-\infty}^{+\infty} \frac{1}{\left(n-\frac{1}{4}\right)\left(n-\frac{3}{4}\right)} = 16\sum_{n=-\infty}^{+\infty} \frac{1}{(4n-1)(4n-3)} = 16\cdot 2\sum_{n=1}^{+\infty} \frac{1}{(4n-1)(4n-3)} = 16\cdot 2\cdot\frac{\pi}{8} = 4\pi.$$

Example 8.12 *Find the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2 - \frac{1}{9}}.$$

Also, find the sum of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(3n)^2 - 1}.$$

Let

$$f(z) = \frac{P(z)}{Q(z)} = \frac{1}{z^2 - \frac{1}{9}}$$

where

$$P(z) = 1 \qquad \text{and} \qquad Q(z) = z^2 - \frac{1}{9}.$$

The denominator has a degree which is 2 bigger than the degree of the numerator, and the zeros of the denominator are $z = \pm\frac{1}{3} \notin \mathbb{Z}$. We conclude that the series is convergent, and that is sum can be found by a residuum formula,

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \frac{1}{n^2 - \frac{1}{9}} &= -\pi \left\{ \operatorname{res}\left(\frac{\cot(\pi z)}{z^2 - \frac{1}{9}} ; \frac{1}{3} \right) + \operatorname{res}\left(\frac{cot(\pi z)}{z^2 - \frac{1}{9}} ; -\frac{1}{3} \right) \right\} \\
&= -\pi \left\{ \frac{\cot \frac{\pi}{3}}{\frac{2}{3}} + \frac{\cot\left(-\frac{\pi}{3}\right)}{-\frac{2}{3}} \right\} = -\frac{3\pi}{2} \cdot 2 \cot \frac{\pi}{3} = -\pi\sqrt{3}.
\end{aligned}$$

Then by a small rearrangement,

$$-\pi\sqrt{3} = \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 - \frac{1}{9}} = -\frac{1}{\frac{1}{9}} + 2\sum_{n=1}^{+\infty} \frac{1}{n^2 - \frac{1}{9}} = -9 + 18\sum_{n=1}^{+\infty} \frac{1}{9n^2 - 1},$$

so

$$\sum_{n=1}^{+\infty} \frac{1}{(3n)^2 - 1} = \sum_{n=1}^{+\infty} \frac{1}{9n^2 - 1} = \frac{1}{2} - \frac{\pi\sqrt{3}}{18} \approx 0,1977.$$

Example 8.13 *Given the function*

$$f(z) = \frac{2z^2 + (2i-1)z - i}{(16z^4 - 1)(z^2+1)^2}.$$

(a) *Determine the singular points in $\mathbb{C} \cup \{\infty\}$ of $f(z)$ and their types. Then find the residuum of $f(z)$ at $z = \infty$.*

(b) *Compute the complex line integral*

$$\oint_{|z|=3} f(z)\, dz.$$

(c) *Find the sum of the series*

$$\sum_{n=1}^{+\infty} \frac{1}{16n^4 - 1}.$$

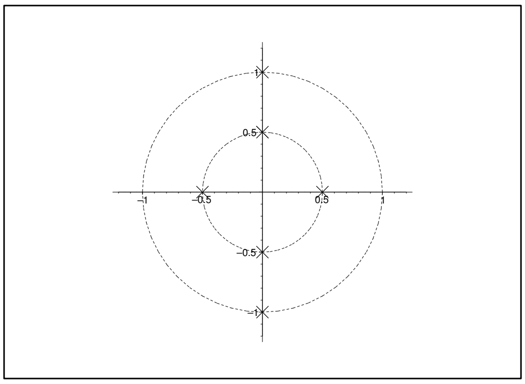


Figure 21: The four simple zeros and the two double zeros of the denominator of $f(z)$.

(a) The denominator

$$\left(16z^4 - 1\right)\left(z^2+1\right)^2 = \left(4z^2+1\right)\left(4z^2-1\right)\left(z^2+1\right)^2$$

is zero for $z = \pm i$ (double zeros), and for $z = \pm \frac{1}{2}$ and $z = \pm \frac{1}{2}\, i$ (simple zeros). The numerator can be written

$$2z^2 + (2i-1)z - i = 2\left(z^2 + iz\right) - (z+i) = (2z-1)(z+i),$$

i.e. the numerator has the simple zeros $z = \frac{1}{2}$ and $z = -i$. We therefore see that $f(z)$ can be reduced in the following way,

$$\begin{aligned} f(z) &= \frac{2z^2 + (2i-1)z - i}{(16z^4-1)(z^2+1)^2} = \frac{(2z-1)(z+i)}{(4z^2+1)(2z-1)(2z+1)(z+i)^2(z-i)^2} \\ &= \frac{1}{(2z+i)(2z-i)(2z+1)(z+i)(z-i)^2}. \end{aligned}$$

