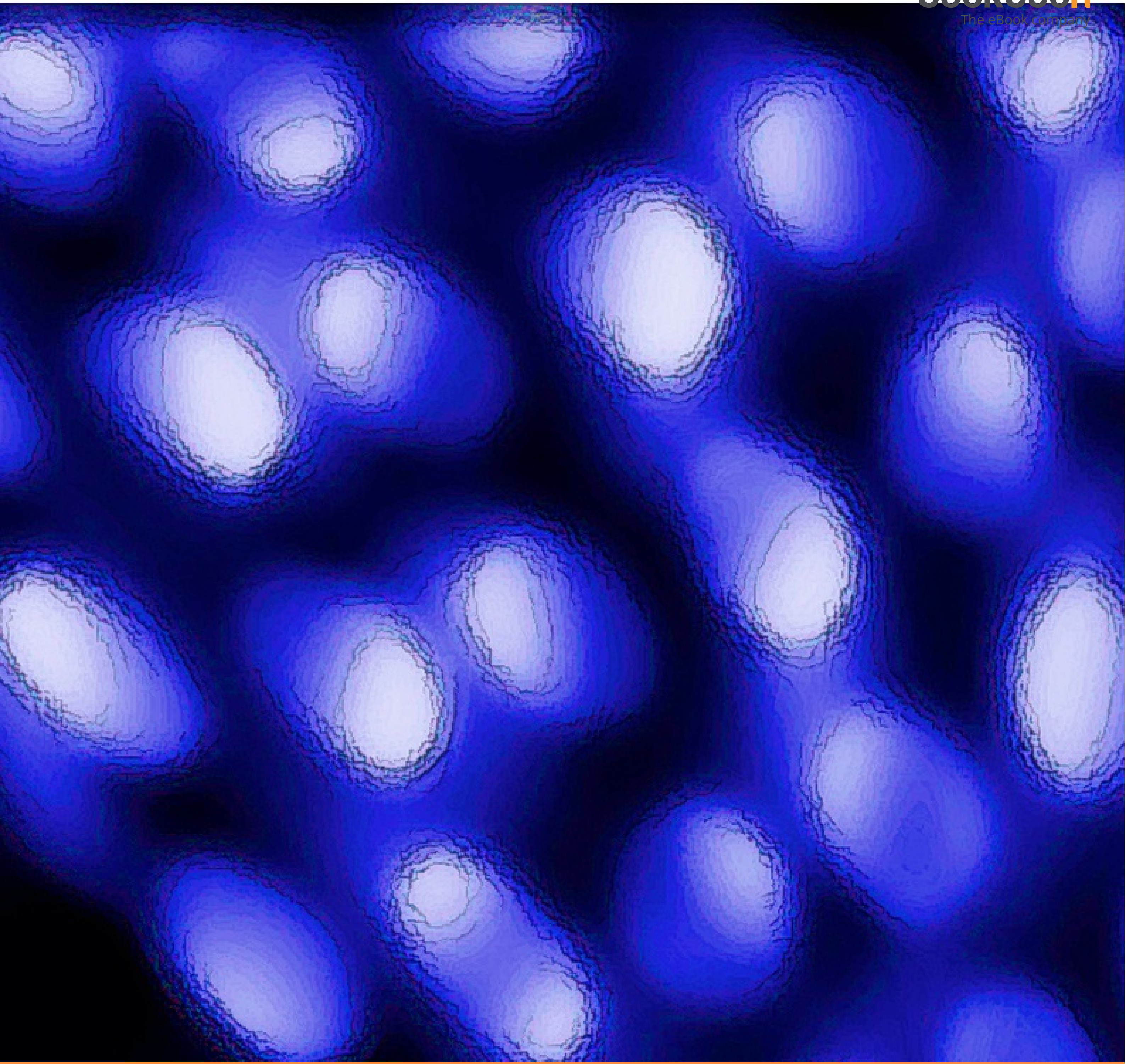

Analytic Aids

Probability Examples c-7

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Probability Examples c-7 – Analytic Aids

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Contents

Introduction

This is the eight book of examples from the *Theory of Probability*. In general, this topic is not my favourite, but thanks to my former colleague, Ole Jørsboe, I somehow managed to get an idea of what it is all about. We shall, however, in this volume deal with some topics which are closer to my own mathematical fields.

The prerequisites for the topics can e.g. be found in the *Ventus: Calculus 2* series and the *Ventus: Complex Function Theory* series, and all the previous *Ventus: Probability c1-c6*.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
27th October 2014

1 Generating functions; background

1.1 Definition of the generating function of a discrete random variable

The *generating functions* are used as analytic aids of random variables which only have values in $\mathbb{N}_0$, e.g. *binomial distributed* or *Poisson distributed* random variables.

In general, a *generating function* of a *sequence* of real numbers $(a_k)_{k=0}^{+\infty}$ is a function of the type

$$A(s) := \sum_{k=0}^{+\infty} a_k\, s^k, \qquad \text{for } |s| < \varrho,$$

provided that the series has a non-empty interval of convergence $]-\varrho,\varrho[$, $\varrho > 0$.

Since a generating function is defined as a *convergent power series*, the reader is referred to the *Ventus: Calculus 3* series, and also possibly the *Ventus: Complex Function Theory* series concerning the theory behind. We shall here only mention the most necessary properties, because we assume everywhere that $A(s)$ is defined for $|s|\varrho$.

A generating function $A(s)$ is *always* of class $C^{\infty}(]-\varrho,\varrho[)$. One may always *differentiate* $A(s)$ *term by term* in the interval of convergence,

$$A^{(n)}(s) = \sum_{k=n}^{+\infty} k(k-1)\cdots(k-n+1)a_k s^{k-n}, \qquad \text{for } s \in]-\varrho,\varrho[.$$

We have in particular

$$A^{(n)}(0) = n! \cdot a_n, \quad \text{i.e.} \quad a_n = \frac{A^{(n)}(0)}{n!} \qquad \text{for every } n \in \mathbb{N}_0.$$

Furthermore, we shall need the well-known

Theorem 1.1 Abel's theorem. *If the convergence radius $\varrho > 0$ is finite, and the series $\sum_{k=0}^{+\infty} a_k\, \varrho^k$ is convergent, then*

$$\sum_{k=0}^{+\infty} a_k\, \varrho^k = \lim_{s \to \varrho-} A(s).$$

In the applications all elements of the sequence are typically bounded. We mention:

1) If $|a_k| \le M$ for every $k \in \mathbb{N}_0$, then

$$A(s) = \sum_{k=0}^{+\infty} a_k\, s^k \qquad \text{convergent for } s \in]-\varrho, \varrho[, \text{ where } \varrho \ge 1.$$

This means that $A(s)$ is defined and a C^∞ function in at least the interval $]-1,1[$, possibly in a larger one.

2) If $a_k \ge 0$ for every $k \in \mathbb{N}_0$, and $\sum_{k=0}^{+\infty} a_k = 1$, then $A(s)$ is a C^∞ function in $]-1,1[$, and it follows from *Abel's theorem* that $A(s)$ can be extended *continuously* to the closed interval $[-1,1]$.
This observation will be important in the applications her, because the sequence (a_k) below is chosen as a sequence (p_k) of probabilities, and the assumptions are fulfilled for such an extension.

If X is a *discrete* random variable of values in $\mathbb{N}_0$ and of the probabilities

$$p_k = P\{X = k\}, \qquad \text{for } k \in \mathbb{N}_0,$$

then we define the *generating function of* X as the function $P : [0,1] \to \mathbb{R}$, which is given by

$$P(s) = E\left\{s^X\right\} := \sum_{k=0}^{+\infty} p_k\, s_k.$$

The reason for introducing the generating function of a discrete random variable X is that it is often easier to find $P(s)$ than the probabilities themselves. Then we obtain the probabilities as the coefficients of the series expansion of $P(s)$ from 0.

1.2 Some generating functions of random variables

We shall everywhere in the following assume that $p \in]0,1[$ and $q := 1 - p$, and $\mu > 0$.

1) If X is *Bernoulli distributed*, $B(1,p)$, then

$$p_0 = 1 - p = q \qquad \text{and} \qquad p_1 = p, \qquad \text{and} \qquad P(s) = 1 + p(s-1).$$

2) If X is *binomially distributed*, $B(n,p)$, then

$$p_k = \binom{n}{k} p^k q^{n-k}, \qquad \text{and} \qquad P(s) = \{1 + p(s-1)\}^n.$$

3) If X is *geometrically distributed*, $\mathrm{Pas}(1,p)$, then

$$p_k = pq^{k-1}, \qquad \text{and} \qquad P(s) = \frac{ps}{1-qs}.$$

4) If X is *negative binomially distributed*, $\mathrm{NB}(\kappa,p)$, then

$$p_k = (-1)^k \binom{-\kappa}{k} p^\kappa q^k, \qquad \text{and} \qquad P(s) = \left\{\frac{p}{1-qs}\right\}^\kappa .$$

5) If X is *Pascal distributed*, $\mathrm{Pas}(r,p)$, then

$$p_k = \binom{k-1}{r-1} p^r q^{k-r}, \qquad \text{and} \qquad P(s) = \left\{\frac{ps}{1-qs}\right\}^r .$$

6) If X is *Poisson distributed*, $P(\mu)$, then

$$p_k = \frac{\mu^k}{k!}\, e^{-\mu}, \qquad \text{and} \qquad P(s) = \exp(\mu(s-1)).$$

1.3 Computation of moments

Let X be a random variable of values in $\mathbb{N}_0$ and with a generating function $P(s)$, which is continuous in $[0,1]$ (and C^∞ in the interior of this interval).

The random variable X has a *mean*, if and only the derivative $P'(1) := \lim_{s\to 1-} P'(s)$ exists and is finite. When this is the case, then

$$E\{X\} = P'(1).$$

The random variable X has a *variance*, if and only if $P''(1) := \lim_{s\to 1-} P''(s)$ exists and is finite. When this is the case, then

$$V\{X\} = P''(1) + P'(1) - \{P'(1)\}^2 .$$

In general, the n-th moment $E\{X^n\}$ exists, if and only if $P^{(n)}(1) := \lim_{s\to 1-} P^{(n)}(s)$ exists and is finite.

1.4 Distribution of sums of mutually independent random variables

If X_1, X_2, …, X_n are mutually independent discrete random variables with corresponding generating functions $P_1(s)$, $P_2(s)$, …, $P_n(s)$, then the generating function of the sum

$$Y_n := \sum_{i=1}^n X_i$$

is given by

$$P_{Y_n}(s) = \prod_{i=1}^n P_i(s), \qquad \text{for } s \in [0,1].$$

1.5 Computation of probabilities

Let X be a discrete random variable with its generating function given by the series expansion

$$P(s)=\sum_{k=1}^{+\infty} p_k s^k.$$

Then the probabilities are given by

$$P\{X=k\}=p_k=\frac{P^{(k)}(0)}{k!}.$$

A slightly more sophisticated case is given by a sequence of mutually independent identically distributed discrete random variables X_n with a given generating function $F(s)$. Let N be another discrete random variable of values in $\mathbb{N}_0$, which is independent of all the X_n. We denote the generating function for N by $G(s)$.
The generating function $H(s)$ of the sum

$$Y_N := X_1+X_2+\cdots+X_N,$$

where the number of summands N is also a random variable, is then given by the composition

$$P_{Y_N}(s) := H(s)=G(F(s)).$$

Notice that if follows from $H'(s)=G'(F(s))\cdot F'(s)$, that

$$E\left\{Y_N\right\}=E\{N\}\cdot E\left\{X_1\right\}.$$

1.6 Convergence in distribution

Theorem 1.2 The continuity theorem. *Let X_n be a sequence of discrete random variables of values in $\mathbb{N}_0$, where*

$$p_{n,k} := P\left\{X_n=k\right\}, \textit{for } n\in\mathbb{N} \textit{ and } k\in\mathbb{N}_0,$$

and

$$P_n(s) := \sum_{k=0}^{+\infty} p_{n,k}\, s^k, \qquad \textit{for } s\in[0,1] \textit{ og } n\in\mathbb{N}.$$

Then

$$\lim_{n\to+\infty} p_{n,k}=p_k \qquad \textit{for every } k\in\mathbb{N}_0,$$

if and only if

$$\lim_{n\to+\infty} P_n(s)=P(s) \qquad \left(=\sum_{k=0}^{+\infty} p_k\, s^k\right) \qquad \textit{for all } s\in[0,1[.$$

If furthermore*,*

$$\lim_{s\to1-} P(s)=1,$$

then $P(s)$ is the generating function of some random variable X, and the sequence (X_n) converges in distribution *towards X.*

2 The Laplace transformation; background

2.1 Definition of the Laplace transformation

The *Laplace transformation* is applied when the random variable X only has values in $[0, +\infty[$, thus it is non-negative.

The *Laplace transform* of a non-negative random variable X is defined as the function $L : [0, +\infty[\to \mathbb{R}$, which is given by

$$L(\lambda) := E\left\{e^{-\lambda X}\right\}.$$

The most important special results are:

1) If the non-negative random variable X is discrete with $P\{x_i\} = p_i$, for all $x_i \geq 0$, then

$$L(\lambda) := \sum_i p_i\, e^{-\lambda\, x_i}, \qquad \text{for } \lambda \geq 0.$$

2) If the non-negative random variable X is continuous with the frequency $f(x)$, (which is 0 for $x < 0$), then

$$L(\lambda) := \int_0^{+\infty} e^{-\lambda x}\, f(x)\, dx \qquad \text{for } \lambda \geq 0.$$

We also write in this case $L\{f\}(\lambda)$.

In general, the following hold for the Laplace transform of a non-negative random variable:

1) We have for every $\lambda \geq 0$,

$$0 < L(\lambda) \leq 1, \qquad \text{with } L(0) = 1.$$

2) If $\lambda > 0$, then $L(\lambda)$ is of class C^∞ and the n-th derivative is given by

$$(-1)^n L^{(n)}(\lambda) = \begin{cases} \sum_i x_i^n \, e^{-\lambda x_i} \, p_i, & \text{when } X \text{ is discrete,} \\ \int_0^{+\infty} x^n \, e^{-\lambda x} \, f(x) \, dx, & \text{when } X \text{ is continuous.} \end{cases}$$

Assume that the non-negative random variable X has the Laplace transform $L_X(\lambda)$, and let a, $b \geq 0$ be non-negative constants. Then the random variable

$$Y := aX + b$$

is again non-negative, and its Laplace transform $L_Y(\lambda)$ is, expressed by $L_X(\lambda)$, given by

$$L_Y(\lambda) = E\left\{e^{-\lambda(aX+b)}\right\} = e^{-\lambda b} \, L_X(a\lambda).$$

Theorem 2.1 Inversion formula. *If X is a non-negative random variable with the distribution function $F(x)$ and the Laplace transform $L(\lambda)$, then we have at every point of continuity of $F(x)$,*

$$F(x) = \lim_{\lambda \to +\infty} \sum_{k=0}^{[\lambda x]} \frac{(-\lambda)^k}{k!} \, L^{(k)}(\lambda),$$

where $[\lambda x]$ denotes the integer part of the real number λx. This result implies that a distribution is uniquely determined by its Laplace transform.

Concerning other inversion formulæ the reader is e.g. referred to the *Ventus: Complex Function Theory* series.

2.2 Some Laplace transforms of random variables

1) If X is $\chi^2(n)$ distributed of the frequency

$$f(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \, x^{n/2-1} \exp\left(-\frac{x}{2}\right). \qquad x > 0,$$

then its Laplace transform is given by

$$L_X(\lambda) = \left\{\frac{1}{2\lambda+1}\right\}^{\frac{n}{2}}.$$

2) If X is *exponentially distributed*, $\Gamma\left(1\,,\,\frac{1}{a}\right)$, $a > 1$, of the frequency

$$f(x) = a\,e^{-ax} \qquad \text{for } x > 0,$$

then its Laplace transform is given by

$$L_X(\lambda) = \frac{a}{\lambda + a}.$$

3) If X is *Erlang distributed*, $\Gamma(n, \alpha)$ of frequency

$$\frac{1}{(n-1)!\,\alpha^n}\,x^{n-1}\exp\left(-\frac{x}{\alpha}\right), \qquad \text{for } n \in \mathbb{N},\ \alpha > 0 \text{ and } x > 0,$$

then its Laplace transform is given by

$$L_X(\lambda) = \left\{\frac{1}{\alpha\lambda + 1}\right\}^n.$$

4) If X is *Gamma distributed*, $\Gamma(\mu, \alpha)$, with the frequency

$$\frac{1}{\Gamma(\mu)\,\alpha^\mu}\,x^{\mu-1}\exp\left(-\frac{x}{\alpha}\right) \qquad \text{for } \mu,\ \alpha > 0 \text{ and } x > 0,$$

then its Laplace transform is given by

$$L_X(\lambda) = \left\{\frac{1}{\alpha\lambda + 1}\right\}^\mu.$$

2.3 Computation of moments

Theorem 2.2 *If X is a non-negative random variable with the Laplace transform $L(\lambda)$, then the n-th moment $E\{X^n\}$ exists, if and only if $L(\lambda)$ is n times continuously differentiable at 0. In this case we have*

$$E\{X^n\} = (-1)^n L^{(n)}(0).$$

In particular, if $L(\lambda)$ is twice continuously differentiable at 0, then

$$E\{X\} = -L'(0), \qquad \text{and} \qquad E\left\{X^2\right\} = L''(0).$$

2.4 Distribution of sums of mutually independent random variables

Theorem 2.3 *Let $X_1, \ldots, X_n$ be non-negative, mutually independent random variable with the corresponding Laplace transforms $L_1(\lambda), \ldots\ L_n(\lambda)$. Let*

$$Y_n = \sum_{i=1}^n X_i \qquad \text{and} \qquad Z_n = \frac{1}{n}\,Y_n = \frac{1}{n}\sum_{i=1}^n X_i.$$

Then

$$L_{Y_n}(\lambda) = \prod_{i=1}^n L_i(\lambda), \qquad \text{and} \qquad L_{Z_n}(\lambda) = L_{Y_n}\left(\frac{\lambda}{n}\right) = \prod_{i=1}^n L_i\left(\frac{\lambda}{n}\right).$$

If in particular X_1 and X_2 are independent non-negative random variables of the frequencies $f(x)$ and $g(x)$, resp., then it is well-known that the frequency of $X_1 + X_2$ is given by a convolution integral,

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\,dt.$$

In this case we get the well-known result,

$$L\{f \star g\} = L\{f\} \cdot L\{g\}.$$

Theorem 2.4 *Let X_n be a sequence of non-negative, mutually independent and identically distributed random variables with the common Laplace transform $L(\lambda)$. Furthermore, let N be a random variable of values in $\mathbb{N}_0$ and with the* generating function *$P(s)$, where N is independent of all the X_n.*
Then $Y_N := X_1 + \cdots + X_N$ has the Laplace transform

$$L_{Y_N}(\lambda) = P(L(\lambda)).$$

2.5 Convergence in distribution

Theorem 2.5 *Let (X_n) be a sequence of non-negative random variables of the Laplace transforms $L_n(\lambda)$.*

1) If the sequence (X_n) converges in distribution *towards a non-negative random variable X with the Laplace transform $L(\lambda)$, then*

$$\lim_{n \to +\infty} L_n(\lambda) = L(\lambda) \qquad \text{for every } \lambda \geq 0.$$

2) If

$$L(\lambda) := \lim_{n \to +\infty} L_n(\lambda)$$

exists for every $\lambda \geq 0$, and if $L(\lambda)$ is continuous at 0, then $L(\lambda)$ is the Laplace transform of some random variable X, and the sequence (X_n) converges in distribution towards X.

3 Characteristic functions; background

3.1 Definition of characteristic functions

The *characteristic function* of any random variable X is the function $k : \mathbb{R} \to \mathbb{C}$, which is defined by

$$k(\omega) := E\left\{e^{i\omega X}\right\}.$$

We have in particular:

1) If X has a *discrete distribution*, $P\{X = x_j\} = p_j$, then

$$k(\omega) = \sum_i p_j e^{i\omega x_j}.$$

2) If X has its values in $\mathbb{N}_0$, then X has also a *generating function* $P(s)$, and we have the following connection between the characteristic function and the generating function,

$$k(\omega) = \sum_{k=0}^{+\infty} p_k \left(e^{i\omega}\right)^k = P\left(e^{i\omega}\right).$$

3) Finally, if X has a continuous distribution with the frequency $f(x)$, then

$$k(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} f(x)\, dx,$$

which is known from *Calculus* as one of the possible definition of the *Fourier transform* of $f(x)$, cf. e.g. Ventus: the *Complex Function Theory* series.

Since the characteristic function may be considered as the Fourier transform of X in some sense, all the usual properties of the Fourier transform are also valid for the characteristic function:

1) For every $\omega \in \mathbb{R}$,

$$|k(\omega)| \leq 1, \qquad \text{in particular, } k(0) = 1.$$

2) By complex conjugation,

$$\overline{k(\omega)} = k(-\omega) \qquad \text{for ever } \omega \in \mathbb{R}.$$

3) The characteristic function $k(\omega)$ of a random variable X is *uniformly continuous* on all of $\mathbb{R}$.

4) If $k_X(\omega)$ is the characteristic function of X, and a, $b \in \mathbb{R}$ are constants, then the characteristic function of $Y := aXS + b$ is given by

$$k_Y(\omega) = E\left\{e^{i\omega(aX+b)}\right\} = e^{i\omega b}\, k_X(a\omega).$$

Theorem 3.1 Inversion formula

1) Let X be a random variable of distribution function *$F(x)$ and characteristic function $k(\omega)$. If $F(x)$ is continuous at both x_1 and x_2 (where $x_1 < x_2$), then*

$$F(x_2) - F(x_1) = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-A}^{A} \frac{e^{-i\omega x_1} - e^{-i\omega x_2}}{i\omega}\, k(\omega)\, d\omega.$$

In other words em a distribution is uniquely determined by its characteristic function.

2) We now assume that the characteristic function $k(\omega)$ of X is absolutely integrable*, i.e.*

$$\int_{-\infty}^{+\infty} |k(\omega)|\, d\omega < +\infty.$$

Then X has a continuous distribution*, and the frequency $f(x)$ of X is given by*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega x}\, k(\omega)\, d\omega.$$

In practice this inversion formula is the most convenient.

3.2 Characteristic functions for some random variables

1) If X is a *Cauchy distributed* random variable, $C(a,b)$, a, $b>0$, of frequency

$$f(x)=\frac{b}{\pi\left\{b^2+(x-a)^2\right\}} \qquad \text{for } x\in\mathbb{R},$$

then it has the characteristic function

$$k(\omega)=\exp(i\,a\,\omega-|\omega|).$$

2) If X is a $\chi^2(n)$ distributed random variable, $n\in\mathbb{N}$ of frequency

$$\frac{1}{\Gamma\left(\frac{n}{2}\right)2^{n/2}}\,x^{n/2-1}\,\exp\left(-\frac{x}{2}\right) \qquad \text{for } x>0,$$

then its characteristic function is given by

$$k(\omega)=\left\{\frac{1}{1-2i\omega}\right\}^{n/2}.$$

3) If X is *double exponentially distributed* with frequency

$$f(x)=\frac{a}{2}\,e^{-a|x|}, \qquad \text{for } x\in\mathbb{R},\ \text{where the parameter } a>0,$$

then its characteristic function is given by

$$k(\omega)=\frac{a^2}{a^2+\omega^2}.$$

4) If X is *exponentially distributed*, $\Gamma\left(1\,,\,\frac{1}{a}\right)$, $a>0$, with frequency

$$f(x)=a\,e^{-ax} \qquad \text{for } x>0,$$

then its characteristic function is given by

$$k(\omega)=\frac{a}{a-i\omega}.$$

5) If X is *Erlang distributed*, $\Gamma(n,\alpha)$, where $n\in\mathbb{N}$ and $\alpha>0$, with frequency

$$f(x)=\frac{x^{n-1}\,\exp\left(-\frac{x}{\alpha}\right)}{(n-1)!\,\alpha^n} \qquad \text{for } x>0,$$

then its characteristic function is

$$k(\omega)=\left\{\frac{1}{1-i\alpha\omega}\right\}^{n}.$$

6) If X is *Gamma distributed*, $\Gamma(\mu, \alpha)$, where μ, $\alpha > 0$, with frequency

$$f(x) = \frac{x^{\mu-1} \exp\left(-\dfrac{x}{\alpha}\right)}{\Gamma(\mu)\,\alpha^{\mu}}, \qquad \text{for } x > 0,$$

then its characteristic function is given by

$$k(\omega) = \left\{\frac{1}{1 - i\alpha\omega}\right\}^{\mu}.$$

7) If X is *normally distributed* (or *Gaußian distributed*), $\mathrm{N}\left(\mu\,,\,\sigma^2\right)$, $\mu \in \mathbb{R}$ and $\sigma > 0$, with frequency

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad \text{for } x \in \mathbb{R},$$

then its characteristic function is given by

$$k(\omega) = \exp\left(i\mu\omega - \frac{\sigma^2\omega^2}{2}\right).$$

8) If X is *rectangularly distributed*, $\mathrm{U}(a, b)$, where $a < b$, with frequency

$$f(x) = \frac{1}{b-a} \qquad \text{for } a < x < b,$$

then its characteristic function is given by

$$k(\omega) = \frac{e^{i\omega b} - e^{i\omega a}}{i\omega(b-a)}.$$

3.3 Computation of moments

Let X be a random variable with the characteristic function $k(\omega)$. *If* the n-th moment exists, then $k(\omega)$ is a C^{ω} function, and

$$k^{(n)}(0) = i^n\, E\left\{X^n\right\}.$$

In particular,

$$k'(0) = i\,E\{X\} \qquad \text{and} \qquad k''(0) = -E\left\{X^2\right\}.$$

We get in the special cases,

1) If X is *discretely distributed* and $E\left\{|X|^n\right\} < +\infty$, then $k(\omega)$ is a C^n function, and

$$k^{(n)}(\omega) = i^n \sum_j x_j^n \, \exp\left(i\omega x_j\right) p_j.$$

2) If X is *continuously distributed* with frequency $f(x)$ and characteristic function

$$k(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x}\, f(x)\, dx,$$

and if furthermore,

$$E\{|X|^n\} = \int_{-\infty}^{+\infty} |x|^n \, f(x)\, dx < +\infty,$$

then $k(\omega)$ is a C^n function, and we get by differentiation of the integrand that

$$k^{(n)}(\omega) = i^n \int_{-\infty}^{+\infty} x^n \, e^{i\omega x} \, f(x)\, dx.$$

3.4 Distribution of sums of mutually independent random variables

Let $X_1, \dots, X_n$ be mutually independent random variables, with their corresponding characteristic functions $k_1(\omega), \dots, k_n(\omega)$. We introduce the random variables

$$Y_n := \sum_{i=1}^n X_i \qquad \text{and} \qquad Z_n = \frac{1}{n}\, Y_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

The characteristic functions of Y_n and Z_n are given by

$$k_{Y_n}(\omega) = \prod_{i=1}^n k_i(\omega) \qquad \text{and} \qquad k_{Z_n}(\omega) = \prod_{i=1}^n k_i\left(\frac{\omega}{n}\right).$$

3.5 Convergence in distribution

Let (X_n) be a sequence of random variables with the corresponding characteristic functions $k_n(\omega)$.

