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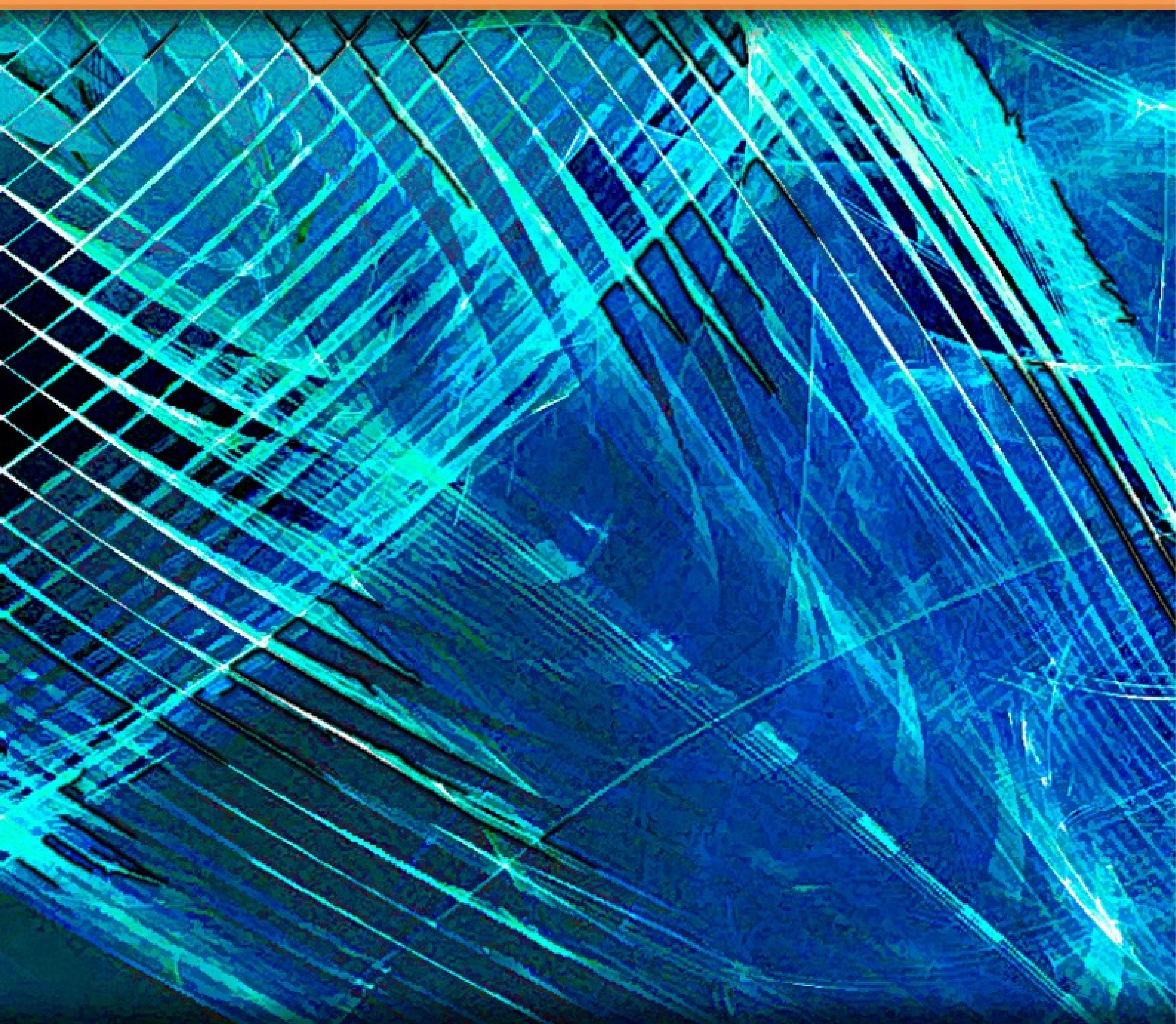
Complex Functions Theory c-11

Leif Mejlbro



Leif Mejlbro

The Laplace Transformation I

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Content

Introduction

In this volume we give some examples of the elementary part of the theory of the *Laplace transformation* as described in *Ventus, Complex Functions Theory a-4, The Laplace Transformation I*. The chapters and the sections will follow the same structure as in the above mentioned book on the theory.

Leif Mejlbro
February 18, 2014

1 The Laplace transformation

1.1 Null sets and null functions; the Lebesgue integral

Example 1.1.1 *Let* $A_0 = [0,1]$*, and let* $A_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ *denote the closed set, which is obtained by removing the open interval in "the middle".*
Then let

$$A_2 := \left[0, \frac{1}{3^2}\right] \cup \left[\frac{2}{3^2}, \frac{3}{3^2}\right] \cup \left[\frac{6}{3^2}, \frac{7}{3^2}\right] \cup \left[\frac{8}{3^2}, \frac{9}{3^2}\right]$$

be the set, which is obtained by removing all the open intervals in "the middle" in each of the two closed subintervals of A_1*.*

Sketch A_1 *and* A_2*.*

Then define the sets A_n *by induction, following the same pattern as described above, always removing the open interval in "the middle" of each subinterval. Let* $A = \bigcap_{n=0}^{+\infty}$*.*

1) Prove that $A \neq \emptyset$*.*

2) Prove that A *is a null set.*

3) Prove that A *contains a non-countable number of points.*

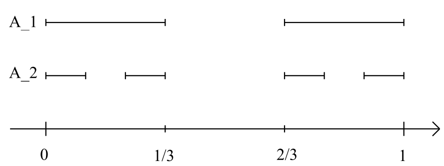

Figure 1: The sets A_1 and A_2.

Each A_n consists of 2^n closed intervals, each of length 3^{-n}. In the transition to A_{n+1} we remove an open interval of length 3^{-n-1} from each interval, so if m denotes the measure, i.e. the sum of all lengths of subintervals in each A_n, then

$$m\left(a_{n+1}\right) = \frac{2}{3}\, m\left(A_n\right).$$

1) Clearly, $0 \in A_n$ for every $n \in \mathbb{N}$, hence also $0 \in A = \bigcap_{n=1}^{+\infty} A_n$, so $A \neq \emptyset$.

2) It follows from

$$m(A) \leq m\left(A_n\right) = \left(\frac{2}{3}\right)^n \qquad \text{for every } n \in \mathbb{N},$$

by taking the limit $n \to +\infty$ that $m(A) = 0$, so A is a null set.

3) Every point $x \in A$ has a triadic expansion,

$$x = 0_3\, x_1 x_2 x_3 \cdots ,$$

where $x_n \in \{0,2\}$ for every $n \in \mathbb{N}$. Thus, $y_n = \frac{1}{2}\, x_n \in \{0,1\}$, so we define an *injective* map $\varphi : A \to [0,1]$ of all triadic numbers in A onto all dyadic numbers in $[0,1]$ by

$$\varphi\left(0_3\, x_1 x_2 x_3 \cdots\right) := 0_2\, y_1 y_2 y_3 \cdots .$$

Since every $y \in [0,1]$ indeed has a dyadic description of some number in $\varphi(A)$, it is also bijective, and the two sets A and $[0,1]$ have the same number of points. Since $m([0,1]) = 1 \neq 0$, it is not a null set, and in particular it contains non-countably many points. The same is true for A which therefore is a null set with non-countable points. $\Diamond$

Example 1.1.2 *We consider the set $\mathbb{R}/\mathbb{Q}$, thus an element $\hat{x} \in \mathbb{R}/\mathbb{Q}$ is a set*

$$\hat{x} := \{x + q \mid q \in \mathbb{Q}\},$$

where x is any representative of the class $\hat{x}$, lying in $x \in [0,1]$. We let $A\,(\subseteq [0,1])$ denote the set of all representatives, chosen in this way.

1) Prove that A is not a countable set.

2) Prove that the splitting

$$\mathbb{R} = \bigcup_{q\in\mathbb{Q}} \{x + q \mid x \in A\} = \bigcup_{q\in\mathbb{Q}} (A + \{q\})$$

of $\mathbb{R}$ is disjoint.

3) Prove that A cannot be a null set.

4) Prove that

$$\bigcup_{q\in\mathbb{Q}\,\cap\,[0,1]} (A + \{q\}) \subseteq [0,2],$$

and apply this inclusion to prove that A cannot have a (Lebesgue) measure $\neq 0$, and conclude that A is a nonmeasurable set.

1) Assume that $\mathbb{R} = A$ is countable. Then also

$$\bigcup_{x\in A} \{x + q \mid q \in \mathbb{Q}\} = \bigcup_{x\in A} \bigcup_{q\in\mathbb{Q}} \{x + q\}$$

would be countable.

However, $\mathbb{R}$ is not a null set, thus in particular not countable, so our assumption is wrong, and A is not countable either.

2) Assume that there are p, $q \in \mathbb{Q}$, such that

$$(A + \{p\}) \cap (A + \{q\}) \neq \emptyset.$$

Then there are x, $y \in A$, and r_1, $r_2 \in \mathbb{Q}$, such that

$$x + r_1 + p = y + r_2 + 1,$$

hence, by a rearrangement,

$$x = y + \{r_2 + q - r_1 - p\} = y + s, \qquad \text{where } s = r_2 + q - r_1 - p \in \mathbb{Q},$$

which proves that $x \sim y$, modulo $\mathbb{Q}$. Since also x, $y \in A$, this is only possible, if $x = y$, which again implies that $p = q$. Therefore, we conclude that the splitting

$$\mathbb{R} = \bigcup_{q\in\mathbb{Q}} (A + \{q\})$$

is disjoint.

3) If A was a (non-countable) null set, then we have above written $\mathbb{R}$ as a countable union of null sets. This would imply that $\mathbb{R}$ also should be a null set, which it is not! We therefore conclude by contraposition that A is not a null set.

4) It is trivial by the geometry that

$$\bigcup_{q\in\mathbb{Q}\,\cap\,[0,1]} (A+\{q\}) \subseteq [0,2].$$

Furthermore, the union on the left hand side is disjoint, so if A had a measure, then we proved above that $m(A) \neq 0$, hence $m(A) > 0$. This implies that

$$m\left(\bigcup_{q\in\mathbb{Q}\,\cap\,[0,1]} (A+\{q\})\right) = +\infty \le 2,$$

which is not possible.

Thus neither $m(A) = 0$ nor $m(A) > 0$, so A cannot be a Lebesgue measurable set. ◊

Example 1.1.3 *Prove that the relation "equal almost everywhere" is an equivalence relation on the class of functions.*

We shall check the three conditions of an equivalence relation for the relation

$$f \sim g, \qquad \text{if and only if } f(x) = g(x) \text{ for almost every } x \in \mathbb{R},$$

i.e. outside a null set.

1) It is obvious that $f \sim f$ for every function.

2) If $f \sim g$, then $f(x) = g(x)$, except on a null set N. Then of course also $g(x) = f(x)$, except on N, so $g \sim f$.

3) *Transitivity.* Assume that $f \sim g$ and $g \sim h$. Then

$$\{x \in \mathbb{R} \mid f(x) \neq h(x)\} \subseteq \{x \in \mathbb{R} \mid f(x) \neq g(x)\} \cup \{x \in \mathbb{R} \mid g(x) \neq h(x)\},$$

so $N := \{x \in \mathbb{R} \mid f(x) \neq h(x)\}$ is contained in a union of two null sets, thus N is also a null set, and we have proved that $f \sim h$.

Summing up we have proved that "equality almost everywhere" $=\sim$ is an equivalence relation.

1.2 The Laplace transformation

Example 1.2.1 *Prove that if $f(t)$ is a bounded function, then $\sigma(f) \leq 0$.*

We assume that $|f(t)| \leq A$ for all $t \geq 0$. Choose any $\sigma > 0$. Then we have the estimate

$$\int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t \leq A \int_0^{+\infty} e^{-\sigma t}\, \mathrm{d}t = \frac{A}{\sigma} < +\infty,$$

so

$$\sigma(f) = \inf\left\{ \sigma \in \mathbb{R} \;\middle|\; \int_0^{+\infty} |f(t)|\, e^{-\sigma t} < +\infty \right\} \leq \inf\{\sigma \in \mathbb{R} \mid \sigma > 0\} = 0,$$

and the claim is proved. $\Diamond$

Example 1.2.2 *Find a continuous function $f \in \mathcal{F} \setminus \mathcal{E}$.*

We shall construct a *continuous* function $f : [0, +\infty[\to \mathbb{R}$, such that for some $\sigma \in \mathbb{R}$,

(1) $\displaystyle\int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t < +\infty,$

i.e. $f \in \mathcal{F}$, and such that also

(2) $\forall\, A \in \mathbb{R}_+ \,\forall\, B \in \mathbb{R} \,\exists\, t \in [0, +\infty[:\; |f(t)| > A\, e^{Bt},$

i.e. $f \notin \mathcal{E}$.

We shall in the construction below explicitly choose $\sigma = 0$, although it is obvious that it can be made for any $\sigma \in \mathbb{R}$. When $\sigma = 0$, then (1) reduces to $f \in L^1(\mathbb{R}_+)$, thus

$$\int_0^{+\infty} |f(t)|\, \mathrm{d}t < +\infty.$$

We choose a continuous function $g : [+, +\infty[\to \mathbb{R}$, which fulfils (2), thus $g \notin \mathcal{E}$. In general, any such function can be chosen, but here we limit ourselves to

$$g(t) := \exp\left(e^t\right), \qquad \text{for } t \in [0, +\infty[,$$

because $g(t)$ then will dominate every exponential e^{At}, so it cannot belong to $\mathcal{E}$, thus $g \notin \mathcal{E}$.

Obviously, also $g \notin \mathcal{F}$, so we shall amend g slightly. First notice that

$$\forall\, A \in \mathbb{R}_+ \,\forall\, B \in \mathbb{R} \,\exists\, n \in \mathbb{N} : |g(n)| = \exp\left(e^n\right) > A\, e^{nB}.$$

This means that every function f which fulfils

(3) $f(n) = g(n) = \exp\left(e^n\right) \qquad \text{for } \mathbb{N},$

while the values $f(t)$ can be anything for $t \in \mathbb{R}_+ \setminus \mathbb{N}$, will satisfy (2) and therefore not belong to $\mathcal{E}$. The idea is then simple. Cut so much in the graph of $g(t)$, that $f(n) = g(n) = \exp\left(e^n\right)$ for all $n \in \mathbb{N}$, that $f(t)$ remains continuous, and such that $f \in L^1(\mathbb{R}_+)$.

Every point $n \in \mathbb{N}$ on the x axis is surrounded by a symmetric interval

$$\left[n - \frac{1}{2^n} \exp\left(-e^n\right), n + \frac{1}{2^n} \exp\left(-e^n\right) \right],$$

of length $2 \cdot 2^{-n} \exp\left(-e^n\right)$, and in this interval f is chosen as the piecewise linear function as indicated on Figure 2, so the graph becomes in this interval a triangle of area

$$\frac{1}{2} \cdot 2 \cdot e^{-n} \exp\left(e^{-n}\right) \cdot \exp\left(e^n\right) = 2^{-n}.$$

Outside these intervals we put $f(t) = 0$, and it is obvious that $f(t)$ then is continuous and by (3) above does not belong to $\mathcal{E}$.

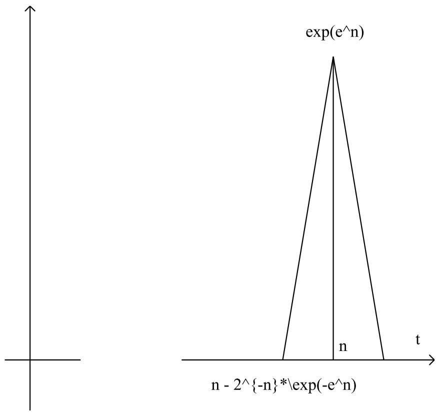

Figure 2: Construction of a function $f \in \mathcal{F} \setminus \mathcal{E}$. The axes have different scales.

It only remains to prove that $f \in \mathcal{F}$. This follows from

$$\int_0^{+\infty} |f(t)|\, e^{-0\cdot t}\, \mathrm{d}t = \int_0^{+\infty} f(t)\, \mathrm{d}t = \sum_{n=1}^{+\infty} \frac{1}{2^n} = 1 < +\infty.$$

Hence, $f \in \mathcal{F} \setminus \mathcal{E}$, as requested. $\Diamond$

Example 1.2.3 *Construct two functions f, $g \in \mathcal{F}$, such that $f \cdot g \notin \mathcal{F}$.*

We choose

$$f(t) = g(t) = \begin{cases} \dfrac{1}{\sqrt{t}} & \text{for } t \in \mathbb{R}_+, \\ 0 & \text{for } t = 0. \end{cases}$$

The functions $f = g$ belong to $\mathcal{F}$, because we for $\sigma = 1$ get

$$\begin{aligned} \int_0^{+\infty} |f(t)|\, e^{-\sigma t}\, \mathrm{d}t &= \int_0^{+\infty} \frac{1}{\sqrt{t}}\, e^{-t}\, \mathrm{d}t = \int_0^1 \frac{1}{\sqrt{t}}\, e^{-t}\, \mathrm{d}t + \int_1^{+\infty} \frac{1}{\sqrt{t}}\, e^{-t}\, \mathrm{d}t \\ &\le \int_0^1 \frac{1}{\sqrt{t}}\, \mathrm{d}t + \int_1^{+\infty} e^{-t}\, \mathrm{d}t = \left[2\sqrt{t}\right]_0^1 + \left[e^{-t}\right]_1^{+\infty} = 2 + \frac{1}{e} < +\infty. \end{aligned}$$

First notice that if $\sigma \le 0$, then

$$\int_\varepsilon^{+\infty} |f(t) \cdot g(t)|\, e^{-\sigma t}\, \mathrm{d}t \ge \int_\varepsilon^{+\infty} |f(t)|^2\, \mathrm{d}t = \int_\varepsilon^{+\infty} \frac{1}{t}\, \mathrm{d}t = +\infty,$$

so it suffices only to consider $\sigma > 0$. We get for every such $\sigma > 0$ and every $\varepsilon \in]0,1[$ that

$$\begin{aligned}\int_{\varepsilon}^{+\infty} |f(t)g(t)|\, e^{-\sigma t}\,\mathrm{d}t &= \int_{\varepsilon}^{+\infty} \frac{1}{t}\, e^{-\sigma t}\,\mathrm{d}t = \int_{\varepsilon}^{1} \frac{1}{t}\, e^{-\sigma t}\,\mathrm{d}t + \int_{1}^{+\infty} \frac{1}{t}\, e^{-\sigma t}\,\mathrm{d}t \\ &\geq \int_{\varepsilon}^{1} \frac{1}{t}\, e^{-\sigma \cdot 1}\,\mathrm{d}t + 0 = e^{-\sigma}\,[\ln t]_{\varepsilon}^{1} = e^{-\sigma}\,\ln\frac{1}{\varepsilon} \to +\infty, \qquad \text{for } \varepsilon \to +\infty,\end{aligned}$$

proving that the improper integral

$$\int_{0}^{+\infty} |f(t)g(t)|\, e^{-\sigma t}\,\mathrm{d}t = +\infty$$

for every $\sigma > \mathbb{R}_+$, hence also for every $\sigma \in \mathbb{R}$, and we have proved that $f \cdot g = f^2 \notin \mathcal{F}$.

Remark 1.2.1 *We chose the argument above to guide the reader through the main steps of the idea of how to construct such an example, but it should also be mentioned that the* continuous $f \in \mathcal{F} \setminus \mathcal{E}$ *constructed in Example 1.2.2 also satisfies that* $f^2 \notin \mathcal{F}$. *The simple proof is left to the reader.* $\Diamond$

Example 1.2.4 *Show that* $\sigma(f) = -\infty$ *for the function* $f(t) = \exp\left(e^t\right)$ *for* $t \geq 0$. $\Diamond$

We get for every $z \in \mathbb{C}$,

$$\int_{0}^{+\infty} |f(t)| \cdot e^{-\Re z \cdot t}\,\mathrm{d}t = \int_{0}^{+\infty} \exp\left(-e^t\right) \cdot e^{-\Re z \cdot t}\,\mathrm{d}t = \int_{0}^{+\infty} e^{-\left(e^t + \Re z \cdot t\right)}\,\mathrm{d}t < +\infty,$$

because $e^t + \Re z \cdot t > t$ for $t \geq T = T(z)$, where the constant $T(z)$ depends on z. Hence,

$$\begin{aligned}\int_{0}^{+\infty} e^{-\left(e^t + \Re z \cdot t\right)}\,\mathrm{d}t &= \int_{0}^{T} e^{-\left(e^t + \Re z \cdot t\right)}\,\mathrm{d}t + \int_{T}^{+\infty} e^{-\left(e^t + \Re z \cdot t\right)}\,\mathrm{d}t \\ &\leq \int_{0}^{T} e^{-\left(e^t + \Re z \cdot t\right)}\,\mathrm{d}t + \int_{T}^{+\infty} e^{-t}\,\mathrm{d}t < +\infty.\end{aligned}$$

Since the improper integral is convergent for every $z \in \mathbb{C}$, we conclude that $\sigma(f) = -\infty$.

Example 1.2.5 *Given a continuous function* $f(t)$ *in a closed bounded interval* $[a,b] \subseteq [0,+\infty[$, *and* $f(t) = 0$ *outside this interval. Prove that* $\sigma(f) = -\infty$.

The function $f(t)$ is assumed to be continuous in a closed bounded interval. It therefore follows from a main theorem that $f(t)$ is bounded, i.e. $|f(t)| \leq A$ for some constant $A > 0$ and all $t \in \mathbb{R}$. Hence, for every $\sigma \in \mathbb{R}$,

$$\int_{0}^{+\infty} e^{-\sigma t}\,|f(t)|\,\mathrm{d}t = \int_{a}^{b} e^{-\sigma t}\,|f(t)|\,\mathrm{d}t \leq (b-a) \cdot A \cdot \max\left\{e^{\sigma a}, e^{-\sigma b}\right\} < +\infty.$$

Then clearly $\sigma(f) = -\infty$. $\Diamond$

Example 1.2.6 *Find the Laplace transforms of*

1) $t\, e^{at}$, *where* $a \in \mathbb{C}$,

2) $t\, \sinh t$,

3) $t\, \cosh t$,

4) $t\, \sin t$,

5) $t\, \cos t$.

1) It follows from

$$\int_0^{+\infty} \left| t\, e^{at} \right| e^{-\sigma t}\, \mathrm{d}t = \int_0^{+\infty} t\, e^{-t(\sigma - \Re a)}\, \mathrm{d}t,$$

that the improper integral is convergent for $\Re z > \Re a$. When this is the case, we get by partial integration,

$$\begin{aligned} \mathcal{L}\left\{t\, e^{at}\right\}(z) &= \int_0^{+\infty} t\, e^{at}\, e^{-zt}\, \mathrm{d}t = \int_0^{+\infty} t\, e^{t(a-z)}\, \mathrm{d}t \\ &= \left[t \cdot \frac{1}{a-z}\, e^{t(a-z)} \right]_0^{+\infty} - \frac{1}{a-z} \int_0^{+\infty} e^{t(a-z)}\, \mathrm{d}t \\ &= 0 - \frac{1}{a-z} \left[\frac{1}{a-z}\, e^{t(a-z)} \right]_0^{+\infty} = \frac{1}{(a-z)^2} \qquad \left[= -\frac{d}{dz}, \mathcal{L}\left\{e^{at}\right\}(z) \right], \end{aligned}$$

so

$$\mathcal{L}\left\{t\, e^{at}\right\}(z) = \frac{1}{(a-z)^2} \qquad \text{for } \Re z > \Re a.$$

The remaining problems are derived from this result by Euler's formulæ.

2) Using the definition of $\sinh t$ and 1) above we get

$$\begin{aligned} \mathcal{L}\{t \cdot \sinh t\}(z) &= \mathcal{L}\left\{ t \cdot \frac{1}{2}\left(e^t - e^{-t}\right) \right\}(z) = \frac{1}{2}\,\mathcal{L}\left\{t\, e^t\right\}(z) - \frac{1}{2}\,\mathcal{L}\left\{t\, e^{-t}\right\}(z) \\ &= \frac{1}{2} \cdot \frac{1}{(1-z)^2} - \frac{1}{2} \cdot \frac{1}{(-1-z)^2} = \frac{1}{2}\left\{ \frac{1}{(z-1)^2} - \frac{1}{(z+1)^2} \right\} \\ &= \frac{1}{2}\, \frac{(z+1)^2 - (z-1)^2}{\left(z^2-1\right)^2} \frac{2z}{\left(z^2-1\right)^2} \qquad \left[= -\frac{d}{dz}\, \mathcal{L}\{\sinh t\}(z) \right], \end{aligned}$$

for $\Re z > \max\{\Re 1, \Re(-1)\} = 1$.

3) Similarly,

$$\begin{aligned} \mathcal{L}\{t\, \cosh t\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{t\, e^t\right\}(z) + \frac{1}{2}\,\mathcal{L}\left\{t\, e^{-t}\right\}(z) = \frac{1}{2}\, \frac{(z+1)^2 + (z-1)^2}{\left(z^2-1\right)^2} \\ &= \frac{z^2+1}{\left(z^2-1\right)^2} \qquad \left[= -\frac{d}{dz}\, \mathcal{L}\{\cosh t\}(z) \right] \qquad \text{for } \Re z > 1. \end{aligned}$$

4) In this case we use Euler's formula,

$$\begin{aligned}\mathcal{L}\{t\,\sin t\}(z) &= \frac{1}{2i}\,\mathcal{L}\left\{t\,e^{it}\right\}(z) - \frac{1}{2i}\,\mathcal{L}\left\{t\,e^{-it}\right\}(z) = \frac{1}{2i}\left\{\frac{1}{(i-z)^2} - \frac{1}{(-i-z)^2}\right\} \\ &= \frac{1}{2i}\cdot\frac{(-i-z)^2-(i-z)^2}{\left(z^2+1\right)^2} = \frac{1}{2i}\cdot\frac{4iz}{\left(z^2+1\right)^2} = \frac{2z}{\left(z^2+1\right)^2} \qquad \left[= -\frac{d}{dz}\,\mathcal{L}\{\sin t\}(z)\right]\end{aligned}$$

for $\Re\, z > \max\{\Re\, i, \Re(-i)\} = 0$.

5) Similarly,

$$\begin{aligned}\mathcal{L}\{t\cdot\cos t\}(z) &= \frac{1}{2}\left\{\frac{1}{(i-z)^2} + \frac{1}{(-i-z)^2}\right\} = \frac{1}{2}\cdot\frac{(-i-z)^2+(i-z)^2}{\left(z^2+1\right)^2} = \frac{i^2+z^2}{\left(z^2+1\right)^2} \\ &= \frac{z^2-1}{\left(z^2+1\right)^2} \qquad \left[= -\frac{d}{dz}\,\mathcal{L}\{\cos t\}(z)\right] \qquad \text{for } \Re\, z > 0. \qquad \Diamond\end{aligned}$$

Example 1.2.7 *For which of the functions below do the Laplace transforms exist?*

1) $\dfrac{1}{1+t}$,

2) $\exp\left(t^2-1\right)$,

3) $\cos\left(t^2\right)$.

1) The function $f(t)=\dfrac{1}{1+t}$ is bounded and continuous in $[0,+\infty[$. Hence, the Laplace transform of $f(t)$ exists and $\varrho(f)=\sigma(f)=0$. The explicit expression cannot be found at this stage of the theory.

2) No matter how we choose any $\sigma\in\mathbb{R}$, we get

$$\int_0^{+\infty} e^{-\sigma t}\, e^{t^2-1}\,\mathrm{d}t=\exp\left(-1-\frac{\sigma^2}{4}\right)\int_0^{+\infty}\exp\left(\left(t-\frac{\sigma}{2}\right)^2\right)\mathrm{d}t=+\infty,$$

so $\exp\left(t^2-1\right)$ does not have a Laplace transform.

3) The function $\cos\left(t^2\right)$ is continuous and bounded, so its Laplace transform exists. However, it cannot be found at this stage of the theory. $\Diamond$

Example 1.2.8 *Find the Laplace transform and the abscissa of convergence for each of the following functions,*

1) t^2+2,

2) $t+e^{-t}+\sin t$,

3) $(1+t)^n$, $n\in\mathbb{N}$.

1) It follows by linearity and by using some of the commonly used tables that

$$\mathcal{L}\left\{t^2+2\right\}(z)=\mathcal{L}\left\{t^2\right\}(z)+\mathcal{L}\{2\}(z)=\frac{2!}{z^3}+2\cdot\frac{1}{z}=2\cdot\frac{z^2+1}{z^3}$$

for $\Re z>0$, and it is easily seen that $\sigma(f)=0$.

2) Similarly, by using linearity and some table,

$$\mathcal{L}\left\{t+e^{-t}+\sin t\right\}(z)=\mathcal{L}\{t\}(z)+\mathcal{L}\left\{e^{-t}\right\}(z)+\mathcal{L}\{\sin t\}(z)=\frac{1}{z^2}+\frac{1}{z+1}+\frac{1}{z^2+1},$$

for $\Re\, z>\max\{0,-1,0\}=0$, thus $\sigma(f)=0$.

3) Since $f(t)$ is a polynomial, which is dominated by the exponential, we get that the improper integral

$$\int_0^{+\infty} (1+t)^n \, e^{-\sigma t} \, \mathrm{d}t, \qquad n \in \mathbb{N},$$

is convergent for every $\sigma \in \mathbb{R}_+$, and divergent for $\sigma = 0$. Hence,

$$\sigma(f) = \sigma\left((1+t)^n\right) = 0.$$

Then we apply the binomial formula and the linearity and some table to get

$$\begin{aligned}\mathcal{L}\left\{(1+t)^n\right\}(z) &= \mathcal{L}\left\{\sum_{j=0}^{n} \binom{n}{j} t^j\right\}(z) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{L}\left\{t^j\right\}(z) \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{j!}{z^{j+1}} = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \cdot \frac{1}{z^{j+1}},\end{aligned}$$

for $\Re\, z > 0$. $\Diamond$

Example 1.2.9 *Find* $\mathcal{L}\left\{\chi_{]1,2[}\right\}(z)$ *and* $\sigma\left(\chi_{]1,2[}\right)$.

It follows immediately from the definition

$$\mathcal{L}\left\{\chi_{]1,2[}\right\}(z) = \int_1^2 e^{-zt} \, \mathrm{d}t$$

that the improper integral is convergent for every $z \in \mathbb{C}$, so we conclude that

$$\sigma\left(\chi_{]1,2[}\right) = -\infty.$$

If $z = 0$, then

$$\mathcal{L}\left\{\left(\chi_{]1,2[}\right)\right\}(0) = 1.$$

If $z \neq 0$, then

$$\mathcal{L}\left\{\left(\chi_{]1,2[}\right)\right\}(z) = \int_1^2 e^{-zt} \, \mathrm{d}t = \left[-\frac{1}{z} \, e^{-zt}\right]_{t=1}^{2} = \frac{e^{-z} - e^{-2z}}{z}.$$

Summing up, $\sigma(f) = -\infty$, and

$$\mathcal{L}\left\{\left(\chi_{]1,2[}\right)\right\}(z) = \begin{cases} e^{-z} \cdot \dfrac{1 - e^{-z}}{z} & \text{for } z \neq 0, \\ 1 & \text{for } z = 0. \end{cases} \qquad \Diamond$$

Example 1.2.10 *Given*

$$f(t) = \min\{t, 1\} \qquad \text{for } t \in [0, +\infty[.$$

Sketch the graph of f, and then find $\mathcal{L}\{f\}(z)$ and $\sigma(f)$.

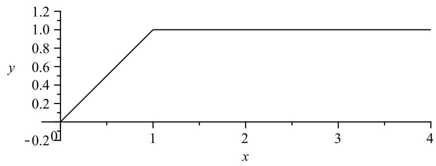

Figure 3: The graph of the function of Example 1.2.10.

Clearly, $\sigma(f) = 0$. Then for $\Re z > 0$,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^1 t\,e^{-zt}\,\mathrm{d}t + \int_1^\infty e^{-zt}\,\mathrm{d}t = \left[-\frac{1}{z}\,t\,e^{-zt}\right]_{t=0}^{1} + \frac{1}{z}\int_0^1 e^{-zt}\,\mathrm{d}t + \left[-\frac{1}{z}\,e^{-zt}\right]_{t=1}^{+\infty} \\ &= -\frac{1}{z}\,e^{-z} + \frac{1}{z^2}\left(1 - e^{-z}\right) + \frac{1}{z}\,e^{-z} = \frac{1 - e^{-z}}{z^2}. \qquad \Diamond\end{aligned}$$

Example 1.2.11 *Compute* $\mathcal{L}\left\{\left(5e^{-2t}-3\right)^2\right\}(z)$.

We get immediately for $\Re z > 0 = \sigma(f)$,

$$\mathcal{L}\left\{\left(5e^{-2t}-3\right)^2\right\}(z) = \mathcal{L}\left\{25\,e^{-4t} - 30\,e^{-2t} + 9\right\}(z) = \frac{25}{z+4} - \frac{30}{z+2} + \frac{9}{z}. \qquad \Diamond$$

Example 1.2.12 *Prove for* $f_{n,a}(t) = e^{-at}\,t^n$, $n \in \mathbb{N}_0$ *and* $a \in \mathbb{C}$, *that*

$$\mathcal{L}\left\{f_{n,a}\right\}(z) = \frac{n!}{(z+a)^{n+1}},$$

and find $\sigma(f)$.

It follows by a straightforward computation that

$$\mathcal{L}\{f_{n,a}\}(z) = \int_0^{+\infty} e^{-at}\,t^n\,e^{-zt}\,\mathrm{d}t = \int_0^{+\infty} t^n\,e^{-(z+a)t}\,\mathrm{d}t = \mathcal{L}\left\{t^n\right\}(z+a) = \frac{n!}{(z+a)^{n+1}},$$

for $\Re z > -\Re a = \sigma(f_{n,a})$. $\Diamond$

Example 1.2.13 *Compute the Laplace transforms of the functions below and find their abscissa of convergence.*

1) $\chi_{\mathbb{R}_+}(t-1) + 3\,e^{-(t+6)}$,

2) $\chi_{\mathbb{R}_+}(t-1) \cdot \sin(t-1)$.

1) It follows from the definition of the Laplace transform that

$$\mathcal{L}\{f\}(z) = \int_1^{+\infty} e^{-zt}\,\mathrm{d}t + 3\,e^{-6}\int_0^{+\infty} e^{-t}\,e^{-zt}\,\mathrm{d}t = \frac{e^{-z}}{z} + \frac{3}{e^6}\cdot\frac{1}{z+1}$$

for $\Re\, z > \sigma(f) = \max\{0,-1\} = 0$.

2) Similarly,

$$\mathcal{L}\{f\}(z) = \int_1^{+\infty} \sin(t-1)\,e^{-zt}\,\mathrm{d}t = e^{-z}\int_0^{+\infty} \sin t \cdot e^{-zt}\,\mathrm{d}t = e^{-z}\,\mathcal{L}\{\sin t\}(z) = \frac{e^{-z}}{z^2+1}$$

for $\Re\, z > 0 = \sigma(f)$. $\Diamond$

Example 1.2.14 *Compute the Laplace transform of* $(\sin t - \cos t)^2$.

