A Semester Course in Trigonometry

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PREFACE

Trigonometry in modern time is an indispensable tool in Physics, engineering, computer science, biology, and in practically all the sciences.
This book consists of my lectures of a freshmen-level mathematics class offered at Arkansas Tech University. This book has been written in a way that can *be read by students.* That is, the text represents a serious effort to produce exposition that is accessible to a student at the freshmen or high school levels.
The chapters of this book are well suited for a one semester course in College Trigonometry.

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Contents

1 Equations and Inequalities

This section illustrates the processes of solving linear and quadratic equations and inequalities. Also, the process of solving absolute value inequalities is discussed.

Solving Linear Equations
By a **linear** equation we mean an equation of the form

$$ax + b = 0$$

where a and b are given numbers and x is the variable to be found, also called the **solution** or **root** of the equation. The process of finding x is referred to as **solving** the given equation.
To solve a linear equation in one variable, isolate the variable on one side of the equation. This can be done thanks to the following two properties of numbers:

Property I: Adding or subtracting the same number to both sides of an equation does not change the solution to the equation.
Property II: Multiplying or dividing both sides of an equation by a nonzero number does not change the solution to the equation.

Remark 1.1
The above two properties apply to any equation and not only for linear equations.

Example 1.1
Solve the equation: $-3x + 20 = 2$.

Solution.
To isolate x, subtract first 20 from both sides of the given equation to obtain $-3x = -18$. Now, divide both sides by -3 to obtain $x = 6$.■

Solving Quadratic Equations
The second type of equations that we discuss here is the so called quadratic equations. By a **quadratic** equation we mean an equation of the form

$$ax^2 + bx + c + 0, \quad a \neq 0,$$

where a, b, and c are given numbers and x is the variable to be found.
There are two methods for finding x.

• **Solving by Factoring**
The process of factoring consists of rewriting the equation in the form

$$a(x-r)(x-s) = 0.$$

Now, by the zero product property, which states that if $u \cdot v = 0$ then either $u = 0$ or $v = 0$, we can conclude that either $x - r = 0$ or $x - s = 0$. That is, $x = r$ or $x = s$.
To factor $ax^2 + bx + c$
1. find two integers that have a product equal to ac and a sum equal to b,
2. replace bx by two terms using the two new integers as coefficients,
3. then factor the resulting four-term polynomial by grouping. Thus, obtaining $a(x-r)(x-s) = 0$.
4. use the zero product property.

Example 1.2
Find the zeros of $f(x) = x^2 - 2x - 8$.

Solution.
We need two numbers whose product is -8 and sum is -2. Such two integers are -4 and 2. Thus,

$$\begin{aligned} x^2 - 2x - 8 &= x^2 + 2x - 4x - 8 \\ &= x(x+2) - 4(x+2) \\ &= (x+2)(x-4) = 0. \end{aligned}$$

Thus, either $x = -2$ or $x = 4$.■

Example 1.3
Find the zeros of $f(x) = 2x^2 + 9x + 4$.

Solution.
We need two integers whose product is $ac = 8$ and sum equals to $b = 9$. Such two integers are 1 and 8. Thus,

$$\begin{aligned} 2x^2 + 9x + 4 &= 2x^2 + x + 8x + 4 \\ &= x(2x+1) + 4(2x+1) \\ &= (2x+1)(x+4). \end{aligned}$$

Hence, the zeros are $x = -\frac{1}{2}$ and $x = -4$.■

• **Solving by Using the Quadratic Formula**:
Many quadratic functions are not easily factored. For example, the function $f(x) = 3x^2 - 7x - 7$. However, the zeros can be found by using the quadratic formula which we derive next:

$$\begin{array}{ccc}
ax^2 + bx + c & = & 0 \ (\textit{subtract } c \textit{ from both sides}) \\
ax^2 + bx & = & -c \ (\textit{multiply both sides by } 4a) \\
4a^2x^2 + 4abx & = & -4ac \ (\textit{add } b^2 \textit{ to both sides}) \\
4a^2x^2 + 4abx + b^2 & = & b^2 - 4ac \\
(2ax + b)^2 & = & b^2 - 4ac \\
2ax + b & = & \pm\sqrt{b^2 - 4ac} \\
x & = & \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{array}$$

provided that $b^2 - 4ac \geq 0$. This last formula is known as the **quadratic formula.** Note that if $b^2 - 4ac < 0$ then the equation $ax^2 + bx + c = 0$ has no solutions.

Example 1.4
Find the zeros of $f(x) = 3x^2 - 7x - 7$.

Solution.
Letting $a = 3, b = -7$ and $c = -7$ in the quadratic formula we have

$$x = \frac{7 \pm \sqrt{133}}{6}.$$■

Example 1.5
Find the zeros of the function $f(x) = 6x^2 - 2x + 5$.

Solution.
Letting $a = 6, b = -2$, and $c = 5$ in the quadratic formula we obtain

$$x = \frac{2 \pm \sqrt{-116}}{12}$$

But $\sqrt{-116}$ is not a real number. Hence, the function has no zeros.■

Solving Linear Inequalities
By a **linear inequality** we mean an inequality of the form

$$ax + b \square 0$$

where $\square$ can be any of the following: $<, >, \leq, \geq$.
To isolate the x, use the following two properties:

Property III: Adding or subtracting the same number to both sides of an inequality does not change the solution to the inequality.
Property IV: Multiplying or dividing both sides of an equality by a nonzero number does not change the solution to the inequality. However, when you multiply or divide by a negative number make sure you reverse the inequality sign.

Example 1.6
Solve the inequality: $x + 4 > 3x + 16$.

Solution.
Add $-x-16$ to both sides of the inequality to obtain $-12 > 2x$ or $2x < -12$. Now divide both sides by 2 to obtain $x < -6$. The solution set is usually represented by an interval. Thus, the interval of solution to the given inequality is $(-\infty, -6)$.■

Solving Quadratic Inequalities
By a **quadratic inequality** we mean an inequality of the form

$$ax^2 + bx + c \square 0,$$

where $\square$ can be any of the following: $<, >, \leq, \geq$.
The process of solving this type of inequalities consists of factoring the quadratic expressions so that we can locate the zeros and then construct a chart of signs which provide the solution interval to the inequality. We illustrate this in the next example.

Example 1.7
Solve the inequality $6x^2 - 4 \leq 5x$.

Solution.
Subtract $5x$ from both sides to obtain $6x^2 - 5x - 4 \leq 0$. Factor $f(x) =$

$6x^2 - 5x - 4 = (3x-4)(2x+1)$. Thus, the zeros of the left-hand side are $x = \frac{4}{3}$ and $x = -\frac{1}{2}$. Next, we construct the following chart of signs:

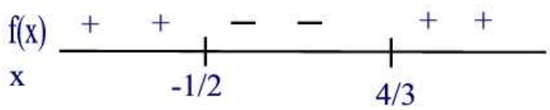


Figure 1.1

According to Figure 1.1, the interval of solution is given by $[-\frac{1}{2}, \frac{4}{3}]$.■

Solving Absolute Value Inequalities
First, we define the **absolute value** of a number x by the formula

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Geometrically, $|x|$ measures the distance from x to the origin. Thus, an inequality of the form $|x| > 5$ indicates that x is more than five units from 0. Any number on the number line to the right of 5 or to the left of -5 is more than five units from 0. So $|x| > 5$ is equivalent to $x < -5$ or $x > 5$. Thus, the interval of solution is given by the union of the intervals $(-\infty, -5)$ and $(5, \infty)$. Symbolically, we will write $(-\infty, -5) \cup (5, \infty)$.
Similarly, the inequality $|x - 9| < 2$ indicates that the distance from x to 9 is less than 2. On a number line, this happens when x is between 7 and 11. That is, the interval of solution is $(7, 11)$.

Example 1.8
Solve $|5 - 3x| \leq 6$.

Solution.
Let $u = 5 - 3x$. Then $|u| \leq 6$. This means that the distance from u to 0 is less than or equal to 6. On a number line, this happens when $-6 \leq u \leq 6$. Thus, $-6 \leq 5 - 3x \leq 6$. Next, we have to isolate the x. Subtract 5 from each part of the inequality to obtain $-11 \leq -3x \leq 1$. Now, divide through by -3 to obtain $-\frac{1}{3} \leq x \leq \frac{11}{3}$. Thus, the interval of solution is $[-\frac{1}{3}, \frac{11}{3}]$.■

Review Problems

Exercise 1.1
Solve each of the following equations:

1. $2x + 10 = 40.$
2. $6(5x - 11) - 12(2x + 5) = 0.$
3. $\frac{3}{5}(x + 5) - \frac{3}{4}(x - 11) = 0.$
4. $\frac{2}{3}x - 5 = \frac{1}{2}x - 3.$
5. $0.08x + 0.12(4000 - x) = 432.$

Exercise 1.2
Solve by using the quadratic formula.

1. $x^2 - 2x - 15 = 0.$
2. $x^2 + x - 2 = 0.$
3. $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0.$
4. $\sqrt{2}x^2 + 3x + \sqrt{2} = 0.$

Exercise 1.3
Solve each of the following equations by factoring.

1. $x^2 - 2x - 15 = 0.$
2. $12x^2 - 41x + 24 = 0.$
3. $(x - 5)^2 - 9 = 0.$

Exercise 1.4
Solve each inequality. Write answers in interval notation.

1. $2x + 3 < 11.$
2. $x + 4 > 3x + 16.$
3. $-3(x + 2) \leq 5x + 7.$
4. $3(x + 7) \leq 5(2x - 8).$

Exercise 1.5
Solve each inequality. Write answers in interval notation.

1. $x^2 + 7x > 0$.
2. $x^2 + 7x + 10 < 0$.
3. $x^2 - 3x \geq 28$.
4. $12x^2 + 8x \geq 15$.

Exercise 1.6
Solve each inequality. Write answers in interval notation.

1. $|x - 1| < 9$.
2. $|2x - 1| > 4$.
3. $|3x - 10| \leq 14$.
4. $|2x - 5| \geq 1$.
5. $|3 - 2x| \leq 5$.

Exercise 1.7
The perimeter of a rectangle is 27 centimeters, and its area is 35 square centimeters. Find the length and the width of the rectangle.

Exercise 1.8
A gardener wishes to use 600 feet of fencing to enclose a rectangular region and subdivide the region into two smaller rectangles. The total enclosed ares is 15,000 square feet. Find the dimensions of the enclosed region.

Exercise 1.9
You can rent a car for the day from company A for $29.00 plus $0.12 a mile. Company B charges $22.00 plus $0.21 a mile. Find the number of miles m (to the nearest mile) per day for which it is cheaper to rent from company A.

Exercise 1.10
Let S be the sum of n consecutive positive integers, i.e.,

$$S = 1 + 2 + 3 + \cdots + n.$$

(a) Find a compact formula for S in terms of n.
(b) How many consecutive positive integers starting with 1 produce a sum of 253?

Exercise 1.11
Write an absolute value inequality to represent all the real numbers within

(a) 8 units of 3.
(b) k units of j (assume $k > 0$).

Exercise 1.12
A ball is thrown directy upward from a height of 32 feet above the ground with initial velocity of 80 feet per second. The position of the ball from the ground after t seconds is given by the equation

$$s(t) = -16t^2 + 80t + 32 \quad ft.$$

Find the time interval during which the ball will be more than 96 feet above the ground.

2 Geometry in the Cartesian System

This section is designed to familiarize students to the Cartesian coordinate system and its many uses in the world of mathematics. The Cartesian coordinate system was developed by the mathematician *René Descartes* in 1637. The **Cartesian coordinate system**, also known as the rectangular coordinate system or the xy-plane, consists of two number scales, called the **x-axis** and the **y-axis**, that are perpendicular to each other at point O called the **origin.** Any point in the system is associated with an **ordered pair** of numbers (x, y) called the **coordinates** of the point. The number x is called the **abscissa** or the **x-coordinate** and the number y is called the **ordinate** or the **y-coordinate.** The abscissa measures the distance from the point to the y-axis whereas the ordinate measures the distance of the point to the x-axis. Positive values of the x-coordinate are measured to the right, negative values to the left. Positive values of the y-coordinate are measured up, negative values down. The origin is denoted as $(0, 0)$.
The axes divide the coordinate system into four regions called **quadrants** and are numbered counterclockwise as shown in Figure 2.1
To **plot a point** $P(a, b)$ means to draw a dot at its location in the xy-plane.

Example 2.1
Plot the point P with coordinates $(5, 2)$.

Solution.
Figure 2.1 shows the location of the point $P(5, 2)$ in the xy-plane. ■

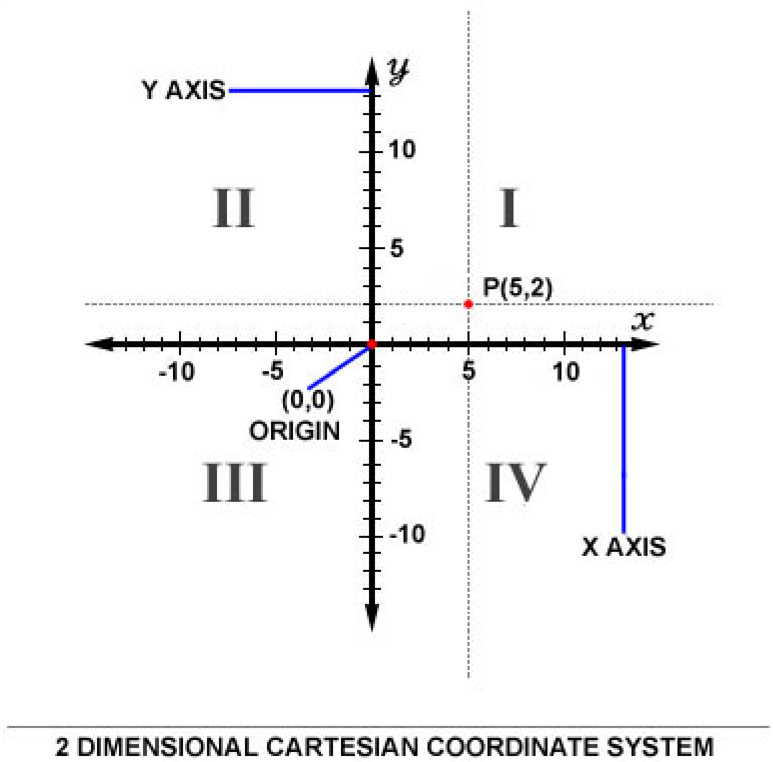


Figure 2.1

Example 2.2
Complete the following table of signs of the coordinates of a point $P(x, y)$.

	x	y
Quadrant I		
Quadrant II		
Quadrant III		
Quadrant IV		
Positive x-axis		
Negative x-axis		
Positive y-axis		
Negative y-axis		

Solution.

	x	y
Quadrant I	+	+
Quadrant II	-	+
Quadrant III	-	-
Quadrant IV	+	-
Positive x-axis	+	0
Negative x-axis	-	0
Positive y-axis	0	+
Negative y-axis	0	-

The Distance Between Two Points
The Distance Formula is a variant of the Pythagorean Theorem that you used back in geometry. Here's how we get from the one to the other: Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let d be the distance between the two points. Construct the right triangle as shown in Figure 2.2.

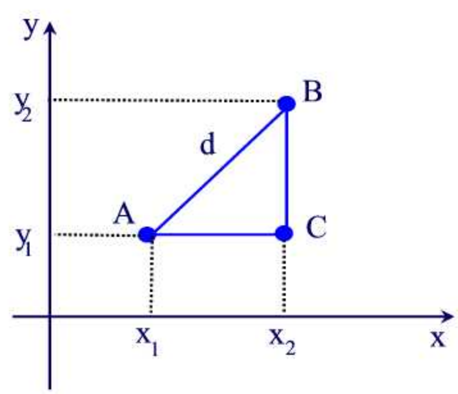


Figure 2.2

By the Pythagorean Theorem we have

$$d^2 = |AC|^2 + |CB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the square root of both sides and recalling that $d > 0$ we obtain the **distance formula**

$$d = d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 2.3
Find the distance between the points $(-5, 8)$ and $(-10, 14)$.

Solution.
Applying the distance formula we find

$$d = \sqrt{(14-8)^2 + (-10-(-5))^2} = \sqrt{36+25} = \sqrt{61}.\blacksquare$$

The Midpoint Formula
The point halfway between the endpoints of a line segment is called the **midpoint**. Thus, a midpoint divides a line segment into two equal parts. Let $M(a, b)$ be the midpoint of the line segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$. See Figure 2.3.

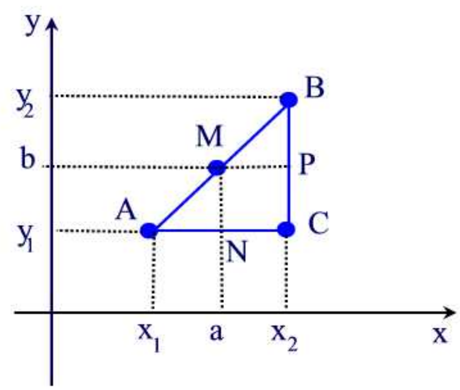


Figure 2.3

The triangles MAN and BMP are similar so that we can write

$$\frac{|MA|}{|BM|} = \frac{|AN|}{|MP|}.$$

But $|MA| = |BM|$ so that $|AN| = |MP|$. Also, $|MP| = |NC|$ so that $|AN| = |NC|$. Thus, N is the midpoint of the line segment with endpoints A and C. It follows that $a - x_1 = x_2 - a$ or $a = \frac{x_1+x_2}{2}$. A similar argument shows that $b = \frac{y_1+y_2}{2}$. Thus, the midpoint M is given by the **midpoint formula**

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example 2.4
Find the midpoint of the line segment with endpoints $A(4, 7)$ and $B(-10, 7)$.

Solution.
Plugging into the midpoint formula we find

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{4+(-10)}{2}, \frac{7+7}{2}\right) \\ &= (-3, 7) \blacksquare \end{aligned}$$

Graph of an Equation
Given an equation involving the two variables x and y. The **graph** of an equation is the set of ordered pairs (x, y) that satisfy the equation.
A typical procedure for graphing an equation is to plot points and then connect them with a continuous curve as shown in the next examples.

Example 2.5
Graph the equation by plotting points: $2x + y = -1$.

Solution.
Writing y in terms of x we find $y = -1 - 2x$. The table below shows some points on the graph of the equation.

x	-2	-1	0	1	2
y	3	1	-1	-3	-5

Next, plot the points and draw a curve through them. See Figure 2.4.■

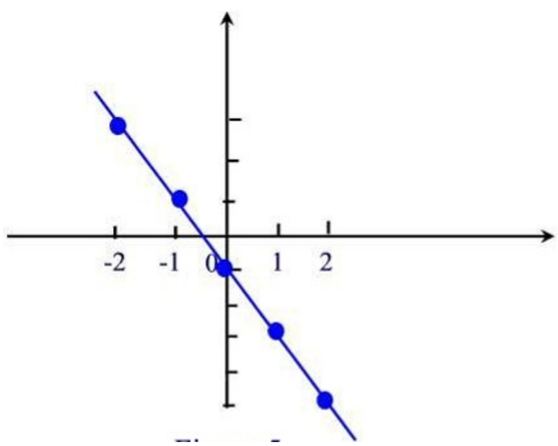


Figure 2.4

Example 2.6
Graph the equation by plotting points: $y = |x + 3| - 2$.

Solution.
The table below shows some points on the graph of the equation.

x	-6	-5	-4	-3	-2	-1	0
y	1	0	-1	-2	-1	0	1

Next, plot the points and draw a curve through them. See Figure 2.5.■

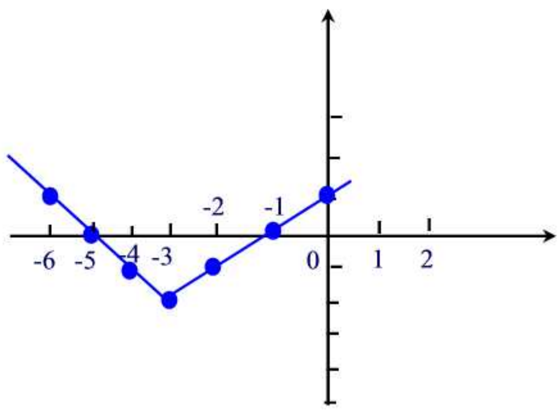


Figure 2.5

Example 2.7
Graph the equation $y = x^2 - 2x - 8$.

Solution.
The table below shows some points on the graph of the equation.

x	-3	- 2	-1	0	1	2	3	4	5
y	7	0	-5	-8	-9	-8	-5	0	7

Next, plot the points and draw a curve through them. See Figure 2.6.■

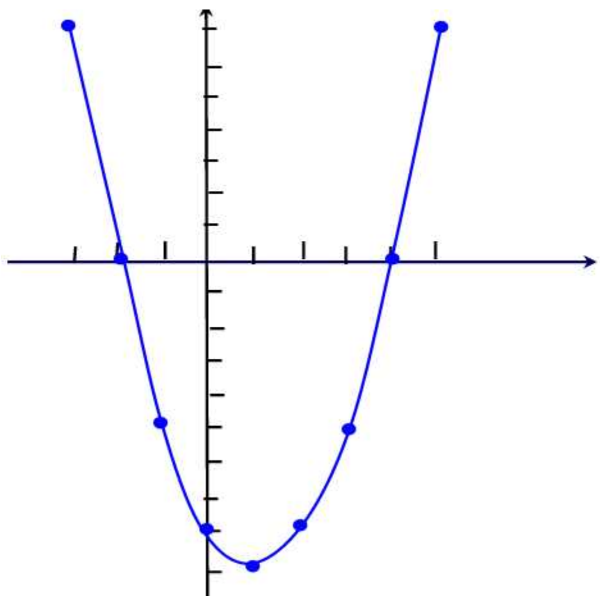


Figure 2.6

Intercepts
A point $(x, 0)$ on the graph of an equation is called the **x-intercept.** Geometrically, the x-intercept is the point where the graph crosses the x-axis. Similarly, a point of the form $(0, y)$ is called the **y-intercept**. This is the point where the graph crosses the y-axis.

Example 2.8
Find the x- and y-intercepts of the graph of $x^2 + y^2 = 4$.

Solution.
Letting $y = 0$ in the given equation we find $x^2 = 4$. Solving for x to obtain $x = \pm 2$. Thus, the x-intercepts are the points $(-2, 0)$ and $(2, 0)$. Similarly, setting $x = 0$ to obtain $y^2 = 4$. Solving for y we obtain $y = \pm 2$. So the points $(0, 2)$ and $(0, -2)$ are the y-intercepts.■

The Equation of a Circle
By a **circle** we mean the collection of all points in the plane that are at an equal distance to a fixed point called the **center** of the circle. The distance of a point on a circle to its center is called the **radius**. The **diameter** of a

circle is the length of a line segment crossing the center and with endpoints on the circle. Thus, the center is the midpoint and as a result a diameter is twice the radius.
Next, we want to find the equation of a circle with center $C(a,b)$ and radius r. For this, let $M(x,y)$ be an arbitrary point on the circle. Then $d(C,M) = r$. By the distance formula, we have

$$(x-a)^2 + (y-b)^2 = r^2.$$

This equation is called the **standard form** of the equation of a circle.

Example 2.9
Determine the center and the radius of the circle with equation: $(x-2)^2 + (y+4)^2 = 25$.

Solution.
The center is the point $(2,-4)$ and the radius is $r = \sqrt{25} = 5.$■

Example 2.10
Find the equation of the circle with center $C(5,-3)$ and radius $r = 4$. Write the answer in standard form.

Solution.
The equation of the circle is given by

$$(x-5)^2 + (y+3)^2 = 16.\blacksquare$$

Example 2.11
Find the equation of the circle with center $C(-2,5)$ and passing through the point $M(1,7)$.

Solution.
The radius of the circle is $r = d(C,M) = \sqrt{(7-5)^2 + (1-(-2))^2} = \sqrt{13}$. Thus, the equation of the circle is

$$(x+2)^2 + (y-5)^2 = 13.\blacksquare$$

Another form of the equation of a circle is known as the **general form** and is given by the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

To find the standard form from the general form we use the process of completing the square as shown in the following example.

Example 2.12
Find the center and the radius of the circle: $x^2 + y^2 - 6x - 4y + 12 = 0$.

Solution.
We use the method of completing the square:

$$\begin{array}{rcl} (x^2 - 6x) + (y^2 - 4y) & = & -12 \\ (x^2 - 6x + 9) + (y^2 - 4y + 4) & = & -12 + 9 + 4 \\ (x - 3)^2 + (y - 2)^2 & = & 1. \end{array}$$

Thus, the center is $(3, 2)$ and the radius is $r = 1$.■

Example 2.13
Find the equation of a circle that has diameter with endpoints $(7, -2)$ and $(-3, 5)$. Write your answer in standard form.

Solution.
The center of the circle is the midpoint of the given diameter. By the midpoint formula, the coordinates of the center are $(\frac{7-3}{2}, \frac{-2+5}{2}) = (2, \frac{3}{2})$. The radius of the circle is the distance between the center and one of the endpoints. This can be found by using the distance formula

$$d = \sqrt{(2-7)^2 + (\frac{3}{2} + 2)^2} = \frac{\sqrt{149}}{2}.$$

The equation of the circle is

$$(x - 2)^2 + (y - \frac{3}{2})^2 = \frac{149}{4}.■$$

Example 2.14
Find an equation of a circle that has its center at $(-2, 3)$ and is tangent to the y-axis. Write your answer in standard form.

Solution.
The radius of the circle is the distance from the center to the y-axis which is the absolute value of the x-coordinate of the the center, i.e. $r = 2$. Hence, the equation of the circle is given by

$$(x + 2)^2 + (y - 3)^2 = 4.■$$

Review Problems

Exercise 2.1
Plot the points whose coordinates are given on a Cartesian coordinate system.

(a) $(2, 4), (0, -3), (-2, 1), (-5, -3)$.
(b) $(-3, -5), (-4, 3), (0, 2), (-2, 0)$.

Exercise 2.2
Find the distance between the points whose coordinates are given.

(a) $(6, 4), (-8, 11)$.
(b) $(5, -8), (0, 0)$.
(c) $(\sqrt{3}, \sqrt{8}), (\sqrt{12}, \sqrt{27})$.
(d) $(x, 4x), (-2x, 3x), \quad x < 0$.

Exercise 2.3
Find the midpoint of the line segment with the following endpoints.

(a) $(1, -1), (5, 5)$.
(b) $(6, -3), (6, 11)$.
(c) $(1.75, 2.25), (-3.5, 5.57)$.

Exercise 2.4
Graph each equation by plotting points that satisfy the equation.

(a) $x - y = 4$.
(b) $y = -2|x - 3|$.
(c) $y = \frac{1}{2}(x - 1)^2$.
(d) $y = x^2 + 2x - 8$.

Exercise 2.5
Find the x- and y-intercepts of each equation.

(a) $2x + 5y = 12$.
(b) $x = |y| - 4$.
(c) $|x| + |y| = 4$.
(d) $|x - 4y| = 8$.

Exercise 2.6
Determine the center and the radius of the circle with the given equation.

(a) $x^2 + y^2 = 36$.
(b) $(x + 2)^2 + (y + 5)^2 = 25$.
(c) $(x - 8)^2 + y^2 = \frac{1}{4}$.

Exercise 2.7
Find the equation of the circle with center $C(4, 1)$ and radius $r = 2$. Write the answer in standard form.

Exercise 2.8
Find the equation of the circle with center $C(0, 0)$ and passing through the point $M(-3, 4)$.

Exercise 2.9
Find the equation of the circle with center $C(1, 3)$ and passing through the point $M(4, -1)$.

Exercise 2.10
Find the center and the radius of each of the following circles.

(a) $x^2 + y^2 - 6x + 5 = 0$.
(b) $4x^2 + 4y^2 + 4x - 63 = 0$.
(c) $x^2 + y^2 - x + 3y - \frac{15}{4} = 0$.

Exercise 2.11
Find the equation of a circle that has diameter with endpoints $(2, 3)$ and $(-4, 11)$. Write your answer in standard form.

Exercise 2.12
Find an equation of a circle that has its center at $(7, 11)$ and is tangent to the x-axis. Write your answer in standard form.

Exercise 2.13
Given the midpoint $M(9, 3)$ of a line segement with endpoints $A(x, y)$ and $B(5, 1)$. Find the coordinates of A.

Exercise 2.14
Find a formula for the set of all points (x, y) for which the distance from (x, y) to $(3, 4)$ is 5.

Exercise 2.15
Find an equation of a circle that is tangent to both axes, has its center in the second quadrant, and has a radius 3.

3 Functions and Function Notation

Functions play a crucial role in mathematics. A function describes how one quantity depends on others. More precisely, when we say that a quantity y is a **function** of a quantity x we mean a rule that assigns to every possible value of x exactly one value of y. We call x the **input** and y the **output**. In **function notation** we write

$$y = f(x).$$

Since y depends on x it makes sense to call x the **independent variable** and y the **dependent variable**.
In applications of mathematics, functions are often representations of real world phenomena. Thus, the functions in this case are referred to as **mathematical models.** If the set of input values is a finite set then the models are known as **discrete** models. Otherwise, the models are known as **continuous** models. For example, if H represents the temperature after t hours for a specific day, then H is a discrete model. If A is the area of a circle of radius r then A is a continuous model.
There are four common ways in which functions are presented and used: By verbal descriptions, by tables, by graphs, and by formulas.

Example 3.1
The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus, we have a relationship between the quantities C and p such that each value of p determines exactly one value of C. In this case, we say that C is a function of p. Describes this function using words, a table, a graph, and a formula.

Solution.
•**Words:** To find the total cost, multiply the price of the item by 0.06 and add the result to the price.
•**Table:** The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

•**Graph:** The graph of the function C is obtained by plotting the data in the above table. See Figure 3.1.
•**Formula:** The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p. \blacksquare$$

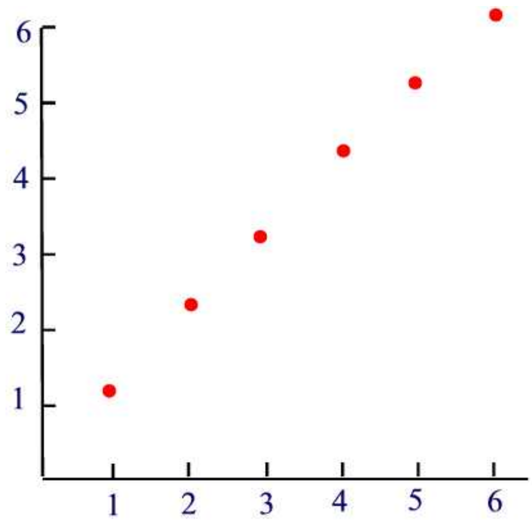


Figure 3.1

Recognizing a Function from a Table
A table can be viewed as a collection of ordered pairs (x, y). Thus, for a collection of data to define a function we need to show that every first component x corresponds to exactly one component y. Thus, if there are ordered pairs with the same x value but different y values then the collection of ordered pairs in *not* a function.

Example 3.2
Identify the set of ordered pairs (x, y) that define y as a function of x.

(a) $\{(5, 10), (3, -2), (4, 7), (5, 8)\}$.
(b) $\{(2, 2), (3, 3), (7, 2)\}$.

Solution.
(a) The first set does not define a function since the ordered pairs $(5, 10)$ and $(5, 8)$ have the same first component with different second components.
(b) This set defines a function since all the first components are different.$\blacksquare$

Recognizing a Function from an Equation
Suppose that an equation in the variables x and y is given. If for a given value of x, you solve the equation for y and you get exactly one value then the equation defines a function.

Example 3.3
Identify the equations that define y as a function of x.

(a) $x^2 - 2y = 2$.
(b) $x^2 + y^2 = 1$.

Solution.
(a) Solving the equation for y we find $y = \frac{x^2}{2} - 1$. Thus, each value of x yields exactly one value of y. This shows that y is a function of x.
(b) Solving for y to obtain $y = \pm\sqrt{1 - x^2}$. Thus, if we let $x = 0$ then $y = \pm 1$. Hence, y is not a function of x.$\blacksquare$

Recognizing a Function from a Graph
Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 3.4
Which of the graphs (a), (b), (c) in Figure 3.2 represent y as a function of x?

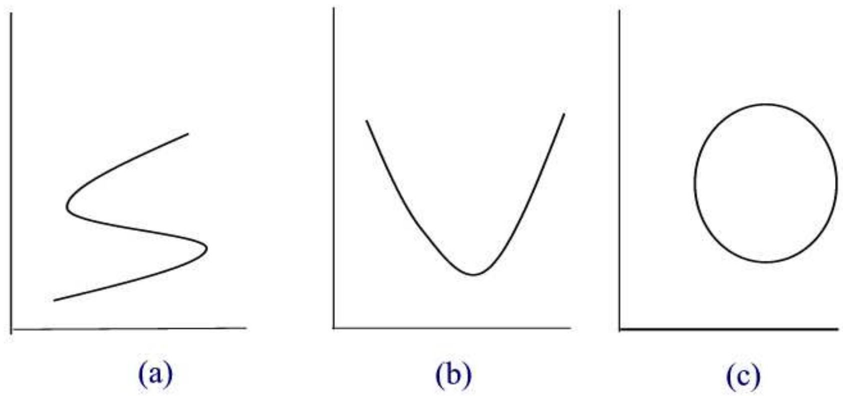


Figure 3.2

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

Evaluating a Function
By evaluating a function, we mean figuring out the output value corresponding to a given input value. Thus, notation like $f(10) = 4$ means that the function's output, corresponding to the input 10, is equal to 4.
If the function is given by a formula, say of the form $y = f(x)$, then to find the output value corresponding to an input value a we replace the letter x in the formula of f by the input a and then perform the necessary algebraic operations to find the output value.

Example 3.5
Let $g(x) = \frac{x^2+1}{5+x}$. Evaluate the following expressions:

(a) $g(2)$ (b) $g(a)$ (c) $g(a) - 2$ (d) $g(a) - g(2)$.

Solution.
(a) $g(2) = \frac{2^2+1}{5+2} = \frac{5}{7}$
(b) $g(a) = \frac{a^2+1}{5+a}$
(c) $g(a) - 2 = \frac{a^2+1}{5+a} - 2\frac{5+a}{5+a} = \frac{a^2-2a-9}{5+a}$
(d) $g(a) - g(2) = \frac{a^2+1}{5+a} - \frac{5}{7} = \frac{7(a^2+1)}{7(5+a)} - \frac{5}{7}\frac{5+a}{5+a} = \frac{7a^2-5a-18}{7a+35}$.■

Domain and Range of a Function
If we try to find the possible input values that can be used in the function $y = \sqrt{x-2}$ we see that we must restrict x to the interval $[2, \infty)$, that is $x \geq 2$. Similarly, the function $y = \frac{1}{x^2}$ takes only certain values for the output, namely, $y > 0$. Thus, a function is often defined for certain values of x and the dependent variable often takes certain values.
The above discussion leads to the following definitions: By the **domain** of a function we mean all possible input values that yield one output value. Graphically, the domain is part of the horizontal axis. The **range** of a function is the collection of all possible output values. The range is part of the vertical axis.
When finding the domain of a function, ask yourself what values can't be used. Your domain is everything else. There are simple basic rules to consider:

• The domain of all polynomial functions, i.e. functions of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is nonnegative integer, is the Real numbers $\mathbb{R}$.
• Square root functions can not contain a negative underneath the radical. Set the expression under the radical greater than or equal to zero and solve for the variable. This will be your domain.
• Fractional functions, i.e. ratios of two functions, determine for which input values the numerator and denominator are not defined and the domain is everything else. For example, make sure not to divide by zero!

Example 3.6
Find, algebraically, the domain and the range of each of the following functions. Write your answers in interval notation:
(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

Solution.
(a) Since the function is a polynomial then its domain is the interval $(-\infty, \infty)$.

To find the range, solve the given equation for x in terms of y obtaining $x = \pm\sqrt{\frac{y}{\pi}}$. Thus, x exists for $y \geq 0$. So the range is the interval $[0, \infty)$.

(b) The domain of $y = \frac{1}{\sqrt{x-4}}$ consists of all numbers x such that $x - 4 > 0$ or $x > 4$. That is, the interval $(4, \infty)$. To find the range, we solve for x in terms of $y > 0$ obtaining $x = 4 + \frac{1}{y^2}$. x exists for all $y > 0$. Thus, the range is the interval $(0, \infty)$.

(c) The domain of $y = 2 + \frac{1}{x}$ is the interval $(-\infty, 0) \cup (0, \infty)$. To find the range, write x in terms of y to obtain $x = \frac{1}{y-2}$. The values of y for which this later formula is defined is the range of the given function, that is, $(-\infty, 2) \cup (2, \infty)$.■

Piecewise Defined Functions

Piecewise-defined functions are functions defined by different formulas for different intervals of the independent variable.

Example 3.7 *(The Absolute Value Function)*
(a) Show that the function $f(x) = |x|$ is a piecewise defined function.
(b) Graph $f(x)$.

Solution.
(a) The absolute value function $|x|$ is a piecewise defined function since

$$|x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

(b) The graph is given in Figure 3.3.■

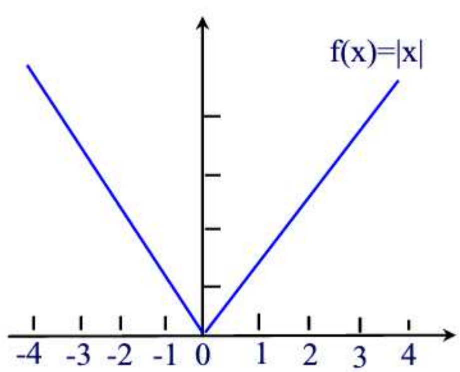


Figure 3.3

Example 3.8 *(The Ceiling Function)*
The Ceiling function $f(x) = \lceil x \rceil$ is the piecewise defined function given by

$$\lceil x \rceil = \text{smallest integer greater than or equal to x.}$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

Solution.
The graph is given in Figure 3.4. An open circle represents a point which is not included.■

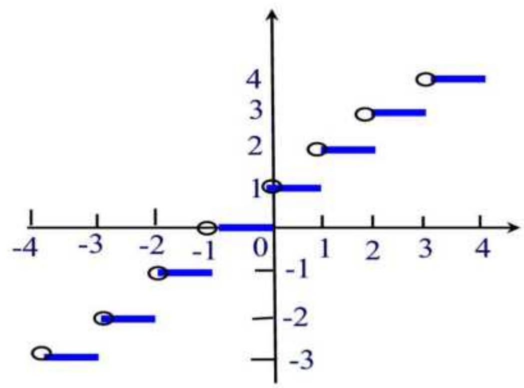


Figure 3.4

Example 3.9 *(The Floor Function)*
The Floor function $f(x) = \lfloor x \rfloor$ is the piecewise defined function given by

$$\lfloor x \rfloor = \text{greatest integer less than or equal to x.}$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

Solution.
The graph is given in Figure 3.5.■

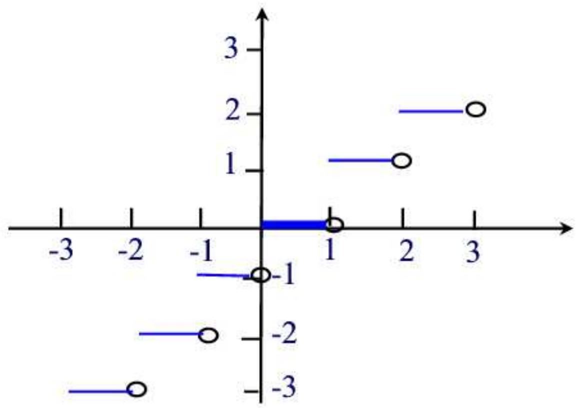


Figure 3.5

Example 3.10
Sketch the graph of the piecewise defined function given by

$$f(x) = \begin{cases} x+4 & \text{for } x \leq -2, \\ 2 & \text{for } -2 < x < 2, \\ 4-x & \text{for } x \geq 2. \end{cases}$$

Solution.
The following table gives values of $f(x)$.

x	-3	-2	-1	0	1	2	3
f(x)	1	2	2	2	2	2	1

The graph of the function is given in Figure 3.6.■

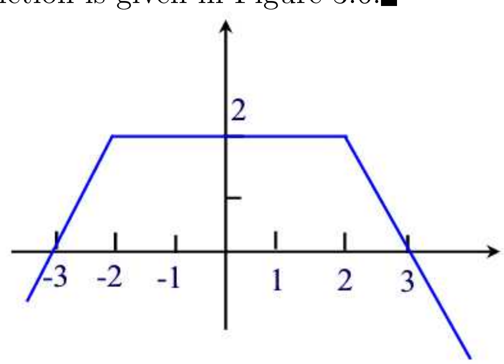


Figure 3.6

We next give a real-world situation where piecewise functions can be used.

Example 3.11
The charge for a taxi ride is \$1.50 for the first $\frac{1}{5}$ of a mile, and \$0.25 for each additional $\frac{1}{5}$ of a mile (rounded up to the nearest $\frac{1}{5}$ mile).

(a) Sketch a graph of the cost function C as a function of the distance traveled x, assuming that $0 \leq x \leq 1$.
(b) Find a formula for C in terms of x on the interval $[0, 1]$.
(c) What is the cost for a $\frac{4}{5} - mile$ ride?

Solution.
(a) The graph is given in Figure 3.7.

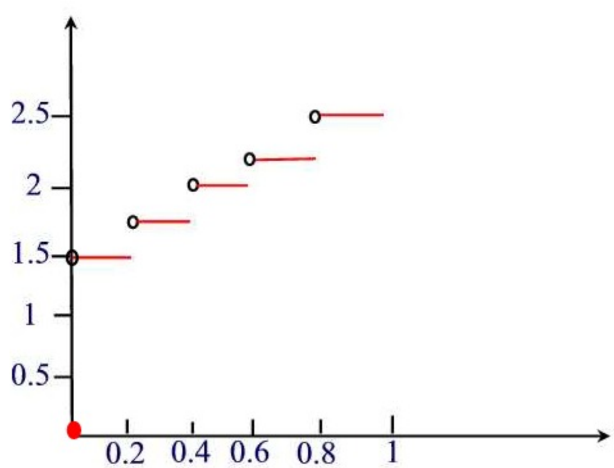


Figure 3.7

(b) A formula of $C(x)$ is

$$C(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1.50 & \text{if } 0 < x \leq \frac{1}{5}, \\ 1.75 & \text{if } \frac{1}{5} < x \leq \frac{2}{5}, \\ 2.00 & \text{if } \frac{2}{5} < x \leq \frac{3}{5}, \\ 2.25 & \text{if } \frac{3}{5} < x \leq \frac{4}{5}, \\ 2.50 & \text{if } \frac{4}{5} < x \leq 1. \end{cases}$$

(c) The cost for a $\frac{4}{5}$ ride is $C(\frac{4}{5}) = \$2.25$.■

Increasing and Decreasing Functions

We say that a function is **increasing** if its graph climbs as x moves from left to right. That is, the function values increase as x increases. It is said to be **decreasing** if its graph falls as x moves from left to right. This means that the function values decrease as x increases.

Example 3.12

Determine the intervals where the function, given in Figure 3.8, is increasing and decreasing.

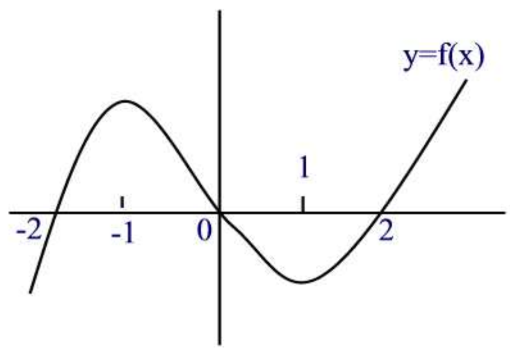


Figure 3.8

Solution.

The function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on the interval $(-1, 1)$.■

One-To-One Functions

We have seen that when every vertical line crosses a curve at most once then the curve is the graph of a function f. We called this procedure the **vertical line test.** Now, if every horizontal line crosses the graph at most once then the function is called **one-to-one**.

Remark 3.1

The test used to identify one-to-one functions which we discussed above is referred to as the **horizontal line test**.

Example 3.13

Use a graphing calculator to decide whether or not the function is one-to-one.

(a) $f(x) = x^3 + 7$. (b) $g(x) = |x|$.

Solution.
(a) Using a graphing calculator, the graph of $f(x)$ is given in Figure 3.9.

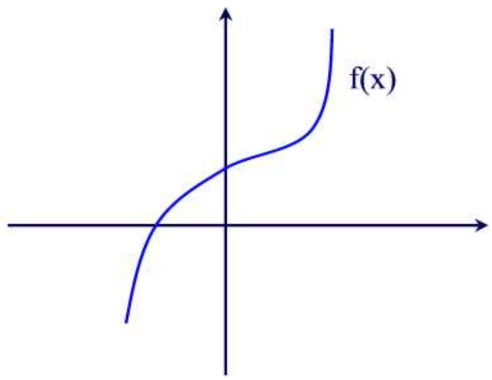


Figure 3.9

We see that every horizontal line crosses the graph once so the function is one-to-one.
(b) The graph of $g(x) = |x|$ (See Figure 3.3) shows that there are horizontal lines that cross the graph twice so that g is not one-to-one.■

Review Problems

Exercise 3.1
Given $f(x) = 3x^2 - 1$, find

(a) $f(-4)$ (b) $f\left(\frac{1}{3}\right)$ (c) $f(-a)$ (d) $f(x+h)$ (e) $f(x+h) - f(x)$.

Exercise 3.2
Given $f(x) = \frac{x}{|x|}$, find

(a) $f(4)$ (b) $f(-2)$ (c) $f(x), x > 0$ (d) $f(x), x < 0$.

Exercise 3.3
Given

$$f(x) = \begin{cases} 3x+1, & \text{if } x < 2 \\ -x^2+11, & \text{if } x \geq 2. \end{cases}$$

Evaluate: (a) $f(-4)$ (b) $f(\sqrt{5})$ (c) $f(x), x < 2$ (d) $f(x+1), x \geq 1$.

Exercise 3.4
Identify the equations that define y as a function of x.

(a) $2x + 3y = 7$.
(b) $-x + y^2 = 2$.
(c) $y = 4 \pm \sqrt{x}$.
(d) $y^2 = x^2$.

Exercise 3.5
Identify the collection of ordered pairs (x, y) that define y as a function of x.

(a) $\{(2,3),(5,1),(-4,3),(7,11)\}$.
(b) $\{(5,10),(3,-2),(4,7),(5,8)\}$.
(c) $\{(1,0),(2,0),(3,0)\}$.

Exercise 3.6
Determine the domain of the function. Write answers in interval notation.

(a) $f(x) = 3x - 4$.
(b) $g(x) = x^2 + 2$.

(c) $h(x) = \frac{4}{x+2}$.
(d) $i(x) = \sqrt{4 - x^2}$.
(e) $j(x) = \frac{1}{\sqrt{x+4}}$.

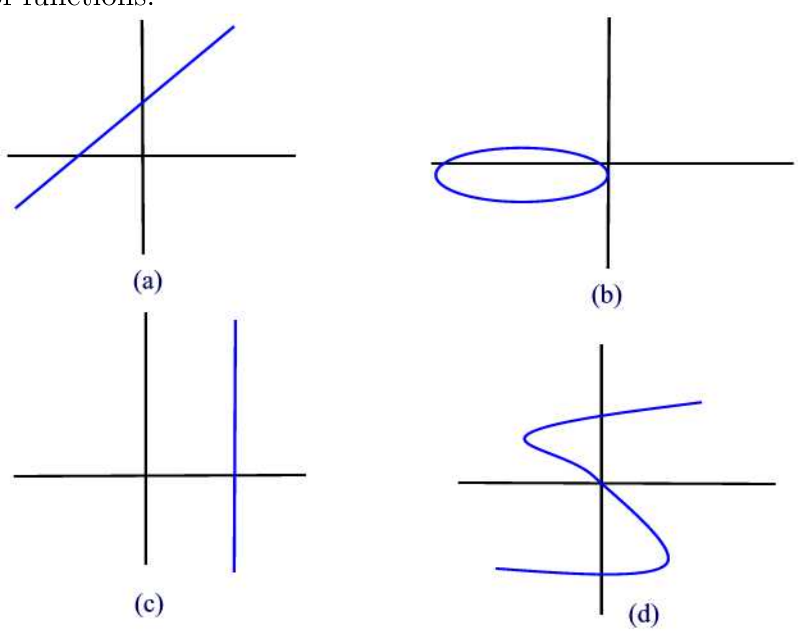
Exercise 3.7
Graph each function. Insert solid circle or hollow circles to indicate the true nature of the function. (a)

$$f(x) = \begin{cases} |x|, & \text{if } x \leq 1 \\ 2, & \text{if } x > 1. \end{cases}$$

(b)

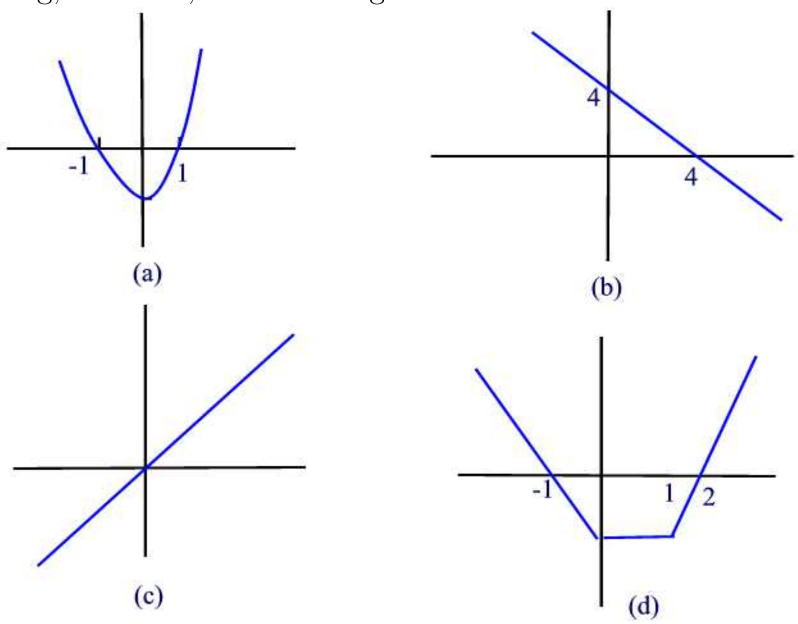
$$g(x) = \begin{cases} 4, & \text{if } x \leq -1 \\ x^2, & \text{if } -1 < x < 1 \\ -x + 5, & \text{if } x \geq 1. \end{cases}$$

Exercise 3.8
Use the vertical line test to determine which of the following graphs are graphs of functions.

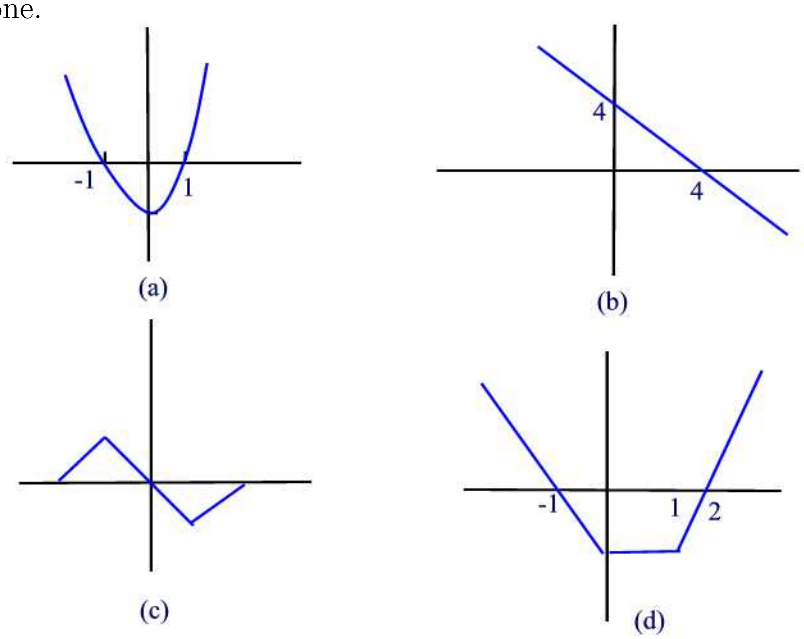


Exercise 3.9
Use the indicated graphs to indentify the intervals over which the function

is increasing, constant, or decreasing.



Exercise 3.10
Use the horizontal line test to determine which of the following functions are one-to-one.



Exercise 3.11
A bus was purchased for \$80,000. Assumeing the bus depreciates linearly at a rate of \$6,500 per year for the first 10 years, write the value v of the bus as a function of the time t (measured in years) for $0 \leq t \leq 10$.

