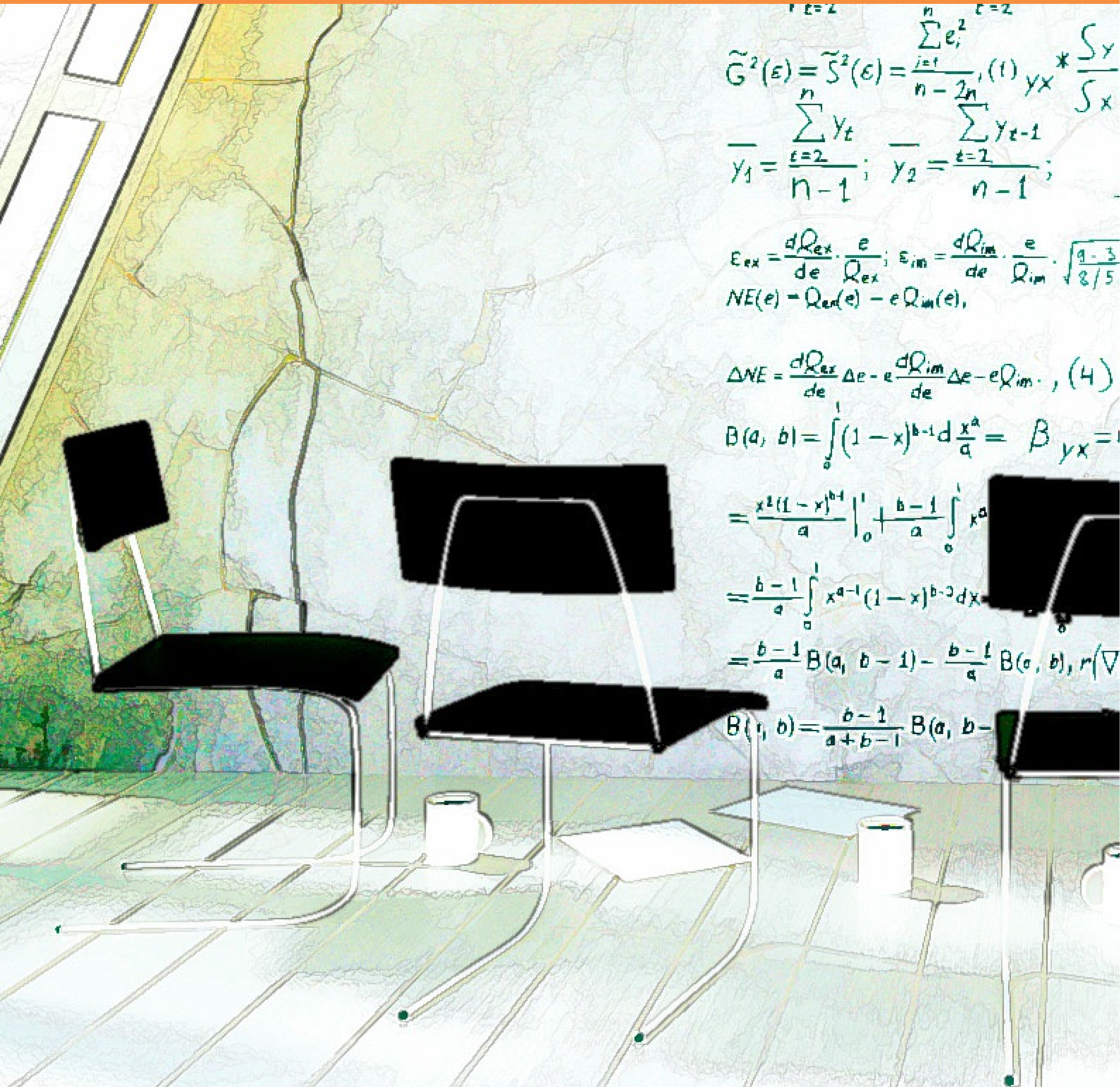

Ordinary differential equations of first order

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LEIF MEJLBRO

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

Ordinary differential equations of first order
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CONTENTS

Preface

In the present volume we present the state-of-the-art of the theory of ordinary differential equations of first order as known in the middle of the twentieth century. The emphasis has been laid on solution formulæ, so the reader can get some idea of how to solve one of these classical equations, while topological methods have been more or less neglected. Some worked out examples illustrate the theory, and sometimes also the difficulties of the methods, in particular when we are forced to use the theorem of implicit given functions. Note, however, that if we use programs like MAPLE in such cases, it is often easy to sketch the solution curves by using a parametric description instead, while an explicit description may not be possible.

In Chapter 1 we collect all the necessary mathematics, like the use of uniformly convergent sequences of functions, used in the iteration processes, Banach's fixed point theorem, and how approximately to solve an implicitly given function in two variables, all needed in the following chapters.

It should be mentioned that the Riccati equation has not been discussed in all details, because it is too closely connected with a class of ordinary differential equations of second order, which are outside the realm of this book. One may say that it belongs to the twilight zone between ordinary differential equations of first and second order.

A short review of all methods and solution formulæ of this book is given in Chapter 10.

July 1, 2017
Leif Mejlbro

1 Some necessary auxiliary results

1.1 Introduction

We shall in this chapter collect some necessary auxiliary results, which all are of independent interest, because they can also be applied on problems which have nothing to do with differential equations. We shall show some of these applications, but the reader must never forget that in the following chapters we shall only use the results of this chapter on differential equations and problems concerned with differential equations.

1.2 Uniformly convergent sequences of functions

Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of (real or complex) functions, all defined in a nonempty inverval I. It is well-known that $\{f_n\}$ converges *pointwisely* towards a function $f(x)$, also defined on I, if for every fixed $x \in I$ the ordinary sequence $\{f_n(x)\}$ of real or complex numbers converges towards the value $f(x)$ of f at this point $x \in I$, when $n \to +\infty$. This can also be expressed in the following way. For every $x \in I$ and every given $\varepsilon > 0$ there exists a constant $N = N(x, \varepsilon) \in \mathbb{N}$, such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } n \geq N(x, \varepsilon).$$

Usually the functions $f_n(x)$ in the sequence are all continuous, and it would therefore be convenient, if the limit function $f(x)$ were also continuous. Unfortunately, this is far from being true, which is demonstrated by the following classical example.

Example 1.1 Consider the sequence of functions $\{f_n(x)\}$, given by

$$f_n(x) = x^n \qquad \text{for } x \in [0,1], \qquad n \in \mathbb{N}.$$

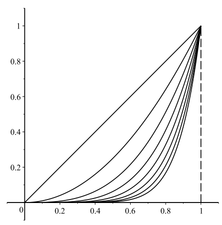


Figure 1.1: *The graphs of the functions* $f_n(x) = x^n$ *for* $n = 1, \dots, 8$.

If $0 \leq x < 1$, then the limit is 0,

$$|f_n(x) - 0| = x^n \to 0 \qquad \text{for } n \to +\infty,$$

and if $x = 1$, then clearly $f_n(1) = 1^n = 1 \to 1$ for $n \to +\infty$. All functions $f_n(x)$, $n \in \mathbb{N}$, are continuous, and the limit function exists and is given by

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

Clearly, $f(x)$ is not continuous. $\Diamond$

In order to secure that continuity is preserved, when we take the limit, we introduce another and stronger concept of convergence.

Definition 1.1 *A sequence* $\{f_n(x)\}_{n\in\mathbb{N}}$ *of functions defined on an interval* I *is said to converge* uniformly *towards a limit function* $f(x)$*, defined on* I*, if, for every* $\varepsilon > 0$ *there exists a constant* $N = N(\varepsilon)$*, not depending on* $x \in I$*, such that*

$$|f(x) - f_n(x)| < \varepsilon \qquad \textit{for all } x \in I, \quad \textit{when } n \geq N(\varepsilon).$$

The importance of uniformly convergent series of functions is shown by the following theorem.

Theorem 1.1 *Let* $\{f_n(x)\}$ *be a uniformly convergent series of* continuous *functions, and let* $f(x)$ *denote the limit function. Then* $f(x)$ *is also continuous.*

Proof. By the assumption of uniform convergence we can for every given $\varepsilon > 0$ find $N = N(\varepsilon) \in \mathbb{N}$, such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3} \qquad \text{for all } x \in I, \text{ and all } n \geq N(\varepsilon).$$

Let $x_0 \in I$ be any point from I. By adding some extra terms, the sum of which is 0, we rewrite $f(x) - f(x_0)$ in the following way,

$$f(x) - f(x_0) = f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)\,,$$

We get by the triangle inequality that

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|\,.$$

If $n \geq N(\varepsilon)$, then

$$(1.1) \qquad |f(x) - f(x_0)| \leq \frac{\varepsilon}{3} + |f_n(x) - f_n(x_0)| + \frac{\varepsilon}{3} = |f_n(x) - f_n(x_0)| + \frac{2\varepsilon}{3}.$$

By assumption, $f_n(x)$ is continuous for every $n > N(\varepsilon)$, in particular continuous at the point x_0, so there exists a $\delta = \delta(x) > 0$, such that

$$(1.2) \qquad |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}, \qquad \text{for } |x - x_0| < \delta.$$

Summing up, it follows from (1.1) and (1.2) that

$$|f(x) - f(x_0)| < \varepsilon \qquad \text{for } |x - x_0| < \delta = \delta(\varepsilon),$$

which proves that the limit function $f(x)$ is continuous at the point $x_0 \in I$. The point x_0 was chosen arbitrarily, so the claim follows. □

Another important result is the following

Theorem 1.2 *Assume that* $\{f_n(x)\}$ *is a sequence of continuous functions defined on a compact, i.e. closed and bounded, interval* I*, which converges uniformly towards the (continuous) function* $f(x)$*,* $x \in I$*. Then*

$$\lim_{n\to+\infty} \int_I f_n(x)\,\mathrm{d}x = \int_I \lim_{n\to+\infty} f_n(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x.$$

In other words, if the sequence of continuous functions $\{f_n\}$ converges uniformly on the closed and bounded interval I towards the continuous function $\lim_{n\to+\infty} f_n(x) = f(x)$, then we can interchange the order of the limit and the integration.

PROOF. We shall prove that to every $\varepsilon > 0$ there exists a constant $N = N(\varepsilon) \in \mathbb{N}$, such that

$$\left| \int_I f(x)\,\mathrm{d}x - \int_I f_n(x)\,\mathrm{d}x \right| < \varepsilon, \qquad \text{for all } n \geq N(\varepsilon).$$

Let $|I| > 0$ denote the length of the bounded interval I, and choose $N = N(\varepsilon) \in \mathbb{N}$, such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{|I|} \qquad \text{for all } x \in I \text{ and all } n \geq N(\varepsilon).$$

Then we get for all $n \geq N(\varepsilon)$,

$$\left| \int_I f(x)\,\mathrm{d}x - \int_I f_n(x)\,\mathrm{d}x \right| \leq \int_I |f(x) - f_n(x)|\,\mathrm{d}x \leq \frac{\varepsilon}{|I|} \cdot |I| = \varepsilon,$$

and the theorem is proved. □

1.3 Banach's fix point theorem

We shall in this section prove a very important result, which is implicitly applied, whenever we use an iterative scheme of approximation. In many cases the reader is hardly aware of this application.

We shall use a general setup and assume that the reader is familiar with metric spaces. So, let (M, d) be a complete metric space. This means that M is the point set, d is the metric, so $d(x, y)$ denotes the distance between the points x, $y \in M$, and the completeness means that every Cauchy sequence from M is convergent with its limit in M.

Let $T : M \to M$ be a map of M into itself. We say that a point $x \in M$ is a *fix point* of the map T, if $Tx = x$.

We call the map $T : M \to M$ a *contraction*, if there exists a constant $\alpha \in [0, 1[$, such that

$$d(Tx, Ty) \leq \alpha\, d(x, y), \qquad \text{for all } x,\ y \in M. \tag{1.3}$$

The important general result is the following

Theorem 1.3 Banach's fix point theorem. *Assume that* (M, d) *is a* complete *metric space, and that the map* $T : M \to M$ *is a contraction. Then* T *has precisely one fix point.*

In the applications the trick is to define a map T of a complete metric space onto itself, such that the solution of our problem is given by this fix point. Also, the iteration process defined in the proof below can often be applied in practice, when we construct a solution by iteration.

PROOF. Let $\alpha \in [0, 1[$ denote the constant in (1.3).

1) **Uniqueness**. Assume that x, $y \in M$ are two fix points of the map T. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha\, d(x, y).$$

Since $0 \leq \alpha < 1$, this is only possible, when $d(x, y) = 0$, in which case $x = y$, and the uniqueness follows.

2) **Existence**. Choose any $x_0 \in M$, and define inductively a sequence of points (x_n) by

$$(1.4) \qquad x_0, \quad x_1 := Tx_0, \quad x_2 := Tx_1, \quad \cdots, \quad x_n := Tx_{n-1}, \quad \cdots .$$

This is a *Cauchy sequence*, because

$$d(x_{m+1}, x_m) = d(Tx_m, Tx_{m-1}) \leq \alpha\, d(x_m, x_{m-1}) \leq \cdots \leq \alpha^n\, d(x_1, x_0)\,,$$

so

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \leq \left\{\alpha^{n+p-1} + \cdots + \alpha^n\right\} d(x_1, x_0) \\ &= \alpha^n \cdot \frac{1-\alpha^p}{1-\alpha}\, d(x_1, x_0)\,, \end{aligned}$$

hence also

$$(1.5) \qquad d(x_{n+p}, x_n) \leq \alpha^n \cdot \frac{1}{1-\alpha}\, d(x_1, x_0) \qquad \text{for all } p \in \mathbb{N}.$$

Since $\alpha^n \to 0$ for $n \to +\infty$, it follows that (x_n) given by (1.4) is a Cauchy sequence. Since M is complete, this Cauchy sequence has a limit, $x_n \to x \in M$ for $n \to +\infty$.

We shall prove that x is a fix point. This is straightforward. When we insert some x_n and use the triangle inequality, we get

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx) \\ &\leq d(x, x_n) + \alpha \cdot d(x_{n-1}, x) \to 0 \qquad \text{for } n \to +\infty, \end{aligned}$$

so we conclude that $d(x, Tx) = 0$, and we have proved that x is a fix point. □

We note the error bound (1.5), i.e. by letting $p \to +\infty$,

$$d(x, x_n) \le \frac{\alpha^n}{1-\alpha}\, d(x_1, x_0)\,.$$

One important application of Banach's fix point theorem is the solution of the *Fredholm integral equation*

$$(1.6) \qquad x(t) - \mu \int_a^b k(t,\tau)x(\tau)\,\mathrm{d}\tau = u(t).$$

Here, μ is a numerically small real constant, and $[a, b]$ is a given compact i.e. closed and bounded interval, and the *kernel* $k(t,\tau)$ is continuous on the compact set $[a, b] \times [a, b]$, and $u(t)$ is a given function. The task is to find a solution formula of the unknown function $x(t)$.

We choose the metric space $C^0([a, b])$ of all continuous functions defined on $[a, b]$, where the metric is given by

$$\mathrm{d}(x, y) := \max_{t\in[a,b]} |x(t) - y(t)|, \qquad \text{for } x,\, y \in C^0([a, b]).$$

Aside we mention that (M, d) is the well-known normed space $\left(C^0([a, b]), \|\cdot\|_\infty\right)$.

Convergence in this metric space is uniform convergence of a sequence of continuous functions, so according to Section 1.2 the limit function of a Cauchy sequence exists as a continuous function on $[a, b]$, proving that the metric space indeed is complete.

We define the map $T : C^0([a, b]) \to C^0([a, b])$ by

$$(1.7) \qquad Tx(t) := u(t) + \mu \int_a^b k(t,\tau)x(\tau)\,\mathrm{d}\tau.$$

Then (1.6) is equivalent to the equation $Tx(t) = x(t)$, i.e. we shall prove that T defined by (1.7) has a fix point for small $|\mu|$.

It suffices to prove that T becomes a contraction, when $|\mu|$ is chosen sufficiently small. For convenience we define

$$c := \max_{t,\tau\in[a,b]} |k(t,\tau)|,$$

and we choose $\mu \in \mathbb{R}$, such that

$$|\mu| < \frac{1}{(b-a)c}.$$

Then

$$\begin{aligned} d(Tx, Ty) &= \max_{t\in[a,b]} |Tx(t) - Ty(t)| = |\mu| \max_{t\in[a,b]} \left| \int_a^b k(t,\tau)\{x(\tau) - y(\tau)\}\,\mathrm{d}\tau \right| \\ &\le |\mu| \max_{t\in[a,b]} \int_a^b |k(t,\tau)| \cdot |x(\tau) - y(\tau)|\,\mathrm{d}\tau \\ &\le |\mu|(b-a)c \max_{t\in[a,b]} |x(t) - y(t)| = |\mu|(b-a)c\, d(x, y). \end{aligned}$$

Since $\alpha = |\mu|(b-a)c < 1$ and $d(Tx, Ty) \le \alpha\, d(x, y)$, we have proved that T is a contraction, provided that $|\mu| < \dfrac{1}{(b-a)c}$. It follows from Banach's fix point theorem that T has a fix point $x = x(t)$, and this is the unique solution of the integral equation (1.6)

Summing up, we have proved

Theorem 1.4 *Let $k(t,\tau)$, $(t,\tau) \in [a,b]^2$, and $u(t)$, $t \in [a,b]$, be continuous functions, and define $c := \max|k(t,\tau)|$.*
If $\mu \in \mathbb{R}$ satisfies $|\mu| < \dfrac{1}{(b-a)c}$, then the integral equation

$$x(t) - \mu \int_a^b k(t,\tau)x(\tau)\, d\tau = u(t)$$

on $C^0([a,b])$ has a unique solution, which can be found by the iteration formula

$$x_{n+1}(t) := u(t) + \mu \int_a^b k(t,\tau)x_n(\tau)\, d\tau, \qquad n \to +\infty,$$

for any given initial function $x_0 \in C^0([a,b])$.

In many other solution formulæ the iteration becomes even easier to apply than Theorem 1.4. In that case we may reformulate the iteration by introducing, what is called a *Neumann series*. For the time being, Theorem 1.4 suffices.

1.4 Implicit given functions

In general, ordinary differential equations are given implicitly like

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0, \qquad G\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0, \qquad \text{etc.},$$

where F, G, … are continuous functions. In order to solve such equations we must first – at least locally – solve these equations with respect to the term of highest order,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = g\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right), \qquad \text{etc.}.$$

This is the reason, why we include the present section on implicit given functions. This subject has of course independent interest, and even if we in the following chapters only shall apply it on differential equations, we here develop the theory in general.

Theorem 1.5 *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ be an open domain, and let $F(\mathbf{x}, y)$, where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y \in \mathbb{R}$, and $(\mathbf{x}, y) \in \Omega$, be a C^1-function, i.e. $\dfrac{\partial F}{\partial x_j}$, $j = 1, \dots, n$, and $\dfrac{\partial F}{\partial y}$ all exist and are continuous functions in Ω. Let $(\mathbf{x}_0, y_0) \in \Omega$ be a point, for which*

$$F(\mathbf{x}_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y}(\mathbf{x}_0, y_0) \neq 0.$$

Then one can find constants $\delta > 0$ and $a > 0$, such that the equation $F(\mathbf{x}, y) = 0$ has a unique solution $y = y(\mathbf{x}) \in [y_0 - a, y_0 + a]$ for every $\mathbf{x} \in \omega$, where

$$\omega := [x_{01} - \delta, x_{01} + \delta] \times [x_{02} - \delta, x_{02} + \delta] \times \cdots \times [x_{0n} - \delta, x_{0n} + \delta],$$

and where we have written $\mathbf{x}_0 = (x_{01}, x_{02}, \dots, x_{n0})$.
The solution is a function $y = f(\mathbf{x})$ for $\mathbf{x} \in \omega$ of class $C^1(\omega)$. Its partial derivatives $\dfrac{\partial y}{\partial x_j}$ can be found by using the formula

$$\frac{\partial F}{\partial x_j}(\mathbf{x}, y) + \frac{\partial F}{\partial y}(\mathbf{x}, y)\,\frac{\partial y}{\partial x_j} = 0, \qquad j = 1, \dots, n. \tag{1.8}$$

Proof. By assumption, $\dfrac{\partial F}{\partial y}(\mathbf{x}_0, y_0) \neq 0$, so we may without loss of generality assume that $\dfrac{\partial F}{\partial y}(\mathbf{x}_0, y_0) > 0$. Otherwise, consider $-F(\mathbf{x}, y)$ instead.

Also by assumption, $\dfrac{\partial F}{\partial y}(\mathbf{x}, y)$ is continuous at the interior point $(\mathbf{x}_0, y_0)$, so we can find a neighbourhood $\tilde{\omega} \subseteq \Omega$, given by e.g.

$$\tilde{\omega} := [x_{01} - \varepsilon, x_{01} + \varepsilon] \times \cdots \times [x_{0n} - \varepsilon, x_{0n} + \varepsilon], \qquad \varepsilon > 0,$$

such that

$$\frac{\partial F}{\partial y}(\mathbf{x}, y) \geq \frac{1}{2}\,\frac{\partial F}{\partial y}(\mathbf{x}_0, y_0) > 0 \qquad \text{for } (\mathbf{x}, y) \in \tilde{\omega}.$$

Fix $\mathbf{x} \in \tilde{\omega}$ and consider the function

$$\varphi_{\mathbf{x}}(y) = F(\mathbf{x}, y), \qquad \mathbf{x} \in \tilde{\omega} \text{ fixed},$$

in y alone. Since $\dfrac{\partial F}{\partial y} > 0$ in $\tilde{\omega} \times [y_0 - b, y_0 + b]$, the function $\varphi_{\mathbf{x}}(y)$ is strictly increasing for every fixed $\mathbf{x} \in \tilde{\omega}$. This is in particular true for $\mathbf{x} = \mathbf{x}_0$, so

$$F(\mathbf{x}_0, y_0 - b) < 0 \qquad \text{and} \qquad F(\mathbf{x}_0, y_0 + b) > 0.$$

Then we use that the two functions $F(\mathbf{x}, y_0 - b)$ and $F(\mathbf{x}, y_0 + b)$ are both continuous at $\mathbf{x} \in \tilde{\omega}$. Hence, there exists a positive constant $\delta\ (\leq \varepsilon)$, such that

$$F(\mathbf{x}, y_0 - b) < 0 \quad \text{and} \quad F(\mathbf{x}, y_0 + b) > 0 \qquad \text{for } \mathbf{x} \in \omega, \tag{1.9}$$

where $\omega := [x_{01} - \delta, x_{01} + \delta] \times [x_{02} - \delta, x_{02} + \delta] \times \cdots \times [x_{0n} - \delta, x_{0n} + \delta]$.

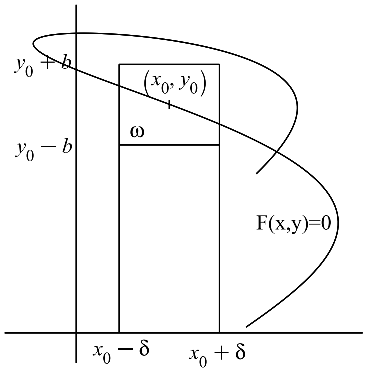


Figure 1.2: *Illustration of the proof of Theorem 1.5 in the case of* $n = 1$*, i.e.* $(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Since $F(\mathbf{x}, y)$ is continuous for $(\mathbf{x}, y) \in \omega \times [y_0 - b, y_0 + b]$, and $\dfrac{\partial F}{\partial y} > 0$ for $(\mathbf{x}, y) \in \omega \times [y_0 - b, y_0 + b]$, it follows from (1.9) that for every $\mathbf{x} \in \omega$ there exists precisely one value $t \in \,]y_0 - b, y_0 + b[$, such that $F(\mathbf{x}, y) = 0$. Therefore, we have described y as a function in $\mathbf{x} \in \omega$ with values in $[y_0 - b, y_0 + b]$, i.e.

$$y = f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \omega.$$

We shall prove that $y = f(\mathbf{x})$, $\mathbf{x} \in \omega$, is continuous.

Let $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$, $|\Delta x_j| < \delta$ for all $j = 1, \dots, n$, be a small increment, such that also $\mathbf{x}_0 + \Delta\mathbf{x} \in \omega$. Then put

$$\Delta y = f(\mathbf{x}_0 + \Delta x) - f(\mathbf{x}_0)\,,$$

where $(\mathbf{x}_0, y_0)$, $(\mathbf{x}_0 + \Delta\mathbf{x}, y_0 + \Delta y) \in \omega \times [y_0 - b, y_0 + b]$ both satisfy the equation $F(\mathbf{x}, y) = 0$. Hence

$$F(\mathbf{x}_0 + \Delta\mathbf{x}, y_0 + \Delta y) - F(\mathbf{x}_0, y_0) = 0.$$

We add and subtract $F\left(\mathbf{x}_0 + \sum_{j=1}^{m} \Delta x_j \mathbf{e}_j, y_0\right)$, $m = 1, \dots, n-1$, to get

$$F(\mathbf{x}_0 + \Delta\mathbf{x}, y_0) - F\left(\mathbf{x}_0 + \sum_{j=1}^{n-1} \Delta x_j \mathbf{e}_j, y_0\right)$$
$$+F\left(\mathbf{x}_0 + \sum_{j=1}^{n-1} \Delta x_j \mathbf{e}_j, y_0\right) - F\left(\mathbf{x}_0 + \sum_{j=1}^{n-2} \Delta x_j \mathbf{e}_j, y_0\right) + \dots - F(\mathbf{x}_0 + \Delta x_1 \mathbf{e}_1, y_0) = 0.$$

These terms can be paired two by two in the order given above. In each of the pairs there is only one variable, and since for every $\varepsilon > 0$ we can choose $\delta > 0$ so small that

$$\left| F\left(\mathbf{x}_0 + \sum_{j=1}^{m} \Delta x_j \mathbf{e}_j, y_0\right) - F\left(\mathbf{x}_0 + \sum_{j=1}^{m-1} \Delta x_j \mathbf{e}_j, y_0\right) \right| < \frac{\varepsilon}{n} \qquad \text{for all } |\Delta x_k| < \delta,$$

we conclude that

$$|F(\mathbf{x}_0 + \Delta\mathbf{x}, y_0) - F(\mathbf{x}_0 + \Delta x_1 \mathbf{e}_1, y_0)| < \varepsilon,$$

when all $|\Delta x_k| < \delta$, and $y = f(x)$ is (locally) continuous.

Formula (1.8) then follows from the *chain rule*. $\Diamond$

2 First order differential equations and differential forms

2.1 Introduction and terminology

Let $F(x,y,z)$ be a continuous function in three variables, defined in some open, nonempty domain. By an *ordinary differential equation of first order* we shall understand an implicit equation of the form

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0, \tag{2.1}$$

where $y = y(x)$ is a C^1-function, and $\frac{\mathrm{d}y}{\mathrm{d}x} = y'(x)$ its derivative, and where we shall solve the equation (2.1) with respect to the function $y = y(x)$.

By a *solution* of the differential equation (2.1) we shall understand a C^1-function $\varphi(x)$, such that

$$F(x, \varphi(x), \varphi'(x)) = 0,$$

i.e. $y = \varphi(x)$ is one of the answers of the problem (2.1).

The *graph* of a solution $y = \varphi(x)$ is called an *integral curve*.

The set of all solutions of (2.1) is called the *total*, or the *complete solution* of (2.1), while each solution is also called a *particular solution*.

The simplest differential equation of first order is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x),$$

the solution of which is found by integration,

$$y = \int_{x_0}^{x} f(t)\,\mathrm{d}t + C, \qquad \text{or} \qquad y = \int f(x)\,\mathrm{d}x \text{ for short,} \tag{2.2}$$

where C is an arbitrary constant. For that reason we in general say that we *integrate* (2.1), when we find a solution. Note in (2.2) that each particular solution is defined by one fixed value of the constant C, and that the complete solution is obtained by letting the constant C vary through the possible values of this constant, usually $\mathbb{R}$ or $\mathbb{C}$.

The general solution of (2.1) is hard to find directly, if possible at all. We shall usually reduce it by an application of the *Theorem of implicit given functions*, i.e. Theorem 1.5, cf. Section 1.4, therefore mostly consider equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \tag{2.3}$$

where the implicit given equation (2.1) (locally) has been solved with respect to the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Assuming that $f(x,y)$ is continuous, and that $y = \varphi(x)$ is a (particular) solution of (2.3), $\varphi'(x) = f(x, \varphi(x))$, where $\varphi(x)$ is continuous, it follows that $\varphi'(x)$ is also continuous, so φ is of class C^1.

An alternative way of formulation of an ordinary differential equation of first order is as a differential form

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0, \tag{2.4}$$

which for $x \neq 0$ and $M(x, y) \neq 0$ is equivalent to (2.3), because it then can be written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{L(x, y)}{M(x, y)}.$$

It is often more convenient to solve (2.4) than (2.3), and then we afterwards have to discuss the cases where either $x = 0$ or $M(x, y) = 0$. This is in general not a problem.

2.2 Geometrical interpretation and isoclines

The geometrical meaning of the equation (2.3) is that at a given point (x, y), the value of the right hand side $f(x, y)$ indicates the slope, i.e. the tangent, of the integral curve through this point, provided that such a solution curve exists. This interpretation can (for first order equations) give a rough idea of the integral curves. In fact, for given constant C draw some curves of equation

$$f(x, y) = C. \tag{2.5}$$

If a point (x, y) lies on one of these curves, and an integral curve passes through this point, then the slope of this particular solution at (x, y) is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = C, \qquad \text{i.e. } \frac{\mathrm{d}y}{\mathrm{d}x} = C.$$

In practice, we sketch some curves of equation (2.5) for some convenient values of the constant C. Sketch tiny tangents of this slope for some points on the curve, and it is not hard so make a crude sketch of the solution curves.

Curves of the form (2.5) are called *isoclines* ("equal slope"), and the procedure above is called the *method of isoclines.* Although it is not 100 % correct, it often gives a good idea of the solution curves.

On Figure 2.1 we have sketched some isoclines for the Riccati equation $y' = x^2 + y^2$. A more systematic way of using the method of isoclines was introduced by Brodetsky [2]. His papers were written long before the era of computers, so it is quite remarkable how much he could obtain without these modern aids. Davis [7] quotes the following from Brodetsky [2]:

> *"Draw the locus of all points at which the required family of curves are parallel to the axis of x; it is of course $f(x, y) = 0$. Draw the locus of points where they are parallel to the axis of y, i.e. $1/f(x, y) = 0$. One or other or both of these loci may not exist in the finite part of the plane; but in any case we get the plane divided up into a number of compartments: in some the required curves have positive dy/dx, in others negative dy/dx. Now calculate d^2y/dx^2 from the given differential equation. This can always be done. Draw the locus of points of inflection, i.e. $d^2y/dx^2 = 0$. We now have a number of compartments, in some of which the curves are concave upward, viz d^2y/dx^2 positive, in others convex downward, viz d^2y/dx^2 negative. We have thus divided up the plane into spaces, in each of which the curves satisfying the differential equation have one of the general forms*
>
> $$(1)\ y' < 0,\ y'' < 0, \quad (2)\ y' > 0,\ y'' < 0, \quad (3)\ y' < 0,\ y'' > 0, \quad (4)\ y' > 0,\ y'' > 0.$$
>
> *Now draw a number of short tangents at a convenient number of points, and the geometrical solution of the differential equation is obtained.*

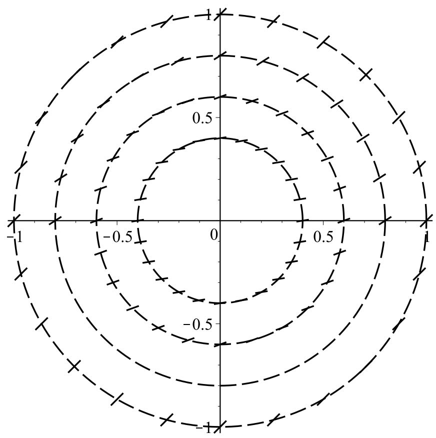


Figure 2.1: *Some isoclines (circles) for the Riccati equation $y' = x^2 + y^2$ and corresponding tangents, namely for $C = 0.16$, 0.36, 0.64 and $C = 1$. It is possible to solve the equation, but not at this stage of the theory. However, it is not hard in principle to sketch some solutions, when we have drawn sufficiently many isoclines.*

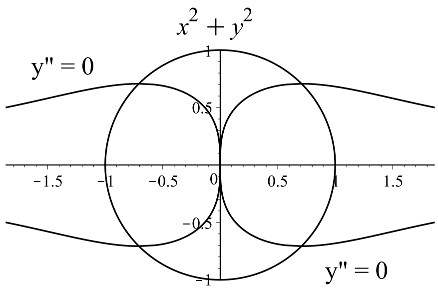


Figure 2.2: *The loci of inflection for the Riccati equation $y' = x^2 + y^2$ and the unit circle.*

In the case of Figure 2.1 we see that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 > 0, \qquad \text{for all } (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\},$$

so in the language of Brodetsky there is only one component, namely the open set $\mathbb{R}^2 \setminus \{(0,0)\}$. Then we see that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2yy' + 2x = 2yx^2 + 2x + 2y^3.$$

The points of inflection are given by $(0,0)$, or, by solving the equation $2yx^2 + 2x + 2y^3 = 0$ with respect to x, when $y \neq 0$,

$$x = \frac{-1 \pm \sqrt{1-4y^4}}{2y}, \qquad \text{when } 0 < |y| \leq \frac{1}{\sqrt{2}}.$$

The curves of inflection are sketched in Figure 2.2. We shall for the time being not go deeper into this example.

2.3 Equipotential curves

Consider the differential equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0,$$

written as a differential form. Its importance in e.g. Physics is, that it can be identified with a two-dimensional *vector field* $(L(x,y), M(x,y))$, where the solutions of the differential equation then are interpreted as the *streamlines* of the field. The orthogonal curves of the streamlines are then called the *equipotential curves*. Since they are perpendicular to the streamlines, they are generated by the orthogonal vector field

$$(2.6) \qquad (-M(x,y), L(x,y)),$$

and the equipotential curves form the complete solution of the equipotential equation

$$(2.7) \qquad -M(x,y)\,\mathrm{d}x + L(x,y)\,\mathrm{d}y = 0,$$

where we interchange $L(x,y)$ and $M(x,y)$, and change the sign on just one of them. Here we have chosen $M(x,y)$, but we might as well, if convenient, change the sign of $L(x,y)$ instead.

2.4 The existence and uniqueness theorem for the ordinary equation of first order

We shall here start with the following important theorem, which we prove in more generality than needed here. In the usual applications it will suffice to apply Corollary 2.1 following after the theorem.

Theorem 2.1 Existence and uniqueness theorem for the ordinary differential equation of first order. (Cauchy). *Consider the differential equation*

$$\frac{dy}{dx} = f(x,y), \tag{2.8}$$

where $f(x,y)$ *is continuous in a neighbourhood* ω *of a point* $(x_0, y_0) \in \omega$*, and furthermore satisfies a* Lipschitz condition*, i.e. there exists a constant* $C > 0$*, such that for all points* (x,y)*,* $(x,\tilde{y}) \in \omega$ *on some vertical line contained in* ω *we have the inequality*

$$|f(x,y) - f(x,\tilde{y})| < C\,|y-\tilde{y}|.$$

Then there is one and only one (continuous) solution $y = y(x)$ *of (2.8), such that* $y(x_0) = y_0$*.*

In other words, when the conditions of the theorem are satisfied, then only one integral curve will pass through the point (x_0, y_0).

Proof. If $f(x,y) \equiv 0$ there is nothing to prove, so assume that $f(x,y) \not\equiv 0$. Assume that $f(x,y)$ is continuous (and not identical 0) in a rectangle defined by

$$|x - x_0| \le a, \qquad |y - y_0| \le b,$$

in which it also satisfies a *Lipschitz condition* of given constant $C > 0$. The closed rectangle is compact, and f is continuous, so $|f(x,y)|$ has a maximum $M > 0$. We choose $h := \min\left\{a, \frac{b}{M}\right\}$, and define ω as the possibly smaller rectangular domain,

$$\omega: \qquad |x - x_0| \le h, \qquad |y - y_0| \le b.$$

We consider the case, where $x \in [x_0, x_0 + b]$, and then define a sequence of functions $y_n(x)$, $n \in \mathbb{N}$, in the following way,

$$y_1(x) := y_0 + \int_{x_0}^{x} f(t, y_0)\,\mathrm{d}t, \tag{2.9}$$

$$y_n(x) := y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t. \tag{2.10}$$

We shall first prove that $\lim_{n\to+\infty} y_n(x) = y(x)$ exists as a continuous function of $x \in [x_0, x_0 + h]$, and afterwards we shall prove that $y(x)$ satisfies the differential equation (2.8). Finally, we shall prove that this solution is unique.

In the first step we prove by induction that if $x \in [x_0, x_0 + h]$, then $|y_n(x) - y_0| \le b$.
It follows from (2.9) that

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0)\,\mathrm{d}t\right| \le M\,(x - x_0) \le b, \tag{2.11}$$

which is the inequality for $n = 1$.

Then assume that for some $n \geq 2$,

$$|y_{n-1}(x) - y_0| \leq b.$$

Then $(x, y_{n-1}(x)) \in \omega$, so $|f(x, y_{n-1}(x))| \leq M$, and it follows from (2.10) that

$$|y_n(x) - y_0| \leq M\,(x - x_0) \leq Mh \leq b,$$

and the claim follows by induction for all $n \in \mathbb{N}$. In particular, if $x \in [x_0, x_0 + h]$, then

$$|f(x, y_n(x))| \leq M \qquad \text{for all } n \in \mathbb{N}.$$

Then by another induction we prove that if $x \in [x_0, x_0, h]$, then

$$(2.12) \qquad |y_n(x) - y_{n-1}(x)| \leq \frac{M\,C^{n-1}}{n!}\,(x - x_0)^n\,.$$

First note that we in (2.11) already have proved (2.12) for $n = 1$.

We assume that for some $n \geq 2$,

$$|y_{n-1}(x) - y_{n-2}(x)| \leq \frac{M\,C^{n-2}}{(n-1)!}\,(x - x_0)^{n-1}\,.$$

As mentioned already, this is true for $n = 2$. When we use (2.10), we get in the next step

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &= \left|\int_{x_0}^{x} \{f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))\}\,\mathrm{d}t\right| \\ &< C \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\,\mathrm{d}t < \frac{M\,C^{n-1}}{n!}\,(x - x_0)^n\,, \end{aligned}$$

where we have used the assumption.

Repeating the same argument for $x \in [x_0 - h, x_0]$, only with $|x - x_0|^n$ instead of $(x - x_0)^n$, we get the same estimates, so we have proved that

$$(2.13) \qquad |y_n(x) - y_{n-1}(x)| < \frac{M\,C^{n-1}\,|x - x_0|}{n!} \qquad \text{for } |x - x_0| \leq h \text{ and } n \in \mathbb{N}.$$

This implies that

$$\left| y_0 + \sum_{n=1}^{+\infty} \{y_n(x) - y_{n-1}(x)\} \right| \leq |y_0| + \frac{M}{C} \sum_{n=1}^{+\infty} \frac{1}{n!}(Ch)^n < +\infty,$$

so the series

$$y_0 + \sum_{n=1}^{+\infty} \{y_n(x) - y_{n-1}(x)\} = y_0 + \lim_{n \to \infty} \sum_{j=1}^{n} \{y_j(x) - y_{j-1}(x)\} = \lim_{n \to +\infty} y_n(x),$$

is absolutely and uniformly convergent in the interval $[x_0 - h, x_0 + h]$. Since each term is continuous, the sum function

$$(2.14) \qquad y(x) = \lim_{n \to +\infty} y_n(x) = y_0 + \lim_{n \to +\infty} \{y_j(x) - y_{j-1}(x)\}$$

is continuous.

We shall then prove that $y(x)$, defined by (2.14) is a solution of the differential equation (2.8).

First note that

$$\left|\int_{x_0}^{x}\{f(t,y(t))-f(t,y_{n-1}(t))\}\,\mathrm{d}t\right|<C\int_{x_0}^{x}|y(t)-y_{n-1}(t)|\,\mathrm{d}t<C\varepsilon_n\,|x-x_0|<C\varepsilon_n h,$$

where $\varepsilon_n>0$ is independent of $x\in[x_0-h,x_0+h]$ and tends towards zero for $n\to+\infty$. This implies that the limit and the integration can be interchanged, so

$$\lim_{n\to+\infty}\int_{x_0}^{x}f(t,y_{n-1}(t))\,\mathrm{d}t=\int_{x_0}^{x}\lim_{n\to+\infty}f(t,y_{n-1}(t))\,\mathrm{d}t=\int_{x_0}^{x}f(t,y(t))\,\mathrm{d}t,$$

from which follows that $y(t)$ satisfies the integral equation

$$y(x)=y_0+\int_{x_0}^{x}f(t,y(t))\,\mathrm{d}t.$$

Then clearly,

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\int_{x_0}^{x}f(t,y(t))\,\mathrm{d}t=f(x,y(x)),$$

and we have proved that $y(x)$ defined by (2.14) is a solution of (2.8), and it is obvious that $y(x_0)=y_0$.

We shall finally prove the uniqueness. Assume that $\tilde{y}(x)$ and $y(x)$ are two distinct solutions, where $\tilde{y}(x_0)=y(x_0)=y_0$. We assume that $\tilde{y}(x)$ is continuous for $x\in\left[x_0,x_0+\tilde{h}\right]$, where $\tilde{h}\le h$, and where $\tilde{h}$ is chosen, such that

$$|\tilde{y}(x)-y_0|<b\qquad\text{for } x\in\left[x_0,x_0+\tilde{h}\right].$$

Since also

$$\tilde{y}(x)=y_0+\int_{x_0}^{x}f(t,\tilde{y})\,\mathrm{d}t,$$

it follows that

$$\tilde{y}(x)-y_n(x)=\int_{x_0}^{x}\{f(t,\tilde{y}(t))-f(t,y_{n-1}(t))\}\,\mathrm{d}t.$$

Using induction once more, we prove as above that

$$|\tilde{y}(x)-y_n(x)|<\frac{C^n b\,(x-x_0)^n}{n!},$$

and therefore by taking the limit,

$$\tilde{y}(x)=\lim_{n\to+\infty}y_n(x)=y(x),$$

so $\tilde{y}(x)=y(x)$ for all $x\in\left[x_0,x_0+\tilde{h}\right]$. The argument is similar for $x\in[x_0-h^*,x_0]$ for some $h^*>0$. This proves the uniqueness in a neighbourhood of the point (x_0,y_0), and the theorem is proved. $\Diamond$

If $f(x,y)$ is of class C^1, then f in particular satisfies a Lipschitz condition, so we have the following

Corollary 2.1 *Let $f(x, y)$ be of class C^1 in an open domain $\Omega \subseteq \mathbb{R}^2$. The initial value problem*

$$\begin{cases} \dfrac{dy}{dx} + f(x, y) = 0, \\ y(x_0) = y_0, \qquad \text{where } (x_0, y_0) \in \Omega, \end{cases}$$

has one and only one solution contained in Ω.

Another version is

Corollary 2.2 *Let the functions $L(x, y)$ and $M(x, y)$ be of class C^1 in an open domain $\Omega \subseteq \mathbb{R}^2$. If*

$$(L(x_0, y_0), M(x_0, y_0)) \neq (0, 0),$$

then the differential equation

$$L(x, y)\, dx + M(x, y)\, dy = 0 \tag{2.15}$$

has a unique solution through the point $(x_0, y_0) \in \Omega$.

In the latter case, the only places where we cannot expect existence, or uniqueness, of solutions, are at points $(x, y) \in \Omega$, where

$$L(x, y) = 0 \qquad \text{and} \qquad M(x, y) = 0.$$

We shall in the following call these points for *singular points* of the equation (2.15). We shall see in the examples that the identification of such singular points helps a lot in the analysis.

2.5 Differential forms

The equation (2.3) does not allow solution curves of vertical slopes. A more general equation is given by the *differential form*

(2.16) $\quad L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0,$

where $L(x,y)$ and $M(x,y)$ form a pair of functions, which are of class C^1 in some domain. We note immediately that in domains, where also $M(x,y) \neq 0$, this equation (2.16) is equivalent to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{L(x,y)}{M(x,y)} = f(x,y),$$

i.e. to (2.5), so it is only when $M(x,y) = 0$ that we get something new. In the language of isoclines, the equation $M(x,y) = 0$ defines the isoclines, where the tangents of the solution curves are vertical. Similarly, $L(x,y) = 0$ is the equation of the (set of) isoclines, where the solution curves have horizontal tangents.

We also note that if $L(x,y)$ and $M(x,y)$ are of class C^1, and (x_0,y_0) is a point, for which both

$$L(x_0,y_0) = 0 \qquad \text{and} \qquad M(x_0,y_0) = 0,$$

then the assumptions of Theorem 2.1 are not met, so in such cases we do not have uniqueness at the point (x_0,y_0). We may have none, one or several solution curves – even infinitely many – passing through this point. Such points are in the present books called *singular points* of the underlying differential form, or equation. In case of a differential form it is a good solution strategy always first to look at the possible singular points, because in a neighbourhood of a singular point "something strange may occur".

In the case of (2.16) the solution curves are parametrized by some parameter t, so a particular solution has for some open interval I the structure

$$\{(f(t), g(t)) \mid t \in I\},$$

where

$$L(f(t),g(t))f'(t) + M(f(t),g(t))g'(t) = 0, \qquad \text{for all } t \in I.$$

The formulation (2.16) of course invites to a consideration of a differential form

$$\mathrm{d}F(x,y) = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y = 0,$$

in which case

$$L(x,y) = \frac{\partial F}{\partial x} \qquad \text{and} \qquad M(x,y) = \frac{\partial F}{\partial y}.$$

Whenever the pair $(L(x,y), M(x,y))$ has this structure, the total solution is obviously given by

$$F(x,y) = C, \qquad C \text{ an arbitrary constant.}$$

Unfortunately, the pair $(L(x,y), M(x,y))$ does not always have this structure, where the complete solution is immediately found. But whenever $(L(x,y), M(x,y) \neq (0,0)$, it defines a vector field which

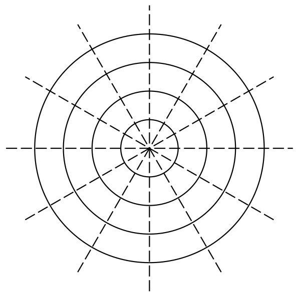


Figure 2.3: *The field $(L(x,y), M(x,y)) = (x,y)$ defines the dashed straight lines through the origin. Clearly, $(L,M) = (0,0)$ only at $(0,0)$, which is a singular point in this case. It is geometrically obvious that the solid circles $x^2+y^2=r^2$ are the only curves, which are perpendicular to the dashed straight lines. We conclude that the complete solution of $x\,dx + y\,dy = 0$ is given by the circles $x^2+y^2=r^2$. This result is also easily derived by the following method of separation on the variables. In this case none of the solution curves passes through $(0,0)$, while the perpendicular field of dashed lines, corresponding to the differential form $y\,dx - x\,dy = 0$ has all its solutions passing though the singular point.*

is always perpendicular to the solution curves of (2.16). This geometrical property may also be of some help in a first analysis, when we just sketch the structure of the solution curves.

If

$$\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x},$$

we call $L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y$ a *closed differential form*. A closed differential form defined in a *simply connected domain* ω is called an *exact differential form*. An exact differential form can always be integrated, i.e. there exists a function $F(x,y)$, such that

$$\mathrm{d}F(x,y) = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y,$$

so the equation $L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0$ is reduced to $\mathrm{d}F(x,y) = 0$, the total solution of which is $F(x,y) = C$, where C is an arbitrary constant.

Example 2.1 *We mention the well-known fact that there exist closed differential forms, which are not integrable. The classical example is*

$$\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy, \qquad \textit{for } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

The domain $\mathbb{R} \setminus \{(0,0)\}$ is not simply connected. But if we restrict a closed differential form to a *simply connected domain*, it becomes exact, so it can be integrated in this smaller domain.

If we multiply by x^2+y^2, we get

$$-y\left(x^2+y^2\right)\,\mathrm{d}x + x\left(x^2+y^2\right)\,\mathrm{d}y = 0, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

This differential form is defined in all of $\mathbb{R}^2$, but it clearly has a singular point in the sense above at $(0,0)$, so we have to remove this point in the solution process. And then we are back to a domain, which is not simply connected.

Let us assume that $x > 0$, i.e. we consider the open right halfplane, which is simply connected. Then

$$\frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left\{-\frac{y}{x^2}\,\mathrm{d}x + \frac{1}{x}\,\mathrm{d}y\right\} = \frac{\mathrm{d}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \mathrm{d}\arctan\left(\frac{y}{x}\right),$$

proving that the differential form is exact in the open right halfplane, because the differential form can be written as one single differential. Similarly in the open left halfplane.

If instead $y \neq 0$, we divide by y^2 in the numerators and denominators and obtain that the differential form is exact in the open upper halfplane as well as in the open lower halfplane. Pairwise these halfplanes are either disjoint, or overlap in an open quadrant, in which the integrals only differ by a constant, so the integral can be extended. This can be done even for three of the halfplanes, so we have an integral in $\mathbb{R}^2$ without a closed halfline. But we cannot extend the integral over this halfline, because the integral here becomes discontinuous, and the differential form is *not* exact in $\mathbb{R}^2 \setminus \{(0,0)\}$. It is exact in every domain, which is obtained by removing a closed halfline from $(0,0)$ to ∞.

ALTERNATIVELY, we may use *polar coordinates* instead,

$$x = r\cos\theta, \qquad \mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta,$$

$$y = r\sin\theta, \qquad \mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta.$$

Then

$$\begin{aligned}\frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y &= -\frac{r\sin\theta}{r^2}(\cos\theta\,\mathrm{d}r + r\sin\theta\,\mathrm{d}\theta) + \frac{r\cos\theta}{r^2}(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)\\ &= \left(-\frac{\sin\theta\cos\theta}{r} + \frac{\cos\theta\sin\theta}{r}\right)\mathrm{d}r + \left(\sin^2\theta + \cos^2\theta\right)\mathrm{d}\theta = \mathrm{d}\theta = 0,\end{aligned}$$

so we get by integration, $\theta = C$, where C is an arbitrary constant. ♢

Assume that $L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y$ is exact in some open domain ω. Then the integral $F(x,y)$ can be found by using the following standard method, which we shall explain.

We use the notation ∂x instead of $\mathrm{d}x$, when we consider the other variable y as a constant, not depending on x, i.e. a socalled partial integration with respect to x. Then clearly,

$$F(x,y) = \int L(x,y)\partial x + \varphi(y), \tag{2.17}$$

for some function $\varphi(y)$, which does not depend on x. Similarly,

$$F(x,y) = \int M(x,y)\partial y + \psi(x),$$

where the task is either to find $\varphi(y)$ or $\psi(x)$.

If we substitute (2.17) in the equation

$$\frac{\partial F}{\partial y} = M(x,y),$$

we obtain

$$\frac{\partial}{\partial y}\left\{\int L(x,y)\partial x+\varphi(y)\right\}=M(x,y),$$

hence by a rearrangement,

$$\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y}=M(x,y)-\frac{\partial}{\partial y}\int L(x,y)\partial x.$$

Since the left hand side only depends on y, we conclude that

$$\frac{\partial}{\partial x}\left\{M(x,y)-\frac{\partial}{\partial y}\int L(x,y)\partial x\right\}=0.$$

Hence,

$$\varphi(y)=\int\left\{M(x,y)-\frac{\partial}{\partial y}\int L(x,y)\partial x\right\}\mathrm{d}y,$$

where the integration with respect to y is no longer partial.

Similarly,

$$\psi(x)=\int\left\{L(x,y)-\frac{\partial}{\partial x}\int M(x,y)\partial y\right\}\mathrm{d}x.$$

With some skill it is easier alternatively to apply the rules of calculation of differential forms,

$$\mathrm{d}(F \pm G) = \mathrm{d}F \pm \mathrm{d}G,$$

$$\mathrm{d}(F \cdot G) = G\,\mathrm{d}F + F\,\mathrm{d}G,$$

$$\mathrm{d}\left(\frac{F}{G}\right) = \frac{1}{G}\,\mathrm{d}F - \frac{F}{G^2}\,\mathrm{d}G, \qquad G(x,y) \neq 0,$$

read from the right to the left. The strategy is always to pair terms which look similar to each other and then apply one of the rules above.

2.6 Examples of exact differential forms

Example 2.2 *Find the complete solution of the equation*

$$\left(3x^2 - 8xy + 6y^2\right)\,dx + \left(12xy - 4x^2 - 6y^2\right)\,dy = 0.$$

The shortcut here is to skip the proof of the differential form being exact, and instead try directly to reduce the given differential form by using the rules above from the right to the left. First we separate all terms and then start pairing them. It will be convenient to put the zero to the left.

$$\begin{aligned}
0 &= \left(3x^2 - 8xy + 6y^2\right)\,\mathrm{d}x + \left(12xy - 4x^2 - 6y^2\right)\,\mathrm{d}y \\
&= 3x^2\,\mathrm{d}x - 4y \cdot 2x\,\mathrm{d}x + 6y^2\,\mathrm{d}x + 6x \cdot 2y\,\mathrm{d}y - 4x^2\,\mathrm{d}y - 6y^2\,\mathrm{d}y \\
&= \mathrm{d}\left(x^3\right) - 4y\,\mathrm{d}\left(x^2\right) + 6y^2\,\mathrm{d}x + 6x\,\mathrm{d}\left(y^2\right) - 4x^2\,\mathrm{d}y - 2\,\mathrm{d}\left(y^3\right) \\
&= d\left(x^3\right) - 2\,\mathrm{d}\left(y^3\right) - 4\left\{y\,\mathrm{d}\left(x^2\right) + x^2\,\mathrm{d}y\right\} + 6\left\{y^2\,\mathrm{d}x + x\,\mathrm{d}\left(y^2\right)\right\} \\
&= d\left(x^3 - 2y^3\right) - 4\,\mathrm{d}\left(x^2y\right) + 6\,\mathrm{d}\left(xy^2\right) \\
&= \mathrm{d}\left(x^3 - 4x^2y + 6xy^2 - 2y^3\right),
\end{aligned}$$

so the total solution is implicitly given by

$$x^3 - 4x^2y + 6xy^2 - 2y^3 + C = 0,$$

where C is an arbitrary constant.

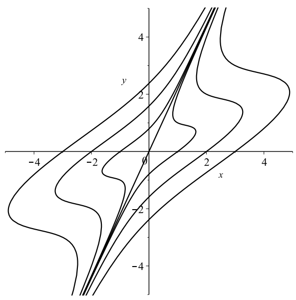


Figure 2.4: *Some solution curves of the equation* $\left(3x^2 - 8xy + 6y^2\right)\,dx + \left(12xy - 4x^2 - 6y^2\right)\,dy = 0$.

Alternatively, we write $L(x,y) = 3x^2 - 8xy + 6y^2$ and $M(x,y) = 12xy - 4x^2 - 6y^2$. Since

$$\frac{\partial L}{\partial y} = -8x + 12y = \frac{\partial M}{\partial x}, \qquad \text{in } \mathbb{R}^2,$$

the differential form is closed in the simply connected domain $\mathbb{R}^2$, hence also exact.

By the formulæ above from the standard procedure we first get

$$F(x,y) = \int L(x,y)\partial x + \varphi(y) = \int \left(3x^2 - 8xy + 6y^2\right)\partial x + \varphi(y) = x^3 - 4x^2y + 6xy^2 + \varphi(y).$$

Then

$$\begin{aligned}\varphi(y) &= \int \left\{ M(x,y) - \frac{\partial}{\partial y}\int L(x,y)\partial x \right\} \mathrm{d}y = \int \left\{ 12xy - 4x^2 - 6y^2 - \frac{\partial}{\partial y}\left(x^2 - 4x^2y + 6xy^2\right) \right\} \mathrm{d}y \\ &= \int \left\{12xy - 4x^2 - 6y^2 + 4x^2 - 12xy\right\} \mathrm{d}y = -\int 6y^2\,\mathrm{d}y = -2y^3 + C.\end{aligned}$$

Finally, by insertion,

$$F(x,y) = x^3 - 4x^2y + 6xy^2 - 2y^3 + C = 0.$$

It is left to the reader to prove that $(0,0)$ is the only singular point of the differential equation.

It should here be noted, that the equation is also homogeneous of degree 2, so it can also be solved by using the methods of Chapter 7. $\Diamond$

Example 2.3 *Find the complete solution of the differential equation*

$$\left(x^3y^4 + xy^2\right)\,dx + \left(x^4y^3 + x^2y\right)\,dy = 0.$$

It is trivial that $x = 0$ and $y = 0$ are both solutions. Since these two lines intersect, their intersection point $(0,0)$ must be a singular point. It is easy to prove – left to the reader – that it is the only singular point.

We guess that the differential form is exact. We can prove this by showing that it is closed in a simply connected domain, so we shall just show that $\dfrac{\partial L}{\partial y} = \dfrac{\partial M}{\partial x}$. We may also say, which we shall do here, that the terms can be paired, so the differential form is equivalent to $\mathrm{d}f(x,y) = 0$, by using some manipulation. This is equally good, but requires some skill.

The strategy is to collect terms of the same degree and then use the rule of differentiation of a product in the opposite direction of the usual one. In the present case we rewrite the equation in the following way,

$$\begin{aligned}0 &= \left(x^3y^4\,\mathrm{d}x + x^4y^3\,\mathrm{d}y\right) + \left(xy^2\,\mathrm{d}x + x^2y\,\mathrm{d}y\right) \\ &= \frac{1}{4}\left\{y^4\,\mathrm{d}\left(x^4\right) + x^4\,\mathrm{d}\left(y^4\right)\right\} + \frac{1}{2}\left\{y^2\,\mathrm{d}\left(x^2\right) + x^2\,\mathrm{d}\left(y^2\right)\right\} = \mathrm{d}\left(\frac{1}{4}\,x^4y^4 + \frac{1}{2}\,x^2y^2\right).\end{aligned}$$

Then by integration,

$$\frac{1}{4}\,x^4y^4 + \frac{1}{2}\,x^2y^2 = \frac{1}{4}\left\{x^4y^4 + 2x^2y^2 + 1\right\} - \frac{1}{4} = \frac{1}{4}\left(x^2y^2 + 1\right)^2 - \frac{1}{4} = \frac{1}{4}(C-1),$$

where $C = c^2 \geq 1$ is an arbitrary constant. Finally, by a rearrangement,

$$\left(x^2y^2 + 1\right)^2 = C = c^2, \qquad \text{where } c \geq 1 \text{ is an arbitrary constant,}$$

i.e.

$$x^2y^2 = c - 1 \geq 0 \qquad \text{for } c \geq 1.$$

If $c = 1$, then either $x = 0$ or $y = 0$, already found above.

If $c > 1$, then $x \neq 0$ and $y \neq 0$, and the solutions are given by

$$y = \pm\frac{\sqrt{c-1}}{x} = \frac{a}{x} \qquad \text{for } a \neq 0.$$

The complete solution is obtained by adding the axes, $x = 0$ and $y = 0$.

CHECK. If $y = \dfrac{a}{x}$, then $\mathrm{d}y = -\dfrac{a}{x^2}\,\mathrm{d}x$, and we get by insertion,

$$\begin{aligned}
&\left(x^3y^4 + xy^2\right)\,\mathrm{d}x + \left(x^4y^3 + x^2y\right)\,\mathrm{d}y \\
&\quad = \left(x^3 \cdot \frac{a^4}{x^4} + x \cdot \frac{a^2}{x^2}\right)\,\mathrm{d}x + \left(x^4 \cdot \frac{a^3}{x^3} + x^2 \cdot \frac{a}{x}\right) \cdot \left(-\frac{a}{x^2}\right)\,\mathrm{d}x \\
&\quad = \left(\frac{a^4}{x} + \frac{a^2}{x}\right)\,\mathrm{d}x - \left(\frac{a^4}{x} + \frac{a^2}{x}\right)\,\mathrm{d}x = 0.
\end{aligned}$$

It follows that every $y = \dfrac{a}{x}$, $x \neq 0$, is indeed a solution ◊

Example 2.4 *Prove that the left hand side of the differential equation*

$$y(2x + y + 1)\,dx + \left(x^2 + 2xy + 4y + x - 2\right)\,dy = 0$$

is exact, and find the complete solution.

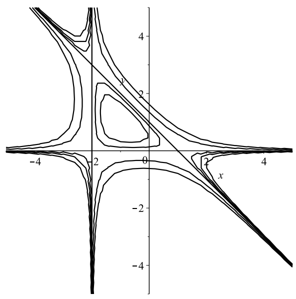


Figure 2.5: *Some solution curves of the equation* $y(2x + y + 1)\,dx + \left(x^2 + 2xy + 4y + x - 2\right)\,dy = 0$. *The* implicitplot *command of MAPLE has been used, which explains why the figure is slightly shaken.*

This is only a matter of manipulation, where we use the rules of the differential,

$$\mathrm{d}u \pm \mathrm{d}v = \mathrm{d}(u \pm v), \qquad \text{and} \qquad v\,\mathrm{d}u + u\,\mathrm{d}v = \mathrm{d}(u \cdot v),$$

and where we pair two similar terms into one differential. Starting with the right hand side of the equation we get

$$\begin{aligned} 0 &= y(2x+y+1)\,\mathrm{d}x + \left(x^2+2xy+4y+x-2\right)\,\mathrm{d}y \\ &= y\,\mathrm{d}\left(x^2\right) + \left(y^2+y\right)\,\mathrm{d}x + x^2\,\mathrm{d}y + x\,\mathrm{d}\left(y^2\right) + 2\,\mathrm{d}\left(y^2\right) + x\,\mathrm{d}y - 2\,\mathrm{d}y \\ &= \mathrm{d}\left(yx^2\right) + \mathrm{d}\left(xy^2\right) + \mathrm{d}(xy) + \mathrm{d}\left(2y^2-2y\right) = \mathrm{d}\left(xy^2+2y^2+x^2y+xy-2y\right) \\ &= \mathrm{d}\left(y\left\{xy+2y+x^2+x-2\right\}\right) = \mathrm{d}(y(x+2)(y+x-1)). \end{aligned}$$

Clearly, $y=0$, $x=-2$ and $y+x-1=0$ are rectilinear solutions. The remaining solutions are given by

$$y(x+2)(y+x-1)=C, \qquad \text{where } C \in \mathbb{R} \text{ is an arbitrary constant.}$$

Note that we get for $C=0$ the rectilinear solutions.

The singular points are the solutions of the equations

$$y(2x+y+1)=0 \qquad \text{and} \qquad x^2+2xy+4y+x-2=0.$$

One possibility is $y=0$, which implies that $x^2+x-2=0$, so either $x=1$ or $x=-2$, and $(1,0)$ and $(-2,0)$ are singular point.

Assume that $y \neq 0$. Then $y=-2x-1$, thus by insertion,

$$0 = x^2-4x^2-2x-8x-4+x-2 = -3x^2-9x-6 = -3\left(x^2+3x+2\right) = -3(x+1)(x+2).$$

If $x=-2$, then $y=4-1=3$, so $(-2,3)$ is a singular point.
If $x=-1$, then $y=2-1=1$, so $(-1,1)$ is a singular point.
The singular points are

$$(1,0), \qquad (-2,0), \qquad (-1,1) \quad \text{and} \quad (-2,3). \qquad \Diamond$$

Example 2.5 *Prove that the differential form*

$$\left(6x^2+3y^2-4x-2\right)\,dx+(6xy-6y)\,dy$$

is exact, and then find the complete solution of the differential equation

$$\left(6x^2+3y^2-4x-2\right)\,dx+(6xy-6y)\,dy=0.$$

The possible singular points are the solutions of the two equations

$$6x^2+3y^2-4x-2=0 \qquad \text{and} \qquad 6xy-6y=6y(x-1)=0,$$

so either $x=1$, in which case

$$0=6+3y^2-4-2=3y^2, \qquad \text{i.e.} \qquad y=0,$$

or $y=0$, in which case

$$0=6x^2-4x-2=2\left(3x^2-2x-1\right)=2(x-1)(3x+1).$$

We conclude that there are two singular points,

$$(1,0) \qquad \text{and} \qquad \left(-\frac{1}{3},0\right).$$

Using the hint that the differential form is exact we go straight to the manipulation of the differential form, using the rules of the differential,

$$\begin{aligned}
0 &= \left(6x^2+3y^2-4x-2\right)\,\mathrm{d}x+(6xy-6y)\,\mathrm{d}y \\
&= 6x^2\,\mathrm{d}x+3y^2\,\mathrm{d}x-4x\,\mathrm{d}x-2\,\mathrm{d}x+6xy\,\mathrm{d}y-6y\,\mathrm{d}y \\
&= 2\,\mathrm{d}\left(x^3\right)-2\,\mathrm{d}\left(x^2\right)-2\,\mathrm{d}x+3\left(y^2\,\mathrm{d}x+2xy\,\mathrm{d}y\right)-3\,\mathrm{d}\left(y^2\right) \\
&= \mathrm{d}\left(2x^3-2x^2-2x+3xy^2-3y^2\right),
\end{aligned}$$

from which we get the complete solution by integration,

$$2x^3-2x^2-2x+3xy^2-3y^2=C, \qquad \text{where } C\in\mathbb{R} \text{ is an arbitrary constant.}$$

If we change the constant C by adding 2 to both sides it is not hard to prove that the general solution can also be written

$$(x-1)\left(2x^2+3y^2-2\right)=C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

If we choose $C=0$, we see that the vertical line $x=1$ is a rectilinear solution, and the ellipse $2x^2+3y^2=2$ is another solution. $\Diamond$

Example 2.6 *Find the complete solution of the differential equation*

$$\left(x-y^2\right)\,dx-2xy\,dy=0.$$

The only singular point is $(0,0)$, and $x=0$ is trivially a solution. The differential form is exact, because

$$0=\left(x-y^2\right)\,\mathrm{d}x-2xy\,\mathrm{d}y=x\,\mathrm{d}x-y^2\,\mathrm{d}x-x\,\mathrm{d}\left(y^2\right)=\frac{1}{2}\,\mathrm{d}\left(x^2\right)-\,\mathrm{d}\left(xy^2\right)=\frac{1}{2}\,\mathrm{d}\left(x^2-2xy^2\right),$$

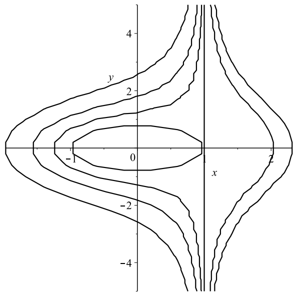


Figure 2.6: *Some solution curves of the equation* $\left(6x^2+3y^2-4x-2\right)\,dx+(6xy-6y)\,dy=0$. *The* implicitplot *command of MAPLE has been used, which explains why the figure is slightly shaken.*

proving that it is exact. Then by integration,

$$x^2-2xy^2=c, \qquad \text{or} \qquad y=\pm\sqrt{\frac{x^2-c}{2x}} \quad \text{for } x\neq 0,$$

where we must assume that $x\left(x^2-c\right)>0$.