First compute

$$(\sin t - \cos t)^2 = \sin^2 t + \cos^2 t - 2\,\sin t \cdot \cos t = 1 - \sin 2t.$$

Then we get for $\Re\, z > 0 = \sigma(f)$, that

$$\mathcal{L}\left\{(\sin t - \cos t)^2\right\}(z) = \mathcal{L}\{1 - \sin 2t\}(z) = \frac{1}{z} - \frac{2}{z^2+4}. \qquad \Diamond$$

Example 1.2.15 *Compute the Laplace transform of* $\cosh^2 4t$.

We get for $\Re\, z > 8 = \sigma(f)$,

$$\mathcal{L}\left\{\cosh^2 4t\right\}(z) = \mathcal{L}\left\{\frac{1+\cosh 8t}{2}\right\}(z) = \frac{1}{2}\cdot\frac{1}{z} + \frac{1}{2}\cdot\frac{z}{z^2-64} = \frac{z^2-32}{z\,(z^2-64)}. \qquad \Diamond$$

Example 1.2.16 *Compute the Laplace transform of* $\chi_{[0,\pi]}(t)\cdot\sin t$.

We write for short $f(t) = \chi_{[0,\pi]}(t)\cdot\sin t$. Then for $z \in \mathbb{C}$,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^{+\infty} \chi_{[0,\pi]}(t)\cdot\sin t\cdot e^{-zt}\,\mathrm{d}]t = \int_0^{\pi} \frac{1}{2i}\left\{e^{it} - e^{-it}\right\}\cdot e^{-zt}\,\mathrm{d}t \\ &= \frac{1}{2i}\int_0^{\pi} e^{(i-z)t}\,\mathrm{d}t - \frac{1}{2i}\int_0^{\pi} e^{-(i+z)t}\,\mathrm{d}t.\end{aligned} \tag{4}$$

The two integrals are convergent for every $z \in \mathbb{C}$, so $\sigma(f) = -\infty$.

If $z \in \mathbb{C} \setminus \{-i, i\}$, then

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{2i}\left[\frac{1}{i-z}\,e^{(i-z)t}\right]_{t=0}^{\pi} - \frac{1}{2i}\left[-\frac{1}{i+z}\,e^{-(i+z)t}\right]_{t=0}^{\pi} \\
&= \frac{1}{2i}\cdot\frac{1}{i-z}\left\{e^{i\pi-\pi z}-1\right\} + \frac{1}{2i}\cdot\frac{1}{i+z}\left\{e^{-i\pi-\pi z}-1\right\} = -\frac{1+e^{-\pi z}}{2i}\left\{\frac{1}{i-z}+\frac{1}{i+z}\right\} \\
&= -\frac{1+e^{-\pi z}}{2i}\cdot\frac{i+z+i-z}{i^2-z^2} = +\frac{e^{-zpiz}+1}{z^2+1}, \qquad \text{for } z \in \mathbb{C}\setminus\{-i,i\}.
\end{aligned}$$

If $z = i$, then it follows from (4) that

$$\mathcal{L}\{f\}(i) = \frac{1}{2i}\int_0^{\pi} e^0\,\mathrm{d}t - \frac{1}{2i}\int_0^{\pi} e^{-2it}\,\mathrm{d}t = \frac{\pi}{2i},$$

which can also by obtained by taking the limit

$$\lim_{z\to i}\mathcal{L}\{f\}(z) = \lim_{z\to i}\frac{e^{-\pi z}+1}{z^2+1} = \lim_{z\to i}\frac{-\pi\,e^{-\pi z}}{2z} = \frac{-\pi(-1)}{2i} = \frac{\pi}{2i}.$$

If $z = -i$, then it follows from (4) that

$$\mathcal{L}\{f\}(-i) = \frac{1}{2i}\int_0^{\pi} e^{2it}\,\mathrm{d}t - \frac{1}{2i}\int_0^{\pi} e^0\,\mathrm{d}t = -\frac{\pi}{2i},$$

which can also by obtained by taking the limit

$$\lim_{z\to -i}\mathcal{L}\{f\}(z) = \lim_{z\to -i}\frac{e^{-\pi z}+1}{z^2+1} = \lim_{z\to -i}\frac{-\pi\,e^{-\pi z}}{2z} = \frac{-\pi(-1)}{-2i} = -\frac{\pi}{2i},$$

and we see (which should not be a surprise) that $z = i$ and $z = -i$ are removable singularities of the analytic function

$$\mathcal{L}\{f\}(z) = \frac{e^{-\pi z}+1}{z^2+1}, \qquad z \in \mathbb{C}. \qquad \Diamond$$

Example 1.2.17 *Prove that neither* $\sin z$ *nor* $\cos z$ *can be the Laplace transform of any function* $f \in \mathcal{F}$.

If $f \in \mathcal{F}$, then

$$\mathcal{L}\{f\}(x) \to 0 \qquad \text{for } \Re z \to +\infty.$$

Therefore, the claim follows if we can prove that neither $\sin z$ nor $\cos z$ satisfy this condition. This is obvious, because not even the restrictions $\sin x$ and $\cos x$ to the real axis have a limit value for $x \in \mathbb{R}$ and $x \to +\infty$. $\Diamond$

Example 1.2.18 *Compute the Laplace transform of $t^n\,e^t$, either by the definition, or by the rule of multiplication by t^n, or by using the shifting property.*

1) *Application of the definition.* Due to the principle of different magnitudes there exists for every $\varepsilon > 0$ a constant $A_\varepsilon > 0$, such that

$$\left|t^n\,e^t\right| \le A_\varepsilon\,e^{(1+\varepsilon)t}, \qquad \text{for all } t \in [0,+\infty[.$$

Furthermore, the improper integral

$$\int_0^{+\infty} \left|t^n\,e^t\right|\,e^{-1\cdot t}\,\mathrm{d}t = \int_0^{+\infty} t^n\,\mathrm{d}t = +\infty$$

is divergent. We therefore conclude that

$$\sigma(f) = \varrho(f) = 1 \qquad \text{and} \qquad f \in \mathcal{E}.$$

Then by the definition for $\Re\,z > 1$,

$$\begin{aligned}\mathcal{L}\left\{t^n\,e^t\right\}(z) &= \int_0^{+\infty} t^n\,e^t\,e^{-zt}\,\mathrm{d}t = \int_0^{+\infty} t^n\,e^{-(z-1)t}\,,dt \\ &= \left[-\frac{1}{z-1}\,t^n\,e^{-(z-1)t}\right]_{t=0}^{+\infty} + \frac{n}{z-1}\int_0^{+\infty} t^{n-1}\,e^t\,e^{-zt}\,\mathrm{d}t \\ &= \frac{n}{z-1}\,\mathcal{L}\left\{t^{n-1}\,e^t\right\}(z).\end{aligned}$$

Finally, we get by recursion,

$$\mathcal{L}\left\{t^n\,e^t\right\}(z) = \frac{n}{z-1}\,\mathcal{L}\left\{t^{n-1}\,e^t\right\}(z) = \cdots = \frac{n!}{(z-1)^{n+1}} \qquad \text{for } \Re\,z > 1.$$

2) *Rule of multiplication by t^n.* First we get

$$\mathcal{L}\left\{e^t\right\}(z) = \int_0^{+\infty} e^t\,e^{-zt}\,\mathrm{d}t = \frac{1}{z-1} \qquad \text{for } \Re\,z > 1.$$

Then by the rule of multiplication by t^n,

$$\mathcal{L}\left\{t^n\,e^t\right\}(z) = (-1)^n\,\frac{d^n}{dz^n}\left\{\frac{1}{z-1}\right\} = (-1)^n\,(-1)^n\cdot\frac{n!}{(z-1)^{n+1}} = \frac{n!}{(z-1)^{n+1}}.$$

3) *Shifting rule.* First we get from a table that

$$\mathcal{L}\left\{t^n\right\}(z) = \frac{n!}{z^{n+1}} \qquad \text{for } \Re\,z > 0.$$

Then an application of the *shifting rule* with $f(t) = t^n$ and $a = 1$ gives

$$\mathcal{L}\left\{t^n\,e^t\right\}(z) = \mathcal{L}\left\{t^n\right\}(z-1) = \frac{n!}{(z-1)^{n+1}} \qquad \text{for } \Re\,z > 0+1 = 1. \qquad \Diamond$$

Example 1.2.19 *Given* $f(t) = t \cdot \cos at$. *Prove that*

$$\mathcal{L}\{f\}(z) = \frac{z^2 - a^2}{\left(z^2 + a^2\right)^2}, \qquad \textit{and} \qquad \sigma(f) = |\Im\, a|.$$

Using various rules of computation we get

$$\begin{aligned}
\mathcal{L}\{t \cdot \cos at\}(z) &= -\frac{d}{dz}\,\mathcal{L}\{\cos at\}(z) = -\frac{d}{dz}\left\{\frac{z}{z^2+a^2}\right\} \\
&= -\frac{1 \cdot \left(z^2+a^2\right) - 2z \cdot z}{\left(z^2+a^2\right)^2} = \frac{z^2-a^2}{\left(z^2+a^2\right)^2} \qquad \text{for } \Re\, z > |\Im\, a| = \sigma(f),
\end{aligned}$$

where $\sigma(f) = |\Im\, a|$ follows from that the analytic function $\mathcal{L}\{f\}(z)$ has the poles $z = \pm ia$.

Alternatively it follows from

$$\begin{aligned}
\mathcal{L}\{t \cdot \cos at\}(z) &= \int_0^{+\infty} t \cdot \frac{1}{2}\left\{e^{iat} + e^{-iat}\right\} \cdot e^{-zt}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{+\infty} t \cdot e^{-(z-ia)t}\,\mathrm{d}t + \frac{1}{2}\int_0^{+\infty} t \cdot e^{-(z+ia)t}\,\mathrm{d}t,
\end{aligned}$$

that the conditions of convergence are

$$\Re(z - ia) > 0 \qquad \text{and} \qquad \Re(z + ia) > 0,$$

hence

$$\Re z + \Im a > 0 \qquad \text{and} \qquad \Re z - \Im a > 0,$$

from which $\Re z > |\Im a|$, so $\sigma(f) = |\Im a|$. $\Diamond$

Example 1.2.20 *Given $f(t) = t \cdot \sin at$. Compute $\mathcal{L}\{f\}(z)$ and $\sigma(f)$.*

This is similar to Example 1.2.19, so we get immediately

$$\sigma(f) = \sigma(\sin at) = |\Im a|,$$

and then for $\Re z > |\Im a|$,

$$\mathcal{L}\{t \cdot \sin at\}(z) = -\frac{d}{dz}\left\{\frac{a}{z^2+a^2}\right\} = \frac{2az}{\left(z^2+a^2\right)^2}. \qquad \Diamond$$

Example 1.2.21 *Given $f(t) = 4t^2 - 3\cos 2t + 5e^{-t}$. Find $\mathcal{L}f(z)$ and $\sigma(f)$.*

It follows immediately that $\sigma(f) = \max\{0, 0, -1\} = 0$. If $\Re z > 0$, then by the linearity,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= 4\,\mathcal{L}\left\{t^2\right\}(z) - 3\,\mathcal{L}\{\cos 2t\}(z) + 5\,\mathcal{L}\left\{e^{-t}\right\}(z)\\ &= 4 \cdot \frac{2!}{z^3} - 3 \cdot \frac{z}{z^2+4} + 5 \cdot \frac{1}{z+1} = \frac{8}{z^3} + \frac{5}{z+1} - \frac{3z}{z^2+4}. \qquad \Diamond\end{aligned}$$

Example 1.2.22 *Compute the Laplace transforms of*

1) $t^2\,\cos^2 t$,

2) $\left(t^2 - 3t + 2\right)\sin 3t$.

1) By a simple computation,

$$\begin{aligned}\mathcal{L}\left\{t^2\,\cos^2 t\right\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{t^2(\cos 2t + 1)\right\}(z) = \frac{1}{2}\,\frac{d^2}{dz^2}\left\{\frac{z}{z^2+4}\right\} + \frac{1}{2}\cdot\frac{2}{z^3}\\ &= \frac{1}{2}\,\frac{d}{dz}\left\{\frac{z^2+4-2z^2}{\left(z^2+4\right)^2}\right\} + \frac{1}{z^3} = \frac{1}{2}\,\frac{d}{dz}\left\{\frac{4-z^2}{\left(z^2+4\right)^2}\right\} + \frac{1}{z^3}\\ &= \frac{1}{2}\cdot\frac{-2z}{\left(z^2+4\right)^2} + \frac{1}{2}\cdot\frac{\left(4-z^2\right)\cdot 2\cdot 2z}{\left(z^2+4\right)^3} + \frac{1}{z^3} = -\frac{z}{\left(z^2+4\right)^2} + \frac{2z\left(4-z^2\right)}{\left(z^2+4\right)^3} + \frac{1}{z^3}\\ &= \frac{z}{\left(z^2+4\right)^3}\left\{-z^2 - 4 + 8 - 2z^2\right\} + \frac{1}{z^3} = \frac{z\left(4-3z^2\right)}{\left(z^2+4\right)^3} + \frac{1}{z^3}.\end{aligned}$$

2) We get analogously,

$$\begin{aligned}
\mathcal{L}\left\{(t^2-3t+2\right\}(z) &= \frac{d^2}{dz^2}\left\{\frac{3}{z^2+9}\right\}+3\,\frac{d}{dz}\left\{\frac{3}{z^2+9}\right\}+2\cdot\frac{3}{z^2+9}\\
&= 3\,\frac{d}{dz}\left\{\frac{-2z}{(z^2+9)^2}\right\}+9\cdot\frac{-2z}{(z^2+9)^2}+\frac{6}{z^2+9}\\
&= -\frac{6}{(z^2+9)^2}+6z\cdot\frac{2\cdot 2z}{(z^2+9)^3}-\frac{18z}{(z^2+9)^2}+\frac{6}{z^2+9}\\
&= \frac{6}{(z^2+9)^3}\left\{-z^2-9+4z^2-3z^3-27z+z^4+18z^2+81\right\}\\
&= \frac{6\left(z^4-3z^3+21z^2-27z+72\right)}{(z^2+9)^3}. \qquad \Diamond
\end{aligned}$$

Example 1.2.23 *Use the rule of periodicity to compute* $\mathcal{L}\{\sin t\}(z)$ *and* $\mathcal{L}\{\cos t\}(z)$.

Both $\sin t$ and $\cos t$ have the period 2π, hence by the rule of periodicity for $\Re\, z>0$,

$$\begin{aligned}
\mathcal{L}\{\sin t\}(z) &= \frac{1}{1-e^{-2\pi z}}\int_0^{2\pi} e^{-zt}\,\sin t\,\mathrm{d}t = \frac{1}{1-e^{-2\pi z}}\cdot\frac{1}{2i}\int_0^{2\pi}\left(e^{it}-e^{-it}\right)e^{-zt}\,\mathrm{d}t\\
&= \frac{1}{1-e^{-2\pi z}}\cdot\frac{1}{2i}\int_0^{2\pi}\left\{e^{(i-z)t}-e^{-(i+z)t}\right\}\mathrm{d}t\\
&= \frac{1}{1-e^{-2\pi z}}\cdot\frac{1}{2i}\left[\frac{1}{i-z}\,e^{(i-z)t}+\frac{1}{i+z}\,e^{-(i+z)t}\right]_{t=0}^{2\pi}\\
&= \frac{1}{1-e^{-2\pi z}}\cdot\frac{1}{2i}\left\{\frac{1}{i-z}\,\left(e^{-2\pi z}-1\right)+\frac{1}{i+z}\,\left(e^{-2\pi z}-1\right)\right\}\\
&= -\frac{1}{2i}\left\{\frac{1}{i-z}+\frac{1}{i+z}\right\} = -\frac{1}{2i}\cdot\frac{i+z+i-z}{i^2-z^2} = \frac{1}{z^2+1},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\{\cos t\}(z) &= \frac{1}{1-e^{-2\pi z}} \int_0^{2\pi} e^{-zt} \cos t \, \mathrm{d}t = \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2} \int_0^{2\pi} \left(e^{it} + e^{-it}\right) e^{-zt} \, \mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2} \int_0^{2\pi} \left\{ e^{(i-z)t} + e^{-(i+z)t} \right\} \mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2} \left[\frac{1}{i-z}\, e^{(i-z)t} - \frac{1}{i+z}\, e^{-(i+z)t} \right]_{t=0}^{2\pi} \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2} \left\{ \frac{1}{i-z} \left(e^{-2\pi z} - 1\right) - \frac{1}{i+z} \left(e^{-2\pi z} - 1\right) \right\} \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2} \left(e^{-2\pi z} - 1\right) \left\{ \frac{1}{i-z} - \frac{1}{i+z} \right\} \\
&= -\frac{1}{2} \cdot \frac{i+z-i+z}{i^2-z^2} = \frac{1}{2} \cdot \frac{2z}{z^2+1} = \frac{z}{z^2+1}.
\end{aligned}$$

In both cases we get the well-known results. Notice that the denominator $1 - e^{-2\pi z} \neq 0$ for $\Re\, z > 0$. $\Diamond$

Example 1.2.24 *Given the function $f(t) = (-1)^{[t]}$ for $t \in [0, +\infty[$, where $[t]$ denotes the entire part of $t \in \mathbb{R}$, i.e. the largest number $n \in \mathbb{N}_0$, for which $[t] = n \leq t$.*
Compute $\mathcal{L}\{f\}(z)$.

It follows immediately that

$$f(t) = (-1)^n \qquad \text{for } t \in [n, n+1[, \qquad n \in \mathbb{N}_0,$$

so $f(t)$ is periodic of period 2.

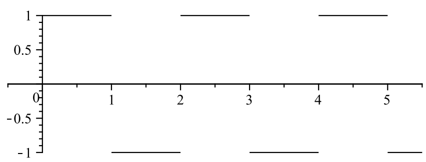

Figure 4: The graph of the function $f(t) = (-1)^{[t]}$ of Example 1.2.24.

We get by using the rule of periodicity,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{1-e^{-2z}}\left\{\int_0^1 e^{-zt}\,\mathrm{d}t - \int_1^2 e^{-zt}\,\mathrm{d}t\right\} \\
&= \frac{1}{1-e^{-2z}}\left\{\left[-\frac{1}{z}\,e^{-zt}\right]_{t=0}^{1} - \left[-\frac{1}{z}\,e^{-zt}\right]_{t=1}^{2}\right\} \\
&= \frac{1}{1-e^{-2z}}\left\{\frac{1}{z} - \frac{e^{-z}}{z} - \frac{e^{-z}}{z} + \frac{e^{-2z}}{z}\right\} \\
&= \frac{1}{z}\cdot\frac{\left(1-e^{-z}\right)^2}{\left(1-e^{-z}\right)\left(1+e^{-z}\right)} = \frac{1}{z}\cdot\frac{1-e^{-z}}{1+e^{-z}}. \qquad \Diamond
\end{aligned}$$

Example 1.2.25 *Given the function $f(t) = t - [t]$ for $t \in [0, +\infty[$, where the entire part $[t]$ of $t \in \mathbb{R}$ is defined in Example 1.2.24, i.e. $[t]$ is the largest $n\mathbb{N}_0$, for which $[t] = n \leq t$. Compute its Laplace transform $\mathcal{L}\{f\}(z)$.*

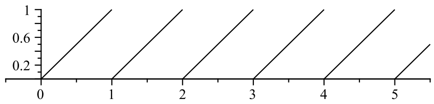

Figure 5: The graph of the sawtooth function $f(t) = t - [t]$ of Example 1.2.25.

The function is a sawtooth function, cf. Figure 5. It is periodic of period 1, hence by the rule of periodicity,

$$\begin{aligned}
\mathcal{L}\{f\}(x) &= \frac{1}{1-e^{-z}} \int_0^1 t\, e^{-zt}\, dt = \frac{1}{1-e^{-z}} \left\{ \left[-\frac{1}{z}\, t\, e^{-zt} \right]_{t_0}^1 + \frac{1}{z} \int_0^1 e^{-zt}\, dt \right\} \\
&= \frac{1}{1-e^{-z}} \left\{ -\frac{1}{z}\, e^{-z} + \left[-\frac{1}{z^2}\, e^{-zt} \right]_{t=0}^1 \right\} = \frac{1}{1-e^{-z}} \left\{ -\frac{1}{z}\, e^{-z} + \frac{1}{z^2} - \frac{1}{z^2}\, e^{-z} \right\} \\
&= \frac{1}{z^2} - \frac{z\, e^{-z}}{1-e^{-z}}. \qquad \Diamond
\end{aligned}$$

Example 1.2.26 *Given $f(t) = [t]$, where the entire part $[t]$ is defined in Example 1.2.24 and Example 1.2.25.*

Clearly, $f(t) = n$ for $t \in [n, n+1[$ and $n \in \mathbb{N}$. Hence, for $\Re\, z > 0$,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \int_0^{+\infty} f(t)\, e^{-zt}\, \mathrm{d}t = \sum_{n=0}^{+\infty} \int_n^{n+1} n\, e^{-zt}\, \mathrm{d}t = \sum_{n=0}^{+\infty} n \left[-\frac{1}{z}\, e^{-zt} \right]_{t=0}^{n+1} \\
&= \frac{1}{z} \sum_{n=0}^{+\infty} n \left\{ e^{-nz} - e^{-(n+1)z} \right\} = \frac{1-e^{-z}}{z} \sum_{n=1}^{+\infty} n\, e^{-nz} \\
&= \frac{1-e^{-z}}{z} \sum_{n=1}^{+\infty} n \left(e^{-z} \right)^n .
\end{aligned}$$

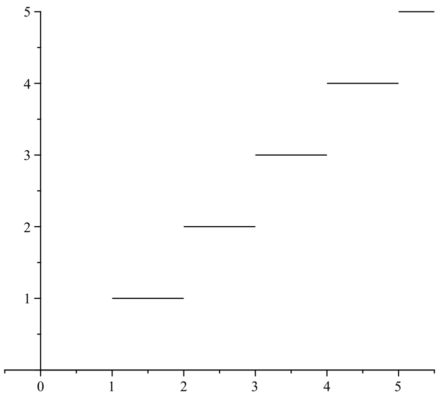

Figure 6: The graph of the entire part $f(t) = [t]$ of Example 1.2.26.

When we differentiate

$$\frac{1}{1-w} = \sum_{n=0}^{+\infty} w^n \qquad \text{for } |w| < 1,$$

and multiply the result by z we get

$$\frac{w}{(1-w)^2} = \sum_{n=1}^{+\infty} n\, w^n \qquad \text{for } |w| < 1.$$

Substituting $w = e^{-z}$ for $|e^{-z}| < 1$, i.e. for $\Re\, z > 0$, we finally get

$$\mathcal{L}\{f\}(z) = \frac{1-e^{-z}}{z} \cdot \frac{e^{-z}}{\left(1-e^{-z}\right)^2} = \frac{e^{-z}}{z\,(1-e^{-z})} = \frac{1}{z\,(e^z-1)}. \qquad \Diamond$$

Example 1.2.27 *Let* $[t]$ *denote the entire part of* $t \in \mathbb{R}$, *defined in Example 1.2.24. Compute* $\mathcal{L}\left\{[t]^2\right\}(z)$.

We first notice that

$$f(t) = [t]^2 = n^2 \qquad \text{for } t \in [n, n+1[, \quad n \in \mathbb{N}_0.$$

Then it follows for $\Re\, z > 0$ from the definition

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \int_0^{+\infty} f(t)\, e^{-zt}\, \mathrm{d}t = \sum_{n=0}^{+\infty} \int_n^{n+1} n^2\, e^{-zt}\, \mathrm{d}t = \sum_{n=0}^{+\infty} n^2 \left[-\frac{1}{z}\, e^{-zt}\right]_n^{n+1} \\ &= \frac{1}{z} \sum_{n=0}^{+\infty} n^2 \left(e^{-nz} - e^{-z}\, e^{-nz}\right) = \frac{1-e^{-z}}{z} \sum_{n=0}^{+\infty} n^2 \left(e^{-z}\right)^n,\end{aligned}$$

which is convergent, because the assumption $\Re\, z > 0$ implies that $|e^{-z}| < 1$.

Then we shall find the sum function of the series. If we put

$$\varphi(w)=\frac{1}{1-w}=\sum_{n=0}^{+\infty} w^n \qquad \text{for } |w|<1,$$

we get

$$w\,\varphi'(w)=\frac{w}{(1-w)^2}=\sum_{n=0}^{+\infty} n\,w^n, \qquad \text{for } |w|<1,$$

hence

$$w\,\frac{d}{dw}\left\{\frac{w}{(1-w)^2}\right\}=\sum_{n=0}^{+\infty} n^2\,w^n = w\left\{\frac{1}{(1-w)^2}+\frac{2w}{(1-w)^3}\right\}=\frac{w(1+w)}{(1-w)^3}.$$

Then by choosing $w=e^{-z}$, $\Re\, z>0$,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{1-e^{-z}}{z}\sum_{n=0}^{+\infty} n^2\left(e^{-z}\right)^n=\frac{1-e^{-z}}{z}\cdot\frac{e^{-z}\left(1+e^{-z}\right)}{\left(1-e^{-z}\right)^3}\\ &= \frac{e^{-z}}{z}\cdot\frac{1+e^{-z}}{\left(1-e^{-z}\right)^2}=\frac{e^z+1}{z\left(e^z-1\right)^2}. \qquad \lozenge\end{aligned}$$

Example 1.2.28 *Compute the Laplace transform of* $\sin^3 t$.

First by Euler's formulæ,

$$\begin{aligned}\sin^3 t &= \left\{\frac{e^{it}-e^{-it}}{2i}\right\}^3=\frac{1}{-8i}\left\{e^{3it}-3e^{it}+3e^{-it}-e^{-3it}\right\}\\ &= -\frac{1}{8i}\left\{2i\,\sin 3t-6i\,\sin t\right\}=-\frac{1}{4}\,\sin 3t+\frac{3}{4}\,\sin t.\end{aligned}$$

Using this result we get for $\Re\, z>0$,

$$\mathcal{L}\left\{\sin^3 t\right\}(z)=-\frac{1}{4}\cdot\frac{3}{z^2+9}+\frac{2}{3}\cdot\frac{1}{z^2+1}=\frac{3}{4}\cdot\frac{8}{(z^2+1)\,(z^2+9)}=\frac{6}{(z^2+1)\,(z^2+9)}. \qquad \lozenge$$

Example 1.2.29 *Let $f(t)$ be a periodic function of period 2 given by $f(t+2) = f(t)$ for $t \geq 0$, where*

$$f(t) = \begin{cases} t & \text{for } t \in [0,1[, \\ 2-t & \text{for } t \in [1,2[. \end{cases}$$

Sketch the graph of $f(t)$ and then compute the Laplace transform $\mathcal{L}\{f\}(z)$.

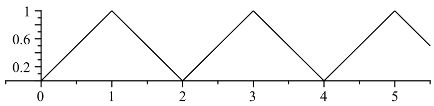

Figure 7: The graph of the the function $f(t)$ of Example 1.2.29.

Using the rule of periodicity, period 2, we get for $\Re z > 0$,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{1-e^{-2z}} \int_0^2 e^{-zt} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2z}} \left\{ \int_0^1 t\, e^{-zt}\,\mathrm{d}t + \int_1^2 (2-t) e^{-zt}\,\mathrm{d}t \right\} \\
&= \frac{1}{1-e^{-2z}} \left\{ \left[-\frac{t}{z}\, e^{-zt} \right]_{y=0}^1 \frac{1}{z} \int_0^1 e^{-zt}\,\mathrm{d}t + \left[\frac{2-t}{-z}\, e^{-zt} \right]_{t=1}^2 - \frac{1}{z} \int_1^2 e^{-zt}\,\mathrm{d}t \right\} \\
&= \frac{1}{1-e^{-2z}} \left\{ -\frac{1}{z}\, e^{-z} - \frac{1}{z^2} \left(e^{-z} - 1 \right) + \frac{1}{z}\, e^{-z} + \frac{1}{z^2} \left(e^{-2z} - e^{-z} \right) \right\} \\
&= \frac{1}{1-e^{-2z}} \cdot \frac{1}{z^2} \left(1 - 2e^{-z} + e^{-2z} \right) = \frac{\left(1-e^{-z}\right)^2}{z^2 \left(1-e^{-z}\right)\left(1+e^{-z}\right)} = \frac{1-e^{-z}}{z^2 \left(1+e^{-z}\right)}. \qquad \Diamond
\end{aligned}$$

Example 1.2.30 *Compute the Laplace transform of the function*

$$f(t) = \begin{cases} \cos t, & \textit{for } t \in [0, \pi[, \\ \sin t, & \textit{for } t \in [\pi, +\infty[. \end{cases}$$

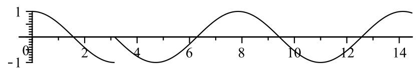

Figure 8: The graph of the the function $f(t)$ of Example 1.2.30.

We get for $\Re z > 0$,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \int_0^{\pi} \cos t \cdot e^{-zt}\,\mathrm{d}t + \int_{\pi}^{+\infty} \sin t \cdot e^{-zt}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{\pi} e^{-(z-i)t}\,\mathrm{d}t + \frac{1}{2}\int_0^{\pi} e^{-(z+i)t}\,\mathrm{d}t + \int_0^{+\infty} \sin(t+\pi)\, e^{-z(t+\pi)}\,\mathrm{d}t \\
&= \frac{1}{2}\left[-\frac{e^{-(z.i)t}}{z-i}\right]_{t=0}^{\pi} + \frac{1}{2}\left[-\frac{e^{-(z+i)t}}{z+i}\right]_{t=0}^{\pi} - e^{-\pi z}\,\mathcal{L}\{\sin t\}(z) \\
&= \frac{1}{2}\left\{-\frac{-e^{-\pi z}}{z-i} + \frac{1}{z-i}\right\} + \frac{1}{2}\left\{-\frac{e^{-\pi z}}{z+i} + \frac{1}{z+i}\right\} - \frac{e^{-\pi z}}{z^2+1} \\
&= \frac{z}{z^2+1} + \frac{1}{2}\cdot\frac{1}{z^1}\left\{(z+i)e^{-\pi z} + (z-i)e^{-\pi z}\right\} - \frac{e^{-\pi z}}{z^2+1} \\
&= \frac{z}{z^2+1} + e^{-\pi z}\cdot\frac{z-1}{z^2+1}. \qquad \Diamond
\end{aligned}$$

Example 1.2.31 *Prove for* $f(t) = e^{-at}\cos bt$ *that*

$$\mathcal{L}\{f\}(z) = \frac{z+a}{(z+a)^2+b^2}.$$

Then find $\sigma(f)$*.*

It follows from the rules of computation that

$$\mathcal{L}\{f\}(z) = \mathcal{L}\left\{e^{-at}\cos bt\right\}(z) = \mathcal{L}\{\cos bt\}(z+a) = \frac{z+a}{(z+a)^2+b^2}.$$

The poles of the fractional function $\dfrac{z+a}{(z+a)^2+b^2}$ are $z = -a \pm ib$, hence

$$\sigma(f) = \max\{\Re(-a+ib), \Re(-a-ib)\} = -\Re\, a + |\Im\, b|. \qquad \Diamond$$

Example 1.2.32 *Compute the Laplace transform of the function*

$$f(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & \text{for } t > \dfrac{2\pi}{3}, \\[2mm] 0, & \text{for } t \le \dfrac{2\pi}{3}. \end{cases}$$

It follows from the shifting rule that

$$\mathcal{L}\{f\}(z) = \exp\left(-\frac{2\pi}{3}\,z\right)\mathcal{L}\{\cos t\}(z) = \exp\left(-\frac{2\pi}{3}\,z\right)\cdot\frac{z}{z^2+1}. \qquad \Diamond$$

Example 1.2.33 *Compute the Laplace transforms of*

1) $e^{-t}\sin^2 t$,

2) $\left(1+t\,e^{-t}\right)^3$.

1) We get for $\Re z > -1$, by the rules of computation,

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\sin^2 t\right\}(z) &= \mathcal{L}\left\{\sin^2 t\right\}(z+1) = \frac{1}{2}\,\mathcal{L}\{1-\cos 2t\}(z+1) \\ &= \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{z+1}{(z+1)^2+4} = \frac{1}{2}\cdot\frac{(z+1)^2+4-(z+1)^2}{(z+1)\left\{(z+1)^2+4\right\}} \\ &= \frac{2}{(z+1)\left\{(z+1)^2+4\right\}}.\end{aligned}$$

2) We first compute

$$\left(1+t\,e^{-t}\right)^3 = 1+3t\,e^{-t}+3t^2\,e^{-2t}+t^3\,e^{-3t}.$$

Hence, by the rules of computation,

$$\mathcal{L}\left\{\left(1+t\,e^{-t}\right)^3\right\}(z) = \frac{1}{z}+\frac{3}{(z+1)^2}+\frac{6}{(z+2)^3}+\frac{6}{(z+3)^4}. \qquad \Diamond$$

Example 1.2.34 *Compute* $\int_0^{+\infty} t\,e^{-3t}\sin t\,dt$.

First notice that

$$\int_0^{+\infty} t\,e^{-zt}\sin t\,\mathrm{d}t = -\frac{d}{dz}\,\mathcal{L}\{\sin t\}(z) = -\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\} = \frac{2z}{\left(z^2+1\right)^2}.$$

Then choose $z=3$ in this formula to get

$$\int_0^{+\infty} t\,e^{-3t}\sin t\,\mathrm{d}t = \frac{6}{100} = \frac{3}{50}. \qquad \Diamond$$

Example 1.2.35 *Compute the improper integral* $\int_0^{+\infty} t^3\,e^{-t}\sin t\,dt$.