Exercise 3.12
A manufacturer produces a product at a cost of \$22.80 per unit. The manufacturer has a fixed cost of \$400,000 per day. Each unit retails for \$37.00. Let x represent the number of units produced in a 5-day period.

(a) Write the total cost C as a function of x.
(b) Write the revenue R as a function of x.
(c) Write the profit P as a function of x.

Exercise 3.13
An open box is to be made from a square piece of cardboard having dimensions 30 inches by 30 inches by cutting out squares of area x^2 from each corner.

(a) Express the volume V of the box as a function of x.
(b) State the domain of V.

Exercise 3.14
If $f(x) = x^2 - x - 5$ and $f(c) = 1$, find the value of c.

Exercise 3.15
Determine whether 1 is in the range of $f(x) = \frac{x-1}{x+1}$.

Exercise 3.16
Determine whether 0 is in the range of $f(x) = \frac{1}{x-3}$.

4 Transformations of Graphs

Throughout this section we consider the relationship between changes made to the formula of a function and the corresponding changes made to its graph. The resulting changes in the graph will consist of shifting, fliping, compressing, and stretching of the original graph.

Reflections and Symmetry
Reflections occur when either the input or the output of a function is multiplied by -1.

Reflection About the x-Axis
For a given function $f(x)$, the points $(x, f(x))$ and $(x, -f(x))$ are on opposite sides of the x-axis. So the graph of the new function $-f(x)$ is the reflection of the graph of $f(x)$ about the x-axis.

Example 4.1
Graph the functions $f(x) = 2^x$ and $-f(x) = -2^x$ on the same axes.

Solution.
The graphs of both $f(x) = 2^x$ and $-f(x)$ are shown in Figure 4.1.■

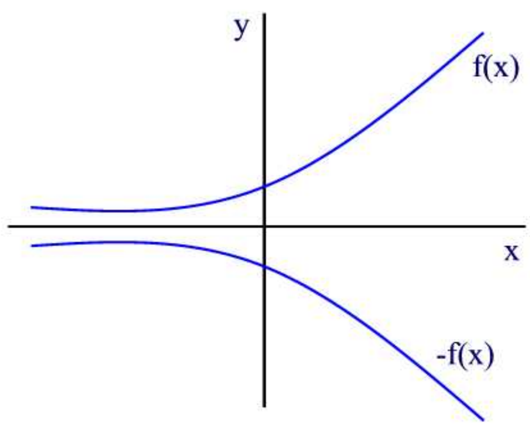


Figure 4.1

Reflection About the y-Axis
We know that the points $(x, f(x))$ and $(-x, f(x))$ are on opposite sides of the y-axis. So the graph of the new function $f(-x)$ is the reflection of the graph of $f(x)$ about the y-axis.

Example 4.2
Graph the functions $f(x) = x^3$ and $f(-x) = -x^3$ on the same axes.

Solution.
The graphs of both $f(x)$ and $f(-x)$ are shown in Figure 4.2.■

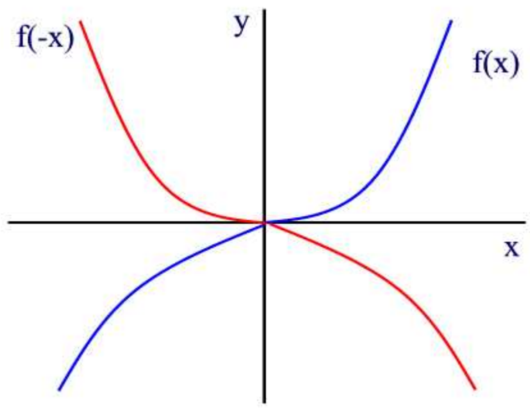


Figure 4.2

Symmetry About the y-Axis
When constructing the graph of $f(-x)$ sometimes you will find that this new graph is the same as the graph of the original function. That is, the reflection of the graph of $f(x)$ about the y-axis is the same as the graph of $f(x)$,e.g., $f(-x) = f(x)$. In this case, we say that the graph of $f(x)$ is symmetric about the y-axis. We call such a function an **even** function.

Example 4.3
(a) Using a graphing calculator show that the function $f(x) = (x - x^3)^2$ is even.
(b) Now show that $f(x)$ is even algebraically.

Solution.
(a) The graph of $f(x)$ is symmetric about the y-axis so that $f(x)$ is even. See Figure 4.3.

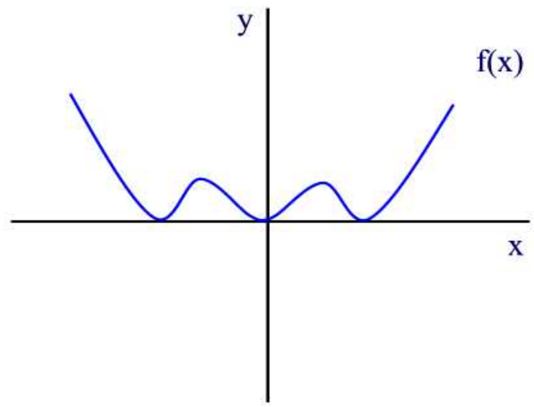


Figure 4.3

(b) Since $f(-x) = (-x-(-x)^3)^2 = (-x+x^3)^2 = [-(x-x^3)]^2 = (x-x^3)^2 = f(x)$ then $f(x)$ is even.■

Symmetry About the Origin
Now, if the images $f(x)$ and $f(-x)$ are of opposite signs i.e, $f(-x) = -f(x)$, then the graph of $f(x)$ is symmetric about the origin. In this case, we say that $f(x)$ is **odd.** Alternatively, since $f(x) = -f(-x)$, if the graph of a function is reflected first across the y-axis and then across the x-axis and you get the graph of $f(x)$ again then the function is odd.

Example 4.4
(a) Using a graphing calculator show that the function $f(x) = \frac{1+x^2}{x-x^3}$ is odd.
(b) Now show that $f(x)$ is odd algebraically.

Solution.
(a) The graph of $f(x)$ is symmetric about the origin so that $f(x)$ is odd. See Figure 4.4.

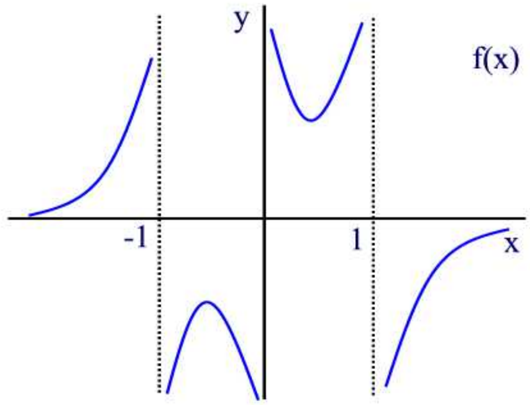


Figure 4.4

(b) Since $f(-x) = \frac{1+(-x)^2}{(-x)-(-x)^3} = \frac{1+x^2}{-x+x^3} = \frac{1+x^2}{-(x-x^3)} = -f(x)$ then $f(x)$ is odd.■

A function can be either even, odd, or neither.

Example 4.5
(a) Show that the function $f(x) = x^2$ is even but not odd.
(b) Show that the function $f(x) = x^3$ is odd but not even.
(c) Show that the function $f(x) = x + x^2$ is neither odd nor even.
(d) Is there a function that is both even and odd? Explain.

Solution.
(a) Since $f(-x) = (-x)^2 = x^2 = f(x)$ and $f(-x) \neq -f(x)$ then $f(x)$ is even but not odd.
(b) Since $f(-x) = (-x)^3 = -x^3 = -f(x)$ and $f(-x) \neq f(x)$ then $f(x)$ is odd but not even.
(c) Since $f(-x) = -x + x^2 \neq \pm f(x)$ then $f(x)$ is neither even nor odd.
(d) We are looking for a function such that $f(-x) = f(x)$ and $f(-x) = -f(x)$. This implies that $f(x) = -f(x)$ or $2f(x) = 0$. Dividing by 2 to obtain $f(x) = 0$. This function is both even and odd. This is the only function that is both even and odd.■

Vertical and Horizontal Shifts
Given the graph of a function, by shifting this graph vertically or horizontally one gets the graph of a new function. In this section we want to find the

formula for this new function using the formula of the original function.

Vertical Shifts
We start with an example of a vertical shift.

Example 4.6
Let $f(x) = x^2$.
(a) Use a calculator to graph the function $g(x) = x^2 + 1$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Use a calculator to graph the function $h(x) = x^2 - 1$. How does the graph of $h(x)$ compare to the graph of $f(x)$?

Solution.
(a) In Figure 4.5 we have included the graph of $g(x) = x^2 + 1 = f(x) + 1$. This shows that if $(x, f(x))$ is a point on the graph of $f(x)$ then $(x, f(x)+1)$ is a point on the graph of $g(x)$. Thus, the graph of $g(x)$ is obtained from the old one by moving it up 1 unit.

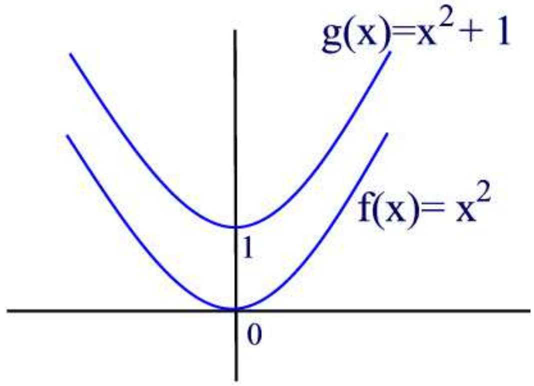


Figure 4.5

(b) Figure 4.6 shows the graph of both $f(x)$ and $h(x)$. Note that $h(x) = f(x) - 1$ and the graph of $h(x)$ is obtained from the graph of $f(x)$ by moving it 1 unit down.■

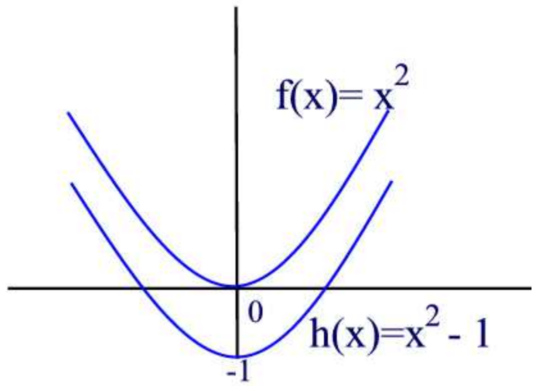


Figure 4.6

In general, if $c > 0$, the graph of $f(x) + c$ is obtained by shifting the graph of $f(x)$ upward a distance of c units. The graph of $f(x) - c$ is obtained by shifting the graph of $f(x)$ downward a distance of c units.

Horizontal Shifts
This discussion parallels the one earlier in this section. Follow the same general directions.

Example 4.7
Let $f(x) = x^2$.
(a) Use a calculator to graph the function $g(x) = (x+1)^2 = f(x+1)$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Use a calculator to graph the function $h(x) = (x-1)^2 = f(x-1)$. How does the graph of $h(x)$ compare to the graph of $f(x)$?

Solution.
(a) In Figure 4.7 we have included the graph of $g(x) = (x+1)^2$. We see that the new graph is obtained from the old one by shifting to the left 1 unit. This is as expected since the value of x^2 is the same as the value of $(x+1)^2$ at the point 1 unit to the left.

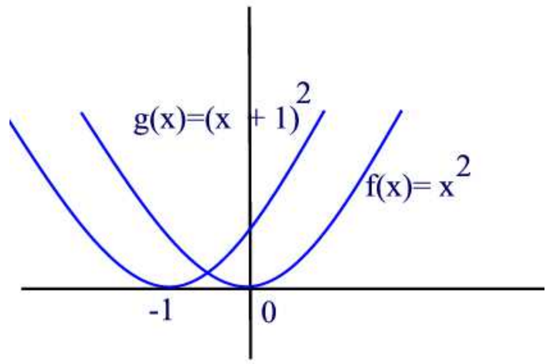


Figure 4.7

(b) Similar to (a), we see in Figure 4.8 that we get the graph of $h(x)$ by moving the graph of $f(x)$ to the right 1 unit.■

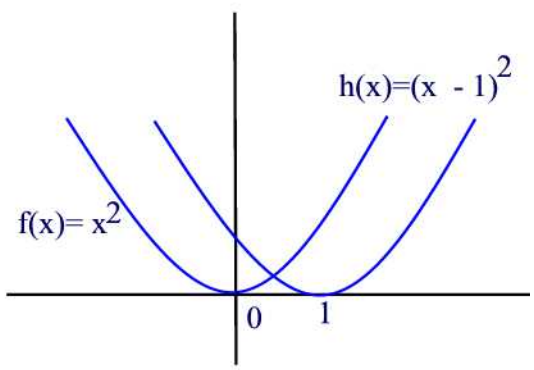


Figure 4.8

In general, if $c > 0$, the graph of $f(x + c)$ is obtained by shifting the graph of $f(x)$ to the left a distance of c units. The graph of $f(x - c)$ is obtained by shifting the graph of $f(x)$ to the right a distance of c units.

Remark 4.1
Be careful when translating graph horizontally. In determing the direction of

horizontal shifts we look for the value of x that would cause the expression between parentheses equal to 0. For example, the graph of $f(x-5) = (x-5)^2$ is the graph of $f(x) = x^2$ shifted 5 units to the right since $+5$ would cause the quantity $x-5$ to equal 0. On the other hand, the graph of $f(x+5) = (x+5)^2$ is the graph of $f(x) = x^2$ shifted 5 units to the left since -5 would cause the expression $x + 5$ to equal 0.

Combinations of Vertical and Horizontal Shifts
One can use a combination of both horizontal and vertical shifts to create new functions as shown in the next example.

Example 4.8
Let $f(x) = x^2$. Let $g(x)$ be the function obtained by shifting the graph of $f(x)$ two units to the right and then up three units. Find a formula for $g(x)$ and then draw its graph.

Solution.
The formula of $g(x)$ is $g(x) = f(x-2) + 3 = (x-2)^2 + 3 = x^2 - 4x + 7$. The graph of $g(x)$ consists of a horizontal shift of x^2 of two units to the right followed by a vertical shift of three units upward. See Figure 4.9.■

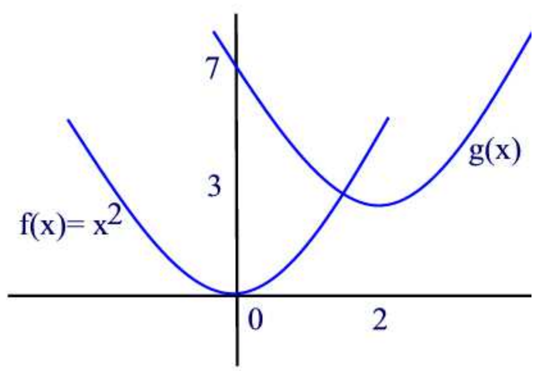


Figure 4.9

Combinations of Shifts and Reflections
We can obtain more complex functions by combining the horizontal and vertical shifts with the horizontal and vertical reflections.

Example 4.9
Let $f(x) = 2^x$.
(a) Suppose that $g(x)$ is the function obtained from $f(x)$ by first reflecting about the y-axis, then translating down three units. Write a formula for $g(x)$.
(b) Suppose that $h(x)$ is the function obtained from $f(x)$ by first translating up two units and then reflecting about the x-axis. Write a formula for $h(x)$.

Solution.
(a) $g(x) = f(-x) - 3 = 2^{-x} - 3$.
(b) $h(x) = -(f(x) + 2) = -2^x - 2$.■

Vertical Stretches and Compressions
We have seen that for a positive k, the graph of $f(x) + k$ is a vertical shift of the graph of $f(x)$ upward and the graph of $f(x) - k$ is a vertical shift down. In this section we want to study the effect of multiplying a function by a constant k. This will result by either a vertical stretch or vertical compression. A **vertical stretching** is the stretching of the graph away from the x-axis. A **vertical compression** is the squeezing of the graph towards the x-axis.

Example 4.10
(a) Complete the following tables

x	$y = x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = 2x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = 3x^2$
-3	
-2	
-1	
0	
1	
2	
3	

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

Solution.
(a)

x	$y = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

x	$y = 2x^2$
-3	18
-2	8
-1	2
0	0
1	2
2	8
3	18

x	$y = 3x^2$
-3	27
-2	12
-1	3
0	0
1	3
2	12
3	27

(b) Figure 4.10 shows that the graphs of $2f(x)$ and $3f(x)$ are vertical stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively. ■

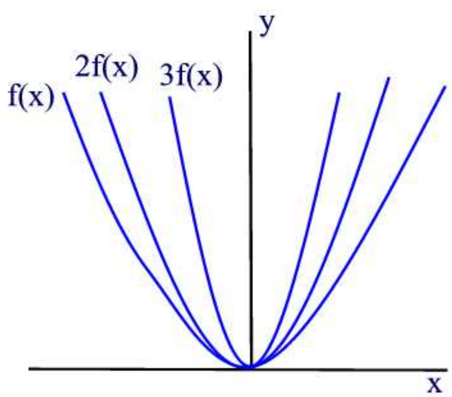


Figure 4.10

Example 4.11
(a) Complete the following tables

x	$y = x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = \frac{1}{2}x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = \frac{1}{3}x^2$
-3	
-2	
-1	
0	
1	
2	
3	

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

Solution.
(a)

x	$y = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

x	$y = \frac{1}{2}x^2$
-3	4.5
-2	2
-1	0.5
0	0
1	0.5
2	2
3	4.5

x	$y = \frac{1}{3}x^2$
-3	3
-2	$\frac{4}{3}$
-1	$\frac{1}{3}$
0	0
1	$\frac{1}{3}$
2	$\frac{4}{3}$
3	3

(b) Figure 4.11 shows that the graphs of $\frac{1}{2}f(x)$ and $\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respectively.■

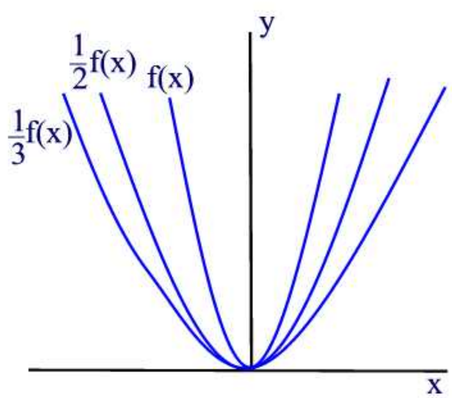


Figure 4.11

Summary
It follows that if a function $f(x)$ is given, then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of k for $k > 1$, and a vertical compression for $0 < k < 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ followed by a reflection about the x-axis. If $0 < |k| < 1$ then the graph of $kf(x)$ is a vertical compression of the graph of $f(x)$ by a factor of k followed by a reflection about the x-axis.

Example 4.12
(a) Use a graphing calculator to graph the functions $f(x) = x^2$, $-2f(x)$, and

$-3f(x)$ on the same axes.
(b) Use a graphing calculator to graph the functions $f(x) = x^2$, $-\frac{1}{2}f(x)$, and -$\frac{1}{3}f(x)$ on the same axes.

Solution.
(a) Figure 4.12 shows that the graphs of $-2f(x)$ and $-3f(x)$ are vertical stretches followed by reflections about the x-axis of the graph of $f(x)$

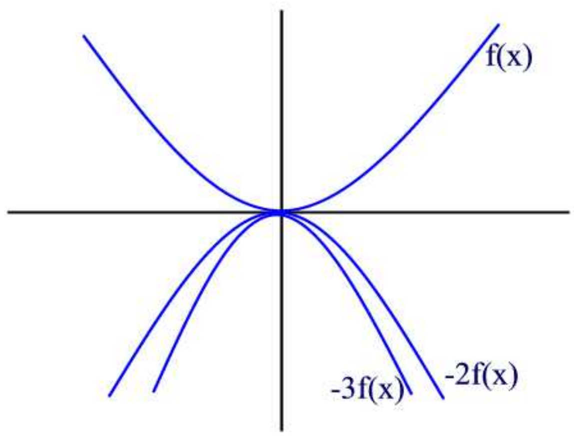


Figure 4.12

(b) Figure 4.13 shows that the graphs of $-\frac{1}{2}f(x)$ and $-\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$.■

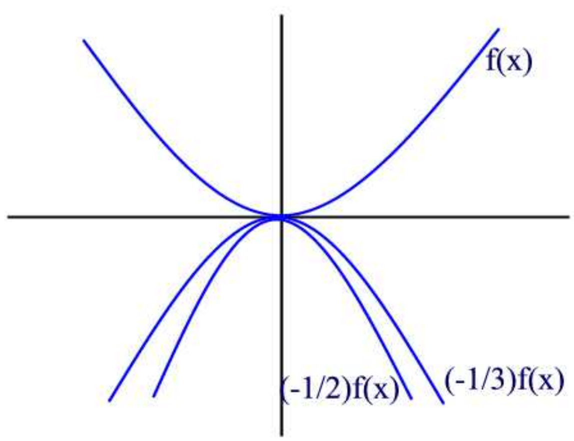


Figure 4.13

Combinations of Shifts
Any transformations of vertical, horizontal shifts, reflections, vertical stretches or compressions can be combined to generate new functions. In this case, always work from inside the parentheses outward.

Example 4.13
How do you obtain the graph of $g(x) = -\frac{1}{2}f(x+3) - 1$ from the graph of $f(x)$?

Solution.
The graph of $g(x)$ is obtained by first shifting the graph of $f(x)$ to the left by 3 units then the resulting graph is compressed vertically by a factor of $\frac{1}{2}$ followed by a reflection about the x-axis and finally moving the graph down by 1 unit.■

Horizontal Stretches and Compressions
A vertical stretch or compression results from multiplying the outside of a function by a constant k. In this section we will see that multiplying the inside of a function by a constant k results in either a horizontal stretch or compression.
A **horizontal stretching** is the stretching of the graph away from the y-axis. A **horizontal compression** is the squeezing of the graph towards the y-axis.
We consider first the effect of multiplying the input by $k > 1$.

Example 4.14
(a) Complete the following tables

x	-3	-2	-1	0	1	2	3
$y = x^2$							
$y = (2x)^2$							
$y = (3x)^2$							

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

Solution.
(a)

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$y = (2x)^2$	36	16	4	0	4	16	36
$y = (3x)^2$	81	36	9	0	9	36	81

(b) Figure 4.14 shows that the graphs of $f(2x) = (2x)^2 = 4x^2$ and $f(3x) = (3x)^2 = 9x^2$ are horizontal compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respecitvely.■

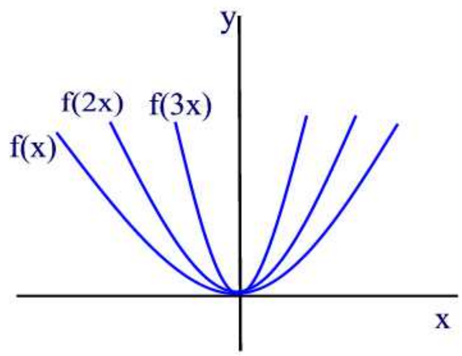


Figure 4.14

Next, we consider the effect of multiplying the input by $0 < k < 1$.

Example 4.15
(a) Complete the following tables

x	-3	-2	-1	0	1	2	3
$y = x^2$							
$y = (\frac{1}{2}x)^2$							
$y = (\frac{1}{3}x)^2$							

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

Solution.
(a)

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$y = (\frac{1}{2}x)^2$	$\frac{9}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$\frac{9}{4}$
$y = (\frac{1}{3}x)^2$	1	$\frac{4}{9}$	$\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{4}{9}$	1

(b) Figure 4.15 shows that the graphs of $f(\frac{x}{2})$ and $f(\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively.■

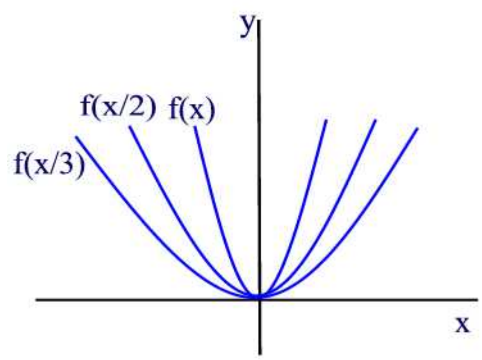


Figure 4.15

Summary
It follows from the above two examples that if a function $f(x)$ is given, then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ for $0 < k < 1$, and a horizontal compression for $k > 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $f(kx)$ is a horizontal compression of the graph of $f(x)$ followed by a reflection about the y-axis. If $0 < |k| < 1$ then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ followed by a reflection about the y-axis.

Example 4.16
(a) Use a graphing calculator to graph the functions $f(x) = x^3, f(-2x)$, and $f(-3x)$ on the same axes.
(b) Use a graphing calculator to graph the functions $f(x) = x^3, f(-\frac{x}{2})$, and $f(-\frac{x}{3})$ on the same axes.

Solution.
(a) Figure 4.16 shows that the graphs of $f(-2x)$ and $f(-3x)$ are vertical stretches followed by reflections about the y-axis of the graph of $f(x)$

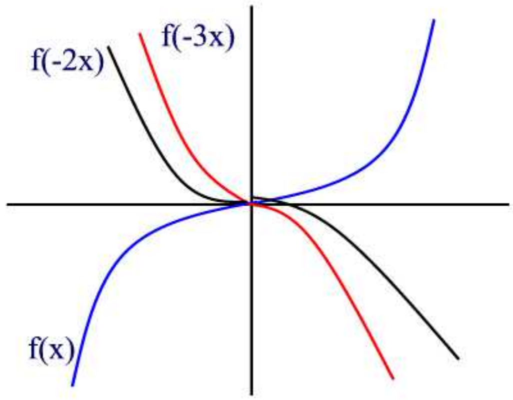


Figure 4.16

(b) Figure 4.17 shows that the graphs of $f(-\frac{x}{2})$ $f(-\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$.■

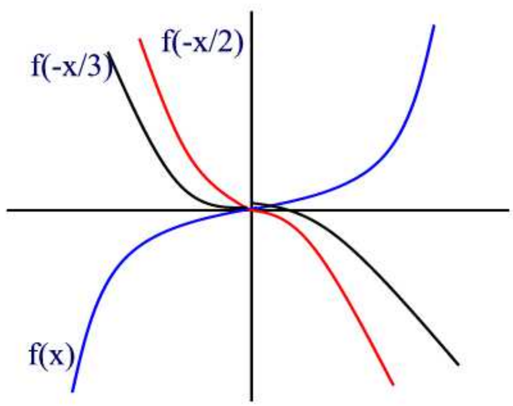
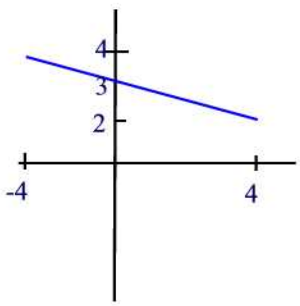
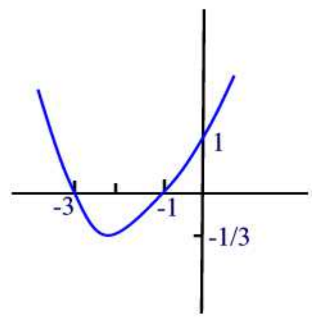


Figure 4.17

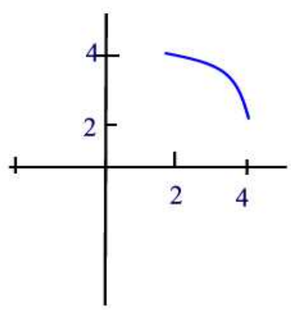
Review Problems

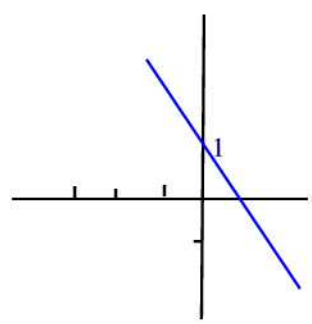
Exercise 4.1
Sketch a graph that is symmetric to the given graph with respect to the x-axis.



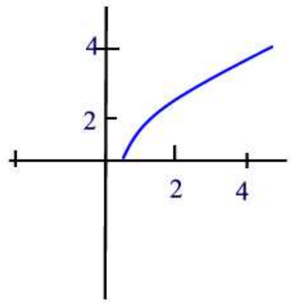


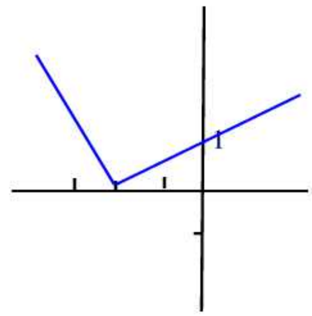
Exercise 4.2
Sketch a graph that is symmetric to the given graph with respect to the y-axis.





Exercise 4.3
Sketch a graph that is symmetric to the given graph with respect to the origin.





Exercise 4.4
Determine whether the graph of each equation is symmetric with respect to (a) x-axis (b) y-axis.

(a) $y = 2x^2 - 5$ (b) $y = x^5 - 3x$ (c) $x^2 + y^2 = 9$ (d) $xy = 8$.

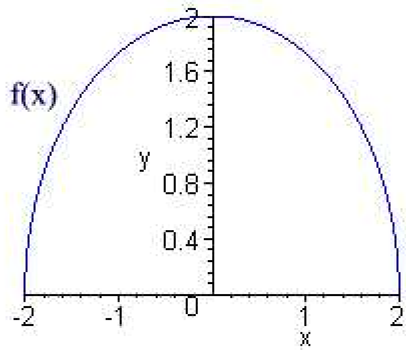
Exercise 4.5
Determine whether the graph of each equation is symmetric with respect to the origin.

(a) $y = 3x - 2$ (b) $y = x^3 - x$ (c) $x^2 + y^2 = 1$ (d) $y = \frac{x}{|x|}$.

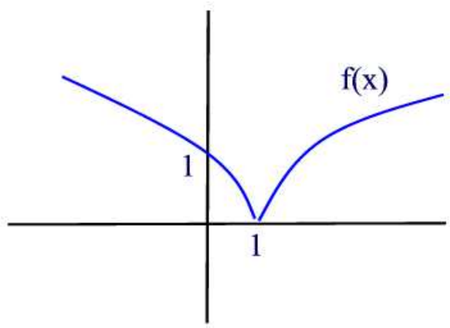
Exercise 4.6
Identify whether the given function is even, odd, or neither.

(a) $f(x) = x^2 - 7$ (b) $g(x) = x^5 + x^3$ (c) $h(x) = 3|x|$ (d) $j(x) = 4 + \sqrt[3]{x}$.

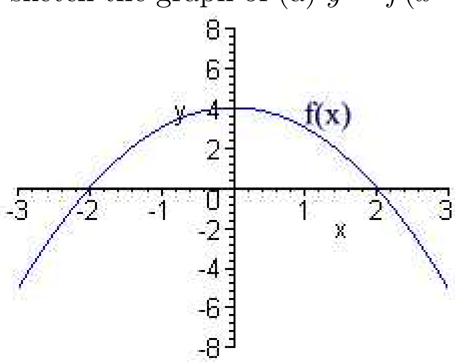
Exercise 4.7
Use the graph of f to sketch the graph of (a) $y = f(x) + 3$ (b) $y = f(x - 3)$.



Exercise 4.8
Use the graph of f to sketch the graph of (a) $y = f(x+2)$ (b) $y = f(x)+2$.



Exercise 4.9
Use the graph of f to sketch the graph of (a) $y = f(x-1)$ (b) $y = f(x)-1$.

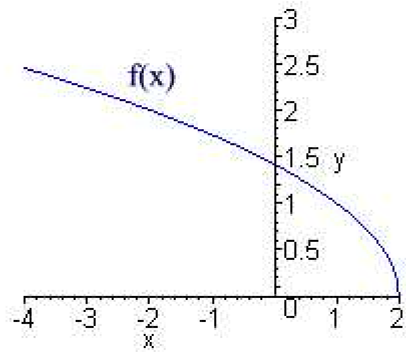


Exercise 4.10
Let f be a function such that $f(-2) = 5$, $f(0) = -2$, and $f(1) = 0$. Give the

coordinates of three points on the graph of

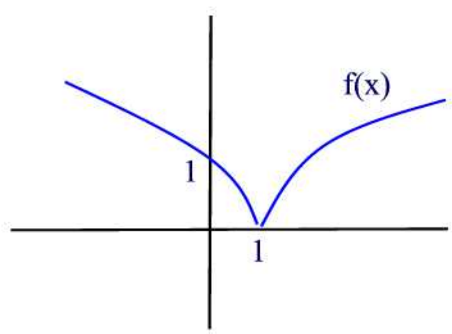
(a) $y = f(x + 3)$ (b) $y = f(x) + 1$.

Exercise 4.11
Use the graph of f to sketch the graph of (a) $y = f(-x)$ (b) $y = -f(x)$.



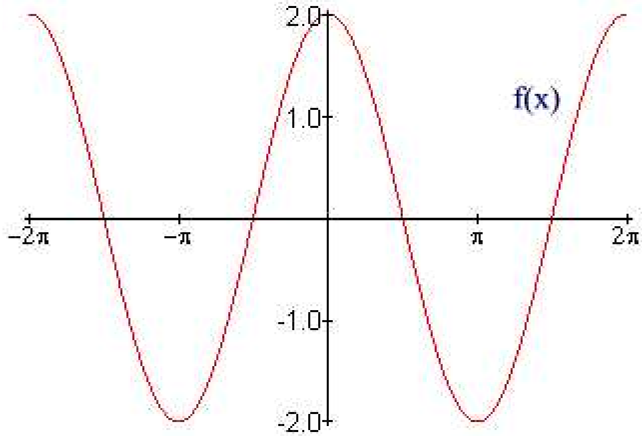
Exercise 4.12
Let f be a function such that $f(-1) = 3$ and $f(2) = -4$. Give the coordinates of two points on the graph of (a) $y = f(-x)$ (b) $y = -f(x)$.

Exercise 4.13
Use the graph of f to sketch the graph of (a) $y = f(-x)$ (b) $y = -f(x)$.



Exercise 4.14
Use the graph of $m(x) = x^2 - 2x - 3$ to sketch the graph of $y = -\frac{1}{2}m(x) + 3$.

Exercise 4.15
Use the graph of f to sketch the graph of (a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$.



5 Combining Functions

In this section we are going to construct new functions from old ones using the operations of addition, subtraction, multiplication, division, and composition.
Let $f(x)$ and $g(x)$ be two given functions. Then for all x in the common domain of these two functions we define new functions as follows:

• **Sum**: $(f+g)(x) = f(x) + g(x)$.
• **Difference**: $(f-g)(x) = f(x) - g(x)$.
• **Product**: $(f \cdot g)(x) = f(x) \cdot g(x)$.
• **Division**: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

Example 5.1
Let $f(x) = x + 1$ and $g(x) = \sqrt{x+3}$. Find the common domain and then find a formula for each of the functions $f+g, f-g, f \cdot g, \frac{f}{g}$.

Solution.
The domain of $f(x)$ consists of all real numbers whereas the domain of $g(x)$ consists of all numbers $x \geq 3$. Thus, the common domain is the interval $[-3, \infty)$. For any x in this domain we have

$$\begin{aligned}
(f+g)(x) &= x+1+\sqrt{x+3} \\
(f-g)(x) &= x+1-\sqrt{x+3} \\
(f \cdot g)(x) &= x\sqrt{x+3}+\sqrt{x+3} \\
\left(\frac{f}{g}\right)(x) &= \frac{x+1}{\sqrt{x+3}} \textit{ provided } x > -3. \blacksquare
\end{aligned}$$

Example 5.2
Let $f(x) = x^2 - 3x + 2$ and $g(x) = 2x - 4$. Evaluate the indicated function.

(a) $(f+g)\left(\frac{1}{2}\right)$ (b) $(f-g)(-1)$ (c) $(fg)\left(\frac{2}{5}\right)$ (d) $\left(\frac{f}{g}\right)(11)$.

Solution.
(a) $f(\frac{1}{2}) = \frac{3}{4}$ and $g(\frac{1}{2}) = -3$ so that $(f+g)\left(\frac{1}{2}\right) = \frac{3}{4} - 3 = -\frac{9}{4}$.
(b) $f(-1) = 6$ and $g(-1) = -6$ so that $(f-g)(-1) = 6 - (-6) = 12$.
(c) $f(\frac{2}{5}) = \frac{39}{25}$ and $g(\frac{2}{5}) = -\frac{16}{5}$ so that $(fg)\left(\frac{2}{5}\right) = -\frac{624}{125}$.
(d) $f(11) = 90$ and $g(11) = 18$ so that $\left(\frac{f}{g}\right)(11) = \frac{90}{18} = \frac{10}{3}$.$\blacksquare$

Difference Quotient
Difference quotients are what they say they are. They involve a difference and a quotient. Geometrically, a difference quotient is the slope of a secant line between two points on a curve. The formula for the difference quotient is:

$$\frac{f(x+h)-f(x)}{h}.$$

Example 5.3
Find the difference quotient of the function $f(x) = x^2$.

Solution.
Since $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$ then

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x^2+2hx+h^2)-x^2}{h} \\ &= \frac{2hx+h^2}{h} = \frac{h(2x+h)}{h} \\ &= 2x+h. \blacksquare \end{aligned}$$

Composition of Functions
Suppose we are given two functions f and g such that the range of g is contained in the domain of f so that the output of g can be used as input for f. We define a new function, called the **composition** of f with g, by the formula

$$(f \circ g)(x) = f(g(x)).$$

Using a Venn diagram (See Figure 5.1) we have

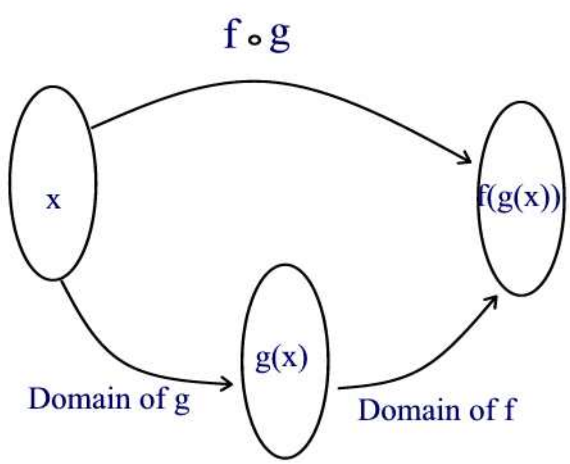


Figure 5.1

Example 5.4
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f \circ g$ and $g \circ f$.
(b) Calculate $(f \circ g)(5)$ and $(g \circ f)(-3)$.
(c) Are $f \circ g$ and $g \circ f$ equal?

Solution.
(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$. Similarly, $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.
(b) $(f \circ g)(5) = 2(5)^2 - 5 = 45$ and $(g \circ f)(-3) = 4(-3)^2 + 4(-3) - 2 = 22$.
(c) $f \circ g \neq g \circ f$.■

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

Example 5.5
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $(f \circ f)(x)$.
(b) Find $[f \circ (f \circ g)](x)$.

Solution.
(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.
(b) $[f \circ (f \circ g)](x) = f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9$.■

Review Problems

Exercise 5.1
Use the given functions f and g to find $f+g, f-g, fg$, and $\frac{f}{g}$. State the domain of each.

(a) $f(x) = x^2 - 2x - 15, g(x) = x + 3.$
(b) $f(x) = x^3 - 2x^2 + 7x, g(x) = x.$
(c) $f(x) = 2x^2 + 4x - 7, g(x) = 2x^2 + 3x - 5.$
(d) $f(x) = \sqrt{4 - x^2}, g(x) = 2 + x.$

Exercise 5.2
Evaluate the indicated function, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x - 4$.

(a) $(f+g)(5)$ (b) $(f+g)\left(\frac{2}{3}\right)$ (c) $(f-g)(-3)$ (d) $(fg)\left(\frac{2}{5}\right)$ (e) $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$.

Exercise 5.3
Find the difference quotient of the given function.

(a) $f(x) = 2x + 4.$
(b) $g(x) = x^2 - 6.$

Exercise 5.4
Find $f \circ g$ and $g \circ f$.

(a) $f(x) = 3x + 5, g(x) = 2x - 7.$
(b) $f(x) = x^3 + 2x, g(x) = -5x.$
(c) $f(x) = \frac{2}{x+1}, g(x) = 3x - 5.$
(d) $f(x) = \frac{1}{x^2}, g(x) = \sqrt{x-1}.$
(e) $f(x) = \frac{3}{|5-x|}, g(x) = -\frac{2}{x}.$

Exercise 5.5
Evaluate each composite function where $f(x) = 2x + 3, g(x) = x^2 - 5x$, and $h(x) = 4 - 3x^2$.

(a) $(f \circ g)(-3)$ (b) $(h \circ g)\left(\frac{2}{5}\right)$ (c) $(g \circ f)(\sqrt{3})$ (d) $(g \circ f)(2c)$.

6 Inverse Functions

An important feature of one-to-one functions is that they can be used to build new functions. So suppose that f is a one-to-one function. A new function, called the **inverse function** (denoted by f^{-1}), is defined such that if f takes an input x to an output y then f^{-1} takes y as its input and x as its output. That is

$$f(x) = y \text{ if and only if } f^{-1}(y) = x.$$

When a function has an inverse then we say that the function is **invertible.**

Example 6.1
Find the inverse function of (a) $f(x) = \log x$ (b) $g(x) = e^x$.

Solution.
(a) $f^{-1}(x) = 10^x$ (b) $g^{-1}(x) = \ln x.$ ■

Remark 6.1
It is important not to confuse between $f^{-1}(x)$ and $(f(x))^{-1}$. The later is just the reciprocal of $f(x)$, that is, $(f(x))^{-1} = \frac{1}{f(x)}$ whereas the former is how the inverse function is represented.

Domain and Range of an Inverse Function
Figure 6.1 shows the relationship between f and f^{-1}.

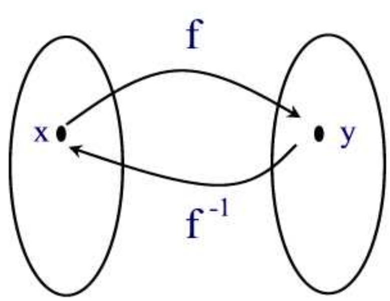


Figure 6.1

This figure shows that we get the inverse of a function by simply reversing the direction of the arrows. That is, the outputs of f are the inputs of f^{-1} and the outputs of f^{-1} are the inputs of f. It follows that

$$\text{Domain of } f^{-1} = \text{Range of f} \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of f}.$$

Example 6.2
Consider the function $f(x) = \sqrt{x-4}$.

(a) Find the domain and the range of $f(x)$.
(b) Use the horizontal line test to show that $f(x)$ has an inverse.
(c) What are the domain and range of f^{-1}?

Solution.
(a) The function $f(x)$ is defined for all $x \geq 4$. The range is the interval $[0, \infty)$.
(b) Graphing $f(x)$ we see that $f(x)$ satisfies the horizontal line test and so f has an inverse. See Figure 6.2.
(c) The domain of f^{-1} is the range of f, i.e. the interval $[0, \infty)$. The range of f^{-1} is the domain of f, that is, the interval $[4, \infty)$.■

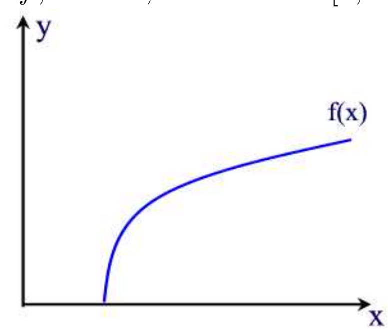


Figure 6.2

Finding a Formula for the Inverse Function
How do you find the formula for f^{-1} from the formula of f? The procedure consists of the following steps:

1. Replace $f(x)$ with y.
2. Interchange the letters x and y.
3. Solve for y in terms of x.
4. Replace y with $f^{-1}(x)$.

Example 6.3
Find the formula for the inverse function of $f(x) = x^3 + 7$.

Solution.
From Figure 16 and the horizontal line test w see that $f(x)$ is invertible. We

find its inverse as follows:

1. Replace $f(x)$ with y to obtain $y = x^3 + 7$.
2. Interchange x and y to obtain $x = y^3 + 7$.
3. Solve for y to obtain $y^3 = x - 7$ or $y = \sqrt[3]{x-7}$.
4. Replace y with $f^{-1}(x)$ to obtain $f^{-1}(x) = \sqrt[3]{x-7}$.■

Compositions of f and its Inverse
Suppose that f is an invertible function. Then the expressions $y = f(x)$ and $x = f^{-1}(y)$ are equivalent. So if x is in the domain of f then

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

and for y in the domain of f^{-1} we have

$$f(f^{-1}(y)) = f(x) = y$$

It follows that for two functions f and g to be inverses of each other we must have $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for x in the domain of f.

Example 6.4
Check that the pair of functions $f(x) = \frac{x}{4} - \frac{3}{2}$ and $g(x) = 4(x + \frac{3}{2})$ are inverses of each other.

Solution.
The domain and range of both functions consist of the set of all real numbers. Thus, for any real number x we have

$$f(g(x)) = f(4(x + \frac{3}{2})) = f(4x + 6) = \frac{4x + 6}{4} - \frac{3}{2} = x.$$

and

$$g(f(x)) = g(\frac{x}{4} - \frac{3}{2}) = 4(\frac{x}{4} - \frac{3}{2} + \frac{3}{2}) = x.$$

So f and g are inverses of each other.■

Review Problems

Exercise 6.1
Given $f(3) = 7$, find $f^{-1}(7)$.

Exercise 6.2
Given $h^{-1}(-3) = -4$, find $h(-4)$.

Exercise 6.3
If 3 is in the domain of f^{-1}, find $f[f^{-1}(3)]$.

Exercise 6.4
If f is a one-to-one function and $f(0) = 5, f(1) = 2$, and $f(2) = 7$, find

(a) $f^{-1}(5)$ (b) $f^{-1}(2)$.

Exercise 6.5
Use composition of functions to determine whether f and g are inverses of one another.

(a) $f(x) = 4x, g(x) = \frac{x}{4}$.
(b) $f(x) = 4x - 1, g(x) = \frac{1}{4}x + \frac{1}{4}$.
(c) $f(x) = -\frac{1}{2}x - \frac{1}{2}, g(x) = -2x + 1$.

Exercise 6.6
Find $f^{-1}(x)$. State any restrictions on the domain of $f^{-1}(x)$.

(a) $f(x) = 2x + 4$.
(b) $f(x) = \frac{2x}{x-1}, x \neq 1$.
(c) $f(x) = \frac{x-1}{x+1}, x \neq -1$.

Exercise 6.7
Find $f^{-1}(x)$. State any restrictions on the domain of $f^{-1}(x)$.

(a) $f(x) = x^2 - 4, x \geq 0$.
(b) $f(x) = \sqrt{x-2}, x \geq 2$.
(c) $f(x) = x^2 + 4x - 1, x \leq -2$.

In this chapter we introduce the trigonometric functions. These functions can be viewed in two different but equivalent ways. The first way is to view them as functions of real numbers, the other as functions of angles. In both ways, the functions assign the same value to a given real number. The difference is that in the first way, the real number is the length of an arc along the unit circle whereas in the second the number is the measure of an angle. The reason of studying both approaches is due to the fact that different applications require that we view these functions differently. For example, the first approach is needed when modeling harmonic motion. The second is needed when measuring the sides of a triangle.

7 Angles and Arcs

As stated in the introduction above, the two approaches of defining trigonometric functions involve the notions of angles and arcs.
In this section you will learn (1) to identify and classify angles, (2) to measure angles in both degrees and radians, (3) to convert between the units, (4) to find the measures of arcs spanned by angles, (5) to find the area of a circular sector, and (6) to measure linear and angular speeds, given a situation representing a circular motion.

Angles appear in a lot of applications. Let's mention one situation where angles can be very useful. Suppose that you are standing at a point 100 feet away of the Washington monument and you would like to approximate the height of the monument. Assuming that your height is negligeable compared to the height of the monument so that you can be identified by a point on the horizontal line. If you know the amount of opening between the line of sight, i.e. the line connecting you to the top of the monument, and the horizontal line then by applying a specific trigonometric function to that opening you will be able to estimate the height of the monument. The "opening" between the line of sight and the horizontal line gives an example of an angle.
An **angle** is determined by rotating a ray (or a half-line) from one position, called the **initial side**, to a terminal position, called the **terminal side**, as shown in Figure 7.1 below. The point V is called the **vertex** of the angle.

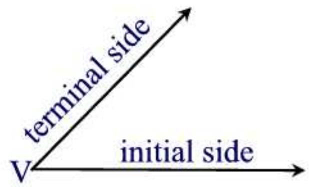


Figure 7.1

If the initial side is the positive x-axis then we say that the angle is in **standard position.** See Figure 7.2.

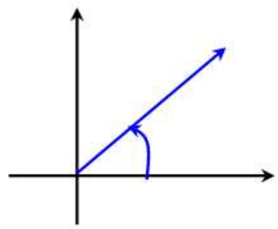


Figure 7.2

Angles that are obtained by a counterclockwise rotation of the initial side are considered **positive** and those that are obtained clockwise are **negative** angles. See Figure 7.3.

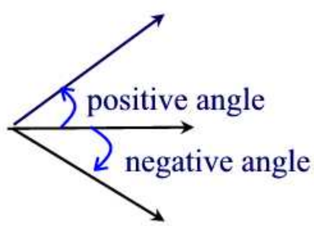


Figure 7.3

Most of the time, we will use Greek lowercase letters such as α (alpha), β (beta), γ (gamma) , etc. to denote angles. If α is an angle obtained by rotating an initial ray $\overrightarrow{OA}$ to a terminal ray $\overrightarrow{OB}$ then we sometimes denote that by writing $\alpha = \angle AOB$.

Angle Measure

The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side, this is how much the angle "opens". There are two commonly used measures of angles: **degrees** and **radians**

• **Degree Measure:**
If we rotate counterclockwise a ray about a fixed vertex and then return back to its initial position then we say that we have a one complete **revolution.** The angle in this case is said to have measure of 360 degrees, in symbol $360°$. Thus, $1°$ is $\frac{1}{360}$th of a revolution. See Figure 7.4)

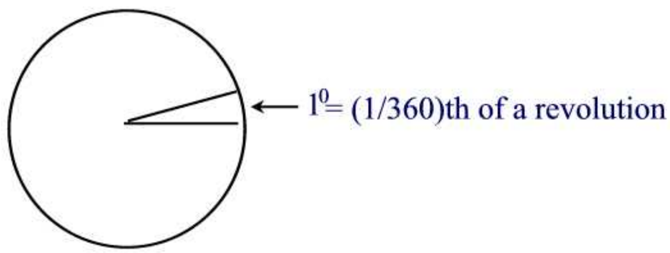


Figure 7.4

Example 7.1
Draw each of the following angles in standard positions: (a) $225°$ (b) $-90°$ (c) $180°$.

Solution.
The specified angles are drawn in Figure 7.5 below

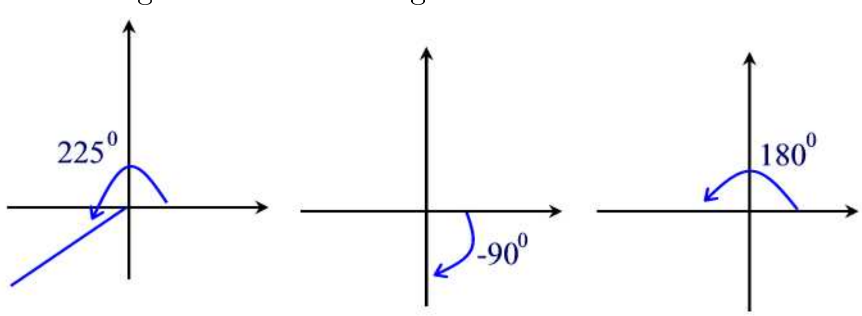


Figure 7.5

Remark 7.1
A protractor can be used to measure angles given in degrees or to draw an angle given in degree measure.

Now, each degree can be divided into 60 equal parts, each called a **minute**. Thus,

$$1^\circ = 60' \ and \ \ 1' = (\frac{1}{60})^\circ.$$

Similarly, each minute can be divided into 60 equal parts, called **seconds**. Thus,

$$1' = 60'' \ \ and \ \ 1'' = (\frac{1}{60})' = (\frac{1}{3600})^\circ.$$

By introducing the minutes and seconds units one can now convert a decimal degree to a degree-minute-second format (DMS) as shown in the next example.

Example 7.2
Convert 32.519° to the form $D^\circ M'S''$.