1) **Necessary condition**. *If* the sequence (X_n) *converges in distribution* towards the random variable X of characteristic function $k(\omega)$, then

$$\lim_{n\to+\infty} k_n(\omega) = k(\omega) \qquad \text{for every } \omega \in \mathbb{R}.$$

2) **Sufficient condition**. *If*

$$k(\omega) = \lim_{n\to+\infty} k_n(\omega)$$

exists for every $\omega \in \mathbb{R}$, and *if also* $k(\omega)$ is continuous at 0, *then* $k(\omega)$ is the characteristic function of some random variable X, and the sequence (X_n) *converges in distribution* towards X.

4 Generating functions

Example 4.1 *Let X be geometrically distributed,*

(1) $P\{X=k\}=pq^{k-1}, \qquad k \in \mathbb{N},$

where $p>0$, $q>0$ and $p+q=1$.
Find the generating function of X.
Let X_1, X_2, ..., X_r be mutually independent, all of distribution given by (1), and let

$$Y_r = X_1 + X_2 + \cdots + X_r.$$

Find the generating function of Y_r, and prove that Y_r has the distribution

$$P\left\{Y_r=k\right\}=\binom{k-1}{r-1} p^r q^{k-r}, \qquad k=r,\, r+1,\, \dots .$$

It follows by insertion that

$$P_X(s)=E\left\{s^X\right\}=\sum_{n=1}^{\infty} pq^{n-1}s^n = ps\sum_{n=1}^{\infty}(qs)^{n-1}=\frac{ps}{1-qs}, \qquad s\in[0,1].$$

The generating function $Q_r(s)$ for $Y_r = X_1 + X_2 + \cdots + X_r$ is

$$\begin{aligned} Q_r(s) &= \prod_{i=1}^{r} P_{X_i}(s) = \left(\frac{ps}{1-qs}\right)^r = p^r s^r (1-qs)^{-s} = p^r s^r \sum_{m=0}^{\infty}\binom{-r}{m}(-1)^m q^m s^m \\ &= \sum_{m=0}^{\infty}\binom{r+m-1}{m} p^r q^m s^{m+r} = \sum_{n=r}^{\infty}\binom{n-1}{r-1} p^r q^{n-r} s^r \qquad \text{for } s\in[0,1]. \end{aligned}$$

Since also

$$Q_r(s)=\sum_n P\left\{Y_r=n\right\} s^n,$$

we conclude that

$$P\left\{Y_r=n\right\}=\binom{n-1}{r-1} p^r q^{n-r}, \qquad n=r,\, r+1,\, \dots .$$

Example 4.2 *Given a random variable X of values in $\mathbb{N}_0$ of the probabilities $p_k = P\{X = k\}$, $k \in \mathbb{N}_0$, and with the generating function $P(s)$. We put $q_k = P\{X > k\}$, $k \in \mathbb{N}_0$, and*

$$Q(s) = \sum_{k=0}^{\infty} q^k s^k, \qquad s \in [0,1[.$$

Prove that

$$Q(s) = \frac{1-P(s)}{1-s} \qquad \textit{for } s \in [0,1[.$$

We have

$$q_k = P\{X > k\} = \sum_{n=k+1}^{\infty} P\{X=n\} = \sum_{n=k+1}^{\infty} p_n = 1 - \sum_{n=0}^{k} p_n.$$

Thus if $s \in [0,1[$, then

$$\begin{aligned} Q(s) &= \sum_{k=0}^{\infty} q_k s^k = \sum_{k=0}^{\infty} s^k - \sum_{k=0}^{\infty}\sum_{n=0}^{k} p_n s^k = \frac{1}{1-s} - \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} p_n s^k \\ &= \frac{1}{1-s} - \sum_{n=0}^{\infty} p_n \cdot \frac{s^n}{1-s} = \frac{1}{1-s}\left\{1 - \sum_{n=0}^{\infty} p_n s^n\right\} = \frac{1-P(s)}{1-s}. \end{aligned}$$

Example 4.3 *We throw a coin, where the probability of obtaining head in a throw is p, where $p \in]0,1[$. We let the random variable X denote the number of throws until we get the results head–tail in the given succession (thus we have $X = n$, if the pair head–tail occurs for the first time in the experiments of numbers $n-1$ and n).*
Find the generating function of X and use it to find the mean and variance of X. For which value of p is the mean smallest?

If $n = 2, 3, \dots$ and $p \neq \dfrac{1}{2}$, then

$$\begin{aligned} P\{X=n\} &= P\{X_i = \text{ head}, i=1,\dots,\, X_n = \text{ tail}\} \\ &\quad + P\{X_1 = \text{ tail}, X_i = \text{ head}, i=2,\dots,\, n-1,\, X_n = \text{ tail}\} \\ &\quad + P\{X_j = \text{ tail},\, j=1,2;\, X_i = \text{ head},\, i=3,\dots,n-1,\, X_n = \text{ tail}\} \\ &\quad + \cdots + P\{X_j = \text{ tail},\, j=1,\dots,n-2;\, X_{n-1} = \text{ head},\, X_n = \text{tail}\} \\ &= p^{n-1}(1-p) + (1-p)p^{n-2}(1-p) + (1-p)^2 p^{n-3}(1-p) \\ &\qquad \cdots + (1-p)^{n-2} p(1-p) \\ &= \sum_{j=1}^{n-1} p^{n-j}(1-p)^j = p^{n-1}(1-p)\sum_{j=1}^{n-1}\left\{\frac{1-p}{p}\right\}^{j-1} \\ &= p^{n-1}(1-p) \cdot \frac{1 - \left(\dfrac{1-p}{p}\right)^{n-1}}{1 - \dfrac{1-p}{p}} = p(1-p) \cdot \frac{p^{n-1} - (1-p)^{n-1}}{2p-1} \\ &= \frac{p(1-p)}{2p-1}\left\{p^{n-1} - (1-p)^{n-1}\right\}, \qquad n \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

If $p = \frac{1}{2}$ then we get instead

$$P\{X = n\} = \sum_{j=1}^{n-1} \left(\frac{1}{2}\right)^{n-j} \left(\frac{1}{2}\right)^{j} = \frac{n-1}{2^n},$$

which can also be obtained by taking the limit in the result above for $p \neq \frac{1}{2}$.

We have to split into the two cases **1.** $p = \frac{1}{2}$ and **2.** $p \neq \frac{1}{2}$.

1) If $p = \frac{1}{2}$, then the generating function becomes

$$\begin{aligned} P(s) &= \sum_{n=2}^{\infty} \frac{n-1}{2^n}\, s^n = \left(\frac{s}{2}\right)^2 \sum_{n=1}^{\infty} n \left(\frac{s}{2}\right)^{n-1} = \left(\frac{s}{2}\right)^2 \cdot \frac{1}{\left(1-\frac{s}{2}\right)^2} = \left(\frac{s}{2-s}\right)^2 \\ &= \left(\frac{2}{2-s} - 1\right)^2 = \frac{4}{(2-s)^2} - \frac{4}{2-s} + 1 \qquad \text{for } s \in [0,2[. \end{aligned}$$

2) If $p \in]0,1[$ and $p \neq \frac{1}{2}$, we get instead

$$\begin{aligned} P(s) &= \sum_{n=2}^{\infty} \frac{p(1-p)}{2p-1} \left\{p^{n-1} - (1-p)^{n-1}\right\} s^n = \frac{p(1-p)}{2p-1} \cdot s \left\{\sum_{n=1}^{\infty} (ps)^n - \sum_{n=1}^{\infty} (1-p)^n s^n\right\} \\ &= \frac{p(1-p)}{2p-1} \cdot s \left\{\frac{ps}{1-ps} - \frac{(1-p)s}{1-(1-p)s}\right\} = \frac{p(1-p)}{2p-1} \cdot s \left\{\frac{1}{1-ps} - \frac{1}{1-(1-p)s}\right\} \\ &= \frac{1-p}{2p-1} \cdot \frac{ps}{1-ps} - \frac{p}{2p-1} \cdot \frac{(1-p)s}{1-(1-p)s} \\ &= \frac{1-p}{2p-1} \cdot \frac{1}{1-ps} - \frac{1-p}{2p-1} \cdot - \frac{p}{2p-1} \cdot \frac{1}{1-(1-p)s} + \frac{p}{2p-1} \\ &= 1 + \frac{1-p}{2p-1} \cdot \frac{1}{1-ps} - \frac{p}{2p-1} \cdot \frac{1}{1-(1-p)s}, \end{aligned}$$

for $s \in \left[0, \min\left\{\frac{1}{p}, \frac{1}{1-p}\right\}\right]$.

In both cases $P^{(n)}(1)$ exists for all n. It follows from

$$E\{X\} = P'(1) \quad \text{and} \quad V\{X\} = P''(1) + P'(1) - \{P'(1)\}^2,$$

that

1) If $p = \frac{1}{2}$, then

$$P'(s) = \frac{8}{(2-s)^3} - \frac{4}{(2-s)^2}$$

and

$$P''(s) = \frac{24}{(2-s)^4} - \frac{8}{(2-s)^3},$$

hence

$$E\{X\} = P'(1) = 4,$$

and

$$V\{X\} = P''(1) + P'(1) - \{P'(1)\}^2 = 16 + 4 - 16 = 4.$$

2) If $p \in]0,1[$, $p \neq \frac{1}{2}$, then

$$P'(s) = \frac{(1-p)p}{2p-1}\left\{\frac{1}{(1-ps)^2} - \frac{1}{\{1-(1-p)s\}^2}\right\},$$

hence

$$\begin{aligned} E\{X\} &= \frac{(1-p)p}{2p-1}\left\{\frac{1}{(1-p)^2} - \frac{1}{\{1-(1-p)\}^2}\right\} = \frac{1}{2p-1}\left\{\frac{p}{1-p} - \frac{1-p}{p}\right\} \\ &= \frac{1}{2p-1}\cdot\frac{2p-1}{(1-p)p} = \frac{1}{p(1-p)}. \end{aligned}$$

Furthermore,

$$P''(s) = \frac{2(1-p)p}{2p-1}\left\{\frac{p}{(1-ps)^3} - \frac{1-p}{\{1-(1-p)s\}^3}\right\},$$

thus

$$\begin{aligned} V\{X\} &= \frac{2}{2p-1}\left\{\left(\frac{p}{1-p}\right)^2 - \left(\frac{1-p}{p}\right)^2\right\} + \frac{1}{p(1-p)} - \frac{1}{p^2(1-p)^2} \\ &= \frac{2}{2p-1}\left\{\frac{p}{1-p} + \frac{1-p}{p}\right\}\left\{\frac{p}{1-p} - \frac{1-p}{p}\right\} + \frac{1}{p(1-p)} - \frac{1}{p^2(1-p)^2} \\ &= \frac{4p^2-4p+2+p-p^2-1}{p^2(1-p)^2} = \frac{3p^2-3p+1}{p^2(1-p)^2} = \frac{p^3+(1-p)^3}{p^2(1-p)^2} = \frac{p}{(1-p)^2} + \frac{1-p}{p^2}. \end{aligned}$$

Now, $p(1-p)$ has its maximum for $p = \frac{1}{2}$ (corresponding to $E\{X\} = 4$), so $p = \frac{1}{2}$ gives the minimum of the mean, which one also intuitively would expect.

An ALTERNATIVE solution which uses quite another idea, is the following: Put

$p_n = P\{\text{HT occurs in the experiments of numbers } n-1 \text{ and } n\}$,

$f_n = P\{\text{HT occurs for the } \textit{first time} \text{ in the experiments of numbers } n-1 \text{ and } n\}$.

Then

(2) $p_n = f_2 p_{n-2} + f_3 p_{n-3} + \cdots + f_{n-2} p_2 + f_n$.

We introduce the generating functions

$$P(s) = \sum_{n=2}^{\infty} p_n s^n = pq \sum_{n=2}^{\infty} s^n pq \cdot \frac{s^2}{1-s}, \qquad s \in [0,1],$$

$$F(s) = \sum_{n=2}^{\infty} f_n s^n.$$

When (2) is multiplied by s^n, and we sum with respect to n, we get alternatively

$$\begin{aligned} P(s) &= \sum_{n=2}^{\infty} p_n s^n = \sum_{n=2}^{\infty}\left\{\sum_{k=2}^{n-2} f_k p_{n-k}\right\} s^n + \sum_{n=2}^{\infty} f_n s^n = \sum_{k=2}^{\infty} f_k \left\{\sum_{n=k+2}^{\infty} p_{n-k} s^{n-k}\right\} s^k + F(s) \\ &= \sum_{k=2}^{\infty} f_k s^k \cdot P(s) + F(s) = F(s)\{P(s)+1\}, \end{aligned}$$

and we derive that

$$\begin{aligned} F(s) &= \frac{P(s)}{P(s)+1} = 1 - \frac{1}{P(s)+1} = 1 - \frac{1}{pq\,\dfrac{s^2}{1-s}+1} = 1 - \frac{1-s}{pqs^2+1-s} \\ &= 1 - \frac{1-s}{(1-ps)(1-qs)} = 1 + \frac{1}{pq}\,\frac{s-1}{\left(s-\dfrac{1}{p}\right)\left(s-\dfrac{1}{q}\right)} \\ &= 1 + \frac{1}{pq}\left\{\frac{\dfrac{1}{p}-1}{\dfrac{1}{p}-\dfrac{1}{q}} \cdot \frac{1}{s-\dfrac{1}{p}} + \frac{\dfrac{1}{q}-1}{\dfrac{1}{q}-\dfrac{1}{p}} \cdot \frac{1}{s-\dfrac{1}{q}}\right\}. \end{aligned}$$

By differentiation,

$$
\begin{aligned}
F'(s) &= \frac{1}{pq}\left\{\frac{\frac{1}{p}-1}{\frac{1}{q}-\frac{1}{p}}\cdot\frac{1}{\left(s-\frac{1}{p}\right)^2}-\frac{\frac{1}{q}-1}{\frac{1}{q}-\frac{1}{p}}\cdot\frac{1}{\left(s-\frac{1}{q}\right)^2}\right\}\\
&= \frac{1}{p-q}\left\{\frac{1-p}{p}\cdot\frac{1}{\left(s-\frac{1}{p}\right)^2}-\frac{1-q}{q}\cdot\frac{1}{\left(s-\frac{1}{q}\right)^2}\right\}\\
&= \frac{pq}{p-q}\left\{\frac{1}{(1-ps)^2}-\frac{1}{(1-qs)^2}\right\},
\end{aligned}
$$

hence

$$E\{X\}=F'(1)=\frac{pq}{p-q}\left\{\frac{1}{q^2}-\frac{1}{p^2}\right\}=\frac{pq}{p-q}\cdot\frac{p^2-q^2}{p^2q^2}=\frac{1}{pq}=\frac{1}{p(1-p)}.$$

Now, $p(1-p)$ is largest for $p=\frac{1}{2}$, where $E\{X\}$ is smallest, corresponding to $E\{X\}=4$.

Furthermore,

$$F''(s)=\frac{pq}{p-q}\left\{\frac{2p}{(1-ps)^3}-\frac{2q}{(1-qs)^3}\right\},$$

so

$$
\begin{aligned}
F''(1) &= \frac{pq}{p-q}\left\{\frac{2p}{q^3}-\frac{2q}{p^3}\right\}=\frac{2}{p^2q^2}\cdot\frac{p^4-q^4}{p-q}\cdot\frac{p^2-q^2}{p^2-q^2}\\
&= \frac{2}{p^2q^2}\cdot\left(p^2+q^2\right)=\frac{2\left\{(p+q)^2-2pq\right\}}{p^2q^2}=\frac{2(1-2pq)}{p^2q^2},
\end{aligned}
$$

and

$$V\{X\}=F''(1)+F'(1)-\{F'(1)\}^2=\frac{2-4pq}{p^2q^2}+\frac{pq}{p^2q^2}-\frac{1}{p^2q^2}=\frac{1-3pq}{p^2q^2},$$

which can be reduced to the other possible descriptions

$$\frac{p}{q^2}+\frac{q}{p^2}=\frac{p}{(1-p)^2}+\frac{1-p}{p^2}.$$

Example 4.4 *1) The distribution of a random variable X is given by*

$$P\{X=k\}=(-1)^k\binom{-\alpha}{k}p^\alpha q^k, \qquad k\in\mathbb{N}_0,$$

where $\alpha\in\mathbb{R}_+$, $p\in]0,1[$ and $q=1-p$. (Thus $X\in NB(\alpha,p)$.) Prove that the generating function of the random variable X is given by

$$P(s)=p^\alpha(1-qs)^{-\alpha}, \qquad s\in[0,1],$$

and use it to find the mean of X.

2) Let X_1 and X_2 be independent random variables

$$X_1\in NB(\alpha_1,p), \quad X_2\in NB(\alpha_2,p), \quad \alpha_1,\,\alpha_2\in\mathbb{R}_+, \quad p\in]0,1[.$$

Find the distribution function of the random variable X_1+X_2.

3) Let $(Y_n)_{n=3}^\infty$ be a sequence of random variables, where $Y_n\in NB\left(n,1-\dfrac{2}{n}\right)$. Prove that the sequence (Y_n) converges in distribution towards a random variable Y, and find the distribution function of Y.

4) Compute $P\{Y>4\}$ (3 decimals).

1) The generating function for X for $s\in[0,1]$ is given by

$$P(s)=\sum_{k=0}^\infty(-1)^k\binom{-\alpha}{k}p^\alpha q^k s^k=p^\alpha\sum_{k=0}^\infty\binom{-\alpha}{k}(-qs)^k=\frac{p^\alpha}{(1-qs)^\alpha}.$$

It follows from

$$P'(s)=\frac{\alpha\,q\,p^\alpha}{(1-qs)^{\alpha+1}},$$

that

$$E\{X\}=P'(1)=\frac{\alpha\,q\,p^\alpha}{(1-q)^{\alpha+1}}=\frac{\alpha\,p^\alpha q}{p^{\alpha+1}}=\alpha\cdot\frac{q}{p}.$$

2) Since X_1 and X_2 are independent, the generated function for X_1+X_2 is given by

$$P_{X_1+X_2}(s)=\left\{\frac{p}{1-qs}\right\}^{\alpha_1}\cdot\left\{\frac{p}{1-qs}\right\}^{\alpha_2}=\left\{\frac{p}{1-qs}\right\}^{\alpha_1+\alpha_2},$$

and we conclude that $X_1+X_2\in NB(\alpha_1+\alpha_2,p)$, thus the distribution is given by

$$P\{X_1+X_2=k\}=(-1)^k\binom{-\alpha_1-\alpha_2}{k}p^{\alpha_1+\alpha_2}q^k, \qquad k\in\mathbb{N}_0,$$

3) The generating function $P_n(s)$ for Y_n is according to **1.** given by

$$P_n(s) = \frac{\left(1-\frac{2}{n}\right)^n}{\left(1-\frac{2}{n}\,s\right)^n} \to \frac{e^{-2}}{e^{-2s}} = e^{-2(1-s)} = P(s) \qquad \text{for } n \to \infty.$$

Now, $\lim_{s\to 1-} P(s) = e^0 = 1$, so it follows from the continuity theorem that (Y_n) converges in distribution towards a random variable Y of generating function

$$P(s) = e^{-2(1-s)} = e^{-2}e^{2s} = e^{-2}\sum_{n=0}^{\infty}\frac{2^n}{n!}\,s^n = \sum_{n=0}^{\infty} P\{Y=n\}\,s^n.$$

When we identify the coefficients of s^n, we see that the distribution is given by

$$P\{Y=n\} = \frac{2^n}{n!}\,e^{-2}, \qquad n \in \mathbb{N}_0,$$

which we recognize as a Poisson distribution, $Y \in P(2)$.

4) Finally,

$$\begin{aligned} P\{Y>4\} &= 1 - P\{Y=0\} - P\{Y=1\} - P\{Y=2\} - P\{Y=3\} - P\{Y-4\} \\ &= 1 - e^{-2}\left\{1+2+2+\frac{4}{3}+\frac{2}{3}\right\} = 1 - \frac{7}{e^2} \approx 0.05265. \end{aligned}$$

Example 4.5 *Consider a random variable X with its distribution given by*

$$P\{X=k\}=\frac{1}{e^a-1}\,\frac{a^k}{k!},\qquad k\in\mathbb{N},$$

where a is a positive constant.

1. *Find the generating function for X and find the mean of X.*

Let X_1 and X_2 be independent random variables, both having the same distribution as X.

2. *Find the generating function for X_1+X_2, and then find the distribution of X_1+X_2.*

The distribution of X is a truncated Poisson distribution*.*

1) The generating function $P(s)$ is

$$P(s)=\sum_{k=1}^{\infty}P\{X=k\}\,s^k=\frac{1}{e^a-1}\sum_{k=1}^{\infty}\frac{(as)^k}{k!}=\frac{e^{as}-1}{e^a-1}.$$

It follows from

$$P'(s)=\frac{a\,e^{as}}{e^a-1},$$

that

$$E\{X\}=P'(s)=\frac{a\,e^{a}}{e^a-1}.$$

2) Since X_1 and X_2 are independent, both of the same distribution as X, the generating function is given by

$$P(s)=P_{X_1+X_2}(s)=P_1(s)\cdot P_2(s)=\frac{1}{(e^a-1)^2}\,(e^{as}-1)\,,\qquad s\in[0,1].$$

Then we perform a power expansion of those terms *which contain s*,

$$\begin{aligned}P(s)&=\frac{1}{(e^a-1)^2}\left(e^{2as}-2\,e^{as}+1\right)=\frac{1}{(e^a-1)^2}\sum_{k=1(2)}^{\infty}\frac{1}{k!}\left\{(2a)^k-2a^k\right\}s^k\\&=\frac{1}{(e^a-1)^2}\sum_{k=2}^{\infty}\frac{a^k}{k!}\left(2^k-2\right)s^k=\sum_{k=2}^{\infty}P\left\{X_1+X_2=k\right\}s^k.\end{aligned}$$

By identification of the coefficients it follows that X_1+X_2 has the distribution

$$P\left\{X_1+X_2=k\right\}=\frac{1}{(e^a-1)^2}\,\frac{a^k}{k!}\left(2^k-2\right),\qquad k=2,\,3,\,4,\,\dots .$$

Remark 4.1 This result can - though it is very difficult – also be found in the traditional way by computation and reduction of

$$P\left\{X_1+X_2=k\right\}=\sum_{i=1}^{k-1}P\left\{X_1=i\right\}\cdot P\left\{X_2=k-i\right\}.\qquad\Diamond$$

Example 4.6 *A random variable X has the values 0, 2, 4, ... of the probabilities*

$$P\{X = 2k\} = p\,q^k, \qquad k \in \mathbb{N}_0,$$

where $p > 0$, $q > 0$ and $p + q = 1$.

1. *Find the generating function for X.*

2. *Find, e.g. by applying the result of* **1.**, *the mean $E\{X\}$.*

We define for every $n \in \mathbb{N}$ a random variable Y_n by

$$Y_n = X_1 + X_2 + \cdots + X_n,$$

where the random variables X_i are mutually independent and all of the same distribution as X.

3. *Find the generating function for Y_n.*

Given a sequence of random variables $(Z_n)_{n=1}^{\infty}$, where for every $n \in \mathbb{N}$ the random variable Z_n has the same distribution as Y_n corresponding to

$$p = 1 - \frac{1}{2n}, \qquad q = \frac{1}{2n}.$$

4. *Prove, e.g. by applying the result of* **3.** *that the sequence (Z_n) converges in distribution towards a a random variable Z, and find the distribution of Z.*

5. *Is it true that $E\{Z_n\} \to E\{Z\}$ for $n \to \infty$?*

1) The generating function is

$$P_X(s) = \sum_{k=0}^{\infty} p\,q^k s^{2k} = p \sum_{k=0}^{\infty} \left(q\,s^2\right)^k = \frac{p}{1-qs^2} \qquad \text{for } s \in [0,1].$$

2) It follows from

$$P_X'(s) = \frac{2qps}{\left(1-qs^2\right)^2},$$

that

$$E\{X\} = P_X'(1) = \frac{2pq}{p^2} = \frac{2q}{p}.$$

Alternatively we get by the traditional computation that

$$E\{X\} = \sum_{k=1}^{\infty} 2kpq^k = 2pq \sum_{k=1}^{\infty} kq^{k-1} = \frac{2pq}{p^2} = \frac{2q}{p}.$$

3) The generating function for $Y_n = \sum_{i=1}^{n} X_i$ is

$$P_{Y_n} = \{P_X(s)\}^n = \left(\frac{p}{1-qs^2}\right)^2 \qquad \text{for } s \in [0,1].$$

4) If we put $p = 1 - \frac{1}{2n}$, $q = \frac{1}{2n}$, then Z_n has according to **3.** the generating function

$$P_{Z_n}(s) = \frac{\left(1 - \frac{1}{2n}\right)^n}{\left(1 - \frac{s^2}{2n}\right)^n}.$$

Since $\left(1 + \frac{a}{n}\right)^n \to e^a$ for $n \to \infty$, we get

$$P_{Z_n}(s) \to \frac{\exp\left(-\frac{1}{2}\right)}{\exp\left(-\frac{s^2}{2}\right)} = \exp\left(\frac{1}{2}\left(s^2 - 1\right)\right), \qquad \text{for } n \to \infty,$$

where the limit function is continuous. This means that (Z_n) converges in distribution towards a random variable Z, the generating function of which is given by

$$P_Z(s) = \exp\left(\frac{1}{2}\left(s^2 - 1\right)\right).$$

We get by expanding this function into a power series that

$$P_Z(s) = \frac{1}{\sqrt{e}} \exp\left(\frac{1}{2}\, s^2\right) = \frac{1}{\sqrt{e}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2}\right)^k s^{2k}.$$

It follows that Z has the distribution

$$P\{Z = 2k\} = \frac{1}{k!} \left(\frac{1}{2}\right)^k \frac{1}{\sqrt{e}} \qquad \text{for } k \in \mathbb{N}_0,$$

thus $\frac{Z}{2}$ is Poisson distributed with parameter $\frac{1}{2}$.

5) From

$$E\{Z_n\} = n \cdot \frac{2 \cdot \frac{1}{2n}}{1 - \frac{1}{2n}} = \frac{1}{1 - \frac{1}{2n}} \to 1 = E\{Z\} \quad \text{for } n \to \infty,$$

follows that the answer is “yes”.

Example 4.7 *A random variable U, which is not causally distributed, has its distribution given by*

$$P\{U=k\}=p_k, \qquad k\in\mathbb{N}_0,$$

and its generating function is

$$P(s)=\sum_{k=0}^{\infty}p_k s^k, \qquad s\in[0,1].$$

The random variable U_1 has its distribution given by

$$P\{U_1=0\}=0, \qquad P\{U_1=k\}=\frac{p_k}{1-p_0}, \qquad k\in\mathbb{N}.$$

1. *Prove that U_1 has its generating function $P_1(s)$ given by*

$$P_1(s)=\frac{P(s)-p_0}{1-p_0}, \qquad s\in[0,1].$$

We assume that the number of persons per household residential neighbourhood is a random variable X with its distribution given by

$$P\{X=k\}=\frac{3^k}{k!\,(e^3-1)}, \qquad k\in\mathbb{N},$$

(a truncated Poisson distribution).

2. *Compute, e.g. by using the result of* **1.**, *the generating function for X. Compute also the mean of X.*

Let the random variable Y be given by $Y=\left(\frac{1}{2}\right)^X$.