It follows that $z = i$ is a double pole, that

$$z = -i,\ -\frac{i}{2},\ \frac{i}{2},\ -\frac{1}{2}$$

are simple poles, and that $z = \frac{1}{2}$ is a removable singularity. Finally, we have a zero of order 6 in ∞. In particular,

$$\text{res}(f;\infty) = 0.$$

(b) All finite singularities lie inside $|z| = 3$, so

$$\oint_{|z|=3} f(z)\,dz = -\oint^{\star}_{|z|=3} f(z)\,dz = -2\pi i\,\text{res}(f;\infty) = 0,$$

where $\oint^{\star}$ denotes a closed line integral with the opposite direction of the orientation of the plane, i.e. $\oint^{\star} = -\oint$.

(c) We have $16z^4 - 1 = 0$ for

$$z = \frac{1}{2}, \quad -\frac{1}{2}, \quad \frac{i}{2}, \quad -\frac{i}{2},$$

where none of the roots belongs to $\mathbb{Z}$. Furthermore, we have a zero of order 4 at ∞, so the series is convergent, and we can use a residuum formula,

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{1}{16n^4 - 1} &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} \frac{1}{16n^4 - 1} - \frac{1}{2} \cdot \frac{1}{16 \cdot 0^4 - 1} = \frac{1}{2} + \frac{1}{32} \sum_{n=-\infty}^{+\infty} \frac{1}{n^4 - \left(\frac{1}{2}\right)^4} \\ &= \frac{1}{2} - \frac{\pi}{32} \sum_{n=0}^{3} \text{res}\left(\frac{\cot(\pi z)}{z^4 - \left(\frac{1}{2}\right)^4} \,;\, \frac{1}{2}\, i^n \right).\end{aligned}$$

Then consider the function

$$\frac{\cot(\pi z)}{z^4 - \left(\frac{1}{2}\right)^4} \qquad \text{for } z_n = \frac{1}{2}\, i^n.$$

It follows that $\frac{1}{2}$ and $-\frac{1}{2}$ are removable singularities, and since also

$$\text{res}\left(\frac{\cot(\pi z)}{z^4 - \left(\frac{1}{2}\right)^4} \,;\, z_n \right) = \frac{\cot(\pi z_n)}{4z_n^3} = \frac{z_n \cot(\pi z_n)}{4z_n^4} = \frac{z_n \cot(\pi z_n)}{4 \cdot \frac{1}{16}} = 4\, z_n \, \cot(\pi z_n),$$

we find for $z_n = \pm \frac{i}{2}$ that

$$\text{res}\left(\frac{\cot(\pi z)}{z^4 - \left(\frac{1}{2}\right)^4} \,;\, \pm\frac{i}{2} \right) = 4 \cdot \frac{i}{2} \cdot \cot\left(\pi \, \frac{i}{2}\right) = 2i \cdot \frac{\cosh \frac{\pi}{2}}{\sinh \frac{\pi}{2}} = 2 \coth \frac{\pi}{2},$$

hence by insertion,

$$\sum_{n=1}^{+\infty} \frac{1}{16n^4 - 1} = \frac{1}{2} - \frac{\pi}{32}\left\{ 2 \coth \frac{\pi}{2} + 2 \coth \frac{\pi}{2} \right\} = \frac{1}{2} - \frac{\pi}{8} \coth \frac{\pi}{2} \approx 0,0718.$$

Example 8.14 *Given the function*

$$g(z)=\frac{f(z)}{\left(z^2+1\right)^2},$$

where $f(z)$ is analytic in a neighbourhood of the points $z=\pm i$. Furthermore, it is also given that

$$f(-i)=-f(i)\neq 0, \qquad \text{and} \qquad f'(-i)=f'(i).$$

1) Show that $g(z)$ has poles at the points $z=\pm i$, and indicate in both cases the order of the pole.

2) Prove that

$$res(g(z);i)=res(g(z);-i).$$

3) Show that the series

$$\sum_{n=0}^{+\infty}\frac{1}{\left(n^2+1\right)^2}$$

is convergent and find its sum.

1) The denominator $\left(z^2+1\right)^2=(z-i)^2(z+i)^2$ has the double roots $\pm i$, and since $f(-i)=-f(i)\neq 0$, and $f(z)$ is analytic in a neighbourhood of the points $z=\pm i$, we conclude that

$$g(z)=\frac{f(z)}{\left(z^2+1\right)^2}$$

has double poles at $z=\pm i$.