Solution.

$$\begin{array}{rcl}
32.519^\circ & = & 32^\circ + 0.519^\circ \\
& = & 32^\circ + (1^\circ)(0.519) \\
& = & 32^\circ + (60')(0.519) \\
& = & 32^\circ + 31.14' \\
& = & 32^\circ 31' + 0.14' \\
& = & 32^\circ 31' + (1')(0.14) \\
& = & 32^\circ 31' + (60'')(0.14) \\
& = & 32^\circ 31' 8.4'' \ \blacksquare
\end{array}$$

Example 7.3
Convert $50^\circ 6' 21''$ to the nearest ten-thousandth of a degree.

Solution.

$$\begin{array}{rcl}
50^\circ 6' 21'' & = & 50^\circ + 6(1') + 21(1'') \\
& = & 50^\circ + (\frac{6}{60})^\circ + (\frac{21}{3600})^\circ \\
& \approx & 50^\circ + .1^\circ + .0058^\circ \\
& = & 50.1058^\circ \ \blacksquare
\end{array}$$

Remark 7.2
Angles represented in the DMS form are very useful in applications. For example, latitude describes the position of a point on the earth's surface in relation to the equator. A point on the equator has latitude of 0°. The north pole has a latitude of 90°. For example, New York City has latitude of $40^\circ 45' N$.

• **Radian Measure:**
A more natural method of measuring angles used in calculus and other branches of mathematics is the **radian** measure. The amount an angle opens is measured along the arc of the unit circle with its center at the vertex of the angle.(An angle whose vertex is the center of a circle is called a **central angle**.) One **radian**, abbreviated **rad**, is defined to be the measure of a central angle that intercepts an arc s of length one unit. See Figure 7.6.

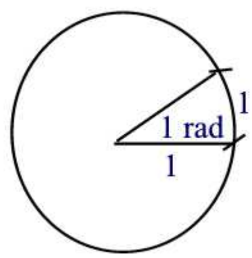


Figure 7.6

Since one complete revolution measured in radians is 2π radians and measured in degrees is 360° then we have the following conversion formulas:

$$1^\circ = \frac{\pi}{180}\ rad \approx 0.01745\ rad \quad and \quad 1\ rad = (\frac{180}{\pi})^\circ \approx 57.296^\circ.$$

Example 7.4
Complete the following chart.

degree	30°	45°	60°	90°	180°	270°
radian						

Solution.

degree	30°	45°	60°	90°	180°	270°
radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

By the conversion formulas, we have, for example $30° = 30(1°) = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$. In a similar way we convert the remaining angles. ■

Example 7.5
Convert each angle in degrees to radians: (a) $150°$ (b) $-45°$.

Solution.
(a) $150° = 150(1°) = 150(\frac{\pi}{180}) = \frac{5\pi}{6} rad.$
(b) $-45° = -45(1°) = -45(\frac{\pi}{180}) = -\frac{\pi}{4} rad.$ ■

Example 7.6
Convert each angle in radians to degrees: (a) $-\frac{3\pi}{4}$ (b) $\frac{7\pi}{3}$.

Solution.
(a) $-\frac{3\pi}{4} = -\frac{3\pi}{4}(1\ rad) = -\frac{3\pi}{4}(\frac{180}{\pi})° = -135°.$
(b) $\frac{7\pi}{3} = \frac{7\pi}{3}(\frac{180}{\pi})° = 420°$ ■

Remark 7.3
When no unit of an angle is given then the angle is assumed to be measured in radians.

Classification of Angles
Some types of angles have special names:(See Figure 7.7)

1. A $90°$ angle is called a **right** angle.
2. A $180°$ angle is called a **straight** angle.
3. An angle between $0°$ and $90°$ is called an **acute** angle.
4. An angle between $90°$ and $180°$ is called an **obtuse** angle.
5. Two acute angles are **complementary** if their sum is $90°$.
6. Two positive angles are **supplementary** if their sum is $180°$.
7. Angles in standard positions with terminal sides that lie on a coordinate axis are called **quadrantal angles.** Thus, the angles $0°, \pm 90°, \pm 180°, etc$ are quadrantal angles.

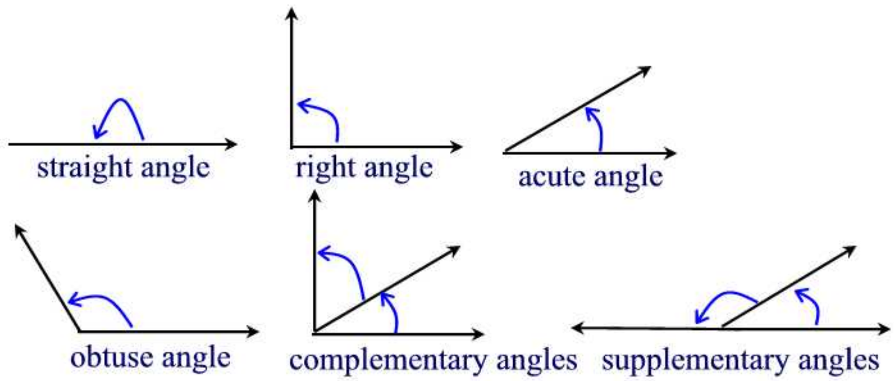


Figure 7.7

Example 7.7
Prove the **Vertical Angle Theorem**: The angles shown in Figure 7.8 are equal.

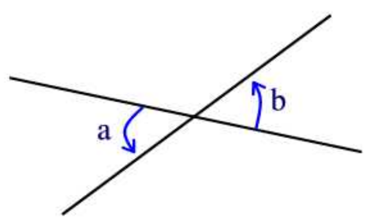


Figure 7.8

Solution.
Let c be the angle shown in Figure 7.9. Then a and c are supplementary, i.e. $a + c = 180°$. Similarly, $b + c = 180°$. It follows that $a = 180° - c = 180° - (180° - b) = b$. ■

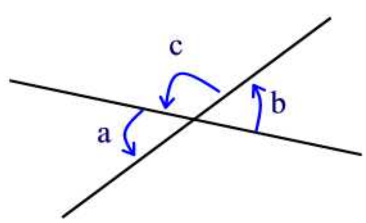


Figure 7.9

Example 7.8
Prove the following theorem: The interior corresponding angles formed by a line that goes through both parallel lines are equal. See Figure 7.10.

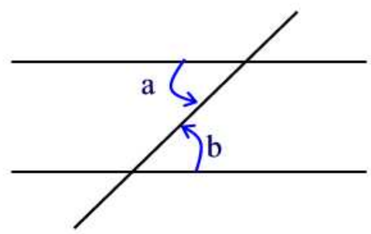


Figure 7.10

Solution.

Let c be the angle shown in the Figure 7.11. Then $a+c = 90°$ and $b+c = 90°$. Thus, $a = b$. ■

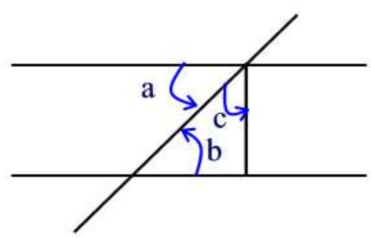


Figure 7.11

Combining the previous two exercises we see that the angles a and b given in Figure 7.12 are equal.

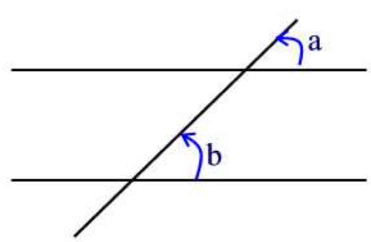


Figure 7.12

Example 7.9
Suppose that $\theta = 41°28'$. Determine the measure of an angle that is:
(a) Complementary to θ (b) Supplementary to θ.

Solution.
(a) $90° - 41°28' = 89°60' - 41°28' = 48°32'$.
(b) $180° - 41°28' = 179°60' - 41°28' = 138°32'$. ■

Remark 7.4
Non quadrantal angles are classified according to the quadrant that contains the terminal side. For example, when we say that an angle is in Quadrant III then by that we mean that the terminal side of the angle lies in the third quadrant.

Two angles in standard positions with the same terminal side are called **coterminal**.(See Figure 7.13) We can find an angle that is coterminal to a given angle by adding or subtracting one revolution. Thus, a given angle has many coterminal angles. For instance, $\alpha = 36°$ is coterminal to all of the following angles: $396°, 756°, -324°, -684°$

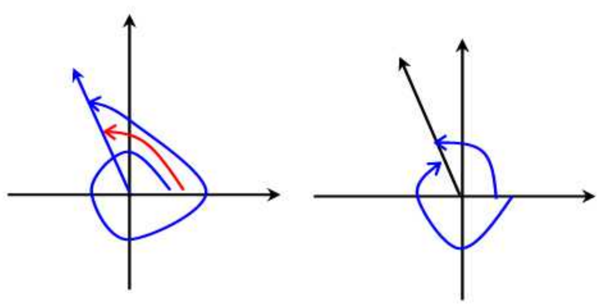


Figure 7.13

Example 7.10
Find a coterminal angle for the following angles, given in standard positions:
(a) $530°$ (b) $-400°$.

Solution.
(a) A positive angle coterminal with $530°$ is obtained by adding a multiple of $360°$. For example, $530° + 360° = 890°$. A negative angle coterminal with $530°$ is obtained by subtracting a multiple of $360°$. For example,

$530° - 720° = -190°$.
(b) A positive angle is $-400° + 720° = 320°$ and a negative angle is $-400° + 360° = -40°$. ■

Length of a Circular Arc
A circular arc swept out by a central angle is the portion of the circle which is opposite an interior angle. We discuss below a relationship between a central angle θ, measured in radians, and the length of the arc s that it intercepts.

Theorem 7.1
For a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by the formula:

$$s = r\theta$$

Proof.
Suppose that $r > 1$. (A similar argument holds for $0 < r < 1$.) Draw the unit circle with the same center C (See Figure 7.14).

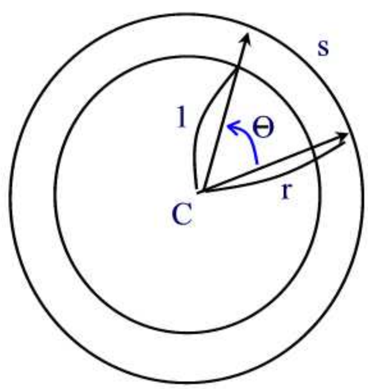


Figure 7.14

By definition of radian measure, the length of the arc determined by θ on the unit circle is also θ. From elementary geometry, we know that the ratio of the measures of the arc lengths are the same as the ratio of the corresponding radii. That is,

$$\frac{r}{1} = \frac{s}{\theta}.$$

Now the formula follows by cross-multiplying. ■

The above formula allows us to define the radian measure using a circle of any radius r. (See Figure 7.15).

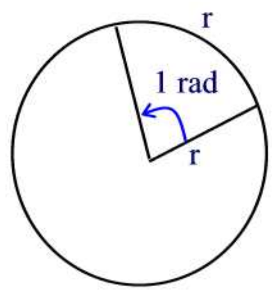


Figure 7.15

Example 7.11
Find the length of the arc of a circle of radius 2 meters subtended by a central angle of measure 0.25 radians.

Solution.
We are given that $r = 2\ m$ and $\theta = 0.25\ rad$. By the previous theorem we have:

$$s = r\theta = 2(0.25) = 0.5\ m \ \blacksquare$$

Example 7.12
Suppose that a central angle of measure 30° is subtended by an arc of length $\frac{\pi}{2}$ feet. Find the radius r of the circle.

Solution.
Substituting in the formula $s = r\theta$ we find $\frac{\pi}{2} = r\frac{\pi}{6}$. Solving for r to obtain $r = 3feet$. $\blacksquare$

Circular Motion
Consider an object moving along a circle of radius r with a constant speed. Let s denote the distance traveled in time t along this circle and let θ be the central angle, measured in radians, corresponding to s. There are two ways to describe the motion of the object- linear and angular speed. The **linear speed** v of the object is the rate at which the distance traveled is changing. It is defined by the formula

$$v = \frac{s}{t}$$

The **angular speed** ω is the rate at which the central angle is changing. It is given by

$$\omega = \frac{\theta}{t}.$$

Since $s = r\theta$ then we have the following relationship between v and ω

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\omega.$$

Example 7.13
The second hand of a clock is 10.2 centimeters long. Find the linear speed of the tip of the second hand.

Solution.
The distance traveled by the tip of the second hand in one revolution is

$$s = 2\pi(10.2) = 20.4\pi\ cm.$$

Therefore, the linear speed is

$$v = \frac{20.4\pi}{60} \approx 1.068cm/sec \blacksquare$$

Example 7.14
A hard disk in a computer rotates at 3600 revolutions per minute. Find the angular speed of the disk in radians per second.

Solution.
We have

$$\begin{aligned} 3600\ rev/minute &= \frac{3600\ rev}{1\ minute}\left(\frac{2\pi\ rad}{1\ rev}\right)\left(\frac{1\ minute}{60\ seconds}\right) \\ &= \frac{120\pi\ radians}{1\ second} \approx 377\ radians/second \end{aligned}$$

Area of a circular sector
A circular sector swept out by an interior angle is the portion of the interior of the circle which is between the two radii, and the circular arc. The area of a circle with radius r is known to be πr^2. This area corresponds to an arc of length $2\pi r$. Let θ be a central angle subtended by an arc of length $r\theta$. See Fig 7.16. The area of the circular sector corresponding to this arc is then

$$A = \frac{\pi r^3\theta}{2\pi r} = \frac{1}{2}r^2\theta.$$

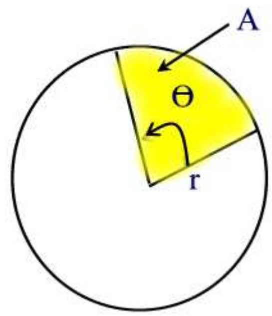


Figure 7.16

Example 7.15
Find the area of a circular sector of radius 10 meters and with central angle $\theta = \frac{\pi}{3} rad.$

Solution.
Substituting in the formula of A yields

$$A = \frac{1}{2}(10)^2(\frac{\pi}{3}) = \frac{50\pi}{3}m^2 \blacksquare$$

Review Problems

Exercise 7.1
(a) Explain the difference between a positive angle and a negative angle.
(b) How is the radian measure of an angle defined?
(c) How is the degree measure of an angle defined?
(d) How do you convert from degrees to radians? from radians to degrees?

Exercise 7.2
(a) When is an angle in standard position?
(b) When are two angles coterminal?
(c) When are two angles complementary?
(d) When are two angles supplementary?

Exercise 7.3
(a) What is the length s of an arc of a circle of radius r that subtends a central angle of θ radians?
(b) What is the area A of a sector of a circle of radius r and central angle θ radians?

Exercise 7.4
Find the radian measure that corresponds to the degree measure.

(a) $-330°$ (b) $5°$ (c) $750°$.

Exercise 7.5
Find the degree measure that corresponds to the given radian measure.

(a) $\frac{9\pi}{4}$ (b) $-\frac{\pi}{6}$ (c) 3.1

Exercise 7.6
Find the length of an arc of a circle of radius 8 m if the arc subtends a central angle of 1 rad.

Exercise 7.7
Draw the following angles in standard position.
(a) $30°$ (b) $45°$ (c) $-270°$.

Exercise 7.8
Convert each DMS measure to its equivalent decimal measure to the nearest ten-thousandth of a degree:
(a) $25°25'12''$ (b) $211°46'48''$.

Exercise 7.9
Convert to the form $D°M'S''$: (a) $24.46°$ (b) $3.402°$.

Exercise 7.10
Convert each angle in degrees to radians.
(a) $165°$ (b) $-270°$ (c) $585°$.

Exercise 7.11
Convert each angle in radians to degrees.
(a) $\frac{9\pi}{2}$ (b) $2\ rad$ (c) $-\frac{2\pi}{3}$.

Exercise 7.12
Find the number of radians in $\frac{3}{8}$ revolution.

Exercise 7.13
Classify each angle by quadrant, and state the measure of the positive angle with measure less than $360°$ that is coterminal with the given angle:
(a) $765°$ (b) $-975°$ (c) $2456°$.

Exercise 7.14
Find two positive angles and two negative angles that are coterminal with the given angles.

(a) $\frac{13\pi}{6}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{2\pi}{3}$ (d) $-45°$ (e) $135°$.

Exercise 7.15
The measures of two angles in standard positions are given. Determine whether the angles are coterminal.

(a) $70°, 340°$
(b) $\frac{5\pi}{6}, \frac{17\pi}{6}$
(c) $155°, 875°$.

Exercise 7.16
Find an angle between $0°$ and $360°$ that is coterminal with the given angle.

(a) $733°$ (b) $-100°$ (c) $1270°$ (d) $-800°$.

Exercise 7.17
Find an angle between 0 and 2π radians that is coterminal with the given angle.

(a) $\frac{17\pi}{6}$ (b) $-\frac{7\pi}{3}$ (c) 10 (d) $\frac{51\pi}{2}$.

Exercise 7.18
Determine the complement and the supplement of each angle:
(a) $87°$ (b) $56°33'15''$ (c) $\frac{4\pi}{3}$.

Exercise 7.19
Determine the length of an arc of a circle of radius 4 centimeters that subtends a central angle of measure 2.3 radians.

Exercise 7.20
Suppose that a central angle of a circle of radius 12 meters subtends an arc of length 14 meters. Find the radian measure of the angle.

Exercise 7.21
Find the length of an arc that subtends a central angle of $45°$ in a circle of radius 10 m.

Exercise 7.22
A central angle θ in a circle of radius 5 m is subtended by an arc of length 6 m. Find the measure of θ in degrees and in radians.

Exercise 7.23
Suppose that the wheels on a tractor have a radius of 3 feet and that the angular speed of the tires is 20 radians per second. What is the linear speed of the tractor?

Exercise 7.24
A wheel is rotating at 50 revolutions per minute. Find the angular speed in radians per second.

Exercise 7.25
The diameter of each wheel of a bicycle is 26 inches. If you are traveling at a speed of 35 miles per hour on this bicycle, through how many revolutions per minute are the wheels turning?

Exercise 7.26
The windshield wiper of a car is 18 inches long. How many inches will the tip of the wiper trace out in $\frac{1}{3}$ revolution?

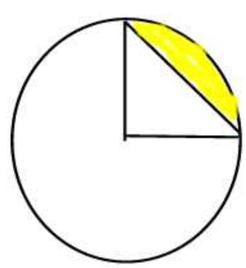
Exercise 7.27
A person is seated on the end of a see-saw whose total length is 5 m. The see-saw moves up and down through a 28° angle every 3 seconds. Through what distance does the person move in a minute?

Exercise 7.28
Assuming the Earth to be a sphere of 6,372 km, find the distance of a point in latitude 36 North from the equator.

Exercise 7.29
Each tire on a car has a radius of 15 inches. The tires are rotating at 450 revolutions per minute. Find the linear speed of the automobile to the nearest mile per hour.

Exercise 7.30
Find the area of the circular sector of radius 15 feet and with an arc of length 12 feet that intercepts a central angle θ.

Exercise 7.31
Find the area of the shaded portion of the circle. The radius of the circle is 9 inches.



Exercise 7.32
Find the measure of a central angle θ in a circle of radius 5 ft if the angle is subtended by an arc of length 7 ft.

Exercise 7.33
How many revolutions will a car wheel of diameter 28 in. make over a period of half an hour if the car is traveling at 60 mph?

Exercise 7.34
Find the area of a cicular sector with central angle 2 rad in a circle of radius 5 m.

8 Trigonometric Functions of Acute Angles

In this section you will learn (1) how to find the trigonometric functions using right triangles, (2) compute the values of these functions for some special angles, and (3) solve model problems involving the trigonometric functions.
First, let's review some of the features of right triangles. A triangle in which one angle is $90°$ is called a **right triangle.** The side opposite to the right angle is called the **hypotenuse** and the remaining sides are called the **legs** of the triangle.
Suppose that we are given an acute angle θ as shown in Figure 8.1. Note that $a \neq 0$ and $b \neq 0$.

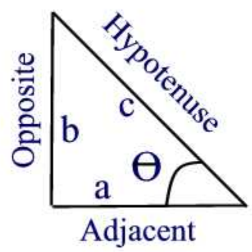


Figure 8.1

Associated with θ are three lengths, the hypotenuse , the opposite side, and the adjacent side. We define the values of the trigonometric functions of θ as ratios of the sides of a right triangle:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{b}{r} \qquad \cos\theta = \frac{adjacent}{hypotenuse} = \frac{a}{r} \qquad \tan\theta = \frac{opposite}{adjacent} = \frac{b}{a}$$
$$\csc\theta = \frac{hypotenuse}{opposite} = \frac{r}{b} \qquad \sec\theta = \frac{hypotenuse}{adjacent} = \frac{r}{a} \qquad \cot\theta = \frac{adjacent}{opposite} = \frac{a}{b}$$

where $r = \sqrt{a^2 + b^2}$ (Pythagorean formula).
The symbols sin, cos, sec, csc, tan, and cot are abbreviations of **sine, cosine, secant, cosecant, tangent,** and **cotangent.** The above ratios are the same regardless of the size of the triangle. That is, the trigonometric functions defined above depend only on the angle θ. To see this, consider Figure 8.2.

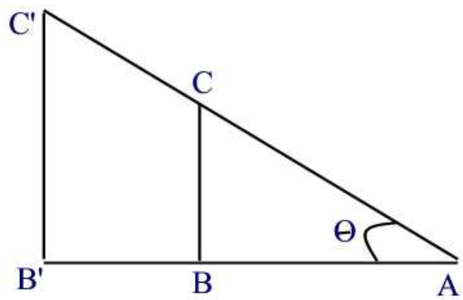


Figure 8.2

The triangles ΔABC and $\Delta AB'C'$ are similar. Thus, $\frac{|AB|}{|AB'|} = \frac{|AC|}{|AC'|} = \frac{|BC|}{|B'C'|}$. For example, using the cosine function we have

$$\cos\theta = \frac{|AB|}{|AC|} = \frac{|AB'|}{|AC'|}.$$

The following identities, known as **reciprocal identities**, follow from the definition given above.

$$\begin{array}{lll} \sin\theta = \frac{1}{\csc\theta}, & \cos\theta = \frac{1}{\sec\theta}, & \tan\theta = \frac{1}{\cot\theta}, \\ \csc\theta = \frac{1}{\sin\theta}, & \sec\theta = \frac{1}{\cos\theta}, & \cot\theta = \frac{1}{\tan\theta}. \end{array}$$

Example 8.1
Find the exact value of the six trigonometric functions of the angle θ shown in Figure 8.3.

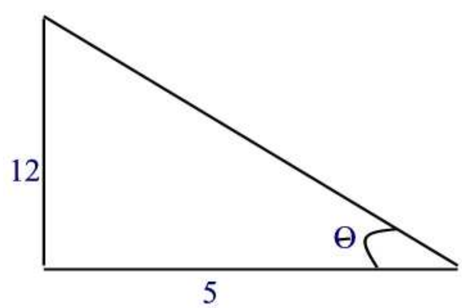


Figure 8.3

Solution.
By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{144 + 25} =$

$\sqrt{169} = 13$. Thus,

$$\begin{array}{lll} \sin\theta = \frac{12}{13} & \cos\theta = \frac{5}{13} & \tan\theta = \frac{12}{5} \\ \csc\theta = \frac{13}{12} & \sec\theta = \frac{13}{5} & \cot\theta = \frac{5}{12}. \blacksquare \end{array}$$

Given the value of one trigonometric function, it is possible to find the values of the remaining trigonometric functions of that angle.

Example 8.2
Suppose that θ is an acute angle for which $\cos\theta = \frac{5}{7}$. Determine the values of the other five trigonometric functions.

Solution.
Since $\cos\theta = \frac{5}{7}$ then the adjacent side has length 5 and the hypotenuse has length 7. See Figure 8.4. Using the Pythagorean theorem, the opposite side has length $\sqrt{49-25} = 2\sqrt{6}$. Thus,

$$\begin{array}{rcl} \sin\theta = \frac{2\sqrt{6}}{7} & ; & \cos\theta = \frac{5}{7} \\ \sec\theta = \frac{7}{5} & ; & \tan\theta = \frac{2\sqrt{6}}{5} \\ \csc\theta = \frac{7}{2\sqrt{6}} = \frac{7\sqrt{6}}{12} & ; & \cot\theta = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}. \blacksquare \end{array}$$

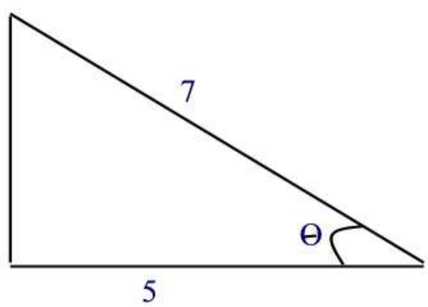


Figure 8.4

Example 8.3
Solve for y given that $\tan 30° = \frac{\sqrt{3}}{3}$.(See Figure 8.5)

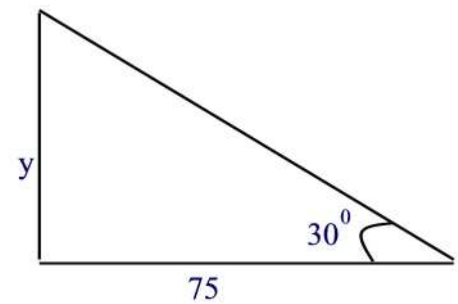


Figure 8.5

Solution.
According to Figure 8.5, $y = 75 \tan 30° = 75(\frac{\sqrt{3}}{3}) = 25\sqrt{3}$. ■

Trigonometric Functions of Special Angles
Next, we compute the trigonometric functions of some special angles. It's useful to remember these special trigonometric ratios because they occur often.

Example 8.4
Determine the values of the six trigonometric functions of the angle 45°. See Figure 8.6.

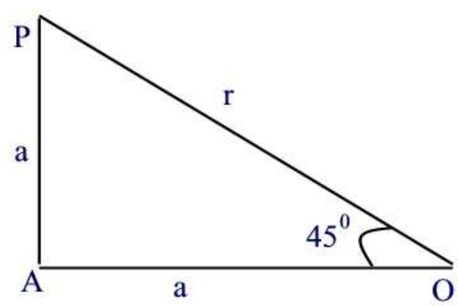


Figure 8.6

Solution.
Using Figure 8.6, the triangle OAP is a right isosceles triangle. By the Pythagorean theorem we find that $r^2 = 2a^2$ or $r = a\sqrt{2}$. Thus,

$$\begin{array}{rclcrcl} \sin 45° &=& \frac{\sqrt{2}}{2} &;& \csc 45° &=& \sqrt{2} \\ \cos 45° &=& \frac{\sqrt{2}}{2} &;& \sec 45° &=& \sqrt{2} \\ \tan 45° &=& 1 &;& \cot 45° &=& 1. \blacksquare \end{array}$$

Example 8.5
Determine the trigonometric functions of the angles
(a) $\theta = 30°$
(b) $\theta = 60°$.

Solution.
(a) Let ABC be an equilateral triangle with side of length a. Let P be the midpoint of the side AC and h the height of the triangle. See Figure 8.7. Using the Pythagorean theorem we find $h = a\frac{\sqrt{3}}{2}$. Thus,

$$\begin{array}{rclcrcl} \sin 30° & = & \frac{1}{2} & ; & \csc 30° & = & 2 \\ \cos 30° & = & \frac{\sqrt{3}}{2} & ; & \sec 30° & = & \frac{2\sqrt{3}}{3} \\ \tan 30° & = & \frac{\sqrt{3}}{3} & ; & \cot 30° & = & \sqrt{3}. \end{array}$$

(b) Similarly,

$$\begin{array}{rclcrcl} \sin 60° & = & \frac{\sqrt{3}}{2} & ; & \csc 60° & = & \frac{2\sqrt{3}}{3} \\ \cos 60° & = & \frac{1}{2} & ; & \sec 60° & = & 2 \\ \tan 60° & = & \sqrt{3} & ; & \cot 60° & = & \frac{\sqrt{3}}{3}.\blacksquare \end{array}$$

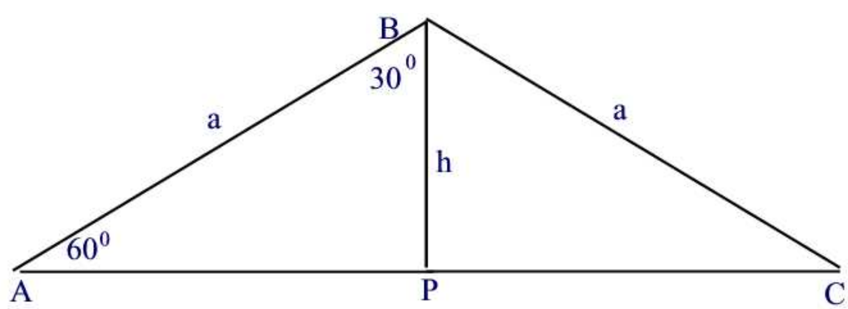


Figure 8.7

Example 8.6
Find the exact value of $2\sin 60° - \sec 45° \tan 60°$.

Solution.
Using the results of the previous two problems we find that

$$2\sin 60° - \sec 45° \tan 60° = 2(\frac{\sqrt{3}}{2}) - \sqrt{2}\sqrt{3} = \sqrt{3} - \sqrt{6}.\blacksquare$$

We follow the convention that when we write a trigonometric function, such as $\sin t$, then it is assumed that t is in radians. If we want to evaluate the trigonometric function of an angle measured in degrees we will use the degree notation such as $\cos 30°$.

Angles of Elevation and Depression

If an observer is looking at an object, then the line from the observer's eye to the object is known as the **line of sight**. If the object is above the horizontal then the angle between the line of sight and the horizontal is called the **angle of elevation.** If the object is below the horizontal then the angle between the line of sight and the horizontal is called the **angle of depression**. See Figure 8.8.

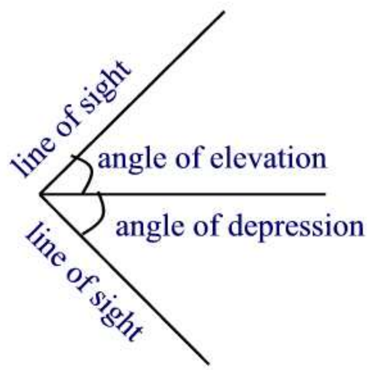


Figure 8.8

Example 8.7

From a point 115 feet from the base of a redwood tree, the angle of elevation to the top of the tree is 64.3°. Find the height of the tree to the nearest foot. See Figure 8.9.

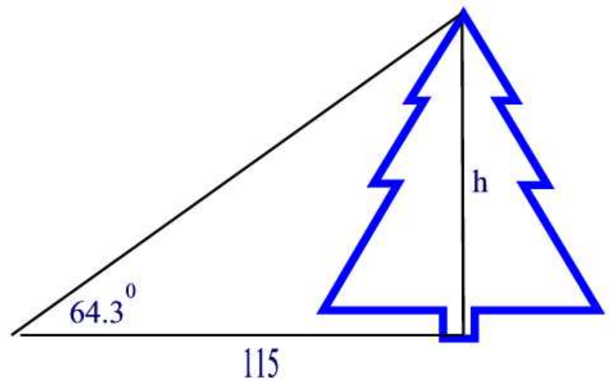


Figure 8.9

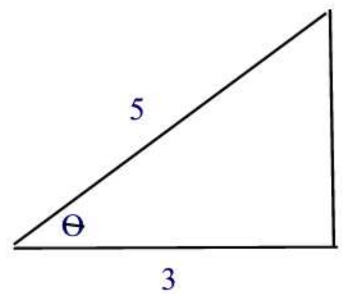
Solution.
According to Figure 8.9, we use the tangent function to find the height h of the tree: $\tan 64.3^\circ = \frac{h}{115}$ so that $h = 115 \tan 64.3^\circ \approx 238.952\ ft.$ ■

Evaluating trigonometric functions with a calculator
When evaluating trigonometric functions using a calculator, you need to set the calculator to the desired mode of measurement (degrees or radians). The functions sine, cosine, and tangent have a key in a standard scientific calculator. For the remaining three trigonometric functions the key x^{-1} is used in the process. For example, to evaluate $\sec \frac{\pi}{8}$, set the calculator to radian mode, then apply the following sequence of keystrokes: $\pi, \div, 8, \cos, x^{-1}$ and enter to obtain $\sec \frac{\pi}{8} \approx 1.0824$.

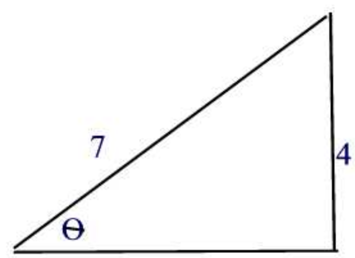
Review Problems

Exercise 8.1
If θ is an acute angle in a right triangle, define the six trigonometric ratios in terms of the adjacent, opposite, and the hypotenuse.

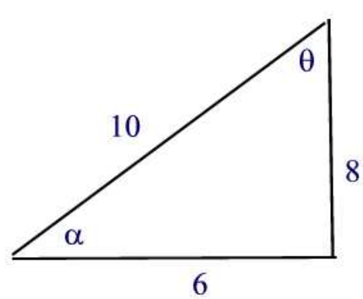
Exercise 8.2
Find the exact value of the six trigonometric functions of the angle θ.



Exercise 8.3
Find the exact value of the six trigonometric functions of the angle θ.



Exercise 8.4
Determine the values of the six trigonometric functions of α and θ.



Exercise 8.5
Let θ be an acute angle of a right triangle and $\sin\theta = \frac{3}{5}$. Find:
(a) $\tan\theta$ (b) $\sec\theta$ (c) $\cos\theta$.

Exercise 8.6
Let θ be an acute angle of a right triangle and $\tan\theta = \frac{4}{3}$. Find:
(a) $\sin\theta$ (b) $\cot\theta$ (c) $\sec\theta$.

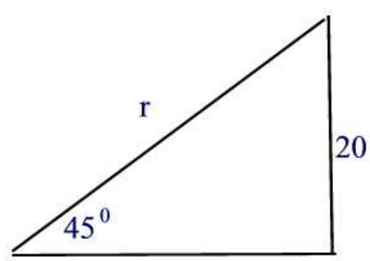
Exercise 8.7
Let θ be an acute angle of a right triangle and $\sec\theta = \frac{13}{12}$. Find:
(a) $\cos\theta$ (b) $\sec\theta$ (c) $\csc\theta$.

Exercise 8.8
Use a calculator to find the value of the trigonometric function to three decimal places.

(a) $\cos 63°20'$ (b) $\cot 55°50'$ (c) $\tan 81.3^{|circ}$ (d) $csc1.2$.

Exercise 8.9
The top of a 13-ft ladder is leaning against a wall of height 12 ft. Find the six trigonometric functions of the angle the ladder makes with the ground.

Exercise 8.10
Solve for r.



Exercise 8.11
A 6-foot person standing 15 feet from a streetlight casts an 8-foot shadow. What is the height of the streetlight?

Exercise 8.12
If $\sin\theta = \frac{1}{3}$, find the exact value of
(a) $\cos(90^\circ - \theta)$.
(b) $\cos^2\theta$.
(c) $\csc\theta$.

Exercise 8.13
Use a calculator in radian mode to complete the following table:

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\sin\theta$								
$\frac{\sin\theta}{\theta}$								

What can you conclude about the ratio $\frac{\sin\theta}{\theta}$ as θ approaches zero?

Exercise 8.14
Use a calculator in radian mode to complete the following table

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\cos\theta - 1$								
$\frac{\cos\theta-1}{\theta}$								

What can you conclude about the ratio $\frac{\cos\theta-1}{\theta}$ as θ approaches 0?

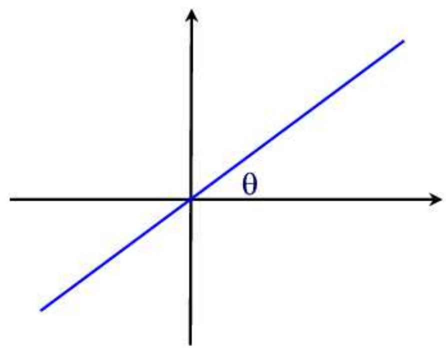
Exercise 8.15
Without using a calculator, find the exact value of each expression:

(a) $\sec 30^\circ \cos 30^\circ - \tan 60^\circ \cot 60^\circ$.
(b) $\sec 45^\circ \cot 30^\circ + 3\tan 60^\circ$
(c) $\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \tan\frac{\pi}{4}$.
(d) $2\csc\frac{\pi}{4} - \sec\frac{\pi}{3}\cos\frac{\pi}{6}$.

Exercise 8.16
Find the exact value of each of the following expressions.

(a) $\sin 30^\circ \cos 60^\circ + \tan 45^\circ$.
(b) $\sec 30^\circ \cos 30^\circ - \tan 60^\circ \cot 60^\circ$.
(c) $\sec\frac{\pi}{3}\cos\frac{\pi}{3} - \tan\frac{\pi}{6}$.

Exercise 8.17
Show that the slope of a line that makes an angle θ with the positive x-axis equals to $\tan\theta$.



Exercise 8.18
From a point A on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is 42.0°. From a point 100 feet away and on the same line, the angle to the top is 37.8°. Find the approximate height of the Washington Monument.

Exercise 8.19
Let B denote the base of a clock tower. The angle of elevation from a point A, on the ground, to the top of the tower is 56.3°. On a line on the ground that is perpendicular to AB and 25 feet from A, the angle of elevation is 53.3°. Find the height of the clock tower.

Exercise 8.20
Show that the area A of the triangle given in Figure 8.10 is $A = \frac{1}{2}ab\sin\theta$.

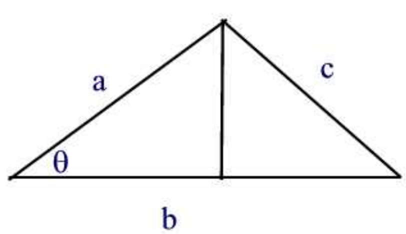


Figure 8.10

Exercise 8.21
The local fire department's longest ladder measures 72 feet. If the angle between the ground and the ladder must be $60°$, how high can the ladder reach? How far from a building should the foot of the ladder be?

Exercise 8.22
A giant redwood tree casts a shadow 532 feet long. Find the height of the tree if the angle of elevation is $25.7°$.

Exercise 8.23
A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

Exercise 8.24
Find the values of the six trigonometric functions of the angle θ in standard position if the point $P(-5, 12)$ is on the terminal side of θ.

9 Trigonometric Functions of Any Angle

Right triangles are useful when trying to calculate the trigonometric functions of acute angles. What about angles that are not acute angles?
In this section you will learn (1) of how to compute the trigonometric functions of any angle, not just acute angles, (2) the sign of the trigonometric functions in each quadrant of the coordinate plane, and (3) the use of reference angles which reduce the question of finding the trigonometric functions of an angle to that of finding the trigonometric functions of the special angles $30°, 45°$, and $60°$.

Let θ be an angle in standard position as shown in Figure 9.1.

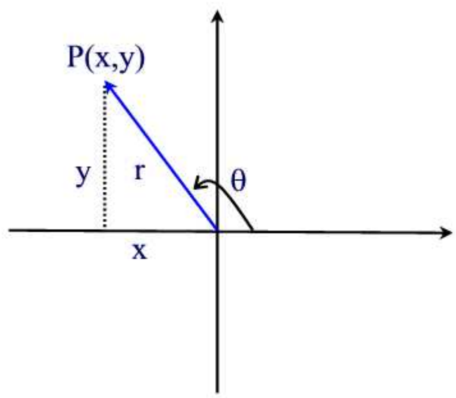


Figure 9.1

Let $P(x, y)$ be any point on the terminal side. If r is the distance from the origin to the point P then by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. We define the trigonometric functions of θ to be

$$\begin{array}{lllllllll} \sin\theta & = & \frac{y}{r} & \cos\theta & = & \frac{x}{r} & \tan\theta & = & \frac{y}{x} \\ \csc\theta & = & \frac{r}{y} & \sec\theta & = & \frac{r}{x} & \cot\theta & = & \frac{x}{y} \end{array}$$

where $x \neq 0$ and $y \neq 0$. If $\theta = k\pi$, where k is an integer, then the functions $\csc\theta$ and $\cot\theta$ are undefined since a point on the terminal side has components $P(x, 0)$. Similarly, if $\theta = (2k+1)\frac{\pi}{2}$ then the functions $\sec\theta$ and $\tan\theta$ are undefined since $P(0, y)$.

Question: Do the above definitions depend on the choice of the point P?

The answer is no. To see this, let $Q(x', y')$ be any other point on the terminal side of θ. See Figure 9.2.

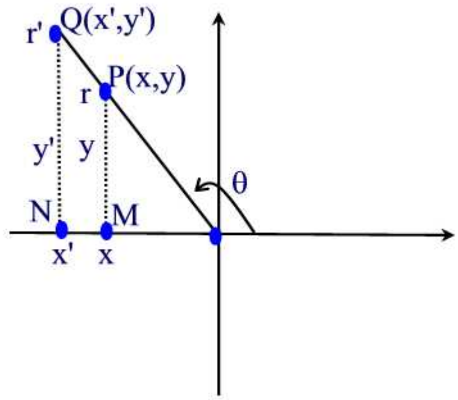


Figure 9.2

Then the right triangles ΔOPM and ΔOQN are similar triangles since corresponding angles are equal. This implies that the ratios of the corresponding sides are equal. Thus

$$\frac{r}{r'} = \frac{x}{x'} = \frac{y}{y'}$$

or equivalently

$$\frac{x}{r} = \frac{x'}{r'} = \frac{y}{r} = \frac{y'}{r'} = \frac{y}{x} = \frac{y'}{x'}$$

Thus,

$$\begin{array}{lllllllll} \sin\theta & = & \frac{y'}{r'} & \cos\theta & = & \frac{x'}{r'} & \tan\theta & = & \frac{y'}{x'} \\ \csc\theta & = & \frac{r'}{y'} & \sec\theta & = & \frac{r'}{x'} & \cot\theta & = & \frac{x'}{y'}. \end{array}$$

This shows that the trigonometric functions are independent of the point chosen on the terminal side of the angle.

Example 9.1
Complete the following chart, using the definitions introduced above.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$						
$90°$						
$180°$						
$270°$						

Solution.
For $\theta = 0°$ we choose the point $P(1,0)$. For $\theta = 90°$ we choose $P(0,1)$; for $\theta = 180°$, we choose $P(-1,0)$ and finally for $\theta = 270°$ we choose $P(0,-1)$. Then by the above definitions we have

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	1	0	undefined	1	undefined
90°	1	0	undefined	1	undefined	0
180°	0	-1	0	undefined	-1	undefined
270°	-1	0	undefined	-1	undefined	0

Example 9.2
Find the exact value of the six trigonometric functions of an angle θ if $P(4,-3)$ is a point on its terminal side.

Solution.
First, note that $r = \sqrt{16+9} = 5$. Thus,

$$\sin\theta = \frac{-3}{5}; \cos\theta = \frac{4}{5}; \tan\theta = \frac{-3}{4}$$
$$\csc\theta = \frac{-5}{3}; \sec\theta = \frac{5}{4}; \cot\theta = \frac{-4}{3}. \blacksquare$$

Example 9.3
Given that $\cos\theta = -\frac{2}{3}, \frac{\pi}{2} < \theta < \pi$, find the exact value of each of the remaining trigonometric functions.

Solution.
Since θ is in Quadrant II then $\sin\theta > 0, \csc\theta > 0, \sec\theta < 0, \tan\theta < 0, \cot\theta < 0$. Thus, $x = -2, r = 3$ and by the Pythagorean formula $y = \sqrt{r^2 - x^2} = \sqrt{9-4} = \sqrt{5}$. It follows that $\sin\theta = \frac{\sqrt{5}}{3}$. So $\csc\theta = \frac{3\sqrt{5}}{5}$, $\tan\theta = -\frac{\sqrt{5}}{2}$, $\cot\theta = -\frac{2\sqrt{5}}{5}$, and finally $\sec\theta = -\frac{3}{2}$. $\blacksquare$

Example 9.4
Complete the following chart of signs of the six trigonometric functions.

Q	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
I						
II						
III						
IV						

Solution.

Q	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
I	+	+	+	+	+	+
II	+	-	-	-	-	+
III	-	-	+	+	-	-
IV	-	+	-	-	+	-

Reference angles
The values of trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding angles called reference angles.
For an angle θ in standard position, the acute angle θ' between the terminal side of θ and either the positive or negative x-axis is called the **reference angle** of θ. Figure 9.3 illustrates the reference angles for some general angles θ.

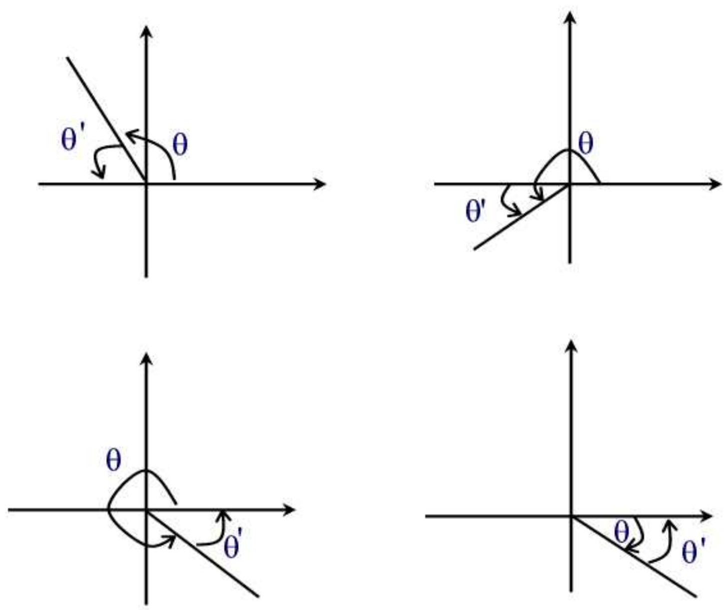


Figure 9.3

Example 9.5
Determine the reference angles for the following given angles:
(a) $-70°$ (b) $255°$ (c) $\frac{5\pi}{3}$ rad

Solution.
(a) $\theta' = 70°$ (b) $\theta' = 75°$ (c) $\theta' = \frac{\pi}{3}$. ■

Remark 9.1
Note that if θ is a nonacute angle and θ' is its reference angle then the trigonometric values of θ are equal to the trigonometric values of θ' with the appropriate sign which depends on the quadrant in which θ lies.

Example 9.6
Evaluate the following:
(a) $\cos\frac{4\pi}{3}$ (b) $\tan(-210°)$ (c) $\csc\frac{11\pi}{3}$.

Solution.
(a) $\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$.
(b) $\tan(-210°) = -\tan 210° = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$.
(c) $\csc\frac{11\pi}{3} = \csc(2\pi + \frac{3\pi}{4}) = \csc\frac{3\pi}{4} = \csc\frac{\pi}{4} = \sqrt{2}$. ■

Example 9.7
Referring to Figure 9.4, answer the following questions.
(a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.
(b) Express the area of ΔOBD in terms of $\tan\theta$.
(c) Use parts (a) and (b) to show

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

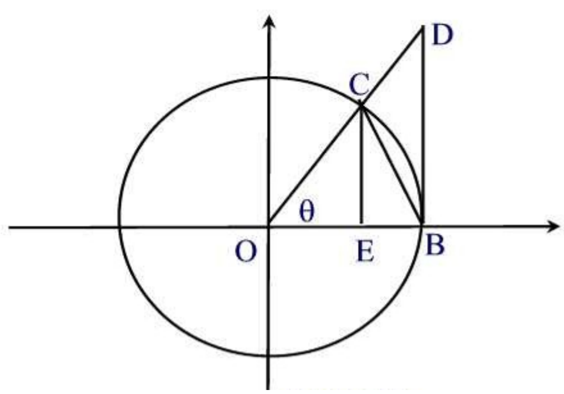


Figure 9.4

Solution.
(a) $Area\ \Delta OBC = \frac{1}{2}|EC||OB| = \frac{1}{2}|EC| = \frac{1}{2}\sin\theta$ since $\sin\theta = \frac{|EC|}{|OC|} = |EC|$.
(b) $Area\ \Delta OBD = \frac{1}{2}|BD||OB| = \frac{1}{2}|BD| = \frac{1}{2}\tan\theta$ since $\tan\theta = \frac{|DB|}{|OB|} = |DB|$.
(c) Using Figure 9.4 we see that

$$Area\ \Delta OBC < Area\ circular\ sector\ OBC < Area\ \Delta OBD.$$

But the area of the circular sector OBC is $\frac{1}{2}\theta$. Hence,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

Multiplying through by $\frac{2}{\sin\theta}$ to obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Note that according to the given figure the angle is assumed to be in the first quadrant so that the sine function is positive there.■

Review Problems

Exercise 9.1
If θ is an angle in standard position, $P(x, y)$ is a point on the terminal side, and r is the distance from the origin to P, write expressions for the six trigonometric functions of θ.

Exercise 9.2
If θ is an angle in standard position, what is its reference angle θ'?

Exercise 9.3
Find the value of each of the six trigonometric functions for the angle whose terminal side passes through the given point:
(a) $P(2, 3)$ (b) $P(-8, -5)$ (c) $P(-2, 3)$.

Exercise 9.4
Let θ be an angle in standard position. State the quadrant in which the terminal side of θ lies:

(a) $\sin\theta > 0$ and $\cos\theta > 0$.
(b) $\cos\theta > 0$ and $\tan\theta < 0$.
(c) $\sin\theta < 0$ and $\cos\theta < 0$.
(d) $\tan\theta < 0$ and $\cos\theta < 0$.

Exercise 9.5
Find the exact value of: (a) $\sin\left(-\frac{13\pi}{6}\right)$ (b) $\cos\left(\frac{19\pi}{6}\right)$.

Exercise 9.6
Given $\sin\theta = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2\sqrt{5}}{5}$. Find the exact values of the four remaining trigonometric functions of θ.

Exercise 9.7
Given that $\sin\theta = \frac{2}{3}$ and $\tan\theta < 0$. Find the exact value of each of the remaining trigonometric functions of θ.

Exercise 9.8
Find the exact value of: $(a) \csc\left(-\frac{\pi}{3}\right)$ (b) $\tan(-30°)$.

Exercise 9.9
Find the exact value without using a calculator:
(a) $\sin^2 30^\circ + \cos^2 30^\circ$.
(b) $\tan 40^\circ - \frac{\sin 40^\circ}{\cos 40^\circ}$.
(c) $\sin 210^\circ - \cos 330^\circ \tan 330^\circ$.
(d) $\sin\left(\frac{3\pi}{2}\right) \tan\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)$.

Exercise 9.10
Find the reference angle θ' :
(a) $\theta = 300^\circ$ (b) $\theta = 2.3\ rad$ (c) $\theta = -135^\circ$.

Exercise 9.11
Find the exact values of each of the following trigonometric functions using reference angles:
(a) $\sin 135^\circ$ (b) $\cos 240^\circ$ (c) $\tan\left(-\frac{\pi}{3}\right)$.

Exercise 9.12
Given that $\cos\theta = -\frac{2}{3}, \frac{\pi}{2} < \theta < \pi$, find the exact value of each of the remaining trigonometric functions.

Exercise 9.13
Without using a calculator, find the exact value of the expression:

$$\sin 35^\circ \csc 55^\circ - \tan 35^\circ \cot 55^\circ.$$

Exercise 9.14
Find the exact value of the sum:

$$\sin 1^\circ + \sin 2^\circ + \cdots + \sin 358^\circ + \sin 359^\circ.$$

10 Trigonometric Functions of Real Numbers

In this section, you will (1) study the trigonometric functions of real numbers, (2) their properties, and (3) some of the identities that they satisfy.

Consider the **unit circle**, i.e. the circle with center at the point $O(0,0)$ and radius 1. Such a circle has the equation $x^2 + y^2 = 1$. Let t be any real number. Start at the point A(1,0) on the unit circle and move on the circle
• counter-clockwise, if $t > 0$, a distance of t units, arriving at some point $P(a,b)$ on the circle;
• clockwise, if $t < 0$, a distance of t units, arriving at some point $P(a,b)$ on the circle.
We define the **wrapping function** W of t to be the point $P(a,b)$. In function notation, we write $W(t) = P(a,b)$.
Now, note that the arc $\widehat{AP}$ subtends a central angle θ. See Figure 10.1. Thus, from the previous section we have $\sin\theta = \frac{b}{r} = b$ and $\cos\theta = \frac{a}{r} = a$ since $r = 1$. We also know that $t = r\theta$. Thus, $t = \theta$. That is, on the unit circle, *the measure of a central angle and the length of its arc are represented by the same real number t.*
So, the trigonometric functions of θ in radians with respect to the unit circle can be viewed as functions of arc length t, which is a REAL NUMBER.
These trigonometric functions of angles are now called **circular functions** and instead of using θ, we write:

$$\begin{array}{lllllllll} \sin t & = & b & \cos t & = & a & \tan t & = & \frac{b}{a} \\ \csc t & = & \frac{1}{b} & \sec t & = & \frac{1}{a} & \cot t & = & \frac{a}{b} \end{array}$$

where $a \neq 0$ and $b \neq 0$. If $a = 0$ then the functions $\sec t$ and $\tan t$ are undefined. If $b = 0$ then the functions $csct$ and $\cot t$ are undefined.

