Methods for finding Zeros in Polynomials

Leif Mejlbro

bookboon
The eBook company

Leif Mejlbro

Methods for finding (Real or Complex) Zeros in Polynomials

Methods for finding (Real or Complex) Zeros in Polynomials

ISBN 978-87-7681-900-2

Contents

Introduction

The class of polynomials is an extremely important class of functions, both in theoretical and in applied mathematics. The definition of a polynomial is so simple that one may believe that everything is trivial for polynomials. Of course, this is far from the truth. For instance, how can one numerically find the solution of the equation

(1) $3z^{87} - z^3 + 1 = 0$

within a given (small) error ε? An application of *Rouché's theorem* (cf. Section 3.5) shows that all 87 roots lie in the narrow annulus $0.96 \le |z| \le 1$, so it will be very crowded in this annulus concerning the roots. In principle we set up some guidelines in this book so the roots can be found. The task, however, is far beyond the scope of the present volume, so it is left to the few interested readers to use the methods given in the following and a computer in order to find the 86 complex roots remaining after we have found the only real root in Section 4.1.

In the first chapter we describe some results on polynomials in general, before we in the next three chapters proceed with the main subject of this book, namely to find the zeros of a polynomial. The topics are (mainly following the contents of the chapters, but not strictly)

1) Explicit solution formulæ

- The fundamental theorem of algebra
- The binomial equation
- The equation of second degree
- Rational roots
- Multiple roots
- The Euclidean algorithm, i.e. common roots of two polynomials

2) Position of roots of polynomials in a complex plane (classical results)

- Descartes's theorem
- Fourier-Budan's theorem
- Sturm's theorem
- Rouché's theorem
- Hurwitz polynomials

3) Approximation methods

- Newton's iteration method
- Graeffe's root-squaring method.

We shall occasionally in a few topics assume some knowledge of *Complex Functions Theory.*

All topics of this book have been known in the literature for more than a century. Nevertheless, it is the impression of the author that they are no longer common knowledge. One example is *Graeffe's root-squaring method* to find numerically roots why lie very close to each other in absolute size. It can in principle be used to find the 86 complex roots of (1), but the work will be so large that it cannot be included here.

Rouché's theorem and *Hurwitz's criterion* of stability and their applications are well-known in *Stability Theory* and among mathematicians, but in general, engineers do not know them. This is a pity, because the can often be used to limit the domain, in which the roots of a polynomial are situation. It is, e.g., by two very simple applications of *Rouché's theorem* that we can conclude that all the roots of (1) lie in the open annulus $0.96 < |z| < 1$.

Another extremely important theorem, which to the author's experience is not commonly known by engineers, is *Weierstraß's approximation theorem.* It states that *every continuous function $f(t)$ defined in a closed bounded interval I can be uniformly approximated by a sequence of polynomials.*

More explicitly, for every given $\varepsilon > 0$ and every given continuous function $f(t)$ on I one can *explicitly* find a polynomial $P(t)$, such that

(2) $|f(t) - P(t)| \leq \varepsilon \qquad \text{for every } t \in I.$

This means in practice that if the tolerated uncertainty is a given $\varepsilon > 0$, then we are allowed to

replace the continuous function $f(t)$ by the polynomial $P(t)$, given by *Weierstraß's approximation theorem.* This is very fortunate for the use of computers, which strictly speaking are limited to only work with polynomials, because only a finite number of constants can be stored in a computer. As indicated above there even exists an explicit construction *(Bernstein polynomials)* of such polynomials $P(t)$, when $f(t)$ and $\varepsilon > 0$ are given, such that (2) in fulfilled. One can prove that these Bernstein polynomials are not the optimum choice, but in general they are "very close" to be it.

Since we want to emphasize this very important theorem of Weierstraß, although it is not needed in the text itself, it has been described in a section of the Appendix.

Errors are unavoidable, so the author just hopes that there will not be too many of them.

October 5, 2014

Leif Mejlbro

1 Complex polynomials in general

1.1 Polynomials in one variable.

A complex function of the form

(3) $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0, \dots, a_n \in \mathbb{C},\ a_0 \neq 0,$ constants,

in the complex variable $z \in \mathbb{C}$ is called a *polynomial* of *degree* n.

When the polynomial is restricted to the real axis, we shall often write $P(x)$, $x \in \mathbb{R}$, instead of $P(z)$, though we may in the later chapters also from time to time use the notation $P(x)$ for $x \in \mathbb{C}$ complex. Sometimes we shall allow ourselves to omit "$x \in \mathbb{R}$" or "$x \in \mathbb{C}$", etc., where it is obvious, whether x is real or complex.

The following two results are well-known.

Proposition 1.1.1 *A polynomial $P(z)$ is continuous everywhere in $\mathbb{C}$.*

PROOF. It suffices to prove that every monomial z^n is continuous at every fixed $z_0 \in \mathbb{C}$. It follows from the *binomial formula*, cf. Appendix 5.1, that

$$(4)\ \ (z_0+\Delta z)^n - z_0^n = \sum_{j=1}^{n} \binom{n}{j} z_0^{n-j}\,\Delta z^j = \Delta z \sum_{j=0}^{n-1} \binom{n}{j+1} z_0^{n-1-j}\,\Delta z^j,$$

so $(z_0+\Delta z)^n - z_0^n \to 0$ for $\Delta z \to 0$, i.e. $(z_0+\Delta z)^n \to z_0^n$ for $\Delta z \to 0$, and the proposition follows. □

Proposition 1.1.2 *A polynomial $P(z)$ of degree n is continuously differentiable everywhere in $\mathbb{C}$, and its derivative $P'(z)$ is a polynomial of degree $n-1$.*

PROOF. It suffices again just to consider a monomial z^n. Then by (4),

$$\frac{(z_0+\Delta z)^n - z_0^n}{\Delta z} = \sum_{j=0}^{n-1} \binom{n}{j+1} z_0^{n-1-j}\Delta z^j = n\,z_0^{n-1} + \Delta z \sum_{j=0}^{n-2} \binom{n}{j+2} z_0^{n-2-j}\Delta z^j,$$

hence,

$$\lim_{\Delta z\to 0} \frac{(z_0+\Delta z)^n - z_0^n}{\Delta z} = n\,z_0^{n-1}.$$

Using this result and the linearity it follows that the derivative of $P(z)$ given by (3) is

$$P'(z) = n\,a_0\,z^{n-1} + (n-1)a_1\,z^{n-2} + \cdots + a_{n-1},$$

which is a polynomial of degree $n-1$, and the proposition is proved. □

It is very important that the description (3) of a polynomial $P(z)$ is unique. This follows from

Theorem 1.1.1 The identity theorem. *Two complex polynomials $P(z)$ and $Q(z)$ which are equal to each other for every $z \in \mathbb{C}$, have the same degree n and the same coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$ as given in (3).*

PROOF. Assume that

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad \text{and} \qquad Q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n.$$

If necessary, we have here supplied with zero terms, so that the $n+1$ coefficients become $a_0, \dots, a_n$ and $b_0, \dots, b_n$, even if e.g. $b_0 = 0$, etc.. We shall prove that if $P(z) = Q(z)$ for all $z \in \mathbb{C}$, then $a_j = b_j$ for every $j = 0, 1, \dots, n$.

If we choose $z = 0$, then we get $a_n = P(0) = Q(0) = b_n$, thus $a_n = b_n$, and it follows by a reduction that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z = b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z \qquad \text{for all } z \in \mathbb{C}.$$

When $z \neq 0$, this equation is equivalent to

$$(5)\ \ a_0 z^{n-1} + a_1 z^{n.2} + \cdots + a_{n-1} = b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}, \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

However, due to the continuity, (5) also holds for $z = 0$.

Repeating this process we get successively $a_{n-1} = b_{n-1}, \dots, a_0 = b_0$, and the theorem is proved. □

Remark 1.1.1 A similar argument shows that if two convergent power series (same point of expansion) are equal in their common domain of convergence, then they have the same coefficients. See also Appendix 5.2. $\Diamond$

It is convenient to define the *zero polynomial* as the function $Q(z) = 0$ and use the polynomial description

$$Q(z) = 0 \cdot z^n + 0 \cdot z^{n-1} + \cdots + 0 \cdot z + 0,$$

whenever necessary, although this is not in agreement with the definition (3). We shall also say that the zero polynomial has the degree $-\infty$. We obtain by this convention that the degree of a product of polynomials is equal to the sum of the degrees of the polynomials, i.e.

$$\deg(P(z) \cdot Q(z)) = \deg P(z) + \deg Q(z),$$

even if one of them is the zero polynomial. Here, $\deg P(z)$ denotes the degree of the polynomial $P(z)$.

1.2 Transformations of real polynomials.

If all coefficients $a_0, \dots, a_n$ of (3) are real, we say that $P(z)$ is a *real polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0, \dots, a_n \in \mathbb{R} \text{ and } z \in \mathbb{C}.$$

This is of course an abuse of the language, because only the *coefficients* are real, and $P(z)$ is not a real number for general $z \in \mathbb{C}$. However, if $z = x \in \mathbb{R}$, then $P(x)$ is always real.

We shall in the following show some simple transformation rules of real polynomials in a real variable $x \in \mathbb{R}$. These rules also hold for a complex variable $z \in \mathbb{C}$, but for clarity we shall only consider $P(x)$, $x \in \mathbb{R}$, in the discussions below.

1.2.1 Translations.

The chance of variable is here given by $x = y + k$, where $k \in \mathbb{R}$ is some real constant. If $P(x)$ has degree n, then it follows by a Taylor expansion from k, cf. Appendix 5.4.1, that

$$Q_1(y) := P(y+k) = \frac{P^{(n)}(k)}{n!}\, y^n + \frac{P^{(n-1)}(k)}{(n-1)!}\, y^{n-1} + \cdots + \frac{P'(k)}{1!}\, y + P(k).$$

The most commonly used translation is given by

$$k = -\frac{a_1}{n\, a_0},$$

by which the coefficient b_1 of y^{n-1} of $Q_1(y)$ becomes 0.

There is a reason why one usually only uses the translation above. In principle, we can set up a set of equations, such that any given b_j, $j = 2, \dots, n$, in $Q_1(y)$ becomes zero. Unfortunately, the equations in the unknown translation parameter k will in general be increasingly difficult to solve, i.e.

$$b_j = \frac{P^{(n-j)}(k)}{(n-j)!} = 0 \qquad \text{for } j = 2, \dots, n.$$

For $j = n$ we see that we shall solve $P(k) = 0$, i.e. find a zero of the polynomial, and we know that this is in general not possible to find in all cases by an exact solution formula.

Example 1.2.1 A polynomial can always be *normalized* by dividing it by $a_0 \neq 0$. We may therefore assume that $a_0 = 1$, so let us consider the polynomial of third degree,

$$P(x) = x^3 + a_1 x^2 + a_2 x + a_3, \qquad a_1,\, a_2,\, a_3 \in \mathbb{R}, \qquad x \in \mathbb{R}.$$

Let $k \in \mathbb{R}$. Then by the translation $x = y + k$,

$$\begin{aligned} P(x) &= x^3 + a_1 x^2 + a_2 x + a_3 = (y+k)^3 + a_1(y+k)^2 + a_2(y+k) + a_3 \\ &= y^3 + \{3k + a_1\}\, y^2 + \left\{3k^2 + 2ka_1 + a_2\right\} y + \left\{k^3 + a_1 k^2 + a_2 k + a_3\right\} \qquad (6) \\ &= y^3 + b_1 y^2 + b_2 y + b_3. \end{aligned}$$

The identity theorem gives

$$b_1 = 3k + a_1, \qquad b_2 = 3k^2 + 2ka_1 + a_2, \qquad b_3 = k^3 + a_1 k^2 + a_2 k + a_3.$$

1) Choosing $b_1 = 0$ we get $k = -\dfrac{a_1}{3}$.

2) Choosing $b_2 = 0$ we get $3k^2 + 2ka_1 + a_2 = 0$, hence

$$k = \frac{-2a_1 \pm \sqrt{4a_1^2 - 12a_2}}{6} = \frac{1}{3}\left\{-a_1 \pm \sqrt{a_1^2 - 3a_2}\right\}.$$

3) Choosing $b_3 = 0$ it follows that this is the same as finding the zeros of the polynomial.

$\Diamond$

1.2.2 Similarities.

For given $k \in \mathbb{R} \setminus \{0\}$, the *similarity* of factor k is defined as the change of variable

$$y = kx, \qquad \text{thus} \qquad x = \frac{y}{k}.$$

In this case the transformed polynomial becomes equivalent to

$$Q_2(y) := k^n\, P\left(\frac{y}{k}\right) = a_0 y^n + a_1 k\, y^{n-1} + a_2 k^2\, y^{n-2} + \cdots + a_{n-1}\, k^{n-1}\, y + a_n k^n,$$

where we for convenience have multiplied by k^n.

Clearly, if x_0 is a root of $P(x)$, i.e. $P(x_0) = 0$, then $y_0 = k\, x_0$ is a root of $Q_2(y)$, so similarities may be used to scaling.

If all coefficients $a_0, \dots, a_n \in \mathbb{Q}$ of $P(x)$ are rational numbers, then we can choose $k \in \mathbb{N}$ so large that the equivalent polynomial

$$\tilde{Q}_2(y) := \frac{1}{a_0}\, Q_2(y) = y^n + \frac{a_1 k}{a_0}\, y^{n-1} + \frac{a_2 k^2}{a_0}\, y^{n-2} + \cdots + \frac{a_n k^n}{a_0}$$

is normalized, $b_0 = 1$, and all other coefficients $b_1, \dots, b_n \in \mathbb{Z}$ are integers.

A polynomial is called *normalized*, if its coefficient $b_0 = 1$ of the term of highest degree. We see that every polynomial of rational coefficients can be transformed into a normalized polynomial by a similarity after a division by $a_0 \neq 0$.

1.2.3 Reflection in 0.

The reflection is given by the change of variable $y = -x$, $x = -y$, so

$$Q_3(y) := P(-y) = (-1)^n a_0 y^n + (-1)^{n-1} a_1 y^{n-1} + \cdots + a_{n-2} y^2 - a_{n-1} y + a_n.$$

This can of course also be considered as a similarity where $k = -1$. All coefficients of odd index change sign, while all coefficients of even index are unchanged. If x_0 is a (real) root of $P(x)$, then $y_0 = -x_0$ is a (real) root of $Q_3(y)$, so all real roots change their sign by a reflection.

We shall later give some criteria concerning real *positive* roots. Reflection can be used to obtain similar results for real *negative* roots.

1.2.4 Inversion.

The *inversion* is given by the change of variable $x = \frac{1}{y}$, $y = \frac{1}{x}$, where we must require that $x \neq 0$ and $y \neq 0$. When we multiply by $y^n \neq 0$, we obtain the equivalent polynomial

$$Q_4(y) := y^n \, P\left(\frac{1}{y}\right) = a_n \, y^n + a_{n-1} \, y^{n-1} + \cdots + a_1 \, y + a_0,$$

so the coefficients are here given in the reversed order.

If $x_0 \neq 0$ is a root of $P(x)$, then $y_0 = 1/x_0$ is a root of $Q_4(y)$, and if $y_0 \neq 0$ is a root of $Q_4(y)$, then $x_0 = 1/y_0$ is a root of $P(x)$.

1.3 The fundamental theorem of algebra.

It is very difficult, if possible at all, to make a serious investigation of the polynomials without being able to refer to the *Fundamental theorem of Algebra*. We shall therefore in this section prove this important theorem, before we start on other deeper results.

We have already used the terminology that $z_0 \in \mathbb{C}$ is a *root* or a *zero* (both names are used in the following) of a polynomial $P(z)$, if $P(z_0) = 0$.

Theorem 1.3.1 The fundamental theorem of algebra. *Every polynomial $P(z)$ of degree ≥ 1 has at least one root $z_0 \in \mathbb{C}$.*

The following proof is often called *Cauchy's proof* in spite of the fact that it is actually due to Argand, 1815. The first attempt of a proof goes back to d'Alembert in 1746, and the theorem is therefore also called *d'Alembert's theorem.*

PROOF. Consider the polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad n \geq 1 \text{ and } a_0 \neq 0.$$

Clearly, $P(0) = a_0$, so if $a_n = 0$, then $z = 0$ is a root. We may therefore in the following assume that also $a_n \neq 0$.

Let $|z| = r > 0$. Then we have the estimate

$$|P(z)| \geq |a_0| \, r^n - |a_1| \, r^{n-1} - |a_2| \, r^{n-2} - \cdots - |a_{n-1}| \, r - |a_n|$$

$$(7) \qquad = r^n \left\{ |a_0| - \left(\frac{|a_1|}{r} + \frac{|a_2|}{r^2} + \cdots + \frac{|a_{n-1}|}{r^{n-1}} + \frac{|a_n|}{r^n} \right) \right\}.$$

When $r \to +\infty$, the right hand side of (7) tends towards $+\infty$. In particular, $P(z) \neq 0$, if $|z| = r \geq A$ is sufficiently large. We choose A, such that $|P(z)| > |a_n|$, if $|z| = r \geq A$.

The real function $|P(z)|$ is continuous on the closed bounded disc $\{z \in \mathbb{C} \mid |z| \leq A\}$, so by one of the main theorems of continuous functions, $|P(z)|$ must have a *minimum* in this disc, so there exists a $z_0 \in \mathbb{C}$, where $|z_0| \leq A$, such that

$$|P(z_0)| \leq |P(z)| \qquad \text{for every } z \in \mathbb{C}, \text{ for which } |z| \leq A.$$

It follows in particular that

$$|P(z_0)| \leq |P(0)| = |a_n|,$$

and since $|P(z)| > |a_n|$ for $z| = A$, we conclude that $|z_0| < A$, so z_0 lies in the interior of this disc.

We shall prove that $P(z_0) = 0$. This is done contrariwise, i.e. we assume instead that $P(z_0) \neq 0$, and then we derive a contradiction.

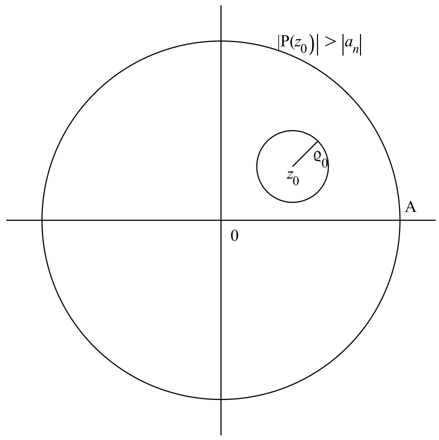

Figure 1: The discs in the proof of Theorem 1.3.1.

Choose ϱ_0, such that $0 < \varrho_0 < A - |z_0|$, cf. Figure 1. Then

$$B := \{z \in \mathbb{C} \mid |z - z_0| < \varrho_0\} \subset \{z \in \mathbb{C} \mid |z| \leq A\},$$

so B is an open subset of the closed disc defined by $|z| \leq A$. We put $z = z_0 + h$ for $z \in B$, so $|h| < \varrho_0$. Then

$$P(z_0 + h) = b_0 h^n + b_1 h^{n-1} + \cdots + b_{n-1} h + b_n, \qquad b_n = P(z_0) \neq 0 \text{ and } b_0 = a_0 \neq 0.$$

Choose $j \in \{0, 1, \ldots, n-1\}$, such that $b_j \neq 0$ and $b_k = 0$ for $k = j+1, \ldots, n-1$. Then, since $P(z_0) = b_n$,

$$P(z_0 + h) = P(z_0) + b_0 h^n + \cdots + b_j h^{n-j}, \qquad \text{where } b_j \neq 0.$$

We write h in polar coordinates,

$$h = \varrho\, e^{i\Theta}, \qquad \text{where } 0 < \varrho < \varrho_0 < A - |z_0|,$$

thus $|z_0 + h| < A$.

Also, write $P(z_0)$ and b_j in polar coordinates,

$$P(z_0) = |P(z_0)|\, e^{i\varphi} \qquad \text{and } b_j = |b_j|\, e^{i\psi}.$$

Then

$$P(z_0 + h) = |P(z_0)|\, e^{i\varphi} + |b_j|\, \varrho^{n-j}\, e^{i(\psi + (n-j)\Theta)} + b_{j-1}\, h^{n-j+1} + \cdots + b_0\, h^n.$$

Choose Θ, i.e. the angle of h, such that $\psi + (n-j)\Theta = \varphi + \pi$. Then

$$P(z_0 + h) = \left\{|P(z_0)| - |b_j|\, \varrho^{n-j}\right\} e^{i\varphi} + b_{j-1}\, h^{n-j+1} + \cdots + b_0\, h^n.$$

Then choose ϱ so small that also $|b_j|\ \varrho^{n-j} \le |P(z_0)|$, so we get the estimates

$$\begin{aligned} |P(z_0+h)| &\le |P(z_0)| - |b_j|\ \varrho^{n-j} + |b_{j-1}|\ \varrho^{n-j+1} + \cdots + |b_0|\ \varrho^n \\ &= |P(z_0)| - \varrho^{n-j}\left\{|b_j| - |b_{j-1}|\ \varrho - \cdots - |b_0|\ \varrho^j\right\}. \qquad (8) \end{aligned}$$

For some smaller $\varrho > 0$ we can obtain that also

$$|b_j| - |b_{j-1}|\ \varrho - \cdots - |b_0|\ \varrho^j > 0,$$

hence, for such a ϱ we have

$$|P(z_0+h)| < |P(z_0)|,$$

which is contradicting the assumption that $|P(z_0)|$ was a minimum. Hence, the other assumption that $P(z_0) \ne 0$ must be wrong, and we finally conclude that $P(z_0) = 0$. □

Corollary 1.3.1 The fundamental theorem of algebra. *Every polynomial $P(z)$ of degree $n \ge 1$ is, apart from the order of the factors, uniquely factorized in the following way,*

(9) $P(z) = a(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n), \qquad a \ne 0.$

In particular, if $P(z)$ has degree n, then $P(z)$ has precisely n roots (counted by their multiplicities).

It is obvious that Corollary 1.3.1 implies Theorem 1.3.1. We shall prove that Theorem 1.3.1 also implies Corollary 1.3.1, so the two results are indeed equivalent.

PROOF. 1) *Existence.* Assume that the polynomial $P(z)$ has degree $n \ge 1$. Then it follows from Theorem 1.3.1 that it has a root α_1, thus $P(\alpha_1) = 0$.

By a Taylor expansion from α_1, cf. Appendix 5.3, we get

$$P(z) = \frac{P'(\alpha_1)}{1!}(z-\alpha_1) + \frac{P''(\alpha_1)}{2!}(z-\alpha_1)^2 + \cdots + \frac{P^{(n)}(\alpha_1)}{n!}(z-\alpha_1)^n = (z-\alpha_1)\cdot P_1(z),$$

where

$$P_1(z) := \frac{P'(\alpha_1)}{1!} + \frac{P''(\alpha_2)}{2!}(z-\alpha_1) + \cdots + \frac{P^{(n)}(\alpha_1)}{n!}(z-\alpha_1)^{n-1}$$

is a polynomial of degree $n-1$.

When we apply the same method on $P_1(z)$ we get similarly

$$P_1(z) = (z-\alpha_2)\cdot P_2(z), \qquad \text{i.e.} \qquad P(z) = (z-\alpha_1)(z-\alpha_2)\,P_2(z),$$

where $P_2(z)$ is a polynomial of degree $n-2$.

We proceed in this way, and after n steps we have obtained (9).

2) *Uniqueness.* Assume that we have two representations of $P(z)$,

$$P(z) = a(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n) = b(z-\beta_1)(z-\beta_2)\cdots(z-\beta_m),$$

where $a \neq 0$ and $b \neq 0$.

The terms of highest order are az^n and bz^m, respectively, so it follows from Theorem 1.1.1, *The identity theorem*, than $n = m$ and $a = b$.

If we choose $z = \alpha_1$, the left hand side becomes zero, $P(\alpha_1) = 0$. Hence, α_1 must also be a root of the right hand side, i.e. α_1 must be one of the n numbers $\beta_1, \dots . \beta_n$. Changing indices, if necessary, we may assume that $\alpha_1 = \beta_1$.

When $z \neq \alpha_1$, it follows from a division by $a(z - \alpha_1)$ that

$$(z - \alpha_2) \cdots (z - \alpha_n) = (z - \beta_2) \cdots (z - \beta_n), \qquad z \in \mathbb{C} \setminus \{\alpha_1\}.$$

Due to the continuity this equation also holds for $z = \alpha_1$. Then proceed as above, i.e. the root α_2 on the left hand side must be one of the remaining numbers $\beta_2, \dots, \beta_n$, so $\alpha_2 = \beta_2$ after another change of index, etc.. Continue in this way n times, until we get the triviality $1 = 1$, and the corollary is proved. □

The n numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ of Corollary 1.3.1 are all roots of $P(z)$. They need not be mutually different; some of them may be multiple roots. In some situations it is better to preserve (9), i.e.

$$P(z) = a\,(z-\alpha_1)\,(z-\alpha_2)\cdots(z-\alpha_n)\,,$$

even if there are repetitions among the factors, but in most cases we prefer to collect identical factors, so

$$(10)\quad P(z) = a\,(z-\alpha_1)^{n_1}\cdots(z-\alpha_r)^{n_r}\,, \qquad n_1+\cdots+n_r = n,$$

where $n_j \in \mathbb{N}$ is called the *multiplicity* of the root α_j, and where $\alpha_1, \dots, \alpha_r$ in (10) are the *mutually different* roots of $P(z)$.

If $n_j = 1$, then the corresponding root α_j is called a *simple root*. If $n_j > 1$, we say that α_j is a *multiple root*. In case of $n_1 = 2$ we also call α_j a *double root*.

Finally, (9) is also true for constants $\neq 0$, i.e. for polynomials of degree 0, because there is no factor of degree 1 in this case. This corresponds to the obvious fact that a constant polynomial $\neq 0$ does not have any root.

1.4 Vieti's formulæ

Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ denote the n roots of the polynomial

$$(11)\quad P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0, \dots, a_n \in \mathbb{C}, \quad a_0 \neq 0.$$

Then also

$$\begin{aligned} P(z) &= a\,(z-\alpha_1)\,(z-\alpha_2)\cdots(z-\alpha_n) \\ (12)\qquad &= a_0 z^n - a_0\,(\alpha_1+\alpha_2+\cdots+\alpha_n)\,z^{n-1} + a_0\,(\alpha_1\alpha_2+\alpha_1\alpha_3+\cdots+\alpha_{n-1}\alpha_n)\,z^{n-2} \\ &\quad - \cdots + (-1)^n a_0 \alpha_1\alpha_2\cdots\alpha_n. \end{aligned}$$

When we identify the coefficients of the two representations (11) and (12) of $P(z)$, we get *Vieti's formulæ* in the n complex variables $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$(13)\quad \begin{cases} b_1 = \dfrac{a_1}{a_0} = -\left\{\alpha_1+\alpha_2+\cdots+\alpha_n\right\}, \\[2mm] b_2 = \dfrac{a_2}{a_0} = +\left\{\alpha_1\alpha_2+\alpha_1\alpha_3+\cdots+\alpha_{n-1}\alpha_n\right\} \\[2mm] b_3 = \dfrac{a_3}{a_0} = -\left\{\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\cdots+\alpha_{n-2}\alpha_{n-1}\alpha_n\right\}, \\[2mm] \qquad \dots \\[2mm] b_n = \dfrac{a_n}{a_0} = (-1)^n\,\alpha_1\alpha_2\cdots\alpha_n. \end{cases}$$

The formulæ of (13) are also called the *elementary s ymmetric polynomials* in the n complex variables $\alpha_1, \dots, \alpha_n$.

Using (13 we easily prove

Theorem 1.4.1 *Assume that the polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0,\, a_1,\, \dots,\, a_n \in \mathbb{R}, \quad a_0 \neq 0,$$

has real *coefficients. If*

(14) $a_1^2 - 2a_0 a_2 < 0$,

then $P(z)$ *has complex, non-real roots.*

Proof. It follows from (13) and (14) that

$$\begin{aligned} 0 \quad > \quad & \frac{a_1^2 - 2a_0a_2}{a_0^2} = \left\{\frac{a_1}{a_0}\right\}^2 - 2 \cdot \frac{a_2}{a_0} = b_1^2 - 2b_2 \\ = \quad & \{-(\alpha_1 + \alpha_2 + \cdots + \alpha_n)\}^2 - 2\{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n\} = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2. \end{aligned}$$

Since the sum of the squares of all roots is negative, they cannot all be real numbers. □

Remark 1.4.1 It follows actually from the proof that it suffices only to require that b_1 and b_2 are real, and then of course (14). The remaining coefficients may be complex. ◊

Example 1.4.1 Every polynomial of the special form

$$P(z) = a_0 z^n + 0 \cdot z^{n-1} + a_2 z^{n-2} + \cdots + a_n, \qquad a_0,\, a_2,\, \dots,\, a_n \in \mathbb{R},$$

where $a_1 = 0$, and *where* a_0 *and* a_2 *have the same sign*, i.e. $a_0 a_2 > 0$, must necessarily have complex roots. This follows immediately from Theorem 1.4.1, because then

$$a_1^2 - 2a_0a_2 = -2a_0a_2 < 0.$$

If in particular we choose $a_0 = a_2 = 1$ and $a_1 = 0$, then it follows that every polynomial of the form

$$z^n + z^{n-2} + a_3 z^{n-3} + \cdots + a_n, \qquad a_3, \dots, a_n \in \mathbb{C},$$

must have complex roots. One simple example is the well-known $z^2 + 1$. ◊

1.5 Rolle's theorems.

In this section we show some variants of the well-known Rolle's theorem, when it is restricted to polynomials. We shall first prove the general r esult.

Theorem 1.5.1 Rolle's theorem. *Assume that* $f(t)$ *is a real continuous function defined in a closed, bounded interval* $[a,b]$. *Furthermore, assume that* f *is continuously differentiable in the interior interval* $]a,b[$. *If* $f(a) = f(b) = 0$, *then there exists at least one point* $\xi \in]a,b[$, *such that* $f'(\xi) = 0$.

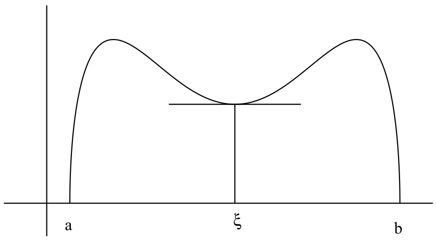

Figure 2: Rolle's theorem.

Proof. We shall give a proof which is very similar to the proof of Taylor's formula in Appendix 5.3. It follows from $f(a) = 0$ that

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t = \int_a^x f'(t)\,\mathrm{d}t \qquad \text{for } a \le x \le b.$$

We get in particular for $x = b$,

(15) $f(b) = 0 = \int_a^b f'(t = \,\mathrm{d}t.$

If $f'(t) \equiv 0$ in $[a, b]$, there is nothing to prove.