If $c=0$, then either $x=0$ or $x=2y^2$. $\Diamond$

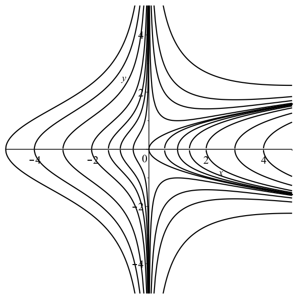


Figure 2.7: *Some solution curves of the equation* $\left(x-y^2\right)\,dx-2xy\,dy=0$.

Example 2.7 *Prove that the differential form*

$$\left(2xy^2+2x\right)\,dx+\left(2x^2y+3y^2\right)\,dy$$

is exact, and find the complete solution of the differential equation

$$\left(2xy^2+2x\right)\,dx+\left(2x^2y+3y^2\right)\,dy=0.$$

The possible singular points are the solution of the two equations

$$2xy^2+2x=2x\left(y^2+1\right)=0 \qquad \text{and} \qquad 2x^2y+3y^2=y\left(2x^2+3y\right)=0.$$

The first equation gives $x=0$, which inserted into the second one gives $y=0$, so $(0,0)$ is the only singular point.

We get by some simple manipulations,

$$\begin{aligned}&\left(2xy^2+2x\right)\,\mathrm{d}x+\left(2x^2y+3y^2\right)\,\mathrm{d}y=y^2\,\mathrm{d}\left(x^2\right)+\,\mathrm{d}\left(x^2\right)+x^2\,\mathrm{d}\left(y^2\right)+\,\mathrm{d}\left(y^3\right)\\&\quad=\,\mathrm{d}\left(x^2y^2\right)+\,\mathrm{d}\left(x^2\right)+\,\mathrm{d}\left(y^3\right)=\,\mathrm{d}\left(x^2y^2+x^2+y^3\right),\end{aligned}$$

proving that the differential form is exact with the integral $f(x,y)=x^2y^2+y^3+x^2=x^2\left(y^2+1\right)+y^3$. The complete solution of the differential equation is

$$x^2y^2+y^3+x^2=x^2\left(y^2+1\right)+y^3=c, \qquad \text{where } c \text{ is an arbitrary constant.} \qquad \diamond$$

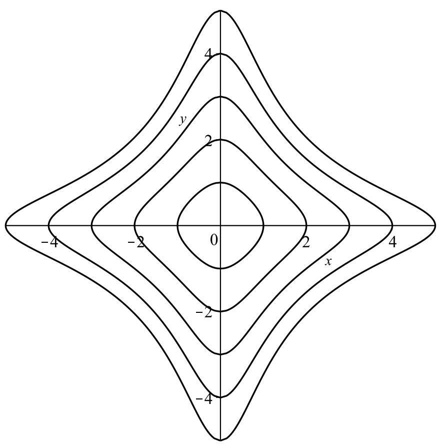


Figure 2.8: *Some solution curves of the equation*

$$\left(2xy^2+2x\right)\,\mathrm{d}x+\left(2x^2y+3y^2\right)\,\mathrm{d}y=0.$$

Example 2.8 *Prove that the differential form*

$$\left(\frac{-2x}{1-x^2+y^2}+2x\right)\,dx+\frac{2y}{1-x^2+y^2}\,dy,\qquad \text{for } x^2-y^2<1,$$

is exact, and find the complete solution of the differential equation

$$\left(\frac{-2x}{1-x^2+y^2}+2x\right)\,dx+\frac{2y}{1-x^2+y^2}\,dy=0,\qquad \text{for } x^2-y^2<1.$$

It follows from an easy manipulation that

$$\begin{aligned}\left(\frac{-2x}{1-x^2+y^2}+2x\right)\,\mathrm{d}x+\frac{2y}{1-x^2+y^2}\,\mathrm{d}y &= -\frac{\mathrm{d}\left(x^2\right)}{1-x^2+y^2}+\mathrm{d}\left(x^2\right)+\frac{\mathrm{d}\left(y^2\right)}{1-x^2+y^2}\\ &= \frac{\mathrm{d}\left(1-x^2+y^2\right)}{1-x^2+y^2}+\mathrm{d}\left(x^2\right)=\mathrm{d}\left(x^2+\ln\left(1-x^2+y^2\right)\right),\end{aligned}$$

so the differential form is exact, and the complete solution of the differential equation is

$$x^2+\ln\left(1-x^2+y^2\right)=c,\qquad \text{where } c \text{ is an arbitrary constant.}\qquad \diamond$$

Example 2.9 *Prove that the differential form*

$$\frac{x-x^3-xy^2}{1+x^2+y^2}\,dx-\frac{2x^2y+2y^3}{1+x^2+y^2}\,dy$$

is exact, and find its integral.

The denominators are never 0, so the possible singular points are the solutions of the two equations

$$x\left(1-x^2-y^2\right)=0, \qquad \text{and} \qquad 2y\left(x^2+y^2\right)=0.$$

If $x=0$, then $y=0$, so $(0,0)$ is a singular point. If $1-x^2-y^2=0$, i.e. $x^2+y^2=1$, then the second equation implies that $y=0$, which again implies that $x=\pm 1$. We conclude that we have the three singular points

$$(-1,0), \qquad (0,0), \qquad (1,0),$$

We shall prove that the differential form is exact. Instead of using the standard procedure we apply the rules of computation of differential forms. We note as above that the numerators in both cases can be factorized. By some simple manipulations,

$$\begin{aligned}
&\frac{x-x^3-xy^2}{1+x^2+y^2}\,\mathrm{d}x-\frac{2x^2y+2y^3}{1+x^2+y^2}\,\mathrm{d}y = x\,\frac{1-x^2-y^2}{1+x^2+y^2}\,\mathrm{d}x-2y\,\frac{x^2+y^2}{1+x^2+y^2}\,\mathrm{d}y\\
&\quad=\frac{1}{2}\,\frac{2-\left(1+x^2+y^2\right)}{1+x^2+y^2}\,\mathrm{d}\left(x^2\right)-\frac{1+x^2+y^2-1}{1+x^2+y^2}\,\mathrm{d}\left(y^2\right)\\
&\quad=\frac{\mathrm{d}\left(x^2\right)}{1+x^2+y^2}-\frac{1}{2}\,\mathrm{d}\left(x^2\right)-\mathrm{d}\left(y^2\right)+\frac{\mathrm{d}\left(y^2\right)}{1+x^2+y^2}=\mathrm{d}\left(\ln\left(1+x^2+y^2\right)-\frac{1}{2}\,x^2-y^2\right),
\end{aligned}$$

so the differential form is exact, and its integral is

$$f(x,y)=\ln\left(1+x^2+y^2\right)-\frac{1}{2}\,x^2-y^2. \qquad \Diamond$$

Example 2.10 *Prove that*

$$\frac{y+y^3-x^2y}{\left(1+x^2+y^2\right)^2}\,dx+\frac{x+x^3-xy^2}{\left(1+x^2+y^2\right)^2}\,dy$$

is exact, and then find the complete solution of the differential equation

$$\frac{y+y^3-x^2y}{\left(1+x^2+y^2\right)^2}\,dx+\frac{x+x^3-xy^2}{\left(1+x^2+y^2\right)^2}\,dy=0.$$

One may of course prove that $\dfrac{\partial L}{\partial y}=\dfrac{\partial M}{\partial x}$, but that task does not look nice. Instead we try some manipulation, pairing "similar terms" and then reducing. If the equation in this way can be transformed into $\mathrm{d}(\cdots)=0$, then we conclude that the differential form is indeed exact. In the particular case we get

$$\begin{aligned}
0 &= \frac{y+y^3-x^2y}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}x+\frac{x+x^3-xy^2}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}y = y\,\frac{1+y^2-x^2}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}x+x\,\frac{1+x^2-y^2}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}y\\
&= \frac{y\,\mathrm{d}x+x\,\mathrm{d}y}{1+x^2+y^2}-xy\,\frac{2x\,\mathrm{d}x+2y\,\mathrm{d}y}{\left(1+x^2+y^2\right)^2}=\frac{\mathrm{d}(xy)}{1+x^2+y^2}-xy\,\frac{\mathrm{d}\left(1+x^2+y^2\right)}{\left(1+x^2+y^2\right)^2}\\
&= \frac{\mathrm{d}(xy)}{1+x^2+y^2}+xy\,\mathrm{d}\left(\frac{1}{1+x^2+y^2}\right)=\mathrm{d}\left(\frac{xy}{1+x^2+y^2}\right),
\end{aligned}$$

so the differential form is exact, and the complete solution is given by

$$\frac{xy}{1+x^2+y^2} = C, \qquad \text{or} \qquad C\left(1+x^2+y^2\right) = xy.$$

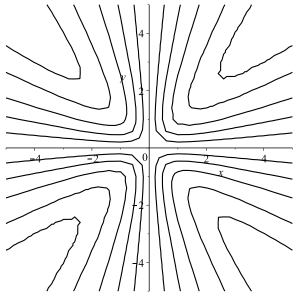


Figure 2.9: *Some solution curves of the equation*

$$\frac{y+y^3-x^2y}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}x + \frac{x+x^3-xy^2}{\left(1+x^2+y^2\right)^2}\,\mathrm{d}y = 0.$$

If $C = 0$, we get the two axes. If $C \neq 0$, then we get in polar coordinates,

$$1+r^2 = \frac{1}{C}\,r^2\,\cos\theta\sin\theta = \frac{1}{2C}\,r^2\sin 2\theta,$$

hence the solution is in polar coordinates given by

$$r = \sqrt{\frac{2C}{\sin 2\theta - 2C}},$$

whenever the radicand is > 0. $\diamondsuit$

Example 2.11 *Prove that the differential form*

$$\left(-\frac{3}{4}+\frac{2x}{1-x^2-y^2}\right)\,dx + \left(y+\frac{2y}{1-x^2-y^2}\right)\,dy, \qquad \textit{for } x^2+y^2<1,$$

is exact in the open unit disc, and find the complete solution of the differential equation

$$\left(-\frac{3}{4}+\frac{2x}{1-x^2-y^2}\right)\,dx + \left(y+\frac{2y}{1-x^2-y^2}\right)\,dy = 0 \qquad \textit{for } x^2+y^2<1.$$

The hint is that the differential form is exact, so we go straight to the manipulations,

$$\begin{aligned} 0 &= \left(-\frac{3}{4}+\frac{2x}{1-x^2-y^2}\right)\,\mathrm{d}x + \left(y+\frac{2y}{1-x^2-y^2}\right)\,\mathrm{d}y \\ &= -\frac{3}{4}\,\mathrm{d}x + \frac{1}{2}\,\mathrm{d}\!\left(y^2\right) + \frac{\mathrm{d}\!\left(x^2+y^2\right)}{1-x^2-y^2} = \mathrm{d}\!\left(-\frac{3}{4}\,x+\frac{1}{2}\,y^2+\ln\!\left(1-x^2-y^2\right)\right), \end{aligned}$$

so the differential form is exact, and the complete solution is obtained by an integration,

$$-\frac{3}{4}\,x+\frac{1}{2}\,y^2+\ln\left(1-x^2-y^2\right)=C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

It can be proved that the equation has the singular point $\left(\frac{1}{9},0\right)$, but the solution curves are hard to sketch, so we shall not bring them here. ♢

Example 2.12 *Prove that the differential form*

$$\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\,dx+\frac{xy}{\sqrt{x^2+y^2}}\,dy \qquad \textit{for } (x,y)\neq(0,0),$$

is exact, and find the complete solution of the differential equation

$$\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\,dx+\frac{xy}{\sqrt{x^2+y^2}}\,dy=0 \qquad \textit{for } x\neq 0.$$

Clearly, the differential equation does not have any singular point in the halfplane $x>0$. By using some simple manipulation,

$$\begin{aligned}\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{xy}{\sqrt{x^2+y^2}}\,\mathrm{d}y &= \frac{x^2+\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{xy}{\sqrt{x^2+y^2}}\,\mathrm{d}y\\ &= \left\{\frac{x^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}\right\}\mathrm{d}x+\frac{xy}{\sqrt{x^2+y^2}}\,\mathrm{d}y\\ &= \sqrt{x^2+y^2}\,\mathrm{d}x+x\left\{\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y\right\}\\ &= \sqrt{x^2+y^2}\,\mathrm{d}x+x\,\mathrm{d}\sqrt{x^2+y^2}=\,\mathrm{d}\left(x\sqrt{x^2+y^2}\right),\end{aligned}$$

so the differential form is exact in $\mathbb{R}\setminus(0,0)$. Its integral is $f(x,y)=x\sqrt{x^2+y^2}$, and the complete solution of differential equation is $x\sqrt{x^2+y^2}=c$, which can also be written

$$y=\pm\frac{\sqrt{c^2-x^4}}{x}, \qquad \text{where } 0<|x|\leq\sqrt{|c|}, \text{ and } c \text{ an arbitrary constant.}$$

If $c=0$, we get the solution $x=0$.

The equation is homogeneous of degree 1, so it can also be solved by the methods given in Chapter 7. ♢

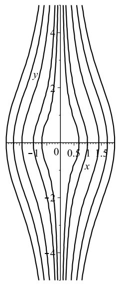


Figure 2.10: *Some solution curves of the equation*

$$\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\,\mathrm{d}x+\frac{xy}{\sqrt{x^2+y^2}}\,\mathrm{d}y=0.$$

Example 2.13 *Prove that the differential form*

$$\left(\frac{2x}{\sqrt{x^2+y^2}}-\frac{y}{x^2+y^2}\right)dx+\left(\frac{2y}{\sqrt{x^2+y^2}}+\frac{x}{x^2+y^2}\right)dy,\qquad \textit{for } x>0,$$

is exact, and find its integral.
What can be said about the complete solution of the corresponding differential equation

$$\left(\frac{2x}{\sqrt{x^2+y^2}}-\frac{y}{x^2+y^2}\right)dx+\left(\frac{2y}{\sqrt{x^2+y^2}}+\frac{x}{x^2+y^2}\right)dy=0,\qquad \textit{for } (x,y)\in\mathbb{R}\setminus\{(0,0)\}?$$

By pairing terms which look alike we get by some manipulation,

$$\begin{aligned}
&\left(\frac{2x}{\sqrt{x^2+y^2}}-\frac{y}{x^2+y^2}\right)\mathrm{d}x+\left(\frac{2y}{\sqrt{x^2+y^2}}+\frac{x}{x^2+y^2}\right)\mathrm{d}y\\
&\quad=\left(\frac{2x\,\mathrm{d}x}{\sqrt{x^2+y^2}}+\frac{2y\,\mathrm{d}y}{\sqrt{x^2+y^2}}\right)-\frac{y\,\mathrm{d}x}{x^2+y^2}+\frac{x\,\mathrm{d}y}{x^2+y^2}\\
&\quad=\frac{\mathrm{d}\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}+\frac{1}{1+\left\{\frac{y}{x}\right\}^2}\left\{-\frac{y}{x^2}\,\mathrm{d}x+\frac{1}{x}\,\mathrm{d}y\right\}=\mathrm{d}\sqrt{x^2+y^2}+\frac{1}{1+\left\{\frac{y}{x}\right\}^2}\,\mathrm{d}\left(\frac{y}{x}\right)\\
&\quad=\mathrm{d}\left(\sqrt{x^2+y^2}+\arctan\left(\frac{y}{x}\right)\right),
\end{aligned}$$

so the differential form is exact for $x>0$, and its integral is given by

$$f(x,y)=\sqrt{x^2+y^2}+\arctan\left(\frac{y}{x}\right).$$

Note, however, that the differential form is *not* exact in $\mathbb{R}^2\setminus\{(0,0)\}$.

ALTERNATIVELY, we may use polar coordinates instead,

$$x=r\cos\theta,\qquad \mathrm{d}x=\cos\theta\,\mathrm{d}r-r\sin\theta\,\mathrm{d}\theta,$$

$$y=r\sin\theta,\qquad \mathrm{d}y=\sin\theta\,\mathrm{d}r+r\cos\theta\,\mathrm{d}\theta.$$

Then

$$\begin{aligned}
&\left(\frac{2x}{\sqrt{x^2+y^2}}-\frac{y}{x^2+y^2}\right)\mathrm{d}x+\left(\frac{2y}{\sqrt{x^2+y^2}}+\frac{x}{x^2+y^2}\right)\mathrm{d}y\\
&\quad=\left(2\cos\theta-\frac{\sin\theta}{r}\right)(\cos\theta\,\mathrm{d}r-r\sin\theta\,\mathrm{d}\theta)+\left(2\sin\theta+\frac{\cos\theta}{r}\right)(\sin\theta\,\mathrm{d}r+r\cos\theta\,\mathrm{d}\theta)\\
&\quad=\left(2\cos^2\theta-\frac{\sin\theta\cos\theta}{r}+2\sin^2\theta+\frac{\sin\theta\cos\theta}{r}\right)\mathrm{d}r\\
&\qquad\qquad+\left(-2r\sin\theta\cos\theta+\sin^2\theta+2r\sin\theta\cos\theta+\cos^2\theta\right)\mathrm{d}\theta\\
&\quad=2\,\mathrm{d}r+\mathrm{d}\theta=\mathrm{d}(2r+\theta).
\end{aligned}$$

The complete solution of the differential equation $\mathrm{d}(2r+\theta)=0$ is

$$2r+\theta=2C,\qquad\text{or}\qquad r=C-\frac{\theta}{2},\quad\text{for }\theta<2C.\qquad\Diamond$$

Example 2.14 *Prove that the differential form*

$$\left(x-\frac{y}{\sqrt{1-xy}}\right)dx+\left(y-\frac{x}{\sqrt{1-xy}}\right)dy,\qquad \textit{for } x\cdot y<1,$$

is exact, and find the complete solution of the differential equation

$$\left(x-\frac{y}{\sqrt{1-xy}}\right)dx+\left(y-\frac{x}{\sqrt{1-xy}}\right)dy=0,\qquad \textit{for } x\cdot y<1.$$

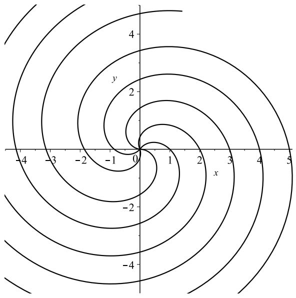


Figure 2.11: *Some solution curves of the equation*

$$\left(\frac{2x}{\sqrt{x^2+y^2}}-\frac{y}{x^2+y^2}\right)\mathrm{d}x+\left(\frac{2y}{\sqrt{x^2+y^2}}+\frac{x}{x^2+y^2}\right)\mathrm{d}y=0.$$

By using the usual technique, where we pair similar terms and apply the rules of the differential, we get straight away

$$\begin{aligned}\left(x-\frac{y}{\sqrt{1-xy}}\right)\mathrm{d}x+\left(y-\frac{x}{\sqrt{1-xy}}\right)\mathrm{d}y &= x\,\mathrm{d}x+y\,\mathrm{d}y-\frac{y\,\mathrm{d}x+x\,\mathrm{d}y}{\sqrt{1-xy}}\\ &= \mathrm{d}\left(\frac{1}{2}\,x^2+\frac{1}{2}\,y^2\right)+\frac{\mathrm{d}(1-xy)}{\sqrt{1-xy}}=\mathrm{d}\left(\frac{1}{2}\,x^2+\frac{1}{2}\,y^2+2\sqrt{1-xy}\right),\end{aligned}$$

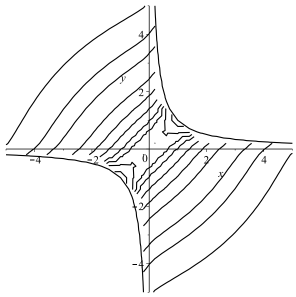


Figure 2.12: *Some solution curves of the equation*

$$\left(x-\frac{y}{\sqrt{1-xy}}\right)\,dx+\left(y-\frac{x}{\sqrt{1-xy}}\right)\,dy=0.$$

The implicitplot *command of MAPLE has been used, which explains why the figure is slightly shaken close to the origo* $(0,0)$. *All curves lie inside the domain given by* $xy<1$.

so the differential form is exact, and the complete solution of the corresponding differential equation is given by (multiply by 2)

$$x^2+y^2+4\sqrt{1-xy}=C \qquad \text{for } xy<1, \text{ where } xy<1. \qquad \Diamond$$

Example 2.15 *Prove that the differential form*

$$\left(\ln\left(1-x^2\right)-\frac{2x^2}{1-x^2}\right)\frac{y}{\sqrt{1-y^2}}\,dx+\frac{x\ln\left(1-x^2\right)}{\left(1-y^2\right)\sqrt{1-y^2}}\,dy$$

is exact in the open square $]-1,1[\times]-1,1[$*, and find its integral.*

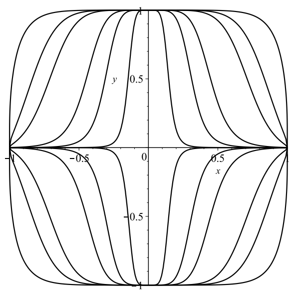


Figure 2.13: *Some solution curves of the equation*

$$\left(\ln\left(1-x^2\right)-\frac{2x^2}{1-x^2}\right)\frac{y}{\sqrt{1-y^2}}\,dx+\frac{x\ln\left(1-x^2\right)}{\left(1-y^2\right)\sqrt{1-y^2}}\,dy=0.$$

All curves lie inside the open square $]-1,1[\times]-1,1[$.

The idea is of course that one should check that $\dfrac{\partial L}{\partial y}=\dfrac{\partial M}{\partial x}$, which will cause a lot of calculations. It is, however, easier to use the rules of calculation of the differential forms. We first note that

$$\mathrm{d}\left(x\,\ln\left(1-x^2\right)\right)=\left(\ln\left(1-x^2\right)-\frac{2x^2}{1-x^2}\right)\mathrm{d}x,$$

and

$$\mathrm{d}\left(\frac{y}{\sqrt{1-y^2}}\right)=\left(\frac{1}{\sqrt{1-y^2}}-\frac{1}{2}\,\frac{-y\cdot 2y}{\left(1-y^2\right)\sqrt{1-y^2}}\right)\mathrm{d}y=\frac{\mathrm{d}y}{\left(1-y^2\right)\sqrt{1-y^2}},$$

hence, by insertion,

$$\begin{aligned}&\left(\ln\left(1-x^2\right)-\frac{2x^2}{1-x^2}\right)\frac{y}{\sqrt{1-y^2}}\,\mathrm{d}x+\frac{x\,\ln\left(1-x^2\right)}{\left(1-y^2\right)\sqrt{1-y^2}}\,\mathrm{d}y\\ =&\quad\frac{y}{\sqrt{1-y^2}}\,\mathrm{d}\left(x\,\ln\left(1-x^2\right)\right)+x\,\ln\left(1-x^2\right)\,\mathrm{d}\left(\frac{y}{\sqrt{1-y^2}}\right)=\mathrm{d}\left(\frac{xy\ln\left(1-x^2\right)}{\sqrt{1-y^2}}\right),\end{aligned}$$

so we conclude that the differential form is exact with the integral

$$f(x,y)=\frac{xy\ln\left(1-x^2\right)}{\sqrt{1-y^2}}.\qquad\Diamond$$

2.7 Integrating factors

Consider again the differential equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0,$$

given as a differential form, where $L(x,y)$ and $M(x,y)$ are C^1 functions. A C^1 function $f(x,y) \neq 0$ is called an *integrating factor*, if the left hand side of the equation above becomes a closed differential form, when it is multiplied by $f(x,y)$, i.e. if

$$f(x,y)\cdot L(x,y)\,\mathrm{d}x + f(x,y)\cdot M(x,y)\,\mathrm{d}y$$

is a closed differential form, hence also locally exact. If $L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y$ already is closed, then the integrating factor can be chosen as any constant $\neq 0$. Otherwise, we see that if $f(x,y)$ is an integrating function, then

$$\frac{\partial}{\partial y}\{f(x,y)L(x,y)\} = \frac{\partial}{\partial x}\{f(x,y)M(x,y)\},$$

which can also be written as the following linear partial differential equation of first order,

$$(2.18)\qquad M(x,y)\frac{\partial f}{\partial x} - L(x,y)\frac{\partial f}{\partial y} - f(x,y)\left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} = 0.$$

It is obvious that if the differential equation already is closed, then the third term of the left hand side of (2.18) is zero, and e.g. $f \equiv 1$ is trivially a solution, i.e. an integrating factor.

Assume that the differential form is not closed. How do we solve the equation (2.18) in order to get an integrating function? This problem is handled in the following way. We introduce a new C^1 function in three variables, $F(x, y, z) = f(x, y) - z$. The equation

$$F(x, y, z) = f(x, y) - z = 0$$

describes a solution surface, $z = f(x, y)$, of gradient

$$\bigtriangledown F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right).$$

Since $\bigtriangledown F \neq \mathbf{0}$, the gradient is always perpendicular to the surface $z = f(x, y)$ in $\mathbb{R}^3$. This means that we can write the differential equation in the following way,

$$\begin{aligned} 0 &= M(x, y)\frac{\partial f}{\partial x} - L(x, y)\frac{\partial f}{\partial y} - \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} f(x, y) \\ &= \bigtriangledown F(x, y, z) \cdot \left(M(x, y), -L(x, y), \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} f(x, y)\right), \end{aligned}$$

so the vector field

$$\left(M(x, y), -L(x, y), \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} f(x, y)\right)$$

can be interpreted as a tangent vector field on the unknown surface $z = f(x, y)$. It defines a curve $(x(t), y(t), z(t))$ on the surface, where

$$\text{(2.19)} \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \left(M(x, y), -L(x, y), \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} f(x, y)\right),$$

where t is some parameter. Hence, *if* we can solve the following system of three usually nonlinear ordinary differential equations

$$\text{(2.20)} \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= M(x, y), \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -L(x, y), \\[2ex] \dfrac{1}{z}\dfrac{\mathrm{d}z}{\mathrm{d}t} &= \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}, \end{cases}$$

and we can eliminate the parameter t, such that $z = f(x, y)$, then this solution is indeed an integrating factor.

Note the words *"if"* and *"and"*, because it is in general far from obvious that this can be done in practice.

2.8 The equation $\{y + xF(x^2 + y^2)\}\ \mathrm{d}x - \{x - yF(x^2 + y^2)\}\ \mathrm{d}y = 0$.

An integrating factor is for $(x, y) \neq (0, 0)$ given by $\mu = \dfrac{1}{x^2 + y^2}$. When we multiply the equation by μ, we get

$$0 = \frac{y + xF(x^2 + y^2)}{x^2 + y^2}\,\mathrm{d}x + \frac{-x + yF(x^2 + y^2)}{x^2 + y^2}\,\mathrm{d}y = \left\{\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2 + y^2}\right\} + \frac{1}{2}\,\frac{F(x^2 + y^2)}{x^2 + y^2}\,\mathrm{d}(x^2 + y^2).$$

The first term is closed, but not exact in $\mathbb{R}^2 \setminus \{(0,0)\}$. In the open upper halfplane $y > 0$, which is simply connected, we get by integration

$$\arctan\left(\frac{x}{y}\right) + \frac{1}{2}\int_{v=x^2+y^2} \frac{F(v)}{v}\,\mathrm{d}v = C, \qquad \text{for } y > 0.$$

2.9 The equation $yf(xy)\,\mathbf{d}x + xg(xy)\,\mathbf{d}y = 0$

Clearly, $x = 0$ and $y = 0$ are trivial solutions. We put $v = xy$. If $f(xy) = g(xy)$, the equation is trivial. Assume that $xy\{f(xy) - g(xy)\} \neq 0$. Then

$$\begin{aligned} 0 &= \frac{yf(xy)\,\mathrm{d}x + xg(xy)\,\mathrm{d}y}{xy\{f(xy) - g(xy)\}} = \frac{y\{f(xy) - g(xy)\}\,\mathrm{d}x + g(xy)\{y\,\mathrm{d}x + x\,\mathrm{d}y\}}{xy\{f(xy) - g(xy)\}} \\ &= \frac{\mathrm{d}x}{x} + \frac{g(xy)}{xy\{f(xy) - g(xy)\}}\,\mathrm{d}(xy) = \frac{\mathrm{d}x}{x} + \frac{g(v)}{v\{f(v) - g(v)\}}\,\mathrm{d}v, \end{aligned}$$

so it follows by integration that

$$\ln|x| + \int_{v=xy} \frac{g(v)}{v\{f(v) - g(v)\}}\,\mathrm{d}v = C.$$

2.10 Examples of integrating factors

Example 2.16 *To illustrate the theory above we consider the linear differential equation*

$$\frac{dy}{dx} + f(x)y + g(x) = 0.$$

We write this equation in the following slightly more general form,

$$\{f(x)y + g(x)\}\,\mathrm{d}x + \,\mathrm{d}y = 0,$$

so by identification,

$$L(x,y) = f(x)y + g(x) \qquad \text{and} \qquad M(x,y) = 1.$$

The system (2.20) then becomes

$$(2.21) \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = M(x,y) = 1, \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -L(x,y) = -f(x)y - g(x), \\[2mm] \dfrac{1}{z}\,\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x} = f(x), \end{cases}$$

so $x = t$ and $\dfrac{1}{z}\,\dfrac{\mathrm{d}z}{\mathrm{d}t} = f(t)$, from which by integration followed by the exponential, $z = \exp\left(\int f(t)\,\mathrm{d}t\right)$, which is the well-known integrating factor of the linear differential equation, so in this case the method was successful.

It is less successful in the given case, if the equation is not homogeneous, $g(x) \neq 0$, and we want to find the equipotential curves of the linear equation. The equation of the equipotential curves is then

$$-\,\mathrm{d}x + \{f(x)y + g(x)\}\,\mathrm{d}y = 0,$$

so $L(x,y) = -1$ and $M(x,y) = f(x)y + g(x)$. The system (2.20) is then

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &=& M(x,y) = f(x)y + g(x), \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} &=& -L(x,y) = 1, \\[2mm] \dfrac{1}{z}\,\dfrac{\mathrm{d}z}{\mathrm{d}t} &=& \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x} = -f'(x)y - g'(x), \end{cases}$$

from which we only get $y = t$, and we are left with the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &=& f(x)t + g(x), \\[2mm] \dfrac{1}{z}\,\dfrac{\mathrm{d}z}{\mathrm{d}t} &=& -f'(x)t - g'(x), \end{cases}$$

and we cannot get further. $\Diamond$

Example 2.17 *Consider the differential equation*

$$\left(y^2 - 4x^2e^{y^2} + 4\right)\,dx + \left(xy - 2x^3ye^{y^2}\right)\,dy = 0.$$

Prove that this equation has an integrating factor of the form $\varphi(x)$, $\varphi(1) = 1$, depending only on x, and then find the complete solution of the differential equation.

We first identify

$$L(x,y) = y^2 - 4x^2e^{y^2} + 4 \qquad \text{and} \qquad M(x,y) = xy - 2x^3ye^{y^2},$$

so

$$\frac{\partial}{\partial y}\{\varphi(x)\,L(x,y)\} = \varphi(x)\cdot\left\{2y - 8x^2ye^{y^2}\right\},$$

and

$$\frac{\partial}{\partial x}\{\varphi(x)\,M(x,y)\} = \varphi(x)\cdot\left\{y - 6x^2ye^{y^2}\right\} + \varphi'(x)\cdot\left\{xy - 2x^3ye^{y^2}\right\}.$$

These two expressions are equal, so by equating them, followed by a rearrangement we get

$$\varphi(x)\cdot\left\{2y - 8x^2ye^{y^2} - y + 6x^2ye^{y^2}\right\} = \varphi'(x)\cdot\left\{xy - 2x^3ye^{y^2}\right\},$$

i.e.

$$\varphi(x)\cdot\left\{y - 2x^2ye^{y^2}\right\} = \varphi'(x)\cdot x\cdot\left\{y - 2x^2ye^{y^2}\right\},$$

which can be reduced to

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{1}{x}, \quad \text{and} \quad \varphi(1) = 1, \qquad \text{from which } \varphi(x) = x.$$

When the differential equation is multiplied by x, followed by some manipulation, we find

$$\begin{aligned} 0 &= x\left(y^2 - 4x^2 e^{y^2} + 4\right) \mathrm{d}x + x\left(xy - 2x^3 y e^{y^2}\right) \mathrm{d}y \\ &= xy^2 \, \mathrm{d}x - e^{y^2} \, \mathrm{d}\left(x^4\right) + 4x \, \mathrm{d}x + x^2 y \, \mathrm{d}y - x^4 \, \mathrm{d}\left(e^{y^2}\right) \\ &= \mathrm{d}\left(\frac{1}{2} x^2 y^2\right) + \mathrm{d}\left(2x^2\right) - \mathrm{d}\left(x^4 e^{y^2}\right) = \mathrm{d}\left(\frac{1}{2} x^2 y^2 + 2x^2 - x^4 e^{y^2}\right). \end{aligned}$$

For convenience this equation is multiplied by 2. Then the complete solution is found by integration,

$$4x^2 + x^2 y^2 - 2x^4 e^{y^2} = x^2 \left\{4 + y^2 - 2x^2 e^{y^2}\right\} = C, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \diamond$$

Example 2.18 *Solve the differential equation*

$$3x^2(y-x)^2 \frac{dy}{dx} + \left\{\sin x - x \cos x - 3x^2 (y-x)^2\right\} = 0.$$

We first note that the equation can be written

$$3x^2(y-x)^2 \frac{\mathrm{d}}{\mathrm{d}x}(y-x) = x^2 \frac{\mathrm{d}}{\mathrm{d}x}(y-x)^3 = x \cos x - \sin x.$$

For $x = 0$ the equation is trivially fulfilled, but this vertical line is not a function in y. For $x \neq 0$ an obvious integrating factor is $\dfrac{1}{x^2}$, in which case we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(y-x)^3 = \frac{1}{x}\,\cos x - \frac{1}{x^2}\,\sin x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right).$$

Then by integration,

$$(y-x)^3 = C + \frac{\sin x}{x},$$

and the complete solution is given by

$$y = x + \sqrt[3]{C + \frac{\sin x}{x}}, \qquad C \text{ arbitrary constant.} \qquad \Diamond$$

Example 2.19 *Prove that* $\cos x\,\cos y$ *is an integrating factor for the differential equation*

$$\left(2x\tan y\sec x + y^2\sec y\right)\,dx + \left(2y\tan x\sec y + x^2\sec x\right)\,dy = 0,$$

and then find the complete solution of the equation.

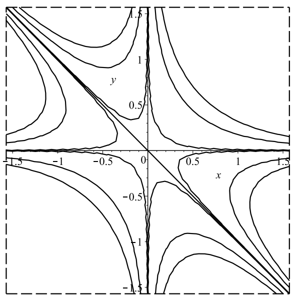


Figure 2.14: *The equation* $\left(2x\tan y\sec x + y^2\sec y\right)\,dx + \left(2y\tan x\sec y + x^2\sec x\right)\,dy = 0$*, in the square* $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\,\times\,\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$*, which is one of its many connected subdomains of definition.*

We recall that $\sec x = 1/\cos x$ for $x,\, y \neq \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. Clearly, $x = 0$ and $y = 0$ are both solutions. The easiest solution method is to multiply by the so-called integrating factor and then pair the terms as above to get a differential of the form $\mathrm{d}F(x,y)$ for some function $F(x,y)$. This function $F(x,y)$ is then the integral of the given differential form. We note that $\cos x \cdot \cos y \neq 0$ in the domains where the differential form is defined. We shall restrict ourselves to solve the equation in the connected domain $(x,y) \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. When we in this domain multiply the differential form by $\cos x \cdot \cos y \neq 0$, we get the equivalent equation

$$\begin{aligned}
0 &= \left(2x\,\sin y + y^2\,\cos x\right)\,\mathrm{d}x + \left(2y\,\sin x + x^2\,\cos y\right)\,\mathrm{d}y \\
&= \sin y\,\mathrm{d}\!\left(x^2\right) + y^2\,\mathrm{d}\sin x + \sin x\,\mathrm{d}\!\left(y^2\right) + x^2\,\mathrm{d}\sin y \\
&= \mathrm{d}\!\left(x^2\sin y\right) + \mathrm{d}\!\left(y^2\sin x\right) = \mathrm{d}\!\left(x^2\sin y + y^2\sin x\right),
\end{aligned}$$

so the differential form is exact, and the complete solution is given by

$$x^2\sin y + y^2\sin x = C, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \Diamond$$

Example 2.20 *Recall* Coulomb's law *(Charles A. de Coulomb, French engineer, 1736–1806). The electrostatic interaction between two charged particles is proportional to their charges and to the inverse of the square of the distance between them, and its direction is along the line joining the two charges. Thus,*

$$F = K \cdot \frac{q_1 q_2}{r^2},$$

where q_1 and q_2 are the two charges and r their distance from each other, while K is some constant depending on the chosen physical units.

Given a charge e_1 at the point $(-a, 0)$, and a charge e_2 at the point $(a, 0)$. Find the field of the electric force in the plane.

This means that we shall consider a unit charge 1 at point (x, y) and derive a differential equation of the curves of force.

We first note that the tangent at (x, y) of the curve of force passing through it has the same direction as the resulting electrostatic force.

Denote by

$$r_1 = \sqrt{(x+a)^2 + y^2}, \qquad r_2 = \sqrt{(x-a)^2 + y^2},$$

the distances of the point (x, y) to $(-a, 0)$, resp. $(a, 0)$. Then using that the tangent and the resulting electrostatic force have the same direction,

$$\left(\frac{e_1}{r_1^3} + \frac{e_2}{r_2^3}\right) y - \left(e_1 \frac{x+a}{r_1^3} + e_2 \frac{x-a}{r_2^3}\right) \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so by a rearrangement,

$$0 = \frac{e_1}{r_1^3}\left\{y - (x+a)\frac{\mathrm{d}y}{\mathrm{d}x}\right\} + \frac{e_2}{r_2^3}\left\{y - (x-a)\frac{\mathrm{d}y}{\mathrm{d}x}\right\}.$$

It is not hard to see that y is an integrating factor, and since the resulting differential form is exact, we can find its integral by integrating along a step curve. Here we shall ALTERNATIVELY just see that by a small calculation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+a}{r_1}\right) &= \frac{1}{r_1} - \frac{x+a}{r_1^3}\left\{y\frac{\mathrm{d}y}{\mathrm{d}x} + (x+a)\right\} = \frac{1}{r_1^3}\left\{(x+a)^2 + y^2 - y\frac{\mathrm{d}y}{\mathrm{d}x}(x+a) - (x+a)\right\} \\ &= \frac{1}{r_1^3}\, y\left\{y - (x+a)\frac{\mathrm{d}y}{\mathrm{d}x}\right\},\end{aligned}$$

and similarly for $\dfrac{x-a}{r_2}$, so when we multiply the original equation by y we get

$$\begin{aligned} 0 &= y\,\frac{e_1}{r_1^3}\left\{y - (x+a)\frac{\mathrm{d}y}{\mathrm{d}x}\right\} + y\,\frac{e_2}{r_2^3}\left\{y - (x-a)\frac{\mathrm{d}y}{\mathrm{d}x}\right\} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+a}{r_1}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-a}{r_2}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+a}{r_1} + \frac{x-a}{r_2}\right).\end{aligned}$$

Finally, the equation of the field of force curves is found by an integration,

$$c = \frac{x+a}{r_1} + \frac{x-a}{r_2} = \frac{x+a}{\sqrt{(x+a)^2 + y^2}} + \frac{x+a}{\sqrt{(x-a)^2 + y^2}}. \qquad \Diamond$$

2.11 Additional cases, where an integrating factor is known

We mention a couple of more rare cases, in which an integrating factor is known.

Theorem 2.2 *Consider the differential equation*

$$L(x,y)\,dx + M(x,y)\,dy = 0, \tag{2.22}$$

where $L(x,y)$ and $M(x,y)$ in some open region are functions of class C^2. If the expression

$$\frac{1}{M(x,y)}\left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} = g(x)$$

is a function $g(x)$ in x alone, then

$$\mu(x) = \exp\left(\int g(x)\,dx\right)$$

is an integrating factor of the differential equation (2.22). Here, μ can be multiplied by any constant $K \neq 0$.

Proof. The criterion of μ being an integrating factor is

$$\frac{\partial(\mu L)}{\partial y} = \frac{\partial(\mu M)}{\partial x}. \tag{2.23}$$

In the present case μ is assumed only to depend on x, thus (2.23) is reduced to

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{\mu(x)}{M(x,y)} \cdot \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} = \exp\left(\int g(x)\,\mathrm{d}x\right) \cdot g(x),$$

which obviously is true. □

Theorem 2.2 is then extended in the following way,

Theorem 2.3 *Consider the differential equation (2.22), i.e.*

$$L(x,y)\,dx + M(x,y)\,dy = 0,$$

where $L(x,y)$ and $M(x,y)$ are functions of class C^2 in some open region. If for some C^1 function $\varphi = \varphi(x,y)$ there exists another function $g = g(t)$, such that

$$\frac{\dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}}{M\dfrac{\partial \varphi}{\partial x} - L\dfrac{\partial \varphi}{\partial y}} = g(\varphi), \tag{2.24}$$

where we only require that the denominator of (2.24) is $\neq 0$, then

$$\mu(x,y) = \exp\left(\int_{t=\varphi(x,y)} g(t)\,dt\right)$$

is an integrating factor of the differential equation.

PROOF. Assuming that μ above is an integrating factor, equation (2.23) must hold. This condition is here reduced to

$$\mu \frac{\partial L}{\partial y} + L \frac{\mathrm{d}\mu}{\mathrm{d}\varphi} \frac{\partial \varphi}{\partial y} = \mu \frac{\partial M}{\partial x} + M \frac{\mathrm{d}\mu}{\mathrm{d}\varphi} \frac{\partial \varphi}{\partial x},$$

thus by a rearrangement,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\varphi} \left\{ L \frac{\partial \varphi}{\partial y} - M \frac{\partial \varphi}{\partial x} \right\} = \mu \cdot \left\{ \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right\},$$

hence by (2.24), followed by the usual solution formula,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\varphi} = \mu\, g(\varphi), \qquad \text{i.e.} \qquad \mu = K \cdot \exp\left(\int_{t=\varphi(x,y)} g(t)\, \mathrm{d}t \right),$$

and the claim is proved. □

We note that Theorem 2.3 contains Theorem 2.2 as a special case. Just choose $\varphi(x,y) = x$.

Note in general that one may use (2.24) to obtain criteria for certain integrating factors. Choose e.g. $\varphi(x,y) = xy$. Then (2.24) becomes

$$\frac{\dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}}{y\, M - x\, L} = g(xy).$$

Likewise,

$$\frac{\dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}}{M - L} = g(x+y)$$

is the criterion for $\varphi(x,y) = x + y$.

However, this method is in general of limited use in practice.

3 Separation of the variables

3.1 Theoretical explanation

The separation of the variables is the most important solution method. Even the solution of the linear differential equation of first order, discussed later in Chapter 4, is derived by introducing a so-called integrating factor and then separate the variables, though one never thinks of the solution procedure in this way. In all the following special cases of differential equations of known solution formula or method, the trick is always to transform the original equation to some equation, where the variables can be separated. Although this is the general idea, it is convenient also to mention some special cases in the following chapters.

We say that the variables are *separated* , when the differential equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0$$

takes the special form

$$(3.1) \qquad f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y = 0,$$

i.e. when $L(x,y) = f(x)$ is a function in x alone, and $M(x,y) = g(y)$ is a function in y alone. Clearly, the differential form is exact. If we put $F(x) := \int f(x)\,\mathrm{d}x$ and $G(y) := \int g(y)\,\mathrm{d}y$, and finally $H(x,y) := F(x) + G(y)$, then

$$\mathrm{d}H = \{F'(x) + 0\}\;\mathrm{d}x + \{0 + G'(y)\}\;\mathrm{d}y = f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y = 0,$$

and we conclude from the above that the complete solution of (3.1) is given by

$$(3.2) \qquad H(x,y) = F(x) + G(y) = \int f(x)\,\mathrm{d}x + \int g(y)\,\mathrm{d}y = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

In the case of Figure 2.3 the variables are separated, $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$, so we get by integrating the variables separately that the complete solution is given by $\frac{1}{2}\,x^2 + \frac{1}{2}\,y^2 = C$, which is reduced to the usual form $x^2 + y^2 = r^2$.

3.2 Examples

Example 3.1 *Discuss the differential equation*

$$\frac{dx}{dt} = \frac{t}{x}, \qquad \textit{for } x > 0 \textit{ and } t \in \mathbb{R},$$

and find its solutions.

Clearly, $t = 0$, i.e. the positive x-axis is the 0-isocline. If $\alpha \neq 0$ then the α-isocline is given by $\frac{t}{x} = \alpha$, i.e. the line $x = \frac{1}{\alpha}\,t$. There is no need to draw these simple isoclines.

As a routine we compute

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{1}{x} - \frac{t}{x^2} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x} - \frac{t^2}{x^3} = \frac{1}{x^3}\left(x^2 - t^2\right), \qquad \text{for } x > 0,$$

so the possible points of inflection must lie on the two straight lines $x = |t|$, $t \neq 0$, in the upper halfplane. However, a simple check shows that both these two lines are solutions.

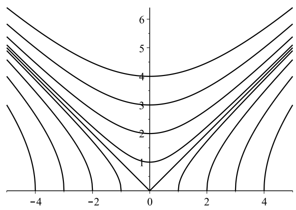


Figure 3.1: *Some solution curves of the equation* $\frac{dx}{dt} = \frac{t}{x}$ *for* $x > 0$.

Then we separate the variables,

$$2x\,\mathrm{d}x = \mathrm{d}\left(x^2\right) = 2t\,\mathrm{d}t = \mathrm{d}\left(t^2\right),$$

hence by integration,

$$x^2 = t^2 + c, \qquad t^2 > -c, \quad c \text{ arbitrary constant},$$

so the solutions are given by

$$x = \begin{cases} \sqrt{t^2 + c}, & \text{for } t \in \mathbb{R}, & \text{if } c > 0, \\ t & \text{for } t > 0, & \text{if } c = 0, \\ -t & \text{for } t < 0, & \text{if } c = 0, \\ \sqrt{t^2 + c} & \text{for } |t| > \sqrt{-c}, & \text{if } c < 0. \end{cases}$$

The equation is also homogeneous of degree 0, so it can be solved by methods given in Chapter 2. $\Diamond$

Example 3.2 *We shall discuss the differential equation*

$$\frac{dx}{dt} = \frac{5t}{x}, \qquad x \neq 0,$$

and find its solutions.

REMARK. This differential equation is also of the homogeneous type, cf. Chapter 7 in the following. Therefore, it can be solved in many different standard ways. We shall here stick to the method of separation of the variables, which is straight forward. $\Diamond$

First note that the isoclines are given by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{5t}{x} = \alpha, \qquad \text{i.e. } x = \frac{5}{\alpha}\,t, \qquad (t,x) \neq (0,0) \text{ and } \alpha \neq 0,$$

which are equations of straight lines through the point of exception $(0,0)$, except for the axes.

The possible points of inflection are described by the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{5}{x} - \frac{5t}{x^2}\,\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{5}{x} - \frac{25t^2}{x^3} = \frac{5}{x^3}\left(x^2 - 5t^2\right),$$

provided that $x \neq 0$, so the points of inflection are lying on the two straight lines defined by

$$x^2 - 5t^2 = 0, \qquad \text{i.e.} \qquad x = \pm\sqrt{5}\,t, \qquad t \neq 0.$$

It should, however, be noted that these two lines are also solutions of the equation.

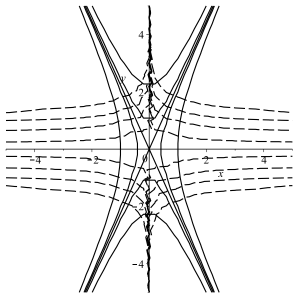


Figure 3.2: *Some solution curves for the equation* $\dfrac{dx}{dt} = \dfrac{5t}{x}$ *(solid curves), and some equipotential curves of this system (dashed curves).*

The equation is easily solved by separation of the variables, where we also multiply by 2,

$$2x\,\mathrm{d}x = 2 \cdot 5t\,\mathrm{d}t, \qquad \text{i.e.} \qquad \mathrm{d}\left(x^2\right) = \mathrm{d}\left(5t^2\right).$$

Then by an integration,

$$x^2 - 5t^2 = c, \qquad c \text{ an arbitrary constant.}$$

If $c = 0$, we get the afore mentioned straight lines (solid lines on Figure 3.2)

$$x = \pm\sqrt{5}\,t.$$

If $c \neq 0$, the solution curves are arcs of the hyperbola (dashed curves on Figure 3.2)

$$x^2 - 5t^2 = 5c, \qquad x \neq 0, \quad c \quad \text{an arbitrary constant,}$$

i.e.

$$x = \pm\sqrt{5\,(t^2 + c)}, \qquad \begin{cases} t \in \mathbb{R}, & \text{if } c > 0, \\ |t| > |c|, & \text{if } c < 0, \end{cases}$$

As a simple exercise we add here the discussion of the corresponding system of equipotential curves. The original equation is written in its differential formulation,

$$x\,\mathrm{d}x - 5t\,\mathrm{d}t = 0,$$

so the differential equation of the equipotential curves is

$$5t\,\mathrm{d}x + x\,\mathrm{d}t = 0,$$

of rectilinear solutions $t = 0$ (not relevant) and $x = 0$. When we separate the variables, we get

$$5\,\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}t}{t} = 0.$$

Integration of this equation, followed by the exponential and incorporation of the possible sign in the constant, gives that the equipotential curve system is implicitly given by

$$x \cdot y^5 = C, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \Diamond$$

Example 3.3 *Find the complete solution of the equation*

$$\frac{dy}{dx} = \frac{y(y+1)}{x^2}.$$

We must of course assume that $x \neq 0$. Then $y = 0$ and $y = -1$ are trivial solutions. If $y \neq 0$, $y \neq -1$ and $x \neq 0$, we can separate the variables,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y(y+1)} = \left(\frac{1}{y} - \frac{1}{y+1}\right) \mathrm{d}y,$$

so by an integration,

$$\ln\left|\frac{y}{y+1}\right| = k - \frac{1}{x}, \qquad \text{or} \qquad \frac{y}{y+1} = C \cdot \exp\left(-\frac{1}{x}\right),$$

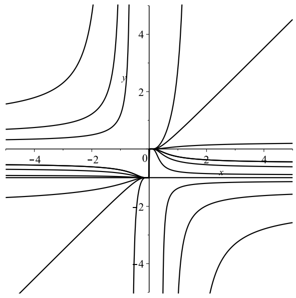


Figure 3.3: *Some solution curves of the equation* $\dfrac{dy}{dx} = \dfrac{y(y+1)}{x^2}$.

hence

$$\frac{1}{y} = C \cdot \exp\left(\frac{1}{x}\right) - 1,$$

or

$$y = \frac{1}{C \cdot \exp\left(\frac{1}{x}\right) - 1}, \qquad \text{for } x \neq -\frac{1}{\ln C}, \text{ if } C > 0 \text{ and } C \neq 1,$$

supplied with $y = 0$ and $y = -1$.

If we multiply the equation by $x^2\,\mathrm{d}x$, then $x = 0$ becomes a rectilinear solution, and the singular points are $(0, 0)$ and $(0, -1)$.

The equation can also be solved as a Bernoulli differential equation, cf. Chapter 5. The details are left to the reader. ◊

Example 3.4 *Find all solutions of the differential equation*

$$\frac{dx}{dt} = (2x-3)(t+1), \qquad \textit{for } t \in \mathbb{R} \textit{ and } x \in \mathbb{R}.$$

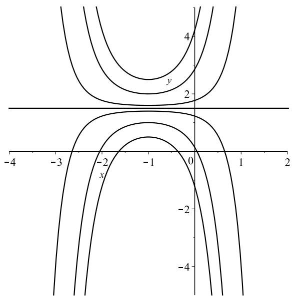


Figure 3.4: *Some solution curves of the equation* $\dfrac{dx}{dt} = (2x-3)(t+1)$.

The right hand side is factorized. Obviously, $x = \dfrac{3}{2}$ is a constant solution. When $x \neq \dfrac{3}{2}$, we get by separation of the variables

$$\frac{\mathrm{d}x}{x-\frac{3}{2}} = \mathrm{d}\ln\left|x-\frac{3}{2}\right| = 2(t+1)\,\mathrm{d}t = d\left\{(t+1)^2\right\},$$

hence by integration,

$$\ln\left|x-\frac{3}{2}\right| = (t+1)^2 + c, \qquad c \text{ arbitrary constant.}$$

Taking the exponential, and building the sign of $x - \frac{3}{2}$ into the new constant C, where $|C| = e^c$, we get the solution

$$x = \frac{3}{2} + C \cdot \exp\left((t+1)^2\right).$$

Note the symmetry with respect to the point $\left(\frac{3}{2}, -1\right)$, where the right hand side of the equation is 0. $\diamondsuit$

Example 3.5 *Find all solutions of the differential equation*

$$\frac{dx}{dt} = \frac{2t}{e^x}, \qquad \textit{for } t \in \mathbb{R} \textit{ and } x \in \mathbb{R}.$$

Separation of the variables gives

$$e^x\,\mathrm{d}x = \mathrm{d}(e^x) = 2t\,\mathrm{d}t = \mathrm{d}\left(t^2\right),$$

hence by integration, $e^x = t^2 + c$, for $|t| > \sqrt{-c}$, when $c < 0$, so the solutions are given by

$$x = \begin{cases} \ln\left(t^2+c\right), & \text{for } t \in \mathbb{R}, & \text{when } c > 0, \\ \ln\left(t^2\right), & \text{for } t \neq 0, & \text{when } c = 0, \\ \ln\left(t^2+c\right), & \text{for } |t| > \sqrt{-c}, & \text{when } c < 0. \end{cases} \qquad \diamondsuit$$

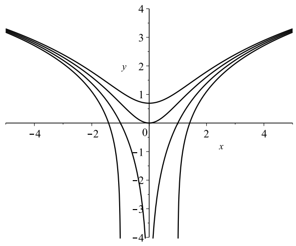


Figure 3.5: Some solution curves of the equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{2t}{e^x}$.

Example 3.6 *Find all solutions of the differential equation*

$$\frac{dx}{dt} = \frac{e^{-x}}{1+t^2}, \qquad \textit{for } x \in \mathbb{R} \textit{ and } t \in \mathbb{R}.$$

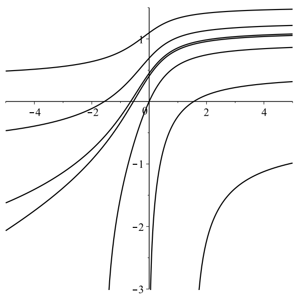


Figure 3.6: *Some solution curves of the equation* $\dfrac{dx}{dt} = \dfrac{e^{-x}}{1+t^2}$.

Separating the variables we get

$$e^x\,\mathrm{d}x = \mathrm{d}(e^x) = \frac{\mathrm{d}t}{1+t^2} = \mathrm{d}\arctan t,$$

hence by integration, $e^x = \arctan t + c$, provided that $\arctan t + c > 0$. Since $\arctan t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ for all $t \in \mathbb{R}$, and is increasing, we get

$$\arctan t + c \quad \begin{cases} > 0 & \text{for } t \in \mathbb{R}, & \text{if } c \geq \frac{\pi}{2}, \\ > 0 & \text{for } t \in \left]-\tan c, +\infty\right[, & \text{if } c \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \\ < 0 & \text{for all } t \in \mathbb{R}, & \text{if } x \leq -\frac{\pi}{2}. \end{cases}$$

Therefore, the solution is given by

$$x = \ln(\arctan t + c) \qquad \text{for} \begin{cases} t \in \mathbb{R}, & \text{if } c \geq \frac{\pi}{2}, \\ t \in \left]-\tan c, +\infty\right[, & \text{if } c \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \end{cases} \qquad \Diamond.$$

Example 3.7 *Find the solutions of the differential equation*

$$\frac{dx}{dt} = \frac{1+x^2}{1+t^2}.$$

A separation of the variables gives

$$\frac{\mathrm{d}x}{1+x^2} = \frac{\mathrm{d}t}{1+t^2},$$

hence by an integration and a rearrangement,

$$\arctan x - \arctan t = c.$$

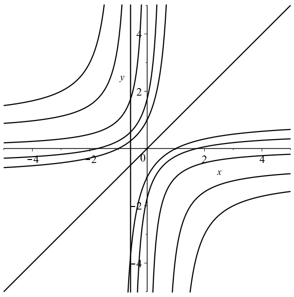


Figure 3.7: *Some solution curves of the equation* $\dfrac{dx}{1+x^2} = \dfrac{dt}{1+t^2}$ *for* $t \in \mathbb{R}$.

When we for $c \neq \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, apply tan on this equation, we get

$$\tan(\arctan x - \arctan t) = \frac{x-t}{1+xt} = \tan c.$$

Solving for x we get

$$x = \frac{t + \tan c}{1 - t \cdot \tan c}, \qquad \text{for } t \neq \cot c,$$

which can be written

$$x = \frac{t \cdot \cos c + \sin c}{\cos c - t \cdot \sin c}, \qquad \text{for } t \neq \cot c.$$

We note that we in the latter form no longer have the restriction $c \neq \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. Furthermore, the constant c is now restricted to e.g. $c \in]-\pi, \pi]$.

For $c = 0$ we get the solution $x = t$. For $c = \pm\dfrac{\pi}{2}$ we get the solution $x = -\dfrac{1}{t}$. Other simple solutions are $x = \dfrac{t+1}{1-t}$ for $t \neq 1$, and $x = \dfrac{t-1}{1+t}$, for $t \neq -1$. ◊

Example 3.8 *Discuss the differential equation*

$$\frac{dx}{dt} = 4t\sqrt{x}, \qquad \textit{for } x > 0, \ \textit{and } t \in \mathbb{R},$$

and find its solutions.

The equation of the isoclines is

$$4t\sqrt{x} = \alpha,$$

i.e. the positive vertical axes $t = 0$, $x > 0$, for $\alpha = 0$ is the 0-isocline, and if $\alpha \neq 0$, the α-isocline is given by

$$x = \frac{\alpha^2}{16t^2}, \qquad \text{the } t\text{-interval defined by } \alpha \cdot t > 0,$$

i.e. $t \in \mathbb{R}_-$ for $\alpha < 0$, and $t \in \mathbb{R}_+$ for $\alpha > 0$.

From

$$\frac{d^2x}{dt^2} = 4\sqrt{x} + \frac{2t}{\sqrt{x}} \cdot \frac{dx}{dt} = 4\sqrt{x} + \frac{2t}{\sqrt{x}} \cdot 4t\sqrt{x} = 4\sqrt{x} + 8t^2 > 0 \qquad \text{for } x > 0,$$

follows that we have no inflection points.

The isocline diagram is very much distorted in this case, so we cannot get too much information from it, and we leave it out.

We separate the equation by rewriting it in the following way,

$$\frac{dx}{2\sqrt{x}} = 2t\,dt, \qquad \text{i.e.} \qquad d\left(\sqrt{x}\right) = d\left(t^2\right) \qquad \text{for } x > 0.$$

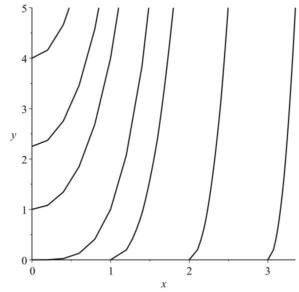


Figure 3.8: *Some solution curves for the equation* $\frac{dx}{dt} = 4t\sqrt{x}$.

By integration the solution is given by

$$\sqrt{x} = t^2 + c, \qquad \text{for } t^2 + c \geq 0, \qquad c \quad \text{arbitrary constant.}$$

Hence, the solution is given by

$$x = \left(t^2 + c\right)^2 \qquad \begin{cases} t \in \mathbb{R} & \text{for } c > 0, \\ t \in \mathbb{R} \setminus \{0\} & \text{for } c = 0, \\ |t| > \sqrt{-c} & \text{for } c < 0. \end{cases}$$

Note in particular that $x = t^4$ for $t \neq 0$ are two branches of a solution. ◊

Example 3.9 *Discuss the differential equation*

$$\frac{dx}{dt} = \left(\sqrt[3]{x}\right)^2 \sin t, \qquad \textit{for } t \in \mathbb{R} \textit{ and } x \in \mathbb{R},$$

and find its solutions.

Clearly, $x = 0$ is a solution. Furthermore, the vertical line $t = 0$ is the 0-isocline. When $\alpha \neq 0$, the α-isocline is given by

$$\alpha = x^{\frac{2}{3}} \sin t, \qquad \text{i.e.} \qquad x = \left(\frac{\alpha}{\sin t}\right)^{\frac{3}{2}}, \qquad t \neq p\pi, \quad p \in \mathbb{Z}.$$

Then we calculate

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{2}{3} \frac{1}{\sqrt[3]{x}} \frac{\mathrm{d}x}{\mathrm{d}t} \cdot \sin t + x^{\frac{2}{3}} \cos t = \frac{2}{3} \sqrt[3]{x} \cdot \sin^2 t + \left(\sqrt[3]{x}\right)^2 \cos t,$$

so possible inflection points must satisfy the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \sqrt[3]{x} \left\{ \frac{2}{3} \cdot \sin^2 t + \sqrt[3]{x} \cdot \cos t \right\} = 0.$$

Here, $x = 0$ is a solution, so when $x \neq 0$, the points of inflection lie on the curves of the equation

$$x = -\left(\frac{2\sin^2 t}{\cos t}\right)^3, \qquad t \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z}.$$

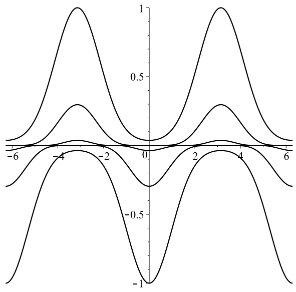


Figure 3.9: *Some solution curves of the equation* $\dfrac{dx}{dt} = \left(\sqrt[3]{x}\right)^2 \sin t$.

Then we separate the variables,

$$\frac{1}{3}\, x^{-\frac{2}{3}}\, \mathrm{d}x = \frac{1}{3}\, \sin t \, \mathrm{d}t, \qquad \text{i.e.} \qquad \mathrm{d}\left(\sqrt[3]{x}\right) = -\frac{1}{3}\, \mathrm{d}\cos t,$$

so by integration,

$$\sqrt[3]{x} = -\frac{1}{3}(\cos t + c), \qquad c \quad \text{arbitrary constant,}$$

and hence,

$$x = -\frac{1}{27}(\cos t + c)^3. \qquad \Diamond$$

Example 3.10 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{y^2}{x^2}, \qquad \textit{for } x \neq 0.$$

Obviously, $y = 0$ is a solution. Assume that both $x \neq 0$ and $y \neq 0$. Then the equation is equivalent to

$$\frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x^2},$$

where the variables are separated. By integration and rearrangement,

$$\frac{1}{x} - \frac{1}{y} = C, \quad \text{or} \quad y - x = Cxy, \quad \text{i.e.} \quad y = \frac{x}{1 - Cx},$$

where C is an arbitrary constant.

Note that the equation is also homogeneous of degree 0, so we may alternatively apply methods from Chapter 7. $\Diamond$

Example 3.11 *Find the complete solution of the equation*

$$\frac{dy}{dx} + y + y^2 = 0.$$

Clearly, $y = 0$ and $y = -1$ are solutions. When $y \neq 0, -1$, the equation can be written

$$0 = \frac{\mathrm{d}y}{y + y^2} + \mathrm{d}x = \frac{\mathrm{d}y}{y(y+1)} + \mathrm{d}x = \left(\frac{1}{y} - \frac{1}{y+1}\right) \mathrm{d}y + \mathrm{d}x,$$

from which by integration,

$$k = \ln\left|\frac{y}{y+1}\right| + x, \qquad \text{i.e.} \qquad \frac{y}{y+1} = \frac{1}{C}\, e^{-x},$$

hence

$$y = \frac{\frac{1}{C}\, e^{-x}}{1 - \frac{1}{C}\, e^{-x}} = \frac{1}{C\, e^{x} - 1}, \qquad C \text{ arbitrary constant,}$$

supplied with $y = 0$ and $y = -1$.

We note that the equation is also a Bernoulli equation, so it can be solved by using methods from Chapter 5. $\Diamond$

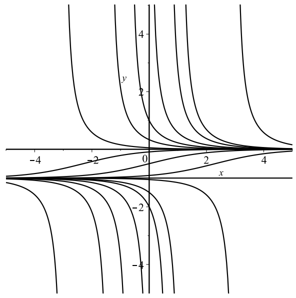


Figure 3.10: *Some solution curves of the equation* $\frac{dy}{dx} + y + y^2 = 0$*. Note that the solution curves either are lying above the line* $y = 0$*, or between the two lines* $y = 0$ *and* $y = -1$*, or below the line* $y = -1$

Example 3.12 *Discuss the differential equation*

$$\frac{1}{x}\,\frac{dx}{dt} = 2^t, \qquad \text{for } t \in \mathbb{R} \text{ and } x > 0,$$

and find its solutions.

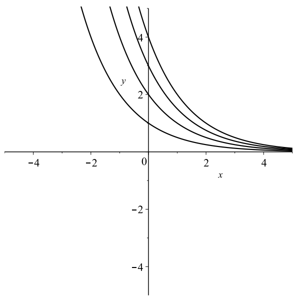


Figure 3.11: *The isoclines for* $\alpha = 1, 2, 3, 4$ *of the equation* $\frac{1}{x} \cdot \frac{dx}{dt} = 2^t$.