2) Then we find that

$$\text{res}(g(z);i)=\frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left(\frac{f(z)}{(z+i)^2}\right)=\lim_{z\to i}\left\{\frac{f'(z)}{(z+i)^2}-2\,\frac{f(z)}{(z+i)^3}\right\}=-\frac{1}{4}\,f'(i)-\frac{i}{4}\,f(i),$$

and

$$\begin{aligned}\text{res}(g(z);-i) &= \frac{1}{1!}\lim_{z\to -i}\frac{d}{dz}\left(\frac{f(z)}{(z-i)^2}\right)=\lim_{z\to -i}\left\{\frac{f'(z)}{(z-i)^2}-2\,\frac{f(z)}{(z-i)^3}\right\}\\ &= -\frac{1}{4}\,f'(-i)-\frac{i}{4}\,f(-i)=-\frac{1}{4}\,f'(i)-\frac{i}{4}\,f(i)=\text{res}(g(z);i).\end{aligned}$$

3) The poles of $\dfrac{1}{\left(z^2+1\right)^2}$ are $z=\pm i$ (double poles), and none of them lies in $\mathbb{Z}$. Since we have a zero of order $4>1$ at ∞, it follows that the series is convergent, and we can find the sum by a residuum formula,

$$\sum_{n=-\infty}^{+\infty}\frac{1}{\left(1+n^2\right)^2}=-\pi\left\{\text{res}\left(\frac{\cot(\pi z)}{\left(z^2+1\right)^2}\,;\,i\right)+\text{res}\left(\frac{\cot(\pi z)}{\left(z^2+1\right)^2}\,;\,-i\right)\right\}.$$

If we put $f(z) = \cot(\pi z)$, then

$$f'(z) = -\frac{1}{\sin^2(\pi z)},$$

and

$$f(i) = \cot(\pi i) = -i\,\coth\pi \neq 0,$$

$$f(-i) = \cot(-\pi i) = i\,\coth\pi = -f(i),$$

$$f'(i) = -\frac{\pi}{\sin^2(\pi i)} = -\frac{\pi}{(i\,\sinh\pi)^2} = \frac{\pi}{\sinh^2\pi} = f'(-i),$$

which is precisely the case of (1) and (2). Hence we get

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \frac{1}{(1+n^n)^2} &= -\pi\cdot 2\,\text{res}\left(\frac{\cot(\pi z)}{(z^2+1)^2}\,;\,i\right) = -2\pi\cdot\left(-\frac{1}{4}\right)\cdot\{f'(i)+i\,f(i)\} \\ &= \frac{\pi}{2}\left\{\frac{\pi}{\sinh^2\pi} + i\cdot(-i\,\coth\pi)\right\} = \frac{\pi^2}{2}\,\frac{1}{\sinh^2\pi} + \frac{\pi}{2}\,\coth\pi.\end{aligned}$$

Then finally,

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{1}{\left(1+n^2\right)^2} &= \frac{1}{2}\left\{\sum_{n=0}^{+\infty}\frac{1}{\left(1+n^2\right)^2}+\sum_{n=-\infty}^{0}\frac{1}{\left(1+n^2\right)^2}\right\}=\frac{1}{2}\left\{\sum_{n=-\infty}^{+\infty}\frac{1}{\left(1+n^2\right)^2}+1\right\}\\ &= \frac{\pi^2}{4}\cdot\frac{1}{\sinh^2\pi}+\frac{\pi}{4}\,\coth\pi+\frac{1}{2}.\end{aligned}$$

Example 8.15 *Prove that*

$$f(t)=\sum_{n=-\infty}^{+\infty}\frac{1}{(\pi n+t)^2}$$

is convergent for every fixed $t\in\mathbb{C}\setminus\{p\pi\mid p\in\mathbb{Z}\}$.
Then find $f(t)$ *for every* $t\in\mathbb{C}\setminus\{p\pi\mid p\in\mathbb{Z}\}$, *expressed by elementary functions.*

We define for every fixed $t\in\mathbb{C}\setminus\{p\pi\mid p\in\mathbb{Z}\}$, i.e. for $-\dfrac{t}{\pi}\notin\mathbb{Z}$, a function $F(z;t)$ by

$$f(z;t)=\frac{1}{(\pi z+t)^2}.$$

Then $F(z;t)$ is analytic for $z\in\mathbb{C}\setminus\left\{-\dfrac{t}{\pi}\right\}$.
Since for $z\neq 0$,

$$z^2F(z;t)=\frac{z^2}{(\pi z+t)^2}=\frac{1}{\pi^2}\left(\frac{1}{1+\dfrac{t}{\pi z}}\right)^2\to\frac{1}{\pi^2}\qquad\text{for } z\to\infty,$$

we conclude that there exists a constant R_t for every $c>\dfrac{1}{\pi^2}$, such that $\left|z^2F(z;t)\right|\leq c$ for $|z|\geq R_t$, i.e.