We get, using Euler's formulæ,

$$\begin{aligned}\int_0^{+\infty} t^3\,e^{-t}\sin t\,\mathrm{d}t &= \frac{1}{2i}\int_0^{+\infty} t^3\,e^{-(1-i)t}\,\mathrm{d}t - \frac{1}{2i}\int_0^{+\infty} t^3\,e^{-(1+i)t}\,\mathrm{d}t \\ &= \frac{1}{2i}\,\mathcal{L}\left\{t^3\right\}(1-i) - \frac{1}{2i}\,\mathcal{L}\left\{t^3\right\}(1+i) = \frac{1}{2i}\left\{\frac{3!}{(1-i)^4}-\frac{3!}{(1+i)^4}\right\} \\ &= \frac{6}{2i}\left\{\frac{1}{-4}-\frac{1}{-4}\right\} = 0.\end{aligned}$$

Alternatively, we get for $\Re z > 0$,

$$\begin{aligned}\int_0^{+\infty} t^3\, e^{-zt}\, \sin t\, \mathrm{d}t &= -\frac{d^3}{dz^3}\,\mathcal{L}\{\sin t\}(z) = -\frac{d^3}{dz^3}\left\{\frac{1}{z^2+1}\right\} = -\frac{d^2}{dz^2}\left\{\frac{-2z}{\left(z^2+1\right)^2}\right\} \\ &= \frac{d^2}{dz^2}\left\{\frac{2z}{\left(z^2+1\right)^2}\right\} = \frac{d}{dz}\left\{\frac{z}{\left(z^2+1\right)^2} - \frac{8z^2}{\left(z^2+1\right)^3}\right\} \\ &= -\frac{8z}{\left(z^2+1\right)^3} - \frac{16z}{\left(z^2+1\right)^3} + \frac{48z^3}{\left(z^2+1\right)^4}.\end{aligned}$$

Finally, we choose $z = 1$ to get

$$\int_0^{+\infty} t^3\, e^{-t}\, \sin t\, \mathrm{d}t = \frac{8}{2^3} - \frac{16}{2^3} + \frac{48}{2^4} = -3 + 3 = 0. \qquad \Diamond$$

Example 1.2.36 *Prove*

The rule of integration. Given a function $f \in \mathcal{F}$, which is also piecewise continuous. Then

$$g(t) := \int_0^t f(\tau)\,\mathrm{d}\tau \in \mathcal{E},$$

and

$$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\}(z) = \frac{1}{z}\,\mathcal{L}\{f\}(z) \qquad \textit{for } \Re z > \max\{0, \sigma(f)\}.$$

PROOF. Choose a $\sigma > \max\{0, \sigma(f)\}$, and then put

$$A = \int_0^{+\infty} e^{-\sigma t}\,|f(t)|\,\mathrm{d}t.$$

Then

$$|g(t)| = \left|\int_0^t f(\tau)\,\mathrm{d}\tau\right| \le \left|\int_0^t e^{\sigma\tau}\,e^{-\sigma\tau}\,|f(\tau)|\,\mathrm{d}\tau\right| \le e^{\sigma t}\int_0^{+\infty} e^{-\sigma t}\,|f(t)|\,\mathrm{d}t = A\,e^{\sigma t},$$

which shows that $g(t)$ is exponentially bounded.

Since f is piecewise continuous, we conclude that $g \in \mathcal{E}$, and

$$\varrho(g) \le \max\{\sigma(f), 0\}.$$

Furthermore, $g'(t) = f(t)$ with the exception of the points of discontinuity of $f(t)$. Finally, $g(0) = 0$.

When we apply the rule of differentiation on $g(t)$, we get for $\Re z > \max\{0, \sigma(f)\}$,

$$\mathcal{L}\{f\}(z) = \mathcal{L}\{g'\}(z) = z\,\mathcal{L}\{g\}(z) - g(0) = z\,\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\}(z),$$

and the rule of integration follows, when we divide this equation by z. □

Example 1.2.37 *Compute the improper integrals given below.*

1) $\int_0^{+\infty} t\,e^{-2t}\cos t\,\mathrm{d}t$. *Hint: Consider* $\mathcal{L}\{t \cdot \cos t\}(z)$.

2) $\int_0^{+\infty} \frac{1}{t}\left\{e^{-t} - e^{-3t}\right\}\mathrm{d}t$. *Hint: Consider* $\mathcal{L}\left\{e^{-t} - e^{-3t}\right\}(z)$.

1) Using the rule of multiplication by t on the hint we get for $\Re > 0$ that

$$\mathcal{L}\{t\,\cos t\}(z) = -\frac{d}{dz}\,\mathcal{L}\{\cos t\}(z) = -\frac{d}{dz}\left\{\frac{z}{z^2+1}\right\} = -\frac{z^2+1-2z^2}{\left(z^2+1\right)^2} = \frac{z^2-1}{\left(z^2+1\right)^2}.$$

Hence, for $z = 2$,

$$\int_0^{+\infty} t\,e^{-2t}\cos t\,\mathrm{d}t = \mathcal{L}\{t\,\cos t\}(2) = \frac{2^2-1}{\left(2^2+1\right)^2} = \frac{3}{25}.$$

Alternatively, one may of course compute the improper integral by elementary calculus. This is left as an exercise for the reader.

2) We shall again use the hint. Then we get for $\Re z > -1$,

$$\mathcal{L}\left\{e^{-t} - e^{-3t}\right\}(z) = \frac{1}{z+1} - \frac{1}{z+3}.$$

Then by *l'Hospital's rule*,

$$\lim_{t\to 0+} \frac{e^{-t} - e^{-3t}}{t} = \lim_{t\to 0+} \frac{d}{dt}\left\{e^{-t} - e^{-3t}\right\} = -1 + 3 = 2.$$

We therefore get by the rule of division by t for real $x > -1$,

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}(x) &= \int_x^{+\infty} \mathcal{L}\left\{e^{-t} - e^{-3t}\right\}(\tau)\,\mathrm{d}\tau = \int_x^{+\infty}\left\{\frac{1}{\tau+1} - \frac{1}{\tau+3}\right\}\mathrm{d}\tau \\ &= \left[\ln\left(\frac{\tau+1}{\tau+3}\right)\right]_x^{+\infty} = -\ln\left(\frac{x+1}{x+3}\right) = \ln\left(\frac{x+3}{x+1}\right).\end{aligned}$$

We get in particular for $x = 0 > -1$,

$$\int_0^{+\infty} \frac{e^{-t} - e^{-3t}}{t}\,\mathrm{d}t = \mathcal{L}\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}(0) = \ln\left(\frac{0+3}{0+1}\right) = \ln 3.$$

In this case it is not possible to compute the improper integral by only using elementary calculus. ◊

Example 1.2.38 *Compute* $\mathcal{L}\left\{\frac{1-\cos t}{t}\right\}(x)$ *for* $x \in \mathbb{R}_+$. *Does the function* $\frac{\cos t}{t}$ *belong to the class of functions* $\mathcal{F}$*?*

We get by using the rule of division by t,

$$\begin{aligned}\mathcal{L}\left\{\frac{1-\cos t}{t}\right\}(x) &= \int_x^{+\infty} \mathcal{L}\{1-\cos t\}(u)\,\mathrm{d}u = \int_x^{+\infty}\left\{\frac{1}{u} - \frac{u}{u^2+1}\right\}\mathrm{d}u \\ &= \left[\ln u - \frac{1}{2}\left(u^2+1\right)\right]_{u=x}^{+\infty} = \frac{1}{2}\left[\ln\left(\frac{u^2}{u^2+1}\right)\right]_{u=x}^{+\infty} \\ &= \frac{1}{2}\ln\left(\frac{x^2+1}{x^2}\right) = \frac{1}{2}\ln\left(1+\frac{1}{x^2}\right),\end{aligned}$$

because $\frac{1-\cos t}{t} \to 0$ for $t \to 0$, proving that the improper integral is convergent.

If $\frac{\cos t}{t}$ was a function from $\mathcal{F}$, then also $\frac{1}{t}$ would be a function in $\mathcal{F}$, because the difference $\frac{1-\cos t}{t}$ lies in $\mathcal{F}$. However, it is well-known that $\frac{1}{t}$ does not belong to $\mathcal{F}$, so we conclude that $\frac{\cos t}{t} \notin \mathcal{F}$. ◊

Example 1.2.39 *Prove that*

$$\mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\right\}(z)=\frac{1}{2}\operatorname{Log}\left(\frac{z^2+b^2}{z^2+a^2}\right),$$

where Log *denotes the principal logarithm.*

It follows from e.g. a table that

$$\mathcal{L}\{\cos at-\cos bt\}(z)=\frac{z}{z^2+a^2}-\frac{z}{z^2+b^2}\qquad\text{for } \Re z>\max\{|\Im a|,|\Im b|\}.$$

Then we note that

$$\lim_{t\to 0+}\frac{\cos at-\cos bt}{t}=0.$$

If $x>\max\{|\Im a|,|\Im b|\}$ is real, then it follows from the rule of division by t that

$$\begin{aligned}\mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\right\}(x)&=\int_x^{+\infty}\left\{\frac{t}{t^2+a^2}-\frac{t}{t^2+b^2}\right\}\mathrm{d}t\\&=\left[\frac{1}{2}\operatorname{Log}\left(t^2+a^2\right)-\frac{1}{2}\operatorname{Log}\left(t^2+b^2\right)\right]_x^{+\infty}=\frac{1}{2}\left\{\operatorname{Log}\left(t^2+a^2\right)-\operatorname{Log}\left(t^2+a^2\right)\right\}\\&=\frac{1}{2}\operatorname{Log}\left(\frac{x^2+b^2}{x^2+a^2}\right)\qquad\text{for } x>\max\{|\Im a|,|\Im b|\}.\end{aligned}$$

Here we have used that

$$\lim_{t\to+\infty}\left\{\operatorname{Log}\left(t^2+a^2\right)-\operatorname{Log}\left(t^2+b^2\right)\right\}=0,$$

and that both x^2+a^2 and x^2+b^2 lie in the right half plane, when $x>\max\{|\Im a|,|\Im b|\}$, so the principal arguments lie in the interval $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

The Laplace transform is an analytic function, so we get by an analytic extension that

$$(5)\quad \mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\right\}(z)=\frac{1}{2}\operatorname{Log}\left(\frac{z^2+b^2}{z^2+a^2}\right),$$

in the domain where the right hand side of (5) is defined, i.e. when

$$\frac{z^2+b^2}{z^2+a^2}\notin\mathbb{R}_-\cup\{0\},\qquad\text{and}\qquad z\neq\pm ia.$$

Now, if $\lambda>0$ is real and positive, then

$$\frac{z^2+b^2}{z^2+a^2}=-\lambda,\qquad\text{if and only if}\qquad(1+\lambda)z^2=\lambda\left(-a^2\right)+\left(-b^2\right),$$

i.e. for

$$z^2=\frac{\lambda}{1+\lambda}\left(-a^2\right)+\frac{1}{1+\lambda}\left(-b^2\right),\qquad\text{for } \lambda\in\mathbb{R}_+.$$

The right hand side is the parametric description of the line segment between $-a^2$ and $-b^2$, exclusive $-a^2$, which we however may add, because $z = \pm ia$ are also exceptional points. We therefore conclude that the formula above holds in the set

$$\left\{ z \in \mathbb{C} \mid z^2 \notin \left[-a^2, -b^2\right] \right\},$$

where $\left[-a^2, -b^2\right]$ is a short hand for the line segment in the complex plane given by

$$\left\{ w = \mu \left(-a^2\right) + (1-\mu)\left(-b^2\right) \mid \mu \in [0,1] \right\}. \qquad \Diamond$$

Example 1.2.40 *Compute the improper integral*

$$\int_0^{+\infty} \frac{\cos 6t - \cos 4t}{t}\,\mathrm{d}t.$$

First we note that

$$\lim_{t\to 0+} \frac{\cos 6t - \cos 4t}{t} = 0.$$

Then it follows from *Ventus, Complex Functions Theory a-2* that the improper integral is convergent, and that its value can be found by some convenient choice of the path of integration.

Then we apply the result of Exercise 1.2.39 with $a = 6$ and $b = 4$, followed by the limit $x \to 0+$, to get

$$\begin{aligned}\int_0^{+\infty} \frac{\cos 6t - \cos 4t}{t}\,\mathrm{d}t &= \lim_{x\to 0+} \mathcal{L}\left\{\frac{\cos 6t - \cos 4t}{t}\right\}(x) \\ &= \lim_{x\to 0+} \frac{1}{2}\,\mathrm{Log}\left(\frac{x^2+4^2}{x^2+6^2}\right) \frac{1}{2}\,\ln\left(\frac{4^2}{6^2}\right) = \ln\frac{4}{6} = \ln\frac{2}{3}. \qquad \Diamond\end{aligned}$$

Example 1.2.41 *Find*

$$\mathcal{L}\left\{\frac{e^t-1}{t}\right\}(x) \qquad \textit{for real } x > 1.$$

Since $\lim_{t\to 0+} \dfrac{e^t-1}{t} = 1$, we can apply the rule of division by t. We first notice that

$$\mathcal{L}\left\{e^t - 1\right\}(x) = \frac{1}{x-1} - \frac{1}{x} \qquad \text{for } x > 1.$$

Hence,

$$\begin{aligned}\int_0^{+\infty} \frac{e^t-1}{t}\,e^{-xt}\,\mathrm{d}t &= \mathcal{L}\left\{\frac{e^t-1}{t}\right\}(x) = \int_x^{+\infty} \left\{\frac{1}{\xi-1} - \frac{1}{\xi}\right\}\mathrm{d}\xi \\ &= \left[\ln\left(\frac{\xi-1}{\xi}\right)\right]_x^{+\infty} = \ln\left(\frac{x}{x-1}\right) = \ln\left(1 + \frac{1}{x-1}\right).\end{aligned}$$

In general, $\dfrac{1}{\zeta}$ lies in the right half plane, when ζ lies in the right half plane. Hence, by an analytic extension of the result above,

$$\mathcal{L}\left\{\frac{e^t-1}{t}\right\}(z) = \mathrm{Log}\left(\frac{z}{z-1}\right) \qquad \text{for } \Re\, z > 1. \qquad \Diamond$$

Example 1.2.42 *Prove that*

$$\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}(z)=\operatorname{Log}\left(\frac{z+b}{z+a}\right),$$

and then find the value of $\int_0^{+\infty}\frac{1}{t}\left\{e^{-3t}-e^{-6t}\right\}\,\mathrm{d}t$.

Clearly,

$$\lim_{t\to 0+}\frac{e^{-at}-e^{-bt}}{t}=\lim_{t\to 0+}\frac{(1-at+\cdots)-(1-bt+\cdots)}{t}=b-a$$

exists, and since it follows from e.g. a table that

$$\mathcal{L}\left\{e^{-at}-e^{-bt}\right\}(z)=\frac{1}{z+a}-\frac{1}{z+b},$$

we get from the rule of division by t that if the real $x>\max\{\Re(-a),\Re(-b)\}$, then

$$\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}(x)=\int_x^{+\infty}\left\{\frac{1}{t+a}-\frac{1}{t+b}\right\}\mathrm{d}t=[\operatorname{Log}(t+a)-\operatorname{Log}(t+b)]_x^{+\infty}=\operatorname{Log}\left(\frac{x+b}{x+b}\right).$$

Hence, by an analytic extension,

$$\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}(z)=\operatorname{Log}\left(\frac{z+b}{z+a}\right),$$

which is true if $z\neq -a$ and $z\neq -b$ and $\dfrac{z+b}{z+a}\notin\mathbb{R}_-$. This means that z must not lie on the line segment in $\mathbb{C}$ between the points $-a$ and $-b$.

When $a=3$ and $b=6$, then we get for $z=0>\max\{-3,-6\}=-3$, that

$$\int_0^{+\infty}\frac{e^{-3t}-e^{-6t}}{t}\,\mathrm{d}t=\mathcal{L}\left\{\frac{e^{-3t}-e^{-6t}}{t}\right\}(0)=\operatorname{Log}\left(\frac{0+6}{0+3}\right)=\ln 2.\qquad\Diamond$$

Example 1.2.43 *Compute*

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\}(x)\qquad\textit{and}\qquad\mathcal{L}\left\{\frac{\cosh t-1}{t}\right\}(x)$$

for real $x>1$.

We apply the rule of division by t to get

$$\begin{aligned}\mathcal{L}\left\{\frac{\sinh t}{t}\right\}(x)&=\int_x^{+\infty}\mathcal{L}\{\sinh t\}(\xi)\,\mathrm{d}\xi=\int_x^{+\infty}\frac{\mathrm{d}\xi}{\xi^2-1}=\frac{1}{2}\int_x^{+\infty}\left\{\frac{1}{\xi-1}-\frac{1}{\xi+1}\right\}\mathrm{d}\xi\\&=\frac{1}{2}\left[\ln\left(\frac{\xi-1}{\xi+1}\right)\right]_x^{+\infty}=N\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\left\{\frac{\cosh t-1}{t}\right\}(x) &= \int_x^{+\infty}\mathcal{L}\{\cosh t-1\}(\xi)\,\mathrm{d}\xi = \int_x^{+\infty}\left\{\frac{\xi}{\xi^2-1}-\frac{1}{\xi}\right\}\mathrm{d}\xi \\ &= \frac{1}{2}\left[\ln\left(\xi^2-1\right)-2\,\ln\xi\right]_x^{+\infty} = \frac{1}{2}\,\ln\left(\frac{x^2}{x^2-1}\right). \qquad \Diamond\end{aligned}$$

Example 1.2.44 *Compute*

$$\int_0^{+\infty}\frac{e^{-t}\,\sin t}{t}\,\mathrm{d}t.$$

We first list the well-known results

$$\mathcal{L}\{\sin t\}(z) = \frac{1}{z^2+1} \qquad \text{and} \qquad \lim_{t\to 0+}\frac{\sin t}{t} = 1.$$

Then we get by the rule of division by t for real $x > 0$ that

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\}(x) = \int_0^{+\infty}\frac{\sin t}{t}\,e^{-xt}\,\mathrm{d}t = \int_x^{+\infty}\frac{\mathrm{d}\tau}{1+\tau^2} = \operatorname{Arccot} x.$$

Finally, we choose $x = 1$ to get

$$\int_0^{+\infty}\frac{e^{-t}\,\sin t}{t}\,\mathrm{d}t = \operatorname{Arccot}\,1 = \frac{\pi}{4}. \qquad \Diamond$$

Example 1.2.45 *Prove that*

$$\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}(z) = \frac{\pi}{2}+\frac{z}{2}\operatorname{Log}\left(\frac{z^2}{z^2+1}\right)-\operatorname{Arctan}\,z.$$

Then explain why the improper integral

$$\int_0^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t$$

is convergent, and find its value.

From

$$\mathcal{L}\{1-\cos t\}(z) = \frac{1}{z}-\frac{z}{z^2+1} \qquad \text{for } \Re\,z > 0,$$

follows (also for $\Re\,z > 0$) by using the rule of division by t applied twice that

$$\mathcal{L}\left\{\frac{1-\cos t}{t}\right\}(z) = \int_z^{+\infty}\left\{\frac{1}{\zeta}-\frac{\zeta}{\zeta^2+1}\right\}\mathrm{d}\zeta = \frac{1}{2}\operatorname{Log}\left(1+z^2\right)-\operatorname{Log}\,z,$$

and

$$\begin{aligned}\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}(z) &= \int_z^{+\infty}\left\{\frac{1}{2}\operatorname{Log}\left(1+\zeta^2\right)-\operatorname{Log}\zeta\right\}\,\mathrm{d}\zeta \\ &= \left[\frac{1}{2}\,\zeta\operatorname{Log}\left(1+\zeta^2\right)-\zeta\operatorname{Log}\right]_z^{+\infty}-\int_z^{+\infty}\left\{\frac{1}{2}\,\zeta\cdot\frac{2\zeta}{1+\zeta^2}-\frac{\zeta}{\zeta}\right\}\,\mathrm{d}\zeta \\ &= -\frac{z}{2}\operatorname{Log}\left(\frac{z^2}{1+z^2}\right)-\int_z^{+\infty}\left\{-\frac{1}{1+\zeta^2}\right\}\,\mathrm{d}\zeta \\ &= \frac{z}{2}\operatorname{Log}\left(\frac{z^2}{1+z^2}\right)+\frac{\pi}{2}-\operatorname{Arctan} z,\end{aligned}$$

where we have applied that

$$\frac{1-\cos t}{t^2}=\frac{1}{2}+\cdots\to\frac{1}{2}\qquad\text{for } t\to 0.$$

The result follows by combining these two equations.

It follows in particular that the improper integral $\int_0^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t$ is convergent, and its value is given by

$$\int_0^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t=\lim_{x\to 0+}\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}(x)=\frac{\pi}{2}.\qquad\Diamond$$

Example 1.2.46 *Compute the improper integral*

$$\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\cdot\sinh t\cdot\sin t}{t}\,\mathrm{d}t.$$

Using the definition $\sinh t = \frac{1}{2}\left(e^t - e^{-t}\right)$ we get

$$\begin{aligned}\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\cdot\sinh t\cdot\sin t}{t}\,dt &= \frac{1}{2}\int_0^{+\infty} e^{-(\sqrt{2}-1)t}\cdot\frac{\sin t}{t}\,\mathrm{d}t - \frac{1}{2}\int_0^{+\infty} e^{-(\sqrt{2}-1)t}\cdot\frac{\sin t}{t}\,\mathrm{d}t\\ &= \frac{1}{2}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}\left(\sqrt{2}-1\right) - \frac{1}{2}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}\left(\sqrt{2}+1\right).\end{aligned}$$

Then we apply the rule of division by t for $x>0$ to find

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\}(x) = \int_x^{+\infty}\mathcal{L}\{\sin t\}(\xi)\,\mathrm{d}\xi = \int_x^{+\infty}\frac{\mathrm{d}\xi}{1+\xi^2} = [\text{Arctan }\xi]_x^{+\infty} = \frac{\pi}{2} - \text{Arctan } x.$$

Since $\sqrt{2}-1>0$ and $\sqrt{2}+1>0$, these points lie in the given domain, so we get by insertion,

$$\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\cdot\sinh t\cdot\sin t}{t}\,\mathrm{d}t = \frac{1}{2}\left\{\text{Arctan}\left(\sqrt{2}+1\right) - \text{Arctan}\left(\sqrt{2}-1\right)\right\}$$

$$= \frac{1}{2}\,\text{Arctan}\left(\tan\left(\text{Arctan}\left(\sqrt{2}+1\right) - \text{Arctan}\left(\sqrt{2}-1\right)\right)\right)$$

$$= \frac{1}{2}\,\text{Arctan}\left(\frac{\left(\sqrt{2}+1\right)-\left(\sqrt{2}-1\right)}{1+\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}\right) = \frac{1}{2}\,\text{Arctan } 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8},$$

and we have proved that

$$\int_0^{+\infty} \frac{e^{-\sqrt{2}\,t}\cdot\sinh t\cdot\sin t}{t}\,\mathrm{d}t = \frac{\pi}{8}. \qquad \Diamond$$

Example 1.2.47 *Prove that*

$$\mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,\mathrm{d}u\right\}(z) = \frac{1}{z}\,\text{Log}\left(1+\frac{1}{z}\right).$$

Since $\lim_{t\to 0+}\frac{1-e^{-u}}{u} = 1$ exists and is finite, we conclude that $\frac{1-e^{-t}}{t}\in\mathcal{E}$, so we get from the rule of integration and the rule of divisions by t, that if $\Re\, z>0$, then

$$\begin{aligned}\mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,\mathrm{d}u\right\}(z) &= \frac{1}{z}\,\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}(z) = \frac{1}{z}\int_{\Gamma_z}\mathcal{L}\left\{1-e^{-t}\right\}(z)\,\mathrm{d}z\\ &= \frac{1}{z}\int_{\Gamma_z}\left\{\frac{1}{z}-\frac{1}{z+1}\right\}\mathrm{d}z = \frac{1}{z}\left\{-\text{Log}\left(\frac{z}{z+1}\right)\right\} = \frac{1}{z}\,\text{Log}\left(1+\frac{1}{z}\right). \qquad \Diamond\end{aligned}$$

Example 1.2.48 *Prove that*

$$\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right\}\mathrm{d}t = \frac{\pi}{4}.$$

This is an exercise in the definition of the Laplace transformation in the rule of integration, and in the rule of division by t, and the shifting property:

$$\int_0^{+\infty} e^{-t}\left\{\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right\}\,\mathrm{d}t = \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right\}(1) = \frac{1}{1}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}(1) = \int_1^{+\infty} \mathcal{L}\{\sin t\}(x)\,\mathrm{d}x$$

$$= \int_1^{+\infty} \frac{\mathrm{d}x}{x^2+1} = [\text{Arctan }x]_1^{+\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \qquad \Diamond$$

Example 1.2.49 *Assume that $g \in \mathcal{E}$, and define*

$$f(t) := \begin{cases} t\,g(t) & \text{for } t \in [1,+\infty[, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\mathcal{L}\{f(t)\}(z) = -\frac{d}{dz}\left\{e^{-z}\,\mathcal{L}\{g(t+1)\}(z)\right\}.$$

We get by combining the rule of differentiation and the rule of delay or shifting property, that

$$\mathcal{L}\{f(t)\}(z) = -\frac{d}{dz}\,\mathcal{L}\left\{g_1(t)\right\}(z) = -\frac{d}{dz}\left\{e^{-z}\,\mathcal{L}\{g(t+1)\}(z)\right\}. \qquad \Diamond$$

Example 1.2.50 *Given that*

$$\mathcal{L}\left\{f''(t)\right\}(z) = \text{Arctan }\frac{1}{z}, \qquad f(0)=2 \quad \textit{and} \quad f'(0)=-1.$$

Find the Laplace transform $\mathcal{L}\{f\}(z)$.

It follows from the rule of differentiation that

$$\text{Arctan }\frac{1}{z} = \mathcal{L}\left\{f''(t)\right\}(z) = z^2\,\mathcal{L}\{f\}(z) - z\cdot f(0) - f'(0) = z^2\,\mathcal{L}\{f\}(z) - 2z + 1,$$

hence by a rearrangement,

$$\mathcal{L}\{f\}(z) = \frac{1}{z^2}\,\text{Arctan }\frac{1}{z} + \frac{2z-1}{z^2}.$$

Example 1.2.51 *Prove that*

$$\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}(z) = \frac{1}{4}\,\mathrm{Log}\left(1+\frac{4}{z^2}\right).$$

Apply the result to find the value of the improper integral

$$\int_0^{+\infty} \frac{e^{-t}\,\sin^2 t}{t}\,\mathrm{d}t.$$

We get by using the rules of computation,

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}(z) &= \frac{1}{2}\,\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\}(z) = \frac{1}{2}\int_{\Gamma_z} \mathcal{L}\{1-\cos 2t\}(\zeta)\,\mathrm{d}\zeta \\ &= \frac{1}{2}\int_{\Gamma_z}\left\{\frac{1}{\zeta}-\frac{\zeta}{\zeta^2+4}\right\}\mathrm{d}\zeta = \frac{1}{2}\left[\frac{1}{2}\,\mathrm{Log}\,\frac{\zeta^2}{\zeta^2+4}\right]_z^{+\infty} = \frac{1}{4}\,\mathrm{Log}\left(\frac{z^2+4}{z^2}\right) \\ &= \frac{1}{4}\,\mathrm{Log}\left(1+\frac{4}{z^2}\right), \qquad \text{for } \Re\, z > 0.\end{aligned}$$

If we in particular choose $z = 1$, then

$$\int_0^{+\infty} \frac{e^{-t}\,\sin^2 t}{t}\,\mathrm{d}t = \mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}(1) = \frac{1}{4}\,\ln\left(1+\frac{4}{1}\right) = \frac{\ln 5}{4}. \qquad \Diamond$$

Example 1.2.52 *Given $f(t) = t^2$ for $t \in [0,2]$, and then $f(t+2) = f(t)$ for every $t \geq 0$. Compute $\mathcal{L}\{f\}(z)$.*

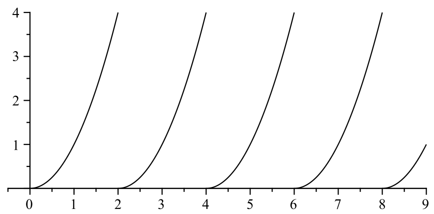

Figure 9: The graph of the the function $f(t)$ of Example 1.2.52.

It follows from the rule of periodicity that

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{1-e^{-2z}}\int_0^2 t^2\,e^{-zt}\,\mathrm{d}t = \frac{1}{1-e^{-2z}}\,\frac{d^2}{dz^2}\int_0^2 e^{-zt}\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2z}}\,\frac{d^2}{dz^2}\left[-\frac{e^{-zt}}{z}\right]_{t=0}^2 = \frac{1}{1-e^{-2z}}\,\frac{d^2}{dz^2}\left\{\frac{1}{z}-\frac{e^{-2z}}{z}\right\} \\
&= \frac{1}{1-e^{-2z}}\,\frac{d}{dz}\left\{-\frac{1}{z^2}+\frac{e^{-2z}}{z^2}+2\,\frac{e^{-2z}}{z}\right\} \\
&= \frac{1}{1-e^{-2z}}\left\{\frac{2}{z^3}-2\,\frac{e^{-2z}}{z^3}-2\,\frac{e^{-2z}}{z^2}-2\,\frac{e^{-2z}}{z^2}-4\,\frac{e^{-2z}}{z}\right\} \\
&= \frac{1}{1-e^{-2z}}\cdot\frac{2}{z^3}\left\{1-e^{-2z}-2z\,e^{-2z}-2z^2\,e^{-2z}\right\} \\
&= \frac{2}{z^3}-\frac{4}{z^2}\cdot\frac{(z+1)e^{-2z}}{1-e^{-2z}} = \frac{2}{z^3}-\frac{4}{z^2}\cdot\frac{z+1}{e^{2z}-1}. \qquad \Diamond
\end{aligned}$$

Example 1.2.53 *Given* $f(t) = |\sin t|$. *Compute* $\mathcal{L}\{f\}(z)$.

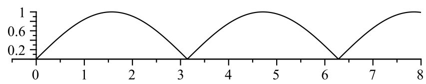

Figure 10: The graph of the the function $f(t)$ of Example 1.2.53.

The function is periodic of period π, so we shall use the rule of periodicity for $\Re z > 0$. We first compute

$$\begin{aligned}
\int_0^{\pi} e^{-zt}\,|\sin t|\,\mathrm{d}t &= \int_0^{\pi} e^{-zt}\cdot\frac{1}{2i}\left\{e^{it}-e^{-it}\right\}\,\mathrm{d}t = \frac{1}{2i}\int_0^{\pi}\left\{e^{(i-z)t}-e^{-(i+z)t}\right\}\,\mathrm{d}t \\
&= \frac{1}{2i}\left[\frac{1}{i-z}\,e^{(i-z)t}+\frac{1}{i+z}\,e^{-(i+z)t}\right]_{t=0}^{\pi} \\
&= \frac{1}{2i}\left\{\frac{1}{i-z}\,e^{i\pi-z\pi}+\frac{1}{i+z}\,e^{-i\pi-z\pi}-\frac{1}{i-z}-\frac{1}{i+z}\right\} \\
&= \frac{1}{2i}\left\{\frac{1}{i-z}+\frac{1}{i+z}\right\}\left(-e^{-z\pi}-1\right) = -\frac{1}{2i}\cdot\frac{i+z+i-z}{i^2-z^2}\cdot\left(1+e^{-z\pi}\right) \\
&= \frac{1+e^{-z\pi}}{z^2+1}.
\end{aligned}$$

Then we get by the rule of periodicity,

$$\mathcal{L}\{|\sin t|\}(z) = \frac{1}{1-e^{-z\pi}}\int_0^{\pi} e^{-zt}\,|\sin t|\,\mathrm{d}t = \frac{1}{z^2+1}\cdot\frac{1+e^{-z\pi}}{1-e^{-z\pi}} = \frac{1}{z^2+1}\cdot\coth\left(z\cdot\frac{\pi}{2}\right). \qquad \Diamond$$

Example 1.2.54 *Given the function* $f(t) = \max\{0, \sin t\}$. *Compute its Laplace transform* $\mathcal{L}\{f\}(z)$.

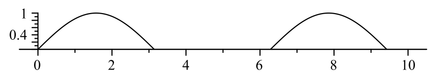

Figure 11: The graph of the the function $f(t)$ of Example 1.2.54.

The function is periodic of period 2π, and since $f(t) = 0$ for $t \in [\pi, 2\pi]$, we get

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{1-e^{-2\pi z}} \int_0^{\pi} e^{-zt}\, \sin t \, \mathrm{d}t = \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2i} \left\{ \int_0^{\pi} e^{-(z-i)t}\, \mathrm{d}t - \int_0^{-\pi} e^{-(z+i)t}\, \mathrm{d}t \right\} \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2i} \left\{ \left[-\frac{1}{z-i} \right]_{t=0}^{\pi} - \frac{1}{2i} \left[-\frac{e^{-(z+i)t}}{z+i} \right]_{t=0}^{\pi} \right\} \\
&= \frac{1}{1-e^{-2\pi z}} \cdot \frac{1}{2i} \left\{ \frac{1}{z-i} \left(e^{-z\pi} + 1 \right) - \frac{1}{z+i} \left(e^{-z\pi} + 1 \right) \right\} \\
&= \frac{1}{\left(1-e^{-\pi z}\right)\left(1+e^{-\pi z}\right)} \cdot \frac{1}{2i} \left(e^{-z\pi} + 1 \right) \cdot \frac{2i}{z^2+1} \\
&= \frac{1}{z^2+1} \cdot \frac{1}{1-e^{-\pi z}}. \qquad \Diamond
\end{aligned}$$

Example 1.2.55 *Given the Laplace transform*

$$\mathcal{L}\{f(t)\}(z) = \frac{z^2-z+1}{(2z+1)^2(z-1)}.$$

Find $\mathcal{L}\{f(2t)\}(z)$.

We shall apply the rule of scaling with $k = \frac{1}{2}$, from which

$$\begin{aligned}\mathcal{L}\{f(2t)\}(z) &= \frac{1}{2}\,\mathcal{L}\{f(t)\}\left(\frac{z}{2}\right) = \frac{1}{2}\cdot\frac{\left(\frac{z}{2}\right)^2 - \frac{z}{2}+1}{\left(2\cdot\frac{z}{2}+1\right)^2\left(\frac{z}{2}-1\right)} = \frac{1}{8}\cdot\frac{z^2-2z+4}{(z+1)^2(z-2)\cdot\frac{1}{2}}\\ &= \frac{1}{4}\cdot\frac{z^2-2z+4}{(z+1)^2(z-2)}. \qquad \Diamond\end{aligned}$$

Example 1.2.56 *Given the Laplace transform*

$$\mathcal{L}\{t\,f(t)\}(z) = \frac{1}{z\,(z^2+1)}.$$

Find $\mathcal{L}\left\{e^{-t}\,f(2t)\right\}(z)$.