The equation of the α-isocline is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot 2^t = \alpha, \qquad \text{i.e.} \qquad x = \alpha \cdot 2^{-t} \qquad \text{where } \alpha > 0.$$

Furthermore,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \ln 2 \cdot x \cdot 2^t + 2^t\,\frac{\mathrm{d}x}{\mathrm{d}t} = \ln 2 \cdot x \cdot 2^t + x \cdot 2^t = x \cdot 2^t \left(\ln 2 + 2^t\right) > 0,$$

so there is no point of inflection.

Since the equation is already separated,

$$\frac{\mathrm{d}\ln x}{\mathrm{d}t} = 2^t = e^{t\ln 2}, \qquad \text{for } t \in \mathbb{R}, \quad x > 0,$$

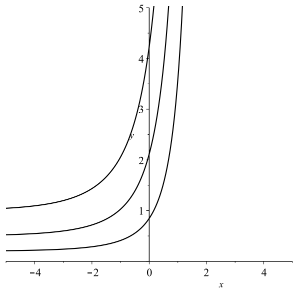


Figure 3.12: *Some solution curves,* $c = 0.2,\ 0.5,\ 1$, *of the equation* $\dfrac{1}{x}\,\dfrac{dx}{dt} = 2^t$.

it follows by integration that

$$\ln x = \int e^{t\ln 2}\,\mathrm{d}t + c = \frac{2^t}{\ln 2} + c,$$

hence, with a new arbitrary constant $C > 0$,

$$x = C \cdot \exp\left(\frac{1}{\ln 2} \cdot 2^t\right), \qquad \text{for } t \in \mathbb{R}, \quad C > 0. \qquad \Diamond$$

Example 3.13 *Discuss the differential equation*

$$\frac{dx}{dt} = 4\left(\sqrt[4]{x}\right)^3, \qquad \textit{for } t \in \mathbb{R}, \ \textit{and } x \geq 0,$$

and find its solutions.

Clearly, the boundary line $x = 0$ is a solution.

When $x > 0$, the α-isocline, $\alpha > 0$, has the equation

$$4\left(\sqrt[4]{x}\right)^3 = \alpha, \qquad \text{i.e.} \qquad x = \left(\sqrt[3]{\frac{\alpha}{4}}\right)^4, \qquad \alpha > 0.$$

All the α-isoclines are straight lines parallel to the t-axis.

The information above is sufficient, but for completeness we compute

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 3 \cdot \frac{1}{\sqrt[4]{x}} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = 12\sqrt{x} > 0, \qquad \text{for } x > 0,$$

proving that there is no inflection point.

Assuming that $x > 0$ we separate the variables,

$$\frac{1}{4}\,x^{-\frac{3}{4}}\,\mathrm{d}x = \mathrm{d}\left(x^{\frac{1}{4}}\right) = \mathrm{d}t,$$

so by integration,

$$\sqrt[4]{x} = t + c, \qquad \text{for } t > -c, \qquad c \quad \text{arbitrary.}$$

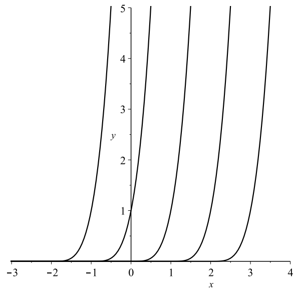


Figure 3.13: *Some solution curves,* $c = -2,\, -1,\, 0\, 1,\, 2$*, of the equation* $\dfrac{dx}{dt} = 4(\sqrt[4]{x})^3$.

The solutions are

$$\begin{cases} x = 0, & \text{for } t \in \mathbb{R}, \\ x = (t+c)^4, & \text{for } t > -c, \end{cases}$$

Note that we also have the concatenated solutions

$$x = \begin{cases} (t+c)^4, & \text{for } t > -c, \\ 0 & \text{for } t \leq -c. \end{cases} \qquad c \in \mathbb{R}. \qquad \Diamond$$

Example 3.14 *Discuss the differential equation*

$$\frac{dx}{dt} + 2t\, e^x = 0 \qquad \textit{for } t \in \mathbb{R} \textit{ and } x \in \mathbb{R},$$

and find its solutions.

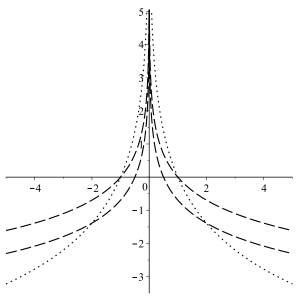


Figure 3.14: *The isoclines (dashed) for* $\alpha = -2,\, -1,\, 0,\, 1,\, 2$ *and the curves of inflection points (dotted) of the equation* $\dfrac{dx}{dt} + 2t\, e^x = 0$.

Clearly, $t = 0$ is the 0-isocline. When $\alpha \neq 0$ the α-isocline is given by

$$-2t\, e^x = \alpha, \qquad \text{i.e.} \qquad x = \ln\left(-\frac{\alpha}{2t}\right) \quad \text{for } t \cdot \alpha < 0.$$

From the rearrangement

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2t\,e^{x}$$

follows that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -2t \cdot e^{x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} - 2e^{x} = 4t^2\,e^{2x} - 2e^{x} = 2e^{x}\left\{2t^2\,e^{x} - 1\right\},$$

so the equation of inflection points, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0$, is reduced to

$$x = \ln\left(\frac{1}{2t^2}\right) = -\ln\left(2t^2\right), \qquad t \neq 0.$$

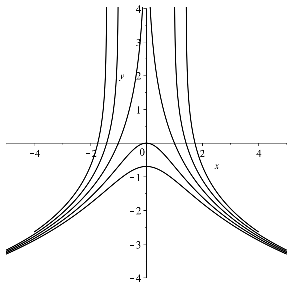


Figure 3.15: *Some solution curves, $c = -2,\,-1,\,0\,1,\,2,$ of the equation* $\frac{dx}{dt} + 2t\,e^{x} = 0.$

When we separate the variables, we get

$$-e^{-x}\,\mathrm{d}x = 2t\,\mathrm{d}t, \qquad \text{i.e.} \qquad \mathrm{d}\left(e^{-x}\right) = \mathrm{d}\left(t^2\right),$$

so by integration,

$$e^{-x} = t^2 + c, \qquad \text{for } t^2 + c > 0,$$

and we get the solutions

$$x = -\ln\left(t^2 + c\right) \qquad \text{for } t^2 + c > 0.$$

If in particular $c = 0$, then $x = -2\ln|t|$ for $t \neq 0$. $\lozenge$

Example 3.15 *Discuss the differential equation*

$$\frac{dx}{dt} + x \cdot \tan t = 0, \qquad \textit{for } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

and find its solutions.

The equation is also linear and homogeneous and can be solved by using a formula form Chapter 4.

A trivial solution is $x = 0$. We assume in the following analysis that $x \neq 0$.

It follows from

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x \cdot \tan t,$$

that the vertical line $t = 0$ is the 0-isocline. If $\alpha \neq 0$, we get the α-isoclines

$$-x \cdot \tan t = \alpha, \qquad \text{i.e.} \qquad x = -\alpha \cdot \cot t, \quad t \neq 0.$$

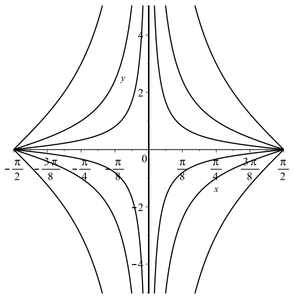


Figure 3.16: *Some α-isoclines, $\alpha = 0, \pm\frac{1}{3}, \pm 1 \pm 3$, of the equation* $\frac{dx}{dt} + x\,\tan t = 0$ *for* $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Furthermore,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\mathrm{d}x}{\mathrm{d}t} \cdot \tan t - x \cdot \left(1 + \tan^2 t\right) = x \cdot \tan^2 t - x - x\,\tan^2 t = -x,$$

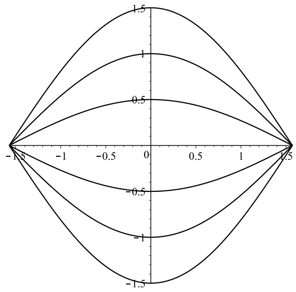


Figure 3.17: *Some solution curves,* $c = 0, \pm\frac{1}{2}, \pm 1 \pm \frac{3}{2}$, *of the equation* $\frac{dx}{dt} + x\,\tan t = 0$ *for* $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

so $x'' = 0$ only for $x = 0$, which is a solution.

When we separate the variables, we get for $x \neq 0$,

$$\frac{1}{x}\,\mathrm{d}x = \mathrm{d}(\ln|x|) = -\tan t\,\mathrm{d}t = -\frac{\sin t}{\cos t}\,\mathrm{d}t = \mathrm{d}(\ln|\cos t|),$$

so by integration, $\ln|x| = \ln|\cos t| + c$, from which with a new constant $C \neq 0$,

$$x = C \cdot \cos t,$$

which for $C = 0$ agrees with the solution $x = 0$. $\Diamond$

Example 3.16 *Discuss the differential equation*

$$\frac{dx}{dt} = \frac{1}{2}\sqrt[3]{\frac{x}{t}}, \qquad \textit{for } x > 0 \textit{ and } t > 0,$$

and find its solutions.

The right hand side of the equation is always positive in the open first quadrant, so the α-isoclines do not exist for $\alpha \leq 0$. If $\alpha > 0$, then the α-isocline is given by

$$\frac{1}{2}\sqrt[3]{\frac{x}{t^2}} = \alpha, \qquad \text{i.e.} \qquad x = (2\alpha)^3 t^2, \quad \text{for } t > 0.$$

These are parts of parabolas, and it is left to the reader to sketch them.

By routine,

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{1}{2}\cdot\frac{1}{3}\,x^{-\frac{2}{3}}t^{-\frac{2}{3}}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{2}\cdot\left(-\frac{2}{3}\right)\cdot t^{-\frac{5}{3}}\,x^{-\frac{1}{3}} = \frac{1}{6}\,x^{-\frac{2}{3}}t^{-\frac{2}{3}}\cdot\frac{1}{2}\,x^{\frac{1}{3}}t^{-\frac{2}{3}} + \frac{1}{3}\cdot t^{-\frac{5}{3}}x^{\frac{1}{3}} \\ &= \frac{1}{12}\,x^{-\frac{1}{3}}t^{-\frac{4}{3}} + \frac{1}{3}\,x^{\frac{1}{3}}t^{-\frac{5}{3}} > 0,\end{aligned}$$

so there are no inflection points in the first quadrant.

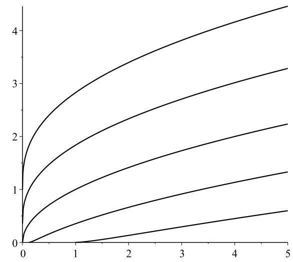


Figure 3.18: *Some solution curves of the equation* $\dfrac{dx}{dt} = \dfrac{1}{2}\sqrt[3]{\dfrac{x}{t}}$ *for* $x > 0$ *and* $t > 0$.

Then we separate the variables,

$$\frac{2}{3}\,x^{-\frac{1}{3}}\,\mathrm{d}x = \frac{1}{3}\,t^{-\frac{2}{3}}\,\mathrm{d}t, \qquad \text{i.e.} \qquad \mathrm{d}\left(x^{\frac{2}{3}}\right) = \mathrm{d}\left(t^{\frac{1}{3}}\right),$$

hence after integration,

$$x^{\frac{2}{3}} = t^{\frac{1}{3}} + c, \qquad \text{provided that } t^{\frac{1}{3}} + c > 0, \qquad c \text{ arbitrary constant.}$$

We have assumed that $x > 0$ and $t > 0$, so the solution is given by

$$x = \left(\sqrt[3]{t} + c\right)^{\frac{3}{2}}, \qquad \text{for } t^{\frac{1}{3}} + c > 0, \text{ i.e. } t > \max\left\{-c^3, 0\right\}.$$

For $c = 0$ we of course get the simpler expression $x = \sqrt{t}$, $t > 0$.

The equation is homogeneous of degree 0, so it can also be solved by methods given in Chapter 2. $\Diamond$

Example 3.17 *Find the solutions of the differential equation*

$$e^{\sqrt{x}}\,\frac{dx}{dt} = 4t\sqrt{x}, \qquad \textit{for } x > 0 \textit{ and } t \in \mathbb{R},$$

i.e. in the open right halfplane.

Apart from the positive x-axis, which is the 0-isocline, the α-isoclines are in general difficult to describe. One may of course express t as a function of x, i.e. the α-isocline is given by

$$t = \frac{\alpha}{4\sqrt{x}}\,e^{\sqrt{x}}, \qquad x > 0.$$

It is hard to get any information out of these isoclines.

A division by $2\sqrt{x} > 0$ separates the variables,

$$2t\,\mathrm{d}t = \mathrm{d}\left(t^2\right) = \frac{1}{2\sqrt{x}}\,e^{\sqrt{x}}\,\mathrm{d}x = e^{\sqrt{x}}\,\mathrm{d}\sqrt{x} = \mathrm{d}\left(e^{\sqrt{x}}\right),$$

and hence by integration,

$$e^{\sqrt{x}} = t^2 + c > 0 \qquad \text{for } t^2 > -c,$$

i which case

$$\sqrt{x} = \ln\left(t^2 + c\right) > 0,$$

so we must furthermore require that $t^2 + c > 1$

$$x = \left\{\ln\left(t^2 + c\right)\right\}^2 \qquad \text{for } t^2 > -c, \quad c \text{ arbitrary.}$$

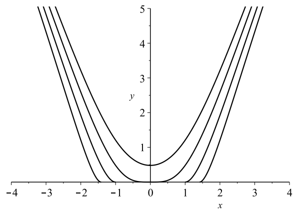


Figure 3.19: *Some solution curves of the equation* $e^{\sqrt{x}}\,\dfrac{dx}{dt} = 4t\sqrt{x}$ *for* $x > 0$ *and* $t \in \mathbb{R}$.

Writing $c = \pm a^2$, $a \geq 0$, we get more explicitly,

$$x = \begin{cases} \left\{\ln\left(t^2 + a^2\right)\right\}^2, & \text{for } t \in \mathbb{R}, & \text{when } a \geq 1 \text{ and } c = a^2 > 0, \\ \left\{\ln\left(t^2 + a^2\right)\right\}^2, & \text{for } |t| > \sqrt{1 - a^2}, & \text{when } 0 < a < 1 \text{ and } c = a^2 > 0, \\ 4\{\ln t\}^2, & \text{for } t > 1, & \text{when } a = 0, \\ 4\{\ln(-t)\}^2, & \text{for } t < -1, & \text{when } a = 0, \\ \left\{\ln\left(t^2 - a^2\right)\right\}^2 & \text{for } |t| > \sqrt{a^2 + 1}, & \text{when } a > 0 \text{ and } c = -a^2 < 0. \end{cases}$$

We note that in this case the full solution is complicated to describe. $\Diamond$

Example 3.18 *Find the solutions of the differential equation*

$$\frac{dx}{dt} = 3x^3\sqrt{t}, \qquad \textit{for } t > 0 \textit{ and } x > 0.$$

We separate the variables,

$$-\frac{2}{x^3}\,\mathrm{d}x = \mathrm{d}\left(\frac{1}{x^2}\right) = -6\sqrt{t}\,\mathrm{d}t = -6\cdot\frac{2}{3}\,\mathrm{d}\left(t^{\frac{3}{2}}\right) = -4\,\mathrm{d}\left(t^{\frac{3}{2}}\right).$$

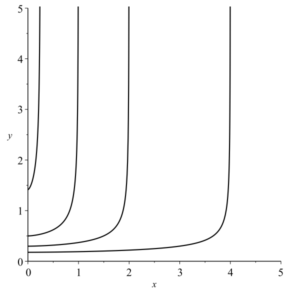


Figure 3.20: Some solution curves of the equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 3x^3\sqrt{t}$ for $t \in \mathbb{R}$.

Then by integration,

$$\frac{1}{x^2} = c - 4t\sqrt{t},$$

so we must choose $c > 0$, $c = 4a\sqrt{a}$, say, for some $a > 0$. Then

$$\frac{1}{x^2} = 4a\sqrt{a} - 4t\sqrt{t} = 4(a\sqrt{a} - t\sqrt{t}), \qquad \text{for } 0 < t < a,$$

and since we had assumed that $x > 0$,

$$x = \frac{1}{2}\,\frac{1}{\sqrt{a\sqrt{a} - t\sqrt{t}}}, \qquad \text{for } 0 < t < a, \quad a > 0 \text{ arbitrary constant.} \qquad \Diamond$$

Example 3.19 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}, \qquad \textit{for } y \in\,]-1,1[.$$

It is straight forward to separate the variables,

$$\frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{\mathrm{d}x}{1+x^2}.$$

Then integrate,

$$\arcsin(y) = \arctan(x) + C,$$

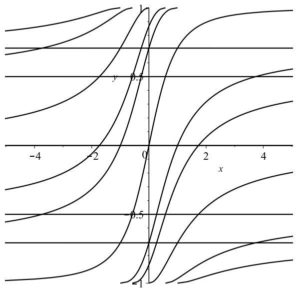


Figure 3.21: *Some solution curves of the equation* $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$. *The solution curves are – due to the differential equation – all increasing.*

which is the complete solution of the differential equation.

One should note that it is not straightforward, though possible, to find y as a function of x and c, because arcsin is not a nice function. Due to the differential equation, all solution curves must be increasing, but if one just apply sin on the equation and reduce, then we get some false solutions, which are decreasing. We shall not here include this discussion, but refer to Figure 3.21. $\diamond$

Example 3.20 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = (x+y)^2.$$

We put $y = -x + u$ and get

$$-1 + \frac{\mathrm{d}u}{\mathrm{d}x} = u^2, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2.$$

We separate the variables,

$$\frac{\mathrm{d}u}{1+u^2} = \mathrm{d}x,$$

and then get by integration,

$$\arctan u = x + k,$$

thus

$$u = \tan(x+k),$$

and the complete solution is given by

$$y = -x + u = -x + \tan(x+k), \qquad k \text{ an arbitrary constant.} \qquad \diamond$$

We shall in the following give a couple of examples from Physics, where we first must derive the differential equation.

Example 3.21 *Find the atmospheric pressure at height h above the ground surface, where we assume constant temperature, and neglect the small internal gravitational effect of the air particles.*

The idea is to use the law of gravitation and *Boyle's law* to express the atmospheric pressure p at height h by a differential equation.

Let r denote the radius of the Earth, and let $0 < h_1 < h_2$. Let f denote a part of the surface of the Earth, which is so small that it can be considered as flat. We erect above f a vertical cylinder.

Let V_1 and V_2 denote the volumes of this cylinder at height h_1 and h_2, resp.. Furthermore, let P_1 and P_2 denote the *forces* on the upper surface of the cylinder at height h_1 and h_2, resp., and let p_1 and p_2 be the corresponding *pressures.* This means that

$$p_1 = \frac{P_1}{f}, \qquad p_2 = \frac{P_2}{f} \quad \text{and} \quad P_1 - P_2 = (V_2 - V_1)\, DG,$$

where the function D converges for $h_2 \to h_1$ towards the atmospheric density at height h_1, and where G denotes the gravitational constant at the height h lying between h_1 and h_2 above the surface. According to *Boyle's law* there exists a constant k, such that

$$\lim_{h_2 \to h_1} D = \frac{p_1}{k}.$$

Furthermore,

$$\lim_{h_2 \to h_1} G = \left(\frac{r}{r + h_1} \right)^2 g,$$

where g is the gravitational constant, $g = 981$ cm/s^2, at sea level.

It follows that

$$\frac{p_2 - p_1}{h_2 - h_1} = \frac{P_2 - P_1}{f\,(h_2 - h_1)} = \frac{P_2 - P_1}{V_2 - V_1} = -DG \to -\frac{p_1}{k} \left(\frac{r}{r + h_1} \right)^2 g, \qquad \text{for } h_2 \to h_1.$$

Hence, by taking this limit,

$$\frac{\mathrm{d}p}{\mathrm{d}h} = -\frac{r^2 g}{k} \cdot \frac{p}{(r+h)^2}.$$

This is a differential equation of first order, in which the variables can be separated,

$$\frac{\mathrm{d}p}{p} = -\frac{r^2 g}{k} \cdot \frac{\mathrm{d}h}{(r+h)^2}.$$

By integration, including an arbitrary constant C,

$$\ln |p| = C + \frac{r^2 g}{k} \cdot \frac{1}{r+h} = C + \frac{gr}{k} \cdot \frac{r}{r+h} = C + \frac{gr}{k} \cdot \left(1 - \frac{h}{r+h} \right) = C + \frac{gr}{k} - \frac{grh}{k(r+h)},$$

so when we apply the exponential and rename the constant,

$$p = p_0 \exp\left(-\frac{grh}{k(r+h)} \right),$$

where p_0 is the atmospheric pressure at the surface of the Earth, $h = 0$. When h is not too large, we may approximate $\dfrac{r}{r+h}$ by 1, and we get the so-called *barometer formula*

$$p = p_0 \exp\left(-\frac{gh}{k}\right). \qquad \diamond$$

Example 3.22 *Given a cylindric vessel of inner radius r. We assume that it is filled with water up to the height H. In the bottom of the vessel there is a circular hole of radius ϱ. Find the time it takes to empty the vessel.*

We shall again first derive the differential equation which governs this situation. Let $g = 981\ \text{cm/s}^2$ be the gravitation constant at sea level. If we neglect the internal friction of the water molecules, and if the surface of water is at height h cm above the bottom, then the water pours out with the speed $v = \sqrt{2gh}$. In practice, however, due to the neglected friction above, the speed can be measured to be $v = c\sqrt{2gh}$, where $c \approx 0.6$.

The height h of the water surface is a function in time t. The water surface is lowered by the velocity given by $-\dfrac{\mathrm{d}h}{\mathrm{d}t}$. The proportion between this velocity and the velocity of the water pouring out of the vessel is equal to the proportion between the area of the hole in the bottom and the area of the cylindric bottom itself. This gives us the equation

$$-\frac{1}{v}\,\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\varrho^2}{r^2},$$

which can be written

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -C\sqrt{h}, \qquad \text{where } C = \frac{1}{r^2}\,c\varrho^2\sqrt{2g}.$$

Here we can separate the variables,

$$\mathrm{d}t = -\frac{1}{C}\frac{\mathrm{d}h}{\sqrt{h}}, \qquad \text{hence } t = \frac{2}{C}\left(\text{const.} - \sqrt{h}\right).$$

Let us assume that the height is H for $t = 0$. Then,

$$t = \frac{2}{C}\left(\sqrt{H} - \sqrt{h}\right), \qquad \text{for } 0 < h \leq H.$$

When $h \to 0$, the vessel is emptied, so if T denotes the time needed for emptying the vessel, then

$$T = \lim_{h\to 0} t = \frac{2}{C}\lim_{h\to 0}\left(\sqrt{H} - \sqrt{h}\right) = \frac{2\sqrt{H}}{C} = \frac{r^2}{\varrho^2}\cdot\frac{1}{C}\sqrt{\frac{H}{2g}}. \qquad \Diamond$$

3.3 The differential equation $y' = f(Ax + By + C)$

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C), \qquad \text{where } A,\ B,\ C \text{ are constants,}$$

and $f(u)$ is a continuous function. Such an equation is solved by a variant of the method of separating the variables.

We first note that if $B = 0$, then the solution follows trivially by integration,

$$y = \int f(Ax + C)\,\mathrm{d}x + \ \text{const.}.$$

We assume in the following that $B \neq 0$, and introduce a new function $u(x)$ by

$$u(x) := Ax + By(x) + C.$$

If we can find all possible $u(x)$, then we also get all solutions

$$y(x) = \frac{1}{B}\,(u(x) - Ax - C),$$

so we shall find a solvable differential equation in u.

When we differentiate $u(x)$, we get from the above that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = A + Bf(u),$$

with the initial condition

$$u(\xi) = A\xi + B\eta + C,$$

if the original initial condition is $y(\xi) = \eta$. This means that we have proved the following theorem,

Theorem 3.1 *Let $f(u)$ be continuous for $r < u < s$. The differential equation*

$$\frac{dy}{dx} = f(Ax + By + C), \qquad B \neq 0, \quad A,\ B,\ C \text{ arbitrary constants},$$

is solved in the strip

$$r < Ax + By + C < s, \qquad u = Ax + By + C,$$

by the formula

$$x = \text{ const.} + \int \frac{du}{A + Bf(u)}, \qquad u = Ax + By + C,$$

provided that $A + Bf(u) \neq 0$ in the domain of integration.

It can be proved that the integral curves are defined, when

$$\inf_{r<u<s} \int_{A\xi+B\eta+C} \frac{\mathrm{d}u}{A + Bf(u)} < x - \xi < \sup_{r<u<s} \int_{A\xi+B\eta+C} \frac{\mathrm{d}u}{A + Bf(u)},$$

where $(x, y) = (\xi, \eta)$ are the initial values.

Example 3.23 The simplest example of an equation of this type is the linear inhomogeneous equation, cf. Chapter 4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y, \qquad \text{where } A = B = 1 \text{ and } C = 0,$$

where we in the following chapter also develop an alternative solution method.

In this case $f(u) = u = x + y$ is continuous, and by the solution formula,

$$x = \text{const.} + \int \frac{\mathrm{d}u}{1+u} = \text{const.} + \ln|1 + u|, \qquad \text{where } u \neq -1,$$

or by a new constant,

$$1 + u = C \cdot e^x, \qquad \text{i.e.} \qquad 1 + x + y = C \cdot e^x,$$

so

$$y = -(x + 1) + C \cdot e^x, \qquad C \text{ arbitrary constant.} \qquad \Diamond$$

Example 3.24 A less trivial example is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y)^2, \qquad \text{where } A = B = 1 \text{ and } C = 0,$$

so $u(x) = x + y(x)$, and $f(u) = u^2$ is continuous. It follows that

$$x = \text{const.} + \int \frac{\mathrm{d}u}{1+u^2} = \text{const.} + \arctan(u),$$

from which

$$u = x + y = \tan(x + C), \qquad \text{or} \qquad y = \tan(x + C) - x,$$

where C is an arbitrary constant.

A simple check shows that this indeed is a solution for every arbitrary constant C. $\Diamond$

3.4 Trajectories

Let $f(x,y)$ be a continuous function in the domain $\Omega \subseteq \mathbb{R}^2$. Consider the differential equation of first order

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \qquad \text{for } (x,y) \in \Omega.$$

Then $f(x,y)$ can be interpreted as the slope of the solution curve passing through the point $(x,y) \in \Omega$, so a tangent of this curve is represented by the vector $(1, f(x,y)$: If α denotes the angle between the positive x-axis and the line in the direction of the tangent $(1, f(x,y))$, then clearly $f(x,y) = \tan\alpha$.

We say that the equation defines a *direction field*.

Now, given the continuous function $f(x,y)$ and a fixed angle $\gamma \in]0,\pi[$, and assume that we for some reason want to find a function $g(x,y)$, such that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y)$$

defines another direction field, such that at every point $(x,y) \in \Omega$ the angle between the two direction fields is given by the fixed angle γ. The unknown function $g(x,y)$ is related to an angle β, where $\beta - \alpha = \gamma$ is the angle between the two direction fields at (x,y), by the formula $g(x,y) = \tan\beta$.

Since we have chosen $0 < \gamma < \pi$, we shall not apply tan, but cot instead, so we put for convenience $c := \tan\gamma$, which then is a given constant. Then by the addition formula for tan,

$$c = \cot\gamma = \frac{1}{\tan\gamma} = \frac{1}{\tan(\beta-\alpha)} = \frac{1+\tan\beta\cdot\tan\alpha}{\tan\beta - \tan\alpha} = \frac{1+g(x,y)\cdot f(x,y)}{g(x,y)-f(x,y)}.$$

When we solve with respect to the unknown function $g(x,y)$, we get

$$g(x,y) = \frac{1+c\,f(x,y)}{c-f(x,y)}, \qquad \text{where } f(x,y) \neq c = \tan\gamma.$$

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y) = \frac{1+c\,f(x,y)}{c-f(x,y)}, \qquad \text{where } f(x,y) \neq c = \tan\gamma,$$

is called the differential equations of the *trajectories* for given angle $\gamma \in]0,\pi[$ of the original equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y).$$

At every point $(x,y) \in \Omega$ the angle between the two systems of solutions is equal to $\gamma = \operatorname{arccot} c$.

Although we can define trajectories corresponding to every angle $0 < \gamma < \pi$, the most interesting case is of course the *orthogonal trajectories*, where $\gamma = \pi/2$, and consequently $c = 0$, so the differential equation of the orthogonal trajectories is in particular simple,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{f(x,y)}.$$

In this particular case we can give the systems of solution curves a physical interpretation. If e.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

is interpreted as the differential equation of the *streamlines* of a flow, then the the orthogonal trajectories are interpreted as the corresponding *equipotential curves*.

Example 3.25 *Given the system $y = x^2 + ax$, $a \in \mathbb{R}$ an arbitrary parameter, of parabolas. Find the orthogonal trajectories of this system.*

We first eliminate the arbitrary constant a. If $x \neq 0$, then

$$a = \frac{y}{x} - x,$$

so by a differentiation,

$$0 = \frac{1}{x^2}\left\{x\,\frac{\mathrm{d}y}{\mathrm{d}x} - y\right\} - 1.$$

When we multiply by x^2 and rearrange, we get

$$x\,\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2, \qquad \text{from which} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + x \quad \text{for } x \neq 0.$$

It follows that

$$f(x,y) = \frac{y}{x} + x = \frac{y + x^2}{x},$$

so the differential equation of the orthogonal trajectories becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{f(x,y)} = -\frac{x}{y + x^2},$$

which is more conveniently written in the form

$$0 = \left(y + x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + x.$$

It is not hard to find the integrating factor e^y. When we apply this, we get

$$\begin{aligned} 0 &= \left(y+x^2\right) e^{2y}\,\mathrm{d}y + x\,e^{2y}\,\mathrm{d}x = y\,e^{2y}\,\mathrm{d}y + \frac{1}{2}\,x^2\,\mathrm{d}\left(e^{2y}\right) + \frac{1}{2}\,e^{2y}\,\mathrm{d}\left(x^2\right) \\ &= \frac{1}{2}\,\mathrm{d}\left\{\left(y-\frac{1}{2}\right)e^{2y}\right\} + \frac{1}{2}\,\mathrm{d}\left(x^2 e^{2y}\right) = \frac{1}{2}\,\mathrm{d}\left\{\left(x^2+y-\frac{1}{2}\right)e^{2y}\right\}, \end{aligned}$$

hence by integration,

$$\left(x^2+y-\frac{1}{2}\right)e^{2y} = C, \qquad c \text{ arbitrary constant.}$$

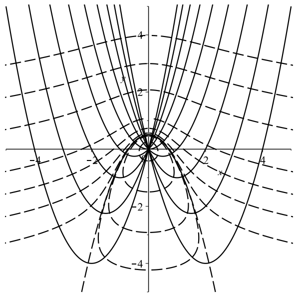


Figure 3.22: *Some parabolas $y = x^2 + ax$, (solid curves), and some of the corresponding orthogonal trajectories (dashed curves).*

An ALTERNATIVE solution method is the following. If we write $t = x^2$, then we get the differential form

$$0 = (y+t)\,\mathrm{d}y + x\,\mathrm{d}x = (y+t)\,\mathrm{d}y + \frac{1}{2}\,\mathrm{d}t.$$

If $t = t(y)$ is considered as a function in y, then this equation can be written

$$\frac{\mathrm{d}t}{\mathrm{d}y} + 2t + 2y = 0,$$

which is linear in t, so either we use the elementary solution formula from a Calculus course, or we apply Chapter 4 to get

$$t = x^2 = C\,e^{-2y} + e^{-2y}\int 2ye^{2y}\,\mathrm{d}y = C\,e^{-2y} + e^{-2y}\left\{ye^{2y} - \int e^{2y}\,\mathrm{d}y\right\} = C\cdot e^{-2y} + y - \frac{1}{2},$$

and we get by a rearrangement that the complete solution is implicitly given by

$$\left(x^2+y-\frac{1}{2}\right)e^{2y} = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

ALTERNATIVELY we express x as a function of y,

$$x = \pm\sqrt{\frac{1}{2} - y + C\cdot e^{-2y}}, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \Diamond$$

Example 3.26 *Given the system of hyperbolas $y^2 - x^2 = a$, where $a \in \mathbb{R}$ is an arbitrary constant. We shall find the system of trajectories, which intersect this system of hyperbolas at the fixed angle $\gamma \in]0, \pi[$.*

We eliminate the constant a by a differentiation,

$$2y\,\frac{\mathrm{d}y}{\mathrm{d}x} - 2x, \qquad \text{i.e.} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{x}{y} \quad \text{for } y \neq 0.$$

Then put $c := \cot\gamma$, and the differential equation of the trajectories is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + c\,f(x,y)}{c - f(x,y)} = \frac{1 + c\,\frac{x}{y}}{c - \frac{x}{y}} = \frac{y + cx}{cy - x},$$

from which

$$0 = (cy - x)\,\mathrm{d}y - (y + cx)\,\mathrm{d}x = cy\,\mathrm{d}y - cx\,\mathrm{d}x - (x\,\mathrm{d}y + y\,\mathrm{d}x) = \mathrm{d}\left\{\frac{c}{2}\left(y^2 - x^2\right) - xy\right\},$$

which has been proved to be an exact differential form. If $c = 0$, corresponding to the orthogonal trajectories, we get the well-known system of hyperbolas,

$$xy = k, \qquad k \in \mathbb{R} \text{ an arbitrary constant}, \qquad \gamma = \frac{\pi}{2}.$$

If instead $\gamma \neq \frac{\pi}{2}$, then $c \neq 0$, and we get by an integration and a rearrangement

$$y^2 - x^2 - \frac{2}{c}\,xy = k, \qquad k \in \mathbb{R} \text{ is an arbitrary constant},$$

which is another system of hyperbolas with the asymptotes

$$y = \frac{1 \pm \sqrt{1 + c^2}}{c}\,x, \qquad c \neq 0,$$

where these asymptotes are also trajectories. ♢

Example 3.27 *Find the system of curves, which is orthogonal to the system om hyperbolas,*

$$x^2 - y^2 + 2cxy = 1, \qquad c \textit{ arbitrary constant.}$$

We shall first find the differential equation for the system of hyperbolas, i.e. we shall eliminate the constant c. From

$$2c = \frac{1 - x^2 + y^2}{xy}$$

follows by differentiation,

$$0 = \frac{1}{x^2y^2}\left\{(-2x + 2yy')xy - \left(1 - x^2 + y^2\right)(y + xy')\right\},$$

hence by a small calculation,

$$\begin{aligned} 0 &= 2yy' \cdot xy - 2x^2y - y\left(1 - x^2 + y^2\right) - x\left(1 - x^2 + y^2\right)y' \\ &= x\left(2y^2 - 1 + x^2 - y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(2x^2 + 1 - x^2 + y^2\right) \\ &= x\left(x^2 + y^2 - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(x^2 + y^2 + 1\right), \end{aligned}$$

or, as a differential form,

$$x\left(x^2+y^2-1\right)\,\mathrm{d}y-y\left(x^2+y^2+1\right)\,\mathrm{d}x=0.$$

The differential equation of the orthogonal curves is then

$$x\left(x^2+y^2-1\right)\,\mathrm{d}x+y\left(x^2+y^2+1\right)\,\mathrm{d}y=0.$$

This can be written (multiply by 2 and reduce)

$$\begin{aligned}0 &= \left(x^2+y^2-1\right)\,\mathrm{d}\left(x^2\right)+\left(x^2+y^2+1\right)\,\mathrm{d}\left(y^2\right)\\ &= y^2\,\mathrm{d}\left(x^2\right)+x^2\,\mathrm{d}\left(y^2\right)+\left(x^2-1\right)\,\mathrm{d}\left(x^2\right)+\left(y^2+1\right)\,\mathrm{d}\left(y^2\right)\\ &= \mathrm{d}\left(x^2\cdot y^2+\frac{1}{2}\left(x^2-1\right)^2+\frac{1}{2}\left(y^2+1\right)^2\right).\end{aligned}$$

Multiply once more by 2 and integrate, (for technical reasons we use the constant $C+2$),

$$C+2=2x^2y^2+x^4-2x^2+1+y^4+2y^2+1=\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)+2,$$

which is reduced to

$$\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=C, \qquad C \text{ arbitrary constant.}$$

This is a system of *Cassini curves*. ◊

4 The linear differential equation of first order

4.1 Theoretical explanations

The linear differential equation of first order has the structure

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x),$$

where $f(x)$ and $g(x)$ are given functions. Its solution is the best commonly known of all equations. Let $F(x) := \int f(x)\,\mathrm{d}x$. Then $e^{F(x)}$ is an integrating factor, because when the equation is multiplied by this function, we get

$$e^{F(x)}\,g(x) = e^{F(x)}\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)e^{F(x)}y = \frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{F(x)}\,y\right\},$$

hence by integration,

$$e^{F(x)}\,y = \int e^{F(x)}\,g(x)\,\mathrm{d}x + C,$$

followed by a rearrangement,

$$y = e^{-F(x)}\int e^{F(x)}\,g(x)\,\mathrm{d}x + C\cdot e^{-F(x)},$$

which is the well-known formula.

4.2 Examples

Since this is the most commonly used case we include quite a few simple examples in the following.

Example 4.1 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + x = t.$$

Since $f(t) = 1$, the integrating factor is $\exp\left(\int f(t)\,\mathrm{d}t\right) = e^t$, so

$$t\,e^t = e^t\,\frac{\mathrm{d}x}{\mathrm{d}t} + e^t\,x = e^t\,\frac{\mathrm{d}x}{\mathrm{d}t} + x\,\frac{e^t}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x\,e^t\right).$$

Integration gives

$$x\,e^t = \int t\,e^t\,\mathrm{d}t = (t-1)e^t + c,$$

so by a rearrangement,

$$x = t - 1 + c\,e^{-t}, \qquad \text{for } x \in \mathbb{R}, \text{ and } c \text{ an arbitrary constant.}$$

It was mentioned in Chapter 2 that it is possible to set up the equation of the equipotential curves for linear differential equations, but in general this equation is not easy to solve, unless the original

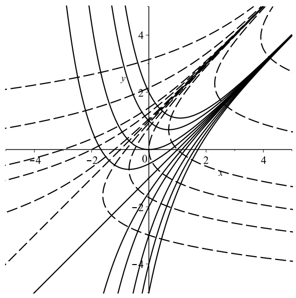


Figure 4.1: *Some solution curves of the equation* $\frac{dx}{dt} + x = t$ *(solid lines) and their equipotential curves (dashed).*

equation is homogeneous. The present case is one of the few exceptions. In fact, when we write the differential equation as a differential form, we get

$$\mathrm{d}x + (x - t)\,\mathrm{d}t = 0.$$

The equation of the equipotential curves is

$$\mathrm{d}t + (t - x)\,\mathrm{d}x = 0,$$

which is precisely the same equation, only with t and x interchanged. The equipotential curves are therefore given by

$$t = x - 1 + C\,e^{-x}, \qquad \text{for } t \in \mathbb{R}, \text{ and } C \text{ an arbitrary constant.} \qquad \diamondsuit$$

Example 4.2 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - 2t\,x = 2t, \qquad \textit{for } t \in \mathbb{R}.$$

Here, $f(t) = -2t$, so the integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(-\int 2t\,\mathrm{d}t\right) = \exp\left(-t^2\right).$$

From this we get

$$\exp\left(-t^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} - 2t\,\exp\left(-t^2\right)x = \frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\exp\left(-t^2\right)\right) = 2t\,e^{-t^2},$$

and then by integration,

$$x\,e^{-t^2} = \int 2t\,e^{-t^2}\,\mathrm{d}t + c = -e^{-t^2} + c,$$

so the compete solution is given by

$$x = -1 + c \cdot e^{t^2}, \qquad \text{for } t \in \mathbb{R}, \text{ and } c \text{ an arbitrary constant.}$$

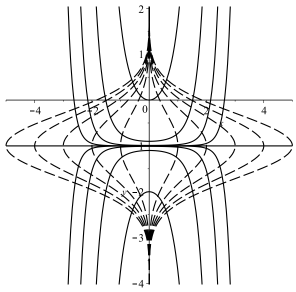


Figure 4.2: *Some solution curves of the equation* $\dfrac{dx}{dt} - 2tx = 2t$ *(solid curves), and some equipotential curves (dashed).*

It is immediately seen that $x = -1$ is a solution.

The differential equation is linear and *homogeneous* in the translated variable $y = x + 1$. Therefore, it is not hard to find the equipotential curves in this case. In fact, the original equation has the corresponding differential form $\mathrm{d}x - 2r(x+1)\,\mathrm{d}t$, so the differential equation of the equipotential curves, written as a differential form, is

$$2t(x+1)\,\mathrm{d}x + \mathrm{d}t = 0,$$

where we get by separating the variables and integrating, $(x+1)^2 = \ln|t| + c$, or, by solving with respect to t,

$$t = C\,\exp\bigl(-(x+1)^2\bigr)\,, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \diamond$$

Example 4.3 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + t^2 x = t^3 + 1, \qquad t \in \mathbb{R}.$$

From $f(t) = t^2$ follows that the integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(\int t^2\,\mathrm{d}t\right) = \exp\left(\frac{1}{3}\,t^3\right),$$

so when we multiply the original equation with this expression, we get

$$\exp\left(\frac{1}{3}\,t^3\right)\frac{\mathrm{d}x}{\mathrm{d}t} + t^2\exp\left(\frac{1}{3}\,t^3\right)x = \frac{\mathrm{d}}{\mathrm{d}t}\left\{x\,\exp\left(\frac{1}{3}\,t^3\right)\right\} = \left(t^3+1\right)\exp\left(\frac{1}{3}\,t^3\right),$$

from which by integration

$$x\,\exp\left(\frac{1}{3}\,t^3\right) = \int \left(t^3+1\right)\exp\left(\frac{1}{3}\,t^3\right)\,\mathrm{d}t + c, \qquad c \text{ arbitrary constant.}$$

This integral does not look nice. The usual procedure is to perform a series of partial integrations, but we may also inspect the original equation. All coefficients are polynomials, so a good idea is to

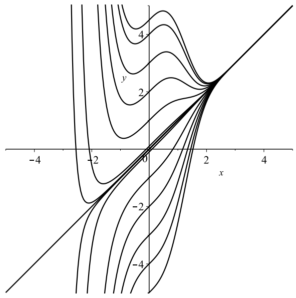


Figure 4.3: *Some solution curves of the equation* $\dfrac{dx}{dt} + t^2\, x = t^3 + 1$.

guess a partial solution x_0 as a polynomial as well, because then we shall stay within the realm of polynomials. The right hand side is of degree 3, and due to the term $t^2 x$ our guess should be at most a polynomial $x_0 = at + b$ of degree 1 in t. We then immediately see that $x_0 = t$ is a partial solution. Finally, using the linearity of the equation, we conclude that the complete solution is given by

$$x = t + c \cdot \exp\left(-\frac{1}{3}\, t^3\right), \qquad \text{for } t \in \mathbb{R} \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

Example 4.4 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - 2t\,x = 2t^3 - 2t^2 + 1, \qquad t \in \mathbb{R}.$$

All coefficients are polynomials, so a good strategy is first to try to guess a polynomial. The right hand side is a polynomial of third degree, so the term $-2tx$ on the left hand side forces us to guess a polynomial of second degree, i.e.

$$x_0 = at^2 + bt + k, \qquad \text{where} \qquad \frac{\mathrm{d}x_0}{\mathrm{d}t} = 2at + b.$$

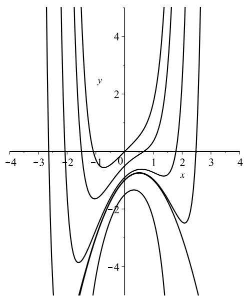


Figure 4.4: *Some solution curves of the equation* $\dfrac{dx}{dt} - 2t\,x = 2t^3 - 2t^2 + 1.$

By insertion,

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = 2at + b - 2at^3 - 2bt^2 - 2kt = -2at^3 - 2bt^2 + 2(a-k)t + b,$$

which is equal to $2t^3 - 2t^2 + 1$, if

$$-2a = 2, \qquad -2b = -2, \qquad 2(a-k) = 0, \qquad b = 1,$$

and we see that $a = -1$, $b = 1$ and $k = -1$ solve the problem, so

$$x_0(t) = -t^2 + t - 1$$

is a particular solution.

Since $f(t) = -2t$, the integrating factor is $\exp(\int(-2t)\,\mathrm{d}t) = \exp\left(-t^2\right)$, so the inverse is a solution of the corresponding homogeneous equation. The complete solution is then

$$x = -t^2 + t - 1 + c \cdot \exp\left(t^2\right), \qquad \text{for } t \in \mathbb{R}, \text{ and } c \text{ an arbitrary constant.}\Diamond$$

Example 4.5 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + 3t^2\,x = t^2, \qquad \textit{for } t \in \mathbb{R}.$$

Clearly, the constant function $x = \dfrac{1}{3}$ is a particular solution. The homogeneous equation has the solution $c \cdot \exp\left(-t^3\right)$, so the complete solution is

$$x = \frac{1}{3} + c \cdot e^{-t^3}, \qquad \text{for } t \in \mathbb{R}, \text{ and } c \text{ an arbitrary constant.}$$

Here, we do not present a figure, because e^{-t^3} for even small $|t|$ is either very large, or very small.

Alternatively, we introduce a new dependent variable, $y = x - \dfrac{1}{3}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 3t^2\,y = \frac{\mathrm{d}x}{\mathrm{d}t} + 3t^2\,x - t^2 = 0,$$

which is an homogeneous equation in y. Hence,

$$y = x - \frac{1}{3} = c \cdot \exp\left(-t^3\right),$$

and thus by a rearrangement,

$$x = \frac{1}{3} + c \cdot e^{-t^3} \text{ for } t \in \mathbb{R} \text{ and } c \text{ and arbitrary constant.} \qquad \diamondsuit$$

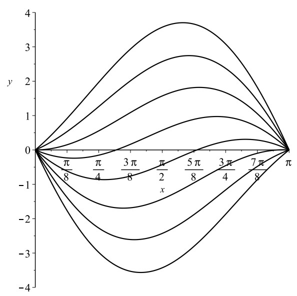


Figure 4.5: *Some solution curves of the equation* $\dfrac{dx}{dt} - \dfrac{\cos t}{\sin t}\,x = \sin t$.

Example 4.6 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + x = \cos t, \qquad \textit{for } t \in \mathbb{R}.$$

We immediately see that e^t is an integrating factor, so

$$e^t\,\frac{\mathrm{d}x}{\mathrm{d}t} + e^t\,x = \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^t\right\} = e^t\,\cos t,$$

hence by integrating and using that $\cos t = \Re\left(e^{it}\right)$,

$$\begin{aligned} e^t x &= \int e^t \cos t \, \mathrm{d}t + c = \Re \int e^{t(1+i)} \, \mathrm{d}t + c = \Re\left\{\frac{1}{1+i}\, e^{t(1+i)}\right\} + c \\ &= \frac{1}{2}\, e^t\, \Re\left\{(1+i)e^{it}\right\} + c = \frac{1}{2}\, e^t \{\cos t + \sin t\} + c, \end{aligned}$$

and the complete solution is

$$x = \frac{1}{2}(\cos t + \sin t) + c \cdot e^{-t}, \qquad \text{for } t \in \mathbb{R}, \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

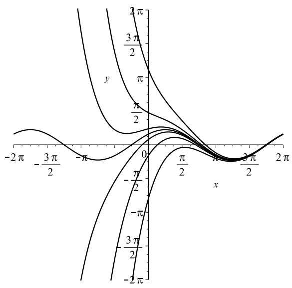


Figure 4.6: *Some solution curves of the equation* $\dfrac{dx}{dt} + x = \cos t$.

Example 4.7 *Try to find with the methods described in this section the complete solution of the differential equation*

$$\frac{dx}{dt} + t \cdot x = t^2 \qquad \textit{for } t \in \mathbb{R}.$$

Since $f(t) = t$, it follows that an integrating factor is

$$\exp\left(\int f(t)\, \mathrm{d}t\right) = \exp\left(\int t \, \mathrm{d}t\right) = \exp\left(\frac{1}{2}\, t^2\right),$$

and the differential equation is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(\frac{1}{2}\, t^2\right) \cdot x\right\} = t^2 \cdot \exp\left(\frac{1}{2}\, t^2\right).$$

When we integrate, we get

$$\exp\left(\frac{1}{2}\, t^2\right) \cdot x = c + \int t^2 \cdot \exp\left(\frac{1}{2}\, t^2\right) \, \mathrm{d}t,$$

where the integral can be expressed by means of the *error function*, which has not yet been introduced. Anyway, we have formally that the complete solution is given by

$$x = \exp\left(-\frac{1}{2}\, t^2\right) \int t^2 \cdot \exp\left(\frac{1}{2}\, t^2\right) \, \mathrm{d}t + c \cdot \exp\left(-\frac{1}{2}\, t^2\right).$$

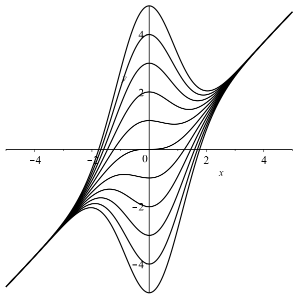


Figure 4.7: *Some solution curves of the equation* $\frac{dx}{dt} + t \cdot x = t^2$. *We have used that the solution can be expressed in terms of the error function, which has not yet been introduced here.*

There are different methods to find the value of the particular solution, which here is given by an integral. One of them is to assume that x has a convergent series expansion,

$$x = \sum_{n=0}^{+\infty} a_n \, x^n,$$

and then insert this series expansion into the differential equation and find the a_n by means of some difference equation. We shall not here go further into this theory. $\Diamond$

Example 4.8 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \frac{1}{t}\,x = -2t^2. \qquad \textit{for } t > 0.$$

Here, $f(t) = \dfrac{1}{t}$, $t > 0$, so the integrating factor is $\exp(\int f(t)\,\mathrm{d}t) = \exp(\ln t) = t$, and we get

$$t\,\frac{\mathrm{d}x}{\mathrm{d}t} + x = \frac{\mathrm{d}}{\mathrm{d}t}\,(t\,x) = -2t^3.$$

Hence, by integration,

$$t\,x = -\frac{2}{4}\,t^4 + c = -\frac{1}{2}\,t^4 + c,$$

and the complete solution is given by

$$x = -\frac{1}{2}\,t^3 + \frac{c}{t}, \qquad \text{for } t > 0, \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

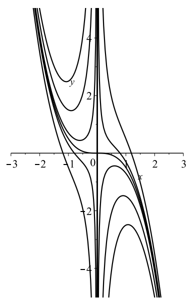


Figure 4.8: *Some solution curves of the equation* $\dfrac{dx}{dt} + \dfrac{1}{t}\,x = -2t^2$.

Example 4.9 *Find the complete solution of the differential equation*

$$t\,\frac{dx}{dt} - 2x = t^5 \qquad \textit{for } t \in \mathbb{R}.$$

We first norm this equation, i.e. we divide by $t \neq 0$, so the coefficient of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ becomes 1,

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t}\,x = t^4, \qquad \text{for } t \neq 0.$$

From $f(t) = -\dfrac{2}{t}$ follows that the integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(-\int \frac{2}{t}\,\mathrm{d}t\right) = \exp(-2\ln(-t)) = \frac{1}{t^2}, \qquad \text{for } t \neq 0.$$

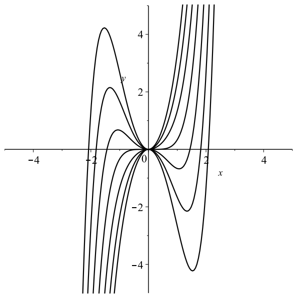


Figure 4.9: *Some solution curves of the equation* $t\,\dfrac{dx}{dt} - 2x = t^5$. *Notice in particular that all solution curves go through the point* $(0,0)$. *This is a singular point, because the coefficient* t *of the highest order term,* $\dfrac{dx}{dt}$, *is 0 at this point, and when* $t = 0$ *the remaining part* $2x + t^2 = 2x = 0$ *for* $x = 0$.

Then

$$\frac{1}{t^2}\,\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t^3}\,x = \frac{1}{t^2}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t^2}\right)\cdot x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{t^2}\right) = t^2,$$

hence by integration,

$$\frac{x}{t^2} = \frac{1}{3}\,t^3 + c, \qquad \text{for } t \neq 0,$$

from which by continuous continuation to $t = 0$,

$$x = \frac{1}{3}\,t^5 + c\,t^2, \qquad \text{for } t \in \mathbb{R} \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

Example 4.10 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - \frac{1}{t}\,x = 1 \qquad \textit{for } t > 0.$$

We divide by $t > 0$ and get

$$\frac{1}{t} = \frac{1}{t}\,\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{t^2}\,x = \frac{1}{t}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\right)\cdot x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{y}\right).$$

Then integrate this equation,

$$\frac{x}{t} = \ln t + c, \qquad \text{hence} \qquad x = t\cdot\ln t + c\cdot t \quad \text{for } t > 0,$$

where c is an arbitrary constant.

Note also that the equivalent equation $\mathrm{d}x = \left(1 + \frac{x}{t}\right)\mathrm{d}t$ is homogeneous of degree 0, so it can be solved by using methods from Chapter 7. $\Diamond$

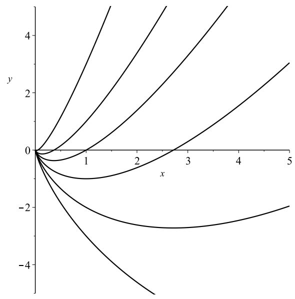


Figure 4.10: *Some solution curves of the equation* $\dfrac{dx}{dt} - \dfrac{1}{t}\,x = 1.$

Example 4.11 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \left(1 + \frac{1}{t}\right) x = \frac{1}{t}, \qquad \textit{for } t > 0.$$

An integrating factor is given by

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(\int \left(1 + \frac{1}{t}\right)\,\mathrm{d}t\right) = \exp(t + \ln t) = t \cdot e^t, \qquad \text{for } t > 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{t \cdot e^t \cdot x\right\} = \frac{1}{t} \cdot t\,e^t = e^t.$$

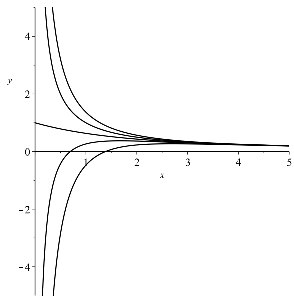


Figure 4.11: *Some solution curves of the equation* $\dfrac{dx}{dt} + \left(1 + \dfrac{1}{t}\right) x = \dfrac{1}{t}.$

By an integration,

$$t \cdot e^t \cdot x = e^t + c,$$

and the complete solution is given by

$$x = \frac{1}{t} + c \cdot \frac{1}{t}\,e^{-t}.$$

Choosing $c = -1$ we get the special solution

$$x = \frac{1}{t}\left(1 - e^{-t}\right) \to 1 \text{ for } t \to 0+,$$

so this solution is bounded. All other solutions, i.e. when $c \neq -1$, are unbounded in the neighbourhood of $t = 0$. $\lozenge$

Example 4.12 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - \frac{2}{t}\,x = 2t + 5, \qquad t \neq 0.$$

Here, $f(t) = -\dfrac{2}{t}$, $t \neq 0$, so an integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(\frac{2}{t}\,\mathrm{d}t\right) = \exp(-2\ln|t|) = \frac{1}{t^2}. \qquad \text{for } t \neq 0.$$

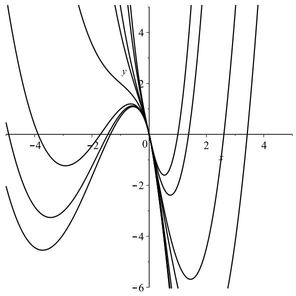


Figure 4.12: *Some solution curves of the equation* $\dfrac{dx}{dt} - \dfrac{2}{t}\,x = 2t + 5$. *Notice in particular that all solution curves are continued continuously through the point* $(0,0)$. *This is a singular point, because it is a singular point for the corresponding nonnormed equation* $t\,dx = \left(2x + 2t^2 + 5t\right)\,dt$.

By a reformulation,

$$\frac{1}{t^2}\,\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t^3}\,x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t^2}\right) = \frac{2}{t} + \frac{5}{t^2}, \qquad \text{for } t \neq 0.$$

Then by integration,

$$\frac{x}{t^2} = 2\ln|t| - \frac{5}{t} + c = \ln\left(t^2\right) - \frac{5}{t} + c \qquad \text{for } t \neq 0.$$

When this equation is multiplied by t^2, we see that it can be continued continuously to $t = 0$, so we finally get

$$x = 2t^2\ln\left(t^2\right) - 5t + c\,t^2, \qquad \text{for } t \in \mathbb{R} \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

Example 4.13 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \left(2t - \frac{1}{t}\right)x = 2t^2, \qquad \textit{for } t > 0.$$

From $f(t) = 2t - \dfrac{1}{t}$, $t > 0$, follows that

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(\int\left(2t - \frac{1}{t}\right)\,\mathrm{d}t\right) = \exp\left(t^2 - \ln t\right) = \frac{1}{t}\,e^{t^2},$$

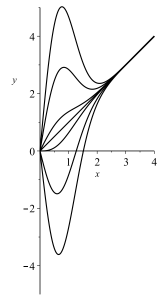


Figure 4.13: *Some solution curves of the equation* $\dfrac{dx}{dt} + \left(2t - \dfrac{1}{t}\right) x = 2t^2$.

is an integrating factor, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\,e^{t^2}\cdot x\right) = 2t\,e^{t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t^2}\right).$$

By integration,

$$\frac{1}{t}\,e^{t^2}\cdot x = e^{t^2} + c,$$

so

$$x = t + c\cdot t\,e^{-t^2}, \qquad \text{for } t > 0 \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

Example 4.14 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \frac{2}{1-t^2}\,x = 1 - t, \qquad \textit{for } t \in \,]-1,1[.$$

Here, the integrating factor is

$$\begin{aligned}\exp\left(\int f(t)\,\mathrm{d}t\right) &= \exp\left(\int \frac{2}{1-t^2}\,\mathrm{d}t\right) = \exp\left(\int \left\{\frac{1}{1-t} + \frac{1}{1+t}\right\}\mathrm{d}t\right)\\ &= \exp\left(\ln\left|\frac{1+t}{1-t}\right|\right) = \exp\left(\ln\left(\frac{1+t}{1-t}\right)\right) = \frac{1+t}{1-t} \quad \text{for } |t| < 1,\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1+t}{1-t}\cdot x\right\} = \frac{1+t}{1-t}\cdot(1-t) = 1+t, \qquad \text{for } t \in \,]-1,1[.$$

Integration gives

$$\frac{1+t}{1-t}\cdot x = \frac{1}{2}(1+t)^2 + c,$$

hence

$$x = \frac{1}{2}\left(1-t^2\right) + c\cdot\frac{1-t}{1+t}, \qquad \text{for } t \in \,]-1,1[, \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

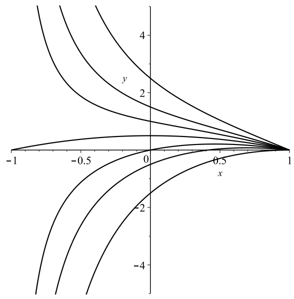


Figure 4.14: *Some solution curves of the equation* $\frac{dx}{dt} + \frac{2}{1-t^2}\,x = 1 - t$.*(Not to scale.)*

Example 4.15 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \frac{2t}{1+t^2}\,x = 1, \qquad \text{for } t \in \mathbb{R}.$$

By inspection, $1 + t^2$ is an integrating factor,

$$1 + t^2 = \left(1+t^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} + 2t \cdot x = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1+t^2\right)x\right\},$$

hence by integration,

$$\left(1+t^2\right)x = t + \frac{1}{3}\,t^3 + c.$$

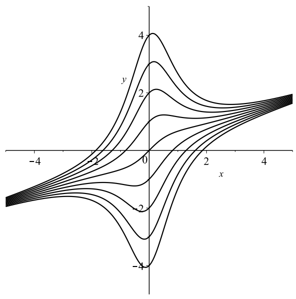


Figure 4.15: *Some solution curves of the equation* $\frac{dx}{dt} + \frac{2t}{1+t^2}\,x = 1$.

The complete solution is

$$x = \frac{t}{1+t^2} + \frac{1}{3}\,\frac{t^3}{1+t^2} + c\,\frac{1}{1+t^2} = \frac{t}{3} \cdot \frac{t^2+3}{t^2+1} + \frac{c}{t^2+1} \qquad \text{for } t \in \mathbb{R},$$

where c is an arbitrary constant. $\Diamond$.

Example 4.16 *Find the complete solution of the differential equation*

$$\frac{dx}{dt}+\frac{2t}{1+t^2}\,x=\frac{1}{2t^2+1},\qquad \textit{for } t\in\mathbb{R}.$$

As in the previous example, $1+t^2$ is an integrating factor, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1+t^2\right)x\right\}=\frac{1+t^2}{2t^2+1}=\frac{1}{2}+\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{1+(\sqrt{2}t)^2}.$$

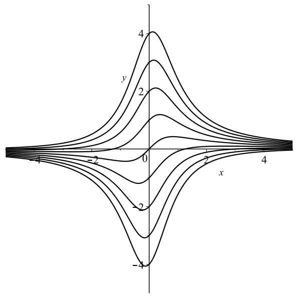


Figure 4.16: *Some solution curves of the equation* $\dfrac{dx}{dt}+\dfrac{2t}{1+t^2}\,x=\dfrac{1}{2t^2+1}$.

Hence, by integration,

$$\left(1+t^2\right)x=\frac{1}{2}\,t+\frac{1}{2\sqrt{2}}\,\arctan(\sqrt{2}\,t)+c,$$

and the complete solution is given by

$$x=\frac{1}{2}\,\frac{t}{1+t^2}+\frac{1}{2\sqrt{2}}\,\frac{\arctan(\sqrt{2}\,t)}{1+t^2}+\frac{c}{1+t^2},\qquad \text{for } t\in\mathbb{R},$$

where c is an arbitrary constant. ◊

Example 4.17 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\, y = -2x.$$

We must of course assume that $x \neq \pm 1$. The equation is linear in y, so it can either be solved in the standard way, or we see "by a divine inspiration" that $1/\left(1-x^2\right)$ is an integrating factor for $x \neq \pm 1$. In fact, reading the equation from the right to the left,

$$-\frac{2x}{1-x^2} = \frac{2x}{x^2-1} = \frac{1}{1-x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2x}{\left(1-x^2\right)^2}\, y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{1-x^2}\right),$$

so by integration,

$$\frac{y}{1-x^2} = \int \frac{2x}{x^2-1}\,\mathrm{d}x + C = \ln\left|x^2-1\right| + C,$$

and the complete solution is

$$y = \left(1-x^2\right)\ln\left|1-x^2\right| + C\cdot\left(1-x^2\right),$$

where C is an arbitrary constant. $\Diamond$

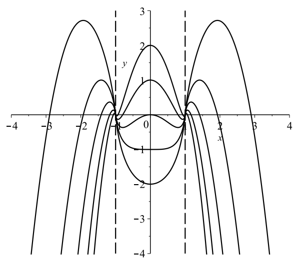


Figure 4.17: *Some solution curves of the equation* $\dfrac{dy}{dx} + \dfrac{2x}{1-x^2}\, x = -2x.$.

Example 4.18 *Find the complete solution of the equation*

$$3xy^2\,\frac{dy}{dx} = 2y^3 + 3x^5.$$

This equation looks bad, but a closer look reveals that it is actually linear in $v = y^3$, because

$$3xy^2\,\frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}\left(y^3\right)}{\mathrm{d}x} = x\,\frac{\mathrm{d}v}{\mathrm{d}x},$$

and the equation is equivalent to the following linear equation in v,

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} - 2v = 3x^5.$$

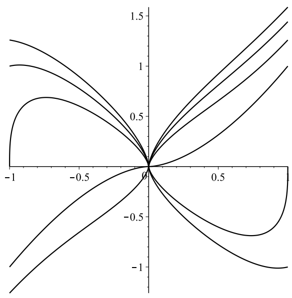


Figure 4.18: *Some solution curves of the equation* $3xy^2\,\dfrac{dy}{dx} = 2y^3 + 3x^5$. *The equation itself is not linear, though it is solved by means of the theory of linear equations. This may explain why the solution curves look rather strange.*

We either solve this equation by the standard method, or we notice that $1/x^3$ is an integrating factor, in which case we get

$$3x^2 = \frac{1}{x^2}\,\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{2}{x^3}\,v = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v}{x^2}\right).$$

Hence by integration,

$$\frac{v}{x^2} = x^3 + C, \qquad \text{i.e.} \qquad y^3 = x^5 + Cx^2,$$

and the complete solution is

$$y = \left\{x^5 + Cx^2\right\}^{\frac{1}{3}}.$$

where C is an arbitrary constant. ◊

Example 4.19 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{1}{2x+y}, \qquad 2x + y \neq 0.$$

When $2x + y \neq 0$, this is equivalent to

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 2x + y,$$

which is linear in x, where y is the independent variable. We get by the usual solution formula,

$$x = c \cdot e^{2y} + e^{2y} \int y \cdot e^{-2y}\,\mathrm{d}y = -\frac{1}{2}\,y - \frac{1}{4} + c \cdot e^{2y}, \qquad c \text{ arbitrary constant.} \qquad \Diamond$$

Example 4.20 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + \frac{1+\tan^2 t}{\tan t}\, x = 1, \qquad \textit{for } t \in \left]0, \frac{\pi}{2}\right[.$$

From

$$f(t)\,\mathrm{d}t = \frac{1+\tan^2 t}{\tan t}\,\mathrm{d}t = \frac{1}{\tan t}\,\mathrm{d}\tan t = \,\mathrm{d}\ln\tan t, \qquad \text{for } t \in \left]0, \frac{\pi}{2}\right[,$$

follows that an integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp(\ln\tan t) = \tan t,$$

so the differential equation is transformed into

$$\tan t = \tan t \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \left(1+\tan^2 t\right)\cdot x = \frac{\mathrm{d}}{\mathrm{d}t}\{x \cdot \tan t\}.$$

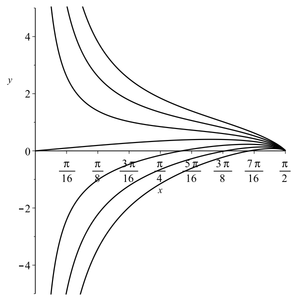


Figure 4.19: *Some solution curves of the equation* $\dfrac{dx}{dt} + \dfrac{1+\tan^2 t}{\tan t}\, x = 1$ *in the interval* $\left]0, \dfrac{\pi}{2}\right[$.