$$|F(z;t)|\leq\frac{c}{|z|^2}\qquad\text{for } |z|\geq R_t.$$

If $-\dfrac{t}{\pi}\notin\mathbb{Z}$, then it follows directly that the series

$$f(t)=\sum_{n=-\infty}^{+\infty}\frac{1}{(\pi n+t)^2}$$

is *convergent*, and we can find its sum by a residuum formula,

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{+\infty} F(n;t) = -\pi \operatorname{res}\left(\cot(\pi z) F(z;t)\,;\, -\frac{t}{\pi}\right) \\ &= -\pi \operatorname{res}\left(\frac{\cot(\pi z)}{(\pi z+t)^2}\,;\, -\frac{t}{\pi}\right) = -\pi \cdot \frac{1}{1!} \lim_{z \to -\frac{t}{\pi}} \frac{d}{dz}\left\{\left(z+\frac{t}{\pi}\right)^2 \cdot \frac{\cot(\pi z)}{(\pi z+t)^2}\right\} \\ &= -\frac{\pi}{\pi^2} \lim_{z \to -\frac{t}{\pi}} \frac{d}{dz} \cot(\pi z) = -\frac{1}{\pi} \lim_{z \to -\frac{t}{\pi}} \left\{-\frac{\pi}{\sin^2(\pi z)}\right\} \\ &= \frac{1}{\sin^2(-t)} = \frac{1}{\sin^2 t}, \end{aligned}$$

where we have applied Rule I with $q = 2$. Note that we shall use the factor $\left(z+\frac{t}{\pi}\right)^2$, and *not* $(\pi z+t)^2$, in the denominator. It is of course also possible directly to prove the convergence.

Example 8.16 **(a)** *Prove that*

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2}$$

is convergent for every fixed $t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}$.

(b) *Express* $f(t)$ *for every* $t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}$*, by elementary functions.*

(c) *Finally, find*

$$g(t) = \sum_{n=1}^{+\infty} \frac{t}{t^2 - \pi^2 n^2}, \qquad t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\},$$

expressed by elementary functions.

(a) Since $t^2 - \pi^2 n^2 \neq 0$ for every $n \in \mathbb{Z}$, when $t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}$, it follows that $\frac{t}{t^2 - \pi^2 n^2}$ is defined for $n \in \mathbb{Z}$. Furthermore,

$$\left|\frac{t}{t^2 - \pi^2 n^2}\right| \leq \frac{|t|}{n^2} \qquad \text{for } |n| \geq N_t,$$

so we conclude that

$$\left|\sum_{n=-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2}\right| \leq \sum_{n=-\infty}^{+\infty} \left|\frac{t}{t^2 - \pi^2 n^2}\right| \leq \sum_{n=-N}^{N} \left|\frac{t}{t^2 - \pi^2 n^2}\right| + 2 \sum_{n=N+1}^{+\infty} \frac{|t|}{n^2} < +\infty,$$

and we see that we could give a direct proof of the convergence.

Alternatively we check the *assumptions* of the residuum formula, because they at the same time assures the convergence, and we also obtain the sum.

Consider for every *fixed* $t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}$ the function

$$F(z;t) = \frac{t}{t^2 - \pi^2 z^2} = -\frac{t}{\pi^2} \cdot \frac{1}{z^2 - \left(\frac{t}{\pi}\right)^2}.$$

This is analytic for $z \in \mathbb{C} \setminus \left\{\frac{t}{\pi}\,;\, -\frac{t}{\pi}\right\}$. Since we have assumed that $\pm\frac{t}{\pi} \notin \mathbb{Z}$, and that $F(z)$ is a rational function with a zero of second order at ∞, there exist constants $c > \left|\frac{t}{\pi^2}\right|$ and $T > \left|\frac{t}{\pi}\right|$, such that

$$|F(z;t)| \le \frac{c}{|z|^2} \qquad \text{for } |z| \ge R,$$

and the conditions for the convergence and the residuum formula are fulfilled. Hence

$$\sum_{n=-\infty}^{+\infty} F(n;t) = \sum_{-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2} = f(t)$$

is convergent.