If $f'(t)$ is not identically 0, then $f'(t)$ must have both positive and negative values in $]a, b[$, since otherwise (15) could not be satisfied. By assumption, $f'(t)$ is continuous in $]a, b[$, so there must exist (at least) one $\xi \in]a, b[$, such that $f'(\xi) = 0$. □

We shall in the following choose $f = P$ as a real polynomial (i.e. of real coefficients) of degree n in the real variable $x \in \mathbb{R}$,

(16) $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \qquad a_0, \dots, a_n \in \mathbb{R}, \quad a_0 \neq 0, \quad x \in \mathbb{R}.$

Theorem 1.5.2 *Let $P(x)$ be gives as in (16), and let $a < b$ be two succeeding real zeros of $P(x)$. If the roots are counted according to their multiplicity, then the derivative $P'(x)$ has always an odd number of zeros in the interval $]a, b[$.*

Proof. We first assume that all roots of $P'(x)$ are simple. Since $P(x)$ does not have zeros in $]a, b[$, we may assume that $P(x) > 0$ in $]a, b[$.

The zeros of $P'(x)$ divide $]a, b[$ into open subintervals, in which $P'(x)$ is alternatively positive and negative. Since $P(x) > 0$, we must have $P'(x) > 0$ in the subinterval, which has $x = a$ as its left bound, and $P'(x) < 0$ for x in the subinterval, which has $x = b$ as its right bound.

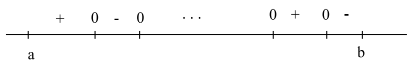

Figure 3: The sign of $P'(x)$ in $[a,b]$.

It follows that we have the following variation of the sign of $P'(x)$,

$$+,\, 0,\, -,\, 0,\, +,\, 0,\, -,\, \cdots,\, -,\, 0,\, +,\, 0,\, -.$$

We notice that between two successive plus signs there are always two zeros of $P'(x)$. This implies that the final zero to the right cannot be paired with another zero. The sequence ends with $+$, 0, $-$, and the number of zeros of $P'(x)$ must therefore be odd in this case.

Then allow that $P'(x)$ has a higher order of zero in the case of either $+$, 0, $-$, or $-$, 0, $+$. This implies that the order of the zero must necessarily be odd so in the count of the zeros we replace 1 by some odd number. This does not change the conclusion of the theorem.

Finally, if the variation of sign of $P'(x)$ is either $+$, 0, $+$, or $-$, 0, $-$, then the zero must be of *even order*. In this case we replace the variation $+$, 0, $+$ by $+$ alone, and the variation $-$, 0, $-$ by $-$ alone, where we in both cases add an *even* number to the number of zeros. This process will not change the conclusion either, and the theorem is proved. □

Theorem 1.5.3 *Let $P(x)$ be the polynomial (16), and let $\alpha < \beta$ be two successive real zeros of the derivative $P'(x)$.*

1) If $P(\alpha) \cdot P(\beta) > 0$, then $P(x)$ has no zero in $]\alpha, \beta[$.

2) If $P(\alpha) \cdot P(\beta) < 0$, then $P(x)$ has precisely one zero in $]\alpha, \beta[$.

PROOF. 1) Assume that $P(\alpha)$, $P(\beta) > 0$, and that there is a $\xi \in]\alpha, \beta[$, such that $P(\xi) = 0$, so we aim at getting to a contradiction.

It follows from the assumption that there exists a $\gamma \in]\alpha, \xi[$, such that $P'(\gamma) < 0$, and a $\mu \in]\xi, \beta[$, such that $P'(\mu) > 0$. Since $P'(x)$ is continuous, there also exists a $\nu \in]\gamma, \mu[$, such that $P'(\nu) = 0$. Then we have also $\nu \in]\alpha, \beta[$, which contradicts the assumption that α and β are successive zeros of $P'(x)$. Hence, $P(x) \neq 0$ for every $x \in]\alpha, \beta[$.

2) Assume that $P(\alpha) \cdot P(\beta) < 0$, so $P(\alpha)$ and $P(\beta)$ have different signs. The continuity of $P(x)$ implies that there exists a zero $\xi \in]\alpha, \beta[$ for $P(x)$, thus $P(\xi) = 0$.

Assume that there exists another zero in the interval, e.g. $\mu \in]\alpha, \xi[$, such that $P(\mu) = P(\xi) = 0$. Then Theorem 1.5.2 implies that there exists another zero $\nu \in]\mu, \xi[$ of $P'(x)$. Since $\nu \in]\alpha, \beta[$, this contradicts the assumption that α and β are two successive zeros of $P'(x)$. □

Corollary 1.5.1 *Let $P(x)$ be the polynomial (16). If the derivative $P'(x)$ has p complex roots (i.e. non-real roots), then the polynomial $P(x)$ itself has at least p complex roots.*

PROOF. The degree of $P(x)$ is n, so $P'(x)$ has degree $n - 1$, and $P'(x)$ has by assumption $n - 1 - p$ real roots.

It follows from Theorem 1.5.2 and Theorem 1.5.3 that the polynomial $P(x)$ has *at most* one extra real root, thus at most $n - p$ real roots of $P(x)$, and hence at least $n - (n - p) = p$ complex roots of $P(x)$. □

Corollary 1.5.2 *Let $P(x)$ be the polynomial (16). The number of complex roots of the derivatives $P^{(j)}(x)$, $j = 0, 1, \dots, n$, is a weakly decreasing function in j.*
Here we have put $P^{(0)}(x) := P(x)$.

Proof. This follows immediately by successive applications of Corollary 1.5.1. □

Example 1.5.1 If we can find the roots of the derivative $P'(x)$ of a polynomial of real coefficients, then Rolle's theorems can be applied to find where the real roots of $P(x)$ are situated on the real axis. This is always possible, if $P(x)$ has degree 3, or if $P(x)$ has degree 4, where the term of degree 3 is missing. The latter condition can always be obtained by using a translation, cf. Section 1.2.1.

We shall illustrate this in the following by some examples.

1) Consider the polynomial $P(x) = x^3 - 2x - 5$. Then $P'(x) = 3x^2 - 2$, which has the roots $\pm\frac{\sqrt{6}}{3}$. We find by insertion the following variation of sign of $P(x)$,

$\to -\infty$	$-\frac{\sqrt{6}}{3}$	$+\frac{\sqrt{6}}{3}$	$\to +\infty$
$-$	$-$	$-$	$+\infty$

from which we conclude that there is only one real root and that it is $> \frac{\sqrt{6}}{3}$.

2) The polynomial $P(x) = x^3 + x^2 - 5x + 3$ has the derivative $P'(x) = 3x^2 + 2x - 5$, which has the roots $-\frac{5}{3}$ and $+1$. We get by insertion the following variation of sign of $P(x)$.

$\to -\infty$	$-\frac{5}{3}$	1	$\to +\infty$
$-$	$+$	0	$+$

from which follows that $x = 1$ is a double root and that there also is a real root $< -\frac{5}{3}$. This is easy to find by Vieti's formulæ, because the sum of the roots, $1 + 1 + \alpha$, must be $-a_1 = -1$, hence $\alpha = -3$.

3) Finally, let $P(x) = x^4 + 12x^2 + 96x - 12$. Then

$$a_1^2 - 2a_0a_2 = -24 < 0,$$

so it follows from Theorem 1.4.1 that we have at least two complex roots. Since $P(0) = -12 < 0$ and $P(x) \to +\infty$ for $x \to \pm\infty$, we must also have at least two real roots. Finally, the total number of roots is 4 by the *Fundamental theorem of algebra*, so we conclude that we have two real and two complex roots.

♢

2 Some solution formulæ of roots of polynomials

There are very few exact solution formulæ of a polynomial equation $P(z) = 0$. The reason is of course *Niels Henrik Abel's* result that $P(z) = 0$ in general cannot be solved by root signs, if $\deg P \geq 5$. (It may of course occasionally be solvable). We shall in this chapter give the exact solution formulæ in the cases of the *binomial equation* and the *equation of second degree.*

There exist exact solution formulæ for *equations of third and fourth degree*, but these are absolutely not of any reasonable computational value, so although they are classical, we shall not give them here.

Finally, we give some useful partial results, assuming either that we have a rational root or a multiple root.

2.1 The binomial equation.

The simplest possible non-trivial polynomial equation is the *binomial equation* in polar coordinates,

$$z^n = a = r \cdot \exp(i\{\Theta + 2p\pi\}), \qquad r \geq 0 \text{ and } p \in \mathbb{Z}.$$

Its n roots are given by

$$(17) \quad z = \sqrt[n]{r} \cdot \exp\left(i\,\frac{\Theta + 2p\pi}{n}\right) = \sqrt[n]{r} \cdot \left\{\cos\left(\frac{\Theta + 2p\pi}{n}\right) + i \cdot \sin\left(\frac{\Theta + 2p\pi}{n}\right)\right\}, \qquad p = 0, 1, \dots, n-1.$$

That everyone of the n numbers of (17) are roots, follows by insertion. That they are mutually different for $r > 0$ follows from the fact that they all lie on a circle of radius $\sqrt[n]{r}$ with the angle $\dfrac{2\pi}{n}$ between two adjacent roots, cf. Figure 4.

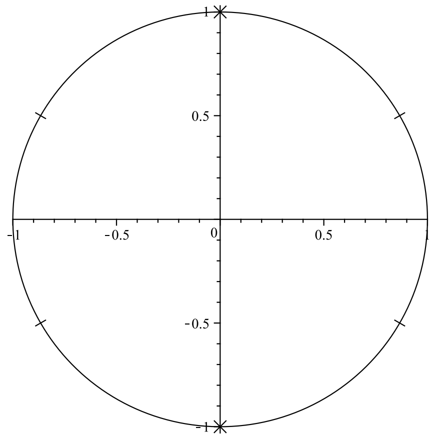

Figure 4: The six roots of $z^6 = -1$.

Finally, it follows from the *Fundamental theorem of algebra*, cf. Corollary 1.3.1, that the equation has precisely n roots.

The geometry of (17), cf. Figure 4, can be exploited in the following way: Rewrite (17) as follows,

$$z = \sqrt[n]{r} \cdot \exp\left(i\, \frac{\Theta}{n} \right) \cdot \exp\left(\frac{2ip\pi}{n} \right) = z_0 \cdot \exp\left(\frac{2ip\pi}{n} \right), \qquad p = 0, 1, \dots, n-1,$$

where $z_0 = \sqrt[n]{r} \cdot \exp\left(i\, \frac{\Theta}{n} \right)$ is anyone of the n possible solutions. Then the other roots are found, when we successively multiply z_0 by

$$\exp\left(\frac{2i\pi}{n} \right) = \cos\left(\frac{2\pi}{n} \right) + i \cdot \sin\left(\frac{2\pi}{n} \right)$$

$1, 2, \dots, n-1$ times.

Example 2.1.1 We shall solve the binomial equation

$$z^3 = -2 - 2i.$$

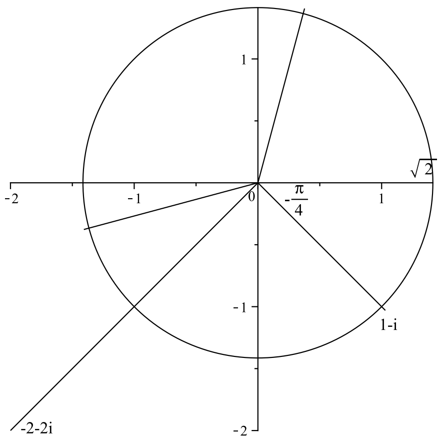

Figure 5: The three roots of $z^3 = -2 - 2i$.

Since $|a| = |-2-2i| = 2\sqrt{2} = \left\{\sqrt{2}\right\}^3$, it follows that all three roots lie on a circle of centre 0 and radius $\sqrt{2}$.

Then it follows from $\Theta = \text{Arg } a = \text{Arg}(-2-2i) = -\dfrac{3\pi}{4}$ that one of the solutions is given by

$$z_1 = \sqrt{2} \cdot \exp\left(-i\,\frac{\pi}{4}\right) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\,\sin\left(-\frac{\pi}{4}\right)\right\} = 1 - i.$$

The remaining two roots are either found geometrically, cf. Figure 5, or by multiplication by $\exp\left(\dfrac{2\pi i}{3}\right)$ and $\exp\left(\dfrac{4\pi i}{3}\right)$, resp., thus

$$z_2 = z_1 \exp\left(\frac{2\pi i}{3}\right) = (1-i)\left\{-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right\} = \frac{\sqrt{3}-1}{2} + i\,\frac{\sqrt{3}+1}{2},$$

and

$$z_3 = z_1 \exp\left(\frac{4\pi i}{3}\right) = (1-i)\left\{-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right\} = -\frac{\sqrt{3}+1}{2} - i\,\frac{\sqrt{3}-1}{2}. \qquad \Diamond$$

It is customary in the solutions to give the exact values of $\cos\left(\dfrac{2\pi}{n}\right)$ and $\sin\left(\dfrac{2\pi}{n}\right)$ for $n = 2, 3, 4, 6, 8, 12$, cf. Table 1, where we have also added $n = 5$ and $n = 10$ for completeness. It should be mentioned that such exact expressions using square roots do *not* exist for $n = 7, 9, 11, 13, 14$, but again for $n = 15, 16, 17$, where, however, they are too complicated to have any practical use.

n	$\cos\left(\frac{2\pi}{n}\right)$	$\sin\left(\frac{2\pi}{n}\right)$
2	$\cos(\pi) = -1$	$\sin(\pi) = 0$
3	$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$	$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$
4	$\cos\left(\frac{\pi}{2}\right) = 0$	$\sin\left(\frac{\pi}{2}\right) = 1$
5	$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$	$\sin\left(\frac{2\pi}{5}\right) = \frac{\sqrt{10+2\sqrt{5}}}{4}$
6	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
8	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
10	$\cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5}+1}{4}$	$\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4}$
12	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Table 1: Table of some exact values of $\cos\left(\frac{2\pi}{n}\right)$ and $\sin\left(\frac{2\pi}{n}\right)$.

If in particular the exponent is $n = 2$ in the binomial equation, then it is possible to give exact solution formulæ in the rectangular coordinates without using polar coordinates.

Theorem 2.1.1 *Given the equation*

$$z^2 = (x+iy)^2 = a = \alpha + i\beta.$$

1) If $\beta > 0$, then the solutions are

$$z = \pm\left\{\sqrt{\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}} + i\sqrt{\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}}\right\}.$$

2) If $\beta < 0$, then the solutions are

$$z = \pm\left\{\sqrt{\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}} - i\sqrt{\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}}\right\}.$$

3) If $\beta = 0$, then the solutions are

$$z = \pm\left\{\sqrt{\frac{|\alpha|+\alpha}{2}} + i\sqrt{\frac{|\alpha|-\alpha}{2}}\right\}.$$

In all three cases we define the square root of a positive number as a positive number.

PROOF. It suffices to prove 1), because 2) and 3) are proved similarly. We shall clearly only check the candidates of the solutions. The plus/minus sign gives us two possible solutions, and a squaring finally gives

$$\left(\pm\left\{\sqrt{\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}}+i\sqrt{\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}}\right\}\right)^2$$

$$=\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}-\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}+2i\sqrt{\frac{\alpha^2+\beta^2-\alpha^2}{4}}=\alpha+i\beta=a,$$

and 1) is proved. □

Example 2.1.2 In practice, Theorem 2.1.1 rarely gives “nice” solutions, though it occurs in special cases. Clearly, the equation

$$z^2 = 3+4i \qquad \text{has the solutions } \pm(2+i),$$

and the equation

$$z^2 = 5+12i \qquad \text{has the solutions } \pm(3+2i).$$

In general, the formulæ of Theorem 2.1.1 become messy. ◊

2.2 The equation of second degree.

The usual solution formula of the polynomial equation of second degree with real coefficients is still valid, when the coefficients are complex. The only modification is that we shall choose one of the two possibilities of the square root $\sqrt{b^2-4ac}$, which is one of the solutions of the binomial equation $z^2 = b^2-4ac$ of degree 2.

Theorem 2.2.1 *The solutions of the polynomial equation of second degree*

$$azr + bz + c = 0,$$

where $a \in \mathbb{C}\setminus\{0\}$*, and* b*,* $c \in \mathbb{C}$ *are constants, are given by*

$$(18) \quad z = \frac{1}{2a}\left\{-b \pm \sqrt{b^2-4ac}\right\}.$$

PROOF. Let $\sqrt{b^2-4ac}$ denote one of the two solutions of the binomial equation $w^2 = b^2-4ac$. The number of candidates of the solutions is two, so it suffices to check the candidates in the original equation. A rearrangement of (18) gives

$$z+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a},$$

hence, by a squaring,

$$z^2+\frac{b}{a}\,z+\frac{b^2}{4a^2} = \frac{b^2-4ac}{4a^2} = \frac{b^2}{4a^2}-\frac{c}{a},$$

which is reduced to

$$z^2 + \frac{b}{a}\,z + \frac{c}{a} = 0.$$

Finally, we multiply by a to get the original equation. □

Example 2.2.1 The equation $z^2 - 2z - 2 - 4i = 0$ has the solutions

$$z = \frac{1}{2}\left\{2 \pm \sqrt{4 - 4(-2-4i)}\right\} = 1 \pm \sqrt{1 + (2+4i)} = 1 \pm \sqrt{3+4i} = 1 \pm (2+i) = \begin{cases} \;\;3+i, \\ -1-i, \end{cases}$$

where we have used that $\pm\sqrt{3+4i} = \pm(2+i)$, cf. Example 2.1.2. A check shows that

$$\alpha_1 + \alpha_2 = 2 = -a_1 \qquad \text{and} \qquad \alpha_1\alpha_2 = -3 + 1 - 4i = -2 - 4i = a_2. \qquad \Diamond$$

2.3 Rational roots.

If all coefficients $a_0, \dots, a_n \in \mathbb{Q}$ of the polynomial $P(z)$ are rational numbers, then there is a limited set, which is easy to find, of possible rational roots, where we just have to check each one to see, if it indeed *is* a root of $P(z)$. first notice that we showed in Section 1.2.2 that if $P(z)$ has only rational coefficients, then we could find an equivalent polynomial with only integer coefficients, and even obtain that $b_0 = 1$. We shall not need this stronger result here, so in the following it is sufficient to assume that all coefficients are integers.

We introduce the following notation. Assume that p and $q \in \mathbb{Z}$, where $q \neq 0$. We say that q is a *divisor* of p and write $q|p$, if there is an $r \in \mathbb{Z}$, such that

$$p = q \cdot r.$$

Theorem 2.3.1 *Assume that the polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0,\, a_1,\, \dots,\, a_n \in \mathbb{Z},$$

has integer coefficients, where $a_0 \neq 0$.
Assume that $z = \dfrac{p}{q} \in \mathbb{Q}$ *is a rational root of* $P(z)$, *where* $p \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *do not have other common divisors from* $\mathbb{Z}$ *than* ± 1. *Then*

$$p|a_n \qquad \text{and} \qquad q|a_0.$$

PROOF. We assume that $P\left(\dfrac{p}{q}\right) = 0$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ do not have other common divisors than ± 1. Then

$$0 = q^n\, P\left(\frac{p}{q}\right) = a_0 p^n + a_1 p^{n-1} q + \cdots + a_{n-1} p q^{n-1} + a_n q^n,$$

hence by some rearrangements,

$$p\left\{a_0 p^{n-1} + a_1 p^{n-2} q + \cdots + a_{n-1} q^{n-1}\right\} = -a_n q^n,$$

and

$$q\left\{a_1 p^{n-1} + a_2 p^{n-2} q + \cdots + a_n q^{n-1}\right\} = -a_0 p^n.$$

Since $a_0, \dots, a_n$, p, $q \in \mathbb{Z}$, and p and q have only the trivial common divisors, it follows from the first equation that $p|a_n$, and from the second one that $q|a_0$. $\square$

In practice Theorem 2.3.1 is applied in the following way. Assume that the polynomial equation $P(z) = 0$ has only integer coefficients. Let $\{q_1, \dots, q_\ell\}$ be all mutually different (positive) divisors in a_0, and let $\{p_1, \dots, p_k\}$ be all mutually different (positive) divisors in a_n. Then the *possible* rational roots (if any) must belong to the set

$$\left\{ \pm \frac{p_i}{q_j} \;\middle|\; i = 1, \dots, k \text{ and } j = 1, \dots, \ell \right\}.$$

Finally, we check all these at most $2k\ell$ possibilities.

Example 2.3.1 We shall solve the equation

$$P(z) = z^3 + 3z - 4 = 0$$

of integer coefficients. First notice that since $a_1 = 0$ and $a_0 \cdot a_2 > 0$, it follows from Example 1.4.1 that we must have two complex conjugated roots, and there is precisely one real root. If this root is rational, it must be one of the elements of the set

$$\{\pm 1, \pm 2, \pm 4\},$$

because $a_0 = 1$ and $a_n = -4$. It follows by inspection that $z = 1$ is a root. Then we get by a division,

$$P(z) = z^3 + 3z - 4 = (z-1)\left(z^2 + z + 4\right).$$

Solving $z^2 + z + 4 = 0$ we get the remaining two roots. Summing up, the three roots are

$$z_1 = 1, \qquad z_2 = -\frac{1}{2} + i\,\frac{\sqrt{15}}{2} \quad \text{and} \quad z_3 = -\frac{1}{2} - i\,\frac{\sqrt{15}}{2}.$$

The equation is of third degree, so it could in principle be solved by *Cardano's formula*, which has been omitted here. We shall here without details show why this is not done. In fact, if we instead of the above apply Cardano's formula, then we get after some very long and tedious computations that the three roots are given by

$$\tilde{z}_1 = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}},$$

$$\tilde{z}_2 = \left\{-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right\}\sqrt[3]{2+\sqrt{5}} + \left\{-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right\}\sqrt[3]{2-\sqrt{5}},$$

$$\tilde{z}_3 = \left\{-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right\}\sqrt[3]{2+\sqrt{5}} + \left\{-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right\}\sqrt[3]{2-\sqrt{5}}.$$

It is far from obvious that $\{z_1, z_2, z_3\}$ and $\{\tilde{z}_1, \tilde{z}_2, \tilde{z}_3\}$ describe the same set of points.

Since solutions by *Cardano's formula* usually have the complicated structure of $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$ above, we have decided here *not* to bring *Cardano's formula* to avoid that the reader would be tempted to use it. $\Diamond$

Remark 2.3.1 Theorem 2.3.1 does *not* assure that a polynomial of integer coefficients has rational roots. In case of $P(z) = z^2 + 1$ we have $a_0 = 1$, $a_1 = 0$ and $a_2 = 1$, so the *candidates* of rational roots are ± 1. However, none of these is a root, the roots being the complex numbers $\pm i$. $\Diamond$

Remark 2.3.2 Assume that

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_0,\, a_1,\, \dots,\, a_n \in \mathbb{C}, \quad a_0 \neq 0.$$

Assume furthermore that every coefficient $a_j \in \mathbb{C}$ has rational real and imaginary parts,

$$a_j = \alpha_j + i\,\beta_j, \qquad \alpha_j,\, \beta_j \in \mathbb{Q}, \qquad j = 0,\, 1,\, \dots,\, n.$$

Then the method above of finding *rational roots* $z_k \in \mathbb{Q}$ still applies. In fact, if

$$P\left(\frac{p}{q}\right) = 0, \qquad p \in \mathbb{Z} \text{ and } q \in \mathbb{N},$$

where p and q only have the common factors ± 1, then we get by splitting $q^n \, P\left(\frac{p}{q}\right) = 0$ into the real and imaginary parts that

$$\alpha_0 p^n + \alpha_1 p^{n-1} q + \cdots + \alpha_{n-1} p q^{n-1} + \alpha_n q^n = 0,$$

and

$$\beta_0 p^n + \beta_1 p^{n-1} q + \cdots + \beta_{n-1} p q^{n-1} + \beta_n q^n = 0.$$

Multiplying by some constant from $\mathbb{N}$ we may assume that $\alpha_0, \dots, \alpha_n$ and $\beta_0, \dots, \beta_n$ are all integers, so it follows from Theorem 2.3.1 that

$$p|\alpha_n \text{ and } p|\beta_n \qquad \text{as well as} \qquad q|\alpha_0 \text{ and } q|\beta_0.$$

Then it is easy to find all *candidates* of a rational root and then check it in the original equation. $\Diamond$

Remark 2.3.3 Whenever the task is to find the roots of a polynomial it will always be a good strategy first to check if the methods of this section apply. $\Diamond$

Example 2.3.2 We shall find all roots – if possible – of the polynomial

$$P(x) = x^6 - 10x^5 + 40x^4 - 82x^3 + 91x^2 - 52x + 12$$

of integer coefficients.

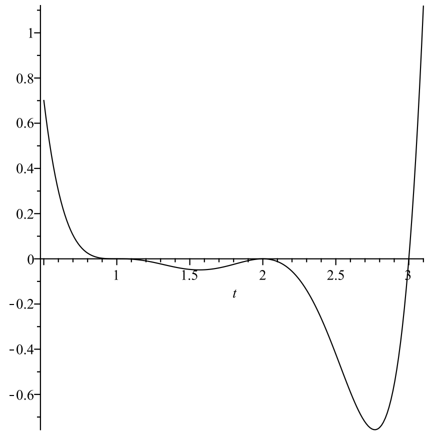

Figure 6: The graph of $P(x)$, which suggests that 1, 2, 3 are roots, though this is not a proof in itself.

Here, $a_0 = 1$ and $a_6 = 12 = 1 \cdot 2 \cdot 2 \cdot 3$, so by Theorem 2.3.1 the *candidates* of possible rational roots are

$$\pm 1,\ \pm 2,\ \pm 3, \pm 4, \pm 6, \pm 12.$$

Then notice that $P(x) > 0$ for $x \leq 0$, so every real root must be positive. This leaves us the possibilities

$$1,\ 2,\ 3,\ 4,\ 6,\ 12.$$

We get by insertion,

$$P(1) = 0, \quad P(2) = 0, \quad P(3) = 0, \quad P(4) = 108, \quad P(6) = 6,000, \quad P(12) = 1,197,900,$$

so $z = 1,\ 2,\ 3$ are roots, and

$$(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$$

is a divisor in $P(x)$, and we get by a division,

$$P(x) = (x-1)(x-2)(x-3)\left(x^3 - 4x^2 + 5x - 2\right).$$

The candidates of rational roots of the latter factor are $x = 1,\ 2$. Notice that they necessarily must be included in the previous set. We get by insertion,

$$1^3 - 4 \cdot 1^2 + 5 \cdot 1 - 2 = 0 \quad \text{and} \quad 2^3 - 4 \cdot 2^2 + 5 \cdot 2 - 2 = 8 - 16 + 10 - 2 = 0,$$

so both $x = 1$ and $x = 2$ are roots in the latter factor. A division by $(x-1)(x-2) = x^2 - 3x + 2$ gives

$$\begin{aligned} P(x) &= (x-1)(x-2)(x-3)\left(x^3 - 4x^2 + 5x - 2\right) \\ &= (x-1)(x-2)(x-3)\{(x-1)(x-2)\cdot(x-1)\} \\ &= (x-1)^3(x-2)^2(x-3), \end{aligned}$$

and the six roots are

$$1, \quad 1, \quad 1, \quad 2, \quad 2, \quad 3.$$

This example has been chosen as simple as possible. In general the computations are not that easy. At the same time it is illustrated that we get more information of $P(x)$ in the factorized form

$$P(x) = (x-1)^3(x-2)^2(x-3),$$

than in the original form

$$P(x) = x^6 - 10x^5 + 40x^4 - 82x^3 + 91x^2 - 52x + 12,$$

so a rule of thumb is to keep a factorization of a polynomial as long as possible. $\Diamond$

Example 2.3.3 We shall find all roots of the polynomial

$$P(x) = x^5 + 2x^4 - 2x^3 + 2x^2 - 3x$$

of integer coefficients.

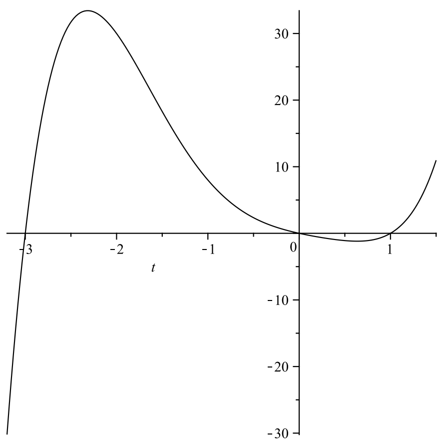

Figure 7: The graph of $P(x)$, which suggests that 0, 1, -3 are roots.

Obviously, $x = 0$ is a root, and we have $P(x) = x \cdot F(x)$, where

$$F(x) = x^4 + 2x^3 - 2x^2 + 2x - 3$$

is a polynomial of integer coefficients.

Here, $a_0 = 1$ and $a_4 = 3$, so it follows from Theorem 2.3.1 that the *candidates* of possible rational roots are ± 1, ± 3. We get by insertion,

$$F(1) = 1 + 2 - 2 + 2 - 3 = 0, \qquad F(-1) = 1 - 2 - 2 - 2 - 3 = -8 \neq 0,$$

$$F(3) = 81054.18 + 6 - 3 \neq 0, \qquad F(-3) = 81 - 54 - 18 - 6 - 3 = 0,$$

and we have proved, what was indicated on Figure 7 that $x = 0$, $x = 1$ and $x = -3$ are roots of $P(x)$.

A division by $(x-1)(x+3) = x^2 + 2x - 3$ gives the following factorization of $P(x)$,

$$P(x) = x \cdot F(x) = x \cdot \left(x^2 + 2x - 3\right)\left(x^2 + 1\right) = x(x-1)(x+3)(x-i)(x+i),$$

and the five complex roots are

$$0, \quad 1, \quad -3, \quad i, \quad -i. \qquad \diamond$$

2.4 The Euclidean algorithm.

We shall here shortly describe how we divide a polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_0, \dots, a_n \in \mathbb{C}, \qquad a_0 \neq 0,$$

by another one,

$$Q(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m, \qquad b_0, \dots, b_m \in \mathbb{C}, \qquad b_0 \neq 0,$$

with remainder term, where we assume that $m \leq n$. Usually even $m < n$ in this division algorithm, and the remainder term is a polynomial $R(z)$ of degree $< \deg Q = m$.

If one does not use a computer, the best way is to use a so-called "gallows construction",

$$\underline{Q(z)|}\, P(z)\, \underline{|\cdots}.$$

For the given polynomials above the construction starts in the following way,

$$\begin{array}{ll}
\underline{b_0 z^m + b_1 z^{m-1} + \cdots + b_m |} & \underline{a_0 z^n + a_1 z^{n-1} + \cdots + a_n | \,(a_0/b_0)\, z^{n-m}} \\
(a_0/b_0)\, z^{n-m} \cdot Q(z) = & \underline{a_0 z^n + a_0 \cdot \dfrac{b_1}{b_0}\, z^{n-1} + \cdots} \\
\text{Subtraction gives:} & \left(a_1 - a_0 \cdot \dfrac{b_1}{b_0}\right) z^{n-1} + \cdots\,,
\end{array}$$

and then proceed similarly, until the bottom polynomial (the remainder term) has lower degree than $Q(z)$. Since the degree is lowered by at least 1 at each step, this construction contains at most $n-m+1$ steps.

Example 2.4.1 We shall find a and b such that x^2+x+1 is a divisor of $x^4+3x^3+5x^2+ax+b$.