We get by a straightforward computation, using various rules of computation,

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\,f(2t)\right\}(z) &= \mathcal{L}\{f(2t)\}(z+1) = \frac{1}{2}\,\mathcal{L}\{f(t)\}\left(\frac{z+1}{2}\right)\\ &= \frac{1}{2}\,\mathcal{L}\left\{\frac{t\,f(t)}{t}\right\}\left(\frac{z+1}{2}\right) = \frac{1}{2}\int_{\Gamma_{(z+1)/2}}\mathcal{L}\{t\,f(t)\}(\zeta)\,\mathrm{d}\zeta\\ &= \frac{1}{2}\int_{(z+1)/2}^{+\infty}\left\{\frac{1}{\zeta}-\frac{\zeta}{\zeta^2+1}\right\}\mathrm{d}\zeta = \frac{1}{2}\left[\operatorname{Log}\left(\frac{\zeta^2+1}{\zeta^2}\right)\right]_{+\infty}^{(z+1)/2}\\ &= \frac{1}{2}\operatorname{Log}\left(\frac{(z+1)^2+4}{(z+1)^2}\right) = \frac{1}{2}\operatorname{Log}\left(\frac{z^2+2z+5}{z^2+2z+1}\right)\\ &= \frac{1}{2}\operatorname{Log}\left(1+\frac{4}{(z+1)^2}\right). \qquad \Diamond\end{aligned}$$

Example 1.2.57 *Let* $r > 0$ *be a constant and put* $r^t := \exp(t\cdot\ln r)$*. Prove that*

$$\mathcal{L}\left\{r^t\,f(t)\right\}(z) = \mathcal{L}f(z-\ln r).$$

Let $\Re\, z > \sigma(f) + \ln r$. Then by a straightforward computation

$$\begin{aligned}\mathcal{L}\left\{r^t\,f(t)\right\}(z) &= \int_0^{+\infty} e^{-zt}\,r^t\,f(t)\,\mathrm{d}t = \int_0^{+\infty} e^{-zt}\,e^{t\cdot\ln r}\,f(t)\,\mathrm{d}t\\ &= \int_0^{+\infty} e^{-(z-\ln r)t}\,f(t)\,\mathrm{d}t = \mathcal{L}\{f\}(z-\ln r). \qquad \Diamond\end{aligned}$$

1.3 The complex inversion formula I

Example 1.3.1 *Find the inverse Laplace transforms of*

1) $\dfrac{1}{\left(z^2+1\right)^2}$,

2) $\dfrac{1}{z^4-1}$,

3) $\dfrac{z^2}{z^3-1}$.

1) It follows from the rule of convolution that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\left(z^2+1\right)^2}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{z^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{z^2+1}\right\}(t) = (\sin * \sin)(t) \\ &= \int_0^t \sin u \cdot \sin(t-u)\,\mathrm{d}u = \frac{1}{2}\int_0^t \{\cos(2u-t)-\cos t\}\,\mathrm{d}u \\ &= \frac{1}{2}\left[\frac{1}{2}\sin(2u-t)\right]_{u=0}^{t} - \frac{1}{2}\,t\,\cos t = \frac{1}{2}\sin t - \frac{1}{2}\,t\,\cos t.\end{aligned}$$

2) In this case we apply a decomposition,

$$\mathcal{L}^{-1}\left\{\frac{1}{z^4-1}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{z^2-1}-\frac{1}{2}\cdot\frac{1}{z^2+1}\right\}(t)=\frac{1}{2}\,\sinh t-\frac{1}{2}\,\sin t.$$

3) Here we use a partial decomposition to get

$$\begin{aligned}\frac{z^2}{z^3-1} &= \frac{z^2}{(z-1)\,(z^2+z+1)}=\frac{1}{3}\cdot\frac{1}{z-1}+\frac{1}{3}\cdot\frac{3z^2-z^2-z-1}{(z-1)\,(z^2+z+1)}\\ &= \frac{1}{3}\cdot\frac{1}{z-1}+\frac{1}{3}\cdot\frac{2z+1}{z^2+z+1}=\frac{1}{3}\cdot\frac{1}{z-1}+\frac{2}{3}\cdot\frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2},\end{aligned}$$

from which

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{z^2}{z^3-1}\right\}(t) &= \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{z-1}\right\}(t)+\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}(t)\\ &= \frac{1}{3}\,e^t+\frac{2}{3}\,exp\left(-\frac{1}{2}\,t\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right). \qquad \Diamond\end{aligned}$$

Example 1.3.2 *Compute the inverse Laplace transforms of*

1) $\dfrac{1}{z(z+3)^2}$,

2) $\dfrac{1}{(z+1)(z-2)^2}$,

3) $\dfrac{z}{(z+1)^3(z-1)^2}$.

1) Here we use the residuum formula,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{z(z+3)^2}\right\}(t) &= \text{res}\left(\frac{e^{zt}}{z(z+3)^2};0\right)+\text{res}\left(\frac{e^{zt}}{z(z+3)2};-3\right)\\ &= \frac{1}{9}+\lim_{z\to-3}\frac{d}{dz}\left\{\frac{e^{zt}}{z}\right\}=\frac{1}{9}+\lim_{z\to-3}\left\{t\cdot\frac{e^{zt}}{z}-\frac{e^{zt}}{z^2}\right\}\\ &= \frac{1}{9}-t\cdot\frac{e^{-3t}}{3}-\frac{1}{9}\,e^{-3t}=\frac{1}{9}\left\{1-(3t+1)e^{-3t}\right\}.\end{aligned}$$

2) Again we apply the residuum formula,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(z+1)(z-2)^2}\right\}(t) &= \operatorname{res}\left(\frac{e^{zt}}{(z+1)(z-2)^2};-1\right)+\operatorname{res}\left(\frac{e^{zt}}{(z+1)(z-2)^2};2\right)\\ &= \frac{e^{-t}}{(-3)^2}+\lim_{z\to 2}\frac{d}{dz}\left\{\frac{e^{zt}}{z+1}\right\}=\frac{1}{9}\,e^{-t}+\lim_{z\to 2}\left\{t\cdot\frac{e^{zt}}{z+1}-\frac{e^{zt}}{(z+1)^2}\right\}\\ &= \frac{1}{9}\,e^{-t}+\frac{1}{3}\,t\cdot e^{2t}-\frac{1}{9}\,e^{2t}=\frac{1}{9}\left\{e^{-t}+(3t-1)e^{2t}\right\}.\end{aligned}$$

3) Finally,

$$\mathcal{L}^{-1}\left\{\frac{z}{(z+1)^3(z-1)^2}\right\}(t)=\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^3(z-1)^2};-1\right)+\operatorname{res}\left(\frac{z\,e^{zt}}{(z+1)^3(z-1)^2};1\right)$$

$$=\frac{1}{2}\lim_{z\to-1}\frac{d^2}{dz^2}\left\{\frac{z\,e^{zt}}{(z-1)^2}\right\}+\lim_{z\to 1}\frac{d}{dz}\left\{\frac{z\,e^{zt}}{(z+1)^3}\right\}$$

$$=\frac{1}{2}\lim_{z\to-1}\frac{d}{dz}\left\{\frac{e^{zt}}{(z-1)^2}+t\cdot\frac{z\,e^{zt}}{(z-1)^2}-2\,\frac{z\,e^{zt}}{(z-1)^3}\right\}+\lim_{z\to 1}\left\{\frac{e^{zt}(1+tz)}{(z+1)^3}-\frac{3z\,e^{zt}}{(z+1)^4}\right\}$$

$$=\frac{1}{2}\lim_{z\to-1}\left\{-\frac{2e^{zt}}{(z-1)^3}+2t\,\frac{e^{zt}}{(z-1)^2}+t^2\,\frac{ze^{zt}}{(z-1)^2}-3t\,\frac{ze^{zt}}{(z-1)^3}-2\,\frac{e^{zt}(1+zt)}{(z-1)^3}+6\,\frac{ze^{zt}}{(z-1)^4}\right\}$$
$$+\frac{1}{8}\,e^t(1+t)-\frac{3}{16}\,e^t$$

$$e^{-t}\left\{-\frac{1}{(-8)}+\frac{t}{4}-\frac{1}{2}\,t^2\cdot\frac{1}{4}-t\cdot\frac{1}{8}+\frac{1-t}{8}-\frac{3}{6}\right\}+\frac{e^t}{16}\{2t-1\}$$

$$=\frac{e^t}{16}\{2t-1\}+\frac{e^{-t}}{16}\left\{1-2t^2\right\}.\qquad\Diamond$$

Example 1.3.3 *Compute the inverse Laplace transforms of*

1) $\dfrac{1}{z^3+1}$,

2) $\dfrac{1}{z^4+4}$.

1) Choose any $z_0\in\left\{-1,\frac{1}{2}+i\,\frac{\sqrt{3}}{2},\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right\}$, so $z_0^3=-1$. Then

$$\operatorname{res}\left(\frac{e^{zt}}{z^3+1};z_0\right)=\frac{e^{z_0t}}{3z_0^2}=-\frac{1}{3}\,z_0\,e^{z_0t},$$

and it then follows from the complex residuum inversion formula that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{z^3+1}\right\}(t) &= -\frac{1}{3}\left\{-1\cdot e^{-t}+\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right)\exp\left(\left\{\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right\}t\right)\right.\\ &\qquad\left.+\left(\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right)\exp\left(\left\{\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right\}t\right)\right\}\\ &= \frac{1}{3}\,e^{-t}-\frac{1}{3}\,e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{3}\,e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}\,t\right).\end{aligned}$$

2) Assume that $z_0^4=-4$, i.e. $z_0\in\{\pm1\pm i\}$. Then

$$\operatorname{res}\left(\frac{e^{zt}}{z^4+4};z_0\right)=\frac{e^{z_0 t}}{4z_0^3}=\frac{1}{4}\cdot\frac{z_0\,e^{z_0 t}}{z_0^4}=-\frac{1}{16}\,z_0\,e^{z_0 t},$$

and it follows from the complex residuum inversion formula that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{z^4+4}\right\}(t) &= -\frac{1}{16}\left\{(1+i)e^{(1+i)t}+(1-i)e^{(1-i)t}-(1+i)e^{-(1+i)t}-(1-i)e^{-(1-i)t}\right\}\\ &= -\frac{1}{16}\left\{e^t\cdot 2\cos t+ie^t\cdot 2i\sin t-e^{-t}\cos t-ie^{-t}\cdot 2i\sin t\right\}\\ &= -\frac{1}{8}\left\{e^t\cos t-e^t\sin t-e^{-t}\cos t+e^{-t}\sin t\right\}\\ &= -\frac{1}{4}\,\sinh t\cdot\cos t+\frac{1}{4}\,\sinh t\cdot\sin t\\ &= \frac{1}{4}\,\sinh t\cdot\{\sin t-\cos t\}.\qquad\Diamond\end{aligned}$$

Example 1.3.4 *Compute the inverse Laplace transform of the function*

$$\frac{z}{(z+1)^2\,(z^2+3z-10)},$$

and find $\sigma(f)$.

By a decomposition,

$$\begin{aligned}\frac{z}{(z+1)^2\,(z^2+3z-10)} &= \frac{z}{(z+1)^2(z+5)(z-2)}\\ &= \frac{1}{12}\,\frac{1}{(z+1)^2}+\frac{1}{12}\,\frac{12z-z^2-3z+10}{(z+1)^2(z+5)(z-2)}\\ &= \frac{1}{12}\,\frac{1}{(z+1)^2}+\frac{1}{12}\,\frac{-z+10}{(z+1)(z+5)(z-2)}\\ &= \frac{1}{12}\,\frac{1}{(z+1)^2}-\frac{11}{144}\,\frac{1}{z+1}+\frac{2}{63}\,\frac{1}{z-2}+\frac{5}{112}\,\frac{1}{z+5},\end{aligned}$$

from which follows that

$$\mathcal{L}^{-1}\left\{\frac{z}{(z+1)^2\,(z^2+3z-10)}\right\} = \frac{1}{12}\,t\,e^{-t} - \frac{11}{144}\,e^{-t} + \frac{2}{63}\,e^{2t} + \frac{5}{112}\,e^{-5t}.$$

Finally, $\dfrac{z}{(z+1)^2\,(z^2+3z-10)}$ has the poles -1, -5, so

$$\sigma(f) = \max\{-1,-5,2\} = 2. \qquad \Diamond$$

Example 1.3.5 *Compute the inverse Laplace transforms of*

1) $\dfrac{6z-4}{z^2-4z+20}$,

2) $\dfrac{3z+7}{z^2-2z-3}$,

3) $\dfrac{4z+12}{z^2+8z+16}$.

1) By a small rearrangement,

$$\frac{6z-4}{z^2-4z+20} = \frac{6(z-2)+8}{(z-2)^2+4^2} = 6\cdot\frac{z-2}{(z-2)^2+4^2} + 2\cdot\frac{4}{(z-2)^2+4^2},$$

from which follows that

$$\mathcal{L}^{-1}\left\{\frac{6z-4}{z^2-4z+20}\right\}(t) = 6e^{2t}\,\cos 4t + 2\,e^{t}\,\sin 4t.$$

2) In this case we use a plain decomposition

$$\frac{3z+7}{z^2-2z-3} = \frac{3z+7}{(z-3)(z+1)} = \frac{16}{4}\,\frac{1}{z-3} + \frac{4}{(-4)}\,\frac{1}{z+1} = \frac{4}{z-3} - \frac{1}{z+1},$$

from which

$$\mathcal{L}^{-1}\left\{\frac{3z+7}{z^2-2z-3}\right\}(t) = 4\,e^{3t} - e^{-t}.$$

3) Finally, again by a decomposition,

$$\frac{4z+12}{z^2+8z+16} = \frac{4z+16-4}{(z+4)^2} = \frac{4}{z+4} - \frac{4}{(z+4)^2},$$

from which

$$\mathcal{L}^{-1}\left\{\frac{4z+12}{z^2+8z+16}\right\}(t) = 4\,e^{-4t} - 4t\,e^{-4t} = 4(1-t)e^{-4t}. \qquad \Diamond$$

Example 1.3.6 *Compute the inverse Laplace transform of* $\dfrac{1}{z^3\,(z^2+1)}$.

First method. First note that $F8z) = \dfrac{1}{z^3\,(z^2+1)}$ is analytic in the set $\mathbb{C} \setminus \{0, i, -i\}$, and that we have the estimate

$$|F(z)| \le \frac{C}{|z|^5} \qquad \text{for } |z| \ge 2.$$

Hence by the residuum formula

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)};0\right) + \operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)};i\right) + \operatorname{res}\left(\frac{e^{zt}}{z^3\,(z^2+1)};0\right) \\
&= \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{e^{zt}}{z^2+1}\right\} + \lim_{z\to i}\left\{\frac{e^{zt}}{z^3\cdot 2z}\right\} + \lim_{z\to -i}\left\{\frac{e^{zt}}{z^3\cdot 2z}\right\} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left\{\frac{t,e^{zt}}{z^2+1} - \frac{2z\,e^{zt}}{(z^2+1)^2}\right\} + \frac{e^{it}}{2} + \frac{e^{-it}}{2} \\
&= \frac{1}{2}\lim_{z\to 0}\left\{\frac{t^2\,e^{zt}}{z^2+1} - \frac{4z\,t\,e^{zt}}{(z^2+1)^2} - \frac{2\,e^{zt}}{(z^2+1)^2} + \frac{8z^2\,e^{zt}}{(z^2+1)^3}\right\} + \cos t \\
&= \frac{1}{2}\left\{t^2 - 0 - 2 + 0\right\} + \cos t = \frac{1}{2}\,t^2 - 1 + \cos t.
\end{aligned}$$

Second method. We know already that $\dfrac{1}{z^2+1} = \mathcal{L}\{\sin t\}(t)$, so it follows form the rule of integration applied successively three times that

$$\frac{1}{z}\cdot\frac{1}{z^2+1} = \mathcal{L}\left\{\int_0^t \sin\tau\,\mathrm{d}\tau\right\}(z) = \mathcal{L}\left\{[-\cos\tau]_0^t\right\}(z) = \mathcal{L}\{1-\cos t\}(z),$$

$$\frac{1}{z^2}\cdot\frac{1}{z^2+1} = \mathcal{L}\left\{\int_0^t (1-\cos\tau)\,\mathrm{d}\tau\right\}(z) = \mathcal{L}\left\{[\tau-\sin\tau]_0^t\right\}(z) = \mathcal{L}\{t-\sin t\}(z),$$

$$\frac{1}{z^3}\cdot\frac{1}{z^2+1} = \mathcal{L}\left\{\int_0^t (\tau-\sin\tau)\,\mathrm{d}\tau\right\}(z) = \mathcal{L}\left\{\frac{1}{2}\,t^2-1+\cos t\right\}(z),$$

from which follows that

$$f(t) = \frac{1}{2}\,t^2 - 1 + \cos t.$$

Third method. We get by an incomplete decomposition,

$$\begin{aligned}\frac{1}{z^3\,(z^2+1)} &= \frac{1}{z^3} + \left\{\frac{1}{z^3\,(z^2+1)} - \frac{1}{z^3}\right\} = \frac{1}{z^3} - \frac{z^2+1-1}{z^3\,(z^2+1)} = \frac{1}{z^3} - \frac{1}{z\,(z^2+1)}\\ &= \frac{1}{z^3} - \frac{1}{z} + \left\{\frac{-1}{z\,(z^2+1)} + \frac{1}{z}\right\} = \frac{1}{z^3} - \frac{1}{z} + \frac{z}{z^2+1},\end{aligned}$$

so it suffices to apply a table to get

$$f(t) = \frac{1}{2}\,t^2 - 1 + \cos t. \qquad \Diamond$$

Example 1.3.7 *Compute the inverse Laplace transforms of*

1) $\dfrac{2z^2-4}{(z-2)(z-3)(z+1)}$,

2) $\dfrac{5z^2-15z-11}{(z+1)(z-2)^3}$.

1) It follows from the residuum formula that

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{2z^2-4}{(z-2)(z-3)(z+1)}\right\}(t)\\ &\quad= \operatorname{res}\left(\frac{\left(2z^2-4\right)e^{zt}}{(z-2)(z-3)(z+1)};-1\right) + \operatorname{res}\left(\frac{\left(2z^2-4\right)e^{zt}}{(z-2)(z-3)(z+1)};2\right)\\ &\qquad\qquad + \operatorname{res}\left(\frac{\left(2z^2-4\right)e^{zt}}{(z-2)(z-3)(z+1)};3\right)\\ &\quad= \frac{(2-4)e^{-t}}{(-1)(-2)} + \frac{(8-4)e^{2t}}{(-1)\cdot 3} + \frac{(18-4)e^{3t}}{1\cdot 4} = -e^{-t} - \frac{4}{3}\,e^{2t} + \frac{7}{2}\,e^{3t}.\end{aligned}$$

2) We get analogously,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{5z^2-15z-11}{(z+1)(z-2)^3}\right\}(t) &= \text{res}\left(\frac{\left(5z^2-15z-11\right)e^{zt}}{(z+1)(z-2)^3};-1\right)+\text{res}\left(\frac{\left(5z^2-15z-11\right)e^{zt}}{(z+1)(z-2)^3};2\right)\\
&= \frac{(5+15-11)e^{-t}}{(-1)^3}+\frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{\frac{5z^2-15z-11}{z+1}\cdot e^{zt}\right\}\\
&= -9\,e^{-t}+\frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{(5z-20)e^{zt}+\frac{9}{z+1}\,e^{zt}\right\}\\
&= -9e^{-t}+\frac{1}{2}\lim_{z\to 2}\frac{d}{dz}\left\{5e^{zt}+t(5z-20)e^{zt}-\frac{9}{(z+1)^2}\,e^{zt}+\frac{9t}{z+1}\,e^{zt}\right\}\\
&= -9e^{-t}+\frac{1}{2}\left[1+r\,e^{zt}+t^2(5z-20)e^{zt}+\frac{18}{(z+1)^3}\,e^{zt}-\frac{18t}{(z+1)^2}\,e^{zt}+\frac{9t^2}{z+1}\,e^{zt}\right]_{z=2}\\
&= -9e^{-t}+\frac{1}{2}\,e^{2t}\left\{10t-10t^2+\frac{18}{27}-\frac{18}{9}\,t+\frac{9}{3}\,t^2\right\}\\
&= -9e^{-t}+\frac{1}{2}\,e^{2t}\left(-7t^2+8t+\frac{2}{3}\right)\\
&= -9e^{-t}+\left(-\frac{7}{2}\,t^2+4t+\frac{1}{3}\right)e^{2t}. \qquad \Diamond
\end{aligned}$$

Example 1.3.8 *Compute the inverse Laplace transforms of*

1) $\dfrac{3z+1}{(z-1)\left(z^2+1\right)}$,

2) $\dfrac{z^2+2z+3}{\left(z^2+2z+2\right)\left(z^2+2z+5\right)}$.

1) We get by a decomposition

$$\begin{aligned}
\frac{3z+1}{(z-1)\left(z^2+1\right)} &= \frac{2}{z-1}+\left\{\frac{3z+1}{(z-1)\left(z^2+1\right)}-\frac{2}{z-1}\right\}\\
&= \frac{2}{z-1}+\frac{3z+1-2z^2-2}{(z-1)\left(z^2+1\right)}=\frac{2}{z-1}-\frac{2z}{z^2+1}+\frac{1}{z^2+1},
\end{aligned}$$

and it suffices to use a table to obtain

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3z+1}{(z-1)\left(z^2+1\right)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{z-1}\right\}(t)-2\mathcal{L}^{-1}\left\{\frac{z}{z^2+1}\right\}(t)+\mathcal{L}^{-1}\left\{\frac{1}{z^2+1}\right\}(t)\\
&= 2e^t-2\cos t+\sin t.
\end{aligned}$$

2) In this case be get by inspection,

$$\begin{aligned}\frac{z^2+2z+3}{(z^2+2z+2)(z^2+2z+5)} &= \frac{(z+1)^2}{\{(z+1)^2+1\}\{(z+1)^2+4\}} \\ &= \frac{1}{3}\cdot\frac{1}{(z+1)^2+1}+\frac{1}{3}\cdot\frac{2}{(z+1)^2+2^2},\end{aligned}$$

so by the shifting rule,

$$\mathcal{L}\left\{\frac{z^2+2z+3}{(z^2+2z+2)(z^2+2z+5)}\right\}(t) = \frac{1}{2}\,e^{-t}\sin t+\frac{1}{3}\,e^{-t}\sin 2t. \qquad \Diamond$$

Example 1.3.9 *Compute the inverse Laplace transform of*

$$\frac{z^3+5z^2+4z+20}{z^2\left(z^2+9\right)}.$$

It follows from the decomposition

$$\begin{aligned}\frac{z^3+5z^2+4z+20}{z^2\left(z^2+9\right)} &= (z+5)\cdot\frac{z^2+4}{z^2\left(z^2+9\right)}=(z+5)\left\{\frac{4}{9}\cdot\frac{1}{z^2}+\frac{5}{9}\cdot\frac{1}{z^2+9}\right\}\\ &= \frac{4}{9}\cdot\frac{1}{z}+\frac{20}{9}\cdot\frac{1}{z^2}+\frac{5}{9}\cdot\frac{z}{z^2+9}+\frac{25}{27}\cdot\frac{3}{z^2+9},\end{aligned}$$

using tables that

$$\mathcal{L}^{-1}\left\{\frac{z^3+5z^2+4z+20}{z^2\left(z^2+9\right)}\right\}(t)=\frac{4}{9}+\frac{20}{9}\,t+\frac{5}{9}\,\cos 3t+\frac{25}{27}\,\sin 3t. \qquad \Diamond$$

Example 1.3.10 *Compute the inverse Laplace transforms of*

1) $\dfrac{3z-12}{z^2+8}$,

2) $\dfrac{2z+1}{z(z+1)}$.

1) We first note that $8=\left(2\sqrt{2}\right)^2$, so we get by a splitting of the fraction,

$$\begin{aligned}\frac{3z-12}{z^2+8} &= 3\cdot\frac{z}{z^2+\left(2\sqrt{2}\right)^2}-\frac{12}{2\sqrt{2}}\cdot\frac{2\sqrt{2}}{z^2+\left(2\sqrt{2}\right)^2}\\ &= 3\,\mathcal{L}\left\{\cos\left(2\sqrt{2}\,t\right)\right\}(z)-3\sqrt{2}\cdot\mathcal{L}\left\{\sin\left(2\sqrt{2}\,t\right)\right\}(z),\end{aligned}$$

and we conclude that

$$\mathcal{L}^{-1}\left\{\frac{3z-12}{z^2+8}\right\}(t)=3\,\cos\left(2\sqrt{2}\,t\right)-3\sqrt{2}\,\sin\left(2\sqrt{2}\,t\right).$$

2) By first using a simple decomposition we get

$$\frac{2z+1}{z(z+1)}=\frac{1}{z}+\frac{1}{z+1}=\mathcal{L}\{1\}(z)+\mathcal{L}\left\{e^{-t}\right\}(z),$$

hence,

$$\mathcal{L}^{-1}\left\{\frac{2z+1}{z(z+1)}\right\}(t)=1+e^{-t}. \qquad \Diamond$$

Example 1.3.11 *Compute the inverse Laplace transforms of*

1) $\dfrac{z}{(z+1)^5}$,

2) $\dfrac{3z-14}{z^2-4z+8}$.

1) By a simple decomposition,

$$\frac{z}{(z+1)^5}=\frac{z+1-1}{(z+1)^5}=\frac{1}{(z+1)^4}-\frac{1}{(z+1)^5}=\frac{1}{3!}\,\frac{3!}{(z+1)^4}-\frac{1}{4!}\,\frac{4!}{(z+1)^5},$$

we immediately get that

$$\mathcal{L}^{-1}\left\{\frac{z}{(z+1)^5}\right\}(t)=\frac{1}{6}\,t^3\,e^{-t}-\frac{1}{24}\,t^4\,e^{-t},\qquad \text{for } t\geq 0.$$

2) First note that $z^2-4z+8=(z-2)^2+2^2$. Then by a decomposition,

$$\frac{3z-14}{z^2-4z+8}=3\cdot\frac{z-2}{(z-2)^2+2^2}-4\cdot\frac{2}{(z-2)^2+2^2},$$

from which it immediately follows that

$$\mathcal{L}^{-1}\left\{\frac{3z-14}{z^2-4z+8}\right\}(t)=3\,e^{2t}\cos 2t-4\,e^{2t}\sin 2t.\qquad\Diamond$$

Example 1.3.12 *Compute the inverse Laplace transforms of*

1) $\dfrac{8z+20}{z^2-12z+32}$,

2) $\dfrac{3z+2}{4z^2+12z+9}$,

3) $\dfrac{5z-2}{3z^2+4z+8}$.

1) First note that

$$z^2-12z+32=(z-6)^2-2^2=(z-8)(z-4).$$

Then we get by a decomposition

$$\frac{8z+20}{z^2-12z+32}=\frac{8z+20}{(z-8)(z-4)}=\frac{64+20}{4}\,\frac{1}{z-8}-\frac{32+20}{4}\,\frac{1}{z-4}=\frac{21}{z-8}-\frac{13}{z-4},$$

from which finally,

$$\mathcal{L}^{-1}\left\{\frac{8z+20}{z^2-12z+32}\right\}(t)=21\,e^{8t}-13\,e^{4t}.$$

2) It follows from $4z^2+12z+9=(2z+3)^2$ that we have the decomposition

$$\frac{3z+2}{4z^2+12z+9}=\frac{\frac{3}{2}(2z+3)+2-\frac{9}{4}}{(2z+3)^2}=\frac{3}{4}\cdot\frac{1}{z+\frac{3}{2}}-\frac{1}{16}\cdot\frac{1}{\left(z+\dfrac{3}{2}\right)^2},$$

hence,

$$\mathcal{L}^{-1}\left\{\frac{3z+2}{4z^2+12z+9}\right\}(t)=\frac{3}{4}\exp\left(-\frac{3}{2}\,t\right)-\frac{1}{16}\,t\,\exp\left(-\frac{3}{2}\,t\right).$$

3) The roots of $3z^2+4z+8$ are

$$z=\frac{1}{6}\left\{-4\pm\sqrt{16-96}\right\}=\frac{1}{6}\left\{-4\pm i\,4\sqrt{5}\right\}=\frac{2}{3}\,\left\{-1\pm i\sqrt{5}\right\},$$

so we decompose in the following way

$$\begin{aligned}\frac{5z-2}{3z^2+4z+8}&=\frac{1}{3}\cdot\frac{5z-2}{z^2+2\cdot\dfrac{2}{3}\,z+\left\{\dfrac{2}{3}\right\}^2+\dfrac{8}{3}-\dfrac{4}{9}}=\frac{1}{3}\cdot\frac{5\left(z+\dfrac{2}{3}\right)-\dfrac{10}{3}-2}{\left(z+\dfrac{2}{3}\right)^2+\left(\dfrac{\sqrt{20}}{3}\right)^2}\\&=\frac{5}{3}\cdot\frac{z+\dfrac{2}{3}}{\left(z+\dfrac{2}{3}\right)^2+\left(\dfrac{2\sqrt{5}}{3}\right)^2}-\frac{16}{9}\cdot\frac{3}{2\sqrt{5}}\cdot\frac{\dfrac{2\sqrt{5}}{3}}{\left(z+\dfrac{2}{3}\right)^2+\left(\dfrac{2\sqrt{5}}{3}\right)^2},\end{aligned}$$

and we finally get by the shifting rule and the rule of change of scale that

$$\mathcal{L}^{-1}\left\{\frac{5z-2}{3z^2+4z+8}\right\}(t)=\frac{5}{3}\exp\left(-\frac{2}{3}\,t\right)\cos\left(\frac{2\sqrt{5}}{3}\,t\right)-\frac{8\sqrt{5}}{15}\exp\left(-\frac{2}{3}\,t\right)\sin\left(\frac{2\sqrt{5}}{3}\,t\right).\qquad\Diamond$$

Example 1.3.13 *Compute the inverse Laplace transforms of*

1) $\dfrac{z}{(z+1)(z+2)}$,

2) $\dfrac{1}{(z+1)^3}$.

1) By a decomposition,

$$\frac{z}{(z+1)(z+2)}=-\frac{1}{z+1}+2\cdot\frac{1}{z+2},$$

from which

$$\mathcal{L}^{-1}\left\{\frac{z}{(z+1)(z+2)}\right\}(t)=2\,e^{-2t}-e^{-t}.$$

2) It follows immediately from

$$\frac{1}{(z+1)^3} = \frac{1}{2} \cdot \frac{2!}{(z+1)^3} = \frac{1}{2}\,\mathcal{L}\left\{t^2\,e^{-t}\right\}(z)$$

that

$$\mathcal{L}^{-1}\left\{\frac{1}{(z+1)^3}\right\}(t) = \frac{1}{2}\,t^2\,e^{-t}. \qquad \Diamond$$

Example 1.3.14 *Compute the inverse Laplace transforms of*

1) $\dfrac{3z^3-3z^2-40z+36}{\left(z^2-4\right)^2}$,

2) $\dfrac{z}{\left(z^2-2z+2\right)\left(z^2+2z+2\right)}$.

1) We get by a decomposition

$$\begin{aligned}\frac{3z^3-3z^2-40z+36}{\left(z^2-4\right)^2} &= \frac{3z^3-3z^2-40z+36}{(z-2)^2\,(z+2)^2}\\ &= -\frac{2}{(z-2)^2}+\frac{5}{(z+2)^2}+\frac{3z^3-3z^2-40z+36+2(z+2)^2-5(z-2)^2}{(z-2)^2\,(z+2)^2}\\ &= -\frac{2}{(z-2)^2}+\frac{5}{(z+2)^2}+\frac{3z^3-3z^2-40z+36+2z^2+8z+8-5z^2+20z-20}{(z-2)^2\,(z+2)^2}\\ &= -\frac{2}{(z-2)^2}+\frac{5}{(z+2)^2}+\frac{3z^3-6z^2-12z+24}{\left(z^2-4\right)^2}\\ &= -\frac{2}{(z-2)^2}+\frac{5}{(z+2)^2}+\frac{3z-6}{(z-2)(z+2)} = -\frac{2}{(z-2)^2}+\frac{5}{(z+2)^2}+\frac{3}{z+2},\end{aligned}$$

hence

$$\mathcal{L}^{-1}\left\{\frac{3z^3-3z^2-40z+36}{\left(z^2-4\right)^2}\right\}(t) = -2t\,e^{2t}+5t\,e^{-2t}+3\,e^{-2t}.$$

2) It follows from the decomposition

$$\frac{z}{\left(z^2-2z+2\right)\left(z^2+2z+2\right)} = \frac{1}{4}\,\frac{1}{z^2-2z+2}-\frac{1}{4}\,\frac{1}{z^2+2z+2} = \frac{1}{4}\,\frac{1}{(z-1)^2+1}-\frac{1}{4}\,\frac{1}{(z+1)^2+1},$$

that

$$\mathcal{L}^{-1}\left\{\frac{z}{\left(z^2-2z+2\right)\left(z^2+2z+2\right)}\right\}(t))\frac{1}{4}\,e^{t}\,\sin t-\frac{1}{4}\,e^{-t}\,\sin t = \frac{1}{2}\,\sinh t\cdot\sin t. \qquad \Diamond$$

Example 1.3.15 *Compute the inverse Laplace transforms of*

1) $\dfrac{e^{-z}}{z^2+1}$,

2) $\dfrac{z}{(z+1)^2}+\dfrac{e^{-z}}{z}$.

1) It follows from the shifting property that

$$\frac{e^{-z}}{z^2+1}=e^{-z}\,\mathcal{L}\{\sin t\}(z)=\mathcal{L}\{H(t-1)\sin(t-1)\}(z),$$

hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-z}}{z^2+1}\right\}(t)=H(t-1)\sin(t-1).$$

2) We find in this case by decomposition, tables and the shifting property that

$$\frac{z}{(z+1)^2}+\frac{e^{-z}}{z}=\frac{1}{z+1}-\frac{1}{(z+1)^2}+e^{-z}\,\mathcal{L}\{1\}(z),$$

and

$$LL^{-1}\left\{\frac{z}{(z+1)^2}+\frac{e^{-z}}{z}\right\}(t)=e^{-t}-t\,e^{-t}+H(t-1). \qquad \Diamond$$

Example 1.3.16 *Compute the inverse Laplace transforms of*

1) $\dfrac{e^{-5z}}{(z-2)^4}$,

2) $\dfrac{z\,\exp\left(-\frac{4\pi}{5}\,z\right)}{z^2+25}$,

3) $\dfrac{(z+1)e^{-\pi z}}{z^2+z+1}$.

1) We get, using a table,

$$\frac{e^{-5z}}{(z-2)^4}=e^{-5z}\cdot\frac{1}{3!}\cdot\frac{3!}{(z-2)^4}=\frac{e^{-5z}}{6}\,\mathcal{L}\left\{e^{2}t\,t^3\right\}(z),$$

hence by the the shifting property,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5z}}{(z-2)^4}\right\}(t)=\frac{1}{6}\,H(t-5)\,(t-5)^3\,e^{-2(t-5)}.$$

2) Using the same method as above we get

$$\frac{z\,\exp\left(-\frac{4\pi}{5}\,z\right)}{z^2+25} = \exp\left(-\frac{4\pi}{5}\,z\right)\mathcal{L}\{\cos 5t\}(z),$$

hence

$$\mathcal{L}^{-1}\left\{\frac{z\,\exp\left(-\frac{4\pi}{5}\,z\right)}{z^2+25}\right\}(t) = H\left(t-\frac{4\pi}{5}\right)\cos(5t-4\pi) = H\left(t-\frac{4\pi}{5}\right)\cos 5t.$$

3) We get in this case

$$\begin{aligned}\frac{(z+1)e^{-\pi z}}{z^2+z+1} &= e^{-\pi z}\,\frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+e^{-\pi z}\cdot\frac{1}{2}\cdot\frac{2}{\sqrt{3}]}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= e^{-\pi z}\,\mathcal{L}\left\{e^{-\frac{1}{2}\,t}\cos\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z)+\frac{1}{\sqrt{3}}\,e^{-\pi z}\,\mathcal{L}\left\{e^{-\frac{1}{2}\,t}\,\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z),\end{aligned}$$

hence

$$\mathcal{L}^{-1}\left\{\frac{(}{z}+1)e^{-\pi z}z^2+z+1\right\}(t)$$

$$= H(t-\pi)\,e^{-\frac{1}{2}(t-\pi)}\left\{\cos\left(\frac{\sqrt{3}}{2}\{t-\pi\}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\{t-\pi\}\right)\right\}$$

$$= H(t-\pi)\exp\frac{\pi}{2}\,\exp\left(-\frac{t}{2}\right)\left\{\cos\left(\frac{\sqrt{3}}{2}t-\frac{\sqrt{3}}{2}\pi\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t-\frac{\sqrt{3}}{2}\pi\right)\right\}$$

$$= H(t-\pi)\exp\frac{\pi}{2}\,\exp\left(-\frac{t}{2}\right)\cdot\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}t-\left\{\frac{\sqrt{3}}{2}+\frac{1}{6}\right\}\pi\right). \qquad \Diamond$$

Example 1.3.17 *Compute the inverse Laplace transforms of*

1) $\dfrac{e^{-2z}}{z^2}$,

2) $\dfrac{8e^{-3z}}{z^2+4}$,

3) $\dfrac{z\,e^{-2z}}{z^2+3z+2}$.