3. *Compute, e.g. by using the result of* **2.**, *the mean and variance of Y.*

The heat consumption Z per quarter per house (measured in m^3 district heating water) is assumed to depend of the number of persons in the house in the following way:

$$Z=200\left\{1-\left(\frac{1}{2}\right)^X\right\}=200(1-Y).$$

4. *Compute the mean and the dispersion of Z. The answers should be given with 2 decimals.*

1) A direct computation gives

$$P_1(s)=\sum_{k=1}^{\infty}\frac{p_k}{1-p_0}\,s^k=\frac{1}{1-p_0}\left\{\sum_{k=0}^{\infty}p_k s^k-p_0\right\}=\frac{P(s)-p_0}{1-p_0}.$$

2) Also here be direct computation,

$$P_X(s) = \frac{1}{e^3-1}\sum_{k=1}^{\infty}\frac{1}{k!}\,(3s)^k = \frac{e^{3s}-1}{e^3-1}.$$

ALTERNATIVELY we can apply **1.**, though this is far more difficult, because one first have to realize that we shall choose

$$p_k = \frac{1}{e^3}\cdot\frac{3^k}{k!}, \qquad k \in \mathbb{N}_0,$$

with

$$P(s) = e^{3(s-1)}.$$

Then we shall check that these candidates of the probabilities are added up to 1, and then prove that

$$P\left\{U_1 = k\right\} = \frac{p_k}{1-p_0}, \qquad k \in \mathbb{N},$$

and finally insert

$$P_1(s) = P_X(s) = \frac{e^{3(s-1)}-e^{-3}}{1-e^{-3}} = \frac{e^{3s}-1}{e^3-1}.$$

The mean is

$$E\{X\} = P(1) = \left[\frac{3e^{3s}}{e^3-1}\right]_{s=1} = \frac{3e^3}{e^3-1} = 3+\frac{3}{3^3-1} \approx 3.15719.$$

3) We get by the definition,

$$E\left\{s^X\right\} = P_X(s) = \frac{e^{3s}-1}{e^3-1},$$

where we obtain the mean of $Y = \left(\frac{1}{2}\right)^X$ by putting $s = \frac{1}{2}$, thus

$$E\{Y\} = E\left\{\left(\frac{1}{2}\right)^X\right\} = \frac{\exp\left(\frac{3}{2}\right)-1}{e^3-1} = \frac{1}{\exp\left(\frac{3}{2}\right)+1} \approx 0,18243.$$

Analogously,

$$E\left\{Y^2\right\} = E\left\{\left(\frac{1}{2}\right)^{2X}\right\} = E\left\{\left(\frac{1}{4}\right)^X\right\} = P_X\left(\frac{1}{4}\right) = \frac{\exp\left(\frac{3}{4}\right)-1}{e^3-1},$$

hence

$$V\{Y\} = \frac{\exp\left(\frac{3}{4}\right)-1}{e^3-1} - \left\{\frac{\exp\left(\frac{3}{2}\right)-1}{e^3-1}\right\}^2 \approx 0.02525.$$

4) The mean of Z is obtained by a direct computation,

$$E\{Z\} = 200\, E\{Y\} = 163.514.$$

The corresponding dispersion is

$$s = \sqrt{V\{Z\}} = 200\sqrt{V\{Y\}} = 31.7786.$$

Example 4.8 *Let X_1, X_2, ... be mutually independent random variables, all of distribution given by*

$$P\{X_i = k\} = p\,q^{k-1}, \qquad k \in \mathbb{N},$$

where $p > 0$, $q > 0$ and $p + q = 1$.
Furthermore, let N be a random variable, which is independent of the X_i and which has its distribution given by

$$P\{N = n\} = \frac{a^n}{n!}\,e^{-a}, \qquad n \in \mathbb{N}_0,$$

where a is a positive constant.

1. *Find the generating function $P(s)$ for the random variable X_1.*

2. *Find the generating function for the random variable $\sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.*

3. *Find the generating function for the random variable N.*

We introduce another random variable Y by

(3) $Y = X_1 + X_2 + \cdots + X_N$,

where N denotes the random variable introduced above, and where the number of random variables on the right hand side of (3) is itself a random variable (for $N = 0$ we interpret (3) as $Y = 0$).

4. *Prove that the random variable Y has its generating function $P_Y(s)$ given by*

$$P_Y(s) = \exp\left(\frac{a(s-1)}{1-qs}\right), \qquad 0 \le s \le 1.$$

HINT: *One may use that*

$$P\{Y = 0\} = P\{N = 0\},$$

$$P\{Y = k\} = \sum_{n=1}^{\infty} P\{N = n\} \cdot P\{X_1 + X_2 + \cdots + X_n = k\}, \qquad k \in \mathbb{N}.$$

5. *Compute the mean $E\{Y\}$.*

1) The generating function for X_1 is

$$P(s) = \sum_{k=1}^{\infty} pq^{k-1}s^k = ps\sum_{k=1}^{\infty}(qs)^{k-1} = \frac{ps}{1-qs}, \qquad s \in [0,1].$$

2) The generating function for $\sum_{i=1}^{n} X_i$ is

$$P_n(s) = P(s)^n = \left(\frac{ps}{1-qs}\right)^n, \qquad s \in [0,1] \text{ og } n \in \mathbb{N}.$$

3) The generating function for N is

$$Q(s)=\sum_{n=0}^{\infty}\frac{a^n}{n!}\,e^{-a}s^n=e^{-1}\cdot e^{as}=e^{a(s-1)}.$$

4) Now,

$$P\{Y=0\}=P\{N=0\}=e^{-a},$$

so the generating function for Y_N is

$$\begin{aligned}
P_Y(s) &= P\{Y=0\}+\sum_{k=1}^{\infty}P\{Y=k\}\,s^k \\
&= e^{-a}+\sum_{k=1}^{\infty}\left(\sum_{m=1}^{\infty}P\{N=n\}\cdot P\left\{X_1+X_2+\cdots+X_n=k\right\}\right)s^k \\
&= e^{-a}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}P\left\{X_1+X_2+\cdots+X_n=k\right\}s^k\right) \\
&= \sum_{n=0}^{\infty}P\{N=n\}\,(P_n(s))=Q(P(s)) \\
&= Q\left(\frac{ps}{1-qs}\right)=\exp\left(a\left\{\frac{ps}{1-qs}-1\right\}\right) \\
&= \exp\left(a\,\frac{ps-1+qs}{1-qs}\right)=\exp\left(\frac{a(s-1)}{1-qs}\right).
\end{aligned}$$

5) It follows from

$$P_Y'(s)=P_Y(s)\cdot a\left\{\frac{1}{1-qs}+\frac{q(s-1)}{(1-qs)^2}\right\},$$

that the mean is

$$E\{Y\}=P_Y'(1)=P_Y(1)\cdot a\cdot\frac{1}{1-q}=\frac{a}{p}.$$

Example 4.9 *Let X_1, X_2, ... be mutually independent random variables, all of distribution given by*

$$P\{X_i = k\} = \frac{1}{\ln 3} \cdot \frac{1}{k}\left(\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}.$$

Furthermore, let N be a random variable, which is independent of the X_i and Poisson distributed with parameter $a = \ln 9$.

1. *Find the mean of X_1.*

2. *Find the generating function for the random variable X_1.*

3. *Find the generating function for the random variable $\sum_{i=1}^n X_i$, $n \in \mathbb{N}$.*

4. *Find the generating function for the random variable N.*

Introduce another random variable Y by

(4) $Y = X_1 + X_2 + \cdots + X_N$,

where N denotes the random variable introduced above, and where the number of random variables on the right hand side of (4) also is a random variable (for $N = 0$ we interpret (4) as $Y = 0$).

5. *Find the generating function for Y, and then prove that Y is negative binomially distributed.*
HINT: *One may use that*

$$P\{Y = 0\} = P\{N = 0\},$$

$$P\{Y = k\} = \sum_{n=1}^{\infty} P\{N = n\} \cdot P\{X_1 + X_2 + \cdots + X_n = k\}, \qquad k \in \mathbb{N}.$$

6. *Find the mean of Y.*

1) The mean is

$$E\{X_1\} = \frac{1}{\ln 3}\sum_{k=1}^{\infty} k \cdot \frac{1}{k}\left(\frac{2}{3}\right)^k = \frac{1}{\ln 3} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{1}{\ln 3} \cdot \frac{2}{3} \cdot \frac{1}{\frac{1}{3}} = \frac{2}{\ln 3}.$$

2) The generating function for X_1 is

$$P_{X_1}(s) = \frac{1}{\ln 3}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{3}\right)^k s^k = \frac{1}{\ln 3}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2s}{3}\right)^k = \frac{1}{\ln 3}\ln\left(\frac{1}{1 - \frac{2s}{3}}\right) = \frac{1}{\ln 3}\ln\left(\frac{3}{3 - 2s}\right).$$

3) Since the X_i are mutually independent, we get

$$P_n(s) = \{P_{X_1}(s)\}^n = \left\{\frac{1}{\ln 3}\ln\left(\frac{3}{3 - 2s}\right)\right\}^n.$$

4) Since $N \in P(\ln 9)$, we obtain the generating function either by using a table or by the computation

$$P_N(s) = \sum_{n=0}^{\infty} \frac{(\ln 9)^n}{n!}\, e^{-\ln 9} s^n = \frac{1}{9} \sum_{n=0}^{\infty} \frac{1}{n!}\,(s\, \ln 9)^n = \frac{1}{9}\, e^{s\, \ln 9} = 9^{s-1}.$$

5) First compute

$$P\{Y=0\} = P\{N=0\} = \frac{1}{9} \qquad [= P_N(0)].$$

This implies that the generating function for Y is

$$\begin{aligned}
P_Y(s) &= \frac{1}{9}+\sum_{k=1}^{\infty} P\{Y=k\}s^k = \frac{1}{9}+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} P\{N=n\}\cdot P\left\{X_1+\cdots+X_n=k\right\}s^k \\
&= \frac{1}{9}+\sum_{n=1}^{\infty} P\{N=n\}\cdot\sum_{k=1}^{\infty} P\left\{\sum_{i=1}^{n} C_i=k\right\}s^k = \frac{1}{9}+\sum_{n=1}^{\infty} P\{N=n\}\cdot\left(P_{X_1}(s)\right)^n \\
&= \sum_{n=0}^{\infty} P\{N=n\}\left(P_{X_1}(s)\right)^n = P_N\left(P_{X_1}(s)\right) = \frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\ln 9\cdot\frac{1}{\ln 3}\,\ln\left(\frac{3}{3-2s}\right)\right)^n \\
&= \frac{1}{9}\,\exp\left(2\,\ln\left(\frac{3}{3-2s}\right)\right) = \frac{1}{9}\,\frac{1}{\left(1-\frac{2}{3}\,s\right)^2} = \left\{\frac{\frac{1}{3}}{1-\frac{2}{3}\,s}\right\}^2,
\end{aligned}$$

which according to the table corresponds to $Y \in NB\left(2,\frac{1}{3}\right)$.

6) We get by using a table,

$$E\{Y\} = 2\cdot\frac{1-\frac{1}{3}}{\frac{1}{3}} = 4.$$

Alternatively,

$$P_Y'(s) = \frac{1}{9}\cdot 2\frac{2}{3}\cdot\frac{1}{\left(1-\frac{2}{3}\,s\right)^3} = \frac{4}{27}\cdot\frac{1}{\left(1-\frac{2}{3}\,s\right)^3},$$

hence

$$E\{Y\} = P_Y'(1) = \frac{4}{27}\cdot\frac{1}{\left(\frac{1}{3}\right)^3} = 4.$$

Example 4.10 *The number N of a certain type of accidents in a given time interval is assumed to be Poisson distributed of parameter a, and the number of wounded persons in the i-th accident is supposed to be a random variable X_i of the distribution*

(5) $P\{X_i = k\} = (1-q)q^k, \qquad k \in \mathbb{N}_0,$

where $0 < q < 1$. We assume that the X_i are mutually independent and all independent of the random variable N.

1. *Find the generating function for N.*

2. *Find the generating function for X_i and the generating function for $\sum_{i=1}^n X_i$, $n \in \mathbb{N}$.*

The total number of wounded persons is a random variable Y given by

(6) $Y = X_1 + X_2 + \cdots + X_N,$

where N denotes the random variable introduced above, and where the number of random variables on the right hand side of (6) is itself a random variable.

3. *Find the generating function for Y, and find the mean $E\{Y\}$.*

Given a sequence of random variables $(Y_n)_{n=1}^{\infty}$, where for each $n \in \mathbb{N}$ the random variable Y_n has the same distribution as Y above, corresponding to $a = n$ and $q = \dfrac{1}{3n}$.

4. *Find the generating function for Y_n, and prove that the sequence (Y_n) converges in distribution towards a random variable Z.*

5. *Find the distribution of Z.*

1) If $N \in P(a)$, then

$$P\{N = n\} = \frac{a^n}{n!}\, e^{-a}, \qquad n \in \mathbb{N}_0,$$

and its generating function is

$$P_N(s) = \exp(a(s-1)).$$

2) The generating function for X_i is

$$P_{X_i}(s) = \sum_{k=0}^{\infty} (1-q)q^k s^k = (1-q)\sum_{k=0}^{\infty}(qs)^k = \frac{1-q}{1-qs}.$$

The generating function for $\sum_{i=1}^n X_i$ is given by

$$P_{\sum_{i=1}^n X_i}(s) = \left(\frac{1-q}{1-qs}\right)^n.$$

3) Since all the random variables are mutually independent, the generating function for $Y = X_1 + X_2 + \cdots + X_N$ is given by

$$P_Y(s) = P_N\left(P_{X_i}(s)\right) = \exp\left(a\left(\frac{1-q}{1-qs} - 1\right)\right) = \exp\left(aq\,\frac{s-1}{1-qs}\right).$$

4) The generating function for Y_n is given by

$$P_{Y_n}(s) = \exp\left(n \cdot \frac{1}{3n} \cdot \frac{s-1}{1-\frac{s}{3n}}\right) = \exp\left(\frac{1}{3} \cdot \frac{s-1}{1-\frac{s}{3n}}\right).$$

When $n \to \infty$ we see that

$$P_{Y_n}(s) \to P(s) = \exp\left(\frac{s-1}{3}\right).$$

Since $\lim_{s \to 1-} P(s) = 1$, we conclude that $P(s)$ is the generating function for some random variable Z, thus

$$P_Z(s) = \exp\left(\frac{s-1}{3}\right).$$

5) It follows immediately from **4.** that $Z \in P\left(\frac{1}{3}\right)$ is Poisson distributed with parameter $a = \frac{1}{3}$.

Example 4.11 *Let X_1, X_2, X_3, ... be mutually independent random variables, all of distribution given by*

$$P\{X_i = k\} = p_1 (1-p_1)^{k-1}, \qquad k \in \mathbb{N}, \qquad hvor\ p_1 \in]0,1[,$$

and let N be a random variable, which is independent of all the X_i-erne, and which has its distribution given by

$$P\{N = n\} = p_2 (1-p_1)^{n-1}, \qquad n \in \mathbb{N}, \qquad p_2 \in]0,1[.$$

1. *Find the generating function $P_{X_1}(s)$ for X_1 and the generating function $P_N(s)$ for N.*

2. *Find the generating function for the random variable $\sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.*

Introduce another random variable Y by

(7) $Y = X_1 + X_2 + \cdots + X_N$,

where N denotes the random variable introduced above, and where the number of random variables on the right hand side of (7) is itself a random variable.

3. *Find the generating function for Y, and then prove that Y is geometrically distributed.*

4. *Find mean and variance of Y.*

1) We get either by using a table or by a simple computation that

$$P_{X_1}(s) = \sum_{k=1}^{\infty} p_1 (1-p_1)^{k-1} s^k = p_1 s \cdot \sum_{k=1}^{\infty} \{(1-p_1)\, s\}^{k-1} = \frac{p_1 s}{1-(1-p_1)\, s}, \qquad s \in [0,1].$$

We get analogously,

$$P_N(s) = \frac{p_2 s}{1-(1-p_2)\, s} \qquad \text{for } s \in [0,1].$$

2) The generating function for $\sum_{i=1}^{n} X_i$ is

$$(P_{X_1}(s))^n = \left(\frac{p_1 s}{1-(1-p_1)\,s}\right)^n, \qquad s \in [0,1].$$

3) The generating function for Y is

$$\begin{aligned} P_Y(s) &= P_N\left(P_{X_1}(s)\right) = \frac{p_2 \cdot \dfrac{p_1 s}{1-(1-p_1)\,s}}{1-(1-p_2)\cdot \dfrac{p_1 s}{1-(1-p_1)\,s}} = \frac{p_1 p_2 s}{1-(1-p_1)\,s-(1-p_2)\,p_1 s} \\ &= \frac{(p_1 p_2)\,s}{1-(1-p_1 p_2)\,s}, \qquad s \in [0,1]. \end{aligned}$$

This is the generating function for a geometric distribution of parameter $p_1 p_2$, so Y is geometrically distributed.

4) From Y being geometrically distributed of parameter $p_1 p_2$ it follows that

$$E\{Y\} = \frac{1}{p_1 p_2} \qquad \text{and} \qquad V\{Y\} = \frac{1-p_1 p_2}{(p_1 p_2)^2}.$$

Remark 4.2 The distribution of Y may also be found without using the generating function. In fact,

$$P\{Y=k\}=\sum_{n=1}^{k} P\{N=n\}\cdot P\left\{X_1+X_2+\cdots+X_n=k\right\}.$$

Since $X_1+X_2+\cdots+X_n \in \operatorname{Pas}(n,p_1)$, we get

$$\begin{aligned}
P\{Y=k\} &= \sum_{n=1}^{k} p_2\left(1-p_2\right)^{n-1}\binom{k-1}{n-1} p_1^n\left(1-p_1\right)^{k-n} \\
&= p_1p_2\left(1-p_1\right)^{k-1}\sum_{n=1}^{k}\binom{k-1}{n-1}\left\{p_1\left(\frac{1-p_2}{1-p_1}\right)\right\}^{n-1} \\
&= p_1p_2\left(1-p_1\right)^{k-1}\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\left\{\frac{p_1\left(1-p_2\right)}{1-p_1}\right\}^{\ell} \\
&= p_1p_2\left(1-p_1\right)^{k-1}\left\{1+\frac{p_1\left(1-p_2\right)}{1-p_1}\right\}^{k-1} \\
&= p_1p_2\left\{1-p_1+p_1-p_1p_2\right\}^{k-1}=\left(p_1p_2\right)\cdot\left(1-p_1p_2\right)^{k-1},
\end{aligned}$$

and we have given an ALTERNATIVE PROOF of the claim that Y is geometrically distributed of parameter p_1p_2. $\square$

Example 4.12 **1.** *Let U be a random variable with values only in $\mathbb{N}_0$, and let $V = 3U$. Prove the following connection between the generating functions of U and V,*

$$P_V(s) = P_U\left(s^3\right), \qquad 0 \le s \le 1.$$

Let the random variable X have its distribution given by

$$P\{X = 3k\} = p(1-p)^{k-1}, \qquad k \in \mathbb{N},$$

where p is a constant, $0 < p < 1$.

2. *Prove, e.g. by using the result of* **1.** *that X has the generating function*

$$p_X(s) = \frac{ps^3}{1-(1-p)s^3}, \qquad 0 \le s \le 1,$$

and then find the Laplace transform $L_X(\lambda)$ of X.

A sequence of random variables $(X_n)_{n=1}^{\infty}$ is defined by X_n taking the values $\frac{3}{n}$, $\frac{6}{n}$, $\frac{9}{n}$, $\dots$ of the probabilities

$$P\left\{X_n = \frac{3k}{n}\right\} = \frac{1}{3n}\left(1 - \frac{1}{3n}\right)^{k-1}, \qquad k \in \mathbb{N}.$$

3. *Find the Laplace transform $L_{X_n}(\lambda)$ of the random variable X_n.*

4. *Prove that the sequence (X_n) converges in distribution towards some random variable Y, and find the distribution function of Y.*

1) By the definition,

$$P_U(s) = \sum_{k=0}^{\infty} P\{U = k\}\, s^k.$$

From $V = 3U$ follows that

$$P_V(s) = \sum_{k=0}^{\infty} P\{V = 3U = 3s\}\, s^{3k} = \sum_{k=0}^{\infty} P\{U = k\}\, s^{3k} = P_U\left(s^3\right).$$

2) Let $Y \in \text{Pas}(1,p)$ be geometrically distributed. Then

$$P_Y(s) = \frac{ps}{1-qs} = \frac{ps}{1-(1-p)s}.$$

From $X = 3Y$ and **1.** we get

$$P_X(s) = \frac{ps^3}{1-(1-p)s^3}.$$

The Laplace transform of X is

$$\begin{aligned} L_X(\lambda) &= \sum_{k=1}^{\infty} P\{X = 3k\}\, e^{-3k\lambda} = \sum_{k=1}^{\infty} p(1-o)^{k-1} e^{-3k\lambda} \\ &= p \cdot e^{-3\lambda} \sum_{k=1}^{\infty} \left\{(1-p)e^{-3\lambda}\right\}^{k-1} = \frac{p\, e^{-3\lambda}}{1-(1-p)e^{-3\lambda}}. \end{aligned}$$

3) We derive the Laplace transform of X_n from the Laplace transform of X by putting $p = \dfrac{1}{3n}$ and by replacing λ by $\dfrac{\lambda}{n}$, thus

$$L_{X_n}(\lambda) = \frac{\dfrac{1}{3n}\,\exp\left(-\dfrac{3\lambda}{n}\right)}{1-\left(1-\dfrac{1}{3n}\right)\exp\left(-\dfrac{3\lambda}{n}\right)} = \frac{\dfrac{1}{3n}}{\exp\left(+\dfrac{3\lambda}{n}\right)-1+\dfrac{1}{3n}}.$$

4) Now,

$$\exp\left(\frac{3\lambda}{n}\right)-1+\frac{1}{3n} = 1+\frac{3\lambda}{n}+\frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)-1+\frac{1}{3n} = \frac{1}{3n}\,(1+9\lambda)+\frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right),$$

so

$$L_{X_n}(\lambda) = \frac{1}{1+9\lambda+\varepsilon\left(\dfrac{1}{n}\right)} \to \frac{1}{1+9\lambda} = L_Z(\lambda),$$

where $Z \in \Gamma\left(1,\dfrac{1}{9}\right)$ is exponentially distributed, thus (X_n) converges in distribution towards $Z \in \Gamma\left(1,\dfrac{1}{9}\right)$.

Example 4.13 *A football team shall play 5 tournament matches. The coach judges that in each match there is the probability* $\frac{2}{5}$ *for victory,* $\frac{2}{5}$ *for defeat, and* $\frac{1}{5}$ *for draw, and that the outcome of a match does not influence on the probabilities of the following matches.*
A victory gives 2 points, a draw gives 1 point, and a defeat gives 0 point.
Let the random variable X indicate the number of victories in the 5 matches, and let Y indicate the number of obtained points in the 5 matches. Then we can also write

$$X = \sum_{i=1}^{5} X_i \qquad \text{and} \qquad Y = \sum_{i=1}^{5} Y_i,$$

where

$$X_i = \begin{cases} 1, & \text{if victory in match number } i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Y_i = \begin{cases} 2, & \text{if victory in match number } i, \\ 1, & \text{if draw in match number } i, \\ 0, & \text{if defeat in match number } i. \end{cases}$$

1) Compute $P\{X = k\}$, $k = 0$, 1, 2, 3, 4, 5, and the mean $E\{X\}$.

2) Find the mean and variance of Y.

3) Compute $P\{Y = 10\}$.

4) Compute $P\{Y = 8\}$.

5) Find the generating function for Y_i, and then find (use a pocket calculator) the generating function for

$$Y = \sum_{i=1}^{5} Y_i.$$

Compute also the probabilities $P\{Y = k\}$, $k = 0$, 1, 2, ..., 10.

6) In the Danish tournament league a victory gives 3 points, a draw gives 1 point, and a defeat gives 0 point. Let Z denote the number of obtained points in the 5 matches (all other assumptions are chosen as the same as above). Then Z can as value have all integers between 0 and 15, with one exception (which one?). Find all the probabilities by using generating functions in the same way as in **5.**.

1) Since $X \in B\left(5, \frac{2}{5}\right)$ is binomially distributed, we get

$$p_k = P\{X = k\} = \binom{5}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{5-k}, \qquad k = 0, 1, 2, 3, 4, 5,$$

We get more explicitly,

$$p_0 = \left(\frac{3}{4}\right)^5 \qquad = \frac{243}{3125},$$

$$p_1 = 5 \cdot \frac{2}{5}\left(\frac{3}{5}\right)^4 \qquad = \frac{810}{3125} = \frac{162}{625},$$

$$p_2 = 10 \cdot \left(\frac{2}{5}\right)^2 \left(\frac{3}{5}\right)^3 \qquad = \frac{1080}{2125} = \frac{216}{625},$$

$$p_3 = 10 \cdot \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^2 \qquad = \frac{720}{3125} = \frac{144}{625},$$

$$p_4 = 5 \cdot \left(\frac{2}{5}\right)^3 4 \cdot \frac{3}{5} \qquad = \frac{240}{3125} = \frac{48}{625},$$

$$p_5 = \left(\frac{2}{5}\right)^5 \qquad = \frac{32}{3125}.$$

The mean is

$$E\{X\} = 5 \cdot \frac{2}{5} = 2.$$

2) The mean of Y_i is

$$E\left\{Y_i\right\} = 2 \cdot \frac{2}{5} + 1 \cdot \frac{1}{5} + 0 \cdot \frac{2}{5} = 1 \qquad \text{for } i = 1, \dots, 5,$$

and since

$$E\left\{Y_i^2\right\} = 4 \cdot \frac{2}{5} + 1 \cdot \frac{1}{5} + 0 \cdot \frac{2}{5} = \frac{9}{5} \qquad \text{for } i = 1, \dots, 5,$$

the variances are

$$V\left\{Y_i\right\} = \frac{9}{5} - 1^2 = \frac{4}{5}.$$

Now the Y_i are mutually independent, so it follows that

$$E\{Y\} = \sum_{i=1}^{5} E\left\{Y_i\right\} = 5 \qquad \text{and} \qquad V\{Y\} = \sum_{i=1}^{5} V\left\{Y_i\right\} = 4.$$

3) If $Y = 10$, then the team must have won all 5 matches, thus

$$P\{Y = 10\} = P\{X = 5\} = \left(\frac{2}{5}\right)^5 = \frac{32}{3125}.$$

4) The case $Y = 8$ occurs if either we have 4 victories and 1 defeat, or 3 victories and 2 draws. Hence

$$P\{Y = 8\} = 5 \cdot \left(\frac{2}{5}\right)^4 \cdot \frac{2}{5} + \left(\begin{array}{c} 5 \\ 3 \end{array}\right) \left(\frac{2}{5}\right)^3 \left(\frac{1}{5}\right)^2 = \frac{5 \cdot 2^5 + 10 \cdot 2^3}{5^5} = \frac{240}{3125} = \frac{48}{625}.$$

5) From

$$p_0 = \frac{2}{5}, \qquad p_1 = \frac{1}{5} \qquad \text{and} \qquad p_2 = \frac{2}{5},$$

follows that the generating function for each Y_i is given by

$$a(s) = \frac{2}{5}\, s^2 + \frac{1}{5}\, s + \frac{2}{5} = \frac{1}{5}\, \left(2s^2 + s + 2\right).$$

This implies that the generating function for $Y = \sum_{i=1}^{5} Y_i$ is given by (either by using a pocket calculator or MAPLE)

$$\begin{aligned} P_Y(s) &= a(s)^5 = \left(\frac{2}{5}\, s^2 + \frac{1}{5}\, s + \frac{2}{5}\right)^5 \\ &= \frac{32}{3125}\, s^{10} + \frac{16}{625}\, s^9 + \frac{48}{625}\, s^8 + \frac{72}{625}\, s^7 + \frac{114}{625}\, s^6 + \frac{561}{3125}\, s^5 + \frac{114}{625}\, s^4 + \frac{72}{625}\, s^3 \\ &\qquad + \frac{48}{625}\, s^2 + \frac{16}{625}\, s + \frac{32}{3125}. \end{aligned}$$

It follows that $P\{Y = k\}$ is the coefficient of s^k.