(b) Now, $\pm\frac{t}{\pi}$ are at most simple poles, so we bet by the residuum formula and RULE II,

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2} = -\pi \left\{ \operatorname{res}\left(-\frac{t}{\pi^2}\,\frac{\cot(\pi z)}{z^2 - \left(\frac{t}{\pi}\right)^2}\,;\, \frac{t}{\pi}\right) + \operatorname{res}\left(-\frac{t}{\pi^2}\,\frac{\cot(\pi z)}{z^2 - \left(\frac{t}{\pi}\right)^2}\,;\, -\frac{t}{\pi}\right)\right\} \\ &= -\pi \cdot \left(-\frac{t}{\pi^2}\right) \cdot \left\{ \lim_{z \to \frac{t}{\pi}} \frac{\cot(\pi z)}{2z} + \lim_{z \to -\frac{t}{\pi}} \frac{\cot(\pi z)}{2z} \right\} \\ &= \frac{t}{\pi} \cdot \left\{ \frac{\cot\left(\pi \cdot \frac{t}{\pi}\right)}{2 \cdot \frac{t}{\pi}} + \frac{\cot\left(\pi \cdot \left(-\frac{t}{\pi}\right)\right)}{2 \cdot \left(-\frac{t}{\pi}\right)} \right\} = \frac{t}{\pi} \cdot \left\{ \frac{\pi}{2t} \cdot \cot t + \frac{\pi}{2t} \cdot \cot t \right\} = \cot t, \end{aligned}$$

and we have proved that

$$\cot t = \sum_{n=-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2}, \qquad t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}.$$

(c) Since $F(-z) = F(z)$, it follows from **(b)** for $t \in \mathbb{C} \setminus \{p\pi \mid p \in \mathbb{Z}\}$ that

$$\begin{aligned} g(t) &= \sum_{n=1}^{+\infty} \frac{t}{t^2 - \pi^2 n^2} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{t}{t^2 - \pi^2 n^2} + \frac{1}{2} \sum_{n=-\infty}^{-1} \frac{t}{t^2 - \pi^2 n^2} \\ &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} \frac{t}{t^2 - \pi^2 n^2} - \frac{1}{2} \cdot \frac{t}{t^2 - \pi^2 0^2} = \frac{1}{2}\left\{ \cot t - \frac{1}{t} \right\}. \end{aligned}$$

Example 8.17 *Find the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^4+4}.$$

The corresponding analytic function

$$f(z) = \frac{1}{z^4+4}$$

has the simple poles $\{1+i, \, -1+i, \, -1-i, \, 1-i\}$, none of which lies in $\mathbb{Z}$. Furthermore, $f(z)$ is a rational function with a zero of fourth order at ∞, hence the series is convergent, and its sum is given by a residuum formula,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{z^4+4} = -\pi \sum_{z_0^4=-4} \text{res}\left(\frac{\cot(\pi z)}{z^4+4}\,;\, z_0\right).$$

Since $z_0^4 = -4$ for every pole z_0, it follows by RULE II that

$$\begin{aligned} \text{res}\left(\frac{\cot(\pi z)}{z^4+4}\,;\, z_0\right) &= \cot(\pi z_0)\,\text{res}\left(\frac{1}{z^4+4}\,;\, z_0\right) = \cot(\pi z_0)\cdot\frac{1}{4z_0^3} = \frac{z_0}{4z_0^4}\cdot\cot(\pi z_0) \\ &= -\frac{1}{16}\, z_0\, \cot(\pi z_0). \end{aligned}$$

Then by insertion,

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \frac{1}{z^4+4} &= \pi \sum_{z_0^4=-4} \operatorname{res}\left(\frac{\cot(\pi z)}{z^4+4}\,;\, z_0\right) = \frac{\pi}{16} \sum_{z_0^4=-4} z_0 \cot\left(\pi z_0\right) \\ &= \frac{\pi}{16}\{(1+i)\cot(\pi+i\pi)+(1-i)\cot(\pi-i\pi)+(-1+i)\cot(-\pi+i\pi)+(-1-i)\cot(-\pi-i\pi)\} \\ &= \frac{\pi}{16}\{2(1+i)\cot(\pi+i\pi)+2(1-i)\cot(\pi-i\pi)\} = \frac{\pi}{8}\{(1+i)\cot(i\pi)+(1-i)\cot(-i\pi)\} \\ &= \frac{\pi}{8}\cdot 2i\cdot\frac{\cosh\pi}{i\,\sinh\pi} = \frac{\pi}{4}\cdot\coth\pi,\end{aligned}$$

thus

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^4+4} = \frac{\pi}{4}\cdot\coth\pi.$$