1) It follows from

$$\frac{e^{-2z}}{z^2}=e^{-2z}\,\mathcal{L}\{t\}(z)$$

and the shifting property that

$$\mathcal{L}^{-1}\left\{\frac{e^{-2z}}{z^2}\right\}(t)=\begin{cases} t-2 & \text{for } t\geq 2,\\ 0 & \text{for } t<2.\end{cases}$$

2) In this case we find

$$\frac{8\,e^{-3z}}{z^2+4}=e^{-3z}\cdot 4\cdot\frac{2}{z^2+4}=e^{-3z}\,\mathcal{L}\{4\,\sin 2t\}(z),$$

hence by the shifting property

$$\mathcal{L}^{-1}\left\{\frac{8e^{-3z}}{z^2+4}\right\}(t)=\begin{cases} 4\,\sin(2\{t-3\}) & \text{for } t\geq 3,\\ 0 & \text{for } t<3.\end{cases}$$

3) Using the same method we get in this case

$$\frac{z\,e^{-2z}}{z^2+3z+2} = e^{-2z}\cdot\frac{z}{z^2+3z+2} = e^{-2z}\left\{\frac{2}{z+2}-\frac{1}{z+1}\right\}$$
$$= e^{-2z}\,\mathcal{L}\left\{2\,e^{-2t}-e^{-t}\right\}(z),$$

from which by the shifting property

$$\mathcal{L}^{-1}\left\{\frac{z\,e^{-2z}}{z^2+3z+2}\right\}(t) = \begin{cases} 2\,e^{-2(t-2)}-e^{-(t-2)} & \\ \qquad = 2\,e^4\,e^{-2t}-e^2\,e^{-t} & \text{for } t \geq 2, \\ 0 & \text{for } t < 2. \end{cases} \qquad \Diamond$$

Example 1.3.18 *Compute the inverse Laplace transform of*

$$\frac{e^{-z}\,(1-e^{-z})}{z\,(z^2+1)}.$$

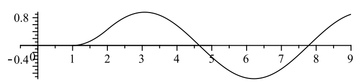

Figure 12: The graph of the the solution $f(t)$ of Example 1.3.18.

First, rewrite in the following way,

$$\frac{e^{-z}\,(1-e^{-z})}{z\,(z^2+1)} = \left(e^{-z}-e^{-2z}\right)\left\{\frac{1}{z}-\frac{z}{z^2+1}\right\} = e^{-z}\,\mathcal{L}\{1-\cos t\}(z)-e^{-2z}\,\mathcal{L}\{1-\cos t\}(z),$$

from which we get

$$f(t)=\mathcal{L}^{-1}\left\{\frac{e^{-z}\,(1-e^{-z})}{z\,(z^2+1)}\right\}(t) = \begin{cases} \cos(t-2)-\cos(t-1) & \text{for } t \geq 2, \\ 1-\cos(t-1) & \text{for } 1 \leq t < 2, \\ 0 & \text{for } t < 1. \end{cases} \qquad \Diamond$$

Example 1.3.19 *Compute the inverse Laplace transform of*

$$\frac{e^{-2z}}{(z-1)^4}.$$

It follows from

$$\frac{e^{-2z}}{(z-1)^4} = e^{-2z}\cdot\frac{1}{6}\cdot\mathcal{L}\left\{e^t\,t^3\right\}(z),$$

that

$$\mathcal{L}^{-1}\left\{\frac{e^{-2z}}{(z-1)^4}\right\}(t) = \begin{cases} \dfrac{1}{6}\,e^{-2}\,e^t\,(t-2)^3 & \text{for } t \ge 2, \\ 0 & \text{for } t < 2. \end{cases} \qquad \Diamond.$$

Example 1.3.20 *Compute the inverse Laplace transform of the function*

$$F(z) := \text{Log}\left(\frac{z+2}{z+1}\right),$$

by first finding $\mathcal{L}^{-1}\left\{F'(z)\right\}(t)$.

It follows from the computation

$$F'(z) = \frac{1}{z+2} - \frac{1}{z+1} = \mathcal{L}\left\{e^{-2t}\right\}(z) - \mathcal{L}\left\{e^{-t}\right\}(z),$$

that

$$\mathcal{L}^{-1}\left\{F'(z)\right\}(t) = e^{-2t} - e^{-t},$$

so

$$\mathcal{L}\left\{e^{-2t} - e^{-t}\right\}(z) = F'(z).$$

Then it follows from the rule of division by t that

$$\mathcal{L}\left\{\frac{e^{-2t}-e^{-t}}{t}\right\}(z) = \int_{\Gamma_z} F'(\zeta)\,\mathrm{d}\zeta = -F(z),$$

from which we derive that

$$\mathcal{L}^{-1}\{F(z)\}(t) = \mathcal{L}^{-1}\left\{\text{Log}\left(\frac{z+2}{z+1}\right)\right\}(t) = \frac{e^{-t}-e^{-2t}}{t}. \qquad \Diamond$$

Example 1.3.21 *Compute the inverse Laplace transform of*

$$F(z) := \text{Arctan}\left(\frac{2}{z^2}\right)$$

by first finding $\mathcal{L}^{-1}\left\{F'(z)\right\}(t)$.

We first compute

$$\begin{aligned}
F'(z) &= \frac{1}{1+\left\{\frac{2}{z^2}\right\}^2}\cdot\left(-\frac{4}{z^3}\right) = -\frac{4z}{z^4+4} = -\frac{4z}{(z^4+4z^2+4)-4z^2} \\
&= -\frac{4z}{\left(z^2+2\right)^2-(2z)^2} = -\frac{4z}{\left(z^2+2z+2+2\right)\left(z^2-2z+2\right)} \\
&= \frac{1}{z^2+2z+2} - \frac{1}{z^2-2z+2} = \frac{1}{(z+1)^2+1} - \frac{1}{(z-1)^2+1} \\
&= \mathcal{L}\left\{e^{-t}\,\sin t - e^{t}\,\sin t\right\}(z) = \mathcal{L}\{-2\sinh t\cdot\sin t\}(z).
\end{aligned}$$

Then we apply the rule of division by t to get

$$F(z) = -\int_{\Gamma_z} F'(\zeta)\,\mathrm{d}\zeta = \mathcal{L}\left\{\frac{2\,\sinh t\cdot\sin t}{t}\right\}(z),$$

from which finally,

$$\mathcal{L}^{-1}\left\{\text{Arctan}\left(\frac{2}{z^2}\right)\right\}(t) = \frac{2\,\sinh t\cdot\sin t}{t}. \qquad \Diamond$$

Example 1.3.22 *Compute the inverse Laplace transform of*

$$F(z) := \text{Log}\left(\frac{z^2+a^2}{z^2+b^2}\right),$$

by first finding $\mathcal{L}^{-1}\left\{F'(z)\right\}(t)$.

It follows from

$$F'(z) = \frac{2z}{z^2+a^2} - \frac{2z}{z^2+b^2}$$

that

$$\mathcal{L}^{-1}\left\{F'(z)\right\}(t) = 2\,\cos at - 2\,\cos bt.$$

Then by the rule of division by t,

$$\mathcal{L}\left\{2\,\frac{\cos bt - \cos at}{t}\right\}(z) = -\int_{\Gamma_z} F'(z)\,\mathrm{d}z = F(z) = \text{Log}\left(\frac{z^2+a^2}{z^2+b^2}\right),$$

from which

$$\mathcal{L}^{-1}\left\{\text{Log}\left(\frac{z^2+a^2}{z^2+b^2}\right)\right\}(t) = 2\cdot\frac{\cos bt - \cos at}{t}. \qquad \Diamond$$

Example 1.3.23 *Find the inverse Laplace transform of* $\exp\left(\frac{1}{z}\right) - 1$.

The function $\exp\left(\frac{1}{z}\right) - 1$ has the convergent Laurent series expansion

$$\exp\left(\frac{1}{z}\right) - 1 = \sum_{n=1}^{+\infty}\frac{1}{n!}\,\frac{1}{z^n} = \sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\,\frac{1}{z^{n+1}} \qquad \text{for } z \in \mathbb{C}\setminus\{0\},$$

and it satisfies the estimate

$$\left|\exp\left(\frac{1}{z}\right) - 1\right| \le \frac{C}{|z|} \qquad \text{for } |z| > R,$$

for some positive constants C and R. Hence the inverse Laplace transform is given by

$$\mathcal{L}^{-1}\left\{\exp\left(\frac{1}{z}\right) - 1\right\} = \sum_{n=0}^{+\infty}\frac{1}{(n+1)!\,n!}\,t^n.$$

Remark 1.3.1 If we use, which is the topic of *Ventus, Complex Functions Theory c-12, The Laplace Transform II*, that

$$J_1(u) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{u}{2}\right)^{2n+1} = \frac{u}{2}\sum_{n=0}^{+\infty}\frac{1}{(n+1)!\,n!}\left(-\frac{u^2}{4}\right)^n,$$

and put $t = -\dfrac{u^2}{4}$, then $u = 2i\sqrt{t}$, hence by insertion,

$$\mathcal{L}^{-1}\left\{\exp\left(\frac{1}{z}\right) - 1\right\}(t) = \sum_{n=0}^{+\infty} \frac{1}{n!\,(n+1)!}\, t^n = \frac{1}{i\sqrt{t}}\, J_1\left(2i\sqrt{t}\right),$$

where we use that the Bessel function is an analytic function in $\mathbb{C}$. $\Diamond$

Example 1.3.24 *Find the inverse Laplace transform of* $\sinh\left(\dfrac{1}{z}\right)$.

It follows from the Laurent series expansion

$$\sinh\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)!}\,\frac{1}{z^{2n+1}}, \qquad \text{for } z \in \mathbb{C}\setminus\{0\},$$

and the estimate

$$\left|\sinh\frac{1}{z}\right| \le \frac{C}{|z|} \qquad \text{for } |z| > R,$$

that

$$\mathcal{L}^{-1}\left\{\sinh\left(\frac{1}{z}\right)\right\}(t) = \sum_{n=0}^{+\infty} \frac{1}{(2n)!(2n+1)!}\, t^{2n}. \qquad \Diamond$$

Example 1.3.25 *Find the inverse Laplace transform of* $\cosh\left(\dfrac{1}{z}\right) - 1$.

The Laurent series expansion is given by

$$\cosh\left(\frac{1}{z}\right) - 1 = \sum_{n=1}^{+\infty} \frac{1}{(2n)!}\,\frac{1}{z^{2n}} = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)!}\,\frac{1}{z^{(2n+1)+1}}, \qquad \text{for } z \in \mathbb{C}\setminus\{0\},$$

and we clearly have the estimate

$$\left|\cosh\left(\frac{1}{z}\right) - 1\right| \le \frac{C}{|z|^2} \qquad \text{for } |z| > R,$$

so the inverse Laplace transform is

$$\mathcal{L}^{-1}\left\{\cosh\left(\frac{1}{z}\right) - 1\right\}(t) = \sum_{n=0}^{+\infty} \frac{1}{(2n+2)!}\,\frac{1}{(2n+1)!}\, t^{2n+1}. \qquad \Diamond$$

Example 1.3.26 *Find the inverse Laplace transforms of*

1) $\dfrac{1}{z}\,\sin\dfrac{1}{z}$,

2) $\dfrac{1}{z}\,\cos\dfrac{1}{z}$.

1) The Laurent series expansion is for $z \neq 0$ given by

$$F(z) = \frac{1}{z}\,\sin\frac{1}{z} = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{1}{z^{2n+1}} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{1}{z^{(2n+1)+1}},$$

hence

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{\{(2n+1)!\}^2}\,t^{2n+1}.$$

2) In this case, the Laurent series expansion is given by

$$F(z) = \frac{1}{z}\,\cos\frac{1}{z} = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,\frac{1}{z^{2n}} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,\frac{1}{z^{2n+1}},$$

hence

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{\{(2n)!\}^2}\,t^{2n}. \qquad \Diamond$$

Example 1.3.27 *Find the inverse Laplace transform of*

$$\text{Log}\left(1+\frac{1}{z^2}\right), \qquad \textit{for } |z|>1.$$

We get for $|z|>1$ the convergent Laurent series expansion,

$$F(z) = \text{Log}\left(1+\frac{1}{z^2}\right) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{1}{z^{(2n-1)+1}},$$

Hence

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F\}(t) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n(2n-1)!}\,t^{2n-1} = -2\sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n-1} \\ &= -\frac{2}{t}\left\{\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n}-1\right\} = \frac{2}{t}\,(1-\cos t). \end{aligned}$$

Remark 1.3.2 If we compare with Example 1.2.39 we see by putting $a = 0$ and $b = 1$ that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\}(z) = \mathcal{L}\left\{\frac{1 - cost}{t}\right\}(z) = \frac{1}{2}\,\mathrm{Log}\left(\frac{z^2+b^2}{z^2+a^2}\right) = \frac{1}{2}\,\mathrm{Log}\left(1+\frac{1}{z^2}\right),$$

so

$$\mathcal{L}^{-1}\left\{\mathrm{Log}\left(1+\frac{1}{z^2}\right)\right\}(t) = \frac{2}{t}\,(1-\cos t),$$

and we have given an alternative proof of the above. $\diamond$

Example 1.3.28 *Find the inverse Laplace transform of*

$$\frac{1}{z}\,\mathrm{Log}\left(1+\frac{1}{z^2}\right), \qquad \textit{for } |z| > 1.$$

By a Laurent series expansion,

$$F(z) = \frac{1}{z}\,\mathrm{Log}\left(1+\frac{1}{z^2}\right) = \frac{1}{z}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{1}{z^{2n}} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{1}{z^{2n+1}},$$

which is convergent for $|z| > 1$, we get

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n \cdot (2n)!}\, t^{2n}. \qquad \Diamond$$

Example 1.3.29 *Find the Laplace transform of the function*

$$\frac{\cos\sqrt{t}}{\sqrt{t}}.$$

We get for $t > 0$ the series expansion

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}\, t^n = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}\, t^{n-\frac{1}{2}}.$$

This gives by a formal termwise Laplace transformation,

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}\, \frac{\Gamma\left(n+\frac{1}{2}\right)}{z^{n+\frac{1}{2}}}.$$

Then we compute

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)(2n-3)\cdots 1}{2\cdot 2\cdots 2}\,\sqrt{\pi} = \frac{(2n)!}{4^n\, n!}\,\sqrt{\pi}.$$

Hence by insertion, where the *series* is seen to be convergent for $z \neq 0$,

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}(z) = \frac{1}{\sqrt{z}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{(2n)!}{4^n\, n!}\,\sqrt{\pi} \cdot \frac{1}{z^n} = \frac{\sqrt{\pi}}{\sqrt{z}} \sum_{n=0}^{+\infty} \frac{1}{n!}\left\{-\frac{1}{4z}\right\}^n = \frac{\sqrt{\pi}}{\sqrt{z}}\,\exp\left(-\frac{1}{4z}\right).$$

Thus we have proved that if we confine ourselves to the half plane $\Re\, z > 0$, such that the square root is also defined uniquely, then

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}(z) = \sqrt{\frac{\pi}{z}}\,\exp\left(-\frac{1}{4z}\right), \qquad \text{for } \Re\, z > 0. \qquad \Diamond$$

1.4 Convolutions

1.4.1 Convolutions, general

Example 1.4.1 *Prove that if f, $g \in \mathcal{E}$, then also $f \star g \in \mathcal{E}$.*

We assume that f and g are both piecewise continuous and that they fulfil the estimates

$$|f(t)| \le A\, e^{Bt} \qquad \text{and} \qquad |g(t)| \le A\, e^{Bt}, \qquad \text{for } t \ge 0,$$

where we clearly can choose the same constants in both cases.

Then

$$\begin{aligned} |(f \star g)(t)| &= \left|\int_0^t f(\tau) g(t-\tau)\, \mathrm{d}\tau\right| \le \int_0^t A\, e^{B\tau} \cdot A\, e^{B(t-\tau)}\, \mathrm{d}\tau \\ &= A^2\, t\, e^{Bt} \le A^2\, e^{(B+1)t}, \end{aligned}$$

where we have used that $0 \le t \le e^t$ for $t \ge 0$.

We have furthermore,

$$\begin{aligned} |(f \star g)(t) - (f \star g)(t_0)| &= \left|\int_0^t f(\tau) g(t-\tau)\, \mathrm{d}\tau - \int_0^{t_0} f(tau) g(t_0 - \tau)\, \mathrm{d}\tau\right| \\ &\le \int_0^{t_0} |f(\tau)| \cdot |g(t-\tau) - g(t_0-\tau)| + \int_{t_0}^t |f(\tau)| \cdot |g(t-\tau)|\, \mathrm{d}\tau \\ &\le A\, e^{Bt_0} \int_0^{t_0} |g(t-\tau) - g(t_0-\tau)|\, \mathrm{d}\tau + A^2\, e^{Bt} \cdot |t - t_0|\,. \end{aligned}$$

The latter term tends trivially towards 0 for $t \to t_0$, and since g is piecewise continuous, the former term also tends towards 0 for $t \to t_0$ for almost every t_0, so $f \star g$ is also piecewise continuous, and we have proved that the convolution $f \star g \in \mathcal{E}$. $\Diamond$

Example 1.4.2 *Use the theorem of convolution to find the inverse Laplace transform of*

$$\frac{1}{(z+3)(z-1)}.$$

This is an alternative way of computation to ordinary decomposition. We get straight away,

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\frac{1}{(z+3)(z-1)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{z+3}\right\} \star \mathcal{L}^{-1}\left\{\frac{1}{z-1}\right\}(t) = \left(e^{-3\tau} \star e^{\tau}\right)(t) \\ &\quad = \int_0^t e^{-3\tau} \cdot e^{t-\tau}\, \mathrm{d}\tau = e^t \int_0^t e^{-4\tau}\, \mathrm{d}\tau = \frac{1}{4} e^t \left[e^{-4\tau}\right]_0^t = \frac{1}{4} e^{-3t} - \frac{1}{4} e^t. \qquad \Diamond \end{aligned}$$

Example 1.4.3 *Compute the explicit function* $\left(\chi_{\mathbb{R}_+} \star \cdots \star \chi_{\mathbb{R}_+}\right)(t)$, *where the convolution is taken* n *times.*

It follows from $\mathcal{L}\left\{\chi_{\mathbb{R}_+}\right\}(z) = \dfrac{1}{z}$ that

$$\mathcal{L}\left\{\chi_{\mathbb{R}_+} \star \cdots \star \chi_{\mathbb{R}_+}\right\}(z) = \frac{1}{z^n} = \frac{1}{(n-1)!}\,\mathcal{L}\left\{t^{n-1}\right\}(z),$$

from which we immediately get

$$\left(\chi_{\mathbb{R}_+} \star \cdots \star \chi_{\mathbb{R}_+}\right)(t) = \frac{t^{n-1}}{(n-1)!}. \qquad \Diamond.$$

Example 1.4.4 *Apply convolution to compute the inverse Laplace transform of*

$$F(z) = \frac{1}{z^2(z+1)^2}.$$

We note that

$$\frac{1}{z^2} = \mathcal{L}\{t\}(z) \qquad \text{and} \qquad \frac{1}{(z+1)^2} = \mathcal{L}\left\{t\,e^{-t}\right\}(z),$$

so by the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{z^2(z+1)^2}\right\}(t) &= \left(\tau\,e^{-\tau} \star \tau\right)(t) = \int_0^t \tau\,e^{-\tau}(t-\tau)\,\mathrm{d}\tau = t\int_0^t \tau e^{-\tau}\,\mathrm{d}\tau - \int_0^t \tau^2 e^{-\tau}\,\mathrm{d}\tau \\
&= t\,\left[-\tau\,e^{-\tau}\right]_0^t + t\int_0^t e^{-\tau}\,\mathrm{d}\tau + \left[\tau^2\,e^{-\tau}\right]_0^t - 2\int_0^t \tau\,e^{-\tau}\,\mathrm{d}\tau \\
&= -t^2\,e^{-t} + t\,\left[-e^{-\tau}\right]_0^t + t^2\,e^{-t} + 2\,\left[\tau\,e^{-\tau}\right]_0^t - 2\int_0^t e^{-\tau}\,\mathrm{d}\tau \\
&= t - t\,e^{-t} + 2t\,e^{-t} + 2\,\left\{e^{-t} - 1\right\} = (t+2)e^{-t} + t - 2. \qquad \Diamond
\end{aligned}$$

Example 1.4.5 *Apply the convolution theorem to find a solution formula of the inverse Laplace transform of*

$$\frac{\mathcal{L}\{f\}(z)}{1+z^2} \qquad \textit{for given } f \in \mathcal{F}.$$

By a straightforward computation, using the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f\}(z)}{z^2+1}\right\}(t) &= (\sin \star f)(t) = \int_0^t f(u)\cdot\sin(t-u)\,\mathrm{d}u \\
&= \sin t\int_0^t f(u)\cos u\,\mathrm{d}u - \cos t\int_0^t \sin u\,\mathrm{d}u.
\end{aligned}$$

Example 1.4.6 *Apply the convolution theorem to prove that*

$$\int_0^t \sin u \cdot \cos(t-u)\,\mathrm{d}u = \frac{1}{2}\,t\,\sin t.$$

We get by the Laplace transformation,

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{2}\,t\,\sin t\right\}(z) &= -\frac{1}{2}\,\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\} = -\frac{1}{2}\cdot\frac{-2z}{\left(z^2+1\right)^2} = \frac{1}{z^2+1}\cdot\frac{z}{z^2+1} \\ &= \mathcal{L}\{\sin\star\cos\}(z).\end{aligned}$$

Hence, by the inverse Laplace transformation,

$$\frac{1}{2}\,t\,\sin t = (\sin\star\cos)(t) = \int_0^t \sin u\cdot\cos(t-u)\,\mathrm{d}u. \qquad \Diamond$$

Example 1.4.7 *Prove that*

$$(6)\quad \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(u)\,\mathrm{d}u\,\mathrm{d}t_{n-1}\cdots \mathrm{d}t_1 = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} f(u)\,\mathrm{d}u = \left(\frac{t^{n-1}}{(n-1)!}\star f\right)(t).$$

In other words, equation (6) means that n successive integrations all starting at 0 can be expressed as a convolution integral.

Let I denote the integration operator defined by

$$I(f)(t) := \int_0^t f(u)\,\mathrm{d}u.$$

Then (6) means that

$$I^n(f)(t) = \frac{1}{(n-1)!}\left(t^{n-1}\star f\right)(t).$$

First method. It follows from the rule of integration that

$$\mathcal{L}\left\{\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(u)\,du\,dt_{n-1}\cdots dt_1\right\}(z) = \frac{1}{z^n}\,\mathcal{L}\{f\}(z).$$

It also follows from the rule of convolution that

$$\mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\star f\right\}(z) = \mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\right\}(z)\cdot\mathcal{L}\{f\}(z) = \frac{1}{(n-1)!}\cdot\frac{(n-1)!}{z^n}\,\mathcal{L}\{f\}(z) = \frac{1}{z^n}\,\mathcal{L}\{f\}(z).$$

When we combine these two equations and then apply the inverse Laplace transformation, we get (6).

Second method. *Induction.* If $n=1$, then both sides of (6) are equal to $\int_0^t f(u)\,\mathrm{d}u$, so the claim is true for $n=1$.

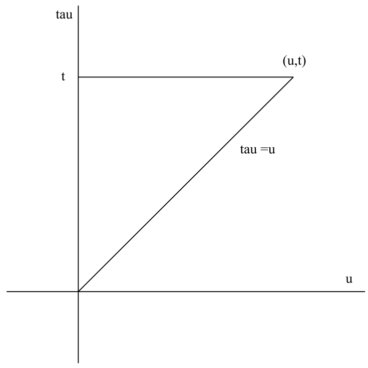

Figure 13: The domain of integration in Example 1.4.7, second method.

Then assume that the claim is true for some $n \in \mathbb{N}$, and consider $n+1$ integrations. Then it follows from the assumption of induction followed by an interchange of the order of integration, cf. Figure 13, that

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_n} f(u)\,\mathrm{d}u\,\mathrm{d}t_n \cdots \mathrm{d}t_1 = \int_0^t \left\{ \int_0^{tau} \frac{(\tau-u)^{n-1}}{(n-1)!} f(u)\,\mathrm{d}u \right\} \mathrm{d}\tau$$

$$= \int_0^t \left\{ \int_u^t \frac{(\tau-u)^{n-1}}{(n-1)!}\,\mathrm{d}\tau \right\} f(u)\,\mathrm{d}u = \int_0^t \left[\frac{(\tau-u)^n}{n!} \right]_{\tau=u}^t f(u)\,\mathrm{d}u = \int_0^t \frac{(t-u)^n}{n!} f(u)\,\mathrm{d}u,$$

proving that then the assumption also holds for $n+1$. Since it already holds for $n=1$, it follows by induction that it holds in general.

Remark 1.4.1 It is then only a matter of interpretation to write

$$(7)\quad I^n(f) = \frac{1}{(n-1)!}\left(t^{n-1} \star f\right)(t) = \frac{1}{\Gamma(n)}\left(t^{n-1} \star f\right)(t) \qquad \text{for } n \in \mathbb{N},$$

instead of (6). This structure also invites one to guess that in general,

$$I^\alpha(f) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1} \star f\right)(t) \qquad \text{for } \alpha \in \mathbb{R}_+,$$

which indeed also is true, though we shall not prove it here. $\Diamond$

1.4.2 Convolution equations

Example 1.4.8 *Solve the convolution equation*

$$\int_0^t f(u) f(t-u)\,\mathrm{d}u = 2f(t) + t - 2, \qquad \textit{for } t \in \mathbb{R}_+,$$

where the unknown function f is assumed to be continuous.

If we put $F(z) = \mathcal{L}\{f\}(z)$, then it follows from the Laplace transformation and the rule of convolution that

$$F(z)^2 = 2\,F(z) + \frac{1}{z^2} - \frac{2}{z},$$

thus

$$\{F(z)-1\}^2 = F(z)^2 - 2\,F(z) + 1 = 1 - \frac{2}{z} + \frac{1}{z^2} = \left(1 - \frac{1}{z}\right)^2,$$

and hence

$$F(z) = 1 \pm \left(1 - \frac{1}{z}\right) = \begin{cases} 2 - \dfrac{1}{z}, & \not\to 0 \text{ for } \Re z \to +\infty, \\[2ex] \dfrac{1}{z} & \to 0 \text{ for } \Re z \to +\infty. \end{cases}$$

Since f is assumed to be continuous, $F(z) = 2 - \frac{1}{z}$ cannot be possible, so we conclude that

$$F(z) = \frac{1}{z} = \mathcal{L}\{1\}(z),$$

hence $f(t) = 1$, which is also easily checked,

$$\int_0^t f(u)f(t-u)\,\mathrm{d}u = \int_0^t 1 \cdot 1\,\mathrm{d}u = t,$$

and

$$2f(t) + t - 2 = 2 + t - 2 = t. \qquad \diamond$$

Example 1.4.9 *Solve the convolution equation*

$$f(t) = t^2 + \int_0^t f(u)\sin(t-u)\,\mathrm{d}u, \qquad t \in \mathbb{R}_+.$$

We write for short $F(z) = \mathcal{L}\{f\}(z)$. Then it follows by an application of the Laplace transformation, followed by the rule of convolution, that

$$F(z) = \frac{2!}{z^3} + F(z) \cdot \frac{1}{z^2+1},$$

thus by solving in $F(z)$,

$$F(z) = \frac{1}{1 - \frac{1}{z^2+1}} \cdot \frac{2}{z^3} = \frac{z^2+1}{z^2} \cdot \frac{2}{z^3} = \frac{2}{z^3} + \frac{2}{z^5} = \frac{2!}{z^3} + \frac{1}{12} \cdot \frac{4!}{z^5},$$

and we get by the inverse Laplace transformation that

$$f(t) = t^2 + \frac{1}{12}\,t^4. \qquad \diamond$$

Example 1.4.10 *Solve the convolution equation*

$$f(t) = t + 2\int_0^t f(u)\cos(t-u)\,\mathrm{d}u, \qquad \textit{for } t \in \mathbb{R}_+.$$

We assume that $f \in \mathcal{F}$. Then it follows from the linearity and the rule of convolution that

$$\mathcal{L}\{f\}(z) = \mathcal{L}\{t\}(z) + 2\mathcal{L}\{f\}(z) \cdot \mathcal{L}\{\cos t\}(z) = \frac{1}{z^2} + \frac{2z}{z^2+1}\,\mathcal{L}\{f\}(z),$$

thus

$$\left\{1 - \frac{2z}{z^2+1}\right\}\mathcal{L}\{f\}(z) = \frac{z^2-2z+1}{z^2+1}\,\mathcal{L}\{f\}(z) = \frac{(z-1)^2}{z^2+1}\,\mathcal{L}\{f\}(z) = \frac{1}{z^2},$$

and hence

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{z^2+1}{z^2(z-1)^2} = \frac{1}{(z-1)^2} + \left\{\frac{1}{z(z-1)}\right\}^2 \\
&= \frac{1}{(z-1)^2} + \left\{-\frac{1}{z} + \frac{1}{z-1}\right\}^2 = \frac{1}{z^2} + \frac{2}{(z-1)^2} - \frac{2}{z(z-1)} \\
&= \frac{1}{z^2} + \frac{2}{(z-1)^2} + \frac{2}{z} - \frac{2}{z-1}.
\end{aligned}$$

Finally, it follows from the inverse Laplace transformation that

$$f(t) = 2 + t - 2\,e^t + 2t, e^t. \qquad \diamond$$

Example 1.4.11 *Prove that the convolution equation*

$$f(t) = t + \frac{1}{6}\int_0^t (t-u)^3 f(u)\,\mathrm{d}u, \qquad \textit{for } t \in \mathbb{R}_+,$$

has the solution $f(t) = \frac{1}{2}(\sin t + \sinh t)$.
Is this solution unique?