6) Clearly, $P\{Z = 14\} = 0$. In fact, 5 victories gives 15 points, and the second best result is described by 4 victories and 1 draw, corresponding to $k = 4 \cdot 3 + 1 \cdot 1 = 13$.
In this new case the generating function for each Z_i is given by

$$b(s) = \frac{2}{5}\, s^3 + \frac{1}{5}\, s + \frac{2}{5} = \frac{1}{5}\, \left(2s^3 + s + 2\right),$$

where we have replaced s^2 by s^3.
Thus the generating function for $Z = \sum_{i=1}^{5} Z_i$ is given by

$$\begin{aligned} P_Z(s) &= b(s)^5 = \left(\frac{2}{5}\, s^3 + \frac{1}{5}\, s + \frac{2}{5}\right)^5 \\ &= \frac{32}{3125}\, s^{15} + 0 \cdot s^{14} + \frac{16}{625}\, s^{13} + \frac{32}{625}\, s^{12} + \frac{16}{625}\, s^{11} + \frac{64}{625}\, s^{10} + \frac{72}{625}\, s^9 + \frac{48}{625}\, s^8 \\ &\qquad + \frac{98}{625}\, s^7 + \frac{16}{125}\, s^6 + \frac{241}{3125}\, s^5 + \frac{66}{625}\, s^4 + \frac{8}{125}\, s^3 + \frac{16}{625}\, s^2 0 \frac{16}{625}\, s + \frac{32}{3125}, \end{aligned}$$

which can also be written in the following way, in which it is easier to evaluate the magnitudes of the coefficients,

$$\begin{aligned} P_Z(s) &= \frac{1}{3125}\, \left\{32s^{15} + 80s^{13} + 160s^{12} + 80s^{11} + 320s^{10}\right. \\ &\qquad \left. + 360s^9 + 240s^8 + 490s^7 + 400s^6 + 241s^5 + 330s^4 + 200s^3 + 80s^2 + 80s + 32\right\}. \end{aligned}$$

Since $P\{Z = k\}$ is the coefficient of s^k in $P_Z(s)$, we conclude that under the given assumptions there is the biggest chance for obtaining 7 points,

$$P\{Z = 7\} = \frac{490}{3125} = \frac{98}{625}.$$

5 The Laplace transformation

Example 5.1 *Let X be exponentially distributed of the frequency*

$$f(x) = \begin{cases} a\,e^{-ax}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Find $L_X(\lambda)$, and use it to find $E\{X\}$ and $V\{X\}$.

We first note that

$$L_X(\lambda) = \int_0^\infty a\,e^{-ax} e^{-\lambda x}\,dx = a \int_0^\infty e^{-(\lambda+a)x}\,dx = \frac{a}{\lambda + a}.$$

Hence

$$E\{X\} = \left[-L_X'(\lambda)\right]_{\lambda=0} = \left[-\left(-\frac{a}{(\lambda+a)^2}\right)\right]_{\lambda=0} = \frac{a}{a^2} = \frac{1}{a},$$

and

$$E\left\{X^2\right\})\left[L_X''(\lambda)\right]_{\lambda=0} = \left[\frac{2a}{(\lambda+a)^3}\right]_{\lambda=0} = \frac{2a}{a^3} = \frac{2}{a^2},$$

from which

$$V\{X\} = E\left\{X^2\right\} - (E\{X\})^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2},$$

in accordance with previous results.

Example 5.2 *Let X_1, X_2, … be mutually independent random variables, where X_k is Gamma distributed with form parameter k and scale parameter 1, thus $X_k \in \Gamma(k,1)$, $k \in \mathbb{N}$. Define*

$$Y_n = \sum_{k=1}^{n} X_k \quad \textit{and} \quad Z_n = \frac{1}{n^2}\, Y_n, \qquad n \in \mathbb{N}.$$

1) *Find the means $E\{Y_n\}$ and $E\{Z_n\}$.*

2) *Find the Laplace transform of Y_n and the Laplace transform of Z_n.*

3) *Prove, e.g. by using the result of* **2.**, *that the sequence $(Z_n)_{n=1}^{\infty}$ converges in distribution towards a random variable Z, and find the distribution function of Z.*

We get from $X_k \in \Gamma(k,1)$ that

$$E\{X_k\} = k \qquad \text{and} \qquad L_{X_k}(\lambda) = \left(\frac{1}{1+\lambda}\right)^k .$$

1) The means are

$$E\{Y_n\} = \sum_{k=1}^{n} E\{X_k\} = \sum_{k=1}^{n} k = \frac{1}{2}\, n(n+1),$$

$$E\{Z_n\} = \frac{1}{n^2}\, E\{Y_n\} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

2) From

$$Y_n \in \Gamma\left(\sum_{k=1}^{n} k\,,\, 1\right) = \Gamma\left(\frac{n(n+1)}{2}\,,\, 1\right),$$

follows that

$$L_{Y_n}(\lambda) = \left(\frac{1}{1+\lambda}\right)^{\frac{n(n+1)}{2}} .$$

Alternatively,

$$L_{Y_n}(\lambda) = \prod_{k=1}^{n} L_{X_k}(\lambda) = \prod_{k=1}^{n} \left(\frac{1}{1+\lambda}\right)^k = \left(\frac{1}{1+\lambda}\right)^{\frac{n(n+1)}{2}},$$

thus the same result.

Since $L_{Z_n}(\lambda)$ is obtained from $L_{Y_n}(\lambda)$ by replacing λ by $\dfrac{\lambda}{n^2}$, we get

$$L_{Z_n}(\lambda) = L_{Y_n}\left(\frac{\lambda}{n^2}\right) = \frac{1}{\left(1+\dfrac{\lambda}{n^2}\right)^{\frac{n(n+1)}{2}}}.$$

3) Since the denominator converges for $n \to \infty$,

$$\left(1+\frac{\lambda}{n^2}\right)^{\frac{n(n+1)}{2}} = \left\{\left(1+\frac{\lambda}{n^2}\right)^{n^2} \cdot \left(1+\frac{\lambda}{n^2}\right)^{n}\right\}^{\frac{1}{2}} \to \left(e^{\lambda}\cdot 1\right)^{\frac{1}{2}} = \exp\left(\frac{\lambda}{2}\right) \qquad \text{for } n \to \infty,$$

we get

$$L_{Z_n}(\lambda) \to \exp\left(-\frac{\lambda}{2}\right) = L_Z(\lambda) \qquad \text{for } n \to \infty,$$

so (Z_n) converges in distribution towards a causally distributed random variable Z with the distribution function

$$F_Z(z) = \begin{cases} 0 & \text{for } z < \dfrac{1}{2}, \\ 1 & \text{for } z \ge \dfrac{1}{2}. \end{cases}$$

Example 5.3 *A random variable Z has the values 1, 2, ... with the probabilities*

$$P\{Z=k\} = -\frac{1}{\ln p}\cdot\frac{q^k}{k},$$

where $p>0$, $q>0$ and $p+q=1$. We say that Z has a logarithmic distribution.

1. *Find the Laplace transform $L_Z(\lambda)$ of Z.*

2. *Find the mean of the random variable Z.*

We consider a sequence of random variables $(X_n)_{n=2}^{\infty}$, where X_n has the values 1, 2, ... of the probabilities

$$P\{X_n=k\} = -\frac{1}{\ln p_n}\cdot\frac{q_n^k}{k},$$

where $q_n = \dfrac{1}{n}$ and $p_n+q_n=1$.

3. *Prove that the sequence (X_n) converges in distribution towards a random variable X, and find the distribution function of X.*

1) The Laplace transform is

$$L_Z(\lambda) = \sum_{n=1}^{\infty} P\{Z=n\}e^{\lambda n} = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\frac{q^n}{n}\,e^{-\lambda n} = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\frac{\left(qe^{-\lambda}\right)^n}{n} = \frac{\ln\left(1-qe^{-\lambda}\right)}{\ln p}.$$

2) By a straightforward computation,

$$E\{Z\} = -\frac{1}{\ln p}\sum_{k=1}^{\infty} k\cdot\frac{q^k}{k} = -\frac{1}{\ln p}\cdot\frac{q}{1-q} = -\frac{q}{p\,\ln p}.$$

Alternatively,

$$E\{Z\} = -L_Z'(0) = -\frac{1}{\ln p}\cdot\left[\frac{qe^{-\lambda}}{1-qe^{-\lambda}}\right]_{\lambda=0} = -\frac{1}{\ln p}\cdot\frac{q}{1-q} = -\frac{q}{p\,\ln p}.$$

3) It follows from **1.** that

$$L_{X_k}(\lambda) = \frac{\ln\left(1-q_ke^{-\lambda}\right)}{\ln p_k} = \frac{\ln\left(1-\dfrac{1}{k}\,e^{-\lambda}\right)}{\ln\left(1-\dfrac{1}{k}\right)}.$$

For every fixed $\lambda>0$ we get by l'Hospital's rule, where we put $x=\dfrac{1}{k}$,

$$\lim_{k\to\infty} L_{X_k}(\lambda) = \lim_{k\to\infty}\frac{\ln\left(1-\dfrac{1}{k}\,e^{-\lambda}\right)}{\ln\left(1-\dfrac{1}{k}\right)} = \lim_{x\to 0}\frac{\ln\left(1-x\,e^{-\lambda}\right)}{\ln(1-x)} = \lim_{x\to 0}\frac{\dfrac{-e^{-\lambda}}{1-x\,e^{-\lambda}}}{-\dfrac{1}{1-x}} = e^{-\lambda}.$$

If $\lambda = 0$, then $L_{X_k} = e^{-0}$ for every k, so

$$L_X(\lambda) = \begin{cases} & \text{for } \lambda > 0, \\ 1 & \text{for } \lambda = 0, \end{cases}$$

and $L_X(\lambda)$ exists for all $\lambda \ge 0$, and it is continuous at $\lambda = 0$. This implies that (X_n) converges in distribution towards some random variable X, which has the Laplace transform $L_X(\lambda) = e^{-\lambda}$, from which we conclude that X is causally distributed with $a = 1$, thus $P\{X = 1\} = 1$.

Example 5.4 *A random variable X has the values 1, 2, ... of the probabilities*

$$P\{X = k\} = pq^{k-1}, \qquad hvor\ p > 0,\ q > 0,\ p + q = 1.$$

1. *Find the Laplace transform of X.*

We consider a sequence of random variables $(X_n)_{n=1}^{\infty}$, where X_n has the values $\frac{1}{n}$, $\frac{2}{n}$, ... of the probabilities

$$P\left\{X_n = \frac{k}{n}\right\} = \frac{a}{n}\left(1 - \frac{a}{n}\right)^{k-1}, \qquad k \in \mathbb{N}$$

(here $a \in]0,1[$ is a constant).

2. *Prove that the mean of X_n does not depend on n.*

3. *Find the Laplace transform of X_n.*

4. *Prove that the sequence (X_n) converges in distribution towards a random variable Y, and find the distribution function of Y.*

1. The Laplace transform is

$$L_X(\lambda) = \sum_{n=1}^{\infty} e^{-\lambda n} pq^{n-1} = \frac{p}{q}\sum_{n=1}^{\infty}\left(qe^{-\lambda}\right)^n = \frac{p}{q}\cdot\frac{q\cdot e^{-\lambda}}{1-qe^{-\lambda}} = \frac{p\,e^{-\lambda}}{1-qe^{-\lambda}}.$$

2. and 3. The Laplace transform of X_n is

$$\begin{aligned}
L_X(\lambda) &= \sum_{n=1}^{\infty}\exp\left(-\lambda\,\frac{k}{n}\right)\cdot\frac{a}{n}\left(1-\frac{a}{n}\right)^{k-1} = \frac{\frac{a}{n}}{1-\frac{a}{n}}\sum_{k=1}^{\infty}\left\{\exp\left(-\frac{\lambda}{n}\left(1-\frac{a}{n}\right)\right)\right\}^k \\
&= \frac{\frac{a}{k}}{1-\frac{a}{n}}\cdot\frac{\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)}{1-\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)} = \frac{\frac{a}{n}\exp\left(-\frac{\lambda}{n}\right)}{1-\lambda\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)} \\
&= \frac{\frac{a}{n}}{1-\frac{a}{n}}\cdot\frac{1}{1-\left(1-\frac{a}{n}\right)\exp\left(-\frac{-\lambda}{n}\right)} - \frac{\frac{a}{n}}{1-\frac{a}{n}},
\end{aligned}$$

hence

$$
\begin{aligned}
E\{X_n\} &= -L'_{X_n}(0) = -\frac{\frac{a}{n}}{1-\frac{a}{n}}\cdot\left.\frac{\left(1-\frac{a}{n}\right)\cdot\left(-\frac{1}{n}\right)\exp\left(-\frac{\lambda}{n}\right)}{\left\{1-\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)\right\}^2}\right]_{\lambda=0} \\
&= \frac{\frac{a}{n}}{1-\frac{a}{n}}\cdot\frac{\left(1-\frac{a}{n}\right)\cdot\frac{1}{n}}{\left(\frac{a}{n}\right)^2} = \frac{1}{a},
\end{aligned}
$$

which is independent of n.

4. It follows by l'Hospital's rule that

$$
\begin{aligned}
\lim_{n\to\infty} L_{X_n}(\lambda) &= \lim_{n\to\infty}\frac{\frac{a}{n}\exp\left(-\frac{\lambda}{n}\right)}{1-\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)} = \lim_{x\to 0}\frac{a\,x\,e^{-\lambda x}}{1-(1-a\,x)e^{-\lambda x}} \\
&= a\lim_{x\to 0}\frac{e^{-\lambda x}-\lambda\,x\,e^{-\lambda x}}{\lambda(1-a\,x)e^{-\lambda x}+a\,e^{-\lambda x}} = a\lim_{x\to 0}\frac{1-\lambda\,x}{\lambda(1-a\,x)+a} = \frac{a}{\lambda+a} = L_Y(\lambda),
\end{aligned}
$$

and we get by using a table that $Y\in\Gamma\left(1,\frac{1}{a}\right)$ is exponentially distributed. This proves that (X_n) converges in distribution towards Y.

Example 5.5 *A random variable X has the values 1, 2, … of the probabilities*

$$P\{X=k\} = \frac{a^{k-1}}{(k-1)!}\,e^{-a}, \qquad k \in \mathbb{N},$$

where a is some positive constant.

1. *Find the Laplace transform of X.*

2. *Find the mean of X.*

We consider a sequence of random variables $(Y_n)_{n=1}^{\infty}$, where for each $n \in \mathbb{N}$ the random variable Y_n has its distribution given by

$$P\left\{Y_n = \frac{k}{2n}\right\} = \frac{(2n)^{k-1}}{(k-1)!}\,e^{-2n}, \qquad k \in \mathbb{N}.$$

3. *Find the Laplace transform of Y_n.*

4. *Prove, e.g. by using the result of* **3.**, *that the sequence $(Y_n)_{n=1}^{\infty}$ converges in distribution towards a random variable Y, and find the distribution function of Y.*

5. *Is it true that $E\{Y_n\} \to E\{Y\}$ for $n \to \infty$?*

1) The Laplace transform of X is

$$\begin{aligned} L_X(\lambda) &= \sum_{k=1}^{\infty} \frac{a^{k-1}}{(k-1)!}\,e^{-a}\cdot e^{-\lambda k} = e^{-a}\cdot e^{-\lambda}\sum_{k=0}^{\infty}\frac{a^k}{k!}\,\left(e^{-\lambda}\right)^k = e^{-a-\lambda}\cdot\exp\left(a\,e^{-\lambda}\right) \\ &= \exp\left(-a-\lambda+a\,e^{-\lambda}\right) = \exp\left(a\left(e^{-\lambda}-1\right)-\lambda\right), \qquad \lambda \geq 0. \end{aligned}$$

2) The mean is

$$E\{X\} = \sum_{k=1}^{\infty} k\cdot\frac{a^{k-1}}{(k-1)!}\,e^{-a} = e^{-a}\sum_{k=0}^{\infty}\frac{k+1}{k!}\,a^k = e^{-a}\left\{\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}+\sum_{k=0}^{\infty}\frac{1}{k!}\,a^k\right\} = e^{-a}(a+1)e^{a} = a+1.$$

Alternatively,

$$L_X'(\lambda) = \left(-1-a\,e^{-\lambda}\right)\exp\left(-a-\lambda+a\,e^{-\lambda}\right),$$

så

$$E\{X\} = -L_X'(0) = 1+a.$$

3) The Laplace transform of X_n with $a=2n$ is

$$L_X\left(\frac{\lambda}{2n};a=2n\right) = \exp\left(-2n-\lambda+2n\,\exp\left(-\frac{\lambda}{2n}\right)\right) = \exp\left(2n\left\{\exp\left(-\frac{\lambda}{2n}\right)-1\right\}-\lambda\right).$$

Since $X_n = 2nY_n$, the Laplace transform of Y_n is given by

$$L_{Y_n}(\lambda) = L_{X_n}\left(\frac{\lambda}{2n}\right) = \exp\left(2n\left\{\exp\left(-\frac{\lambda}{2n}\right)-1\right\}-\frac{\lambda}{2n}\right), \qquad \lambda \geq 0.$$

4) It follows from

$$\begin{aligned} L_{Y_n}(\lambda) &= \exp\left(2n\left\{1-\frac{\lambda}{2n}+\frac{\lambda}{2n}\,\varepsilon\left(\frac{\lambda}{2n}\right)-1\right\}-\frac{\lambda}{2n}\right) \\ &= \exp\left(-\lambda+\lambda\,\varepsilon\left(\frac{\lambda}{2n}\right)-\frac{\lambda}{2n}\right)\to e^{-\lambda} \qquad \text{for } n\to\infty, \end{aligned}$$

that $Y_n \xrightarrow{D} Y$, where Y has the distribution function

$$F_Y(y)=\begin{cases} 1 & \text{for } y\ge 1,\\ 0 & \text{for } y<1.\end{cases}$$

5) Since

$$E\{Y_n\}=\frac{1}{2n}(2n+1)=1+\frac{1}{2n}\to 1=E\{Y\},$$

we conclude that the answer is "yes".

Example 5.6 *A random variable X has the values 1, 3, 5, ... of probabilities*

$$P\{X=2k+1\}=p(1-p)^k, \qquad k\in\mathbb{N}_0,$$

where p is a constant, $0<p<1$.

1. *Find the Laplace transform $L_X(\lambda)$ of the random variable X.*

2. *Find the mean of the random variable X.*

We consider a sequence of random variables $(X_n)_{n=1}^{\infty}$, where X_n has the values $\frac{1}{n}, \frac{3}{n}, \frac{5}{n}, \dots$ of the probabilities

$$P\left\{X_n=\frac{2k+1}{n}\right\}=\frac{1}{2n}\left(1-\frac{1}{2n}\right)^k, \qquad k\in\mathbb{N}_0.$$

3. *Find the Laplace transform $L_{X_n}(\lambda)$ of the random variable X_n.*

4. *Find the mean of the random variable X_n.*

5. *Prove that the sequence (X_n) converges in distribution towards a random variable Y, and find the distribution function of Y.*

1) The Laplace transform is

$$\begin{aligned} L_X(\lambda) &= \sum_{k=0}^{\infty} p(1-p)^k\exp(-\lambda(2k+1)) = p\,e^{-\lambda}\sum_{k=0}^{\infty}\left\{(1-p)e^{-2\lambda}\right\}^k \\ &= \frac{p\,e^{-\lambda}}{1-(1-p)e^{-2\lambda}}=\frac{p\,e^{\lambda}}{e^{2\lambda}-(1-p)}. \end{aligned}$$

2) The mean is

$$\begin{aligned} E\{X\} &= \sum_{k=0}^{\infty}(2k+1)p(1-p)^k = 2p(1-p)\sum_{k=1}^{\infty}k(1-p)^{k-1} + p\sum_{k=0}^{\infty}(1-p)^k \\ &= 2p(1-p)\cdot\frac{1}{\{1-(1-p)\}^2} + p\cdot\frac{1}{1-(1-p)} \\ &= \frac{2p(1-p)}{p^2} + \frac{p}{p} = 2\,\frac{1-p}{p} + 1 = \frac{2}{p} - 1. \end{aligned}$$

Alternatively,

$$L_X'(\lambda) = L_X(\lambda)\cdot\left\{\frac{p\,e^{\lambda}}{e^{2\lambda}-(1-p)} - \frac{2p\,e^{\lambda}}{\{e^{2\lambda}-(1-p)\}^2}\right\},$$

thus

$$E\{X\} = -L_X'(0) = -1 + \frac{2p}{p^2} = \frac{2}{p} - 1.$$

3) If we put $p = \dfrac{1}{2n}$, then we get $L_{X_n}(\lambda)$ from $L_X(\lambda)$ by replacing λ by $\dfrac{\lambda}{n}$, thus

$$L_{X_n}(\lambda) = L_X\left(\frac{\lambda}{n}\right) = \frac{\frac{1}{2n}\,\exp\left(\frac{\lambda}{n}\right)}{\exp\left(\frac{2\lambda}{n}\right) - \left(1-\frac{1}{2n}\right)} = \frac{\exp\left(\frac{\lambda}{n}\right)}{2n\left\{\exp\left(\frac{2\lambda}{n}\right)-1\right\}+1}.$$

4) It follows from

$$L_{X_n}'(\lambda) = \frac{1}{n}\,mL_{X_n}(\lambda) - \frac{\exp\left(\frac{\lambda}{n}\right)}{\left\{2n\left(\exp\left(\frac{2\lambda}{n}\right)-1\right)+1\right\}^2}\cdot 2n\cdot\frac{2}{n},$$

that

$$E\{X_n\} = -L_{X_n}'(0) = -\frac{1}{n} + 4 = 4 - \frac{1}{n}.$$

Alternatively,

$$\begin{aligned} E\{X_n\} &= \sum_{k=0}^{\infty}\frac{2k+1}{n}\cdot\frac{1}{2n}\left(1-\frac{1}{2n}\right)^k \\ &= \frac{1}{n^2}\left(1-\frac{1}{2n}\right)\sum_{k=1}^{\infty}k\left(1-\frac{1}{2n}\right)^{k-1} + \frac{1}{n}\cdot\frac{1}{2n}\sum_{k=0}^{\infty}\left(1-\frac{1}{2n}\right)^k \\ &= \frac{1}{n^2}\left(1-\frac{1}{2n}\right)\cdot\frac{1}{\left\{1-\left(1-\frac{1}{2n}\right)\right\}^2} + \frac{1}{2n^2}\cdot\frac{1}{1-\left(1-\frac{1}{2n}\right)} \\ &= \frac{1}{n^2}\left(1-\frac{1}{2n}\right)\cdot\frac{1}{\left(\frac{1}{2n}\right)^2} + \frac{1}{2n^2}\cdot\frac{1}{\frac{1}{2n}} = 4\left(1-\frac{1}{2n}\right) + \frac{1}{n} = 4 - \frac{1}{n}. \end{aligned}$$

5) It follows from

$$2n\left\{\exp\left(\frac{2\lambda}{n}\right)-1\right\}+1=2n\left\{1+\frac{2\lambda}{n}+\frac{2\lambda}{n}\,\varepsilon\left(\frac{2\lambda}{n}\right)-1\right\}+1=1+4\lambda+4\lambda\,\varepsilon\left(\frac{2\lambda}{n}\right),$$

that

$$L_{X_n}(\lambda)=\frac{\exp\left(\frac{\lambda}{n}\right)}{2n\left\{\exp\left(\frac{2\lambda}{n}\right)-1\right\}+1}=\frac{\exp\left(\frac{\lambda}{n}\right)}{1+4\lambda+4\lambda\,\varepsilon\left(\frac{2\lambda}{n}\right)}\to\frac{1}{1+4\lambda}\qquad\text{for } n\to\infty.$$

Now, $\dfrac{1}{1+4\lambda}$ is continuous for $\lambda\in[0,\infty[$. Hence (X_n) converges in distribution towards a random variable Y, where $L_Y(\lambda)=\dfrac{1}{1+4\lambda}$ corresponds to $Y\in\Gamma(1,4)$, i.e. an exponential distribution of frequency

$$f_Y(y)=\begin{cases}\dfrac{1}{4}\exp\left(-\dfrac{y}{4}\right) & \text{for } y>0,\\[2mm] 0 & \text{for } y\le 0.\end{cases}$$

Example 5.7 *The random variables X_1, X_2 and X_3 are assumed to be mutually independent and each of them following a rectangular distribution over the interval* $]0,1[$.
Let X denote the random variable

$$X = X_1 + X_2 + X_3.$$

1) *Find the mean and variance of the random variable X.*
   HINT: *Find first the frequency of $X_1 + X_2$.*

2) *Find the Laplace transform $L(\lambda)$ of the random variable X, and prove that*

$$L(\lambda) = 1 - \frac{3}{2}\,\lambda + \frac{5}{4}\,\lambda^2 + \lambda^2 \varepsilon(\lambda).$$

1) We conclude from

$$E\{X_1\} = E\{X_2\} = E\{X_3\} = \frac{1}{2},$$

that

$$E\{X\} = E\{X_1\} + E\{X_2\} + E\{X_3\} = \frac{3}{2}.$$

Since

$$V\{X_1\} = V\{X_2\} = V\{X_3\} = \frac{1}{12},$$

and X_1, X_2 and X_3 are mutually independent, we get

$$V\{X\} = V\{X_1\} + V\{X_2\} + V\{X_3\} = 3 \cdot \frac{1}{12} = \frac{1}{4}.$$

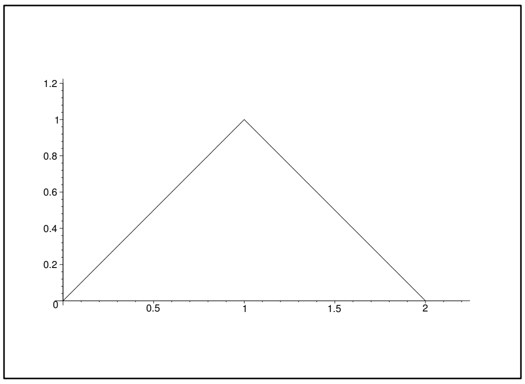

Figure 1: The graph of $g(y)$.

2) The frequency $g(y)$ of $Y = X_1 + X_2$ is 0 for $y \notin]0,2[$. If $0 < y < 2$, then

$$g(y) = \int_0^y f(y-s)f(s)\,ds.$$

Hence, for $0 < y < 1$,

$$g(y) = \int_0^y f(y-s)f(s)\,ds = \int_0^y 1\cdot 1\,ds = y.$$

If $1 \le y < 2$, then we get instead

$$g(y) = \int_0^y f(y-s)f(s)\,ds = \int_{y-1}^1 1\cdot 1\,ds = 2-y.$$

Summing up, the frequency of $Y = X_1 + X_2$ is given by

$$g(y) = \begin{cases} y & \text{for } y\,]0,1[, \\ 2-y & \text{for } y \in [1,2[\\ 0 & \text{otherwise.} \end{cases}$$

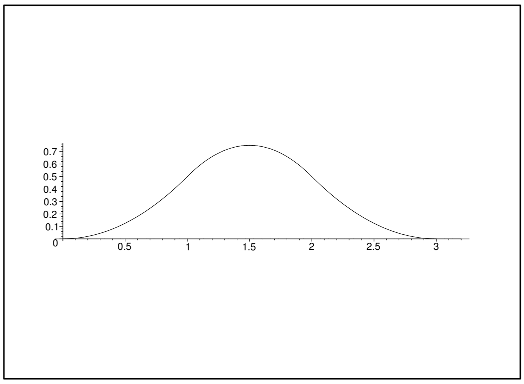

Figure 2: The graph of $h(x)$.