Remark 8.5 In a VARIANT we have the following estimates for e.g. $|z| \geq 2$,

$$|f(z)| = \frac{1}{|z^4+4|} \leq \frac{1}{|z|^4-4} = \frac{1}{|z|^4}\cdot\frac{1}{1-\dfrac{4}{|z|^4}} \leq \frac{1}{|z|^4}\cdot\frac{1}{1-\dfrac{4}{16}} = \frac{4}{3}\cdot\frac{1}{|z|^4},$$

so in particular, $C = \dfrac{4}{3}$ and $a = 4 \geq 2$ for $|z| \geq 2$. $\Diamond$

Example 8.18 *Compute the sum of the series*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n+\dfrac{1}{3}\right)\left(n+\dfrac{2}{3}\right)}.$$

If we put

$$f(z) = \frac{1}{\left(z+\dfrac{1}{3}\right)\left(z+\dfrac{2}{3}\right)},$$

then $f(z)$ is analytic in $\mathbb{C}\setminus\left\{-\dfrac{1}{3}\,,\,-\dfrac{2}{3}\right\}$, where $-\dfrac{1}{3}, -\dfrac{2}{3} \notin \mathbb{Z}$. Furthermore,

$$\left|z^2 f(z)\right| \to 1 \qquad \text{for } z\to\infty,$$

so we have checked the conditions for the convergence of the series and the sum can be found by a residuum formula. Hence,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\left(n+\frac{1}{3}\right)\left(n+\frac{2}{3}\right)} = -\pi\left\{\text{res}\left(\frac{\cot(\pi z)}{\left(z+\frac{1}{3}\right)\left(z+\frac{2}{3}\right)}\,;\,-\frac{1}{3}\right)+\text{res}\left(\frac{\cot(\pi z)}{\left(z+\frac{1}{3}\right)\left(z+\frac{2}{3}\right)}\,;\,-\frac{2}{3}\right)\right\}$$

$$= -\pi\left\{\frac{\cot\left(-\frac{\pi}{3}\right)}{-\frac{1}{3}+\frac{2}{3}}+\frac{\cot\left(-\frac{2\pi}{3}\right)}{-\frac{2}{3}+\frac{1}{3}}\right\} = -\pi\left\{\frac{-\cot\frac{\pi}{3}}{\frac{1}{3}}+\frac{\cot\frac{\pi}{3}}{-\frac{1}{3}}\right\} = \pi\cdot 3\cdot 2\cdot\cot\frac{\pi}{3}$$

$$= \pi\cdot 3\cdot 2\cdot\frac{1}{\sqrt{3}} = 2\sqrt{3}\,\pi,$$

where we have used that $-\frac{1}{3}$ and $-\frac{2}{3}$ are simple poles of $f(z)$. Thus we have proved that

$$\sum_{-\infty}^{+\infty} \frac{1}{\left(n+\frac{1}{3}\right)\left(n+\frac{2}{3}\right)} = 2\sqrt{3}\,\pi.$$

Example 8.19 *1) Determine the singular points in* $\mathbb{C}$ *of* f*, defined by*

$$(17)\quad f(z) = \frac{1}{z^{2p}\left(e^z - 1\right)}, \qquad p \in \mathbb{N},$$

and find the residuum of f *at every singular point.*
HINT: *When* $\operatorname{res}(f(z);0)$ *shall be found one may without proof apply the following Taylor series expansion*

$$\frac{z}{e^z-1} = \sum_{b=0}^{+\infty} \frac{B_n}{n!}\, z^n \qquad \text{for } |z| < 2\pi,$$

where the left hand side of the equality is replaced by 1 for $z = 0$.

2) Let K_n *denote in the* (x,y)*-plane the boundary of the square*

$$[-r_n\,,\, r_n] \times [-r_n\,,\, r_n]\,, \qquad \text{where } r_n = \pi + 2n\pi, \text{ and } n \in \mathbb{N}.$$

Prove for every (fixed) $p \in \mathbb{N}$ *that*

$$\oint_{K_n} f(z)\, dz \to 0 \qquad \text{for } n \to +\infty \quad \text{on } \mathbb{N},$$

where f *is the function given by (17).*

3) Apply Cauchy's residuum theorem on the square with the boundary K_n *and then apply the results of* **(1)** *and* **(2)** *above, and the limit* $n \to +\infty$ *to prove that*

$$\sum_{n=1}^{+\infty} \frac{1}{n^{2p}} = (-1)^{p+1}\,\frac{(2\pi)^{2p} B_{2p}}{2(2p)!} \qquad \text{for every } p \in \mathbb{N}.$$

Prove also that B_2, B_4, $\dots$, B_{2p}, $\dots$ *have alternating signs.*

1) The singular points are $z = 2i\,n\,\pi$, $n \in \mathbb{Z}$. Of these, $z = 0$ is a $(2n+1)$-tuple pole, while all the others are simple poles.