Put $F(z) = \mathcal{L}\{f\}(z)$. Then by the Laplace transformation,

$$F(z) = \frac{1}{z^2} + \frac{1}{6}\cdot\frac{3!}{z^4}\,F(z) = \frac{1}{z^2} + \frac{1}{z^4}\,F(z),$$

thus by a rearrangement,

$$F(z) = \frac{1}{1-\frac{1}{z^4}}\cdot\frac{1}{z^2} = \frac{z^2}{z^4-1} = \frac{z^2}{(z^2-1)(z^2+1)} = \frac{1}{2}\cdot\frac{1}{z^2-1} + \frac{1}{2}\cdot\frac{1}{z^2+1},$$

and hence by the inverse Laplace transformation,

$$f(t) = \frac{1}{2}\,\sinh t + \frac{1}{2}\,\sin t.$$

The solution is unique in the class of functions $\mathcal{F}$. $\Diamond$

Example 1.4.12 *Solve the integro differential equation*

$$\int_0^t f(u)\cos(t-u)\,\mathrm{d}u = f(t), \qquad t \in \mathbb{R}_+ \textit{ and } f(0) = 1.$$

We put for short $F(z) = \mathcal{L}\{f\}(z)$. Then by the Laplace transformation of the equation,

$$F(z)\cdot\frac{z}{z^2+1} = z\cdot F(z) - 1,$$

thus

$$F(z) = \frac{-1}{\frac{z}{z^2+1} - z} = -\frac{1}{z}\cdot\frac{z^2+1}{1-z^2-1} = \frac{z^2+1}{z^3} = \frac{1}{z^3} + \frac{1}{2}\cdot\frac{2!}{z^3},$$

and hence by the inverse Laplace transformation,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{z} + \frac{1}{z^2}\right\}(t) = 1 + \frac{1}{2}\,t^2. \qquad \Diamond$$

Example 1.4.13 *Find a function $f(t)$ in $\mathcal{F}$, such that*

$$\int_0^t u\, f(u)\cos(t-u)\,\mathrm{d}u = t\,e^{-t} - \sin t, \qquad \textit{for } t \in \mathbb{R}_+.$$

Write for short, $F(z) = \mathcal{L}\{f\}(z)$. Then by the Laplace transformation of the equation,

$$\mathcal{L}\left\{\int_0^t u\, f(u)\cos(t-u)\,\mathrm{d}u\right\}(z) = \mathcal{L}\{t\,f(t)\}(z)\cdot\mathcal{L}\{\cos t\}(z) = -\frac{d}{dz}\,F(z)\cdot\frac{z}{z^2+1}$$

$$= \mathcal{L}\left\{t\,e^{-t}\right\}(z) - \mathcal{L}\{\sin t\}(z) = \mathcal{L}\{t\}(z+1) - \frac{1}{z^2+1} = \frac{1}{(z+1)^2} - \frac{1}{z^2+1},$$

thus

$$\frac{dF}{dz} = \frac{z^2+1}{z}\left\{\frac{1}{z^2+1} - \frac{1}{(z+1)^2}\right\} = \frac{1}{z} - \frac{z^2+1}{z(z+1)^2} = \frac{z^2+2z+1-z^2-1}{z(z+1)^2} = \frac{2}{(z+1)^2},$$

and hence by an integration,

$$F(z) = -\frac{2}{z+1} + c.$$

We have assumed that $F(z) = \mathcal{L}\{f\}(z)$, where $f \in \mathcal{F}$. Therefore, we must require that $F(z) \to 0$ for $\Re\, z \to +\infty$, so $c = 0$, and we get

$$f(t) = \mathcal{L}^{-1}\left\{-\frac{2}{z+1}\right\}(t) = -2\,e^{-t},$$

which even belongs to $\mathcal{E} \subset \mathcal{F}$. $\Diamond$

Example 1.4.14 *Find all functions $f \in \mathcal{F}$, for which*

$$\int_0^x (t) f(x-t)\,\mathrm{d}t = 8(\sin x - x\cdot\cos x), \qquad \textit{for } x \in \mathbb{R}_+.$$

As usual, write for short $F(z) = \mathcal{L}\{f\}(z)$. Then by an application of the Laplace transformation,

$$\begin{aligned} F(z)^2 &= 8(\mathcal{L}\{\sin x\}(z) - \mathcal{L}\{x\cdot\cos x\}(z)) = 8\left\{\frac{1}{1+z^2} - (-1)\,\mathrm{d}dz\left(\frac{z}{1+z^2}\right)\right\} \\ &= 8\left\{\frac{1}{1+z^2} + \frac{1}{1+z^2} - \frac{2z^2}{(1+z^2)^2}\right\} = 16\cdot\frac{1+z^2-z^2}{(1+z^2)^2} = \left\{\frac{4}{1+z^2}\right\}^2, \end{aligned}$$

and hence

$$F(z) = \pm\frac{4}{1+z^2} = \pm 4\,\mathcal{L}\{\sin x\}(z).$$

Finally, the inverse Laplace transformation gives

$$f(x) = \pm 4\,\sin x,$$

and we see that the sign $\pm$ is quite obvious. $\Diamond$

Example 1.4.15 *Solve the integro differential equation*

$$f'(t)+5\int_0^t f(u)\cos(2\{t-u\})\,\mathrm{d}u=10,\qquad \text{for } t\in\mathbb{R}_+\cup\{0\} \text{ and } f(0)=2.$$

When we successively apply the Laplace transformation, the rule of differentiation and the rule of convolution, then we get

$$\begin{aligned}\frac{10}{z} &= z\,\mathcal{L}\{f\}(z)-f(0)+5\cdot\mathcal{L}\{f\}(z)\cdot\mathcal{L}\{\cos 2t\}(z)\\ &= z\,\mathcal{L}\{f\}(z)-2+5\cdot\frac{z}{z^2+4}\cdot\mathcal{L}\{f\}(z)=z\cdot\frac{z^2+4+5}{z^2+4}\,\mathcal{L}\{f\}(z)-2,\end{aligned}$$

hence by a rearrangement,

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{z^2+4}{z\,(z^2+9)}\left\{\frac{10}{z}+2\right\}=\frac{z^2+4}{(z^2+9)}\cdot(2z+10)\\ &= 2(z+5)\left\{\frac{4}{9}\cdot\frac{1}{z^2}+\frac{5}{9}\cdot\frac{1}{z^2+9}\right\}\\ &= \frac{8}{9}\cdot\frac{1}{z}+\frac{40}{9}\cdot\frac{1}{z^2}+\frac{10}{9}\cdot\frac{z}{z^2+9}+\frac{50}{27}\cdot\frac{3}{z^2+9}.\end{aligned}$$

Finally, we get by the inverse Laplace transformation that

$$f(t)=\frac{8}{9}+\frac{40}{9}\,t+\frac{10}{9}\,\cos 3t+\frac{50}{27}\,\sin 3t,$$

which clearly belongs to $\mathcal{E}\subset\mathcal{F}$. $\Diamond$

1.4.3 Bessel functions

Example 1.4.16 *Apply the series expansion*

$$\cos(t\cdot\sin\Theta)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n}\,\sin^{2n}\Theta$$

to prove that

$$J_0(t)=\frac{1}{\pi}\int_0^{\pi}\cos(t\cdot\sin\Theta)\,\mathrm{d}\Theta.$$

The series expansion follows immediately from the series expansion of $\cos u$, $u=t\cdot\sin\Theta$. For fixed t the argument $t\cdot\sin\Theta\in[-|t|,|t|]$ is bounded, so the series expansion is for fixed t uniformly convergent in Θ. Hence, we can interchange summation and integration. This interchange will give us

$$(8)\quad \varphi(t):=\frac{1}{\pi}\int_0^{\pi}\cos(t\cdot\sin\Theta)\,\mathrm{d}\Theta=\frac{1}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n}\int_0^{\pi}\sin^{2n}\Theta\,\mathrm{d}\Theta.$$

We shall compute the trigonometric integral. It follows for $n \in \mathbb{N}$ by partial integration that

$$\begin{aligned}
\int_0^{2n} \Theta \, d\Theta &= \int_0^{\pi} \sin^{2n-1}\Theta \cdot \sin\Theta \, d\Theta \\
&= \left[-\cos\Theta \cdot \sin^{2n-1}\Theta\right]_0^{\pi} + (2n-1)\int_0^{\pi} \sin^{2n-2}\Theta \cdot \cos^2\Theta \, d\Theta \\
&= (2n-1)\int_0^{\pi} \sin^{2n-2}\Theta \, d\Theta - (2n1)\int_0^{\pi} \sin^{2n}\Theta \, d\Theta.
\end{aligned}$$

This gives by a rearrangement and recursion,

$$\begin{aligned}
\int_0^{\pi} \sin^{2n}\Theta \, d\Theta &= \frac{2n-1}{2n}\int_0^{\pi} \sin^{2n-2}\Theta \, d\Theta = \cdots = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\int_0^{\pi} 1 \, d\Theta \\
&= \frac{(2n)!}{\{2^n \, n!\}^2}\,\pi,
\end{aligned}$$

thus by insertion into (8),

$$\varphi(t) = \frac{1}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\,t^{2n}\cdot\frac{(2n)!}{\{2^n \, n!\}^2}\cdot\pi = \sum_{n00}^{+\infty}\frac{(-1)^n}{\{2^n \, n!\}^2}\,t^{2n} = J_0(t),$$

and the claim is proved.

An *alternative proof* is the following. It follows from (8) that

$$\varphi(0) = \frac{1}{\pi} \cdot 1 \cdot 1 \cdot \pi = 1,$$

and

$$\varphi'(t) = \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-1)!}\, t^{2n-1} \int_{:} 0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta,$$

and

$$\varphi''(t) = \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-2)!}\, t^{2n-2} \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta,$$

so by insertion into the Bessel equation,

$$\begin{aligned}
&t^2\, \varphi''(t) + t\, \varphi'(t) + t^2\, \varphi(t) \\
&\quad = \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}\, t^{2n+2} \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta + \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-1)!}\, t^{2n} \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta \\
&\qquad\qquad + \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-2)!}\, t^{2n} \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta \\
&\quad = \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-1)!}\, t^{2n} \left\{ -(2n-1) \int_0^{\pi} \sin^{2n-2} \Theta \, \mathrm{d}\Theta \right. \\
&\qquad\qquad \left. + \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta + (2n-1) \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta \right\} \\
&\quad = \frac{1}{\pi} \sum_{n=1}]^{+\infty} \frac{(-1)^n}{(2n-1)!}\, t^{2n} \left\{ -(2n-1) \int_0^{\pi} \sin^{2n-2} \Theta \, \mathrm{d}\Theta + 2n \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta \right\} \\
&\quad = \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n-1)!}\, t^{2n} \left\{ -(2n-1) \int_0^{\pi} \sin^{2n-2} \Theta \, \mathrm{d}\Theta + 2n \cdot \frac{2n-1}{2n} \int_0^{\pi} \sin^{2n} \Theta \, \mathrm{d}\Theta \right\} \\
&= 0,
\end{aligned}$$

proving that φ fulfils the Bessel equation of order 0.

The series expansion of φ is clearly convergent in $\mathbb{C}$, so there exists a constant $c \in \mathbb{C}$, such that $\varphi(t) = c\, J_0(t)$. Finally, it follows from $\varphi(0) = 1 = J_0(0)$ that $c = 1$, and we have proved that

$$J_0(t) = \frac{1}{\pi} \int_0^{\pi} \cos(t \cdot \sin \Theta) \, \mathrm{d}\Theta. \qquad \Diamond$$

Example 1.4.17 *Prove by using the series expansion of $J_(t)$ that*

$$\int_0^t J_0(t-\tau)J_0(\tau)\,\mathrm{d}\tau = (J_0 \star J_0)(t) = \sin t, \qquad \textit{for } t \geq 0.$$

Using the convergent series expansion

$$J_0(t) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \left\{\frac{t}{2}\right\}^{2n}, \qquad \text{for } t \in \mathbb{R},$$

it follows that

$$\begin{aligned}
J_0(t-\tau)J_0(\tau) &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \cdot \frac{1}{2^{2n}} \cdot (t-\tau)^{2n} \cdot \sum_{m=0}^{+\infty} \frac{(-1)^m}{(m!)^2} \cdot \frac{1}{2^{2m}} \cdot \tau^{2m} \\
&= \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{p=0}^{2n} \frac{(-1)^n}{(n!)^2} \cdot \frac{1}{2^{2n}} \cdot \frac{(-1)^m}{(m!)^2} \cdot \frac{1}{2^{2m}} \left(\begin{array}{c} 2n \\ p \end{array}\right) t^{2n-p} \cdot (-1)^p\,\tau^p \cdot \tau^{2m} \\
&= \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{p=0}^{2n} \frac{(-1)^{n+m+p}}{4^{n+m}(n!)^2(m!)^2} \left(\begin{array}{c} 2n \\ p \end{array}\right) t^{2n-p}\,\tau^{2m+p}.
\end{aligned}$$

The series are uniformly convergent, so we can integrate termwise. This gives

$$\int_0^t J_0(t-\tau)J_0(\tau)\,\mathrm{d}\tau = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{p=0}^{2n} \frac{(-1)^{n+m+p}}{4^{n+m}(n!)^2(m!)^2} \cdot \frac{1}{2m+p+1} \left(\begin{array}{c} 2n \\ p \end{array}\right) t^{2n+2m+1}.$$

Then apply the following change of indices, $q = n + m$, i.e. $m = q - n$ to get that this expression is equal to

$$\begin{aligned}
&\sum_{q=0}^{+\infty}\sum_{n=0}^{q}\sum_{p=0}^{2n} \frac{(-1)^{q+p}}{4^q(n!)^2\{(q-n)!\}^2} \cdot \frac{1}{2q-2n+p+1} \cdot \frac{(2n)!}{(2n-p)!p!}\,t^{2q+1} \\
&\quad = \sum_{q=0}^{+\infty} \frac{(-1)^q}{(2q+1)!}\,t^{2q+1} \left\{ \frac{(2q+1)!}{4^q} \sum_{n=0}^{q}\sum_{p=0}^{2n} \frac{(-1)^p\,(2n)!}{(n!)^2\{(q-n)!\}^2(2q-2n+p+1)(2n-p)!p!} \right\} \\
&\quad = \sum_{q=0}^{+\infty} \frac{(-1)^q}{(2q+1)!}\,t^{2q+1} \left\{ \frac{(2q+1)!}{4^q} \sum_{n=0}^{q} \frac{1}{(n!)^2\{(q-n)!\}^2} \left(\sum_{p=0}^{2n} \left(\begin{array}{c} 2n \\ p \end{array}\right) \frac{(-1)^p}{2q-2n+p+1} \right) \right\},
\end{aligned}$$

so it "only" remains to prove that the expression $\{\cdots\}$ is equal to 1.

We consider the well-known finite series expansion

$$(1-t)^{2n} = \sum_{p=0}^{2n} \left(\begin{array}{c} 2n \\ p \end{array}\right) (-1)^p\,t^p,$$

so

$$t^{2q-2n}(1-t)^{2n} = \sum_{p=0}^{2n} \left(\begin{array}{c} 2n \\ p \end{array}\right) (-1)^p t^{2q-2n+p},$$

hence by an integration, using the Beta function,

$$\int_0^1 t^{2q-2n}(1-t)^{2n}\,\mathrm{d}t = \sum_{p=0}^{2n} \binom{2n}{p} \frac{(-1)(^p}{2q-2n+p+1} = B(2q-2n+1,2n+1)$$
$$= \frac{(2q-2n)!(2n)!}{(2q+1)!}.$$

Then we get by insertion into $\{\cdots\}$,

$$\frac{(2q+1)!}{4^q}\sum_{n=0}^{q} \frac{1}{(n!)^2\{(q-n)!\}^2} \cdot \frac{(2q-2n)!(2n)!}{(2q+1)!} = \frac{1}{4q}\sum_{n=0}^{q} \binom{2n}{n}\binom{2q-2n}{q-n},$$

which we shall prove is equal to 1.

We remind the reader of the general binomial expansion

$$(1-4x)^{-\frac{1}{2}} = \sum_{q=0}^{+\infty} \binom{-\frac{1}{2}}{k}(-4x)^q = \sum_{q=0}^{+\infty} \binom{2q}{q} x^q \qquad \text{for } |x| < \frac{1}{4}.$$

It follows by a Cauchy multiplication that the coefficient of x^q in

$$(1-4x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} = (1-4x)^{-1}$$

is equal to both

$$\sum_{n=0}^{q} \binom{2n}{n}\binom{2q-2n}{q-n} \qquad \text{and} \qquad 4^q,$$

because

$$(1-4x)^{-1} = \sum_{q=0}^{+\infty} 4^q\, x^q \qquad \text{for } |x| < \frac{1}{4},$$

so

$$\sum_{n=0}^{q} \binom{2n}{n}\binom{2q-2n}{q-n} = 4^q,$$

which finally by insertion gives

$$\frac{1}{4^q}\sum_{n=0}^{q} \binom{2n}{n}\binom{2q-2n}{q-n} = 1,$$

and the claim is proved. $\Diamond$

1.5 Linear ordinary differential equations

1.5.1 Linear differential equation of constant coefficients

Example 1.5.1 *Solve the initial value problem*

$$f''(t) + f(t) = t, \qquad f(0) = 1, \quad f'(0) = -2.$$

There is no need to use the Laplace transformation in this case, because it follows by inspection that $f(t) = t$ is a particular integral, so the complete solution is

$$f(t) = t + a\,\cos t + b\,\sin t.$$

It follows from the initial conditions that $a = 1$ and $b = -3$, so the wanted solution is

$$f(t) = t + \cos t - 3\sin t.$$

Alternatively we put $F(z) = \mathcal{L}\{f\}(z)$ and then apply the Laplace transformation and the rule of differentiation on the differential equation,

$$\mathcal{L}\{t\}(z) = \frac{1}{z^2} = \mathcal{L}\left\{f''\right\}(z) + \mathcal{L}\{f\}(z) = \left(z^2+1\right)F(z) - z + 2,$$

so we get by a rearrangement,

$$F(z) = \frac{1}{z^2+1}\left\{\frac{1}{z^2}+z-2\right\} = \frac{z}{z^2+1} - \frac{2}{z^2+1} + \frac{1}{z^2} - \frac{1}{z^2+1} = \frac{z}{z^2+1} - \frac{3}{z^2+1} + \frac{1}{z^2}.$$

Finally, we get by the inverse Laplace transformation that

$$f(t) = t + \cos t - 3\sin t.$$

Remark 1.5.1 It is seen that both methods are applicable, but that the traditional one is the easiest one in this case. ◊

Example 1.5.2 *Solve the initial value problem*

$$f''(t) - 3f'(t) + 2f(t) = 4\,e^{2t}, \qquad f(0) = -3, \quad f'(0) = 5.$$

We write for short, $F(z) = \mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,

$$\begin{aligned}\mathcal{L}\left\{4e^{2t}\right\}(z) &= \frac{4}{z-2} = \mathcal{L}\left\{f''\right\}(z) - 3\mathcal{L}\left\{f'\right\}(z) + 2\mathcal{L}\{f\}(z)\\ &= z^2 <, F(z) - z\cdot(-3) - 5 - 3z\,F(z) + (-3) + 2\,F(z)\\ &= \left(z^2-3z+2\right)F(z) + 3z - 8,\end{aligned}$$

hence by a rearrangement,

$$\begin{aligned}F(z) &= \frac{1}{(z-1)(z-2)}\left\{\frac{4}{z-2} - 3z + 8\right\} = \frac{4}{(z-1)(z-2)^2} + \frac{-3z+8}{(z-1)(z-2)}\\ &= \frac{4}{z-1} + \frac{4}{z-1}\left\{\frac{1-(z-2)^2}{(z-2)^2}\right\} - \frac{5}{z-1} + \frac{2}{z-2} = -\frac{1}{z-1} + \frac{2}{z-2} - 4\cdot\frac{z-3}{(z-2)^2}\\ &= -\frac{1}{z-1} + \frac{2}{z-2} - \frac{4}{z-2} + \frac{4}{(z-2)^2} = -\frac{1}{z-1} - \frac{2}{z-2}] + \frac{4}{(z-2)^2}.\end{aligned}$$

We finally get by the inverse Laplace transformation,

$$f(t) = -e^t - 2e^{2t} + 4\,t\,e^{2t}. \qquad \Diamond$$

Example 1.5.3 *Solve the initial value problem*

$$f''(t) + 2f'(t) + 5f(t) = e^{-t}\sin t, \qquad f(0) = 0, \quad f'(0) = 1.$$

As usual we write for short $F(z) = \mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\sin t\right\}(z) &= \mathcal{L}\{\sin t\}(z+1) = \frac{1}{(z+1)^2+1}\\ &= z^2F(z) - 0\cdot z - 1 + 2\cdot zF(z) - 0 + 5F(z) = \left(z^2+2z+5\right)F(z) - 1,\end{aligned}$$

so by a rearrangement,

$$\begin{aligned} F(z) &= \frac{1}{(z+1)^2+2^2} + \frac{1}{\{(z+1)^2+4\}\{(z+1)^2+1\}} \\ &= \frac{1}{(z+1)^2+2^2} - \frac{1}{3}\,\frac{1}{(z+1)^2+2^2} + \frac{1}{3}\,\frac{1}{(z+1)^2+1} = \frac{1}{3}\,\frac{2}{(z+1)^2+2^2} + \frac{1}{3}\,\frac{1}{(z+1)^2+1}. \end{aligned}$$

We finally get by the inverse Laplace transformation,

$$f(t) = \frac{1}{3}\,e^{-t}\,\sin 2t + \frac{1}{3}\,e^{-t}\,\sin t. \qquad \Diamond$$

Example 1.5.4 *Solve the initial value problem*

$$f^{(3)}(t) - 3f''(t) + 3f'(t) - f(t) = t^2\,e^t, \qquad f(0)=1, \quad f'(0)=0, \quad f''(0)=-2.$$

It follows from the rule of differentiation and the initial conditions that

$$\begin{aligned} \mathcal{L}\left\{f^{(3)}\right\}(z) &= z^3\mathcal{L}\{f\}(z) - z^2 f(0) - z f'(0) - f''(0) &&= z^3\mathcal{L}\{f\}(z) - z^2 + 2, \\ \mathcal{L}\left\{f''\right\}(z) &= z^2\mathcal{L}\{f\}(z) - z f(0) - f'(0) &&= z^2\mathcal{L}\{f\}(z) - z, \\ \mathcal{L}\left\{f'\right\}(z) &= z\mathcal{L}\{f\}(z) - f(0) &&= z\mathcal{L}\{f\}(z) - 1, \\ \mathcal{L}\{f)(z) &= \mathcal{L}\{f\}(z) &&= 1\cdot\mathcal{L}\{f\}(z), \end{aligned}$$

thus

$$\mathcal{L}\left\{f^{(3)}(t) - 3f''(t) + 3f'(t) - f(t)\right\}(z) = \left(z^3 - 3x^2 + 3z - 1\right)\mathcal{L}\{f\}(z) - z^2 + 3z - 1,$$

and since

$$\mathcal{L}\left\{t^2 e^t\right\}(z) = \mathcal{L}\left\{t^2\right\}(z-1) = \frac{2!}{(z-1)^3},$$

we obtain that

$$(z-1)^3\,\mathcal{L}\{f\}(z) = \frac{2}{(z-1)^3} + z^2 - 3z + 1,$$

hence

$$\begin{aligned} \mathcal{L}\{f\}(z) &= \frac{2}{(z-1)^6} + \frac{z^2-3z+1}{(z-1)^3} = \frac{2}{(z-1)^6} + \frac{(z-1)^2-(z-1)-1}{(z-1)^3} \\ &= \frac{2}{5!}\cdot\frac{5!}{(z-1)^6} - \frac{1}{2}\cdot\frac{2!}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{z-1}. \end{aligned}$$

We finally get by the inverse Laplace transformation,

$$f(t) = \frac{1}{60}\,t^5\,e^t - \frac{1}{2}\,t^2\,e^t - t\,e^t + e^t.$$

Alternatively, (sketch) the equation is easily solved by inspection, if only we get the idea of multiplying it by its integrating factor e^{-1}. Then the equation can be reduced to (details are left to the reader)

$$\frac{d^3}{dt^3}\left\{e^{-t}f(t)\right\} = t^2,$$

so we get by three successive integrations that the complete solution is

$$f(t) = \frac{1}{60}\,t^5\,e^t + a\,t^2\,e^t + b\,t\,e^t + c\,e^t,$$

and then we find the constants a, b and c from the given initial conditions. $\Diamond$

Example 1.5.5 *Solve the* boundary value problem

$$f''(t) + 9f(t) = \cos 2t, \qquad f(0) = 1, \quad f\left(\frac{\pi}{2}\right) = -1.$$

We write for short $F(z) = \mathcal{L}\{f\}(z)$, and put $f'(0) = c$, which is the missing initial condition. Then by the Laplace transformation,

$$\mathcal{L}\{\cos 2t\}(z) = \frac{z}{z^2+4} = z^2F(z) - z - c + 9F(z),$$

hence by a rearrangement,

$$F(z) = \frac{1}{z^2+9}\cdot\frac{z}{z^2+4} + \frac{z+c}{z^2+9} = -\frac{1}{5}\,\frac{z}{z^2+9} + \frac{1}{5}\,\frac{z}{z^2+4} + \frac{z}{z^2+9} + \frac{c}{3}\cdot\frac{3}{z^2+9}.$$

We then apply the inverse Laplace transformation to get

$$f(t) = \frac{4}{5}\,\cos 3t + \frac{1}{5}\,\cos 2t + \frac{c}{3}\,\sin 3t.$$

For the boundary condition at $t = \frac{\pi}{2}$ we get

$$f\left(\frac{\pi}{2}\right) = -1 = -\frac{1}{5} - \frac{c}{3},$$

so

$$\frac{c}{3} = 1 - \frac{1}{5} = \frac{4}{5},$$

and hence

$$f(t) = \frac{4}{5}\,\cos 3t + \frac{1}{5}\,\cos 2t + \frac{4}{5}\,\sin 3t. \qquad \Diamond$$

Example 1.5.6 *Use the Laplace transformation to find a solution formula for the initial problem*

$$f''(t) + a^2 f(t) = g(t), \qquad f(0) = 1, \quad f'(0) = 2,$$

where the constant a and the function $g \in \mathcal{F}$ are given.

We assume that $f \in \mathcal{E}$, and also for the time being that $g \in \mathcal{E}$. Then by the rule of differentiation,

$$\mathcal{L}\left\{f''\right\}(z) = z^2\mathcal{L}\{f\}(z) - z \cdot f(0) - f'(0) = z^2\mathcal{L}\{f\}(z) - z - 2,$$

so when we apply the Laplace transformation operator on the equation we get

$$z^2\mathcal{L}\{f\}(z) - z - 2 + a^2\mathcal{L}\{f\}(z) = \mathcal{L}\{g\}(z),$$

hence by a rearrangement

$$\left(z^2 + a^2\right)\mathcal{L}\{f\}(z) = \mathcal{L}\{g\}(z) + z + 2,$$

and we get

$$\mathcal{L}\{f\}(z) = \frac{z}{z^2+a^2} + \frac{2}{a} \cdot \frac{a}{z^2+a^2} + \frac{1}{a} \cdot \frac{a}{z^2+a^2} \cdot \mathcal{L}\{g\}(z).$$

Then by the inverse Laplace transformation, assuming that $a \neq 0$,

$$(9) \quad f(t) = \cos(at) + \frac{2}{a}\,\sin(at) + \frac{1}{a}\int_0^t g(\tau)\sin(a(t-\tau))\,\mathrm{d}\tau.$$

We have derived (9) assuming that also $g \in \mathcal{E}$, but a trivial check of the solution (9) in the original equation shows that it is valid in general.

Finally, if $a = 0$, the original problem is reduced to

$$f''(t) = g(t), \qquad f(0) = 1, \quad f'(0) = 2,$$

from which successively,

$$f'(t) = 2 + \int_0^t g(\tau)\,\mathrm{d}\tau,$$

and

$$f(t) = 1+2t+\int_0^t \left\{\int_0^u g(\tau)\,\mathrm{d}\tau\right\}\mathrm{d}u = 1+2t+\int_0^t \left\{\int_\tau^t g(\tau)\,\mathrm{d}u\right\}\mathrm{d}\tau = 1+2t+\int_0^t (t-\tau)g(\tau)\,\mathrm{d}\tau. \quad \Diamond$$

Example 1.5.7 *Given $g \in \mathcal{F}$. Apply the Laplace transformation on the equation*

$$f''(t) - a^2\, f(t) = g(t)$$

to obtain a general solution formula for $a \neq 0$.

When we apply the Laplace transformation on the equation we get

$$z^2\,\mathcal{L}\{f\}(z) - z\, f(0) - f'(0) - a^2\,\mathcal{L}\{f\} = \mathcal{L}\{g\}(z),$$

from which

$$\left(z^2 - a^2\right)\mathcal{L}\{f\}(z) = \mathcal{L}\{g\}(z) + z\, f(0) + f'(0),$$

and thus

$$\begin{aligned}\mathcal{L}\{f\}(z) &= \frac{\mathcal{L}\{g\}(z)}{z^2-a^2} + \frac{z}{z^2-a^2}\, f(0) + \frac{1}{z^2-a^2}\, f'(0)\\ &= \frac{1}{a}\cdot\frac{a}{z^2-a^2}\,\mathcal{L}\{g\}(z) + f(0)\cdot\frac{z}{z^2-a^2} + \frac{f'(0)}{a}\cdot\frac{a}{z^2-a^2}.\end{aligned}$$

Then by the inverse Laplace transformation,

$$\begin{aligned}f(x) &= \frac{1}{a}\cdot(\sin at \star g)(x) + f(0)\cdot\cosh(ax) + \frac{f'(0)}{a}\cdot\sinh(ax)\\ &= \frac{1}{a}\int_0^x \sinh(a\{x-t\})g(t)\,\mathrm{d}t + f(0)\cdot\cosh ax + \frac{f'(0)}{a}\cdot\sinh ax\\ &= \frac{1}{a}\,\sinh(ax)\int_0^x \cosh(at)\cdot g(t)\,\mathrm{d}t - \frac{1}{a}\,\cosh(ax)\int_0^x \sinh(at)\cdot g(t)\,\mathrm{d}t\\ &\quad + f(0)\cdot\cosh(ax) + \frac{f'(0)}{a}\cdot\sinh(ax),\end{aligned}$$

which is the well-known Wroński solution formula with respect to the basis $\cosh(ax)$ and $\sinh(ax)$ of the solution space of the corresponding homogeneous equation. $\Diamond$

Example 1.5.8 *Solve the initial value problem*

$$f''(t) + 4f(t) = 9t, \qquad f(0) = 0, \quad f'(0) = 7.$$

We write for short $F(z) = \mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,

$$z^2 F(z) - 0 \cdot z - 7 + 4\,F(z) = \frac{9}{z^2},$$

hence

$$\begin{aligned} F(z) &= \frac{1}{z^2+4}\left\{\frac{9}{z^2}+7\right\} = \frac{7}{z^2+4} + 9 \cdot \frac{1}{z^2\,(z^2+4)} \\ &= \frac{7}{z^2+2^2} + \frac{9}{4}\cdot\frac{1}{z^2} - \frac{9}{4}\cdot\frac{1}{z^2+4} = \frac{9}{4}\cdot\frac{1}{z^2} + \frac{19}{8}\cdot\frac{2}{z^2+2^2}. \end{aligned}$$

Then by the inverse Laplace transformation,

$$f(t) = \frac{9}{4}\,t + \frac{19}{8}\,\sin 2t,$$

where it is easy to check this solution. ♢

Example 1.5.9 *Solve the initial problem*

$$f''(t) - 4f'(t) + 5f(t) = 125t^2, \qquad f(0) = f'(0) = 0.$$

Write for short $F(z) = \mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,,

$$\left(z^2 - 4z + 5\right) F(z) = 125 \cdot \frac{2}{z^3} = 250 \cdot \frac{1}{z^3},$$

hence when we solve with respect to $F(z)$ and reduce,

$$\begin{aligned} F(z) &= 250 \cdot \frac{1}{z^3} \cdot \frac{1}{z^2-4z+5} = \frac{50}{z^3} + 50 \cdot \frac{1}{z^3} \cdot \frac{1}{z^2-4z+5}\left(5 - z^2 - 4z - 5\right) \\ &= 25\,\mathcal{L}\left\{t^2\right\}(z) - 50 \cdot \frac{1}{z^2} \cdot \frac{z+4}{z^2-4z+5} \\ &= 25\,\mathcal{L}\left\{t^2\right\}(z) - 40 \cdot \frac{1}{z^2} + \frac{10}{z^2} \cdot \frac{1}{z^2-4z+5}\left(4z^2 - 16z + 20 - 5z - 20\right) \\ &= \mathcal{L}\left\{25t^2 - 40t\right\}(z) + \frac{10}{z} \cdot \frac{4z-21}{z^2-4z+5} \\ &= \mathcal{L}\left\{25t^2 - 40t\right\}(z) - \frac{42}{z} + \frac{1}{z} \cdot \frac{1}{z^2-4z+5}\left(40z - 210 + 42z^2 - 168z + 210\right) \\ &= \mathcal{L}\left\{25t^2 - 40t - 42\right\}(z) + \frac{42z - 128}{(z-2)^2+1^2} \\ &= \mathcal{L}\left\{25t^2 - 40t - 42\right\}(z) + 42 \cdot \frac{z-2}{(z-2)^2+1^2} - 44 \cdot \frac{1}{(z-2)^2+1^2}. \end{aligned}$$

We finally apply the inverse Laplace transformation to get

$$f(t)=25t^2-40t-42+42e^{2t}\cos t-44e^{2t}\sin t. \qquad \diamond$$

Example 1.5.10 *Solve the initial value problem*

$$f''(t)+f(t)=8\cos t, \qquad f(0)=1, \quad f'(0)=1.$$

We write as usual, $F(z)=\mathcal{L}\{f\}(z)$. Then by the Laplace transformation of the differential equation,

$$8\cdot\frac{z}{z^2+1}=z^2\,F(z)-z-1+F(z)=\left(z^2+1\right)F(z)-z-1,$$

from which

$$\begin{aligned}F(z) &= 8\cdot\frac{z}{z^2+1}\cdot\frac{1}{z^2+1}+\frac{z}{z^2+1}+\frac{1}{z^2+1}=-4\,\frac{d}{dz}\left\{\frac{1}{z^2+1}\right\}+\frac{z}{z^2+1}+\frac{1}{z^2+1}\\ &= \mathcal{L}\{4t\sin t\}(z)+\mathcal{L}\{\cos t\}(z)+\mathcal{L}\{\sin t\}(z),\end{aligned}$$

and we conclude that

$$f(t)=4\,t\,\sin t+\cos t+\sin t. \qquad \diamond$$

Example 1.5.11 *Solve the initial value problem*

$$f^{(3)}(t)-f(t)=e^t, \qquad f(0)=f')0)=f''(0)=0.$$

We write for short $F(z)=\mathcal{L}\{f\}(z)$. Then by the Laplace transformation of the equation,

$$\left(z^3-1\right)F(z)=\frac{1}{z-1},$$

thus

$$\begin{aligned}F(z) &= \frac{1}{(z-1)\left(z^3-1\right)}=\frac{1}{(z-1)^2\left(z^2+z+1\right)}\\ &= \frac{1}{3}\cdot\frac{1}{(z-1)^2}+\frac{1}{3}\cdot\frac{1}{(z-1)^2\left(z^2+z+1\right)}\left\{3-z^2-z-1\right\}\\ &= \frac{1}{3}\cdot\frac{1}{(z-1)^2}-\frac{1}{3}\cdot\frac{z+2}{(z-1)\left(z^2+1+1\right)}\\ &= \frac{1}{3}\cdot\frac{1}{(z-1)^2}-\frac{1}{3}\cdot\frac{1}{z-1}+\frac{1}{3}\cdot\frac{1}{(z-1)\left(z^2+z+1\right)}\left\{z^2+z+1-z-2\right\}\\ &= \frac{1}{3}\cdot\frac{1}{(z-1)^2}-\frac{1}{3}\cdot\frac{1}{z-1}+\frac{1}{3}\cdot\frac{z+1}{z^2+z+1}\\ &= \frac{1}{3}\cdot\frac{1}{(z-1)^2}-\frac{1}{3}\cdot\frac{1}{z-1}+\frac{1}{3}\cdot\frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{6}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}.\end{aligned}$$

Finally, we use the inverse Laplace transformation to get

$$f(t)=\frac{1}{3}\,t-\frac{1}{3}+\frac{1}{3}\,\exp\left(-\frac{1}{2}\,t\right)\cos\frac{\sqrt{3}}{2}\,t+\frac{\sqrt{3}}{9}\,\exp\left(-\frac{1}{2}\,t\right)\sin\frac{\sqrt{3}}{2}\,t. \qquad \Diamond$$

Example 1.5.12 *Solve the initial value problem*

$$f^{(4)}(t)+2f^{(2)}(t)+f(t)=\sin t, \qquad f(0)=f'(0)=f''(0)=f^{(3)}(0)=0.$$

We write for short $F(z)=\mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,

$$\left(z^4+2z^2+1\right)F(z)=\frac{1}{z^2+1},$$

so

$$F(z)=\frac{1}{\left(z^2+1\right)^3}.$$

In this case we shall use the residuum formula over the inverse Laplace transformation,

$$\begin{aligned}
f(t) &= \operatorname{res}\left(\frac{e^{zt}}{\left(z^2+1\right)^3};i\right)+\operatorname{res}\left(\frac{e^{zt}}{\left(z^2+1\right)^3};-i\right)\\
&= \frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}\left\{\frac{e^{zt}}{(z+i)^2}\right\}+\frac{1}{2}\lim_{z\to -i}\frac{d^2}{dz^2}\left\{\frac{e^{zt}}{(z+-)^2}\right\}\\
&= \frac{1}{2}\lim_{z\to i}\frac{d}{dz}\left\{-\frac{2e^{zt}}{(z+i)^3}+t\cdot\frac{e^{zt}}{(z+i)^2}\right\}+\frac{1}{2}\lim_{z\to -i}\frac{d}{dz}\left\{-\frac{2e^{zt}}{(z-i)^3}+t\cdot\frac{e^{zt}}{(z-i)^2}\right\}\\
&= \frac{1}{2}\lim_{z\to i}\left\{6\,\frac{e^{zt}}{(z+i)^4}-4t\,\frac{e^{zt}}{(z+i)^3}+t^2\,\frac{e^{zt}}{(z+i)^2}\right\}\\
&\qquad +\frac{1}{2}\lim_{z\to -i}\left\{6\,\frac{e^{zt}}{(z-i)^4}-4t\,\frac{e^{zt}}{(z-i)^3}+t^2\,\frac{e^{zt}}{(z-i)^2}\right\}\\
&= \frac{1}{2}\left\{6\,\frac{e^{it}}{(2i)^4}-4t\,\frac{e^{it}}{(2i)^3}+t^2\,\frac{e^{it}}{(2i)^2}\right\}+\frac{1}{2}\left\{6\,\frac{e^{-it}}{(-2i)^4}-4t\,\frac{e^{-it}}{(-2i)^3}+t^2\,\frac{e^{-it}}{(-2i)^2}\right\}\\
&= \frac{6}{16}\,\cos t-\frac{4t}{-8}\,\sin t+\frac{t^2}{-4}\,\cos t=\frac{3}{8}\,\cos t+\frac{1}{2}\,t\,\sin t-\frac{1}{4}\,t^2\,\cos t. \qquad \Diamond
\end{aligned}$$

Example 1.5.13 *Solve the* boundary value problem

$$f''(t)+9f(t)=18t, \qquad f(0)=0, \quad f\left(\frac{\pi}{2}\right)=0.$$

This equation is easily solved by inspection, because $f_0(t)=2t$ is clearly a solution of the inhomogeneous equation, so the complete solution is given by

$$f(t)=2t+a\,\cos 3t+b\,\sin 3t.$$

From $f(0)=0$ follows that $a=0$, so

$$f\left(\frac{\pi}{2}\right)=0=2\cdot\frac{\pi}{2}-b=\pi-b,$$

and we conclude that

$$f(t)=2t+\pi\cdot\sin 3t.$$

Alternatively we put the unknowns $F(z) = \mathcal{L}\{f\}(z)$ and $f'(0) = c$. Then by the Laplace transformation,

$$z^2 F(z) - c + 9F(z) = \frac{18}{z^2},$$

thus by a rearrangement,

$$F(z) = \frac{c}{z^2+9} + \frac{18}{z^2(z^2+9)} = \frac{c}{z^2+9} + \frac{2}{z^2} - \frac{2}{z^2+9} = \frac{2}{z^2} + \frac{c-2}{z^2+3^2}.$$

Then apply the inverse Laplace transformation to get

$$f(t) = 2t + \frac{c-2}{3}\sin 3t,$$

for which (the boundary condition)

$$f\left(\frac{\pi}{2}\right) = 0 = \pi + \frac{c-2}{3}\sin\left(\frac{3\pi}{2}\right) = \pi - \frac{c-2}{3},$$

hence

$$f(t) = 2t + \pi \cdot \sin 3t. \qquad \Diamond$$

Example 1.5.14 *Solve the initial value problem*

$$f^{(4)}(t) + f^{(3)}(t) = 2\sin t, \qquad f(0) = f'(0) = 0, \quad f''(0) = 1, f^{(3)} = -2.$$

We write as usual $F(z) = \mathcal{L}\{f\}(z)$. Then by the Laplace transformation of the differential equation,

$$\begin{aligned}\mathcal{L}\{2\sin t\}(z) &= 2\cdot\frac{1}{z^2+1} = z^4 F(z) - 0\cdot z^3 - 0\cdot z^2 - z + 2 + z^3 F(z) - 0\cdot z^2 - 0\cdot z - 1 \\ &= (z+1)z^3 F(z) - z - 1,\end{aligned}$$

from which we can compute $F(z)$,

$$\begin{aligned}F(z) &= \frac{1}{(z+1)z^3}\left\{z+1+2\cdot\frac{1}{z^2+1}\right\} = \frac{1}{z^3} + \frac{2}{(z+1)z^3(z^2+1)} \\ &= \frac{1}{z^3} + \frac{2}{z^3} + 2\frac{1-(z+1)\left(z^2+1\right)}{(z+1)z^3(z^2+1)} = \frac{3}{z^3} + 2\frac{z-1-\left(z^2-1\right)\left(z^2+1\right)}{(z^2-1)z^3(z^2+1)} \\ &= \frac{3}{z^3} + 2\frac{z-z^4}{z^3(z^2-1)(z^2+1)} = \frac{3}{z^3} + \frac{2}{z^2(z^2-1)(z^2+1)} - \frac{2z}{(z^2-1)(z^2+1)} \\ &= \frac{3}{z^3} - \frac{2}{z^2} + \frac{1}{z^2-1} + \frac{1}{z^2+1} - \frac{z}{z^2-1} - \frac{z}{z^2+1} \\ &= \frac{3}{z^3} - \frac{2}{z^2} - \frac{1}{z+1} + \frac{1}{z^2+1} - \frac{z}{z^2+1}.\end{aligned}$$

Finally, by the inverse Laplace transformation,

$$f(t) = \frac{3}{2}t^2 - 2t - e^{-t} + \cos t - \sin t. \qquad \Diamond$$

Example 1.5.15 *Check if the Laplace transformation can be applied when we solve the equation*

$$f''(t) + f(t) = \frac{1}{\cos t}.$$

Since $\dfrac{1}{\cos t} \notin \mathcal{F}$, the Laplace transform cannot be applied on the right hand side of the differential equation, so we must use more traditional methods.