The frequency $h(x)$ of $X = X_1 + X_2 + X_3 = Y + X_3$ is 0 for $x \notin]0,3[$.
If $0 < x < 3$, then

$$h(x) = \int_0^x g(s)f(x-s)\,ds = \int_0^x g(x-s)f(s)\,ds.$$

We shall now split the investigation into the cases of the three intervals $]0,1[$, $[1,2[$ and $[2,3[$.

a) If $x \in]0,1[$, then

$$h(x) = \int_0^x g(x-s)\cdot 1\,ds = \int_0^x (x-s)\,ds = \left[-\frac{1}{2}(x-s)^2\right]_{s=0}^x = \frac{x^2}{2}.$$

b) If $x \in [1,2[$, then

$$\begin{aligned}
h(x) &= \int_0^x g(x-s)f(s)\,ds = \int_0^1 g(x-s)\cdot 1\,ds \\
&= \int_0^{x-1} g(x-s)\,ds + \int_{x-1}^1 g(x-s)\,ds \\
&= \int_0^{x-1} \{2-(x-s)\}\,ds + \int_{x-1}^1 (x-s)\,ds \\
&= \left[\frac{1}{2}\,(2-x+s)^2\right]_{s=0}^{x-1} + \left[-\frac{1}{2}\,(x-s)^2\right]_{s=x-1}^{2} \\
&= \frac{1}{2}\,\left\{(2-x+x-1)^2-(2-x)^2+(x-x+1)^2-(x-1)^2\right\} \\
&= \frac{1}{2}\,\left\{1-(x-s)^2+1-(x-1)^2\right\} \\
&= \frac{1}{2}\,\left\{2-x^2+4x-4-x^2+2x-1\right\} = \frac{1}{2}\,\left\{-2x^2+6x-3\right\} \\
&= \frac{3}{4}-\left(x-\frac{3}{2}\right)^2.
\end{aligned}$$

c) If $x \in [2,3[$, then

$$\begin{aligned}
h(x) &= \int_0^x g(x-s)f(s)\,ds = \int_0^1 g(x-s)\cdot 1\,ds = \int_{x-1}^x g(t)\,dt = \int_{x-1}^2 g(t)\,dt \\
&= \int_{x-1}^2 (2-t)\,dt = \left[-\frac{1}{2}\,(2-t)^2\right]_{x-1}^2 = \frac{1}{2}\,\frac{(2-x+1)^2}{2} = \frac{1}{2}\,(3-x)^2.
\end{aligned}$$

Summing up, the frequency $h(x)$ of X is given by

$$h(x) = \begin{cases} \dfrac{1}{2}\,x^2 & \text{for } x \in\,]0,1[, \\[2ex] \dfrac{3}{4}-\left(x-\dfrac{3}{2}\right)^2 & \text{for } x \in [1,2[, \\[2ex] \dfrac{1}{2}\,(3-x)^2 & \text{for } x \in [2,3[, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

3) When $\lambda \geq 0$ and $i = 1,\ 2,\ 3$, then

$$L_{X_i}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt = \int_0^1 e^{-\lambda t}\,dt = \left[-\frac{1}{\lambda}\,e^{-\lambda t}\right]_0^1 = \frac{1-e^{-\lambda}}{\lambda}.$$

Since X_1, X_2 and X_3 are mutually independent, we get

$$\begin{aligned}
L_X(\lambda) &= \left(\frac{1-e^{-\lambda}}{\lambda}\right)^3 = \frac{1}{\lambda^3}\left\{1-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\lambda^n\right\}^3 = \left\{\frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\,\lambda^n\right\}^3 \\
&= \left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\,\lambda^{n-1}\right\}^3 = \left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\,\lambda^n\right\}^3 = \left\{1-\frac{\lambda}{2}+\frac{\lambda^2}{6}+\lambda^2\varepsilon(\lambda)\right\}^3 \\
&= \left\{1+\frac{\lambda^4}{4}-\lambda+\frac{\lambda^2}{3}+\lambda^2\varepsilon(\lambda)\right\}\cdot\left\{1-\frac{\lambda}{2}+\frac{\lambda^2}{6}+\lambda^2\varepsilon(\lambda)\right\} \\
&= \left\{1-\lambda+\frac{7}{12}\,\lambda^2+\lambda^2\varepsilon(\lambda)\right\}\cdot\left\{1-\frac{\lambda}{2}+\frac{\lambda^2}{6}+\lambda^2\varepsilon(\lambda)\right\} \\
&= 1-\left(\frac{1}{2}+1\right)\lambda+\left(\frac{1}{6}+\frac{1}{2}+\frac{7}{12}\right)\lambda^2+\lambda^2\varepsilon(\lambda) = 1-\frac{3}{2}\,\lambda+\frac{2+6+7}{12}\,\lambda^2+\lambda^2\varepsilon(\lambda) \\
&= 1-\frac{3}{2}\,\lambda+\frac{5}{4}\,\lambda^2+\lambda^2\varepsilon(\lambda).
\end{aligned}$$

Example 5.8 *A random variable Y has the frequency*

$$f(y) = \begin{cases} a^2 y\, e^{-ay}, & y \geq 0 \\ 0, & y < 0, \end{cases}$$

where a is a positive constant.

1. *Find the Laplace transform $L_Y(\lambda)$ of the random variable Y.*

2. *Find the mean of the random variable Y.*

A random variable Y has the values 0, 1, 2, 3, ... of the probabilities

$$P\{X = k\} = (k+1)p^2 q^k,$$

where $p > 0$, $q > 0$, $p + q = 1$.

3. *Find the Laplace transform $L_X(\lambda)$ of X.*

Find the mean of X.

A sequence of random variables (X_n) is given by X_n having the values 0, $\frac{1}{n}$, $\frac{2}{n}$, ... of the probabilities

$$P\left\{X_n = \frac{k}{n}\right\} = (k+1)\left(\frac{a}{n}\right)^2\left(1 - \frac{a}{n}\right)^k,$$

where a is a constant, $0 < a < 1$.

5. *Find the Laplace transform of X_n.*

6. *Find the mean of the random variable X_n.*

7. *Prove that the sequence (X_n) converges in distribution towards a random variable Y (as defined above).*

8. *Prove that $E\{X_n\} \to E\{Y\}$ for $n \to \infty$.*

1) If $\lambda \geq 0$, then

$$L_Y(\lambda) = \int_0^\infty a^2 y\, e^{-ay} e^{-\lambda y}\, dy = \frac{a^2}{(\lambda + a)^2} \int_0^\infty (a+\lambda)^2 y\, e^{-(a+\lambda y)}\, dy = \frac{a^2}{(\lambda+a)^2} = \frac{1}{\left(1 + \frac{\lambda}{a}\right)^2}.$$

2) The mean is

$$E\{Y\} = -L_Y'(0) = -\left.\frac{-2}{\left(1+\frac{\lambda}{a}\right)^3} \cdot \frac{1}{a}\right]_{\lambda=0} = \frac{2}{a}.$$

3) If $\lambda \geq 0$, then

$$L_X(\lambda) = \sum_{n=0}^{\infty} e^{-\lambda n}(n+1)p^2q^n = \sum_{n=0}^{\infty}(n+1)p^2\left(q\,e^{-\lambda}\right)^n = \frac{p^2}{\left(1-q\,e^{-\lambda}\right)^2}.$$

4) The mean is

$$E\{X\} = -L_X'(0) = -\lim_{\lambda\to 0}\frac{-2p^2}{\left(1-q\,e^{-\lambda}\right)^3}\cdot\left(-q\,e^{-\lambda}\right)\cdot(-1) = \frac{2p^2q}{(1-q)^3} = 2\,\frac{q}{p}.$$

5) If X_n, then

$$L_{X_n}(\lambda) = \sum_{k=0}^{\infty}\exp\left(-\lambda\,\frac{k}{n}\right)\,(k+1)\left(\frac{a}{n}\right)^2\left(1-\frac{a}{n}\right)^k = \frac{\left(\frac{a}{n}\right)^2}{\left\{1-\exp\left(-\frac{\lambda}{n}\right)\left(1-\frac{a}{n}\right)\right\}^2}.$$

6) The mean is

$$\begin{aligned}E\{X_n\} &= -L_{X_n}'(0) = -\lim_{\lambda\to 0}\left(\frac{a}{n}\right)^2\cdot(-2)\,\frac{-\left(1-\frac{a}{n}\right)\exp\left(-\frac{\lambda}{n}\right)\cdot\left(-\frac{1}{n}\right)}{\left\{1-\exp\left(-\frac{\lambda}{n}\right)\left(1-\frac{a}{n}\right)\right\}^3}\\ &= \left(\frac{a}{n}\right)^2\cdot\frac{2}{\left\{1-\left(1-\frac{a}{n}\right)\right\}^3}\cdot\left(1-\frac{a}{n}\right)\cdot\frac{1}{n} = \frac{2}{n}\cdot\frac{1-\frac{a}{n}}{\frac{a}{n}} = \frac{2}{a}-\frac{2}{n}.\end{aligned}$$

7) We get by a rearrangement,

$$L_{X_n}(\lambda) = \frac{\left(\frac{a}{n}\right)^2}{\left\{1-\exp\left(-\frac{\lambda}{n}\right)\left(1-\frac{a}{n}\right)\right\}^2} = \frac{a^2}{\left\{n-\exp\left(-\frac{\lambda}{n}\right)(n-a)\right\}^2},$$

where

$$\begin{aligned}n-\exp\left(-\frac{\lambda}{n}\right)\cdot(n-a) &= n\left\{1-\exp\left(-\frac{\lambda}{n}\right)\right\}+a\cdot\exp\left(-\frac{\lambda}{n}\right)\\ &= n\left(1-\left\{1-\frac{\lambda}{n}+\frac{\lambda}{n}\,\varepsilon\left(\frac{\lambda}{n}\right)\right\}\right)+a\,\exp\left(-\frac{\lambda}{n}\right)\\ &= \lambda+n\cdot\frac{\lambda}{n}\,\varepsilon\left(\frac{\lambda}{n}\right)+a\cdot\exp\left(-\frac{\lambda}{n}\right)\to\lambda+a \qquad \text{for } n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}L_{X_n}(\lambda) = \frac{a^2}{(\lambda+a)^2} = L_Y(\lambda).$$

Since $L_Y(\lambda)$ is continuous at 0, it follows that $\{X_n\}$ converges in distribution towards Y.

8) The claim follows trivially from

$$\lim_{n\to\infty} E\{X_n\} = 2\lim_{n\to\infty}\frac{1}{a}\left(1-\frac{a}{n}\right) = \frac{2}{a} = E\{Y\}.$$

Example 5.9 *A random variable X has the frequency*

$$f_X(x) = \begin{cases} a\,e^{-ax}, & x \ge 0,\\[2mm] 0, & x < 0,\end{cases}$$

where a is a positive constant.

1) Find for every $n \in \mathbb{N}$ the mean $E\{X^n\}$.

2) Find the Laplace transform $L_X(\lambda)$ of X and show that it is given by

$$L_X(\lambda) = 1 - \frac{\lambda}{a} - \left(\frac{\lambda}{a}\right)^2 - \left(\frac{\lambda}{a}\right)^3 + \left(\frac{\lambda}{a}\right)^4 + \lambda^4\varepsilon(\lambda).$$

3) A random variable Y is given by $U = kX$, where k is a positive constant. Find the distribution function of Y.

4) Let U and V be independent random variables of the frequencies

$$f_U(u) = \begin{cases} 2a\,e^{-2au}, & u \ge 0,\\[2mm] 0, & u < 0,\end{cases} \qquad f_V(v) = \begin{cases} 3a\,e^{-3av}, & v \ge 0,\\[2mm] 0, & v < 0.\end{cases}$$

The random variable Z is given by $Z = 2U + 3V$.
Find the frequency of Z.

1) We get by a straightforward computation,

$$E\{X^n\} = \int_0^\infty a\,x^n e^{-ax}\,dx = \frac{1}{a^n}\int_0^\infty t^n e^{-n}\,dt = \frac{n!}{a^n}.$$

2) If $\lambda \ge 0$, then

$$L_X(\lambda) = \int_0^\infty a\,e^{-ax}e^{-\lambda x}\,dx = a\int_0^\infty e^{-(a+\lambda)x}\,dx = \frac{a}{a+\lambda} = \frac{1}{1+\dfrac{\lambda}{a}}.$$

If $0 \le \lambda < a$, then

$$L_X(\lambda) = \frac{1}{1+\dfrac{\lambda}{a}} = 1 - \frac{\lambda}{a} + \left(\frac{\lambda}{a}\right)^2 - \left(\frac{\lambda}{a}\right)^3 + \left(\frac{\lambda}{a}\right)^4 + \lambda^4\varepsilon(\lambda).$$

3) The distribution of Y for $y > 0$ is given by

$$P\{Y \le y\} = P\{kX \le y\} = \left\{X \le \frac{y}{k}\right\} = \int_0^{\frac{y}{k}} a\, a^{-ax}\, dx = 1 - \exp\left(-\frac{a}{k}\, y\right),$$

hence the frequency is

$$f_Y(y) = \begin{cases} \dfrac{a}{k}\, \exp\left(-\dfrac{a}{k}\, y\right) & \text{for } y \ge 0, \\ 0 & \text{for } y < 0. \end{cases}$$

4) It follows from **3.** that $2U$ has the frequency $f_X(u)$, and that $3V$ has the frequency $f_X(v)$. (In the former case $k = 2$, and in the latter case $k = 3$).
This means that $2U,\, 3V \in \Gamma\left(1, \frac{1}{a}\right)$, so

$$Z = 2U + 3V \in \Gamma\left(1+1, \frac{1}{a}\right) = \Gamma\left(2, \frac{1}{a}\right),$$

and the frequency of Z is given by

$$f_Z(z) = \begin{cases} a^2 z\, e^{-az} & \text{for } z > 0, \\ 0 & \text{for } z \le 0. \end{cases}$$

Example 5.10 *Given a sequence of random variables $(X_n)_{n=1}^{\infty}$, where X_n has the distribution function*

$$F_n(x) = \begin{cases} 0 & \text{for } x < 0, \\ n^2 x^2 & \text{for } 0 \le x \le \dfrac{1}{n}, \\ 1 & \text{for } x > \dfrac{1}{n} \end{cases}$$

1) *Find for every $n \in \mathbb{N}$ the mean $E\{X_n\}$ and variance $V\{X_n\}$.*

2) *Prove that the sequence (X_n) converges in probability towards a random variable X, and find the distribution function of X.*

3) *Find the Laplace transform $L_n(\lambda)$ of the random variable X_n.*
Is the sequence of functions $(L_n(\lambda))$ convergent?

4) *Find the distribution function of $Y_n = X_n^2$.*

5) *Assuming that the random variables X_1 and X_2 are independent, we shall find the frequency of the random variable $Z = X_1 + X_2$.*

1) The frequencies are obtained by differentiation,

$$f_n(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 2n^2 x & \text{for } 0 < x < \dfrac{1}{n}, \\ 0 & \text{for } x \ge \dfrac{1}{n}, \end{cases}$$

hence

$$E\{X_n\} = \int_0^{\frac{1}{n}} 2n^2 x^2\, dx = 2n^2 \left[\frac{x^3}{3}\right]_0^{\frac{1}{n}} = \frac{2}{3n},$$

and

$$E\left\{X_n^2\right\} = \int_0^{\frac{1}{n}} 2n^2 x^3\, dx = 2n^2 \left[\frac{x^4}{4}\right]_0^{\frac{1}{n}} = \frac{1}{2n^2},$$

whence

$$V\{X_n\} = E\left\{X_n^2\right\} - (E\{X_n\}) = \frac{1}{2n^2} - \frac{4}{9n^2} = \frac{1}{18n^2}.$$

2) If $x \le 0$, then of course $F_n(x) = 0 \to 0$ for $n \to \infty$.
If $x > 0$, then there is an N, such that $x > \dfrac{1}{n}$ for every $n \ge N$, thus $F_n(x) = 1$ for $n \ge N$, and $F_n(x) \to 1$ for $n \to \infty$. We conclude that $(F_n(x))$ converges in distribution towards the causal distribution

$$F(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 1. \end{cases}$$

3) If $\lambda > 0$, then

$$\begin{aligned} L_n(\lambda) &= \int_0^\infty e^{-\lambda x} f_n(x)\, dx = 2n^2 \int_0^{\frac{1}{n}} e^{-\lambda x} x\, dx = 2n^2 \left[-\frac{x}{\lambda}\, e^{-\lambda x} \right]_0^{\frac{1}{n}} + \frac{2n^2}{\lambda} \int_0^{\frac{1}{n}} e^{-\lambda x}\, dx \\ &= -2n^2 \cdot \frac{1}{\lambda n} \cdot \exp\left(-\frac{\lambda}{n} \right) + \frac{2n^2}{\lambda} \left[-\frac{1}{\lambda}\, e^{-\lambda x} \right]_0^{\frac{1}{n}} = -\frac{2n}{\lambda} \exp\left(-\frac{\lambda}{n} \right) + \frac{2n^2}{\lambda^2} \left(1 - \exp\left(-\frac{\lambda}{n} \right) \right). \end{aligned}$$

Then by a series expansion,

$$\begin{aligned} L_n(\lambda) &= -\frac{2n}{\lambda} \left\{ 1 - \frac{\lambda}{n} + \frac{1}{2!} \frac{\lambda^2}{n^2} + \frac{\lambda^2}{n^2}\, \varepsilon\left(\frac{\lambda}{n} \right) \right\} + \frac{2n^2}{\lambda^2} \left\{ 1 - \left(1 - \frac{\lambda}{n} + \frac{1}{2} \cdot \frac{\lambda^2}{n^2} + \frac{\lambda^2}{n^2}\, \varepsilon\left(\frac{\lambda}{n} \right) \right) \right\} \\ &= -\frac{2n}{\lambda} + 2 - \frac{\lambda}{n} + \frac{\lambda}{n}\, \varepsilon\left(\frac{\lambda}{n} \right) + \frac{2n}{\lambda} - 1 + 2\,\varepsilon\left(\frac{\lambda}{n} \right) = 1 - \frac{\lambda}{n} + \varepsilon_1\left(\frac{\lambda}{n} \right), \end{aligned}$$

and we conclude that $L_n(\lambda) \to 1$ for $\lambda \to 0+$ and $n \to \infty$.

4) If $y > 0$, then

$$P\{Y_n \le y\} = P\left\{ (X_n)^2 \le y \right\} = P\{X_n \le \sqrt{y}\} = F_n(\sqrt{y}),$$

hence

$$P\{Y_n \le y\} = \begin{cases} 0 & \text{for } y \le 0, \\ n^2 y & \text{for } 0 \le y \le \dfrac{1}{n^2}, \\ 1 & \text{for } y > \dfrac{1}{n^2}. \end{cases}$$

5) We first note that

$$f_Z(z) = \int_0^\infty f_1(x) f_2(z-x)\, dx = \int_0^1 2x \cdot f_2(z-x)\, dx.$$

If $f_Z(z) \ne 0$, then $z - x \in \left[0, \dfrac{1}{2} \right]$, thus $x \in [0,1] \cap \left[z - \dfrac{1}{2}, z \right]$.
In particular, $f_Z(z) = 0$ if either $z \le 0$ or $z \ge \dfrac{3}{2}$.

If $z \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} f_Z(z) &= \int_0^z 2x \cdot 2 \cdot 4 \cdot (z-x)\, dx = 16 \int_0^z \left(zx - x^2\right)\, dx \\ &= 16 \left[z \cdot \frac{x^2}{2} - \frac{x^3}{3} \right]_{x=0}^{z} = 16 \left(\frac{z^3}{2} - \frac{z^3}{3} \right) = \frac{8}{3}\, z^3. \end{aligned}$$

If $z \in \left[\frac{1}{2}, 1\right]$, then

$$\begin{aligned} f_Z(z) &= \int_{z-\frac{1}{2}}^{z} 16 \left(zx - x^2\right)\, dx = 16 \left[z \cdot \frac{x^2}{2} - \frac{x^3}{3} \right]_{z-\frac{1}{2}}^{z} \\ &= \frac{8}{3}\, z^3 - 8 \left(z - \frac{1}{2} \right)^2 z + \frac{16}{3} \left(z - \frac{1}{2} \right)^3 \\ &= \frac{8}{3}\, z^3 - 8z^3 + 8z^2 - 2z + \frac{16}{3}\, z^3 - 8z^2 + 4z - \frac{2}{3} = 2z - \frac{2}{3}. \end{aligned}$$

Finally, if $z \in \left[1, \frac{3}{2}\right]$, then

$$\begin{aligned} f_Z(z = &= \int_{z-\frac{1}{2}}^{1} 16 \left(zx - x^2\right)\, dx = 16 \left[z\, \frac{x^2}{2} - \frac{x^3}{3} \right]_{z-\frac{1}{2}}^{1} \\ &= 16 \left(\frac{1}{2}\, z - \frac{1}{3} \right) - 16 \left[\frac{z}{2} \left(z - \frac{1}{2} \right)^2 - \frac{1}{3} \left(z - \frac{1}{2} \right)^3 \right] \\ &= 8z - \frac{16}{3} - 8z^3 + 8z^2 - 2z + \frac{16}{3}\, z^3 - 8z^2 + 4z - \frac{2}{3} \\ &= -\frac{8}{3}\, z^3 + 10z - 6. \end{aligned}$$

Summing up,

$$f_Z(z) = \begin{cases} \dfrac{8}{3}\, z^3 & \text{for } z \in \left[0, \dfrac{1}{2}\right], \\[2ex] 2z - \dfrac{2}{3} & \text{for } z \in \left[\dfrac{1}{2}, 1\right], \\[2ex] -\dfrac{8}{3}\, z^3 + 10z - 6 & \text{for } z \in \left[1, \dfrac{3}{2}\right], \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Example 5.11 *Let X_1, X_2, ... be mutually independent and identically distributed random variables of values in $[0,\infty[$, and let $L(\lambda)$ denote the Laplace transform of X_i.*
Let N be a random variable, independent of all the X_i-erne and of values in $\mathbb{N}_0$, and let $P(s)$ be the generating function of N.
Let the random variable Y_N be given by

$$Y_N = X_1 + X_2 + \cdots + X_N$$

(where the number of random variables on the right hand side is itself a random variable).

1. *Prove that Y_N has the Laplace transform $L_{Y_N}(\lambda)$ given by*

$$L_{Y_N}(\lambda) = P(L(\lambda)), \qquad \lambda \geq 0.$$

Assume in particular that all X_i are exponentially distributed of parameter a, and let N be Poisson distributed of parameter b.

2. *Find in this special case $L_{Y_N}(\lambda)$, and the mean and variance of Y_N.*

3. *Find also in this special case the distribution function of Y.*

1) We apply

$$(8)\ P\{Y_N \leq y\} = \sum_{n=0}^{\infty} P\{N=n\} \cdot P\{Y_n \leq y\}.$$

Then

$$\begin{aligned} L_{Y_N}(\lambda) &= \int_0^{\infty} e^{-\lambda y} \frac{d}{dy} P\{Y_N \leq y\}\, dy = \int_0^{\infty} e^{-\lambda y} \sum_{n=0}^{\infty} P\{N=n\} \cdot \frac{d}{dy} P\{Y_n \leq y\}\, dy \\ &= \sum_{n=0}^{\infty} P\{N=n\} \int_0^{\infty} e^{-\lambda y} f_n(y)\, dy = \sum_{n=0}^{\infty} P\{N=n\} \left(\int_0^{\infty} e^{-\lambda y} f(y)\, dy \right)^n \\ &= \sum_{n=0}^{\infty} P\{N=n\} \, (L(\lambda))^n = P(L(\lambda)). \end{aligned}$$

2) When $X_i \in \Gamma\left(1, \frac{1}{a}\right)$, then

$$L(\lambda) = \frac{a}{\lambda + a}.$$

When $N \in P(b)$, then

$$P(s) = \exp(b\{s-1\}).$$

Then it follows from **1.** that

$$L_{Y_N}(\lambda) = P(L(\lambda)) = \exp\left(b \left(\frac{a}{\lambda + a} - 1 \right) \right) = \exp\left(-b \cdot \frac{\lambda}{\lambda + a} \right).$$

Since

$$L'_{Y_N}(\lambda) = -\frac{ba}{(\lambda+a)^2}\ \exp\left(-b\cdot\frac{\lambda}{\lambda+a}\right),$$

we get

$$E\{X\} = -K'_{Y_N}(0) = \frac{ba}{a^2} = \frac{b}{a}.$$

From

$$L''_{Y_N}(\lambda) = \left(\frac{ba}{(\lambda+a)^2}\right)^2 \exp\left(-b\cdot\frac{\lambda}{\lambda+a}\right) + \frac{2ba}{(\lambda+a)^3}\ \exp\left(-b\cdot\frac{\lambda}{\lambda+a}\right),$$

follows that

$$E\left\{X^2\right\} = L''_{Y_N}(0) = \frac{b^2}{a^2} + \frac{2b}{a^2},$$

and we conclude that

$$V\{X\} = \frac{2b}{a^2}.$$

3) This question is underhand, because one is led to consider $L_{Y_N}(\lambda)$, which does not give easy computation. We shall instead apply that if $y > 0$, then

$$G(y) = P\{Y \le y\} = P\{N=0\} + \sum_{k=1}^{\infty} P\{N=k\}\cdot P\left\{X_1+\cdots+X_k \le y\right\}.$$

We see that $G(y)$ has a jump at $y=0$ of the size

$$P\{N=0\}=e^{-b},$$

and that $G(y)$ for $y>0$ is differentiable with the derivative

$$G'(y)=f_{Y_n}(y)=\sum_{n=1}^{\infty} P\{N=n\}\cdot f_{Y_n}(y).$$

Since $N \in P(b)$, we get

$$P\{N=n\}=\frac{b^n}{n!}\,e^{-b}.$$

Since

$$Y_n=\sum_{j=1}^{n} X_j \in \Gamma\left(n,\frac{1}{a}\right),$$

we get

$$f_{Y_n}(y)=\frac{a^n}{(n-1)!}\,y^{n-1}e^{-ay}.$$

Hence, Y has a *jump at* $y=0$ *of the size* e^{-b}, and if $y>0$, then

$$G'(y)=f_{Y_N}(y)=\sum_{n=1}^{\infty}\frac{b^n}{n!}\,e^{-b}\cdot\frac{a^n}{(n-1)!}\,y^n e^{-ay}.$$

Example 5.12 *Let X_1, X_2, X_3, … be mutually independent random variables, all of the distribution given by*

$$P\{X_i = k\} = \frac{a^k}{k!}\,e^{-a}, \qquad k \in \mathbb{N}_0; \quad i \in \mathbb{N}$$

(here a is a positive constant).
Let N be another random variable, which is independent of all the X_i and which has its distribution given by

$$P\{N = n\} = p\,q^{n-1}, \qquad n \in \mathbb{N},$$

where $p > 0$, $q > 0$, $p + q = 1$.

1. *Find the Laplace transform $L(\lambda)$ of the random variable X_1.*

2. *Find the Laplace transform of the random variable $\sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.*

3. *Find the generating function $P(s)$ of the random variable N.*

We introduce another random variable Y by

(9) $Y = X_1 + X_2 + \cdots + X_N$,

where N denotes the random variable introduced above, and where the number of random variables on the right hand side of (9) is also a random variable.