Integration gives

$$x \cdot \tan t = c + \int \frac{\sin t}{\cos t}\,\mathrm{d}t = c - \ln\cos t.$$

The complete solution is given by

$$x = c\cdot\cot t - \cot t\cdot\ln\cos t \qquad \text{for } t \in \left]0, \frac{\pi}{2}\right[, \text{ and } c \text{an arbitrary constant.} \qquad \Diamond$$

Example 4.21 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - \tan t \cdot x = t, \qquad \textit{for } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

By inspection, $\cos t > 0$ is an integrating factor, and

$$\cos t \cdot \frac{\mathrm{d}x}{\mathrm{d}t} - \sin t \cdot x = \frac{\mathrm{d}}{\mathrm{d}t}\{\cos t \cdot x\} = t \cdot \cos t.$$

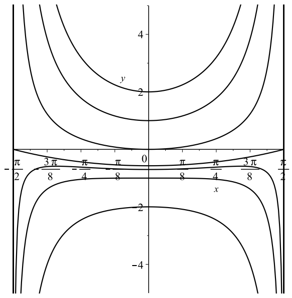


Figure 4.20: *Some solution curves of the equation* $\dfrac{dx}{dt} - \tan t \cdot x = t$.

We get by integration,

$$\cos t \cdot x = c + \int t \cos t \,\mathrm{d}t = c + t \sin t - \int \sin t \,\mathrm{d}t = c + t \sin t + \cos t.$$

The complete solution is

$$x = 1 + t \cdot \tan t + \frac{c}{\cos t} \qquad \text{for } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \text{ and } c \text{ arbitrary.}$$

As seen on Figure 4.20 there is only one bounded solution. It is given by the constant $c = -\dfrac{\pi}{2}$. $\Diamond$

Example 4.22 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - \frac{\cos t}{\sin t}\, x = \sin t, \qquad \textit{for } t \in \,]0, \pi[.$$

When this equation is divided by $\sin t > 0$ for $t \in \,]0, \pi[$, we get

$$1 = \frac{1}{\sin t}\,\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\cos t}{\sin^2 t}\, x = \frac{1}{\sin t}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sin t}\right) x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\sin t}\right),$$

so an integration gives

$$\frac{x}{\sin t} = t + c, \qquad \text{for } t \in \,]0, \pi[,$$

and the complete solution is

$$x = t \cdot \sin t + c \cdot \sin t \qquad \text{for } t \in \,]0, \pi[, \text{ and } c \text{ an arbitrary constant.} \qquad \Diamond$$

Example 4.23 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} + (1 + \tan t)x = \cos t \qquad \textit{for } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

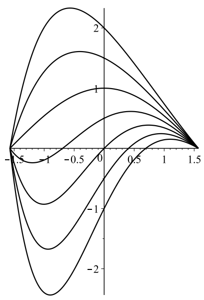


Figure 4.21: *Some solution curves of the equation* $\dfrac{dx}{dt} + (1 + \tan t)x = \cos t.$

Here, $f(t) = 1 + \tan t$, so an integrating factor is

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(\int \left\{1 + \frac{\sin t}{\cos t}\right\} \mathrm{d}t\right) = e^t \cdot \exp(-\ln|\cos t|) = \frac{e^t}{\cos t},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{e^t}{\cos t} \cdot x\right\} = \frac{e^t}{\cos t} \cdot \cos t = e^t.$$

Integration of this equation gives

$$\frac{e^t}{\cos t} \cdot x = e^t + c,$$

so the complete solution is

$$x = \cos t + c \cdot e^{-t} \cos t, \qquad \text{for } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \text{ and } c \text{ an arbitrary constant.}$$

Example 4.24 *Find the complete solution of the differential equation*

$$\frac{dx}{dt} - \frac{1}{t}\,x = \frac{1}{t+1}\sqrt{\frac{t-1}{t+1}}, \qquad \textit{for } t > 1.$$

From $f(t) = -\dfrac{1}{t}$, $t > 1$, follows that

$$\exp\left(\int f(t)\,\mathrm{d}t\right) = \exp\left(-\int \frac{\mathrm{d}t}{t}\right) = \exp(-\ln t) = \frac{1}{t}, \qquad \text{for } t > 1,$$

is an integrating factor. Then,

$$\frac{1}{t}\,\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{t^2}\,x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{t}\right) = \frac{1}{t}\cdot\frac{1}{t+1}\sqrt{\frac{t-1}{t+1}}.$$

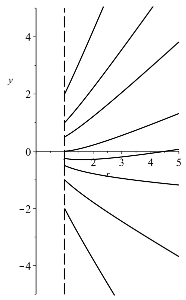


Figure 4.22: *Some solution curves of the equation* $\dfrac{dx}{dt} - \dfrac{1}{t}\,x = \dfrac{1}{t+1}\sqrt{\dfrac{t-1}{t+1}}$.

Using the strategy always to give a nasty expression another name we apply in the resulting integral below the monotonous substitution

$$u = \sqrt{\frac{t-1}{t+1}}, \qquad \text{i.e.} \qquad u^2 = \frac{t-1}{t+1} \in]0,1[\qquad \text{for } t > 1,$$

hence

$$t = \frac{1+u^2}{1-u^2} = \frac{2}{1-u^2} - 1 \qquad \text{and} \qquad \mathrm{d}t = \frac{4u}{(1-u^2)^2}\,\mathrm{d}u,$$

and we get for $t > 1$ (and $u \in]0,1[$),

$$\begin{aligned}
\frac{x}{t} &= c + \int \frac{1}{t}\cdot\frac{1}{t+1}\sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t = c + \int \frac{1-u^2}{1+u^2}\cdot\frac{1-u^2}{2}\cdot u\cdot\frac{4u}{(1-u^2)^2}\,\mathrm{d}u, \qquad u = \sqrt{\frac{t-1}{t+1}} \\
&= c + \int \frac{2u^2}{1+u^2}\,\mathrm{d}u = c + \int\left\{2 - \frac{2}{1+u^2}\right\}\mathrm{d}u = c + 2u - 2\arctan u, \qquad u = \sqrt{\frac{t-1}{t+1}} \\
&= c + 2\sqrt{\frac{t-1}{t+1}} - 2\arctan\left(\sqrt{\frac{t-1}{t+1}}\right), \qquad t > 1,
\end{aligned}$$

and the complete solution is given by

$$x = ct + 2t\sqrt{\frac{t-1}{t+1}} - 2t\,\arctan\left(\sqrt{\frac{t-1}{t+1}}\right), \qquad t > 1 \text{ and } c \text{ an arbitrary constant. } \Diamond$$

Example 4.25 Consider an electrical circuit of resistance R and inductance L, but without a capacity C. The impressed voltage is assumed to be the following function in time,

$$E(t) = E_0\,\sin\omega t, \qquad E_0,\,\omega > 0 \text{ positive constants.}$$

The current $I(t)$ then satisfies the following linear differential equation

$$E = I \cdot R + L\,\frac{\mathrm{d}I}{\mathrm{d}t},$$

which after a small rearrangement is written in the usual form of a linear differential equation of first order,

$$\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{R}{L}\,I = \frac{E_0}{L}\,\sin\omega t.$$

At time $t = 0$ the current is I_0, and it follows from the solution formula that

$$I(t) = \exp\left(-\frac{R}{L}\,t\right) \cdot \left\{ I_0 + \frac{E_0}{L}\int_0^t \exp\left(\frac{r}{L}\,t\right)\sin\omega t\,\mathrm{d}t \right\}.$$

The integral can be evaluated in various ways. For instance two partial integrations will lead to the result. Here we shall instead use complex functions. Using that $\sin\omega t = \Im\exp(i\omega t)$, where $\Im$ denotes the imaginary part, we get

$$\begin{aligned}
\int_0^t \exp\left(\frac{R}{L}\,t\right)\sin\omega t\,\mathrm{d}t &= \Im\int_0^t \exp\left(\frac{R}{L}\,t\right)\exp(i\omega t)\,\mathrm{d}t = \Im\int_0^t \exp\left(\left\{\frac{R}{L} + i\omega\right\} t\right)\,\mathrm{d}t \\
&= \Im\int_0^t \exp\left(\frac{R+iL\omega}{L}\,t\right) t = \Im\left[\frac{1}{R+iL\omega}\,\exp\left(\frac{R+iL\omega}{L}\,t\right)\right]_0^t \\
&= \frac{L}{R^2+L^2\omega^2}\left[\exp\left(\frac{R}{L}\,t\right)\,\Im\{(R-iL\omega)(\cos\omega t + i\sin\omega t)\}\right]_0^t \\
&= \frac{L}{R^2+L^2\omega^2}\left\{\exp\left(\frac{R}{L}\,t\right)\cdot\{R\sin\omega t - L\omega\cos\omega t\} + L\omega\right\},
\end{aligned}$$

so

$$I(t) = \exp\left(-\frac{R}{L}\,t\right)\cdot\left\{I_0 + \frac{LE_0\omega}{R^2+L^2\omega^2}\right\} + E_0\left\{\frac{R}{R^2+L^2\omega^2}\,\sin\omega t - \frac{L\omega}{R^2+L^2\omega^2}\,\cos\omega t\right\}.$$

Clearly, there exists $\gamma \in [0, 2\pi[$, such that

$$\cos\gamma = \frac{R}{\sqrt{R^2+L^2\omega^2}} \qquad \text{and} \qquad \sin\gamma = \frac{L\omega}{\sqrt{R^2+L^2\omega^2}}.$$

Using this γ the solution can also be written

$$I(t) = \exp\left(-\frac{R}{L}\,t\right)\cdot\left\{I_0 + \frac{LE_0\omega}{R^2+L^2\omega^2}\right\} + \frac{E_0}{\sqrt{R^2+L^2\omega}}\,\sin(\omega t - \gamma).$$

The first term is dying out in time, while the second term is a sinus oscillation with a phase shift and a change of the amplitude, so

$$I(t) \approx \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \gamma) \qquad \text{for large } t. \qquad \Diamond$$

5 Bernoulli's equation

5.1 Theoretical considerations

This equation, first studied by *Jacob Bernoulli (1654–1705)*, in 1695, has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + g(x)y^{\alpha}, \qquad \text{where } \alpha \neq 0 \text{ and } \alpha \neq 1. \tag{5.1}$$

Note that if $\alpha = 1$, then the equation is linear and homogeneous, and if $\alpha = 0$, then the equation is linear and inhomogeneous. This is the reason for excluding these trivial values of α.

The idea is to perform a transformation, which carries the Bernoulli equation into a linear differential equation of first order in a new variable. We shall see below, how this is done.

If $\alpha > 0$, then $y = 0$ is trivially a solution. If $\alpha < 0$, then $y(x) = 0$ is excluded from the domain.

Assume that $y \neq 0$. The trick is to divide (5.1) by y^{α} and reduce, i.e.

$$y^{-\alpha}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y^{1-\alpha} + g(x). \tag{5.2}$$

Put $z = y^{1-\alpha}$ and multiply by $1-\alpha$. Then (5.2) is transformed into

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-\alpha)f(x)z + (1-\alpha)g(x),$$

which is a linear and inhomogeneous equation in z, so if we put $F(x) = \int f(x)\,\mathrm{d}x$, the solution is

$$z = y^{1-\alpha} = (1-\alpha)e^{(1-\alpha)F(x)}\int g(x)e^{-(1-\alpha)F(x)}\,\mathrm{d}x + C\cdot e^{(1-\alpha)F(x)}, \tag{5.3}$$

where C is an arbitrary constant. Finally, y is found from (5.3) by extracting the $1-\alpha$ "root". In most cases this puts some restrictions on the possible values of the constant C and of the interval of definition.

We mention that the equipotential curves of course exist, but if we want to find explicitly the complete solution of the equipotential curves, then we must require that either $f(x) = g(x)$, or $f(x)$ and $g(x)$ are both constants. In fact, written as a differential form the original equation is

$$\mathrm{d}y - \{f(x)y + g(x)y^{\alpha}\}\,\mathrm{d}x = 0,$$

from which the differential equation of the equipotential curves becomes

$$\{f(x)y + g(x)y^{\alpha}\}\,\mathrm{d}y + \mathrm{d}x = 0.$$

If $f(x)$ and $g(x)$ are both constants, then the variables are already separated, so the equation can be integrated right away.

If instead $g(x) = f(x) \neq 0$, then the equation is written

$$f(x)\{y + y^{\alpha}\}\,\mathrm{d}y + \mathrm{d}x = 0,$$

so when we divide by $f(x)$ the variables are separated,

$$\{y + y^{\alpha}\}\,\mathrm{d}y + \frac{1}{f(x)}\,\mathrm{d}x = 0.$$

5.2 Examples

Example 5.1 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} = y^2,$$

Formally this is a Bernoulli equation, but at the same time the variables can be separated. Clearly, $y = 0$ is a solution. Assume that $y \neq 0$. Then we divide by y^2 to get

$$1 = \frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right), \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) = -1.$$

By integration,

$$\frac{1}{y} = -x + C, \qquad \text{i.e.} \qquad y = \frac{1}{C - x} \quad \text{for } x \neq C,$$

supplied with the rectilinear solution $y = 0$.

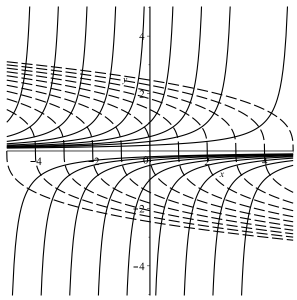


Figure 5.1: *Some solution curves of the differential equation* $\frac{dy}{dx} = y^2$ *(solid), and some of its equipotential curves (dashed).*

In this case it is easy to find the equipotential curves. First we write the equation above in its differential form, $\mathrm{d}y - y^2\,\mathrm{d}x = 0$, so the equation of the equipotential curves is

$$y^2\,\mathrm{d}y + \mathrm{d}x = 0, \qquad \text{or} \qquad \mathrm{d}\left(y^3\right) + 3\,\mathrm{d}x = 0.$$

We get by integration, $y^3 = k - 3x$, so the system of equipotential curves is given by

$$y = \sqrt[3]{k - 3x}, \qquad \text{for } x \neq \frac{k}{3},$$

where k is an arbitrary constant. ◊

Example 5.2 *Find the complete solution of the Bernoulli equation*

$$\left(1-x^2\right)\frac{dy}{dx}-xy=xy^2 \qquad \textit{for } x\in\,]-1,1[.$$

Clearly, $y=0$ is a solution. If $y\neq 0$, we divide by $-\left(1-x^2\right)y^2\neq 0$ to get

$$-\frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{1-x^2}\,\frac{1}{y}=-\frac{x}{1-x^2},$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right)+\frac{x}{1-x^2}\,\frac{1}{y}=-\frac{x}{1-x^2}.$$

Then

$$\begin{aligned}\frac{1}{y} &= \exp\left(-\int\frac{x}{1-x^2}\,\mathrm{d}x\right)\left\{C-\int\frac{x}{1-x^2}\,\exp\left(\frac{x}{1-x^2}\,\mathrm{d}x\right)\,\mathrm{d}x\right\}\\ &= \exp\left(\frac{1}{2}\ln\left(1-x^2\right)\right)\cdot\left\{C-\int\frac{x}{1-x^2}\,\exp\left(-\frac{1}{2}\ln\left(1-x^2\right)\right)\,\mathrm{d}x\right\}\\ &= \sqrt{1-x^2}\cdot\left\{C-\int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x\right\}=C\sqrt{1-x^2}-\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\\ &= -1+C\sqrt{1-x^2},\end{aligned}$$

so the complete solution is $y=0$ supplied with the family of curves

$$y=\frac{1}{C\sqrt{1-x^2}-1} \qquad \text{for } x\in\,]-1,1[\text{ and } |x|\neq\sqrt{1-\frac{1}{C^2}},$$

where C is an arbitrary constant.

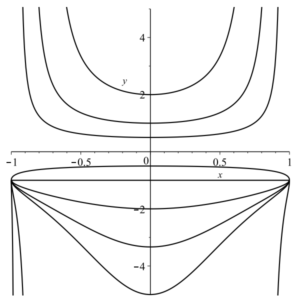


Figure 5.2: *Some solution curves of the differential equation* $\left(1-x^2\right)\dfrac{dy}{dx}-xy=xy^2$.

Alternatively it follows from the differential form of the equation,

$$\left(1-x^2\right)\,\mathrm{d}y-x\left(y+y^2\right)\,\mathrm{d}x=0,$$

that we can directly separate the variables,

$$0 = \frac{\mathrm{d}y}{y+y^2} - \frac{x\,\mathrm{d}x}{1-x^2} = \left(\frac{1}{y} - \frac{1}{1+y}\right)\mathrm{d}y + \frac{1}{2}\,\frac{\mathrm{d}\left(1-x^2\right)}{1-x^2} = \mathrm{d}\ln\left(\left|\frac{y}{1+y}\right|\frac{1}{1-x^2}\right),$$

from which by an integration followed by the exponential and a discussion of the sign of the constant,

$$\frac{y}{1+y}\cdot\frac{1}{\sqrt{1-x^2}} = C,$$

where C is an arbitrary constant. It is left to the reader to find y as a function in x.

It is possible in this case to find an implicit given expression of the equipotential curves. First we write the original equation in its differential form,

$$\left(1-x^2\right)\mathrm{d}y - x\left(y+y^2\right)\mathrm{d}x = 0.$$

This gives us the equation of the equipotential curves,

$$x\left(y+y^2\right)\mathrm{d}y + \left(1-x^2\right)\mathrm{d}x = 0,$$

from which

$$\mathrm{d}\left(\frac{1}{2}\,y^2 + \frac{1}{3}\,y^3\right) + \left(\frac{1}{x} - x\right)\mathrm{d}x = 0.$$

Then by integration,

$$\frac{1}{2}\,y^2 + \frac{1}{3}\,y^3 + \ln|x| - \frac{1}{2}\,x^2 = k,$$

where k is an arbitrary constant. Clearly, this expression is fairly complicated, so if we want to sketch the equipotential curves, we must use some implicit plot program. We shall, not do it here. ♢

Example 5.3 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} - \frac{3}{x}\,y = \frac{3}{x}\,y^{\frac{2}{3}}, \qquad \textit{for } x \neq 0.$$

Clearly, $y = 0$ is a solution. It is also the set of all singular points. In fact, since $\alpha = \frac{2}{3} < 1$, the equation does not fulfil a Lipschitz condition for $y = 0$, so to write down all solutions will be quite a job. First we take any solution curve for $y < 0$. When it touches the line $y = 0$, we may continue for a while along some segment of $y = 0$, before we continue continuously along a curve for $y > 0$.

We then assume that $x \neq 0$ and $y \neq 0$. We divide by $y^{\frac{2}{3}}$ (the standard method for solution or the Bernoulli equation) to get

$$\frac{1}{y^{\frac{2}{3}}}\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}\,y^{\frac{1}{3}} = \frac{3}{x},$$

which can be written as a linear inhomogeneous equation in $u = \sqrt[3]{y}$,

$$\frac{\mathrm{d}\sqrt[3]{y}}{\mathrm{d}x} - \frac{1}{x}\,\sqrt[3]{y} = \frac{1}{x},$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}\,u = \frac{1}{x}.$$

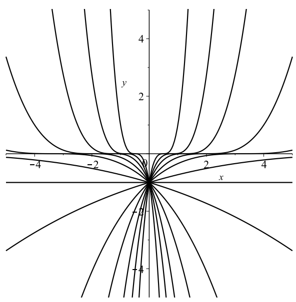


Figure 5.3: *Some solution curves of the differential equation* $\frac{dy}{dx} - \frac{3}{x}\,y = \frac{3}{x}\,y^{\frac{2}{3}}$*, for* $x \neq 0$*. Since* $\alpha = \frac{2}{3} < 1$ *the equation does not fulfil a Lipschitz condition for* $y = 0$*. This means that the solutions can be put together with any one solution to the left, until it touches the line* $y = 0$*, then we insert a segment of the line* $y = 0$ *before we continue along another solution curve. It is a hard task to write down all possibilities.*

By a rearrangement and a division by x,

$$0 = \frac{1}{x}\,\frac{\mathrm{d}(u+1)}{\mathrm{d}x} - \frac{1}{x^2}\,(u+1) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u+1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt[3]{y}+1}{x}\right),$$

hence, by integration,

$$\frac{\sqrt[3]{y}+1}{x} = C, \qquad \text{or} \qquad \sqrt[3]{y} = -1 + Cx,$$

so the complete solution is $y = 0$ supplied with the family

$$y = (Cx - 1)^3 \qquad \text{for } x \neq 0 \text{ and } y \neq 0,$$

where C is an arbitrary constant.

We note that we for $C = 0$ get another rectilinear solution, $y = -1$.

Alternatively, the equation can be written in its differential form

$$x\,\mathrm{d}y - 3\left(y + y^{\frac{2}{3}}\right)\,\mathrm{d}x = 0,$$

where all points of the line $y = 0$ clearly are singular point, as well as $y = 0$ is a rectilinear solution. It is not hard to see that $(0, -1)$ is another singular point, which is also clear from the figure.

When we separate the variables, we get

$$\frac{1}{3y^{\frac{2}{3}}}\,\frac{\mathrm{d}y}{\sqrt[3]{y}+1} - \frac{\mathrm{d}x}{x} = 0,$$

so

$$\mathrm{d}\ln|\sqrt[3]{y}+1| - \mathrm{d}\ln|x| = \mathrm{d}\left|\frac{\sqrt[3]{y}+1}{x}\right| = 0,$$

from which we get by integration, followed by the exponential, and finally building the sign into the arbitrary constant,

$$\frac{\sqrt[3]{y}+1}{x} = C, \qquad \text{i.e.} \qquad \sqrt[3]{y} = Cx - 1, \qquad \text{or} \qquad y = (Cx-1)^3, \qquad C \text{ arbitrary constant,}$$

supplied with $y = 0$.

It is here possible implicitly to solve the corresponding equation of the equipotential curves,

$$3\left(y + y^{\frac{2}{3}}\right)\,\mathrm{d}y + x\,\mathrm{d}x = 0,$$

in which the variables are already separated. We get by integration with an arbitrary constant k,

$$\frac{3}{2}\,y^2 + \frac{9}{5}\,y^{\frac{5}{3}} + \frac{1}{2}\,x^2 = \frac{k}{2},$$

so

$$x = \pm\sqrt{k - 3y^2 - \frac{18}{5}\,y^{\frac{5}{3}}}, \qquad \text{whenever this expression is defined.} \qquad \Diamond$$

Example 5.4 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{x}{2y} \qquad \textit{for } x \neq 0 \textit{ and } y \neq 0.$$

The standard procedure is to divide by $\frac{1}{2y}$, which is the same as multiplying by $2y$. When doing so we get

$$2y\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}\,y^2 = \frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} - \frac{1}{x}\,y^2 = x.$$

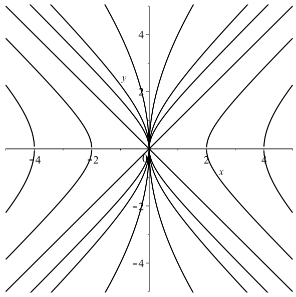


Figure 5.4: *Some solution curves of the differential equation* $\frac{dy}{dx} - \frac{y}{2x} = \frac{x}{2y}$, *for* $x \neq 0$ *and* $y \neq 0$.

This equation is linear in $u = y^2$. When we divide by x, we get

$$1 = \frac{1}{x}\,\frac{\mathrm{d}\big(y^2\big)}{\mathrm{d}x} - \frac{1}{x^2}\,y^2 = \frac{1}{x}\,\frac{\mathrm{d}\big(y^2\big)}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)\cdot y^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2}{x}\right).$$

By integration,

$$\frac{y^2}{x} = x - 2c, \qquad \text{i.e.} \qquad y^2 = x^2 - 2cx = (x-c)^2 - c^2,$$

which is rearranged as the usual equation of hyperbolas,

$$(x-c)^2 - y^2 = c^2.$$

For $c = 0$ we get the two lines $y = x$ and $y = -x$, where we formally have to exclude the point $(x, y) = (0, 0)$, which does not belong to the domain. However, the origo plays the role as a singular point, which is also seen from the figure.

ALTERNATIVELY, the equation is also homogeneous of degree 0, so we can use a different solution method. If we put $v = \frac{y}{x}$, or $y = v \cdot x$, the the equation is transformed into

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v - \frac{1}{2}\,v = \frac{1}{2v}, \qquad \text{for } x \neq 0 \text{ and } v \neq 0,$$

We multiply by 2v to get

$$1 = x\,\frac{\mathrm{d}\big(v^2\big)}{\mathrm{d}x} + v^2 = \frac{\mathrm{d}\big(x \cdot v^2\big)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2}{x}\right).$$

We get by integration,

$$x - 2c = \frac{y^2}{x}, \qquad \text{i.e.} \qquad (x-c)^2 - y^2 = c^2, \qquad \text{where } c \text{ is an arbitrary constant.} \qquad \diamond$$

Example 5.5 *Find the complete solution of the Bernoulli differential equation*

$$x\,\frac{dy}{dx} = y + \frac{3}{2}\,x^2\sqrt{y} \qquad \textit{for } x > 0 \textit{ and } y \geq 0.$$

Clearly, the boundary $y = 0$ may formally be considered as a solution. At the same time, the Lipschitz condition is not fulfilled for $y = 0$, so segments of the line $y = 0$ can be added to the solution curves in the open first quadrant, giving a fairly complicated solution to describe. We shall not do it here, and only solve the equation in the open first quadrant.

Let $y > 0$. We rearrange the equation and divide by $2\sqrt{y}$ to get

$$\frac{3}{4}\,x^2 = x\,\frac{1}{2\sqrt{y}}\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2}\,\sqrt{y} = x\,\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} - \frac{1}{2}\,\sqrt{y},$$

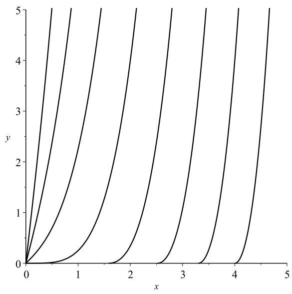


Figure 5.5: *Some solution curves of the differential equation* $x\,\dfrac{dy}{dx} = y + \dfrac{3}{2}\,x^2\sqrt{y}$ *for* $x > 0$ *and* $y \geq 0$.

which is linear in $u = \sqrt{y}$. When we divide by $x\sqrt{x}$, we get

$$\frac{3}{4}\sqrt{x} = \frac{1}{\sqrt{x}}\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} - \frac{1}{2}\frac{1}{x^{\frac{3}{2}}}\sqrt{y} = \frac{1}{\sqrt{x}}\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{x}}\right)\cdot\sqrt{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\frac{y}{x}}\right),$$

hence, by integration,

$$\sqrt{\frac{y}{x}} = \frac{3}{4}\int\sqrt{x}\,\mathrm{d}x + c = \frac{1}{2}x\sqrt{x} + c,$$

so

$$\sqrt{y} = \frac{1}{2}x^2 + c\sqrt{x} \qquad \text{for } x > 0 \text{ and } y > 0,$$

i.e.

$$y = \left\{\frac{1}{2}x^2 + c\sqrt{x}\right\}^2 \qquad \text{for } \begin{cases} x > 0 & \text{if } c \geq 0, \\ x > (-2c)^{\frac{2}{3}} & \text{if } c < 0. \end{cases} \qquad \Diamond$$

Example 5.6 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{2x^2}{y}.$$

We must here assume that $x \neq 0$ and $y \neq 0$. Then multiply by $2y$ to get

$$-4x^2 = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y^2 = \frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} - \frac{4}{x}y^2,$$

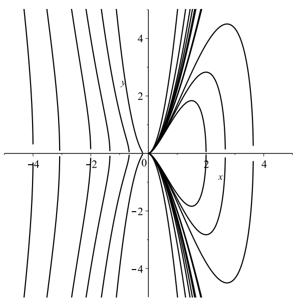


Figure 5.6: *Some solution curves of the differential equation* $\dfrac{dy}{dx} - \dfrac{2}{x}y = \dfrac{2x^2}{y}$.

which is a linear equation in $u = y^2$. We either use the solution formula, or simply divide by x^4 to get

$$-\frac{4}{x^2} = \frac{1}{x^4}\frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} - \frac{4}{x^5}y^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{y^2}{x^4}\right\},$$

from which by an integration,

$$\frac{y^2}{x^4} = -4 \int \frac{\mathrm{d}x}{x^2} = \frac{4}{x} + c,$$

hence

$$y^2 = 4x^3 + cx^4, \qquad \text{or} \qquad y = \pm\sqrt{4x^3 + cx^4} = \pm x\sqrt{4x + cx^2}, \qquad c \text{ an arbitrary constant,}$$

whenever $4x + cx^2 = x(4 + cx) > 0$. $\diamond$

Example 5.7 *Find the complete solution of the Bernoulli differential equaton*

$$\frac{dy}{dx} + \frac{2}{x+1}\,y = -\frac{2x+4}{x^2(x+1)}\,\sqrt{y}.$$

Clearly, $y = 0$ is a solution, and since $\sqrt{y}$ occurs, where the exponent $\frac{1}{2} < 1$, the equation does not fulfil a Lipschitz condition for $y = 0$, so there are lots of possible concatenations of solutions on the x-axis.

We assume in the following that $y > 0$, and that $x \neq 0$, $x \neq -1$. (Note that if we multiply by $x^2(x+1)$, then both $x = 0$ and $x = -1$ are solutions.) Divide by $2\sqrt{y}$ to get

$$-\frac{x+2}{x^2(x+1)} = \frac{1}{2\sqrt{y}}\,\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x+1}\,\sqrt{y} = \frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} + \frac{1}{x+1}\,\sqrt{y},$$

which is a linear differential equation in the new variable $u = \sqrt{y}$. This equation is multiplied by $x + 1$,

$$-\frac{x+2}{x^2} = (x+1)\,\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} + 1 \cdot \sqrt{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{(x+1)\sqrt{y}\right\},$$

hence, by an integration

$$(x+1)\sqrt{y} = c - \int \frac{x+2}{x^2}\,\mathrm{d}x = x - \ln|x| + \frac{2}{x},$$

so

$$\sqrt{y} = \frac{c}{x+1} - \frac{\ln|x|}{x+1} + \frac{2}{(x+1)x}, \qquad \text{provided that } \frac{1}{x+1}\left\{c - \ln|x| + \frac{2}{x}\right\} > 0,$$

in which case

$$y = \frac{1}{x+1^2}\left\{c - \ln|x| + \frac{2}{x}\right\}^2.$$

More specifically, if $x > -1$, then the condition is

$$\ln|x| - \frac{2}{x} > c,$$

and if $x < -1$, then the condition is

$$\ln|x| - \frac{2}{x} < c. \qquad \diamond$$

Example 5.8 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx}\,\cos x + y\,\sin x + \frac{1}{2}\,y^2 = 0 \qquad \textit{for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Clearly, $y = 0$ is a solution.

When $y \neq 0$, we divide by $-y^2$ and get after a rearrangement,

$$\frac{1}{2} = -\frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}\,\cos x - \frac{1}{y}\,\sin x = \cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) + \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)\cdot\frac{1}{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{y}\right),$$

which is easy to integrate

$$\frac{\cos x}{y} = \frac{1}{2}\,x + \frac{1}{2}\,c = \frac{x+c}{2}.$$

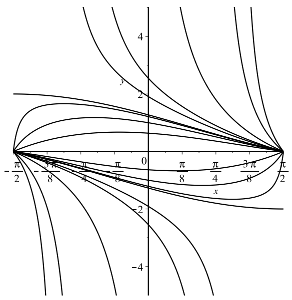


Figure 5.7: *Some solution curves of the differential equation* $\dfrac{dy}{dx}\,\cos x + y\,\sin x + \dfrac{1}{2}\,y^2 = 0.$

Finally, we solve this equation with respect to y,

$$y = \frac{2\cos x}{x+c}, \qquad \text{where } c \text{ is an arbitrary constant,}$$

supplied with the rectilinear solution $y = 0$. $\diamond$

Example 5.9 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} + \frac{1}{x}\,y = \frac{1}{(\ln x)^2}\,y^2, \qquad \textit{for } 0 < x < 1 \textit{ and } x > 1.$$

Clearly, $y = 0$ is a solution. When $y \neq 0$, we divide by $-y^2$ to get the following linear equation in $u = \dfrac{1}{y}$,

$$-\frac{1}{(\ln x)^2} = -\frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}\cdot\frac{1}{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) - \frac{1}{x}\cdot\frac{1}{y}.$$

This equation is then divided by x,

$$-\frac{1}{x(\ln x)^2} = \frac{1}{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) - \frac{1}{x^2}\cdot\frac{1}{y} = \frac{1}{x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)\cdot\frac{1}{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{xy}\right),$$

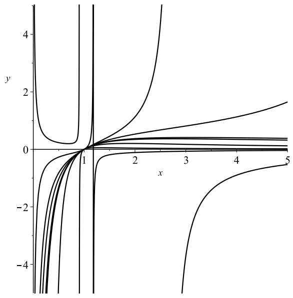


Figure 5.8: *Some solution curves of the differential equation* $\frac{dy}{dx} + \frac{1}{x}\,y = \frac{1}{(\ln x)^2}\,y^2$.

which is readily integrated,

$$\frac{1}{xy} = c - \int \frac{\mathrm{d}x}{x(\ln x)^2} = c - \int \frac{\mathrm{d}\ln x}{(\ln x)^2} = c + \frac{1}{\ln x} = \frac{c\ln x + 1}{\ln x},$$

so the complete solution is given by

$$y = \frac{\ln x}{x(c\ln x + 1)}, \qquad c \text{ arbitrary constant,}$$

where $x > 0$, $x \neq 1$ (and $x \neq \exp\left(-\frac{1}{c}\right)$, for $c \neq 0$), supplied with the line $y = 0$. ◊

Example 5.10 *Find the complete solution of the Bernoulli differential equation*

$$x\,\frac{dy}{dx}+y=2x^{\frac{1}{3}}y^{\frac{2}{3}}.$$

Clearly, $y=0$ is a solution. Since the exponent $\dfrac{2}{3}<1$, the equation does not satisfy a Lipschitz condition for $y=0$, so this line consists of singular points, and we have the possibility of concatenating various solution curves joining on the x-axis. We shall not go through this discussion in all details.

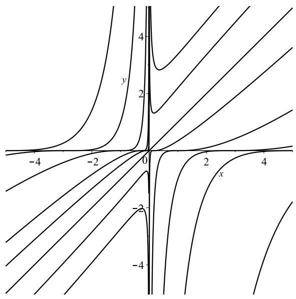


Figure 5.9: *Some solution curves of the differential equation* $x\,\dfrac{dy}{dx}+y=2x^{\frac{1}{3}}y^{\frac{2}{3}}$.

For $y\neq 0$ the equation is divided by $3y^{\frac{2}{3}}$,

$$\frac{2}{3}\,x^{\frac{1}{3}}=x\cdot\frac{1}{3}\,y^{-\frac{2}{3}}\,\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{3}\,y^{\frac{1}{3}}=x\,\frac{\mathrm{d}}{\mathrm{d}x}\,y^{\frac{1}{3}}+\frac{1}{3}\,y^{\frac{1}{3}}.$$

This equation, which is linear in $y^{\frac{1}{3}}$, is then for $x\neq 0$ divided by $x^{\frac{2}{3}}$, so

$$\frac{2}{3}\,x^{-\frac{1}{3}}=x^{\frac{1}{3}}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{\frac{1}{3}}\right)+\frac{1}{3}\,x^{-\frac{2}{3}}\,y^{\frac{1}{3}}=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{3}}y^{\frac{1}{3}}\right),$$

hence by integration,

$$\sqrt[3]{xy}=c+\frac{2}{3}\int x^{-\frac{1}{3}}\,\mathrm{d}x=c+x^{\frac{2}{3}},$$

from which

$$xy=\left(c+x^{\frac{2}{3}}\right)^3,$$

or

$$y=\left\{\frac{c}{\sqrt[3]{x}}+\sqrt[3]{x}\right\}^3,\qquad \text{where } c \text{ is an arbitrary constant,}$$

supplied with the trivial solution $y=0$. Note that $y=x$ for $c=0$.

The equivalent differential form $x\,\mathrm{d}y+\left(y-2x^{\frac{1}{3}}y^{\frac{2}{3}}\right)\mathrm{d}x=0$ is homogeneous of degree 1, so we may also solve the equation by applying methods from Chapter 7. $\Diamond$

Example 5.11 *Find the complete solution of the Bernoulli differential equation*

$$\frac{dy}{dx} + \frac{y}{1-x^2} = \frac{1-x}{(1+x)^2 y}.$$

We must of course assume that $y \neq 0$ and $x \neq \pm 1$. When we multiply by $2y$ we get

$$2\,\frac{1-x}{(1+x)^2} = 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}\,y^2 = \frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} + \frac{2}{1-x^2}\,y^2,$$

which is a linear, inhomogeneous differential equation in the new unknown variable $u = y^2$. We first compute

$$\int f(x)\,\mathrm{d}x = \int \frac{2}{1-x^2}\,\mathrm{d}x = \int \left\{\frac{1}{x+1} - \frac{1}{x-1}\right\}\mathrm{d}x = \ln\left|\frac{x+1}{x-1}\right|.$$

When the equation is multiplied by $\dfrac{x+1}{x-1}$, we get

$$\begin{aligned}\frac{x+1}{x-1}\,\frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} + \frac{2}{1-x^2}\cdot\frac{x+1}{x-1}\,y^2 &= \frac{x+1}{x-1}\,\frac{\mathrm{d}\left(y^2\right)}{\mathrm{d}x} + \frac{2}{(x-1)^2}\,y^2 \\ = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x+1}{x-1}\,y^2\right\} &= \frac{x+1}{x-1}\cdot 2\,\frac{1-x}{(1+x)^2} = -\frac{2}{x+1},\end{aligned}$$

hence by integration, where c is an arbitrary constant,

$$\frac{x+1}{x-1}\,y^2 = c - 2\ln|x+1|.$$

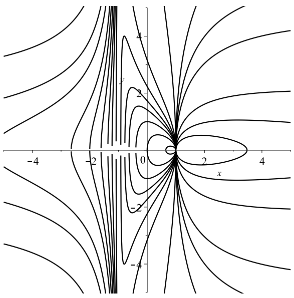


Figure 5.10: *Some solution curves of the differential equation* $\dfrac{dy}{dx} + \dfrac{y}{1-x^2} = \dfrac{1-x}{(1+x)^2 y}$.

We solve with respect to y^2,

$$y^2 = \frac{x-1}{x+1}\{c - 2\ln|x+1|\}, \qquad \text{for } x \neq \pm 1 \text{ and } Y^2 > 0,$$

i.e. for

$$\left(x^2-1\right)c > 2\left(x^2-1\right)\ln|x+1|.$$

There are two possibilities,

1) If $-1 < x < 1$, then $2\ln|x+1| > c$.

2) If $|x| > 1$, then $2\ln|x-1| < c$.

Whenever the expression is defined, the solution is given by

$$y = \pm\sqrt{\frac{x-1}{x+1}\{c - 2\ln|x+1|\}}. \qquad \diamond$$

Example 5.12 *Find for $x \neq 1$ the complete solution of the Bernoulli differential equation*

$$(1-x)\frac{dy}{dx} + 2y = (x-1)(4x-1)y^2,$$

and check, if any of the solutions can be extended to $x = 1$.

Clearly, $y = 0$ is a rectilinear solution, which is also defined for $x = 0$. If the differential equation is written as a differential form,

$$(1-x)\,\mathrm{d}y + y\{2 + (1-x)(4x-1)y\}\,\mathrm{d}x = 0,$$

it follows easily that $(1,0)$ is the only singular point, and that both $y = 0$ and $x = 1$ are (rectilinear) solutions to this extended problem. Therefore, possible extensions can only take place through the singular point $(1,0)$.

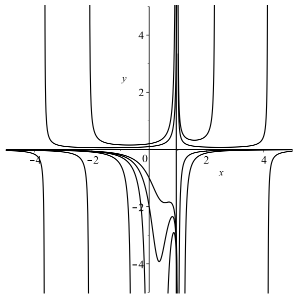


Figure 5.11: *Some solution curves of the differential equation* $(1-x)\frac{dy}{dx} + 2y = (x-1)(4x-1)y^2$.

When $y \neq 0$ we divide by $-y^2$ and obtain

$$-(x-1)(4x-1) = (x-1)\left(-\frac{1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{y} = (x-1)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) - 2\,\frac{1}{y},$$

which is a linear and inhomogeneous equation in the variable $u = \dfrac{1}{y}$. Since we have assumed that also $x \neq 1$, we can divide by $(x-1)^2$. Then

$$-\frac{4x-1}{x-1} = -4 - \frac{3}{x-1} = \frac{1}{x-1}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right) - \frac{2}{(x-1)^2}\,\frac{1}{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(x-1)y}\right).$$

Hence, by integration,

$$\frac{1}{(x-1)y} = -4x - 3\ln|x-1| + c,$$

and we get by a rearrangement, solving with respect to y,

$$y = \frac{1}{(x-1)(c-4x-3\ln|x-1|)}, \qquad \text{where } c \text{ is an arbitrary constant.}$$

It follows from this expression that all solution curves of this structure are repelled from the singular point $(1,0)$, so only the trivial solution $y = 0$ can be extended. $\diamond$

Example 5.13 *Find the complete solution of the Bernoulli differential equation*

$$\cosh x \cdot \frac{dy}{dx} - 2y\sinh x + \frac{2\sqrt{y}\cosh^2 x}{1+e^x} = 0 \qquad \textit{for } x \in \mathbb{R} \textit{ and } y \geq 0.$$

Clearly, $y = 0$ is a solution. Then assume that $y > 0$. We divide by $2\sqrt{y}$ to get

$$\cosh x \cdot \frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} - \sinh x \cdot \sqrt{y} + \frac{2\cosh^2 x}{1+e^x} = 0,$$

which is linear in $u = \sqrt{y}$. Divide the equation by $\cosh^2 x > 0$

$$-\frac{2}{1+e^x} = \frac{1}{\cosh x}\,\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} - \frac{\sinh x}{\cosh^2 x}\,\sqrt{y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{y}}{\cosh x}\right).$$

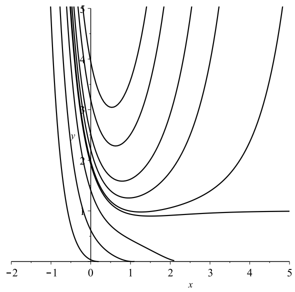


Figure 5.12: *Some solution curves of the differential equation* $\cosh x \cdot \dfrac{dy}{dx} - 2y \sinh x + \dfrac{2\sqrt{y}\cosh^2 x}{1+e^x} = 0$.

Hence, by integration

$$\frac{\sqrt{y}}{\cosh x} = 2c - \int \frac{2\,\mathrm{d}x}{1+e^x} = 2c - \int \frac{2e^x\,\mathrm{d}x}{e^x\,(1+e^x)} = 2c - 2\ln\left(\frac{e^x}{e^x+1}\right) = 2c + 2\ln\left(1+e^{-x}\right),$$

where we must require that $2c + 2\ln(1+e^{-x}) > 0$, i.e. $1 + e^{-x} > e^{-c}$, thus $e^{-x} > e^{-c} - 1$. If $c \geq 0$, this is always true. If instead $c < 0$, then $x < -\ln(e^{-c} - 1)$. Then finally,

$$\sqrt{y} = 2\cosh x \cdot \left\{c + \ln\left(1+e^{-x}\right)\right\},$$

so the complete solution is

$$y = 4\cosh^2 x \cdot \left\{c + \ln\left(1+e^{-x}\right)\right\}^2 \qquad \text{for} \begin{cases} \text{all } x \in \mathbb{R}, \text{ when } c \geq 0 \\ \\ x < -\ln(e^{-c}-1)\,, \text{ when } c < 0. \end{cases} \qquad \Diamond$$

6 Riccati's equation

6.1 Complete solution, when a particular solution is known.

In connection with *Bernoulli's equation* of Chapter 5 we briefly mention *Riccati's equation*

$$(6.1)\qquad \frac{\mathrm{d}y}{\mathrm{d}x}+f(x)y=g(x)y^2+h(x).$$

It was introduced by count Jacopo Riccati (1676–1754), in the special form

$$\frac{\mathrm{d}y}{\mathrm{d}x}=ay^2+bx^\alpha,$$

though he could not add much to the solution of this equation.

We shall later return to this important equation. Unfortunately, its solution is not an easy matter, so here we shall only consider (6.1) in the case, where we already *know* a solution z, in which case we can take a shortcut. So we assume that we are given a function $z=z(x)$, such that

$$(6.2)\qquad \frac{\mathrm{d}z}{\mathrm{d}x}+f(x)z=g(x)z^2+h(x).$$

We put $y:=u+z$ into (6.1) in order to get

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}z}{\mathrm{d}x}+f(x)\cdot u+f(x)\cdot z=g(x)\{u+z\}^2+h(x).$$

When we subtract (6.2) from this equation, we get

$$\frac{\mathrm{d}u}{\mathrm{d}x}+f(x)\cdot u=g(x)\cdot\left\{(u+z)^2-z^2\right\}=g(x)\cdot\left\{u^2+2z\cdot u\right\},$$

which is reduced to the following *Bernoulli equation* in u,

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\{f(x)-2z(x)\cdot g(x)\}u=g(x)\cdot u^2,$$

where $n=2$. So either $u=0$, in which case $y=u+z=z$ is the given solution, or $u\neq 0$, in which case a division by $-u^2$ gives

$$(6.3)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right)+\{2z(x)\cdot g(x)-f(x)\}\cdot\frac{1}{u}=-g(x).$$

Equation (6.3) is a linear inhomogeneous equation in $\frac{1}{u}$, which is solved in the well-known way. Put

$$(6.4)\qquad F(x):=\int\{2z(x)\cdot g(x)-f(x)\}\,\mathrm{d}x.$$

When (6.3) is multiplied by $e^{F(x)}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{e^{F(x)}}{u}\right\}=-e^{F(x)}\,g(x),$$

hence by an integration,

$$\frac{e^{F(x)}}{u} = -\int e^{F(x)}\, g(x)\, \mathrm{d}x + C,$$

so

$$\frac{1}{u} = C \cdot e^{-F(x)} - e^{-F(x)} \int e^{F(x)}\, g(x)\, \mathrm{d}x = e^{-F(x)} \left\{ C - \int e^{F(x)}\, g(x)\, \mathrm{d}x \right\},$$

and hence,

$$u = \frac{e^{F(x)}}{C - \int e^{F(x)}\, g(x)\, \mathrm{d}x},$$

and the solution of the Riccati equation is $y = z$, or $y = u + z$, i.e.

$$y = \begin{cases} z = z(x), \\[2mm] z + \dfrac{e^{F(x)}}{C - \int e^{F(x)}\, g(x)\, \mathrm{d}x} = \dfrac{C \cdot z + e^{F(x)} - z \int e^{F(x)}\, g(x)\, \mathrm{d}x}{C - \int e^{F(x)}\, g(x)\, \mathrm{d}x}, \qquad C \in \mathbb{R} \text{ arbitrary}, \end{cases}$$

whenever the function is defined. Note that we for $C \to +\infty$ obtain $y = z$.

The method above is a shortcut, when a solution z of (6.1) is known. We shall later give another method of solution, but for the time being we shall try to guess a solution.

6.2 The structure of the solutions of the Riccati differential equation.

We shall prove the following theorem, which describes the structure of the general solution of the Riccati differential equation. In special cases the theorem also contains the structure of the complete solutions of the linear differential equation of first order, and the Bernoulli differential equation.

Theorem 6.1 *Let $a(x)$, $b(x)$, $c(x)$ and $d(x)$ be C^1-functions all defined in a given open interval I. We assume that the determinant*

$$\begin{vmatrix} a(x) & b(x) \\ c(x) & d(x) \end{vmatrix} = a(x)d(x) - b(x)c(x) \neq 0.$$

Then the set of functions

$$y = \frac{a(x) + k \cdot b(x)}{c(x) + k \cdot d(x)} \qquad \text{for } k \in \mathbb{R},$$

supplied with $y = \frac{b(x)}{d(x)}$, if $d(x) \neq 0$, is the general solution of a Riccati differential equation

$$\frac{dy}{dx} = A(x)y + B(x)y^2 + C(x),$$

where the coefficients $A(x)$, $B(x)$ and $C(x)$ are uniquely determined for given $a(x)$, $b(x)$, $c(x)$ and $d(x)$.
If $B(x) = 0$, the equation is linear. If $C(x) = 0$, the equation is a Bernoulli differential equation.

Conversely, given a Riccati differential equation of the form

$$\frac{dy}{dx} = A(x)y + B(x)y^2 + C(x),$$

then there exist functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$, such that the complete solution is given by

$$y = \frac{a(x) + k \cdot b(x)}{c(x) + k \cdot d(x)} \qquad \text{for } k \in \mathbb{R},$$

supplied with $y = \frac{b(x)}{d(x)}$, if $d(x) \neq 0$, is the general solution of a Riccati differential equation

$$\frac{dy}{dx} = A(x)y + B(x)y^2 + C(x).$$

Proof. Given the set of functions

$$y = \frac{a(x) + k \cdot b(x)}{c(x) + k \cdot d(x)} \qquad \text{for } k \in \mathbb{R}.$$

We shall find the differential equation, which has these functions as solutions, i.e. we shall eliminate the constant $k \in \mathbb{R}$.

When we solve with respect to the constant k, we get

$$k = \frac{c(x) \cdot y - a(x)}{b(x) - d(x) \cdot y}, \qquad k \in \mathbb{R}.$$

Then differentiate with respect to x, which gives 0 on the left hand side. When we also multiply by $\{b(x)-d(x)\cdot y\}^2$, we get with the shorthand $a' = \dfrac{\mathrm{d}a}{\mathrm{d}x}$, etc.

$$\begin{aligned} 0 &= \{b(x)-d(x)y\}\cdot\{c(x)y'+c'(x)y-a'(x)\}+\{c(x)y-a(x)\}\cdot\{d(x)y'-b'(x)\} \\ &= \{b(x)c(x)-a(x)d(x)\}y'+\{c'(x)b(x)-c(x)b'(x)\}y+\{c(x)d'(x)-c'(x)d(x)\}y^2 \\ &\qquad +\{a'(x)d(x)-a(x)d'(x)\}y+\{a(x)b'(x)-a'(x)b(x)\}, \end{aligned}$$

so if we put

$$D(x) := a(x)d(x)-b(x)c(x), \qquad [\neq 0 \text{ by assumption}],$$

$$P(x) := \{c'(x)b(x)-c(x)b'(x)\}-\{a(x)d'(x)-a'(x)d(x)\},$$

$$Q(x) = c(x)d'(x)-c'(x)d(x), \qquad R(x) := a(x)b'(x)-a'(x)b(x),$$

then the given set of functions all satisfy the Riccati differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P(x)}{D(x)}\,y+\frac{Q(x)}{D(x)}\,y^2+\frac{R(x)}{D(x)},$$

where

$$A(x) = \frac{P(x)}{D(x)} = \frac{\{c'(x)b(x)-c(x)b'(x)\}-\{a(x)d'(x)-a'(x)d(x)\}}{a(x)d(x)-b(x)c(x)},$$

$$B(x) = \frac{Q(x)}{D(x)} = \frac{c(x)d'(x)-c'(x)d(x)}{a(x)d(x)-b(x)c(x)}, \qquad C(x) = \frac{R(x)}{D(x)} = \frac{a(x)b'(x)-a'(x)b(x)}{a(x)d(x)-b(x)c(x)}.$$

If $B(x)\equiv 0$, then the equation is linear, and if $C(x)\equiv 0$, then the equation is a Bernoulli differential equation. If the equation is Riccati, but neither linear nor Bernoulli, then we also obtain a solution, when we let $k\to+\infty$, namely $y=\dfrac{b(x)}{d(x)}$.

Then let

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A(x)y+B(x)y^2+C(x), \qquad A,\,B,\,C\in C^0,$$

be given. If $B(x)\equiv 0$, the equation is linear, so its complete solution has the structure,

$$y = a(x)+k\cdot b(x) = \frac{a(x)+k\cdot b(x)}{1+k\cdot 0},$$

and if $C(x)\equiv 0$, then the equation is a Bernoulli equation, which is transformed into a linear differential equation in $\frac{1}{y}$, so its general solution is

$$y = \frac{1}{a(x)+k\cdot b(x)} = \frac{1+k\cdot 0}{a(x)+k\cdot b(x)},$$

i.e. of the right structure.

Let us then assume that $B(x)$ and $C(x)\neq 0$. According to the existence theorem there exists a solution $y_0 = y_0(x)$, so

$$\frac{\mathrm{d}y_0}{\mathrm{d}x} = A(x)y_0+B(x)y_0^2+C(x).$$

The trick is to put $y = y_0 + u$, where u is the new unknown variable. We get by insertion

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx} + \frac{dy_0}{dx} = \frac{du}{dx} + A(x)y_0 + B(x)y_0^2 + C(x)\\ &= A(x)\{y_0+u\} + B(x)\cdot\{y_0+u\}^2 + C(x)\\ &= A(x)\cdot y_0 + B(x)\cdot y_0^2 + C(x) + \{A(x) + 2B(x)y_0(x)\}\,u + B(x)u^2,\end{aligned}$$

from which by a reduction,

$$\frac{du}{dx} = \{A(x) + 2B(x)y_0(x)\}\,u + B(x)u^2.$$

By assumption, $B(x) \neq 0$. The solution $u = 0$ just gives $y = y_0$, so assume that $u \neq 0$ in the following, and then divide by $-u^2$ and put $v = \frac{1}{u}$ to get the linear equation

$$\frac{dy}{dx} = -\{A(x) + 2B(x)y_0(x)\}\,v - B(x).$$

The structure of the solution is well-known, $v = v_0 + k\cdot\varphi$, so

$$y = y_0 + u = y_0 + \frac{1}{v} = y_0 + \frac{1}{v_0 + k\cdot\varphi} = \frac{y_0v_0 + 1 + k\cdot y_0\varphi}{v_0 + k\cdot\varphi},$$

and the solution has the wanted structure. □

6.3 Examples where it is easy to guess a particular solution.

It is possible to give a qualified guess of a particular solution of Riccati's differential equation in the following cases:

1) The Riccati equation (monomials)

$$\frac{dy}{dx} + a\cdot x^n y^2 = b\cdot x^{-n-2}, \qquad \text{for } x > 0,$$

where for convenience $(n+1)^2 + 4ab \geq 0$, unless we allow complex constants in the result. The natural guess is $y_0 = c\cdot x^{-n-1}$.

2) The Riccati equation (exponentials)

$$\frac{dy}{dx} + a\,e^{nx}\,y^2 = b\,e^{-nx},$$

where we must require that $n^2 + 4ac \geq 0$, if we restrict ourselves to real solutions. The natural guess is $y_0 = c\cdot e^{-nx}$.

3) The Riccati equations (trigonometric or hyperbolic functions)

a) Cosine in the denominator,

$$\frac{dy}{dx} + a\,\sin x\cdot y^2 = b\cdot\frac{\sin x}{\cos^2 x}, \qquad \left(\text{guess } y_0 = \frac{c}{\cos x}\right),$$

b) Sine in the denominator,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a\,\cos x \cdot y^2 = b \cdot \frac{\cos x}{\sin^2 x}, \qquad \left(\text{guess } y_0 = \frac{c}{\sin x}\right),$$

c) Hyperbolic cosine in the denominator,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a\,\sinh(x)\,y^2 = b \cdot \frac{\sinh x}{\cosh^2 x}, \qquad \left(\text{guess } y_0 = \frac{c}{\cosh x}\right),$$

d) Hyperbolic sine in the denominator,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a\,\cosh(x)\,y^2 = b \cdot \frac{\cosh x}{\sinh^2 x}, \qquad \left(\text{guess } y_0 = \frac{c}{\sinh x}\right).$$

Example 6.1 *Solve the Riccati equation*

$$x\,\frac{dy}{dx} = y - (y-x)^2, \qquad \textit{for } (x,y) \in \mathbb{R}_+ \times \mathbb{R}.$$

It is immediately seen that $y = x$ is a particular solution. Also, the other rectilinear solution $y = x + 1$ is not hard to find either.

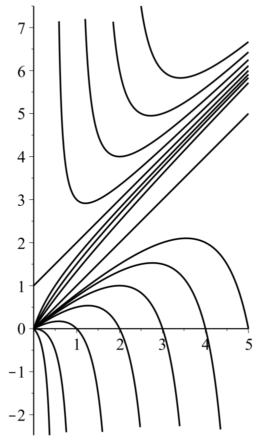


Figure 6.1: *Some solution curves for the Riccati equation* $x\,\dfrac{dy}{dx} = y - (y-x)^2$, *for* $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$.

Then we put $y = u + z = u + x$ and get by insertion for the left hand side,

$$x\,\frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}}{\mathrm{d}x}(u+x) = x\,\frac{\mathrm{d}u}{\mathrm{d}x} + x,$$

and for the right hand side,

$$y - (y-x)^2 = u + x - (u + x - x)^2 = u - u^2 + x.$$

These two expressions are equal, if

$$x\,\frac{\mathrm{d}u}{\mathrm{d}x} = u - u^2,$$

where we can separate the variables,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{u-u^2} = \left\{\frac{1}{u} - \frac{1}{u-1}\right\}\mathrm{d}u, \qquad x > 0.$$

Since $x > 0$, it follows by an integration for $u \neq 0$ and $u \neq 1$, that $\ln x = \ln\left|\dfrac{u}{u-1}\right| + c$ for some constant c. Taking the exponential and noticing that we can get rid of the absolute value sign by choosing the new constant $-C \in \mathbb{R}$, instead of $e^c \in \mathbb{R}_+$, we get $x = -C \cdot \dfrac{u}{u-1}$, i.e. $\dfrac{-C}{x} = \dfrac{u-1}{u} = 1 - \dfrac{1}{u}$, or by a rearrangement, $\dfrac{1}{u} = 1 + \dfrac{C}{x} = \dfrac{x+C}{x}$, so finally, $u = \dfrac{x}{x+C}$. Summing up, the solution is

$$y = \begin{cases} x, & \\ x + \dfrac{x}{x+C}, & C \in \mathbb{R} \text{ arbitrary,} \end{cases}$$

Note, that $u = 0$ corresponds to the solution $y = x$, and the excepted value $u = 1$ corresponds to $y = x + 1$, which by a simple check is seen to be a solution as well. $\Diamond$

Example 6.2 *Find the complete solution of the Riccati differential equation*

$$\frac{dy}{dx} = y + e^{2x} - y^2,$$

by first guessing a more or less obvious particular solution.

It should not be a surprise that $y = e^x$ is a particular solution.

Then we put $y = u + e^x$ and get by insertion,

$$\frac{\mathrm{d}u}{\mathrm{d}x} + e^x = u + e^x + e^{2x} - u^2 - 2e^x\, u - e^{2x},$$

which is reduced to

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (2e^x - 1)\, u = -u^2.$$

Here, $u = 0$ is of course trivially a solution, because it corresponds to the particular solution $y = u + e^x = e^x$ of the Riccati equation. So we assume that $u \neq 0$ in the following. When we divide by $-u^2 \neq 0$, we get

$$1 = -\frac{1}{u^2}\,\frac{\mathrm{d}u}{\mathrm{d}x} - (2e^x - 1)\,\frac{1}{u} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) - (2e^x - 1)\,\frac{1}{u}.$$

Here

$$\int (2e^x - 1)\;\mathrm{d}x = 2e^x - x,$$

so

$$\begin{aligned}
\frac{1}{u} &= c \cdot \exp(2e^x - x) + \exp(2e^x - x) \int 1 \cdot \exp(-2e^x + x)\;\mathrm{d}x \\
&= c \cdot e^{-x} \cdot \exp(2e^x) + e^{-x}\,\exp(2e^x) \int e^x\,\exp(-2e^x)\;\mathrm{d}x \\
&= e^{-x} \cdot \exp(2e^x) \cdot \left\{ c + \int_{u=e^x} e^{-2u}\,\mathrm{d}u \right\} = e^{-x} \cdot \exp(2e^x) \cdot \left\{ c - \frac{1}{2}\,\exp(-2e^x) \right\} \\
&= c \cdot e^{-x}\,\exp(2e^x) - \frac{1}{2}\,e^{-x} = -\frac{1}{2}\,e^{-x}\,\{1 - 2c \cdot \exp(2e^x)\}\,,
\end{aligned}$$

from which

$$u = \frac{-2e^x}{1 - 2c \cdot \exp(2e^x)} = \frac{2e^x}{2c\,\exp(2e^x) - 1},$$

and the complete solution is $y = e^x$, supplied with

$$y = e^x + \frac{2e^x}{2c\,\exp(2e^x) - 1} = e^x \cdot \frac{2c\,\exp(2e^x) + 1}{2c\,\exp(2e^x) - 1},$$

where c is an arbitrary constant. We note that we get formally $y = e^x$ in the limit, if we for fixed x let $c \to \pm\infty$. $\Diamond$

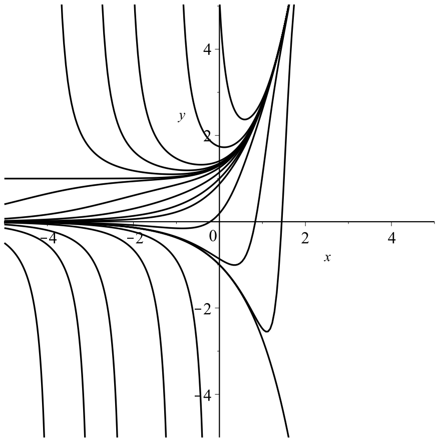


Figure 6.2: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} = y + e^{2x} - y^2$.

Example 6.3 *Find the complete solution of the Riccati equation*

$$\frac{dy}{dx} + y = \frac{1}{x}\,y^2 + 1 - \frac{1}{x}.$$

We start by guessing the particular solution $y = 1$, and then put $y = u + 1$. By insertion,

$$\frac{\mathrm{d}u}{\mathrm{d}x} + u + 1 = \frac{1}{x}(u+1)^2 + 1 - \frac{1}{x} = \frac{u^2}{x} + \frac{2u}{x} + \frac{1}{x} + 1 - \frac{1}{x},$$

so we get the derived Bernoulli equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \left(1 - \frac{2}{x}\right)u = \frac{u^2}{x}.$$

Dividing by $-u^2$ we get

$$-\frac{1}{u^2}\,\frac{\mathrm{d}u}{\mathrm{d}x} - \left(1 - \frac{2}{x}\right)\frac{1}{u} = -\frac{1}{x}.$$

or put in another way,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) - \left(1 - \frac{2}{x}\right)\frac{1}{u} = -\frac{1}{x},$$

which is linear in $v = \dfrac{1}{u}$, i.e.

$$\begin{aligned}\frac{1}{u} &= \frac{1}{x^2}\,e^x\left\{c - \int \frac{1}{x}\cdot x^2\,e^{-x}\,\mathrm{d}x\right\} = \frac{1}{x^2},e^x\left\{c - \int xe^{-x}\,\mathrm{d}x\right\}\\ &= \frac{1}{x^2}\,e^x\left\{c + (x+1)e^{-x}\right\} = \frac{ce^x + x + 1}{x^2},\end{aligned}$$

so the complete solution is

$$y = u + 1 = \frac{x^2}{ce^x + x + 1} + 1 = \frac{ce^x + x^2 + x + 1}{ce^x + x + 1}, \qquad c \text{ arbitrary constant,}$$

supplied with the line $y = 0$. $\Diamond$

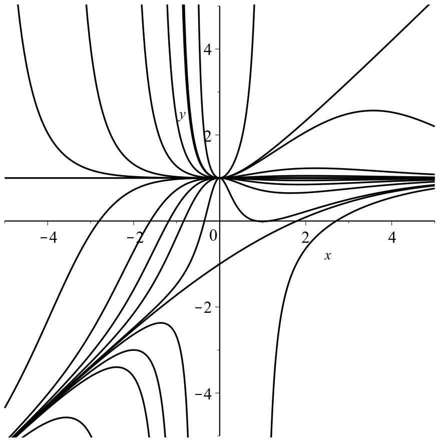


Figure 6.3: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} + y = \dfrac{1}{x}\, y^2 + 1 - \dfrac{1}{x}$.

Example 6.4 *Prove that the Riccati differential equation*

$$x\,\frac{dy}{dx} + (x-1)y = y^2 - 2x^2$$

has a polynomial of degree 1 as a particular solution. Then find the complete solution.

Let us first find the singular points. We write the equation in its differential form,

$$x\,\mathrm{d}y + \left\{(x-1)y - y^2 + 2x^2\right\}\,\mathrm{d}x,$$

so the singular points satisfy the two equations

$$x = 0 \qquad \text{and} \qquad (x-1)y - y^2 + 2x^2 = 0,$$

from which $x = 0$ and $-y - y^2 = -y(y+1) = 0$. We have the two singular points $(0,0)$ and $(0,-1)$.