2) Putting $r_n = \pi + 2n\pi$ we get the following estimates,

$$\begin{aligned}
&\left|\oint_{K_n} \frac{dz}{z^{2p}\left(e^z-1\right)}\right| \\
&\quad \le \int_{-r_n}^{r_n} \frac{dy}{\left|r_n+iy\right|^{2p}\left|e^{r_n+iy}-1\right|} + \int_{-r_n}^{r_n} \frac{dy}{\left|-r_n+iy\right|^{2p}\left|e^{-r_n+iy}-1\right|} \\
&\qquad\quad + \int_{-r_n}^{r_n} \frac{dx}{\left|x+ir_n\right|^{2p}\left|e^{x+ir_n}-1\right|} + \int_{-r_n}^{r_n} \frac{dx}{\left|x-ir_n\right|^{2p}\left|e^{x-ir_n}-1\right|} \\
&\quad \le \frac{2r_n}{\left|r_n\right|^{2p}\cdot\frac{1}{2}} + \frac{2r_n}{\left|r_n\right|^{2p}\cdot\frac{1}{2}} + \frac{2r_n}{\left|r_n\right|^{2p}} + \frac{2r_n}{\left|r_n\right|^{2p}} = \frac{1}{2} r_n^{2p-1} \to 0
\end{aligned}$$

for $n \to +\infty$.

3) Now, $z = 2in\pi$ is a simple pole for $n \in \mathbb{Z} \setminus \{0\}$, so we have the following computation of the residuum,

$$\begin{aligned}\operatorname{res}(f(z); 2in\pi) &= \lim_{z \to 2in\pi} \frac{1}{z^{2p}} \frac{1}{\frac{d}{dz}(e^z - 1)} = \lim_{z \to 2i\pi} \frac{1}{z^{2p}} \cdot \frac{1}{e^z} \\ &= \frac{1}{(2in\pi)^{2p}} = (-1)^p \frac{1}{(2\pi)^{2p}} \cdot \frac{1}{n^{2p}}.\end{aligned}$$

Since

$$\frac{z}{e^z - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} z^n,$$

it follows by a division by z^{2p+1},

$$\frac{1}{z^{2p}(e^z - 1)} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} z^{-n-2p-1}.$$

We find the term $\dfrac{a_{-1}}{z}$ by choosing $n = 2p$, so

$$\operatorname{res}(f(z); 0) = a_{-1} = \frac{B_{2p}}{(2p)!}.$$

Then by Cauchy's residuum theorem,

$$\frac{1}{2\pi i} \oint_{K_n} f(z)\, dz = \sum_{k=-n}^{n} \operatorname{res}(f(z); 2ik\pi) = \frac{B_{2p}}{(2p)!} + 2 \cdot (-1)^p \cdot \frac{1}{(2\pi)^{2p}} \sum_{k=1}^{n} \frac{1}{k^{2p}},$$

hence by a rearrangement,

$$\sum_{k=1}^{n} \frac{1}{k^{2p}} = \frac{(-1)^p}{2} \cdot (2\pi)^{2p} \cdot \frac{1}{2\pi i} \oint_{K_n} f(z)\, dz + (-1)^{p+1} \frac{(2\pi)^{2p} B_{2p}}{2(2p)!}, \qquad n \in \mathbb{N}.$$

Since $p \in \mathbb{N}$, the left hand side converges, when $n \to +\infty$. Then by **(2)** it follows from taking this limit,

$$\sum_{n=1}^{+\infty} \frac{1}{n^{2p}} = (-1)^{p+1} \cdot \frac{(2\pi)^{2p} B_{2p}}{2(2p)!} \qquad \text{for every } p \in \mathbb{N}.$$

The left hand side is always positive, so the factor $(-1)^{p+1}$ on the right hand side causes that

$$B_{2p} \quad \begin{cases} > 0 & \text{for } p \text{ odd}, \\ < 0 & \text{for } p \text{ even}, \end{cases}$$

and the sequence B_2, B_4, $\dots$, B_{2p}, $\dots$ has alternating sign.

Example 8.20 *Given the function*

$$f(z) = \frac{1}{z^2 \sin z}.$$

1) Find all the isolated singularities in $\mathbb{C}$ *of the function* f*, and determin the type for each of them.*

2) Find in a neighbourhood of $z_0 = 0$ *the principal part of the Laurent series of* f*, i.e. that part of the series which contains terms of the type*

$$\frac{b_n}{z^n}, \qquad n > 0.$$

*(*HINT. *Use the Taylor series of* $\sin z$ *with* $z_0 = 0$ *as expansion point).*

3) Find the residues in the isolated singularities of f.