It follows by the division algorithm that

$$\begin{array}{l}
\underline{x^2+x+1}|x^4+3x^3+5x^2+ax+b|\underline{x^2+2x+2} \\
\qquad\qquad\quad \underline{x^4+\ x^3+\ x^2+\qquad\quad} \\
\qquad\qquad\quad 2x^2+4x^2+ax+b \\
\qquad\qquad\quad \underline{2x^3+2x^2+2x} \\
\qquad\qquad\qquad\quad 2x^2+(a-2)x+b \\
\qquad\qquad\qquad\quad \underline{2x^2+2x+2} \\
\qquad\qquad\qquad\quad (a-4)x+b-2
\end{array}$$

If x^2+x+1 is a divisor, then the remainder term must be 0, thus $a=4$ and $b=2$, and we get

$$x^4+3x^3+5x^2+4x+2=\left(x^2+x+1\right)\left(x^2+2x+2\right).$$

The roots are

$$x=-\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2} \quad \text{and} \quad x=-1\pm i. \qquad \Diamond$$

It is easy to understand the principle of the division algorithm above. However, if the coefficients are not integers, the computations become usually very hard and tedious. In some cases one is only interested in a *constant* times the remainder and not in the quotient itself. If the coefficients are integers, then the third line in the gallows will usually have rational coefficients and not integers. Multiply this third line by a constant, such that this new fourth line has integers as coefficients and proceed in this way. Of course, in this case it does not make sense to indicate the quotient. Only the remainder term times a convenient constant is here of interest.

In theoretical considerations one argues on the pure division algorithm as in the *Euclidean algorithm* described in the following.

Let $P_1(z)$ and $P_2(z)$ be two polynomials, where $\deg P_2 \le \deg P_1$. Then by the division algorithm we get a unique quotient $Q_1(z)$ and a unique remainder term $P_3(z)$, such that

$$P_1(z) = P_2(z) \cdot Q_1(z) + P_3(z), \qquad \deg P_3 < \deg P_2.$$

Then repeat this process with $P_2(z)$ and $P_3(z)$, thus

$$P_2(z) = P_3(z) \cdot Q_2(z) + P_4(z), \qquad \deg P_4 < \deg P_3,$$

where $Q_2(z)$ and $P_4(z)$ again are uniquely determined polynomials.

Proceed in this way, until we obtain an equation, in which the remainder is the zero polynomial. Thus,

$$(19) \quad \begin{cases} P_1 = P_2 Q_1 + P_3, & \deg P_3 < \deg P_2, \\ P_2 = P_3 Q_2 + P_4, & \deg P_4 < \deg P_3, \\ \quad \cdots & \quad \cdots \\ P_{m-2} = P_{m-1} Q_{m-2} + P_m, & \deg P_m < \deg P_{m-1}, \\ P_{m-1} = P_m Q_{m-1}. \end{cases}$$

It follows from the first line of (19) that a common divisor of P_1 and P_2 must also be a divisor of P_3. Then the second line of (19) implies that this common divisor is also a divisor of P_4, etc., so it must be a divisor of P_m.

Conversely, it follows immediately from (19) that $P_m | P_{m-1}$, hence also $P_m | P_{m-2}$, etc. so P_m must be a divisor in both P_1 and P_2.

Hence we have proved

Theorem 2.4.1 *Given two polynomials $P_1(z)$ and $P_2(z)$. There exists precisely one* normalized *polynomial $D(z)$, i.e. the coefficient of the term of highest degree in $D(z)$ is 1, such that all common divisors of $P_1(z)$ and $P_2(z)$ are precisely all divisors of $D(z)$.*
We call $D(z)$ the greatest common divisor *of $P_1(z)$ and $P_2(z)$, and we denote it by*

$$D = (P_1, P_2)\,.$$

One usually finds $D(z) = P_m(z)$ by means of the Euclidean algorithm *(19) above.*

Remark 2.4.1 If we are only interested in the remainder polynomials P_k of (19) and all coefficients are rational, it may be convenient to apply the modified division algorithm described after Example 2.4.1. $\Diamond$

Corollary 2.4.1 *Let* $(P_1, P_2) = D$. *If* Q *is a* normalized *polynomial, then*

$$(P_1 \cdot Q, P_2 \cdot Q) = D \cdot Q.$$

Proof. This follows from the fact that the Euclidean algorithm for $P_1 Q$ and $P_2 Q$ is obtained from the Euclidean algorithm for P_1 and P_2 by multiplying every division equation by Q. □

Example 2.4.2 We shall find the greatest common divisor of the two polynomials

$$x^4 - 3x^3 + 5x^2 + x - 4 \qquad \text{and} \qquad x^5 + 7x^4 - 8x^3 + 5x^2 - 4x - 1.$$

This example will demonstrate that without a computer the Euclidean algorithm gives some very tough and tedious computation to carry out by hand. We shall therefore not follow (19) strictly, but use some shortcuts, whenever possible.

The problem can in fact be reduced, if we start by checking the possible rational roots ± 1. Of these only $x = 1$ is a root of the latter polynomial, and $x = 1$ is also a root of the former polynomial, we may reduce the problem considerably by a division by $x - 1$.

We shall in the following assume that we have *not* noticed that $x = 1$ is a common root of the two polynomials. Then by the division algorithm,

$$\begin{array}{l|l|l}
\underline{x^4 - 3x^3 + 5x^2 + x - 4} & x^5 + 7x^4 - 8x^3 + 5x^2 - 4x - 1 & \underline{x + 10} \\
 & \underline{x^5 - 3x^4 + 5x^3 + \;\; x^2 - 4x} & \\
 & \quad 10x^4 - 13x^3 + 4x^2 \qquad\qquad - 1 & \\
 & \quad \underline{10x^4 - 30x^3 + 50x^2 + 10x - 40} & \\
 & \qquad\qquad 17x^3 - 46x^2 - 10x + 39 &
\end{array}$$

Then multiply the first divisor $x^4 - 3x^3 + 5x^2 + x - 4$ by 17 and divide the result by the remainder term $17x^3 - 46x^2 - 10x + 39$,

$$\begin{array}{l|l|l}
\underline{17x^3 - 46x^2 - 10x + 39} & 17x^4 - 51x^3 + 85x^2 + 17x - 68 & \underline{x} \\
 & \underline{17x^4 - 46x^3 - 10x^2 + 39} & \\
 & \qquad -5x^3 + 95x^2 - 22x - 68 &
\end{array}$$

The next division is not nice, so we use the modified algorithm, multiplying the remainder by -17, before we proceed,

$$\begin{array}{l|l|l}
\underline{17x^3 - 46x^2 - 10x + 39} & 85x^3 - 1615x^2 + 374x + 1156 & \underline{5} \\
 & \underline{85x^3 - \;\; 230x^2 - \;\; 50x + \;\; 195} & \\
 & \qquad -1385x^2 + 424x + \;\; 961 &
\end{array}$$

We change the sign of the remainder term, and the theory then tells us that we shall divide $1385x^2 - 424x - 961$ into some multiple of $17x^3 - 46x^2 - 10x + 39$, so the factor should be chosen as 1385. This does not look too nice, so instead we notice that $1385x^2 - 424x - 961$ is a polynomial of second degree with a known solution formula. The roots are

$$1 \qquad \text{and} \qquad -\frac{961}{1385},$$

so

$$1385x^2 - 424x - 961 = (x-1)(1385x+961).$$

It follows by insertion that $x = 1$ is also a root of $17x^3 - 46x^2 - 10x + 39$, and it follows by the division algorithm that

$$\begin{array}{l}
\underline{x-1}|17x^3 - 46x^2 - 10x + 39|\underline{17x^2 - 29x - 39} \\
\quad\quad \underline{17x^3 - 17x^2\quad\quad} \\
\quad\quad\quad\quad -29x^2 - 10x + 39 \\
\quad\quad\quad\quad \underline{-29x^2 + 29x\quad} \\
\quad\quad\quad\quad\quad\quad -39x + 39 \\
\quad\quad\quad\quad\quad\quad \underline{-39x + 39} \\
\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}$$

where the remainder is 0, so

$$17x^3 - 46x^2 - 10x + 39 = (x-1)\left(17x^2 - 29x - 39\right).$$

The possible rational roots of $17x^2 - 29x - 39$ are

$$\pm 1, \quad \pm 3, \quad \pm 13, \quad \pm 39, \quad \pm\frac{1}{17}, \quad \pm\frac{3}{17}, \quad \pm\frac{13}{17}, \quad \pm\frac{39}{17}.$$

Clearly, none of these is equal to $-\frac{961}{1385}$, so the only common root is $x = 1$, and the largest common divisor is

$$D(x) = x - 1. \qquad \diamondsuit$$

2.5 Roots of multiplicity > 1.

It is sometimes possible by applying the Euclidean algorithm to find the roots of multiple multiplicity.

Theorem 2.5.1 *Given a polynomial $P(z)$. The roots of multiple degree of $P(z)$ are the roots of the greatest common divisor $D_2 = (P, P')$ of the polynomial $P(z)$ and its derivative $P'(z)$. Each of the roots of D_2 has a multiplicity which is 1 smaller than its multiplicity in the original polynomial $P(z)$.*

PROOF. Write

$$(20) \quad P(z) = a\,(z-\alpha_1)^{p_1}\cdots(z-\alpha_r)^{p_r}, \qquad p_1, \dots, p_r \ge 1,$$

where $\alpha_1, \dots, \alpha_r$ denote the r mutually different roots. Then

$$\begin{aligned}
P'(z) &= a \cdot p_1\,(z-\alpha_1)^{p_1-1}\,(z-\alpha_2)^{p_2}\cdots(z-\alpha_r)^{p_r} \\
&\quad + a \cdot p_2\,(z-\alpha_1)^{p_1}\,(z-\alpha_2)^{p_2-1}\cdots(z-\alpha_r)^{p_r} \\
&\quad + \cdots \\
&\quad + a \cdot p_r\,(z-\alpha_1)^{p_1}\cdots(z-\alpha_{r-1})^{p_{r-1}}\cdots(z-\alpha_r)^{p_r-1} \\
(21) \quad &= a\,(z-\alpha_1)^{p_1-1}\cdots(z-\alpha_r)^{p_r-1}\left\{p_1\,(z-\alpha_2)\cdots(z-\alpha_r) + \cdots + p_r\,(z-\alpha_1)\cdots(z-\alpha_{r-1})\right\}.
\end{aligned}$$

If we put $z = \alpha_j$ into the latter factor of (21), all terms disappear with the exception of

(22) $p_j \left(\alpha_j - \alpha_1\right) \cdots \left(\alpha_j - \alpha_{j-1}\right) \cdot \left(\alpha_j - \alpha_{j+1}\right) \cdots \left(\alpha_j - \alpha_r\right) \neq 0.$

It follows from (20) and (22) that

$$D_2 = (P, P') = (z - \alpha_1)^{p_1 - 1} \cdots (z - \alpha_r)^{p_r - 1},$$

where we of course remove all factors where the exponent is $p_j = 1$. □

Example 2.5.1 We shall find all $n \in \mathbb{N}$, for which the polynomial $(z+1)^{n+1} + z^{n+1} + 1$ has roots of multiplicity > 1, i.e. we shall find $n \in \mathbb{N}$, such that

$$P(z) = (z+1)^{n+1} + z^{n+1} + 1 \qquad \text{and} \qquad P'(z) = (n+1)\left\{(z+1)^n + z^n\right\}$$

have common roots, i.e. the greatest common divisor $D(z) = (P, P')$ is a polynomial of degree ≥ 1.

Using the Euclidean algorithm we get

$$\begin{array}{l} \underline{(z+1)^n+z^n}|(z+1)\cdot(z+1)^n + \qquad z\cdot z^n+1|\underline{z+1} \\ \qquad\qquad \underline{(z+1)\cdot(z+1)^n+(z+1)\cdot z^n} \\ \qquad\qquad\qquad\qquad -z^n+1 \end{array}$$

so the task is reduced to find common roots of the polynomials $(z+1)^n+z^n$ and z^n-1. This is of course equivalent to find the common roots of

$$\{(z+1)^n+z^n\}-\{z^n-1\}=(z+1)^n+1 \qquad \text{and} \qquad z^n-1.$$

The roots of $(z+1)^n+1$ lie on a circle of centre -1 and radius 1.
The roots of z^n-1 lie on a circle of centre 0 and radius 1.

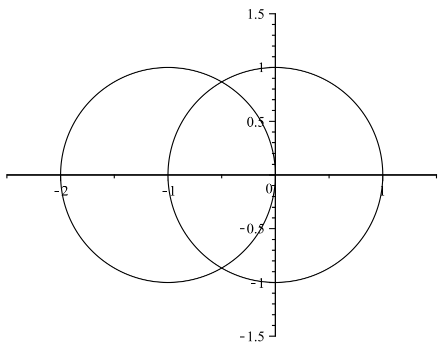

Figure 8: The roots lie on both circles.

The only possibilities are, cf. Figure 8,

$$-\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2}=\exp\left(\pm\frac{2i\pi}{3}\right).$$

It only remains to find $n\in\mathbb{N}$, such that these two numbers are roots in both $(z+1)^n$ and z^n-1.

Notice that if

$$z_0=-\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2}=\exp\left(\pm\frac{2i\pi}{3}\right), \qquad \text{then} \qquad z_0+1=\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2}=\exp\left(\pm\,\frac{i\pi}{3}\right)$$

with corresponding signs. Hence,

$$z_0^n=\exp\left(\pm\frac{2in\pi}{3}\right)=1 \qquad \text{for } n=3p, \quad p\in\mathbb{N},$$

and

$$(z_0+1)^n=\exp\left(\pm\frac{in\pi}{3}\right)=-1 \qquad \text{for } n=3(2p+1), p\in\mathbb{N},$$

and we conclude that the possible exponents must have the structure

(23) $n = 3(2p+1), \qquad p \in \mathbb{N}_0.$

Then by insertion,

$$P(z_0) = (z_0+1)^{n+1} + z_0^{n+1} + 1 = -(z_0+1) + z_0 + 1 = 0,$$

and

$$P'(z_0) = (n+1)\left\{(z_0+1)^n + z_0^n\right\} = (n+1)\{-1+1\} = 0,$$

thus it follows for the given exponents (23) that

$$z_0 = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2} = \exp\left(\pm\frac{2i\pi}{3}\right)$$

are indeed roots of multiplicity > 1, and there is no other possibility.

Finally, it follows from

$$\begin{aligned} P''(z_0) &= (n+1)n\left\{(z_0+1)^{n-1} + z_0^{n-1}\right\} = (n+1)n\left\{-\exp\left(\mp\frac{i\pi}{3}\right) + \exp\left(\mp\frac{2i\pi}{3}\right)\right\} \\ &= (n+1)n\left\{-\left(\frac{1}{2} \mp i\,\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} \mp i\frac{\sqrt{3}}{2}\right)\right\} = -(n+1)n \neq 0 \end{aligned}$$

that the multiplicity is 2 in both cases. ◊

Returning to Theorem 2.5.1 it follows that we can repeat the process on D_2. This means that the roots of $D_3 := (D_2, D_2')$ are the roots of $P(z)$ of at least multiplicity 3, and their multiplicities in D_3 are their multiplicities in $P(z)$ minus 2. If D_3 is not a constant, then proceed with $D_4 := (D_3, D_3')$, etc.

Summing up, we get a sequence of polynomials of decreasing degrees,

$$(24)\quad \begin{cases} D_1 = P & \text{all roots of } P, \\ D_2 = (P, P') & \text{all multiple roots of } P, \\ D_3 = (D_2, D_2') & \text{all roots of } P \text{ of at least multiplicity 3}, \\ \vdots & \vdots \\ D_j = \left(D_{j-1}, D_{j-1}'\right) & \text{all roots of } P \text{ of at least multiplicity } j. \end{cases}$$

Obviously, this process stops after a finite number of steps.

We immediately get from the above

Theorem 2.5.2 *Let the D_j be defined by (24). If $D_{j+1}(z) \neq 0$, then the roots of the quotient $D_j(z)/D_{j+1}(z)$ are all simple. They are the roots of $P(z)$ of precisely multiplicity j.*

Theorem 2.5.2 is convenient in the sense that the multiple roots of $P(z)$ of multiplicity j are the simple roots of the simple polynomial D_j/D_{j+1}, where one could hope for more efficient solution methods, because the degree of D_j/D_{j+1} is smaller. The disadvantage is of course that it cannot be used, when $P(z)$ has only simple roots, because then $D_2 = (P, P') = 1$.

Example 2.5.2 Given the polynomial

$$P(z) = z^6 - 3z^5 + 7z^4 - 10z^3 + 8z^2 - 5z + 2.$$

We shall factorize $P(z)$ in the following way,

$$P(z) = G_1(z) \cdot G_2^2(z) \cdots G_m^m(z),$$

where $G_q(z)$ is the product of all factors $z - a_i$, corresponding to the roots of multiplicity q, and where m is the highest multiplicity of roots in $P(z)$.

We first check for possible rational roots, cf. Section 2.3. These can only be one of the numbers ± 1 and ± 2. Since the terms of $P(z)$ have alternating signs, -1 and -2 are not possible roots. Then we get by insertion,

$$P(1) = 1 - 3 + 7 - 10 + 8 - 5 + 1 = 0,$$

and

$$P(2) = 64 - 3 \cdot 32 + 7 \cdot 16 - 10 \cdot 8 + 8 \cdot 4 - 5 \cdot 2 + 2 = 8 + 36 - 10 + 2 = 36.$$

We conclude that $z = 1$ is a root.

Since

$$P'(z) = 6z^5 - 15z^4 + 28z^3 - 30z^2 + 16z - 5, \qquad \text{and} \qquad P'(1) = 0,$$

and

$$P''(z) = 30z^4 - 60z^3 + 84z^2 - 60z + 16, \qquad \text{and} \qquad P''(1) = 10 \neq 0,$$

we conclude that $z = 1$ is a root of multiplicity 2, and $(z-1)^2 = z^2 - 2z + 1$ must be a divisor in $P(z)$. We get by division,

$$\begin{array}{l}
\underline{z^2 - 2z + 1} | z^6 - 3z^5 + 7z^4 - 10z^3 + 8z^2 - 5z + 2 | \underline{z^4 - z^3 + 4z^2 - z + 2} \\
\underline{z^6 - 2z^5 + z^4} \\
-z^5 + 6z^4 - 10z^3 + 8z^2 - 5z + 2 \\
\underline{-z^5 + 2z^4 - z^3} \\
4z^4 - 9z^3 + 8z^2 - 5z + 2 \\
\underline{4z^4 - 8z^3 + 4z^2} \\
-z^3 + 4z^2 - 5z + 2 \\
\underline{-z^3 + 2z^2 - z} \\
2z^2 - 4z + 2 \\
\underline{2z^2 - 4z + 2} \\
0
\end{array}$$

We conclude that

$$(25) \qquad P(z) = z^6 - 3z^5 + 7z^4 - 10z^3 + 8z^2 - 5z + z = (z-1)^2 \left\{ z^4 - z^3 + 4z^2 - z + 2 \right\},$$

so the investigation is then reduced to the polynomial

$$P_1(z) = z^4 - z^3 + 4z^2 - z + 2, \qquad \text{where} \qquad P_1'(z) = 4z^3 - 3z^2 + 8z - 1.$$

We divide $P_1'(z)$ into $16P_1(z)$,

$$\begin{array}{r|l|l} \underline{4z^3 - 3z^2 + 8z - 1} & 16z^4 - 16z^3 + 64z^2 - 16z + 32 & \underline{4z - 1} \\ & \underline{16z^4 - 12z^3 + 32z^2 - 4z} & \\ & -4z^3 + 32z^2 - 12z + 32 & \\ & \underline{-4z^3 + 3z^2 - 8z + 1} & \\ & 29z^2 - 4z + 31 & \end{array}$$

The remainder $20z^2 - 4z + 31$ has the roots

$$\frac{2 \pm i\sqrt{895}}{29},$$

which by insertion are seen not to be roots of $4z^3 - 3z^2 + 8z - 1$. We therefore conclude that (25) is the factorization with

$$G_1(z) = z^4 - z^3 + 4z^2 - z + 2 \qquad \text{and} \qquad G_2(z) = z - 1.$$

The simple roots (two pairs of pairwise complex conjugated roots) of $G_1(z)$ cannot be found with the methods known so far in this investigation. $\Diamond$

Example 2.5.3 Let $P(z)$ be a polynomial of rational coefficients. Assume that α is the only root of multiplicity ν. We shall prove that α is rational.

From Theorem 2.5.2 follows that we can write

$$P(z) = G_1(z)G_2^2(z) \cdots G_m^m(z),$$

where by assumption,

$$G_\nu^\nu(z) = (z - \alpha)^\nu,$$

and where in general, $G_j(z)$ has the simple roots which are precisely the roots of $P(z)$ of multiplicity j. We get by the Euclidean algorithm,

$$\begin{array}{lllll}
D_1 & = & (P, P') & = & G_2 G_3^2 \cdots G_\nu^{\nu-1} \cdots G_n^{n-1} \\
D_2 & = & (D_1, D_1') & = & G_3 G_4^2 \cdots G_\nu^{\nu-2} \cdots G_n^{n-2}, \\
\vdots & & \vdots & & \vdots \\
D_{\nu-1} & = & \left(D_{\nu-2}, D_{\nu-2}'\right) & = & G_\nu \cdots G_n^{n-\nu+1}, \\
D_\nu & = & \left(D_{\nu-1}, D_{\nu-1}'\right) & = & G_{\nu+1} \cdots G_n^{n-\nu}, \\
\vdots & & \vdots & & \vdots \\
D_{n-1} & = & G_n. & &
\end{array}$$

Since rational coefficients are preserved by the Euclidean algorithm, G_m must have rational coefficients. Then also

$$P_1(z) = P(z)/G_n^n(z) = G_1(z)G_2^2(z) \cdots G_{m-1}^{m-1}(z)$$

must have rational coefficients, and we can repeat the procedure from the very beginning of this example on $P_1(z)$. The conclusion is, that $G_{m-1}(z)$ must have rational coefficients, etc., so every $G_j(z)$ must have rational coefficients. For $j = \nu$ we get $G_\nu(z) = z - \alpha$, because no other root has multiplicity ν. We therefore conclude that $\alpha \in \mathbb{Q}$ is rational. $\Diamond$

Example 2.5.4 We shall reconsider the polynomial

$$P(z) = z^6 - 10z^5 + 40z^4 - 82z^3 + 91z^2 - 52z + 12$$

of Example 2.3.2. This time we shall find its roots by using Theorem 2.5.2 instead.

We get

$$\begin{aligned}
P'(z) &= 6z^5 - 50z^4 + 160z^3 - 246z^2 + 182z - 52 \\
&= 2\left\{3z^5 - 25z^4 + 80z^3 - 123z^2 + 91z - 26\right\}.
\end{aligned}$$

If we divide $P(z)$ by $P'(z)$ we get a quotient of the form $Q_0(z) = \alpha z + \beta$ and a remainder term. This shows that it would be better to divide

$$3^2 P(z) = 9z^6 - 90z^5 + 360z^4 - 738z^3 + 819z^2 - 468z + 108$$

by

$$\frac{1}{2}\,P'(z) = 3z^5 - 25z^4 + 80z^3 - 123z^2 + 91z - 26.$$

At this step we can leave the division to the reader for exercise. The result becomes

$$3^2\,P(z) = (3z-5)\cdot\frac{1}{2}\,P'(z) - \left\{5z^4 - 31z^3 + 69z^2 - 65z + 22\right\}.$$

Then we divide

$$5^2\cdot\frac{1}{2}\,P'(z) = 75z^5 - 625z^4 - 2000z^3 - 3075z^2 + 2275z - 650$$

by

$$\tilde{R}_0(z) = 5z^4 - 31z^3 + 69z^2 - 65z + 22.$$

The result is

$$\begin{aligned}\frac{25}{2}\,P'(z) &= (15z-32)\tilde{R}_0(z) - 27z^3 + 108z^2 - 135z + 54\\ &= (15z-32)\tilde{R}_0(z) - 27\left\{z^3 - 4z^2 + 5z - 2\right\}.\end{aligned}$$

Then divide $\tilde{R}_0(z)$ above by the modified remainder

$$\tilde{R}_1(z) = z^3 - 4z^2 + 5z - 2$$

to get

$$\tilde{R}_0(z) = (5z-11)\tilde{R}_1(z),$$

so we conclude that the greatest common divisor is

$$D_1(z) = (P(z), P'(z)) = \tilde{R}_1(z) = z^3 - 4z^2 + 5z - 2.$$

We check if $D_1(z)$ has multiple roots. First,

$$D_1'(z) = 3z^2 - 8z + 5,$$

so we get by a polynomial division,

$$3^2\,D_1(z) = (3z-4)D_1'(z) - 2(z-1),$$

where the remainder is $\tilde{R}_2(z) = z - 1$. Then finally,

$$D_1'(z) = (3z-5)\tilde{R}_2(z),$$

and we conclude that

$$D_2(z) = (D_1(z), D_1'(z)) = z - 1.$$

Summing up, we have proved that

$$(26)\quad \begin{cases} P(z) & = & z^6 - 10z^5 + 40z^4 - 82z^3 + 91z^2 - 52z + 12 \\ D_1(z) & = & z^3 - 4z^2 + 5z - 2 \\ D_2(z) & = & z - 1. \end{cases}$$

It follows from $D_2(z) = z - 1$ that $z = 1$ is a root of multiplicity 3. By a division,

$$\frac{P(z)}{(z-1)^3} = z^3 - 7z^2 + 16z + 12$$

and

$$\frac{D_1(z)}{(z-1)^3} = z - 2,$$

so $z = 2$ must be a root of multiplicity 2 of $P(z)$.
Finally

$$\frac{P(z)}{(z-1)^3(z-2)^2} = z - 3,$$

and we conclude that

$$P(z) = (z-1)^3(z-2)^2(z-3).$$

An *alternative* method is to use (26) to get

$$\frac{P(z)}{D_1(z)} = z^3 - 6z^2 + 11z - 6,$$

where the simple roots of this quotient are all mutually different roots of $P(z)$. Furthermore,

$$\frac{D_1(z)}{D_2(z)} = z^2 - 3z + 2 = (z-1)(z-2),$$

so $z = 1$ and $z = 2$ are the roots of multiplicity ≥ 2. Finally,

$$\frac{P(z)}{D_1(z)} : \frac{D_1(z)}{D_2} = \frac{P(z) \cdot D_2(z)}{D_1(z)^2} = z - 3,$$

where we have removed all factors of higher multiplicity. Hence, $z = 3$ is the only simple root, and $(z-1)^2(z-2)^2(z-3)$ must be a factor of $P(z)$. It follows again from the division

$$\frac{P(z)}{(z-1)^2(z-2)^2(z-3)} = z - 1$$

that

$$P(z) = (z-1)^3(z-2)^2(z-3). \qquad \Diamond$$

3 Position of roots of polynomials in the complex plane

3.1 Complex roots of a real polynomial.

Recall that a polynomial is called real if all its coefficients are real. The following well-known theorem is here included for completeness.

Theorem 3.1.1 *Let $P(z)$ be a real polynomial. If $\alpha + i\beta$, $\beta \neq 0$, is a complex root, then the complex conjugated $\alpha - i\beta$ is also a complex root. In particular,*

$$(z-(\alpha+i\beta))(z-(\alpha-i\beta)) = (z-\alpha)^2+\beta^2 = z^2-2\alpha z+\alpha^2+\beta^2,$$

is a divisor of $P(z)$ of real coefficients, so

$$P(z) = \left(z^2-2\alpha z+\alpha^2+\beta^2\right) Q(z),$$

where $Q(z)$ is also a real polynomial.

PROOF. Assume that

$$P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n, \qquad a_0,\, a_1,\, \dots,\, a_n \in \mathbb{R},$$

has the complex root $\alpha+i\beta$, where $\alpha,\,\beta\in\mathbb{R}$ and $\beta\neq 0$. Then

$$P(\alpha+i\beta)=a_0(\alpha+i\beta)^n+a_1(\alpha+i\beta)^{n-1}+\cdots+a_{n-1}(\alpha+i\beta)+a_n=0.$$

Since the a_j are all real, it follows by complex conjugation that also

$$P(\alpha-i\beta)=a_0(\alpha-i\beta)^n+a_1(\alpha-i\beta)^{n-1}+\cdots+a_{n-1}(\alpha-i\beta)+a_n=\overline{0}=0,$$

hence $\alpha-i\beta$ is also a root of $P(z)$, and since $\beta\neq 0$, we have $\alpha-i\beta\neq\alpha+i\beta$, so

$$(z-(\alpha+i\beta))(z-(\alpha-i\beta))=z^2-2\alpha z+\alpha^2+\beta^2$$

is a divisor of $P(z)$ of real coefficients, so the quotient $Q(z)$ is also a real polynomial, and $\deg Q=\deg P-2$.

If we argue similarly on $Q(z)$, it follows immediately that if $\alpha+i\beta$ is a root of order j, then $\alpha-i\beta$ is a root of the same order j. □

We mention in this connection a similar theorem with square roots instead of the imaginary "i", so such a conclusion of a "twin solution" is not restricted to the complex case alone.

Theorem 3.1.2 *Assume that all coefficients of $P(z)$ are rational numbers, $a_0,\, a_1,\, \dots,\, a_n\in\mathbb{Q}$. If $P(z)$ has the root $\alpha+\sqrt{\beta}$, where $\alpha,\,\beta\in\mathbb{Q}$ and $\sqrt{\beta}\notin\mathbb{Q}$, then $\alpha-\sqrt{\beta}$ is also a root of $P(z)$.*

PROOF. Assume that $\alpha+\sqrt{\beta}$ is a root, where $\alpha,\,\beta\in\mathbb{Q}$ and $\sqrt{\beta}\notin\mathbb{Q}$. Then

$$\begin{aligned} P(\alpha+\sqrt{\beta}) &= a_0\left(\alpha+\sqrt{\beta}\right)^n+a_1\left(\alpha+\sqrt{\beta}\right)^{n-1}+\cdots+a_{n-1}\left(\alpha+\sqrt{\beta}\right)+a_n \\ &= a_0\sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}\beta^{j/2}+a_1\sum_{j=0}^{n-1}\binom{n-1}{j}\alpha^{n-j-1}\beta^{j/2}+\cdots+a_{n-1}\left(\alpha+\sqrt{\beta}\right)+a_n \\ &= a_0\left\{\alpha^n+n\cdot\alpha^{n-1}\sqrt{\beta}+\binom{n}{2}\alpha^{n-2}\beta+\cdots\right\}+\cdots+a_n \\ &= Q_1(\alpha.\beta)+\sqrt{\beta}\cdot Q_2(\alpha,\beta)=0. \end{aligned}$$

Since both $Q_1(\alpha,\beta)$, $Q_2(\alpha,\beta)\in\mathbb{Q}$ and $\sqrt{\beta}\notin\mathbb{Q}$, we must have $Q_1(\alpha,\beta)=0$ and $Q_2(\alpha,\beta)=0$.