Then assume that $y = ax+b$ is a solution, where a and b are unknown constants. We get by insertion,

$$x\,a + (x-1)(ax+b) = a\,x^2 + (a-a+b)x - b = a\,x^2 + b\,x - b,$$

and

$$y^2 - 2x^2 = (ax+b)^2 - 2x^2 = \left(a^2-2\right)x^2 + 2ab\,x + b^2.$$

These two expressions are equal, if and only if

$$a^2 - 2 = a, \qquad \text{and} \qquad 2ab = b \qquad \text{and} \qquad b^2 = -b.$$

It follows from the first equation that

$$a^2 - a - 2 = 0, \qquad \text{i.e.} \qquad a = \frac{1}{2} \pm \sqrt{2+\frac{1}{4}} = \frac{1}{2} \pm \frac{3}{2} = \left\{ \begin{array}{c} 2 \\ -1 \end{array} \right. .$$

Then the remaining two equations are only fulfilled for $b = 0$, so we have found the particular solutions are either

$$y = 2x \qquad \text{or} \qquad y = -x.$$

We choose in the following analysis $y = 2x$, when we reduce to a Bernoulli differential equation, i.e. we put $y = v + 2x$, where v is the new unknown. By insertion,

$$\begin{aligned} 0 &= x\,\frac{\mathrm{d}y}{\mathrm{d}x} + (x-1)y - y^2 + 2x^2 = x\left\{\frac{\mathrm{d}v}{\mathrm{d}x} + 2\right\} + (x-1)(v+2x) - (v+2x)^2 + 2x^2 \\ &= x\,\frac{\mathrm{d}v}{\mathrm{d}x} + 2x + xv - v + 2x^2 - 2x - v^2 - 4xv - 4x^2 - 4x^2 + 2x^2 = x\,\frac{\mathrm{d}v}{\mathrm{d}x} - (3x+1)v - v^2, \end{aligned}$$

which is a Bernoulli differential equation. The solution $v = 0$ corresponds to the particular solution $y = 2x$ found previously.

Assume in the following that $v \neq 0$. We divide by $-v^2$ to get

$$0 = -x\,\frac{1}{v^2}\,\frac{\mathrm{d}v}{\mathrm{d}x} + (3x+1)\,\frac{1}{v} + 1 = x\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) + (3x+1)\,\frac{1}{v} + 1,$$

which is linear in the new unknown variable $u := \dfrac{1}{v}$. Assuming $x \neq 0$, this can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) + \left(3 + \frac{1}{x}\right)\frac{1}{v} = -\frac{1}{x},$$

so by the usual solution formula,

$$\begin{aligned} \frac{1}{v} &= \exp\left(-\int\left(3+\frac{1}{x}\right)\mathrm{d}x\right)\left\{\frac{c}{3} - \int \frac{1}{x}\,\exp\left(\int\left(3+\frac{1}{x}\right)\mathrm{d}x\right)\mathrm{d}x\right\} \\ &= \frac{e^{-3x}}{x}\left\{\frac{c}{3} - \int e^{3x}\,\mathrm{d}x\right\} = \frac{e^{-3x}}{x}\left\{\frac{c}{3} - \frac{1}{3}\,e^{3x}\right\} = \frac{1}{3x\,e^{3x}}\left\{c - e^{3x}\right\}. \end{aligned}$$

By inversion we get

$$v = \frac{3xe^{3x}}{c - e^{3x}}, \qquad \text{for } e^{3x} \neq c.$$

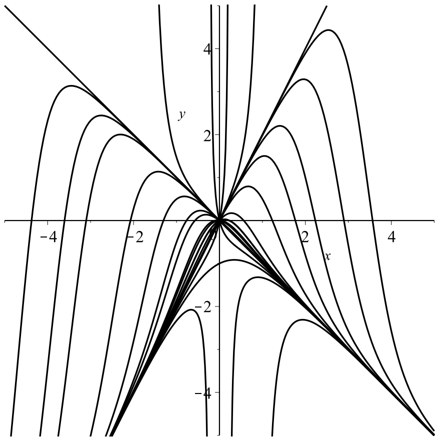


Figure 6.4: *Some solution curves for the Riccati equation* $x\,\frac{dy}{dx}+(x-1)y=y^2-2x^2$. *Only the curve corresponding to* $c=1$ *passes through the singular point* $(0,-1)$.

Then finally,

$$y=v+2x=\frac{3xe^{3x}}{c-e^{3x}}+2x=\frac{x}{c-e^{3x}}\left\{3e^{3x}+2c-2e^{3x}\right\}=\frac{x\left\{e^{3x}+2c\right\}}{ce^{3x}},$$

provided that $e^{3x}\neq c$, supplied with $y=2x$. Note that $y=-x$ is obtained for $c=0$. $\Diamond$

Example 6.5 *Given the Riccati equation*

$$\frac{dy}{dx}+\frac{2}{x}\,y=\frac{1}{x^3}\,y^2+4x \qquad \textit{for } x>0.$$

Find $k\in\mathbb{R}$, *such that* $y=kx^2$ *is a solution, and use it to find the complete solution.*

We first consider the extended corresponding differential form version of the equa5ion,

$$x^3\,\mathrm{d}y+\left(2x^2y-y^2-4x^4\right)\,\mathrm{d}x, \qquad \text{defined for } x\in\mathbb{R}.$$

The only singular point is $(0,0)$, and also $x=0$ is a solution curve of this extended problem. When restricted to the halfplane $x>0$, the solution curves are the same in both cases.

Then we use the hint. After a rearrangement of the equation we get by insertion of $y=kx^2$,

$$0=2kx+2kx-k^2x-4x=-\left(k^2-4k+4\right)x=-(k-2)^2x,$$

which is fulfilled for all $x>0$, if and only if $k=2$, and we have proved that $y=2x^2$ is a particular solution.

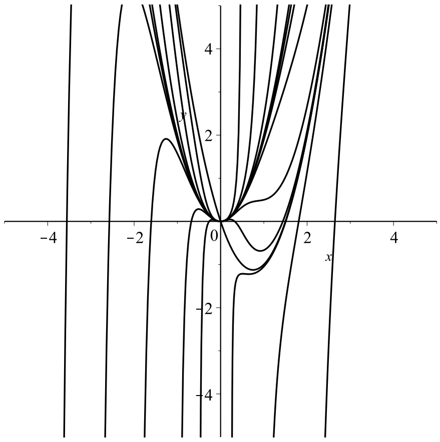


Figure 6.5: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} + \dfrac{2}{x}\,y = \dfrac{1}{x^3}\,y^2 + 4x$. *We have here allowed* $x \in \mathbb{R}$ *and not just restricted the variable to the halvplane* $x > 0$.

Then put $y = u + 2x^2$. Assuming that $u \neq 0$ we get by insertion,

$$\begin{aligned} 0 &= \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}\,y - \frac{1}{x^3}\,y^2 - 4x = \frac{\mathrm{d}u}{\mathrm{d}x} + 4x + 4x - \frac{1}{x^3}\left(u^2 + 4ux^2 + 4x^4\right) - 4x \\ &= \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u^2}{x^2} - \frac{4u}{x} = -u^2\left\{-\frac{1}{u^2}\,\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{4}{x}\,\frac{1}{u} + \frac{1}{x^2}\right\} = -u^2\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) + \frac{4}{x}\,\frac{1}{u} + \frac{1}{x^2}\right\} \\ &= -\frac{u^2}{x^4}\left\{x^4\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) + 4x^3\,\frac{1}{u} + x^2\right\} = -\frac{u^2}{x^4}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^4}{u}\right) + x^2\right\}. \end{aligned}$$

Since $u \neq 0$ and $x \neq 0$ by assumption, we have reduced the equation to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^4}{u}\right) = -x^2,$$

hence by integration,

$$\frac{x^4}{u} = \frac{1}{3}\left(c - x^3\right), \qquad \text{i.e.} \qquad u = \frac{3x^4}{c - x^3} \quad \text{for } x \neq \sqrt[3]{c}.$$

The complete solution is $y = 2x^2$, supplied with the family

$$y = 2x^2 + \frac{3x^4}{c - x^3} = \frac{2cx^2 + 3x^4 - 2x^5}{c - x^3} = \frac{x^2}{c - x^3}\left\{2c + 3x^2 - 2x^3\right\} \qquad \text{for } x \neq \sqrt[3]{c},$$

where c is an arbitrary constant.

We note that $y = 2x^2 - 3x = x(2x - 3)$ for $c = 0$, and that $y \sim 2x^2$ for $c \to \pm\infty$. ♢

Example 6.6 *Find the complete solution of the Riccati differential equation,*

$$\frac{dy}{dx} + y^2 = \frac{2}{x^2}, \qquad \textit{for } x \neq 0.$$

First consider the differential form of the equation

$$x^2\,\mathrm{d}y + \left(x^2y^2 - 2\right)\,\mathrm{d}x = 0.$$

Clearly $x = 0$ is a rectilinear solution, and there is no singular point.

We guess a solution of the structure $y = \dfrac{a}{x}$ for some constant a. By insertion,

$$-\frac{a}{x^2} + \frac{a^2}{x^2} = \frac{2}{x^2}, \qquad \text{i.e.} \qquad a^2 - a - 2 = 0.$$

When we solve this equation, we get either $a = -1$ or $a = 2$, so $y = -\dfrac{1}{x}$ and $y = \dfrac{2}{x}$ are both solutions.

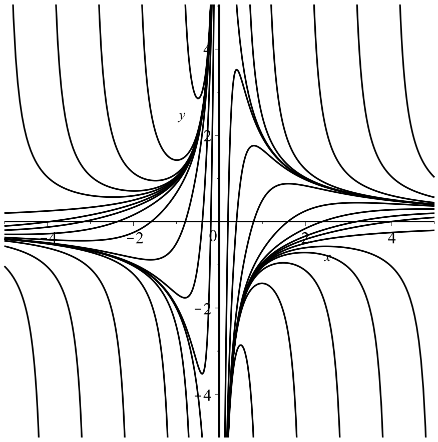


Figure 6.6: *Some solution curves for the Riccati equation* $\frac{dy}{dx} + y^2 = \frac{2}{x^2}$.

We choose one of them, $-\frac{1}{x}$ and change variable to $y = v - \frac{1}{x}$. By insertion,

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{1}{x^2} + v^2 - \frac{2}{x}\,v + \frac{1}{x^2} = \frac{2}{x^2},$$

which is reduced to the Bernoulli differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{2}{x}\,v + v^2 = 0.$$

When $v = 0$, we get the aforementioned solution $y = -\frac{1}{x}$. When $v \neq 0$, we divide by $-v^2$,

$$0 = -\frac{1}{v^2}\,\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2}{x}\,\frac{1}{v} - 1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) + \frac{2}{x}\,\frac{1}{v} - 1 = 0.$$

Then we multiply this equation by x^2 and perform a small rearrangement to get

$$x^2 = x^2\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) + 2x \cdot \frac{1}{v} = x^2\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) + \frac{\mathrm{d}\left(x^2\right)}{\mathrm{d}x} \cdot \frac{1}{v} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{v}\right).$$

This equation is integrated straight away,

$$\frac{x^2}{v} = \frac{1}{3}\,x^3 + \frac{c}{3} = \frac{x^3 + c}{3},$$

from which

$$v = \frac{3x^2}{x^3 + c},$$

and the complete solution is for an arbitrary constant c,

$$y = v - \frac{1}{x} = \frac{3x^2}{x^3 + c} - \frac{1}{x} = \frac{3x^3 - x^3 - c}{x^4 + cx} = \frac{2x^3 - c}{x\left(x^3 + c\right)},$$

supplied with $y = -\frac{1}{x}$. Note that the choice $c = 0$ gives $y = \frac{2}{x}$. $\Diamond$

Example 6.7 *(Monomials) Find the complete solution of the Riccati differential equation*

$$\frac{dy}{dx} - x^2 y^2 = \frac{2}{x^4}, \qquad \textit{for } x \neq 0.$$

Here $n = 2$, so we guess $(-n-1 = -3)$ the particular solution $y_0 = c \cdot y^{-3}$. We get by insertion,

$$0 = -3c \cdot x^{-4} - c^2 x^{-4} - 2 = -x^{-4}\left(c^2 + 3c + 2\right) = -x^{-4}(c+1)(c+2),$$

from which follows that $y_0 = c \cdot x^{-3}$ is a particular solution for $c = -1$ or $c = -2$.

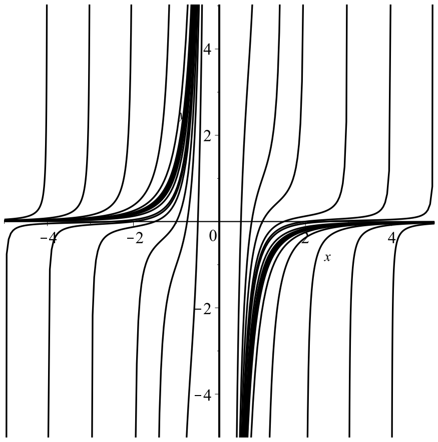


Figure 6.7: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} - x^2 y^2 = \dfrac{2}{x^4}$.

Then by the standard method we put $y = u - x^{-3}$, $(c = -1)$, and get by insertion in the equation,

$$\begin{aligned} 0 &= \frac{\mathrm{d}u}{\mathrm{d}x} + 3x^{-4} - x^2\left(u^2 - 2x^{-3}u + x^{-6}\right) - 2x^{-4} = \frac{\mathrm{d}u}{\mathrm{d}x} + 3\,x^{-4} - x^2u^2 + 2x^{-1}u - x^{-4} - 2x^{-4} \\ &= \frac{\mathrm{d}u}{\mathrm{d}x} + 2x^{-1}u - x^2u^2. \end{aligned}$$

We divide by $-u^2$ to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) - \frac{2}{x}\,\frac{1}{u} = -x^2,$$

which has the complete solution

$$\frac{1}{u} = c\,x^2 - x^3 = x^2(c - x), \qquad \text{where } c \text{ is an arbitrary constant.}$$

It follows that the complete solution of the original Riccati equation is given by

$$y = u - \frac{1}{x^3} = \frac{1}{x^2(c-x)} - \frac{1}{x^3} = \frac{2x - c}{x^3(c-x)}, \qquad \text{where } c \text{ is an arbitrary constant,}$$

supplied with the guessed solution $y_0 = -x^{-3}$. $\Diamond$

Example 6.8 *(Exponential) Find the complete solution of the Riccati differential equation*

$$\frac{dy}{dx} + e^{2x} y^2 = -e^{-2x}.$$

We guess a solution of the form $y_0 = c\, e^{-2x}$. By insertion,

$$-2ce^{-2x} + c^2 e^{-2x} = -e^{-2x}, \qquad \text{so} \qquad c^2 - 2c + 1 = (c-1)^2 = 0,$$

so $c = 2$, and $y_0 = e^{-2x}$ is a particular solution.

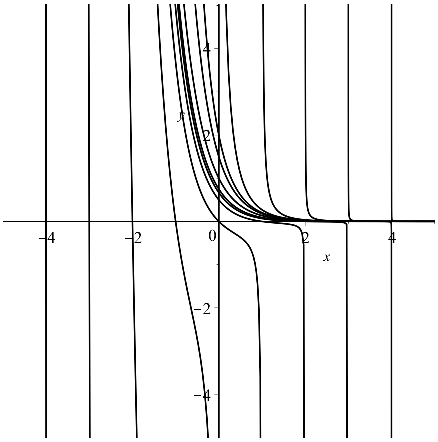


Figure 6.8: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} + e^{2x} y^2 = -e^{-2x}$.

Using the standard method we put $y = u + e^{-2x}$. Then by insertion

$$0 = \frac{\mathrm{d}u}{\mathrm{d}x} - 2e^{-2x} + e^{2x}\left\{u^2 + 2e^{-2x}u + e^{-4x}\right\} + e^{-2x} = \frac{\mathrm{d}u}{\mathrm{d}x} + e^{2x}u^2 + 2u,$$

where we divide by $-u^2$ to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) - 2 \cdot \frac{1}{u} = e^{2x}.$$

The complete solution of this equation is

$$\frac{1}{u} = c\, e^{2x} + x\, e^{2x} = e^{2x}(x + c), \qquad \text{so} \qquad u = \frac{e^{-2x}}{x + c},$$

and the complete solution of the Riccati differential equation is

$$y = \frac{e^{-2x}}{x + c} + e^{-2x} = e^{-2x} \cdot \frac{c + x + 1}{c + x}, \qquad \text{where } c \text{ is an arbitrary constant,}$$

supplied with the previously guessed solution, $y_0 = e^{-2x}$. $\Diamond$

We shall finally in the following example demonstrate that Riccati equations have successfully been used as models in practice. We shall with some very necessary modifications follow Davis [7], who himself is following Carson et al. [4], concerning parachute jumps in 1942. The author remembers that he had a simplified model as homework as a student in the early 1960s, so it must have been known at that time in the academic circles. We first set up the model.

Example 6.9 *Consider a body of mass m which falls under gravity in a medium which offers resistance to its motion proportional to the square of the velocity of the body.*

If x denotes the distance passed over by the body in time t, then the motion is defined by the equation

$$m\,\frac{d^2x}{dt^2} = mg - K\left(\frac{dx}{dt}\right)^2.$$

This is a nonlinear differential equation of second order, which we have not considered yet. However, if we let $v := \frac{dx}{dt}$ denote the velocity, then we get the following Riccati differential equation in v,

$$m\,\frac{dv}{dt} = mg - K\cdot v^2, \qquad \textit{and} \qquad v = \frac{dx}{dt}.$$

The choice of solution method is not unique. Since the variable t does not occur on the right hand side of the equation, we can actually separate the variables. This variant is left to the interested reader. Another method, which we shall follow here, is the traditional one, i.e. we guess a particular solution and then reduce the problem to solving an auxiliary Bernoulli differential equation. In fact, let us see if we have any constant solution $v = V$. By insertion,

$$0 = mg - K\cdot V^2, \qquad \text{from which is seen that } v = V := \sqrt{\frac{mg}{K}} \text{ is a constant solution.}$$

The standard method is then to put $v = u + V$ and derive a Bernoulli differential equation in u. We get by insertion,

$$\begin{aligned} 0 &= \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{K}{m}(u+V)^2 - g = \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{K}{m}\left(u^2 + 2Vu + V^2\right) - g \\ &= \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{2KV}{m}\cdot u + \frac{K}{m}u^2 + \frac{K}{m}\cdot\frac{mg}{K} - g = \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{2KV}{m}u + \frac{K}{m}u^2. \end{aligned}$$

Thus, we have derived a Bernoulli differential equation. However, the solution of this is again left to the reader as an exercise, because we also see that the variables without difficulty can be separated. We shall use the latter method, where we first write

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{K}{m} \cdot u(u+2V),$$

thus by separation and decomposition,

$$\frac{K}{m}\,\mathrm{d}t = -\frac{\mathrm{d}u}{u(u+2V)} = \left(-\frac{1}{2V}\cdot\frac{1}{u} + \frac{1}{2V}\cdot\frac{1}{u+2V}\right)\mathrm{d}u.$$

We integrate and get with an arbitrary constant k_1,

$$\frac{K}{m}\,t + k_1 = \frac{1}{2V}\ln\left|\frac{u+2V}{u}\right|,$$

or put in another way, assuming that u (and V) are positive,

$$\ln\left(1+\frac{2V}{u}\right) + k_2 = \frac{2kV}{m}\,t,$$

and we get with a third arbitrary constant C that

$$C\left(1+\frac{2V}{u}\right) = \exp\left(\frac{2KV}{m}\,t\right), \qquad \text{i.e.} \qquad \frac{1}{u}\left\{\exp\left(\frac{2KV}{m}\,t\right) - C\right\}.$$

We find that

$$u = \frac{2VC}{\exp\left(\frac{2KV}{m}\,t\right) - C},$$

and therefore

$$v = u + V = V\cdot\left\{\frac{2C}{\exp\left(\frac{2KV}{m}\,t\right) - C} + 1\right\} = V\cdot\frac{\exp\left(\frac{2KV}{m}\,t\right) + C}{\exp\left(\frac{2KV}{m}\,t\right) - C}.$$

We see in particular for every fixed constant C that

$$\lim_{t\to+\infty} v(t) = V = \sqrt{\frac{mg}{K}},$$

and when $C = 0$, we get the constant solution V.

Let $v(0) = v_0$ be a given initial velocity. Then

$$v_0 = V\cdot\frac{1+C}{1-C}, \qquad \text{i.e.} \qquad u_0 := \frac{v_0}{V} = \frac{1+C}{1-C},$$

from which $C = \dfrac{u_0 - 1}{u_0 + 1}$. By insertion,

$$\begin{aligned} v &= V \cdot \frac{\exp\left(\frac{2KV}{m}t\right) + C}{\exp\left(\frac{2KV}{m}t\right) - C} = V \cdot \frac{(u_0+1)\exp\left(\frac{KV}{m}t\right) + (u_0-1)\exp\left(-\frac{KV}{m}t\right)}{(u_0+1)\exp\left(\frac{KV}{m}t\right) - (u_0-1)\exp\left(-\frac{KV}{m}t\right)} \\ &= V \cdot \frac{u_0\cosh\left(\frac{KV}{m}t\right) + \sinh\left(\frac{KV}{m}t\right)}{u_0\sinh\left(\frac{KV}{m}t\right) + \cosh\left(\frac{KV}{m}t\right)}. \end{aligned}$$

Finally, we assume that $x(0) = 0$ for $t = 0$. Then $x = x(t)$ is determined as follows,

$$\begin{aligned} x(t) &= \int_0^t v(\tau)\,\mathrm{d}\tau = V\int_0^t \frac{u_0\cosh\left(\frac{KV}{m}t\right) + \sinh\left(\frac{KV}{m}t\right)}{u_0\sinh\left(\frac{KV}{m}t\right) + \cosh\left(\frac{KV}{m}t\right)} \\ &= \frac{m}{K}\int_0^{\frac{KV}{m}t} \frac{u_0\cosh\xi + \sinh\xi}{u_0\sinh\xi + \cosh\xi}\,\mathrm{d}\xi = \frac{m}{K}\left[\ln(u_0\sinh\xi + \cosh\xi)\right]_{\xi=0}^{\frac{KV}{m}t} \\ &= \frac{m}{K}\ln\left(u_0\sinh\left(\frac{KV}{m}t\right) + \cosh\left(\frac{KV}{m}t\right)\right). \end{aligned}$$

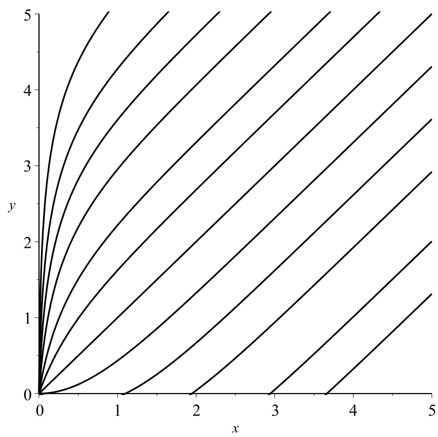


Figure 6.9: *Some principal solution curves of the structure* $y = \ln(c \cdot \sinh t + \cosh t)$. *As expected, the curves are eventually almost parallel.*

Finally, $V^2 = \dfrac{mg}{K}$, so $\dfrac{KV}{m} = \dfrac{KV^2}{mV} = \dfrac{K}{mV} \cdot \dfrac{mg}{K} = \dfrac{g}{V}$, and $u_0 = \dfrac{v_0}{V}$, and $\dfrac{m}{K} = \dfrac{V^2}{g}$, so the solution is written

$$x(t) = \frac{V^2}{g}\ln\left(\frac{v_0}{V}\sinh\left(\frac{gt}{V}\right) + \cosh\left(\frac{gt}{V}\right)\right),$$

where we assume that we have estimated by measurements the limit value of the velocity.

As already mentioned there are some variants, which are left to the reader. It was also mentioned in the beginning of the example that this model was set up in 1942 by Carlson et al. [4] concerning parachuting. The authors concluded that there are four factors involved in free fall,

1) the altitude, or air density,

2) the weight of the falling body,

3) the position in the air of the body, i.e. its geometry,

4) the amount of spinning and tumbling.

They also concluded that the air-density factor is probably the most important, since the average velocity seems to increase with elevation.

Someone from the same university then tested the model in the case, when the average weight of the jumpers with their equipment was 261.2 pounds, where $g = 32.2$ feet/second^2, where $V = 265.7$ feet/second, and $K = 0.1191$ measured in feet-pound units, all measured. Furthermore, $x = 29,300$ feet. It turns up, that the model gives a good approximation of the measured values. $\Diamond$

Example 6.10 *Find the complete solution of the Riccati equation*

$$\frac{dy}{dx} + y^2 = \frac{1}{x^4}.$$

We shall first guess a solution. If we try $y = \frac{1}{x^2}$, then we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = -\frac{2}{x^3} + \frac{1}{x^4} = \frac{1}{x^2} - \frac{2}{x^3} + \frac{1}{x^4} - \frac{1}{x^2} = \left(-\frac{1}{x} + \frac{1}{x^2}\right)^2 - \frac{1}{x^2}.$$

This leads to the adjusted guess

$$y_0 = \frac{1}{x} + \frac{1}{x^2}.$$

We get by insertion,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = -\frac{1}{x^2} - \frac{2}{x^3} + \left(\frac{1}{x} + \frac{1}{x^2}\right)^2 = -\frac{1}{x^2} - \frac{2}{x^3} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4} = \frac{1}{x^4},$$

and we have proved that $y_0 = \frac{1}{x} + \frac{1}{x^2}$ is a solution.

By the standard procedure we shall put

$$y = u + y_0 = u + \frac{1}{x} + \frac{1}{x^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x^2} - \frac{2}{x^3},$$

and

$$y^2 = \left(u + \frac{1}{x} + \frac{1}{x^2}\right)^2 = u^2 + 2u\left(\frac{1}{x} + \frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{x^2}\right)^2,$$

hence,

$$\begin{aligned} 0 &= \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - \frac{1}{x^4} = \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x^2} - \frac{3}{x^3} + u^2 + 2u\left(\frac{1}{x} + \frac{1}{x^2}\right) + \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4} - \frac{1}{x^4} \\ &= \frac{\mathrm{d}u}{\mathrm{d}x} + u^2 + 2\left(\frac{1}{x} + \frac{1}{x^2}\right)u, \end{aligned}$$

which is a Bernoulli differential equation.

Clearly, $u = 0$ corresponds to the solution $y_0 = \frac{1}{x} + \frac{1}{x^2}$ above, so we assume in the following that $u \neq 0$. When we divide by $-u^2$, we get

$$0 = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x} - 1 - 2\left(\frac{1}{x} + \frac{1}{x^2}\right)\frac{1}{u} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) - 2\left(\frac{1}{x} + \frac{1}{x^2}\right)\frac{1}{u} - 1,$$

which is linear in $\frac{1}{u}$. We compute

$$\exp\left(2\int\left(\frac{1}{x} + \frac{1}{x^2}\right)\mathrm{d}x\right) = \exp\left(\ln(x^2) - \frac{2}{x}\right) = x^2\exp\left(-\frac{2}{x}\right).$$

Then by the solution formula,

$$\begin{aligned}\frac{1}{u} &= x^2\exp\left(-\frac{2}{x}\right)\left\{c+\int\frac{1}{x^2}\exp\left(\frac{2}{x}\right)\,\mathrm{d}x\right\}\\ &= x^2\exp\left(-\frac{2}{x}\right)\left\{c-\frac{1}{2}\exp\left(\frac{2}{x}\right)\right\}=c\cdot x^2\exp\left(-\frac{2}{x}\right)-\frac{x^2}{2},\end{aligned}$$

so

$$y=u+\frac{1}{x}+\frac{1}{x^2}=\frac{1}{\left\{c\exp\left(-\frac{2}{x}\right)-\frac{1}{2}\right\}x^2}+\frac{1+x}{x^2}=\frac{1+(1+x)\left\{C\exp\left(-\frac{2}{x}\right)-\frac{1}{2}\right\}}{\left\{C\exp\left(-\frac{2}{x}\right)-\frac{1}{2}\right\}x^2},$$

where C is an arbitrary constant. $\Diamond$

Example 6.11 *Find a solution procedure for a differential equation of the type*

$$\frac{dy}{dx}=\left(y^2-a^2\right)f(x)+(y+a)g(x).$$

Since $y+a$ is a common factor on the right hand side, $y=-a$ is clearly a (constant) solution, and we choose the transformation $y(x)=u(x)-a$. Then we get

$$\frac{\mathrm{d}u}{\mathrm{d}x}=(u-2a)uf(x)+ug(x)=u^2f(x)+u\{g(x)-2af(x)\},$$

which is a Bernoulli equation. If we put $z=\frac{1}{u}$, then we get the following linear equation in z,

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\{g(x)-2a\}f(x)\cdot z+f(x),$$

which e.g. is solved by the usual solution formula. $\Diamond$

6.4 Relationship between the Riccati equation and the linear homogeneous differential equation of second order

The importance of the Riccati equations is due to their connection with linear homogeneous differential equations of second order. If we can solve a Riccati equation, then we can also solve its corresponding linear homogeneous differential equation of second order, and *vice versa*. We shall here describe this relationship.

First consider the general Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+Q(x)y+R(x)y^2=P(x),$$

where we assume that $R(x)\neq 0$. In fact, if $R(x)\equiv 0$, then the equation degenerates to a linear differential equation.

We shall use the transformation

$$y:=\frac{1}{R(x)\cdot u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x},\qquad\text{i.e.}\qquad\frac{\mathrm{d}\ln|u|}{\mathrm{d}x}=R(x)\cdot y.$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{R(x)\cdot u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{1}{R(x)\cdot u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{R'(x)}{R(x)^2u}\frac{\mathrm{d}u}{\mathrm{d}x},$$

and

$$R(x)y^2 = \frac{1}{R(x)\cdot u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2,$$

so by reading the equation from the right to the left,

$$P(x) = \frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y + R(x)y^2 = \frac{1}{R(x)u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{R'(x)}{R(x)^2u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{Q(x)}{R(x)u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

When we multiply by $R(x)^2u$ and rearrange, we get

$$R(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \{R(x)Q(x) - R'(x)\}\frac{\mathrm{d}u}{\mathrm{d}x} - P(x)R(x)^2u = u,$$

which indeed is a linear homogeneous differential equation of second order.

If we know a solution $y(\neq 0)$ of the Riccati equation, then we can find a solution u of the corresponding linear homogeneous differential equation of second order from integration of

$$\frac{\mathrm{d}\ln|u|}{\mathrm{d}x} = R(x)y(x), \qquad \text{i.e.} \qquad u_0(x) = \exp\left(\int R(x)y(x)\,\mathrm{d}x\right).$$

Conversely, if $u = u(x) \neq 0$ is a solution of the derived linear homogeneous differential equation of second order, then

$$y = \frac{u'(x)}{R(x)u(x)}$$

is a solution of the Riccati equation.

If instead we are given the linear homogeneous differential equation of second order,

$$A(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + B(x)\frac{\mathrm{d}u}{\mathrm{d}x} + C(x)u = 0, \qquad \text{where } A(x) \neq 0,$$

we only have to identify $P(x)$, $Q(x), R(x)$ of the unknown Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y + R(x)y^2 = P(x).$$

This identification is not unique, but we shall choose the simplest method. We found above that such a Riccati equation was related to the differential equation of second order,

$$R(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \{R(x)Q(x) - R'(x)\}\frac{\mathrm{d}u}{\mathrm{d}x} - P(x)R(x)^2u = u.$$

When we identify the coefficients, we get the three equations

$$R(x) = A(x), \qquad R(x)Q(x) - R'(x) = B(x), \qquad P(x)R(x)^2 = -C(x),$$

from which

$$R(x) = A(x), \qquad Q(x) = \frac{A'(x) + B(x)}{A(x)}, \qquad \text{and} \qquad P(x) = -\frac{C(x)}{A(x)^2},$$

and we have shown the relationship between a Riccati equation and a linear homogeneous differential equation of second order.

Let us return to the general Ricatti equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y + R(x)y^2 = P(x),$$

and make the transformation $y = S(x)z$ for some factor $S(x) \neq 0$. Then by insertion,

$$P(x) = S(x)\,\frac{\mathrm{d}z}{\mathrm{d}x} + \{S'(x) + Q(x)S(x)\}z + R(x)S(x)^2 z^2.$$

Then we choose $S(x)$, such that $S'(x) + Q(x)S(x) = 0$, i.e.

$$S(x) = \exp\left(-\int Q(x)\,\mathrm{d}x\right).$$

By this transformation the Riccati equation is reduced to

$$\frac{\mathrm{d}z}{\mathrm{d}x} + R(x)S(x)z^2 = \frac{P(x)}{S(x)},$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + R(x)\exp\left(-\int Q(x)\,\mathrm{d}x\right) z^2 = P(x)\exp\left(\int Q(x)\,\mathrm{d}x\right),$$

where the linear term has disappeared. We can therefore, if necessary, limit our investigations of the Riccati equation to the simplified equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + R_1(x)y^2 = P_1(x)$$

with no linear term.

The drawback is of course that there exist no general solution formulæ for neither the Riccati equation, nor for the linear differential equation of second order, although solution formulæ and procedures are known in many cases.

6.5 The cross-ratio for the Riccati equation

When we look at the solutions of the previous examples of the Riccati equation it is striking that apart from the given, or guessed, solution $\neq 0$ they all have a structure like

$$y = \frac{c \cdot f(x) + g(x)}{c \cdot h(x) + k(x)}, \qquad \text{where } c \text{ is an arbitrary constant,}$$

and where the numerator and denominator are not proportional, a condition which we here most conveniently write as $f(x)k(x) \neq g(x)h(x)$. We shall use this condition in the following.

If we solve the above equation with respect to c, we get

$$c = \frac{g(x) - yk(x)}{yh(x) - f(x)}.$$

In order to derive a differential equation of y we eliminate the constant c by differentiation. When we also multiply by the common denominator $\{yh(x) - f(x)\}^2$ of the differential quotient, we get

$$\begin{aligned}
0 &= \{yh(x) - f(x)\} \cdot \{g'(x) - y'k(x) - yk'(x)\} - \{y'h(x) + yh'(x) - f'(x)\} \cdot \{g(x) - yk(x)\} \\
&= f(x)k(x)y' + g'(x)h(x)y + f(x)k'(x)y - h(x)k'(x)y^2 - f(x)g'(x) \\
&\qquad -g(x)h(x)y' - g(x)h'(x)y - f'(x)k(x)y + h'(x)k(x)y^2 + f'(x)g(x) \\
&= \{f(x)k(x) - g(x)h(x)\}\frac{\mathrm{d}y}{\mathrm{d}x} + \{f(x)k'(x) + g'(x)h(x) - f'(x)k(x) - g(x)h'(x)\}y \\
&\qquad +\{h'(x)k(x) - h(x)k'(x)\}y^2 + \{f'(x)g(x) - f(x)g'(x)\}.
\end{aligned}$$

By assumption, $f(x)k(x) \neq g(x)h(x)$, so the coefficient of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not 0. Now, if $h'(x)k(x) = h(x)k'(x)$, then $h(x)$ and $k(x)$ are proportional, and the equation is linear. Finally, if $f'(x)g(x) = f(x)g'(x)$, i.e. $f(x)$ and $g(x)$ are proportional, then the equation is reduced to a Bernoulli differential equation. In all remaining cases y, given by the four functions and an arbitrary constant, is a solution of a Riccati equation.

Let us return to the general structure of the solutions of a Riccati equation, i.e.

$$y = \frac{c \cdot f(x) + g(x)}{c \cdot h(x) + k(x)}, \qquad c \text{ arbitrary constant,}$$

where $f(x)$, $g(x)$, $h(x)$, $k(x)$ are given functions, and where $h(x)$ and $k(x)$ are not proportional.

Let y_1, y_2, y_3, y_4, corresponding to four different constants c_1, c_2, c_3, c_4, resp., be four linearly independent solutions. Then for $i \neq j$,

$$\begin{aligned} y_i - y_j &= \frac{c_i \cdot f(x) + g(x)}{c_i \cdot h(x) + k(x)} - \frac{c_j \cdot f(x) + g(x)}{c_j \cdot h(x) + k(x)} \\ &= \frac{c_i c_j f(x)h(x) + c_j g(x)h(x) + c_i f(x)k(x) + g(x)k(x)}{\{c_i \cdot h(x) + k(x)\}\{c_j h(x) + k(x)\}} \\ &\qquad - \frac{c_j c_i f(x)h(x) + c_i g(x)h(x) + c_j f(x)k(x) + g(x)k(x)}{\{c_j \cdot h(x) + k(x)\}\{c_i h(x) + k(x)\}} \\ &= \frac{(c_j - c_i)\, g(x)h(x) + (c_i - c_j)\, f(x)k(x)}{\{c_i \cdot h(x) + k(x)\}\{c_j h(x) + k(x)\}} = \frac{(c_i - c_j)\{f(x)k(x) - g(x)h(x)\}}{\{c_i \cdot h(x) + k(x)\}\{c_j h(x) + k(x)\}}. \end{aligned}$$

The cross-ratio of the four solutions y_1, y_2, y_3, y_4 is defined as

$$R := \frac{y_1 - y_3}{y_3 - y_2} : \frac{y_1 - y_4}{y_y - y_2} = \frac{y_1 - y_3}{y_1 - y_4} \cdot \frac{y_2 - y_4}{y_2 - y_3}.$$

We note that the factor $f(x)k(x) - g(x)h(x)$ occurs twice in the numerator and twice in the denominator, hence, they all cancel. Furthermore, the factor $c_i h(x) + k(x)$, $i = 1, 2, 3, 4$, occurs once in the numerator and once in the denominator of the cross-ration, so they also all cancel. Hence, the value of the cross-ration of four linearly independent solutions of a given Riccati equation is the constant

$$R = \frac{c_1 - c_3}{c_1 - c_4} \cdot \frac{c_2 - c_4}{c_2 - c_3} \neq 0.$$

6.6 Discussion of the equation $y' = ay^2 + bx^\alpha$.

We shall here return to the discussion of the special Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay^2 + bx^\alpha$$

for some special values of the exponent α, where the Riccati equation can be solved by means of elementary functions.

This equation may seem rather special. However, it can be solved by using the method of convergent power series. For some values of the exponent n it even has elementary functions as solutions. Furthermore, the equation is also connected with the Bessel functions, J_ν and $Y_\nu(x)$, so it indeed has some interesting and unexpected properties.

We shall in this section allow complex constants, and we shall also use the method of solving a linear homogeneous differential equation of second order with polynomial coefficients by means of convergent power series. This method is included in all elementary calculus courses on linear differential equations, so the reader should be familiar with it.

6.6.1 The case $\alpha = 0$.

In this case the equation is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay^2,$$

which is solved by separation,

$$-\frac{1}{y^2}\,\mathrm{d}y = -a\,\mathrm{d}x,$$

and we get by integration,

$$\frac{1}{y} = -a(x-c) = a(c-x),$$

i.e.

$$y = \frac{1}{a}\cdot\frac{1}{c-c}, \qquad x \neq c, \text{ where } c \text{ is an arbitrary constant.}$$

6.6.2 The case $\alpha = -2$.

When $\alpha = -2$, the differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay^2 + \frac{b}{x^2}, \qquad x \neq 0.$$

For convenience we restrict ourselves to one of the four open quadrants, namely the first quadrant, $x > 0$ and $y > 0$. The other quadrants are treated likewise (left to the reader).

We apply the transformation

$$z(x) = \frac{1}{y(x)}, \qquad \text{i.e.} \qquad y(x) = \frac{1}{z(x)},$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{z(x)^2}\,\frac{\mathrm{d}z}{\mathrm{d}x} = ay^2 + \frac{b}{x^2} = \frac{a}{z(x)^2} + \frac{b}{x^2},$$

from which by a rearrangement,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + a + b\left(\frac{z}{x}\right)^2 = 0 \qquad \text{for } x > 0 \text{ and } z > 0.$$

This is an homogeneous differential equation, cf. Chapter 7. We first note that possible rectilinear solutions $z = cx$ must satisfy the equation

$$bc^2 + c + a = 0,$$

so we have two, one or none rectilinear solutions.

In general we use the transformation

$$z = x \cdot u, \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = x\,\frac{\mathrm{d}u}{\mathrm{d}x} + u,$$

so we get the equation

$$x\,\frac{\mathrm{d}u}{\mathrm{d}x} + u + a + bu^2 = 0,$$

thus by separating the variables,

$$\frac{\mathrm{d}u}{bu^2 + u + a} = -\frac{\mathrm{d}x}{x},$$

which is easily integrated, once the constants a and b are given.

6.6.3 The Riccati equation $y' + ay^2 = bx^n$ in general.

Let us consider the general Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ay^2 = bx^n.$$

We first make the transformation

$$y = \frac{u'}{a \cdot u},$$

on the special Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ay^2 = bx^n.$$

This gives the result

$$0 = \frac{1}{a \cdot u}\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{1}{a \cdot u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{a}{a^2\,u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - b\,x^n = \frac{1}{a\,u}\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - b\,x^n.$$

By a rearrangement,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - c^2\,x^n\,u(x) = 0, \qquad \text{where } c^2 := ab.$$

One of the tricks of solving such equations is to find the right transformation, which carries the equation into another one, where solution procedures are known. The proper one here is fairly hard to guess immediately, so we just claim that

$$u = \exp(Ax^p) \cdot v, \qquad \text{for some constants } A \text{ and } p \text{ to be found,}$$

is the right one.

By insertion into the linear second order differential equation we get

$$\begin{aligned}0 &= \frac{\mathrm{d}^2u}{\mathrm{d}x^2} - c^2x^n u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\exp(Ax^p)\,\frac{\mathrm{d}v}{\mathrm{d}x} + Apx^{p-1}\exp(Ax^p)\right\} - c^2x^n\exp(Ax^p)\cdot v \\ &= \exp(Ax^p)\,\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2Apx^{p-1}\exp(Ax^p)\,\frac{\mathrm{d}v}{\mathrm{d}x} \\ &\qquad + \left\{Ap(p-1)x^{p-2} + A^2p^2x^{2(p-1)}\right\}\exp(Ax^p)\,v - c^2x^n\exp(Ax^p)\,v,\end{aligned}$$

which is reduced to

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2Apx^{p-1}\,\frac{\mathrm{d}v}{\mathrm{d}x} + \left\{A^2p^2x^{2(p-1)} + Ap(p-1)x^{p-2} - c^2x^n\right\}v = 0.$$

when we choose

$$A = \frac{c}{p} \qquad \text{and} \qquad p = \frac{n}{2} + 1, \text{ i.e. } n = 2(p-1), \qquad \text{assuming } p \neq 0, \text{ i.e. } n \neq -2,$$

and keep p in the following, we get

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2cx^{p-1}\,\frac{\mathrm{d}v}{\mathrm{d}x} + c(p-1)x^{p-2}v = 0.$$

Let us assume that this equation has a solution of the following form of a formal power series,

$$v := \sum_{m=0}^{+\infty} a_m\,x^{mp},$$

where the unknowns are the coefficients a_m, $m \in \mathbb{N}_0$. By formal insertion into the equation,

$$\begin{aligned}0 &= \frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2cx^{p-1}\frac{\mathrm{d}v}{\mathrm{d}x} + c(p-1)x^{p-2}v \\ &= \sum_{m=1}^{+\infty} mp(mp-1)a_m x^{mp-2} + \sum_{m=0}^{+\infty} 2cmpa_m x^{mp+p-2} + c(p-1)\sum_{m=0}^{+\infty} a_m x^{mp+p-2} \\ &= \sum_{m=1}^{+\infty} mp(mp-1)a_m x^{mp-2} + \sum_{m=0}^{+\infty}\{2cmp + c(p-1)\}a_m x^{m(p+1)-2} \\ &= \sum_{m=0}^{+\infty}(m+1)p((m+1)p-1)a_{m+1}x^{m(p+1)-2} + \sum_{m=0}^{+\infty} c\{2mp+p-1\}a_m x^{m(p+1)-2} \\ &= \sum_{m=0}^{+\infty}\{(m+1)p((m+1)p-1)a_{m+1} + c((2m+1)p-1)a_m\}\,x^{m(p+1)-2}.\end{aligned}$$

If this equation holds, we necessarily must have that all coefficients are 0, so we get the recursion formula

$$(m+1)p\{(m+1)p-1\}a_{m+1} + c\{(2m+1)p-1\}a_m = 0, \qquad \text{for all } m \in \mathbb{N}_0.$$

We may here note that if for some $m_0 \in \mathbb{N}_0$,

$$(2m_0+1)\,p - 1 = 0, \qquad \text{i.e.} \qquad p = \frac{1}{2m_0+1} \quad \text{for some } m_0 \in \mathbb{N}_0,$$

then $a_{m_0+1} = 0$, and $a_{m_0+q} = 0$ for all $q \in \mathbb{N}$ by induction, so the series terminates, and we have only a finite number of terms

$$v = \sum_{m=0}^{m_0} a_m \, x^{mp}.$$

Since for this value of p,

$$n = 2(p-1) = 2\left\{\frac{1}{2m_0+1} - 1\right\} = -\frac{4m_0}{2m_0+1},$$

we conclude that the series is finite, when the exponent

$$n \in \left\{ -\frac{4m}{2m+1} \;\middle|\; m \in \mathbb{N}_0 \right\}.$$

If instead $(m_0+1)\,p - 1 = 0$, i.e.

$$p = \frac{1}{m_0+1} \qquad \text{and} \qquad n = 2(p-1) = 2\left\{\frac{1}{m_0+} - 1\right\} = -\frac{2m_0}{m_0+1},$$

then $a_0 = \cdots = a_{m_0} = 0$ by recursion, and a_{m_0+1} can be chosen equal to 1.

In general, when $(m+1)p \neq 1$ and $p \neq 0$, we get the recursion formula,

$$a_{m+1} = -c\,\frac{(2m+1)p-1}{(m+1)p\{(m+1)p-1\}}\,a_m.$$

We put for convenience $a_0 = 1$. Then

$$a_1 = -c\,\frac{p-1}{p(p-1)} \qquad \left(= -\frac{c}{p}\right),$$

$$a_2 = -c\,\frac{3c-1}{2p(2p-1)} \cdot a_1 = c^2\,\frac{(p-1)(3p-1)}{p(p-1)2p(2p-1)} \qquad \left(= c^2\,\frac{3p-1}{2p^2(2p-1)}\right),$$

$$a_3 = -c\,\frac{5p-1}{3p(3p-1)} \cdot a_2 = (-c)^3 \cdot \frac{(p-1)(3p-1)(5p-1)}{p(p-1)2p(2p-1)3p(3p-1)},$$

and we conclude by induction that in general,

$$a_m = (-c)^m \cdot \frac{\prod_{j=1}^m \{(2j-1)p-1\}}{m!p^m \cdot \prod_{j=1}^m (jp-1)}.$$

From this we formally get

$$v(x) = V(x,c) = \sum_{m=0}^{+\infty} (-c)^m \cdot \frac{\prod_{j=1}^m \{(2j-1)p-1\}}{m!p^m \prod_{j=1}^m (jp-1)}\, x^{mp}, \qquad p = \frac{n}{2}+1.$$

When $p \in \mathbb{N}_0$, this series solution converges for all $x \in \mathbb{R}$. If $p \in \mathbb{Z} \setminus \mathbb{N}_0$, then the series converges for $x \in \mathbb{R} \setminus \{0\}$. If $p \notin \mathbb{Z}$, then the series converges at least for $x \in \mathbb{R}_+$. When $p \geq 0$, then the series converges at least for $x \geq 0$. Finally, if $p \in \mathbb{Q}$ has an odd denominator, then the series converges for $x \neq 0$.

It follows from the definition of the transformation, i.e. $u = v \cdot \exp\left(\frac{c}{p} \cdot x^p\right)$, that the solution of the reduced equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - c^2\, x^n u = 0, \qquad \text{where } c^2 = a \cdot b,$$

is given by

$$u_1(x) = \exp\left(\frac{c}{p} \cdot x^p\right) \cdot V(x,c), \qquad \text{where } p = \frac{n}{2} + 1.$$

Then note that the equation is not changed, if the constant c is replaced by its opposite $-c$, so

$$u_2(x) = \exp\left(-\frac{c}{p} \cdot x^p\right) V(x,-c)$$

is another, linearly independent solution of the equation. Hence, $\lambda u_2(x)$ and $\lambda \{u_1(x) + c u_2(x)\}$ are the complete solutions of the second order linear homogeneous differential equation, where λ and c are arbitrary constants. Using this, and the correspondence $y = \dfrac{u'(x)}{a \cdot u(x)}$, we conclude that the complete solution of the original Riccati equation is either

$$y = \frac{u_2'(x)}{a\, u_2(x)}, \qquad \text{or} \qquad y = \frac{u_1'(x) + c\, u_2'(x)}{a\,\{u_1(x) + c\, u_2(x)\}}, \qquad \text{where } c \text{ is an arbitrary constant.}$$

Note in particular that if $c \in \mathbb{C} \setminus \mathbb{R}$ is complex, then $u_2(x) = \overline{u_1(x)}$, so $u_1(x)$ and $u_2(x)$ are complex conjugated, and it is possible to restrict ourselves to real solutions only.

Summary. We shall here give some guidelines of how to solve the Riccati differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a\,y^2 = b\,x^n,$$

where a, b, $n \in \mathbb{R}$ are given constants.

1) Choose $c \in \mathbb{C}$ as one of the possible two solutions of the equation $c^2 = ab$. We note that if $a = 0$ or $b = 0$, then the problem is reduced to either an integration or to a separation of the variables, followed by an integration. We therefore assume in the following that $c^2 = ab \neq 0$.

2) Put for convenience, $p := \frac{n}{2} + 1$. We derive a linear homogeneous auxiliary differential equation of second order with solutions given by $v = \sum_{m=0}^{+\infty} a_m x^{mp}$, where the a_m satisfy the recursion formula

$$(m+1)p\{(m+1)p-1\}a_{m+1} + c\{(2m+1)p-1\}a_m = 0, \qquad \text{for } m \in \mathbb{N}_0.$$

   One should here always check if the recursion formula contains any zeros, which then must be dealt with separately. We mention that if $(2m_0+1)\,p = 1$ for some $m_0 \in \mathbb{N}$, then the series is reduced to the finite sum $\sum_{m=0}^{m_0} a_m\,x^{mp}$, and if instead $(m_0+1)\,p = 1$, then the first m_0 terms are 0, i.e. $a_0 = \cdots = a_{m_0} = 0$, and the expansion starts from $m = m_0 + 1$.

3) Define in general (tacitly including the modifications mentioned above),

$$v = V(x,c) := \sum_{m=0}^{+\infty} (-c)^m \frac{\prod_{j=1}^{m}\{(2j-1)p-1\}}{m!p^m \prod_{j=1}^{m}(jp-1)}\, x^{mp},$$

   (where $jp - 1 \neq 0$ for $j = 0, \dots, m$, etc.).

4) Define the linearly independent solutions of the auxiliary linear differential equation of second order,

$$u_1 = \exp\left(\frac{c}{p}\,x^p\right) V(x,c) \qquad \text{and} \qquad u_2 = \exp\left(-\frac{c}{p}\,x^p\right) V(x,-c).$$

   These will span the set of all solutions of this equation.

5) Finally, the complete solution of the original Riccati equation is given by either

$$\tilde{y} = \frac{u_2'(x)}{a\,u_2(x)}, \qquad \text{or} \qquad y_c = \frac{u_1'(x) + c\,u_2'(x)}{a\,\{u_1(x) + c\,u_2(x)\}},$$

   where c is an arbitrary constant.

Example 6.12 *Find the complete solution of the Riccati equation*

$$\frac{dy}{dx} + y^2 = x^{-\frac{8}{5}}.$$

In the present case, $a = b = 1$, so $c^2 = ab = 1$, and we can choose $c = 1$. Furthermore, $n = -\frac{8}{5}$, so $p = \frac{n}{2} + 1 = \frac{1}{5}$. It follows that $(2m_0+1)\,p = 1$ for $m_0 = 2$, so the v-series is reduced to the finite sum

$$v = \sum_{m=0}^{2} a_m\,x^{\frac{m}{5}},$$

where

$$a_0 = 1, \qquad a_1 = -\frac{c}{p} = -5, \qquad \text{and} \qquad a_2 = -\frac{\frac{3}{5} - 1}{\frac{2}{5}\left(\frac{2}{5} - 1\right)}\, a_1 = -\frac{-2}{-\frac{6}{25}} = \frac{25}{3},$$

hence

$$v = V(x,1) = \sum_{m=0}^{2} a_m\, x^{\frac{m}{5}} = 1 - 5\sqrt[5]{x} + \frac{25}{3}\,\sqrt[5]{x^2}.$$

We put temporarily $z := x^p = \sqrt[5]{x}$ and note that $z' = \frac{1}{5}\,x^{-4}$. Then

$$V(x,1) = 1 - 5z + \frac{25}{3}\,z^2, \qquad \text{and} \qquad V(x,-1) = 1 + 5z + \frac{25}{3}\,z^2.$$

Two linearly independent solutions of the auxiliary linear homogeneous differential equation of second order are then given by

$$u_1(x) = \exp\left(\frac{c}{p}\,x^p\right) V(x,1) = \left\{1 - 5z + \frac{25}{3}\,z^2\right\} e^{5z},$$

and

$$u_2(x) = \left\{1 + 5z + \frac{25}{3}\,z^2\right\} e^{-5z}.$$

Then we use that $z' = \frac{1}{5}\,z^{-4}$ to get

$$\begin{aligned} u_1'(x) &= \left\{-z^{-4} - \frac{10}{3}\,z^{-3}\right\} e^{5z} + \left\{z^{-4} - 5z^{-3} + \frac{25}{3}\,z^{-2}\right\} e^{5z} \\ &= \left\{-\frac{25}{3}\,z^{-3} + \frac{25}{3}\,z^{-2}\right) e^{5z} = \frac{25}{3z^3}(z-1)e^{5z}, \end{aligned}$$

and similarly,

$$\begin{aligned} u_2'(x) &= \left\{z^{-4} - \frac{10}{3}\,z^{-3}\right\} e^{-5z} - \left\{z^{-4} + 5x^{-3} + \frac{25}{3}\,z^{-2}\right\} e^{-5z} \\ &= \left\{-\frac{25}{3}\,z^{-3} - \frac{25}{3}\,z^{-2}\right\} e^{-5z} = -\frac{25}{3z^3}(z+1)e^{-5z}. \end{aligned}$$

Then

$$y_0(x) = \frac{u_2'(x)}{a\,u_2(x)} = -\frac{\frac{25}{3}\,z^{-3}(z+1)e^{-5z}}{\left(1 + 5z + \frac{25}{3}\,z^2\right) e^{-5z}} = -\frac{25(z+1)}{z^3\,(3 + 15z + 25z^2)} = -\frac{25\,(1 + \sqrt[5]{x})}{3\sqrt[5]{x^3} + 15\sqrt[5]{x^4} + 25x},$$

and the remaining solutions are, expressed by an arbitrary constant c, using that $a = 1$,

$$\begin{aligned} y_c &= \frac{u_1' + c\,u_2'}{u_1 + c\,u_2} = \frac{-\frac{25}{3}\,z^{-3}(z-1)e^{5z} - c\,\frac{25}{3}\,z^{-3}(z+1)e^{-5z}}{-\frac{1}{3}\,(3 - 15z + 25z^2)\,e^{5z} - c\,\frac{1}{3}\,(3 + 15z + 25z^2)\,e^{-5z}} \\ &= -25\,\frac{(\sqrt[5]{x} - 1)\,e^{5\sqrt[5]{x}} + c\,(\sqrt[5]{x} + 1)\,e^{-5\sqrt[5]{x}}}{\left(3\sqrt[5]{x^3} - 15\sqrt[5]{x^4} + 25x\right) e^{5\sqrt[5]{x}} + c\left(3\sqrt[5]{x^3} + 15\sqrt[5]{x^4} + 25x\right) e^{-5\sqrt[5]{x}}}, \end{aligned}$$

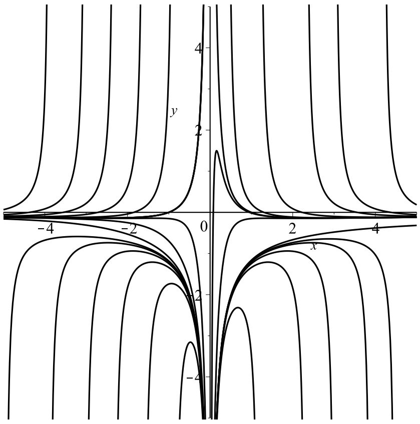


Figure 6.10: *Some solution curves for the Riccati equation* $\dfrac{dy}{dx} + y^2 = x^{-\frac{8}{5}}$.

where c is an arbitrary constant. If necessary, we can "further reduce" by using that

$$e^{5\sqrt[5]{x}} = \frac{1}{2}\cosh\left(5\sqrt[5]{x}\right) + \frac{1}{2}\sinh\left(5\sqrt[5]{x}\right),$$

and

$$e^{-5\sqrt[5]{x}} = \frac{1}{2}\cosh\left(5\sqrt[5]{x}\right) - \frac{1}{2}\sinh\left(5\sqrt[5]{x}\right),$$

but not much is gained in this case. This method is better, when c is imaginary. $\Diamond$

7 The homogeneous case

7.1 Theoretical explanations

A function $F(x,y)$ is called *homogeneous of degree* n, if it satisfies the following condition

(7.1) $\quad F(\lambda x, \lambda y) = \lambda^n F(x,y), \qquad \text{for all } \lambda \in \mathbb{R}.$

When (7.1) is differentiated with respect to λ, we get

$$x\,\frac{\partial F}{\partial x}(\lambda x, \lambda y) + y\,\frac{\partial F}{\partial y}(\lambda x, \lambda y) = n\,\lambda^{n-1} F(x,y).$$

When we choose $\lambda = 1$, if follows that we have proved

Theorem 7.1 Euler's theorem. *Assume that $F(x,y)$ is a homogeneous function of degree n. Then*

$$n\,F(x,y) = x\,\frac{\partial F}{\partial x}(x,y) + y\,\frac{\partial F}{\partial y}(x,y).$$

We mention – the proof is left to the reader – that we have the following extension.

Theorem 7.2 Euler's extended theorem. *Assume that the function $F(x,y;u,v)$ is homogeneous of degree m in the variables x and y, and homogeneous of degree n in the variables u and v. Then $F(x,y;u,v)$ satisfies the equation*

$$(n-m)F = \left(u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}\right)\left(x\,\frac{\partial F}{\partial u} + y\,\frac{\partial F}{\partial v}\right) - \left(x\,\frac{\partial}{\partial u} + y\,\frac{\partial}{\partial v}\right)\left(u\,\frac{\partial F}{\partial x} + v\,\frac{\partial F}{\partial y}\right).$$

We shall not need this result in the following.

Consider the equation

(7.2) $\quad L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0,$

where we assume that $L(x,y)$ and $M(x,y)$ are both homogeneous functions of the *same degree* n.

Concerning isoclines, such equations are particular nice, because every straight line $y = \alpha \cdot x$, $\alpha \in \mathbb{R}$ a constant, supplied with the vertical line $x = 0$ is an isocline with the fixed slope

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{L(x,y)}{M(x,y)} = -\frac{x^n\,L\left(1,\dfrac{y}{x}\right)}{x^n\,M\left(1,\dfrac{y}{x}\right)} = -\frac{L(1,\alpha)}{M(1,\alpha)} \qquad \text{along the line } y = \alpha\cdot x,$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{L(0,y)}{M(0,y)} = -\frac{y^n\,L(0,1)}{y^n\,M(0,1)} = -\frac{L(0,1)}{M(0,1)} \qquad \text{along the line } x = 0.$$

This structure gives us the hint that we must have some similarity of the solution curves with respect to the point $(0,0)$, which then ought to be a singular point.

We note that if furthermore $\frac{dy}{dx} = \alpha$ along the line $y = \alpha \cdot x$, i.e. all the tangents have the same direction as the line itself, then $y = \alpha \cdot x$ is a rectilinear solution. The condition for this is

$$\alpha = -\frac{L(1,\alpha)}{M(1,\alpha)}, \qquad \text{i.e.} \qquad L(1,\alpha) + \alpha \cdot M(1,\alpha) = 0,$$

which is then solved with respect to α.

Then we just add an inspection of the line $x = 0$ to see if it is a solution curve. This must be done separately, because it is not included in the above. The condition is of course

$$M(0,y) = 0.$$

When $(0,0)$ is the only singular point, the rectilinear solutions, if any, divide the plane into sectors limiting the other solution curves. These can only "leave" such a sector through a singular point, where the uniqueness of the solution does not hold, i.e. points (x,y) for which $L(x,y) = 0$ and $M(x,y) = 0$.

Once the possible rectilinear solutions have been found, the standard way of solving an homogeneous equation is the following. First of all we have above checked if the vertical line $x = 0$ is a solution or not. Then we substitute $y = v \cdot x$ for $x \neq 0$, where $dy = v\,dx + x\,dv$, in (7.2),

$$\begin{aligned} 0 &= L(x,y)\,dx + M(x,y)\,dy = L(x,xv)\,dx + M(x,xv)\{v\,dx + x\,dv\} \\ &= x^n\,L(1,v)\,dx + x^n\,M(1,v)\{v\,dx + x\,dv\}, \end{aligned}$$

which is reduced to

$$\{L(1,v) + v\,M(1,v)\}\,dx + x\,M(1,v)\,dv = 0.$$

1) If $L(1,v) + v\,M(1,v) = 0$, this differential form is reduced to $x\,M(1,v)\,dv = 0$, so either $x = 0$, or $M(1,v) = 0$, which implies that also $L(1,v) = 0$, and the differential form degenerates.

2) If instead $L(1,v) + v\,M(1,v) \neq 0$, the variables can be separated, so either $x = 0$, or, when $x \neq 0$,

$$\frac{dx}{x} + \frac{M(1,v)}{L(1,v) + v\,M(1,v)}\,dv = 0,$$

hence by integration,

$$\ln|x| + \int_{v=y/x} \frac{M(1,v)}{L(1,v) + v\,M(1,v)}\,dv = C, \qquad \text{for } x \neq 0.$$

A special case is the equation

$$f\left(\frac{y}{x}\right)\,dx - dy = 0, \qquad \text{or} \qquad \frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

which has homogeneous coefficients, both of degree 0. We must assume that $x \neq 0$. Again, we write $v = \frac{y}{x}$, i.e. $y = v \cdot x$, and $dy = v\,dx + x\,dv$, and we get by insertion,

$$0 = f\left(\frac{y}{x}\right)\,dx - dy = f(v)\,dx - v\,dx - x\,dv = \{f(v) - v\}\,dx - x\,dv,$$

where the variables can be separated. We first note, that if v_0 is a solution of the equation $f(v)-v=0$, then the corresponding straight line $y=v_0\,x$ is a solution of the equation for $x \neq 0$. If $f(v) \neq v$, we get by separating the variables,

$$0 = \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}v}{f(v)-v},$$

hence by integration,

$$\ln|x| - \int_{v=y/x} \frac{\mathrm{d}v}{f(v)-v} = C,$$

which is an implicit form of the general solution. Note that one should not forget the solutions $y = vx$, where v is a solution of $f(v) = v$.

Solutions of equations of this type are extremely easy to sketch by the method of *isoclines*. In fact, the alternative way of writing the equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right) = f(v),$$

shows again that all straight lines $y = vx$ are isoclines, corresponding to the slope $f(v)$.

7.2 A more general equation

In this section we show that a general equation of the structure

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right),$$

where a, b, c, α, β, γ are given real constants, can be reduced either to an homogeneous equation, or to a differential equation, which can be solved by already known methods.

There are two cases. Either

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0 \qquad \text{or} \qquad \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0.$$

1) If the determinant is not zero,

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0,$$

then the two lines

$$ax+by+c=0 \qquad \text{and} \qquad \alpha x+\beta y+\gamma = 0,$$

have one and only one intersection point (ξ, η), where by Cramèr's formula

$$\xi = -\frac{\begin{vmatrix} c & b \\ \gamma & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}}, \qquad \eta = -\frac{\begin{vmatrix} a & c \\ \alpha & \gamma \end{vmatrix}}{\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}}.$$

If we move the coordinate system to a new origo at (ξ, η), and write $x_1 = x - \xi$ and $y_1 = y - \eta$, then the equation is written

$$\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = f\left(\frac{ax_1 + by_1}{\alpha x_1 + by_1}\right),$$

where the right hand side is a homogeneous function in (x_1, y_1) or order 0.

Then we proceed as described in the previous section.

2) Assume that

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0.$$

a) If $\beta = 0$, then either $\alpha = 0$ or $b = 0$.

i) If $\alpha = 0$, then we must have $\gamma \neq 0$, and the equation is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a}{\gamma}x + \frac{b}{\gamma}y + \frac{c}{\gamma}\right),$$

i.e. of a type already discussed in Chapter 3.

ii) If $\beta = 0$ and $\alpha \neq 0$, then $b = 0$, in which case the equation is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax + c}{\alpha x + \gamma}\right),$$

the solution of which is found directly by integration and addition of an arbitrary constant, since y does not occur on the right hand side of the equation.

b) If instead $a = 0$, we just interchange x and y, and consider the function $g(w) := f(1/w)$. Apart from the change of notation the analysis is the same as above.

c) Finally, we assume that $a \neq 0$ and $\beta \neq 0$. We introduce the denominator as a new variable,

$$v = \alpha x + \beta y + \gamma.$$

It follows from the linearity and the assumption $\beta \neq 0$, that if we can find all solutions v, then also all solutions y.

By assumption,

$$0 = \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = a\beta - \alpha b, \qquad \text{i.e.} \qquad a\beta = \alpha b.$$

Therefore,

$$\begin{aligned} \frac{ax+by+c}{\alpha x+\beta y+\gamma} &= \frac{a\beta x+b\beta y+c\beta}{\beta v} = \frac{b\alpha x+b\beta y+c\beta}{\beta v} \\ &= \frac{b(\alpha x+\beta y+\gamma)+c\beta-b\gamma}{\beta v} = \frac{bv+c\beta-b\gamma}{\beta v}. \end{aligned}$$

It furthermore follows from the definition $v = \alpha x + \beta y + \gamma$, that

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \alpha + \beta\,\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha + \beta\, f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right) = \alpha + \beta\, f\left(\frac{bv+c\beta+b\gamma}{\beta v}\right),$$

where the variables can be separated, so the implicit solution is given by

$$\int \frac{\mathrm{d}v}{\alpha+\beta\, f\left(\dfrac{bv+c\beta+b\gamma}{\beta v}\right)} = x + C, \qquad C \text{ arbitrary constant},$$

where $v = \alpha x + \beta y + \gamma$.