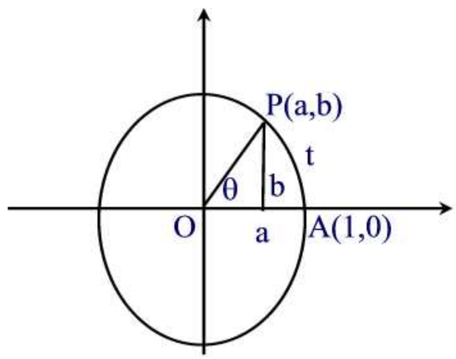


Figure 10.1

Thus, for any real number t, $W(t) = (\cos t, \sin t)$.

Remark 10.1
It follows from the above discussion that the value of a trigonometric function of a real number t is its value at the angle t radians.

Example 10.1
Find the values of x and y such that $W(\frac{\pi}{6}) = P(x, y)$.

Solution.
From Section 8, we have $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin \frac{\pi}{6} = \frac{1}{2}$. Thus, $W(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ ■

Remark 10.2
The difference between the domain of the trigonometric functions defined in the previous section and the ones defined here is the following: the domain of each of the trigonometric functions of the previous section consists of *angles* whereas the domain of each of the functions of this section consists of the set of all *real numbers*.

Properties of the Trigonometric Functions of Real Numbers
First, recall that a function $f(t)$ is **even** if and only if $f(-t) = f(t)$. In this case, the graph of f is symmetric about the y-axis. A function f is said to be **odd** if and only if $f(-t) = -f(t)$. The graph of an odd function is symmetric about the origin.

Theorem 10.1
The functions $\sin t, \csc t, \tan t$ and $\cot t$ are odd functions. The functions $\cos t$ and $\sec t$ are even. That is,

$$\begin{array}{rclcrcl} \sin(-t) & = & -\sin t & & \tan(-t) & = & -\tan t \\ \csc(-t) & = & -\csc t & & \cot(-t) & = & -\cot t \\ \cos(-t) & = & \cos t & & \sec(-t) & = & \sec t \end{array}$$

Proof.
Let $P(a, b)$ be the point on the unit circle such that the arc $\widehat{AP}$ has length t. Then the arc $\widehat{AP'}$, where $P'(a, -b)$, has length t and subtends a central angle $-t$. See Figure 10.2. It follows that

$$\begin{array}{rclclcrclcl} \sin(-t) & = & -b & = & -\sin t & & \tan(-t) & = & \frac{-b}{a} & = & -\tan t \\ \csc(-t) & = & -\frac{1}{b} & = & -\csc t & & \cot(-t) & = & -\frac{a}{b} & = & -\cot t \\ \cos(-t) & = & a & = & \cos t & & \sec(-t) & = & \frac{1}{a} & = & \sec t. \blacksquare \end{array}$$

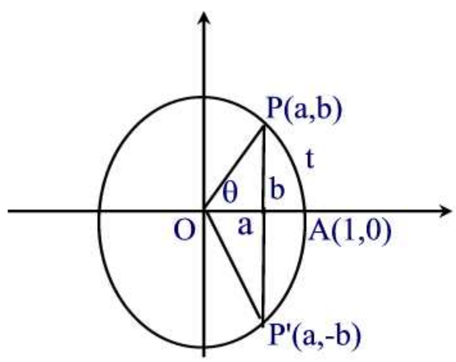


Figure 10.2

Example 10.2
Is the function $f(t) = t - \cos t$ even, odd, or neither?

Solution.
Since $f(-t) = -t - \cos(-t) = -t - \cos t \neq \pm f(t)$ then $f(t)$ is neither even nor odd.$\blacksquare$

We say that a function f is **periodic** of period p if and only if p is the smallest positive number such that $f(t+p) = f(t)$. Graphically, this means that if the graph of f is shifted horizontally by p units, the new graph is identical to the original.

Theorem 10.2
(a) The functions $\sin t, \cos t, \sec t$, and $\csc t$ are periodic functions of period 2π. That is, for any real number t in the domain of these functions

$$\begin{array}{rclcrcl} \sin(t+2\pi) &=& \sin t & \quad & \cos(t+2\pi) &=& \cos t \\ \csc(t+2\pi) &=& \csc t & \quad & \sec(t+2\pi) &=& \sec t. \end{array}$$

(b) The functions $\tan t$ and $\cot t$ are periodic of period π. That is, for any real number t in the domain of these functions

$$\tan(t+\pi) = \tan t \quad \text{and} \quad \cot(t+\pi) = \cot t.$$

Proof.
(a) Since the circumference of the unit circle is 2π then $W(t+2\pi) = W(t)$. That is $(\cos(t+2\pi), \sin(t+2\pi)) = (\cos t, \sin t)$. This implies the following

$$\sin(t+2\pi) = \sin t \quad \text{and} \quad \cos(t+2\pi) = \cos t.$$

Also,

$$\begin{array}{rclclcl} \sec(t+2\pi) &=& \frac{1}{\cos(t+2\pi)} &=& \frac{1}{\cos t} &=& \sec t \\ \csc(t+2\pi) &=& \frac{1}{\sin(t+2\pi)} &=& \frac{1}{\sin t} &=& \csc t \end{array}$$

We show that 2π is the smallest positive number such that the above equalities hold. We prove the result for the sine function. Let $0 < c < 2\pi$ be such that $\sin(x+c) = \sin x$ for all real numbers x. In particular if $x = 0$ then $\sin c = 0$ and consequently $c = k\pi$ for some positive integer k. Thus, $0 < k\pi < 2\pi$ and this implies $k = 1$. Now if $x = \frac{\pi}{2}$ then $\sin(\frac{\pi}{2}+\pi) = \sin\frac{\pi}{2} = 1$. But $\sin(\frac{\pi}{2}+\pi) = -1$, a contradiction. It follows that 2π is the smallest positive number such that $\sin(x+2\pi) = \sin x$. This shows that $\sin x$ is periodic of period 2π. A similar proof holds for the cosine function. Since $\sec t = \frac{1}{\cos t}$ and $\csc t = \frac{1}{\sin t}$ then these functions are of period 2π.
(b) We have that $W(t) = P(a,b)$ and $W(t+\pi) = P(-a,-b)$. Thus,

$$\begin{array}{rclclcl} \tan(t+\pi) &=& \frac{-b}{-a} &=& \frac{b}{a} &=& \tan t \\ \cot(t+\pi) &=& \frac{1}{\tan(t+\pi)} &=& \frac{1}{\tan t} &=& \cot t. \end{array}$$

Now, if $0 < c < \pi$ is such that $\tan(c+x) = \tan x$ for all real numbers x then in particular, for $x = 0$ we have $\tan c = 0$ and this implies that $c = k\pi$ for some positive integer k. Thus, $0 < k\pi < \pi$ i.e. $0 < k < 1$ which is a contradiction. It follows that π is the smallest positive integer such that $\tan(x+\pi) = \tan x$. Hence, the tangent function is of period π. Since $\cot x = \frac{1}{\tan x}$ then the cotangent function is also of period π.■

Theorem 10.3
The domain of $\sin t$ and $\cos t$ consists of all real numbers whereas the range consists of the interval $[-1, 1]$.

Proof.
For any real number t we can find a point $P(a, b)$ on the unit circle such that $W(t) = P(a, b)$. That is, $\cos t = a$ and $\sin t = b$. Hence, the domain of $\sin t$ and $\cos t$ consists of all real numbers. Since P is on the unit circle then $-1 \le a \le 1$ and $-1 \le b \le 1$. That is, $-1 \le \cos t \le 1, -1 \le \sin t \le 1$. So the range consists of the closed interval $[-1, 1]$.■

Theorem 10.4
(a) The domain of $\tan t$ and $\sec t$ consists of all real numbers except the numbers $(2n+1)\frac{\pi}{2}$, where n is an integer.
(b) The range of $\tan t$ consists of all real numbers.
(c) The range of $\sec t$ is $(-\infty, -1] \cup [1, \infty)$.

Proof.
(a) Since $\tan t = \frac{b}{a}$ and $\sec t = \frac{1}{a}$ then the domain consists of those real numbers where $a \neq 0$. But $a = 0$ at $P(0, 1)$ and $P(0, -1)$. i.e. t is an odd multiple of $\frac{\pi}{2}$. That is, the domain of the secant function and the tangent function consists of all real numbers different from $(2n+1)\frac{\pi}{2}$ where n is an integer.
(b) We next determine the range of the tangent function. Let t be any real number. Let $P(a, b)$ be the point on the unit circle that corresponds to an angle θ such that $\tan\theta = \frac{b}{a} = t$. This implies that $b = at$. Since $a^2 + b^2 = 1$ then $a^2(1+t^2) = 1$. Thus, $a = \pm\frac{1}{\sqrt{1+t^2}}$ and $b = \pm\frac{t}{\sqrt{1+t^2}}$. What we have shown here is that, given any real number t there is an angle θ such that $\tan\theta = t$. This proves that the range of the tangent function is the interval $(-\infty, \infty)$, i.e. the set of all real numbers.
(c) If $t \neq (2n+1)\frac{\pi}{2}$, i.e. $a \neq 0$, then $|\sec t| = \frac{1}{|a|} \ge 1$ (since $|a| \le 1$) and this is equivalent to $\sec t \le -1$ or $\sec t \ge 1$. Thus, the range of the secant function is the interval $(-\infty, -1] \cup [1, \infty)$.■

Theorem 10.5
(a) The domain of $\cot t$ and $\csc t$ consists of all real numbers except the numbers $n\pi$, where n is an integer.
(b) The range of $\cot t$ consists of all real numbers.
(c) The range of $\csc t$ is the interval $(-\infty, -1] \cup [1, \infty)$.

Proof.
(a) Since $\cot t = \frac{a}{b}$ and $\csc t = \frac{1}{b}$ then the domain consists of those real numbers where $b \neq 0$. But $b = 0$ at $P(1, 0)$ and $P(-1, 0)$. i.e. t is a multiple of π. That is, the domain of the cosecant function and the cotangent function consists of all real numbers different from $n\pi$ where n is an integer.
(b) Similar argument to part (b) of the previous theorem.
(c) If $t \neq n\pi$, then $b \neq 0$ and therefore $\csc t = \frac{1}{|b|} \geq 1$. This is equivalent to $\csc t \leq -1$ or $\csc t \geq 1$. Thus, the range of the cosecant function is the set $(-\infty, -1] \cup [1, \infty)$. ■

Example 10.3
Find the domain of the function $f(x) = \cot(2x - \frac{\pi}{4})$.

Solution.
The tangent function is defined for all real numbers such that $2x - \frac{\pi}{4} \neq n\pi$. That is, $x \neq (4n + 1)\frac{\pi}{8}$, where n is an integer. ■

Example 10.4
Find the domain of the function $f(x) = \csc \frac{x}{2}$.

Solution.
The function $f(x)$ is defined for all x such that $\frac{x}{2} \neq n\pi$. That is, $x \neq 2n\pi$, where n is an integer. ■

Some Fundamental Trigonometric Identities
By an **identity** we mean an equality of the form $f(x) = g(x)$ which is valid for any real number x in the common domain of f and g.
Now, if $P(a, b)$ is the point on the unit circle such that $W(t) = P(a, b)$ then the trigonometric functions are defined by:

$$\begin{array}{lllllllll} \cos t & = & a & \sin t & = & b & \tan t & = & \frac{b}{a} \\ \sec t & = & \frac{1}{a} & \csc t & = & \frac{1}{b} & \cot t & = & \frac{a}{b}. \end{array}$$

From these definitions, we have the following **reciprocal identities**:

$$\csc t = \frac{1}{\sin t} \quad ; \quad \sec t = \frac{1}{\cos t} \quad ; \quad \cot t = \frac{1}{\tan t}.$$

Also, we have the following **quotient identities:**

$$\tan t = \frac{\sin t}{\cos t} \quad ; \quad \cot t = \frac{\cos t}{\sin t}$$

Example 10.5
Given $\sin\theta = \frac{2\sqrt{2}}{3}$ and $\cos\theta = -\frac{1}{3}$. Find the exact values of the four remaining trigonometric functions.

Solution.

$$\begin{array}{rclcrcl} \sec\theta & = & -3 & ; & \csc\theta & = & \frac{3\sqrt{2}}{4} \\ \tan\theta & = & -2\sqrt{2} & ; & \cot\theta & = & -\frac{\sqrt{2}}{4} \end{array} \blacksquare$$

Since $a^2 + b^2 = 1$ then we can derive the following **Pythagorean identities:**

$$\cos^2 t + \sin^2 t = 1 \tag{1}$$

Dividing both sides of (1) by $\cos^2 t$ to obtain

$$1 + \tan^2 t = \sec^2 t \tag{2}$$

Finally, dividing both sides of (1) by $\sin^2 t$ we obtain

$$1 + \cot^2 t = \csc^2 t \tag{3}$$

Example 10.6
Given $\cos\theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$. Find the remaining trigonometric functions.

Solution.
Using the identity $\cos^2\theta + \sin^2\theta = 1$ to obtain

$$\sin^2\theta + \frac{1}{9} = 1.$$

Solving for $\sin\theta$ and using the fact that $\sin\theta > 0$ in Quadrant II we find $\sin\theta = \frac{2\sqrt{2}}{3}$. It follows that $\sec\theta = -3, \csc\theta = \frac{3\sqrt{2}}{4}, \tan\theta = -2\sqrt{2}$, and $\cot\theta = -\frac{\sqrt{2}}{4}$. $\blacksquare$

Review Problems

Exercise 10.1
—rm Find $W(t)$ for each t : (a) $t = \frac{7\pi}{6}$ (b) $t = -\frac{7\pi}{4}$ (c) $t = \frac{11\pi}{6}$.

Exercise 10.2
Find the exact value of each function:

(a) $\tan\left(\frac{11\pi}{6}\right)$.
(b) $\csc\left(-\frac{\pi}{3}\right)$.
(c) $\sec\left(-\frac{7\pi}{6}\right)$.

Exercise 10.3
Find each value.

(a) $\cos\frac{2\pi}{3}$ (b) $\tan\left(-\frac{\pi}{3}\right)$ (c) $\sin\frac{19\pi}{4}$.

Exercise 10.4
Use the even-odd property of the trigonometric functions to determine each value.

(a) $\sin\left(-\frac{\pi}{6}\right)$ (b) $\cos\left(-\frac{\pi}{4}\right)$.

Exercise 10.5
Determine whether the function defined by each equation is even, odd, or neither:

(a) $f(x) = \sin x + \cos x$.
(b) $g(x) = \tan x + \sin x$.
(c) $h(x) = \frac{\sin x}{x}$.

Exercise 10.6
Let $P(a, b)$ be the point on the unit circle and terminal side of a central angle θ. Find the six trigonometric functions of the angle $\theta + \pi$.

Exercise 10.7
Let $P(a, b)$ be the point on the unit circle and terminal side of a central angle θ. Find the six trigonometric functions of the angle $\pi - \theta$.

Exercise 10.8
Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to the angles $t = \theta$ and $t = \frac{\pi}{2} - \theta$ respectively. Identify the symmetry of the points (x_1, y_1) and (x_2, y_2) and then find the six trigonometric functions of the angle $\frac{\pi}{2} - \theta$.

Exercise 10.9
Find the positive angle between the positive x-axis and the line $y = \sqrt{3}x + 2$.

Exercise 10.10
Let $P(a, b)$ be the point on the unit circle and the terminal side of an angle θ. Calculate $\sin^2\theta + \cos^2\theta$.

Exercise 10.11
Find the domain of the function $f(x) = \tan(3x - \frac{\pi}{4})$.

Exercise 10.12
Find the domain of the function $f(x) = \sec\frac{x}{2}$.

Exercise 10.13
Show that for any integer n we have

$$\begin{aligned} \tan(x + n\pi) &= \tan x, \\ \cot(x + n\pi) &= \cot x. \end{aligned}$$

Exercise 10.14
Show that for any integer n we have

$$\begin{aligned} \cos(x + 2n\pi) &= \cos x; \\ \sec(x + 2n\pi) &= \sec x; \\ \sin(x + 2n\pi) &= \sin x; \\ \csc(x + 2n\pi) &= \csc x. \end{aligned}$$

Exercise 10.15
Establish the identity:

$$(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = 1.$$

Exercise 10.16
Use the trigonometric identities to write each expression in terms of a single trigonometric function or a constant.

(a) $\tan t \cos t$.
(b) $\frac{\csc t}{\cot t}$.
(c) $\frac{1-\cos^2 t}{\tan^2 t}$.
(d) $\frac{1}{1-\sin t} + \frac{1}{1+\sin t}$.
(e) $\sin^2 t(1 + \cot^2 t)$.

Exercise 10.17
Write $\sin t$ in terms of $\cos t, 0 < t < \frac{\pi}{2}$.

Exercise 10.18
Factor each expression:

(a) $\cos^2 t - \sin^2 t$.
(b) $2\sin^2 t - \sin t - 1$.
(c) $\cos^4 t - \sin^4 t$.

Exercise 10.19
A function f is periodic with a period of 3. If $f(2) = -1$, determine $f(14)$.

Exercise 10.20
(a) What is an even function?
(b) Which trigonometric functions are even?
(c) What is an odd function?
(d) Which trigonometric functions are odd?

Exercise 10.21
(a) State the reciprocal identities.
(b) State the Pythagorean identities.

Exercise 10.22
(a) What is a periodic function?
(b) What are the periods of the six trigonometric functions?

Exercise 10.23
What are the domain and range of each of the six trigonometric functions?

Exercise 10.24
Show that the point $P(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle.

Exercise 10.25
If $\sin t = -\frac{8}{17}$ and the terminal side for t is in Quadrant IV, find $\csc t + \sec t$.

The graph of a function gives us a better idea of its behavior. In this and the next two sections we are going to graph the six trigonometric functions as well as transformations of these functions. These functions can be graphed on a rectangular coordinate system by plotting the points whose coordinates belong to the function.

11 Graphs of the Sine and Cosine Functions

In this section, you will learn how to graph the two functions $y = \sin x$ and $y = \cos x$. The graphing mechanism consists of plotting points whose coordinates belong to the function and then connecting these points with a smooth curve, i.e. a curve with no holes, jumps, or sharp corners.

Recall from Section 10 that the domain of the sine and cosine functions is the set of all real numbers. Moreover, the range is the closed interval $[-1, 1]$ and each function is periodic of period 2π. Thus, we will sketch the graph of each function on the interval $[0, 2\pi]$ (i.e one **cycle**) and then repeats it indefinitely to the right and to the left over intervals of lengths 2π of the form $[2n\pi, (2n + 2)\pi]$ where n is an integer.

Graph of $y = \sin x$
We begin by constructing the following table

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	-$\frac{1}{2}$	-1	-$\frac{1}{2}$	0

Plotting the points listed in the above table and connecting them with a smooth curve we obtain the graph of one period (also known as one **cycle**) of the sine function as shown in Figure 11.1.

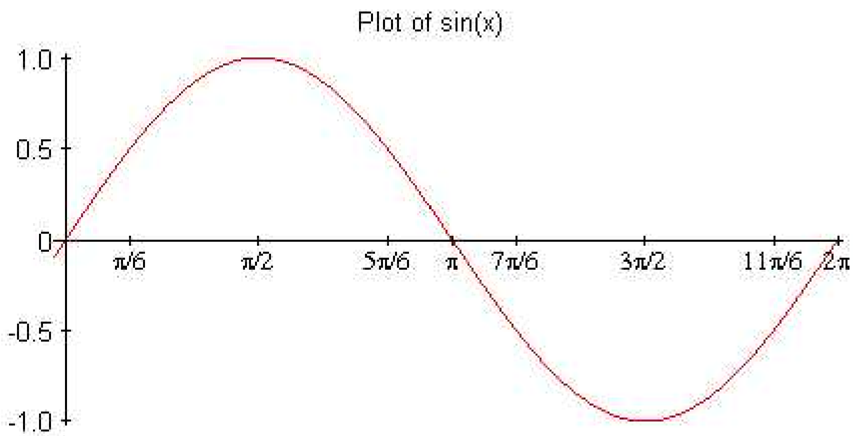


Figure 11.1

Now to obtain the graph of $y = \sin x$ we repeat the above cycle in each direction as shown in Figure 11.2.

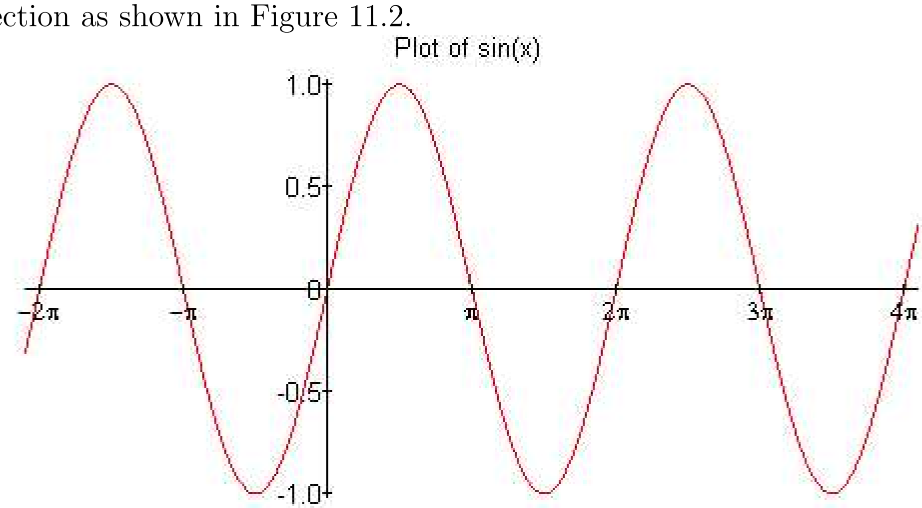


Figure 11.2

Graph of $y = \cos x$

We proceed as we did with the sine function by constructing the table below.

x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-$\frac{1}{2}$	0	$\frac{1}{2}$	1

A one cycle of the graph is shown in Figure 11.3.

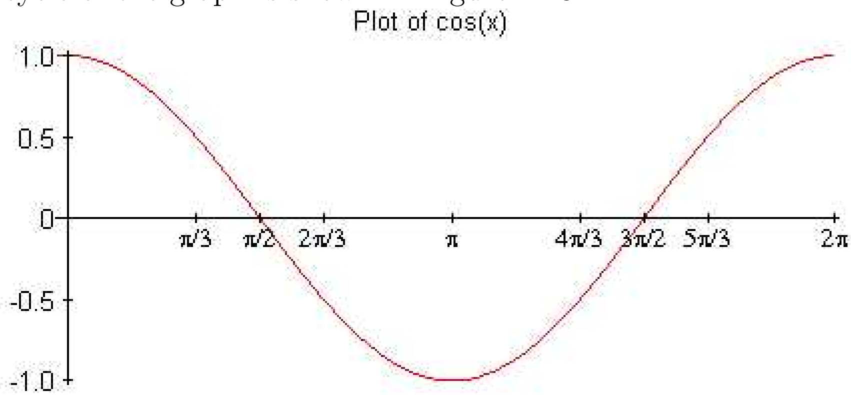


Figure 11.3

A complete graph of $y = \cos x$ is given in Figure 11.4

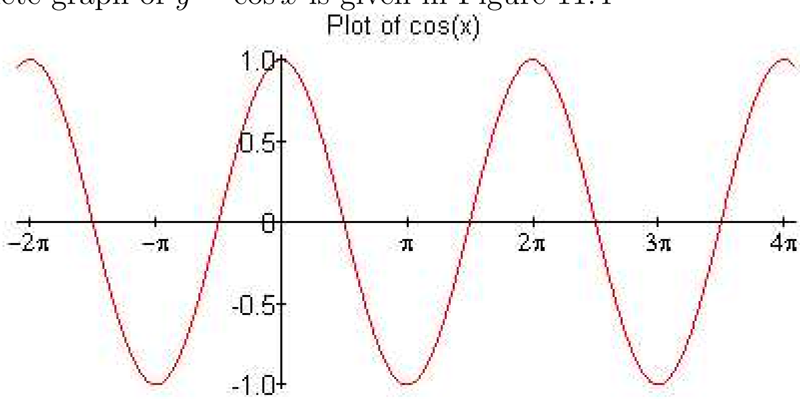


Figure 11.4

Amplitude and period of $y = a\sin(bx), y = a\cos(bx), b > 0$

We now consider graphs of functions that are transformations of the sine and cosine functions.

• **The parameter a:** This is outside the function and so deals with the output (i.e. the y values). Since $-1 \leq \sin(bx) \leq 1$ and $-1 \leq \cos(bx) \leq 1$ then $-a \leq a\sin(bx) \leq a$ and $-a \leq a\cos(bx) \leq a$. So, the range of the function $y = a\sin(bx)$ or the function $y = a\cos(bx)$ is the closed interval $[-a, a]$. The number $|a|$ is called the **amplitude**. Graphically, this number

describes how tall the graph is. The amplitude is half the distance from the top of the curve to the bottom of the curve. If $b = 1$, the amplitude $|a|$ indicates a vertical stretch of the basic sine or cosine curve if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then a reflection about the x-axis is required.
Figure 11.5 shows the graph of $y = 2\sin x$ and the graph of $y = 3\sin x$.

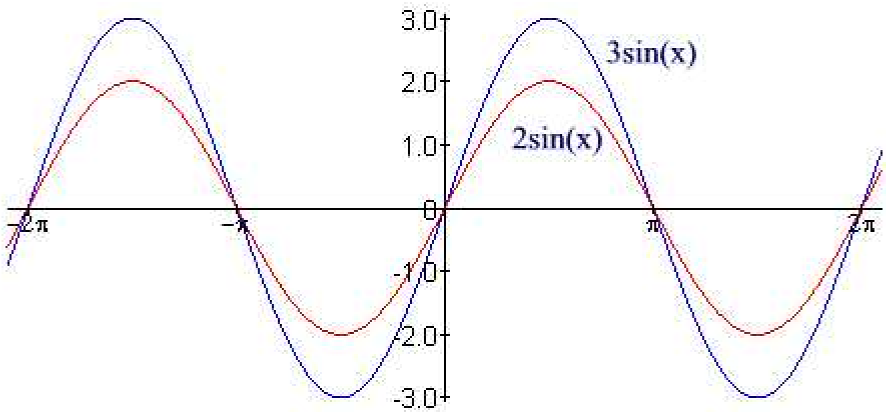


Figure 11.5

• **The parameter b:** This is inside the function and so effects the input (i.e. x values). Now, the graph of either $y = a\sin(bx)$ or $y = a\cos(bx)$ completes one period from $bx = 0$ to $bx = 2\pi$. By solving for x we find the interval of one period to be $[0, \frac{2\pi}{b}]$. Thus, the above mentioned functions have a period of $\frac{2\pi}{b}$. The number b tells you the number of cylces in the interval $[0, 2\pi]$. Graphically,b either stretches (if $b < 1$) or compresses (if $b > 1$) the graph horizontally.
Figure 11.6 shows the function $y = \sin x$ with period 2π and the function $y = \sin(2x)$ with period π.

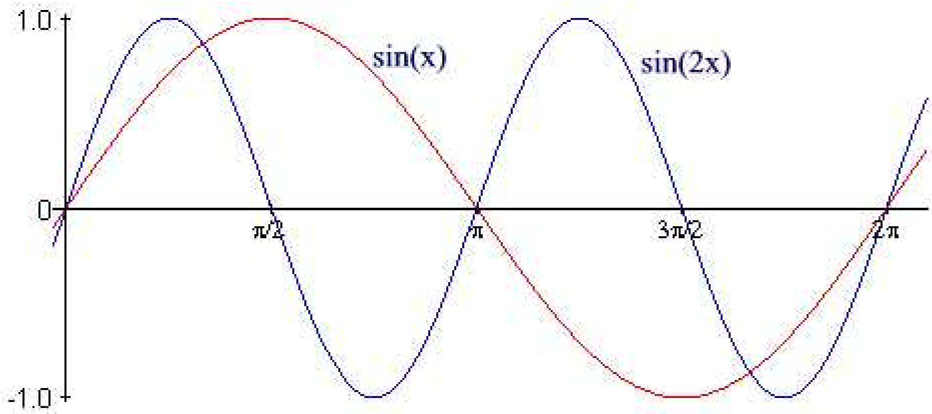


Figure 11.6

Guidelines for Sketching Graphs of Sine and Cosine Functions
To graph $y = a\sin(bx)$ or $y = a\cos(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{2\pi}{b}$. Start at 0 on the x-axis, and lay off a distance of $\frac{2\pi}{b}$.
2. Divide the interval into four equal parts by means of the points: 0, $\frac{\pi}{2b}$, $\frac{\pi}{b}$, $\frac{3\pi}{2b}$, and $\frac{2\pi}{b}$.
3. Evaluate the function for each of the five x-values resulting from step 2. The points will be maximum points, minimum points and x-intercepts.
4. Plot the points found in step 3, and join them with a sinusoidal curve with amplitude $|a|$.
5. Draw additional cycles of the graph, to the right and to the left, as needed.

Example 11.1
(a) What are the zeros of $y = a\sin(bx)$ on the interval $[0, \frac{2\pi}{b}]$?
(b) What are the zeros of $y = a\cos(bx)$ on the interval $[0, \frac{2\pi}{b}]$?

Solution.
(a) The zeros of the sine function $y = a\sin(bx)$ on the interval $[0, 2\pi]$ occur at $bx = 0, bx = \pi$, and $bx = 2\pi$. That is, at $x = 0, x = \frac{\pi}{b}$, and $x = \frac{2\pi}{b}$. The maximum value occurs at $bx = \frac{\pi}{2}$ or $x = \frac{\pi}{2b}$. The minimum value occurs at $bx = \frac{3\pi}{2}$ or $x = \frac{3\pi}{2b}$.
(b) The zeros of the cosine function $y = a\cos(bx)$ occur at $bx = \frac{\pi}{2}$ and

$bx = \frac{3\pi}{2}$. That is, at $x = \frac{\pi}{2b}$ and $x = \frac{3\pi}{2b}$.
The maximum value occurs at $bx = 0$ or $bx = 2\pi$. That is, at $x = 0$ or $x = \frac{2\pi}{b}$. The minimum value occurs at $bx = \pi$ or $x = \frac{\pi}{b}$.■

Example 11.2
Sketch one cycle of the graph of $y = 2\cos x$.

Solution.
The amplitude of $y = 2\cos x$ is 2 and the period is 2π. Finding five points on the graph to obtain

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	2	0	-2	0	2

The graph is a vertical stretch by a factor of 2 of the graph of $\cos x$ as shown in Figure 11.7. ■

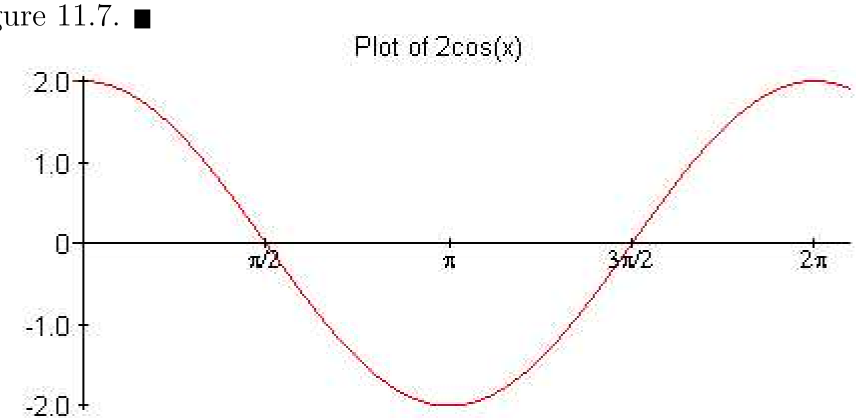


Figure 11.7

Example 11.3
Sketch one cycle of the graph of $y = \cos \pi x$.

Solution.
The amplitude of the function is 1 and the period is $\frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	1	0	-1	0	1

The graph is a horizontal compression by a factor of $\frac{1}{\pi}$ of the graph of $\cos x$ as shown in Figure 11.8. ■

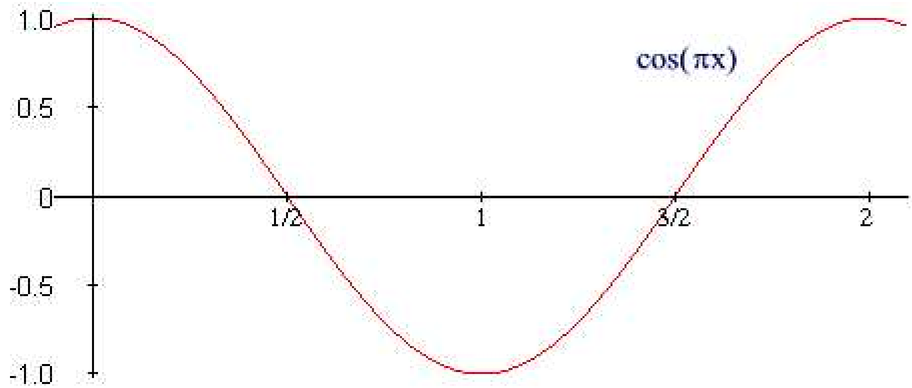


Figure 11.8

Example 11.4
Sketch the graph of the function $y = |\cos x|$ on the interval $[0, 2\pi]$.

Solution.
Since $|\cos x| = \cos x$ when $\cos x \geq 0$ and $|\cos x| = -\cos x$ for $\cos x < 0$ then the graph of $y = |\cos x|$ is the same as the graph of $\cos x$ on the intervals where $\cos x \geq 0$ and is the reflection of $\cos x$ about the x-axis on the intervals where $\cos x < 0$. One cycle of the graph is shown in Figure 11.9.■

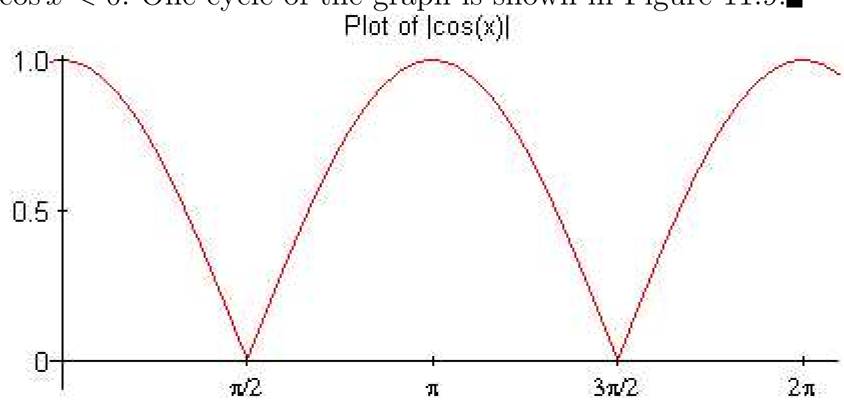


Figure 11.9

Review Problems

Exercise 11.1
State the amplitude and the period of the function defined by each equation:

(a) $y = 2\sin x$.
(b) $y = \frac{1}{2}\sin 2\pi x$.
(c) $y = 2\cos\frac{\pi x}{3}$.
(d) $y = -3\cos\frac{2x}{3}$.

Exercise 11.2
Graph one full cycle of the function defined by each equation:

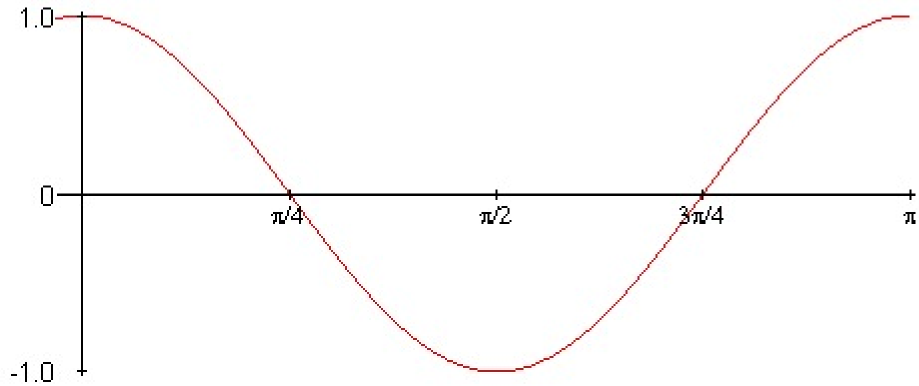
(a) $y = \frac{1}{2}\sin x$.
(b) $y = -\frac{7}{2}\cos x$.
(c) $y = \cos 3x$.
(d) $y = \sin\frac{3\pi}{4}x$.

Exercise 11.3
Graph one full cycle of the function defined by each equation:

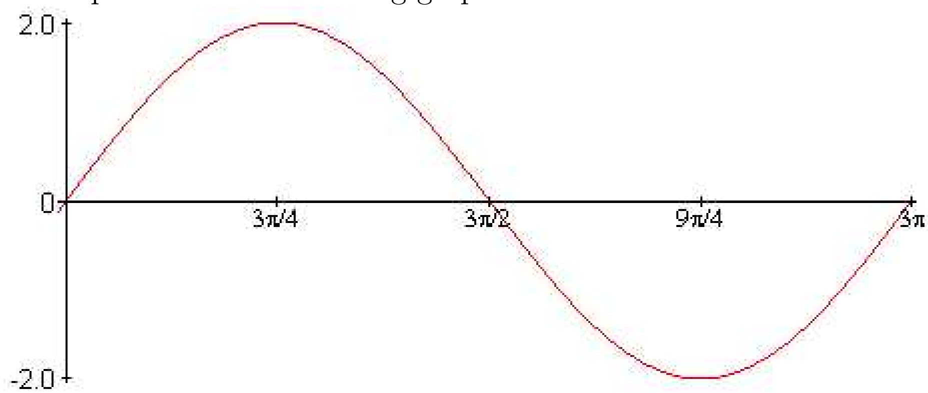
(a) $y = 2\sin \pi x$.
(b) $y = 4\sin\frac{2\pi x}{3}$.
(c) $y = \cos\frac{3\pi}{4}x$.

Exercise 11.4
Graph one full cycle of the function defined by each equation:
(a) $y = \left|2\sin\frac{x}{2}\right|$.
(b) $y = |-2\cos 3x|$.
(c) $y = -\left|2\sin\frac{x}{2}\right|$.

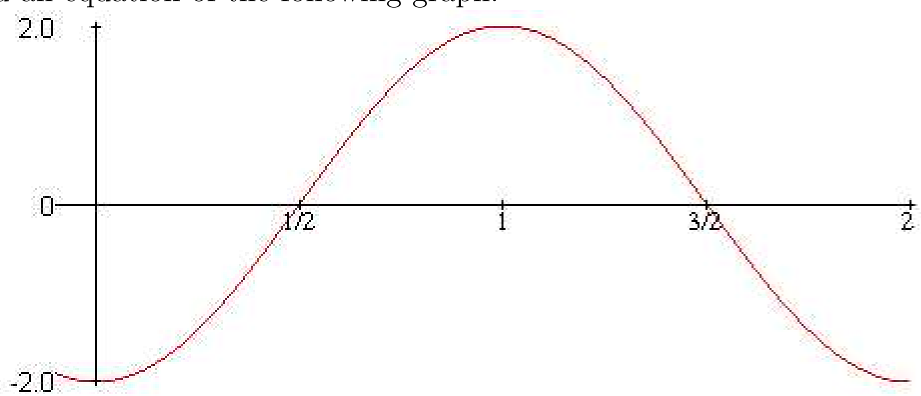
Exercise 11.5
Find an equation of the following graph.



Exercise 11.6
Find an equation of the following graph.

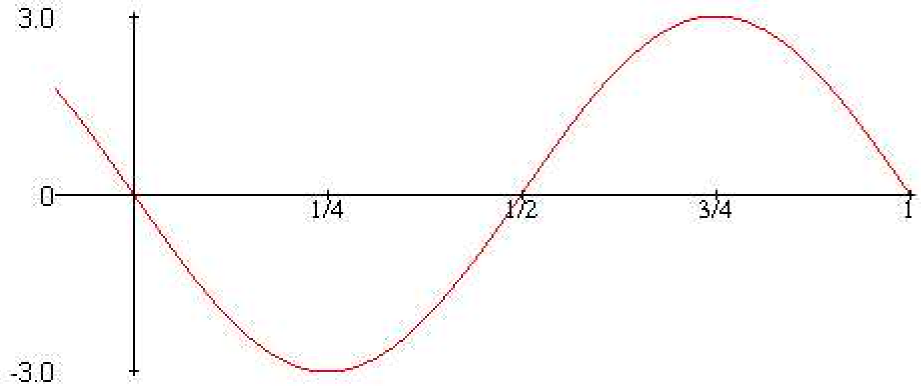


Exercise 11.7
Find an equation of the following graph.



Exercise 11.8

Find an equation of the following graph.



Exercise 11.9
Sketch the graph of $y = 2\sin\frac{2x}{3}, -3\pi \leq x \leq 6\pi$.

Exercise 11.10
Sketch the graphs of $y_1 = 2\cos\frac{x}{2}$ and $y_2 = 2\cos x$ on the same axes for $-2\pi \leq x \leq 4\pi$.

Exercise 11.11
Write an equation for a sine function with amplitude $= 5$ and period $= \frac{2\pi}{3}$.

Exercise 11.12
Write an equation for a cosine function with amplitude $= 3$ and period $= \frac{\pi}{2}$.

Exercise 11.13
A tidal wave that is caused by an earthquake under the ocean is called a **tsunami**. A model of a tsunami is given by $f(t) = A\cos Bt$. Find the equation of a tsunami that has an amplitude of 60 feet and a period of 20 seconds.

Exercise 11.14
The temperature of a chemical reaction oscillated between a low of $30^{circ}C$ and a high of $110°C$. The temperature is at its lowest point when $t = 0$ and completes one cycle over a five hour period.

(a) Sketch a graph of the temperature T, against the elapsed time, t, over a ten-hour period.
(b) Find the period and the amplitude of the graph you drew in part (a).

Exercise 11.15
The function $f(x) = \frac{\sin x}{x}$ is important in calculus. Graph this function using a graphing calculator. Comment on its behavior when x is close to 0.

Exercise 11.16
The function $f(x) = a \sin bx$ has an amplitude of 3 and a period of 4. Find the possible values of a and b.

Exercise 11.17
Determine the domain and the range of the function $f(x) = (\sin x)^{\cos x}$. What is its amplitude?

Exercise 11.18
Graph one full period of $y = 2 - \sin \frac{x}{2}$.

Exercise 11.19
Graph the functions $y = |\sin x|$ and $y = \sin |x|$ on the same coordinate axes.

Exercise 11.20
Explain how the graph of $y = \cos 2x$ differs from the graph of $y = \cos x$.

12 Graphs of the Other Trigonometric Functions

In this section, you will learn how to sketch the graphs of the functions $\tan x, \cot x, \sec x$, and $\csc x$ and transformations of these functions. We are going to use the same method we used for $\sin x$ and $\cos x$. We will use a table of values to plot some of the points. However, the functions of this section are not continuous everywhere like the $\sin x$ and $\cos x$ functions; what this means is that there will be some "breaks" in the graphs- each of them will have vertical asymptotes.

Graph of $y = \tan x$
Recall that the domain of the tangent function consists of all numbers $x \neq (2n+1)\frac{\pi}{2}$, where n is any integer. The range consists of the interval $(-\infty, \infty)$. Also, the tangent function is periodic of period π. Thus, we will sketch the graph on an interval of length π and then complete the whole graph by repetition. The interval we consider is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. First, we will consider the behavior of the tangent function near both $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. For this purpose, we construct the following table:

x	$-\frac{\pi}{2}$	-1.57	-1.5	-1.4	0	1.4	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	undefined	-1255.77	-14.10	-5.80	0	5.8	14.10	1255.77	undefined

It follows that as x approaches $-\frac{\pi}{2}$ from the right the tangent function decreases without bound whereas it increases without bound when x gets closer to $\frac{\pi}{2}$ from the left. We say that the vertical lines $x = \pm\frac{\pi}{2}$ are **vertical asymptotes.** In general, the vertical asymptotes for the graph of the tangent function consist of the zeros of the cosine function, i.e. the lines $x = (2n+1)\frac{\pi}{2}$, where n is an integer.
Next, we construct the following table that provides points on the graph of the tangent function:

x	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan x$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Plotting these points and connecting them with a smooth curve we obtain one period of the graph of $y = \tan x$ as shown in Figure 12.1.

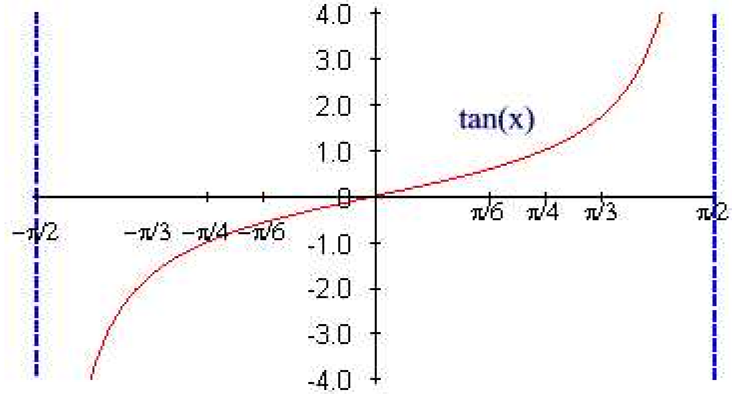


Figure 12.1

We obtain the complete graph by repeating the one cycle over intervals of lengths π as shown in Figure 12.2.

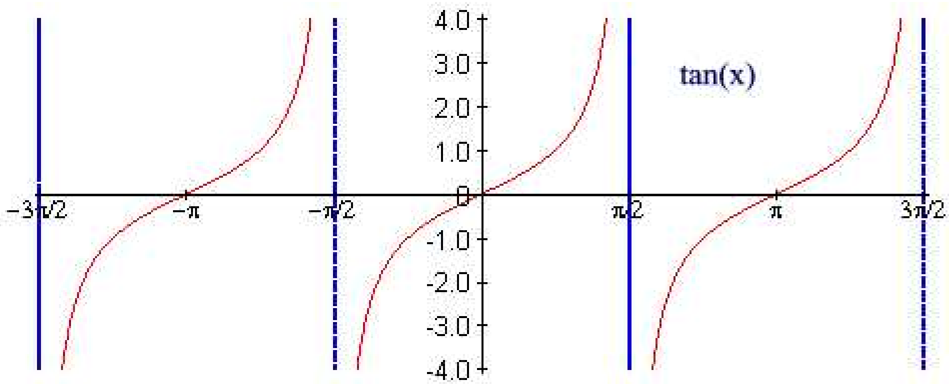


Figure 12.2

Example 12.1
What are the x-intercepts of $y = \tan x$?

Solution.
The x-intercepts of $y = \tan x$ are the zeros of the sine function. That is, the numbers $x = n\pi$ where n is any integer. ■

Graph of $y = \cot x$

The graph of the cotangent function is similar to the graph of the tangent function. Since

$$\cot x = \frac{\cos x}{\sin x}$$

then the vertical asymptotes occur at $x = n\pi$ where n is any integer.
Figure 12.3 shows two periods of the graph of the cotangent function.

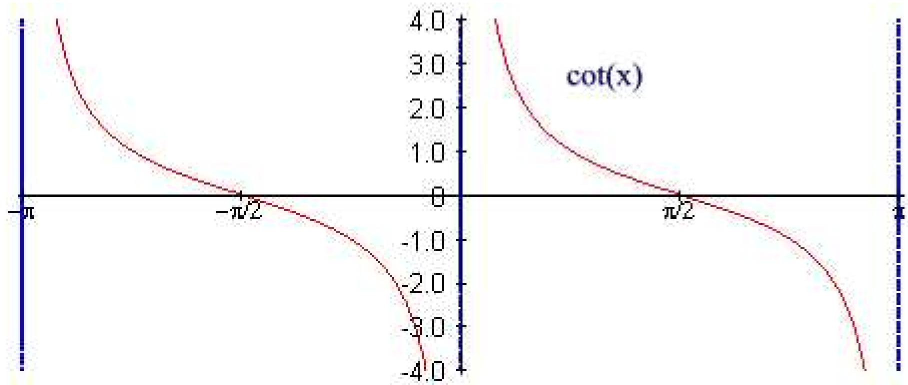


Figure 12.3

The Functions $y = a\tan(bx)$ **and** $y = a\cot(bx), b > 0$

• Note that since the graphs of the tangent function and the cotangent function have no maximum or minimum then these functions have no amplitude.
• The parameter $|a|$ indicates a vertical stretching of the basic tangent or cotangent function if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then reflection about the x-axis is required.
• Since the function $y = \tan x$ (respectively $y = \cot x$) completes one cycle on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (respectively, on $(0, \pi)$) then the function $y = a\tan(bx)$ (respectively, $y = a\cot(bx)$) completes one cycle on the interval $(-\frac{\pi}{2b}, \frac{\pi}{2b})$ (respectively, on the interval $(0, \frac{\pi}{b})$). Thus, these functions are periodic of period $\frac{\pi}{b}$.

Guidelines for Sketching Graphs of Tangent and Cotangent Functions

To graph $y = a\tan(bx)$ or $y = a\cot(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{\pi}{b}$.
2. Graph the asymptotes:

• $x = -\frac{\pi}{2b}$ and $x = \frac{\pi}{2b}$, for the tangent function.
• $x = 0$ and $x = \frac{\pi}{b}$ for the cotangent function.

3. Divide the interval into four equal parts by means of the points:
• $-\frac{\pi}{4b}, 0, \frac{\pi}{4b}$ (for the tangent function).
• $\frac{\pi}{4b}, \frac{\pi}{2b}, \frac{3\pi}{4b}$ (for the cotangent function).

4. Evaluate the function for each of the three x-values resulting from step 3.
5. Plot the points found in step 4, and join them with a smooth curve.
6. Draw additional cycles of the graph, to the right and to the left, as needed.

Example 12.2
Find the period of the function $y = 2\tan\left(\frac{x}{2}\right)$ and then sketch its graph.

Solution.
The period is $\frac{\pi}{\frac{1}{2}} = 2\pi$. Finding some points on the graph

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
y	-2	0	2

The graph of one cycle is given in Figure 12.4.■

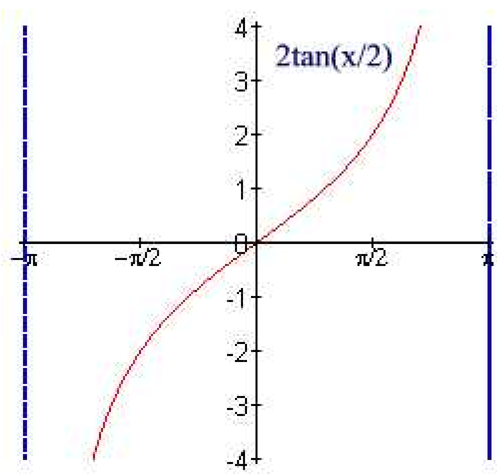


Figure 12.4

Example 12.3
Sketch the graph of $\cot 3x$ through two periods.