4. *Prove that the random variable Y has its Laplace transform $L_Y(\lambda)$ given by the composite function*

$$L_Y(\lambda) = P(L(\lambda)),$$

and find explicitly $L_Y(\lambda)$.
Hint: *One may use that we have for $k \in \mathbb{N}_0$,*

$$P\{Y = k\} = \sum_{n=1}^{\infty} P\{N = n\} \cdot P\{X_1 + X_2 + \cdots + X_n = k\}.$$

5. *Compute the mean $E\{Y\}$.*

1) The Laplace transform of $X_1 \in P(a)$ is given by

$$L(\lambda) = \sum_{k=0}^{\infty} \frac{a^k}{k!}\,e^{-a} \cdot e^{-k\lambda} = e^{-a} \sum_{k=0}^{\infty} \frac{1}{k!}\,\left(a\,e^{-\lambda}\right)^k = \frac{\exp\left(a\,e^{-\lambda}\right)}{\exp(a)} = \exp\left(a\left\{e^{-\lambda} - 1\right\}\right).$$

2) The Laplace transform of $\sum_{i=1}^{n} X_i$ is given by

$$\{L(\lambda)\}^n = \exp\left(na\left\{e^{-\lambda} - 1\right\}\right).$$

3) The generating function for $N \in \text{Pas}(1,p)$ is found by means of a table,

$$P(s) = \frac{ps}{1-qs}.$$

ALTERNATIVELY,

$$P(s) = p\sum_{n=1}^{\infty} q^{n-1}s^n = ps\sum_{n=1}^{\infty}(qs)^{n-1} = \frac{ps}{1-qs}.$$

4) It follows from

$$P\{Y=k\} = \sum_{n=1}^{\infty} P\{N=n\}\cdot P\left\{X_1+X_2+\cdots+X_n=k\right\},$$

that

$$\begin{aligned} L_Y(\lambda) &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} P\{N=n\}\cdot P\left\{X_1+X_2+\cdots+X_n=k\right\}\cdot e^{-k\lambda} \\ &= \sum_{n=1}^{\infty} P\{N=n\}\sum_{k=0}^{\infty} P\left\{X_1+X_2+\cdots+X_n=k\right\}e^{-\lambda k} \\ &= \sum_{n=1}^{\infty} P\{N=n\}\cdot(L(\lambda))^n = P(L(\lambda)) = \frac{p\cdot\exp\left(a\left(e^{-\lambda}-1\right)\right)}{1-q\cdot\exp\left(a\left(e^{-\lambda}-1\right)\right)} \\ &= \frac{p}{q}\cdot\frac{q\cdot\exp\left(a\left(e^{-\lambda}-1\right)\right)-1+1}{1-q\cdot\exp\left(a\left(e^{-\lambda}-1\right)\right)} = \frac{p}{q}\cdot\frac{1}{1-q\cdot\exp\left(a\left(e^{-\lambda}-1\right)\right)}-\frac{p}{q}. \end{aligned}$$

5) Since

$$L_Y'(\lambda) = -\frac{p}{q}\cdot\frac{1}{\left\{1-q\,\exp\left(a\left(e^{-\lambda}-1\right)\right)\right\}^2}\cdot q\,\exp\left(a\left(e^{-\lambda}-1\right)\right)\cdot a\,e^{-\lambda},$$

the mean is

$$E\{X\} = -L_Y'(0) = \frac{pa}{(1-q)^2} = \frac{pa}{p^2} = \frac{a}{p}.$$

Example 5.13 *Let X_1, X_2, X_3, ... be mutually independent random variables, all with the frequency*

$$f(x) = \begin{cases} 4x\,e^{-2x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Let N be another random variable, which is independent of all the X_i, and which has its distribution given by

$$P\{N = n\} = \frac{3}{4} \cdot \left(\frac{1}{4}\right)^{n-1}, \qquad n \in \mathbb{N}.$$

1. *Find the Laplace transform $L(\lambda)$ of the random variable X_1.*

2. *Find the Laplace transform of the random variable $\sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.*

3. *Find the generating function of the random variable N.*

Then introduce a random variable Y by

(10) $Y = X_1 + X_2 + \cdots + X_N$,

where N denotes the random variable introduced above, and where the number of random variables on the right hand side in (10) also is a random variable.

4. *Find the Laplace transform of Y and the mean $E\{X\}$.*

5. *Prove that the frequency of Y is given by*

$$g(y) = \begin{cases} k\left\{e^{-y} - e^{-3y}\right\}, & y > 0, \\ 0, & y \le 0, \end{cases}$$

and find k.

1) Since $X \in \Gamma\left(2, \frac{1}{2}\right)$, get by using a table that

$$L(\lambda) = \left\{\frac{1}{\frac{1}{2}\lambda + 1}\right\}^2 = \left(\frac{2}{\lambda+2}\right)^2.$$

Alternatively,

$$L(\lambda) = \int_0^\infty 4x\,e^{-2x}e^{-\lambda x}\,dx = 4\int_0^\infty x\,e^{-(\lambda+2)x}\,dx = \frac{4}{(\lambda+2)^2}.$$

2) Since the X_i are mutually independent and identically distributed, the Laplace transform of $\sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$, is given by

$$(L(\lambda))^n = \left(\frac{2}{\lambda+2}\right)^{2n}.$$

3) Since $N \in \text{Pas}\left(1, \dfrac{3}{4}\right)$, we get from a table that the generating function is

$$P(s) = \frac{\frac{3}{4}\,s}{1-\frac{1}{4}\,s} = \frac{3s}{4-s}.$$

ALTERNATIVELY,

$$P(s) = \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} s^n = \frac{3s}{4}\sum_{n=1}^{\infty}\left(\frac{s}{4}\right)^{n-1} = \frac{\frac{3s}{4}}{1-\frac{s}{4}} = \frac{3s}{4-s}.$$

4) The Laplace transform of Y is given by (cf. e.g. the previous examples)

$$\begin{aligned} L_Y(\lambda) &= P(L(\lambda)) = \frac{3\left(\dfrac{2}{\lambda+2}\right)^2}{4-\left(\dfrac{2}{\lambda+2}\right)^2} = \frac{12}{4(\lambda+2)^2-4} \\ &= \frac{3}{(\lambda+1)(\lambda+3)} = \frac{3}{2}\,\frac{1}{\lambda+1} - \frac{3}{2}\,\frac{1}{\lambda+3}. \end{aligned}$$

Now,

$$L'_Y(\lambda) = -\frac{3}{2} \cdot \frac{1}{(\lambda+1)^2} + \frac{3}{2} \cdot \frac{1}{(\lambda+3)^2},$$

so the mean is

$$E\{X\} = -L'_Y(0) = \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{9} = \frac{3}{2} - \frac{1}{6} = \frac{4}{3}.$$

5) Since $g(y)$ is the frequency of some random variable $\tilde{Y}$, where

$$\begin{aligned} L_{\tilde{Y}}(\lambda) &= k \int_0^{\infty} \left\{ e^{-y} - e^{-3y} \right\} e^{-2y}\, dy = k \int_0^{\infty} e^{-(\lambda+1)y}\, dy - k \int_0^{\infty} e^{-(\lambda+3)y}\, dy \\ &= k \left\{ \frac{1}{\lambda+1} - \frac{1}{\lambda+3} \right\} \end{aligned}$$

has the same structure as $L_Y(\lambda)$, we conclude from the uniqueness that $Y = \tilde{Y}$ and that $k = \frac{3}{2}$, and the frequency of Y is $g(y)$ with $k = \frac{3}{2}$.

TEST:

$$\int_{-\infty}^{\infty} g(y)\, dy = k \int_0^{\infty} \left\{ e^{-y} - e^{-3y} \right\}\, dy = k \left\{ 1 - \frac{1}{3} \right\} = \frac{2}{3}\, k = 1$$

for $k = \frac{3}{2}$. $\Diamond$

Example 5.14 *Let X be a normally distributed random variable of mean 0 and variance 1.*

1. *Find the frequency and mean of X^2.*

2. *Find the Laplace transform of X^2.*

Now let X_1, X_2, ... be mutually independent random variables, $X_i \in N(0,1)$, and let a_1, a_2, ... be given constants, and define

$$Y_n = \sum_{k=1}^{n} a_k X_k^2, \qquad n \in \mathbb{N}.$$

3. *Find the Laplace transform of Y_n.*

4. *Prove that the sequence $\{Y_n\}_{n=1}^{\infty}$ converges in distribution towards a random variable Y, if and only if*

$$\lim_{n\to\infty} E\{Y_n\} < \infty.$$

By the assumption the frequency of X is given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \qquad x \in \mathbb{R}.$$

1) The distribution function of $Y = X^2$ is 0 for $y \le 0$.
If $y > 0$, then

$$P\left\{X^2 \le y\right\} = P\{-\sqrt{y} \le X \le \sqrt{y}\} = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\,\Phi(\sqrt{y}) - 1.$$

When $y > 0$, the corresponding frequency is found by differentiation,

$$f(y) = 2\,\Phi'(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{y}}\,\varphi(\sqrt{y}) = \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{1}{2}y\right).$$

The mean is

$$\begin{aligned} E\left\{X^2\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{2}x^2\right) dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} x\,\mathrm{d}\left(-\exp\left(-\frac{1}{2}x^2\right)\right) \\ &= \frac{1}{\sqrt{2\pi}} \left[-x \exp\left(-\frac{1}{2}x^2\right)\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) dx = 0 + 1 = 1. \end{aligned}$$

2) Since $X^2 \ge 0$, we can find its Laplace transform. If $\lambda \ge 0$, then

$$\begin{aligned} L_{X^2}(\lambda) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{1}{2}y\right) \exp(-\lambda y)\,dy = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{1}{2}\left(\lambda + \frac{1}{2}\right) y\right) d\left(\sqrt{y}\right) \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{1}{2} \cdot \frac{t^2}{\frac{1}{2\lambda+1}}\right) dt = \frac{1}{\sqrt{2\lambda+1}} \sqrt{\frac{2\lambda+1}{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(2\lambda+1)t^2\right) dt \\ &= \frac{1}{\sqrt{2\lambda+1}}. \end{aligned}$$

3) We get the Laplace transform of $a\,X^2 = a\,Y_1$ from $L_X(\lambda)$ by replacing λ by $a\lambda$, i.e.

$$LaX^2(\lambda) = L_{X^2}(a\,\lambda) = \frac{1}{\sqrt{2\lambda a+1}}.$$

Now, the X_k are mutually randomly independent, so

$$L_{Y_n}(\lambda) = \prod_{k=1}^{n} L_{a_k X_k^2}(\lambda) = \prod_{k=1}^{n} L_{X^2}\left(a_k\lambda\right) = \frac{1}{\sqrt{\prod_{k=1}^{n}\left(1+2\lambda a_k\right)}}.$$

4) We get by using the result of **1.**,

$$E\left\{Y_n\right\} = \sum_{k=1}^{n} a_k E\left\{X_k^2\right\} = \sum_{k=1}^{\infty} a_k,$$

thus

$$\lim_{n\to\infty} E\left\{Y_n\right\} = \sum_{k=1}^{\infty} a_k.$$

Then we get for $\lambda \geq 0$,

$$\ln \prod_{k=1}^{n}\left(1+2\lambda a_k\right) = \sum_{k=1}^{n} \ln\left(1+2\lambda\,a_k\right) = \sum_{k=1}^{n}\left(2\lambda a_k + \lambda a_k \varepsilon\left(\lambda a_k\right)\right),$$

where we by considering a graph can get more precisely that

$$0 \leq \sum_{k=1}^{n} \ln\left(1+2\lambda a_k\right) \leq \sum_{k=1}^{n} 2\lambda\,a_k,$$

and

$$\sum_{k=1}^{\infty} \ln\left(1+2\lambda\,a_k\right) \sim \sum_{k=1}^{\infty} 2\lambda\,a_k.$$

It follows from the equivalence of the two series that

$$1 \leq \prod_{k=1}^{\infty}\left(1+2\lambda\,a_k\right) < \infty, \quad \text{if and only if} \quad \sum_{k=1}^{\infty} a_k < \infty.$$

If therefore

$$\lim_{n\to\infty} E\left\{Y_n\right\} < \infty,$$

then in particular $\lim_{n\to\infty}\left\{-L_{Y_n}'(\lambda)\right\}$ is convergent and continuous for $\lambda \geq 0$, hence by rewriting the expression, followed by a reduction, $\sum_{k=1}^{\infty} a_k < \infty$, which according to the above implies that

$$\lim_{n\to\infty} L_{Y_n}(\lambda) = \frac{1}{\sqrt{\prod_{k=1}^{\infty}\left(1+2\lambda\,a_k\right)}}$$

is continuous for $\lambda \geq 0$. Then (Y_n) converges in distribution towards a random variable Y.

Conversely, if $\lim_{n\to\infty} E\{Y_n\} = \infty$, then we get by the same argument that $\sum_{k=1}^{\infty} a_k = \infty$ implies that $\prod_{k=1}^{\infty}(1+2\lambda\, a_k) = \infty$ for $\lambda > 0$, and of course 1 for $\lambda = 0$, hence

$$\lim_{n\to\infty} L_{Y_n}(\lambda) = \begin{cases} 1 & \text{for } \lambda = 0, \\ \\ 0 & \text{for } \lambda > 0, \end{cases}$$

corresponding to the zero function, which is not the Laplace transform of any random variable. This shows that (X_n) does not converge in distribution.

Example 5.15 *We say that a function $\varphi :]0,\infty[\to \mathbb{R}$ is* completely monotone*, if φ is a C^{∞} function, and*

$(-1)^n \varphi^{(n)}(\lambda) \geq 0$ *for every $n \in \mathbb{N}_0$ and every $\lambda > 0$.*

Prove that if X is a non-negative random variable, then the Laplace transform $L(\lambda)$ of X is completely monotone.

Remark 5.1 Conversely, it can be proved that if $\varphi :]0,\infty[\to \mathbb{R}$ is completely monotone, and

$\lambda_{\lambda\to 0+}\varphi(\lambda) = 1$,

then $\varphi(\lambda)$ is the Laplace transform of some random variable X.

When X is non-negative, its Laplace transform exists, and

1) $L(\lambda) = \sum_i p_i e^{-\lambda x_i}$, (discrete),
2) $L(\lambda) = \int_0^\infty e^{-\lambda x} f(x)\, dx$, (continuous),
3) $L(\lambda) = E\left\{e^{-\lambda x}\right\}$, (in general).

Due to the exponential function and the law of magnitudes we may for $\lambda > 0$ differentiate 1) under the sum, 2) under the integral, and 3) under the symbol E, with respect to λ. Hence we get in general [i.e. in case 3)] for $\lambda > 0$ and $n \in \mathbb{N}_0$,

$$(-1)^n L^{(n)}(\lambda) = \lambda^n E\left\{X^n e^{-\lambda X}\right\}.$$

Since $X^n e^{-\lambda X} \geq 0$, the right hand side is always ≥ 0, and the claim is proved.

Clearly,

$$L(0) = \lim_{\lambda \to 0+} L(\lambda) = \lim_{\lambda \to 0+} E\left\{e^{-\lambda X}\right\} = E\{1\} = 1,$$

and

$$0 < L(\lambda) = E\left\{e^{-\lambda X}\right\} \leq E\{1\} = 1,$$

because $0 \leq e^{-\lambda X} \leq 1$, når $X \geq 0$.

A loose argument shows that the last claim follows from the fact, that if $(-1)^n \varphi^{(n)}(\lambda) \geq 0$ for all $n \in \mathbb{N}$, then we get in e.g. the continuous case that

$$\int_0^\infty e^{-\lambda x} x^n f(x)\, dx \geq 0 \text{ for all } \lambda > 0 \text{ and all } n \in \mathbb{N}_0,$$

thus

$$x^n f(x) \geq 0 \text{ for all } n \in \mathbb{N}_0 \text{ and } x \geq 0,$$

and hence $f(x) \geq 0$. Finally,

$$\int_0^\infty f(x)\, dx = \lim_{\lambda \to 0+} \varphi(\lambda) = 1.$$

Example 5.16 *A random variable X has the values 2, 3, 4, ... of the probabilities*

$$P\{X = k\} = (k-1)p^2(1-p)^{k-2},$$

where $0 < p < 1$, thus $X \in \text{Pas}(2,p)$.

1. *Find the generating function and the Laplace transform of X.*

2. *Find the mean of X.*

Given a sequence of random variable $(X_n)_{n=1}^{\infty}$, where X_n has the values $\frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \dots$ of the probabilities

$$P\left\{X_n = \frac{k}{n}\right\} = (k-1)\left(\frac{1}{3n}\right)^2\left(1-\frac{1}{3n}\right)^{k-2}.$$

3. *Find the Laplace transform of X_n.*

4. *Prove that the sequence (X_n) converges in distribution towards a random variable Y, which is Gamma distributed, and find its frequency of Y.*

1) The generating function of X is given by

$$\begin{aligned} P(s) &= \sum_{k=2}^{\infty} P\{X=k\}s^k = \sum_{k=2}^{\infty}(k-1)p^2(1-p)^{k-2}s^k \\ &= p^2s^2\sum_{k=2}^{\infty}(k-1)\{(1-p)s\}^{k-2} = p^2s^2\sum_{\ell=1}^{\infty}\ell\{(1-p)s\}^{\ell-1} \\ &= p^2s^2\cdot\frac{1}{\{1-(1-p)s\}^2} = \left\{\frac{ps}{1-(1-p)s}\right\}^2 \qquad \text{for } s\in[0,1]. \end{aligned}$$

Then by a simple substitution,

$$L(\lambda) = P\left(e^{-\lambda}\right) = \left\{\frac{p\,e^{-\lambda}}{1-(1-p)e^{-\lambda}}\right\}^2 = \left\{\frac{p}{e^{\lambda}-(1-p)}\right\}^2.$$

2) Here there are several possibilities, of which we indicate four:

First variant. It follows from

$$P'(s) = 2\cdot\frac{ps}{1-(1-p)s}\cdot\frac{\{1-(1-p)s\}p + p(1-p)s}{\{1-(1-p)s\}^2},$$

that

$$E\{X\} = P'(1) = 2\cdot 1\cdot\frac{p}{p^2} = \frac{2}{p}.$$

Second variant. It follows from

$$L'(\lambda)=p^2(-1)\left\{e^{\lambda}-(1-p)\right\}^{-3}\cdot e^{\lambda},$$

that

$$E\{X\}=-L'(0)=\frac{2p^2}{p^3}=\frac{2}{p}.$$

Third variant. By a straightforward computation,

$$\begin{aligned}E\{X\} &= \sum_{k=2}^{\infty} k\,P\{X=k\}=\sum_{k=2}^{\infty}k(k-1)p^2(1-p)^{k-2}\\ &= p^2\sum_{k=2}^{\infty}k(k-1)(1-p)^{k-2}=p_2\cdot\frac{2}{\{1-(1-p)\}^3}=\frac{2}{p}.\end{aligned}$$

Fourth variant. (The easiest one!) Since $X\in \operatorname{Pas}(2,p)$, er have of course $E\{X\}=\dfrac{2}{p}$.

3) If we put $p=\dfrac{1}{3n}$, then nX_n has the same distribution as X. Now, X_n is obtained by diminishing the values by a factor $\dfrac{1}{n}$, so X_n has the Laplace transform

$$L_{X_n}(\lambda)=\left\{\frac{\frac{1}{3n}}{e^{\lambda/n}-\left(1-\frac{1}{3n}\right)}\right\}^2=\frac{1}{\left\{3n\left(\exp\left(\frac{\lambda}{n}\right)-1\right)+1\right\}^2}.$$

4) It follows from

$$\exp\left(\frac{\lambda}{n}\right)=1+\frac{\lambda}{n}+\frac{\lambda}{n}\,\varepsilon\left(\frac{\lambda}{n}\right),$$

that

$$L_{X_n}(\lambda)=\frac{1}{\left\{3n\left(\frac{\lambda}{n}+\frac{\lambda}{n}\,\varepsilon\left(\frac{\lambda}{n}\right)\right)+1\right\}^2}=\frac{1}{\left\{3\lambda+3\lambda\,\varepsilon\left(\frac{\lambda}{n}\right)+1\right\}^2}\to\frac{1}{(3\lambda+1)^2}\qquad\text{for } \lambda\geq 0.$$

Clearly, the limit function is continuous, so it follows that the sequence (X_n) converges in distribution towards Y, where Y has the Laplace transform

$$L_Y(\lambda)=\frac{1}{(3\lambda+1)^2},\qquad \lambda\geq 0.$$

If $Y\in\Gamma(\mu,\alpha)$, then its Laplace transform is

$$\frac{1}{(\alpha\lambda+1)^{\mu}}.$$

Then by comparison $\alpha=3$ and $\mu=2$, so $Y\in\Gamma(2,3)$, and Y has the frequency

$$f(y)=\begin{cases}\dfrac{1}{9}\,y\,\exp\left(-\dfrac{y}{3}\right), & y>0,\\[2mm] 0, & y\leq 0.\end{cases}$$

Example 5.17 *A random variable X has the values 0, 2, 4, … of the probabilities*

$$P\{X=2k\}=p(1-p)^k, \qquad k\in\mathbb{N}_0,$$

where p is a constant, $0<p<1$.

1. *Find the Laplace transform $L_X(\lambda)$ of the random variable X.*

2. *Find the mean of the random variable X.*

A sequence of random variables $(X_n)_{n=1}^{\infty}$ is determined by that X_n has the values 0, $\frac{2}{n}$, $\frac{4}{n}$, … of the probabilities

$$P\left\{X_n=\frac{2k}{n}\right\}=\frac{1}{4n}\left(1-\frac{1}{4n}\right)^k, \qquad k\in\mathbb{N}_0.$$

3. *Find the Laplace transform $L_{X_n}(\lambda)$ of the random variable X_n.*

4. *Find the mean of the random variable X_n.*

5. *Prove that the sequence (X_n) converges in distribution towards a random variable Y, and find the distribution function of Y.*

1) The Laplace transform is

$$\begin{aligned} L_X(\lambda) &= \sum_{k=0}^{\infty} P\{X=2k\}\,e^{-2\lambda k}=\sum_{k=0}^{\infty} p(1-p)^k e^{-2\lambda k} \\ &= p\sum_{k=0}^{\infty}\left\{(1-p)e^{-2\lambda}\right\}^k=\frac{p}{1-(1-p)e^{-2\lambda}}, \qquad \lambda\ge 0. \end{aligned}$$

2) The mean can be found in two ways:

a) By the usual definition,

$$E\{X\}=\sum_{k=1}^{\infty}2kp(1-p)^k=2p(1-p)\sum_{k=1}^{\infty}k(1-p)^{k-1}=2p(1-p)\,\frac{1}{p^2}=2\,\frac{1-p}{p}.$$

b) By means of the Laplace transform,

$$E\{X\}=-L_X'(0)=\left[\frac{p}{\{1-(1-p)e^{-2\lambda}\}^2}\cdot 2(1-p)e^{-2\lambda}\right]_{\lambda=0}=\frac{2p(1-p)}{p^2}=2\,\frac{1-p}{p}.$$

3) The Laplace transform of X_n is obtained from the Laplace transform of X by replacing λ by $\frac{\lambda}{n}$, and p by $\frac{1}{4n}$,

$$L_{X_n}(\lambda)=\frac{\frac{1}{4n}}{1-\left(1-\frac{1}{4n}\right)\exp\left(-2\,\frac{\lambda}{n}\right)}=\frac{1}{4n\left(1-\exp\left(-\frac{2\lambda}{n}\right)\right)+\exp\left(-2\,\frac{\lambda}{n}\right)}.$$

4) Since

$$-L'_{X_n}(\lambda) = \frac{8\exp\left(-\frac{2\lambda}{n}\right) - \frac{2}{n}\exp\left(-\frac{2\lambda}{n}\right)}{\left\{4n\left(1-\exp\left(-2\,\frac{\lambda}{n}\right)\right)+\exp\left(-2\,\frac{\lambda}{n}\right)\right\}^2},$$

we get the mean

$$E\{X\} = -L'_{X_n}(0) = \frac{1}{\{0+1\}^2}\cdot\left\{8-\frac{2}{n}\right\} = 8\left(1-\frac{1}{4n}\right).$$

5) Then by a Taylor expansion, $e^t = 10t + t\,\varepsilon(t)$, so it follows from **3.** that

$$\begin{aligned} L_{X_n}(\lambda) &= \frac{1}{4n\left\{1-1+\frac{2\lambda}{n}+\frac{2\lambda}{n}\,\varepsilon\left(\frac{2\lambda}{n}\right)\right\}+\exp\left(-2\,\frac{\lambda}{n}\right)} = \frac{1}{8\lambda+8\lambda\,\varepsilon\left(\frac{2\lambda}{n}\right)+\exp\left(-2\,\frac{\lambda}{n}\right)} \\ &\to \frac{1}{8\lambda+1} \qquad \text{for } n\to\infty. \end{aligned}$$

Since $\frac{1}{8\lambda+1}$ is continuous, this shows that (X_n) converges in distribution towards a random variable Y, where the Laplace transform of Y is $L_Y(\lambda) = \frac{1}{8\lambda+1}$, hence

$$Y \in \Gamma(1,8).$$

Thus the frequency of Y is

$$f_Y(y) = \begin{cases} \frac{1}{8}\exp\left(-\frac{y}{8}\right), & y>0, \\ 0, & y\le 0, \end{cases}$$

so we have obtained an exponential distribution.

6 The characteristic function

Example 6.1 *Find the characteristic function for a random variable, which is Poisson distributed of mean a.*

It follows from

$$P\{X = k\} = \frac{a^k}{k!}\, e^{-a}, \qquad k \in \mathbb{N}_0,$$

that the characteristic function for X is given by

$$k(\omega) = \sum_{k=0}^{\infty} e^{i\omega k}\, \frac{a^k}{k!}\, e^{-a} = e^{-a} \sum_{k=0}^{\infty} \frac{1}{k!}\, \left\{a\, e^{i\omega}\right\}^k = e^{-a} \exp\left(a\, e^{i\omega}\right) = \exp\left(a\left(e^{i\omega} - 1\right)\right).$$

Example 6.2 *Let X have the frequency*

$$f(x)=\begin{cases}1-|x|, & |x|<1,\\[2mm] 0, & |x|\ge 1.\end{cases}$$

Find the characteristic function for X.
Let X_1 and X_2 be independent random variables, which are rectangularly distributed over $\left]-\frac{1}{2},\frac{1}{2}\right[$.
Prove that X has the same distribution as X_1+X_2,

1) by a straightforward computation of the frequency of X_1+X_2,

2) by using characteristic functions.

The characteristic function for $\omega\neq 0$ is

$$\begin{aligned}k(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t}f(t)\,dt=\int_{-1}^{1}\{\cos\omega x+i\,\sin\omega x\}(1-|x|)\,dx=2\int_0^1\cos\omega x\cdot(1-|x|)\,dx\\ &= 2\left[\frac{1}{\omega}(1-x)\sin\omega x\right]_0^1+\frac{2}{\omega}\int_0^1\sin\omega x\,dx=\frac{2}{\omega}\left[-\frac{\cos\omega x}{\omega}\right]_0^1=\frac{2}{\omega^2}(1-\cos\omega).\end{aligned}$$

If $\omega=0$, then $k(0)=1$.