7.3 Examples

Example 7.1 *Find the complete solution of the differential equation*

$$\left(x^2 - y^2\right)\, dx + 2xy\, dy = 0.$$

This equation is homogeneous of degree 2. Possible singular points are the solutions of the two equations

$$x^2 - y^2 = 0 \qquad \text{and} \qquad 2xy = 0.$$

Clearly, $(0,0)$ is the only singular point. Furthermore, $x = 0$ is a solution. Possible rectilinear solutions are found by inserting $y = \alpha \cdot x$, $\alpha \in \mathbb{R}$ a constant. We find

$$x^2\left(1-\alpha^2\right)\,\mathrm{d}x + 2x^2\alpha^2\,\mathrm{d}x = x^2\left\{1-\alpha^2+2\alpha^2\right\}\,\mathrm{d}x = \left\{1+\alpha^2\right\}x^2\,\mathrm{d}x = 0.$$

Since $1 + \alpha^2 > 0$, the only rectilinear solution is $x = 0$.

Then we put $y = v \cdot x$, $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$, so the differential equation becomes

$$\begin{aligned} 0 &= \left(x^2-y^2\right)\,\mathrm{d}x + 2xy\,\mathrm{d}y = \left(x^2-v^2x^2\right)\,\mathrm{d}x + 2vx^2(v\,\mathrm{d}x + x\,\mathrm{d}v) \\ &= \left(x^2-v^2x^2+2v^2x^2\right)\,\mathrm{d}x + 2vx^3\,\mathrm{d}v = x^2\left(1+v^2\right)\,\mathrm{d}x + 2vx^3\,\mathrm{d}v = \left(1+v^2\right)x^2\,\mathrm{d}x + x^3\,\mathrm{d}\!\left(v^2\right). \end{aligned}$$

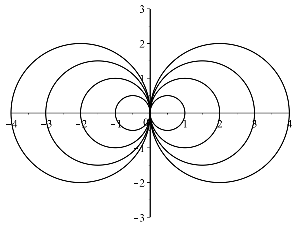


Figure 7.1: *Some solution curves of the equation* $\left(x^2-y^2\right)\,dx+2xy\,dy=0$, *describing an ideal dipole.*

When $x \neq 0$, we get by separating the variables,

$$0=\frac{\mathrm{d}x}{x}+\frac{\mathrm{d}\left(v^2\right)}{1+v^2}=\mathrm{d}\left(\ln|x|+\ln\left(1+v^2\right)\right).$$

When we integrate this equation, we get with an arbitrary constant

$$c=\ln|x|+\ln\left(1+v^2\right)=\ln|x|+\ln\left(\frac{x^2+y^2}{x^2}\right)=\ln\left(x^2+y^2\right)-\ln|x|,$$

hence, by the exponential,

$$\frac{x^2+y^2}{|x|}=e^c,$$

or, allowing the constant C also to be negative, $|2C|=e^c\neq 0$,

$$x^2+y^2=2Cx,\qquad \text{i.e.}\qquad (x-C)^2+y^2=C^2.$$

These are circles of centre $(C,0)$ and radius $|C|$, $C\neq 0$, supplied with the vertical line $x=0$.

Check. Since $2xy\,\mathrm{d}y=x\,\mathrm{d}\left(y^2\right)$ and $y^2=2Cx-x^2$, we get by insertion,

$$\left(x^2-y^2\right)\,\mathrm{d}x+2xy\,\mathrm{d}y=\left(x^2-2Cx+x^2\right)\,\mathrm{d}x+x(2C-2x)\,\mathrm{d}x=\left(2x^2-2Cx\right)\,\mathrm{d}x+\left(2Cx-2x^2\right)\,\mathrm{d}x=0,$$

so these curves (circles) are indeed solutions.

Alternatively, we put $z=y^2$. Then the equation can be written in the form

$$x\,\frac{\mathrm{d}z}{\mathrm{d}x}=z-x^2,$$

which is linear in z, so we can use the solution formula from Chapter 4. ◊

Example 7.2 *Find the complete solution of the differential equation*

$$\left(x^2-2xy-y^2\right)\,dx+\left(x^2+2xy-y^2\right)\,dy=0.$$

The equation is homogeneous of degree 2. Assume that $y=\alpha x$ is a rectilinear solution. Then the constant α must satisfy

$$0=\left(1-2\alpha-\alpha^2\right)\cdot 1+\left(1+2\alpha-\alpha^2\right)\cdot\alpha=-\alpha^3+\alpha^2-\alpha+1=-(\alpha-1)\left(\alpha^2+1\right),$$

so $\alpha=1$ is the only real solution, and $y=x$ is the only rectilinear solution.

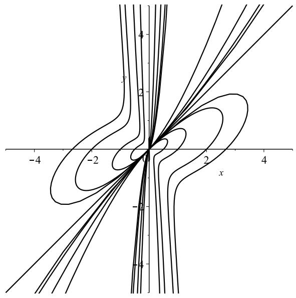


Figure 7.2: *Some solution curves of the differential equation*

$$\left(x^2-2xy-y^2\right)\,dx+\left(x^2+2xy-y^2\right)\,dy=0.$$

Then put $y = v \cdot x,\ \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$, into the equation

$$\begin{aligned} 0 &= x^2\left(1-2v-v^2\right)\mathrm{d}x + x^2 \cdot v\left(1+2v-v^2\right)\mathrm{d}x + x^3\left(1+2v-v^2\right)\mathrm{d}v \\ &= x^2\left(1-v+v^2-v^3\right)\mathrm{d}x + x^3\left(1+2v-v^2\right)\mathrm{d}v. \end{aligned}$$

We separate the variables,

$$\begin{aligned} 0 &= \frac{\mathrm{d}x}{x} + \frac{1+2v-v^2}{1-v+v^2-v^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{v^2-2v-1}{(v-1)\left(v^2+1\right)}\,\mathrm{d}v \\ &= \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}v}{v-1} + \frac{v^2-2v-1+v^2+1}{(v-1)\left(v^2+1\right)} = \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}v}{v-1} - \frac{2\,\mathrm{d}v}{v^2+1}, \end{aligned}$$

so we get by integration with an arbitrary constant c,

$$c = \ln|x| - \ln|v-1| - 2\arctan v = \ln\left|\frac{x^2}{y-x}\right| - 2\arctan\!\left(\frac{y}{x}\right). \qquad \Diamond$$

Example 7.3 *Find the complete solution of the differential equation*

$$\left(y^2 - 6xy + \frac{5}{2}x^2\right)dx + xy\,dy = 0.$$

The equation is homogeneous of degree 2. Clearly, $x = 0$ is a rectilinear solution. Other possible rectilinear solutions $y = \alpha \cdot x$ must satisfy the equation

$$0 = \alpha^2 - 6\alpha + \frac{5}{2} + \alpha^2 = 2\alpha^2 - 6\alpha + \frac{5}{2} = 2\left(\alpha^2 - 3\alpha + \frac{5}{4}\right) = 2\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{5}{2}\right).$$

We conclude that the rectilinear solutions are

$$x = 0, \qquad y = \frac{1}{2}x \quad \text{and} \quad y = \frac{5}{2}x.$$

Then put $y = v \cdot x,\ \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$, and insert

$$\begin{aligned} 0 &= \left(v^2 - 6v + \frac{5}{2}\right)x^2\,\mathrm{d}x + x^2 v \cdot (x\,\mathrm{d}v + v\,\mathrm{d}x) = x^2\left\{2v^2 - 6 + \frac{5}{2}\right\}\mathrm{d}x + v \cdot x^3\,\mathrm{d}v \\ &= x^2\left\{2\left(v-\frac{1}{2}\right)\left(v-\frac{5}{2}\right)\mathrm{d}x + v \cdot x\,\mathrm{d}v\right\}, \end{aligned}$$

where the variables can be separated,

$$0 = 2\,\frac{\mathrm{d}x}{x} + \frac{v}{\left(v-\frac{1}{2}\right)\left(v-\frac{5}{2}\right)}\,\mathrm{d}v = 2\,\frac{\mathrm{d}x}{x} - \frac{1}{4}\,\frac{\mathrm{d}v}{v-\frac{1}{2}} + \frac{5}{4}\,\frac{\mathrm{d}v}{v-\frac{5}{2}},$$

which can be rewritten as

$$0 = 8\,\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}v}{v-\frac{1}{2}} + 5\,\frac{\mathrm{d}v}{v-\frac{5}{2}}.$$

An integration gives,

$$
\begin{aligned}
k &= \ln x^8 - \ln\left|v-\frac{1}{2}\right| + 5\ln\left|v-\frac{5}{2}\right| = \ln x^8 + \ln\left|\frac{v-\frac{5}{2}}{v-\frac{1}{2}}\right| + 4\ln\left|v-\frac{5}{2}\right| \\
&= \ln x^4 + \ln\left|\frac{2y-5x}{2y-x}\right| + \ln\left(y-\frac{5}{2}\,x\right)^4 .
\end{aligned}
$$

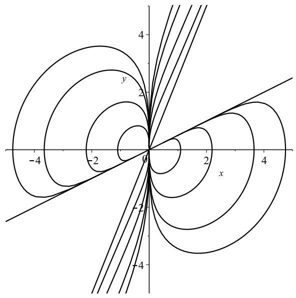


Figure 7.3: *Some solution curves of the differential equation* $\left(y^2-6xy+\frac{5}{2}\,x^2\right)\,dx+xy\,dy=0.$

Finally, we take the exponential and define a new constant $C\in\mathbb{R}$, which also includes the possible change of sign,

$$x^4\cdot\frac{(2y-5x)^5}{2y-x}=C, \qquad C\in\mathbb{R} \text{ an arbitrary constant.}$$

When we choose $C=0$, we get the two rectilinear lines $x=0$ and $2y-5x=0$, so we just have to add the third rectilinear solution $2y-x=0$. ♢

Example 7.4 *Find the complete solution of the differential equation*

$$\frac{dy}{dx}=\frac{2x^2+y^2}{-2xy+3y^2}.$$

The equation is homogeneous of degree 0. It is not defined for $-2xy+3y^2=0$, i.e. it is not defined for $y=0$ or $y=\frac{2}{3}\,x$, so we shall in the final solution strictly speaking exclude the points of these two lines.

The meaning of the equation in the book, where I found this example, is of course that we shall apply one of the standard solution methods. This is, however, not necessary in this case, because when we consider the corresponding differential form defined in the whole plane,

$$0=\left(2x^2+y^2\right)\mathrm{d}x+\left(2xy+3y^2\right)\,\mathrm{d}y,$$

then it turns up, that this is exact and can be directly integrated. In fact,

$$
\begin{aligned}
0 &= \left(2x^2+y^2\right)\mathrm{d}x+\left(2xy+3y^2\right)\,\mathrm{d}y = 2x^2\,\mathrm{d}x+\left(y^2\,\mathrm{d}x+2xy\,\mathrm{d}y\right)+3y^2\,\mathrm{d}y \\
&= \mathrm{d}\left(\frac{2x^3}{3}\right)+\mathrm{d}\left(xy^2\right)+\mathrm{d}\left(y^3\right) = \mathrm{d}\left(\frac{2}{3}\,x^3+xy^2+y^3\right),
\end{aligned}
$$

so it is indeed an exact differential form. When we multiply by 3 and integrate we get the complete solution

$$3y^3 + 3xy^2 + 2x^3 = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

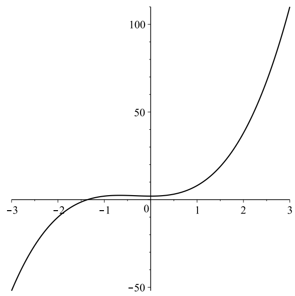


Figure 7.4: *The graph of the function* $f(\alpha) = 3\alpha^3 + 3\alpha^2 + 2$. *It follows that this function has only one root, which approximately can be found to be* $\alpha \approx -1.36029$.

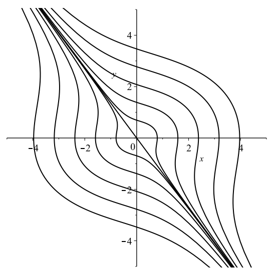


Figure 7.5: *Some solution curves of the equation* $\dfrac{dy}{dx} = \dfrac{2x^2 + y^2}{-2xy + 3y^2}$.

Homogeneous polynomial equations of even degree always have at least one rectilinear solution. We shall show this in the present case. When we check the guess $y = \alpha \cdot x$, we see that α must satisfy the equation

$$0 = 2 + \alpha^2 + 2\alpha^2 + 3\alpha^3 = 3\alpha^3 + 3\alpha^2 + 2.$$

This equation of third degree has either 1 or 3 real solutions, which all define a rectilinear solution, when they exist.

In the present case it is not hard to see by considering the graph that there is only one real root. This can approximately be found to be $\alpha \approx -1.36029$. $\Diamond$

Example 7.5 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{x}{2y}.$$

This equation is homogeneous of degree 0. We must assume that $x \neq 0$ and $y \neq 0$. Outside the two exceptional lines $x = 0$ and $y = 0$, i.e. the axes, the differential equation is equivalent to the differential form

$$(7.3) \qquad 2xy\,\mathrm{d}y = \left(y^2 + x^2\right)\mathrm{d}x,$$

which is homogeneous of degree 2, so extended to the whole plane this has at least one rectilinear solution. It is immediately seen that the y-axis, $x = 0$, is a rectilinear solution of (7.3), but this is excluded from the original domain.

There is still a possibility of two extra rectilinear solutions, so we put $y = \alpha \cdot x$ and see that the constant α then must satisfy the equation $2\alpha^2 = \alpha^2 + 1$, i.e. $\alpha^2 = 1$, and we conclude that we have the two real solutions, $\alpha = \pm 1$. Hence,

$$y = \pm x \qquad \text{for } x \neq 0,$$

are two more rectilinear solutions.

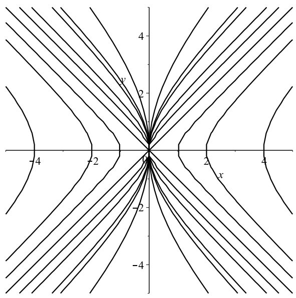


Figure 7.6: *Some solution curves of the differential equation* $\dfrac{dy}{dx} - \dfrac{y}{2x} = \dfrac{x}{2y}$.

The equation (7.3) can of course be solved by the standard procedure, but since we have assumed $x \neq 0$, it is easier to apply that x^{-2} is an integrating factor. We then get by a rearrangement,

$$0 = \frac{1}{x^2}\left\{2xy\,\mathrm{d}y - y^2\,\mathrm{d}x - x^2\,\mathrm{d}x\right\} = \left\{\frac{1}{x}\,\mathrm{d}\!\left(y^2\right) - \frac{y^2}{x^2}\,\mathrm{d}x\right\} - \mathrm{d}x = \mathrm{d}\!\left(\frac{y^2}{x^2} - x\right) = \mathrm{d}\!\left(\frac{y^2 - x^2}{x}\right), \quad x \neq 0,$$

hence, by integration,

$$\frac{y^2 - x^2}{x} = 2C, \qquad \text{i.e.} \qquad y^2 - (x+C)^2 = -C^2, \qquad \text{for } x \neq 0.$$

ALTERNATIVELY, we put $z = y^2$, from which we obtain the linear equation

$$x\,\frac{\mathrm{d}z}{\mathrm{d}x} = z + x^2,$$

which can be solved by the solution formula of Chapter 4. ◊

Example 7.6 *Find the complete solution of the differential equation*

$$y\left(3x^2 + y^2\right) = 2x^3\,\frac{dy}{dx}.$$

This equation is homogeneous of degree 3 with the singular point $(0,0)$. Its equivalent differential form is

$$y\left(3x^2 + y^2\right)\,\mathrm{d}x = 2x^3\,\mathrm{d}y,$$

which has trivially the two rectilinear solutions $x = 0$ and $y = 0$, where the former is not a solution of the original differential equation. We have the possibility of two more rectilinear solutions, so we insert $y = \alpha \cdot x$ into the equation. This gives us

$$\alpha\left(3 + \alpha^2\right) = 2\alpha, \qquad \text{i.e.} \qquad \alpha\left(\alpha^2 + 1\right) = 0.$$

Hence, $y = 0$ is the only rectilinear solution for $x \neq 0$.

When we insert $y = v \cdot x,\ \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$, into the differential form, we get

$$vx\left(3x^2 + 3v^2x^2\right)\,\mathrm{d}x = 2x^3(x\,\mathrm{d}v + v\,\mathrm{d}x) = 2x^4\,\mathrm{d}v + 2x^3 v\,\mathrm{d}x,$$

which for $x \neq 0$ is reduced to

$$2x\,\mathrm{d}v = \left(3v + 3v^3 - 2v\right)\,\mathrm{d}x = \left(3v^3 + v\right)\,\mathrm{d}x = v\left(3v^2 + 1\right)\mathrm{d}x.$$

We separate the variables,

$$\begin{aligned}\frac{\mathrm{d}x}{x} &= \frac{2\,\mathrm{d}v}{v\left(3v^2+1\right)} = \left\{\frac{2}{v} + \frac{2}{v\left(3v^2+1\right)} - \frac{2}{v}\right\}\mathrm{d}v = \left\{\frac{2}{v} + \frac{2}{v}\cdot\frac{1}{3x^2+1}\left(1 - 3v^2 - 1\right)\right\}\mathrm{d}v \\ &= \left\{\frac{2}{v} - \frac{6v}{3v^2+1}\right\}\mathrm{d}v = \mathrm{d}\ln\!\left(\frac{v^2}{3v^2+1}\right) = \mathrm{d}\ln\!\left(\frac{y^2}{3y^2+x^2}\right),\end{aligned}$$

so by an integration and a rearrangement,

$$c = \ln\left|\frac{y^2}{x\left(3y^2+x^2\right)}\right|, \qquad \text{where } c \text{ is an arbitrary constant.}$$

Finally, we take the exponential and allow the new constant to include the sign of x,

$$\frac{y^2}{x\left(3y^2+x^2\right)} = C, \qquad \text{i.e.} \qquad y^2 = Cx\left(3y^2+x^2\right).$$

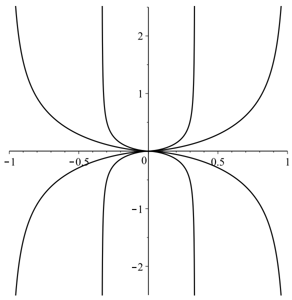


Figure 7.7: *Some solution curves of the differential equation* $y\left(3x^2+y^2\right)=2x^3\,\dfrac{dy}{dx}$.

Since $y = 0$ is a rectilinear solution, we have here assumed that $y \neq 0$. This implies that temporarily $C \neq 0$, and we must have $Cx \geq 0$. Then it follows from

$$y^2(1-3Cx)=Cx^3>0, \qquad \text{so} \qquad 0<3Cx<1,$$

and the solution is given by either $y = 0$, or

$$y=\pm\sqrt{\frac{Cx^3}{1-3Cx}} \qquad \text{for } 0<3Cx<1, \quad \text{i.e.} \quad 0<|x|<\frac{1}{3|C|}. \qquad \Diamond$$

Example 7.7 *Find the complete solution of the differential equation*

$$2x^3\,dx + \left(y^3 + 3x^2 y\right)\,dy = 0.$$

The equation is homogeneous of degree 3, and its singular point is $(0,0)$. Neither $x = 0$ nor $y = 0$ is a rectilinear solution.

Assume that $y = \alpha \cdot x$ is a rectilinear solution. Then the constant α must satisfy the equation

$$0 = 2 + \alpha\left(\alpha^2 + 3\alpha\right) = \alpha^4 + 3\alpha^2 + 2 = \left(\alpha^2+1\right)\left(\alpha^2+2\right),$$

which has no real root, so there is not rectilinear solution in this case.

Then we use the standard procedure. So let $y = vx$, $\mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$, which by insertion give

$$0 = 2x^3\,\mathrm{d}x + x^3\left(v^3+3v\right)\left(v\,\mathrm{d}x + x\,\mathrm{d}v\right) = x^3\left\{\left(v^4+3v^2+2\right)\,\mathrm{d}x + x\left(v^3+3v\right)\,\mathrm{d}v\right\}.$$

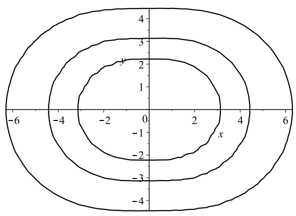


Figure 7.8: *Some solution curves of the differential equation* $2x^3\,dx + \left(y^3+3x^2y\right)\,dy = 0$. *There is no solution curve passing through the singular point in this case. It is also called a centre.*

We then separate the variables,

$$\begin{aligned} 0 &= \frac{\mathrm{d}x}{x} + \frac{v^2+3}{v^4+3v^2+2}\cdot v\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{v^2+3}{\left(v^2+1\right)\left(v^2+2\right)}\cdot\frac{1}{2}\,\mathrm{d}\!\left(v^2\right) \\ &= \frac{\mathrm{d}x}{x} + \frac{1}{2}\left\{\frac{2}{v^2+1} - \frac{1}{v^2+2}\right\}\mathrm{d}\!\left(v^2\right) = \frac{1}{2}\left\{2\frac{\mathrm{d}x}{x} + \left(\frac{2}{v^2+1} - \frac{1}{v^2+2}\right)\mathrm{d}\!\left(v^2\right)\right\}. \end{aligned}$$

Then by integration,

$$c = \ln\left(x^2\right) + \ln\left(\left(v^2+1\right)^2\right) - \ln\left(v^2+2\right) = \ln\left(\frac{x^2\left(v^2+1\right)^2}{v^2+2}\right) = \ln\left(\frac{\left(y^2+x^2\right)^2}{y^2+2x^2}\right).$$

Taking the exponential and redefining the constant we get

$$\frac{\left(y^2+x^2\right)^2}{y^2+2x^2} = C, \qquad \text{i.e} \qquad \left(y^2+x^2\right)^2 = C\left(y^2+2x^2\right).$$

In this case the arbitrary constant must be chosen positive, $C > 0$. ◊

Example 7.8 *Find the complete solution of the differential equation*

$$\left(y^3 - 3yx^2\right)\,dx + \left(x^3 - 3y^2x\right)\,dy = 0.$$

The equation is homogeneous of degree 3. Clearly, $x = 0$ and $y = 0$ are rectilinear solutions. Possible other rectilinear solutions, $y = \alpha \cdot x$, must satisfy the equation

$$0 = \alpha^3 - 3\alpha + \alpha\left(1 - 3\alpha^2\right) = -2\alpha^3 - 2\alpha = -2\alpha\left(\alpha^2 + 1\right),$$

so $\alpha = 0$ is the only possible solution, and we have already found $y = 0$ (and $x = 0$).

Then put $y = v \cdot x$, $\mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$. By insertion,

$$\begin{aligned} 0 &= \left(v^3 - 3v\right) x^3\,\mathrm{d}x + x^3\left(1 - 3v^2\right)(x\,\mathrm{d}v + v\,\mathrm{d}x) \\ &= x^3\left\{\left(-2v^3 - 2v\right)\,\mathrm{d}x + x\left(1 - 3v^2\right)\,\mathrm{d}v\right\}, \end{aligned}$$

so when we separate the variables,

$$0 = \frac{\mathrm{d}x}{x} + \frac{3v^2 - 1}{2v\left(1 + v^2\right)}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{3v^2 - 1}{v^2\left(v^2 + 1\right)} \cdot \frac{1}{4}\,\mathrm{d}\big(v^2\big) = \frac{\mathrm{d}x}{x} + \frac{1}{4}\left(-\frac{1}{v^2} + \frac{4}{v^2 + 1}\right)\,\mathrm{d}\big(v^2\big),$$

or,

$$0 = 4\,\frac{\mathrm{d}x}{x} + \left(\frac{4}{v^2 + 1} - \frac{1}{v^2}\right)\,\mathrm{d}\big(v^2\big).$$

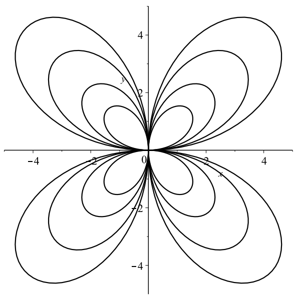


Figure 7.9: *Some solution curves of the differential equation*

$$\left(y^3 - 3yx^2\right)\,dx + \left(x^3 - 3y^2x\right)\,dy = 0.$$

We integrate to get with an arbitrary constant k,

$$k = \ln\left(x^4\right) + 4\ln\left(v^2 + 1\right) - \ln\left(v^2\right),$$

so by taking the exponential and introducing a new arbitrary constant $C > 0$,

$$C = x^4 \cdot \left(v^2 + 1\right)^4 \cdot \frac{1}{v^2} = x^4\left(\frac{y^2 + x^2}{x^2}\right)^4 \cdot \frac{x^2}{y^2} = \frac{\left(x^2 + y^2\right)^4}{x^2y^2},$$

which is written

$$\left(x^2 + y^2\right)^4 = Cx^2y^2,$$

or in polar coordinates,

$$r^2 = C \cdot \cos^2\theta \cdot \sin^2\theta = \frac{C}{4}\,\sin^2 2\theta.$$

When we introduce another arbitrary constant $\tilde{C}$, this equation can be reduced to

$$r = \tilde{C}\,\sin 2\theta, \quad (\geq 0). \qquad \Diamond$$

Example 7.9 *Find the complete solution of the differential equation*

$$2x^3\,\frac{dy}{dx} = y\left(3x^2+y^2\right).$$

The only singular point is $(0,0)$, and $y=0$ is clearly a rectilinear solution. For the corresponding differential form of the equation,

$$y\left(3x^2+y^2\right)\,\mathrm{d}x = 2x^3\,\mathrm{d}y,$$

the vertical line $x=0$ is also a rectilinear solution.

For $x \neq 0$ and $y \neq 0$ we multiply the differential form by $2xy$, so

$$0 = 2x^4\cdot 2y\,\mathrm{d}y - y^2\left(3x^2+y^2\right)\cdot 2x\,\mathrm{d}x = 2\left(x^2\right)^2\,\mathrm{d}\left(y^2\right) - y^2\left(3x^2+y^2\right)\,\mathrm{d}\left(x^2\right).$$

Then put $u = x^2$ and $v = y^2$, to get

$$0 = 2u^2\,\mathrm{d}v - v(3u+v)\,\mathrm{d}u,$$

which is homogeneous of degree 2 in (u,v). When we put $v = z\cdot u$, we get

$$0 = 2u^2(u\,\mathrm{d}z + z\,\mathrm{d}u) - zu(3u+zu)\,\mathrm{d}u = u^2\{2u\,\mathrm{d}z + 2z\,\mathrm{d}u - z(3+z)\,\mathrm{d}u\} = u^2\{2u\,\mathrm{d}z - z(z+1)\,\mathrm{d}u\}.$$

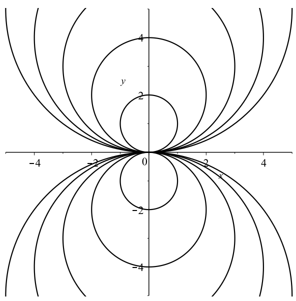


Figure 7.10: *Some solution curves of the differential equation* $2x^3\,\dfrac{dy}{dx} = y\left(3x^2+y^2\right)$.

Then we separate the variables,

$$0 = \frac{\mathrm{d}z}{z(z+1)} - \frac{\mathrm{d}u}{2u} = \left(\frac{1}{z} - \frac{1}{z+1}\right)\,\mathrm{d}z - \frac{\mathrm{d}u}{2u},$$

hence by integration, ($u > 0$ and $v > 0$, thus also $z > 0$)

$$k = \ln z - \ln(z+1) - \frac{1}{2}\ln u = \ln\left(\frac{z}{z+1}\cdot\frac{1}{\sqrt{u}}\right).$$

Taking the exponential we get with a new arbitrary constant $\tilde{C}$,

$$\tilde{C} = \frac{z}{z+1}\cdot\frac{1}{y} = \frac{v}{v+u}\cdot\frac{1}{y} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2},$$

so by a rearrangement,

$$0 = x^2 + y^2 - \frac{1}{\tilde{C}}\,y = x^2+y^2-2cy+c^2-c^2 = x^2+(y-c)^2-c^2, \qquad 2c\tilde{C} = 1,$$

and the complete solution is the system of circles of centre $(0,c)$ and radius $|c|$ for $c \neq 0$, supplied with the line $y = 0$. ♢

Example 7.10 *Find the complete solution of the differential equation*

$$2xy^3\,\frac{dy}{dx} = 4x^4 - x^2y^2 + 2y^4.$$

This equation is homogeneous of degree 4. We consider instead the (almost) equivalent differential form

$$\left(4x^4 - x^2y^2 + 2y^4\right)\,\mathrm{d}x - 2xy^3\,\mathrm{d}y = 0.$$

Its singular point is $(0,0)$. The degree 4 is even, so we have 1, 3 or 5 rectilinear solutions. The y-axis, $x = 0$, is trivially a rectilinear solution. Other possibilities are found by insertion of $y = \alpha x$, which gives the following equation in the constant α,

$$2x^4\alpha^4 = 4x^4 - x^4\alpha^2 + 2x^4\alpha^4,$$

which for $x \neq 0$ is reduced to $0 = 4 - \alpha^2$, i.e. $\alpha = \pm 2$. We conclude that $y = \pm 2x$ are two rectilinear solutions.

By the standard procedure, $y = v\cdot x$, $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$. Insertion into the differential form gives

$$0 = x^4\left(4 - v^2 + 2v^4\right)\,\mathrm{d}x - 2x^4v^3(v\,\mathrm{d}x + x\,\mathrm{d}v) = x^4\left\{\left(4-v^2\right)\,\mathrm{d}x - 2xv^3\,\mathrm{d}v\right\},$$

i.e. for $x \neq 0$,

$$\frac{\mathrm{d}x}{x} = -\frac{2v^3}{v^2-4}\,\mathrm{d}v = -\frac{v^2}{v^2-4}\,\mathrm{d}\left(v^2\right) = -\frac{v^2-4+4}{v^2-4}\,\mathrm{d}\left(v^2\right) = -\,\mathrm{d}\left(v^2\right) - \frac{4}{v^2-4}\,\mathrm{d}\left(v^2\right),$$

hence by an integration and a rearrangement,

$$-c = v^2 + 4\ln\left|v^2-4\right| + \ln|x| = \left(\frac{y}{x}\right)^2 + 4\ln\left|\left(\frac{y}{x}\right)^2 - 4\right| + \ln|x| = \left(\frac{y}{x}\right)^2 + 4\ln\left|y^2-4x^2\right| - 7\ln|x|,$$

where $-c$ is an arbitrary constant, and $x \neq 0$, and $y \neq \pm 2x$. The complete solution is either the two rectilinear functions

$$y = \pm 2x,$$

or, it is implicitly given by the equation

$$y^2 + 4x^2\ln\left|y^2-4x^2\right| - 7x^2\ln|x| + cx^2 = 0,$$

where c is an arbitrary constant. ♢

Example 7.11 *Find the complete solution of the differential equation*

$$8xy^3\,dx = \left(3x^4 + 6x^2y^2 - y^4\right)\,dy.$$

The equation is homogeneous of degree 4. It is easily seen that $(0,0)$ is the only singular point. Possible rectilinear solutions $y = \alpha \cdot x$ satisfy the equation

$$8\alpha^3 = \alpha\left(3 + 6\alpha^2 - \alpha^4\right),$$

thus either $\alpha = 0$, or by a small rearrangement,

$$0 = \alpha^4 + 2\alpha^2 - 3 = \left(\alpha^2+1\right)^2 - 4 = \left(\alpha^2+3\right)\left(\alpha^2-1\right),$$

from which we get the other possibilities, $\alpha = \pm 1$. We conclude that we have three rectilinear solutions,

$$y = 0, \qquad y = x, \quad \text{and} \quad y = -x.$$

We assume in the following that $y = v \cdot x$, where $v \neq 0,\ 1,\ -1$. Then $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$, and we get by insertion,

$$\begin{aligned} 0 &= 8xy^3\,\mathrm{d}x + \left(y^4 - 6x^2y^2 - 3x^4\right)\,\mathrm{d}y \\ &= x^4\left\{8v^3 + v\left(v^4 - 6v^2 - 3\right)\right\}\,\mathrm{d}x + x^5\left(v^4 - 6v^2 - 3\right)\,\mathrm{d}v \\ &= x^4\left\{v^5 + 2v^3 - 3v\right\}\,\mathrm{d}x + x^5\left\{v^4 - 6v^2 - 3\right\}\,\mathrm{d}v. \end{aligned}$$

When we separate the variables, we get

$$0 = \frac{\mathrm{d}x}{x} + \frac{v^4 - 6v^2 - 3}{v^5 + 2v^3 - 3v}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{v^4 - 6v^2 - 3}{v^2\left(v^2-1\right)\left(v^2+3\right)}\,\frac{1}{2}\,\mathrm{d}\left(v^2\right).$$

We multiply by 2 and decompose

$$\begin{aligned} 0 &= 2\frac{\mathrm{d}x}{x}+\left\{\frac{1}{v^2}-\frac{1}{v^2-1}+\frac{2}{v^2+3}\right\}\mathrm{d}\big(v^2\big) \\ &= \mathrm{d}\ln\big(x^2\big)+\mathrm{d}\ln!\,\big(v^2\big)-\mathrm{d}\ln\big(v^2-1\big)^2+\mathrm{d}\ln\big(v^2+3\big)^2 \\ &= \mathrm{d}\left(x^2\cdot v^2\cdot\left(\frac{v^2+3}{v^2-1}\right)^2\right)=\mathrm{d}\ln\left(y^2\left(\frac{y^2+3x^2}{y^2+x^2}\right)^2\right). \end{aligned}$$

The complete solution is then obtained by integration,

$$y^2\left(\frac{y^2+3x^2}{y^2-x^2}\right)^2=k, \qquad \text{where } k\geq 0 \text{ is an arbitrary constant,}$$

i.e. when we take the square root and allow the constant to be both positive and negative,

$$y\cdot\frac{y^2+3x^2}{y^2-x^2}=C, \qquad C \text{ arbitrary constant,}$$

supplied with the two rectilinear solutions $y=\pm x$, because $y=0$ corresponds to $y=0$.

Example 7.12 *Find the complete solution of the differential equation*

$$\frac{dy}{dx}+\frac{y}{x}-\left(\frac{y}{x}\right)^2=-3, \qquad \textit{for } x\neq 0.$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y=\alpha\cdot x$ must satisfy the equation $\alpha+\alpha-\alpha^2=-3$, hence $\alpha=-1$ or $\alpha=3$, and there are two rectilinear solutions, $y=-x$ and $y=3x$.

Put $y=v\cdot x$. Then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=x\dfrac{\mathrm{d}v}{\mathrm{d}x}+v$, and we get by insertion,

$$x\frac{\mathrm{d}v}{\mathrm{d}x}+v+v-v^2=-3, \qquad \text{i.e.} \qquad x\frac{\mathrm{d}v}{\mathrm{d}x}=v^2-2v-3=(v+1)(v-3),$$

and we get by separating the variables,

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}v}{(v+1)(v-3)}=-\frac{1}{4}\frac{\mathrm{d}v}{v+1}+\frac{1}{4}\frac{\mathrm{d}v}{v-3}, \qquad \text{or} \qquad 4\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}v}{v-3}-\frac{\mathrm{d}v}{v+1}.$$

Then we integrate,

$$4\ln|x|+k=\ln x^4+k=\ln|v-3|-\ln|v+1|=\ln\left|\frac{v-3}{v+1}\right|=\ln\left|\frac{y-3x}{y+x}\right|=k.$$

Apply the exponential and let the sign go into the constant to get the complete parametrized solution

$$\frac{y-3x}{y+x}=C\,x^4, \qquad \text{or} \qquad y=x\cdot\frac{3+Cx^4}{1-Cx^4},$$

where $C\in\mathbb{R}$ is an arbitrary constant. When $C=0$, we get the rectilinear solution $y=3x$. The other rectilinear solution $y=-x$ is not included, so it must be added.

If $C>0$, we must require that $x\neq\pm\sqrt[4]{\dfrac{1}{C}}$. If instead $C\leq 0$, the solution is defined for all $x\in\mathbb{R}$. $\Diamond$

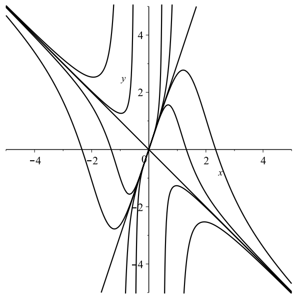


Figure 7.11: *Some solution curves of the differential equation* $\frac{dy}{dx}+\frac{y}{x}-\left(\frac{y}{x}\right)^2=-3$, *where we must assume that* $x \neq 0$.

Example 7.13 *Find the complete solution of the differential equation*

$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+\frac{y}{x}-1 \qquad \textit{for } x>0.$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y=\alpha\cdot x$ must satisfy the equation

$$\alpha=\alpha^2+\alpha-1, \qquad \text{i.e.} \qquad \alpha^2=1, \qquad \text{hence} \qquad \alpha=\pm 1.$$

We conclude that $y=x$ and $y=-x$ are two rectilinear solutions.

Then we put $y=v\cdot x$, $\frac{\mathrm{d}y}{\mathrm{d}x}=x\,\frac{\mathrm{d}v}{\mathrm{d}x}+v$, thus by insertion,

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x}+v=v^2+v-1, \qquad \text{i.e.} \qquad x\,\frac{\mathrm{d}v}{\mathrm{d}x}=v^2-1.$$

We separate the variables,

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}v}{v^2-1}=\frac{1}{2}\left(\frac{1}{v-1}-\frac{1}{v+1}\right)\mathrm{d}v, \quad \text{or more conveniently} \quad 2\,\frac{\mathrm{d}x}{x}=\left(\frac{1}{v-1}-\frac{1}{v+1}\right)\mathrm{d}v,$$

and it follows by integration, that

$$\ln x^2+k=\ln\left|\frac{v-1}{v+1}\right|=\ln\left|\frac{y-x}{y+x}\right|.$$

By the exponential and allowing the constant to be real we get the parametrized solution

$$-Cx^2=\frac{y-x}{y+x}, \qquad \text{i.e. when solving with respect to } y, \qquad y=x\,\frac{1-Cx^2}{1+Cx^2}.$$

If $C=0$ we get the rectilinear solution $y=x$. The other one, $y=-x$ must be added to this parametrized solution.

If $C<0$, we must assume that $x\neq\sqrt{\frac{1}{-C}}$. If $C\geq 0$, then $x\in\mathbb{R}_+$. $\Diamond$

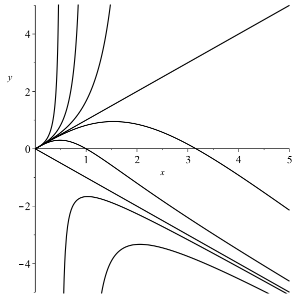


Figure 7.12: *Some solution curves of the differential equation* $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$*, where* $x > 0$.

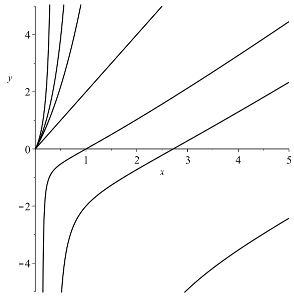


Figure 7.13: *Some solution curves of the differential equation* $\frac{dy}{dx} = 1 + \frac{1}{4}\left(\frac{y}{x}\right)^2$*, where* $x > 0$.

Example 7.14 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = 1 + \frac{1}{4}\left(\frac{y}{x}\right)^2 \qquad \textit{for } x > 0.$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y = \alpha \cdot x$ must satisfy

$$\alpha = 1 + \frac{1}{4}\,\alpha^2, \qquad \text{i.e.} \qquad 0 = \alpha^2 - 4\alpha + 4 = (\alpha - 2)^2.$$

We conclude that $y = 2x$ is the only rectangular solution.

Put $y = v \cdot x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v$. Then by insertion,

$$\frac{\mathrm{d}x}{x} - 4\,\frac{\mathrm{d}v}{(v-2)^2} = 0, \qquad \text{i.e.} \qquad C = \ln x + \frac{4}{v-2} = \ln x + \frac{4x}{y - 2x},$$

and the complete solution for $x > 0$ is either $y = 2x$, or

$$\ln x + \frac{4x}{y - 2x} = C, \qquad \text{where } C \text{ is an arbitrary constant,}$$

or

$$y = 2x + \frac{4x}{C - \ln x}, \quad \text{for } x \neq e^C. \qquad \Diamond$$

Example 7.15 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$

This equation is homogeneous of degree 0, but it is not of polynomial type, so we can expect strange phenomena. It is not defined for $x = 0$, or for $\frac{y}{x} = \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, i.e. we shall avoid all lines $y = \left(\frac{1}{2} + p\right)\pi x$, for $p \in \mathbb{Z}$, and $x = 0$.

Possible rectilinear solutions $y = \alpha \cdot x$ must satisfy the equation $\alpha = \alpha + \tan\alpha$, so $\tan\alpha = 0$, which means that $\alpha = p\pi$, $p \in \mathbb{Z}$.

Then we use the standard method, which still applies, so we put $y = v \cdot x$, $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$, from which

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\,\frac{\mathrm{d}v}{\mathrm{d}x}.$$

By insertion,

$$v + x\,\frac{\mathrm{d}v}{\mathrm{d}x} = v + \tan v, \qquad \text{i.e.} \qquad x\,\frac{\mathrm{d}v}{\mathrm{d}x} = \tan v.$$

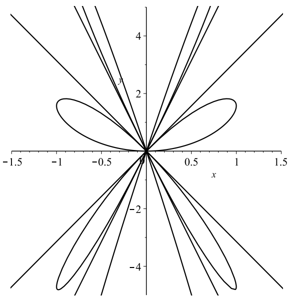


Figure 7.14: *Some solution curves of the differential equation* $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$. *They clearly behave wildly and are difficult to find, because all the rectilinear solutions are of the form* $y = p\pi x$, *where* $p \in \mathbb{Z}$, *and all other solution curves are lying in between. Here we have only sketched the rectilinear solutions* $y = -2\pi x$, $y = -\pi x$, $y = \pi x$ *and* $y = 2\pi x$, *and a few other solution curves in between.*

Then we separate the variables,

$$\frac{\mathrm{d}x}{x} = \cot v \,\mathrm{d}v = \frac{\cos v}{\sin v}\,\mathrm{d}v = \mathrm{d}\ln|\sin v|,$$

so by the usual procedure – integration followed by a rearrangement, then taking the exponential, and finally building the sign into the constant,

$$C = \frac{\sin v}{x} = \frac{1}{x}\sin\left(\frac{y}{x}\right), \qquad \text{or} \qquad \sin\left(\frac{y}{x}\right) = Cx. \qquad \Diamond$$

Example 7.16 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{y}{x}\ln\frac{y}{x}.$$

The differential equation is only defined for $\frac{y}{x} > 0$, i.e. for (x, y) in either the open first or the open third quadrant. In each of these two open domains it is homogeneous of degree 0. Possible rectilinear solutions $y = \alpha \cdot x$ must satisfy the equation $\alpha = \alpha \cdot \ln\alpha$, so either $\alpha = 0$, which is not possible, or $\alpha = e$. Thus, $y = e \cdot x$ is the only rectilinear solution.Then put $y = v \cdot x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = x \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + x$, $v > 0$ and $v \neq e$, into the equation,

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = v\ln x, \qquad \text{i.e.} \qquad x\,\mathrm{d}v = v(\ln v - 1)\,\mathrm{d}x.$$

Then we separate the variables,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}v}{v(\ln v - 1)} = \frac{\mathrm{d}\ln v}{\ln v - 1} = \mathrm{d}\ln|\ln v - 1|,$$

so by integration for some constant k,

$$k = -\ln|x| + \ln|\ln v - 1| = \ln\left|\frac{\ln v - 1}{x}\right|.$$

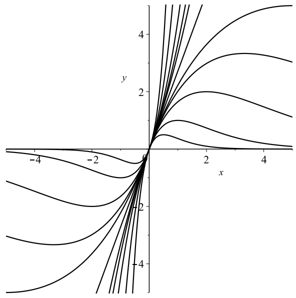


Figure 7.15: *Some solution curves of the differential equation*

$$\frac{dy}{dx} = \frac{y}{x}\,\ln\frac{y}{x}.$$

Taking the exponential and allowing the arbitrary constant to be negative we get

$$\frac{\ln\left(\dfrac{y}{x}\right) - 1}{x} = C, \qquad \text{i.e.} \qquad \ln|y| - \ln|x| - 1 = Cx,$$

so by a rearrangement and the exponential,

$$y = x \cdot \exp(1 + Cx), \qquad \text{for } C \text{ an arbitrary constant.} \qquad \Diamond$$

Example 7.17 *Find the complete solution in the angular domain given by* $0 < y < x$ *of the differential equation*

$$\frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}.$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y = \alpha \cdot x$ must satisfy

$$\alpha = \frac{\alpha}{1 - \sqrt{\alpha}},$$

the only solution of which is $\alpha = 0$. However, this is not possible, because $y = 0$ is excluded of the domain.

We put $y = v \cdot x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,\dfrac{\mathrm{d}v}{\mathrm{d}x} + v$, where $0 < v < 1$ by assumption. Then by insertion,

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{v}{1 - \sqrt{v}},$$

i.e.

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1 - \sqrt{v}} - v = \frac{v - v + v\sqrt{v}}{1 - \sqrt{v}} = \frac{v\sqrt{v}}{1 - \sqrt{v}}.$$

We separate the variables,

$$\frac{\mathrm{d}x}{x} = \frac{1-\sqrt{v}}{v\sqrt{v}}\,\mathrm{d}v = \frac{1}{v\sqrt{v}}\,\mathrm{d}v - \frac{\mathrm{d}v}{v} = -2\,\mathrm{d}\left(\frac{1}{\sqrt{v}}\right) - \frac{\mathrm{d}v}{v},$$

or by a small rearrangement,

$$0 = \frac{\mathrm{d}x}{x} + 2\,\mathrm{d}\left(\frac{1}{\sqrt{v}}\right) + \frac{\mathrm{d}v}{v}.$$

We integrate for $0 < y < x$ and $0 < v < 1$ to get with an arbitrary constant C,

$$C = \ln x + \frac{2}{\sqrt{v}} + \ln v = \ln y + 2\sqrt{\frac{x}{y}},$$

Solving with respect to x we get

$$x = y\left(\frac{C-\ln y}{2}\right)^2, \qquad \text{where } y > 0 \text{ and } |C - \ln y| > 2. \qquad \Diamond$$

Example 7.18 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = -1 + 2\sqrt{\frac{y}{x}} \qquad \textit{in the open first quadrant.}$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y = \alpha x$ must satisfy

$$\alpha = -1 + 2\sqrt{\alpha}, \qquad \text{thus} \qquad \left(\sqrt{\alpha}-1\right)^2 = 0, \qquad \text{i.e.} \qquad \alpha = 1,$$

because $\alpha \geq 0$. Hence, $y = x$ is a rectilinear solution.

We put with an unknown function $v > 0$, $y = v \cdot x$, where also $v \neq 1$. Then by insertion,

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v = -1 + 2\sqrt{v}, \qquad \text{i.e.} \qquad x\,\mathrm{d}v = -\left(\sqrt{v}-1\right)^2\,\mathrm{d}x,$$

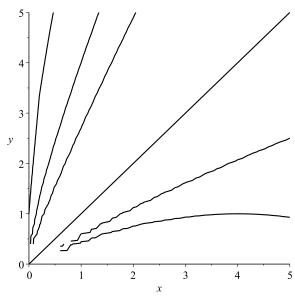


Figure 7.16: *Some solution curves of the differential equation* $\frac{dy}{dx} = -1 + 2\sqrt{\frac{y}{x}}$. *We have used the command, implicitplot, which explains why some of the curves are not quite correct.*

where we separate the variables,

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}v}{(\sqrt{v}-1)^2} = -\frac{2\sqrt{v}\,\mathrm{d}\sqrt{v}}{(\sqrt{v}-1)^2} = -\frac{2}{(\sqrt{v}-1)^2}\left(\sqrt{v}-1+1\right)\,\mathrm{d}\sqrt{v} = -\frac{2\,\mathrm{d}\sqrt{v}}{\sqrt{v}-1} - \frac{2\,\mathrm{d}\sqrt{v}}{(\sqrt{v}-1)^2},$$

so we get by integration (where $x > 0$ and $y > 0$ by assumption)

$$\ln x + k = -2\ln\left|\sqrt{v}-1\right| + \frac{2}{\sqrt{v}-1} = -2\ln\left|\sqrt{\frac{y}{x}}-1\right| + \frac{2\sqrt{x}}{\sqrt{y}-\sqrt{x}},$$

which with a new arbitrary constant c can be reduced to

$$\frac{\sqrt{x}}{\sqrt{y}-\sqrt{x}} - \ln\left|\sqrt{y}-\sqrt{x}\right| = c,$$

supplied with the rectilinear solution $y = x$. $\diamond$

Example 7.19 *Find the complete solution of the differential equation*

$$\frac{dy}{dx} = \frac{y}{x}\,\ln\frac{y}{x}, \qquad \textit{in the open first quadrant.}$$

The equation is homogeneous of degree 0. Possible rectilinear solutions $y = \alpha x$ must satisfy

$$\alpha = \alpha\cdot\ln\alpha, \qquad \text{i.e.} \qquad \alpha = e, \qquad (\alpha = 0 \text{ not possible}),$$

so $y = e\cdot x$ is a rectilinear solution.

Then put $y = v\cdot x$. We get by insertion

$$x\,\frac{\mathrm{d}v}{\mathrm{d}x} + v = v\cdot\ln v, \qquad \text{i.e.} \qquad x\,\frac{\mathrm{d}v}{\mathrm{d}x} = v\cdot\{\ln v - 1\}.$$

Hence, by separating the variables,

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}v}{v \cdot \{\ln v - 1\}} = \frac{\mathrm{d}\ln v}{\ln v - 1} = \mathrm{d}\ln|\ln v - 1|.$$

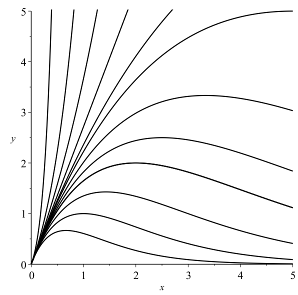


Figure 7.17: *Some solution curves of the differential equation* $\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x}$, *in the first open quadrant.*

By integration,

$$\ln|\ln v - 1| = \ln x + k,$$

hence with a new constant, which can also be negative,

$$\ln v - 1 = c, \qquad \text{i.e.} \qquad \ln v = 1 + cx.$$

so we finally get

$$v = \frac{y}{x} = e^{1+cx}, \qquad \text{i.e.} \qquad y = x \cdot e^{1+cx}.$$

For $c = 0$ we get the rectilinear solution $y = e \cdot x$. $\Diamond$

Example 7.20 *A dog and his master are standing opposite to each other on each side of a river of width* $a > 0$*, which flows with the speed* $c > 0$*. At time* $t = 0$ *the dog jumps into the water and starts swimming at a constant speed* $v > 0$*, always in the direction of his master.*

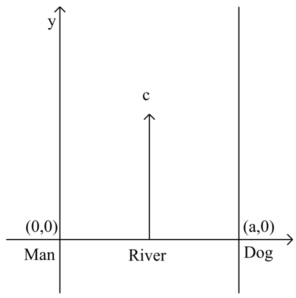
Assume at time $t = 0$ *that the coordinates of the master is* $(0, 0)$*, and of the dog* $(a, 0)$*, and that the flow of the river is parallel to the* y*-axis.*

1) Derive the differential equations, which describe the coordinates $(x(t), y(t))$ *of the dog as functions in time* t*.*

2) Use the above to derive the differential equation of $y = y(x)$ *as a function in* x*, and find the solution.*

3) Finally, find the time it takes the dog to swim across the river, whenever this is possible.

Clearly, $0 < x \leq a$, and due to the sidewards drifting, $y \geq 0$, where $y = 0$ is only possible for $x = a$ or $x = 0$.



1) Assume that the dog at time $t \geq 0$ is at point (x, y) in the river. The speed is v, and its direction is defined by $(-x, -y)$, so its velocity vector at point (x, y) is

$$\frac{v}{\sqrt{x^2+y^2}}\,(-x,-y).$$

The drifting of the dog is equal to the velocity $(0, c)$ of the river.[1] When we add these two velocities we obtain the combined velocity vector field of the dog,

$$(x'(t), y'(t)) = \left(-\frac{vx}{\sqrt{x^2+y^2}}, c - \frac{vy}{\sqrt{x^2+y^2}}\right),$$

so the differential equations in t are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{vx}{\sqrt{x^2+y^2}}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{vy - c\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}.$$

2) Since $\dfrac{\mathrm{d}x}{\mathrm{d}t} < 0$ for all $0 < x \leq a$, we can eliminate t to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{vy - c\sqrt{x^2+y^2}}{vx},$$

which is an homogeneous differential equation of order 0. Using the standard procedure we put

$$y = x \cdot u, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}u}{\mathrm{d}x} + u, \qquad u = \frac{y}{x}.$$

Then

$$x\,\frac{\mathrm{d}u}{\mathrm{d}x} + u = \frac{vu - c\sqrt{1+u^2}}{v} = u - \frac{c}{v}\,\sqrt{x^2+y^2},$$

which is reduced to

$$x\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{c}{v}\,\sqrt{1+u^2}.$$

[1] There is a small disturbing error in [9], where this example was found. The coordinates are not $(c, 0)$ as indicated, and the following equations of [9] are also wrong. The result, however, is correct.

There is no rectilinear solution, because $c/v > 0$ and $\sqrt{1+u^2} \geq 1 \neq 0$. We get by separating the variables,

$$\frac{\mathrm{d}u}{\sqrt{1+u^2}} = -\frac{c}{v}\,\frac{\mathrm{d}x}{x},$$

and by integration,

$$\operatorname{arsinh} u = -\frac{c}{v}\,\ln x + k_1 = \frac{c}{v}\,\ln\left(\frac{k}{x}\right),$$

where $k \in \mathbb{R}_+$ is an arbitrary constant. Hence,

$$u = \frac{y}{x} = \sinh\left(\frac{c}{v}\,\ln\left(\frac{k}{x}\right)\right),$$

and the complete solution is

$$y = x \cdot \sinh\left(\frac{c}{v}\,\ln\left(\frac{k}{x}\right)\right), \qquad k \text{ arbitrary constant.}$$

The initial value is given by the dog's position at $t = 0$, i.e. $(x(0), y(0)) = (a, 0)$, from which we conclude that $k = a$, and the searched solution is given by

$$y = x\,\sinh\left(\frac{c}{v}\,\ln\left(\frac{a}{x}\right)\right), \qquad 0 < x < a.$$

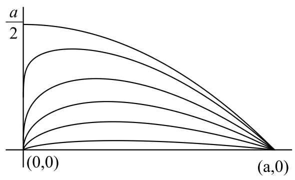


Figure 7.18: *Some solution curves for various values of $c/v \leq 1$. The upper curve corresponds to $c = v$, and the curves below corresponds to decreasing values of c/v.*

3) The solution is also written

$$\begin{aligned} y &= \frac{x}{2}\left\{\exp\left(\frac{c}{v}\,\ln\left(\frac{a}{x}\right)\right) - \exp\left(-\frac{c}{v}\,\ln\left(\frac{a}{x}\right)\right)\right\} \\ &= \frac{x}{2}\left\{\left(\frac{a}{x}\right)^{c/v} - \left(\frac{x}{a}\right)^{c/v}\right\} \approx \frac{a^{c/v}}{2}\cdot x^{1-c/v} \qquad \text{for } x > 0 \text{ small.} \end{aligned}$$

It follows that we only get $\lim_{x\to 0+} y(x) = 0$, when $c < v$. If so, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{vx}{\sqrt{x^2+y^2}} = -\frac{v}{\cosh\left(\frac{c}{v}\,\ln\left(\frac{a}{x}\right)\right)},$$

so by separating the variables,

$$\begin{aligned} dt &= -\frac{1}{v}\cosh\left(\frac{c}{v}\ln\left(\frac{a}{x}\right)\right) dx = -\frac{1}{2v}\left\{\left(\frac{a}{x}\right)^{c/v} + \left(\frac{x}{a}\right)^{c/v}\right\} dx \\ &= -\left\{\frac{1}{2v}a^{c/v}\cdot x^{-c/v} + \frac{1}{2v}a^{-c/v}\cdot x^{c/v}\right\} dx, \end{aligned}$$

and it follows by integration that

$$\begin{aligned} t + k_2 &= -\frac{1}{2v}\left\{a^{c/v}\cdot\frac{1}{1-\frac{c}{v}}x^{1-\frac{c}{v}} + a^{-c/v}\cdot\frac{1}{1+\frac{c}{v}}x^{1+\frac{c}{v}}\right\} \\ &= -\frac{1}{2}\left\{\frac{a^{c/v}}{v-c}x^{1-\frac{c}{v}} + \frac{a^{-c/v}}{v+c}x^{1+\frac{c}{v}}\right\}. \end{aligned}$$

When $t = 0$ we have $x = a$, so the constant k_2 is given by

$$k_2 = -\frac{1}{2}\left\{\frac{a^{c/v}}{v-c}\cdot a^{1-c/v} + \frac{a^{-c/v}}{v+c}\cdot a^{1+c/v}\right\} = -\frac{a}{2}\left\{\frac{1}{v-c} + \frac{1}{v+c}\right\} = -\frac{av}{v^2-c^2},$$

and we get by insertion and rearrangement,

$$t = \frac{av}{v^2-c^2} - \frac{1}{2}\left\{\frac{a^{c/v}}{v-c}x^{1-\frac{c}{v}} + \frac{a^{-c/v}}{v+c}x^{1+\frac{c}{v}}\right\}.$$

The dog wil reach the opposite bank at time

$$T = \lim_{x\to 0+} t(x) = \frac{av}{v^2-c^2}, \qquad \text{when } 0 \le c < v.$$

We note the limiting case $c = v$, where the solution is given by

$$y = x \sinh\left(\ln\left(\frac{a}{x}\right)\right) = \frac{x}{2}\left\{\frac{a}{x} - \frac{x}{a}\right\} = \frac{1}{2}\left\{a - \frac{1}{a}\,x^2\right\},$$

or, by another rearrangement,

$$2ay = a^2 - x^2,$$

which is a parabola. The dog will reach the bank at the point $\left(0, \frac{a}{2}\right)$ at time $t = +\infty$, i.e. not in finite time. ♢

Example 7.21 *Find the system of curves $y = f(x)$, which satisfies the following condition. If (x, y) is a point on one of the curves from this system, then (x, y) has the same distance from $(0, 0)$ as the intersection point of its tangent with the y-axis.*If (x_0, y_0) is a point on a solution curve, then its tangent is given by

$$y - y_0 = f'(x_0) \cdot (x - x_0)\,.$$

Its intersection point with the y-axis is given by $x = 0$, so

$$y = y_0 - f'(x_0) \cdot x_0,$$

the absolute value of which should be equal to $\sqrt{x^2 + y^2}$.

Writing (x, y) instead of (x_0, y_0) we get the condition

$$\left| y - x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \right| = \sqrt{x^2 + y^2}.$$

For $x \neq x$ we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x}\right)^2},$$

which is homogeneous of order 0, so we put

$$y = x \cdot u, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = x\,\frac{\mathrm{d}u}{\mathrm{d}x} + u, \qquad u = \frac{y}{x},$$

and get after the usual reduction,

$$x\,\frac{\mathrm{d}u}{\mathrm{d}x} = \pm\sqrt{1 + u^2}.$$

We separate the variables,

$$\frac{\mathrm{d}u}{\sqrt{1 + u^2}} = \pm\frac{\mathrm{d}x}{x}.$$

By integration,

$$\operatorname{arsinh} u = k \pm \ln |x|,$$

from which

$$u = \frac{y}{x} = \sinh(k \pm \ln|x|) = \sinh k \cdot \cosh(\ln|x|) \pm \cosh k \cdot \sinh(\ln|x|).$$

By a small tedious argument it can be shown that this is a system of parabolas,

$$y = \frac{x^2}{2C} - \frac{C}{2}, \qquad C \text{ an arbitrary constant.} \qquad \Diamond$$

8 The homogeneous case where $L(x, y)$ and $M(x, y)$ are polynomials of degree 1

8.1 Theoretical background

A special important case occurs, when the differential form is of degree 1 and has polynomial coefficients. The reason is the following. One is often interested in the behaviour of the solutions in a small neighbourhood of a singular point, where some unexpected phenomena may by expected. So let us consider the differential form

$$L(x, y)\,\mathrm{d}x + M(x, y)\,\mathrm{d}y = 0,$$

where $L(x, y)$ and $M(x, y)$ for the time being are C^∞ functions. Its singular points are here defined as the solutions of the two equations

$$L(x, y) = 0 \qquad \text{and} \qquad M(x, y) = 0.$$

There may be no finite singular points. A trivial example is

$$(1 + x + y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y = 0,$$

because the two lines $1 + x + y = 0$ and $x + y = 0$ never intersect. It is, however, easy to see that the solution is given by $2x + x^2 + 2xy + y^2 = c$, where c is an arbitrary constant.

Then let (x_0, y_0) be a singular point. Using the translation $\tilde{x} = x - x_0$ and $\tilde{y} = y - y_0$ we see that we may assume that the singular point is $(0, 0)$, and we shall hereafter again write (x, y) instead of $(\tilde{x}, \tilde{y})$.

By a Taylor expansion,

$$L(x, y) = Ax + By + \cdots, \qquad \text{and} \qquad M(x, y) = Cx + Dy + \cdots,$$

where the dots indicate higher order terms which are small in the neighbourhood of $(0, 0)$ compared with $Ax + By$ and $Cx + Dy$. Here, A, B, C and D are real constants, not all of them 0. Thus, in a neighbourhood of the singular point $(0, 0)$ it suffices to consider the approximate homogeneous system

$$(8.1) \qquad (Ax + By)\,\mathrm{d}x + (Cx + Dy)\,\mathrm{d}y = 0,$$

for x, y small. Of course one shall also consider (8.1) for large x and y, but then the approximation is bad for large x and y. However, such an extended investigation may shed some light on the structure of the solution, so we shall also do it here and only give the warning that in general it is not a good model of what is going on far away from the singular point.

We shall in practice recommend the *standard procedure*, which can always be applied in the homogenous case, because it is so easy to remember, which solution formulæ in general are not. Let us briefly sketch the method, already given in Chapter 7. We put with a new unknown variable

$$y = v \cdot x, \qquad \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x,$$

where we formally must assume that $x \neq 0$.

Using the standard procedure above on the equation (8.1) we get

$$x(A + Bv)\,\mathrm{d}x + x(C + Dv)(x\,\mathrm{d}v + v\,\mathrm{d}x) = 0,$$

which for $x \neq 0$ is reduced to

$$\left\{A + (B+C)v + Dv^2\right\} \mathrm{d}x + x(C+Dv)\,\mathrm{d}v = 0,$$

The variables can here be separated,

$$\frac{\mathrm{d}x}{x} + \frac{C+Dv}{A+(B+C)v+Dv^2}\,\mathrm{d}v = 0.$$

In order to proceed we must use the possible roots of the denominator and then decompose the result, before we integrate.

Then we turn to the theory in order to explain what we may expect. Here we do not necessarily use the standard method given above, and the reader may of course *alternatively* in practice use formulæ developed in the following.

We shall first consider the various cases, in which at least one of the constants A, B, C, D is 0 in the equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = (Ax+By)\,\mathrm{d}x + (Cx+Dy)\,\mathrm{d}y = 0.$$

1) If $A = B = C = D = 0$, the differential form is the zero form, and every C^1 function $y = y(x)$ is trivially a solution of $0\,\mathrm{d}x + 0\,\mathrm{d}y = 0$.

2) If just one of the constants is $\neq 0$, the solution is also trivial.

   a) If $A \neq 0$, then $Ax\,\mathrm{d}x = 0$ has the solution $\frac{A}{2}\,x^2 =$ constant, which can be reduced to $x =$ arbitrary constant.

   b) If $B \neq 0$, then $By\,\mathrm{d}x = 0$ has the solutions $x =$ arbitrary constant, supplied with $y = 0$,

   c) If $C \neq 0$, then $Cx\,\mathrm{d}y = 0$ has the solutions $y =$ arbitrary constant, supplied with $x = 0$.

   d) If $D \neq 0$, then $Dy\,\mathrm{d}y = 0$ has the complete solution $y =$ arbitrary constant.

3) Then assume that precisely two of the constants are 0. We have the following possibilities.

   a) If $A = B = 0$, while C, $D \neq 0$, the differential equation $(Cx+Dy)\,\mathrm{d}y = 0$ has the solutions $Cx + Dy = 0$ and $y =$ an arbitrary constant.

   b) If $A = C = 0$, while B, $D \neq 0$, the differential equation $By\,\mathrm{d}x + Dy\,\mathrm{d}y = y\{B\,\mathrm{d}x + D\,\mathrm{d}y\} = 0$ has the solutions $Bx + Dy =$ constant, supplied with $y = 0$.

   c) If $A = D = 0$, which B, $C \neq 0$, the differential equation $By\,\mathrm{d}x + Cx\,\mathrm{d}y = 0$ has the complete solution $|x|^B|y|^C =$ constant. The proof is easy. Just apply the standard procedure described previously in Chapter 7.

   d) If $B = C = 0$, while A, $D \neq 0$, the differential form is exact, so the solutions of $Ax\,\mathrm{d}x + Dy\,\mathrm{d}y = 0$ is given by (where we have integrated and then multiplied by 2),

$$Ax^2 + Dy^2 = \text{ arbitrary constant.}$$

   e) If $B = D = 0$, and A, $C \neq 0$, then the differential equation $Ax\,\mathrm{d}x + Cx\,\mathrm{d}y = x\{A\,\mathrm{d}x + C\,\mathrm{d}y\} = x\,\mathrm{d}(Ax+Cy) = 0$ has the complete solution $x = 0$ and $Ax + Cy =$ arbitrary constant.

   f) If $C = D = 0$, while A, $B \neq 0$, then the differential equation $(Ax+By)\,\mathrm{d}x = 0$ has the soluions $x =$ constant, supplied with $Ax + By = 0$.