Solution.
The given function is of period $\frac{\pi}{b} = \frac{\pi}{3}$. Finding points for one cycle

x	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
y	1	0	-1

Two cycles of the graph is shown in Figure 12.5.■

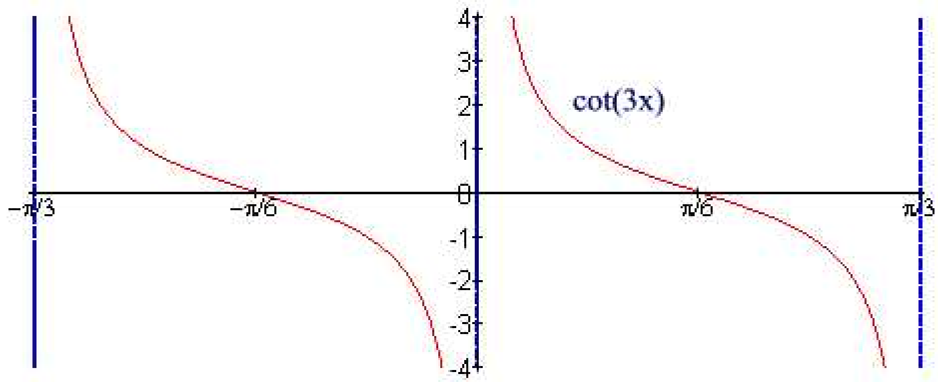


Figure 12.5

Graph of the Secant Function
Recall that the domain of the secant function consists of all numbers $x \neq (2n+1)\frac{\pi}{2}$, where n is any integer. So the graph has vertical asymptotes at $x = (2n+1)\frac{\pi}{2}$. The range consists of the interval $(-\infty, -1] \cup [1, \infty)$. Also, the secant function is periodic of period 2π. Thus, we will sketch the graph on an interval of length 2π and then complete the whole graph by repetition. Note that the value of $\sec x$ at a given number x equals the reciprocal of the corresponding value of $\cos x$. Thus, to sketch the graph of $y = \sec x$, we first sketch the graph of $y = \cos x$. On the same coordinate system, we plot, for each value of x, a point with height equal the reciprocal of $\cos x$. The accompanying table gives some points to plot.

x	$\sec x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{4}$	1.414
0	1
$\frac{\pi}{4}$	1.414
$\frac{\pi}{2}$	undefined
$\frac{3\pi}{4}$	-1.414
π	-1
$\frac{5\pi}{4}$	-1.414
$\frac{3\pi}{2}$	undefined

Plotting these points and connecting them with a smooth curve we obtain the graph of $y = \sec x$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ as shown in Figure 12.6.

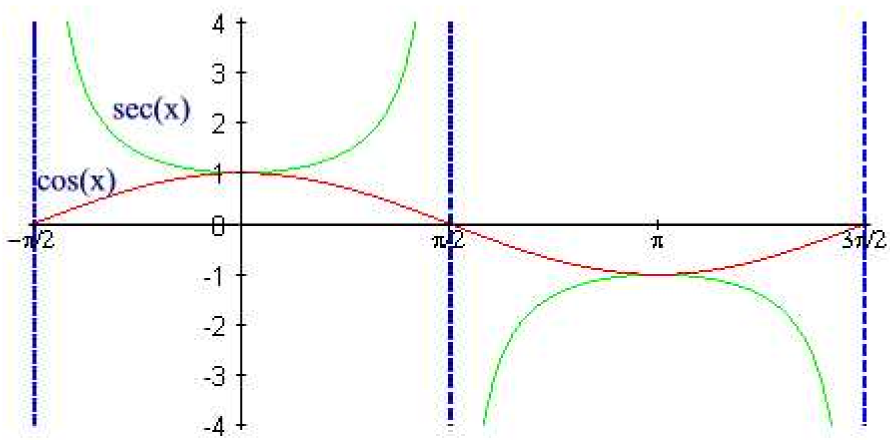


Figure 12.6

Example 12.4
What are the x-intercepts of $y = \sec x$?

Solution.
There are no x-intercepts since either $\sec x \leq -1$ or $\sec x \geq 1$. ■

Graph of $y = cscx$
The graph of $y = \csc x$ may be graphed in a manner similar to $\sec x$. The resulting graph is shown in Figure 12.7. Note that the vertical asymptotes occur at $x = n\pi$, where n is an integer since the domain consists of all real numbers different from $n\pi$.

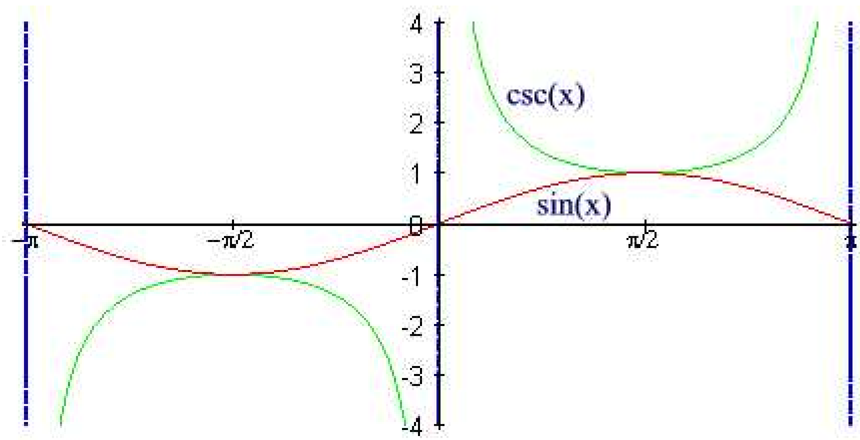


Figure 12.7

Finally, note that in comparing the graphs of secant and cosecant functions with those of the sine and the cosine functions, the "hills" and "valleys" are interchanged. For example, a hill on the cosine curve corresponds to a valley on the secant curve and a valley corresponds to a hill.

Guidelines for Sketching Graphs of $y = a\sec(bx)$ and $y = a\csc(bx)$
To graph $y = a\sec(bx)$ or $y = a\csc(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{2\pi}{b}$.
2. Graph the asymptotes:
• $x = -\frac{\pi}{2b}, x = \frac{\pi}{2b}$, and $x = \frac{3\pi}{2b}$, for the secant function.
• $x = -\frac{\pi}{b}, x = 0$, and $x = \frac{\pi}{b}$ for the cosecant function.

3. Divide the interval into four equal parts by means of the asymptotes and of the points:
• $0, \frac{\pi}{b}$ (for the secant function).
• $-\frac{\pi}{2b}, \frac{\pi}{2b}$ (for the cosecant function).

4. Evaluate the function for each of the two x-values resulting from step 3. One of the point is the lowest of the "valley" and the other is the highest of the "hill."
5. Plot the points found in step 4, and join them with a smooth curve.
6. Draw additional cycles of the graph, to the right and to the left, as needed.

Example 12.5
Sketch the graph of $y = \sec 2x$.

Solution.
The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$. Finding some of the points on the graph

x	0	$\frac{\pi}{2}$
y	1	-1

Figure 12.8 shows one period of the graph.■

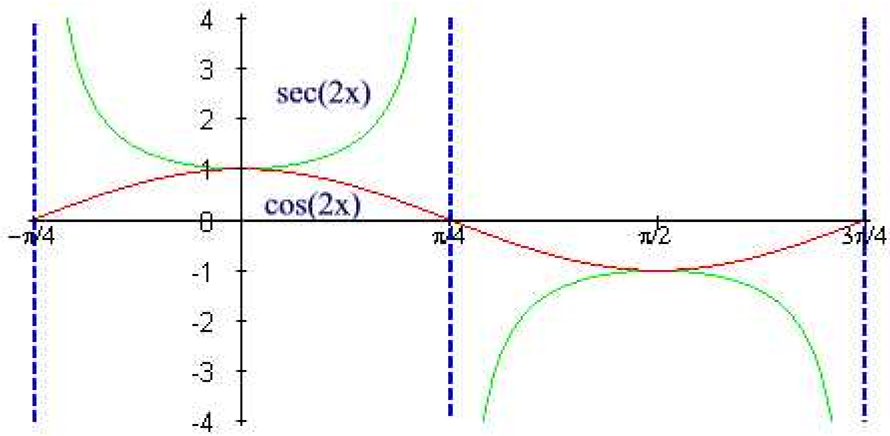


Figure 12.8

Review Problems

Exercise 12.1
For what values of x is $y = \tan x$ undefined?

Exercise 12.2
For what values of x is $y = \cot x$ undefined?

Exercise 12.3
State the period of each function:

(a) $y = \frac{1}{2} \cot 2x$.
(b) $y = -\tan 3x$.
(c) $y = -3 \cot \frac{2x}{3}$.

Exercise 12.4
Sketch one full cycle of the graph of each function:

(a) $y = 3 \tan x$.
(b) $y = 4 \cot x$.
(c) $y = -3 \tan 3x$.
(d) $y = -3 \cot \frac{x}{2}$.
(e) $y = \frac{1}{2} \cot 2x$.

Exercise 12.5
Graph $y = 3 \tan \pi x$ from -2 to 2.

Exercise 12.6
Graph $y = \cot \frac{\pi x}{2}$ from -4 to 4.

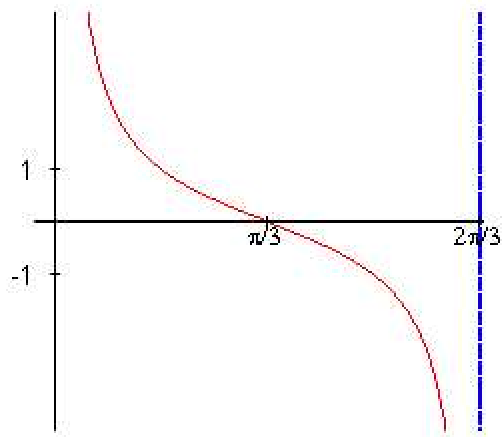
Exercise 12.7
Sketch the graph of $y = |\tan x|$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Exercise 12.8
Sketch the graph of $y = |\cot x|$ on the interval $(0, \pi)$.

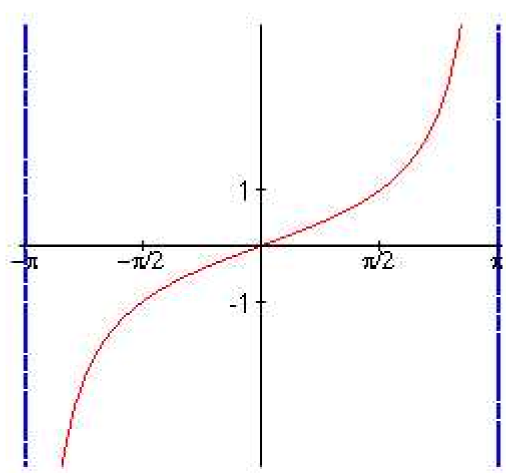
Exercise 12.9
Find the value of b if the function $y = \tan bx$ has period $\frac{\pi}{3}$.

Exercise 12.10
Find the value of b if the function $y = \cot bx$ has period 2.

Exercise 12.11
Find an equation of the graph



Exercise 12.12
Find an equation of the graph



Exercise 12.13
For what values of x is $y = \sec x$ undefined?

Exercise 12.14
For what values of x is $y = \csc x$ undefined?

Exercise 12.15
State the period of each function:

(a) $y = \csc 3x$.
(b) $y = \csc \frac{x}{2}$.
(c) $y = -3 \sec \frac{x}{4}$.
(d) $y = 2 \csc \frac{\pi x}{2}$.

Exercise 12.16
Sketch one full cycle of the graph of each function:

(a) $y = -2 \csc \frac{x}{3}$.
(b) $y = \frac{1}{2} \sec \frac{x}{2}$.
(c) $y = 3 \csc \frac{\pi x}{2}$.

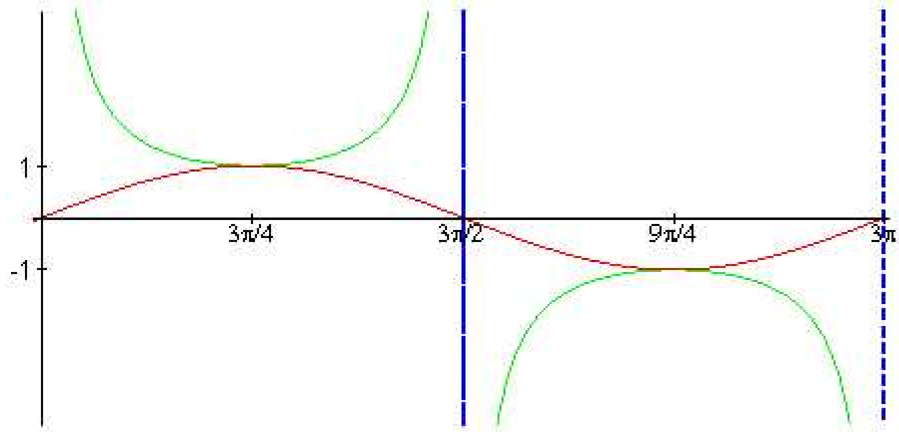
Exercise 12.17
Graph $y = 3 \sec \pi x$ from -2 to 4.

Exercise 12.18
Graph $y = \csc \frac{\pi x}{2}$ from -4 to 4.

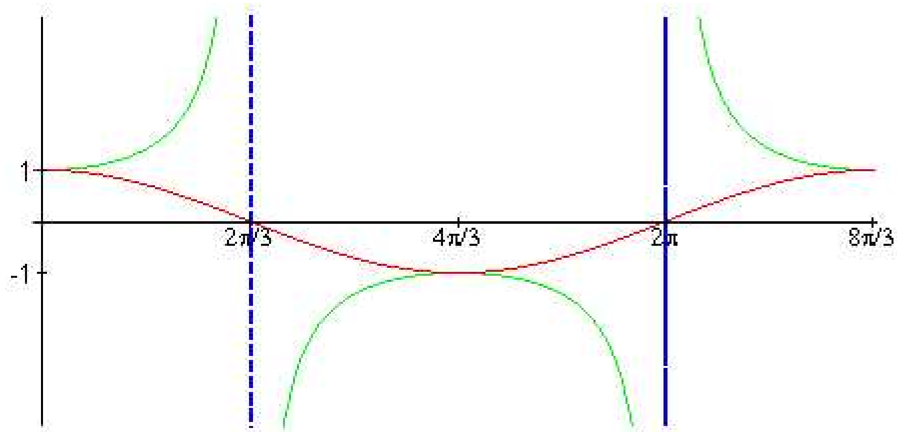
Exercise 12.19
Find the value of b if the function $y = \sec bx$ has period $\frac{3\pi}{4}$.

Exercise 12.20
Find the value of b if the function $y = \csc bx$ has period $\frac{5\pi}{2}$.

Exercise 12.21
Find an equation of the graph



Exercise 12.22
Find an equation of the graph



Exercise 12.23
Sketch the graph of $y = |\csc x|$.

Exercise 12.24
Sketch the graph of $y = |\sec x|$.

Exercise 12.25
Graph one full cycle of $y = \tan x$ and $x = \tan y$ on the same coordinate axes.

Exercise 12.26
The functions $y = \tan x$ and $y = \tan(-x)$ have the same period. Find that period.

13 Translations of Trigonometric Functions

In this section, we will rely heavily on our knowledge of transformations to develop an efficient way of graphing periodic functions. Essentially we will be concerned with translations of the basic trigonometric graphs.
Recall the following translations of graphs(See Section 5):

• To get the graph of $y = f(x - c)$ with $c > 0$, move the graph of $y = f(x)$ to the right by c units.
• To get the graph of $y = f(x + c)$ with $c > 0$, move the graph of $y = f(x)$ to the left by c units.
• To get the graph of $y = f(x) + c$ with $c > 0$, move the graph of $y = f(x)$ upward by c units.
• To get the graph of $y = f(x) - c$ with $c > 0$, move the graph of $y = f(x)$ downward by c units.
• The graph of $y = -f(x)$ is a reflection of the graph of $f(x)$ about the x-axis.
• The graph of $y = f(-x)$ is a reflection of the graph of $f(x)$ about the y-axis.
• The graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically stretched (respectively compressed) by a factor of c, if $c > 1$ (respectively $0 < c < 1$). If $c < 0$ then either the vertical stretch or compression must be followed by a reflection about the x-axis.
• The graph of $y = f(cx)$ is the graph of $y = f(x)$ horizontally stretched (respectively compressed) by a factor of c, if $0 < c < 1$ (respectively $c > 1$). If $c < 0$ then either the horizontal stretch or compression must be followed by a reflection about the y-axis.

Graphs of $y = a\sin(bx + c) + d, b > 0$
We will discuss transformations of the sine function of the form $y = a\sin(bx + c) + d, b > 0$. Similar arguments apply for the remaining five trigonometric functions.
Let's look closely at the effects of each of the parameters a, b, c, and d.

• **The value a**.
This is outside the function and so deals with the output (i.e. the y values). This constant will change the amplitude of the graph, or how tall the graph is. The amplitude, $|a|$, is half the distance from the top of the curve to the

bottom of the curve. Multiplying the sine function by a results in a vertical stretch or compression (followed by a reflection about the x-axis if $a < 0$.)

• **The value b**.
This is inside the function and so effects the input or domain (i.e. the x values). This constant will stretch or compress the graph horizontally. However, it will not change the period directly. For example the function $y = \sin(2x)$ does not have period 2. The period is given by the fraction $\frac{2\pi}{b}$ (i.e. the original period divided by the constant b). So for example the function $y = \sin(2x)$ will have period $\frac{2\pi}{2} = \pi$. b tells you the number of the cycles of the sine function on an interval of length 2π. Thus, the graph of $y = \sin 2x$ consists of two cycles of the sine function on an interval like $[0, 2\pi]$.

• **The value d**.
This again is outside and so will effect the y values of the graph. This constant will vertically shift the graph up and down (depending on if d is positive or negative).

• **The constant c**.
This is on the inside and deals with moving the function horizontally left/right. For example the curve $y = \sin(x - 2)$ is the graph of $y = \sin(x)$ shifted horizontally to the right 2 units. Note that $b = 1$ in this example. For $b \neq 1$, the shift is $-\frac{c}{b}$. To see why this is so, recall that one cycle of $y = a\sin(bx + c)$ is completed for

$$0 \leq bx + c \leq 2\pi.$$

Solving for x we find

$$\begin{array}{lll} -c & \leq bx & \leq -c + 2\pi \\ -\frac{c}{b} & \leq x & \leq -\frac{c}{b} + \frac{2\pi}{b}. \end{array}$$

So basically, the graph of $y = a\sin(bx + c)$ is a horizontal shift of the graph of $y = a\sin(bx)$ by $-\frac{c}{b}$ units. We call $-\frac{c}{b}$ the **phase shift**.

Guidelines for Graphing $y = a\sin(bx + c) + d, b > 0$
To sketch the graph of $y = a\sin(bx + c) + d$ follow these steps.

1. Find the period $\frac{2\pi}{b}$.
2. Find the phase shift $-\frac{c}{b}$.
3. Find the points: $-\frac{c}{b}, \frac{\pi}{2b} - \frac{c}{b}, \frac{\pi}{b} - \frac{c}{b}, \frac{3\pi}{2b} - \frac{c}{b}, \frac{2\pi}{b} - \frac{c}{b}$.

4. Compute the sine of the angles in step 3.
5. Multiply the numbers in step 4 by a.
6. Add the number d to the values obtained in step 5.
7. Plot the points in Step 6 and connect them with a smooth curve to obtain one full cycle of the graph.

Example 13.1
Sketch one full cycle of the graph of $y = -\sin x + 1, 0 \leq x \leq 2\pi$.

Solution.
Starting with the basic sine function we use the points

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	-1	0

Find some plotting points (see the guidelines above)

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	1	2	1

The graph consists of a reflection of the graph of $\sin x$ about the x-axis and then a vertical shift upward by 1 unit as shown in Figure 13.1.■

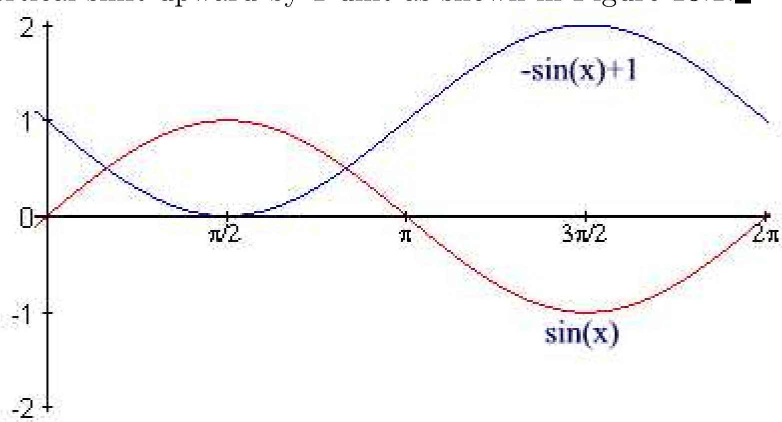


Figure 13.1

Example 13.2
Sketch one full cycle of the graph of the function $y = \sin\left(x - \frac{\pi}{4}\right)$.

Solution.
Find some plotting points as suggested by the guideline.

x	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$
y	0	1	0	-1	0

The graph consists of a horizontal shift of $\sin x$ by $\frac{\pi}{4}$ units to the right as shown in Figure 13.2.■

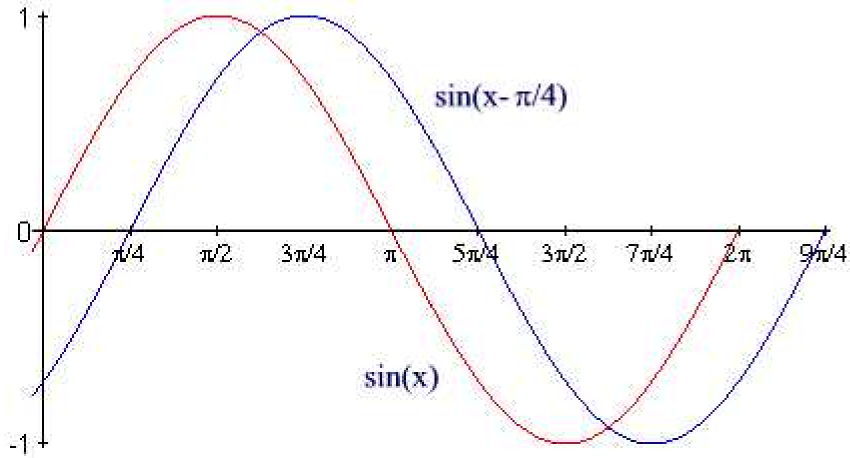


Figure 13.2

Example 13.3
Sketch one full cycle of the graph of $y = \frac{1}{2}\sin(x - \frac{\pi}{3})$.

Solution.
The amplitude is $\frac{1}{2}$, the period is 2π, and the phase shift is $\frac{\pi}{3}$. Find some plotting points.

x	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$
y	0	$\frac{1}{2}$	0	-$\frac{1}{2}$	0

Figure 13.3 shows one period of the graph on the interval $[\frac{\pi}{3}, \frac{7\pi}{3}]$.■

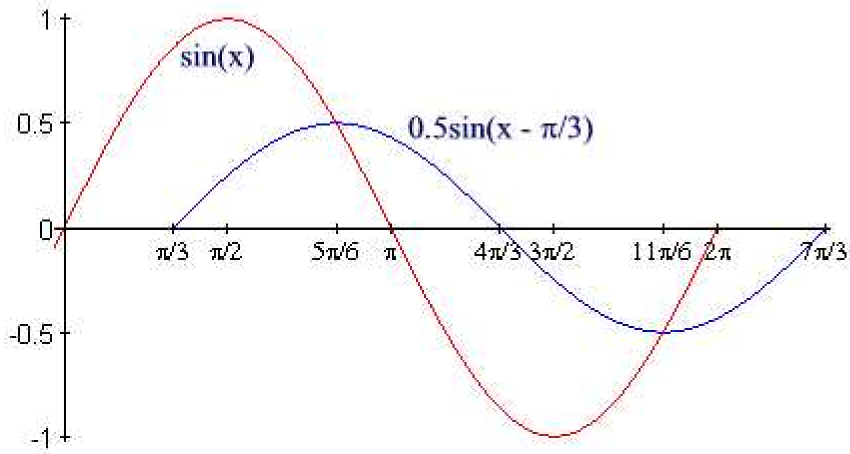


Figure 13.3

Example 13.4
Sketch the graph of $y = -3\cos(2\pi x + 4\pi)$.

Solution.
Find some plotting points.

x	-2	$-\frac{7}{4}$	$-\frac{3}{2}$	-$\frac{5}{4}$	-1
y	-3	0	3	0	-3

The amplitude is 3, the period is $\frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$, and the phase shift is $-\frac{c}{b} = -2$. Figure 13.4 shows two cycles of the graph.■

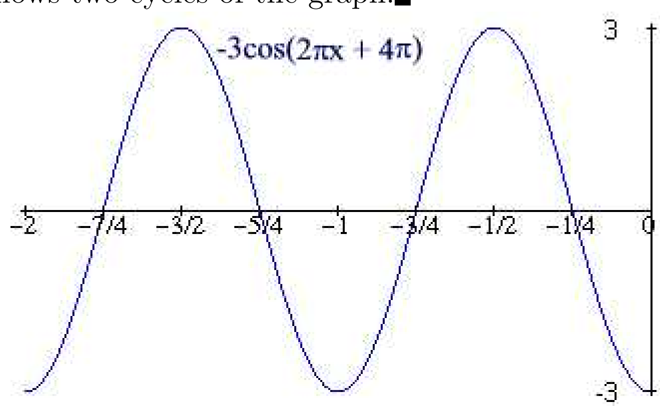


Figure 13.4

Review Problems

Exercise 13.1
Find the amplitude, period, and phase shift for the graph of each function:

(a) $y = -4\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right)$.
(b) $y = \frac{5}{4}\cos(3x - 2\pi)$.

Exercise 13.2
Find the phase shift and period for the graph of each function:

(a) $y = 2\tan\left(2x - \frac{\pi}{4}\right)$.
(b) $y = -3\cot\left(\frac{x}{4} + 3\pi\right)$.

Exercise 13.3
Find the phase shift and period for the graph of each function:

(a) $y = 2\sec\left(2x - \frac{\pi}{8}\right)$.
(b) $y = -3\csc\left(\frac{x}{3} + \pi\right)$.

Exercise 13.4
Graph one full cycle of each function:

(a) $y = \cos\left(2x - \frac{\pi}{3}\right)$.
(b) $y = -2\sin\left(\frac{x}{3} - \frac{2\pi}{3}\right)$.

Exercise 13.5
Graph one full cycle of each function:

(a) $y = \tan(x - \pi)$.
(b) $y = \frac{3}{2}\cot\left(3x + \frac{\pi}{4}\right)$.

Exercise 13.6
Graph one full cycle of each function:

(a) $y = \csc(2x + \pi)$.
(b) $y = \sec(2x + \frac{\pi}{6})$.

Exercise 13.7
Graph one full cycle of each function:

(a) $y = 2\sin\left(\frac{\pi}{2}x + 1\right) - 2.$
(b) $y = -3\cos(2\pi x - 3) + 1.$

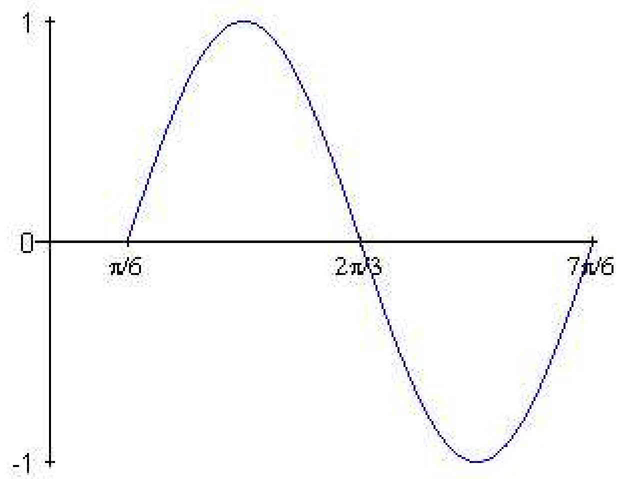
Exercise 13.8
Graph one full cycle of each function:

(a) $y = \csc\frac{x}{3} + 4.$
(b) $y = \sec\left(x - \frac{\pi}{2}\right) + 1.$

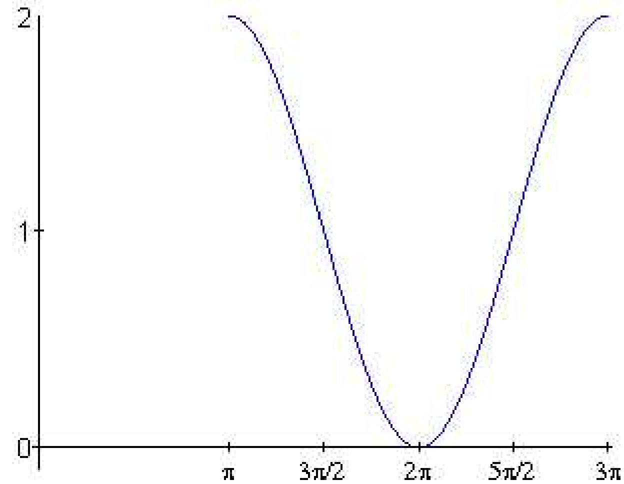
Exercise 13.9
Graph one full cycle of each function:

(a) $y = \tan\frac{x}{2} - 4.$
(b) $y = \cot 2x + 3.$

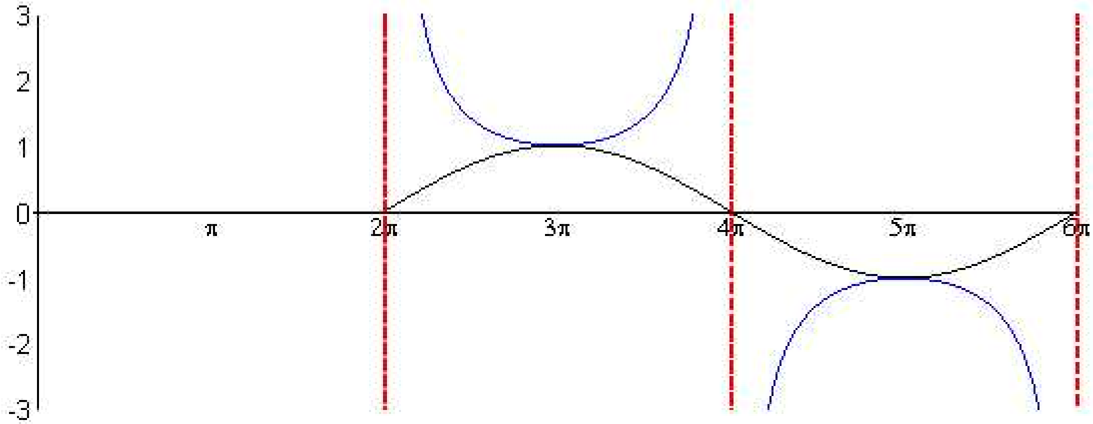
Exercise 13.10
Find an equation of the graph



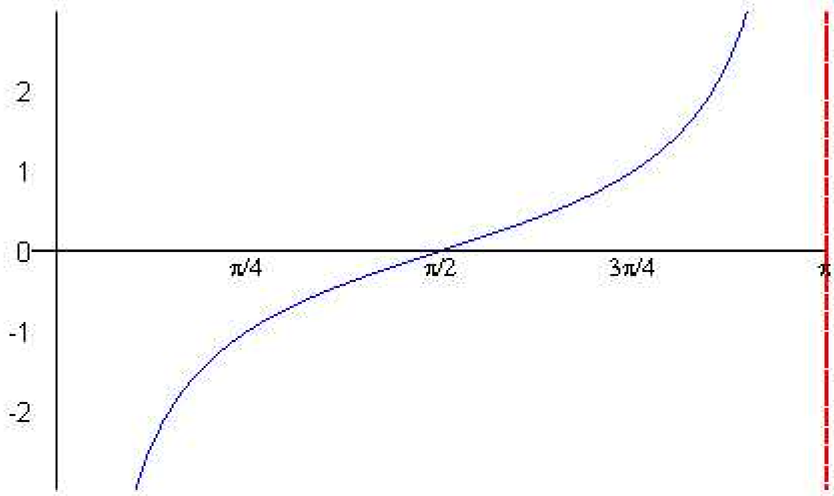
Exercise 13.11
Find an equation of the graph



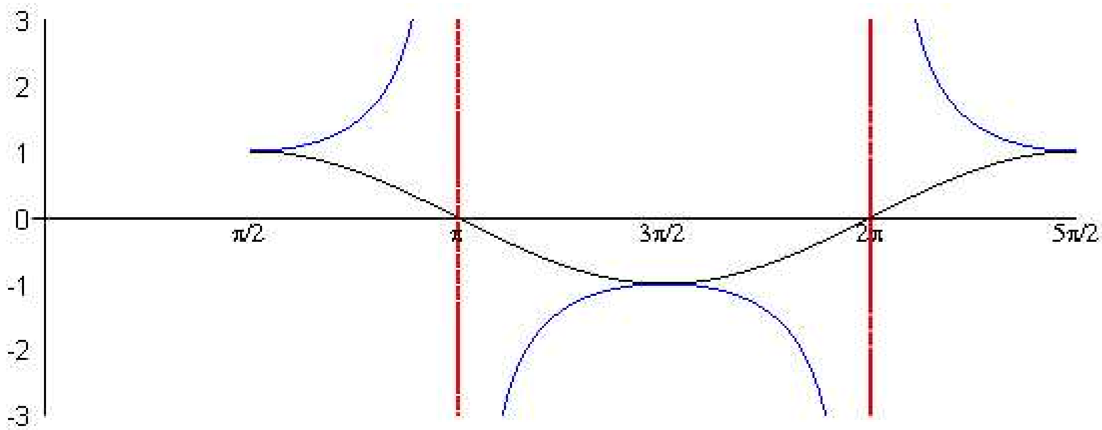
Exercise 13.12
Find an equation of the graph



Exercise 13.13
Find an equation of the graph



Exercise 13.14
Find an equation of the graph



Exercise 13.15
Find an equation of the sine function with amplitude 2, period π, and phase shift $\frac{\pi}{3}$.

Exercise 13.16
Find an equation of the cosine function with amplitude 3, period 3π, and phase shift $-\frac{\pi}{4}$.

Exercise 13.17
Find an equation of the tangent function with period 2π and phase shift $\frac{\pi}{2}$.

Exercise 13.18
Find an equation of the cotangent function with period $\frac{\pi}{2}$ and phase shift $-\frac{\pi}{4}$.

Exercise 13.19
Find an equation of the secant function with period 4π and phase shift $\frac{3\pi}{4}$.

Exercise 13.20
Find an equation of the cosecant function with period $\frac{3\pi}{2}$ and phase shift $\frac{\pi}{4}$.

Exercise 13.21
(a) Find the period and phase shift of the function $y = 2\cot\left(x - \frac{\pi}{2}\right)$.
(b) Sketch one full cycle of the the graph.

Exercise 13.22
(a) Find the period and phase shift of the function $y = 4\csc(2x + \pi)$.
(b) Sketch one full cycle of the the graph.

Exercise 13.23
(a) Find the period and phase shift of the function $y = -4\sec 4\pi x$.
(b) Sketch one full cycle of the the graph.

Exercise 13.24
(a) Find the period and phase shift of the function $y = \tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$.
(b) Sketch one full cycle of the the graph.

Exercise 13.25
(a) Find the period, amplitude and phase shift of the function $y = 10\sin\left(x - \frac{\pi}{2}\right)$.
(b) Sketch one full cycle of the the graph.

Exercise 13.26
(a) Find the period, amplitude and phase shift of the function $y = \cos 2\left(x - \frac{\pi}{2}\right)$.
(b) Sketch one full cycle of the the graph.

Exercise 13.27
Write a sentence to explain how to obtain the graph of $y = 2\sin\left(2x - \frac{\pi}{2}\right) - 1$ from the graph of $y = 2\sin 2x$.

Exercise 13.28
State the period and phase shift for the function $y = 2\cot\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$.

Exercise 13.29
State the amplitude, the period, and the phase shift of the function $y = -3\cos\left(2x + \frac{\pi}{2}\right)$.

Exercise 13.30
Graph one full cycle of the function $y = 3\sin\left(4x - \frac{2\pi}{3}\right) - 3$.

Exercise 13.31
Graph one full cycle of the function $y = -\cos\left(3x + \frac{\pi}{2}\right) + 2$.

Exercise 13.32
Graph one full cycle of the function $y = 2\csc\left(x - \frac{\pi}{4}\right) - 3$.

Exercise 13.33
Graph one full cycle of the function $y = \sec\left(x - \frac{\pi}{2}\right) + 1$.

Exercise 13.34
Graph one full cycle of the function $y = \cot(2x + 3)$.

Exercise 13.35
Graph one full cycle of the function $y = \tan\frac{1}{2} - 4$.

Exercise 13.36
The owner of a shoe store finds that the number of pairs of shoes S, in hundreds, that it sells can be modeled by the function

$$S = 2.7\cos\left(\frac{\pi}{6}t - \frac{7}{12}\pi\right) + 4$$

where t is measured in months, with $t = 0$ representing January 1.

(a) Find the phase shift and the period of S.
(b) Graph one period of S.
(c) Use the graph to determine in which month the store sells the most shoes.

Exercise 13.37
The maximum value of $y = 3\sin x + 4$ is 7. Do you agree? Explain.

Exercise 13.38

The function $bp(t) = 32\cos\left(\frac{10\pi}{3}t - \frac{\pi}{3}\right) + 112,\ 0 \le t \le 20$ gives the blood pressure in millimeters of mercury (mm Hg), a patient during a 20-second interval.

(a) Find the phase shift and the period of *pb*.
(b) Graph one period of *bp*.
(c) What are the patient's maximum (*systolic*) and minimum (*diastolic*) blood pressure during the given time interval?

14 Simple Harmonic Motion

In this section, we use our knowledge of trigonometric functions to describe motion that repeats itself periodically, such as the up-and-down motion of a mass attached to a spring or the back-and-forth motion of a simple pendulum. These phenomena are described by the **sinusoidal** functions, which are the sine and cosine functions or a combination of these functions.
The motion of body is called **a simple harmonic motion** if the body oscillates about an initial state known as the **equilibrium position**. Examples of such motion are the motion of a pendulum swinging back and forth, a spring compressing or stretching, radio waves and television signals.
Variables that describe the periodic nature of the motion are: amplitude, period, and frequency.

• **Amplitude**
In a simple harmonic motion a body generally goes back and forth between two extreme points; the points of maximum displacement from the equilibrium point. The point of maximum displacement is known as the **amplitude** of the motion. For example, if a pendulum is displaced 1 cm from equilibrium and then allowed to oscillate we can say that the amplitude of oscillation is 1 cm.

• **Period**
In a simple harmonic motion, a particle completes a round trip in a certain period of time. This time, p , which denotes the time it takes for the particle to return to its initial position, is called the **period** or **cycle** of the motion.

• **Frequency**
Another concept related to time is the frequency. Frequency, denoted by f , is defined as the number of cycles per unit time and is related to period as such:

$$f = \frac{1}{p}$$

Period is measured in seconds, while frequency is measured in Hertz (or Hz), where 1 Hz = 1 cycle/second.

Modeling Simple Harmonic Motions
As pointed out earlier in the section, a simple harmonic motion is modeled

by either the function $f(t) = a\sin(bt)$ or the function $f(t) = a\cos(bt)$. But the period of either function is known to be $p = \frac{2\pi}{b}$. Solving for b we find

$$b = \frac{2\pi}{p} = 2\pi f.$$

Thus, a simple harmonic motion can be modeled by one of the following functions:

$$y = a\cos(2\pi f)t \quad or \quad y = a\sin(2\pi f)t$$

where $|a|$ is the amplitude.

Remark 14.1
1. If maximum displacement occurs at $t = 0$ then the motion is modeled by the cosine function.
2. If zero displacement occurs at $t = 0$ then the motion is modeled by the sine function.

Example 14.1
Find the amplitude, period and frequency of the simple harmoninc motion described by the equation

$$y = 3\cos\frac{2}{3}t.$$

Solution.
The amplitude is $|a| = |3| = 3$. The period is $= \frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = 3\pi$. The frequency is $f = \frac{1}{p} = \frac{1}{3\pi}$.■

Example 14.2
Find an equation of a simple harmoninc motion with frequency $f = 1.5$ cycles per second and amplitude 4 inches. Assume that maximum displacement occurs at $t = 0$.

Solution.
Since maximum displacement occurs at $t = 0$ then $y = a\cos 2\pi ft$. But $f = 1.5$ and $a = 4$ so that $y = 4\cos 3\pi t$.■

Example 14.3
Find an equation of a simple harmoninc motion with frequency $f = 1$ cycles per second and amplitude 2 cm. Assume zero displacement occurs at $t = 0$.

Solution.
Since zero displacement occurs at $t = 0$ then $y = a\sin 2\pi ft$. Since $a = 2$ and $f = 1$ then $y = 2\sin 2\pi t$.■

Review Problems

Exercise 14.1
Find the amplitude, period, and frequency of the simple harmonic motion:

(a) $y = \frac{2}{3}\cos\frac{t}{3}$.
(b) $y = 4\sin 3t$.

Exercise 14.2
Find the amplitude, period, and frequency of the simple harmonic motion:

(a) $y = 2\sin\frac{\pi t}{3}$.
(b) $y = 5\cos 2\pi t$.

Exercise 14.3
Find an equation of a simple harmoninc motion with frequency $f = 0.8$ cycles per second and amplitude 4 cm. Assume maximum displacement occurs at $t = 0$. Sketch a full cycle of the equation.

Exercise 14.4
Find an equation of a simple harmoninc motion with frequency $f = 0.6$ cycles per second and amplitude 1 m. Assume maximum displacement occurs at $t = 0$. Sketch a full cycle of the equation.

Exercise 14.5
Find an equation of a simple harmoninc motion with amplitude equals to 4 inches and period equals to $\frac{\pi}{2}$. Assume zero displacement at $t = 0$. Graph one full cycle of the equation.

Exercise 14.6
Find an equation of a simple harmoninc motion with amplitude equals to 4 inches and frequency equals to 4 seconds. Assume zero displacement at $t = 0$. Graph one full cycle of the equation.

Exercise 14.7
Find an equation of a simple harmoninc motion with frequency $f = \frac{1}{\pi}$ cycles per second and amplitude 3 inches. Assume maximum displacement occurs at $t = 0$.

Exercise 14.8
Find an equation of a simple harmoninc motion with frequency $f = 0.5$ cycles per second and amplitude 5 inches. Assume maximum displacement occurs at $t = 0$.

Exercise 14.9
Find an equation of a simple harmoninc motion with period $p = 5$ seconds and amplitude 5 cm. Assume maximum displacement occurs at $t = 0$.

Exercise 14.10
Find an equation of a simple harmoninc motion with period $p = \pi$ seconds and amplitude 2 cm. Assume maximum displacement occurs at $t = 0$.

Exercise 14.11
A point P moving in simple harmoninc motion completes 8 cycles every second. If the amplitude of the motion is 50 cm, find an equation that describes the motion P as a function of time. Assume the point P is at its maximum displacement when $t = 0$.

Exercise 14.12
A mass suspended from a spring oscillates in simple harmonic motion at a frequency of 4 cycles per second. The distance from the highest point to the lowest point of the oscillation is 100 cm. Find an equation of that describes the distance of the mass from its rest position as a function of time. Assume the mass is at its lowest point when $t = 0$.

Exercise 14.13
Find the amplitude, the period, and the frequency of the harmonic motion given by $y = 2.5 \sin 50t$.

15 Verifying Trigonometric Identities

In this section, you will learn how to use trigonometric identities to simplify trigonometric expressions.
Equations such as

$$(x-2)(x+2) = x^2 - 4 \quad \textit{or} \quad \frac{x^2-1}{x-1} = x+1$$

are referred to as identities. An **identity** is an equation that is true for all values of x for which the expressions are defined. For example, the equation

$$(x-2)(x+2) = x^2 - 4$$

is defined for all real numbers x. The equation

$$\frac{x^2-1}{x-1} = x+1$$

is true for all real numbers $x \neq 1$.
We have already seen many trigonometric identities. For the sake of completeness we list these basic identities:

Reciprocal Identities

$$\begin{array}{rclrcl} \sin x & = & \frac{1}{\csc x} & \cos x & = & \frac{1}{\sec x} \\ \csc x & = & \frac{1}{\sin x} & \sec x & = & \frac{1}{\cos x} \\ \tan x & = & \frac{1}{\cot x} & \tan x & = & \frac{1}{\cot x} \end{array}$$

quotient identities

$$\tan t = \frac{\sin t}{\cos t} \quad ; \quad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean identities

$$\begin{array}{rcl} \cos^2 x + \sin^2 x & = & 1 \\ 1 + \tan^2 x & = & \sec^2 x \\ 1 + \cot^2 x & = & \csc^2 x \end{array}$$

Even-Odd identities

$$\begin{array}{rclrcl} \sin(-x) & = & -\sin x & \cos(-x) & = & \cos x \\ \csc(-x) & = & -\csc x & \sec(-x) & = & \sec x \\ \tan(-x) & = & -\tan x & \cot(-x) & = & -\cot x \end{array}$$

Simplifying Trigonometric Expressions

Some algebraic expressions can be written in different ways. Rewriting a complicated expression in a much simpler form is known as **simplifying** the expression. There are no standard steps to take to simplify a trigonometric expression. Simplifying trigonometric expressions is similar to factoring polynomials: by trial and error and by experience, you learn what will work in which situations. To simplify algebraic expressions we used factoring, common denominators, and other formulas. We use the same techniques with trigonometric expressions together with the fundamental trigonometric identities listed earlier in the section.

Example 15.1

Simplify the expression $\frac{\sec^2\theta-1}{\sec^2\theta}$.

Solution.

Using the identity $1+\tan^2\theta=\sec^2\theta$ we find

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{1+\tan^2\theta-1}{\sec^2\theta}\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta=\sin^2\theta \blacksquare\end{aligned}$$

Example 15.2

Simplify the expression: $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}$.

Solution.

Taking common denominator and using the identity $\cos^2\theta+\sin^2\theta=1$ we find

$$\begin{aligned}\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} &= \frac{(1+\cos\theta)^2+\sin^2\theta}{\sin\theta(1+\cos\theta)}\\ &= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\ &= 2\csc\theta\blacksquare\end{aligned}$$

Example 15.3

Simplify the expression: $(\sin x-\cos x)(\sin x+\cos x)$.

Solution.

Multiplying we find

$$(\sin x-\cos x)(\sin x+\cos x)=\sin^2 x-\cos^2 x \blacksquare$$

Example 15.4
Simplify $\cos x + \tan x \sin x$.

Solution.
Using the quotient identity $\tan x = \frac{\sin x}{\cos x}$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ we find

$$\begin{aligned}\cos x + \tan x \sin x &= \cos x + \frac{\sin x}{\cos x}\sin x\\ &= \frac{\cos^2 x + \sin^2 x}{\cos x}\\ &= \frac{1}{\cos x} = \sec x. \blacksquare\end{aligned}$$

Establishing Trigonometric Identities
A trigonometric identity is a trigonometric equation that is valid for all values of the variable for which the expressions in the equation are defined. How do you show that a trigonometric equation is *not* an identity? All you need to do is to show that the equation does not hold for some value of the variable. For example, the equation

$$\sin x + \cos x = 1$$

is not an identity since for $x = \frac{\pi}{4}$ we have

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1.$$

To verify that an equation is an identity, we start by simplifying one side of the equation and end up with the other side.
One of the common methods for establishing trigonometric identities is to start with the side containing the more complicated expression and, using appropriate basic identities and algebraic manipulations, such as taking a common denominator, factoring and multiplying by a conjugate, to arrive at the other side of the equality.

Example 15.5
Establish the identity: $\frac{1+\sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1-\cos\theta}$.

Solution.
Using the identity $\cos^2\theta + \sin^2\theta = 1$ we have

$$\begin{aligned}\frac{\sin^2\theta}{1-\cos\theta} &= \frac{1-cos^2\theta}{1-\cos\theta}\\ &= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}\\ &= 1+\cos\theta = \cos\theta(1+\sec\theta)\\ &= \frac{1+\sec\theta}{\sec\theta}\blacksquare\end{aligned}$$

Example 15.6
Show that $\sin\theta = \cos\theta$ is not an identity.

Solution.
Letting $\theta = \frac{\pi}{2}$ we get $1 = \sin\frac{\pi}{2} \neq \cos\frac{\pi}{2} = 0.$ ■

Example 15.7
Verify the identity: $\cos x(\sec x - \cos x) = \sin^2 x$.

Solution.
The left-hand side looks more complex then the right-hand side, so we start with it and try to transform it to the right-hand side.

$$\begin{aligned}\cos x(\sec x - \cos x) &= \cos x \sec x - \cos^2 x \\ &= \cos x \frac{1}{\cos x} = \cos^2 x \\ &= 1 - \cos^2 x = \sin^2 x. \blacksquare\end{aligned}$$

Example 15.8
Verify the identity: $2\tan x \sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$.

Solution.
Starting from the right-hand side to obtain

$$\begin{aligned}\frac{1}{1-\sin x} - \frac{1}{1+\sin x} &= \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)} \\ &= \frac{2\sin x}{1-\sin^2 x} \\ &= \frac{2\sin x}{\cos^2 x} \\ &= 2\frac{\sin x}{\cos x}\frac{1}{\cos x} = 2\tan x \sec x \blacksquare\end{aligned}$$

Example 15.9
Verify the identity: $\frac{\cos x}{1-\sin x} = \sec x + \tan x$.

Solution.
Using the conjugate of $1 - \sin x$ to obtain

$$\begin{aligned}\frac{\cos x}{1-\sin x} &= \frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)} \\ &= \frac{\cos x+\cos x \sin x}{1-\sin^2 x} \\ &= \frac{\cos x+\cos x \sin x}{\cos^2 x} \\ &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x. \blacksquare\end{aligned}$$

Review Problems

Exercise 15.1
Explain the difference between an equation and an identity.

Exercise 15.2
How do you prove a trigonometric identity?

Exercise 15.3
Simplify: $\frac{\sin x \sec x}{\tan x}$.

Exercise 15.4
Simplify: $\cos^3 x + \sin^2 x \sec x$.

Exercise 15.5
Simplify: $\frac{1+\cos x}{1+\sec x}$.

Exercise 15.6
Simplify: $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$.

Exercise 15.7
Simplify: $\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$.

Exercise 15.8
Simplify: $\frac{\cos x}{\sec x+\tan x}$.

Exercise 15.9
Establish the following identities:

(a) $\frac{4\sin^2 x-1}{2\sin x+1} = 2\sin x - 1$.
(b) $(\sin x - \cos x)(\sin x + \cos x) = 1 - 2\cos^2 x$.

Exercise 15.10
Establish the following identities:

(a) $\frac{1}{\sin x} - \frac{1}{\cos x} = \frac{\cos x - \sin x}{\sin x \cos x}$.
(b) $\frac{\cos x}{1-\sin x} = \sec x + \tan x$.

Exercise 15.11
Establish the following identities:

(a) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$.
(b) $\frac{2\sin x \cot x + \sin x - 4\cot x - 2}{2\cot x + 1} = \sin x - 2$.

Exercise 15.12
Establish the following identities:

(a) $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \csc^2 x \sec^2 x$.
(b) $\frac{\frac{1}{\sin x} + \frac{1}{\cos x}}{\frac{1}{\sin x} - \frac{1}{\cos x}} = \frac{\cos^2 x - \sin^2 x}{1 - 2\cos x \sin x}$.

Exercise 15.13
Establish the following identities:

(a) $\frac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \frac{2}{\sec^2 x}$.
(b) $\frac{1+\sin x}{\cos x} - \frac{\cos x}{1 - \sin x} = 0$.

Exercise 15.14
Establish the following identities:

$$\frac{1 + \tan x}{1 - \tan x} = \frac{\cos x + \sin x}{\cos x - \sin x}.$$

Exercise 15.15
Express $\cos x$ in terms of $\sin x$.

Exercise 15.16
Express $\tan x$ in terms of $\cos x$.

Exercise 15.17
Express $\sec x$ in terms of $\sin x$.

Exercise 15.18
Express $\csc x$ in terms of $\sec x$.

Exercise 15.19
Making the indicated trigonometric substitutions in the given algebraic expression and simplify. Assume that $0 \le \theta \le \frac{\pi}{2}$.

(a) $\frac{x}{\sqrt{1-x^2}}, \ x = \sin\theta.$
(b) $\sqrt{1+x^2}, \ x = \tan\theta.$
(c) $\sqrt{x^2-1}, \ x = \sec\theta.$
(d) $\frac{x^2}{\sqrt{4+x^2}}, \ x = 2\tan\theta.$

Exercise 15.20
Show that $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x$ is not an identity.

Exercise 15.21
Show that $\tan^4 x - \sec^4 x = \tan^2 x + \sec^2 x$ is not an identity.

Exercise 15.22
Show that $\tan^4 x - 1 = \sec^2 x$ is not an identity.

Exercise 15.23
Verify the identity: $\frac{1+\tan x}{1-\tan x} = \tan\left(\frac{\pi}{4} + x\right).$

Exercise 15.24
The identity $\cos^2 x + \sin^2 x = 1$ is one of the Pythagorean identities. What are the other two Pythagorean identities and how are they derived?