Similarly,

$$\begin{aligned} P(\alpha-\sqrt{\beta}) &= a_0\left(\alpha-\sqrt{\beta}\right)^n+a_1\left(\alpha-\sqrt{\beta}\right)^{n-1}+\cdots+a_{n-1}\left(\alpha-\sqrt{\beta}\right)+a_n \\ &= a_0\sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}(-1)^j\beta^{j/2}+a_1\sum_{j=0}^{n-1}\binom{n-1}{j}\alpha^{n-j-1}(-1)^j\beta^{j/2}+\cdots \\ &\qquad +a_{n-1}\left(\alpha-\sqrt{\beta}\right)+a_n. \end{aligned}$$

The terms in which $\sqrt{\beta}$ does not occur must correspond to even j, in which case $(-1)^j = 1$. Summing up, we get the same $Q_1(\alpha, \beta)$ as above.

The terms in which $\sqrt{\beta}$ does occur as an extra factor must correspond to odd j, in which case $(-1)^j = -1$. Summing up, these terms add up to $-\sqrt{\beta} \cdot Q_2(\alpha, \beta)$, where $Q_2(\alpha, \beta)$ is given as above. Since $Q_1(\alpha, \beta)$ and $Q_2(\alpha, \beta)$ are the same in the two cases, we get

$$P\left(\alpha - \sqrt{\beta}\right) = Q_1(\alpha, \beta) - \sqrt{\beta} \cdot Q_2(\alpha, \beta) = 0.$$

It follows that

$$\left(z - \left(\alpha + \sqrt{\beta}\right)\right)\left(z - \left(\alpha - \sqrt{\beta}\right)\right) = (z - \alpha)^2 - \beta = z^2 - 2\alpha z + \alpha^2 - \beta$$

of rational coefficients is a divisor in $P(z)$,

$$P(z) = \left\{z^2 - 2\alpha z + \alpha^2 - \beta\right\} Q(z),$$

where $Q(z)$ has rational coefficients. □

3.2 Descartes's theorem.

We have previously from time to time used that if a polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_0, \qquad x \in \mathbb{R},$$

has only positive coefficients, $a_j > 0$ for all j, then $P(x)$ cannot have positive roots, and if the coefficients are alternating, $a_j = (-1)^j |a_j|$ (or $a_j = (-1)^{j+1} |a_j|$) for all j, then $P(x)$ cannot have negative roots.

We shall in this section derive some improved results concerning where the real roots are lying on the real axis. We assume that the real polynomial is normalized, i.e. that $a_0 = 1$.

Theorem 3.2.1 *If $P(x)$ is of even degree, then $P(x)$ has an even number of real roots (including the possibility of no root at all).*
If $P(x)$ is of odd degree, then $P(x)$ has an odd number of real roots. All roots are here counted according to their multiplicities.

Proof. The theorem follows from Theorem 3.1.1, which states that for real polynomials, non-real roots are always given in complex conjugated pairs, so the number of non-real roots is an even number. The fundamental theorem of algebra states that the degree of a polynomial is equal to its number of roots, counted by their multiplicity, so the theorem follows by parity. □

Theorem 3.2.2 *Given the real normalized polynomial*

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \qquad a_1, \dots, a_n \in \mathbb{R}.$$

Define constants H and L by

$$-H := \min\{0, a_1, \dots, a_n\} \qquad \text{and} \qquad -L = \min\{0, -a_1, +a_2, \dots, (-1)^n a_n\}.$$

If $P(x)$ has a real root, then it must lie in the interval $[-1-L, 1+H]$.

Proof. If $x > 1 + H \geq 1$, then we get the estimates

$$\begin{aligned} P(x) &\geq x^n - H\left\{x^{n-1} + x^{n-2} + \cdots + 1\right\} = x^n - H \cdot \frac{x^n - 1}{x-1} \\ &= \frac{(x-1)x^n - H \cdot x^n + H}{x-1} > \frac{H \cdot x^n - H \cdot x^n + H}{x-1} = \frac{H}{x-1} > 0, \end{aligned}$$

from which follows that $P(x)$ has no real root $> 1 + H$.

A similar estimate shows that the polynomial

$$(-1)^n P(-x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^n a_n$$

does not have any root $-x > 1 + L$. Thus we conclude that $P(x)$ does not have any root $x < -1 - L$, and the theorem is proved. □

If $a_n \neq 0$, then 0 is not a root, and there is a neighbourhood of 0 which does not contain any root from $P(x)$.

Corollary 3.2.1 *Given the real normalized polynomial*

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \qquad a_1, \dots, a_n \in \mathbb{R} \text{ and } a_n \neq 0.$$

Define N and M by

$$-N = \min\left\{0, \frac{a_1}{a_n}, \dots, \frac{a_n}{a_n}\right\} \qquad \text{and} \qquad -M = \min\left\{0, (-1)^{n-1}\frac{a_1}{a_n}, (-1)^{n-2}\frac{a_2}{a_n}, \dots, -\frac{a_{n-1}}{a_n}, \frac{a_n}{a_n}\right\}.$$

If $P(x)$ has a real root, then it must lie in one of the two intervals

$$\left]-\infty, -\frac{1}{1+M}\right] \qquad \text{and} \qquad \left[\frac{1}{1+N}, +\infty\right[.$$

Proof. Put $x = \dfrac{1}{y}$. Then

$$y^n P\left(\frac{1}{y}\right) = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + 1 = a_n \left\{y^n + \frac{a_{n-1}}{a_n} y^{n-1} + \cdots + \frac{a_1}{a_n} y + 1\right\}.$$

It follows from Theorem 3.2.2 that any real root y, if it exists, must lie in $[-1 - M, 0[\cup]0, 1 + N]$. Since $y = \dfrac{1}{x}$, we conclude that any real root must lie in the union

$$\left]-\infty, -\frac{1}{1+M}\right] \cup \left[\frac{1}{1+N}, +\infty\right[. \qquad \square$$

Example 3.2.1 Let

$$P(x)=x^{19}-4x^{13}+6x^{10}+20x^5-2x^3-2x^2-4.$$

Then

$$-H=\min\{0,-4,6,20,-2,-2,-4\}=-4, \qquad \text{hence } H=4,$$

and

$$-L=\min\{0,-4,-6,20,-2,2,-4\}=-6, \qquad \text{hence } L=6,$$

because $a_{19-13}=a_6=-4$, $a_{19-10}=a_9=6$, $a_{19-5}=a_{14}=20$, $a_{19-3}=a_{16}=-2$, $a_{19-2}=a_{17}=-2$ and $a_{19}=-4$. If α is a real root, then $\alpha\in[-1-L,1+H]=[-7,5]$.

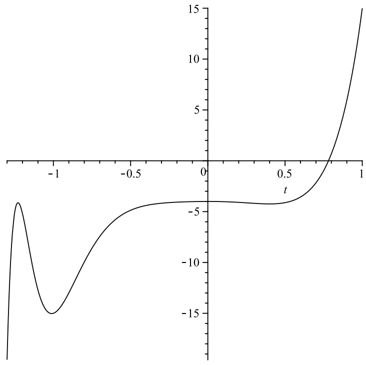

Figure 9: The graph of $P(x) = x^{19} - 4x^{13} + 6x^{10} + 20x^5 - 2x^3 - 2x^2 - 4$.

Furthermore,

$$-\frac{1}{4}\,y^{19}\,P\left(\frac{1}{y}\right) = y^{19} + \frac{1}{2}\,y^{17} - 5y^{14} - \frac{3}{2}\,y^9 + y^6 - \frac{1}{4},$$

so

$$-N = \min\left\{0, 1, \frac{1}{2}, -5, -\frac{3}{2}, 1, -\frac{1}{4}\right\} = -5, \qquad \text{thus } N = 5,$$

and

$$-M = \min\left\{0, \frac{1}{2}, 5, \frac{3}{2}, -1, \frac{1}{4}\right\} = -1, \qquad \text{thus } M = 1,$$

so possible real roots must also lie in $\left]-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{6}, +\infty\right[$.

Combining these results we see that possible real roots are limited to

$$\left[-7, -\frac{1}{2}\right] \cup \left[\frac{1}{6}, 5\right].$$

It follows from Figure 9 that there is only one real root and it lies in the interval $[0.7, 0.8] \subset \left[\frac{1}{6}, 5\right]$. $\Diamond$

Theorem 3.2.3 Descartes's theorem (1637). *Let*

$$P(x) = x^n + a_1 x^{n-1} + \cdots a_{n-1}x + a_n, \qquad a_1, \dots, a_n \in \mathbb{R},$$

be a real normalized polynomial.

- *The number of positive roots of $P(x)$, counted by multiplicity, is at most equal to the number of changes in sign in the sequence $1, a_1, \dots, a_n$.*
- *The difference between the two numbers is an even number.*

In the count of changes of sign we only consider coefficients $a_j \neq 0$ which are not zero.

PROOF. If $P(0) = 0$, then the root $x = 0$ has some multiplicity $k \in \mathbb{N}$. Since 0 is not positive, we can divide by x^k to get $P_1(x) = P(x) \cdot x^{-k}$ for $x > 0$. Then by continuity, $P_1(0) \neq 0$, and $P_1(x)$ and $P(x)$ must have the same positive roots, all of the same multiplicity.

We may therefore in the following assume that $a_n = P(0) \neq 0$.

Theorem 3.2.3 clearly follows, if we can prove

Lemma 3.2.1 *Let $P(x)$ be a normalized polynomial of real coefficients, where $P(0) \neq 0$. If $P(x)$ has r positive roots, then there are $r + 2p$ changes of sign in the sequence $1, a_1, \dots, a_n$, where p is some nonnegative integer, $p \in \mathbb{N}_0$.*

PROOF. *Induction after r.*

1) If $r = 0$, then $P(x)$ has no positive roots. Then $P(x)$ has constant sign for $x \geq 0$. Since $P(x) \to +\infty$ for $x \to +\infty$, this sign must be $+$, and we conclude that $a_n = P(0) > 0$.
   The sequence $1, a_1, \dots, a_n$ then starts and ends with positive terms. If a_k is the first negative term, then we search for the first following positive term a_ℓ, $\ell > k$. It follows that we have two changes in the subsequence $1, \dots, a_\ell$.
   Proceed in the same way with the subsequence $a_\ell, \dots, a_n$, where both a_ℓ and a_n are positive, etc.. After a finite number of steps, each adding 2 to the count of changes of sign, we finally reach a_n, and the claim follows for $r = 0$.

2) Assume that the lemma holds for some $r_0 \in \mathbb{N}_0$. We shall prove that it also holds for its successor $r = r_0 + 1$.

   We assume that the polynomial $P(x)$ has $r_0 + 1$ positive roots, and we choose one of them, $\alpha > 0$. Then

   (27) $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n = (x - \alpha) P_0(x)$,

   where

   $$P_0(x) = x^{n-1} + b_1 x^{n-1} + \cdots + b_{n-1}$$

   must have r_0 positive roots, because we by division have removed one positive root α from $P(x)$.

   Using the assumption of induction above, $P_0(x)$ has $r_0 + 2p_0$ for some $p_0 \in \mathbb{N}_0$ changes of sign in its sequence of coefficients.

   Then consider more closely the sequence of coefficients $1, b_1, \dots, b_{n-1}$ of $P_0(x)$. Let b_{λ_1} be the first negative of these, then b_{λ_2} the first positive of them after b_{λ_1}, etc., up to $b_{\lambda_{r_0+2p_0}}$, which represents the last change of sign.

   We have schematically,

   $$\begin{array}{cccccccccc} 1, & \cdots, & b_{\lambda_1}, & \cdots, & b_{\lambda_2}, & \cdots, & \cdots, & b_{\lambda_{r_0+2p_0}}, & \cdots \\ + & \geq 0 & - & \leq 0 & + & \geq & & (-1)^{r_0} & \end{array}$$

Then notice that the coefficients $a_0 = 1$, $a_1, \dots, a_n$ of $P(x)$ are found from the coefficients $b_0 = 1$, $b_1, \dots, b_{n-1}$ of $P_0(x)$ from (27) by the equations

$$a_0 = 1, \quad a_1 = b_1 - \alpha b_0, \quad a_2 = b_2 - \alpha b_1, \quad \cdots, \quad a_{n-1} = b_{n-1} - \alpha b_{n-2}, \quad a_n = -\alpha b_{n-1}.$$

In particular,

$$\begin{cases} a_0 = 1, & \text{positive,} \\ a_{\lambda_1} = b_{\lambda_1} - \alpha b_{\lambda_1 - 1}, & \text{negative,} \\ a_{\lambda_2} = b_{\lambda_2} - \alpha b_{\lambda_2 - 1}, & \text{positive,} \\ \vdots & \vdots \\ a_{\lambda_{r_0+2p_0}} = b_{\lambda_{r_0+2p_0}} - \alpha b_{\lambda_{r_0+2p_0}-1}, & (-1)^{r_0}, \\ a_n = -\alpha b_{n-1}, & (-1)^{r_0}. \end{cases}$$

In the last equation we use that $b_{n-1} \neq 0$ must have the same sign as $b_{\lambda_{r_0+2p_0}}$.

Then consider the sequence $1, a_1, \dots, a_n$ of coefficients. According to the analysis above there must be an odd number of changes of sign between 1 and a_{λ_1}, an odd number of changes of sign between a_{λ_1} and a_{λ_2}, etc., until we obtain an odd number of changes of sign between $a_{\lambda_{r_0}+2p_0}$ and a_n. In the total count we therefore get that the number of changes of sign is $r_0 + 2p_0 + 1$ plus an even number ≥ 0. This is precisely $r_0 + 1 + 2p$ for some $p \in \mathbb{N}_0$.

Lemma 3.2.1 now follows by induction, and then Theorem 3.2.3 is trivial. □

Example 3.2.2 1) We have trivially one change of sign in $P(x) = x - 1$, so we must have a positive root. It is of course $x = 1$.

2) The polynomial $P(x) = x^2 - x + 1$ has two changes of sign, so we either have none or two positive roots. Since the discriminant is negative, we must have none real root.

3) The polynomial $P(x) = x^4 - 7x^2 + 6x - 1$ has no rational root. Its sequence of coefficients has three changes of sign, so the polynomial has either one or three positive roots.
The polynomial $P(-x) = x^4 - 7x^2 - 6x - 1$ has one change of sign, so $P(x)$ has one negative root, cf. also Figure 10.

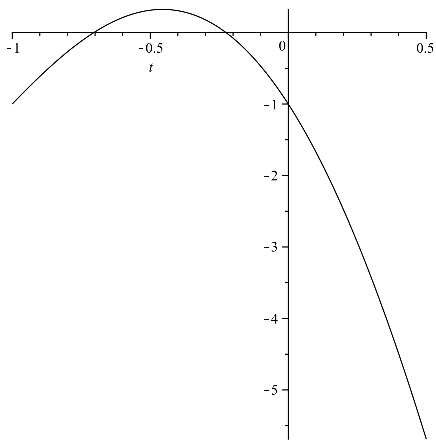

Figure 10: The graph of $P(x) = x^4 - 7x^2 + 6x - 1$.

4) The polynomial

$$P(x) = x^{19} - 4x^{13} + 6x^{10} + 20x^5 - 2x^3 - 2x^2 - 4,$$

also considered in Example 3.2.1, cf. Figure 9, page 49, has three changes of sign in its sequence of coefficients, so we have either one or three positive roots.
The polynomial

$$P(-x) - x^{19} + 4x^{13} + 6x^{10} - 20x^5 + 2x^3 - 2x^2 - 4$$

has four changes of sign in its sequence of coefficients, so $P(x)$ has either none, two or four negative roots. ◊

Example 3.2.3 Consider the polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad a_1, \dots, a_n \in \mathbb{R}, \quad a_n \neq 0,$$

of degree ≥ 3. Assume that two successive coefficients are 0, thus $a_i = a_{i+1} = 0$ for some $i \in \{1, \dots, n-2\}$. The sequence of coefficients for $P(x)$ is then

$$1,\, a_1,\, a_2,\, \dots,\, a_{i-1},\, 0,\, 0,\, a_{i+2},\, \dots,\, a_n,$$

(at most $n-1$ numbers $\neq 0$), and the sequence of coefficients for $(-1)^n P(-x)$ is

$$1,\, -a_1,\, a_2,\, \dots,\, (-1)^{i-1},\, 0,\, 0,\, (-1)^{i+2}a_{i+2},\, \dots,\, (-1)^n a_n.$$

We get an estimate of the number of positive roots of $P(x)$ from the former sequence, and an estimate of the number of negative roots of $P(x)$ from the latter sequence. Since $a_n \neq 0$, it follows that 0 is not a root.

Choose j, such that a_j and $a_{j+1} \neq 0$, and then combine the two sequences above,

$$\begin{array}{cc} a_j, & a_{j+1}, \\ (-1)^j a_j, & (-1)^{j+1} a_{j+1}. \end{array}$$

It follows that we have just one (horizontal) change of sign in this group of coefficients.

If one of a_j or a_{j+1} is zero, we of course have no such change. Thus, combining the two sequences, where we in each horizontal sequence have at most $n-1$ coefficients $\neq 0$, we conclude that we can at most have $(n-1)-1 = n-2$ changes of signs in total in the two sequences. Hence, the polynomial $P(x)$ has at most $n-2$ real roots, and thus at least one pair of complex conjugated roots.

For $n = 3$ we have only the possibility of $x^3 + a_3$, where the roots are found by solving a binomial equation, cf. Section 2.1, hence the only real root is $\sqrt[3]{-a_3}$.

For $n = 4$ we have the two possibilities

$$x^4 + a_1 x^3 + a_4 \qquad \text{and} \qquad x^4 + a_3 x + a_4.$$

Notice that e.g. $x^4 - 2x^2 + 1$ can be considered as a polynomial of degree 2 in the new variable $z = x^2$, and that we have two changes of sign. However, it has the four real roots $1, 1, -1, -1$, so both 1 and -1 are double roots. It is therefore in the result above essential that we assume that two *successive* coefficients are 0. Otherwise the conclusion may be wrong. $\diamond$

3.3 Fourier-Budan's theorem.

Given a real normalized polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n, \qquad a_1, \dots, a_n \in \mathbb{R},$$

and put $x = x_0 + t$ for any real x_0. Then by a Taylor expansion,

$$P(x_0 + t) = P(x_0) + \frac{P'(x_0)}{1!}\, t + \cdots + \frac{P^{(n)}(x_0)}{n!}\, t^n,$$

where always $P^{(n)}(x_0) = n!$. It follows from Theorem 3.2.3, *Descartes's theorem*, that the number of roots of $P(x)$ in the interval $]x_0, +\infty[$ (counted by multiplicity) *at most* is equal to the number $V(x_0)$ of changes of sign in the sequence

$$P(x_0), \, P'(x_0), \, \dots, \, P^{(n)}(x_0) = n!,$$

and that the difference between the two numbers is even.

Given any half open interval $]\alpha, \beta]$, closed to the right, the number of roots of $P(x)$ in $]\alpha, \beta]$ (counted by their multiplicities) must for some nonnegative integers A and B be

$$\{V(\alpha) - 2A\} - \{V(\beta) - 2B\} = V(\alpha) - V(\beta) - 2(A - B),$$

so this number is $V(\alpha) - V(\beta)$, modulo some even number. We shall prove that we always have $A \geq B$.

Theorem 3.3.1 Fourier-Budan's theorem. (Mentioned at lectures in 1797; published in 1820.) *Let $P(x)$ be a real normalized polynomial of degree n.*

1) The number $V(x)$ of changes of sign in the sequence

$$P(x), \, P'(x), \, \dots, \, P^{(n)}(x)$$

is a monotone decreasing function, which is half continuous from the right.

2) For every interval $]\alpha, \beta]$ the number of roots in this interval is at most $V(\alpha) - V(\beta)$, and the difference between the two numbers is even.

3) If x is negative and numerically large, then $V(x) = n$, and if x is positive and large, then $V(x) = 0$.

Proof. Let $x_1, \dots, x_N$ be all real numbers, which are roots in at least one of the polynomials $P(x)$, $P'(x)$, $\dots$, $P^{(n)}(x)$, and let $x_1, \dots, x_N$ be increasingly ordered. Then each of the polynomials $P(x)$, $P'(x)$, $\dots$, $P^{(n-1)}(x)$ must have constant sign in each of the intervals

$$]-\infty, x_1[, \quad]x_1, x_2[, \quad \dots, \quad]x_{N-1}, x_N[, \quad]x_N, +\infty[,$$

which is supplemented with the trivial $P^{(n)}(x) = n! > 0$ for every $x \in \mathbb{R}$. Thus, $V(x)$ must be constant in each of these intervals.

Since clearly,

$$P(x) \to +\infty, \quad P'(x) \to +\infty, \quad \dots, \quad P^{(n-1)}(x) \to +\infty \qquad \text{for } x \to +\infty,$$

we conclude that $P(x)$, $P'(x)$, $\dots$, $P^{(n)}(x)$ must all be positive in $]x_N, +\infty[$, so $V(x) = 0$ in $]x_N, +\infty[$.

Then notice that

$$(-1)^n P(x) \to +\infty, \quad (-1)^{n-1} P'(x) \to +\infty, \quad \dots, \quad -P^{(n-1)}(x) \to +\infty \qquad x \to -\infty,$$

which implies that $P(x)$, $P'(x)$, $\dots$, $P^{(n)}(x)$ are successively positive and negative in $]-\infty, x_1[$, so we conclude that $V(x) = n$ in $]-\infty, x_1[$.

Theorem 3.3.1 follows, if we can prove

Lemma 3.3.1 *Given any $i = 1, \dots, N$. Then $V(x_i) = V(x)$ for all $x \in \,]x_i, x_{i+1}[$, and $V(x_i) \le V(x)$ for all $x \in \,]x_{i-1}, x_i[$. In the latter case the difference is an even number, provided that x_i is not a root of $P(x)$. If on the other hand x_i is a root of $P(x)$, then this difference is instead equal to the multiplicity of the root x_i in $P(x)$ plus some even nonnegative number.*

PROOF. Consider the sequence $P(x)$, $P'(x)$, $\dots$, $P^{(n)}(x)$ at the point x_i and in the two adjacent intervals.

If x_i is a root of multiplicity m in $P(x)$, then the first m numbers of $P(x_i), P'(x_i), \dots, P^{(n)}(x_i)$ must all be zero, and then $P^{(m)}(x_i) \ne 0$. Its value can be positive as indicated in Table 2, or negative as indicated in the *alternative* (Alt.) of Table 2.

	$x<x_i$ (altern. signs)	x_i	$x>x_i$ (const. sign)		Alt. $x<x_i$	Alt. x_i	Alt. $x_i>x$
$P(x)$	$\pm$	0	$+$		$\pm$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$P^{(m-1)}(x)$	$-$	0	$+$		$+$	0	-
$P^{(m)}(x)$	$+$	$+$	$+$		-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$

Alternatives

						x_i			x_i			x_i	
$P^{(k)}(x)$	$+$	$+$	$+$		-	-	-	$+$	$+$	$+$	-	-	-
$P^{(k+1)}(x)$	$\pm$	0	$+$		$\pm$	0	-	$\pm$	0	-	$\pm$	0	$+$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P^{[\ell-1)}(x)$	-	0	$+$		$+$	0	-	$+$	0	-	-	0	$+$
$P^{(\ell)}(x)$	$+$	$+$	$+$		-	-	-	-	-	-	$+$	$+$	$+$
$\vdots$	$\vdots$	$\vdots$	$\vdots$										
$P^{(n)}(x)$	$+$	$+$	$+$										

Table 2: Possible variations of signs in the neighbourhood of x_i in the proof of Fourier-Budan's theorem.

We shall take into account the possibility that there later on may be zeros in the sequence $P(x_i)$, $P'(x_i)$, ..., $P^{(n)}(x_i)$. One such example is given in Table 2 for the sequence $P^{k+1)}(x_i)$, ..., $P^{(\ell-1}(x_i)$, where $P^{(k)}(x_i)$ and $P^{(\ell)}(x_i)$ are chosen positive in the main case, while the possible alternatives are -, -, or +, -, or -, +.

Whenever $P^{(j)}(x_i) \neq 0$, the function $P^{(j)}$ must necessarily have the same sign in the two adjacent intervals.

Thus, if both $P^{(j)}(x_i)$ and $P^{(j+1)}(x_i)$ are $\neq 0$, then the pair $\left(P^{(j)}(x), P^{(j+1)}(x)\right)$, will contribute with the same number (either 0, or 1) in the two adjacent intervals as at the point x_i itself.

Assume that x_i is a root of $P(x)$. Then by Taylor's formula,

$$\begin{cases} P(x) & = \dfrac{P^{(m)}(x_i)}{m!}(x-x_i)^m + \cdots, & \text{thus } \dfrac{P(x)}{(x-x_i)^m} & \to & \dfrac{P^{(m)}(x_i)}{m!} \\ \vdots & \qquad \vdots & \qquad\vdots & & \vdots \\ P^{(m-2)}(x) & = \dfrac{P^{(m)}(x_i)}{2!}(x-x_i)^2 + \cdots, & \text{thus } \dfrac{P^{(m-2)}(x)}{(x-x_i)^2} & \to & \dfrac{P^{(m)}(x_i)}{2!} \\ P^{(m-1)}(x) & = \dfrac{P^{(m)}(x_i)}{1!}(x-x_i) + \cdots, & \text{thus } \dfrac{P^{(m-1)}(x)}{x-x_i} & \to & \dfrac{P^{(m)}(x_i)}{1!}. \end{cases}$$

for $x \to x_i$. It follows in this case that $P(x)$, ..., $P^{(m-1)}(x)$ in the interval to the right of x_i must have the same sign as $P^{(m)}(x_i)$, while $P^{(m-1)}(x)$ in the interval to the left of x_i has the opposite sign

x	$-\infty$	-2	-1	0	1	2	$+\infty$
$P(x) = x^5 - x^4 - 3x^3 + 2x + 5$	-	-	+	+	+	+	+
$P'(x) = 5x^4 - 4x^3 - 9x^2 + 2$	+	+	+	+	-	+	+
$P''(x) = 20x^3 - 12z^2 - 18x$	-	-	-	0	-	+	+
$P^{(3)}(x) = 60x^2 - 24x - 18$	+	+	+	-	+	+	+
$P^{(4)}(x) = 120x - 24$	-	-	-	-	+	+	+
$P^{(5)}(x) = 120$	+	+	+	+	+	+	+
$V(x)$	5	5	4	2	2	0	0

Table 3: Table of $P(x), \dots, P^{(5)}(x)$ and $V(x)$ for $x = -\infty, -2, -1, 0, 1, 2, +\infty$ in Example 3.3.1

of $P^{(m)}(x_i)$, and $P^{(m-2)}(x_i)$ has the same sign as $P^{(m)}(x_i)$, etc.. Hence, there is no change of sign in the subsequence $P(x), \dots, P^{(m)}(x)$ at x_i as well as in the interval to the right of x_i, and there are m changes of sign in the interval to the left of x_i.

Concerning the possible subsequence $P^{(k)}(x), \dots, P^{(\ell)}(x)$ as in Table 2, a similar argument shows that $P^{(k+1)}(x), \dots, P^{(\ell-1)}(x)$ have the same sign of $P^{(\ell)}(x_i)$ in the interval to the right of x_i, while in the interval to the left of x_i the signs are alternating. Hence, there are just as many changes of sign in the interval to the right of x_i of the subsequence $P^{(k)}(x), \dots, P^{(\ell)}(x)$ as at x_i itself, namely none, if the combination of signs of $P^{(k)}(x_i)$ and $P^{(\ell)}(x_i)$ is either +, +, or -, -, and it is 1, if this combination of signs is either +, -, or -, +.

In the interval to the left of x_i the number of changes of sign is even, if the combination of signs is either +, +, or -, -, in fact = the largest even number $\leq \ell - k$, and odd if the combination of signs is either +, -, or -, +, in fact = the largest odd number $\leq \ell - k$.

It follows from the discussion above that we have the same number of changes of sign in the interval to the right of x_i, and when we look at the interval immediately to the left of x_i we have found a loss of m changes of sign, when x_i is a root of $P(x)$ of multiplicity m, and an even number of changes of sign of each subsequence of the form $P^{(k)}(x), \dots, P^{(\ell)}(x)$, and Lemma 3.3.1 is proved, hence as a consequence also Theorem 3.3.1. □

Example 3.3.1 Given the polynomial $P(x) = x^5 - 4^4 - 3x^3 + 2x + 5$, we obtain Table 3 of the signs of $P(x), P'(x), \dots, P^{(5)}(x)$, so we can compute the value of $V(x)$ for various values of x.

The columns of $-\infty$ and $+\infty$ correspond to large negative and large positive x. It follows that all real roots of $P(z)$ must lie in the interval $]-2, 2]$, and that we have one root in $]-2, -1[$, none or two roots in $]-1, 0]$, no root in $]0, 1]$, and none or two roots in $]1, 2]$.

It follows from Figure 11 that we have one real root in $]-2, -1]$, no root in $]-1, 1]$ and two real roots in $]1, 2]$. The remaining two roots must be complex conjugated.◊

Example 3.3.2 Consider the polynomial $P(x) = x^4 - 7x^2 + 6x - 1$ of Example 3.2.2. We choose $x_i = -4, -3, \dots, 2, 3$, and then set up Table 4.

We conclude from this table that we have one root in the interval $]-4, -3]$, another one in the interval $]2, 3]$, and none or two real roots in the interval $]0, 1]$, and none in $]-3, 0]$. ◊

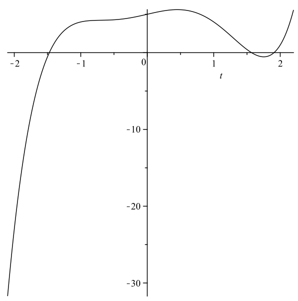

Figure 11: The graph of $P(x) = x^5 - x^4 - 3x^3 + 2x + 5$ of Example 3.3.1.

x	$-\infty$	-4	-3	-2	-1	0	1	2	3	$+\infty$
$P(x) = x^4 - 7x^2 + 6x - 1$	+	+	-	-	-	-	-	-	+	+
$P'(x) = 4x^3 - 14x + 6$	-	-	-	+	+	+	-	+	+	+
$P''(x) = 12x^2 - 14$	+	+	+	+	-	-	-	+	+	+
$P^{(3)}(x) = 24x$	-	-	-	-	-	0	+	+	+	+
$P^{(4)}(x) = 24$	+	+	+	+	+	+	+	+	+	+
$V(x)$	4	4	3	3	3	3	1	1	0	0

Table 4: Table illustrating Fourier's theorem in the case of Example 3.3.2.