1) The frequency for both X_1 and X_2 is given by

$$f(t)=\begin{cases}1 & \text{for } t\in\left]-\frac{1}{2},\frac{1}{2}\right[,\\[2mm] 0 & \text{otherwise,}\end{cases}$$

hence the frequency of X_1+X_2 is given by

$$g(s)=\int_{-\infty}^{\infty}f(t)f(s-t)\,dt=\int_{-\frac{1}{2}}^{\frac{1}{2}}f(s-t)\,dt.$$

If $s\notin\,]-1,0[$, then $g(s)=0$.
If $s\in\,]-1,0]$, then

$$g(s)=\int_{-\frac{1}{2}}^{\frac{1}{2}}f(s-t)\,dt=\int_{-\frac{1}{2}}^{s+\frac{1}{2}}1\,dt=s+1=1-|s|.$$

If $s\in\,]0,1[$, then

$$g(s)=\int_{-\frac{1}{2}}^{\frac{1}{2}}f(s-t)\,dt=\int_{s-\frac{1}{2}}^{\frac{1}{2}}1\,dt=1-s=1-|s|,$$

and the claim follows.

2) If $\omega \neq 0$, then we get the characteristic function for X_i,

$$\begin{aligned} h(\omega) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\omega t}\,dt = \frac{1}{i\omega}\left\{\exp\left(i\,\frac{\omega}{2}\right) - \exp\left(-i\,\frac{\omega}{2}\right)\right\} \\ &= \frac{2}{\omega}\cdot\frac{1}{2i}\left\{\exp\left(i\,\frac{\omega}{2}\right) - \exp\left(-i\,\frac{\omega}{2}\right)\right\} = \frac{2}{\omega}\,\sin\frac{\omega}{2}. \end{aligned}$$

Hence, the characteristic function for $X_1 + X_2$ is

$$\{h(\omega)\}^2 = \frac{4}{\omega^2}\,\sin^2\frac{\omega}{2} = \frac{4}{\omega^2}\cdot\frac{1-\cos\omega}{2} = \frac{2}{\omega^2}\,(1-\cos\omega) = k(\omega).$$

Since X and $X_1 + X_2$ have the same characteristic function, they are identical.

Example 6.3 *Let X have the frequency*

$$f(x) = \frac{a}{\pi\left(a^2+x^2\right)}, \qquad x \in \mathbb{R},$$

where a is a positive constant.
Prove by applying the inversion formula that X has the characteristic function

$$k(\omega) = e^{-a|\omega|}.$$

Then prove that if $X_1, X_2, \dots, X_n$ are mutually independent all of the frequency $f(x)$, then

$$Z_n = \frac{1}{n}\,(X_1 + \cdots + X_n)$$

also has the frequency $f(x)$.

When we apply the inversion formula on $k(\omega)$, we get

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}e^{-a|\omega|}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{(a-ix)\omega}\,d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-(a+ix)\omega}\,d\omega \\ &\quad= \frac{1}{2\pi}\left[\frac{e^{(a-ix)\omega}}{a-ix}\right]_{-\omega}^{0} + \frac{1}{2\pi}\left[\frac{e^{-(a+ix)\omega}}{a-ix}\right]_{0}^{\infty} = \frac{1}{2\pi}\left(\frac{1}{a-ix}+\frac{1}{a+ix}\right) \\ &\quad= \frac{1}{2\pi}\cdot\frac{a+ix+a-ix}{a^2+x^2} = \frac{a}{\pi\left(a^2+x^2\right)}, \end{aligned}$$

and the claim follows from the uniqueness of the characteristic function.

The characteristic function for

$$Y_n = \frac{1}{n}\,(X_1 + \cdots + X_n)$$

is

$$k_{Y_n}(\omega) = \prod_{i=1}^{n} k_i\left(\frac{\omega}{n}\right) = \prod_{i=1}^{n}\exp\left(-a\left|\frac{\omega}{n}\right|\right) = e^{-a|\omega|} = k_X(\omega),$$

showing that Y_n has the same frequency as X.

Example 6.4 *Let X_1, X_2, ... be mutually independent, identically distributed random variables all of mean μ. Let*

$$Z_n = \frac{1}{n}\left(X_1 + \cdots + X_n\right), \qquad n \in \mathbb{N}.$$

Prove that the sequence (Z_n) converges in distribution towards μ.

Given $\mu = E\{X\}$ exists, we must have the following

$$(11) \quad \int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty,$$

which shall be used later.
Let $k(\omega)$ denote the characteristic function for X_i. Then the characteristic function for Z_n is given by

$$k_n(\omega) = \left\{k\left(\frac{\omega}{n}\right)\right\}^n .$$

It follows from (11) that

$$k(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\, d(x)\, dx \quad \text{and} \quad k'(\omega) = i\int_{-\infty}^{\infty} e^{i\omega x}\, x\, f(x)\, dx$$

are both defined and bounded.

It follows from

$$k(\omega)=k(0)+\frac{1}{1!}\,k'(0)\,\omega+\omega\,\varepsilon(\omega)=1+i\,\mu\cdot\omega+\omega\,\varepsilon(\omega),$$

that

$$k_n(\omega)=\left\{k\left(\frac{\omega}{n}\right)\right\}^n=\left\{1+\frac{i\,\mu\,\omega}{n}+\frac{\omega}{n}\,\varepsilon\left(\frac{\omega}{n}\right)\right\}^n=\left(1+\frac{1}{n}\left\{i\,\mu\,\omega+\omega\,\varepsilon\left(\frac{\omega}{n}\right)\right\}\right)^n.$$

Hence, by taking the limit,

$$\lim_{n\to\infty}k_n(\omega)=e^{i\,\mu\,\omega},$$

which is the characteristic function for the causal distribution μ.
In particular, $e^{i\,\mu\,\omega}$ is continuous at $\omega=0$. Hence it follows that the sequence (Z_n) converges in distribution towards μ.

Example 6.5 *Let X have the mean 0 and variance σ^2.*
Prove that

$$k(\omega)=1-\frac{1}{2}\,\sigma^2\omega^2+\omega^2\varepsilon(\omega)\qquad \textit{for } \omega\to 0.$$

Then prove the following special case of the Central Limit Theorem:
Let $X_1,\,_2,\,\dots$ be mutually independent, identically distributed random variables of mean 0 and variance σ^2. Define

$$Z_n=\frac{1}{2}\sigma\,\sqrt{n}\,\left(X_1+\cdots+X_n\right),\qquad n\in\mathbb{N}.$$

Then for every $z\in\mathbb{R}$,

$$P\left\{Z_n\le z\right\}\to\Phi(z)\qquad \textit{for } n\to\infty.$$

We see that

$$\begin{aligned}k(\omega)&=\textstyle\int_{-\infty}^{\infty}e^{i\omega x}f(x)\,dx,\\ k('\omega)&=\textstyle\int_{-\infty}^{\infty}e^{i\omega x}i\,x\,f(x)\,dx,\\ k''(\omega)&=-\textstyle\int_{-\infty}^{\infty}x^2e^{i\omega x}f(x)\,dx,\end{aligned}$$

are all absolutely convergent, and

$$k(0)=1,\qquad\qquad k'(0)=i\int_{-\infty}^{\infty}x\,f(x)\,dx=i\,\mu=0,$$

$$k''(0)=-\int_{-\infty}^{\infty}x^2f(x)\,dx=-\int_{-\infty}^{\infty}(x-\mu)^2f(x)\,dx=\sigma^2,$$

hence by a Taylor expansion,

$$\begin{aligned}k(\omega)&=k(0)+\frac{1}{1!}\,k'(0)\,\omega+\frac{1}{2!}\,k''(0)\,\omega^2+\omega^2\varepsilon(\omega)\\ &=1-\frac{\sigma^2\omega^2}{2}+\omega^2\varepsilon(\omega).\end{aligned}$$

The characteristic function $k_n(\omega)$ for Z_n is given by

$$\begin{aligned} k_n(\omega) &= E\left\{e^{i\omega Z_n}\right\} = E\left\{\exp\left(i\,\omega\sum_{k=1}^{n}\frac{1}{\sigma\sqrt{n}}\,X_n\right)\right\} = \prod_{k=1}^{n} E\left\{\exp\left(\frac{i\,\omega}{\sigma\sqrt{n}}\,X_k\right)\right\} \\ &= \left(E\left\{\exp\left(\frac{i\,\omega}{\sigma\sqrt{n}}\,X\right)\right\}\right)^n, \end{aligned}$$

where

$$\begin{aligned} E\left\{\exp\left(\frac{i\,\omega}{\sigma\sqrt{n}}\,X\right)\right\} &= \int_{-\infty}^{\infty}\exp\left(i\,\omega\,\frac{x}{\sigma\sqrt{n}}\right) f(x)\,dx = k\left(\frac{\omega}{\sigma\sqrt{n}}\right) = 1-\frac{\sigma^2}{2}\,\frac{\omega^2}{\sigma^2 n}+\frac{\omega^2}{\sigma^2 n}\,\varepsilon\left(\frac{\omega}{\sigma\sqrt{n}}\right) \\ &= 1-\frac{1}{n}\cdot\frac{\omega^2}{2}+\frac{\omega^2}{\sigma^2 n}\,\varepsilon\left(\frac{\omega}{\sigma\sqrt{n}}\right). \end{aligned}$$

Hence by insertion,

$$\begin{aligned} k_n(\omega) &= \left\{1-\frac{\sigma^2}{2}\,\frac{\omega^2}{\sigma^2 n}+\frac{\omega^2}{\sigma^2 n}\,\varepsilon\left(\frac{\omega}{\sigma\sqrt{n}}\right) = 1-\frac{1}{n}\cdot\frac{\omega^2}{2}+\frac{\omega^2}{\sigma^2 n}\,\varepsilon\left(\frac{\omega}{\sigma\sqrt{n}}\right)\right\}^n \\ &\to \exp\left(-\frac{\omega^2}{2}\right) \quad \text{for } n\to\infty. \end{aligned}$$

Now, $\exp\left(-\frac{\omega^2}{2}\right)$ is the characteristic function for $\Phi(x)$, so we conclude that (Z_n) converges in distribution towards the normal distribution,

$$\lim_{n\to\infty} P\left\{Z_n \le x\right\} = \Phi(x).$$

Example 6.6 *1) A random variable X has the frequency*

$$f(x) = \frac{1}{\pi\left(1+x^2\right)}, \qquad x \in \mathbb{R}.$$

Prove by e.g. applying the inversion formula that X has the characteristic function

$$k(\omega) = e^{-|\omega|}.$$

2) A random variable Y has the frequency

$$g(y) = \frac{a}{\pi\left(a^2+(y-b)^2\right)}, \qquad y \in \mathbb{R},$$

where $a > 0$ and $b \in \mathbb{R}$. Find the characteristic function for Y.

3) Let (Y_j) be a sequence of mutually independent random variables, where each random variable Y_j has the frequency

$$g_j(y) = \frac{a_j}{\pi\left(a_j^2+(y-b_j)^2\right)}, \qquad y \in \mathbb{R},$$

where $a_j > 0$ and $b_j \in \mathbb{R}$, and let Z_n denote the random variable

$$Z_n = \sum_{j=1}^{n} Y_j.$$

Find the characteristic function for Z_n.

4) Find a necessary and sufficient condition, which the constants a_j and b_j must fulfil in order that the sequence $(Z_n)_{n=1}^{\infty}$ converges in distribution. In case of convergence, find the limit distribution.

1) It follows by the inversion formula that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,\omega\,x} e^{-|\omega|}\,d\omega &= \frac{1}{2\pi}\int_{-\omega}^{0} e^{(1-ix)\omega}\,d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(1+ix)\omega}\,d\omega \\
&= \frac{1}{2\pi}\left[\frac{e^{(1-ix)\omega}}{1-ix}\right]_{-\omega}^{0} + \frac{1}{2\pi}\left[\frac{e^{-(1+ix)\omega}}{-(1+ix)}\right]_{0}^{\infty} = \frac{1}{2\pi}\left\{\frac{1}{1-ix}+\frac{1}{1+ix}\right\} \\
&= \frac{1}{2\pi}\cdot\frac{1+ix+1-ix}{1+x^2}\frac{1}{\pi}\cdot\frac{1}{1+x^2} = f(x).
\end{aligned}$$

This shows that $k(\omega) = e^{-|\omega|}$ is the characteristic function for

$$f(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2}.$$

2) The characteristic function for Y is

$$\begin{aligned}
k_Y(\omega) &= \int_{-\infty}^{\infty} e^{i\omega y}\cdot\frac{1}{\pi}\cdot\frac{a}{a^2+(y-b)^2}\,dy = e^{i\omega b}\int_{-\infty}^{\infty} e^{i\omega y}\cdot\frac{1}{\pi}\cdot\frac{a}{a^2+y^2}\,dy \\
&= e^{i\omega b}\int_{-\infty}^{\infty} e^{i\,a\,\omega\cdot\frac{1}{a}\,y}\cdot\frac{1}{\pi}\cdot\frac{1}{1+\left(\frac{y}{a}\right)^2}\,d\left(\frac{y}{a}\right) = e^{i\omega\,b}k(a\omega) = e^{i\omega\,b}e^{-a|\omega|}.
\end{aligned}$$

3) It follows from **2.** that

$$k_{Z_n}(\omega) = \prod_{j=1}^{n} k_{Y_j}(\omega) = \prod_{j=1}^{n} e^{i\,\omega\,b_j} \cdot e^{-a_j|\omega|} = \exp\left(i\,\omega \sum_{j=1}^{n} b_j\right) \cdot \exp\left(-|\omega| \sum_{j=1}^{n} a_j\right).$$

4) The sequence (Z_n) converges min distribution if and only if $\lim_{n\to\infty} k_{Z_n}(\omega)$ is convergent for all ω with a limit function $h(\omega)$, d which is continuous at 0.
Clearly, the only possible *candidate* is

$$h(\omega) = \exp\left(i\,\omega \sum_{n=1}^{\infty} b_n\right) \cdot \exp\left(-|\omega| \sum_{n=1}^{\infty} a_n\right).$$

It is in fact the limit function, if the right hand side is convergent for every $\omega \in \mathbb{R}$. This is fulfilled, if and only if

$$(12) \quad \sum_{n=1}^{\infty} a_n = a \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n = b$$

are both convergent. When this is the case, then

$$h(\omega) = e^{i\,\omega\,b} e^{-a|\omega|} = k_Y(\omega)$$

by **2.**.
This shows that (Z_n) converges in distribution towards a random variable Y, if and only if the series of (12) are convergent, and when this is the case, the frequency of Y is

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (y-b)^2}, \qquad y \in \mathbb{R}.$$

Example 6.7 *Let $X_1, X_2, \dots$ be mutually independent random variables. where*

$$P\left\{X_j = \sqrt{j}\right\} = P\left\{X_j = -\sqrt{j}\right\} = \frac{1}{2}, \qquad j \in \mathbb{N},$$

and let

$$Z_n = \frac{1}{n}\sum_{j=1}^{n} X_j, \qquad n \in \mathbb{N}.$$

Prove that the sequence $(Z_n)_{n=1}^{\infty}$ converges in distribution, and find the limit distribution

1) either by applying the Central Limit Theorem;

2) or by computing $\lim_{n\to\infty} k_n(\omega)$, where $k_n(\omega)$ is the characteristic function for Z_n.

HINT: *Use that*

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\,\varepsilon(x) \qquad \textit{for } x \to 0$$

and

$$-\ln(1-x) = x + \frac{x^2}{2} + x^2\,\varepsilon(x) \qquad \textit{for } x \to 0.$$

1) From $E\{X_j\} = 0$ follows that

$$E\{Z_n\} = \frac{1}{n}\sum_{j=1}^{n} E\{X_j\} = 0,$$

and

$$\begin{aligned} s_n^2 &= V\{Z_n\} = \frac{1}{n^2}\sum_{j=1}^{n} V\{X_j\} = \frac{1}{n^2}\sum_{j=1}^{n}\left(E\left\{X_j^2\right\} - \left(E\{X_j\}\right)^2\right) = \frac{1}{n^2}\sum_{j=1}^{n} E\left\{X_j^2\right\} \\ &= \frac{1}{n^2}\sum_{j=1}^{n}\left\{\left(\sqrt{j}\right)^2\cdot\frac{1}{2} + \left(-\sqrt{j}\right)^2\cdot\frac{1}{2}\right\} = \frac{1}{n^2}\sum_{j=1}^{n} j = \frac{1}{n^2}\cdot\frac{1}{2}\,n(n+1) = \frac{1}{2}\,\frac{n+1}{n}. \end{aligned}$$

Now,

$$\frac{Z_n - E\{Z_n\}}{s_n} = \frac{Z_n}{\sqrt{\dfrac{n+1}{n}}} = \sqrt{\frac{2n}{n+1}}\cdot Z_n,$$

so by the Central Limit Theorem,

$$\lim_{n\to\infty} P\left\{Z_n \le x\sqrt{\frac{n+1}{2n}}\right\} = \Phi(x) \qquad \text{for every } x \in \mathbb{R}.$$

We get from $\sqrt{\dfrac{n+1}{2n}} \to \dfrac{1}{\sqrt{2}}$ for $n \to \infty$ that

$$F_Z(x) = \lim_{n\to\infty} P\{Z_n \le x\} = \Phi\left(\sqrt{2}\cdot x\right),$$

hence $Z = \dfrac{1}{\sqrt{2}}\,Y$, where $Y \in N(0,1)$.

2) It follows from

$$\begin{aligned} k_{Z_n}(\omega) &= \prod_{j=1}^n E\left\{\exp\left(i\,\frac{\omega}{n}\,X_j\right)\right\} = \prod_{j=1}^n \left\{\frac{1}{2}\,\exp\left(i\,\frac{\omega}{n}\,\sqrt{j}\right) + \frac{1}{2}\,\exp\left(i\,\frac{\omega}{n}\,\left(-\sqrt{j}\right)\right)\right\} \\ &= \prod_{j=1}^n \cos\left(\frac{\sqrt{j}}{n}\,\omega\right), \end{aligned}$$

by taking the logarithm and using the Taylor expansions given in the hint,

$$\begin{aligned} \ln k_{Z_n}(\omega) &= \sum_{j=1}^n \ln\left(\cos\left(\frac{\sqrt{j}}{n}\,\omega\right)\right) \\ &= \sum_{j=1}^n \ln\left\{1 - \frac{1}{2}\left(\frac{\sqrt{j}\cdot\omega}{n}\right)^2 + \frac{1}{24}\left(\frac{\sqrt{j}\cdot\omega}{n}\right)^4 + \left(\frac{\sqrt{j}\cdot\omega}{n}\right)^4 \varepsilon\left(\frac{\sqrt{j}\cdot\omega}{n}\right)\right\} \\ &= \sum_{j=1}^n \ln\left\{1 - \frac{\omega^2}{2}\cdot\frac{j}{n^2} + \frac{\omega^4}{24}\cdot\frac{j^2}{n^4} + \omega^4\cdot\frac{j^2}{n^4}\,\varepsilon\left(\frac{\sqrt{j}\cdot\omega}{n}\right)\right\} \\ &= -\sum_{j=1}^n \left\{\left(\frac{\omega^2}{2}\cdot\frac{j}{n^2} - \frac{\omega^4}{24}\cdot\frac{j^2}{n^4} + \frac{\omega^4 j^2}{n^4}\,\varepsilon\left(\frac{\sqrt{j}\,\omega}{n}\right)\right)\right. \\ &\qquad \left. + \frac{1}{2}\left(\frac{\omega^2}{2}\cdot\frac{j^2}{n^4} - \frac{\omega^4}{24}\cdot\frac{j^2}{n^4} + \frac{\omega^4 j^2}{n^4}\,\varepsilon\left(\frac{\omega\sqrt{j}}{n}\right)\right)^2 + \frac{\omega^4 j^2}{n^4}\,\varepsilon\left(\frac{\omega\sqrt{j}}{n}\right)\right\} \\ &= -\sum_{j=1}^n \frac{\omega^2}{2n^2}\cdot j + \frac{\omega^4}{24n^4}\sum_{j=1}^n j^2 + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right) + \frac{1}{2}\,\frac{\omega^4}{4n^4}\sum_{j=1}^n j^2 + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right) \\ &= -\frac{\omega^2}{2n^2}\cdot\frac{1}{2}\,n(n+1) + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right) = -\frac{\omega^2}{4} + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right) \to -\frac{1}{2}\left(\frac{\omega}{\sqrt{2}}\right)^2 \qquad \text{for } n \to \infty. \end{aligned}$$

Hence,

$$k_{Z_n}(\omega) \to \exp\left(-\frac{1}{2}\left(\frac{\omega}{\sqrt{2}}\right)^2\right) \qquad \text{for } n \to \infty.$$

If Y is normally distributed, then of course

$$k_Y(\omega) = \exp\left(-\frac{1}{2}\,\omega^2\right),$$

and thus

$$Z = \frac{1}{\sqrt{2}}\,Y \in N\left(0, \frac{1}{\sqrt{2}}\right).$$

Example 6.8 *A random variable X has the frequency*

$$f(x) = \begin{cases} \dfrac{1}{\pi} \cdot \dfrac{1-\cos x}{x^2}, & x \neq 0, \\[2mm] \dfrac{1}{2\pi}, & x = 0. \end{cases}$$

1. *Prove by using the inversion formula that X has the characteristic function*

$$k(\omega) = \begin{cases} 1-|\omega|, & |\omega| \leq 1, \\[2mm] 0, & |\omega| > 1. \end{cases}$$

2. *Prove by e.g. using the result of* **1.** *that X does not have a mean.*

Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables, where each X_n has the frequency

$$f_n(x) = n\,f(nx) = \begin{cases} \dfrac{1}{\pi}\,\dfrac{1-\cos nx}{nx^2}, & x \neq 0, \\[2mm] \dfrac{n}{2\pi}, & x = 0, \end{cases} \qquad n \in \mathbb{N}.$$

3. *Find the characteristic function $k_n(\omega)$ for X_n.*

4. *Show, e.g. by using the result of* **3.** *that the sequence (X_n) converges in distribution towards a random variable Y, and find the distribution function of Y.*

1) According to the inversion formula we shall only prove that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,x\,\omega} k(\omega)\,d\omega = f(x).$$

Now, $1-|\omega|$, $|\omega| \leq 1$, is an even function, hence by insertion,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,x\,\omega} k(\omega)\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\,x\,\omega}(1-|\omega|)\,dx = \frac{1}{\pi}\int_0^1 (1-\omega)\cos\omega\,x\,d\omega.$$

We find for $x=0$,

$$\frac{1}{\pi}\int_0^1 (1-\omega)\,d\omega = \frac{1}{\pi}\left(1-\frac{1}{2}\right) = \frac{1}{2\pi} = f(0).$$

If $x \neq 0$, then we get by partial integration,

$$\begin{aligned} \frac{1}{\pi}\int_0^1 (1-\omega)\cos\omega\,x\,dx &= \frac{1}{\pi}\left[(1-\omega)\frac{\sin\omega\,x}{x}\right]_0^1 + \frac{1}{\pi\,x}\int_0^1 \sin\omega\,x\,dx = \frac{1}{\pi\,x}\left[-\frac{\cos\omega\,x}{x}\right]_0^1 \\ &= \frac{1-\cos x}{\pi\,x^2} = f(x), \end{aligned}$$

and the claim is proved.

2) We know that *if* $E\{X\}$ exists, then $k(\omega)$ *is* differentiable at 0.
Since, however, $k(\omega)$ is *not* differentiable at $\omega = 0$, we conclude by contraposition that $E\{X\}$ does *not* exist, so we conclude that X does not have a mean.

3) Then by a simple transformation,

$$\begin{aligned} k_n(\omega) &= \int_{-\infty}^{\infty} e^{i\,\omega\,x} f_n(x)\,dx = \int_{-\omega}^{\infty} e^{i\,\omega\,x} f(nx)n\,dx = \int_{-\infty}^{\infty} \exp\left(i\,\frac{\omega}{n}\,t\right)\,f(t)\,dt = k\left(\frac{\omega}{n}\right) \\ &= \begin{cases} 1 - \left|\dfrac{\omega}{n}\right|, & |\omega| \le n, \\ 0, & |\omega| > n. \end{cases} \end{aligned}$$

4) It follows from **3.** that

$$\lim_{n\to\infty} k_n(\omega) = 1 = k_0(\omega) \qquad \text{for ethvert } \omega \in \mathbb{R},$$

where $k_0(\omega) \equiv 1$ is the characteristic function for the causal distribution $P\{Y = 0\} = 1$.
Since $k_0(\omega) = 1$ is continuous, it follows that (X_n) converges in distribution towards the causal distribution Y.

Remark 6.1 In *Distribution Theory*, which is a mathematical discipline dealing with generalized functions, one expresses this by $(f_n) \to \delta$, where δ is Dirac's δ "function". $\Diamond$

Example 6.9 *A random variable Y has the frequency*

$$f(y) = \frac{a}{2}\, e^{-a|y|}, \qquad y \in \mathbb{R},$$

where $a > 0$ is a positive constant.

1. *Find the characteristic function for Y.*

2. *Find the mean and variance of Y.*

A random variable X has the values ± 1, ± 2, $\dots$ of the probabilities

$$P\{X = k\} = P\{X = -k\} = \frac{1}{2}\, p\, q^{k-1}, \qquad k \in \mathbb{N},$$

where $p > 0$, $q > 0$, $p + q = 1$.