Exercise 15.25
Verify the identity: $\frac{1-\tan^4 x}{\sec^2 x} = 1 - \tan^2 x.$

Exercise 15.26
Verify the identity: $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x.$

Exercise 15.27
Verify the identity: $\sin^2 x - \cos^2 x = 2\sin^2 x - 1.$

Exercise 15.28
Verify the identity: $\frac{\frac{1}{\sin x} + \csc x}{\frac{1}{\sin x} - \sin x} = 2\sec^2 x.$

Exercise 15.29
Verify the identity: $\sin^4 x - \cos^4 x = 1 - 2\sin^2 x.$

16 Sum and Difference Identities

In this section, you will learn how to apply identities involving the sum or difference of two variables.

Formulas for $\sin(x+y)$ **and** $\sin(x-y)$

Let x and y be two angles as shown in Figure 16.1.

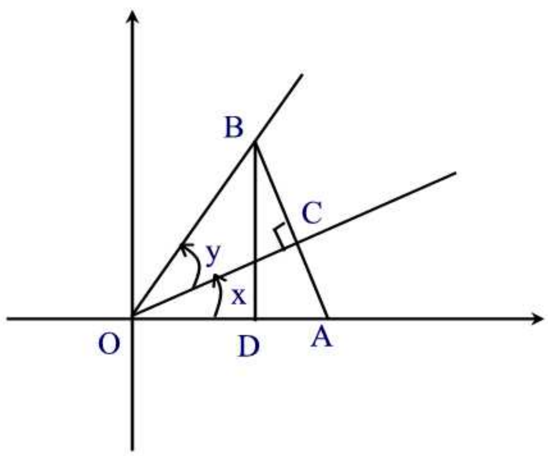


Figure 16.1

Let A be the point on the x-axis such that $|OA| = 1$. From A drop the perpendicular to the terminal side of x. From B drop the perpendicular to the x-axis. Then

$$Area\ \Delta OAB = Area\ \Delta OAC + \ area \Delta OCB.$$

But

$$Area\ \Delta OAC = \frac{1}{2}|OC||AC| = \frac{1}{2}\sin x \cos x.$$

$$Area\ \Delta OCB = \frac{1}{2}|OC||BC| = \frac{1}{2}|OB|^2 \sin y \cos y.$$

$$Area\ \Delta OAB = \frac{1}{2}|BD||OA| = \frac{1}{2}|OB| \sin(x+y).$$

Hence,

$$\frac{1}{2}|OB|\sin(x+y) = \frac{1}{2}\sin x \cos x + \frac{1}{2}|OB|^2 \sin y \cos y.$$

Mutliplying both sides by $\frac{2}{|OB|}$ and using the fact that $|OB| = \frac{\cos x}{\cos y}$ one obtains the addition formula for the sine function:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

To find the difference formula for the sine function we proceed as follows:

$$\begin{array}{rcl} \sin(x-y) & = & \sin(x+(-y)) \\ & = & \sin x \cos(-y) + \cos x \sin(-y) \\ & = & \sin x \cos y - \cos x \sin y \end{array}$$

where we use the fact that the sine function is odd and the cosine function is even.

Example 16.1
Find the exact value of $\sin 75°$.

Solution.
Notice first that $75° = 30° + 45°$. Thus,

$$\begin{array}{rcl} \sin 75° & = & \sin(45° + 30°) \\ & = & \sin 45° \cos 30° + \cos 45° \sin 30° \\ & = & \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \blacksquare \end{array}$$

Example 16.2
Find the exact value of $\sin \frac{\pi}{12}$.

Solution.
Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3}$, the difference formula for sine gives

$$\begin{array}{rcl} \sin \frac{\pi}{12} & = & \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ & = & \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\ & = & \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} \blacksquare \end{array}$$

Example 16.3
Show that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ using the difference formula of the sine function.

Solution.
Since the sine function is an odd function then we can write

$$\begin{array}{rcl}\sin x &=& -\sin(-x) = -\sin\left[\left(\frac{\pi}{2}-x\right)-\frac{\pi}{2}\right]\\ &=& -[\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}\right)]\\ &=& \cos\left(\frac{\pi}{2}-x\right) \blacksquare\end{array}$$

Theorem 16.1 *(***Cofunctions Identities)**
For any angle x, measured in radians, we have

$$\begin{array}{rclrcl}\sin\left(\frac{\pi}{2}-x\right) &=& \cos x & \cos\left(\frac{\pi}{2}-x\right) &=& \sin x\\ \sec\left(\frac{\pi}{2}-x\right) &=& \csc x & \csc\left(\frac{\pi}{2}-x\right) &=& \sec x\\ \tan\left(\frac{\pi}{2}-x\right) &=& \cot x & \cot\left(\frac{\pi}{2}-x\right) &=& \tan x\end{array}$$

Proof.
Recall that $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$.

$$\begin{array}{rcl}\sin\left(\frac{\pi}{2}-x\right) &=& \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x = \cos x\\ \cos\left(\frac{\pi}{2}-x\right) &=& \sin x \quad (See\ Example\ 16.3)\\ \sec\left(\frac{\pi}{2}-x\right) &=& \frac{1}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{1}{\sin x} = \csc x\\ \csc\left(\frac{\pi}{2}-x\right) &=& \frac{1}{\sin\left(\frac{\pi}{2}-x\right)} = \frac{1}{\cos x} = \sec x\\ \tan\left(\frac{\pi}{2}-x\right) &=& \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{\cos x}{\sin x} = \cot x\\ \cot\left(\frac{\pi}{2}-x\right) &=& \frac{1}{\tan\left(\frac{\pi}{2}-x\right)} = \frac{1}{\cot x} = \tan x \blacksquare\end{array}$$

Formulas for $\cos(x+y)$ **and** $\cos(x-y)$
Since $\sin x$ and $\cos x$ are cofunctions of each other then

$$\begin{array}{rcl}\cos(x+y) &=& \sin\left(\frac{\pi}{2}-(x+y)\right) = \sin\left[\left(\frac{\pi}{2}-x\right)-y\right]\\ &=& \sin\left(\frac{\pi}{2}-x\right)\cos y - \cos\left(\frac{\pi}{2}-x\right)\sin y\\ &=& \cos x\cos y - \sin x\sin y\end{array}$$

For the difference formula we have

$$\begin{array}{rcl}\cos(x-y) &=& \cos(x+(-y))\\ &=& \cos x\cos(-y) - \sin x\sin(-y)\\ &=& \cos x\cos y + \sin x\sin y\end{array}$$

where we have used the fact that the sine function is odd and the cosine is even.

Example 16.4
Find the exact value of $\cos \frac{7\pi}{12}$.

Solution.

$$\begin{aligned}
\cos \tfrac{7\pi}{12} &= \cos\left(\tfrac{\pi}{4} + \tfrac{\pi}{3}\right)\\
&= \cos \tfrac{\pi}{4} \cos \tfrac{\pi}{3} - \sin \tfrac{\pi}{4} \sin \tfrac{\pi}{3}\\
&= \tfrac{\sqrt{2}}{2}\tfrac{1}{2} - \tfrac{\sqrt{2}}{2}\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{2}-\sqrt{6}}{4} \blacksquare
\end{aligned}$$

Example 16.5
Find the exact value of: $\sin 42^\circ \cos 12^\circ - \cos 42^\circ \sin 12^\circ$.

Solution.
$\sin 42^\circ \cos 12^\circ - \cos 42^\circ \sin 12^\circ = \sin(42^\circ - 12^\circ) = \sin 30^\circ = \frac{1}{2}$. $\blacksquare$

Example 16.6
Suppose that α and β are both in the third quadrant and that $\sin \alpha = -\frac{\sqrt{3}}{2}$ and $\sin \beta = -\frac{1}{2}$. Determine the value of $\cos(\alpha + \beta)$.

Solution.
Since α and β are in the third quadrant then $\cos \alpha = -\sqrt{1 - \sin^2 \alpha} = -\frac{1}{2}$ and $\cos \beta = -\sqrt{1 - \sin^2 \beta} = -\frac{\sqrt{3}}{2}$. Thus,

$$\begin{aligned}
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\\
&= (-\tfrac{1}{2})(-\tfrac{\sqrt{3}}{2}) - (-\tfrac{\sqrt{3}}{2})(-\tfrac{1}{2}) = 0 \blacksquare
\end{aligned}$$

Formulas for $\tan(x + y)$ and $\tan(x - y)$
Using the sum formulas for the sine and the cosine functions we have

$$\begin{aligned}
\tan(x + y) &= \frac{\sin(x+y)}{\cos(x+y)}\\
&= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\\
&= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}\\
&= \frac{\tan x + \tan y}{1 - \tan x \tan y}
\end{aligned}$$

For the difference formula we have

$$\begin{aligned}
\tan(x - y) &= \tan(x + (-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)}\\
&= \frac{\tan x - \tan y}{1 + \tan x \tan y}
\end{aligned}$$

since $\tan(-x) = -\tan x$.

Example 16.7
Establish the identity: $\tan(\theta + \pi) = \tan\theta$.

Solution.
$\tan(\theta + \pi) = \frac{\tan\theta + \tan\pi}{1 - \tan\theta\tan\pi} = \tan\theta$ since $\tan\pi = 0$. ■

Review Problems

Exercise 16.1
(a) State the addition formulas for sine, cosine, and tangent.
(b) State the subtraction formulas for sine, cosine, and tangent.

Exercise 16.2
Find the exact value of the expression

(a) $\sin(45° + 30°)$.
(b) $\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$.
(c) $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

Exercise 16.3
Find the exact value of the expression

(a) $\cos 212° \cos 122° + \sin 212° \sin 122°$.
(b) $\sin 167° \cos 107° - \cos 167° \sin 107°$.

Exercise 16.4
Find the exact value of the expression

(a) $\sin \frac{5\pi}{12} \cos \frac{\pi}{4} - \cos \frac{5\pi}{12} \sin \frac{\pi}{4}$.
(b) $\cos \frac{\pi}{12} \cos \frac{\pi}{4} - \sin \frac{\pi}{12} \sin \frac{\pi}{4}$.

Exercise 16.5
Find the exact value of the expression

(a) $\frac{\tan \frac{7\pi}{12} - \tan \frac{\pi}{4}}{1 + \tan \frac{7\pi}{12} \tan \frac{\pi}{4}}$.
(b) $\frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{3}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{3}}$.

Exercise 16.6
Write each expression in terms of a single trigonometric function.

(a) $\sin x \cos 3x + \cos x \sin 3x$.
(b) $\sin 7x \cos 3x - \cos 7x \sin 3x$.

Exercise 16.7
Write each expression in terms of a single trigonometric function.

(a) $\cos 4x \cos(-2x) - \sin 4x \sin(-2x)$.
(b) $\frac{\tan 3x + \tan 4x}{1 - \tan 3x \tan 4x}$.
(c) $\frac{\tan 2x - \tan 3x}{1 + \tan 2x \tan 3x}$.

Exercise 16.8
Given $\tan\alpha = \frac{24}{7}$, α in Quadrant I, and $\sin\beta = -\frac{8}{17}$, β in Quadrant II, find the exact value of

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\tan(\alpha-\beta)$.

Exercise 16.9
Given $\sin\alpha = -\frac{4}{5}$, α in Quadrant III, and $\cos\beta = -\frac{12}{13}$, β in Quadrant II, find the exact value of

(a) $\sin(\alpha-\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\tan(\alpha+\beta)$.

Exercise 16.10
Given $\cos\alpha = -\frac{3}{5}$, α in Quadrant III, and $\sin\beta = \frac{5}{13}$, β in Quadrant I, find the exact value of

(a) $\sin(\alpha-\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\tan(\alpha+\beta)$.

Exercise 16.11
Establish the following identities:

(a) $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$.
(b) $\csc(\pi - \theta) = \csc\theta$.

Exercise 16.12
Establish the following identities:

(a) $\sin 6x \cos 2x - \cos 6x \sin 2x = 2\sin 2x \cos 2x$.
(b) $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$.

Exercise 16.13
Establish the following identity: $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{1+\cot\alpha\tan\beta}{1-\cot\alpha\tan\beta}$.

Exercise 16.14
Write the given expression as a function of only $\sin\theta, \cos\theta$, or $\tan\theta$. (k is a given integer)

(a) $\cos(\theta + 3\pi)$ (b) $\cos[\theta + (2k+1)\pi]$ (c) $\sin(\theta + 2k\pi)$.

Exercise 16.15
Establish the identity

$$\frac{\sin(x+h) - \sin x}{h} = \cos x \frac{\sin h}{h} + \sin x \left(\frac{\cos h - 1}{h} \right).$$

Exercise 16.16
Establish the identity

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \frac{\sin h}{h}.$$

Exercise 16.17
Find the exact value of: $\sin\frac{\pi}{12}\cos\frac{\pi}{4} - \sin\frac{\pi}{12}\sin\frac{\pi}{4}$.

Exercise 16.18
Write the following expressions in terms of a single trigonometric function.

(a) $\sin 7x \cos 2x - \cos 7x \sin 2x$
(b) $\cos x \cos 2x + \sin x \sin 2x$
(c) $\cos 4x \cos 2x - \sin 4x \sin 2x$
(d) $\sin 7x \cos 3x + \cos 7x \sin 3x$
(e) $\frac{\tan 2x - \tan 3x}{1 + \tan 2x \tan 3x}$.

Exercise 16.19
Given $\tan\alpha = -\frac{4}{3}$, α in Quadrant II, and $\tan\beta = \frac{15}{8}$, β in Quadrant III, find the exact value of

(a) $\sin(\alpha - \beta)$ (b) $\cos(\alpha + \beta)$ (c) $\tan(\alpha - \beta)$.

Exercise 16.20
Given $\sin\alpha = \frac{24}{25}$, α in Quadrant II, and $\cos\beta = -\frac{4}{5}$, β in Quadrant III, find the exact value of

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha - \beta)$ (c) $\tan(\alpha + \beta)$.

Exercise 16.21
Given $\cos\alpha = \frac{15}{17}$, α in Quadrant I, and $\sin\beta = -\frac{3}{5}$, β in Quadrant III, find the exact value of

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha-\beta)$ (c) $\tan(\alpha+\beta)$.

17 The Double-Angle and Half-Angle Identities

The sum formulas discussed in the previous section are used to derive formulas for double angles and half angles.
To be more specific, consider the sum formula for the sine function

$$\sin{(x+y)} = \sin x \cos y + \cos x \sin y.$$

Then letting $y = x$ to obtain

$$\sin 2x = 2 \sin x \cos x. \tag{4}$$

This is the first double angle formula. To obtain the formula for $\cos 2x$ we use the sum formula for the cosine function

$$\cos{(x+y)} = \cos x \cos y - \sin x \sin y.$$

Letting $y = x$ we obtain

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{5}$$

Since $\sin^2 x + \cos^2 x = 1$ then there are two alternatives to Eq (5), namely

$$\cos 2x = 2\cos^2 x - 1 \tag{6}$$

and

$$\cos 2x = 1 - 2\sin^2 x. \tag{7}$$

Letting $y = x$ in the sum formula of the tangent function we obtain

$$\tan{(2x)} = \tan{(x+x)} = \frac{2\tan x}{1-\tan^2 x}. \tag{8}$$

Formulas (4) - (8) are examples of **double angle identities.**

Example 17.1
Given $\cos\theta = \frac{5}{13}, \frac{3\pi}{2} < \theta < 2\pi$, find $\sin 2\theta, \cos 2\theta$, and $\tan 2\theta$.

Solution.
Since θ is in quadrant IV then $\sin\theta = -\sqrt{1-\cos^2\theta} = -\frac{12}{13}$. Thus,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta = -\tfrac{120}{169}\\
\cos 2\theta &= 2\cos^2\theta - 1 = -\tfrac{119}{169}\\
\tan 2\theta &= \tfrac{\sin 2\theta}{\cos 2\theta} = \tfrac{120}{119} \blacksquare
\end{aligned}$$

Example 17.2
Develop a formula for $\cot 2\theta$ in terms of θ.

Solution.
Using the formula for $\tan 2\theta$ we have

$$\begin{aligned}
\cot 2\theta &= \tfrac{1}{\tan(2\theta)} = \tfrac{1-\tan^2\theta}{2\tan\theta}\\
&= \tfrac{1}{2}(\tfrac{1}{\tan\theta} - \tan\theta) = \tfrac{1}{2}(\cot\theta - \tan\theta) \blacksquare
\end{aligned}$$

Using Eq (6) we find $2\sin^2 x = 1 - \cos 2x$ and therefore

$$\sin^2 x = \frac{1-\cos 2x}{2}. \qquad (9)$$

Similarly, using Eq (7) to obtain

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad (10)$$

and

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos 2x}{1+\cos 2x}. \qquad (11)$$

Formulas (9) - (11) are known as the **square identities.**

Example 17.3
Show that

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

Solution.

$$\begin{aligned}
\sin^4\theta = (\sin^2\theta)^2 &= (\tfrac{1-\cos 2\theta}{2})^2\\
&= \tfrac{1}{4}(1 + \cos^2 2\theta - 2\cos 2\theta)\\
&= \tfrac{1}{4}(1 + (\tfrac{1+\cos 4\theta}{2}) - 2\cos 2\theta)\\
&= \tfrac{3}{8} - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}\cos 4\theta \blacksquare
\end{aligned}$$

We close this section by deriving identities for the sine, cosine, and tangent for half-angle $\frac{\alpha}{2}$.
Let $\theta = \frac{\alpha}{2}$ in Eq (9) through Eq (11) we obtain

$$\begin{array}{rcl} \sin^2\frac{\alpha}{2} & = & \frac{1-\cos\alpha}{2} \\ \cos^2\frac{\alpha}{2} & = & \frac{1+\cos\alpha}{2} \\ \tan^2\frac{\alpha}{2} & = & \frac{1-\cos\alpha}{1+\cos\alpha}. \end{array}$$

Taking square roots to obtain

$$\begin{array}{rcl} \sin\frac{\alpha}{2} & = & \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ \cos\frac{\alpha}{2} & = & \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ \tan\frac{\alpha}{2} & = & \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}. \end{array}$$

where + or − is determined by the quadrant of the angle $\frac{\alpha}{2}$.
Alternative formulas for $\tan\frac{\alpha}{2}$ can be obtained geometrically by means of Figure 17.1.

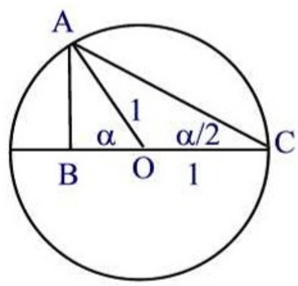


Figure 17.1

Indeed, we have $\cos\alpha = |OB|, \sin\alpha = |AB|$, and

$$\tan\frac{\alpha}{2} = \frac{|AB|}{|BC|} = \frac{\sin\alpha}{1+\cos\alpha}.$$

If we mutliply the top and bottom of the last identity by $1-\cos\theta$ and then using the identity $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$\tan\frac{\theta}{2} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

Example 17.4
Given $\sin\alpha = \frac{3}{5}$ and α in quadrant II. Determine the values of $\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2}$, and $\tan\frac{\alpha}{2}$.

Solution.
Since α is in quadrant II then $\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\frac{4}{5}$. Thus,

$$\begin{array}{rcl}
\sin\frac{\alpha}{2} & = & \sqrt{\frac{1-\cos\alpha}{2}} \\
 & = & \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3\sqrt{10}}{10} \\
\cos\frac{\alpha}{2} & = & -\sqrt{\frac{1+\cos\alpha}{2}} \\
 & = & -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{\sqrt{10}}{10} \\
\tan\frac{\alpha}{2} & = & -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\
 & = & -3 \blacksquare
\end{array}$$

Review Problems

Exercise 17.1
(a) State the double-angle formulas for sine, cosine, and tangent.
(b) State the half-angle formulas for sine, cosine, and tangent.

Exercise 17.2
Write each trigonometric expression in terms of a single trigonometric function.

(a) $1 - 2\sin^2 5\beta$.
(b) $\frac{2\tan 3\alpha}{1-\tan^2 3\alpha}$.

Exercise 17.3
Use the half-angle identities to find the exact value of each trigonometric expression.

(a) $\cos 157.5°$ (b) $\sin 112.5°$.

Exercise 17.4
Use the half-angle identities to find the exact value of each trigonometric expression.

(a) $\tan 67.5°$ (b) $\tan \frac{3\pi}{8}$.

Exercise 17.5
Find the exact value of $\sin 2\theta, \cos 2\theta$, and $\tan 2\theta$ given that $\sin\theta = \frac{8}{17}$ and θ is in Quadrant II.

Exercise 17.6
Find the exact value of $\sin 2\theta, \cos 2\theta$, and $\tan 2\theta$ given that $\tan\theta = -\frac{24}{7}$ and θ is in Quadrant IV.

Exercise 17.7
Find the exact value of $\sin 2\theta, \cos 2\theta$, and $\tan 2\theta$ given that $\cos\theta = \frac{40}{41}$ and θ is in Quadrant IV.

Exercise 17.8
Find the exact value of $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ given that $\sin\theta = \frac{5}{13}$ and θ is in Quadrant II.

Exercise 17.9
Find the exact value of $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ given that $\cos\theta = -\frac{8}{17}$ and θ is in Quadrant III.

Exercise 17.10
Find the exact value of $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ given that $\tan\theta = \frac{4}{3}$ and θ is in Quadrant I.

Exercise 17.11
Find the exact value of $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ given that $\sec\theta = \frac{17}{15}$ and θ is in Quadrant I.

Exercise 17.12
Find the exact value of $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ given that $\cot\theta = \frac{8}{15}$ and θ is in Quadrant III.

Exercise 17.13
Establish the identities:

(a) $\frac{\sin 2x}{1-\sin^2 x} = 2\tan x$.
(b) $\cos^4 x - \sin^4 x = \cos 2x$.

Exercise 17.14
Establish the identities:

(a) $\cos 3x - \cos x = 4\cos^3 x - 4\cos x$.
(b) $\sin^2\frac{x}{2} = \frac{\sec x - 1}{2\sec x}$.

Exercise 17.15
Establish the identities:

(a) $2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$.
(b) $\tan 2x = \frac{2}{\cot x - \tan x}$.

Exercise 17.16
If $\cos x = -\frac{2}{3}$, and x is in quadrant II, find $\cos 2x$ and $\sin 2x$.

Exercise 17.17
Write $\cos 3x$ in terms of $\cos x$.

Exercise 17.18
Verify the identity: $\frac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$.

Exercise 17.19
Express $\sin^2 x \cos^2 x$ in terms of the first powers of cosine.

Exercise 17.20
Use a half-angle formula to find the exact value of $\sin 22.5°$.

Exercise 17.21
Find $\tan \frac{x}{2}$ if $\sin x = \frac{2}{5}$ and x is in quadrant II.

18 Conversion Identities

In this section, you will learn (1) how to restate a product of two trigonometric functions as a sum, (2) how to restate a sum of two trigonometric functions as a product, and (3) how to write a sum of two trigonometric functions as a single function.

Product-To-Sum Identities

By the addition and subtraction formulas for the cosine, we have

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{12}$$

and

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \tag{13}$$

Adding these equations together to obtain

$$2\cos x \cos y = \cos(x+y) + \cos(x-y) \tag{14}$$

or

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)] \tag{15}$$

Subtracting(12) from (13) to obtain

$$2\sin x \sin y = \cos(x-y) - \cos(x+y) \tag{16}$$

or

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]. \tag{17}$$

Now, by the addition and subtraction formulas for the sine, we have

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y. \end{aligned}$$

Adding these equations together to obtain

$$2\sin x \cos y = \sin(x+y) + \sin(x-y) \tag{18}$$

or

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]. \tag{19}$$

Identities (15), (17), and (19) are known as the **product-to-sum identities.**

Example 18.1
Write $\sin 3x \cos x$ as a sum/difference containing only sines and cosines.

Solution.
Using (19) we obtain

$$\begin{array}{rcl}\sin 3x \cos x &=& \frac{1}{2}[\sin(3x+x) + \sin(3x-x)] \\ &=& \frac{1}{2}(\sin 4x + \sin 2x)\blacksquare\end{array}$$

Sum-to-Product Identities

We next derive the so-called **sum-to-product identities.** For this purpose, we let $\alpha = x + y$ and $\beta = x - y$. Solving for x and y in terms of α and β we find

$$x = \frac{\alpha+\beta}{2} \quad \textit{and} \quad y = \frac{\alpha-\beta}{2}.$$

By identity (14) we find

$$\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}). \tag{20}$$

Using identity (16) we find

$$\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2}). \tag{21}$$

Now, by identity (18) we have

$$\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}). \tag{22}$$

Using this last identity by replacing β by $-\beta$ and using the fact that the sine function is odd we find

$$\sin\alpha - \sin\beta = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2}). \tag{23}$$

Formulas (20) - (23) are known as the **sum-to-product formulas**.

Example 18.2
Establish the identity: $\frac{\cos 2x+\cos 2y}{\cos 2x-\cos 2y} = -\cot(x+y)\cot(x-y)$.

Solution.
Using the product-to-sum identities we find

$$\begin{aligned}\frac{\cos 2x+\cos 2y}{\cos 2x-\cos 2y} &= \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{-2\sin\left(\frac{2x+2y}{2}\right)\sin\left(\frac{2x-2y}{2}\right)}\\ &= -\cot(x+y)\cot(x-y) \blacksquare\end{aligned}$$

Writing $a\sin x + b\cos x$ in the Form $k\sin(x+\theta)$.
Let $P(a,b)$ be a coordinate point in the plane and let θ be the angle with initial side the x-axis and terminal side the ray $\overrightarrow{OP}$ as shown in Figure 18.1

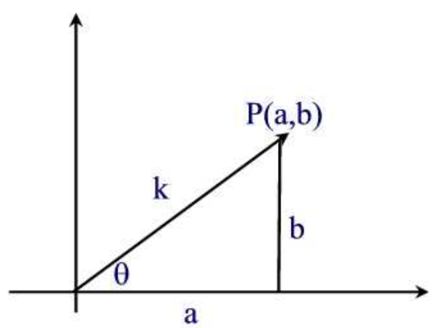


Figure 18.1

Let $k=\sqrt{a^2+b^2}$. Then, according to Figure 91 we have

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad and \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Then in terms of k and θ we can write

$$\begin{aligned}a\sin x + b\cos x &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)\\ &= k(\cos\theta\sin x + \sin\theta\cos x) = k\sin(x+\theta).\end{aligned}$$

Example 18.3
Write $y=\frac{1}{2}\sin x - \frac{1}{2}\cos x$ in the form $y=k\sin(x+\theta)$.

Solution.
Since $a=\frac{1}{2}$ and $b=-\frac{1}{2}$ then $k=\sqrt{(\frac{1}{2})^2+(-\frac{1}{2})^2}=\frac{\sqrt{2}}{2}, \cos\theta=\frac{a}{k}=\frac{\sqrt{2}}{2}, \sin\theta=\frac{b}{k}=-\frac{\sqrt{2}}{2}$. Thus $\theta=-45°$ and

$$y=\frac{\sqrt{2}}{2}\sin(x-45°).\blacksquare$$

Review Problems

Exercise 18.1
(a) State the product-to-sum formulas.
(b) State the sum-to-product formulas.

Exercise 18.2
Write each expression as the sum or difference of two functions.

(a) $2\sin x \cos 2x$ (b) $2\sin 4x \sin 2x$ (c) $\cos 3x \cos 5x$.

Exercise 18.3
Find the exact value of each expression.

(a) $\cos 75^\circ \cos 15^\circ$ (b) $\sin \frac{13\pi}{12} \cos \frac{\pi}{12}$ (c) $\sin \frac{11\pi}{12} \sin \frac{7\pi}{12}$.

Exercise 18.4
Write each expression as the product of two functions.

(a) $\sin 4\theta + \sin 2\theta$
(b) $\cos 3\theta + \cos \theta$.

Exercise 18.5
Write each expression as the product of two functions.

(a) $\sin \frac{\theta}{2} - \sin \frac{\theta}{3}$
(b) $\cos \frac{\theta}{2} - \cos \theta$.

Exercise 18.6
Establish the identity.

(a) $2\cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$.
(b) $2\cos 3x \sin x = 2\sin x \cos x - 8\cos x \sin^3 x$.

Exercise 18.7
Establish the identity.

(a) $\sin 3x - \sin x = 2\sin x - 4\sin^3 x$
(b) $\sin(x + y)\cos(x - y) = \sin x \cos x + \sin y \cos y$.

Exercise 18.8
Establish the identity.

(a) $\frac{\sin 3x - \sin x}{\cos 3x - \cos x} = -\cot 2x$
(b) $\frac{\sin 5x + \sin 3x}{4\sin x \cos^3 x - 4\sin^3 x \cos x} = 2\cos x.$

Exercise 18.9
Write the given equation in the form $y = k\sin(x+\alpha)$, where α is in degrees.

(a) $y = \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x$
(b) $y = \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x.$

Exercise 18.10
Write the given equation in the form $y = k\sin(x+\alpha)$, where α is in degrees.

(a) $y = \pi\sin x - \pi\cos x$
(b) $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x.$

Exercise 18.11
Write the given equation in the form $y = k\sin(x+\alpha)$, where α is in radians.

(a) $y = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$
(b) $y = -10\sin x + 10\sqrt{3}\cos x.$

Exercise 18.12
Graph one full cycle of each equation.

(a) $y = -\sin x - \sqrt{3}\cos x$
(b) $y = \sin x + \sqrt{3}\cos x.$

Exercise 18.13
Graph one full cycle of each equation.

(a) $y = -5\sin x + 5\sqrt{3}\cos x$
(b) $y = 6\sqrt{3}\sin x - 6\cos x.$

Exercise 18.14
Find the amplitude, phase shift, and period, and then graph one full cycle of the function.

$$y = \sin\frac{x}{2} - \cos\frac{x}{2}.$$

Exercise 18.15
Find the amplitude, phase shift, and period, and then graph one full cycle of the function.

$$y = \sqrt{3}\sin 2x - \cos 2x.$$

Exercise 18.16
Find the amplitude, phase shift, and period, and then graph one full cycle of the function.

$$y = \sin \pi x - \sqrt{3}\cos \pi x.$$

Exercise 18.17
Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

Exercise 18.18
Write $\sin 7x + \sin 3x$ as a product of trigonometric functions.

Exercise 18.19
Verify the identity: $\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Exercise 18.20
Write each expression as the sum or difference of two functions.

(a) $2\sin 4x \sin 2x$ (b) $2\sin 5x \cos 3x$ (c) $\cos 6x \sin 2x$

Exercise 18.21
Find the exact value of each expression.

(a) $\sin 105° \cos 15°$ (b) $\sin\frac{\pi}{12}\cos\frac{7\pi}{12}$ (c) $\sin\frac{11\pi}{12}\sin\frac{7\pi}{12}$

Exercise 18.22
Write each expression as the product of two functions.

(a) $\sin 5\theta + \sin 9\theta$
(b) $\cos 3\theta + \cos 5\theta$

Exercise 18.23
Write each expression as the product of two functions.

(a) $\sin 7\theta - \sin 3\theta$
(b) $\cos\frac{\theta}{2} + \cos\theta$

Exercise 18.24
Establish the identity.

(a) $\sin 5x \cos 3x = \sin 4x \cos 4x + \sin x \cos x$.
(b) $2\cos 5x \cos 7x = \cos^2 6x - \sin^2 6x + 2\cos^2 x - 1$.

Exercise 18.25
Write the given equation in the form $y = k\sin(x+\alpha)$, where α is in degrees.

(a) $y = -\sin x - \cos x$
(b) $y = \sqrt{3}\sin x - \cos x$.

Exercise 18.26
Write the given equation in the form $y = k\sin(x+\alpha)$, where α is in radians.

(a) $y = 2\sin x + 2\cos x$
(b) $y = -\sqrt{2}\sin x + \sqrt{2}\cos x$.

Exercise 18.27
Graph one full cycle of each equation.

(a) $y = -\sqrt{3}\sin x + \cos x$
(b) $y = 5\sqrt{2}\sin x + 5\sqrt{2}\cos x$.

Exercise 18.28
Find the amplitude, phase shift, and period, and then graph one full cycle of the function.

$$y = -\sqrt{3}\sin\frac{x}{2} + \cos\frac{x}{2}.$$

19 Inverse Trigonometric Functions

In this and the next section, we will discuss the inverse trigonometric functions. Looking at the graphs of the trigonometric functions we see that these functions are not one-to-one in their domains by the horizontal line test. However, restricted to suitable domains these functions become one- to-one and therefore possess inverse functions.

The Inverse Sine Function

The function $f(x) = \sin x$ is increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. See Figure 19.1. Thus, $f(x)$ is one-to-one and consequently it has an inverse denoted by

$$f^{-1}(x) = \sin^{-1} x.$$

We call this new function the **inverse sine function.**

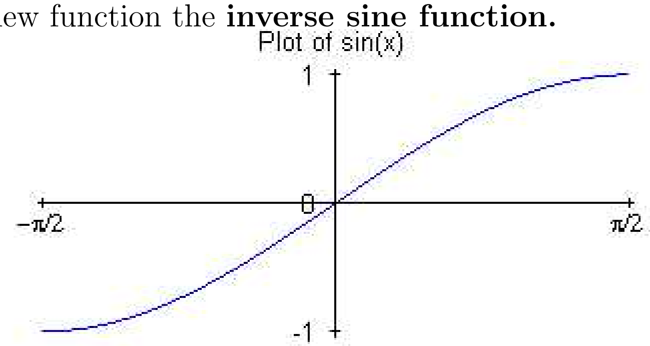


Figure 19.1

From the definition of inverse functions discussed in Section 6, we have the following properties of $f^{-1}(x)$:

(i) $Dom(\sin^{-1} x) = Range(\sin x) = [-1, 1]$.
(ii) $Range(\sin^{-1} x) = Dom(\sin x) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
(iii) $\sin(\sin^{-1} x) = x$ for all $-1 \leq x \leq 1$.
(iv) $\sin^{-1}(\sin x) = x$ for all $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
(v) $y = \sin^{-1} x$ if and only if $\sin y = x$. Using words, the notation $y = \sin^{-1} x$ gives the angle y whose sine value is x.

Remark 19.1

If x is outside the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ then we look for the angle y in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin x = \sin y$. In this case, $\sin^{-1}(\sin x) = y$. For example, $\sin^{-1}(\sin \frac{5\pi}{6}) = \sin^{-1}(\sin \frac{\pi}{6}) = \frac{\pi}{6}$.

The graph of $y = \sin^{-1} x$ is the reflection of the graph of $y = \sin x$ about the line $y = x$ as shown in Figure 19.2.

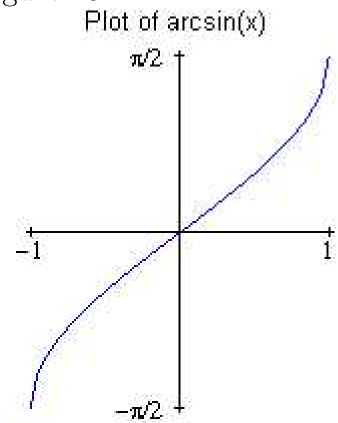


Figure 19.2

Example 19.1
Find the exact value of:
(a) $\sin^{-1} 1$ (b) $\sin^{-1} \frac{\sqrt{3}}{2}$ (c) $\sin^{-1}(-\frac{1}{2})$

Solution.
(a) Since $\sin \frac{\pi}{2} = 1$ then $\sin^{-1} 1 = \frac{\pi}{2}$.
(b) Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ then $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.
(c) Since $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ then $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$. ■

Example 19.2
Find the exact value of:
(a) $\sin(\sin^{-1} 2)$ (b) $\sin^{-1}(\sin \frac{\pi}{3})$.

Solution.
(a) $\sin(\sin^{-1} 2)$ is undefined since 2 is not in the domain of $\sin^{-1} x$.
(b) $\sin(\sin^{-1} \frac{\pi}{3}) = \frac{\Pi}{3}$.■

Next, we will express the trigonometric functions of the angle $\sin^{-1} x$ in terms of x. Let $u = \sin^{-1} x$. Then $\sin u = x$. Since $\sin^2 u + \cos^2 u = 1$ then $\cos u = \pm\sqrt{1 - x^2}$. But $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$ so that $\cos u \ge 0$. Thus

$$\cos(\cos^{-1} x) = \sqrt{1 - x^2}.$$

It follows that for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ we have

$$\begin{array}{rcl}
\sin(\sin^{-1} x) & = & x \\
\cos(\sin^{-1} x) & = & \sqrt{1-x^2} \\
\csc(\sin^{-1} x) & = & \frac{1}{\sin(\sin^{-1} x)} = \frac{1}{x} \\
\sec(\sin^{-1} x) & = & \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}} \\
\tan(\sin^{-1} x) & = & \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}} \\
\cot(\sin^{-1} x) & = & \frac{1}{\tan(\sin^{-1} x)} = \frac{\sqrt{1-x^2}}{x}.
\end{array}$$

Example 19.3
Find the exact value of:
(a) $\cos(\sin^{-1} \frac{\sqrt{2}}{2})$ (b) $\tan(\sin^{-1}(-\frac{1}{2}))$

Solution.
(a) Using the above discussion we find $\cos(\sin^{-1} \frac{\sqrt{2}}{2}) = \sqrt{1-(\frac{\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$.
(b) $\tan(\sin^{-1}(-\frac{1}{2})) = \frac{\frac{-1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{\sqrt{3}}{3}$. ∎

<u>The Inverse Cosine Function</u>
In order to define the inverse cosine function, we will restrict the function $f(x) = \cos x$ over the interval $[0, \pi]$. There the function is always decreasing. See Figure 19.3. Therefore $f(x)$ is one-to-one function. Hence, its inverse will be denoted by

$$f^{-1}(x) = \cos^{-1} x.$$

We call $\cos^{-1} x$ the **inverse cosine function.**

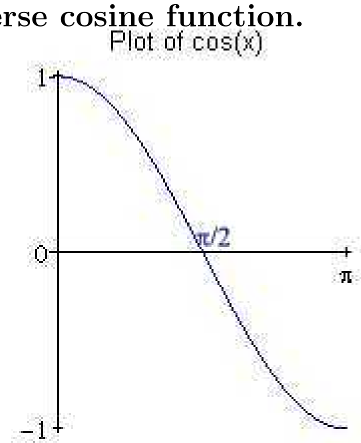


Figure 19.3

The following are consequences of the definition of inverse functions:
(i) $Dom(\cos^{-1} x) = Range(\cos x) = [-1, 1]$.
(ii) $Range(\cos^{-1} x) = Dom(\cos x) = [0, \pi]$.
(iii) $\cos(\cos^{-1} x) = x$ for all $-1 \le x \le 1$.
(iv) $\cos^{-1}(\cos x) = x$ for all $0 \le x \le \pi$.
(v) $y = \cos^{-1} x$ if and only if $\cos y = x$. Using words, the notation $y = \cos^{-1} x$ gives the angle y whose cosine value is x.

Remark 19.2
If x is outside the interval $[0, \pi]$ then we look for the angle y in the interval $[0, \pi]$ such that $\cos x = \cos y$. In this case, $\cos^{-1}(\cos x) = y$. For example, $\cos^{-1}(\cos \frac{7\pi}{6}) = \cos^{-1}(\cos \frac{5\pi}{6}) = \frac{5\pi}{6}$.

The graph of $y = \cos^{-1} x$ is the reflection of the graph of $y = \cos x$ about the line $y = x$ as shown in Figure 19.4.

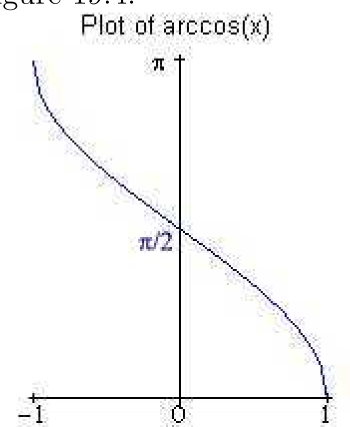


Figure 19.4

Example 19.4
Let $\theta = \cos^{-1} x$. Find the six trigonometric functions of θ.

Solution.
Let $u = \cos^{-1} x$. Then $\cos u = x$. Since $\sin^2 u + \cos^2 u = 1$ then $\sin u =$

$\pm\sqrt{1-x^2}$. Since $0 \leq u \leq \pi$ then $\sin u \geq 0$ so that $\sin u = \sqrt{1-x^2}$. Thus,

$$\begin{array}{rcl}
\sin(\cos^{-1} x) & = & \sqrt{1-x^2} \\
\cos(\cos^{-1} x) & = & x \\
\csc(\cos^{-1} x) & = & \frac{1}{\sin(\cos^{-1} x)} = \frac{1}{\sqrt{1-x^2}} \\
\sec(\cos^{-1} x) & = & \frac{1}{\cos(\cos^{-1} x)} = \frac{1}{x} \\
\tan(\cos^{-1} x) & = & \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{\sqrt{1-x^2}}{x} \\
\cot(\cos^{-1} x) & = & \frac{1}{\tan(cos^{-1} x)} = \frac{x}{\sqrt{1-x^2}}. \blacksquare
\end{array}$$

Example 19.5
Find the exact value of:
(a) $\cos^{-1}\frac{\sqrt{2}}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$.

Solution.
(a) $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
(b) $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$. ■

The Inverse Tangent Function
Although not one-to-one on its full domain, the tangent function is one-to-one when restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ since it is increasing there (See Figure 19.5).

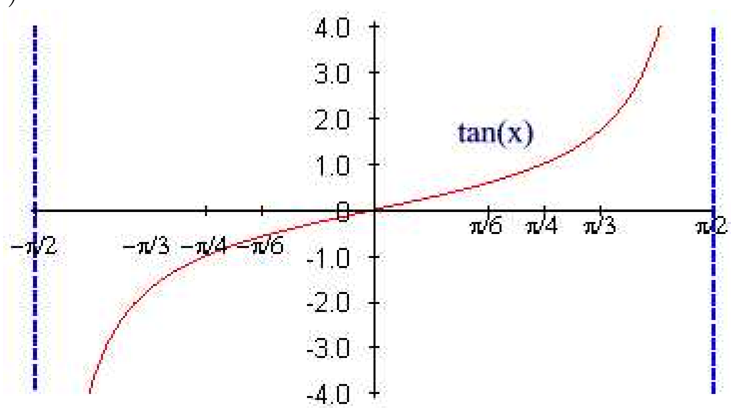


Figure 19.5

Thus, the inverse function exists and is denoted by

$$f^{-1}(x) = \tan^{-1} x.$$

We call this function the **inverse tangent function.**
As before, we have the following properties:
(i) $Dom(\tan^{-1} x) = Range(\tan x) = (-\infty, \infty)$.
(ii) $Range(\tan^{-1} x) = Dom(\tan x) = (-\frac{\pi}{2}, \frac{\pi}{2})$.
(iii) $\tan(\tan^{-1} x) = x$ for all x.
(iv) $\tan^{-1}(\tan x) = x$ for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(v) $y = \tan^{-1} x$ if and only if $\tan y = x$. In words, the notation $y = \tan^{-1} x$ means that y is the angle whose tangent value is x.

Remark 19.3
If x is outside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $x \neq (2n+1)\frac{\pi}{2}$, where n is an integer, then we look for the angle y in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan x = \tan y$. In this case, $\tan^{-1}(\tan x) = y$. For example, $\tan^{-1}(\tan \frac{5\pi}{6}) = \tan^{-1}(\tan(-\frac{\pi}{6})) = -\frac{\pi}{6}$.

The graph of $y = \tan^{-1} x$ is the reflection of $y = \tan x$ about the line $y = x$ as shown in Figure 19.6.

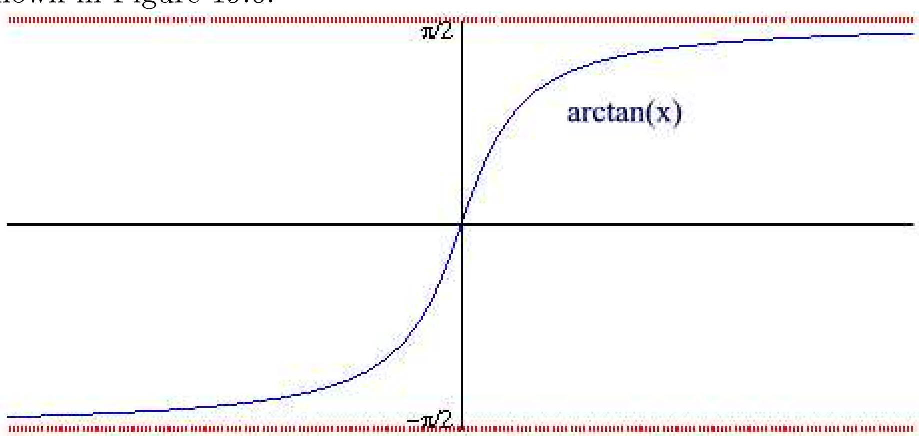


Figure 19.6

Example 19.6
Find the exact value of:
(a) $\tan^{-1}(\tan \frac{\pi}{4})$ (b) $\tan^{-1}(\tan \frac{7\pi}{5})$.

Solution.
(a) $\tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4}$.
(b) $\tan^{-1}(\tan \frac{7\pi}{5}) = \tan^{-1}(\tan(\frac{2\pi}{5})) = \frac{2\pi}{5}$. ■

Example 19.7
Let $u = \tan^{-1} x$. Find the six trigonometric functions of u.

Solution.
Since $1 + \tan^2 u = \sec^2 u$ then $\sec u = \pm\sqrt{1+x^2}$. But $-\frac{\pi}{2} < u < \frac{\pi}{2}$ then $\sec u > 0$ so that $\sec u = \sqrt{1+x^2}$. Also, $\cot u = \frac{1}{\tan u} = \frac{1}{x}$. In summary,

$$\begin{aligned}
\sin(\tan^{-1} x) &= \frac{1}{\csc(\tan^{-1} x)} = \frac{x}{\sqrt{1+x^2}} \\
\cos(\tan^{-1} x) &= \frac{1}{\sec(\tan^{-1} x)} = \frac{1}{\sqrt{1+x^2}} \\
\csc(\tan^{-1} x) &= \frac{\sqrt{1+x^2}}{x} \\
\sec(\tan^{-1} x) &= \sqrt{1+x^2} \\
\tan(\tan^{-1} x) &= x \\
\cot(\tan^{-1} x) &= \frac{1}{x} \blacksquare
\end{aligned}$$

The Inverse Cotangent Function
The function $f(x) = \cot x$ is always decreasing on $(0, \pi)$. See Figure 19.7.

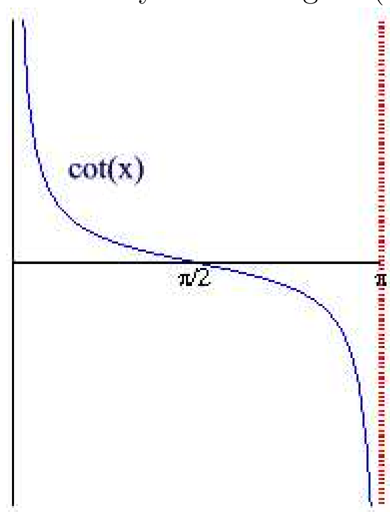


Figure 19.7

Thus, it is one-to-one and has an inverse denoted by

$$f^{-1}(x) = \cot^{-1} x$$

We call $f^{-1}(x)$ the **inverse cotangent function.**

Properties of $y = \cot^{-1} x$:

(i) $Dom(\cot^{-1} x) = Range(\cot x) = (-\infty, \infty)$.
(ii) $Range(\cot^{-1} x) = Dom(\cot x) = (0, \pi)$.
(iii) $\cot(\cot^{-1} x) = x$ for any x.
(iv) $\cot^{-1}(\cot x) = x$ for $0 < x < \pi$.
(v) $y = \cot^{-1} x$ if and only if $\cot y = x$. Thus, $y = \cot^{-1} x$ means that y is the angle whose cotangent value is x.

Remark 19.4
If x is outside the interval $(0, \pi)$ and $x \neq n\pi$, where n is an integer, then we look for the angle y in the interval $(0, \pi)$ such that $\cot x = \cot y$. In this case, $\cot^{-1}(\cot x) = y$. For example, $\cot^{-1}(\cot \frac{7\pi}{5}) = \cot^{-1}(\cot \frac{2\pi}{5}) = \frac{2\pi}{5}$.

The graph of $y = \cot^{-1} x$ is shown in Figure 19.8.

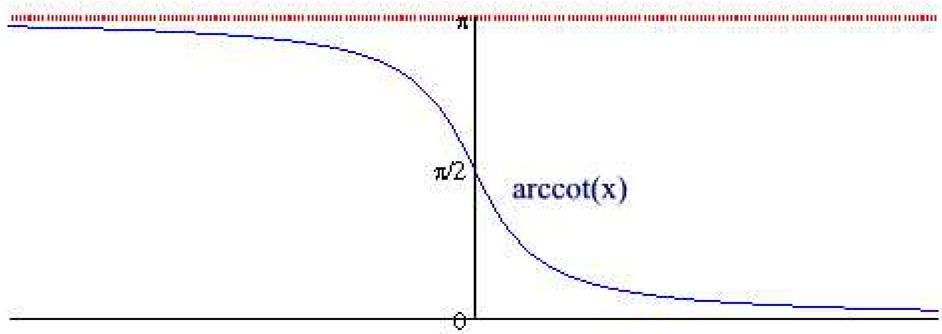


Figure 19.8

Example 19.8
Let $u = \cot^{-1} x$. Find the six trigonometric functions of u.

Solution.
Since $1 + cot^2 u = \csc^2 u$ then $\csc u = \pm\sqrt{1+x^2}$. But $0 < u < \pi$ then $\csc u > 0$ so that $\csc u = \sqrt{1+x^2}$. Also, $\tan u = \frac{1}{\cot u} = \frac{1}{x}$. In summary,

$$\begin{array}{rcl}
\sin(\cot^{-1} x) &=& \frac{1}{\csc(\cot^{-1} x)} = \frac{1}{\sqrt{1+x^2}} \\
\cos(\cot^{-1} x) &=& \frac{1}{\sec(\cot^{-1} x)} = \frac{x}{\sqrt{1+x^2}} \\
\csc(\cot^{-1} x) &=& \sqrt{1+x^2} \\
\sec(\cot^{-1} x) &=& \frac{\sqrt{1+x^2}}{x} \\
\tan(\cot^{-1} x) &=& \frac{1}{x} \\
\cot(\cot^{-1} x) &=& x \blacksquare
\end{array}$$

The Inverse Secant Function

The function $f(x) = \sec x$ is increasing on the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. See Figure 19.9. Thus, $f(x)$ is one-to-one and consequently it has an inverse denoted by

$$f^{-1}(x) = \sec^{-1} x.$$

We call this new function the **inverse secant function.**

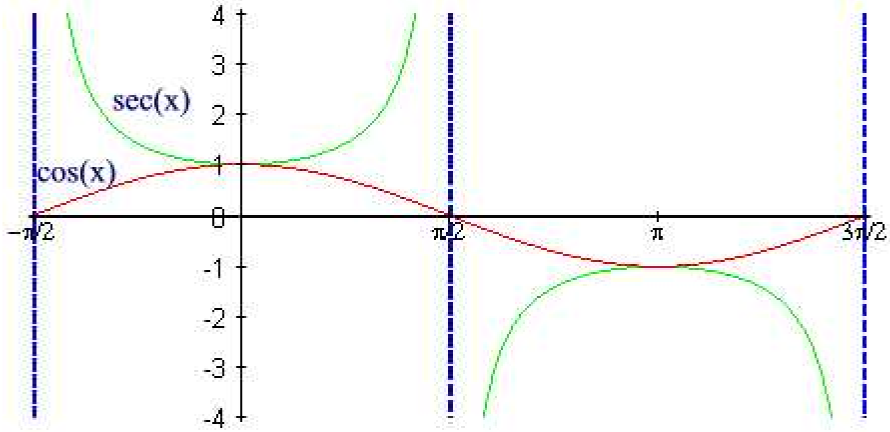


Figure 19.9

We call this new function the **inverse secant function.** From the definition of inverse functions we have the following properties of $f^{-1}(x)$:

(i) $Dom(\sec^{-1} x) = Range(\sec x) = (-\infty, -1] \cup [1, \infty)$.

(ii) $Range(\sec^{-1} x) = Dom(\sec x) = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

(iii) $\sec(\sec^{-1} x) = x$ for all $x \leq -1$ or $x \geq 1$.

(iv) $\sec^{-1}(\sec x) = x$ for all x in $[0, \frac{\pi}{2})$ or x in $(\frac{\pi}{2}, \pi]$.

(v) $y = sec^{-1}x$ if and only if $\sec y = x$.