3.4 Sturm's theorem.

The problem of determining the number of real roots in a given interval was solved in 1829 by the French mathematician Sturm. We shall apply the Euclidean algorithm on $P(x)$ and $P'(x)$. We write in the present case the equations of division in the following way,

$$(28)\quad \left\{\begin{array}{rcl} P(x) &=& P'(x)Q_1(x) - P_2(x), \\ P'(x) &=& P_2(x)Q_2(x) - P_3(x), \\ P_2(x) &=& P_3(x)Q_3(x) - P_4(x), \\ \vdots & & \qquad\vdots \\ P_{m-2}(x) &=& P_{m-1}(x)Q_{m-1}(x) - P_m(x), \\ P_{m-1}(x) &=& P_m(x)Q_m(x) \end{array}\right.$$

This is also written in a more traditional way as

$$\begin{array}{rcl} P(x) &=& P'(x)Q_1(x) + \{P_2(x)\}, \\ P'(x) &=& \{-P_2(x)\} \cdot \{-Q_2(x)\} + \{-P_3(x)\}, \\ -P_2(x) &=& \{-P_3(x)\}\, Q_3(x) + P_4(x), \\ -P_3(x) &=& P_4(x) \cdot \{-Q_4(x)\} + P_5(x), \\ P_4(x) &=& P_5(x)Q_5(x) + \{-P_6(x)\}, \end{array}$$

etc.. Here we note that the first and the fifth equation have the same combination of signs, so these will be repeated cyclically of period 4.

Thus, the changed form means that the quotients of the equations of division have been denoted

$Q_1(x)$, $-Q_2(x)$, $Q_3(x)$, $-Q_4(x)$, $\dots$, and the remainders are $-P_2(x)$, $-P_3(x)$, $P_4(x)$, $P_5(x)$, $\dots$. This means that (P, P') is the normalized polynomial which is associated with $P_m(x)$.

The polynomial $P_m(x)$ is a divisor in all of the polynomials $P(x)$, $P'(x)$, $P_2(x)$, $\dots$, $P_m(x)$.

We define the so-called *Sturm chain* as the sequence of polynomials,

$$H(x) := \frac{P(x)}{P_m(x)},\; H_1(x) := \frac{P'(x)}{P_m(x)},\; H_2(x) := \frac{P_2(x)}{P_m(x)},\; \dots,\; H_{m-1}(x) = \frac{P_{m-1}(x)}{P_m(x)},\; H_m(x) = \frac{P_m(x)}{P_m(x)} = 1.$$

When the equations of (28) are divided by $P_m(x)$, we clearly get

$$(29)\quad \left\{\begin{array}{rcl} H(x) & = & H_1(x)Q_1(x) - H_2(x), \\ H_1(x) & = & H_2(x)Q_2(x) - H_3(x) \\ H_2(x) & = & H_3(x)Q_3(x) - H_4(x) \\ \vdots & & \vdots \\ H_{m-1}(x) & = & H_{m-1}(x)Q_{m-1}(x) - H_m(x) \\ H_{m-1}(x) & = & H_m(x)Q_m(x), \\ H_m(x) & = & 1, \end{array}\right.$$

where all polynomials have real coefficients.

Assuming that $P(x)$ is normalized, we get from the *Fundamental theorem of algebra* that

$$P(x) = (x - \alpha_1)^{\nu_1} \cdots (x - \alpha_s)^{\nu_s},$$

where $\alpha_1, \dots, \alpha_s$ are the mutually different (real or complex) roots of $P(x)$, and $\nu_1, \dots, \nu_s$ are their multiplicities, so

$$\nu_1 + \cdots + \nu_s = n,$$

and the largest common divisor, cf. Theorem 2.4.1, page 33, is

$$(P, P') = (x - \alpha_1)^{\nu_1 - 1} \cdots (x - \alpha_s)^{\nu_s - 1},$$

where we conventionally put $(x - \alpha_j)^0 := 1$.

Since $P_m(x)$ and (P, P') have the same roots of the same multiplicities, we conclude that

$$H(x) = a\,(x - \alpha_1) \cdots (x - \alpha_s), \qquad a \neq 0,$$

thus the polynomials $H(x)$ and $P(x)$ have the same (different) roots. Only in the case of $H(x)$ they are all simple.

After these preparations we formulate

Theorem 3.4.1 Sturm's theorem. *Let $W(x)$ denote the number of changes of sign in the sequence*

$$H(x),\ H_1(x),\ \dots,\ H_m(x).$$

Then $W(x)$ is a monotonically decreasing function in x. It is half continuous from the right.
For every half open interval $]\alpha, \beta]$, the number of mutually different roots of $P(x)$ in $]\alpha, \beta]$ is equal to $W(\alpha) - W(\beta)$.

PROOF. Let $x_1, \dots, x_N$ denote all real numbers, which are roots in at least one of the polynomials $H(x), H_1(x), \dots, H_{m-1}(x)$, where we assume that they form an increasing sequence.

In each of the open intervals $]-\infty, x_1[$, $]x_1, x_2[$, $\dots$, $]x_{N-1}, x_N[$, $]x_N, +\infty[$, each of the polynomials $H(x), H_1(x), \dots, H_{m-1}(x)$ must have constant sign. Furthermore, $H_m(x) = 1 > 0$ for all x. Hence, $W(x)$ is constant in each of the intervals mentioned above.

The theorem follows, if we can prove the following

Lemma 3.4.1 *Let $x_1, \dots, x_N$ be given as above. Fore every $i = 1, \dots, N$, the value of $W(x_i)$ is equal to the value of $W(x)$ in the adjacent interval to the right of x_i.*
In the adjacent interval to the left of x_i the value of $W(x)$ is given by

$$W(x) = \begin{cases} W(x_i), & \text{if } x_i \text{ is not a root of } P(x), \\ W(x_i) + 1, & \text{if } x_i \text{ is a root of } P(x). \end{cases}$$

	Main case				Alternative case		
	$x < x_i$	x_i	$x > x_i$		$x < x_i$	x_i	$x > x_i$
$H(x)$	-	0	+		+	0	-
$H_1(x)$	+	+	+		-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$H_{k-1}(x)$	+	+	+		$-$	$-$	$-$
$H_k(x)$		0				0	
$H_{k+1}(x)$	-	-	-		+	+	+
$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$H_n(x)$	+	+	+				

Table 5: Table of variations of sign at a zero x_i and in the two adjacent intervals.

PROOF. Consider the sequence $H(x)$, $H_1(x)$, $\dots$, $H_m(x)$ at x_i, as well as in the two adjacent intervals of x_i.

Since $H_m(x_i) = 1 \neq 0$, it follows from the equations (29) that two successive numbers of the sequence $H(x_i)$, $H_1(x_i)$, $\dots$, $H_m(x_i)$ cannot be 0 simultaneously.

Assume that x_i is a root of $P(x)$, thus also a root of $H(x)$. Then the sequence must start with $H(x_i) = 0$, hence $H_1(x_i) \neq 0$, and $H_1(x_i)$ is either positive as indicated in the main case of Table 5, or it is negative as indicated in the alternative case of Table 5.

If x_i is not a root of $P(x)$, then it is not a root of $H(x)$ either. Hence, the sequence starts with $H(x_i) \neq 0$.

It is possible that we later get $H_k(x_i) = 0$ in the sequence. Since we never can have two successive zeros in the sequence, both $H_{k-1}(x_i)$ and $H_{k+1}(x_i)$ are $\neq 0$. Using the formula from (29),

$$H_{k-1}(x_i) = H_k(x_i)\, Q_k(x_i) - H_{k+1}(x_i), \qquad \text{and} \qquad H_k(x_i) = 0,$$

we conclude that $H_{k-1}(x_i)$ and $H_{k+1}(x_i)$ must have different signs. The main case in Table 5 has +, -, and the alternative case has -, +.

For every given j, for which $H_j(x_i) \neq 0$, it follows that $H_j(x)$ has the same sign in the two adjacent intervals as at x_i. Hence, for every j, for which both $H_j(x_i)$ and $H_{j+1}(x_i)$ are $\neq 0$, the pair $(H_j(x), H_{j+1}(x))$ will give the same contribution (either 0 or 1) to $W(x)$ in the two adjacent intervals as at x_i.

Assume then that x_i is a root of $P(x)$ of multiplicity m. Then, by Taylor's formula,

$$P(x) = \frac{P^{(m)}(x_i)}{m!}(x - x_i)^m + \cdots \qquad \text{and} \qquad P'(x) = \frac{P^{(m)}(x_i)}{(m-1)!}(x - x_i)^{m-1} + \cdots,$$

so the two polynomials $P(x)$ and $P'(x)$ must in some interval $]x_i, x_i + \varepsilon[$ have the same sign (=

x	$-\infty$	-2	-1	0	1	2	$+\infty$
$P(x) = x^5 - x^4 - 3x^3 + 2x + 5$	-	-	+	+	+	+	+
$P'(x) = 5x^4 - 4x^3 - 9x^2 + 2$	+	+	+	+	-	+	+
$P_2(x) \approx 34x^3 + 9x^2 - 40x - 127$	-	-	-	-	-	+	+
$P_3(x) \approx 79x^2 - 574x + 827$	+	+	+	+	+	-	+
$P_4(x) \approx -3953x + 7578$	+	+	+	+	+	-	-
$P_5(x) =$ some negative const.	-	-	-	-	-	-	-
$W(x)$	4	4	3	3	3	1	1

Table 6: Table of $W(x)$ for $x = -2, -1, 0, 1, 2$.

same sign as $\left.P^{(m)}(x_i)\right)$. Hence, $H(x)$ and $H_1(x)$ must also have the same sign in $]x_i, x_i + \varepsilon[$, and consequently in the whole adjacent interval to the right of x_i.

Since x_i is a simple zero of $H(x)$, the sign of $H(x)$ in the adjacent interval to the left of x_i must be the opposite one of the sign in the interval to the right of x_i. Hence, the pair $(H(x), H_1(x))$ produces no change of sign at x_i and in the adjacent interval to the right of x_i, while we get one chance of sign in the adjacent interval to the left of x_i.

If $H_k(x_i) = 0$, then there is precisely one change of sign in the subsequence $H_{k-1}(x)$, $H_k(x)$, $H_{k+1}(x)$ at x_i and in the the two adjacent intervals, no matter the sign of $H_k(x)$.

Summing up, we have in the adjacent interval to the right of x_i the same number of changes of sign as at x_i. When we consider the adjacent interval to the left of x_i, we have found a loss of one change of sign, when we pass through x_i, if x_i is a root of $P(x)$, and no chance in the number of changes of sign, when x_i is not a root of $P(x)$, and Lemma 3.4.1, hence also Theorem 3.4.1, are proved. □

Notice that if x is not a multiple root of $P(x)$, i.e. not a root in $P_m(x)$, then $W(x)$ is equal to the number of changes of sign in the sequence $P(x)$, $P'(x)$, $P_2(x)$, …, $P_m(x)$, and we can avoid the division by $P_m(x)$, if we are content with finding the number of roots in intervals of endpoints which are not multiple roots.

Remark 3.4.1 It is usually very difficult and tedious to find the polynomials $P_2(x)$, …, $P_m(x)$ by the Euclidean algorithm, which is caused by the denominators of the coefficients. However, in the computation of $W(x)$ we only *count the changes of sign*, so we may, if convenient, multiply every polynomial by a positive constant. We use the symbol $\approx$ to indicate that some polynomial $P_q(x)$ has been multiplied by some positive constant. ♢

Example 3.4.1 Consider the polynomial $P(x) = x^5 - x^4 - 3x^3 + 2x + 5$ of Example 3.3.1, where we illustrated Fourier-Budan's theorem.

Leaving out the tedious computations we end up with Table 6.

Since $P_5(x)$ is a negative constant, we conclude that all roots of $P(x)$ are simple.

The columns corresponding to $-\infty$ and $+\infty$ correspond to large positive and negative x, so the signs are determined by the terms of highest degree.

There are $W(-\infty) - W(+\infty) = 4 - 1 = 3$ real roots. One of these lies in the interval $]-2,-1[$, and two of them in the interval $]1,2[$, cf. Figure 11. $\Diamond$

3.5 Rouché's theorem

We shall prove some general theorems from *Complex Functions Theory* and then apply them to polynomials in order to get the information of how many roots (counted by their multiplicity) a polynomial has in a domain bounded by a simple closed curve. By varying this curve we may obtain information of where the roots are more precisely lying in the complex plane.

We first prove the following general result.

Theorem 3.5.1 *Let $f : [a,b] \to \mathbb{C} \setminus \{0\}$ be a continuous function on a closed, bounded interval $[a,b]$, where $f(t) \neq 0$ for every $t \in [a,b]$. Then f has a* continuous *argument function* $\arg f(t)$.

Proof. a) First assume that there is an $\alpha \in \mathbb{R}$, such that

$$f(t) \in \mathbb{C} \setminus \left\{ z = r\, e^{i\alpha} \mid r \geq 0 \right\} \qquad \text{for every } t \in [a,b],$$

i.e. the image of $[a,b]$ by f does not cross the half line $\left\{ z = e^{i\alpha} \mid r \geq 0 \right\}$.

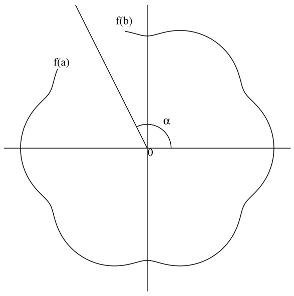

Figure 12: The image $f([a,b])$ does not cross the half line $\left\{ z = r\, e^{i\alpha} \mid r \geq 0 \right\}$.

Choose the argument function of $f(t)$, such that

$$\alpha < \arg f(t) < \alpha + 2\pi.$$

Define the logarithmic function $\text{Log}_\alpha : \mathbb{C} \setminus \left\{ z = r\, e^{i\alpha} \mid r \geq 0 \right\} \to \mathbb{C}$ by

$$\text{Log}_\alpha\, z = \ln |z| + i\, \text{Arg}_\alpha\, z, \qquad \text{where} \qquad \text{Arg}_\alpha\, z \in]\alpha, \alpha + 2\pi[.$$

Then Log_α is continuous, so the composed map $\text{Log}_\alpha \circ f$ is again continuous, hence also the imaginary part

$$\arg f(t) := \text{Arg}_\alpha\, f(t).$$

b) Then assume that no such α exists. We put

$$m = \inf\{|f(t)| \mid t \in [a,b]\}.$$

Since f is continuous on the bounded, closed interval $[a,b]$, and $f : [a,b] \to \mathbb{C} \setminus \{0\}$, we conclude from one of the main theorems for continuous functions that $m > 0$. It also follows from another one of the main theorems that f is *uniformly continuous* on $[a,b]$. Hence, corresponding to $m > 0$ there is a $\delta > 0$, such that

$$|f(s) - f(t)| < m \qquad \text{for all } s,\, t \in [a,b] \text{ for which } |s - t| < \delta.$$

Choose division points $a = t_0 < t_1 < \cdots < t_n = b$, such that $|t_j - t_{j-1}| < \delta$ for $j = 1, \dots, n$. Then to each of the intervals $[t_{j-1}, t_j]$ there is an $\alpha_j \in \mathbb{R}$, such that

$$f(t) \in \mathbb{C} \setminus \{ z = r\, e^{\alpha_j} \mid r \geq 0 \} \qquad \text{for every } t \in [t_{j-1}, t_j]\,.$$

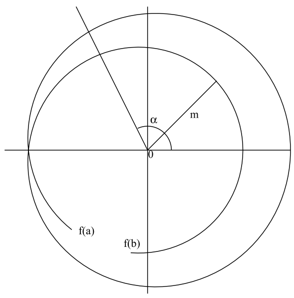

Figure 13: The image $f([a,b])$ crosses every half line $\left\{z = r\,e^{i\alpha} \mid r \geq 0\right\}$.

we may e.g. choose $\alpha_j = \arg f(t_j) + \pi$.

It follows from a) above that there exists a continuous argument function $\arg_j f(t)$ on $[t_{j-1}, t_j]$.

Furthermore, these argument functions can be chosen such that

$$\arg_j f(t_j) = \arg_{j+1} f(t_j) \qquad \text{at } t_j \in [t_{j-1}, t_j] \cap [t_j, t_{j+1}] = \{t_j\}\,.$$

Then the argument function $\arg f(t)$, defined by

$$\arg f(t) := \arg_j f(t) \qquad \text{for } t \in [t_{j-1}, t_j]\,, \quad j = 1, \dots, n,$$

is uniquely determined and continuous. □

Given one continuous argument function $\Theta(t) = \arg f(t)$, any other continuous argument function is given by $\Theta(t) + 2p\pi$ for some $p \in \mathbb{Z}$. In fact, it is obvious that $\Theta(t) + 2p\pi$ is a continuous argument function, and if $\Theta(t)$ and $\Theta_1(t)$ both are continuous argument functions for $f(t)$, then $\Theta_1(t) - \Theta(t)$ is continuous on $[a,b]$. Since arguments differ by $2p\pi$ for some $p \in \mathbb{Z}$, it follows from the continuity that p must be constant in $[a,b]$, and the claim is proved.

It follows in particular from the above that the difference

(30) $\arg f(b) - \arg f(a)$

has the same value for *every continuous argument function* $\arg f(t)$.

We call this difference (30) the *argument variation* of f along $[a,b]$.

If in particular $f(a) = f(b)$, i.e. the continuous curve $z = f(t)$, $t \in [a,b]$, is a closed curve which does not pass through 0, then the argument variation is an integer times 2π,

(31) $\arg f(b) - \arg f(a) = 2n\pi \qquad \text{for some } n \in \mathbb{Z}.$

The number $n \in \mathbb{Z}$ of (31) is called the *winding number* around 0 of the function $f : [a,b] \to \mathbb{C} \setminus \{0\}$, or the closed curve $f([a,b])$ not passing through 0.

The winding number is interpreted as the number of times the curve winds around 0, counted positive in the positive sense of the plane, and negative otherwise. We notice that counting negative loops may cancel some of the counting of positive loops.

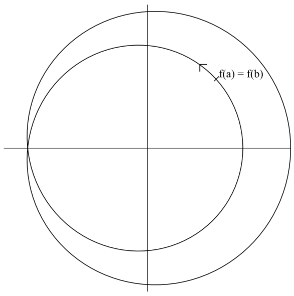

Figure 14: The winding number of a closed curve not passing through 0. In the present example the winding number is 2.

The geometrical interpretation above of the winding number is often very easy in practice. We shall later prove that the winding number is equal to the difference of the number of zeros and the number is equal to the difference of the number of zeros and the number of poles in a domain of an analytic function. For polynomials the number of poles is 0, so we get the number of roots, counted by their multiplicity.

Before we can make the statement more precise we need another theorem.

Theorem 3.5.2 *Given two continuous and complex functions $f : [a,b] \to \mathbb{C} \setminus \{0\}$ and $g : [a,b] \to \mathbb{C}$, for which*

$$f(a) = f(b) \qquad \text{and} \qquad g(a) = g(b).$$

If

$$(32)\quad |g(t)| < |f(t)| \qquad \text{for every } t \in [a,b],$$

then the two functions f and $f + g$ have the same winding number around 0.

Proof. From the estimate

$$|f(t) + g(t)| \geq |f(t)| - |g(t)| > 0 \qquad \text{for every } t \in [a,b],$$

follows that both f and $f + g : [a,b] \to \mathbb{C} \setminus \{0\}$ are continuous and that neither of then is 0 in the interval $[a,b]$. Furthermore,

$$f(a) = f(b) \qquad \text{and} \qquad (f+g)(a) = (f+g)(b),$$

so the images $f([a,b])$ and $(f+g)([a,b])$ are both closed curves, not passing through 0. We write

$$f(t) + g(t) = f(t) \cdot \left\{1 + \frac{g(t)}{f(t)}\right\}, \qquad t \in [a,b].$$

It follows from $\left|\frac{g(t)}{f(t)}\right| < 1$ that

$$\Re\left\{1 + \frac{g(t)}{f(t)}\right\} \geq 1 - \left|\frac{g(t)}{f(t)}\right| > 0,$$

hence $1+\dfrac{g(t)}{f(t)}$ lies in the right half plane for every $t \in [a,b]$. In particular, the principal argument $\operatorname{Arg}\left\{1+\dfrac{g(t)}{f(t)}\right\}$ is continuous for $t \in [a,b]$.

Choose any continuous argument function $\arg^* f$ of f. Then $\arg^* f(t) + \operatorname{Arg}\left\{1+\dfrac{g(t)}{f(t)}\right\}$ must be a continuous argument function $\arg(f+g)$ for $f+g$. Finally, since

$$\operatorname{Arg}\left\{1+\frac{g(a)}{f(a)}\right\} = \operatorname{Arg}\left\{1+\frac{g(b)}{f(b)}\right\},$$

we conclude that

$$\arg(f+g)(b) - \arg(f+g)(a) = \arg^* f(b) - \arg^* f(a) = 2n\pi,$$

and the theorem is proved. □

The importance of Theorem 3.5.2 lies in the fact that it allows us slightly to perturb closed curves without changing their winding numbers.

An obvious extension of the definition of the *winding number* is the following: Let $\Omega \subseteq \mathbb{C}$ be an open domain, and C a simple closed curve in Ω. Let $f : \Omega \to \mathbb{C} \setminus \{0\}$ be a continuous map. Then the image $f(C)$ must be a closed curve in $\mathbb{C} \setminus \{0\}$, and as such it has a *winding number* around 0.

We shall now restrict ourselves to complex functions $f(z)$ which are either polynomials, or fractions of polynomials. Then we introduce

Definition 3.5.1 *Let $f : \Omega \to \mathbb{C}$ be a fraction of two polynomials, $f(z) = \dfrac{P(z)}{Q(z)}$, $Q(z) \neq 0$ for all $z \in \Omega$. We define the* logarithmic derivative *of $f(z)$ by*

$$\frac{f'(z)}{f(z)}, \qquad \textit{for } z \in \{z \in \Omega \mid f(z) \neq 0\}.$$

Hence, the logarithmic derivative of $f(z) = \dfrac{P(z)}{Q(z)}$ is defined in the set

$$\Omega^* := \{z \in \mathbb{C} \mid P(z) \neq 0 \text{ and } Q(z) \neq 0\}.$$

If in particular, $f : \Omega \to \mathbb{C}\,(\mathbb{R}_- \cup \{0\})$ does not have real values ≤ 0 for any $z \in \Omega$, then the principal logarithm $\text{Log}\, f(z)$ of $f(z)$ is analytic in Ω, and its derivative is

$$(33) \quad \frac{d}{dz}\,\text{Log}\, f(z) = \frac{f'(z)}{f(z)}.$$

This is the reason why we in general call the right hand side of (33) the *logarithmic derivative* of $f(z)$, even when $\text{Log}\, f(z)$ is not defined.

Theorem 3.5.3 The argument principle. *Let $f(z) = \dfrac{P(z)}{Q(z)}$ be a quotient of two polynomials $P(z)$ and $Q(z)$. Let C be a simple closed curve in $\Omega = \{z \in \mathbb{C} \mid P(z) \neq 0$ and $Q(z) \neq 0\}$, and let ω be the* bounded *domain of boundary $f(C)$.*
Let $N = N(\omega)$ denote the number of zeros of the numerator $P(z)$ in ω, and $R = R(\omega)$ the number of zeros of the denominator $Q(z)$ in ω, all counted according to their multiplicities.
Then the difference $N - R$ is equal to the winding number of the closed curve $f(C)$ around 0 in the w-plane.
We have more precisely,

$$(34) \quad \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N(\omega) - R(\omega) = \textit{winding number of } f(C) \textit{ around } w_0 = 0.$$

Proof. We first prove that $\dfrac{1}{2\pi i}\oint_C \dfrac{f'(z)}{f(z)}\,\mathrm{d}z$ is the winding number of $f(C)$ around the point $w_0 = 0$ in the w-plane. Assume that C is given by the parametric description $z(t)$, $t \in [a, b]$. Then we define by $g(t) = f(z(t))$ a continuous complex function $g : [a, b] \to \mathbb{C} \setminus \{0\}$ with a continuous argument function $\arg g$.

Use the same construction as in b) in the proof of Theorem 3.5.1 to conclude that to every subinterval $[t_j, t_{j+1}]$ there corresponds a curve C_j, which is a subset of C. Then we get the computations,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=0}^{n-1} \frac{1}{2\pi i}\int_{C_j} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j=0}^{n-1} \frac{1}{2\pi i}\int_{t_j}^{t_{j+1}} \frac{g'(t)}{g(t)}\,\mathrm{d}t = \sum_{j=0}^{n-1} \frac{1}{2\pi i}\left[\ln|g(t)| + i\cdot \arg g(t)\right]_{t_j}^{t_{j+1}}$$

$$= \frac{1}{2\pi i}\sum_{j=0}^{n-1}\{\ln|g(t_{j+1})| - \ln|g(t_j)|\} + \frac{1}{2\pi}\sum_{j=0}^{n-1}\{\arg g(t_{j+1}) - \arg g(t_j)\}$$

$$= \frac{1}{2\pi i}\{\ln|g(t_n)| - \ln|g(t_0)|\} + \frac{1}{2\pi}\{\arg g(t_n) - \arg g(t_0)\}$$

$$= \frac{1}{2\pi}\{\ln|g(b)| - \ln|g(a)|\} + \frac{1}{2\pi}\{\arg g(b) - \arg g(a)\}$$

$$= 0 + \text{winding number of } g([a,b]) = f(C) \text{ around } w_0 = 0.$$

Finally,

$$(35)\quad \frac{f'(z)}{f(z)} = \frac{Q(z)}{P(z)}\cdot\frac{P'(z)\cdot Q(z) - Q'(z)\cdot P(z)}{Q(z)^2} = \frac{P'(z)}{P(z)} - \frac{Q'(z)}{Q(z)},$$

so the claim follows, if only we can prove it for $f(z) = P(z)$, a polynomial.

Given a polynomial, $P(z) = a\cdot(z-z_1)^{n_1}\cdots(z-z_p)^{n_p}$. Then

$$\frac{P'(z)}{P(z)} = \frac{n_1}{z-z_1} + \cdots + \frac{n_p}{z-z_p},$$

so

$$\frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}\,\mathrm{d}z = \sum_{j=1}^{p}\frac{1}{2\pi i}\oint_C \frac{n_j}{z-z_j}\,\mathrm{d}z.$$

We shall without proof use the well-known fact that

$$\frac{1}{2\pi i}\oint_C \frac{\mathrm{d}z}{z-z_0} = \begin{cases} 1 & \text{if } z_0 \text{ lies inside } C,\\ 0 & \text{if } z_0 \text{ lies outside } C.\end{cases}$$

This gives

$$\frac{1}{2\pi i}\oint_C \frac{P'(z)}{P(z)}\,\mathrm{d}z = \text{number of zeros of } P(z) \text{ inside } C.$$

Similarly,

$$\frac{1}{2\pi i}\oint_C \frac{Q'(z)}{Q(z)}\,\mathrm{d}z = \text{number of zeros of } Q(z) \text{ inside } C,$$

so when we integrate (35) the claim follows by insertion. □

Combining Theorem 3.5.2 and Theorem 3.5.3, *the argument principle*, we easily get

Theorem 3.5.4 Rouché's theorem for polynomials. *Let $P(z)$ and $Q(z)$ be polynomials, and let C be a simple closed curve in $\mathbb{C}$. If*

$$|P(z)| > |Q(z)| \qquad \textit{for every } z \in C,$$

then $P(z)$ and $P(z)+Q(z)$ have the same number of zeros inside C.

PROOF. Theorem 3.5.2 tells us that $P(z)$ and $P(z)+Q(z)$ have the same winding number with respect to $w_0 = 0$ in the w-plane. Theorem 3.5.3 tells us for polynomials that the winding number is equal to the number of zeros. □

A simple consequence of Theorem 3.5.4 is another proof of

Corollary 3.5.1 The fundamental theorem of algebra. *Given a polynomial of degree $n \in \mathbb{N}$,*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad \textit{where } a_0 \neq 0.$$

Then $P(z)$ has precisely n roots, counted by multiplicity.

PROOF. It follows from

$$P(z) = z^n \left\{ a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} \right\}, \qquad z \neq 0,$$

that there is an R, such that

$$|a_0|\, r^n > |a_n| + |a_{n-1}|\, r + \cdots + |a_1|\, r^{n-1} \qquad \text{for } r \geq R.$$

Putting $P_1(z) = a_0 z^n$ and $P_2(z) = a_n + a_{n-1} z + \cdots + a_1 z^{n-1}$ we see that

$$|P_1(z)| > |P_2(z)| \qquad \text{for every } z \in \mathbb{C}, \text{ for which } |z| \geq R.$$

It follows from Theorem 3.5.4 that $P_(z) = a_0 z^n$ and $P_1(z) + P_2(z) = P(z)$ have the same number of zeros inside every circle C_r of radius $r \geq R$ and the common centre 0. Then the claim follows, because $P_1(z) = a_0 z^n$ trivially has an n-tuple zero at $z = 0$ and no other zero. □

Example 3.5.1 Rouché's theorem does not preserve the multiplicity of a given zero. Given a complex constant $a \in \mathbb{C}$, where $0 < |a| < 1$. Choose $P(z) = z^n$ and $Q(z) = -a$. If C is the unit circle $|z| = 1$, then

$$|P(z)| = |z^n| = 1 > |a| = |Q(z)| \qquad \text{for } |z| = 1,$$

so it follows from *Rouché's theorem* that $P(z) = z^n$ and $P(z) + Q(z) = z^n - a$ have the same *number* $(= n)$ of zeros in the open unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. However, $P(z)$ has the zero $z_0 = 0$ of multiplicity n, while $P(z) + Q(z)$ has n *simple roots*, all lying on the circle of radius $\sqrt[n]{|a|} \neq 0$. ◊

Example 3.5.2 A typical application of *Rouché's theorem* is the following. Given the polynomial

$$z^3 + 2z^2 - 50z + 100.$$

It has three zeros by the *fundamental theorem of algebra.*

These three roots all lie inside the circle of equation $|z| = 9$. In fact, choose $P_1(z) = z^3$ and $Q_1(z) = 2z^2 - 50z + 100$. Then we have the following estimate for $|z| = 9$,

$$|P_1(z)| = 9^3 = 729 \qquad \text{and} \qquad |Q_1(z)| \le 2 \cdot 9^2 + 50 \cdot 9 + 100 = 712,$$

so $|Q_1(z)| < |P_1(z)|$ for $|z| = 9$, and the claim follows from an application of *Rouché's theorem.*

There is only one root inside the circle $|z| = 4$. We again apply *Rouché's theorem.* However, this time we choose $P_2(z) = -50z$ and $Q_2(z) = z^3 + 2z^2 + 100$ and get the following estimates for $|z| = 4$,

$$|P_2(z)| = |-50z| = 200 \qquad \text{and} \qquad |Q_2(z)| = \left|z^3 + 2z^2 + 100\right| \le 64 + 32 + 100 = 196,$$

so $|P_2(z)| > |Q_2(z)|$ for $|z| = 4$, and the claim follows, because the only root of $P_2(z) = -50z$ inside $|z| = 4$ is $z = 0$.

Finally, there is no root lying inside $|z| = \frac{7}{4}$. In this case we choose

$$P_3(z) = 100 \qquad \text{and} \qquad Q_3(z) = z^3 + 2z^2 - 50z,$$

in which case we get the following estimate for $|z| = \frac{7}{4}$,

$$\begin{aligned} |Q_3(z)| &= \left|z^3 + 2z^2 - 50z\right| \le \left(2 + \frac{7}{4}\right)\left\{\frac{7}{4}\right\}^2 + 50 \cdot \frac{7}{4} = \frac{15}{4} \cdot \frac{49}{19} + \frac{7}{4} \cdot 50 \\ &< \left\{\frac{1}{4} + \frac{7}{4}\right\} \cdot 50 = 100 = |P_3(z)|, \end{aligned}$$

and the claim follows, because the constant $P_3(z) = 100 \neq 0$.