4) Assume that just one of the constants is zero.

a) If $A = 0$, the differential equation becomes $Bx\,\mathrm{d}x + (Cx + Dy)\,\mathrm{d}y = 0$, the complete solution of which is

$$|y|^C |(B + C)x + Dy|^B = \text{ arbitrary constant.}$$

b) If $D = 0$, the differential equation becomes $(Ax + By)\,\mathrm{d}x + Cx\,\mathrm{d}y = 0$, the solution of which is

$$|x|^B |Ax + (B + C)y|^C = \text{ arbitrary constant.}$$

c) The cases, where either $B = 0$ or $C = 0$ are treated as if all four constants were $\neq 0$. See below.

5) Finally, assume that A, $D \neq 0$, and that at least one, possibly both, of the two remaining constants B, C is $\neq 0$.

a) First assume that $B = C$.h en the given differential form is exact,

$$0 = (Ax + By)\,\mathrm{d}x + (Bx + Dy)\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}\left(Ax^2 + 2Bxy + Dy^2\right), \qquad B = C,$$

so the complete solution is implicitly given by

$$Ax^2 + 2Bxy + Dy^2 = k, \qquad \text{where } k \text{ is an arbitrary constant.}$$

If furthermore $D = 0$, then the calculations become trivial (left to the reader). If instead $D \neq 0$, then we write the equation above in the form

$$\left(y - \frac{B}{D}\,x\right)^2 + \frac{AD - B^2}{D^2}\,x^2 = k.$$

Here we have three possibilities.

i) If $AD - B^2 > 0$, i.e. $B^2 < AD$, then $k = a^2$ must be *positive*, and the complete solution is a system of ellipses surrounding the singular point $(0, 0)$. No proper solution curve passes through $(0, 0)$, which is also called a *centre*.

ii) If $AD - B^2 = 0$, i.e. $B^2 = AD$, then $k = a^2$ must be *nonnegative*, i.e. $a \in \mathbb{R}$, and the complete solution is a bundle of parallel lines, $Dy - Bx = Da$, $a \in \mathbb{R}$.

iii) If $AD - B^2 < 0$, i.e. $B^2 > AD$, then we get two straight lines for $k = 0$, intersecting at the singular point $(0, 0)$,

$$Dy = \left(B \pm \sqrt{B^2 - AD}\right)x.$$

If $k \neq 0$, we get hyperbolas, lying in the interior of the angular domains, defined by these lines. The singular point $(0, 0)$ is called a *col* or a *saddle point*.

In the following we assume that $B \neq C$.

We derived previously, when we used the standard procedure,

$$(8.2) \qquad 0 = \frac{\mathrm{d}x}{x} + \frac{C + Dv}{A + (B + C)v + Dv^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{Cx + Dy}{Ax^2 + (B + C)xy + Dy^2}\,\mathrm{d}v.$$

Even though we have eliminated v, but not $\mathrm{d}v$, in this equation, it still gives us the hint that

$$\left\{Ax^2 + (B + C)xy + Dy^2\right\}^{-1}$$

ought to be an integrating factor of (8.1). So let us try this idea and then see what happens. Divide (8.1) by $Ax^2 + (B + C)xy + Dy^2$, assuming tacitly that this polynomial is $\neq 0$. Then we get

$$0 = \frac{(Ax + By)\,\mathrm{d}x + (Cx + Dy)\,\mathrm{d}y}{Ax^2 + (B + C)xy + Dy^2}. \tag{8.3}$$

The almost trivial cases, where A or D is 0, are left to the reader, so we assume in the following that $A \cdot D \neq 0$. Then the denominator can be written

$$Dy^2 + (B + C)xy + Ax^2 = D(y - \lambda_1 x)\,(y - \lambda_2 x)\,,$$

where λ_1 and λ_2 are the two roots of the corresponding quadratic equation, i.e.

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \frac{-(B + C) \pm \sqrt{(B + C)^2 - 4AD}}{2D}.$$

Again there are three possibilities.

1) If $(B + C)^2 > 4AD$, then the roots λ_1, λ_2, are real and distinct.

2) If $(B + C)^2 = 4AD$, then $\lambda_1 = \lambda_2 = \lambda$ is a real double root.

3) If $(B + C)^2 < 4AC$, then λ_1, λ_2 are two complex conjugated roots.

We shall in the following go through these three possibilities.

1) We first assume that the roots $\lambda_1 \neq \lambda_2$ are both real. Then we can find constants μ_1 and μ_2, such that (8.3) can be written in the following way,

$$\begin{aligned} 0 &= 2\,\frac{(Ax+By)\,\mathrm{d}x+(Cx+Dy)\,\mathrm{d}y}{Ax^2+(B+C)xy+Dy^2} = 2\,\frac{(Ax+By)\,\mathrm{d}x+(Cx+Dy)\,\mathrm{d}y}{D(y-\lambda_1 x)\,(y-\lambda_2 x)} \\ &= \frac{\mu_1(y-\lambda_2 x)\,(\,\mathrm{d}y-\lambda_1\,\mathrm{d}x)+\mu_2(y-\lambda_1 x)\,(\,\mathrm{d}y-\lambda_2\,\mathrm{d}x)}{(y-\lambda_1 x)\cdot(y-\lambda_2 x)} \\ &= \mu_1\,\frac{\mathrm{d}(y-\lambda_1 x)}{y-\lambda_1 x}+\mu_2\,\frac{\mathrm{d}(y-\lambda_2 x)}{y-\lambda_2 x}, \end{aligned}$$

where by some tedious computations,

$$\lambda_1 = \frac{-(B+C)+\sqrt{(B+C)^2-4AD}}{2D}, \qquad \lambda_2 = \frac{-(B+C)-\sqrt{(B+C)^2-4AD}}{2D},$$

$$\mu_1 = 1+\frac{C-B}{\sqrt{(B+C)^2-4AC}}, \qquad \mu_2 = 1-\frac{C-B}{\sqrt{(B+C)^2-4AC}}.$$

When we integrate the exact differential form above for $y \neq \lambda_1$ and $y \neq \lambda_2 x$, we get for some arbitrary constant k_1,

$$(8.4) \qquad \mu_1 \ln|y-\lambda_1 x| + \mu_2 \ln|y-\lambda_2 x| = k_1,$$

so we get by the usual way of including the possible sign into the constant,

$$(8.5) \qquad |y-\lambda_1 x|^{\mu_1}\cdot|y-\lambda_2 x|^{\mu_2} = k,$$

where we also allow $k=0$, as well as $y=\lambda_1 x$ and $y=\lambda_2 x$, if the left hand side makes sense.

Since $\mu_1+\mu_2=2$, and μ_1 and μ_2 are real, at least one of them, μ_1 say, is >0. Choosing $k=0$ we conclude that $y-\lambda_1 x=0$ is a rectilinear solution.

If also $\mu_2>0$, we conclude that $y-\lambda_2 x=0$ is also a rectilinear solution. If instead $\mu_2<0$, we just multiply (8.4) by -1, and then (8.5) is replaced by

$$|y-\lambda_1 x|^{-\mu_1}\cdot|y-\lambda_2 x|^{-\mu_2} = \tilde{k},$$

where for $\tilde{k}=0$ and $\mu_2>0$ follows as above that $y-\lambda_2 x=0$ is also a rectilinear solution.

We shall for convenience use (8.5) as the structure equation of the solution, when the roots $\lambda_1 \neq \lambda_2$ are both real. We shall later return to the case, when λ_1 and λ_2 are complex conjugated, because then a modified version of (8.5) can be applied. Note also, that μ_1 and μ_2 may be multiplied by the same arbitrary constant $k \neq 0$ to get more convenient values of the exponents.

The singular point $(0,0)$ is a *col*, also called a *saddle point*, in the case described above.

2) Then we assume that $(B+C)=4AD$, so

$$\lambda_1 = \lambda_2 = \lambda = -\frac{B+C}{2D}$$

is a real double root. In this case it is easiest to use the standard transformation $y = v \cdot x$, in which case we get (8.2),

$$\begin{aligned}
0 &= \frac{\mathrm{d}x}{x} + \frac{C+Dv}{A+(B+C)v+Dv^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{v+\dfrac{C}{D}}{\left(v+\dfrac{B+C}{D}\right)^2}\,\mathrm{d}v \\
&= \frac{\mathrm{d}x}{x} + \frac{v+\dfrac{B+C}{2D}+\dfrac{C-B}{2D}}{\left(v+\dfrac{B+C}{2D}\right)^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{\mathrm{d}v}{v+\dfrac{B+C}{2D}} + \frac{C-B}{2D}\,\frac{\mathrm{d}v}{\left(v+\dfrac{B+C}{2D}\right)^2}.
\end{aligned}$$

Integrate to get

$$\begin{aligned}
k_1 &= \ln|x| + \ln\left|v+\frac{B+C}{2D}\right| + \frac{B-C}{2D}\,\frac{1}{v+\dfrac{B+c}{2D}} \\
&= \ln|2Dy+(B+C)x| - \ln|2D| + \frac{(B-C)x}{2Dy+(B+C)x}.
\end{aligned}$$

Then, with a new arbitrary constant k_2,

$$\ln|2Dy+(B+C)x| + \frac{(B-C)x}{2Dy+(B+C)x} = k_2,$$

which is the general structure in this case.

We mention again that if $B = C$, the complete solution is a parallel bundle of straight lines, $Dy + Bx = k$. If $B \neq C$, every solution passes through the singular point $(0, 0)$, which is called a *node* in this case.

3) Finally, when $(B+C)^2 < 4AD$, the two roots λ_1, λ_2 are complex conjugated, $\lambda_2 = \overline{\lambda_1}$, and (8.4) is replaced by

$$\text{(8.6)} \qquad \mu_1 \log(y-\lambda_1 x) + \mu_2 \log(y-\lambda_2 x) = k_1,$$

where log denotes the many-valued complex logarithm. We still have with a trivial modification,

$$\mu_1 = 1 - \frac{C-B}{\sqrt{4Ac-(B+C)^2}}\,i, \qquad \mu_2 = 1 + \frac{C-B}{\sqrt{4Ac-(B+C)^2}}\,i,$$

so $\mu_2 = \overline{\mu_1}$, and the solution (8.6) becomes

$$\begin{aligned}
k_1 &= \mu_1 \log(y-\lambda_1 x) + \mu_2 \log(y-\lambda_2 x) = 2\Re\left\{\mu_1 \log(y-\lambda_1 x)\right\} \\
&= 2\Re\left\{\left(1-\frac{C-B}{\sqrt{4AD-(B+C)^2}}\right)\cdot\left(\ln|y-\lambda_1 x| + i\,\arg(y-\lambda_1 x)\right)\right\} \\
&= 2\ln|y-\lambda_1 x| + 2\,\frac{C-B}{\sqrt{4AD-(B+C)^2}},\ \arg(y-\lambda_1 x)\,.
\end{aligned}$$

The trick is to write $y - \lambda_1 x = \varrho \cdot e^{i\Theta}$, where this expression is only 0 at the singular point $(0,0)$, because λ_1 is a complex number. Then

$$\frac{k_1}{2} = \ln \varrho + \frac{C-B}{\sqrt{4AD-(B+C)^2}}\,\Theta,$$

so

$$\varrho = |y - \lambda_1 x| = k_3 \cdot \exp\left(-\frac{C-BV}{\sqrt{4AD-(B+C)^2}}\,\Theta\right), \qquad \Theta \in \mathbb{R},$$

which for $B \neq C$ defines a system of spirals. In this case the singular point $(0,0)$ is called a *focus.*

ALTERNATIVELY, we may apply the standard method, described in the beginning, i.e. we put $y = v \cdot x$. Then, by (8.2),

$$\begin{aligned}
0 &= \frac{\mathrm{d}x}{x} + \frac{C+Dv}{A+(B+C)v+Dv^2}\,\mathrm{d}v \\
&= \frac{\mathrm{d}v}{v} + \frac{1}{2}\,\frac{2Dv+(B+C)}{Dv^2+(B+C)v+A}\,\mathrm{d}v + \frac{1}{2}\,\frac{B-C}{Dv^2+(B+C)v+A}\,\mathrm{d}v \\
&= \frac{1}{2}\,\mathrm{d}\big(\ln\big(Dy^2+(B+C)xy+Ax^2\big)\big) + \frac{1}{2}\,\frac{B-C}{D}\,\frac{\mathrm{d}v}{\left(v+\frac{B+C}{2D}\right)^2 + \frac{4AD-(B+C)^2}{4D^2}} \\
&= \frac{1}{2}\,\mathrm{d}\ln\big(Dy^2+(B+C)xy+Ax^2\big) + \frac{B-C}{\sqrt{4AD-(B+C)^2}}\,\mathrm{d}\arctan\left(\frac{2Dv+B+C}{\sqrt{4AD-(B+C)^2}}\right),
\end{aligned}$$

where we assume that $x \neq 0$. When we integrate we get implicitly for $x > 0$ say, the complete solution

$$\ln\big(Dy^2+(B+C)xy+Ax^2\big) + \frac{B-C}{\sqrt{4AD-(B+C)^2}}\arctan\left(\frac{2Dv+B+C}{\sqrt{4AD-(B+C)^2}}\right) = k.$$

Similarly, when $x < 0$.

The differential form, however, is not exact in $\mathbb{R} \setminus \{(0,0)\}$, so the two solutions in the half-planes $x > 0$ and $x < 0$ can only be joined continuously (for appropriate choices of the constants) for either $y > 0$ or for $y < 0$, but not for both open half-axes at the same time.

8.2 Examples

Example 8.1 *Find the complete solution of the differential equation*

$$y\,dx - x\,dy = 0.$$

The equation is homogeneous of degree 1. Clearly, $(0,0)$ is a singular point, and the only one. By the uniqueness theorem there is just one solution curve through every other point in the plane. Furthermore, $A = 0$, $B = 1$, $C = -1$ and $D = 0$, so $(B+C)^2 = 0 = 4AC$, and there exists at least one rectilinear solution.

We then find the possible rectilinear solutions. Clearly, the axes $x = 0$ and $y = 0$ are both solutions. Assume that $y = \alpha \cdot x$, where $\alpha \in \mathbb{R}$ is a constant. By insertion,

$$y \,\mathrm{d}x - x \,\mathrm{d}y = \alpha x \,\mathrm{d}x - x \cdot \alpha \,\mathrm{d}x = x(\alpha - \alpha)\,\mathrm{d}x = 0,$$

so every straight line through $(0,0)$ is a solution, and these form the complete solution.

This is of course a very extreme example.$\diamondsuit$

Example 8.2 *Find the complete solution of the differential equation*

$$(x + y)\, dx + (x + 2y)\, dy = 0.$$

The differential form is homogeneous of degree 1, and $A = 1$, $B = C = 1$ and $D = 2$, so $(B + C)^2 < 4AD$, and there is no rectilinear solutions. It is no need to use the theory of homogeneous differential forms further, because it is already exact, which follows from the calculations

$$\begin{aligned} 0 &= (x + y)\,\mathrm{d}x + (x + 2y)\,\mathrm{d}y = x\,\mathrm{d}x + \{y\,\mathrm{d}x + x\,\mathrm{d}y\} + 2y\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}\left(x^2\right) + d(xy) + d\left(y^2\right) \\ &= \frac{1}{2}\,\mathrm{d}\left(x^2 + 2xy + 2y^2\right). \end{aligned}$$

Then by integration,

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 = a^2, \qquad a > 0,$$

which describes a system of ellipses. $\diamondsuit$

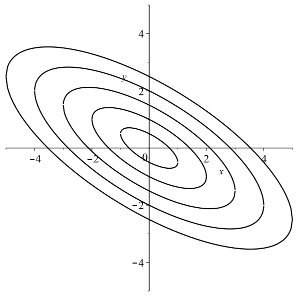


Figure 8.1: *Some solution curves of the equation* $(x+y)\,dx+(x+2y)\,dy=0$*. No solution curve passes through the singular point* $(0,0)$*, which is a centre. Centres are unstable in the sense that if* $B \neq C$*, no matter how small the difference is, then the centre is turned into a focus, and the solution curves spiral away or towards* $(0,0)$*.*

Example 8.3 *Find the complete solution of the differential equation*

$$(x-2y)\,dx+2x\,dy=0.$$

The equation is homogeneous of degree 1, and $A=1$, $B=-2$, $C=2$, and $D=0$, so $(B+C)^2=0=4AD$, and we have one rectilinear solution. This is easily seen by inspection to be $x=0$.

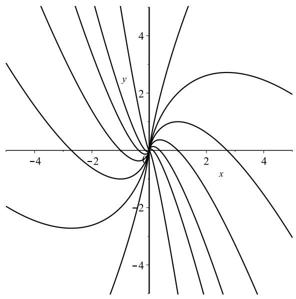


Figure 8.2: *Some solution curves of the equation* $(x-2y)\,dx+2x\,dy=0$*. We see how the solution curves with the rectilinear solution* $x=0$ *as a tangent in the limit, tend towards the singular point* $(0,0)$*, which is a node.*

Instead of using the standard procedure (left to the reader as an exercise) we divide the equation by $2x^2$ for $x \neq 0$. Then we get a closed differential form,

$$0=\frac{1}{2x}\,\mathrm{d}x-y\cdot\frac{1}{x^2}\,\mathrm{d}x+\frac{1}{x}\,\mathrm{d}y=\mathrm{d}\ln\left(\sqrt{|x|}\right)+\mathrm{d}\left(\frac{y}{x}\right),$$

so when $x \neq 0$ we get by integration

$$\frac{y}{x}+\ln\sqrt{|x|}=c, \qquad \text{which is written } y=c\cdot x-x\ln\sqrt{|x|}, \qquad c \text{ arbitrary constant,}$$

supplied with the vertical line $x=0$. $\Diamond$

Example 8.4 *Find the complete solution of the differential equation*

$$(x+y)\,dx + (-x+y)\,dy = 0.$$

The differential form is of degree 1, where $A = 1$, $B = 1$, $C = -1$ and $D = 1$, so $(B+C)^2 = 0 < 4AC = 4$, and there is no rectilinear solution. As $B \neq C$, the singular point $(0,0)$ is a focus, and the solutions spiral either away or towards it.

We divide for $(x,y) \neq (0,0)$ the equation by x^2+y^2. Then we get

$$0 = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2} + \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2+y^2} = \frac{1}{2}\,\frac{\mathrm{d}\left(x^2+y^2\right)}{x^2+y^2} + \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2+y^2}.$$

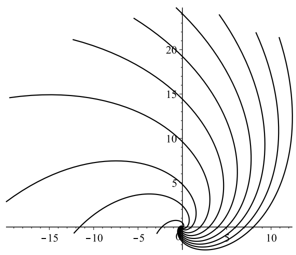


Figure 8.3: *Some solution curves of the equation* $(x+y)\,dx + (-x+y)\,dy = 0$*. All solution curves start from the singular point* $(0,0)$*, which is a focus. Foci are so to speak "disturbed centres".*

This structure clearly invites us to apply polar coordinates instead. So if $x = r\cos\vartheta$ and $y = r\sin\vartheta$, then the equation is written

$$0 = \mathrm{d}r + \frac{r^2\sin\vartheta\,\mathrm{d}\cos\vartheta - \cos\vartheta\,\mathrm{d}\sin\vartheta}{r^2} = \mathrm{d}r - \left(\sin^2\vartheta + \cos^2\vartheta\right)\,\mathrm{d}\vartheta = \mathrm{d}r - \,\mathrm{d}\vartheta,$$

hence by integration, $k = \ln r - \vartheta$, from which we get the spirals, described in polar coordinates,

$$r = C \cdot e^{\vartheta}, \qquad C > 0 \text{ an arbitrary constant.} \qquad \Diamond$$

Example 8.5 *Find the complete solution of the differential equation*

$$(x+6y)\,dx + (6y-x)\,dy = 0.$$

The equation is homogeneous of degree 1, and $(0,0)$ is the only singular point. By identification, $A = 1$, $B = 6$, $C = -1$ and $D = 6$, so

$$(B+C)^2 = 25 \qquad \text{and} \qquad 4AC = 24 < (B+C)^2.$$

We conclude that there must be two rectilinear solutions, $y = \alpha \cdot x$, where the slope α satisfies the equation

$$0 = 1 + 6\alpha + \alpha(6\alpha - 1) = 6\alpha^2 + 5\alpha + 1 = 6\left(\alpha + \frac{1}{2}\right)\left(\alpha + \frac{1}{3}\right).$$

We conclude that $y = -\dfrac{1}{2}\,x$ and $y = -\dfrac{1}{3}\,x$ are the two rectilinear solutions.

Then we put $y = v \cdot x$, $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$, where $v \neq -\dfrac{1}{2}, -\dfrac{1}{3}$. We get by insertion,

$$\begin{aligned} 0 &= x(6v+1)\,\mathrm{d}x + x(6v-1)(v\,\mathrm{d}x + x\,\mathrm{d}v) = x\left\{\left(6v+1+6v^2-v\right)\,\mathrm{d}x + x(6v-1)\,\mathrm{d}v\right\} \\ &= x\left\{\left(6v^2+5v+1\right)\,\mathrm{d}x + x(6v-1)\,\mathrm{d}v\right\}. \end{aligned}$$

Separating the variables,

$$\begin{aligned} 0 &= \frac{\mathrm{d}x}{x} + \frac{6v-1}{6v^2+5v+1}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \frac{6v-1}{6\left(v+\frac{1}{3}\right)\left(v+\frac{1}{2}\right)}\,\mathrm{d}v \\ &= \frac{\mathrm{d}x}{x} + \left\{\frac{1}{6}\,\frac{-2-1}{-\frac{1}{3}+\frac{1}{2}}\,\frac{1}{v+\frac{1}{3}} + \frac{1}{6}\,\frac{-3-1}{-\frac{1}{2}+\frac{1}{3}}\,\frac{1}{v+\frac{1}{2}}\right\}\mathrm{d}v \\ &= \frac{\mathrm{d}x}{x} + \left\{\frac{-3}{v+\frac{1}{3}} + \frac{4}{v+\frac{1}{2}}\right\}\mathrm{d}v = \mathrm{d}\ln\left|x\left(v+\frac{1}{3}\right)^{-3}\left(v+\frac{1}{2}\right)^4\right|. \end{aligned}$$

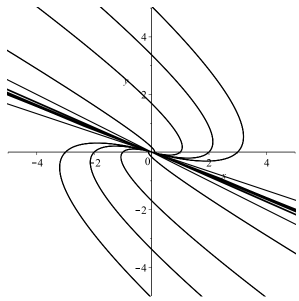


Figure 8.4: *Some solution curves of the equation* $(x+6y)\;dx + (6y-x)\;dy = 0$.

By integration with an arbitrary constant k,

$$k = x\left(v+\frac{1}{3}\right)^{-3}\left(v+\frac{1}{2}\right)^4 = x\cdot 3^3(3v+1)^{-3}\cdot\frac{1}{2^4}(2v+1)^4 = \frac{3^3}{2^4}\,\frac{(2y+x)^4}{(3y+x)^3},$$

so with another constant,

$$\frac{(2y+x)^4}{(3y+x)^3} = C, \qquad \text{where } C \in \mathbb{R} \text{ is an arbitrary constant,}$$

supplied with the rectilinear solution $3y + x = 0$.

In polar coordinates the above solution is written

$$r = C\cdot\frac{(3\sin\theta+\cos\theta)^3}{(2\sin\theta+\cos\theta)^4}, \qquad \text{where } C \text{ is an arbitrary constant.} \qquad \diamond$$

Example 8.6 *Find the complete solution of the differential equation*

$$(4x - 3y)\,dx + (y - 2x)\,dy = 0.$$

The equation is homogeneous of degree 1, and $(0, 0)$ is the singular point. The coefficients are $A = 4$, $B = -3$, $C = -2$ and $D = 1$, so $(B + C)^2 = 25 > 16 = 4AD$, and we conclude that we have two rectilinear solutions. Neither $x = 0$ nor $y = 0$ is one of these, so if $y = \alpha\, x$ is a rectilinear solution, then α must satisfy the equation

$$0 = D\,\alpha^2 + (B + C)\alpha + A = \alpha^2 - 5\alpha + 4 = (\alpha - 1)(\alpha - 4).$$

The roots are $\alpha_1 = 1$ and $\alpha_2 = 4$, so $y = x$ and $y = 4x$ are the two rectilinear solutions.

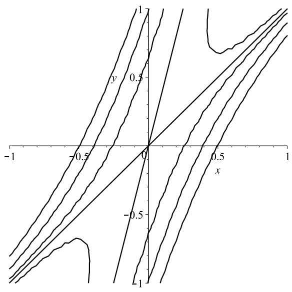


Figure 8.5: *Some solution curves of the equation* $(4x - 3y)\,dx + (y - 2x)\,dy = 0$.

Putting $y = v \cdot x,\ \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$ we get

$$0 = x(4-3v)\,\mathrm{d}x + x(v-2)(x\,\mathrm{d}v + v\,\mathrm{d}x) = x\left\{\left(v^2-5v+4\right)\,\mathrm{d}x + x(v-2)\,\mathrm{d}v\right\},$$

so when we separate the variables,

$$0 = \frac{\mathrm{d}x}{x} + \frac{v-2}{(v-1)(v-4)}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} + \left\{\frac{1}{3}\,\frac{1}{v-1} + \frac{1}{2}\,\frac{1}{v-4}\right\}\mathrm{d}v,$$

and after some reductions,

$$0 = 3\,\frac{\mathrm{d}x}{x} + \mathrm{d}\ln|v-1| + 2\,\mathrm{d}\ln|v-4| = \mathrm{d}\left(\ln\left|x^3(v-1)(v-4)^2\right|\right) = \mathrm{d}\ln\left(\left|(y-x)(y-4x)^2\right|\right).$$

Then by integration, follows by the exponential and building the sign into the constant,

$$(y-x)(y-4x)^2 = C, \qquad \text{where } C \text{ is an arbitrary constant,}$$

and where the two rectilinear solutions, $y = x$ and $y = 4x$, are obtained for $C = 0$. ♢

Example 8.7 *Find the complete solution of the differential equation*

$$(8y+10x)\,dx + (5y+7x)\,dy = 0.$$

The equation is homogeneous of degree 1. We have two rectilinear solutions. Their slopes α satisfy the equation

$$0 = D\,\alpha^2 + (B+C)\alpha + A = 5\alpha^2 + 15\alpha + 10 = 5\left(\alpha^2+3\alpha+2\right) = 5(\alpha+1)(\alpha+2),$$

so the roots are $\alpha = -1$ and $\alpha = -2$, and the rectilinear solutions are

$$y = -x \qquad \text{and} \qquad y = -2x.$$

Then put $y = v \cdot x,\ \mathrm{d}y = x\,\mathrm{d}v + v\,\mathrm{d}x$. By insertion,

$$\begin{aligned} 0 &= (8vx+10x)\,\mathrm{d}x + (5vx+7x)\{x\,\mathrm{d}v + v\,\mathrm{d}x\} \\ &= x^2(5v+7)\,\mathrm{d}v + x\left(8v+10+5v^2+7v\right)\,\mathrm{d}x \\ &= x\left\{x(5v+7)\,\mathrm{d}v + 5\left(v^2+3v+2\right)\,\mathrm{d}x\right\}. \end{aligned}$$

Then we separate the variables,

$$0 = \frac{5}{x}\,\mathrm{d}x + \frac{5v+7}{v^2+3v+2}\,\mathrm{d}v = \frac{5}{x}\,\mathrm{d}x + \frac{5v+7}{(v+1)(v+2)}\,\mathrm{d}v = \frac{5}{x}\,\mathrm{d}x + \left(\frac{2}{v+1} + \frac{3}{v+2}\right)\mathrm{d}v,$$

thus by integration,

$$c = 5\ln|x| + \ln(v+1)^2 + \ln|v+2|^3 = \ln\left|x^2(v+1)^2 \cdot x^3(v+2)^3\right| = \ln\left|(x+y)^2 \cdot (y+2x)^3\right|.$$

Then apply the exponential and build the sign into the constant to get the complete solution

$$(y+x)^2(y+2x)^3 = C, \qquad C \text{ an arbitrary constant,}$$

where $C = 0$ corresponds to the two rectilinear solutions. ♢

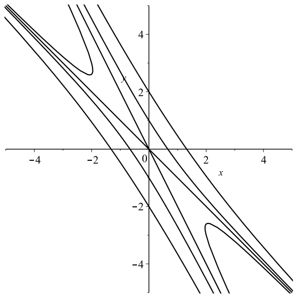


Figure 8.6: *Some solution curves of the equation* $(8y + 10x)\, dx + (5y + 7x)\, dy = 0$.

Example 8.8 *Find the complete solution of the differential equation*

$$(7x - 3y - 7)\, dx + (3x - 7y - 3)\, dy = 0.$$

When we translate the equation, $t := x - 1$, then the equation becomes homogeneous of degree 1 in (t, y),

$$(7t - 3y)\, \mathrm{d}t + (3t - 7y)\, \mathrm{d}y = 0,$$

so the singular point is $(t, y) = (0, 0)$, i.e. $(x, y) = (1, 0)$, and $A = 7$, $B = -3$, $C = 3$ and $D = -7$, so $(B + C)^2 = 0$ and $4AD = -196 < (B + C)^2$, so we conclude that we must have two rectilinear solutions $y = \alpha_1 t$ and $y = \alpha_2 t$, because $x = 0$ is not a solution. The equation

$$0 = D\alpha^2 + (B + C)\alpha + A = -7\alpha^2 + 7 = -7(\alpha - 1)(\alpha + 1)$$

has the two roots $\alpha = \pm 1$, so the two rectilinear solutions are $y = \pm t = \pm(x - 1)$.

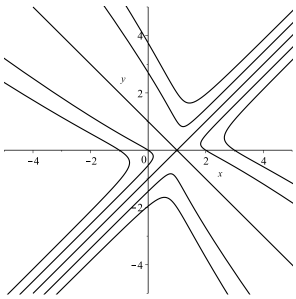


Figure 8.7: *Some solution curves of the equation* $(7x - 3y - 7)\, dx + (3x - 7y - 3)\, dy = 0$.

Then we write $y = v \cdot t$, $\mathrm{d}v = t\, \mathrm{d}v + v\, \mathrm{d}t$ and get by insertion,

$$0 = (7t - 3vt)\, \mathrm{d}t + (3t - 7vt)(v\, \mathrm{d}t + t\, \mathrm{d}v) = t\left\{7\left(1 - v^2\right)\, \mathrm{d}t + t(3 - 7v)\, \mathrm{d}v\right\}.$$

We separate the variables,

$$\begin{aligned} 0 &= 7\,\frac{\mathrm{d}t}{t} + \frac{3 - 7v}{1 - v^2}\, \mathrm{d}v = 7\, \mathrm{d}\ln|t| + \frac{7v - 3}{(v - 1)(v + 1)}\, \mathrm{d}v = \mathrm{d}\ln|t^{\,7} + \left\{\frac{2}{v - 1} + \frac{5}{v + 1}\right\} \mathrm{d}v \\ &= \mathrm{d}\ln|t|^7 + \mathrm{d}\ln(v - 1)^2 + \mathrm{d}\ln\left|(v + 1)^5\right| = \mathrm{d}\ln\left|(y - t)^2 \cdot (y + t)^5\right|. \end{aligned}$$

Then integrate, apply the exponential and build the sign of the expression into the constant to get the complete solution

$$(y-t)^2 \cdot (y+t)^5 = C, \qquad \text{i.e.} \qquad (y-x+1)^2 \cdot (y+x-1)^5 = C, \qquad C \text{ an arbitrary constant,}$$

where the rectilinear solutions are obtained for $C = 0$. $\diamondsuit$

Example 8.9 *Find the complete solution of the differential equation*

$$(-2x+3y-7)\,dx + (4x-5y+13)\,dy = 0.$$

The equations of the singular point are

$$-2x+3y = 7, \qquad \text{and} \qquad 4x-5y = -13,$$

from which we conclude that $(x_0, y_o) = (-2, 1)$ is the only singular point. Then change variables to

$$x_1 = x+2 \qquad \text{and} \qquad y_1 = y-1.$$

Then the equation becomes homogeneous of degree 1 in the new variables (x_1, y_1),

$$(-2x_1+3y_1)\,\mathrm{d}x_1 + (4x_1-5y_1)\,\mathrm{d}y_1 = 0.$$

Here, $A = -2$, $B = 3$, $C = 4$ and $D = -5$, so $(B+C)^2 = 49$, and $4AD = 40 < 49 = (B+C)^2$, so there are two rectilinear solutions $y_1 = \alpha \cdot x_1$, where the slope α satisfies the equation

$$0 = -2+3\alpha+(4-5\alpha)\alpha = -5\alpha^2+7\alpha-2 = -5(\alpha-1)\left(\alpha-\frac{2}{5}\right),$$

so $\alpha = 1$ or $\alpha = \dfrac{2}{5}$. This gives us the two rectilinear solutions

$$y_1 = x_1 \qquad \text{and} \qquad y_1 = \frac{2}{5}\,x_1,$$

which in the original coordinates are written

$$y-x-3 = 0 \qquad \text{and} \qquad 5y-2x-9 = 0.$$

Put $y_1 = v \cdot x_1$, $\mathrm{d}y_1 = x_1\,\mathrm{d}v + v\,\mathrm{d}x_1$. Then by insertion

$$0 = x_1(-2+3v)\,\mathrm{d}x_1 + x_1 v(4-5)\,\mathrm{d}x_1 + x_1^2(4-5v)\,\mathrm{d}v = x_1\left\{\left(-5v^2+7v-2\right)\,\mathrm{d}x_1 + x_1(4-5v)\,\mathrm{d}v\right\},$$

and separation of the variables,

$$0 = \frac{\mathrm{d}x_1}{x_1} - \frac{1}{5}\,\frac{4-5v}{(v-1)\left(v-\dfrac{2}{5}\right)}\,\mathrm{d}v = \frac{\mathrm{d}x_1}{x_1} + \frac{1}{3}\,\frac{\mathrm{d}v}{v-1} + \frac{2}{3}\,\frac{\mathrm{d}v}{v} - \frac{2}{5}.$$

We integrate to get with an arbitrary constant k,

$$\begin{aligned} k &= 3\ln|x_1| + \ln\, v-1| + \ln\left(v-\frac{2}{5}\right)^2 = \ln|y_1-x_1| + \ln\left(y-\frac{2}{5}\right)^2 \\ &= \ln|y_1-x_1| + \ln(5y-2x)^2 - \ln 25. \end{aligned}$$

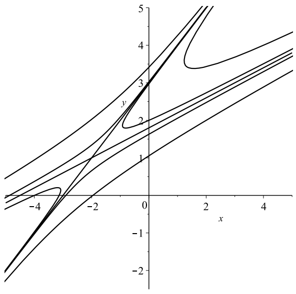


Figure 8.8: *Some solution curves of the equation* $(-2x+3y-7)\,dx+(4x-5y+13)\,dy=0$.

Then apply the exponential and redefine the constant to become real, no matter the sign,

$$(y_1-x_1)\cdot(5y_1-2x_1)^2=C.$$

For $C=0$ we get the two rectilinear solutions.

Finally, we translate beck to the original coordinates, $x_1=x+2$ and $y_1=y-1$, to get the complete solution

$$(y-x-3)\cdot(5y-2x-p)^2=C, \qquad \text{where } C\in\mathbb{R} \text{ is an arbitrary constant.} \qquad \Diamond$$

Example 8.10 *Find the complete solution of the differential equation*

$$(9x-2y-7)\,dx+(-2x+6y-4)\,dy=0.$$

We shall first find the singular point, i.e. solve the equations

$$9x-2y=7 \qquad \text{and} \qquad -2x+6y=4.$$

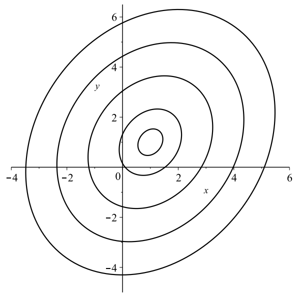


Figure 8.9: *Some solution curves of the equation* $(9x-2y-7)\,dx+(-2x+6y-4)\,dy=0$.

It follows immediately by inspection that $(x_0,y_0)=(1,1)$ is the singular point. Using the transformation $x_1=x-x_0=x-1$ and $y_1=y-y_0=y-1$ we get

$$(9x_1-2y_1)\,\mathrm{d}x_1+(-2x_1+6y_1)\,\mathrm{d}y_1=0,$$

where $A=9$, $B=-2$, $C=-2$ and $D=6$. We clearly have $(B+C)^2<4AD$, so there is no rectilinear solution. Since $B=C=-2$, we can already conclude that the solution curves must be a system of ellipses around the center (singular point in (x_1,y_1)), and the differential form must be exact, and lo and behold,

$$\begin{aligned}0 &= (9x_1-2y_1)\,\mathrm{d}x_1+(-2x_1+6y_1)\,\mathrm{d}y_1=9x_1\,\mathrm{d}x_1-2\,(y_1\,\mathrm{d}x_1+x_1\,\mathrm{d}y_1)+6y_1\,\mathrm{d}y_1\\ &= \mathrm{d}\left(\frac{9}{2}\,x_1^2-2x_1y_1+3y_1^2\right),\end{aligned}$$

and we get by integration

$$C=9x_1^2-4x_1y_1+6y_1^2, \qquad \text{for } C>0 \text{ an arbitrary constant,}$$

or, in the original coordinates

$$C=9(x-1)^2-4(x-1)(y-1)+6(y-1)^2, \qquad \text{for } C>0 \text{ an arbitrary constant,}$$

which of course may be calculated, but this form is the most convenient one. $\Diamond$

Example 8.11 *Let $k \in \mathbb{R}$ be a constant, and consider the differential equation*

$$(kx + ky + y)\, dx - (x + y)\, dy = 0.$$

Specify the values of k, for which the differential equation has none, one or two rectilinear solutions.

Find the complete solution for $k = 0$, and for $k = -2$.

We first note that no matter the choice of $k \in \mathbb{R}$ the y-axis, $x = 0$ is never a rectilinear solution, so the possible rectilinear solutions must have the structure $y = \alpha \cdot x$ for some constant $\alpha \in \mathbb{R}$, where we by insertion see that α must satisfy the equation $0 = x(k + k \cdot \alpha + \alpha)\, \mathrm{d}x - \alpha(1 + \alpha)\, \mathrm{d}x$, which is reduced to

$$k + (k+1)\alpha - \alpha^2 - \alpha = k - \alpha^2 = 0, \qquad \text{or} \qquad \alpha^2 = k.$$

1) If $k < 0$, there is no rectilinear solution.

2) If $k = 0$, there is one rectilinear solution, namely $y = 0$.

3) If $k > 0$, there are two rectilinear solutions, $y = \pm\sqrt{k} \cdot x$.

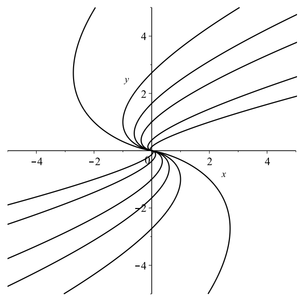


Figure 8.10: *Some solution curves of the equation $y\, dx - (x + y)\, dy = 0$.*

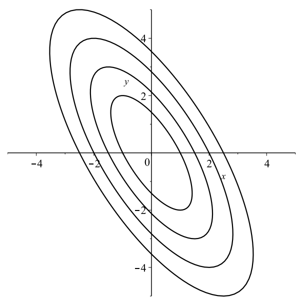


Figure 8.11: *Some solution curves of the equation $(-2x - y)\, dx - (x + y)\, dy = 0$.*

If $k = 0$, we get the equation

$$0 = y\, \mathrm{d}x - (x + y)\, \mathrm{d}y = y\, \mathrm{d}x - x\, \mathrm{d}y - y\, \mathrm{d}y.$$

As mentioned above, $y = 0$ is the only rectilinear solution. When $y \neq 0$, we divide by y^2 and get an exact differential form,

$$0 = \frac{1}{y}\,\mathrm{d}x - x \cdot \frac{1}{y^2}\,\mathrm{d}y = \mathrm{d}\left(\frac{x}{y}\right) - \mathrm{d}\ln|y|,$$

so by an integration,

$$\frac{x}{y} - \ln|y| = C, \qquad \text{or} \qquad x = y\ln|y| + Cy, \qquad \text{where } C \text{ is an arbitrary constant.}$$

If instead $k = -2$, then the equation becomes exact,

$$0 = -2x\,\mathrm{d}x - 2y\,\mathrm{d}x + y\,\mathrm{d}x - x\,\mathrm{d}y - y\,\mathrm{d}y = -\,\mathrm{d}(x^2) - \mathrm{d}\left(\frac{1}{2}\,y^2\right) - d(xy) = -\,\mathrm{d}\left(x^2 + xy + \frac{1}{2}\,y^2\right),$$

so by a trivial change of sign, followed by an integration we get

$$C = x^2 + xy + \frac{1}{2}\,y^2 = \left(x + \frac{1}{2}\,y\right)^2 + \frac{1}{4}\,y^2,$$

so $C > 0$, and the solution curves form a system of ellipses. The singular point $(0, 0)$ is a centre. $\Diamond$

9 Some simple implicit given differential equations.

In this section we list some important differential equations of first order, where a solution formula is known. Some of them are latently based on some integrating factor, but it will not be necessary to indicate the integrating factor in all cases. It should be noted, that when y' in the equation is given implicitly, we may not in all cases get existence or uniqueness theorems.

9.1 The equation $x = g(y')$.

We start with the simple equation, where y itself does not occur, and where x is given as a function of the derivative y' of y. We have the following theorem

Theorem 9.1 *Let $g : \,]t_1, t_2[\, \to \mathbb{R}$ be a strictly monotone C^1-function, and put*

$$a := \inf_{t_1<t<t_2} g(t) \qquad \text{and} \qquad b := \sup_{t_1<t<t_2} g(t).$$

Then the solution of the differential equation

$$x = g\left(\frac{dy}{dx}\right)$$

is defined for $x \in \,]a, b[$, and it is given by the parametric description

$$x = g(t) \qquad \text{and} \qquad y = c + \int t g'(t)\, dt,$$

where c is an arbitrary constant.

PROOF. Since $g(t)$ is of class C^1 and strictly monotone, the map $x = g(t)$ has a uniquely determined inverse $t = h(x)$, which is also of class C^1. In particular, the differential equation can be written $y' = h(x)$, and its solution is given by

$$y = c + \int h(x)\,\mathrm{d}x = c + \int h(g(t)) g'(t)\,\mathrm{d}t = c + \int t \cdot g'(t)\,\mathrm{d}t. \qquad \diamond$$

We note that the solution curves are obtained by translating one solution curve vertically.

Example 9.1 *Find the solutions of the equation*

$$x = \left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx}.$$

In this case, $g(t) = t^3 + t$ is a monotone C^1-function $\left(g'(t) = 3t^2 + 1 \geq 1 > 0\right)$, so we can apply the theorem. We get

$$y = c + \int t \cdot g'(t)\,\mathrm{d}t = c + \int t\left(3t^2+1\right)\,\mathrm{d}t = c + \int \left(3t^3 + t\right)\,\mathrm{d}t = c + \frac{3}{4}t^4 + \frac{1}{2}t^2,$$

and a parametric description $(x(t), y(t))$ of the complete solution is given by

$$x = t^3 + t, \qquad y = \frac{3}{4}t^4 + \frac{1}{2}t^2 + c,$$

where c is an arbitrary constant. $\diamond$

Example 9.2 *Find the complete solution of the differential equation*

$$x = y + \sin y'.$$

In this case $g(t) = t + \sin t$ and $g'(t) = t + \cos t > 0$ for $t \neq \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. We shall only solve the equation for $|t| < \frac{\pi}{2}$, where we have an existence and uniqueness theorem. We find

$$x = t + \sin t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

and

$$y = c + \int t(1+\cos t)\,\mathrm{d}t = c + \frac{1}{2}t^2 + t \cdot \sin t + \cos t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

where $c \in \mathbb{R}$ is an arbitrary constant. ♢

9.2 The differential equation $y = g(y')$

Analogously we may consider the equation $y = g(y')$, where x on the left hand side has been replaced by y. In this case we have the following theorem:

Theorem 9.2 *Let $g(t)$ be a strictly monotone C^1 function of domain $]t_1, t_2[$, and assume that $0 \notin]t_1, t_2[$. Choose any $\xi \in \mathbb{R}$ and $t_0 \in]t_1, t_2[$, and put*

$$\eta = g(t_0), \qquad a_\xi = \xi + \inf_{t_1<t<t_2} \int_{t_0}^{t} \frac{g'(t)}{t}\,dt, \qquad b_\xi = \xi + \sup_{t_1<t<t_2} \int_{t_0}^{t} \frac{g'(t)}{t}\,dt.$$

Then the differential equation

$$y = g\left(\frac{dy}{dx}\right)$$

has precisely one solution $y = \varphi(x)$, for which $(x_0(t_0), y(t_0)) = (\xi, \eta)$, defined in the interval $]a_\xi, b_\xi[$. Its parametric description is given by

$$x(t) = \xi 0 \int_{t_0}^{t} \frac{g'(\tau)}{\tau}\,d\tau, \qquad y(t) = \varphi(x) = g(t).$$

Proof. Since $y = g(t)$ is a strictly monotone C^1-function, its inverse function $t = h(y) \neq 0$ is a uniquely determined C^1-function. Hence, the original differential equation is equivalent to the simpler equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h(y).$$

It follows from the existence and uniqueness theorem that there is precisely one solution curve through the point (ξ, η). Since $h(y) \neq 0$ by assumption, we get by separation of the variables,

$$\mathrm{d}x = \frac{\mathrm{d}y}{h(y)},$$

hence by integration,

$$x = \xi + \int_{\eta}^{y} \frac{\mathrm{d}y}{h(y)}, \qquad \text{for } a_\xi < x < b_\xi.$$

Finally, since $y = g(t)$ and $\mathrm{d}y = g'(t)\,\mathrm{d}t$ and $h(y) = t$,

$$x = \xi + \int_{t_0}^{t} \frac{g'(\tau)}{\tau}\,\mathrm{d}\tau, \qquad \text{supplied with } y = g(t),$$

and the theorem is proved. □

Example 9.3 *Find the complete solution of the differential equation*

$$y = \left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx}.$$

Here $g(t) = t^3 + t$, so $g'(t) = 3t^2 + 1$. Then

$$y = \varphi(x) = g(t) = t^3 + t,$$

and

$$x = c + \int \frac{g'(t)}{t}\,\mathrm{d}t = c + \int \left(3t + \frac{1}{t}\right)\mathrm{d}t = \frac{3}{2}t^2 + \ln|t| + c. \qquad \Diamond$$

Example 9.4 *Find the complete solution of the differential equation*

$$y = \left(\frac{dy}{dx}\right)^2 \sin\left(\frac{dy}{dx}\right).$$

In this case, $g(t) = t^2 \cdot \sin t$, which is strictly increasing and $\neq 0$ for e.g. $t \in \left]0, \frac{\pi}{2}\right[$ as one of the many possibilities. We shall solve the equation in the mentioned interval.

Let us check. The C^1-function $g(t) = t^2 \cdot \sin t$ is clearly $\neq 0$ for $t \in \left]0, \frac{\pi}{2}\right[$, and

$$g'(t) = t^2 \cos t + 2t \cdot \sin t > 0, \qquad \text{for } t \in \left]0, \frac{\pi}{2}\right[,$$

shows that $g(t)$ is strictly increasing in this interval. Then by the theorem, the coordinates of the solution are given by

$$y = t^2 \cdot \sin t,$$

and

$$x = c + \int \frac{g'(t)}{t}\,\mathrm{d}t = c + \int \{t\cos t + 2\sin t\}\,\mathrm{d}t = c + \int \left\{\frac{\mathrm{d}}{\mathrm{d}t}(t\sin t) + \sin t\right\}\mathrm{d}t = c + t \cdot \sin t - \cos t,$$

and a parametric description is given by

$$(x(t), y(t)) = \left(c + t \cdot \sin t - \cos t, t^2 \sin t\right), \qquad t \in \left]0, \frac{\pi}{2}\right[,$$

where c is an arbitrary constant. $\Diamond$

9.3 Goursat's equation

Consider *Goursat's equation*

(9.1) $$y\left(a+\alpha x^m y^n\right) \mathrm{d}x + x\left(b+\beta x^m y^n\right) \mathrm{d}y = 0.$$

If $b\alpha - \beta a = 0$, then (9.1) is reduced to

$$\alpha y \, \mathrm{d}x + \beta x \, \mathrm{d}y = 0,$$

where the variables can be separated.

If $b\alpha - \beta a \neq 0$, we can find uniquely determined constants p and q, such that

$$\mu(x,y) = x^p y^q$$

is an integration function. The values of p and q are given by the equations

$$\begin{cases} bp - aq = a - b, \\ \beta p - \alpha q = \alpha(n+1) - \beta(m+1). \end{cases}$$

Example 9.5 *Consider the differential equation*

$$0 = \left(-3y + 2x^3y^3\right) \, dx + \left(4x - 3x^4y^2\right) \, dy = y\left(-3 + 2x^3y^2\right) \, dx + x\left(4 - 3x^3y^2\right) \, dy.$$

The latter form shows that this is a *Goursat equation.*

By inspection it is seen that the lines (axes) $x = 0$ and $y = 0$ are solutions.

When $x \cdot y \neq 0$ the integrating factor must have the structure $\mu = x^m y^n$. When the equation in the former form is multiplied by this μ, we get

$$0 = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = \left(-3x^m y^{n+1} + 2x^{m+3}y^{n+3}\right)\,\mathrm{d}x + \left(4x^{m+1}y^n - 3x^{m+4}y^{n+2}\right)\,\mathrm{d}y,$$

where

$$\frac{\partial L}{\partial y} = -3(n+1)x^m y^n + 2(n+3)x^{m+3}y^{n+2}, \qquad \frac{\partial M}{\partial x} = 4(m+1)x^m y^n - 3(m+4)x^{m+3}y^{n+2}.$$

The domain $\mathbb{R}^2$ is simply connected, and this differential form is therefore exact, if it is closed, i.e. if the two expressions above are equal,

$$-3(n+1) = 4(m+1), \qquad \text{thus} \qquad 4m + 3n = -7,$$

and

$$2(n+3) = -3(m+4), \qquad \text{thus} \qquad 3m + 2n = -18.$$

By a simple subtraction we see that $m + n = 11$, hence, $m = -7 - 3 \cdot 11 = -40$, and whence $n = 51$, and we have found the probably unexpected integrating factor $x^{-40}y^{51}$.

By insertion into the differential form we get by pairing terms of the same polynomial degree and using the rules of differential of a product in the opposite way of the usual one, i.e. $f\,\mathrm{d}g + g\,\mathrm{d}f = d(f \cdot g)$,

$$\begin{aligned}
0 &= \left(-3x^{-40}y^{52} + 2x^{-37}y^{54}\right)\,\mathrm{d}x + \left(4x^{-39}y^{51} - 3x^{-36}y^{53}\right)\,\mathrm{d}y \\
&= \left\{-3x^{-40}y^{52}\,\mathrm{d}x + 4x^{-39}y^{51}\,\mathrm{d}y\right\} + \left\{2x^{-37}y^{54}\,\mathrm{d}x - 3x^{-36}y^{53}\,\mathrm{d}y\right\} \\
&= \left\{-\frac{3}{-39}\,y^{52}\,\mathrm{d}\left(x^{-39}\right) + \frac{4}{52}\,x^{-39}\,\mathrm{d}\left(y^{52}\right)\right\} + \left\{\frac{2}{-36}\,y^{54}\,\mathrm{d}\left(x^{-36}\right) - \frac{3}{54}\,x^{-36}\,\mathrm{d}\left(y^{54}\right)\right\} \\
&= \frac{1}{13}\left\{y^{52}\,\mathrm{d}\left(x^{-39}\right) + x^{-39}\,\mathrm{d}\left(y^{52}\right)\right\} - \frac{1}{18}\left\{y^{54}\,\mathrm{d}\left(x^{-36}\right) + x^{-36}\,\mathrm{d}\left(y^{54}\right)\right\} \\
&= \frac{1}{13}\,\mathrm{d}\left(x^{-39}y^{52}\right) - \frac{1}{18}\,\mathrm{d}\left(\frac{y^{54}}{x^{36}}\right) = \mathrm{d}\left\{\frac{1}{13}\,\frac{y^{52}}{x^{36}}\right\}.
\end{aligned}$$

Then by integration for some arbitrary constant C_1,

$$\frac{1}{13}\left(\frac{y^4}{x^3}\right)^{13} - \frac{1}{18}\,\frac{y^{54}}{x^{36}} = C_1,$$

so finally by a rearrangement,

$$18y^{52} - 13x^3y^{54} = C \cdot x^{39}.$$

When $C = 0$, we either get $y = 0$, already mentioned above, or $x^3y^2 = \dfrac{18}{13}$. Apart from the case $C = 0$, MAPLE is not too happy with sketching the solutions. $\Diamond$

9.4 Clairaut's equation

A somewhat different implicitly given differential equation was considered by *A. C. Clairaut* (1713-1765). This equation is written

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} + g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right),$$

where $g(t)$ is some C^1 function. If the function $g(t)$ is defined for some constant $t = c$, then it follows immediately by insertion that we trivially have the rectilinear solution

$$y = c \cdot x + g(c), \qquad c \in \mathbb{R}, \text{ and } g(c) \text{ is defined.}$$

The problem is, whether these are all the solutions, or, if there exist other non-rectilinear solutions. We shall, following Kamke [9], prove the following theorem, which answers some of the questions. Davis [7] proves a weaker theorem. The reader is warned that the proof given in the following is fairly long and complicated, and it is using a lot of almost forgotten Mathematics from the past. Kamke's proof has been elaborated, and some of his minor errors have been corrected.

Theorem 9.3 *Let $g(t)$ be a C^1-function, for which the derivative $g'(t)$ is a strictly monotone function in the interval $t \in \,]\tau_1, \tau_2[$. The complete solution of Clairaut's differential equation*

$$y = x\,\frac{dy}{dx} + g\left(\frac{dy}{dx}\right),$$

is given by

1) *either the solution $y = \varphi$ with the parametric description*

$$\begin{cases} x = -g'(t), \\ y = \varphi(x) = -tg'(t) + g(t), \end{cases} \qquad \text{for } \tau_1 < t < \tau_2.$$

The solution $y = \varphi(x)$ is defined for $x \in]a, b[$, where

$$a = \inf_{\tau_1 < t < \tau_2} \{-g'(t)\}, \qquad b = \sup_{\tau_1 < t < \tau_2} \{-g'(t)\};$$

2) *or the straight lines*

$$y = c\,x + g(c), \qquad \text{for } c \in \,]\tau_1, \tau_2[\,,$$

which also are the tangents of the above mentioned solution curves of 1), i.e. the curve in 1) is the envelope of the straight lines;

3) *all curves, which are obtained by concatenating smoothly the curve from 1) with possible halflines of type 2).*

Proof. It was proved previously that the straight lines of 2) are always solutions of Clairaut's differential equation. We shall prove that these straight lines are all tangents of the function given in 1), where we later prove that this function is also a solution.

By assumption, the derivative $g'(t)$ is continuous and strictly monotone, so the first coordinate function $x = -g'(t)$ is a continuous and strictly monotone function. Its inverse function $t = h(x)$ exists and is also continuous and strictly monotone for $x \in]a, b[$. This implies that

$$y = -tg'(t) + g(t) = xh(x) + g(h(x)) = \varphi(x)$$

is an uniquely determined function for $x \in]a, b[$.

Choose any x_1, $x_2 \in]a, b[$, and let t_1, $t_2 \in \,]\tau_1, \tau_2[$ be the uniquely determined t-values, i.e. we have

$$x_1 = -g'(t_1) \qquad \text{and} \qquad x_2 = -g'(t_2)\,.$$

Then by the parametric description,

$$\varphi(x_2) - \varphi(x_1) = t_1 g'(t_1) - g(t_1) - t_2 g'(t_2) - g(t_2) = t_1 \{g'(t_1) - g'(t_2)\} - (t_2 - t_1)\, g'(t_2) - g'(t_1)\,,$$

so using that $x_1 = -g'(t_1)$ and $x_2 = -g'(t_2)$, i.e. $x_2 - x_1 = g'(t_1) - g'(t_2)$, we get

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} = t_1 + \frac{-\,(t_2 - t_1)\, g'(t_2) + (t_2 - t_1)\, g'(\tau)}{g'(t_1) - g'(t_2)},$$

where $\tau \in \,]t_1, t_2[$ is chosen appropriately, so

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} = t_1 + \frac{(t_2-t_1)\{g'(\tau)-g'(t_2)\}}{g'(t_1)-g'(t_2)} = t_1 - (t_2-t_1)\,\theta$$

for some $\theta \in]0,1[$. When we let $x_2 \to x_1$, then by the continuity, $t_2 \to t_1$, so the right hand side tends to t_1, from which follows that $\varphi'(x_1)$ exists and that $\varphi'(x) = t_1$. We conclude from

$$y = -tg'(t) + g(t) = x \cdot t + g(t),$$

that

$$\varphi(x_1) = x_1 \cdot t + g(t_1) = x_1\varphi'(x_1) + g\left(\varphi'\left(x_1\right)\right)$$

for every $x_1 \in]a,b[$, and we have proved that the curve of 1) is a solution.

We also proved above that if $x = -g'(t)$, then $\varphi'(x) = t$. Let $(x_1, \varphi(x_1))$ be a point on the curve. The tangent of the curve at this point is given by the equation

$$y - \varphi(x_1) = (x - x_1)\,\varphi'(x_1) = (x - x_1)\,t_1,$$

so

$$y = t_1 x + \varphi(x_1) - t_1 x_1 = t_1 x - t_1 g'(t_1) + g(t_1) + t_1 g'(t_1) = t_1 x + g(t_1)\,,$$

which is a straight line of type 2), thus every tangent of the curve 1) is a solution curve of type 2). Since $t_1 \in \,]\tau_1, \tau_2[$ is arbitrary, we get all solutions of 2) in this way as a tangent to the curve of 1).

We still have to prove that apart from the trivial concatenations of 3) there is no other solution curve of Clairaut's differential equation. Assume that $y = \varphi(x)$, $x \in]A,B[$, is a solution. Let $x_1,\, x_2 \in]A,B[$, and assume that

$$\varphi'(x_1) = \varphi'(x_2) = p,$$

i.e. we have parallel tangents at the points $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$. Then we get from Clairaut's equation

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} + g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right), \qquad y = \varphi(x),$$

that

$$\varphi(x_1) = p \cdot x_1 + g(p) \qquad \text{and} \qquad \varphi(x_2) = p \cdot x_2 + g(p),$$

from which

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} = p.$$

This implies that the line segment between the two points $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$ must lie on the solution curve. In fact, if this was not true, then we could find $\xi \in \,]x_1, x_2[$, such that $(\xi, \varphi(xi))$ does not lie on this line segment. The function φ is continuous, so there must exist a largest interval $]\alpha, \beta[$, such that no point $(x, \varphi(x))$ for $x \in]\alpha, \beta[$ lies on this line segment. However, the endpoints $(\alpha, \varphi(\alpha))$

and $(\beta, \varphi(\beta))$ must lie on the segment. We apply the mean value theorem to find a point $\gamma \in]\alpha, \beta[$, such that

$$\varphi'(\gamma) = \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha} = p,$$

which is a contradiction, unless $\alpha = \beta$, so $]\alpha, \beta[= \emptyset$. We have proved that the line segment between $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$, where $\varphi'(x_1) = \varphi'(x_2) = p$, is part of the solution curve.

We then prove that $\varphi'(x)$ actually is (not necessarily strictly) monotone for $x \in]A, B[$. Assume contrariwise. Assume that we can find three points $x_1 < x_0 < x_2$, such that

$$\varphi'(x_1) = \varphi'(x_2) \neq \varphi'(x_0) \,.$$

Then we see immediately that according to the above this is not possible.

We have proved that every solution $y = \varphi(x)$ of Clairaut's differential equation has the property that its derivative $\varphi'(x)$ is continuous and monotone.

Let $x_1, \, x_2 \in]A, B[$ be any points from the domain. Then by Clairaut's equation

$$\begin{aligned} \varphi(x_2) - \varphi(x_1) &= x_2\varphi'(x_2) + g\left(\varphi'(x_2)\right) - x_1\varphi'(x_1) - g\left(\varphi'(x_1)\right) \\ &= \left(x_2 - x_1\right)\varphi'(x_2) + x_1\left\{\varphi'(x_2) - \varphi'(x_1)\right\} + g'\left(\varphi(x_2)\right) - g\left(\varphi(x_1)\right). \end{aligned}$$

Using that $\varphi'(x)$ and $g'(t)$ are continuous, we conclude from the mean value theorem for fixed x_1 that there is a function $\Delta(x_2)$, where $\Delta(x_2) \to 0$ for $x_2 \to x_1$, such that

$$g\left(\varphi'(x_2)\right) - g\left(\varphi'(x_1)\right) = \left\{\varphi'(x_2) - \varphi'(x_1)\right\} \cdot \left\{g'\left(\varphi'(x_1)\right) + \Delta(x_2)\right\}.$$

Then

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} = \varphi'(x_2) + \frac{\varphi'(x_2) - \varphi'(x_1)}{x_2 - x_1}\left\{x_1 + g'\left(\varphi'(x_1)\right) + \Delta(x_2)\right\}.$$

When we let $x_2 \to x_1$, the left hand side tends towards $\varphi'(x_1)$, which due to the continuity of $\varphi'(x)$ is also the limit of $\varphi'(x_2)$ for $x_2 \to x_1$. Therefore, when we take this limit, the remaining term must tend towards 0, i.e.

$$\lim_{x_2 \to x_1} \frac{\varphi'(x_2) - \varphi'(x_1)}{x_2 - x_1}\left\{x_1 + g'(\varphi'(x_1)) + \Delta(x_2)\right\} = 0.$$

We conclude that if $x_1 + g'(\varphi'(x_1)) \neq 0$, then $\varphi''(x_1)$ also exists, and its value is $\varphi''(x_1) = 0$.

If

$$x + g'(\varphi'(x)) = 0 \qquad \text{for all } x \in]A, B[,$$

then $g'(\varphi'(x))$, hence also $\varphi'(x)$, is a strictly monotone function of x, and it follows from the differential equation that

$$\varphi(x) = -\varphi'(x)\, g'(\varphi'(x)) + g(\varphi'(x)).$$

This equation shows that the coordinates $(x, \varphi(x))$ are described by the parametric description of 1), where $t = \varphi'(x)$, and our solution curve is precisely given by 1), or is a subgraph of this solution curve.

If instead

$$\xi + g'(\varphi'(\xi)) \neq 0$$

for some ξ, then $\varphi''(\xi) = 0$ by the above, and the solution is a straight line. We use the continuity of $g'(t)$ and $\varphi'(x)$ to conclude that this inequality also holds in a neighbourhood of ξ. Let $]\alpha, \beta[\subseteq]A, B[$ be the largest open subinterval containing ξ, such that the solution is a straight line segment for $x \in]\alpha, \beta[$. Assume that $\alpha \in]A, B[$. Then $x + g'(\varphi'(x)) = 0$, so $(\alpha, \varphi(\alpha))$ is a point of the solution given by 1). Similarly, if $\beta \in]A, B[$. However, α and β cannot both lie in $]A, B[$, because if this was the case, then

$$\alpha + g'(\varphi'(\alpha)) = \beta + g'(\varphi'(\beta)) = 0,$$

and since $\varphi'(\alpha) = \varphi'(\beta)$, we must have $\alpha = \beta$, and $]\alpha, \beta[= \emptyset$.

We have proved that every point of the solution curve $y = \varphi(x)$, $x \in]A, B[$, of Clairaut's differential equation either lies on the curve given in 1), or on a rectilinear solution given in 2).

Summarizing, a solution curve in $]A, B[$ consists of either a rectilinear solution given by 2), or the solution given in 1), possibly supplied with rectilinear extensions at the endpoints of the solution given by 1). □

Many geometrical problems lead to Clairaut's differential equation. Assume that we have given a line bundle

$$y = ax + b,$$

where the constants a and b for some function $G(a, b)$ satisfy the equation,

$$G(a, b) = 0.$$

For every line $y = y(x) = ax + b$ we find

$$a = y'(x), \qquad \text{thus} \qquad b = y - x\,y'(x),$$

and it follows that we have the differential equation

$$G\left(\frac{\mathrm{d}y}{\mathrm{d}x}, y - x\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

If $G(a,b) = 0$ can be explicitly written as the function $b = g(a)$, then the differential equation is written

$$b = g(a) = g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = y - x\,\frac{\mathrm{d}y}{\mathrm{d}x},$$

so by a rearrangement,

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} + g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right),$$

which is a Clairaut differential equation.

In practice, $G(a,b)$ can be assumed to be a C^1-function. At all points, where $\dfrac{\partial G}{\partial b} \neq 0$ it follows from the theorem of implicitly given functions that we *locally* can solve the equation $b = g(a)$, in which case we locally get a Clairaut equation.

Example 9.6 We first choose an example from Davis [7],

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right), \qquad \text{i.e. } g(t) = \cos t.$$

It follows immediately that every line from the line bundle

$$y = cx + g(c) = cx + \cos c,$$

is a solution, where $c \in \mathbb{R}$ is an arbitrary constant.

Let us convince ourselves that this is indeed a solution. It follows that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = c$, so

$$x\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = c \cdot x + \cos x = y,$$

and the claim follows.

The assumption of the theorem is that $g'(t) = -\sin t$ is monotone. This is true for $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, where we cannot extend this domain further. Davis [7] is not too precise here, but he must have assumed that t lies in this interval. We shall not assume this to demonstrate what happens without this restriction.

The other solution of type 1) can here be obtained by eliminating the constant c from the two equations

$$F(x,y,c) = c \cdot x + \cos c - y = 0, \qquad \text{and} \qquad \frac{\partial F}{\partial c} = x - \sin c = 0.$$

It follows from the second equation that we may choose

$$c = \begin{cases} \arcsin x + 2p\pi, & p \in \mathbb{Z}, \\ \pi - \arcsin x + 2q\pi, & q \in \mathbb{Z}, \end{cases} \qquad x \in [-1, 1].$$

It is at this point Davis [7] erroneously claims that the only solution is $c = \arcsin x$, from which he gets the singular solution

$$y = cx + \cos c = x \, \arcsin x + \cos(\arcsin x) = x \cdot \arcsin x + \sqrt{1 - x^2}.$$

Let us prove here that this is a solution. It follows from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \arcsin x,$$

that

$$x \, \frac{\mathrm{d}y}{\mathrm{d}x} + \cos\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x \cdot \arcsin x + \sqrt{1-x^2} = y,$$

so

$$y = x \cdot \arcsin x + \sqrt{1-x^2}$$

is indeed a solution.