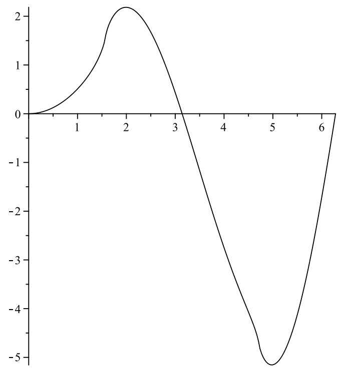

Figure 14: The graph of $f(t) = t \cdot \sin t + \cos t \cdot \log|\cos t|$, $[0, 2\pi]$, in Example 1.5.15.

The set of solutions of the corresponding homogeneous equation is spanned by the two linearly independent solutions

$$\varphi_1(t) = \cos t \qquad \text{and} \varphi_2(t) = \sin t,$$

which have their Wrońskian given by

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ sint & \cos t \end{vmatrix} = 1.$$

For $t \neq \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, the complete solution is given by the usual solution formula, known from elementary Calculus,

$$\begin{aligned} f(t) &= \sin t \int \frac{\cos t}{\cos t}\,\mathrm{d}t - \cos t \int \frac{\sin t}{\cos t}\,\mathrm{d}t + C_1\,\cos t + C_2\,\sin t \\ &= t\,\sin t + \cos t \cdot \log|\cos t| + c_1 \cos t + c_2 \sin t. \end{aligned}$$

We notice that the solution $f(t)$ can be extended continuously to all of $\mathbb{R}$, because $\cos t \cdot \log|\cos t| \to 0$ for $t \to \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. However, this uniquely determined extension is not differentiable at the points $\dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. $\Diamond$

1.5.2 Linear differential equations of simple polynomial coefficients

Example 1.5.16 *Solve the initial value problem*

$$f''(t) - t\,f'(t) + f(t) = 1, \qquad f(0) = 1, \quad f'(0) = 2.$$

We start this subsection with an example which shows that an application of the Laplace transformation may not always be the best method. In fact, when we apply the Laplace transformation on this initial value problem, then we get

$$\begin{aligned}\mathcal{L}\{1\}(z) = \frac{1}{z} &= \left[z^2\mathcal{L}\{f\}(z) - z\cdot f(0) - f'(0)\right] - \left[-\frac{d}{dz}\,\mathcal{L}\left\{f'\right\}(z)\right] + \mathcal{L}\{f\}(z) \\ &= z^2\mathcal{L}\{f\} - z - 2 + \frac{d}{dz}\left[z\mathcal{L}\{f\}(z) - f(0)\right] + \mathcal{L}\{f\}(z) \\ &= z^2\mathcal{L}\{f\}(z) + 2\mathcal{L}\{f\}(z) + z\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) - z - 2.\end{aligned}$$

Writing $\mathcal{L}\{f\}(z) = F(z)$ we see that $F(z)$ must satisfy the differential equation

$$z\,\frac{dF}{dz} + \left(z^2+2\right)F(z) = z + 2 + \frac{1}{z} = \frac{(z+1)^2}{z},$$

where we use a known solution formula from elementary Calculus to get the complete solution

$$F(z) = \mathcal{L}\{f\}(z) = z^2\,\exp\left(-\frac{z^2}{2}\right)\left\{C + \int \exp\left(\frac{z^2}{2}\right)\frac{1}{z^2}\left(\frac{(z+1)^2}{z}\right)\,\mathrm{d}z\right\},$$

which does not look promising when one want to compute the integral explicitly, let alone finding its inverse Laplace transform.

The equation can be solved by using the method of power series instead. Since all coefficients of the equation are polynomials of at most degree 1, and since $f''(t)$ has $1 \neq 0$ as its coefficients, we conclude that the equation has power series solutions which are convergent everywhere.

First it follows from the initial conditions that

$$a_0 = 1 \qquad \text{and} \qquad a_1 = 2,$$

where we assume that

$$f(t) = \sum_{n=0}^{+\infty} a_0\,t^n.$$

Then it follows from the differential equation that

$$\begin{aligned}1 &= f''(t) - t\,f'(t) + f(t) = \sum_{n=2}^{+\infty} n(n-1)a_n\,t^{n-2} - t\sum_{n=1}^{+\infty} na_n\,t^{n-1} + \sum_{n=0}^{+\infty} a_n\,t^n \\ &= \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}\,t^n - \sum_{n=0}^{+\infty} a_n\,t^n \\ &= \sum_{n=0}^{+\infty}\left\{(n+2)(n+1)a_{n+2} - (n-1)a_n\right\}t^n.\end{aligned}$$

It follows from the *identity theorem* that

$$
\begin{aligned}
&2\cdot 1\cdot a_2+a_0=2a_2+1=1, &&\text{for } n=0, &&\text{hence } a_2=0,\\
&3\cdot 2\cdot a_3=0, &&\text{for } n=1, &&\text{hence } a_3=0,\\
&4\cdot 3\cdot a_4-a_2=12a_4-2=0, &&\text{for } n=2, &&\text{hence } a_4=\frac{1}{6}.
\end{aligned}
$$

For $n \geq 2$ we get the recursion formula

$$a_{n+2}=\frac{n-1}{(n+2)(n+1)}, a_n.$$

There is a jump of 2 units in the indices in this formula, and since $a_3 = 0$, we immediately conclude that $a_{2n+1} = 0$ for all $n \in \mathbb{N}$.

Let $n = 2m$ be even. Then the recursion formula is written

$$
\begin{aligned}
a_{2m+2} &= a_{2(m+1)}=\frac{2m-1}{(2m+2)(2m+1)}\,a_{2m}=\frac{2m-1}{(2m+1)(2m+1)}\cdot\frac{2m-3}{2m(2m-1)}\,a_{2(m-1)}=\cdots\\
&= \frac{2m-1}{(2m+2)(2m+1)}\cdot\frac{2m-3}{2m(2m-1)}\cdots\frac{3}{6\cdot 5}\,a_4=\frac{1}{2m+1}\cdot\frac{1}{2(m+1)}\cdot\frac{1}{2m}\cdots\frac{1}{2\cdot 3}\cdot 3\cdot\frac{1}{6}\\
&= \frac{4}{2m+1}\cdot\frac{1}{2^{m+1}}\cdot\frac{1}{(m+1)!},
\end{aligned}
$$

hence by a change of index,

$$a_{2m} = \frac{4}{2m-1} \cdot \frac{1}{2^m} \cdot \frac{1}{m!} \qquad \text{for } m \geq 1,$$

and the real series solution is given by

$$f(t) = 1 + 2t + 4 \sum_{n=1}^{+\infty} \frac{1}{(2n-1)n!} \left(\frac{t^2}{2}\right)^n, \qquad \text{for } t \in \mathbb{R}. \qquad \Diamond$$

Example 1.5.17 *Solve the initial value problem*

$$f''(t) + t\, f'(t) - f(t) = 0, \qquad f(0) = 0, \quad f'(0) = 1.$$

This example is – apart from a change of sign, and different initial conditions – very similar to Example 1.5.16. And yet we this time have no problem in solving the equation.

We put for short $F(z) = \mathcal{L}\{f\}(z)$. Then by the Laplace transformation and the rules of computation,

$$0 = z^2 F(z) - 0 \cdot z - 1 - \frac{d}{dz}\{z\, F(z) - 0\} - F(z) = z^2 - 1 - F(z) - z\, F'(z) - F(z),$$

hence by a rearrangement,

$$z\, F'(z) - \left(z^2 - 2\right) F(z) = -1.$$

The solution of the corresponding homogeneous equation is a constant times

$$\frac{1}{z^2} \exp\left(\frac{z^2}{2}\right),$$

so a particular integral of the inhomogeneous equation is given by

$$-\frac{1}{z^2} \exp\left(\frac{z^2}{2}\right) \int^z \zeta^2 \exp\left(-\frac{\zeta^2}{2}\right) \cdot \frac{1}{\zeta}\, \mathrm{d}\zeta = +\frac{1}{z^2}.$$

The complete solution is

$$F(z) = \frac{1}{z^2} + C \cdot \frac{1}{z^2} \cdot \exp\left(\frac{z^2}{2}\right).$$

The function $F(z)$ must satisfy the condition that $F(z) \to 0$, whenever $\Re\, z \to +\infty$, so we necessarily must have $C = 0$, and we get

$$F(z) = \frac{1}{z^2}.$$

Finally, we get by the inverse Laplace transformation that

$$f(t) = t,$$

which is immediately checked in the original problem. $\Diamond$

Example 1.5.18 *Solve the initial value problem*

$$t\,f''(t) + (1-2t)f'(t) - 2\,f(t) = 0, \qquad f(0) = 1, \quad f'(0) = 2.$$

It follows by an inspection that

$$0 = t\,f''(t) + f'(t) - 2\{t\,f'(t) + f(t)\} = \frac{d}{dt}\left\{t\,f'(t) - 2t\,f(t)\right\},$$

thus by an integration,

$$t\left\{f'(t) - 2\,f(t)\right\} = c \text{ a constant.}$$

For $t = 0$ we trivially get $c = 0$, so the equation is reduced to $f'(t) = 2f(t)$, the solution of which is $f(t) = k \cdot e^{2t}$. Then it follows from any one of the two initial conditions that $k = 1$, and the solution is given by

$$f(t) = e^{2t}.$$

Remark 1.5.2 The problem is *ill-posed*, because we are given two initial conditions at a *singular point*, i.e. $t = 0$, where the coefficient t of the highest order term is 0. If we change one of them to a different value, there will by no solution at all. $\diamond$

Alternatively we shall use the Laplace transformation. Let us assume that $f \in \mathcal{F}$. Then by Laplace transforming the differential equation,

$$\begin{aligned}
0 &= \mathcal{L}\left\{tf''\right\}(z) + \mathcal{L}\left\{f'\right\}(z) - 2\mathcal{L}\left\{tf'\right\}(z) - 2\mathcal{L}\{f\}(z) \\
&= -\frac{d}{dz}\,\mathcal{L}\left\{f''\right\}(z) + z\mathcal{L}\{f\}(z) - f(0) + 2\,\mathrm{d}\mathrm{d}z\,\mathcal{L}\left\{f'\right\}(z) - 2\mathcal{L}\{f\}(z) \\
&= -\frac{d}{dz}\left[z^2\mathcal{L}\{f\}(z) - z\cdot f(0) - f'(0)\right] + z\mathcal{L}\{f\}(z) - f(0) + 2\,\frac{d}{dz}[z\mathcal{L}\{f\}(z) - f(0)] - 2\mathcal{L}\{f\}(z) \\
&= -2z\mathcal{L}\{f\}(z) - z^2\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) + f(0) + z\mathcal{L}\{f\}(z) - f(0) + 2\mathcal{L}\{f\}(z) + 2z\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) - 2\mathcal{L}\{f\}(z) \\
&= \left(-z^2 + 2z\right)\frac{d}{dz}\,\mathcal{L}\{f\}(z) - z\mathcal{L}\{f\}(z).
\end{aligned}$$

It follows for $\Re\,z > 2$ that

$$(10) \quad -(z-2)\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) = \mathcal{L}\{f\}(z),$$

thus

$$\mathcal{L}\{f\}(z) = \frac{c}{z-2},$$

for some constant c, and hence by the inverse Laplace transformation,

$$f(t) = c \cdot e^{2t}.$$

Finally, it follows from $f(0) = 1$ that $c = 1$. It also follows from $f'(0) = n$ that $c = 1$. This shows again that the problem is ill-posed. We also notice that neither f(0) nor $f'(0)$ occur in the equation (10). $\diamond$

Example 1.5.19 *Solve the differential equation*

$$t\, f''(t) + (t-1)f'(t) - f(t) = 0, \qquad f(0) = 5, \quad \lim_{t \to +\infty} f(t) = 0.$$

This is neither a classical initial value problem nor an ordinary boundary value problem, so we cannot apply the usual theorem of existence and uniqueness of the solution. In particular, the condition $f(0) = 5$ is given at a singular point.

When we use the Laplace transformation, we get

$$\begin{aligned} 0 &= -\frac{d}{dz}\left[z^2\mathcal{L}\{f\}(z) - zf(0) - f'(0)\right] - \frac{d}{dz}[z\mathcal{L}\{f\}(z) - f(0)] - z\mathcal{L}\{f\}(z) + f(0) - \mathcal{L}\{f\}(z) \\ &= -2z\mathcal{L}\{f\}(z) - z^2\frac{d}{dz}\mathcal{L}\{f\}(z) - f(0) - \mathcal{L}\{f\}(z) - z\frac{d}{dz}\mathcal{L}\{f\}(z) - z\mathcal{L}\{f\}(z) - \mathcal{L}\{f\}(z) + f(0) \\ &= -z(z+1)\frac{d}{dz}\mathcal{L}\{f\}(z) - (3z+2)\,\mathcal{L}\{f\}(z) + 2 \cdot 5 \\ &= -\frac{1}{z}\left\{\left(z^3+z^2\right)\frac{d}{dz}\mathcal{L}\{f\}(z) + \left(3z^2+2z\right) \cdot \mathcal{L}\{f\} - 10z\right\} \\ &= -\frac{1}{z}\left\{\frac{d}{dz}\left[(z+1)z^2\,\mathcal{L}\{f\}(z)\right] - 10z\right\}, \end{aligned}$$

thus

$$\frac{d}{dz}\left[(z+1)z^2\,\mathcal{L}\{f\}(z)\right] = 10z,$$

and hence by an integration,

$$(z+1)z^2\,\mathcal{L}\{f\}(z) = 5z^2 + c.$$

It follows from the *final value theorem* and the given condition $\lim_{t \to +\infty} f(t) = 0$ that

$$\lim_{x \to 0+} x\,\mathcal{L}\{f\}(x) = 0,$$

so we conclude that $c = 0$, and the problem has been reduced to the equation

$$(z+1)\mathcal{L}\{f\}(z) = 5,$$

i.e.

$$\mathcal{L}\{f\}(z) = \frac{5}{z+1} = \mathcal{L}\left\{5e^{-t}\right\}(z),$$

and it follows from the inverse Laplace transformation that

$$f(t) = 5\, e^{-t}.$$

Alternatively we get by a small rearrangement,

$$0 = t\, f''(t) + (t-1)f'(t) - f(t) = t\left\{f''(t) + f'(t)\right\} - \left\{f'(t) + f(t)\right\},$$

hence, dividing by $t^2 > 0$,

$$\begin{aligned} 0 &= \frac{t\,\frac{d}{dt}\left\{f'(t)+f(t)\right\} - 1\cdot\left\{f'(t)+f(t)\right\}}{t^2} = \frac{d}{dt}\left[\frac{1}{t}\left\{f'(t)+f(t)\right\}\right] \\ &= \frac{d}{dt}\left[\frac{1}{t\,e^t}\left\{e^t\,f'(t) + e^t\,f(t)\right\}\right] = \frac{d}{dt}\left[\frac{1}{t\,e^t}\,\frac{d}{dt}\left\{e^t\,f(t)\right\}\right], \end{aligned}$$

so by an integration, where a is an arbitrary constant,

$$a = \frac{1}{t\,e^t}\,\frac{d}{dt}\left\{e^t\,f(t)\right\},$$

thus

$$\frac{d}{dt}\left\{e^t\,f(t)\right\} = a\,t\,e^t.$$

Then by another integration,

$$e^t f(t) = b + a \int t\, e^t \, \mathrm{d}t = b + a(t-1)e^t,$$

and $f(t)$ is given by

$$f(t) = b\, e^{-t} + a(t-1).$$

From $\lim_{t \to +\infty} f(t) = 0$ follows that $a = 0$, and from $f(0) = b = 5$ we finally get

$$f(t) = 5\, e^{-t}. \qquad \lozenge$$

1.5.3 Linear equations of constant coefficients and discontinuous right hand side

Example 1.5.20 *Solve the initial value problem*

$$f''(t) + 4f(t) = H(t-2), \qquad f(0) = 0, \quad f'(0) = 1,$$

where the right hand side of the equation is discontinuous at a point.

We get using the Laplace transformation

$$\begin{aligned} \mathcal{L}\{H(t-2)\}(z) = e^{-2z}\mathcal{L}\{H\}(z) = \frac{1}{z}\, e^{-2z} &= z^2\mathcal{L}\{f\}(z) - z \cdot f(0) - f'(0) + 4\mathcal{L}\{f\}(z) \\ &= \left(z^2+4\right)\mathcal{L}\{f\}(z) - 1, \end{aligned}$$

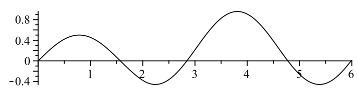

Figure 15: The graph of $f(t) = \frac{1}{2}\,\sin 2t + \frac{1}{2}\,\sin^2(t-2) \cdot H(t-2)$ in Example 1.5.20.

hence by a rearrangement,

$$\begin{aligned}
\mathcal{L}\{f\}(z) &= \frac{1}{z^4+4} + e^{-2z} \cdot \frac{1}{z\left(z^2+4\right)} = \frac{1}{z^2+4} + e^{-2z}\, z \left\{ \frac{1}{z^2\left(z^2+4\right)} \right\} \\
&= \frac{1}{2}\, \mathcal{L}\{\sin 2t\}(z) + e^{-2z} \left\{ \frac{1}{4} \cdot \frac{1}{z} - \frac{1}{4} \cdot \frac{z}{z^2+4} \right\} \\
&= \frac{1}{2}\, \mathcal{L}\{\sin 2t\}(z) + e^{-2z}\, \mathcal{L} \left\{ \frac{1}{4}(1-\cos 2t) \right\} (z) \\
&= \mathcal{L} \left\{ \frac{1}{2}\, \sin 2t + \frac{1}{2}\, \sin^2(t-2) \cdot H(t-2) \right\} (z).
\end{aligned}$$

We finally get by the inverse Laplace transformation,

$$f(t) = \frac{1}{2}\, \sin 2t + \frac{1}{2}\, \sin^2(t-2) \cdot H(t-2). \qquad \Diamond$$

Example 1.5.21 *Solve the initial value problem*

$$f''(t) + 4f(t) = \chi_{[0,1]}(t), \qquad f(0) = 0, \quad f'(0) = 1,$$

where the right hand side is discontinuous at a point.

Write for convenience $\mathcal{L}\{f\}(z) = F(z)$. Then we get by the Laplace transformation,

$$z^2\, F(z) - 0 \cdot z - 1 + 4\, F(z) = \mathcal{L}\left\{\chi_{[0,1]}\right\}(z) = \frac{1}{z} \cdot \left(1 - e^{-z}\right),$$

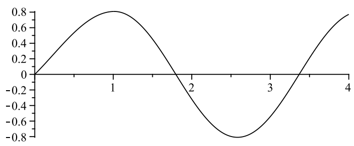

Figure 16: The graph of $f(t) = \frac{1}{2}\, \sin 2t + \frac{1}{2}\, \sin^2 t - \frac{1}{2}\, \sin^2(t-1) \cdot H(t-1)$ in Example 1.5.21.

hence by a rearrangement,

$$\begin{aligned} F(z) &= \frac{1}{z^2+4}\left\{1+\frac{1}{z}\left(1-e^{-z}\right)\right\} = \frac{1}{z^2+4} + \left(1-e^{-z}\right)\cdot\frac{1}{z}\,\left(z^2+4\right) \\ &= \mathcal{L}\left\{\frac{1}{2}\,\sin 2t\right\}(z) + \left(1-e^{-z}\right)\left\{\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{4}\cdot\frac{z}{z^2+4}\right\} \\ &= \mathcal{L}\left\{\frac{1}{2}\,\sin 2t\right\}(z) + \frac{1}{2}\,\mathcal{L}\left\{\sin^2 t\right\}(z) - \frac{1}{2}\,e^{-z}\,\mathcal{L}\left\{\sin^2 t\right\}(z). \end{aligned}$$

Finally, by the inverse Laplace transformation,

$$f(t) = \frac{1}{2}\,\sin 2t + \frac{1}{2}\,\sin^2 t - \frac{1}{2}\,\sin^2(t-1)\cdot H(t-1). \qquad \Diamond$$

Example 1.5.22 *Solve the initial value problem*

$$f''(t) + 9f(t) = H(t-1), \qquad f(0) = f'(0) = 0,$$

where the right hand side has a point of discontinuity.

Write for short, $\mathcal{L}\{f\}(z) = F(z)$. Then we get by the Laplace transformation that

$$\left(z^2+9\right) F(z) = \frac{1}{z}\,e^{-z},$$

so

$$F(z) = e^{-z}\cdot\frac{1}{z\left(z^2+9\right)} = e^{-z}\left\{\frac{1}{9}\cdot\frac{1}{z} - \frac{1}{9}\cdot\frac{1}{z^2+9}\right\} = \frac{1}{9}\,e^{-z}\,\mathcal{L}\{1-\cos 3z\}(z),$$

hence by the inverse Laplace transformation,

$$f(t) = \frac{1}{9}\,\{1-\cos 3(t-1)\}\cdot H(t-1). \qquad \Diamond$$

Example 1.5.23 *Solve the initial value problem*

$$f''(t) + f(t) = H(t-1), \qquad f(0) = f'(0) = 0.$$

When we apply the Laplace transformation, the equation becomes

$$\left(z^2+1\right)\mathcal{L}\{f\}(z) = \frac{1}{z}\,e^{-z},$$

so

$$\mathcal{L}\{f\}(z) = e^{-z}\,\frac{1}{z\left(z^2+1\right)} = e^{-z}\left\{\frac{1}{z} - \frac{z}{z^2+1}\right\} = e^{-z}\mathcal{L}\{1-\cos t\}(z),$$

hence

$$f(t) = \begin{cases} 1-\cos(t-1) & \text{for } t \geq 1, \\ 0 & \text{for } t < 1. \end{cases} \qquad \Diamond$$

Example 1.5.24 *Solve the integro-differential equation*

$$f'(t)+f(t)+\int_0^t f(\tau)\,\mathrm{d}\tau=\chi_{[1,2]}(t),\qquad f(0)=1.$$

We put $F(z)=\mathcal{L}\{f\}(z)$. Then we get by the Laplace transformation,

$$\begin{aligned} z\,F(z)-1+F(z)+\frac{1}{z}\,F(z) &= \mathcal{L}\left\{\chi_{[1,2]}\right\}(z)\\ &= \mathcal{L}\{H(t-t)-H(t-2)\}(z)=\frac{1}{z}\left\{e^{-z}-e^{-2z}\right\},\end{aligned}$$

hence

$$\frac{z^2+z+1}{z}\,F(z)=1+\frac{1}{z}\left\{e^{-z}-e^{-2z}\right\},$$

from which

$$\begin{aligned} F(z) &= \frac{z}{z^2+z+1}+\frac{1}{z^2+z+1}\,e^{-z}-\frac{1}{z^2+z+1}\,e^{-2z}\\ &= \frac{z+\frac{1}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &\qquad +e^{-z}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-e^{-2z}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \mathcal{L}\left\{\exp\left(-\frac{1}{2}\,t\right)\left(\cos\frac{\sqrt{3}}{2}\,t\right)-\frac{1}{\sqrt{3}}\,\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z)\\ &\qquad \frac{2}{\sqrt{3}}\,e^{-z}\,\mathcal{L}\left\{\exp\left(-\frac{1}{2}\,t\right)\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z)\\ &\qquad -\frac{2}{\sqrt{3}}\,e^{-2z}\,\mathcal{L}\left\{\exp\left(-\frac{1}{2}\,t\right)\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\}(z).\end{aligned}$$

Finally we get from the inverse Laplace transformation that the solution is given by

$$\begin{aligned} f(t) &= \exp\left(-\frac{1}{2}\,t\right)\left\{\cos\left(\frac{\sqrt{3}}{2}\,t\right)-\frac{1}{\sqrt{3}}\,\sin\left(\frac{\sqrt{3}}{2}\,t\right)\right\}H(t)\\ &\qquad +\frac{2}{\sqrt{3}}\,\exp\left(-\frac{1}{2}(t-1)\right)\sin\left(\frac{\sqrt{3}}{2}\,(t-1)\right)\cdot H(t-1)\\ &\qquad +\frac{2}{\sqrt{3}}\,\exp\left(-\frac{1}{2}(t-2)\right)\sin\left(\frac{\sqrt{3}}{2}\,(t-2)\right)\cdot H(t-2).\qquad\Diamond\end{aligned}$$

1.6 The two-sided Laplace transformation

Example 1.6.1 *Compute the two-sided Laplace transform of the functions*

1)

$$f_1(t) = -\left(e^t + e^{-2t}\right) H(-t), \qquad t \in \mathbb{R}.$$

2)

$$f_2(t) = \begin{cases} -e^{-t}, & t \in \mathbb{R}_-, \\ e^{-2t}, & t \in \mathbb{R}_+, \end{cases}$$

3)

$$f_3(t) = \left(e^t + e^{-2t}\right) H(t), \qquad t \in \mathbb{R}.$$

Specify in each case the strip of convergence.

Recall that the two-sided Laplace transformation is defined by

$$\int_{-\infty}^{+\infty} e^{-zt}\, f(t)\, \mathrm{d}t,$$

whenever this improper integral is convergent.

1) In this case,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-zt}\, f_1(t)\,\mathrm{d}t &= -\int_{-\infty}^{0} \left(e^{t}+e^{-2t}\right) e^{-zt}\,\mathrm{d}t \\ &= -\int_{-\infty}^{0} \left\{e^{(1-z)t}+e^{-(2+z)t}\right\}\,\mathrm{d}t,\end{aligned}$$

from which follows that the strip of convergence is determined by

$$\Re(1-z) = 1-\Re\, z > 0, \qquad \text{hence} \qquad \Re\, z < 1,$$

and

$$\Re(2+z) = 2+\Re\, z < 0, \qquad \text{hence} \qquad \Re\, z < -2.$$

The domain of convergence is therefore given by $\Re\, z < -2$, and in this we get the two-sided Laplace transform of $f_1(t)$,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-zt}\, f_1(t)\,\mathrm{d}t &= -\left[\frac{1}{1-z}\, e^{(1-z)t} - \frac{1}{2+z}\, e^{-(2+z)t}\right]_{-\infty}^{0} \\ &= -\left\{\frac{1}{1-z}-\frac{1}{2+z}\right\} = \frac{1}{z-1}+\frac{1}{z+2} = \frac{2z+1}{z^2+z-2}.\end{aligned}$$

2) Similarly,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-zt}\, f_2(t)\,\mathrm{d}t &= \int_{-\infty}^{0} e^{-zt}\left(-e^{t}\right)\mathrm{d}t + \int_{0}^{+\infty} e^{-zt}\, e^{-2t}\,\mathrm{d}t \\ &= -\int_{-\infty}^{0} e^{(1-z)t}\,\mathrm{d}t + \int_{0}^{+\infty} e^{-(z+2)t}\,\mathrm{d}t.\end{aligned}$$

The strip of convergence is determined by

$$\Re(1-z) = 1-\Re\, z > 0, \qquad \text{hence} \qquad \Re\, z < 1,$$

and

$$\Re(z+2) = 2+\Re\, z > 0, \qquad \text{hence} \qquad \Re\, z > -2.$$

The strip of convergence is $-2 < \Re\, z < 1$. If z lies in this strip, then we get the two-sided Laplace transform of $f_2(t)$,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-zt}\, f_2(t)\,\mathrm{d}t &= -\left[\frac{1}{1-z}\, e^{(1-z)t}\right]_{-\infty}^{0} + \left[-\frac{1}{z+2}\, e^{-(z+2)t}\right]_{0}^{+\infty} \\ &= -\frac{1}{1-z}+\frac{1}{z+2} = \frac{1}{z-1}+\frac{1}{z+2} = \frac{2z+1}{z^2+z-2}.\end{aligned}$$

Formally the result is of the same structure as in 1), but they are nevertheless different, because their domains of convergence are disjoint.

3) Similarly, the two-sided Laplace transform of $f_3(t)$ is given by

$$\int_{-\infty}^{+\infty} f_3(t)\, e^{-zt}\, \mathrm{d}t = \int_0^{+\infty} \left(e^t + e^{-2t}\right) e^{-zt}\, \mathrm{d}t = \frac{1}{z-1} + \frac{1}{z+2} = \frac{2z+1}{z^2+z-2},$$

in the domain of convergence given by $\Re\, z > 1$.

We see in all three cases that the two-sided Laplace transform is given by

$$\frac{2z+1}{z^2+z-2}.$$

We see that it is very important to distinguish between the domains of convergence, whenever the two-sided Laplace transformation is considered. $\Diamond$

1.7 The Fourier transformation

Example 1.7.1 *Explain why the rule of periodicity in general does not make sense for the Fourier transformation.*

Let us assume that $f(t)$ is periodic, i.e. $f(x+T) = f(x)$ for some $T > 0$ and all $x \in \mathbb{R}$. If $f(x)$ is not the zero function, then we get

$$\int_{nT}^{(n+1)T} f(x)\, e^{-ix\xi}\, \mathrm{d}x = \int_0^T f(x)\, e^{-i(x+n)\xi}\, \mathrm{d}x = e^{-inT\xi} \int_0^T f(x)\, e^{-ix\xi}\, \mathrm{d}x.$$

Since $f(x)$ is *not* the zero function, there exists a set $\Xi \subseteq \mathbb{R}$, where Ξ is not a Lebesgus null set, such that

$$\int_0^T f(x)\, e^{-ix\xi}\, \mathrm{d}x \neq 0 \qquad \text{for every } \xi \in \Xi.$$

Hence, if $\xi \in \Xi$, then the series

$$\int_{-\infty}^{+\infty} f(x)\, e^{-ix\xi}\, \mathrm{d}x = \sum_{-\infty}^{+\infty} \int_{nT}^{(n+1)T} f(x)\, e^{-ix\xi}\, \mathrm{d}x = \int_0^T f(x)\, e^{-ix\xi}\, \mathrm{d}x \cdot \sum_{n=-\infty}^{+\infty} e^{inT\xi}$$

is divergent, because $e^{inT\xi} \nrightarrow 0$ for $n \to +\infty$. This implies that the Fourier transform of f does not exist for a periodic function, unless it is the trivial zero function. $\Diamond$

Example 1.7.2 *Compute the Fourier transform of* $\chi_{[-a,a]}$.