The roots are numerically computed to be approximately

$$-8.889\,794\,306, \qquad 2.658\,473\,477, \qquad 4.231\,320\,828.$$

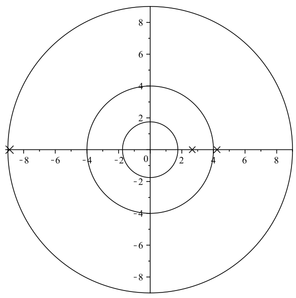

Figure 15: The roots of $z^3 + 2z^2 - 50z + 100$ and Rouché's theorem.

It should be noted that if we change the sign of the term $-50z$, so we instead consider the polynomial $z^3 + 2z^2 + 50z + 100$, then we can without any changes repeat all the arguments above, so we have

- three roots inside $|z| = 9$,
- one root inside $|z| = 4$,
- no root inside $|z| = \frac{7}{4}$.

It is, however, in this case possible to find the roots directly, since we have

$$z^3 + 2z^2 + 50z + 100 = (z+2)\left(z^2 + 50\right),$$

so the roots are -2, $5\sqrt{2}\,i$ and $-5\sqrt{2}\,i$. $\Diamond$

Example 3.5.3 Consider the polynomial $3z^{87} - z^3 + 1$. The degree is 87, so by the *fundamental theorem of algebra* it has 87 roots. They all lie in the open unit disc. In fact, choosing

$$P_1(z) = 3z^{87} \qquad \text{and} \qquad Q_1(z) = -z^3 + 1,$$

we get the following estimate for $|z| = 1$,

$$|Q_1(z)| = \left|-z^3 + 1\right| \le 2 < 3 = \left|3z^{87}\right| = |P_1(z)|\,.$$

Since the dominating term $P_1(z)$ only has the 87-tuple root of $z_0 = 0$ lying inside $|z| = 1$, the claim follows.

Choosing $P_2(z) = 1$ and $Q_2(z) = 3z^{87} - z^3$, we get the estimate, using e.g. a pocket calculator,

$$|Q_2(z)| = \left|3z^{87} - z^3\right| \le 0.98 < 1 = |P_2(z)| \qquad \text{for } |z| = 0.96,$$

so there lies no zero in the slightly smaller disc $|z| \le 0.96$. Hence, all 87 roots z_j must lie in the narrow annulus given by $0.96 < |z_j| < 1$, so it is from a numerical point of view fairly crowded concerning the roots in this narrow annulus. For the time being it does not look too promising to find (numerically) these roots with any prescribed tolerated uncertainty. That it is possible (though we shall not later in this book do it for the given example), follows from an application of *Graeffe's squaring method*, which will be described in Chapter 4. $\Diamond$

3.6 Hurwitz polynomials

In connection with the question of stability of mechanical or electrical systems concerning oscillations, it is of great importance to be able to decide whether a polynomial has all its roots lying an the open left hand side of the plane. Polynomials of this property of only having roots in the open left hand side of the plane are called *Hurwitz polynomials*. In this connection and aiming at proving stability it is, however, of less importance also to find approximate values of these roots. It suffices in most applications that they all have a negative real part.

In order to be more precise concerning what is meant by a *Hurwitz polynomial* we shall start with the following considerations. Given the polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of complex coefficients. Then clearly the polynomial

$$\overline{P}(z) := \overline{a_0}\, z^n + \overline{a_1}\, z^{n-1} + \cdots + \overline{a_{n-1}}\, z + \overline{a_n} = \overline{a_0 \overline{z}^n + a_1 \overline{z}^{n-1} + \cdots + a_{n-1}\overline{z} + a_n},$$

where we have taken the complex conjugated coefficients, must have the complex conjugated roots of those of $P(z)$. Hence, the roots of the product $P(z)\overline{P}(z)$ are either real, or complex conjugated of the same multiplicity, and $P(z)\overline{P}(z)$ must have *real* coefficients. Now, complex conjugation maps the left (or right) half plane into itself, so without loss of generality we may in the remainder of this section restrict ourselves to only consider polynomials of *real* coefficients, $a_0, \dots, a_n \in \mathbb{R}$.

Then we introduce the following more precise definition.

Definition 3.6.1 *We say that a polynomial $P(z)$ of real coefficients is a* Hurwitz polynomial*, if all its zeros lie in the open left half plane $\Re\, z < 0$.*

It follows by the *fundamental theorem of algebra* that

(36) $P(z) = a_0 z^n + \cdots + a_n = a_0 (z - \lambda_1) \cdots (z - \lambda_n)\,, \qquad z \in \mathbb{C},$

where some of the λ_j may be identical. If $P(z)$ is a Hurwitz polynomial, and $\alpha + i\beta$, $\alpha < 0$, $\beta \neq 0$, is a root, then $\alpha - i\beta$ is also a root. Thus,

$$(z - \alpha - i\beta)(z - \alpha + i\beta) = (z - \alpha)^2 + \beta^2 = z^2 - 2\alpha z + \left(\alpha^2 + \beta^2\right)$$

must be a divisor of $P(z)$. By assumption $\alpha < 0$, so this divisor has only positive coefficients.

If $P(z)$ is a Hurwitz polynomial, and λ is a real root, then $\lambda < 0$, so the divisor $z - \lambda$ has trivially positive coefficients.

Hence, if $P(z)$ is a Hurwitz polynomial, then it can be factorized into factors of first or second degree, all of positive coefficients. Then it follows after a multiplication that $P(z)$ itself must have positive coefficients, and we have proved

Theorem 3.6.1 *A* necessary *condition that a polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

of real coefficients is a Hurwitz polynomial, is that all its coefficients have the same sign, either all positive or all negative.

For $n = 1$ and $n = 2$ the condition of Theorem 3.6.1 is also sufficient. This is no longer the case when $n \geq 3$.

Example 3.6.1 The polynomial

$$16z^3 + 8z^2 + 9z + 17$$

satisfies the *necessary condition* of Theorem 3.6.1. Its roots are -1, $\frac{1}{4} + i$ and $\frac{1}{4} - i$, so two of them have positive real part, and the polynomial is not a Hurwitz polynomial, showing that the condition of Theorem 3.6.1 is not sufficient for polynomials of degree ≥ 3. $\Diamond$

Theorem 3.6.2 *A polynomial $P(z)$ of real coefficients is a Hurwitz polynomial, if and only if*

$$\begin{cases} |P(z)| > |P(-z)| & \text{for } \Re\, z > 0, \\ P(iy) \neq 0 & \text{for } y \in \mathbb{R}. \end{cases}$$

Proof. Factorize $P(z)$,

$$P(z) = a_0 z^n + \cdots + a_n = a_0 (z - \lambda_1) \cdots (z - \lambda_n)\,, \qquad z \in \mathbb{C}.$$

Assume that $\lambda_j = \alpha + i\beta$, where $\Re\, \lambda_j = \alpha < 0$. If we write $z = x + iy$, then

$$|z - \lambda_j|^2 = (x - \alpha)^2 + (y - \beta)^2.$$

If $\Re z = x > 0$, then it follows from $\alpha < 0$ that

$$|z - \lambda_j|^2 = (x-\alpha)^2 + (y-\beta)^2 > (-x-\alpha)^2 + (-y+\beta)^2 = \left|-z-\overline{\lambda_j}\right|^2 .$$

If $\lambda_j = \alpha$ is real, i.e. $\beta = 0$, then it follows straightaway that

$$z - \alpha| > |-z-\alpha| \qquad \text{for } \Re z > 0.$$

If instead λ_j is not real, i.e. $\beta \neq 0$, then

$$\left|(z-\lambda_j)\left(z-\overline{\lambda_j}\right)\right| > \left|\left(-z-\overline{\lambda_j}\right)(-z-\lambda_j)\right| \qquad \text{for } \Re z > 0.$$

If therefore $P(z)$ is a Hurwitz polynomial, then

$$|P(z)| > |P(-z)| \qquad \text{for } \Re z > 0.$$

If $x = 0$, then $|P(z)| = |P(-z)| = |P(iy)| = |P(-iy)| \neq 0$, because $P(z)$ is a Hurwitz polynomial, so it has no zero on the imaginary axis.

Conversely, if $P(z)$ is a polynomial of real coefficients satisfying $|P(z)| > |P(-z)|$ for $\Re z > 0$, then in particular $P(z) \neq 0$ for $\Re z > 0$. Adding the condition that also $P(iy) \neq 0$ for $y \in \mathbb{R}$ proves that $P(z)$ is a Hurwitz polynomial, and the theorem is proved. □

Theorem 3.6.3 Schur's criterion. *The polynomial $P(z)$ of real coefficients is a Hurwitz polynomial, if and only if all its coefficients are of the same sign, and the polynomial*

$$Q(z) = \frac{P(1)P(z) - P(-1)P(-z)}{z+1}$$

of lower degree is a Hurwitz polynomial.

PROOF. When $P(z)$ is a Hurwitz polynomial we put

$$R(z) = P(1)P(z) - P(-1)P(-z).$$

Applying Theorem 3.6.2 on $P(z)$ we get $|P(z)| > |P(-z)|$ for $\Re z > 0$, and $|P(1)| > |P(-1)|$, so

$$|P(1)P(z)| > |P(-1)P(-z)| \qquad \text{for } \Re z > 0,$$

and also for $\Re z = 0$, because then $|P(z)| = |P(-z)| > 0$ and $P(1)| > |P(-1)|$. Hence, $R(z)$ is also a Hurwitz polynomial, because $R(z) \neq 0$ for $\Re z \geq 0$.

Trivially,

$$R(-1) = P(1)P(-1) - P(-1)P(1) = 0,$$

so -1 is a root, and $z+1$ is a divisor, i.e. $\dfrac{R(z)}{z+1}$ is a polynomial of lower degree, i.e.

$$(37) \quad Q(z) := \frac{R(z)}{z+1} = \frac{P(1)P(z) - P(-1)P(-z)}{z+1}$$

is a Hurwitz polynomial.

We shall then prove that if $P(z)$ is *not* a Hurwitz polynomial, then $Q(z)$ given by (37) is *not* a Hurwitz polynomial, which will conclude the proof.

Assume that $P(z)$ has a root iy_0 on the imaginary axis. Then $-iy_0$ is also a root of $P(z)$, so

$$Q(iy_0) = \frac{P(1)P(iy_0) - P(-1)P(-iy0)}{iy_0 + 1} = 0,$$

and iy_0 is a root of $Q(iy_0)$. This shows that $Q(z)$ is not a Hurwitz polynomial in this case.

Assume that $P(z_0) = 0$, where $\Re z_0 > 0$. Then

$$(z_0+1)\,Q(z_0) = -P(-1)P(-z_0)\,, \qquad (-z_0+1)\,Q(-z_0) = P(1)P(-z_0)\,.$$

If also $P(-z_0) = 0$, then it immediately follows that $Q(z_0) = 0$, and $Q(z)$ is not a Hurwitz polynomial in this case.

Therefore, the remaining possibility is that $P(z_0) = 0$, while $P(-z_0) \neq 0$. Then apply that $a_0, \ldots, a_n$ all have the same sign, so

$$P(-1)| = \left|(-1)^n a_0 + (-1)^{n-1} a_1 + \cdots + a_n\right| < |a_0 + a_1 + \cdots + a_n| = P(1)|.$$

It follows that

$$|(z_0+1)\,Q(z_0)| = |P(-1)| \cdot |P(-z_0)| < |P(1)| \cdot |P(-z_0)| = |(-z_0+1)\,Q(-z_0)|\,.$$

Now, $\Re z_0 > 0$, so it follows immediately from Theorem 3.6.2 that $(z+1)Q(z)$ is not a Hurwitz polynomial, thus $Q(z)$ is not a Hurwitz polynomial, and Schur's criterion is proved. □

Since $\deg Q < \deg P$, Schur's criterion may be applied at most $n-2$ times before we can conclude whether $P(z)$ is a Hurwitz polynomial or not.

Example 3.6.2 Given the polynomial $P(z) = z^3 + 2z^2 + 3z + 1$ we compute by Schur's criterion,

$$Q(z) = \frac{1}{z+1}\left\{7\left(z^3+2z^2+3z+1\right) - (-1)\left(-z^3+2z^2-3z+1\right)\right\} = 6z^2+10z+8.$$

It follows that $Q(z)$ is a polynomial of second degree of positive coefficients, so by the remark following Theorem 3.6.1, the polynomial $Q(z)$, hence also $P(z)$, is a Hurwitz polynomial. ◊

Example 3.6.3 Given the polynomial $P(z) = z^3 + 2z^2 + z + 3$ we compute by Schur's criterion,

$$Q(z) = \frac{1}{z+1}\left\{7\left(z^3+2z^2+z+3\right) - 3\left(-z^3+2z^2-z+3\right)\right\} = 10z^2-2z+12.$$

Clearly, $Q(z)$ is not a Hurwitz polynomial, so $P(z)$ is not a Hurwitz polynomial either. ◊

We mention without proof the following most commonly used criterion of Hurwitz polynomial. The proof is very long and tedious.

Theorem 3.6.4 Hurwitz's criterion (1895). *Given a polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of positive *coefficients. Then $P(z)$ is a Hurwitz polynomial, if and only if the following system of inequalities is fulfilled,*

$$D_1 = a_1 > 0, \qquad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \qquad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \qquad \dots,$$

$$D_n = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & a_n \end{vmatrix} > 0,$$

where we have put $a_k = 0$ for $k > n$.

Example 3.6.4 *(Cf. also Example 3.6.2).* We get for the polynomial $P(z) = z^3 + 2z^2 + 3z + 1$,

$$D_1 = 2 > 0, \qquad D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 > 0, \qquad D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 5 > 0,$$

so $P(z)$ is a Hurwitz polynomial, and all three roots lie in the left half plane. ◊

Example 3.6.5 *(Cf. also Example 3.6.3).* We get for the polynomial $P(z) = z^3 + 2z^2 + z + 3$,

$$D_2 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 < 0.$$

Hence, there is at least one root satisfying $\Re z \geq 0$. Assuming that $z = iy$ is purely imaginary we get

$$P(iy) = \left(3 - 2y^2\right) + iy\left(1 - y^2\right) \neq 0 \qquad \text{for all } y \in \mathbb{R},$$

so there is at least one root of positive real part. It cannot be real, so we have two complex conjugated roots of positive real part. The approximate values of the roots are

$$-2.174\,559\,41 \qquad \text{and} \qquad 0.087\,279\,7 \pm 1.171\,312\,1\,i. \qquad \diamond$$

4 Approximation methods

4.1 Newton's approximation formula

This is usually derived from *Banach's fix point theorem.*. We shall not formulate and prove the latter theorem in its full generality in a complete metric space, but only consider the space of real numbers $\mathbb{R}$, or the space of complex numbers $\mathbb{C}$.

Definition 4.1.1 *We say that a map* $f : \mathbb{R} \to \mathbb{R}$, *or* $f : \mathbb{C} \to \mathbb{C}$, *is a* contraction, *if there exists a* $\lambda \in [0, 1[$, *the* contraction factor, *such that*

$$|f(y) - f(x)| \le \lambda |y - x| \qquad \text{for all } x,\, y \in \mathbb{R} \text{ (or } \in \mathbb{C}\text{)}.$$

A contraction is clearly continuous, and if we put $f^{\circ n} = f \circ \cdots \circ f$, i.e. a composition n times, then $f^{\circ n}$ is a contraction of contraction factor λ^n. In fact,

$$\begin{aligned} |f^{\circ n}(y) - f^{\circ n}(x)| &= \left| \left(f^{\circ(n-1)}(y) \right) - f\left(f^{\circ(n-1)}(x) \right) \right| \le \lambda \left| f^{\circ(n-1)}(y) - f^{\circ(n-1)}(x) \right| \\ &\le \cdots \le \lambda^n \, |y - x|, \end{aligned}$$

and the claim is proved.

Definition 4.1.2 *Let* $f : \mathbb{R} \to \mathbb{R}$, *or* $f : \mathbb{C} \to \mathbb{C}$, *be given. We say that* $x_0 \in \mathbb{R}\,(\mathbb{C})$ *is a* fix point *of* f, *if* $f(x_0) = x_0$.

We can now prove *Banach's fix point theorem* for $\mathbb{R}$ and $\mathbb{C}$, and we notice that if we follow the same proof with obvious modifications, the theorem is proved in general for complete metric spaces.

Theorem 4.1.1 Banach's fix point theorem in $\mathbb{R}$, or $\mathbb{C}$. *Every contraction on* $\mathbb{R}$ *or* $\mathbb{C}$ *has precisely one fix point* x_0. *If* $x \in \mathbb{R}$*(or* $\mathbb{C}$*) is any given point, then* $f^{\circ n}(x) \to x_0$ *for* $n \to +\infty$.

PROOF. Assume that $f : \mathbb{R} \to \mathbb{R}$ (of $f : \mathbb{C} \to \mathbb{C}$) is a contraction of contraction factor $\lambda \in [0, 1[$.

Uniqueness. Assume that x_1 and x_2 are two fix points of f. Then

$$0 \le |x_1 - x_2| = |f(x_1) - f(x_2)| \le \lambda\, |x_1 - x_2|\,,$$

because f is a contraction. Since $0 \le \lambda < 1$, this is only possible, if $x_1 = x_2$, so the contraction has at most one fix point.

Existence. Choose any fixed $x \in \mathbb{R}$ (or $\mathbb{C}$), and consider the sequence $(f^{\circ n}(x))$. Then for every $n \in \mathbb{N}$,

$$\left| f^{\circ n}(x) - f^{\circ(n+1)}(x) \right| = |f^{\circ n}(x) - f^{\circ n}(f(x))| \le \lambda^n |x - f(x)|,$$

because $f^{\circ n}$ is a contraction of contraction factor λ^n.

Then for any given $n, k \in \mathbb{N}$, it follows from the above that

$$\begin{aligned}\left|f^{\circ n}(x)-f^{\circ(n+k)}(x)\right| &\leq \left|f^{\circ n}(x)-f^{\circ(n+1)}(x)\right|+\cdots+\left|f^{\circ(n+k-1)}(x)-f^{\circ(n+k)}(x)\right| \\ &\leq \left\{\lambda^n+\lambda^{n+1}+\cdots+\lambda^{n+k-1}\right\}|x-f(x)|=\lambda^n\cdot\frac{1-\lambda^k}{1-\lambda}|x-f(x)| \\ &\leq \frac{\lambda^n}{1-\lambda}|x-f(x)|.\end{aligned}$$

From $\lambda \in [0,1[$ follows that

$$\frac{\lambda^n}{1-\lambda}|x-f(x)| \to 0 \qquad \text{for } n \to +\infty,$$

hence $(f^{\circ n}(x))$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$). Since $\mathbb{R}$ and $\mathbb{C}$ are complete, the Cauchy sequence $(f^{\circ n}(x))$ is convergent with the limit x_0, say, thus

$$f^{\circ n}(x) \to x_0 \qquad \text{for } n \to +\infty.$$

Furthermore, since f is continuous,

$$f\left(f^{\circ n}(x)\right)=f^{\circ(n+1)}(x) \to f(x) \qquad \text{for } n \to +\infty.$$

The limit of a convergent sequence in $\mathbb{R}$ (or $\mathbb{C}$) is unique, so we conclude that x_0, constructed in this way as the limit of the sequence $(f^{\circ n}(x))$, is indeed a fix point. □

We are only considering polynomials, so there is no need to formulate *Newton's iteration method* for real C^2 functions. If a polynomial $Q(x)$ has multiple roots, then we know from Chapter 2 how we can find another polynomial $P(x)$ of precisely the same roots, all of them, however, only of multiplicity 1. Therefore, we can without loss of generality in the following assume that the polynomial $P(x)$ has only simple roots, i.e. if $P(x_0)=0$, then $P'(x_0) \neq 0$.

Theorem 4.1.2 Newton's iteration method for polynomials of only simple roots. *Let $P(x)$ be a polynomial of only simple roots, and assume that x_0 is a (real or complex) zero of $P(x)$. If we choose x_1 sufficiently close to x_0 and define*

$$x_{n+1}=f(x_n):=x_n-\frac{P(x_n)}{P'(x_n)} \qquad \textit{for } n \in \mathbb{N},$$

then $x_n \to x_0$ for $n \to +\infty$.

Proof. Since x_0 is a simple zero, we must have $P'(x) \neq 0$ in an open neighbourhood Ω of x_0. Hence the map

$$f(x):=x-\frac{P(x)}{P'(x)}, \qquad \text{for } x \in \Omega,$$

is of class $C^\infty(U)$, and even an analytic function in U, because $P(x)$ is a polynomial. It follows that

$$f'(x)=1-\frac{P'(x)}{P'(x)}+\frac{P(x)P''(x)}{\{P'(x)\}^2}=\frac{P(x)P''(x)}{\{P'(x)\}^2} \qquad \text{in } U.$$

In particular, $f'(x_0) = 0$, because $P(x_0) = 0$. Since

$$\frac{f(x) - f(y)}{x - y} \approx f'(x_0) = 0$$

for x and y in a small neighbourhood of x_0, there exists to every $\lambda \in]0, 1[$ an $r_\lambda > 0$, such that

$$f(x) - f(y)| \leq \lambda |x - y| \qquad \text{for } |x - x_0|, |y - x_0| < r_\lambda,$$

and f is a local contraction on $B(x_0, r_\lambda)$. The theorem then follows from Banach's fix point theorem. □

Although Newton's iteration method usually is very efficient, there is, however, a drawback, because the contraction in the proof is *local*. If one of the elements x_n of the iterative sequence does not lie in $B(x_0, r_\lambda)$, then we cannot conclude anything about its successor. This could easily happen with a bad choice of x_1, because we neither know x_0 (we are going to *find* x_0 by this method) nor the radius r_λ. It is therefore in general a matter of a lucky choice of the starting point x_1, if this method is going to be successful.

Usually one only applies *Newton's iteration method* in the case of a real polynomial, i.e. of real coefficients.

Theorem 4.1.3 *Let $P(x)$ be a polynomial of real coefficients of degree ≥ 3, and assume that it has a simple real root x_0 lying in some interval $]a,b[$.*

1) If $P'(x) \cdot P''(x) > 0$ in $]a,b[$, then the Newton iteration method converges towards x_0, if we choose the right end point $x_1 = b$ as our starting point.

2) If $P'(x) \cdot P''(x) < 0$ in $]a,b[$, then the Newton iteration method converges towards x_0, if we choose the left end point $x_1 = a$ as our starting point.

Proof. We shall prove Theorem 4.1.3 by some simple graphical considerations, cf. Figures 16-19.

1) Assume that $P(x)$ is convex and increasing in a neighbourhood of x_0, i.e.

$$P'(x) > 0 \text{ and } P''(x) > 0 \text{ in a neighbourhood of } x_0.$$

If we choose the starting point x_1 to the right of the root x_0, it follows readily from Figure 16 that

$$x_1 > x_2 > x_3 > \cdots > x_0, \qquad x_n \searrow x_0,$$

because a bounded decreasing sequence is convergent, and x_0 is the only possible limit point.

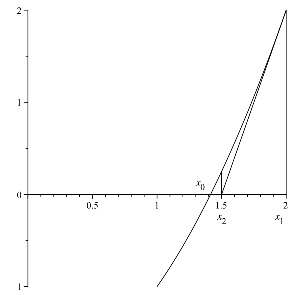

Figure 16: If $P(x)$ is convex and increasing in a neighbourhood of x_0, choose the starting point to the right of x_0 and obtain a decreasing sequence converging towards x_0.

2) Assume that $P(x)$ is convex and increasing in a neighbourhood of the root x_0, i.e.

$$P'(x) < 0 \text{ and } P''(x) > 0 \text{ in a neighbourhood of } x_0.$$

If we choose the starting point x_1 to the left of the root x_0, it follows readily from Figure 17 that

$$x_1 < x_2 < x_3 < \cdots < x_0, \qquad x_n \nearrow x_0,$$

because a bounded increasing sequence is convergent, and x_0 is the only possible limit point.

3) Assume that $P(x)$ is concave and increasing in a neighbourhood of x_0, i.e.

$$P'(x) > 0 \text{ and } P''(x) < 0 \text{ in a neighbourhood of } x_0.$$

If we choose the starting point x_1 to the left of the root x_0, it follows readily from Figure 18 that

$$x_1 < x_2 < x_3 < \cdots < x_0, \qquad x_n \nearrow x_0,$$

because a bounded increasing sequence is convergent, and x_0 is the only possible limit point.

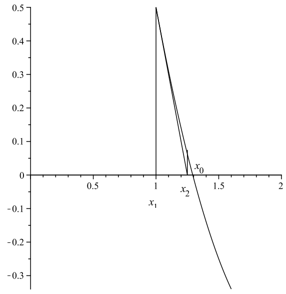

Figure 17: If $P(x)$ is convex and decreasing in a neighbourhood of x_0, choose the starting point to the left of x_0 and obtain an increasing sequence converging towards x_0.

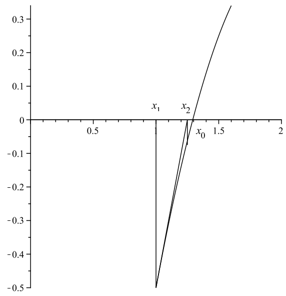

Figure 18: If $P(x)$ is concave and increasing in a neighbourhood of x_0, choose the starting point to the left of x_0 and obtain an increasing sequence converging towards x_0.

4) Assume that $P(x)$ is concave and decreasing in a neighbourhood of x_0, i.e.

$P'(x) < 0$ and $P''(x) < 0$ in a neighbourhood of x_0.

If we choose the starting point x_1 to the right of the root x_0, it follows readily from Figure 19 that

$$x_1 > x_2 > x_3 > \cdots > x_0, \qquad x_n \searrow x_0,$$

because a bounded decreasing sequence is convergent, and x_0 is the only possible limit point. $\square$

Example 4.1.1 We consider the polynomial $P(x) = x^3 - 2x - 5$ from Example 1.5.1 (1). Then we know already that there is a real root $> \dfrac{\sqrt{6}}{3}$. By insertion,

$$P(1) = 1 - 2 - 5 = -6, \qquad P(2) = 8 - 4 - 5 = -1, \qquad P(3) = 27 - 6 - 5 = 16,$$

so the real root lies in the interval $[2, 3]$, which can also be seen from the graph.

It follows from

$$P'(x) = 3x^2 - 2 \qquad \text{and} \qquad P''(x) = 6x$$

that $P'(x) > 3 \cdot 2^2 - 2 = 10$ and $P''(x) > 6 \cdot 2 = 12$, hence $P'(x) \cdot P''(x) > 0$ in this interval, so we choose the right end point $x_1 = 3$ of the interval as our starting point. Notice that even if x_0 apparently lies very close to $x = 2$, the best strategy is to choose $x_1 = 3$.

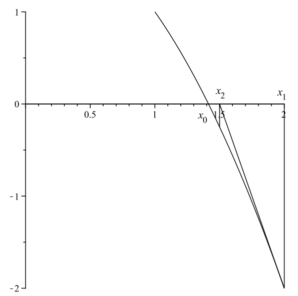

Figure 19: If $P(x)$ is concave and decreasing in a neighbourhood of x_0, choose the starting point to the right of x_0 and obtain a decreasing sequence converging towards x_0.

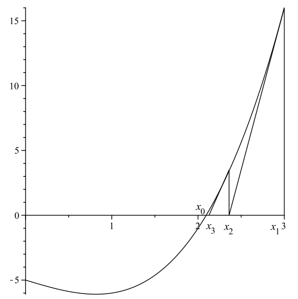

Figure 20: The graph of $P(x) = x^3 - 2x - 5$.

The iteration map is

$$f(x) = x - \frac{P(x)}{P'(x)} = x - \frac{1}{3} \cdot \frac{3x^3 - 6x - 15}{3x^2 - 2} = \frac{2}{3}\,x + \frac{1}{3} \cdot \frac{4x + 15}{3x^2 - 2},$$

so the iteration formula becomes

$$x_{n+1} = \frac{2}{3}\,x_n + \frac{1}{4} \cdot \frac{4x_n + 15}{3x_n^2 - 2}.$$

Using just a simple pocket calculator we get with $x_1 = 3 > x_0$,

$$x_1 = 3, \qquad x_2 = 2.36, \qquad x_3 = 2.12720,$$

$$x_4 = 2.09514, \qquad x_5 = 2.09455, \qquad x_6 = 2.09455,$$

so an approximate value of the real root is $\alpha_1 \approx 2.09455$.

Then using Vieti's formulæ we see that the two complex conjugated roots α_2, α_3 must have the structure $1.047274 \pm iy$, because $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and $\Re\,\alpha_2 = \Re\,\alpha_3$. Furthermore,

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = (-1)^3 \cdot (-5) = 5 = 2.09455 \left\{1.047275^2 + y^2\right\},$$

from which

$$y^2 = \frac{5}{2.09455} - 1.047275^2 = 1.29036,$$

so $y = 1.13594$. The three roots are therefore approximately

$$\alpha_1 = 2.09455, \qquad \alpha_2 = 1.047275 + i \cdot 1.13594, \qquad \alpha_3 = 1.047275 - i \cdot 1.13594. \qquad \Diamond$$

Example 4.1.2 In Example 1.5.1 (3) we considered the polynomial $P(x) = x^4 + 12x^2 + 96x - 12$ and showed that it had two real and two complex conjugated roots. From

$$P(0) = -12 \qquad \text{and} \qquad P(1) = 97$$

follows that we have one real root in the interval $]0, 1[$.

From

$$P(-3) = -111 \qquad \text{and} \qquad P(-4) = 52$$

follows that we have another real root in $]-4, -3[$.

These rough results can also be obtained by considering the graph of Figure 21.

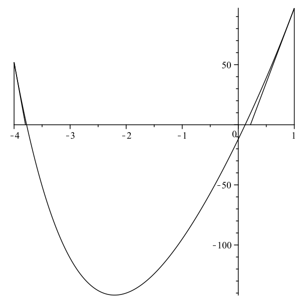

Figure 21: The graph of $P(x) = x^4 + 12x^2 + 96x - 12$.

Then we compute

$$P'(x) = 4x^3 + 24x + 96 = 4\left\{x^3 + 6x + 24\right\},$$

$$P''(x) = 12x^2 + 24 = 12\left\{x^2 + 2\right\}.$$

Clearly, $P''(x) > 0$ for all $x \in \mathbb{R}$, and $P'(x) > 0$ for $x \in]0,1[$. Hence, $P'(x) \cdot P''(x) > 0$ for $x \in]0,1[$, so we choose $x_1 = 1$ as our starting point in this interval.

Then, since $P''(x) > 0$ everywhere, $P'(x)$ is increasing. It follows from

$$P(-3) = 4\{-27 - 18 + 24\} = -84 < 0$$

that $P'(x) \cdot P''(x) < 0$ for $x \in]-4,-3[$. Therefore, the starting point of the iteration in this interval is chosen as $x_1 = -4$.