Remark 19.5

If x is outside the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and $x \neq (2n+1)\frac{\pi}{2}$, where n is an integer, then we look for the angle y in the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ such that $\sec x = \sec y$. In this case, $\sec^{-1}(\sec x) = y$. For example, $\sec^{-1}(\sec \frac{7\pi}{6}) = \sec^{-1}(\sec \frac{5\pi}{6}) = \frac{5\pi}{6}$.

The graph of $y = \sec^{-1} x$ is the reflection of the graph of $y = \sec x$ about the line $y = x$ as shown in Figure 19.10.

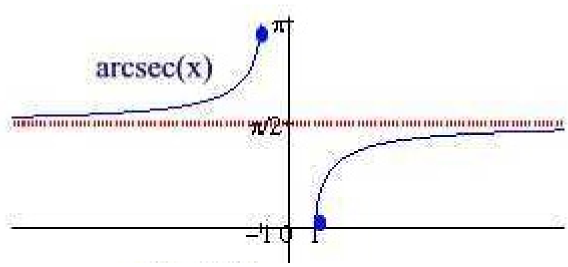


Figure 19.10

Example 19.9
Find the exact value of:
(a) $\sec^{-1}\sqrt{2}$ (b) $\sec^{-1}(\sec\frac{\pi}{3})$.

Solution.
(a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$.
(b) $\sec^{-1}(\sec\frac{\pi}{3}) = \frac{\pi}{3}$. ■

Example 19.10
Let $u = \sec^{-1} x$. Find the six trigonometric functions of u.

Solution.
Since $\sec u = x$ then $\cos u = \frac{1}{x}$. Since $\sin^2 u + \cos^2 u = 1$ and u is in either Quadrant I or Quadrant II where $\sin u > 0$ then $\sin u = \frac{\sqrt{1-x^2}}{|x|}$. Also, $\csc u = \frac{|x|}{\sqrt{1-x^2}}$. In summary,

$$\begin{array}{rcl}
\sin(\sec^{-1} x) & = & \frac{\sqrt{1-x^2}}{|x|} \\
\cos(\sec^{-1} x) & = & \frac{1}{x} \\
\csc(\sec^{-1} x) & = & \frac{|x|}{\sqrt{1-x^2}} \\
\sec(\sec^{-1} x) & = & x \\
\tan(\sec^{-1} x) & = & \frac{x\sqrt{1-x^2}}{|x|} \\
\cot(\sin^{-1} x) & = & \frac{|x|}{x\sqrt{1-x^2}}
\end{array}$$ ■

<u>The inverse cosecant function</u>
In order to define the inverse cosecant function, we will restrict the function $y = \csc x$ over the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. There the function is always

decreasing (See Figure 19.11)and therefore is one-to-one function. Hence, its inverse will be denoted by

$$f^{-}(x) = \csc^{-1} x.$$

We call $\csc^{-1} x$ the **inverse cosecant function.**

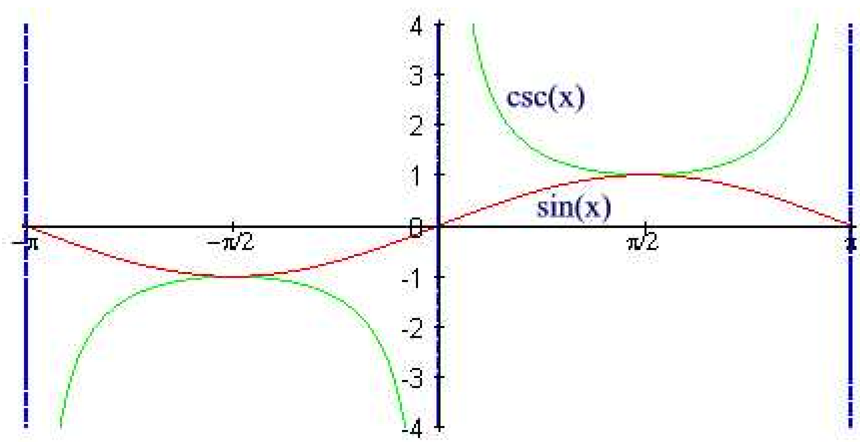


Figure 19.11

The following are consequences of the definition of inverse functions:
(i) $Dom(\csc^{-1} x) = Range(\csc x) = (-\infty, -1] \cup [1, \infty)$.
(ii) $Range(\csc^{-1} x) = Dom(\csc x) = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
(iii) $\csc(\csc^{-1} x) = x$ for all $x \leq -1$ or $x \geq 1$.
(iv) $\csc^{-1}(\csc x) = x$ for all $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$.
(v) $y = \csc x$ if and only if $\csc y = x$.

Remark 19.6
If x is outside the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ and $x \neq n\pi$, where n is an integer, then we look for the angle y in the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ such that $\csc x = \csc y$. In this case, $\csc^{-1}(\csc x) = y$. For example, $\csc^{-1}(\csc(\frac{5\pi}{6})) = \csc^{-1}(\csc\frac{\pi}{6}) = \frac{\pi}{6}$.

The graph of $y = \csc^{-1} x$ is the reflection of the graph of $y = \csc x$ about the line $y = x$ as shown in Figure 19.12.

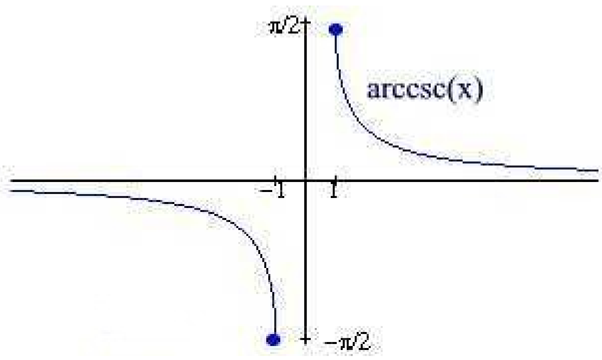


Figure 19.12

Example 19.11
Let $u = \csc^{-1} x$. Find the six trigonometric functions of u.

Solution.
Since $\csc u = x$ then $\sin u = \frac{1}{x}$. Since $\sin^2 u + \cos^2 u = 1$ and u is in either Quadrant I or Quadrant IV then $\cos u > 0$ and $\cos u = \frac{\sqrt{x^2-1}}{|x|}$. Also, $\sec u = \frac{|x|}{\sqrt{x^2-1}}$. In summary,

$$\begin{array}{rcl}
\sin(\csc^{-1} x) & = & \frac{1}{x} \\
\cos(\csc^{-1} x) & = & \frac{\sqrt{x^2-1}}{|x|} \\
\csc(\csc^{-1} x) & = & x \\
\sec(\csc^{-1} x) & = & \frac{|x|}{\sqrt{x^2-1}} \\
\tan(\csc^{-1} x) & = & \frac{|x|}{x\sqrt{x^2-1}} \\
\cot(\csc^{-1} x) & = & \frac{x\sqrt{x^2-1}}{|x|} \blacksquare
\end{array}$$

Example 19.12
Find the exact value of $\cos\left(\frac{\pi}{4} - \csc^{-1} \frac{5}{3}\right)$.

Solution.
We have

$$\begin{array}{rcl}
\cos\left(\frac{\pi}{4} - \csc^{-1} \frac{5}{3}\right) & = & \cos \frac{\pi}{4} \cos\left(\csc^{-1} \frac{5}{3}\right) + \sin \frac{\pi}{4} \sin\left(\csc^{-1} \frac{5}{3}\right) \\
 & = & \frac{\sqrt{2}}{2}\frac{4}{5} + \frac{\sqrt{2}}{2}\frac{3}{5} \\
 & = & \frac{7\sqrt{2}}{10} \blacksquare
\end{array}$$

Example 19.13
Find the exact value of $\sin\left(\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)\right)$.

Solution.
Consider a right triangle with acute angle $\csc^{-1}\frac{2}{\sqrt{3}}$, opposite side $\sqrt{3}$, adjacent side 1 and hypotenuse of length 2. Then

$$\begin{aligned}\sin\left(\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)\right) &= -\sin\left(\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)\right)\\ &= -\frac{\sqrt{3}}{2}\ \blacksquare\end{aligned}$$

Example 19.14
Use a calculator to find the value of $\csc^{-1} 5$, rounded to four decimal places.

Solution.
Let $x = \csc^{-1} 5$ then $\csc x = 5$ and this leads to $\sin x = \frac{1}{5} = 0.2$. Hence, either $x = \sin^{-1} 0.2 \approx 0.2014$ or $x \approx \pi - 0.2014$. $\blacksquare$

Review Problems

Exercise 19.1
Find the exact radian value.

(a) $\sin^{-1} 1$ (b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (c) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ (d) $\cos^{-1}\left(-\frac{1}{2}\right)$.

Exercise 19.2
Find the exact value of the given expression, if it is defined.

(a) $\cos\left(\cos^{-1}\frac{1}{2}\right)$ (b) $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$.

Exercise 19.3
Find the exact value of the given expression, if it is defined.

(a) $\cos^{-1}\left(\sin\frac{\pi}{4}\right)$
(b) $\sin^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$
(c) $\sin\left(\sin^{-1}\frac{2}{3}+\cos^{-1}\frac{1}{2}\right)$.

Exercise 19.4
Solve the equation for x algebraically.

(a) $\sin^{-1}(x-1)=\frac{\pi}{2}$.
(b) $\cos^{-1}\left(x-\frac{1}{2}\right)=\frac{\pi}{3}$.

Exercise 19.5
Solve the equation for x algebraically.

(a) $\sin^{-1}\frac{\sqrt{2}}{2}+\cos^{-1}x=\frac{2\pi}{3}$
(b) $\sin^{-1}x+\cos^{-1}\frac{4}{5}=\frac{\pi}{6}$.

Exercise 19.6
Evaluate each expression.
(a) $y=\cos(\sin^{-1}x)$ (b) $y=\tan(\cos^{-1}x)$ (c) $y=\sec(\sin^{-1}x)$.

Exercise 19.7
Establish the identities.

(a) $\sin^{-1}x+\sin^{-1}(-x)=0$
(b) $\cos^{-1}x+\cos^{-1}(-x)=\pi$.

Exercise 19.8
Solve for y in terms of x.

(a) $2x = \frac{1}{2}\sin^{-1} 2y$
(b) $x - \frac{\pi}{3} = \cos^{-1}(y-3)$.

Exercise 19.9
Find the exact radian value.

(a) $\cot^{-1}\frac{\sqrt{3}}{3}$ (b) $\csc^{-1}(-\sqrt{2})$ (c) $\tan^{-1}\sqrt{3}$ (d) $\sec^{-1}\frac{2\sqrt{3}}{3}$.

Exercise 19.10
Find the exact value of the given expression.

(a) $\tan(\tan^{-1} 2)$ (b) $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Exercise 19.11
Find the exact value of the given expression.

(a) $\tan^{-1}\left(\sin\frac{\pi}{6}\right)$ (b) $\cot^{-1}\left(\cos\frac{2\pi}{3}\right)$.

Exercise 19.12
Solve the equation for x algebraically.

$\tan^{-1}\left(x + \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

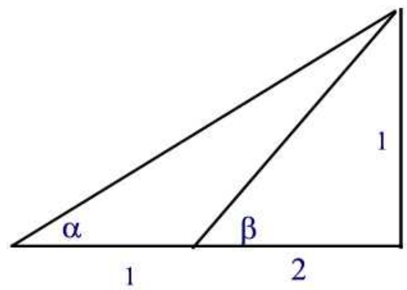
Exercise 19.13
Establish the identities.

(a) $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}, \quad x > 0$.
(b) $\sec^{-1}\frac{1}{x} + \csc^{-1}\frac{1}{x} = \frac{\pi}{2}$.

Exercise 19.14
Solve for y in terms of x.

(a) $5x = \tan^{-1} 3y$
(b) $x + \frac{\pi}{2} = \tan^{-1}(2y-1)$.

Exercise 19.15
Show that $\alpha+\beta=\frac{\pi}{4}$



Exercise 19.16
Find the exact radian value: (a) $\sin^{-1}\frac{\sqrt{2}}{2}$ (b) $\cos^{-1}-\frac{1}{2}$.

Exercise 19.17
Find the exact value of the given expression.

(a) $cos(\cos^{-1}2)$ (b) $\cos\left(\sin^{-1}\frac{5}{13}\right)$ (c) $sin(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right))$.

Exercise 19.18
Find the exact value of the given expression.

(a) $\cos\left(2\sin^{-1}\frac{\sqrt{2}}{2}\right)$
(b) $\sin\left(2\sin^{-1}\frac{4}{5}\right)$
(c) $\cos\left(\sin^{-1}\frac{3}{4}+\cos^{-1}\frac{5}{13}\right)$.

Exercise 19.19
Solve the equation for x algebraically.

(a) $\sin^{-1}x=\cos^{-1}\frac{5}{13}$.
(b) $\cos^{-1}\left(x-\frac{1}{2}\right)=\frac{\pi}{3}$.

Exercise 19.20
Solve the equation for x algebraically.

(a) $\sin^{-1}x+\cos^{-1}\frac{4}{5}=\frac{\pi}{6}$.
(b) $\cos^{-1}x+\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{2}$.

Exercise 19.21
Evaluate each expression.
(a) $y = \tan(\cos^{-1} x)$ (b) $y = \sec(\sin^{-1} x)$.

Exercise 19.22
Establish the identities.

(a) $\sin^{-1} x + \sin^{-1}(-x) = 0$
(b) $\cos^{-1} x + \cos^{-1}(-x) = \pi$.

Exercise 19.23
Find the exact radian value.

(a) $\tan^{-1} \sqrt{3}$ (b) $\sec^{-1} \frac{2\sqrt{3}}{3}$ (c) $\cot^{-1} \sqrt{3}$ (d) $\csc^{-1}(-2)$.

Exercise 19.24
Find the exact value of the given expression.

(a) $\tan(\tan^{-1} \frac{1}{2})$ (b) $\cos(\sec^{-1} 2)$.

Exercise 19.25
Find the exact value of the given expression.

(a) $\cot(\csc^{-1} 2)$ (b) $\sec(\tan^{-1} \frac{12}{5})$.

Exercise 19.26
Find the exact value of the given expression: $\cos(2\tan^{-1} 1)$

Exercise 19.27
Solve the equation for x algebraically.

$\tan^{-1} x = \sin^{-1} \frac{24}{25}$.

Exercise 19.28
Establish the identity.

$$\sec^{-1} \frac{1}{x} + \csc^{-1} \frac{1}{x} = \frac{\pi}{2}.$$

Exercise 19.29
Solve the equation: $1 + \sin x = \cos^2 x$.

20 Trigonometric Equations

An equation that contains trigonometric functions is called a **trigonometric equation**. In this section we will discuss some techniques for solving trigonometric equations. The values that satisfy a trigonometric equation are called **solutions** of the equation. To **solve** a trigonometric equation is to find all its solutions.

Example 20.1
Determine whether $x = \frac{\pi}{4}$ is a solution of the equation

$$\sin x = \frac{1}{2}.$$

Is $x = \frac{\pi}{6}$ a solution?

Solution.
Since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \neq \frac{1}{2}$ then $x = \frac{\pi}{4}$ is not a solution to the given equation. On the contrary, $x = \frac{\pi}{6}$ is a solution since $\sin \frac{\pi}{6} = \frac{1}{2}$. ■

Unless the domain of a variable is restricted, most trigonometric equations have an infinite number of solutions, a fact due to the periodicity property of the trigonometric functions.

Solving the Equation $\sin x = \sin a$
The first set of solutions is given by the formula $x = a + 2k\pi$, where k is an integer. But $\sin a = \sin(\pi - a)$ so that the second set of solutions is given by the formula $x = \pi - a + 2k\pi$.

Example 20.2
Find all the solutions of the equation $2\sin x - 1 = 0$.

Solution.
The given equation is equivalent to $\sin x = \frac{1}{2} = \sin \frac{\pi}{6}$. The solutions to this equation are given by

$$\left\{ \begin{array}{lll} x & = & \frac{\pi}{6} + 2k\pi \\ x & = & \frac{5\pi}{6} + 2k\pi \end{array} \right.$$ ■

Example 20.3
Solve the equation: $\sin x = \frac{1}{3}$.

Solution.
Since $\sin x = \sin(\sin^{-1}\frac{1}{3})$ then the solutions are given by

$$\begin{cases} x &= \sin^{-1}\frac{1}{3} + 2k\pi \\ x &= \pi - \sin^{-1}\frac{1}{3} + 2k\pi \end{cases} \blacksquare$$

Sometimes some standard algebraic techniques such as collecting like terms or factoring are used in solving trigonometric equations.

Example 20.4
Solve the equation: $\sin^2 x - \sin x = 0.$

Solution.
Factoring we find $\sin x(\sin x - 1) = 0$. Thus, either $\sin x = 0$ or $\sin x = 1$. The solutions of the equation $\sin x = 0$ are given by $x = k\pi$ where k is any integer. The solutions of the equation $\sin x = 1$ are given by $x = (2k+1)\frac{\pi}{2}$ where k is an arbitrary integer.$\blacksquare$

Solving the Equation $\cos x = \cos a$
The first set of solutions is given by the formula $x = a + 2k\pi$, where k is an integer. But $\cos a = \cos(-a)$ so that the second set of solutions is given by the formula $x = -a + 2k\pi$.

Example 20.5
Solve the equation: $2\cos^2 x - 7\cos x + 3 = 0.$

Solution.
Factoring the given equation to obtain:

$$(2\cos x - 1)(\cos x - 3) = 0.$$

This equation is satisfied for all values of x such that either $\cos x = \frac{1}{2}$ or $\cos x = 3$. Since $-1 \leq \cos x \leq 1$ then the second equation has no solutions. The solutions to the first equation in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. All the solutions are given by $\frac{\pi}{3} + 2k\pi$ or $\frac{5\pi}{3} + 2k\pi$ where k is an integer. $\blacksquare$

Example 20.6
Solve the equation: $3\cos x + 3 = \sin^2 x.$

Solution.
Using the identity $\sin^2 x + \cos^2 x = 1$ we obtain the quadratic equation $2\cos^2 x + 3\cos x + 1 = 0$ which can be factored into $(2\cos x + 1)(\cos x + 1) = 0$. Thus either $\cos x = -\frac{1}{2}$ or $\cos x = -1$. The solutions to the first equation are given by

$$\begin{cases} x = \frac{2\pi}{3} + 2k\pi \\ x = \frac{4\pi}{3} + 2k\pi. \end{cases}$$

The solutions to the second equation are given by $x = (2k+1)\pi$ where k is an arbitrary integer. ■

Example 20.7
Solve the equation: $\sin 2x - \cos x = 0$.

Solution.
Using the identity $\sin 2x = 2\sin x \cos x$ the given equation can be factored as $\cos x(2\sin x - 1) = 0$. Thus, either $\cos x = 0$ or $\sin x = \frac{1}{2}$. The solutions to the first equation are given by $x = (2k+1)\frac{\pi}{2}$ and those to the second equation are given by

$$\begin{cases} x = \frac{\pi}{6} + 2k\pi \\ x = \frac{5\pi}{6} + 2k\pi \end{cases}$$

where k is an integer. ■

Example 20.8
Solve the equation: $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution.
Squaring both sides of the equation and expanding to obtain

$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$

Using the identity $\sin^2 x + \cos^2 x = 1$, the last equation reduces to

$$2\cos^2 x + 2\cos x = 0.$$

Factoring to obtain $\cos x(2\cos x + 1) = 0$. Thus, either $\cos x = 0$ or $\cos x = -\frac{1}{2}$. The first equation has the solutions $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The second equation has the solution π. Now since we solved this equation by squaring then we must check for extraneous solutions. Substituting the three values found above in to the equation we find that only π and $\frac{\pi}{2}$ satisfy the equation. ■

Solving the Equation $\tan x = \tan a$

The solutions to this equation are given by the formula

$$x = a + k\pi$$

where k is an integer.

Example 20.9

Solve the equation $\tan^2 x - 3 = 0$.

Solution.

Isolating $\tan x$ we find

$$\begin{array}{rcl} \tan^2 x - 3 & = & 0 \\ \tan^2 x & = & 3 \\ \tan x & = & \pm\sqrt{3} \end{array}$$

Solving the equation $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$ we find the solutions $x = \frac{\pi}{3} + k\pi$. Solving the equation $\tan x = -\sqrt{3} = \tan\frac{5\pi}{3}$ we find the solutions $x = \frac{5\pi}{3} + k\pi$ ■

Example 20.10

Find the values of x for which the curves $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

Solution.

The solutions to the equation $\sin x = \cos x$ are the points of intersection of the two curves. The above equation is equivalent to $\tan x = 1 = \tan\frac{\pi}{4}$. The collection of all solutions is given by $\frac{\pi}{4} + k\pi$ where k is an integer. ■

Example 20.11

Solve the equation: $\sin 2x = 1, 0 \le x < 2\pi$.

Solution.

We have $2x = (2k+1)\frac{\pi}{2}$ or $x = (2k+1)\frac{\pi}{4}$, where k is an integer. Since $0 \le x < 2\pi$ then $0 \le (2k+1)\frac{\pi}{4} < 2\pi$ or $0 \le 2k+1 < 8$. Thus $0 \le k < \frac{7}{2}$. This gives the values $k = 0, 1, 2$ and $k = 3$. So the solutions to the equation on the given interval are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5}{4}\pi, \frac{7}{4}\pi$. ■

Review Problems

Exercise 20.1
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$\sec x - \sqrt{2} = 0.$$

Exercise 20.2
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$\sin^2 x - 1 = 0.$$

Exercise 20.3
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$2\sin^2 x + 1 = 3\sin x.$$

Exercise 20.4
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$\sin^4 x = \sin^2 x.$$

Exercise 20.5
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$\tan^2 x + \tan x - \sqrt{3} = \sqrt{3}\tan x.$$

Exercise 20.6
Solve the following equation for exact solutions in the interval $0 \leq x < 2\pi$.

$$2\cos^2 x + 1 = -3\cos x.$$

Exercise 20.7
Solve the following equation for exact solutions in the interval $0^\circ \leq x < 360^\circ$. Round approximate solutions to the nearest tenth of a degree.

$$3\sec x - 8 = 0.$$

Exercise 20.8
Solve the following equation for exact solutions in the interval $0^\circ \leq x < 360^\circ$. Round approximate solutions to the nearest tenth of a degree.

$$3\cos x + 3 = 0.$$

Exercise 20.9
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$3\tan^2 x - 2\tan x = 0.$$

Exercise 20.10
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$2\sin^2 x = 1 - \cos x.$$

Exercise 20.11
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$2\tan^2 x - \tan x - 10 = 0.$$

Exercise 20.12
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$2\sin x\cos x - \sin x - 2\cos x + 1 = 0.$$

Exercise 20.13
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$3\sin^2 x - \sin x - 1 = 0.$$

Exercise 20.14
Solve the following equation for exact solutions in the interval $0° \leq x < 360°$. Round approximate solutions to the nearest tenth of a degree.

$$\cos^2 x - 3\sin x + 2\sin^2 x = 0.$$

Exercise 20.15
Find the exact solutions, in radians, of the equation

$$\tan 2x - 1 = 0.$$

Exercise 20.16
Find the exact solutions, in radians, of the equation

$$\sin 2x - \sin x = 0.$$

Exercise 20.17
Find the exact solutions, in radians, of the equation

$$\sin^2 \frac{x}{2} + \cos x = 1.$$

Exercise 20.18
Find the exact solutions, in radians, where $0 \le x < 2\pi$.

$$\cos 2x = 1 - 3\sin x.$$

Exercise 20.19
Find the exact solutions, in radians, where $0 \le x < 2\pi$.

$$\sin 2x \cos x + \cos 2x \sin x = 0.$$

Exercise 20.20
Find the exact solutions, in radians, where $0 \le x < 2\pi$.

$$\cos 2x \cos x - \sin 2x \sin x = 0.$$

Exercise 20.21
Find the exact solutions, in radians, where $0 \le x < 2\pi$.

$$2\sin x \cos x - 2\sqrt{2}\sin x - \sqrt{3}\cos x + \sqrt{6} = 0.$$

Exercise 20.22
Solve the equation: $2\sin^2 x \cos x - \cos x = 0$, for $0 \le x < 2\pi$.

Exercise 20.23
Solve the equation: $3\cos^2 x - 5\cos x - 4 = 0, 0 \le x < 2\pi$.

Exercise 20.24
Solve the equation $\sin 3x = 1$.

Exercise 20.25
How many solutions does the equation $\sin\left(\frac{1}{x}\right) = 0$ have on the interval $0 < x < \frac{\pi}{2}$?

Exercise 20.26
Solve the eqution: $2\sin 3x - 1 = 0$.

Exercise 20.27
Solve the equation: $\sqrt{3}\tan\frac{x}{2} - 1 = 0$.

Exercise 20.28
Solve the equation: $\tan^2 x - \tan x - 2 = 0$.

Exercise 20.29
Solve the equation: $3\sin x - 2 = 0$.

Exercise 20.30
If a projectile is fired with velocity v_0 at an angle θ, then the maximum height it reaches (in feet) is modeled by the function

$$M(\theta) = \frac{v_0^2 \sin^2\theta}{64}.$$

Suppose that $v_0 = 400 ft/s$.
(a) At what angle θ should the projectile be fired so that the maximum height it reaches is 2000 ft?
(b) Is it possible for the projectile to reach a height of 3000 ft?
(c) Find the angle θ for which the projectile will travel highest.

Exercise 20.31
Approximate the largest value of k for which the equation $\sin x \cos x = k$ has a solution.

21 The Law of Sines

One important use of trigonometry is to solve problems that can be modeled by a triangle. Determining the measures of all the sides and angles of a triangle is referred to as **solving the triangle**. In Section 8, we used trigonometric functions to solve right triangles. These functions can also be used to solve oblique triangles, that is, triangles with no right angles. This can be done by using the Law of Sines to be discussed in this section and the Law of Cosines to be discussed in the next section.
To simplify our discussion, we will agree that in the triangle ΔABC, the vertices are A, B, and C, and the sides opposite these vertices are a, b, and c respectively. See Figure 21.1. Also, when writing $A = 42°$ we will mean that the measure of the angle at vertex A is $42°$.

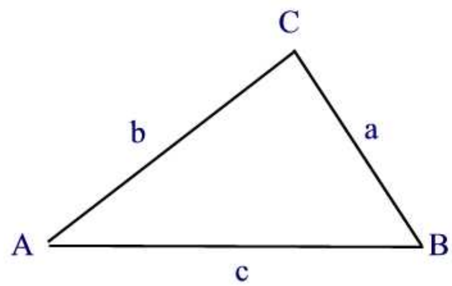


Figure 21.1

The Law of Sines
The Law of Sines is a relationship between the angles and the sides of a triangle. This law requires that either two angles and a side are given (AAS) or two sides and an angle are given (SSA)(note that the angle must not be the angle between the two given sides).
We derive the Law of Sines as follows. Consider a triangle with sides a, b, c and angles A, B, C. Let CD be the altitude drawn from C. See Figure 21.2.

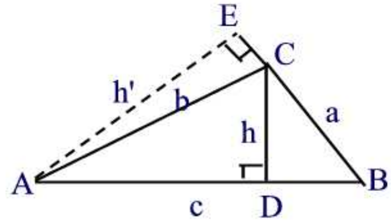


Figure 21.2

From Figure 21.2, we see that $h = b\sin A$ and $h = a\sin B$. Thus, $b\sin A = a\sin B$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$. Next, let AE be the altitude drawn from A. Then $h' = c\sin B$ and $\sin C = \sin(\pi - \angle ECA) = \sin\angle ECA = \frac{h'}{b}$ so that $h' = b\sin C$. Thus, $c\sin B = b\sin C$ or $\frac{\sin B}{b} = \frac{\sin C}{c}$. We conclude that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This relationship is known as the **Law of Sines**.

Example 21.1
Solve the triangle ABC if $a = 74.1, A = 52.1^\circ$, and $C = 35.9^\circ$.

Solution.
Using the Law of Sines we can write the equality

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Solving this for c we find

$$c = \frac{a\sin C}{\sin A} = \frac{(74.1)\sin(35.9^\circ)}{\sin(52.1^\circ)} \approx 55.1.$$

To find the angle B we use the fact that the sum of the interior angles is 180°. Thus, $B = 180^\circ - (A + C) = 180^\circ - (52.1^\circ + 35.9^\circ) = 92.0^\circ$.
To find b, we use the Law of sines again

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

and solving for b we find

$$b = \frac{a\sin B}{\sin A} = \frac{74.1\sin(92.0^\circ)}{\sin(52.1^\circ)} \approx 93.8 \blacksquare$$

Example 21.2
Solve the triangle in Figure 21.3.

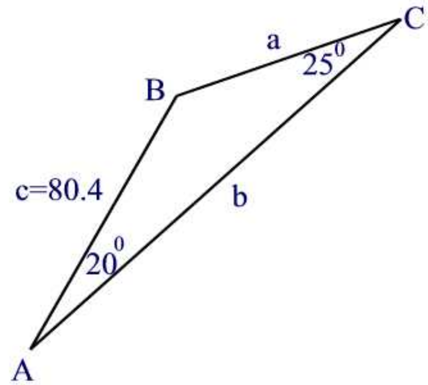


Figure 21.3

Solution.
Note first that $B = 180^\circ - (20^\circ + 25^\circ) = 135^\circ$. By the Law of Sines we have

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Solving this equation for a we find

$$a = \frac{c \sin A}{\sin C} = \frac{80.4 \sin 20^\circ}{\sin 25^\circ} \approx 65.1.$$

Similarly, to find b we use $\frac{b}{\sin B} = \frac{c}{\sin C}$. Solving for b

$$b = \frac{c \sin B}{\sin C} = \frac{80.4 \sin 135^\circ}{\sin 25^\circ} \approx 134.5. \blacksquare$$

The Ambiguous Case (SSA)
Given two sides and the angle opposite one of them. One must compute the angle opposite the other side. Recall that if the sine function is positive, there are two possible answers, a first quadrant angle and a second quadrant angle. Since the sum of all angles of a triangle is 180 degrees, there may be two possible answers. Recall further that the sine function must be less than or equal to one. If the sine exceeds one, there is no solution, and thus no triangle. The three examples below show the three possible scenarios for this case: no solution, one solution, and two solutions.

Example 21.3 *(No Solution)*
If $a = 5.7, b = 8.8$ and $A = 68.7°$, find c, B and C for any possible triangles.

Solution.
This is SSA so we use the Law of Sines where we have ambiguity. We first have $\frac{\sin B}{b} = \frac{\sin A}{a}$ which gives $\sin B = \frac{b \sin A}{a}$. From that we get $\sin B = \frac{8.8 \sin 68.7°}{5.7}$ or $\sin B \approx 1.438$. This is impossible since the sine of an angle can not exceed 1, so there are no possible triangles. See Figure 21.4. ■

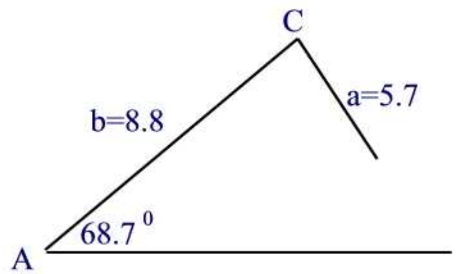


Figure 21.4

Example 21.4 *(One Solution)*
If $a = 2.0, b = 1.4$, and $A = 44.5°$, find c, B, and C for any possible triangles.

Solution.
This is SSA so we use the Law of Sines where we have ambiguity. We first have $\frac{\sin B}{b} = \frac{\sin A}{a}$ which gives $\sin B = \frac{b \sin A}{a}$. From that we get $\sin B = \frac{1.4 \sin 44.5°}{2.0} \approx 0.4906$. Hence, either $B = \arcsin(0.4906) \approx 29.4°$ or $B = 180° - 29.4° \approx 150.6°$ (reject since $44.5° + 150.6° > 180°$). Now, $C = 180° - 44.5° - 29.4° = 106.1°$.
To find c, we use the Law of Sines again, $\frac{\sin A}{a} = \frac{\sin C}{c}$ or $c = \frac{a \sin C}{\sin A} = \frac{(2.0) \sin(106.1°)}{\sin 44.5°} \approx 2.7$. So there is one possible triangle as shown in Figure 21.5. ■

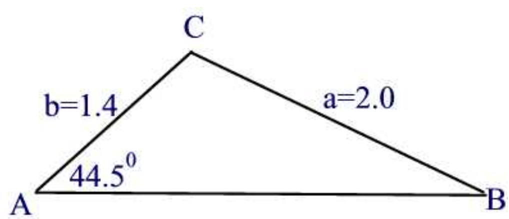


Figure 21.5

Example 21.5 *(Two Solutions)*
If $a = 32.2, b = 20.3 and B = 20°$, *find c, B and C for any possible triangles.*

Solution.
This is an SSA problem so we will use the Law of Sines. Keep in mind that this is the ambiguous case. We first have $\frac{\sin B}{b} = \frac{\sin A}{a}$ which gives $\sin A = \frac{a \sin B}{b}$. From that we get $\sin A = \frac{32.2 \sin 20°}{20.3} \approx 0.5425$.
Now, $A = \arcsin 0.5425 \approx 32.9°$. However, the angle $180° - 32.9° = 147.1°$ gives the same sine value. We therefore can construct two triangles.
Triangle # 1 - $A_1 = 32.9°$. Then $C_1 = 180° - 20° - 32.9° = 127.1°$. By the Law of Sines, $\frac{\sin C_1}{c_1} = \frac{\sin A_1}{a}$, giving $c_1 = \frac{a \sin C_1}{\sin A} \approx 47.3$. This does leave the largest side opposite the largest angle and the smallest side opposite the smallest angle so we suspect our work was probably right.
Triangle # 2 - $A_2 = 147.1°$. Then $C_2 = 180° - 147.1° - 20° = 12.9°$. By the Law of sines, $\frac{\sin C_2}{c_2} = \frac{\sin A_2}{a}$ giving $c_2 = \frac{a \sin C_2}{\sin A_2} \approx 13.2$. This also leaves the largest side opposite the largest angle and the smallest side opposite the smallest angle so we suspect our work was probably right. See Figure 21.6.■

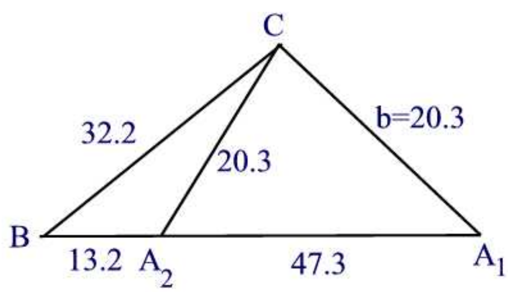


Figure 21.6

Example 21.6
A forest fire is spotted by observers in two fire towers 12 miles apart. Tower B is on a bearing of $S12°10'E$ from Tower A. If the bearing of the fire from Tower A is $S45°40'W$ and from Tower B is $N75°20'W$, how far is the fire from Tower B?

Solution.
Using Figure 21.7 we see that the triangle ABC has interior angles of measure $A = 57°50', B = 63°10'$ and $C = 180° - 57°50' - 63°10' = 59°$. The problem is to find a. By the Law of Sines, $\frac{\sin A}{a} = \frac{\sin C}{c}$. This implies that $a = \frac{c \sin A}{\sin C} =$

$\frac{12\sin 57°50'}{\sin 59°} \approx 11.9$ miles.■

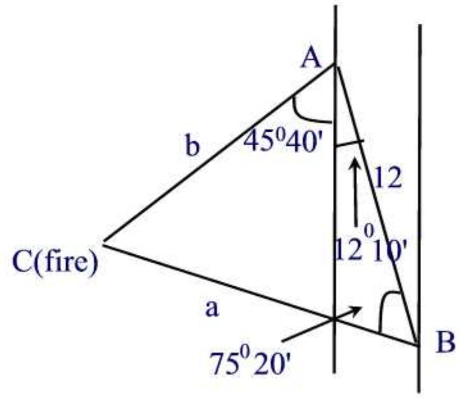


Figure 21.7

Review Problems

Exercise 21.1
Solve the triangle: $A = 42°, B = 61°, a = 12$.

Exercise 21.2
Solve the triangle: $B = 28°, C = 78°, c = 44$.

Exercise 21.3
Solve the triangle: $A = 110°, C = 32°, b = 12$.

Exercise 21.4
Solve the triangle: $A = 82°, B = 65.4°, b = 36.5$.

Exercise 21.5
Solve the triangle: $A = 33.8°, C = 98.5°, c = 102$.

Exercise 21.6
Solve the triangle: $C = 114.2°, c = 87.2, b = 12.1$.

Exercise 21.7
Solve the triangle: $A = 37°, c = 40, a = 28$.

Exercise 21.8
Solve the triangle: $A = 30°, a = 1.0, b = 2.4$.

Exercise 21.9
Solve the triangle: $C = 47.2°, a = 8.25, c = 5.80$.

Exercise 21.10
Solve the triangle: $B = 117.32°, b = 67.25, a = 15.05$.

Exercise 21.11
When the angle of elevation of the sun is $62°$, a telephone pole tilted at an angle of $7°$ away from the sun casts a shadow 30 feet long on the ground (See Figure 21.8). To tenths place, approximate the length of the phone pole.

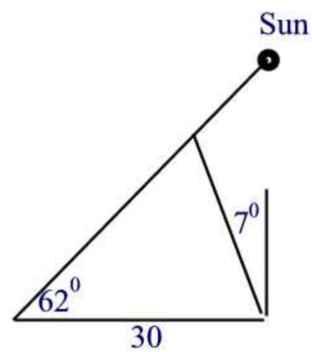


Figure 21.8

Exercise 21.12
To find the distance between two points A and B that are on opposite sides of a river, a surveyor measures a distance on the same side of the river as point A. The distance to this point is 240 feet and call it point C. He then measures the angles from A to B as 62° and measures the angle from C to B as 55°. (See Figure 21.9) To tenths place, approximate the distance from A to B.

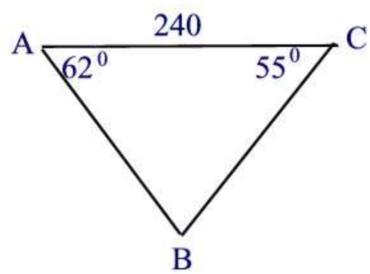
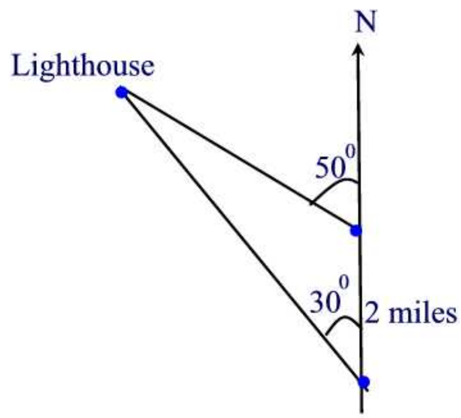


Figure 21.9

Exercise 21.13
Two tracking stations are on an east-west line 110 miles apart. A large forest fire is located on a bearing of $N42°E$ from the western station and a bearing of $N15°E$ from the eastern station. How far is the fire from the western station?

Exercise 21.14
A ship sailing due north spots a lighthouse 30° to the left of its line of travel.

Two miles later, the lighthouse is 50° to the left of its line of travel. How far is the ship from the lighthouse at that point? (Assume the earth is flat.)



Exercise 21.15
Scientists wish to measure the diameter of a large circular meteor crater. Two points A and B on the edge of the pit are 120 feet apart. From a point C on the far side of the pit, the angle between the lines AC and BC is measured to be 8°. Use the Law of Sines to find the diameter of the crater.

Exercise 21.16
Solve the triangle: $A = 45°, a = 7\sqrt{2}, b = 7.$

Exercise 21.17
Solve the triangle: $A = 43.1°, a = 186.2, b = 248.6.$

Exercise 21.18
Solve the triangle: $A = 57°, a = 15, c = 20.$

Exercise 21.19
Solve the triangle: $A = 42°, a = 70, b = 122.$

Exercise 21.20
Solve the triangle: $B = 32°, a = 42, b = 30.$

Exercise 21.21
Solve the triangle: $C = 65°, b = 10, c = 8.0.$

Exercise 21.22
Solve the triangle: $A = 14.8°, c = 6.35, a = 4.80$.

Exercise 21.23
Show that for any triangle ABC one has

$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}.$$

Exercise 21.24
A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Loas Angeles, 340 miles apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?

Exercise 21.25
To find the distance across a river, a surveyor chooses points A and B, which are 200 ft apart on one side of the river. She then chooses a reference point C on the opposite side of the river and finds that $A \approx 82°$ and $B \approx 52°$. Approximate the distance from A to C.

22 The Law of Cosines and Its Applications

The Law of Sines is applicable when either two angles and a side are given or two sides and an angle are given such that the angle is opposite to the angle between the two sides. When two sides and the angle between them are given (SAS) or the three sides are given (SSS) then a triangle is being solved by using the Law of Cosines.
By the **Law of Cosines** we mean one of the following formulas

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{24}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{25}$$

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{26}$$

In words, the Law of Cosines says that the square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those two sides times the cosine of the included angle. Note that if a triangle is a right triangle at A then $\cos A = 0$ and the Law of Cosines reduces to the Pythagorean Theorem $a^2 = b^2 + c^2$. Thus, the Pythagorean Theorem is a special case of the Law of Cosines.
We derive the first formula. The proofs of the other two are quite similar. Consider the triangle given in the Figure 22.1.

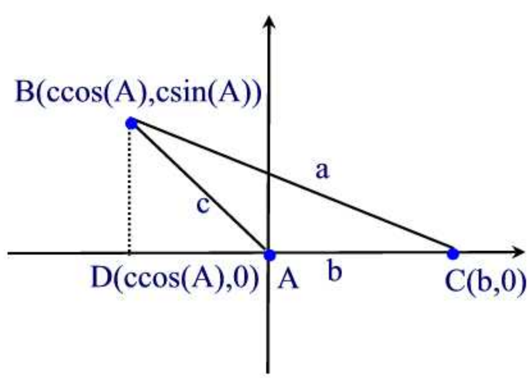


Figure 22.1

Using the distance formula and the identity $\sin^2 A + \cos^2 A = 1$ we have

$$\begin{array}{rcl} a^2 & = & d(B,C) = (c\cos A - b)^2 + (c\sin A - 0)^2 \\ & = & c^2\cos^2 A - 2bc\cos A + b^2 + c^2\sin^2 A \\ & = & c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc\cos A \\ & = & b^2 + c^2 - 2bc\cos A \end{array}$$

The above formulas are useful when trying to solve the SAS problem. To Solve the SSS problem, we use (24) - (26) to write the cosine functions in terms of the sides of the triangle. That is,

$$\begin{array}{rcl} \cos A & = & \frac{b^2+c^2-a^2}{2bc} \\ \cos B & = & \frac{a^2+c^2-b^2}{2ac} \\ \cos C & = & \frac{b^2+a^2-c^2}{2ab}. \end{array}$$

Example 22.1 *(SSS)*
Solve the triangle with sides $a = 3, b = 5, c = 7$.

Solution.
Find the largest angle of the triangle first. This will be C because the longest side is c. Then by the Law of Cosines we have

$$\begin{array}{rcl} c^2 & = & a^2 + b^2 - 2ab\cos C \\ 49 & = & 9 + 25 - 2(3)(5)\cos C \\ 49 & = & 34 - 30\cos C \\ 15 & = & -30\cos C \\ -1/2 & = & \cos C \\ \arccos(-1/2) & = & C \\ 120^\circ & = & C. \end{array}$$

Now that we have an angle, we can switch to the law of sines.(Easier to use) To find B we proceed as follows

$$\begin{array}{rcl} \frac{\sin B}{b} & = & \frac{\sin C}{c} \\ \frac{\sin B}{5} & = & \frac{\sin 120^\circ}{7} \\ 7\sin B & = & 5\sin 120^\circ \\ \sin B & = & \frac{5}{7}\sin 120^\circ \\ \sin B & = & 0.6185895741317 \\ B & = & \arcsin(0.6185895741317) \approx 38.2^\circ \end{array}$$

Finally, $A = 180^\circ - (120^\circ + 38.2^\circ) = 21.8^\circ.\blacksquare$

Example 22.2 *(SAS)*
Solve the triangle if $a = 3, b = 7$ and $C = 37°$.

Solution.
We are given two sides and the included angle. We must find the third side. The missing side is c. By the Law of Cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ c^2 &= 9 + 49 - 2(3)(7)\cos 37° \\ c^2 &= 58 - 42\cos 37° \\ c &= \sqrt{58 - 42\cos 37°} \approx 4.9. \end{aligned}$$

Now use the Law of Sines and find the smallest angle. The smallest angle is definitely an acute angle. The Law of Sines can not distinquish between acute and obtuse because both angles give a positive answer. The smallest angle is opposite side a, the shortest side.

$$\begin{aligned} \frac{\sin A}{3} &= \frac{\sin 37°}{4.9} \\ 4.9\sin A &= 3\sin 37° \\ \sin A &= \frac{3}{4.9}\sin 37° \\ \sin A &= .36845817 \\ A &\approx 21.6° \end{aligned}$$

To find the angle B, $B = 180° - (21.6° + 37°) = 121.4°$.■

Example 22.3
A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in Figure 22.2. Use the surveyor's data to approximate the length of the tunnel.

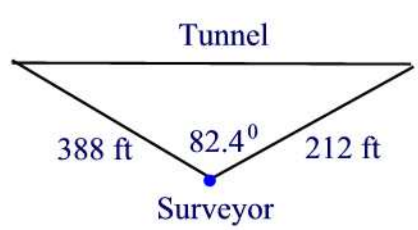


Figure 22.2

Solution.
By the Law of Cosines we have

$$\begin{aligned} c^2 &= \sqrt{a^2 + b^2 - 2ab\cos C} \\ &= \sqrt{388^2 + 212^2 - 2(388)(212)\cos 42.4^\circ} \\ &\approx 416.8ft. \blacksquare \end{aligned}$$

Applications of the Law of Cosines and Law of Sines
The Law of Cosines can be used to derive a formula for finding the area of a triangle given two sides and the included angle. To avoid confusion, we shall use the letter K for the area since A has been used to denote an angle (or a vertex.)
Consider the triangles in Figure 22.3.

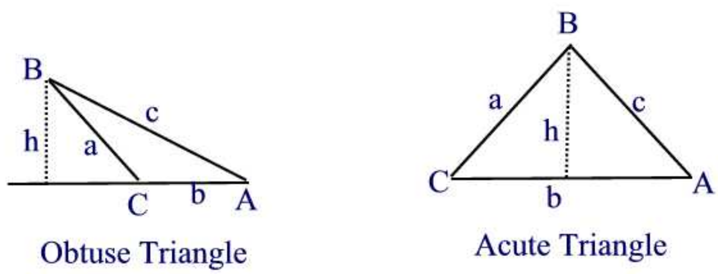


Figure 22.3

Then the area of the triangle is $K = \frac{1}{2}height \times base = \frac{1}{2}hb$. But $\sin A = \frac{h}{c}$ or $h = c\sin A$. Thus,

$$K = \frac{1}{2}bc\sin A.$$

Using similar arguments, one can establish the area formulas

$$K = \frac{1}{2}ac\sin B \quad and \quad K = \frac{1}{2}ab\sin C.$$

Example 22.4
Given $A = 62^\circ, b = 12$ *meters*, and $c = 5.0$ meters, find the area of the triangle ABC.

Solution.
Using the formula for area, we have

$$K = \frac{1}{2}bc\sin A = \frac{1}{2}(12)(5.0)\sin 62^\circ \approx 26\ m^2.\blacksquare$$

Example 22.5
A farmer has a triangular field with sides 120 yards, 170 yards, and 220 yards. Find the area of the field in square yards. Then find the number of acres if 1 acre = 4840 square yards.

Solution.
We need to find an angle so we can use the area formula So, let $a = 120, b = 170, c = 220$. We start by finding C.

$$\begin{array}{ccc} c^2 & = & a^2 + b^2 - 2ab\cos C \\ 48400 & = & 14400 + 28900 - 40800\cos C \\ 5100 & = & -40800\cos C \\ -\frac{5100}{40800} & = & \cos C \\ \arccos\left(\frac{-5100}{40800}\right) & = & C \\ 97.2^\circ & \approx & C \end{array}$$

Now find the area

$$\begin{array}{ccc} K & = & \frac{1}{2}ab\sin C \\ K & = & \frac{1}{2}(120)(170)\sin 97.2^\circ \\ K & = & 10120\ \textit{square yards} \end{array}$$

The number of acres is found by: $\frac{10120}{4840} \approx 2.1\ \textit{acres}.\blacksquare$

The formulas of area requires that two sides and an included angle be given. What if two angles and an included side are given? In this case, the Law of Sines and the Law of Cosines must be used together. To be more precise, suppose that the angles B and C together with the side a are given. Then, by the Law of Sines we can find c :

$$\begin{array}{ccc} \frac{\sin C}{c} & = & \frac{\sin A}{a} \\ c & = & \frac{a\sin C}{\sin A}. \end{array}$$

By the Law of Cosines we have that

$$K = \frac{1}{2}ac\sin B = \frac{1}{2}\frac{a^2\sin C\sin B}{\sin A}.$$

In a similar way, we can derive the formulas

$$K = \frac{1}{2}\frac{b^2 \sin A \sin C}{\sin B} \quad and \quad K = \frac{1}{2}\frac{c^2 \sin A \sin B}{\sin C}.$$

Example 22.6
Given $A = 32°, C = 77°$, and $a = 14$ inches, find the area of the triangle ABC.

Solution.
Since A and C are given then we can find $B = 180° - (32° + 77°) = 71°$. Thus,

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{14^2 \sin 71° \sin 77°}{2 \sin 32°} \approx 170 \textit{ square inches.}\blacksquare$$

Finally, we will find a formula for the area when the three sides of the triangle are given. In what follows, we let $s = \frac{a+b+c}{2}$ (i.e. s is half the perimeter of the triangle). So let's look at Figure 22.4.

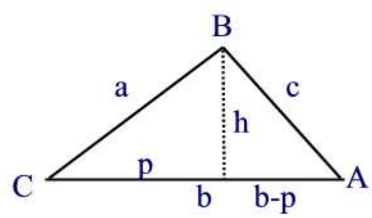


Figure 22.4

Using the Pythagorean theorem we can write $a^2 - p^2 = c^2 - (b - p)^2$ or $a^2 - p^2 = c^2 - b^2 - p^2 + 2bp$. Solving for p we find

$$p = \frac{a^2 + b^2 - c^2}{2b}.$$

A simple arithmetic shows the following

$$\begin{aligned}
h^2 &= a^2 - p^2 = (a-p)(a+p) \\
&= \left(a - \tfrac{a^2+b^2-c^2}{2b}\right)\left(a + \tfrac{a^2+b^2-c^2}{2b}\right) \\
&= \left(\tfrac{2ab-a^2-b^2+c^2}{2b}\right)\left(\tfrac{2ab+a^2+b^2-c^2}{2b}\right) \\
&= \tfrac{c^2-(a-b)^2}{2b}\tfrac{(a+b)^2-c^2}{2b} \\
&= \tfrac{(c-a+b)(c+a-b)(a+b-c)(a+b+c)}{4b^2} \\
&= \tfrac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4b^2} \\
&= \tfrac{(2s)2(s-a)2(s-b)2(s-c)}{4b^2} \\
&= \tfrac{4s(s-a)(s-b)(s-c)}{b^2}
\end{aligned}$$

Thus,

$$h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b}$$

But the area of the triangle is

$$K = \frac{1}{2}bh = \left(\frac{1}{2}b\right)\left(\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b}\right) = \sqrt{s(s-a)(s-b)(s-c)}.$$

This last formula is known as **Heron's formula**.

Review Problems

Exercise 22.1
Find the third side of the triangle: $a = 12, b = 18, C = 44°$.

Exercise 22.2
Find the third side of the triangle: $a = 120, c = 180, B = 56°$.

Exercise 22.3
Find the third side of the triangle: $a = 9.0, b = 7.0, C = 72°$.

Exercise 22.4
Find the third side of the triangle: $a = 25.9, c = 33.4, B = 84.0°$.

Exercise 22.5
Given the three sides of a triangle, find the specified angle: $a = 8.0, b = 9.0, c = 12$. Find C.

Exercise 22.6
Given the three sides of a triangle, find the specified angle: $a = 108, b = 132, c = 160$. Find A.

Exercise 22.7
Given the three sides of a triangle, find the specified angle: $a = 32.5, b = 40.1, c = 29.6$. Find B.

Exercise 22.8
Find the area of the triangle: $A = 105°, b = 12, c = 24$.

Exercise 22.9
Find the area of the triangle: $A = 42°, B = 76°, c = 12$.

Exercise 22.10
Find the area of the triangle: $a = 16, b = 12, c = 14$.