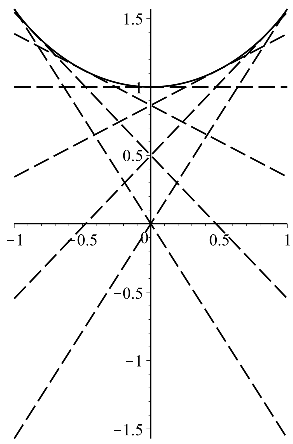


Figure 9.1: *The solution* $y = x \cdot \arcsin x + \sqrt{1-x^2}$ *and some tangent solution* $y = cx + \cos c$ *for* $c = -\frac{\pi}{2}, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, *all restricted to* $x \in [-1, 1]$

A similar check shows that

$$y_p(x) = x \cdot \{\arcsin x + 2p\pi\} + \sqrt{1-x^2}, \qquad p \in \mathbb{Z}, \quad x \in [-1, 1],$$

are also *singular solutions* of Clairaut's equation.

The other possibilities are

$$y = x\{\pi - \arcsin x + 2q\pi\} - \sqrt{1-x^2}, \qquad p \in \mathbb{Z}, \quad x \in [-1,1].$$

Here,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pi - \arcsin x + 2q\pi, \qquad q \in \mathbb{Z}, \quad x \in [-1,1],$$

so

$$x\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x\{\pi - \arcsin x + 2q\pi\} - \sqrt{1-x^2} = y,$$

and these functions are also solutions.

This example illustrates how easy it is to make an error, when one solves equations of this type.

Since we did not solve the equation itself

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

with respect to the highest order term $\frac{\mathrm{d}y}{\mathrm{d}x}$, we cannot directly apply the *Existence and Uniqueness Theorem*, so it is no contradiction that a lot of the solution curves intersect. In fact, each singular solution touches a general solution at everyone of its points.

We also note that there is no solution lying above the curve $y = x \arcsin x + \sqrt{1-x^2}$, $x \in [-1,1]$, so we do not have an existence theorem in this open domain. $\Diamond$

Example 9.7 *Find the complete solution of the Clairaut differential equation*

$$y = x\,\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The corresponding function $g(t) = \sqrt{1+t^2}$ is of class C^∞, and

$$g'(t) = \frac{t}{\sqrt{1+t^2}} \qquad \text{and} \qquad g''(t) = \frac{1}{\left(\sqrt{1+t^2}\right)^3} > 0 \qquad \text{for all } t,$$

so the assumptions of the theorem are fulfilled.

The domain of the envelope curve is the interval $]-1,1[$, because

$$a = \inf_{t\in\mathbb{R}}\left\{-\frac{t}{\sqrt{1+t^2}}\right\} = -1, \qquad \text{and} \qquad b = \sup_{t\in\mathbb{R}}\left\{-\frac{t}{\sqrt{1+t^2}}\right\} = 1.$$

A parametric description of the envelope curve is

$$x = -g'(t) = \frac{t}{\sqrt{1+t^2}} \qquad \text{and} \qquad y = -tg'(t) + g(t) = -\frac{t^2}{\sqrt{1+t^2}} + \sqrt{1+t^2} = \frac{1}{\sqrt{1+t^2}}.$$

The first equation shows that x and t have different signs, so when we solve with respect to t we get

$$t = -\frac{x}{\sqrt{1-x^2}},$$

and then

$$y = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\dfrac{x^2}{1-x^2}}} = \sqrt{1-x^2},$$

so the envelope is the upper half of the unit circle, and the remaining solutions are its tangents, where these can be concatenated with arcs of the circle.

Note also that there is no solution curve going through any point of the open upper unit disc. $\Diamond$

Example 9.8 *Find the complete solution of the Clairaut equation*

$$y = x\,\frac{dy}{dx} - \exp\left(\frac{dy}{dx}\right).$$

We find by identification that $g(t) = -e^t$, so $g'(t) = g''(t) = -e^t \neq 0$, and it follows from the theorem that the envelop solution exists. The endpoints of the x-domain are given by

$$a = \inf_{t\in\mathbb{R}} \{-g'(t)\} = 0 \qquad \text{and} \qquad \sup_{t\in\mathbb{R}} \{-g'(t)\} = +\infty.$$

The parametric description of the solution is given by

$$x = -g'(t) = e^t \text{ for } t \in \mathbb{R}, \qquad \text{with the inverse } t = \ln x \text{ for } x > 0,$$

and

$$y = -tg'(t) + g(t) = te^t - e^t = (t-1)e^t = (\ln x - 1)x \qquad \text{for } x > 0.$$

The rectilinear solutions (the tangents) are given by

$$y = c \cdot x - e^c, \qquad \text{where } c \text{ is an arbitrary constant.} \qquad \Diamond$$

Example 9.9 *Given a bundle of straight lines $y = ax+b$. We assume that each of the lines intersects the positive x-axis and the positive y-axis in such a way, that the distance between these two intersection points is constant $c > 0$ for all lines.*
Find the corresponding Clairaut equation and find the envelope solution.

The intersection points are given by either $x = 0$, where $y = b > 0$, or $y = 0$, where $x = -\frac{b}{a} > 0$. It follows that at least $b > 0$ and $a < 0$, and the two intersection points are then

$$(0, b) \qquad \text{and} \qquad \left(-\frac{b}{a}, 0\right).$$

The distance between these two points is

$$c = \sqrt{\left(-\frac{b}{a}\right)^2 + b^2} = -\frac{b}{a}\sqrt{1+a^2}, \qquad a < 0 \text{ and } b,\, c > 0.$$

We solve this equation with respect to b, because then we get our function $g(t)$ of Clairaut's equation, where we put $a = t$,

$$b = -\frac{ca}{\sqrt{1+a^2}}, \qquad g(t) = -\frac{ct}{\sqrt{1+t^2}}, \qquad t < 0.$$

The corresponding Clairaut differential equation is then

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{c\,\frac{\mathrm{d}y}{\mathrm{d}x}}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}, \qquad \text{where } \frac{\mathrm{d}y}{\mathrm{d}x} < 0.$$

By differentiation,

$$g'(t) = -\frac{c}{\left(\sqrt{1+t^2}\right)^3},$$

which is a strictly monotone function. Then the assumptions of the theorem are fulfilled, and the envelope solution exists. Its parametric description is given by

$$x = -g'(t) = \frac{c}{\left(\sqrt{1+t^2}\right)^3}, \qquad t < 0,$$

and

$$y = -tg'(t) + g(t) = \frac{ct}{\left(\sqrt{1+t^2}\right)^3} - \frac{ct}{\sqrt{1+t^2}} = -\frac{ct^3}{\left(\sqrt{1+t^2}\right)^3}, \qquad t < 0.$$

We eliminate t to get

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{c^2} \cdot \left(\frac{1}{1+t^2} + \frac{t^2}{1+t^2}\right) = \sqrt[3]{c^2},$$

for $0 < x < c$ and $0 < y < c$. This describes the part of the asteroid, which lies in the first quadrant.
$\Diamond$

9.5 d'Alembert's equation

d'Alembert's differential equation

$$y = x\,f\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

is a generalisation of Clairaut's differential equation. In fact, we get Clairaut's equation by choosing $f(t) = t$.

The equation was introduced by d'Alembert (1717–1783). However, in some books it is named after Lagrange (1736–1813) instead.

We note that the c-isocline is the straight line

$$y = xf(c) + g(c), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = c.$$

From this follows that if $f(c) = c$ for some c, then clearly $y = cx + f(c)$ is a rectilinear solution, and *vice versa.*

Since d'Alembert's equation is more general than Clairaut's equation, we may expect some complications in the solution process, and the following theorem is not so nice as the Theorem for Clairaut's equation.

Theorem 9.4 *Let $f(t)$ and $g(t)$ be C^1-functions in the interval $]\tau_1, \tau_2[$. If $f(c) = c$ for some constant c, then $y = cx + g(c)$ is a rectilinear solution.*
Furthermore, we can find all solutions $y = \varphi(x)$ of d'Alembert's differential equation

$$y = x\,f\left(\frac{dy}{dx}\right) + g\left(\frac{dy}{dx}\right)$$

which also satisfy

$$f(\varphi'(x)) \neq \varphi'(x) \qquad \text{and} \qquad xf'(\varphi'(x)) + g'(\varphi'(x)) \neq 0.$$

The solutions are found in the following way: Choose a solution of the linear differential equation

$$\frac{dx}{dt} = \frac{xf'(t) + g'(t)}{t - f(t)},$$

i.e. one of the functions

$$x(t) = \exp\left(\int_{t_0}^{t} \frac{f'(t)}{t - f(t)}\,dt\right)\left\{x_0 + \int_{t_0}^{t} \frac{g'(t)}{t - f(t)} \exp\left(\int_{t_0}^{t} \frac{f'(t)}{f(t) - t}\,dt\right)\,dt\right\},$$

and choose one of the biggest subintervals $]t_1, t_2[$ of $]\tau_1, \tau_2[$, such that both $f(t) \neq t$ and $x'(t) \neq 0$ for $t_1 < t < t_2$. Then a parametric description of the solution is given by

$$x = x(t) \qquad \text{and} \qquad y = xf(t) + g(t) \qquad \text{for } t_1 < t < t_2.$$

The structure of the theorem is strange. We can only find solutions $y = \varphi(x)$, when also

$$f(\varphi'(x)) \neq \varphi'(x) \qquad \text{and} \qquad xf(\varphi'(x)) + g'(\varphi'(x)) \neq 0,$$

so the solution procedure should always be followed by a discussion of the cases, when

$$f(\varphi'(x)) = \varphi'(x) \qquad \text{or} \qquad x f(\varphi'(x)) + g'(\varphi'(x)) = 0.$$

The strange condition $x f(\varphi'(x)) + g'(\varphi'(x)) \neq 0$ is used below, when we prove that the solution $\varphi(x)$ is of class C^2 in its domain.

PROOF. Assume that $y = \varphi(x)$ is a solution, which also fulfils the two restrictions mentioned above. Then, using the differential equation for the two points x_1, x_2 in the domain of $\varphi(x)$,

$$\begin{aligned} \varphi(x_2) - \varphi(x_1) &= x_2 f(\varphi(x_2)) + g(\varphi(x_2)) - x_1 f(\varphi(x_1)) - g(\varphi(x_1)) \\ &= (x_2 - x_1) f(\varphi(x_2)) + x_1 \{f(\varphi(x_2)) - f(\varphi(x_1))\} + g(\varphi(x_2)) - g(\varphi(x_1)) . \end{aligned}$$

By the mean value theorem there exists a ξ between x_1 and x_2, such that

$$\varphi(x_2) - \varphi(x_1) = (x_2 - x_1) f(\varphi(x_2)) + \{\varphi(x_2) - \varphi(x_1)\} \{x_1 f'(\varphi(\xi)) + g'(\varphi(\xi))\} .$$

We rearrange and divide by $x_2 - x_1$,

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \{x_1 f'(\varphi(\xi)) + g'(\varphi'(\xi))\} = \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} - f(\varphi(x_2)) .$$

When $x_2 \to x_1$, the right hand side is convergent, hence also the left hand side, where $\xi \to x_1$ for $x_2 \to x_1$. Therefore,

$$\lim_{x_2 \to x_1} \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \{x_1 f'(\varphi(\xi)) + g'(\varphi'(\xi))\} = \varphi'(x_1) - f(\varphi'(x_1)) .$$

At this step we use the assumption that $x_1 f'(\varphi'(x_1)) + g'(\varphi'(x_1)) \neq 0$, so we can divide the equation by $x_1 f'(\varphi'(x_1)) + g'(\varphi'(x_1))$ and then take the limit to get

$$\lim_{x_2 \to x_1} \frac{\varphi'(x_2) - \varphi'(x_1)}{x_2 - x_1} = \frac{\varphi'(x_1) - f(\varphi'(x_1))}{x_1 f'(\varphi'(x_1))\, g'(\varphi'(x_1))},$$

proving that $\varphi''(x)$ exists everywhere in the domain of $\varphi(x)$, subject to the two restrictions. Then we use the first restriction $f(\varphi'(x)) \neq \varphi'(x)$ to conclude that the right hand side is $\neq 0$, and we conclude that also $\varphi''(x) \neq 0$ everywhere in the domain of $\varphi(x)$.

Assume that $t = \varphi'(x) \in \,]t_1, t_2[$, where $\varphi''(x) \neq 0$. Then the inverse exists, $x = x(t)$, and

$$1 = \frac{\mathrm{d}t}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}x}\varphi'(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \varphi''(x) \cdot x'(t), \qquad \text{i.e. } \varphi''(x) = \frac{1}{x'(t)}.$$

This allows us to differentiate d'Alembert's equation

$$\varphi(x) = x f(\varphi(x)) + g(\varphi'(x)) ,$$

with respect to x,

$$\varphi'(x) = f(\varphi'(x)) + \varphi''(x) \cdot \{x f'(\varphi'(x)) + g'(\varphi'(x))\} = f(\varphi'(x)) + \frac{x f'(\varphi'(x)) + g'(\varphi'(x))}{x'(t)},$$

from which we get

$$x'(t) \{\varphi'(x) - f(\varphi'(x))\} = x f'(\varphi'(x)) + g'(\varphi'(x)) .$$

Using that $\varphi'(x) = 1$ by definition, and $\varphi'(x) \neq f(\varphi'(x))$ by assumption, we get the following linear differential equation in x,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x f'(t) + g'(t)}{t - f(t)},$$

which by the usual solution formula has the complete solution

$$x(t) = \exp\left(\int \frac{f'(t)}{t - f(t)}\,\mathrm{d}t\right)\left\{c + \int \frac{g'(t)}{t - f(t)}\exp\left(\int \frac{f'(t)}{f(t) - t}\,\mathrm{d}t\right)\mathrm{d}t\right\},$$

where c is an arbitrary constant.

It follows from d'Alambert's equation itself that

$$y = \varphi(x) = x f(\varphi'(x)) + g(\varphi'(x)) = x f(t) + g(t),$$

so we have found the parametric description of the solution, fulfilling the two restrictions.

Then assume that $f(t) \neq t$ and $x'(t) \neq 0$ in some interval, where $x(t)$ is a solution of the linear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x f'(t) + g'(t)}{t - f(t)}.$$

It follows from $x'(t) \neq 0$ that the numerator $x f'(t) + g'(t) \neq 0$, and the inverse function $t = t(x)$ exists and satisfies

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{t - f(t)}{x f'(t) + g'(t)}.$$

Using that $y = xf(t) + g(t)$ we get by differentiation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)\,\frac{\mathrm{d}x}{\mathrm{d}t} + xf'(t) + g'(t),$$

so

$$\varphi'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x} = f(t) + \{t - f(t)\} = t,$$

and therefore,

$$y = xf(t) + g(t) = xf\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + g\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right),$$

and we have proved that $y = \varphi(x)$, given by the parametric description and satisfying the restriction, is a solution of d'Alembert's equation. □

Example 9.10 *Discuss the d'Alembert differential equation*

$$y = 2x\,\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2.$$

Even if this equation looks very simple we shall see in the following that its solution is not so simple. First we see that it is a d'Alembert equation with $f(t) = 2t$ and $g(t) = -t^2$. Since $f(t) \neq t$, unless for $t = 0$, this is not a Clairaut equation. However, $f(0) = 0$ implies that $y = xf(0) + g(0) = 0$ is a rectilinear solution.

We then turn to the other solutions described in the theorem above. From the first restriction $f(\varphi'(x)) = 2\varphi'(x) \neq \varphi'(x)$ we conclude that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \varphi'(x) \neq 0$ for all x in the domain.

The second restriction is $xf'(\varphi'(x)) + g'(\varphi'(x)) = 2x - 2\dfrac{\mathrm{d}y}{\mathrm{d}x} \neq 0$, from which we get the condition $\dfrac{\mathrm{d}y}{\mathrm{d}x} \neq x$. We therefore assume in the following that

$$\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} \neq x.$$

The consider the linear differential equation for $x = x(t)$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{xf'(t) + g'(t)}{t - f(t)} = \frac{2x - 2t}{t - 2t} = -\frac{2}{t}\,x + 2,$$

the solution of which is the parametric description of the x-coordinate of the solution,

$$x = \frac{1}{t^2}\left\{c + 2\int t^2\,\mathrm{d}t\right\} = \frac{2}{3}\,t + \frac{c}{t^2},$$

which we supply with the parametric description of the y-coordinate of the solution,

$$y = xf(t) + g(t) = 2xt - t^2 = \frac{4}{3}\,t^2 + \frac{2c}{t} - t^2 = \frac{t^2}{3} + \frac{2c}{t}.$$

The restrictions above require that $f(t) \neq t$, so $t \neq 0$, and furthermore that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{3} - \frac{2c}{t^3} = \frac{2}{3t^3}\left(t^3 - 3c\right) \neq 0.$$

We conclude that for a given constant $c \in \mathbb{R}$ the t interval must *not* contain the points $t = 0$ or $t = \sqrt[3]{3c}$.

1) If $c = 0$, then the only restriction is $t \neq 0$, and the parametric description is

$$x = \frac{2}{3}\,t, \qquad \text{i.e.} \qquad t = \frac{3}{2}\,x, \qquad x \neq 0,$$

and

$$y = 2xt - t^2 = 2x \cdot \frac{3}{2}\,x - \left(\frac{3}{2}\right)^2 = 3x^2 - \frac{9}{4}\,x^2 = \frac{3}{4}\,x^2, \qquad x \neq 0.$$

It is obvious that these two branches of the parabola $y = \frac{3}{4}\,x^2$, for $x < 0$ and for $x > 0$, can be continued through 0. They can also be put together with the positive x-axis for the parabola for negative x, and with the negative x-axis for the parabola for positive x. In particular, we do not have uniqueness at the point $(0,0)$. So we have the solutions

$$y = \frac{3}{4}\,x^2, \quad x \in \mathbb{R}, \qquad y = \begin{cases} \frac{3}{4}\,x^2, & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad y = \begin{cases} 0, & x > 0, \\ \frac{3}{4}\,x^2, & x < 0. \end{cases}$$

2) If $c > 0$, we must consider the three parameter intervals,

$$t < 0, \qquad 0 < t < \sqrt[3]{3c}, \qquad \text{or} \qquad t > \sqrt[3]{3c}.$$

Recall that the parametric descriptions are

$$x = \frac{2}{3}\,t + \frac{c}{t^2}, \text{ with } \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{3} - \frac{2c}{t^3}, \qquad \text{and} \qquad y = \frac{t^2}{3} + \frac{2c}{t} \text{ with } \frac{\mathrm{d}y}{\mathrm{d}x} = t.$$

a) If $t < 0$, then x is monotonously increasing, and the range of $x(t)$ is $\mathbb{R}$. It follows from $\frac{\mathrm{d}y}{\mathrm{d}x} = t$ that $y = \varphi(x)$ is monotonously decreasing.

b) If $0 < t < \sqrt[3]{3c}$, then $x \to +\infty$ for $t \to 0+$, while for $t \to \left(\sqrt[3]{3c}\right)-$,

$$x \to \sqrt[3]{3c} \qquad \text{and} \qquad y \to \sqrt[3]{3c} \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = t \to \sqrt[3]{3c}.$$

c) If $t > \sqrt[3]{3c}$, then $x \to +\infty$ for $t \to +\infty$, and

$$(x,y) \to \left(\sqrt[3]{3c}, \sqrt[3]{9c^2}\right) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \to t \to \sqrt[3]{3c} \qquad \text{for } t \to \sqrt[3]{3c}.$$

We conclude from b) and c) that the solution can be glued together at the point $\left(\sqrt[3]{3c}, \sqrt[3]{9c^2}\right)$ in a cusp lying on the parabola $y = x^2$. The halftangents are pointing in the same direction.

3) If $c < 0$, the discussion is similar. In this case we get the three intervals

$$t < \sqrt[3]{3c}, \qquad \sqrt[3]{3c} < t < 0, \qquad \text{or} \qquad t > 0.$$

a) If $t > 0$, then x is monotonously increasing, and the range of $x(t)$ is $\mathbb{R}$. From $\frac{\mathrm{d}y}{\mathrm{d}x} = t > 0$ follows that $y = \varphi(x)$ is monotonously increasing.

b) If $\sqrt[3]{3c} < t < 0$, then $x \to -\infty$ for $t \to 0-$, and $(x,y) \to \left(\sqrt[3]{3c}, \sqrt[3]{9c^2}\right)$ for $x \to \sqrt[3]{3c} < 0$.

c) Similarly for $t < \sqrt[3]{3c}$. In particular, the solution of c) is continuously glued together in a cusp with the solution of b).

We still have to investigate, if the two conditions $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$, or $\frac{\mathrm{d}y}{\mathrm{d}x} \neq x$ exclude some solution points. We have already seen that the solution $y = \frac{3}{4}\,x^2$ could be extended through $(0,0)$, at which point the uniqueness is not fulfilled.

Then we consider the hypothetical possibility of $\frac{\mathrm{d}y}{\mathrm{d}x} = x$. We first see that the original differential equation

$$y = 2x\,\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

can also be written

$$x^2 - y = \left(x - \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \geq 0,$$

so if for some finite point $\frac{\mathrm{d}y}{\mathrm{d}x} = x$, then this point must lie on the parabola $y = x^2$. However, none of the solution curves found so far have finite limit points, so this possibility is ruled out.

At the same time we conclude that there is no solution curve going through any point above the parable $y = x^2$.

It still could occur that a solution also was the solution of the simpler equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x$. The latter equation has the complete solution $y = \frac{1}{2}\,x^2 + c$. The right hand side of the differential equation gives $2x^2 - x^2 = x^2 \neq \frac{1}{2}\,x^2 + c$ as function, so this possibility is also ruled out.

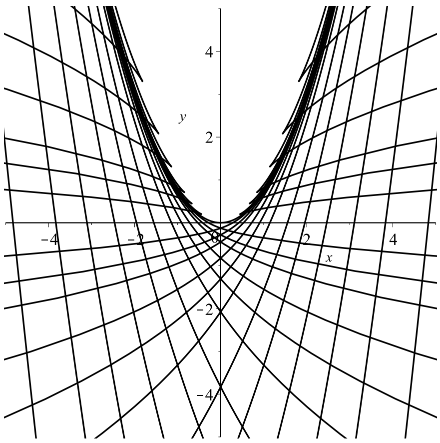


Figure 9.2: *Some solution curves of the differential equation* $y = 2xy' - (y')^2$. *We note the cusps, all lying on the parabola* $y = x^2$. *No solution curve will pass through any point above the parabola* $y = x^2$. *And there is no uniqueness either.*

Summing up, the complete solution of the d'Alembert equation

$$y = 2x\,\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

is split up into the following possibilities.

1) The rectilinear solution $y = 0$, i.e. the x-axis.

2) For constant $c = 0$,

$$y = \frac{3}{4}\,x^2, \quad x \in \mathbb{R}, \qquad y = \begin{cases} \frac{3}{4}\,x^2, & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad y = \begin{cases} 0, & x > 0, \\ \frac{3}{4}\,x^2, & x < 0. \end{cases}$$

3) For constant $c \neq 0$, a parametric description is given by

$$x = \frac{2}{3}\,t + \frac{c}{t^2} \qquad \text{and} \qquad y = \frac{t^2}{3} + \frac{2c}{t}, \qquad \text{for } t \neq 0 \text{ and } t \neq \sqrt[3]{3c}. \qquad \Diamond$$

9.6 Chrystal's equation

This equation

$$\text{(9.2)} \qquad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + A\,x\,\frac{\mathrm{d}y}{\mathrm{d}x} + B\,y + C\,x^2 = 0, \qquad A,\ B,\ C \text{ constants},$$

was in some detail discussed in 1896 by *G. Chrystal, (1851–1911)*. It may also, under certain conditions have singular solutions.

The general strategy is usual to solve such an implicit given equation with respect to its highest order term, here $\frac{\mathrm{d}y}{\mathrm{d}x}$. This is here straightforward,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\,A\,x \pm \frac{1}{2}\sqrt{A^2x^2 - 4By - 4C\,x^2}. \tag{9.3}$$

In order to get rid of the square root we introduce a new variable z by

$$4By = \left(A^2 - 4C - z^2\right)x^2, \qquad \text{i.e. } y = \frac{1}{4B}\left(A^2 - 4C - z^2\right)x^2,$$

hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x^2z}{4B}\,\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{2x}{4B}\left(A^2 - 4C - z^2\right) = -\frac{1}{2}\,Ax \mp \frac{1}{2}\,xz,$$

and by a reduction,

$$xz\,\frac{\mathrm{d}z}{\mathrm{d}x} = A^2 + AB - 4C \pm Bz - z^2.$$

Here, the variables can be separated, so

$$\frac{z\,\mathrm{d}z}{A^2 + AB - 4C \pm Bz - z^2} = \frac{\mathrm{d}x}{x} \tag{9.4}$$

for $x \neq 0$ and $A^2 + AB - 4C \pm Bz - z^2 \neq 0$.
If

$$-(z-a)(z-b) = A^2 + AB - 4C \pm Bz - z^2,$$

i.e.

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = \pm\frac{1}{2}\,B \pm \sqrt{B^2 + 4AB - 16C} = \pm\frac{1}{2}\,B \pm \frac{1}{2}\sqrt{(2A+B)^2 - 16C},$$

and $a \neq b$, then we can rearrange (9.4) in the following way,

$$\frac{z\,\mathrm{d}z}{(z-a)(z-b)} = \left\{\frac{a}{a-b}\,\frac{1}{z-a} - \frac{b}{a-b}\,\frac{1}{z-b}\right\}\mathrm{d}z = -\frac{\mathrm{d}x}{x}.$$

Therefore, if $a \neq b$, i.e. $(2A+B)^2 \neq 16C$, then the solution is given by

$$\frac{a}{a-b}\ln|z-a| - \frac{b}{a-b}\ln|z-b| + \ln|x| = \tilde{c},$$

or, when $c = (a-b)\tilde{c}$,

$$\ln\left(|z-a|^a\right) - \ln\left(|z-b|^b\right) + (a-b)\ln|x| = c.$$

When we take the exponential of this expression, we get

$$x^{a-b}\,\frac{|z-a|^a}{|z-b|^b} = \text{ constant}, \qquad \text{i.e} \qquad \frac{|xz-ax|^a}{|xz-bx|^b} = \text{constant},$$

so

$$(xz)^2 = \left(A^2 - 4C\right) x^2 - 4By, \qquad \text{or} \qquad xz = \pm\sqrt{\left(A^2 - 4C\right) x^2 - 4Bt},$$

where the further discussion still becomes complicated, as long as we do not know the concrete values of the constants A, B and C.

If instead $a = b$, i.e. $(2A + B)^2 = 16C$, then $a = b = \pm\frac{1}{2} B$, and we get the equation

$$-\frac{\mathrm{d}x}{x} = \frac{z\,\mathrm{d}z}{(z-a)^2} = \left\{\frac{1}{z-a} + \frac{a}{(z-a)^2}\right\} \mathrm{d}z,$$

hence by integration and a rearrangement,

$$(xz - ax)\exp\left(-\frac{a}{z-a}\right) = (xz - ax)\exp\left(-\frac{ax}{zx - ax}\right) = \text{constant},$$

where as above,

$$(xz)^2 = \left(A^2 - 4C\right) x^2 - 4By, \qquad \text{or} \qquad xz = \pm\sqrt{\left(A^2 - 4C\right) x^2 - 4Bt}.$$

Since the exponential appears here, the solution of such a system is different from the previous one considered.

In practice, *Chrystal's equation* is not easy to solve, and *Davis* [7] actually misses some of the possibilities. We shall here illustrate the methods of solution by giving two examples, which cover the main cases mentioned above. They are not not included in *Davis* [7],

Example 9.11 Choose $A = 1$, $B = 1$ and $C = 1$ in *Chrystal's equation*, i.e.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\,\frac{\mathrm{d}y}{\mathrm{d}x} + y + x^2 = 0. \tag{9.5}$$

Note that (9.5 can be rearranged in the following way,

$$\left\{\frac{\mathrm{d}y}{\mathrm{d}x} + x\right\}^2 = -y, \tag{9.6}$$

so we immediately get the requirement that $y \le 0$. The x-axis itself, i.e. $y = 0$, cannot be a solution either. Choosing $y = 0$ we see that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -x$, so there is only a *possibility* of concatenating two solutions at the point $(0,0)$ on the boundary, one for $x < 0$, and one for $x > 0$. It will follow from the results below that this is only possible for one more or less trivial pair of solutions, so we shall here assume $y < 0$ and only consider $y = 0$ in the limit.

In this open lower halfplane, $y < 0$, equation (9.6) is split into the following two equations,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{-y}, \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + x = -\sqrt{-y}, \qquad y < 0. \tag{9.7}$$

In either case, the existence theorem applies, so through every point (x_0, y_0), $y_0 < 0$, we have precisely one solution of either of the two equations, including points on the negative y-axis, $(0, y_0)$, $y_0 < 0$.

We mention this here, because we during proof are forced temporarily to assume that $x \neq 0$, i.e. we formally solve the equations separately in each of the two quadrants in the lower halfplane. On the negative y-axis the slope of the unique solution curve through the point $(0, y_0)$, $y_0 < 0$, is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{-y_0}.$$

so there must be a solution curve passing through such a point.

When $x \neq 0$, the idea is to get rid of the additional term x, so we search for a transformation, in which x gives the missing factor x in such a way that this x can be cancelled. Due to the square root, y must contain the factor x^2, so in order to make a transformation reasonable we choose $y = a\,x^2 z^2$, where $a < 0$ is some constant, and z is the new unknown function.

Looking at the theory above we see that apart from the constant we have found the right transformation. So in the following we take $a = -\frac{1}{4}$ from the theory,

$$(9.8) \qquad y = -\frac{1}{4}\,x^2 z^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\,x^2 z\,\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{2}\,x\,z^2,$$

hence by insertion in (9.7),

$$-\frac{1}{2}\,x^2 z\,\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{2}\,x\,z^2 + x = \pm\frac{1}{2}\,x\,z.$$

The point here is that due to the temporary assumption of $x \neq 0$ we can divide this equation by x, so after a rearrangement,

$$xz\,\frac{\mathrm{d}z}{\mathrm{d}x} + z^2 - 2 = \pm z, \qquad \text{or} \qquad xz\,\frac{\mathrm{d}z}{\mathrm{d}x} = 2 \pm z - z^2.$$

The trick here is that the variables can be separated, so (9.7) is replaced by the following two equations,

$$(9.9) \qquad \frac{\mathrm{d}x}{x} = \frac{z\,\mathrm{d}z}{2 + x - z^2}, \qquad \text{or} \qquad \frac{\mathrm{d}x}{x} = \frac{z\,\mathrm{d}z}{2 - z - z^2}, \qquad\qquad x \neq 0.$$

Case I. In the first case we get by a decomposition,

$$\frac{\mathrm{d}x}{x} = \frac{z\,\mathrm{d}z}{2 + z - z^2} = \frac{-z\,\mathrm{d}z}{(z-2)(z+1)} = -\frac{2}{3}\,\frac{\mathrm{d}z}{z-2} - \frac{1}{3}\,\frac{\mathrm{d}z}{z+1},$$

or by a multiplication by 3, followed by a rearrangement,

$$2\,\frac{\mathrm{d}z}{z-2} + \frac{\mathrm{d}z}{z+1} + 3\,\frac{\mathrm{d}x}{x} = 0.$$

We let in the following k denote an unspecified constant, not necessarily the same in the different equations. Then we get by integration,

$$2\ln|z-2| + \ln|z+1| + 3\ln|x| = 2\ln|xz - 2x| + \ln|xz + x| = k.$$

When we apply the exponential we get with a new constant k, which can also be negative, so we get rid of the absolute value signs,

$$(xz - 2x)^2(xz + x) = k, \qquad x \neq 0.$$

Due to (9.8) we have $|xz| = 2\sqrt{-y}$, so

$$\left(\sqrt{-y}-x\right)^2\left(2\sqrt{-y}+x\right)=0, \qquad \text{if } xz>0,$$

$$\left(\sqrt{-y}+x\right)^2\left(-2\sqrt{-y}+x\right)=0, \qquad \text{if } xz<0.$$

If $k=0$ and $x>0$, then either

$$\sqrt{-y}-x=0, \qquad \text{or} \qquad 2\sqrt{-y}+x=0.$$

The latter is not possible, so a possible solution is $y=-x^2$ for $x>0$. A check in (9.5) gives

$$y=-x^2, \quad \frac{dy}{dx}=-2x, \quad \left(\frac{dy}{dx}\right)^2+2x\,\frac{dy}{dx}+y+x^2=0,$$

so $y=-x^2$ is indeed a solution for $x>0$.

If instead $k=0$ and $x<0$, the possible solution is then given by $2\sqrt{-y}+x=0$, i.e. $y=-\frac{1}{4}\,x^2$. A similar routine check as above shows that this is indeed a solution for $x<0$. So we have found two solutions, which are the only ones which can be concatenated,

$$y=\begin{cases}-\frac{1}{4}\,x^2, & \text{for } x<0,\\ 0, & \text{for } x=0,\\ -x^2, & \text{for } x>0.\end{cases}$$

We cannot choose $k<0$. In fact, $\left(\sqrt{-y}\pm x\right)^2>0$, so we get either $2\sqrt{-y}+x>0$ for $x>0$, or $2\sqrt{-y}-x>0$ for $x<0$.

From the rearrangement

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x - \sqrt{-y}$$

follows that the 0-isocline is defined by $-x - \sqrt{-y} = 0$, so $x < 0$ and $y = -x^2$. Note that $y = -x^2$ is a solution for $x > 0$ and the 0-isocline for $x < 0$.

Furthermore, the points of inflection are defined by

$$0 = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1 + \frac{1}{2\sqrt{-y}} \cdot \left\{-x - \sqrt{-y}\right\} = -\frac{1}{2\sqrt{-y}} \left\{3\sqrt{-y} + x\right\},$$

so the points of inflection are lying on the curve

$$y = -\frac{1}{9}\,x^2 \qquad \text{for } x < 0.$$

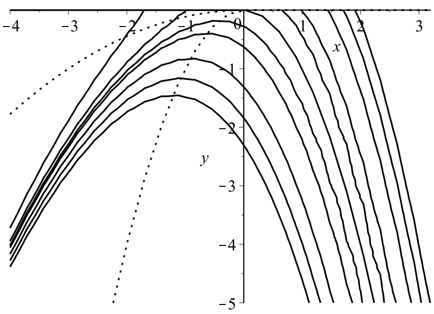


Figure 9.3: *Some solution curves of the equation* $\frac{dy}{dx} + x = -\sqrt{-y}$. *The 0-isocline and the curve of points of inflection are dotted.*

MAPLE. The curves in Figure 9.3 have been drawn by using the following MAPLE commands:

with(plots,implicitplot):

$f := x \to$ piecewise $\left(x < 0, 0, 0 \le x, -x^2\right)$

$g := (x, y) \to \left(\sqrt{-y} - x\right)^2 \cdot \left(2\sqrt{-y} + x\right)$

implicitplot $\Big(\Big[g(x,y) = -5, g(x,y) = -1, g(x,y) = 0.3, g(x,y) = 5, g(x,y) = 7, y = f(x), y = -x^2 - f(x),$
$y = -\frac{x^2 + f(x)}{9}\Big]$, $x = -4..3.2, y = -5..0$, scaling = constrained, color = black,
linestyle = [solid,solid,solid,solid,solid,solid,solid,solid,dot,dot])

Case II. Case I was dealing with the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+x=-\sqrt{-y},\qquad y<0.$$

By the change of variable $x \to -t$ and multiplication by -1 this is transferred into

$$\frac{\mathrm{d}y}{\mathrm{d}t}+t=\sqrt{-y},\qquad y<0,$$

which is Case II. So we can just take the results above and replace x everywhere by $-x$, also in the MAPLE program. This gives the following figure.

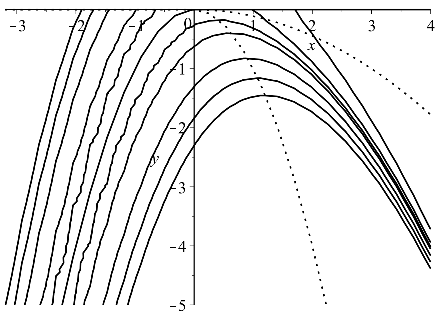


Figure 9.4: *Some solution curves of the equation* $\frac{dy}{dx}+x=\sqrt{-y}$. *The 0-isolcline and the curve of points of inflection are dotted.*

The original equation (9.2), i.e.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2x\,\frac{\mathrm{d}y}{\mathrm{d}x}+y+x^2=0,$$

has both systems of Case I and Case II as solutions. This is of course not contradicting the uniqueness theorem, because the assumptions of this theorem are not met. It is worth mentioning that (9.2) has also the two solutions

$$y=-x^2 \qquad \text{and} \qquad y=-\frac{1}{9}\,x^2 \qquad\qquad \text{for } x\in\mathbb{R},$$

going through the boundary point $(0,0)$, as well as the two concatenated solutions

$$y=\begin{cases}-\frac{1}{9}\,x^2, & \text{for } x<0,\\ 0, & \text{for } x=0,\\ -x^2, & \text{for } x<0,\end{cases} \qquad \text{and} \qquad y=\begin{cases}-x^2, & \text{for } x<0,\\ 0, & \text{for } x=0,\\ -\frac{1}{9}\,x^2, & \text{for } x<0,\end{cases},$$

found previously in Case I and Case II. So we have three concatenated solutions of (9.2) going through the boundary point $(0,0)$, and the uniqueness theorem does not hold at this point. $\Diamond$

In the next example we shall see what happens if one of the roots a or b is 0, because then z in the denominator cancels.

Example 9.12 We consider the *Chrystal equation*

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x\,\frac{\mathrm{d}y}{\mathrm{d}x} + 3y + x^2 = 0.$$

We solve this equation with respect to the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$,

$$\text{(9.10)} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\,x \pm \sqrt{-12y-3x^2} = -\frac{1}{2}\,x \pm \frac{1}{2}\,xz,$$

where we put

$$\text{(9.11)} \qquad -12y-3x^2 = x^2z^2, \qquad \text{for } y \le -\frac{1}{4}\,x^2,$$

i.e.

$$y = -\frac{1}{12}\left(3+z^2\right)x^2 \qquad \text{for } y \le -\frac{1}{4}\,x^2.$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\,x - \frac{1}{6}\,xz^2 - \frac{1}{6}\,x^2z\,\frac{\mathrm{d}z}{\mathrm{d}x},$$

and by insertion in (9.10),

$$-\frac{1}{2}\,x - \frac{1}{6}\,xz^2 - \frac{1}{6}\,x^2z\,\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{2}\,x \pm \frac{1}{2}\,xz,$$

which for $x \neq 0$ is reduced to

$$\text{(9.12)} \qquad z\left\{x\,\frac{\mathrm{d}z}{\mathrm{d}x} + z \pm 3\right\} = 0.$$

If $z=0$, then $y = -\dfrac{1}{4}\,x^2$, where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2}\,x$, and by (9.10) (a check),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\,x \pm \frac{1}{2}\sqrt{-12y-3x^2} = -\frac{1}{2}\,x,$$

so $y = -\dfrac{1}{4}\,x^2$ is indeed a solution.

Then we assume that $z \neq 0$, so we can divide the equation by z. Then separate the variables in the following variant to get

$$0 = x\,\mathrm{d}z + z\,\mathrm{d}x \pm 3\,\mathrm{d}x = d(xz) \pm \mathrm{d}(3x) = \mathrm{d}(xz \pm 3x),$$

hence by integration, for some arbitrary constant C_1,

$$xz \pm 3x = C_1, \qquad \textit{i.e. } xz = \mp(3x + C),$$

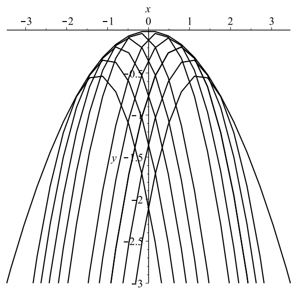


Figure 9.5: *Some solution curves of the equation* $\left(\frac{dy}{dx}\right)^2 + x\,\frac{dy}{dx} + 3y + x^2 = 0$. *Since this equation is equivalent to (9.10), comprising of two differential equations, we get two solution curves through every point* (x, y), *for which* $y < \frac{1}{4}\,x^2$. *We note that the boundary curve* $y = -\frac{1}{4}\,x^2$ *is a (singular) solution of the equation.*

where C is another arbitrary constant. Squaring this equation we get by (9.11) that

$$-12y - 3x^2 = x^2 z^2 = (3x + C)^2,$$

so

$$y = -\frac{1}{4}\,x^2 - \frac{1}{12}(3x + C)^2. \qquad \Diamond$$

The examples above cover most cases, when some derived second order algebraic equation has two different roots. We have seen that the implicit given function in this case behaves like a polynomial, though it is in most cases strictly speaking not a polynomial. The essential is that its factors are all power functions. The situation becomes different, when the derived equation, mentioned above, has a double root. In this case an exponential enters the implicit equation, which therefore becomes transcendental.

In order to illustrate this phenomenon we consider the following example.

Example 9.13 Given the *Chrystal equation*

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x\,\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + x^2 = 0, \tag{9.13}$$

where $A = 1$, $B = 2$ and $C = 1$. In order to get rid of the term $x\,\frac{\mathrm{d}y}{\mathrm{d}x}$ we rewrite (9.13) in the following way,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\,x\right)^2 = -2y - \frac{3}{4}\,x^2, \tag{9.14}$$

from which immediately follows that we must require that

$$y \le -\frac{3}{8}\,x^2,$$

with equality only for $(x, y) = (0, 0)$. Note that the boundary curve $y = -\frac{3}{8}\,x^2$ is not a solution, which follows directly by inserting in (9.14).

Then we check, if (9.9) has any solution of the structure $y = -a\,x^2$, where $a > 0$ is some positive constant. Insertion in (9.14) gives

$$\left(-2a + \frac{1}{2}\right)^2 = 2a - \frac{3}{4},$$

i.e.

$$4a^2 - 2a + \frac{1}{4} - 2a + \frac{3}{4} = 4a^2 - 4a + 1 = (2a - 1)^2 = 0.$$

The only possibility is $a = \frac{1}{2}$, where clearly $-\frac{1}{2} < -\frac{3}{8}$, so the parabola $y = -\frac{1}{2}\,x^2$ lies in the domain where solutions exist. If the reader still is not convinced, a check in (9.14) also shows that $y = -\frac{1}{2}\,x^2$ is a solution, and it is obviously the only curve passing through the boundary point $(0, 0)$.

For later use we take the square root of (9.14) and get the two equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\,x = \sqrt{-2y - \frac{3}{4}\,x^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\,x = -\sqrt{-2y - \frac{3}{4}\,x^2}, \tag{9.15}$$

both defined for $y < -\frac{3}{8}\,x^2$.

The equation of the 0-isocline of (9.13) or (9.14), is $2y + x^2 = 0$, i.e. $y = -\frac{1}{2}\,x^2$. However, $y = -\frac{1}{2}\,x^2$ is also a solution of both (9.13) and (9.14), which follows directly by insertion. The explanation of this apparently strange phenomenon is that (9.13) and (9.14) actually at the same time describe the two differential equations of (9.15), where for the first equation

$$y = -\frac{1}{2}\,x^2 \text{ is a } \begin{cases} \text{solution, when } x < 0, \\ \text{0-isocline, when } x > 0, \end{cases}$$

and for the second equation,

$$y = -\frac{1}{2}\,x^2 \text{ is a } \begin{cases} \text{0-isocline, when } x < 0, \\ \text{solution, when } x > 0. \end{cases}$$

Due to the uniqueness of each of the two equations of (9.15) we have two solution curves through every point (x, y), for which $y < -\frac{3}{8}\,x^2$, so by gluing the two solutions together at the point $(0,0)$ we obtain that $y = -\frac{1}{2}\,x^2$ is indeed a solution of (9.13).

Let us then search for possible inflection points, i.e. the set of points where $\frac{d^2y}{dx^2} = 0$. We write the first equation of (9.15) in the form

$$\frac{dy}{dx} = -\frac{1}{2}\,x + \sqrt{-2y - \frac{3}{4}\,x^2}.$$

Then

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{2} + \frac{1}{2}\,\frac{-2\,\frac{dy}{dx} - \frac{3}{2}\,x}{\sqrt{-2y - \frac{3}{4}\,x^2}} = -\frac{1}{2} + \frac{1}{2}\,\frac{x - \sqrt{-2y - \frac{3}{4}\,x^2} - \frac{3}{2}\,x}{\sqrt{-2y - \frac{3}{4}\,x^2}} \\ &= -\frac{1}{2}\,\frac{1}{\sqrt{-2y - \frac{3}{4}\,x^2}}\left\{\sqrt{-2y - \frac{3}{4}\,x^2} + \frac{1}{2}\,x + \sqrt{-2y - \frac{3}{4}\,x^2}\right\} \\ &= -\frac{1}{\sqrt{-2y - \frac{3}{4}\,x^2}}\left\{\sqrt{-2y - \frac{3}{4}\,x^2} + \frac{1}{4}\,x\right\}. \end{aligned}$$

The equation of the set of inflection points is then

$$\sqrt{-2y - \frac{3}{4}\,x^2} = -\frac{1}{4}\,x,$$

which can only be fulfilled if $x < 0$ and

$$-2y - \frac{3}{4}\,x^2 = \frac{1}{16}\,x^2, \qquad \text{i.e.} \qquad y = -\frac{13}{32}\,x^2,$$

provided that $x < 0$ (and $y < -\frac{3}{8}\,x^2$, which is fulfilled, because $-\frac{13}{32} < -\frac{3}{8}$).

For the second equation we get the curve of inflection points,

$$y = -\frac{13}{32}\,x^2, \qquad \text{for } x > 0,$$

which is seen from the above, because the change of variable $x \to -t$ carries the second equation into the first one.

Following the theory, we introduce for $x \neq 0$ the new variable z by

$$\text{(9.16)} \qquad y = \frac{1}{4B}\left(A^2 - 4C - z^2\right)x^2 = -\frac{1}{8}\left(3 + z^2\right)x^2 = -\frac{3}{8}\,x^2 - \frac{1}{8}\,x^2 z^2,$$

from which

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4}\,x - \frac{1}{4}\,xz^2 - \frac{1}{4}\,xz^2 - \frac{1}{4}\,x^2 z\,\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{4}\,x\left\{3 + z^2 + xz\,\frac{\mathrm{d}z}{\mathrm{d}x}\right\},$$

and

$$-2y - \frac{3}{4}\,x^2 = \frac{3}{4}\,x^2 + \frac{1}{4}\,x^2 z^2 - \frac{3}{4}\,x^2 = \frac{1}{4}\,x^2 z^2 = \left(\frac{1}{2}\,xz\right)^2,$$

so

$$\sqrt{-2y - \frac{3}{4}\,x^2} = \frac{1}{2}\,|xz|. \tag{9.17}$$

The two equations of (9.15) can of course be joined in one equation, allowing $\pm$ to enter,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\,x = \pm\sqrt{-2y - \frac{3}{4}\,x^2},$$

so using the z variable we get

$$-\frac{1}{4}\,x\left\{3 + z^2 + xz\,\frac{\mathrm{d}z}{\mathrm{d}x}\right\} + \frac{1}{2}\,x = \pm\frac{1}{2}\,xz,$$

or, when we for $x \neq 0$ divide by $-\frac{1}{4}\,x$,

$$1 + z^2 + xz\,\frac{\mathrm{d}z}{\mathrm{d}x} = \pm 2z,$$

where the variables can be separated,

$$\frac{\mathrm{d}x}{x} = -\frac{z\,\mathrm{d}z}{z^2 \pm 2z + 1} = -\frac{z\,\mathrm{d}z}{(z \pm 1)^2}. \tag{9.18}$$

Since

$$-\frac{z}{(z+1)^2} = -\frac{(z+1) - 1}{(z+1)^2} = -\frac{1}{z+1} + \frac{1}{(z+1)^2},$$

and

$$-\frac{z}{(z-1)^2} = -\frac{(z-1) + 1}{(z-1)^2} = -\frac{1}{z-1} - \frac{1}{(z-1)^2},$$

integration of (9.18) gives either

$$\ln|x| = -\ln|z+1| - \frac{1}{z+1} + k, \qquad \text{or} \qquad \ln|x| = -\ln|z-1| + \frac{1}{z-1} + k,$$

so either

$$-\frac{1}{z-1} = -\frac{x}{xz - x} = \ln|xz - x| + k, \tag{9.19}$$

or

$$\frac{1}{z+1} = \frac{x}{xz+x} = \ln|xz+x| + k, \tag{9.20}$$

where k just denotes some constant, which may not be the same, whenever it occurs.

Then we shall eliminate z. It follows from (9.17) that

$$|xz| = 2\sqrt{-2y - \frac{3}{4}\,x^2}.$$

In case of (9.19 we get the following two possibilities,

$$2\sqrt{-2y - \frac{3}{4}\,x^2} = \begin{cases} C \cdot \exp\left(-\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} - x}\right) + x, & \text{for } xz > 0, \\[2ex] C \cdot \exp\left(\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} + x}\right) - x, & \text{for } xz < 0. \end{cases} \tag{9.21}$$

In case of (9.20) the two possibilities are

$$2\sqrt{-2y - \frac{3}{4}\,x^2} = \begin{cases} C \cdot \exp\left(\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} + x}\right) - x, & \text{for } xz > 0, \\[2ex] C \cdot \exp\left(-\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} - x}\right) + x, & \text{for } xz < 0, \end{cases} \tag{9.22}$$

so when we glue the solutions from (9.21) and (9.22) together, we get

$$2\sqrt{-2y - \frac{3}{4}\,x^2} = \begin{cases} C \cdot \exp\left(-\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} - x}\right) + x, \\[2ex] C \cdot \exp\left(+\dfrac{x}{2\sqrt{-2y-\frac{3}{4}\,x^2} + x}\right) - x. \end{cases} \tag{9.23}$$

In this case MAPLE is in trouble when plotting some solutions using IMPLICITPLOT. $\Diamond$

10 Summary of solution methods and formulæ

We collect in this chapter in a very short form the solution methods and solution formulæ of this book. The notation is as usual

$$\omega = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y, \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),$$

or similarly, where we assume that $L(x,y)$ and $M(x,y)$, or $f(x,y)$ are all sufficiently often continuously differentiable functions, typically of at least class C^1, occasionally of class C^2, in some given open domain Ω. These assumptions will tacitly be required in the following. The proofs of all formulæ have been given previously in this book.

10.1 Exact and closed differential forms

An exact differential form is easy to integrate. If for some function $f(x,y)$,

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = \mathrm{d}f(x,y),$$

then its integral is $f(x,y)$, and the corresponding differential equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = \mathrm{d}f(x,y) = 0,$$

has the complete solution,

$$f(x,y) = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

A given differential form

$$\omega = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y,$$

is closed in the open domain Ω, if

$$\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x}, \qquad \text{for all } (x,y) \in \Omega.$$

If the differential form ω is closed in a *simply connected domain* Ω, then it is exact and can be reduced to the form $\mathrm{d}f(x,y)$.

If ω is exact, its line integral from a point (x_0,y_0) to another one, (x,y), is independent of the continuous and piecewise C^1 path of integration from (x_0,y_0) to (x,y), where this path should lie inside the open domain Ω. If possible, one may choose the curve consisting of straight line segments

$$(x_0,y_0) \to (x,y_0) \to (x,y),$$

provided that all line segments lie in Ω. If this is not the case, it is not hard geometrically to find an alternative path of integration lying inside Ω.

Alternatively we pair terms of

$$\omega = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y,$$

which "have a similar look", whatever is meant by that. We first isolate all addends in $L(x,y)$, which only depend on x and collect them by addition into one function $\lambda(x)$. If we put $\Lambda(x) = \int \lambda(x)\,\mathrm{d}x$, then clearly $\lambda(x)\,\mathrm{d}x = \mathrm{d}\Lambda(x)$. Terms from $M(x,y)\,\mathrm{d}y$ of the form $\mu(y)\,\mathrm{d}y$ are treated likewise. All remaining terms contain both x and y, probably in the form $\mathrm{d}x$ or $\mathrm{d}y$. These must be paired according to some of the rules of computation below,

$$\mathrm{d}u \pm \mathrm{d}v = d(u \pm v), \qquad v\,\mathrm{d}u + u\,\mathrm{d}v = \mathrm{d}(u \cdot v),$$

$$\frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2} = \mathrm{d}\left(\frac{u}{v}\right), \quad v \neq 0, \qquad f(u)\,\mathrm{d}u = \mathrm{d}\left(\int f(u)\,\mathrm{d}u\right).$$

The latter method is often faster, but it requires some skill.

10.2 Integrating factors

When the differential form

$$\omega = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y,$$

is not closed, it is sometimes possible to find a function $f(x,y) \neq 0$, such that

$$f(x,y)\,\omega = f(x,y)\,L(x,y)\,\mathrm{d}x + f(x,y)\,M(x,y)\,\mathrm{d}y$$

becomes closed. This is the case, when $f(x,y)$ fulfils the following linear partial differential equation of first order,

$$M(x,y)\,\frac{\partial f}{\partial x} - L(x,y)\,\frac{\partial f}{\partial y} = \left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} f(x,y).$$

If ω already is closed, then the right hand side is 0, and we may choose $f(x,y)$ a constant. If ω is not closed, we put temporarily $z = f(x,y)$, and then we (try to) solve the following system of differential equations in the auxiliary parameter t,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= M(x,y), \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -L(x,y), \\[2ex] \dfrac{1}{z}\,\dfrac{\mathrm{d}z}{\mathrm{d}t} &= \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}, \end{cases}$$

and then we eliminate the parameter t to get the function $z = f(x,y)$, where $f(x,y)$ is an integrating function.

10.3 The equation $\{y + x\,F(x^2+y^2)\}\,\mathbf{d}x - \{x - y\,F(x^2+y^2)\}\,\mathbf{d}y = 0$.

The solution formula is

$$\arctan\left(\frac{x}{y}\right) + \frac{1}{2}\int_{v=x^2+y^2} \frac{F(v)}{v}\,\mathrm{d}v = C, \qquad \text{provided that } y > 0.$$

It is left to the reader to derive similar formulæ, when $y < 0$, or $x > 0$, resp. $x < 0$ instead.

10.4 The equation $y\,f(xy)\,\mathbf{d}x + x\,g(xy)\,\mathbf{d}y = 0$.

The solution formula is for $x\,y\{f(x\,y) - g(x\,y)\} \neq 0$ given by

$$\ln|x| + \int_{v=xy} \frac{g(v)}{v\{f(v)-g(v)\}}\,\mathrm{d}v = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

This solution is supplied with the two trivial solutions $x = 0$ and $y = 0$ and a discussion of the case, where $f(xy) = g(x,y) = 0$.

10.5 The case where $\frac{1}{M(x,y)}\left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} = g(x)$, and generalizations.

Given the C^2 differential form

$$\omega = L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y.$$

If

$$\frac{1}{M(x,y)}\left\{\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right\} = g(x)$$

is a function in x alone, then

$$\mu(x) := \exp\left(\int g(x)\,\mathrm{d}x\right)$$

is an integrating factor of ω.

The extended version is:

If for some C^1 function $\varphi = \varphi(x,y)$,

$$\frac{\dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x}}{M(x,y)\,\dfrac{\partial \varphi}{\partial x} - L(x,y)\,\dfrac{\partial \varphi}{\partial y}} = g(\varphi) \qquad [= g(\varphi(x,y))],$$

then

$$\mu(x,y) = \exp\left(\int_{t=\varphi(x,y)} g(t)\,\mathrm{d}t\right)$$

is an integrating factor of ω.

10.6 Separation of the variables

The implicitly most commonly used method. If the differential equation takes the special form

$$f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y = 0,$$

where x and y are separated, then its complete solution is given by integrating each term separately,

$$\int f(x)\,\mathrm{d}x + \int g(y)\,\mathrm{d}y = C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

In many of the solutions strategies we try to shift parameters in such a way that the variables are separated after this shift of variables.

10.7 The differential equation $y' = f(Ax + By + C)$

Let $f(u)$ be a continuous function, and A, B, C constants. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C).$$

If $B = 0$, then the solution is trivially given by

$$y = \int f(Ax + C)\,\mathrm{d}x + \text{ const.}$$

If instead $B \neq 0$, then the complete solution is given by

$$y = \frac{u(x) - Ax - C}{B},$$

where the function $u(x)$ is implicitly given by

$$x = F(u;c) = \int \frac{\mathrm{d}u}{A + B\,f(u)} + c, \qquad c \text{ arbitrary constant.}$$

This means that we shall first find $F(u;c)$ and then the inverse of $F(u;c) = x$ and finally insert these functions $u = u(x;c)$ into the formula of y.

10.8 Orthogonal trajectories

Given the differential equation

$$L(x,y)\,\mathrm{d}x + M(x,y)\,\mathrm{d}y = 0.$$

The differential equation of the orthogonal trajectories is then

$$M(x,y)\,\mathrm{d}x - L(x,y)\,\mathrm{d}y = 0.$$

Note that even if one of these two equations may be easy to solve, the same may not be true concerning the other one.

10.9 The linear differential equation of first order

The complete solution of the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x)$$

is given by

$$y = e^{-F(x)} \int e^{F(x)} g(x)\,\mathrm{d}x + C \cdot e^{-F(x)}, \qquad C \text{ an arbitrary constant,}$$

where

$$F(x) = \int f(x)\,\mathrm{d}x.$$

This is probably the most well-known formula for ordinary differential equations. It should be known by all students who have been through an elementary calculus course at the university.

10.10 Bernoulli's differential equation

Consider Bernoulli's differential equation

$$\frac{\mathrm{d}y}{dax} = f(x)\,y + g(x)\,y^{\alpha},$$

where $f(x)$ and $g(x)$ are continuous functions, and α is a constant.

If $\alpha = 0$, the equation is reduced to the linear inhomogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - f(x)\, y = g(x),$$

solved previously.

If $\alpha = 1$, the equation is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \{f(x) + g(x)\}y,$$

which is solved by separating the variables, so (leaving out the details)

$$y = C \cdot \exp\left(\int \{f(x) + g(x)\}\, \mathrm{d}x \right), \qquad C \text{ arbitrary constant.}$$

Then assume that $\alpha \neq 0$ and $\alpha \neq 1$. If $\alpha > 0$, then $y = 0$ is trivially a solution. If $\alpha < 0$, then the differential equation is not defined for $y = 0$, and $y = 0$ is not a solution. For $y \neq 0$ we define a new variable by $z = y^{1-\alpha}$, and then derive the following linear, inhomogeneous differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1 - \alpha)\, f(x)\, z + (1 - \alpha)\, g(x),$$

which is solved in the usual way. Finally, the solution of Bernoulli's equation is given by

$$y = z^{\frac{1}{1-\alpha}},$$

whenever defined.

We mention in particular the case, when $\alpha = 2$. Then the derived differential equation in z is given by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -f(x)\, z - g(x), \qquad \text{where } y = \frac{1}{z}, \text{ when } z \neq 0.$$

This case occurs in the most common case of the Riccati equation, when a solution of the Riccati equation is known.

10.11 Riccati's differential equation

Given continuous functions $f(x)$, $g(x)$ and $h(x)$. The corresponding Riccati differential equation is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)\, y = g(x)\, y^2 + h(x).$$

The structure of the general solution of the Riccati differential equation is given by

$$y = \frac{a(x) + k \cdot b(x)}{c(x) + k \cdot d(x)} \quad \text{for } k \in \mathbb{R}, \qquad \text{supplied with } y = \frac{b(x)}{d(x)},$$

where $a(x)$, $b(x)$, $c(x)$, $d(x)$ are some in general unknown functions. The additional solution is formally obtained by taking the limit $k \to +\infty$.

If we know (or guess) in advance a solution $z = z(x)$ of Riccati's equation, then the general solution $\neq z$ is given by

$$y = u(x) + z(x),$$

where $u(x)$ is the general solution of the following Bernoulli differential equation,

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \{f(x) - 2z(x) \cdot g(x)\}u = g(x) \cdot u^2,$$

which is transformed into the following linear differential equation in $\frac{1}{u}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) + \{2z(x) \cdot g(x) - f(x)\} \cdot \frac{1}{u} = -g(x).$$

If

$$F(x) := \int \{2z(x) \cdot g(x) - f(x)\} \, \mathrm{d}x,$$

where $z(x)$ is the given solution, then the general solution of the Riccati differential equation is either $y = z(x)$, or

$$y = \frac{C \cdot z(x) + \int e^{F(x)} g(x) \, \mathrm{d}x}{C - \int e^{F(x)} g(x) \, \mathrm{d}x}, \qquad C \text{ arbitrary constant.}$$

The special Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)\, y + R(x)\, y^2 = P(x),$$

where $R(x) \neq 0$, can be solved by transforming it into a linear differential equation of second order, typically by inserting a formal power series solution. We shall here not go further into this matter, because we have restricted ourselves to differential equation of first order.

10.12 Homogeneous differential equations

A function $F(x, y)$ is called homogeneous of degree n, if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y), \qquad \text{for all } \lambda \in \mathbb{R}.$$

A homogeneous differential equation has the structure

$$L(x, y) \, \mathrm{d}x + M(x, y) \, \mathrm{d}y = 0,$$

where $L(x, y)$ and $M(x, y)$ are homogeneous functions of the same degree n.

We first find the possible rectilinear solutions $x = 0$, or y $= \alpha \cdot x$ by insertion and reduction, i.e. we check if $M(0, y) = 0$, when $x = 0$ is a solution. Otherwise, the constant α must satisfy the equation

$$L(1, \alpha) + \alpha \cdot M(1, \alpha) = 0.$$

For every solution α of this equation, $y = \alpha \cdot x$ is a rectilinear solution of the differential equation.

The rectilinear solutions divide the plane into a number of open sectors. In each of these sectors we change variables,

$$y = v \cdot x, \qquad \text{and} \qquad \mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v.$$

This transformation reduces the differential equation to

$$\{L(1,v) + v\,M(1,v)\}\,\mathrm{d}x + x\,M(1,v)\,\mathrm{d}v = 0,$$

where the variables can be separated, and the general solution is given by the implicit expression

$$\ln|x| + \int_{v=y/x} \frac{M(1,v)}{L(1,v) + v\,M(1,v)}\,\mathrm{d}v + C, \qquad C \text{ an arbitrary constant.}$$

10.13 The differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f\left(\dfrac{y}{x}\right)$

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right), \qquad \text{or} \qquad f\left(\frac{y}{x}\right)\mathrm{d}x - \mathrm{d}y = 0, \qquad x \neq 0,$$

is an homogeneous differential equation of degree 0, so by the previous section the general solution is given by

$$\ln|x - \int_{v=y/x} \frac{\mathrm{d}v}{f(v) - v} = C, \qquad C \text{ an arbitrary constant.}$$

10.14 The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right)$

Given the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right),$$

where a, b, c and α, β, γ are given constants. The solution method of this equation depends very much on these constants. We describe below the possibilities.

1) If $\gamma = c = 0$, then the equation is homogeneous of degree 0, and for $x \neq 0$ it is written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+by}{\alpha x+\beta y}\right) = f\left(\frac{a+b\,\frac{y}{x}}{\alpha+\beta\,\frac{y}{x}}\right) = F\left(\frac{y}{x}\right),$$

which is of the type considered in the previous section.

2) Assume that at least one of the constants c and γ is $\neq 0$. Then we have two possibilities.

$$\text{Either} \qquad \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0, \qquad \text{or} \qquad \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0.$$

a) We first assume that $\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0$. Then the two lines $ax+by+c=0$ and $\alpha x+\beta y+\gamma=0$, intersect at the point (ξ,η), where

$$\xi = -\begin{vmatrix} c & b \\ \gamma & \beta \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \qquad \text{and} \qquad \eta = -\begin{vmatrix} a & c \\ \alpha & \gamma \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}.$$

Change variables to $x_1 := x - \xi$ and $y_1 := y - \eta$, and the equation is reduced to

$$\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = f\left(\frac{ax_1+by_1}{\alpha x_1+\beta y_1}\right) = f\left(\frac{a+b\,\frac{y_1}{x_1}}{\alpha+\beta\,\frac{y_1}{x_1}}\right) = F\left(\frac{y_1}{x_1}\right),$$

which was solved in the previous section.

b) Then assume that $\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0$. We have the following possibilities,

$\beta = 0$, and either $\alpha = 0$ or $b = 0$;

$a = 0$, and either $\alpha = 0$ or $b = 0$;

all four constants are $\neq 0$.

i) If $\beta = 0$ and $\alpha = 0$, then the equation is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{a}{\gamma}\,x + \frac{b}{\gamma}\,y + \frac{c}{\gamma}\right),$$

which is a type, which already has been considered.

ii) If $\beta = 0$ and $b = 0$, the equation becomes trivial,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+c}{\alpha x+\gamma}\right),$$

and the complete solution is

$$y = \int f\left(\frac{ax+c}{\alpha x+\gamma}\right) \mathrm{d}x + C, \qquad C \text{ an arbitrary constant.}$$

iii) If $a = 0$, we express $x = x(y)$ as a function in y,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{f\left(\dfrac{by+c}{\alpha x+\beta y+\gamma}\right)} = g\left(\frac{by+c}{\alpha x+\beta y+\gamma}\right),$$

and we shall just repeat the analysis above, when $\beta = 0$.

iv) Finally, assume that a, b, α, $\beta \neq 0$. Put

$$v := \alpha x + \beta y + \gamma.$$

Then the new variable v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \alpha + \beta\, f\left(\frac{bv + c\beta + b\gamma}{\beta v}\right),$$

where the variables can be separated, so the general solution is described by

$$y = \frac{v}{\beta} - \frac{\alpha}{\beta}\, x - \frac{\gamma}{\beta},$$

where v is implicitly given by

$$\int \frac{\mathrm{d}v}{\alpha + \beta\, f\left(\dfrac{bv + c\beta + b\gamma}{\beta v}\right)} = x + C, \qquad C \text{ an arbitrary constant.}$$

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