The iteration map is given by

$$f(x) = x - \frac{P(x)}{P'(x)} = x - \frac{x^4 + 12x^2 + 96x - 12}{4\{x^3 + 6x + 24\}} = \cdots = \frac{3x}{4} - \frac{3}{2} \cdot \frac{x^2 + 12 - 2}{x^3 + 6x + 24},$$

so the iteration formula is given by

$$x_{n+1} = \frac{3}{4} \cdot x_n - \frac{3}{2} \cdot \frac{x_n^2 + 12x_n - 2}{x_n^3 + 6x_n + 24}.$$

Choosing $x_1 = 1$ we get successively,

$$x_2 = \frac{3}{4} - \frac{3}{2} \cdot \frac{1 + 12 - 2}{1 + 6 + 24} = \frac{3}{4} - \frac{3}{2} \cdot \frac{11}{31} = 0.21774,$$

$$x_3 = \frac{3}{4} \cdot 0.21774 - \frac{3}{2} \cdot \frac{0.21774^2 + 12 \cdot 0.21774 - 2}{0.21774^3 + 6 \cdot 0.21774 + 24} = 0.12418,$$

$$x_4 = \frac{3}{4} \cdot 0.12418 - \frac{3}{2} \cdot \frac{0.12418^2 + 12 \cdot 0.12418 - 2}{0.12418^3 + 6 \cdot 0.12418 + 24} = 0.12310,$$

$$x_5 = \frac{3}{4} \cdot 0.12310 - \frac{3}{2} \cdot \frac{0.12310^2 + 12 \cdot 0.12310 - 2}{0.12310^3 + 6 \cdot 0.12310 + 24} = 0.12310,$$

so an approximate value of this real root is $\alpha_1 = 0.12310$.

Then we turn to the root in the interval $[-4, -3]$, where we should choose $x_1 = -4$ as our starting point. However, in order not to make errors in the computations, due to the minus signs, we instead introduce $x_n = -y_n$. Then the iteration formula becomes

$$y_{n+1} = \frac{3}{4}\,y_n + \frac{3}{2} \cdot \frac{y_n^2 - 12y_n - 2}{24 - 6y_n - y_n^3} = \frac{3}{4}\,y_n + \frac{3}{2} \cdot \frac{12y_n + 2 - y_n^2}{y_n^3 + 6y_n + 24}.$$

Choosing $y_1 = 4$ we get

$$y_2 = 3 + \frac{3}{2} \cdot \frac{48 + 2 - 16}{64 + 24 - 24} = 3 + \frac{51}{64} = 3.79688,$$

$$y_3 = \frac{3}{4} \cdot 3.79688 + \frac{3}{2} \cdot \frac{12 \cdot 3.79688 + 2 - 3.79688^2}{3.79688^3 + 6 \cdot 3.79688 - 24} = 3.77668,$$

$$y_4 = \frac{3}{4} \cdot 3.77668 + \frac{3}{2} \cdot \frac{12 \cdot 3.77668 + 2 - 3.77668^2}{3.77668^3 + 6 \cdot 3.77668 - 24} = 3.77649,$$

$$y_5 = \frac{3}{4} \cdot 3.77649 + \frac{3}{2} \cdot \frac{12 \cdot 3.77649 + 2 - 3.77649^2}{3.77649^3 + 6 \cdot 3.77649 - 24} = 3.77649,$$

and we conclude that an approximate value of this real root is $\alpha_2 = -y_5 = -3.77649$.

To find the complex roots we apply Vieti's formulæ. The sum of the roots is $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, so

$$\alpha_3 + \alpha_4 = 3.77649 - 0.012310 = 3.65339.$$

Since these two roots are complex conjugated,

$$\Re\,\alpha_3 = \Re\,\alpha_4 = \frac{1}{2} \cdot 3.65339 = 1.826695,$$

and

$$\alpha_3 = 1.826695 + iy, \qquad \alpha_4 = 1.826695 - iy.$$

The product of the roots is $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 = (-1)^4 \cdot (-12)$, thus

$$-12 = 0.12310 \cdot \{-3.77649\} \cdot \left\{1.826695^2 + y^2\right\},$$

and

$$y^2 + 1.826695^2 = \frac{12}{0.12310 \cdot 3.77649} = 25.81278,$$

from which

$$y^2 = 25.81278 - 1.826695^2 = 22.47597,$$

so $y = 4.74088$.

Summing up, the four roots are approximately given by

$$\alpha 1 = 0.12418, \qquad \alpha_2 = -3.77649, \qquad \left.\begin{array}{c}\alpha_3\\ \alpha_4\end{array}\right\} = 1.826695 \pm i \cdot 4.74088. \qquad \Diamond$$

Example 4.1.3 Cf. also Example 2.4.2. The polynomial $P(x) = x^4 - 3x^3 + 5x^2 + x - 4$ has the real root $x = 1$, and we get by a division that

$$P(x) = (x-1)Q(x), \qquad \text{where} \qquad Q(x) = x^3 - 2x^2 + 3x + 4.$$

The possible rational roots of $Q(x)$ are ± 1, ± 2, ± 4, and it is easily seen that this set of possible roots can be reduced to -1, -2, -4. We finally get by insertion,

$$Q(-1) = -2, \qquad Q(-2) = -18, \qquad Q(-4) < 0,$$

so $Q(x)$ does not have rational roots.

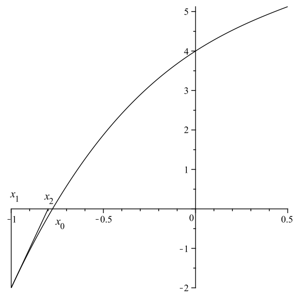

Figure 22: The graph of $P(x) = x^3 - 2x^2 + 3x + 4$.

Since $a_1^2 - 2a_0a_2 = (-2)^2 - 2 \cdot 1 \cdot 3 = 4 - 6 = -2 < 0$, it follows from Theorem 1.4.1 that $Q(x)$ must have non-real roots. These are complex conjugated in pairs, because the coefficients of the polynomial are real. Now, $n = 3$, so we can only have one such pair, and we have precisely one real root. It follows from the continuity and $Q(-1) = -2$ and $Q(0) = 4$ that this real root lies in the interval $]-1, 0[$, which can also be seen from Figure 22.

By differentiation,

$$Q'(x) = 3x^2 - 4x + 3 \qquad \text{and} \qquad Q'(x) > 0 \text{ for } x < 0,$$

$$Q''(x) = 6x - 4 \qquad \text{and} \qquad Q''(x) < 0 \text{ for } x < 0.$$

Since $Q'(x) \cdot Q''(x) < 0$, we choose the left end point $x_1 = -1 < x_0 = \alpha_1$ as our starting point of the iteration below.

The iteration map is

$$f(x) = x - \frac{Q(x)}{Q'(x)} = x - \frac{x^3 - 2x^2 + 3x + 4}{3x^2 - 4x + 3},$$

where it is no help to further reduce the fraction, so the iteration formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n^2 + 3x_n + 4}{3x_n^2 - 4x_n + 3} = x_n + \frac{(-x_n)^3 + 2(-x_n)^2 + 3(-x_n) - 4}{3(-x_n)^2 + 4(-x_n) + 3}.$$

We get using $x_1 = -1$,

$$x_2 = -1 - \frac{(-1)^3 - 2 - 3 + 4}{3 + 4 + 3} = -1 + \frac{2}{10} = -0.8,$$

$$x-3 = -0.8 + \frac{0.8^3 + 2\cdot 0.8^2 + 3\cdot 0.8 - 4}{3\cdot 0.8^2 + 4\cdot 0.8 + 3} = -0.77635,$$

$$x_4 = -0.77635 + \frac{0.77635^3 + 2\cdot 0.77635^2 + 3\cdot 0.77635 - 4}{3\cdot 0.77635^2 + 4\cdot 0.77635 + 3} = -0.77605,$$

$$x_5 = -0.77605 + \frac{0.77605^3 + 2\cdot 0.77605^2 + 3\cdot 0.77605 - 4}{3\cdot 0.77605^2 + 4\cdot 0.77605 + 3} = -0.77605.$$

The real root is approximately given by $\alpha_1 = -0.77605$.

Then use Vieti's formulæ and that $\Re\,\alpha_2 = \Re\,\alpha_3$ to get

$$\alpha_1 + \alpha_2 + \alpha_3 = -(-2) = 2, \qquad \text{thus } \alpha_2 + \alpha_3 = 2.77605,$$

and therefore α_2, $\alpha_3 = 1.388025 \pm iy$. Finally,

$$\alpha_1\alpha_2\alpha_3 = (-1)^3 4 = -4 = -0.77605 \cdot \left\{1.388025^2 + y^2\right\},$$

from which

$$y^2 = \frac{4}{0.77605} - 1.388025^2 = 3.22769, \qquad \text{or} \qquad y = 1.79658,$$

and the three roots are approximately

$$\alpha_1 = -0.77605, \qquad \alpha_2 = 1.388025 + i \cdot 1.79658, \qquad \alpha_3 = 1.388025 - i \cdot 1.79658. \qquad \Diamond$$

Example 4.1.4 Finally, we shall consider the polynomial $P(x) = 3x^{87} - x^3 + 1$ of Example 3.5.3. We know already that it has precisely one real root, and that Rouché's theorem implied that all roots lie in the annulus $\frac{24}{25} < |\alpha| < 1$. Clearly,

$$P(x) = 3x^{87} - x^3 + 1 > 0 \qquad \text{for } 0 < x < 1,$$

so the real root must lie in the interval $]-1, -0.96[$, cf. also Figure 23.

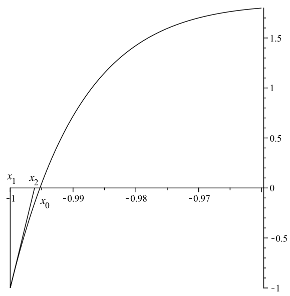

Figure 23: The graph of $P(x) = 3x^{87} - x^3 + 1$.

In order to avoid mistakes concerning the minus signs in the iteration process we put $y = -x$, $x = -y$, so we shall instead find $y \in]0.96, 1[$, such that

$$Q(y) := 3y^{87} - y^3 - 1 = 0, \qquad y \in]0.96, 1[.$$

By differentiation,

$$Q'(y) = 4\left\{87y^{86} - y^2\right\} > 0 \qquad \text{for } y \in]0.96, 1[,$$

$$Q''(y) = 6\left\{3\,741\, y^{85} - y\right\} > 0 \qquad \text{for } y \in]0.96, 1[.$$

Since $Q'(y)Q''(y) > 0$, Theorem 4.1.3 tells us to choose the right end point $y_1 = 1$ of the interval as our starting point for the iteration process.

The iteration map is

$$f(y) = y - \frac{Q(y)}{Q'(y)} = y - \frac{3y^{87} - y^3 - 1}{3\left\{87\, y^{86} - y^2\right\}},$$

hence the iteration formula becomes

$$y_{n+1} = y_n - \frac{3y_n^{87} - y_n^3 - 1}{3\left\{87\, y_n^{86} - y_n^2\right\}}.$$

Choosing $y_1 = 1$, cf. the above, we get

$$y_2 = 1 - \cdot \frac{3-1-1}{3\{87-1\}} = 1 - \frac{1}{3 \cdot 86} = 0.99612,$$

$$y_3 = 0.99612 - \frac{3 \cdot 0.99612^{87} - 0.99612^3 - 1}{3\left\{87 \cdot 0.99612^{86} - 0.99612^2\right\}} = 0.99530,$$

$$y_4 = 0.99530 - \frac{3 \cdot 0.99530^{87} - 0.99530^3 - 1}{5\left\{87 \cdot 0.99530^{86} - 0.99530^2\right\}} = 0.99527,$$

$$y_5 = 0.99527 - \frac{3 \cdot 0.99527^{87} - 0.99527^3 - 1}{3\left\{87 \cdot 0.99527^{86} - 0.99527^2\right\}} = 0.99527,$$

so using this particular pocket calculator we conclude that the real root is approximately $\alpha_1 = -0.99527$. Since the exponent, 87, is very large, one should double check this result, because we do not know the programs used in the pocket calculator. Figure 23 was created in MAPLE, so quite another program. Looking at Figure 23 the found approximate value above looks very plausible. $\Diamond$

4.2 Graeffe's root-squaring process

The most well-known approximation method, when we shall find the roots of a polynomial, is of course *Newton's approximation theorem*, which was treated in Section 4.1. There exists, however, another method, which is less known, and yet it is in some cases superior to Newton's approximation, in particular when the derivative $P'(x)$ is very small in a neighbourhood of a zero. This method is called *Graeffe's root-squaring process* after the Swiss mathematician C. H. Graeffe (1799–1873), who published this method as early as in 1837. Although the method may seem troublesome at the first glance, it may still have some advantages, in particular when one has a computer - or just a pocket calculator - at hand.

4.2.1 Analysis

1) *Description of the squaring process.* Assume first that all the roots of the normalized real polynomial equation

   (38) $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$

   are all real and mutually distinct in absolute value, thus in particular all simple. We shall for convenience, which will become clear later, write them in the form

   $$-r_1,\ -r_2,\ -r_3,\ \dots,\ -r_n, \qquad \text{where} \qquad |-r_1| > |-r_2| > \cdots > |-r_n|,$$

   arranged in decreasing order of the modules from $-r_1$ to $-r_n$.

   We rearrange (38) in such a way that all even powers are on one side of the equality sign, and all the odd powers on the other side of the equality sign,

   $$x^n + a_2 x^{n-2} + \cdots = -\left\{a_1 x^{n-1} + a_3 x^{n-3} + \cdots\right\},$$

and then square both sides,

$$\left\{\begin{array}{l} x^{2n}+a_2^2x^{2n-4}+a_3^2x^{2n-6}+\cdots \\ \quad +2a_2x^{2n-2}+2a_4x^{2n-4}+\cdots \\ \quad +2a_2a_4x^{2n-6}+\cdots \\ \quad +\cdots \end{array}\right\} = \left\{\begin{array}{l} a_1^2x^{2n-2}+a_3^2x^{2n-6}+a_5^2x^{2n-10}+\cdots \\ \quad +2a_1a_3x^{2n-4}+2a_1a_5x^{2n-6}+\cdots \\ \quad +2a_3a_5x^{2n-8}+\cdots \\ \quad \cdots . \end{array}\right.$$

Collecting all terms on the left we get after a reduction,

$$\begin{aligned} &x^{2n}-\left\{a_1^2-2a_2\right\}x^{2n-2}+\left\{a_2^2-2a_1a_3+2a_4\right\}x^{2n-4} \\ (39)\qquad &-\left\{a_3^2-2a_2a_4+2a_1a_5-2a_6\right\}x^{2n-6}+\cdots=0. \end{aligned}$$

The idea is that every root $-r_j$ of (38) is also a root of (39), because if

$$P(-r_j)=P_{\text{even}}(-r_j)+P_{\text{odd}}(-r_j)=0,$$

then the process above is described by

$$P_{\text{even}}(-r_j)=-P_{\text{odd}}(-r_j) \qquad \text{and} \qquad P_{\text{even}}(-r_j)^2=P_{\text{odd}}(-r_j)^2,$$

so (39) is written

$$\begin{aligned} P_{\text{even}}(-r_j)^2-P_{\text{odd}}(-r_j)^2 &= \{P_{\text{even}}(-r_j)+P_{\text{odd}}(-r_j)\}\{P_{\text{even}}(-r_j)-P_{\text{odd}}(-r_j)\} \\ &= P(-r_j)\{P_{\text{even}}(-r_j)-P_{\text{even}}(-r_j)\}=0. \end{aligned}$$

In the next step we let $y:=-x^2$ in (39. Then notice that

$$x^{2n-j}=(-1)^{n-j}y^{n-j}=(-1)^n\cdot(-1)^j\,y^{n-j},$$

so by this substitution (39) becomes

$$\begin{aligned} &y^{2n}-\left\{a_1^2-2a_2\right\}y^{n-1}+\left\{a_2^2-2a_1a_3+2a_4\right\}y^{n-2} \\ (40)\qquad &-\left\{a_3^2-2a_2a_4+2a_1a_5-2a_6\right\}y^{n-3}+\cdots=0. \end{aligned}$$

From $y=-x^2$ follows that the roots of (40) are

$$y_j=-\{-r_j\}^2=-r_j^2, \qquad j=1,\dots,n,$$

so (40) has the same degree as (38), and its roots are obtained from the roots of (38) by a squaring, followed by putting a minus sign in front of them.

Repeating this process we find at step number k some polynomial equation

$$(41)\quad x_k^n+a_{1,k}x_k^{n-1}+a_{2.k}x_k^{n-2}+\cdots+a_{n-1,k}x_k+a_{n,k}=0$$

of the n real roots

$$(42)\quad -r_1^{2^k}, \quad -r_2^{2^k}, \quad \dots, \quad -r_n^{2^k}.$$

The process above is the same also when some of the roots have the same modulus, including the case when we have pairs of conjugated complex roots. It is, however, easiest to describe under the assumption that all roots are real and of mutually different modules. We shall first analyze this simple case.

2) *The case of only simple real roots of mutually different modules.* Since $|r_1| > |r_2| > \cdots > |r_n|$ by assumption, we obtain for "large" k that

(43) $|r_1|^{2^k} \gg |r_2|^{2^k} \gg \cdots \gg |r_n|^{2^k}$,

so the roots $-r_j^{2^k}$ are very different in absolute value. This can be used to find approximate values of each root $-r_j^{2^k}$. To see this we apply Vieti's formulæ on the equation (41) of the n separated (simple) roots (42). We write for short $m = 2^k$. Due to the minus signs in (42) we get

$$(44)\qquad \begin{cases} a_{1,k} = r_1^m + r_2^m + \cdots + r_{n-1}^m + r_n^m, \\ a_{2,k} = r_1^m \left\{r_2^m + \cdots + r_n^m\right\} + r_2^m \left\{r_3^m + \cdots + r_n^m\right\} + \cdots \, , \\ a_{3,k} = r_1^m \left\{r_2^m r_3^m + \cdots\right\} + r_2^m \left\{r_3^m r_4^m + \cdots\right\} + \cdots \, , \\ \quad \vdots \qquad \vdots \\ a_{n,k} = r_1^m r_2^m \cdots r_n^m \end{cases}$$

Applying (43) on (44) we see that we have approximately

$$(45)\qquad \begin{cases} a_{1,k} = r_1^m, \\ a_{2,k} = r_1^m r_2^m, \\ a_{3,k} = r_1^m r_2^m r_3^m, \\ \quad\vdots \qquad \vdots \\ a_{n,k} = r_1^m r_2^m r_3^m \cdots r_n^m, \end{cases}$$

because all of the other finitely many terms of (44) are much smaller than the leading term in (45). It then is very easy to find r_j from (45), because

$$(46)\qquad \begin{cases} r_1^m = a_{1,k}, \\ r_2^m = \dfrac{a_{2,k}}{a_{1,k}} \\ \quad\vdots \qquad \vdots \\ r_j^m = \dfrac{a_{j,k}}{a_{j-1,k}} \\ \quad\vdots \qquad \vdots \\ r_n^m = \dfrac{a_{n,k}}{a_{n-1,k}}, \end{cases} \quad \text{i.e.} \quad \begin{cases} |r_1| = \sqrt[m]{a_{1,k}}, \\ |r_2| = \sqrt[m]{\dfrac{a_{2,k}}{a_{1,k}}}, \\ \quad\vdots \qquad \vdots \\ |r_j| = \sqrt[m]{\dfrac{a_{j,k}}{a_{j-1,k}}}, \\ \quad\vdots \qquad \vdots \\ |r_n| = \sqrt[m]{\dfrac{a_{n,k}}{a_{n-1,k}}}, \end{cases} \qquad \text{where } m = 2^k.$$

Clearly, we cannot determine the sign of the roots by this method, but these can easily be found, either by a graphical consideration, or by the theory of the previous chapters, or by simply inserting $\pm r_1$ into the original polynomial equation (38)

The analysis above was based on the assumption that $|r_1| > |r_2| > \cdots > |r_n|$, and that the roots $-r_1, -r_2, \ldots, -r_n$ are all real.

3) *The case of precisely two roots of equal modulus.* We shall now assume that all roots are real and that just two of them are equal in absolute value, say $|r_3| = |r_4|$, so $r_4 = \pm r_3$. We apply the same method as described above and finally arrive again at (41) and (42), where of course $-r_3^{2^k} = -r_4^{2^k}$ for $k \in \mathbb{N}$, because then all the exponents are even. Recalling that $2^k = m$, formulæ (44) are then written

$$\begin{cases} a_{1,k} = r_1^m, \\ a_{2,k} = r_1^m r_2^m, \\ a_{3,k} = r_1^m r_2^m r_3^m + r_1^m r_2^m r_4^m + \cdots = 2 r_1^m r_2^m r_3^m + \cdots, \\ a_{4,k} = r_1^m r_2^m r_3^m r_4^m + \cdots = r_1^m r_2^m r_3^{2m} + \cdots, \\ \quad\vdots \qquad\qquad \vdots \\ a_{n-1,k} = r_1^m r_2^m r_3^{2m} r_5^m \cdots r_{n-1}^m + \cdots, \\ a_{n,k} = r_1^m r_2^m r_3^{2m} r_5^m \cdots r_n^m, \end{cases}$$

where the dots indicate numerically smaller terms. Hence, (45) is here replaced by

$$(47)\qquad \begin{cases} a_{1,k} = r_1^m, \\[2mm] a_{2,k} = r_1^m r_2^m, \\[2mm] a_{3,k} = 2r_1^m r_2^m r_3^m, \\[2mm] a_{4,k} = r_1^m r_2^m r_3^{2m}, \\ \vdots \\ a_{n-1,k} = r_1^m r_2^m r_3^{2m} r_5^m \cdots r_{n-1}^m, \\[2mm] a_{n,k} = r_1^m r_2^m r_3^{2m} r_5^m \cdots r_n^m. \end{cases}$$

The only difference between (45) and (47), when $|r_3| = |r_4|$ is that $c_3 = 2r_1^m r_2^m r_3^m$, where we get an extra factor 2, because in the following the product $r_3^{2m} = r_3^m r_4^m$ is unchanged from the previous. Therefore, (46) is replaced by

$$(48)\qquad \left.\begin{array}{l} r_1^m = a_{1,k}, \\[2mm] r_2^m = \dfrac{a_{2,k}}{a_{1,k}}, \\[2mm] r_3^m = r_4^m = \dfrac{a_{3,k}}{2a_{2,k}} \\[2mm] r_5^m = \dfrac{a_{5,k}}{a_{4,k}}, \\ \vdots \qquad\quad \vdots \\ r_n^m = \dfrac{a_{n,k}}{a_{n-1,k}}, \end{array}\right\}, \quad \text{thus} \quad \begin{cases} |r_1| = \sqrt[m]{a_{1,k}}, \\[2mm] |r_2| = \sqrt[m]{\dfrac{a_{2,k}}{a_{1,k}}}, \\[2mm] |r_3| = |r_4| = \sqrt[m]{\dfrac{a_{3,k}}{2a_{2,k}}}, \\[2mm] |r_5| = \sqrt[m]{\dfrac{a_{5,k}}{a_{4,k}}}, \\ \vdots \qquad\quad \vdots \\ |r_n| = \sqrt[m]{\dfrac{a_{n,k}}{a_{n-1,k}}}, \end{cases} \qquad \text{where } m = 2^k,$$

so the difference is that $|r_3| = |r_4| = \sqrt[m]{\dfrac{a_{3,k}}{2a_{2,k}}}$. We can for large m estimate the number of equal modulus, because

$$a_{3,k+1} = 2\left\{r_1^{2m} r_2^{2m} r_3^{2m}\right\} = \frac{1}{2}\,\{a_{3,k}\}^2,$$

so we just compare two successive coefficients (with respect to k), if the lower number is squared or not, when we pass from k to $k+1$ for large k. We notice that since $\sqrt[m]{2} \to 1$ for $m \to +\infty$, we may above instead use the simpler estimate

$$|r_3| = |r_4| \approx \sqrt[m]{\frac{a_{3,k}}{,}a_{2,k}},$$

because in the limit, $k \to +\infty$, or $m \to +\infty$, we shall get the right values of $|r_3| = |r_4|$.

4) *The case of precisely p roots of equal modulus.* Similarly, if e.g. $|r_3| = |r_4| = |r_5|$ (three roots of equal modulus), then we get for large $m = 2^k$,

$$|r_3| = |r_4| = |r_5| = \sqrt[m]{\frac{a_{3,k}}{3a_{2,k}}} \approx \sqrt[m]{\frac{a_{3,k}}{a_{2,k}}},$$

and generally, if $|r_j| > |r_{j+1}| = |r_{j+2}| = \cdots = |r_{j+p}| > |r_{j+p+1}|$, then

$$(49) \qquad |r_{j+1}| = |r_{j+2}| = \cdots = |r_{j+p}| = \sqrt[m]{\frac{a_{j+1,k}}{p \cdot a_{j,k}}} \approx \sqrt[m]{\frac{a_{j+1,k}}{a_{j,k}}} \qquad \text{for large } m = 2^k,$$

because $\sqrt[m]{p} \to 1$ for $m \to +\infty$.

We can for large m estimate the number p of roots of equal modulus, because for large k,

$$a_{j+1,k+1} \approx \frac{1}{p} \cdot a_{j+1,k}^2, \qquad \text{or} \qquad p \approx \frac{a_{j+1,k}^2}{a_{j+1,k+1}}.$$

5) *The case of precisely one pair of complex conjugated roots with different modules from all the other roots.* We shall now consider the case where the given equation (38) contains one pair of simple complex conjugated roots. It suffices in the analysis only to consider a polynomial equation of fourth degree of the roots

$$-r_1, \quad -r_2\, e^{i\Theta}, \quad -r_2\, e^{-i\Theta}, \quad -r_3,$$

where we assume that $|r_1| > |r_2| > |r_3|$. Then the polynomial can also be written

$$(x + r_1)\left(x + r_2\, e^{i\Theta}\right)\left(x + r_2\, e^{-i\Theta}\right)(x + r_3) = 0.$$

When we perform k root-squaring operations and put $m = 2^k$, then the resultant equation has the roots

$$-r_1^m, \quad -r_2^m\, e^{im\Theta}, \quad -r_2^m\, e^{-im\Theta}, \quad -r_3^m,$$

so the corresponding polynomial can be written

$$(z + r_1^m)\left(z + r_2^m\, e^{im\Theta}\right)\left(z + r_2^m\, e^{-im\Theta}\right)(z + r_3^m) = 0,$$

which we rearrange as

$$z^4 + \left\{r_1^m + r_2^m\, e^{im\Theta} + r_2^m\, e^{-im\Theta} + r_3^m\right\} z^3$$

$$(50) \qquad + \left\{r_1^m r_2^m\, e^{im\Theta} + r_1 r_2^m\, e^{-im\Theta} + r_1^m r_3^m + r_2^{2m} + r_2^m r_3^m\, e^{im\Theta} + r_2^m r_3^m\, e^{-im\Theta}\right\} z^2$$

$$+ \left\{r_1 r_2^{2m} + r_1^m r_2^m r_3^m e^{im\Theta} + r_1^m r_2^m r_3^m\, e^{-im\Theta} + r_2^{2m} r_3^m\right\} z + r_1^m r_2^{2m} r_3^m = 0.$$

We notice that the first terms in (50) are dominating in the coefficients of z^3 and z. Then we turn to the coefficient of z^2, which is also written

$$2r_1^m r_2^m \cos m\Theta + r_1^m r_3^m + r_2^{2m} + 2r_2^m r_3^m \cos m\Theta.$$

If $\cos m\Theta$ is approximately $+1$ or -1, then $2r_1^m r_2^m \cos m\Theta$ is numerically dominant. If instead $\cos m\Theta$ is approximately 0, either $r_1^m r_3$ or r_2^{2m} become dominant. Therefore, if m increases, then the coefficient of z^2 continuously fluctuates in sign, which is very unlike the coefficients corresponding to real roots, which remain positive. Hence, we can identify complex conjugated roots by this fluctuations of the corresponding coefficient. Clearly, the restriction to a polynomial of fourth degree is of no importance. The observation above holds for general real polynomials.

Once we have identified a pair of simple complex conjugated roots, like in (50), the modules is found in the following way, where we first analyze (50), where r_2^m should be found. For large m we have approximately

$$z^4 + r_1^m z^3 + a_{2,k}\left(r_1, r_2, r_3, \Theta\right) z^2 + r_1^m r_2^{2m} z + r_1^m r_2^{2m} r_3^m = 0,$$

from which we derive that $a_{1,k} = r_1^m$ and $a_{3,k} = r_1^m r_2^m$ and $a_{4,k} = r_1^m r_2^{2m} r_3^m$. We see that we shall neglect the fluctuating coefficient $a_{2,k}\left(r_1, r_2, r_3, \Theta\right)$ and only consider

$$\text{(51)} \qquad \left.\begin{aligned} r_1^m &= a_{1,k}, \\ r_2^{2m} &= \frac{a_{3,k}}{a_{1,k}}, \\ r_3^m &= \frac{a_{4,k}}{a_{3,k}}, \end{aligned}\right\} \quad \text{thus} \quad \left\{\begin{aligned} |r_1| &= \sqrt[m]{a_{1,k}}, \\ |r_2| &= \sqrt[2m]{\frac{a_{3,k}}{a_{1,k}}}, \\ |r_3| &= \sqrt[m]{\frac{a_{4,k}}{c_{3,k}}}, \end{aligned}\right. \qquad \text{where } m = 2^k.$$

The general method is to neglect the fluctuating coefficient $a_{j,k}\left(r_1,\dots,r_n,\Theta\right)$ corresponding to a pair of simple complex conjugated roots, and then the modulus $|r_j|$ of the two complex roots is given by

$$|r_j| = \sqrt[2m]{\frac{a_{j+1,k}}{a_{j,k}}}.$$

Then recall that we have assumed that there is only one pair of complex conjugated roots $u \pm iv$. In order to find u and v we notice that it follows from Vieti's formulæ after a rearrangement that

$$u = -\frac{1}{2}\left\{a_1 + \text{ sum of all real roots in the polynomial}\right\},$$

so the real part is easy to find. Then

$$u^2+v^2 = r_j^2 = \sqrt[m]{\frac{a_{j+1,k}}{a_{j,k}}} \quad \text{implies that} \quad v = \sqrt{r_j^2-u^2},$$

so we have also found the imaginary part. Thus for a single pair of complex conjugated roots $u \pm iv$ the method above should be straightforward.

6) *Several pairs of complex roots.* If there is at least two pairs of complex conjugated roots, the description of the method becomes more complicated. First of all one identifies all modules $|r_j|$ and find all the *real* roots by the previous described method.

   The complex roots are then written in the form

$$u_j \pm iv_j, \qquad \text{where } u_j^2+v_j^2 = r_j^2.$$

   Then we apply some of Vieti's formulæ, starting with $a_1 = a_{1,0}$, and then $a_{n-1} = a_{n-1,0}$ in case of two pairs, etc., and then find some equations of the unknown real parts u_j. One should of course choose the least complicated of Vieti's formulæ in this process, but it must be admitted that if there are many pairs of complex roots, then we are forced to solve a very complicated system of non-linear equations in the u_j. We shall later illustrate this by an example with two pairs.