Exercise 22.11
Find the area of the triangle: $a = 3.6, b = 4.2, c = 4.8$.

Exercise 22.12
Find the area of a triangular piece of land that is bounded by sides of 236 meters, 620 meters, and 814 meters.

Exercise 22.13
A commercial piece of real estate is priced at $2.20 per square foot. Find, to the nearest $1000, the cost of a triangular lot measuring 212 feet by 185 feet by 240 feet.

Exercise 22.14
An engineer wishes to measure the diameter of a hole in the ground, but his tape measure isn't long enough. He places stakes at points A and B on opposite sides of the hole, then places a third stake at a point C at the edge of the hole somewhere between A and C. His tape is long enough to measure the distance from C to A at 28 feet and the distance from C to B at 26 feet. From the point C, the angle between the lines AC and CB is 158°. How far is it from A to B?

Exercise 22.15
The sides of a triangle are $a = 5, b = 8$, and $c = 12$. Find the angles of the triangle.

Exercise 22.16
Solve the triangle with $A = 46.5°, b = 10.5$, and $c = 18.0$.

Exercise 22.17
Find the third side of the triangle: $a = 400, b = 620, C = 116°$.

Exercise 22.18
Find the third side of the triangle: $a = 122, c = 144, B = 48°$.

Exercise 22.19
Given the three sides of a triangle, find the specified angle: $a = 25, b = 32, c = 40$. Find C.

Exercise 22.20
Find the area of the triangle: $A = 116°, B = 34°, c = 8.5$.

Exercise 22.21
Find the area of the triangle: $A = 42°, B = 76°, c = 12$.

Exercise 22.22
Find the area of the triangle: $a = 16, b = 12, c = 14$.

Exercise 22.23
Find the area of the triangle: $a = 3.6, b = 4.2, c = 4.8$.

Exercise 22.24
A pilot sets out from an airport and heads in the direction $N\ 20°E$, flying at $200mph$. After one hour, he makes a course direction and heads in the direction $N\ 40°E$. Half an hour after that, engine trouble forces him to make an emergency landing.

(a) Find the distance between the airport and his final landing point.
(b) Find the bearing from the airport to his final landing point. Notation such as $N\ 40°E$ is known as a **bearing** in navigation.

Exercise 22.25
A businessman wishes to buy a triangular lot in a busy downtown location. The lot frontages on the three adjacent streets are 125, 280, and 315 ft. Find the area of the lot.

23 The Dot Product of Two Vectors

The concept of vectors is widely used in the physical sciences. One important question about vectors is the question of orthogonality. That is, when two vectors are perpendicular. Testing the orthogonality of two vectors can be accomplished by the use of the concept of *dot product.*
We start this section by discussing the notion of a vector. You must have already encountered this concept without noticing that. Remember that the speed and the velocity of a moving object are two completely different concepts. **Speed** is a scalar quantity which refers to "how fast an object is moving." A fast-moving object has a high speed while a slow-moving object has a low speed. An object with no movement at all has a zero speed. The **Velocity** of an object measures the speed together with the direction of the moving object. We represent a velocity by a vector (a concept to be defined below) whose magnitude or length is the speed and whose direction is the direction of the moving object. Thus, velocity is a vector quantity. As such, velocity is "direction-aware." When evaluating the velocity of an object, one must keep track of direction. It would not be enough to say that an object has a velocity of 55 mi/hr. One must include direction information in order to fully describe the velocity of the object. For instance, you must describe an object's velocity as being 55 mi/hr, **east**. This is one of the essential differences between speed and velocity. Speed is a scalar and does not keep track of direction; velocity is a vector and is direction-aware.

A **vector v** is a line segment with a direction as shown in Figure 23.1. If the direction of the vector is switched then we get the opposite vector **-v**. The magnitude of a vector will be denoted by $||\mathbf{v}||$.

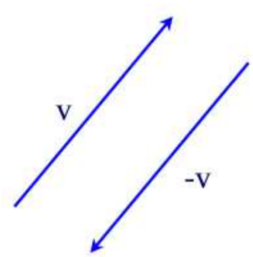


Figure 23.1

By introducing a coordinate plane, it is possible to develop an analytic approach to vectors. So we will assume that the undirected endpoint of a vector coincides with the point $O(0,0)$ as shown in the Figure 23.2. We denote this vector by the ordered pair $\mathbf{v} =< a, b >$. a is called the **first component** and b is called the **second component**.

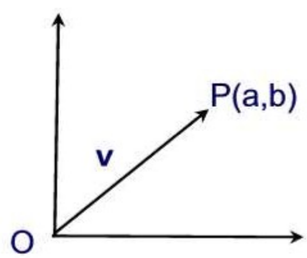


Figure 23.2

Expressing vectors in terms of components provides a convenient method for performing the following operations:

• Magnitude: $||\mathbf{v}|| = d(O,P) = \sqrt{a^2+b^2}$.
• Direction: $\theta = 180° - \alpha = 180° - \arctan\left(\left|\frac{b}{a}\right|\right)$, where θ is the angle between the vector and the positive x-axis and α is the reference angle.
• Sum: $\mathbf{v} + \mathbf{w} =< a, b > + < c, d >=< a + c, b + d >$.
• Scalar Multiplication: $k\mathbf{v} = k < a, b >=< ka, kb >$.

Example 23.1
Given $\mathbf{v} =< -2, 4 >$ and $\mathbf{w} =< -3, -2 >$. Find

(a) The direction of the vector $\mathbf{v}$ (b) $||\mathbf{v} + 2\mathbf{w}||$.

Solution.
(a) $\tan\alpha = \left|\frac{b}{a}\right| = |\frac{4}{-2}| = 2$. So that $\theta = 180° - \arctan 2 \approx 178.9°$.
(b) $||\mathbf{v} + 2\mathbf{w}|| = || < -2, 4 > +2 < -3, -2 > || = || < -8, 0 > || = \sqrt{64+0} = 8.$■

Unit Vectors
A **unit vector** is a vector of magnitude 1. For example, the vector $\mathbf{v} =<$

$-\frac{4}{5}, \frac{1}{5} >$ is a unit vector since

$$||\mathbf{v}|| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = 1.$$

Now, for any given nonzero vector $\mathbf{v} =< a, b >$ we can find a unit vector in the same direction as the vector $\mathbf{v}$. Indeed, if we let $\mathbf{w} = \frac{\mathbf{v}}{||v||} =< \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} >$ then

$$||\mathbf{w}|| = \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = \sqrt{1} = 1.$$

Example 23.2
Find a unit vector in the direction of $\mathbf{v} =< -4, 2 >$.

Solution.
The magnitude or norm of the vector $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$.
Thus,

$$\mathbf{w} = \frac{\mathbf{v}}{||v||} < -\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} > . \blacksquare$$

Two special unit vectors are the vectors $\mathbf{i} =< 1, 0 >$ and $\mathbf{j} =< 0, 1 >$ as shown in Figure 23.3.

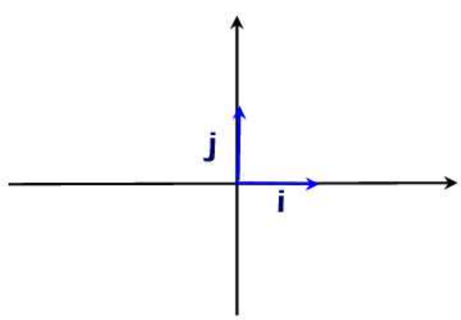


Figure 23.3

Any vector $\mathbf{v} =< a, b >$ can be expressed as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. To see this,

$$\mathbf{v} =< a, b >= a < 1, 0 > +b < 0, 1 >= a\mathbf{i} + b\mathbf{j}.$$

Example 23.3
Given $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$. Find $3\mathbf{u} - 2\mathbf{v}$.

Solution.

$$3\mathbf{u} - 2\mathbf{v} = 3(3\mathbf{i} - 2\mathbf{j}) - 2(-2\mathbf{i} + 3\mathbf{j}) = 9\mathbf{i} - 6\mathbf{j} + 4\mathbf{i} - 6\mathbf{j} = 13\mathbf{i} - 12\mathbf{j}.\blacksquare$$

Horizontal and Vertical Components of a Vector
Now, let $\mathbf{v}$ be a nonzero vector. Suppose we know the direction angle θ of the vector as shown in Figure 23.4.

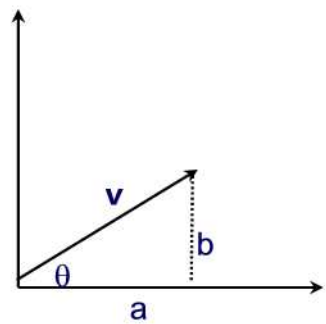


Figure 23.4

Then, we can find the components $< a, b >$ of the vector $\mathbf{v}$ using the definition of the sine and cosine functions as follows:

$$\cos\theta = \frac{a}{||\mathbf{v}||} \quad and \quad \sin\theta = \frac{b}{||\mathbf{v}||}.$$

Thus,

$$a = ||\mathbf{v}||\cos\theta \quad and \quad b = ||\mathbf{v}||\sin\theta||.$$

We call $||\mathbf{v}||\cos\theta$ the **horizontal component** of $\mathbf{v}$ and $||\mathbf{v}||\sin\theta$ the **vertical component**.
Since $\mathbf{v} = a\mathbf{i}+b\mathbf{j} = ||\mathbf{v}||\cos\theta\mathbf{i}+||\mathbf{v}||\sin\theta\mathbf{j} = ||\mathbf{v}||(\cos\theta\mathbf{i}+\sin\theta\mathbf{j}) = ||\mathbf{v}||\mathbf{u}$ where the vector $\mathbf{u} = \cos\theta\mathbf{i}+\sin\theta\mathbf{j}$ is a unit vector since $||\mathbf{u}|| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

Example 23.4
Find the approximate horizontal and vertical components of a vector $\mathbf{v}$ of norm 5 meters and direction angle $\theta = 27°$.

Solution.

$$a = 5\cos 27^\circ \approx 4.46 \quad and\ b = 5\sin 27^\circ \approx 2.27.\blacksquare$$

The Dot Product of Two Vectors

The **dot product**, also called the **scalar product** or **inner product**, of two vectors is a number obtained by performing a specific operation on the vector components. More precisely, the dot product of two vectors is determined by multiplying their x-coordinates, then multiplying their y-coordinates, and finally adding the two products. That is, if $\mathbf{v} =< a, b >$ and $\mathbf{w} =< c, d >$ then

$$\mathbf{v} \cdot \mathbf{w} = ac + bd$$

Example 23.5

Find the dot product of $\mathbf{v} =< 4, 1 >$ and $\mathbf{w} =< -1, 4 >$.

Solution.

$$\mathbf{v} \cdot \mathbf{w} = 4(-1) + 1(4) = 0.\blacksquare$$

Theorem 23.1

Let $\mathbf{u} =< a_1, b_1 >, \mathbf{v} =< a_2, b_2 >$, and $\mathbf{w} =< a_3, b_3 >$ be three vectors and k be a constant number. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u})\mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$.
4. $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.
5. $\mathbf{0} \cdot \mathbf{v} = 0$ where $\mathbf{0} =< 0, 0 >$.
6. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$.
7. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$.

Proof.

1. Using the definition, we see that

$$\mathbf{u} \cdot \mathbf{v} = a_1a_2 + b_1b_2 = a_2a_1 + b_2b_1 = \mathbf{v} \cdot \mathbf{u}.$$

That is, the dot product operation is commutative; it does not matter in which order the operation is performed.
2. We have

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= <a_1,b_1>\cdot<a_2+a_3,b_2+b_3>=a_1(a_2+a_3)+b_1(b_2+b_3)\\ &= a_1a_2+a_1a_3+b_1b_2+b_1b_3\\ &= (a_1a_2+b_1b_2)+(a_1a_3+b_1b_3)\\ &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}\end{aligned}$$

This property says that the dot product is distributive with respect to vector addition.
3. We have

$$\begin{aligned}k(\mathbf{u}\cdot\mathbf{v}) &= k(a_1a_2+b_1b_2)\\ &= (ka_1)a_2+(kb_1)b_2\\ &= (k\mathbf{u})\mathbf{v}\end{aligned}$$

In a similar fashion, one can prove that $k(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(k\mathbf{v})$.
4. We have

$$\mathbf{v}\cdot\mathbf{v}=a_1^2+b_1^2=||\mathbf{v}||^2.$$

5. We have

$$\mathbf{0}\cdot\mathbf{v}=0a_1+0b_1=0.$$

6. $\mathbf{i}\cdot\mathbf{i}=1(1)+0(0)=1$ and $\mathbf{j}\cdot\mathbf{j}=0(0)+1(1)=1$.
7. $\mathbf{i}\cdot\mathbf{j}=1(0)+0(1)=0$ and $\mathbf{j}\cdot\mathbf{i}=0(1)+1(0)=0$. ■

Applications of the Dot Product
In this section you will learn four applications of the dot product of two vectors: (1) finding the angle between two vectors, (2) scalar projection onto a vector, (3) testing orthogonality, and (4) finding the work done by a force.

• Angle Between Two Vectors
Let $\mathbf{v}=<a,b>$ and $\mathbf{w}=<c,d>$ be two vectors and θ be the angle between them as shown in Figure 23.5.

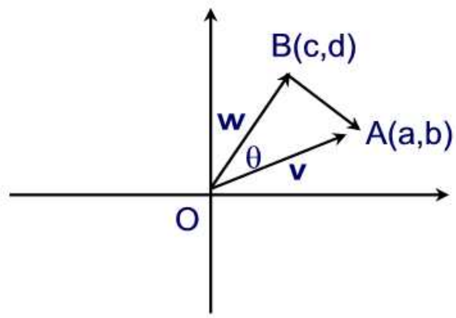


Figure 23.5

According to Figure 23.5, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (c\mathbf{i} + d\mathbf{j}) - (a\mathbf{i} + b\mathbf{j}) = (c-a)\mathbf{i} + (d-b)\mathbf{j}$. By the Law of Cosines for the triangle OAB, we have

$$||\overrightarrow{AB}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2||\mathbf{v}||||\mathbf{w}|| \cos\theta.$$

But, $||\overrightarrow{AB}||^2 = (c-a)^2 + (d-b)^2$, $||\mathbf{v}||^2 = a^2 + b^2$, and $||\mathbf{w}||^2 = c^2 + d^2$. Thus,

$$\begin{array}{ccc}
(c-a)^2 + (d-b)^2 & = & a^2 + b^2 + c^2 + d^2 - 2||\mathbf{v}||||\mathbf{w}|| \cos\theta \\
c^2 - 2ac + a^2 + d^2 - 2bd + d^2 & = & a^2 + b^2 + c^2 + d^2 - 2||\mathbf{v}||||\mathbf{w}|| \cos\theta \\
-2ac - 2bd & = & -2||\mathbf{v}||||\mathbf{w}|| \cos\theta \\
ac + bd & = & ||\mathbf{v}||||\mathbf{w}|| \cos\theta \\
\mathbf{v} \cdot \mathbf{w} & = & ||\mathbf{v}||||\mathbf{w}|| \cos\theta
\end{array}$$

Dividing both sides of this last equality by $||\mathbf{v}||||\mathbf{w}||$ we obtain

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}$$

Thus, choose the smallest non-negative angle satisfying

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}\right)$$

Example 23.6
Find the measure of the smallest positive angle between the vectors $\mathbf{v} =< 2, -1 >$ and $\mathbf{w} =< 3, 4 >$.

Solution.
Using the equation for the angle between two vectors, we have

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}\\ &= \frac{2(3)+(-1)(4)}{\sqrt{2^2+(-1)^2}\sqrt{3^2+4^2}}\\ &= \frac{2}{\sqrt{5}\sqrt{25}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}\end{aligned}$$

Thus,

$$\theta = \arccos\left(\frac{2\sqrt{5}}{25}\right) \approx 79.7^\circ.\blacksquare$$

• **Orthogonal Vectors**
Two vectors are **orthogonal** if and only if the angle between them is 90°. Thus, two vectors are othogonal if and only if their dot product is 0.

Example 23.7
Show that the vectors $\mathbf{v} =< 5, -2 >$ and $\mathbf{w} =< 2, 5 >$ are orthogonal.

Solution.
Since $\mathbf{v}\cdot\mathbf{w} = 5(2) + (-2)(5) = 0$ then the two vectors are orthogonal.■

• **Scalar Projection**
One important use of dot products is in projections. The **scalar projection** of **v** onto **w**, denoted by $\text{proj}_{\mathbf{w}}\mathbf{v}$, is the length of the segment AB shown in Figure 23.6.

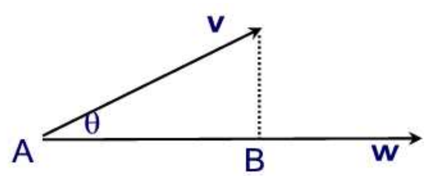


Figure 23.6

According to Figure 23.6 we can write

$$\text{proj}_{\mathbf{w}}\mathbf{v} = ||\mathbf{v}||\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||}.$$

Example 23.8
Given $\mathbf{v} =< 6, 7 >$ and $\mathbf{w} =< 3, 4 >$, find $\text{proj}_{\mathbf{w}}\mathbf{v}$.

Solution.

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||} = \frac{6(3) + 7(4)}{\sqrt{3^2 + 4^2}} = \frac{46}{5}. \blacksquare$$

• **Work Done by a Force**
Suppose you wish to find the work W done in moving a particle from one point to another. From physics we know $W = Fd$, where F is the magnitude of the force moving the particle and d is the distance between the two points. However, this relation is only valid when the force acts in the direction the particle moves. Suppose this is not the case. See Figure 23.7.

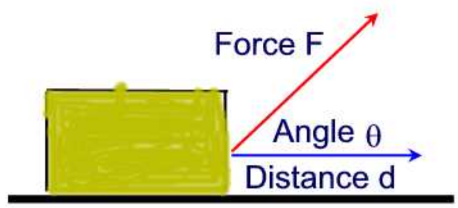


Figure 23.7

Let the force vector be $\mathbf{F}$ and the displacement vector be $\mathbf{d}$. In this case, the work is the product of the distance moved (the magnitude of the displacement vector) and the magnitude of the component of the force that acts in the direction of displacement (the scalar projection of $\mathbf{F}$ onto $\mathbf{d}$):

$$W = ||\mathbf{d}||||\mathbf{F}|| \cos\theta = \mathbf{F}\mathbf{d}.$$

Example 23.9
A force of 40 pounds is exerted in the direction of the handle of the wagon. If the handle makes an angle of $\frac{\pi}{4}$ with the horizontal and the wagon is pulled along a flat surface for 1 mile (5280 feet), find the work done.

Solution.
The work done can be measured by the product of the distance the wagon is

pulled and the component of the force in the direction of the handle of the wagon along the horizontal direction where the wagon is pulled.

$$W = 40\cos\left(\frac{\pi}{4}\right)(5280) = 149,341 \text{ foot} - \text{pounds}\blacksquare$$

Review Problems

Exercise 23.1
Find the magnitude and direction of the given vector. Find the unit vector in the direction of the given vector.

(a) $\mathbf{v} =< -3, 4 >$ (b) $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$.

Exercise 23.2
Find the magnitude and direction of the given vector. Find the unit vector in the direction of the given vector.

(a) $\mathbf{v} =< 6, 10 >$ (b) $\mathbf{v} = 42\mathbf{i} - 18\mathbf{j}$.

Exercise 23.3
Perform the indicated operations where $\mathbf{u} =< -2, 4 >$ and $\mathbf{v} =< -3, -2 >$.

(a) $3\mathbf{u}$ (b) $\frac{2}{3}\mathbf{u} + \frac{1}{6}\mathbf{v}$ (c) $||3\mathbf{u} - 4\mathbf{v}||$.

Exercise 23.4
Perform the indicated operations where $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$.

(a) $6\mathbf{u} + 2\mathbf{v}$ (b) $\frac{2}{3}\mathbf{v} + \frac{3}{4}\mathbf{u}$ (c) $||2\mathbf{v} + 3\mathbf{u}||$.

Exercise 23.5
Find the horizontal and vertical components of the given vector. Write an equivalent vector in the form $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$:
Magnitude = 5 and direction angle = $27°$.

Exercise 23.6
Find the horizontal and vertical components of the given vector. Write an equivalent vector in the form $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$:
Magnitude = 4 and direction angle = $\frac{\pi}{4}$.

Exercise 23.7
Find the horizontal and vertical components of the given vector. Write an equivalent vector in the form $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$:
Magnitude = 2 and direction angle = $\frac{8\pi}{7}$.

Exercise 23.8
Find the dot product of the vectors: $\mathbf{v} = < 3, -2 >; \mathbf{w} = < 1, 3 >$.

Exercise 23.9
Find the dot product of the vectors: $\mathbf{v} = < 4, 1 >; \mathbf{w} = < -1, 4 >$.

Exercise 23.10
Find the dot product of the vectors: $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}; \mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$.

Exercise 23.11
Find the dot product of the vectors: $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j}; \mathbf{w} = -2\mathbf{i} - 3\mathbf{j}$.

Exercise 23.12
Find the angle between the two vectors. State which pair of vectors is orthogonal.

(a) $\mathbf{v} = < 2, -1 >$ and $\mathbf{w} = < 3, 4 >$.
(b) $\mathbf{v} = < 5, -2 >$ and $\mathbf{w} = < 2, 5 >$.

Exercise 23.13
Find the angle between the two vectors. State which pair of vectors is orthogonal.

(a) $\mathbf{v} = 8\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 8\mathbf{j}$.
(b) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 12\mathbf{j}$.

Exercise 23.14
Find $\text{Proj}_{\mathbf{w}}\mathbf{v}$.

(a) $\mathbf{v} = < 6, 7 >$ and $\mathbf{w} = < 3, 4 >$.
(b) $\mathbf{v} = < -7, 5 >$ and $\mathbf{w} = < -4, 1 >$.

Exercise 23.15
Find $\text{Proj}_{\mathbf{w}}\mathbf{v}$.

(a) $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} + 3\mathbf{j}$.
(b) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -6\mathbf{i} + 12\mathbf{j}$.

Exercise 23.16
A 150-pound box is dragged 15 feet along a level floor. Find the work done if a force of 75 pounds at an angle of $32°$ is used.

Exercise 23.17
A rope is being used to pull a box up a ramp that is inclined at $15°$. The rope exerts a force of 75 pounds on the box, and it makes an angle of $30°$ with the plane of the ramp. Find the work done in moving the box 12 feet.

Exercise 23.18
Find the smallest positive angle to the nearest degree between the vectors $\mathbf{v} = < 3, 5 >$ and $\mathbf{w} = < -6, 2 >$.

Exercise 23.19
Find the dot product of $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$.

Exercise 23.20
Find $3\mathbf{u} - 5\mathbf{v}$ given the vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$.

Exercise 23.21
Given $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$. Find $||\mathbf{u}||$.

Exercise 23.22
Find the components of the vector with initial point $A(-2, 4)$ and terminal point $B(3, 7)$.

Exercise 23.23
Find the magnitude and the direction of the vector $\mathbf{u} = < -2, 3 >$.

Exercise 23.24
Find the components of the vector with initial point $A(4, 2)$ and terminal point $B(-3, -3)$.

Exercise 23.25
Find the angle between the two vectors. State which pair of vectors is orthogonal.

(a) $\mathbf{v} = < 1, -5 >$ and $\mathbf{w} = < -2, 3 >$.
(b) $\mathbf{v} = 8\mathbf{i} + \mathbf{j}; \mathbf{w} = -\mathbf{i} + 8\mathbf{j}$.

Exercise 23.26
Find $\text{Proj}_{\mathbf{w}}\mathbf{v}$.

(a) $\mathbf{v} =< -7, 5 >$ and $\mathbf{w} =< -4, 1 >$.
(b) $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$.

Exercise 23.27
Find $\text{Proj}_{\mathbf{w}}\mathbf{v}$.

(a) $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} + 3\mathbf{j}$.
(b) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -6\mathbf{i} + 12\mathbf{j}$.

Exercise 23.28
A 100-pound force is pulling a sled loaded with bricks that weighs 400 pounds. The force is at an angle of 42° with the displacement. Find the work done in moving the sled 25 feet.

24 Introduction to Complex Numbers

Up until now, you've been told that you can't take the square root of a negative number. That's because you had no numbers that, when squared, were negative. So an equation like $x^2 + 1 = 0$ has no real solutions. Trying to solve this last equation, we end up with $x = \pm\sqrt{-1}$. Thus, solving the equation involves using a new number called i, standing for "imaginary", such that $i = \sqrt{-1}$. It follows that $i^2 = -1$.
With the above definition, we are now in a position to find the square root of negative numbers. If a is a positive number then $-a$ is negative and

$$\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}.$$

Example 24.1
Simplify $\sqrt{-18}$.

Solution.
We have $\sqrt{-18} = \sqrt{9 \cdot 2 \cdot (-1)} = 3i\sqrt{2}$. ■

By a **complex number** we mean a number that can be written in the form $a + bi$. We call a the **real part** and b the **imaginary part**. When $a = 0$ we say that the number is **purely imaginary**.

Example 24.2
Write the number $\sqrt{-37} - 3$ in the form $a + bi$.

Solution.
Since $\sqrt{-37} = \sqrt{37(-1)} = (\sqrt{37})i$ then $\sqrt{-37} - 3 = -3 + (\sqrt{37})i$. ■

The Arithmetic of Complex Numbers

When a number system is extended the arithmetic operations must be defined for the new numbers, and the important properties of the operations should still hold. For example, addition of whole numbers is commutative. This means that we can change the order in which two whole numbers are added and the sum is the same: $3 + 5 = 8$ and $5 + 3 = 8$.
We need to define the four arithmetic operations on complex numbers.

• **Equality of Complex Numbers**
Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

Example 24.3
Find x so that $3 + (4 - x)i = 3 + i$.

Solution.
By the equality of complex numbers, we must have $4 - x = 1$. Solving for x we find $x = 3$. ■

• Addition and Subtraction
To add or subtract two complex numbers, you add or subtract the real parts and the imaginary parts.

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i.\\(a + bi) - (c + di) &= (a - c) + (b - d)i\end{aligned}$$

Example 24.4
Perform the indicated operation (a) $(3 - 5i) + (6 + 7i)$ (b) $i - (3 - 4i)$.

Solution.
(a) $(3 - 5i) + (6 + 7i) = (3 + 6) + (-5 + 7)i = 9 + 2i$.
(b) $i - (3 - 4i) = (0 - 3) + (1 - (-4))i = -3 + 5i$. ■

Remark 24.1
The operations of addition and subtraction are the same as combining similar terms in expressions that have a variable. For example, if we were to simplify the expression $(3 - 5x) + (6 + 7x)$ by combining similar terms, then the constants 3 and 6 would be combined, and the terms $-5x$ and $7x$ would be combined to yield $9 + 2x$.

• Multiplication of Numbers
The formula for multiplying two complex numbers is

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

You do not have to memorize this formula, because you can arrive at the same result by treating the complex numbers like expressions with a variable, multiply them as usual by using the FOIL method and then combine like terms. The only difference is that powers of i do simplify (using $i^2 = -1$), while powers of x do not.

Example 24.5
Multiply $(2+3i)(4+7i)$.

Solution.

$$\begin{aligned}(2+3i)(4+7i) &= (2)(4)+(2)(7i)+(4)(3i)+(3i)(7i)\\ &= 8+14i+12i+21(-1)\\ &= (8-21)+(14+12)i = -13+26i. \blacksquare\end{aligned}$$

• **Complex Conjugate**
The **conjugate** (or complex conjugate) of the complex number $a+bi$ is $a-bi$. We denote the conjugate of $a+bi$ by $\overline{a+bi} = a-bi$.

Multiplying $a+bi$ by its conjugate we find

$$(a+bi)(a-bi) = (a^2+b^2)+0i = a^2+b^2.$$

Thus, a complex number times its conjugate is always real; i.e., its imaginary part is zero.

Example 24.6
Find the conjugate of (a) $-3-4i$ and (b) $3+5i$.

Solution.
(a) $\overline{-3-4i} = -3+4i$ (b) $\overline{3+5i} = 3-5i$. $\blacksquare$

• **Division of Complex Numbers**
By the ratio $\frac{a+bi}{c+di}$ we mean a complex number $\alpha+\beta i$ such that

$$\frac{a+bi}{c+di} = \alpha+\beta i \tag{27}$$

Cross multiply and simplify to obtain

$$\begin{aligned}a+bi &= (c+di)(\alpha+\beta i)\\ &= (c\alpha-d\beta)+(d\alpha+c\beta)i\end{aligned}$$

Thus, $c\alpha-d\beta = a$ and $d\alpha+c\beta = b$. Solve this system of two equations for α and β, using the method of elimination, to obtain

$$\alpha = \frac{ac+bd}{c^2+d^2} \quad and \quad \beta = \frac{bc-ad}{c^2+d^2}.$$

One can easily see that the right hand side of (27) is obtained by multiplying $a+bi$ and $c+di$ by the conjugate $c-di$.(Prove that)

Example 24.7
Find $\frac{2+6i}{4+i}$.

Solution.

$$\frac{2+6i}{4+i} = \frac{(2+6i)}{(4+i)}\frac{(4-i)}{(4-i)} = \frac{14+22i}{17} = \frac{14}{17} + \frac{22}{17}i. \blacksquare$$

Solving Equations

With the existence of the square roots of a negative number, it is possible to find the solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ using the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 24.8
What are the solutions of the quadratics: $x^2 + x + 1 = 0$ and $x^2 - 2x + 3 = 0$.

Solution.
Using the usual quadratic formula, we get the solutions as follows

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

and

$$\frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm i\sqrt{8}}{2} = 1 \pm i\sqrt{2}. \blacksquare$$

Review Problems

Exercise 24.1
Write the given complex number in the form $z = a + bi$.

(a) $2+\sqrt{-9}$ (b) $4-\sqrt{-121}$ (c) $-\sqrt{-100}$.

Exercise 24.2
Simplify and then write the complex number in the form $z = a + bi$.

(a) $(2+5i)+(3+7i)$
(b) $(-5-i)+(9-2i)$
(c) $(-3+i)-(-8+2i)$.

Exercise 24.3
Simplify and then write the complex number in the form $z = a + bi$.

(a) $8i-(2-3i)$
(b) $(4i-5)-2$
(c) $3(2+7i)+5(2-i)$.

Exercise 24.4
Simplify and then write the complex number in the form $z = a + bi$.

(a) $(2+3i)(4-5i)$
(b) $(5-3i)(-2-4i)$
(c) $(5+7i)(5-7i)$.

Exercise 24.5
Simplify and then write the complex number in the form $z = a + bi$.

(a) $(8i+11)(-7+5i)$
(b) $(9-12i)(15i+7)$.

Exercise 24.6
Write each expression as a complex number in the form $z = a + bi$.

(a) $\frac{4+i}{3+5i}$ (b) $\frac{1}{-8+i}$ (c) $\frac{1}{7-3i}$.

Exercise 24.7
Write each expression as a complex number in the form $z = a + bi$.

(a) $\frac{2i}{11+i}$ (b) $\frac{6+i}{i}$ (c) $(-5+7i)^2$

Exercise 24.8
Write each expression as a complex number in the form $z = a + bi$.

(a) $(1-i) - 2(4+i)^2$ (b) $(1-i)^3$ (c) $(2i)(8i)$ (d) $(-6i)(-5i)^2$

Exercise 24.9
Simplify and write the complex number as $i, -i$, or -1.

(a) $-i^{40}$ (b) i^{223} (c) i^{2001} (d) i^0 (e) i^{-1}

Exercise 24.10
Simplify each product.

(a) $\sqrt{-1}\sqrt{-4}$ (b) $\sqrt{-3}\sqrt{-121}$

Exercise 24.11
Simplify each product.

(a) $(4+\sqrt{-81})(4-\sqrt{-81})$ (b) $(5+\sqrt{-16})^2$.

Exercise 24.12
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.

(a) $z^2 + 2z + 2 = 0$
(b) $6z^2 - 5z + 5 = 0$.

Exercise 24.13
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.

(a) $2z^2 + z + 3 = 0$
(b) $3z^2 + 2z + 4 = 0$.

Exercise 24.14
The absolute value of a complex number $a + bi$ is the real number

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Find the indicated absolute value of each complex number.

(a) $|5 + 12i|$ (b) $|7 - 4i|$ (c) $|-3i|$

Exercise 24.15
Establish that $|a + bi| = |a - bi|$. That is, the absolute value of a complex number and the absolute value of its conjugate are equal.

Exercise 24.16
Show that $z - \overline{z}$ is purely imaginary and $z + \overline{z}$ is a real number.

Exercise 24.17
Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Show the following:

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.
(c) $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$.

Exercise 24.18
Show that if $x = 1 + i\sqrt{3}$ then $x^2 - 2x + 4 = 0$.

Exercise 24.19
Write the given complex number in the form $z = a + bi$.

(a) $(3 + 5i) + (4 - 2i)$ (b) $(3 + 5i) - (4 - 2i)$ (c) $(3 + 5i)(4 + 2i)$ (d) i^{23}.

Exercise 24.20
Simplify and then write the complex number in the form $z = a + bi$.

(a) $\frac{3+5i}{1-2i}$
(b) $\frac{7+3i}{4i}$.

Exercise 24.21
Simplify and then write the complex number in the form $z = a + bi$.

$$(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$$

Exercise 24.22
Simplify and write the complex number as $i, -i$, or -1.

(a) $-i^{40}$ (b) i^{223} (c) i^{2001} (d) i^0 (e) i^{-1}

Exercise 24.23
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.

(a) $z^2 + 9 = 0$
(b) $x^2 + 4x + 5 = 0$.

Exercise 24.24
Show that the solutions of the equation

$$4x^2 - 24x + 37 = 0$$

are complex conjugate of each other.

Exercise 24.25
Find the indicated absolute value of each complex number.

(a) $|3 + 4i|$ (b) $|8 - 5i|$.

Exercise 24.26
Show that $z \cdot \overline{z}$ is a real number.

Exercise 24.27
Show that $z = \overline{z}$ if and only if z is real.

25 Trigonometric Form of Complex Numbers

In this section, you will learn (1) how to represent a complex number graphically, (2) to compute the absolute value of a complex number, (3) to represent a complex number in trigonometric form, and (4) to represent the product and the quotient of two complex numbers in trigonometric form.

Geometrical Interpretation of a Complex Number

The real numbers can be represented on the number line as shown in Figure 25.1.

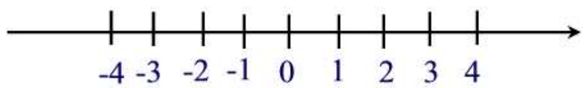


Figure 25.1

Is there a similar representation for the complex numbers?
The definition of a complex number involves two real numbers. But two real numbers give a point on a plane. So complex numbers can be plotted in a plane by using the x-axis for the real part and the y-axis for the imaginary part.
This plane is called The **Complex Plane**. See Figure 25.2.

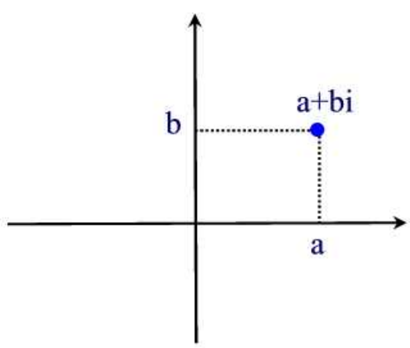


Figure 25.2

Example 25.1
Represent each of the following complex numbers as a point in the complex

plane:
(a) $4 - 3i$ (b) $-3 + 4i$ (c) $-3 - 4i$ (d) $\overline{-2 - 3i}$

Solution.
The points are shown in Figure 25.3.■

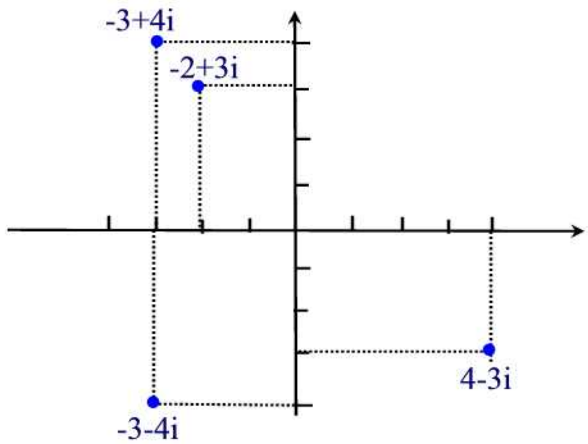


Figure 25.3

The Module of a Complex Number
We define the **absolute value** or the **modulus** of a complex number $a + bi$ as the distance between the point (a, b) to the origin:

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Example 25.2
Determine the absolute value of each of the following complex numbers:
(a) $2 - 3i$ (b) $-5i$ (c) $1 - i$

Solution.
(a) $|2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$
(b) $|-5i| = \sqrt{0^2 + (-5)^2} = 5.$
(c) $|1 - i| = \sqrt{1^2 + 1^2} = \sqrt{2}.$■

Trigonometric Form of a Complex Number
A complex number $z = a + bi$ can be specified by giving the distance, r, of

the point from the origin and the angle, t, between the line joining the point to the origin and the positive x-axis. See Figure 25.4.

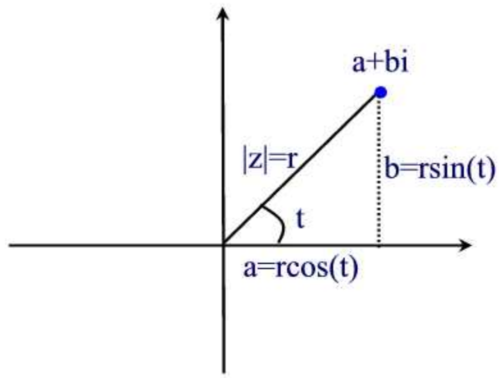


Figure 25.4

By some simple trigonometry it follows that $a = r\cos t$ and $b = r\sin t$. Thus, the complex number z can be written as $z = r\cos t + ir\sin t = rcis(t)$ where $cis(t) = \cos t + i\sin t$. This is known as the **trigonometric form** or the **polar form** of a complex number. r is called the **modulus** of z and t is the **argument** of z.

Example 25.3
Express the complex number $z = 2 + 2i$ in trigonometric form.

Solution.
The modulus of the number z is $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. To find the argument, we use $\tan t = \frac{b}{a} = 1$. Since the complex number is in the first quadrant then $t = \frac{\pi}{4}$. Thus, $z = 2\sqrt{2}cis(\frac{\pi}{4})$.■

Remark 25.1
It is important to remember that the trigonometric form of a complex number is not unique. For example, all expressions of the form $z = 2\sqrt{2}cis(\frac{\pi}{4} + 2n\pi)$, where n is an integer, represent the complex number $z = 2+2i$. An argument in the interval $[-\pi, \pi]$ is called the **principle argument** and we write $t = Arg(z)$.

In some cases calculations in polar form are much simpler so it is important

to be able to work with complex numbers in both forms. There will be times when conversion between these forms is necessary.
Given a modulus r and argument t of a complex number it is easy to find the number in Cartesian coordinate system by following the two steps:

- Evaluate $a = r\cos t$ and $b = r\sin t$.
- Write down the number in the form $a + ib$.

Example 25.4
If a complex number z has modulus of 2 and argument of $-\frac{\pi}{6}$, express z in the form $a + ib$.

Solution.
We have,

$$\begin{array}{rcl} a & = & r\cos t = 2\cos\left(-\frac{\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \\ b & = & r\sin t = 2\sin\left(-\frac{\pi}{6}\right) = 2 \times \frac{-1}{2} = -1 \end{array}$$

Thus, $z = \sqrt{3} - i.$■

The Complex Form of $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$

The multiplication of two complex numbers becomes much easier using the polar form. Take two complex numbers $z_1 = r_1(\cos t_1 + i\sin t_1)$ and $z_2 = r_2(\cos t_2 + i\sin t_2)$ and multiply them together:

$$\begin{array}{rcl} z_1z_2 & = & r_1r_2(\cos t_1 \cos t_2 - \sin t_1 \sin t_2) + ir_1r_2(\sin t_1 \cos t_2 + \cos t_1 \sin t_2) \\ & = & r_1r_2(\cos(t_1 + t_2) + i\sin(t_1 + t_2)) = r_1r_2cis(t_1 + t_2) \end{array}$$

where we have used the trigonometric identities

$$\begin{array}{rcl} \cos(x + y) & = & \cos x \cos y - \sin x \sin y \\ \sin(x + y) & = & \sin x \cos y + \cos x \sin y. \end{array}$$

So the modulus of the product, r_1r_2, is the product of the moduli of z_1 and z_2, namely r_1 and r_2. The argument of the product,$t_1 + t_2$, is the sum of the arguments of z_1 and z_2.
This gives a simple rule for multiplying complex numbers in polar form:

- Multiply the moduli.
- Add the arguments.

Example 25.5
Find the product of $z_1 = 1 - i\sqrt{3}$ and $z_2 = 1 + i$ using the polar form of the complex numbers. Write the final answer in standard form.

Solution.
We have

$$\begin{array}{rcl} z_1 &=& 1 - i\sqrt{3} = 2cis(\frac{5\pi}{3}) \\ z_2 &=& 1 + i = \sqrt{2}cis(\frac{\pi}{4}) \end{array}$$

Thus,

$$\begin{array}{rcl} z_1z_2 &=& 2\sqrt{2}cis\left(\frac{5\pi}{3} + \frac{\pi}{4}\right) = 2\sqrt{2}cis\left(\frac{23\pi}{12}\right) \\ &=& 2\sqrt{2}cis\left(-\frac{\pi}{12}\right) \\ &=& 2\sqrt{2}(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)) \\ &=& 2\sqrt{2}(\frac{\sqrt{2}+\sqrt{6}}{4} + i\frac{\sqrt{2}-\sqrt{6}}{4}) \\ &=& (1+\sqrt{3}) + i(1-\sqrt{3}) \blacksquare \end{array}$$

In a similar way division can be discussed using polar form. If $z_1 = r_1(\cos t_1 + i\sin t_1)$ and $z_2 = r_2(\cos t_2 + i\sin t_2)$ then

$$\begin{array}{rcl} \frac{z_1}{z_2} &=& \frac{r_1(\cos t_1 + i\sin t_1)}{r_2(\cos t_2 + i\sin t_2)} \\ &=& \frac{r_1}{r_2}\frac{(\cos t_1 + i\sin t_1)(\cos t_2 - i\sin t_2)}{(\cos t_2 + i\sin t_2)(\cos t_2 - i\sin t_2)} \\ &=& \frac{r_1}{r_2}\frac{\cos t_1\cos t_2 - \sin t_1\sin t_2 + i(\sin t_1\cos t_2 - \cos t_1\sin t_2)}{(\cos^2 t_2 + \sin^2 t_2)} \\ &=& \frac{r_1}{r_2}\left[\cos(t_1 - t_2) + i\sin(t_1 - t_2)\right] \\ &=& \frac{r_1}{r_2}cis(t_1 - t_2). \end{array}$$

Thus, the modulus for the quotient of two complex numbers in trigonometric form is the quotient of the moduli of the two numbers, and the argument of the quotient is the difference of arguments of the two numbers.

Example 25.6
Find $\frac{z_1}{z_2}$ where $z_1 = 3 - i\sqrt{3}$ and $z_2 = 4 + 4i$ by dividing trigonometric forms. Express the answer in trigonometric form.

Solution.
We have

$$\begin{array}{rcl} z_1 &=& 2\sqrt{3}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) \\ z_2 &=& 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ \frac{z_1}{z_2} &=& \frac{2\sqrt{3}}{4\sqrt{2}}\left[\cos\left(\frac{11\pi}{6} - \frac{\pi}{4}\right) + i\sin\left(\frac{11\pi}{6} - \frac{\pi}{4}\right)\right] \\ &=& \frac{\sqrt{6}}{4}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \blacksquare \end{array}$$

Review Problems

Exercise 25.1
Graph each complex number. Find the absolute value of each complex number.

(a) $z = -2 - 2i$ (b) $z = 1 + i\sqrt{3}$ (c) $z = -2i$ (d) $z = 3 - 5i$

Exercise 25.2
Write each complex number in trigonometric form.

(a) $z = 1 - i$ (b) $z = 1 + i\sqrt{3}$ (c) $z = -2i$ (d) $z = -5$

Exercise 25.3
Write each complex number in the form $z = a + bi$.

(a) $z = 2(\cos 45^\circ + i \sin 45^\circ)$
(b) $z = (\cos 315^\circ + i \sin 315^\circ)$
(c) $z = 5(\cos 120^\circ + i \sin 120^\circ)$.

Exercise 25.4
Write each complex number in the form $z = a + bi$.

(a) $z = 6cis135^\circ$
(b) $z = 8cis0^\circ$
(c) $z = 5cis90^\circ$.

Exercise 25.5
Write each complex number in the form $z = a + bi$.

(a) $z = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
(b) $z = 4\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$
(c) $z = 5(\cos \pi + i \sin \pi)$.

Exercise 25.6
Write each complex number in the form $z = a + bi$.

(a) $z = 8cis\frac{3\pi}{4}$
(b) $z = 9cis\frac{11\pi}{6}$
(c) $z = 2cis2$.

Exercise 25.7
Multiply the complex numbers. Write the answer in trigonometric form.

(a) $(2cis30°)\cdot(3cis225°)$
(b) $[8(\cos 88° + i\sin 88°)]\cdot[12(\cos 112° + i\sin 112°)]$

Exercise 25.8
Multiply the complex numbers. Write the answer in trigonometric form.

(a) $\left[5\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]\cdot\left[2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)\right]$
(b) $(4cis2.4)\cdot(6cis4.1)$.

Exercise 25.9
Divide the complex numbers. Write the answer in the form $z = a + bi$.

(a) $\frac{32cis30°}{4cis150°}$
(b) $\frac{27(\cos 315° + i\sin 315°)}{9(\cos 225° + i\sin 225°)}$.

Exercise 25.10
Divide the complex numbers. Write the answer in the form $z = a + bi$.

(a) $\frac{12cis\frac{2\pi}{3}}{4cis\frac{11\pi}{6}}$.
(b) $\frac{25(\cos 3.5 + i\sin 3.5)}{5(\cos 1.5 + i\sin 1.5)}$.

Exercise 25.11
Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

Exercise 25.12
Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

Exercise 25.13
Write each of the following complex numbers in trigonometric form.

(a) $1 + i$ (b) $-1 + \sqrt{3}i$ (c) $-4\sqrt{3} - 4i$ (d) $3 + 4i$

Exercise 25.14
Let
$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad and \quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$
Find $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$.

26 De Moivre's Theorem

In this section, you will learn how (1) to find the powers of a complex number and (2) to find the roots of a complex number.
If $z = rcis\theta$ then by muliplying z by itself we find that

$$z^2 = r^2cis(2\theta).$$

Now, if we multiply z^2 by z we obtain

$$z^3 = r^3cis(3\theta).$$

So, one might conjecture that

$$z^n = r^ncis(n\theta)$$

for any positive integer n.
In order to prove this result, we use the procedure of mathematical induction:

• The result is true when $n = 1$.
• Induction hypothesis: Assume that the formula is valid for $n \geq 1$. That is, $z^n = r^ncis(n\theta)$.
• Induction conclusion: We must show that the formula is valid for $n + 1$, i.e. $z^{n+1} = r^{n+1}cis[(n + 1)\theta]$.
Indeed,

$$\begin{aligned} z^{n+1} &= z^n \cdot z = [r^ncis(n\theta)]\,(rcis\theta) \\ &= r^{n+1}cis(n\theta + \theta) = r^{n+1}cis(n + 1)\theta \end{aligned}$$

This formulas, is known as **De Moivre's formula**.

Example 26.1
Write $[2(\cos 240° + i \sin 240°)]^4$ in standard form.

Solution.
By De Moivre's formula we have

$$\begin{aligned} [2(\cos 240° + i \sin 240°)]^4 &= 2^4(\cos 4 \cdot 240° + i \sin 4 \cdot 240°) \\ &= 16(\cos 960° + i \sin 960°) \\ &= 16(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}) \\ &= -8 - i(8\sqrt{3}) \blacksquare \end{aligned}$$

Example 26.2
Write $(1+i)^8$ in standard form.

Solution.
Writing $1+i$ in trigonometric form we have $1+i = \sqrt{2}cis\frac{\pi}{4}$. Thus, applying De Moivre's formula we find

$$\begin{aligned}(1+i)^8 &= (\sqrt{2})^8 cis(8\cdot\tfrac{\pi}{4})\\ &= 16cis(2\pi) = 16 \blacksquare\end{aligned}$$

Now, if z and w are complex numbers such that $w^n = z$ then we call w an **nth root** of z.
We next consider how to determine such roots. Write z and w in trigonometric form $z = rcis\theta$ and $w = r'cis\theta'$. Then

$$[r'cis\theta']^n = rcis\theta.$$

By De Moivre's formula,

$$r'^n cis(n\theta') = rcis\theta.$$

Thus,

$$r' = \sqrt[n]{r} \quad and \quad \theta' = \frac{\theta + k\cdot 360^\circ}{n}$$

where $k = 0, 1, 2, \cdots, n-1$.

Remark 26.1
Note that we chose $0 \le k \le n-1$ because past n the roots repeat themselves. That is, a complex number has exactly n complex roots.

Example 26.3
Find all the roots of the equation $x^5 - 32 = 0$.

Solution.
Basically, we are looking for the fifth roots of the complex number $z = 32$. Since $z = 32cis(0)$ then the five fifth roots of z are given by

$$\begin{aligned}w_0 &= 2cis(0) = 2\\ w_1 &= 2cis(\tfrac{360^\circ}{5}) = 2cis(72^\circ)\\ w_2 &= 2cis(\tfrac{2\cdot 360^\circ}{5} = 2cis(144^\circ)\\ w_3 &= 2cis(\tfrac{3\cdot 360^\circ}{5} = 2cis(216^\circ)\\ w_4 &= 2cis(\tfrac{4\cdot 360^\circ}{5} = 2cis(288^\circ) \blacksquare\end{aligned}$$

Review Problems

Exercise 26.1
Find the indicated power. Write the answer in the form $z = a + bi$.

(a) $[2(\cos 30^\circ + i \sin 30^\circ)]^8$.
(b) $(\cos 240^\circ + i \sin 240^\circ)^{12}$.

Exercise 26.2
Find the indicated power. Write the answer in the form $z = a + bi$.

(a) $(2cis225^\circ)^5$
(b) $(4cis\frac{5\pi}{6})^3$.

Exercise 26.3
Find the indicated power. Write the answer in the form $z = a + bi$.

(a) $(1 + i\sqrt{3})^8$
(b) $(2\sqrt{3} - 2i)^5$.

Exercise 26.4
Find the indicated power. Write the answer in the form $z = a + bi$.

(a) $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{12}$
(b) $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{6}$

Exercise 26.5
Find all the indicated roots. Write the answers in standard form.

(a) $z = \sqrt{9}$. (b) $z = \sqrt[6]{64}$ (c) $z = \sqrt[5]{-1}$.

Exercise 26.6
Find all the indicated roots. Write the answers in standard form.

(a) $z = \sqrt[4]{-16}$ (b) $z = \sqrt[3]{1}$ (c) $z = \sqrt[4]{1 + i}$.

Exercise 26.7
Find all the indicated roots. Write the answers in standard form.

(a) $z = \sqrt[5]{-1+i}$ (b) $z = \sqrt[3]{2-2i\sqrt{3}}$ (c) $z = \sqrt{-16+16i\sqrt{3}}$.

Exercise 26.8
Find all the roots of the given equation. Write your answers in trigonometric form.

(a) $x^3 + 8 = 0$
(b) $x^4 + i = 0$
(c) $x^5 + 32i = 0$.

Exercise 26.9
Find all the roots of the given equation. Write your answers in trigonometric form.

(a) $x^4 - (1 - i\sqrt{3}) = 0$
(b) $x^3 + (1 + i\sqrt{3}) = 0$
(c) $x^6 - (4 - 4i) = 0$.

Exercise 26.10
Show that if $z = r(\cos\theta + i\sin\theta)$ then $\overline{z} = r(\cos\theta - i\sin\theta)$.

Exercise 26.11
Show that if $z = r(\cos\theta + i\sin\theta)$ then $z^{-1} = r^{-1}(\cos\theta - i\sin\theta)$.

Exercise 26.12
Show that if $z = r(\cos\theta + i\sin\theta)$ then $z^{-2} = r^{-2}(\cos 2\theta - i\sin 2\theta)$.

Exercise 26.13
Find the six roots of $z = -64$.

Exercise 26.14
Find the three cube roots of $z = 2 + 2i$.

Exercise 26.15
Solve the equation : $z^6 + 64 = 0$.