4) Denote by N *a positive integer, and let* C_N *denote the curve run through in a positive sense, which is bounding the square*

$$\left\{ z = x + iy \;\middle|\; x,\, y \in \left[-\left(N + \frac{1}{2}\right)\pi\,,\, \left(N + \frac{1}{2}\right)\pi \right] \right\}.$$

Compute $\oint_{C_N} f(z)\,dz$.

5) When $z = x + iy$*, then* $|\sin z|^2 = \sin^2 x + \sinh^2 y$*. It follows that*

$$|\sin z| \ge |\sin x| \qquad \text{and} \qquad |\sin z| \ge |\sinh y|.$$

Prove that

$$\oint_{C_N} f(z)\,dz \to 0 \qquad \text{for } N \to +\infty.$$

6) Prove that

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

1) We have poles at $z = p\pi$, $p \in \mathbb{Z}$. When $p = 0$, we see that the pole $z = 0$ is a triple pole; any other pole is simple.

2) Now, $f(z)$ is an odd function with the triple pole at $z = 0$, so the principal part must have the structure

$$\frac{1}{z^2 \sin z} = \frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + \cdots\,,$$

hence

$$\begin{aligned}
1 &= a_{-3} \cdot \frac{\sin z}{z} + a_{-1} \cdot z \sin z + \text{ terms of order } > 3 \\
&= a_{-3} \cdot \left\{ 1 - \frac{z^2}{6} + \cdots \right\} + a_{-1} \left\{ z^2 - \cdots \right\} + \cdots \\
&= a_{-3} + \left\{ a_{-1} - \frac{1}{6}\, a_{-3} \right\} z^2 + \cdots\,.
\end{aligned}$$

Then it follows by the identity theorem that

$$a_{-3} = 1 \qquad \text{og} \qquad a_{-1} = \frac{1}{6},$$

so the principal part is

$$\frac{1}{z^3} + \frac{1}{6}\,\frac{1}{z}.$$

3) We have in the triple pole $z_0 = 0$,

$$\operatorname{res}\left(\frac{1}{z^2 \sin z}\,;\,0\right) = a_{-1} = \frac{1}{6}.$$

When $z_p = p\pi$, $p \in \mathbb{Z} \setminus \{0\}$, the pole is simple, thus

$$\operatorname{res}\left(\frac{1}{z^2 \sin z}\,;\,p\pi\right) = \lim_{z \to p\pi} \frac{1}{z^2} \cdot \frac{1}{\cos z} = \frac{(-1)^p}{\pi^2} \cdot \frac{1}{p^2}.$$

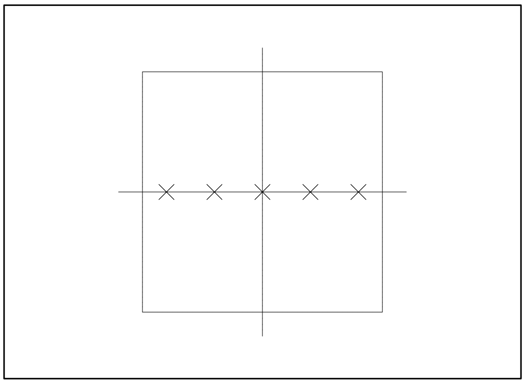


Figure 22: The curve C_N for $N = 2$ and the singularities inside.

4) Using Cauchy's residuum theorem,

$$\oint_{C_N} f(z)\,dz = \sum_{p=-N}^{N} \operatorname{res}(f(z);p\pi) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^n}{n^2}.$$

5) Then we have the estimates

$$\left|\int_{\Gamma_N'} \frac{dz}{z^2 \sin z}\right| \le (2N+1)\cdot \frac{1}{\left(N+\frac{1}{2}\right)^2\pi^2} \to 0 \qquad \text{for } N \to +\infty,$$

where Γ_N' is anyone of the vertical line segments of C_N.
In instead Γ_N'' is one of the horizontal segments, then

$$\left|\int_{\Gamma_N''} \frac{dz}{z^2 \sin z}\right| \le (2N+1)\cdot \frac{1}{\left(N+\frac{1}{2}\right)^2\pi^2} \cdot \frac{1}{\sinh\left(N+\frac{1}{2}\right)\pi} \to 0$$

for $N \to +\infty$.
Summing up we get

$$\lim_{N\to+\infty} \oint_{C_N} f(z)\,dz = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2} = 0.$$

Finally, by a rearrangement,

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$