3. *Prove that the characteristic function for X is given by*

$$k_X(\omega) = \frac{p(\cos\omega - q)}{1 + q^2 - 2q\,\cos\omega}, \qquad \omega \in \mathbb{R}.$$

Then consider a sequence of random variables $(X_n)_{n=1}^{<infty}$, where X_n has the values $\pm\frac{1}{n}$, $\pm\frac{2}{n}$, $\dots$ of the probabilities

$$P\left\{X_n = \frac{k}{n}\right\} = P\left\{X_n = -\frac{k}{n}\right\} = \frac{1}{2} \cdot \frac{1}{3n}\left(1 - \frac{1}{3n}\right)^{k-1}, \qquad k \in \mathbb{N}.$$

4. *Find by using the result of* **3.** *the characteristic function $k_n(\omega)$ for X_n.*

5. *Prove that the sequence (X_n) converges in in distribution towards a random variable Z, and find the frequency of Z.*

1) The characteristic function is

$$\begin{aligned} k_Y(\omega) &= \int_{-\infty}^{\infty} e^{i\,\omega\, y} \cdot \frac{a}{2} \cdot e^{-a|y|}\, dy = \frac{a}{2}\int_{-\infty}^{0} e^{(a+i\omega)y}\, dy + \frac{a}{2}\int_0^{\infty} e^{(-a+i\omega)y}\, dy \\ &= \frac{a}{2}\left[\frac{e^{(a+i\,\omega)y}}{a+i\,\omega}\right]_{-\omega}^{0} + \frac{a}{2}\left[\frac{e^{(-a+i\,\omega)y}}{-a+i\,\omega}\right]_0^{\infty} = \frac{a}{2}\left(\frac{1}{a+i\,\omega} + \frac{1}{a-i\,\omega}\right) = \frac{a^2}{a^2+\omega^2}. \end{aligned}$$

2) By the symmetry, $E\{Y\} = 0$. The variance is then

$$V\{Y\} = E\left\{Y^2\right\} = \frac{a}{2}\int_{-\infty}^{\infty} y^2 e^{-a|y|}\, dy = \frac{1}{a^2}\int_0^{\infty} t^2 e^{-t}\, dt = \frac{2!}{a^2} = \frac{2}{a^2}.$$

3) The characteristic function for X is

$$\begin{aligned}
k_X(\omega) &= \sum_{k=1}^{\infty} P\{X=-k\}\cdot e^{-i\,k\,\omega} + \sum_{k=1}^{\infty} P\{X=k\}\cdot e^{i\,k\,\omega} \\
&= \frac{p}{2}\sum_{k=1}^{\infty} q^{k-1}\cdot\left(e^{-i\,\omega}\right)^k + \frac{p}{2}\sum_{k=1}^{\infty} q^{k-1}\left(e^{i\,\omega}\right)^k \\
&= \frac{p}{2}e^{-i\,\omega}\sum_{k=1}^{\infty}\left(q\,e^{-i\,\omega}\right)^{k-1} + \frac{p}{2}\,e^{i\,\omega}\sum_{k=1}^{\infty}\left(q\,e^{i\,\omega}\right)^k = \frac{p}{2}\cdot\frac{e^{-i\,\omega}}{1-q\,e^{-i\,\omega}} + \frac{p}{2}\cdot\frac{e^{i\,\omega}}{1-q\,e^{i\,\omega}} \\
&= p\,\mathrm{Re}\left\{\frac{e^{i\,\omega}}{1-q\,e^{i\,\omega}}\cdot\left(\frac{1-q\,e^{-i\,\omega}}{1-q\,e^{-i\,\omega}}\right)\right\} = p\,\mathrm{Re}\left\{\frac{e^{i\,\omega}-q}{1-2q\,\cos\omega+q^2}\right\} = \frac{p(\cos\omega-1)}{1+q^2-2q\,\cos\omega}.
\end{aligned}$$

4) We put $p=\dfrac{1}{3n}$ and $q=1-\dfrac{1}{3n}$. The characteristic function for X_n is obtained by replacing ω by $\dfrac{\omega}{n}$, thus

$$k_n(\omega) = \frac{\dfrac{1}{3n}\left(\cos\dfrac{\omega}{n}-1+\dfrac{1}{3n}\right)}{1+\left(1-\dfrac{1}{3n}\right)^2-2\left(1-\dfrac{1}{3n}\right)\cos\left(\dfrac{\omega}{n}\right)}, \qquad n\in\mathbb{N}.$$

5) It follows by insertion of

$$\cos\frac{\omega}{n} = 1-\frac{1}{2}\cdot\frac{\omega^2}{n^2}+\frac{\omega^2}{n^2}\,\varepsilon\left(\frac{\omega}{n}\right),$$

that

$$\begin{aligned}
k_n(\omega) &= \frac{\frac{1}{3n}\left(1-\frac{\omega^2}{2n^2}+\frac{\omega^2}{n^2}\,\varepsilon\left(\frac{\omega}{n}\right)-1+\frac{1}{3n}\right)}{1+1-\frac{2}{3n}+\frac{1}{9n^2}-2\left(1-\frac{1}{3n}\right)\left(1-\frac{\omega^2}{n^2}+\frac{\omega^2}{2n^2}\,\varepsilon\left(\frac{\omega}{n}\right)\right)} = \frac{1}{3n}\,\frac{\frac{1}{3n}+\frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)}{2-\frac{2}{3n}+\frac{1}{9n^2}-2+\frac{2}{3n}+\frac{\omega^2}{n^2}+\frac{\omega^2}{n^2}\,\varepsilon\left(\frac{\omega}{n}\right)} \\
&= \frac{1}{9n^2}\cdot\frac{1+\varepsilon\left(\dfrac{1}{n}\right)}{\dfrac{1}{9n^2}+\dfrac{1}{n^2}\,\omega^2+\dfrac{\omega^2}{n^2}\,\varepsilon\left(\dfrac{\omega}{n}\right)} = \frac{1+\varepsilon\left(\dfrac{1}{n}\right)}{1+9\,\omega^2+\varepsilon\left(\dfrac{\omega}{n}\right)},
\end{aligned}$$

hence

$$\lim_{n\to\infty} k_n(\omega) = \frac{1}{1+9\,\omega^2} = \frac{\dfrac{1}{9}}{\dfrac{1}{9}+\omega^2} = k_y(\omega),$$

where Y is the random variable from **1.**, corresponding to $a=\dfrac{1}{3}$.

Since $k_y(\omega)$ is continuous, (X_n) converges in distribution towards Y for $a=\dfrac{1}{3}$, thus

$$f_Y(y) = \frac{1}{6}\,\exp\left(-\frac{|y|}{3}\right), \qquad y\in\mathbb{R}.$$

Example 6.10 **1.** *Let X be a random variable with the characteristic function $k(\omega)$. Prove that the random varible $Y = -X$ has the characteristic function*

$$k_Y(\omega) = \overline{k(\omega)}.$$

Let X_1 and X_2 be independent random variables, both of the distribution given by

$$P\{X_i = j\} = \left(\frac{1}{2}\right)^j, \qquad j \in \mathbb{N}; \quad i = 1, 2.$$

2. *Find the characteristic function $k_1(\omega)$ for X_1.*

3. *Find the distribution of the random variable $Z = X_1 - X_2$.*

4. *Find, e.g. by using the result of* **1.**, *the characteristic function for Z.*

Let $Z_1, Z_2, \ldots$ be mutually independent random variables, all of the same distribution as Z, and let

$$U_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i, \qquad n \in \mathbb{N}.$$

5. *Prove e.g. by using characteristic functions that the sequence $(U_n)_{n=1}^{\infty}$ converges in distribution towards a random variable U, and find the distribution function of U.*

1) Since X is real, it immediately follows that

$$k_Y(\omega) = E\left\{e^{i\,\omega\,Y}\right\} = E\left\{e^{-i\,\omega\,X}\right\} = \overline{E\left\{e^{i\,\omega\,X}\right\}} = \overline{k_X(\omega)}.$$

Alternatively,

$$\begin{aligned} k_Y(\omega) &= E\{\cos(\omega Y) + i\,\sin(\omega Y)\} = E\{\cos(-\omega X) + i\,\sin(-\omega X)\} \\ &= E\{\cos(\omega X) - i\,\sin(\omega X)\} = \overline{k_X(\omega)}. \end{aligned}$$

2) The characteristic function is

$$k_{X_1}(\omega) = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j e^{i\,\omega\,j} = \sum_{j=1}^{\omega} \left(\frac{e^{i\,\omega}}{2}\right)^j = \frac{\frac{1}{2}\,e^{i\,\omega}}{1 - \frac{1}{2}\,e^{i\,\omega}} = \frac{e^{i\,\omega}}{2 - e^{i\,\omega}}.$$

3) The distribution function is

$$\begin{aligned} F_Z(z) &= P\{X_1 - X_2 \le z\} = \sum\sum\nolimits_{j-k \le [z]} P\{X_1 = j\} \cdot P\{X_2 = k\} \\ &= \sum_{k=\max\{1,1-[z]\}}^{\infty} \sum_{j=1}^{k+[z]} \left(\frac{1}{2}\right)^j \cdot \left(\frac{1}{2}\right)^k = \sum_{k=\max\{1,1-[z]\}}^{\infty} \left(\frac{1}{2}\right)^k \cdot \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{k+[z]+1}}{1 - \frac{1}{2}} \\ &= \sum_{k=\max\{1,1-[z]\}}^{\infty} \left\{\left(\frac{1}{2}\right)^k - \left(\frac{1}{2}\right)^{2k+[z]}\right\}. \end{aligned}$$

If $z<0$, then

$$\begin{aligned} F_Z(z) &= \sum_{k=1-[z]}^{\infty}\left\{\left(\frac{1}{2}\right)^k-\left(\frac{1}{2}\right)^{2k+[z]}\right\}=\sum_{k=1}^{\infty}\left\{\left(\frac{1}{2}\right)^{k-[z]}-\left(\frac{1}{2}\right)^{2k-[z]}\right\} \\ &= \left(\frac{1}{2}\right)^{-[z]}\sum_{k=1}^{\infty}\left\{\left(\frac{1}{2}\right)^k-\left(\frac{1}{4}\right)^k\right\}=\left(\frac{1}{2}\right)^{-[z]}\left(1-\frac{1}{3}\right)=\frac{2}{3}\left(\frac{1}{2}\right)^{-[z]}. \end{aligned}$$

If $z\geq 0$, then

$$F_Z(z)=\sum_{k=1}^{\infty}\left\{\left(\frac{1}{2}\right)^k-\left(\frac{1}{2}\right)^{2k+[z]}\right\}=1-\left(\frac{1}{2}\right)^{[z]}\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k=1-\frac{1}{3}\left(\frac{1}{2}\right)^{[z]}.$$

Summing up,

$$F_Z(z)=\begin{cases} \dfrac{2}{3}\left(\dfrac{1}{2}\right)^{-[z]}, & \text{hvis } z<0, \\ 1-\dfrac{1}{3}\left(\dfrac{1}{2}\right)^{[z]}, & \text{if } z\geq 0, \end{cases} \qquad [z] \text{ integer part of } z.$$

ALTERNATIVELY, $Z = X_1 - X_2$ is its values in $\mathbb{R}$. By the symmetry,

$$P\{Z = k\} = P\{Z = -k\}.$$

If $k \geq 0$, then

$$\begin{aligned} P\{Z=k\} &= P\{Z=-k\} = \sum_{j=1}^{\infty} P\{X_1 = j+k\} \cdot P\{X_2 = j\} = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j+k} \left(\frac{1}{2}\right)^{j} \\ &= \left(\frac{1}{2}\right)^k \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^j = \left(\frac{1}{2}\right)^k \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^k, \qquad k \in \mathbb{N}_0, \end{aligned}$$

where we describe the distribution by the probabilities of the points.

4) It follows from **1.** and **2.** that

$$k_Z(\omega) = \frac{e^{i\,\omega}}{2 - e^{i\,\omega}} \cdot \frac{e^{-i\,\omega}}{2 - e^{-i\,\omega}} = \frac{1}{5 - 4\,\cos\omega}.$$

ALTERNATIVELY, $k_Z(\omega)$ is computed in the following way,

$$\begin{aligned} k_Z(\omega) &= \sum_{k=0}^{\infty} P\{Z=k\} e^{ik\omega} + \sum_{k=0}^{\infty} P\{Z=-k\} e^{-ik\omega} = \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{1}{2}\, e^{i\omega}\right)^k + \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{2}\, e^{-i\omega}\right)^k \\ &= \frac{1}{3} \left\{ \frac{1}{1 - \frac{1}{2} e^{i\omega}} + \frac{\frac{1}{2}\, e^{-i\omega}}{1 - \frac{1}{2} e^{-i\omega}} \right\} = \frac{1}{3} \cdot \frac{1 - \frac{1}{2} e^{-i\omega} + \frac{1}{2} e^{-i\omega} - \frac{1}{4}}{\frac{5}{4} - \cos\omega} = \frac{1}{4} \cdot \frac{1}{\frac{5}{4} - \cos\omega} \\ &= \frac{1}{5 - 4\cos\omega}, \qquad \omega \in \mathbb{R}. \end{aligned}$$

5) The characteristic function for U_n is

$$k_{U_n}(\omega) = \left(k_Z\left(\frac{\omega}{\sqrt{n}}\right)\right)^n = \frac{1}{\left(5 - 4\,\cos\frac{\omega}{\sqrt{n}}\right)^n}.$$

We conclude from

$$\left(5 - 4\,\cos\frac{\omega}{\sqrt{n}}\right)^n = \left(5 - 4\left\{1 - \frac{1}{2}\,\frac{\omega^2}{n} + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)\right\}\right)^n = \left(1 + \frac{2\omega^2}{n} + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)\right)^n,$$

that

$$k_{U_n}(\omega) \to \lim_{n \to \infty} \left\{1 + \frac{2\omega^2}{n} + \frac{1}{n}\,\varepsilon\left(\frac{1}{n}\right)\right\}^{-n} = e^{-2\omega^2} = \exp\left(-\frac{1}{2}\,4\omega^2\right).$$

We see that $k_U(\omega) = \exp\left(-\frac{1}{2} \cdot 4\omega^2\right)$ is continuous, hence $U \in N(0,4)$, and $U_n \to U$ in distribution, where $U \in N(0,4)$ is normally distributed.

ALTERNATIVELY we may use that X_1 and X_2 are both geometrically distributed of variance 2, hence the Z_i have the variance 4. Then it follows from the *Central Limit Theorem* that

$$\frac{1}{2}\,U_n = \frac{1}{2\sqrt{n}}\sum_{i=1}^{n} Z_n$$

for $n \to \infty$ converges in distribution towards $V \in N(0,1)$.
Then

$$U_n \xrightarrow{D} U \in N(0,4).$$

Example 6.11 *Let X_1 and X_2 be independent random variables of distribution given by*

$$P\{X_1 = j\} = P\{X_2 = j\} = p\,q^j, \qquad j \in \mathbb{N}_0,$$

where $p > 0$, $q > 0$, $p + q = 1$, and let $Y = X_1 - X_2$.

1. *Find the mean and variance of Y.*

2. *Find $P\{Y = j\}$ for every $j \in \mathbb{Z}$.*

3. *Find the characteristic function for X_1 and the characteristic function for $-X_2$, and thus this to find the characteristic function for Y.*

Given a sequence of random variables $(Y_n)_{n=1}^{\infty}$, where for each $n \in \mathbb{N}$, the random variable Y_n has a distribution as Y corresponding to $p = \dfrac{1}{2n}$, $q = 1 - \dfrac{1}{2n}$. Let $Z_n = \dfrac{1}{n}\,Y_n$.

4. *Prove, e.g. by using* **3.** *that the sequence $(Z_n)_{n=1}^{\infty}$ converges in distribution towards a random variable Z, and find distribution of Z.*

1) Using that X_1 and X_2 are identically distributed and that both the mean and the variance exist, we get

$$E\{Y\} = E\{X_1\} - E\{X_2\} = 0,$$

andd

$$\begin{aligned}
V\{Y\} &= 2\,V\{X_1\} = 2\,E\left\{X_1^2\right\} = 2\,E\{X_1(X_1-1)\} + 2\,E\{X_1\} \\
&= 2\sum_{j=2}^{\infty} j(j-1)p\,q^j + 2\sum_{j=1}^{\infty} jpq^j = 2pq^2\left(\frac{1}{1-q}\right)^2 + 2pq\cdot\frac{1}{1-q} = 2\left(\frac{q^2}{p^2}+\frac{q}{p}\right) \\
&= 2\,\frac{q}{p^2}\,(q+p) = \frac{2q}{p^2}.
\end{aligned}$$

2) The probability is

$$P\{Y = j\} = \sum_{\substack{\ell-k=j\\ \ell\ge 0,\,k\ge 0}} P\{X_1 = \ell\}\cdot P\{X_2 = k\} = p^2 \sum_{\substack{\ell-k=j\\ \ell\ge 0,\,k\ge 0}} q^j\cdot q^k.$$

If $j \ge 0$, then $\ell = k + j$, hence by the symmetry,

$$P\{Y=j\} = P\{Y=-j\} = p^2\sum_{k=0}^{\infty} q^{k+j}\cdot q^k = p^2 q^j \sum_{k=0}^{\infty}\left(q^2\right)^k = \frac{p^2\cdot q^j}{1-q^2} = \frac{pq^j}{1+q}.$$

3) he characteristic function for X_1 is

$$k_{X_1}(\omega) = \sum_{k=0}^{\infty} P\{X_1 = k\}\, e^{i\,k\,\omega} = p \sum_{k=0}^{\infty} q^k \left(e^{i\,\omega}\right)^k = \frac{p}{1 - q\, e^{i\,\omega}}.$$

The characteristic function for $-X_2$ is

$$K_{-X_2}(\omega) = k_{X_1}(-\omega) = \frac{p}{1 - q\, e^{-i\,\omega}}.$$

The characteristic function for $Y = X_1 - X_2$ is

$$k_Y(\omega) = k_{X_1}(\omega) \cdot k_{-X_2}(\omega) = \frac{p}{1 - q\, e^{i\,\omega}} \cdot \frac{p}{1 - q\, e^{-i\,\omega}} = \frac{p^2}{1 + q^2 - 2q\,\cos\omega}.$$

4) The characteristic function for $Z_n = \dfrac{1}{n} Y_n$ is

$$k_{Z_n}(\omega) = \frac{\left(\dfrac{1}{2n}\right)^2}{1 + \left(-\dfrac{1}{2n}\right)^2 - 2\left(1 - \dfrac{1}{2n}\right)\cos\left(\dfrac{\omega}{n}\right)} = \frac{1}{4n^2 + (2n-1)^2 - 4n(2n-1)\cos\left(\dfrac{\omega}{n}\right)}.$$

Using an expansion of the denominator we get

$$\begin{aligned} &8n^2 - 4n + 1 - \left(8n^2 - 4n\right)\left(1 - \frac{1}{2}\,\frac{\omega^2}{n^2} + \frac{1}{n^2}\,\varepsilon\left(\frac{1}{n}\right)\right) \\ &\qquad = 8n^2 - 4n + 1 - 8n^2 + 4n + 4\omega^2 - 2\,\frac{\omega^2}{n} + \varepsilon\left(\frac{1}{n}\right) = 1 + 4\omega^2 + \varepsilon\left(\frac{1}{n}\right), \end{aligned}$$

hence

$$\lim_{n\to\infty} k_{Z_n}(\omega) = \lim_{n\to\infty} \frac{1}{1 + 4\omega^2 + \varepsilon\left(\dfrac{1}{n}\right)} = \frac{1}{1 + 4\,\omega^2}.$$

Since the double exponentially distributed random variable Z with $a = \dfrac{1}{2}$ has the characteristic function

$$k_Z(\omega) = \frac{\left(\dfrac{1}{2}\right)^2}{\left(\dfrac{1}{2}\right)^2 + \omega^2} = \frac{1}{1 + 4\,\omega^2},$$

we conclude that (Z_n) converges in distribution towards Z.

Example 6.12 *A random variable X has the frequency*

$$f(x) = \begin{cases} \dfrac{1}{\pi}\,\dfrac{\sin^2 x}{x^2}, & x \neq 0, \\ \\ \dfrac{1}{\pi}, & x = 0. \end{cases}$$

1. *Find the median of X.*

It can be shown (shall not be proved) that X has the characteristic function

$$k(\omega) = \begin{cases} 1 - \dfrac{|\omega|}{2}, & |\omega| \le 2, \\ \\ 0, & |\omega| > 2. \end{cases}$$

2. *Prove that X does not have a mean.*

Let X_1, X_2, X_3, ... be mutually independent random variables, all of the same distribution as X. Let

$$Z_n = \frac{1}{n}\sum_{j=1}^{n} X_j, \qquad n \in \mathbb{N}.$$

3. *Find the characteristic function for Z_n.*

4. *Prove that the sequence $(Z_n)_{n=1}^{\infty}$ converges in distribution towards a random variable Z, and find the distribution of Z.*

5. *Compute the probability $P\left\{-\frac{1}{2} < Z < \frac{1}{2}\right\}$.*

1) It follows from $f(-x) = f(x)$ that the median is $\langle X \rangle = 0$.

2) Since $k(\omega)$ is not differentiable at $\omega = 0$, the random variable X does not have a mean.

3) The characteristic function for Z_n is

$$k_{Z_n}(\omega) = \left\{k\left(\frac{\omega}{n}\right)\right\}^n = \begin{cases} \left(1 - \dfrac{|\omega|}{2n}\right)^n & \text{for } |\omega| \le 2n, \\ \\ 0 & \text{for } |\omega| > 2n. \end{cases}$$

4) Now, $k_{Z_n}(\omega) \to \exp\left(-\frac{|\omega|}{2}\right)$ for $n \to \infty$ and every fixed $\omega \in \mathbb{R}$. Since $\exp\left(-\frac{|\omega|}{2}\right)$ is continuous, (Z_n) converges in distribution towards Z. Using a table we see that $Z \in C\left(0, \frac{1}{2}\right)$ is Cauchy distributed of the frequency

$$f_Z(z) = \frac{\frac{1}{2}}{\pi\left(\frac{1}{4} + z^2\right)} = \frac{2}{\pi} \cdot \frac{1}{1 + (2z)^2} \qquad \text{for } z \in \mathbb{R}.$$

5) The probability is

$$P\left\{-\frac{1}{2} < Z < \frac{1}{2}\right\} = \frac{2}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dz}{1+(2z)^2} = \frac{1}{\pi}\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{2}{\pi}\left[\text{Arctan } t\right]_0^1 = \frac{1}{2}.$$

Example 6.13 *We say that a random variable X has a* symmetric distribution*, if X and $-X$ have the same distribution.*
Assume that X has the characteristic function $k_X(\omega)$. Prove that $-X$ has the characteristic function

$$k_{-X}(\omega) = \overline{k_X(\omega)}.$$

Prove that the characteristic function for X is a real function, is and only if X has a symmetric distribution.

The first question is almost trivial,

$$k_{-X}(\omega) = E\left\{e^{-i\,\omega\,X}\right\} = \overline{E\left\{e^{i\,\omega\,X}\right\}} = \overline{k_X(\omega)}.$$

1) If X has a symmetric distribution, then

$$k_{-X}(\omega) = k_X(\omega) = \overline{k_X(\omega)},$$

and we conclude that $k_X(\omega)$ is real.

2) Conversely, if $k_X(\omega)$ is real, then

$$k_{-X}(\omega) = \overline{k_X(\omega)} = k_X(\omega),$$

from which follows that $-X$ and X have the same characteristic function, and hence the same distribution. This proves that X has a symmetric distribution.

Example 6.14 *Prove that the characteristic function for the distribution given by*

$$P\{X=-n\} = P\{X=n\} = \frac{c}{n^2 \ln n}, \qquad n = 2,\, 3,\, \dots,$$

where

$$c \cdot \sum_{n=2}^{+\infty} \frac{1}{n^2 \ln n} = \frac{1}{2},$$

is of class C^1.
HINT: *The problem is to prove that the termwise differentiated series*

$$-2c \sum_{n=2}^{\infty} \frac{\sin n\,\omega}{n\,\ln n}$$

is uniformly convergent on $\mathbb{R}$. *Show this by successively proving that*

1)

$$\left| \sum_{n=p}^{q} \sin n\,\omega \right| \le \frac{1}{\left|\sin \frac{\omega}{2}\right|}, \quad \omega \ne 2m\,\pi, \quad p,\, q \in \mathbb{N}, \quad p < q.$$

2)

$$\left| \sum_{n=p}^{N} \frac{1}{n} \sin n\,\omega \right| \le \pi + 1, \quad \omega \in \mathbb{R}, \quad p,\, N \in \mathbb{N}, \quad p < N.$$

3)

$$\left| \sum_{n=p}^{q} \frac{\sin n\,\omega}{n} \cdot \frac{1}{\ln n} \right| \le (\pi+1) \cdot \frac{1}{\ln p}, \quad \omega \in \mathbb{R}, \quad p,\, q \in \mathbb{N}, \quad 2 \le p < q.$$

Here we shall also use Abel's formula for partial summation*, which is written*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n \left(b_n - b_{n+1}\right) + A_q b_q, \quad \textit{where} \quad A_n = \sum_{k=p}^{n} a_k.$$

Abel's formula above is similar to partial integration; ' here we use sums instead of integrals.

The claim follows easily from the estimate in **3.**, because the right hand side tends towards 0 for $p \to \infty$, independently of $\omega \in \mathbb{R}$.

1) If $p < q$ and $\omega \neq 2m\,pi$, then

$$\begin{aligned}\sum_{n=p}^{q} \sin n\,\omega &= \operatorname{Im} \sum_{n=p}^{q} e^{i\,n\,\omega} = \operatorname{Im} \frac{e^{o\,p\,\omega} - e^{i(q+1)\omega}}{1-e^{i\,\omega}} \\ &= \operatorname{Im} \frac{\exp\left(i\left(q+\frac{1}{2}\right)\omega\right) - \exp\left(i\left(p-\frac{1}{2}\right)\omega\right)}{\frac{1}{2i}\left\{\exp\left(i\,\frac{\omega}{2}\right) - \exp\left(-i\frac{\omega}{2}\right)\right\}\cdot 2i} \\ &= \frac{1}{2\,\sin\frac{\omega}{2}} \cdot \left\{\cos\left(p-\frac{1}{2}\right)\omega - \cos\left(q+\frac{1}{2}\right)\omega\right\},\end{aligned}$$

thus we get the estimate

$$\left|\sum_{n=p}^{q} \sin n\,\omega\right| \leq \frac{1+1}{2\left|\sin\frac{\omega}{2}\right|} = \frac{1}{\left|\sin\frac{\omega}{2}\right|} \quad \text{for } \omega \neq 2m\,\pi, \quad m \in \mathbb{Z}.$$

Notice that the left hand side is 0 for $\omega = 2m\,\pi$, $m \in \mathbb{Z}$.

2) Due to the periodicity it suffices to consider $\omega \in [-\pi,\pi]$. Using that sinus is an odd function, it follows that it even suffices to consider $\omega \in [0,\pi]$. Finally, if follows from **1.** that we can restrict ourselves to $\omega \in\ ]0,\omega_0]$, where

$$\omega_0 = 2\operatorname{Arcsin}\frac{1}{\pi+1}.$$

Let $N > p$, and choose $\omega_p = \dfrac{\pi}{p}$. We group the terms in the following way,

$$\sum_{n=1}^{N} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right) = \sum_{k=0}^{k_0-1}\sum_{n=kp+1}^{(k+1)p} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right) + \sum_{n=k_0p+1}^{N} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right),$$

where

$$k_0 = \left[\frac{N-1}{p}\right]$$

denotes the integer part of $(N-1)/p$. We note that the sequence (in k)

$$\left(\sum_{n=k+1}^{(k+1)p} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right)\right)$$

is alternating and that the corresponding sequence of absolute values tends *decressingly* towards 0. Thus we get the following estimate,

$$\begin{aligned}\left|\sum_{n=p}^{N} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right)\right| &\leq \sum_{n=1}^{p} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right) \leq 2\sum_{n=1}^{\left[\frac{p}{2}\right]} \frac{1}{n}\sin\left(n\,\frac{\pi}{p}\right) 01 \\ &\leq 2\sum_{n=1}^{\left[\frac{p}{2}\right]} \frac{1}{n}\cdot n\,\frac{\pi}{p} + 1 \leq 2\cdot\frac{p}{2}\cdot\frac{\pi}{p} + 1 = \pi + 1.\end{aligned}$$

If $\dfrac{\pi}{p+1} < \omega < \dfrac{\pi}{p}$, then we estimate upwards by

$$\sin n\,\omega < \sin\left(n\,\frac{\pi}{p}\right) \qquad \text{for } n \le \left[\frac{p}{2}\right].$$

Hence

$$\left|\sum_{n=p}^{N} \frac{1}{n}\,\sin n\omega\right| \le \pi+1, \quad \omega \in \mathbb{R}, \quad p,\,N \in \mathbb{N}, \quad p<N.$$

3) Let $2 \le p < q$, and choose

$$a_n = \frac{\sin n\,\omega}{n} \qquad \text{with} \qquad A_n = \sum_{k=p}^{n} \frac{\sin k\,\omega}{k}, \quad |A_n| \le \pi 1$$

according to **2.**. Finally, choose $b_n = \dfrac{1}{\ln n}$. Then it follows by an application of Abelian summation that

$$\sum_{n=p}^{q} \frac{\sin n\,\omega}{n}\cdot\frac{1}{\ln n} = \sum_{n=p}^{q-1} A_n \cdot\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right) + A_q\cdot\frac{1}{\ln q}.$$

Thus we get the estimate

$$\begin{aligned}\left|\sum_{n=p}^{q} \frac{\sin n\,\omega}{n}\cdot\frac{1}{\ln n}\right| &\le \sum_{n=p}^{q-1} |A_n|\cdot\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)+|A_q|\cdot\frac{1}{\ln q}\\ &\le (\pi+1)\left\{\sum_{n=1}^{q-1}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)+\frac{1}{\ln q}\right\} = \frac{\pi+1}{\ln p}\end{aligned}$$

as required.
As mentioned above it then follows that the termwise differentiated series is uniformly convergent, and the characteristic function is of class C^1.

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