We get for $\xi \neq 0$,

$$\mathcal{F}\left\{\chi_{[-a,a]}\right\}(\xi) = \int_{-a}^{a} e^{-ix\xi}\, \mathrm{d}x = \left[\frac{1}{-i\xi}\, e^{-ix\xi}\right]_{-a}^{a} = \frac{1}{i\xi}\left\{e^{ia\xi} - e^{-ia\xi}\right\} = \frac{2\,\sin a\xi}{\xi}.$$

If $\xi = 0$, then we get

$$\mathcal{F}\left\{\chi_{[-a,a]}\right\}(0) = \int_{-a}^{a} 1\, \mathrm{d}x = 2a,$$

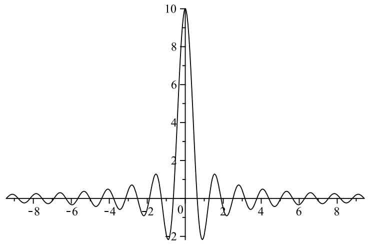

Figure 17: The graph of $f(t) = \frac{2\sin(5\xi)}{\xi}$.

a result which also can be obtained by taking the limit $\xi \to 0$.

Hence we get

$$\mathcal{F}\left\{\chi_{[-a,a]}\right\}(\xi) = \begin{cases} 2\,\dfrac{\sin a\xi}{\xi} & \text{for } \xi \neq 0, \\[2mm] 2a & \text{for } \xi = 0. \end{cases} \qquad \Diamond$$

Example 1.7.3 *Compute the Fourier transform of* $\left(1-x^2\right)\chi_{[-1,1]}(x)$.

Write for short, $f(x) = \left(1-x^2\right)\chi_{[-1,1]}(x)$. Then we get for $\xi \neq 0$,

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{+\infty} e^{-ixi\xi} f(x)\,\mathrm{d}x = \int_{-1}^{1} \left(1-x^2\right) e^{-ix\xi}\,\mathrm{d}x \\ &= \left[\left(1-x^2\right)\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-2x}{-i\xi}\,e^{-ix\xi}\,\mathrm{d}x = \frac{2i}{\xi}\int_{-1}^{1} x\,e^{-ix\xi}\,\mathrm{d}x \\ &= \frac{2i}{\xi}\left[\frac{x}{-i\xi}\,e^{-ix\xi} + \frac{1}{i\xi}\int e^{-ix\xi}\,\mathrm{d}x\right]_{-1}^{1} \\ &= \frac{2i}{-i\xi^2}\left\{e^{-i\xi}+e^{i\xi}\right\} + \frac{2}{\xi^2}\left[\frac{e^{-ix\xi}}{-i\xi}\right]_{x=-1}^{1} \\ &= -\frac{4\cos\xi}{\xi^2} + \frac{4\sin\xi}{\xi^3} = \frac{4}{\xi^3}\left(\sin\xi - \xi\cos\xi\right). \end{aligned}$$

If instead $\xi = 0$, then

$$\hat{f}(0) = \int_{-1}^{1}\left(1-x^2\right)\,\mathrm{d}x = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3}. \qquad \Diamond$$

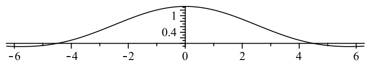

Figure 18: The graph of $f(t) = \frac{4}{\xi^3}(\sin\xi - \xi\cos\xi)$ in Example 1.7.3.

Example 1.7.4 *Compute* $\int_0^{+\infty} \dfrac{u\sin tu}{1+u^2}\,\mathrm{d}u$ *for* $t \in \mathbb{R}$.

We write for convenience,

$$\Phi(t) = \int_0^{+\infty} \frac{u\sin tu}{1+u^2}\,\mathrm{d}u.$$

Then $\Phi(-t) = -\Phi(t)$ and $\Phi(0) = 0$ and we need only consider the case $t > 0$.

Furthermore, $\dfrac{u}{1+u^2} > 0$ for $u > 0$, and the denominator is of degree 1 bigger than the degree of the numerator. For fixed t, the integrand is an even function in $u \in \mathbb{R}$. Hence, it follows from a theorem of improper integrals in Elementary Complex Functions Theory that for fixed $t > 0$,

$$\begin{aligned}\int_0^{+\infty} \frac{u \sin tu}{1+u^2}\,\mathrm{d}u &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{u \sin tu}{1+u^2}\,\mathrm{d}u = \frac{1}{2}\,\Im \int_{-\infty}^{+\infty} \frac{u\,e^{itu}}{1+u^2}\,\mathrm{d}u \\ &= \frac{1}{2}\,\Im\left\{2\pi i \cdot \operatorname{res}\left(\frac{u\,e^{itu}}{1+u^2}; u=i\right)\right\} = \pi\,\Re\left\{\lim_{u\to i}\frac{u\,e^{itu}}{2u}\right\} = \frac{\pi}{2}\,e^{-t}.\end{aligned}$$

Summing up we have proved that

$$\int_0^{+\infty} \frac{u\,\sin tu}{1+u^2}\,\mathrm{d}u = \begin{cases} \overset{\pi}{-}e^t & \text{for } t > 0, \\ 0 & \text{for } t = 0, \\ -\dfrac{\pi}{2}\,e^t & \text{for } t < 0. \end{cases} \qquad \Diamond$$

Example 1.7.5 *Compute* $\int_{-\infty}^{+\infty} \dfrac{\xi\,e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi$ *for* $x \in \mathbb{R}\setminus\{0\}$.

It follows from the estimate

$$\left|\frac{\xi}{\xi^2+1}\right| \le \frac{C}{|\xi|} \qquad \text{for } |\xi| \ge 2,$$

that the improper integral is convergent for $x \in \mathbb{R}\setminus\{0\}$.

If $x > 0$, then it value is given by

$$\int_{-\int}^{+\infty} \frac{\xi\,e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi = 2\pi i \cdot \operatorname{res}\left(\frac{z\,e^{ixz}}{z^2+1}; z=i\right) = 2\pi i \lim_{z\to i}\frac{z\,e^{ixz}}{2z} = \pi i\,e^{-x}.$$

If instead $x < 0$, then

$$\int_{-\infty}^{+\infty} \frac{\xi\,e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi = \overline{\int_{-\infty}^{+\infty} \frac{\xi\,e^{i|x|\xi}}{\xi^2+1}\,\mathrm{d}\xi} = -\pi i\,e^{-|x|}.$$

Summing up we have proved that

$$\int_{-\infty}^{+\infty} \frac{\xi\,e^{ix\xi}}{\xi^2+1}\,\mathrm{d}\xi = \begin{cases} \pi i \cdot e^{-x} & \text{for } x > 0, \\ -\pi i \cdot e^{-x} & \text{for } x < 0. \end{cases}$$

We note that when $x = 0$, the improper integral is divergent. $\Diamond$

Example 1.7.6 *Apply the Fourier transformation with respect to x when solving the partial differential equation (the heat equation)*

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}_+,$$

assuming that $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t)$ tend towards 0 for $x \to +\infty$, and

$$u(0,t) = 0, \quad t \in \mathbb{R}, \qquad u(x,0) = e^{-x}, \quad x \in \mathbb{R}_+.$$

Hint. *Put $u(-x,t) := -u(x,t)$, and then write the solution as an integral.*

We use the hint and extend $u(x,t)$ to the left half plane by

$$u(-x,t) = -u(x,t),$$

which also agrees with that $u(0,t) = 0$. Then we get by the Fourier transformation with respect to x,

$$\begin{aligned} U(\xi,t) &= \int_{-\infty}^{+\infty} e^{-ix\xi}\, u(x,t)\,\mathrm{d}x = \int_0^{+\infty} e^{-ix\xi}\, u(x,t)\,\mathrm{d}x + \int_{-\infty}^{0} e^{-ix\xi}\, u(x,t)\,\mathrm{d}x \\ &= \int_0^{+\infty} e^{-ix\xi}\, u(x,t)\,\mathrm{d}x + \int_{+\infty}^{0} e^{iy\xi}\, u(-y,t)\,(-\mathrm{d}y) \\ &= \int_0^{+\infty} e^{-ix\xi}\, u(x,t)\,\mathrm{d}x - \int_0^{+\infty} e^{ix\xi}\, u(x,t)\,\mathrm{d}x = -2i\int_0^{+\infty} u(x,t)\sin(\xi x)\,\mathrm{d}x. \end{aligned}$$

Using that $U(-\xi,t) = U(\xi,t)$ it follows from the inversion formula that

$$(11)\quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\xi,t)\,e^{ix\xi}\,\mathrm{d}\xi = +\frac{2i}{2\pi}\int_0^{+\infty} U(\xi,t)\,\sin(x\xi)\,\mathrm{d}\xi = \frac{i}{\pi}\int_0^{+\infty} U(\xi,t)\sin(x\xi)\,\mathrm{d}\xi.$$

The heat equation $\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2}$ is then transferred into

$$\int_0^{+\infty} \frac{\partial u}{\partial t}\,\sin(\xi x)\,\mathrm{d}x = 2\int_0^{+\infty} \frac{\partial^2 u}{\partial x^2}\,\sin(\xi x)\,\mathrm{d}x,$$

hence

$$\begin{aligned} \frac{\partial U}{\partial t}(\xi,t) &= 2\int_0^{+\infty} \frac{\partial^2 u}{\partial x^2}\,\sin(\xi x)\,\mathrm{d}x \\ &= 2\left[\frac{\partial u}{\partial x}\,\sin\xi x - \xi u(x,t)\cos(\xi x)\right]_{x=0}^{+\infty} - 2\xi^2\int_0^{+\infty} u(x,t)\sin(\xi x)\,\mathrm{d}x \\ &= 2\xi\,u(0,t) - 2\xi^2\,U(\xi,t) = -2\xi^2\,U(\xi,t), \end{aligned}$$

because $u(x,t) \to 0$ and $\frac{\partial u}{\partial x}(x,t) \to 0$ for $x \to +\infty$.

When we integrate this equation, we get

$$U(\xi,t) = c(\xi)\cdot\exp\left(-2\xi^2 t\right),$$

where we still have to find $c(\xi)$.

It follows from $u(x,0)=e^{-x}$ that

$$\begin{aligned} c(\xi) &= U(\xi,0)=-2i\int_0^{+\infty} u(x,0)\sin(\xi x)\,\mathrm{d}x=-2i\int_0^{+\infty} e^{-x}\,\sin\xi x\,\mathrm{d}x \\ &= \int_0^{+\infty}\left\{e^{-(1+i\xi)x}-e^{-(1-i\xi)x}\right\}\mathrm{d}x=\left[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}-\frac{e^{-(1-i\xi)x}}{-(1-i\xi)}\right]_{x=0}^{+\infty} \\ &= \frac{1}{1+i\xi}-\frac{1}{1-i\xi}=-\frac{2i\xi}{1+\xi^2}, \end{aligned}$$

and we conclude that

$$U(\xi,t)=-\frac{2i\xi}{1+\xi^2}\,\exp\left(-2\xi^2 t\right).$$

Finally, it follows from the inversion formula (11) that

$$\begin{aligned} u(x,t) &= \frac{i}{\pi}\int_0^{+\infty} U(\xi,t)\sin(x\xi)\,\mathrm{d}\xi=\frac{i}{\pi}\int_0^{+\infty}\left\{-\frac{2i\xi}{1+\xi^2}\,\exp\left(-2\xi^2\,t\right)\right\}\sin(x\xi)\,\mathrm{d}\xi \\ &= \frac{2}{\pi}\int_0^{+\infty}\frac{\xi\exp\left(-2\xi^2\,t\right)\sin(x\xi)}{1+\xi^2}\,\mathrm{d}\xi. \qquad \Diamond \end{aligned}$$

1.8 The Laplace transform of a function via a differential equation

Example 1.8.1 *Given* $f(t) = \sin\sqrt{t}$, $t \in \mathbb{R}_+$.

1) Prove that $f(t)$ *satisfies the differential equation*

(12) $4\,t\,f''(t) + 2\,f'(t) + f(t) = 0.$

2) Use (12) to compute $\mathcal{L}\{f\}(z)$

1) First compute for $t \in \mathbb{R}_+$,

$$f'(t) = \frac{1}{2\sqrt{t}}\cos\sqrt{t}, \qquad f''(t) = -\frac{1}{4t}\sin\sqrt{t} - \frac{1}{4t\sqrt{t}}\cos\sqrt{t}.$$

We get by insertion into (12),

$$4tf''(t) + 2f''(t) + f(t) = -\sin\sqrt{t} - \frac{1}{\sqrt{t}}\cos\sqrt{t} + \frac{1}{\sqrt{t}}\cos\sqrt{t} + \sin\sqrt{t} = 0,$$

which proves that $f(t) = \sin\sqrt{t}$, $t > 0$, is a solution of (12).

2) It follows from the computation above that $f'(0)$ is not defined, so we cannot apply the Laplace transformation on (12). However, since $t\,f'(t) \to 0$ for $t \to 0+$, we may instead use that

$$4t\,f''(t) + 2f'(t) + f(t) = 4\,\frac{d}{dt}\{t\,f'(t)\} - 2f'(t) + f(t) = 0.$$

Since $f(0) = 0$ and $\lim_{t\to 0+} t\,f'(t) = 0$, the latter equation can be Laplace transformed, giving

$$\begin{aligned}
0 &= \mathcal{L}\{4t\,f''(t) + 2f'(t) + f(t)\}(z) \\
&= 4\,\mathcal{L}\left\{\frac{d}{dt}(tf'(t))\right\}(z) - 2\,\mathcal{L}\{f'\}(z) + \mathcal{L}\{f\}(z) \\
&= 4\,(z\,\mathcal{L}\{tf'(t)\}(z) - 0) - 2(z\mathcal{L}\{f\}(z) - 0) + \mathcal{L}\{f\}(z) \\
&= 4z\left(-\frac{d}{dz}\,\mathcal{L}\{f'\}(z)\right) + (1-2z)\mathcal{L}\{f\}(z) \\
&= -4z\,\frac{d}{dz}\,(z\,\mathcal{L}\{f\}(z) - f(0)) + (1-2z)\mathcal{L}\{f\}(z) \\
&= -4z^2\,\frac{d}{dz}\,\mathcal{L}\{f\}(z) + (1-6z)\mathcal{L}\{f\}(z).
\end{aligned}$$

We have proved that $\mathcal{L}\{f\}(z)$ for $z \neq 0$ satisfies the differential equation

$$\frac{d}{dz}\,\mathcal{L}\{f\}(z) = \frac{1}{4}\left\{\frac{1}{z^2} - \frac{6}{z}\right\}\mathcal{L}\{f\}(z),$$

from which we conclude that we have for $\Re z > 0$,

$$\mathcal{L}\{f\}(z) = \frac{c}{z\sqrt{z}} \exp\left(-\frac{1}{4z}\right), \qquad \Re z > 0,$$

where we still have to find the constant c.

We cannot find c by the initial or final value theorem. Instead we get by a *formal* series expansion,

$$f(t) = \sin\sqrt{t} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}\, t^{n+\frac{1}{2}} \qquad \text{for } t \geq 0.$$

This gives by a termwise *formal* Laplace transformation,

$$\mathcal{L}\{f\}(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{\Gamma\left(n+\frac{3}{2}\right)}{z^{n+\frac{3}{2}}}.$$

However, if only we can prove that the right hand side of this formal computation is convergent in a right half plan, then it follows that this termwise Laplace transformation is a legal method.

A straightforward computation of the right hand side gives for $\Re z > 0$,

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{\Gamma\left(n+\frac{3}{2}\right)}{z^{n+\frac{3}{2}}} &= \frac{1}{z\sqrt{z}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{z^n} \cdot \frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{(2n+1)!} \\
&= \frac{1}{z\sqrt{z}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{z^n} \cdot \frac{\sqrt{\pi}}{2^{n+1}} \cdot \frac{(2n+1)(2n-1)\cdots 3\cdot 1}{(2n+1)!} \\
&= \frac{1}{z\sqrt{z}} \cdot \frac{\sqrt{\pi}}{2} \sum_{n=0}^{+\infty} \left\{-\frac{1}{2z}\right\}^n \cdot \frac{1}{2n\cdot 2(n-1)\cdots 2\cdot 1} \\
&= \frac{\sqrt{\pi}}{2} \cdot \frac{1}{z\sqrt{z}} \sum_{n=0}^{+\infty} \frac{1}{n!} \left\{-\frac{1}{4z}\right\}^n = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{z\sqrt{z}} \exp\left(-\frac{1}{4z}\right).
\end{aligned}$$

Thus we have justified our formal computation above and found that $c = \dfrac{\sqrt{\pi}}{2}$, hence the Laplace transform is given by

$$\mathcal{L}\left\{\sin\sqrt{t}\right\}(z) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{z\sqrt{z}} \cdot \exp\left(-\frac{1}{4z}\right). \qquad \Diamond$$

2 Appendices

2.1 Trigonometric formulæ

We repeat the formulæ known from e.g. *Ventus, Calculus 1-a, Functions in one Variable.* The *addition formulæ for trigonometric functions* are

(13) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$,

(14) $\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$,

(15) $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$,

(16) $\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$.

Remark 2.1.1 One remembers these important rules by noting that $\cos x$ is even, and $\sin x$ is odd. Therefore, since $\cos(x \pm y)$ is even, the reduction must contain $\cos x \cdot \cos y$ (even times even) and $\sin x \cdot \sin y$ (odd times odd). Then we shall only remember the change of sign in front of $\sin x \cdot \sin y$.

Analogously, $\sin(x \pm y)$ is odd, so the reduction must contain $\sin x \cdot \cos y$ (odd times even) and $\cos x \cdot \sin y$ (even times odd). Here there is no change of sign. $\Diamond$

The antilogarithmic formulæ. These are derived from the *addition formulæ* above.

$$\sin x \cdot \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}, \qquad \text{even},$$

$$\cos x \cdot \cos y = \frac{1}{2}\{\cos(x-y) + \cos(x+y)\}, \qquad \text{even},$$

$$\sin x \cdot \cos y = \frac{1}{2}\{\sin(x-y) + \sin(x+y)\}, \qquad \text{odd}.$$

2.2 Integration of trigonometric polynomials

The task is to find the integral

$$\int \sin^m x \cdot \cos^n x \, \mathrm{d}x, \qquad \text{for } m,\, n \in \mathbb{N}_0.$$

We shall in the following only consider one single term of the the form $\sin^m x \cdot \cos^n x$, where m and $n \in \mathbb{N}_0$, of a trigonometric polynomial, because we in general can find the result by linearity.

We define the *degree* of $\sin^m x \cdot \cos^n x$ as the sum $m+n$.

When we integrate such a single trigonometric product of degree $m+n$, we first must answer the following question: Is it of even or odd degree? These two possibilities are then again subdivided into to subcases, so we have four different variants of method, when we integrate a trigonometric polynomial.

1) The degree $m+n$ is odd.
   a) $m=2p$ is even, and $n=2q+1$ is odd.
   b) $m=2p+1$ is odd, and $n=2q$ is odd.
2) The degree $m+n$ is even.
   a) $m=2p+1$ and $n=2q+1$ are both odd.
   b) $m=2p$ and $n=2q$ are both even.

We shall in the following go through the four possibilities.

1a) $m=2p$ is even and $n=2q+1$ is odd.

Use the substitution $u=\sin x$ (corresponding to $m=2p$ even) and write

$$\cos^{2q+1} x\,\mathrm{d}x = \left(1-\sin^2 x\right)^q \cos x\,\mathrm{d}x = \left(1-\sin^2 x\right)^q\,\mathrm{d}\sin x,$$

thus

$$\int \sin^{2p} x\cdot\cos^{2q+1} x\,\mathrm{d}x = \int \sin^{2p} x\left(1-\sin^2 x\right)^q\,\mathrm{d}\sin x = \int_{u=\sin x} u^{2p}\cdot\left(1-u^2\right)^q\,\mathrm{d}u,$$

and the problem is reduced to an integration of a polynomial, followed by a substitution.

1b) $m=2p+1$ odd and $n=2q$ even.

Apply the substitution $u=\cos x$ (corresponding to $n=2q$ even) and write

$$\sin^{2p+1} x\,\mathrm{d}x = \left(1-\cos^2 x\right)^p \cos x\,\mathrm{d}x = -\left(1-\cos^2 x\right)^p\,\mathrm{d}\cos x,$$

from which

$$\int \sin^{2p+1} x\cdot\cos^{2q} x\,\mathrm{d}x = -\int \left(1-\cos^2 x\right)^p\cdot\cos^{2q} x\,\mathrm{d}\cos x = -\int_{u=\cos x}\left(1-u^2\right)^p\cdot u^{2q}\,\mathrm{d}u,$$

and the problem is again reduced to an integration of a polynomial followed by a substitution.

2) When the degree $m+n$ is even, the trick is to use the double angle, using the formulæ

$$\sin^2 x = \frac{1}{2}(1-\cos 2x),\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x),\qquad \sin x\cdot\cos x = \frac{1}{2}\sin 2x.$$

2a) $m=2p+1$ and $n=2q+1$ are both odd.

Rewrite the integrand in the following way,

$$\sin^{2p+1} x\cdot\cos^{2q+1} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p\left\{\frac{1}{2}(1+\cos 2x)\right\}^q\cdot\frac{1}{2}\sin 2x.$$

This is a reduction to case 1b) above, so by the substitution $u=\cos 2x$ we get

$$\int \sin^{2p+1} x\cdot\cos^{2q+1} x\,\mathrm{d}x = -\frac{1}{2^{p+q+1}}\cdot\frac{1}{2}\int_{u=\cos 2x}(1-u)^p(1+u)^q\,\mathrm{d}u,$$

and the problem is again reduced to an integration of a polynomial followed by a substitution.

2b) $m = 2p$ and $n = 2q$ are both even.

This is the most difficult one of the four cases. First rewrite the integrand in the following way,

$$\sin^{2p} x \cdot \cos^{2q} = \left\{ \frac{1}{2} (1 - \cos 2x) \right\}^p \left\{ \frac{1}{2} (1 + \cos 2x) \right\}^q .$$

The degree of the left hand side is $2p + 2q$ in the pair $(\cos x, \sin x)$, while the right hand side only has the degree $p+q$ in the pair $(\cos 2x, \sin 2x)$ with the double angle as new variable. The problem is that we at the same time by a multiplication get many terms on the right hand side of the equation, which then must be computed separately.

However, since the degree is halved, whenever 2b) is applied, the problem can be solved in a finite number of steps.

We shall illustrate the method of 2b) in the following example.

Example 2.2.1 We shall compute the integral

$$\int \cos^6 x \,\mathrm{d}x.$$

The degree $0+6=6$ is even, and both $m=0$ and $n=6$ are even. Thus we are in case 2b). By using the double angle the integrand becomes

$$\cos^6 x = \left\{\frac{1}{2}(1+\cos 2x)\right\}^3 = \frac{1}{8}(1+3\,\cos 2x + 3\,\cos^2 2x + \cos^3 2x).$$

Integration of the first two terms is straightforward,

$$\frac{1}{8}\int (1+3\cos 2x)\,\mathrm{d}x = \frac{1}{8}\,x + \frac{3}{16}\,\sin 2x.$$

The third term is again of type 2b), so we transform it to the quadruple angle,

$$\frac{1}{8}\int 3\,\cos^2 2x\,\mathrm{d}x = \frac{3}{8}\int \frac{1}{2}(1+\cos 4x)\,\mathrm{d}x = \frac{3}{16}\,x + \frac{3}{64}\,\sin 4x.$$

The last term is of type 1a), so

$$\frac{1}{8}\int \cos^3 2x\,\mathrm{d}x = \frac{1}{8}\int \left(1-\sin^2 2x\right)\cdot\frac{1}{2}\,\mathrm{d}\sin 2x = \frac{1}{16}\,\sin 2x - \frac{1}{48}\,\sin^3 2x.$$

Summing up we get after a reduction,

$$\int \cos^6 x\,\mathrm{d}x = \frac{5}{16}\,x + \frac{1}{4}\,\sin 2x - \frac{1}{48}\,\sin^3 2x + \frac{3}{64}\,\sin 4x. \qquad \Diamond$$

b!

	$f(t)$	$\mathcal{L}\{f\}(z)$	$\sigma(f)$
1	1	$\dfrac{1}{z}$	0
2	t^n	$\dfrac{n!}{z^{n+1}}$	0
3	e^{-at}	$\dfrac{1}{z+a}$	$-\Re\, a$
4	$\sin(at)$	$\dfrac{a}{z^2+a^2}$	$\|\Im\, a\|$
5	$\cos(at)$	$\dfrac{z}{z^2+a^2}$	$\|\Im\, a\|$
6	$\sinh(at)$	$\dfrac{a}{z^2-a^2}$	$\|\Re\, a\|$
7	$\cosh(at)$	$\dfrac{z}{z^2-a^2}$	$\|\Re\, a\|$

Table 1: The simplest Laplace transforms

2.3 Tables of some Laplace transforms and Fourier transforms

The simplest Laplace transforms were already derived in *Ventus, Complex Functions Theory a-4, The Laplace Transformation I.* These are given in Table 1.

We collect in the following tables the results from *Ventus, Complex Functions Theory a-5* where we always can use $\sigma(f) = 0$, so there is no need to specify $\sigma(f)$ in the tables. The first table is ordered according to the simplicity of the function $f(t)$, and the second one is ordered according to the simplicity of $\mathcal{L}\{f\}(z)$. Instead of $\sigma(f)$ we include a reference to where the function is handled in the text.

	$f(t)$	$\mathcal{L}\{f\}(z)$	Reference
1	t^α for $\Re\,\alpha > -1$	$\dfrac{\Gamma(\alpha+1)}{z^{\alpha+1}}$	Complex Functions a-5
2	$\dfrac{1}{t+a}$ for $a>0$	$e^{az}\,\mathrm{Ei}(az)$	Complex Functions a-5
3	$\dfrac{1}{1+t^2}$	$\cos z\cdot\left\{\dfrac{\pi}{2}-\mathrm{Si}(z)\right\}-\sin z\cdot\mathrm{Ci}(z)$	Complex Functions a-5
4	$\ln t$	$-\dfrac{\gamma+\mathrm{Log}\,z}{z}$	Complex Functions a-5
5	$\dfrac{1}{\sqrt{\lvert t-1\rvert}}$	$\sqrt{\dfrac{\pi}{2}}\,e^{-z}\left\{1-i\cdot\mathrm{erf}\left(i\sqrt{z}\right)\right\}$	Complex Functions a-5
6	$\exp\left(-t^2\right)$	$\dfrac{\sqrt{\pi}}{2}\exp\left(\dfrac{z^2}{2}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)$	Complex Function a-5
7	$t^{-\frac{3}{2}}\exp\left(-\dfrac{1}{4t}\right)$	$2\sqrt{\pi}\,e^{-\sqrt{z}}$	Complex Functions a-5
8	$\mathrm{erf}(t)$	$\dfrac{1}{z}\exp\left(\dfrac{z^2}{4}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)$	Complex Functions a-5
9	$\mathrm{erfc}(t)$	$\dfrac{1}{z}\left\{1-\exp\left(\dfrac{z^2}{4}\right)\mathrm{erfc}\left(\dfrac{z}{2}\right)\right\}$	Complex Function a-5
10	$\mathrm{erfc}\left(\sqrt{t}\right)$	$\dfrac{1}{z\sqrt{z+1}}$	Complex Functions a-5
11	$\mathrm{erf}\left(\dfrac{1}{2\sqrt{t}}\right)$	$\dfrac{1-e^{-\sqrt{z}}}{z}$	Complex Functions a-5
12	$\mathrm{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$	$\dfrac{1}{z}\,e^{-\sqrt{z}}$	Complex Functions a-5
13	$\mathrm{Si}(t)$	$\dfrac{1}{z}\,\mathrm{Arctan}\,\dfrac{1}{z}$	Complex Functions a-5
14	$\mathrm{Ci}(t)$	$\dfrac{\mathrm{Log}\left(1+z^2\right)}{2z}$	Complex Functions a-5

Table 2: More advanced Laplace transforms

	$f(t)$	$\mathcal{L}\{f\}(z)$	Reference
15	$\operatorname{Ei}(t)$	$\dfrac{\operatorname{Log}(1+z)}{z}$	Complex Functions a-5
16	$J_n(t)$ for $n \in \mathbb{N}_0$	$\dfrac{\left(\sqrt{z^2+1}-z\right)^n}{\sqrt{z^2+1}}$	Complex Functions a-5
17	$J_0\left(2\sqrt{t}\right)$	$\dfrac{1}{z}\exp\left(-\dfrac{1}{z}\right)$	Complex Functions a-5
18	$\dfrac{1}{\sqrt{t}}J_1\left(2\sqrt{t}\right)$	$1-\exp\left(-\dfrac{1}{z}\right)$	Complex Functions a-5

Table 3: More advanced Laplace transforms, continued

	$F(z)$	$\mathcal{L}^{-1}\{F\}(t)$	Reference
1	$\dfrac{1}{z}$	1	Complex Functions a-4
2	$\dfrac{1}{z+a}$	e^{-at}	Complex Functions a-4
3	z^{-n} for $n \in \mathbb{N}$	$\dfrac{1}{(n-1)!}\,t^{n-1}$	Complex Functions a-4
4	$z^{-\alpha}$, $\Re\,\alpha > 0$	$\dfrac{1}{\Gamma(\alpha)}\,t^{\alpha-1}$	Complex Functions a-5
5	$\dfrac{1}{z^2-a^2}, \quad a \neq 0$	$\dfrac{\sinh(at)}{a}$	Complex Functions a-4
6	$\dfrac{z}{z^2-a^2}$	$\cosh(at)$	Complex Functions a-4
7	$\dfrac{1}{z^2+a^2}, \quad a \neq 0$	$\dfrac{\sin(at)}{a}$	Complex Functions a-4
8	$\dfrac{z}{z^2+a^2}$	$\cos(at)$	Complex Functions a-4

Table 4: Table of inverse Laplace transforms

	$F(z)$	$\mathcal{L}^{-1}\{F\}(t)$	Reference
9	$\dfrac{1}{z\sqrt{z+1}}$	$\operatorname{erf}\left(\sqrt{t}\right)$	Complex Functions a-5
10	$\dfrac{1}{\sqrt{z^2+1}}$	$J_0(t)$	Complex Functions a-4 and Complex Functions a-5
11	$\dfrac{\left(\sqrt{z^2+1}-z\right)^n}{\sqrt{z^2+1}}$ for $n \in \mathbb{N}_0$	$J_n(t)$	Complex Functions a-5
12	$1-\exp\left(-\dfrac{1}{z}\right)$	$\dfrac{1}{\sqrt{t}}\, J_1\left(2\sqrt{t}\right)$	Complex Functions a-5
13	$\dfrac{1}{z}\exp\left(-\dfrac{1}{z}\right)$	$J_0\left(2\sqrt{t}\right)$	Complex Functions a-5
14	$e^{-\sqrt{z}}$	$\dfrac{1}{2t\sqrt{\pi t}}\exp\left(-\dfrac{1}{4t}\right)$	Complex Functions a-5
15	$\dfrac{1}{z}\, e^{-\sqrt{z}}$	$\operatorname{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$	Complex Functions a-5
16	$\dfrac{1}{z}\left\{1-e^{-\sqrt{z}}\right\}$	$\operatorname{erf}\left(\dfrac{1}{2\sqrt{t}}\right)$	Complex Functions a-5
17	$\dfrac{\operatorname{Log} z}{z}$	$-\gamma-\ln t$	Complex Functions a-5
18	$\dfrac{1}{z}\operatorname{Log}(1+z)$	$\operatorname{Ei}(t)$	Complex Functions a-5
19	$\dfrac{1}{z}\operatorname{Log}\left(1+z^2\right)$	$2\operatorname{Ci}(t)$	Complex Functions a-5
20	$\dfrac{1}{z}\operatorname{Arctan}\dfrac{1}{z}$	$\operatorname{Si}(t)$	Complex Functions a-5
21	$\dfrac{1}{2}\operatorname{Log}\left(\dfrac{z+i}{z-i}\right)$	$2i\operatorname{Si}(t)$	Complex Functions a-5

Table 5: Table of inverse Laplace transforms, continued

	$f(t)$	$\mathcal{F}\{f\}(\xi)$
1	$\chi_{[-T,T]}(x), \qquad T>0$	$2\,\dfrac{\sin T\xi}{\xi}$
2	$\left(1-\dfrac{\lvert x\rvert}{T}\right)\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{4}{T\xi^2}\,\sin^2\left(\dfrac{T\xi}{2}\right)$
3	$\dfrac{a}{x^2+a^2}, \qquad \Re\,a>0$	$\pi\,e^{-a\lvert\xi\rvert}$
4	$\dfrac{\sin(Tx)}{x}, \qquad T>0$	$\pi\,\chi_{[-T,T]}(\xi)$
5	$\cos(\omega\,x)\cdot\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{\sin(T(\xi-\omega))}{\xi-\omega}+\dfrac{\sin((T(\xi+\omega))}{\xi+\omega}$
6	$\sin(\omega\,x)\cdot\chi_{[-T,T]}(x), \qquad T>0$	$\dfrac{1}{i}\left\{\dfrac{\sin(T(\xi-\omega))}{\xi-\omega}-\dfrac{\sin(T(\xi+\omega))}{\xi+\omega}\right\}$
7	$e^{-a\lvert x\rvert}, \qquad \Re\,a>0$	$\dfrac{2a}{\xi^2+a^2}$
8	$e^{-ax}\,\chi_{\mathbb{R}_+}(x), \qquad \Re\,a>0$	$\dfrac{1}{a+i\xi}$
9	$e^{ax}\,\chi_{\mathbb{R}_-}(x), \qquad \Re\,a>0$	$\dfrac{1}{a-i\xi}$
10	$\exp\left(-ax^2\right), \qquad a>0$	$\sqrt{\dfrac{\pi}{a}}\cdot\exp\left(-\dfrac{\xi^2}{4a}\right)$
11	1	$2\pi\,\delta$
12	$x^n, \qquad n\in\mathbb{N}_0$	$2\pi\,i^n\,\delta^{(n)}$
13	$e^{ihx}, \qquad h\in\mathbb{R}$	$2\pi\,\delta_{(h)}$
14	$\cosh(hx), \qquad h\in\mathbb{R}$	$\pi\,\delta_{(h)}+\pi\,\delta_{(-h)}$
15	$\sin(hx), \qquad h\in\mathbb{R}$	$-i\,\pi\,\delta_{(h)}+i\,\pi\,\delta_{(-h)}$
16	δ	1
17	$\delta_{(h)}, \qquad h\in\mathbb{R}$	$e^{-ih\xi}$
18	$\delta^{(n)}, \qquad n\in\mathbb{N}_0$	$(i\xi)^n$

Table 6: Some Fourier transforms, $\mathcal{F}\{f\}(\xi)=\int_{-\infty}^{+\infty}e^{-ix\xi}\,f(x)\,\mathrm{d}x$.

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