4.2.2 Template for Graeffe's root-squaring process.

Let

$$P_0(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

be a given normalized polynomial of real coefficients.

1) We shall define the coefficients of the polynomial

$$P_1(y) = y^n + a_1'y^{n-1} + a_2'y^{n-2} + \cdots + a_{n-1}'y + a_n'$$

   in the next step of the root-squaring process. This is described in (52), where we get the a_j' by summing the j-th column.

$$
\text{(52)}\qquad
\begin{array}{ccccccc}
1 & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\
 & & & & & & \\
1^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & \cdots \\
 & -2a_2 & -2a_1a_3 & -2a_2a_4 & -2a_3a_5 & -2a_4a_6 & \cdots \\
 & & 2a_4 & 2a_1a_5 & 2a_2a_6 & 2a_3a_7 & \cdots \\
 & & & -2a_6 & -2a_1a_7 & -2a_2a_8 & \cdots \\
 & & & & 2a_8 & 2a_1a_9 & \cdots \\
 & & & & & -2a_{10} & \cdots \\
 & & & & & & \\
1 & a_1' & a_2' & a_3' & a_4' & a_5' & \cdots
\end{array}
$$

Then repeat (52) on the new coefficients $(1, a_1', a_2', \dots, a_n')$, etc.. It cannot be told in advance how many times this should be done. A rule of thumb is that one should stop when each computed coefficient $a_{j,k+1}$ (with the exceptions of the cases of multiple roots, or complex roots) is roughly the square of the preceding one $a_{j,k}$, i.e. $a_{j,k+1} \approx \{a_{j,k}\}^2$ for all coefficients which are not connected with multiple roots or complex conjugated roots. This happens when the corresponding double products are negligible compared with the square, so in practice it is easy to see when one should stop. This will be evident from the examples in Section 4.2.3.

2) If all coefficients are positive, so all roots are real, and if furthermore they have mutually different modules, then find $|r_1|, \dots, |r_n|$ by (46). Check the sign of the roots, i.e. check the possible solutions $\pm r_1, \dots, \pm r_n$ in the original polynomial $P_0(x)$.

3) If all coefficients are positive, but some of the coefficients are not eventually roughly squared, this is an indication of that we have roots of equal modulus. In this case, apply (48), or (49), whenever needed to find $|r_1|, \dots, |r_n|$ and then check $\pm r_1, \dots, \pm r_n$ in the original polynomial $P_0(x)$.

4) If some of the coefficients fluctuate, this is an indication of a pair of complex conjugated roots. In this case, apply (51), or modifications of (51), to find $|r_1|, \dots, |r_n|$. For these pairs of complex conjugated roots, apply Vieti's formulæ to find them explicitely.

4.2.3 Examples.

We shall give three examples, one with only real roots of mutually different modules, one with only real roots, but where some of the roots have a common modulus, and finally an example, where we have two pairs of complex conjugated roots. In two of the examples it is possible to find the roots directly, so we can compare the results.

Example 4.2.1 We shall find all roots of the polynomial equation

$$P(x) = x^4 - x^3 - 10x^2 - x + 1 = 0.$$

We first notice that it is possible to solve this equation directly, because if x_0 is a root, then $x_0 \neq 0$, and $\dfrac{1}{x_0}$ is also a root. In fact,

$$P\left(\frac{1}{x_0}\right) = \frac{1}{x_0^4}\left\{1 - x_0 - 10x_0^2 - x_0^3 + x_0^4\right\} = \frac{1}{x_0^4}\, P\left(x_0\right) = 0.$$

Since

$$(x-x_0)\left(x-\frac{1}{x_0}\right)=x^2-\left\{x_0+\frac{1}{x_0}\right\}x+1,$$

the polynomial must necessarily have the structure

$$\begin{aligned}P(x) &= x^4-x_3-10x^2-x+1=\left(x^2-ax+1\right)\left(x^2-bx+1\right)\\ &= x^4-(a+b)x^3+(2+ab)x^2-(a+b)x+1.\end{aligned}$$

When we identify the coefficients we get

$$a+b=1, \qquad \text{and} \qquad 2+ab=-10, \quad \text{i.e.} \quad ab=-12,$$

and a and b are the roots of the equation $z^2-(a+b)z+ab=z^2-z-12=0$, thus

$$\left.\begin{array}{c}a\\ b\end{array}\right\}=\frac{1}{2}\left\{1\pm\sqrt{48+1}\right\}=\frac{1}{2}\{1\pm 7\}=\left\{\begin{array}{c}4\\ -3\end{array}\right. ,$$

and we have proved that $P(x)$ can be factorized in the following way,

$$P(x) = x^4 - x^3 - 10x^2 - x + 1 = \left(x^2 - 4x + 1\right)\left(x^2 + 3x + 1\right),$$

from which we immediately get the roots

$$x = 2 \pm \sqrt{2^2 - 1} = 2 \pm \sqrt{3} \qquad \text{and} \qquad x = \frac{1}{2}\left\{-3 \pm \sqrt{3^2 - 4}\right\} = \frac{1}{2}\left\{-3 \pm \sqrt{5}\right\}.$$

Since we shall estimate the efficiency of Graeffe's root-squaring method we notice for later use that the roots are approximately

$$3.7321, \qquad 0.2679, \qquad -0.3820, \qquad -2.6180.$$

It will be convenient also to give the modules of the roots, ordered according to their size,

(53) $|r_1| = 3.7321, \qquad |r_2| = 2.6180, \qquad |r_3| = 0.3820, \qquad |r_4| = 0.2679,$

because this is the order they should occur, when we apply the root-squaring method.

We notice that the four modules are mutually distinct and that all four roots are real, so this should be the easiest case to handle.

It follows from Table 7 that all coefficients are approximately squared, when we pass form one step to the next. We therefore conclude that all four roots are real, and that they are of mutually different modules, in particular they are all simple.

Using $k = 6$ we get $m = 2^k = 64$, and it follows from (46) that

$$r_1^{64} = 4.0239 \cdot 10^{36}, \qquad \text{thus } |r_1| = 3.7321,$$

$$r_2^{64} = \frac{2.2650 \cdot 10^{63}}{4.0239 \cdot 10^{36}}, \qquad \text{thus } |r_2| = 2.6180,$$

$$r_3^{64} = \frac{4.0239 \cdot 10^3 6}{2.2650 \cdot 10^{63}}, \qquad \text{thus } |r_3| = 0.3820,$$

$$r_4^{64} = \frac{1}{4.0239 \cdot 10^{36}}, \qquad \text{thus } |r_4| = 0.2680,$$

in agreement with (53).

Since $r_1 + r_2 + r_3 + r_4 = 1$, we must have $r_1 > 0$ and $r_2 < 0$, and $r_1 + r_2 = 1.1141 = 1 - r_3 - r_4$, or $r_3 + r_4 = -0.1141$. This implies that $r_3 < 0$ and $r_4 > 0$, so we conclude that the roots are

$$r_1 = 3.7321, \qquad r_2 = -2.6180, \qquad r_3 = -0.3820, \qquad r_4 = 0.2680. \qquad \Diamond$$

Example 4.2.2 Let us consider the normalized polynomial

$$P(x) = x^5 + 2x^4 - 5x^3 - 10x^2 + 4x + 8$$

of integer coefficients. Its possible rational roots are ± 1, ± 2, $\pm 4, \pm 8$. A simple check shows that the roots are

$$x_1 = 2, \qquad x_2 = x_3 = -2, \qquad x_4 = 1 \qquad \text{and} \qquad x_5 = -1,$$

k	a_0	a_1	a_2	a_3	a_4
	1	a_1^2 $-2a_2$	a_2^2 $-2a_1a_3$ $2a_4$	a_3^2 $-2a_2a_4$	a_4^2
0	1	-1	-10	-1	1
	1	1 20	100 -2 2	1 20	1
1	1	21	100	21	1
	1	441 -200	10 000 -882 2	441 -200	1
2	1	241	9 120	241	1
	1	5.8081 E4 -1.8240 E4	8.3174 E7 1.1616 E5 $\star$	5.8081 E4 -1.8240 E4	1
3	1	3.9841 E4	8.3058 E7	3.9841 E4	1
	1	1.5873 E9 -1.6612 E8	6.8987 E15 -3.1746 E9	1.5873 E9 -1.6612 E8	1
4	1	1.4212 E9	6.8987 E15	1.4212 E9	1
	1	2.0198 E18 -1.3797 E16	4.7592 E31 $\star$	2.0198 E18 -1.3797 E16	1
5	1	2.0060 E18	4.7592 E31	2.0060 E18	1
	1	4.0240 E36 -9.5184 E31	2.2650 E63	4.0240 E36 -9.5184 E31	1
6	1	4.0239 E36	2.2650 E63	4.0239 E36	1

Table 7: The coefficients of the root-squaring method of Example 4.2.1.

k	$m=2^k$	a_0	a_1	a_2	a_3	a_4	a_5
		1	a_1^2 $-2a_2$	a_2^2 $-2a_1a_3$ $2a_4$	a_3^2 $-2a_2a_4$ $2a_1a_5$	a_4^2 $-2a_3a_5$	a_5^2
0	1	1	2	-5	-10	4	8
		1	4 10	25 40 8	100 40 32	16 160	64
1	2	1	14	73	172	176	64
		1	196 -146	5.329 E3 -4.816 E3 0.358 E3	2.9584 E4 -2.5696 E4 0.1792 E4	3.0976 E4 -2.2016 E4	4.096 E3
2	4	1	50	865	5 680	8 960	4 096
		1	2 500 $-1\,730$	7.4823 E5 -5.6800 E5 0.1792 E5	3.2262 E7 -1.5501 E7 0.0410 E7	8.0282 E7 -4.6531 E7	1.6777 E7
3	8	1	770	1.9815 E5	1.7171 E7	3.3751 E7	1.6777 E7
		1	5.9290 E5 -3.9629 E5	3.9261 E10 -2.6444 E10 0.0067 E10	2.9485 E14 -0.1338 E14 0.0003 E14	1.1391 E14 -5.7617 E14	2.8148 E14
4	16	1	1.9661 E5	1.2885 E10	2.8150 E14	5.6296 E14	2.8148 E14
		1	3.8655 E10 -2.5771 E10	1.6603 E20 -1.1069 E20 $\star$	7.9243 E28 -0.0015 E28 $\star$	3.1693 E29 -1.5847 E29	7.9228 E28
5	32	1	1.2885 E10	5.5340 E19	7.9228 E28	1.5846 E29	7.9228 E28
		1	1.6602 E20 -1.1068 E20	3.0625 E39 -2.0417 E39	6.2771 E57 $\star$	2.5108 E58 -1.2554 E58	6.2771 E57
6	64	1	5.5341 E19	1.0208 E39	6.2771 E57	1.2554 E58	6.2771 57
		1	3.0628 E39 -2.0416 E39	1.0420 E78 0.6948 E78	3.9402 E115 $\star$	1.5760 E116 0.7880 E116	3.9402 E115
7	128	1	1.0212 E39	3.4720 E77	3.9402 E115	7.8800 E115	3.9402 E115
		1	1.0428 E78 -0.6944 E78	1.2055 E155 -0.80475 E155	1.5525 E231 $\star$	6.2094 E231 -3.1050 E231	1.5525 E231
8	256	1	3.4840 E77	4.0075 E154	1.5525 E231	3.1044 E231	1.5525 E231

Table 8: The coefficients of the root-squaring method of Example 4.2.2.

so $x_2 = x_3$ is a double root, x_1, x_2, x_3 have the same modulus 2, and the two simple roots x_4 and x_5 have the same modulus 1. All five roots are real. We shall see how Graeffe's root-squaring method can show that three of the roots have modulus 2, and the remaining two roots have the modulus 1. All derived coefficients in Table 8 are positive, so we conclude that all five roots are real. When we compare the two lines corresponding to $k = 7$ and $k = 8$ we see that

$$a_{1,8} \approx \frac{1}{3}\, a_{1,7}^2, \qquad a_{2,8} \approx \frac{1}{3}\, a_{2,7}^2, \qquad a_{4,8} \approx \frac{1}{2}\, a_{4,7}^2,$$

so three of the roots have the same modulus, and the remaining two have the same modulus, i.e.

$$|x_1| = |x_2| = |x_3| \qquad \text{and} \qquad |x_4| = |x_5|\,.$$

k	$m = 2^k$	$\sqrt[m]{\frac{a_{2,k}}{a_{1,k}}}$	$\sqrt[m]{\frac{a_{3,k}}{a_{2,k}}}$	$\sqrt[m]{\frac{a_{4,k}}{a_{3,k}}}$	$\sqrt[m]{\frac{a_{5,k}}{a_{5,k}}}$
1	2	2.2835	1.5350	1.0116	0.6030
2	4	2.0394	1.6008	1.1207	0.8223
3	8	2.0013	1.7467	1.0881	0.8397
4	16	2.0000	1.8673	1.0443	0.9576
5	32	2.0000	1.9325	1.0219	0.9786
6	64	2.0000	1.9660	1.0109	0.9892
7	128	2.0000	1.9829	1.0054	0.9946
8	256	2.0000	1.9915	1.0027	0.9973

Table 9: Estimates of the modules of the roots in Example 4.2.2.

From Table 9 we derive that $|x_1| = |x_2| = |x_3| = 2$ and $|x_4| = |x_5| \approx 1$. In order to get 1 in the latter case we should proceed with the computations another one or two steps, but most people would already at this stage judge that the modulus is indeed 1. Finally, by inserting ± 1, ± 2 in the original equation we get as before

$$x_1 = 2, \qquad x_2 = x_3 = -2, \qquad x_4 = 1, \qquad x_5 = -1. \qquad \Diamond$$

Example 4.2.3 Finally, we shall find the roots of the polynomial equation

(54) $x^7 + x^6 - 4x^5 - 4x^4 - 2x^3 - 5x^2 - x - 1 = 0,$

by using Graeffe's squaring method.

We first check for possible rational roots. These can only be ± 1, and neither of them are roots for obvious reasons, because the sum of the coefficients is an odd number.

One may also investigate if there are multiple roots. We shall not write down the tedious details, only mention that there are no multiple root in this case.

Then we use the Graeffe's root-squaring method to set up Table 10.

It follows from Table 10 that from row $k = 7$, i.e. $m = 2^7 = 128$, all coefficients are uninfluenced by the product terms with the exception of the fifth and the seventh coefficients, which continually fluctuate in sign. We can therefore terminate the root-squaring process at this stage, concluding that there must be two pairs of complex roots and three real roots. Furthermore, the real roots must be of mutually different modules, because the transition from one step to the next one is approximately a squaring.

It follows from $r_1^{256} \approx 5.6033 \cdot 20^{85}$ that

$$\ln |r_1| = \frac{1}{256} \ln \left(5.6033 \cdot 10^{85}\right) = 0.7713, \qquad \text{hence } |r_1| = 2.1625.$$

It follows from $r_1^{256} \cdot r_2^{256} \approx 1.6806 \cdot 20^{159}$ that

$$\ln |r_2| = \frac{1}{256} \left\{\ln \left(1.6806 \cdot 10^{159}\right) - \ln \left(5.6033 \cdot 10^{85}\right)\right\} = 0.6609, \qquad \text{hence } |r_2| = 1.9365.$$

k	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
	1	a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	a_6^2	a_7^2
		$-2a_2$	$-2a_1a_3$	$-2a_2a_4$	$-2a_3a_5$	$-2a_4a_6$	$-2a_5a_7$	
			$2a_4$	$2a_1a_5$	$2a_2a_6$	$2a_3a_7$		
				$-2a_6$	$-2a_1a_7$			
0	1	1	-4	-4	-2	-5	-1	-1
		1	16	16	4	25	1	1
		$+8$	$+8$	-16	-40	-4	-10	
			-4	-10	$+8$	8		
				$+2$	$+2$			
1	1	9	20	-8	-26	29	-9	1
		81	400	64	676	841	81	1
		-40	144	1040	464	-468	-58	
			-52	522	-360	-16		
				18	-18			
2	1	41	492	1644	762	357	23	1
		1681	2.4206 E5	2.7027 E6	5.8064 E5	1.2745 E5	529	1
		-984	-1.3481 E5	-0.7498 E6	-1.1738 E6	-0.3505 E5	-714	
			0.0152 E5	0.0293 E6	0.0226 E6	0.0329 E5		
				$\star$	$\star$			
3	1	697	1.0877 E5	1.9822 E6	-5.7062 E5	9.5685 E4	-185	1
		4.8581 E5	11.8309 E9	3.9261 E12	32.5607 E10	9.1556 E9	0.3423 E5	1
		-2.1754 E5	-2.7632 E9	0.1241 E12	-37.9334 E10	-0.2111 E9	-1.9137 E5	
			-1.1412 E6	0.1334 E12	-0.0040 E10	0.0040 E9		
4	1	2.6827 E5	9.0666 E9	4.0534 E12	-5.3767 E10	8.9485 E9	-1.5715 E5	1
		7.1969 E10	8.2203 E19	1.6430 E25	0.2891 E21	8.0076 E19	24.6961 E9	
		-1.8133 E10	-0.2175 E19	9.7497 E20	-7.2544 E22	-1.6899 E16	-17.8970 E9	
			$\star$	$\star$	$\star$	$\star$		
5	1	5.3836 E10	8.0028 E19	1.6431 E25	-6.9653 E22	8.0059 E19	6.7991 E9	1
		2.8983 E21	6.4045 E39	2.6998 E50	4.8515 E45	6.4094 E39	4.6228 E19	
		-0.1601 E21	-1.7692 E36	$\star$	-2.6309 E45	$\star$	-16.012 E19	
6	1	2.7383 E21	6.4027 E39	2.6998 E50	2.2206 E45	6.4094 39	-1.1389 E20	1
	1	7.4983 E42	4.0995 E79	7.2889 E100	4.9311 E90	4.1080 E79	1.2971 E40	1
		-0.0128 E42	$\star$	$\star$	-3.4608 E90	$\star$	-1.2819 E38	
7	1	7.4855 E42	4.0995 E79	7.2889 E100	1.4702 E90	4.1080 E79	1-5213 E38	1
		5.6033 E85	1.6806 E159	5.3128 E201	2.1615 E180	1.6876 E159	2.3144 E76	1
		$\star$	$\star$	$\star$	-5.9886 E180	$\star$	-8.2160 E79	
8	1	5.6033 E85	1.6806 E159	5.3128 E201	-3.8271 E180	1.6876 E159	-8.2137 E79	1

Table 10: The coefficients of the root-squaring method of Example 4.2.3.

It follows from $r_3^{256} = \dfrac{a_{3,8}}{a_{2,8}}$ that

$$\ln|r_3| = \frac{1}{256}\left\{\ln\left(5.3128\cdot 10^{201}\right) - \ln\left(1.6806\cdot 10^{159}\right)\right\} = 0.3823, \qquad \text{hence } |r_3| = 1.4625.$$

Since there is only one change of sign in (54) between successive coefficients, only one of the three roots is positive. It follows from $P(2) = -39$ and $P(3) = 1517$ that the positive root must lie between 2 and 3. Thus we conclude that the three real roots are approximately

$$r_1 = 2.1625, \qquad r_2 = -1.9365, \qquad r_3 = -1.4656.$$

Then we compute the modules of the complex roots. In the first case,

$$r_4 \approx \sqrt[512]{\frac{a_{5,8}}{a_{3,8}}} = \sqrt[512]{\frac{1.6876\cdot 10^{159}}{5.3128\cdot 10^{201}}} = 0.8260,$$

and in the second case,

$$r_5 \approx \sqrt[512]{\frac{a_{7,8}}{{}_{5,8}}} = \sqrt[512]{\frac{1}{1.6876\cdot 10^{159}}} = 0.4887.$$

Using Vieta's equations we see that the real parts, u_4 and u_5, satisfy

$$(55)\quad u_4 + u_5 = \frac{1}{2}\left\{-1 - (2.1625 - 1.9365 - 1.4656)\right\} = 0.1150,$$

and

$$a_{n-1} = (-1)^{n-1}\{\text{sum of the products of the roots taken } n-1 \text{ at a time}\}.$$

In the given case, $n = 7$ and $a_6 = -1$, thus

$$\begin{aligned}
-1 \;=\;& r_2 r_3 (u_4+iv_4)(u_4-iv_4)(u_5+iv_5)(u_5-iv_5)\\
&+r_1 r_3 (u_4+iv_4)(u_4-iv_4)(u_5+iv_5)(u_5-iv_5)\\
&+r_1 r_2 (u_4+iv_4)(u_4-iv_4)(u_5+iv_5)(u_5-iv_5)\\
&+r_1 r_2 r_3\{(u_4-iv_4)(u_5+iv_5)(u_5-iv_5)+(u_4+iv_4)(u_5+iv_5)(u_5-iv_5)\}\\
&+r_1 r_2 r_3\{(u_4+iv_4)(u_4-iv_4)(u_5-iv_5)+(u_4+iv_4)(u_4-iv_4)(u_5+iv_5)\}\\
=\;& (r_1r_2+r_2r_3+r_3r_1)\left(u_4^2+v_4^2\right)\left(u_5^2+v_5^2\right)\\
&+r_1r_2r_3\left\{2u_u\left(u_5^2+v_5^2\right)+2u_5\left(u_4^2+v_4^2\right)\right\}\\
=\;& (r_1r_2+r_2r_3+r_3r_1)\, r_4^2\cdot r_5^2 + 2\left(u_4r_5^2+u_5r_4^2\right) r_1r_2r_3.
\end{aligned}$$

It follows by a rearrangement that

$$(56)\qquad 2r_2^2\cdot u_4 + 2r_4^2\cdot u_5 = -\frac{1}{r_1r_2r_3} - r_4^2r_5^2\left\{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right\}.$$

We insert $r_1 = 2.1625$, $r_2 = -1.9365$, $r_3 = -1.4656$, $r_4 = 0.8260$ and $r_5 = 0.4887$ into (56) to get

$$0.4777\cdot u_4 + 1.3646\cdot u_5 = -0.1629 + 0.1200 = -0.0429,$$

so together with (55) we get the system

$$\begin{cases} u_4 + u_5 = 0.1150, \\ 0.4774 \cdot u_4 + 1.3646 \cdot u_5 = -0.0429, \end{cases}$$

so using Cramer's solution formula,

$$u_4 = \frac{\begin{vmatrix} 0.1150 & 1 \\ -0.0429 & 1.3646 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0.4774 & 1.3646 \end{vmatrix}} = \frac{0.1998}{0.8872} = 0.2252,$$

and

$$u_5 = \frac{\begin{vmatrix} 1 & 0.1150 \\ 0.4774 & -0.0429 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0.4774 & 1.3646 \end{vmatrix}} = \frac{-0.0978}{0.8872} = -0.1102.$$

Then finally,

$$v_4 = \sqrt{r_4^2 - u_4^2} = \sqrt{0.8260^2 - 0.2252^2} = 0.7947,$$

and

$$v_5 = \sqrt{r_5^2 - u_5^2} = \sqrt{0.4887^2 - 0.1102^2} = 0.4761,$$

and the complex roots are

$$0.2252 \pm i \cdot 0.7947 \qquad \text{and} \qquad -0.1102 \pm i \cdot 0.4761. \qquad \Diamond$$

The examples above show that the real roots are fairly easy to compute. In case of pairs of complex roots we first find the modulus r_m, and then write $u_m \pm iv_m$, where $r_m^2 = u_m^2 + v_m^2$. Use always Vieti's formula

$$-a_1 = \text{sum of all roots},$$

when we have at least one pair of complex roots. If we have two pairs of complex roots, then we should also include

$$(-1)^{n-1} a_{n-1} = \text{sum of all products of the roots taken } n-1 \text{ at a time.}$$

If there are three pairs of complex roots, we include another one of Vieti's formulæ, in which case the new system becomes more difficult to solve, due to the non-linearity, etc..

The worst case is of course when all roots of the real polynomial of degree $2n$ are complex, in which case we shall include n of $Vieti's$ $2n$ equations, of which only two are linear in the real parts $u_1, \dots, u_n$.

5 Appendix

5.1 The binomial formula

We shall for completeness prove the *binomial formula*

$$(57)\quad (a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j}\, b^j \qquad \text{for } a,\, b \in \mathbb{C}, \qquad n \in \mathbb{N}_0.$$

We use here the well-known notations $c^0 := 1$, and

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \frac{1\cdot 2\cdots n}{(1\cdot 2\cdots j)\cdot(1\cdot 2\cdots(n-j))}, \qquad \text{and } 0! := 1.$$

PROOF. If $a = 0$, we get by the conventions above b^n on both sides of the equality (57).

If $a \neq 0$, it follows by dividing by a^n that it suffices to prove (57) for $a = 1$, i.e. we divide by a^n and replace $\dfrac{b}{a}$ by b to reduce the claim to

$$(58)\quad (1+b)^n = \sum_{j=0}^{n} \binom{n}{j} b^j, \qquad b \in \mathbb{C} \qquad \text{and} \qquad n \in \mathbb{N}_0.$$

We shall prove (58) by induction.

When $n = 0$, we just get $1 = 1$ by the conventions above.

When $n = 1$, we get

$$(1+b)^1 = 1+b \qquad \text{and} \qquad \sum_{j=0}^{1} \binom{1}{j} b^j = \binom{1}{0} b^0 + \binom{1}{1} b^1 = 1+b.$$

Therefore, (58) holds for at least $n = 0$ and $n = 1$.

Assume that (58) holds for some $n \in \mathbb{N}$. Then, using this assumption,

$$\begin{aligned}
(1+b)^{n+1} &= (1+b)\cdot(1+b)^n = (1+b)\sum_{j=0}^{n} \binom{n}{j} b^j \\
&= \sum_{j=0}^{n} \binom{n}{j} b^j + \sum_{j=0}^{n} \binom{n}{j} b^{j+1} = \sum_{j=0}^{n} \binom{n}{j} b^j + \sum_{j=1}^{n+1} \binom{n}{j-1} b^j \\
&= \binom{n}{0} b^0 + \binom{n}{n} b^{n+1} + \sum_{j=1}^{n} \left\{ \binom{n}{j} + \binom{n}{j-1} \right\} b^j.
\end{aligned}$$

We notice that

$$\binom{n}{0} = 1 = \binom{n+1}{0} \qquad \text{and} \qquad \binom{n}{n} = 1 = \binom{n+1}{n+1},$$

and

$$\begin{aligned}\binom{n}{j}+\binom{n}{j-1} &= \frac{n!}{j!(n-j)!}+\frac{n!}{(j-1)!(n+1-j)!}=\frac{n!}{j!(n+1-j)!}\{(n+1-j)+j\}\\ &= \frac{n!(n+1)}{j(n+1-j)!}=\frac{(n+1)!}{j!(n+1-j)!}=\binom{n+1}{j}.\end{aligned}$$

Hence, finally

$$\begin{aligned}(1+b)^{n+1} &= \binom{n}{0}b^0+\sum_{j=1}^{n}\left\{\binom{n}{j}+\binom{n}{j-1}\right\}b^j+\binom{n}{n}b^{n+1}\\ &= \binom{n+1}{0}b^0+\sum_{j=1}^{n}\binom{n+1}{j}b^j+\binom{n+1}{n+1}b^{n+1}=\sum_{j=0}^{n+1}\binom{n+1}{j}b^j,\end{aligned}$$

and (58 follows by induction. □

Remark 5.1.1 It is not hard, using a similar proof, to prove *Euler's rule of differentiation* of a product of two C^n-functions f and g,

$$\text{(59)}\quad \frac{d^n}{dz^n}\{f(z)\cdot g(z)\}=\sum_{j=0}^{n}\binom{n}{j}\frac{d^{n-j}f}{dz^{n-j}}\cdot\frac{d^jg}{dz^j},$$

where we have put $\dfrac{d^0h}{dz^0}(z):=h(z)$. ♢

5.2 The identity theorem for convergent power series

Theorem 5.2.1 *Two complex power series*

$$f(z)=\sum_{n=0}^{+\infty}a_n\,z^n \qquad \textit{and} \qquad g(z)=\sum_{n=0}^{+\infty}b_n\,z^n,$$

which are both convergent in the same nonempty disc $|z|<r$, *and here are equal to each other, have the same coefficients, i.e.*

$$a_n=b_n \qquad \textit{for all } n\in\mathbb{N}_0.$$

PROOF. The proof follows the same patterns as the proof of Theorem 1.1.1. Let $f(z)=g(z)$ for all $|z|<r$. Putting $z=0$ we get

$$a_0=f(0)=g(0)=b_0,$$

so

$$f(z)-a_0=\sum_{n=1}^{+\infty}a_n\,z^n=z\sum_{n=0}^{+\infty}a_{n+1}\,z^n=g(z)-b_0=z\sum_{n=0}^{+\infty}b_{n+1}\,z^n$$

for all $|z| < r$. Hence, for $0 < |z| < r$,

$$\sum_{n=0}^{+\infty} a_{n+1}\, z^n = a_1 + z \sum_{n=0}^{+\infty} a_{n+2}\, z^n = b_1 + z \sum_{n=0}^{+\infty} b_{n+2}\, z^n,$$

so

$$(60)\quad a_1 - b_1 = z \left\{ \sum_{n=0}^{+\infty} b_{n+2}\, z^n - \sum_{n=0}^{+\infty} a_{n+2}\, z^n \right\} \qquad \text{for } 0 < |z| < r.$$

The left hand side of (60) is a constant, while the right hand side tends towards 0 for $z \to 0$. Hence $a_1 = b_1$, and it follows from (60) that

$$\sum_{n=0}^{+\infty} b_{n+2}\, z^n = \sum_{n=0}^{+\infty} a_{n+2}\, z^n \qquad \text{for } 0 < |z| < r.$$

Repeating this argument we get successively,

$$a_2 = b_2, \qquad a_3 = b_3, \qquad \dots, \qquad a_n = b_n, \qquad \dots,$$

and the theorem follows. □

Example 5.2.1 The *power series method.* Using the identity theorem we can describe the power series solution method of differential equations of polynomial coefficients. We shall illustrate this important method by the very simple example

(61) $f'(x) - 2x\,f(x) = 0, \qquad x \in \mathbb{R}, \quad f(0) = 1,$

the solution of which is of course $f(x) = \exp\left(x^2\right)$.

When we insert into (61) the *formal* power series

$$f(x) = \sum_{n=0}^{+\infty} a_n\,x^n \qquad \text{and} \qquad f'(x) = \sum_{n=1}^{+\infty} n\,a_n\,x^{n-1},$$

where the latter is obtained by termwise differentiation of the former, we get *formally*

$$\begin{aligned} 0 &= f'(x) - 2x\,f(x) = \sum_{n=1}^{+\infty} n\,a_n\,x^{n-1} - 2\sum_{n=0}^{+\infty} a_n\,x^{n+1} = \sum_{n=-1}^{+\infty}(n+2)a_{n+2}\,x^{n+1} - \sum_{n=0}^{+\infty} 2a_n\,x^{n+1} \\ &= a_1 + \sum_{n=0}^{+\infty}\{(n+2)a_{n+2} - 2a_n\}\,x^{n+1}. \end{aligned}$$

Since this is the zero polynomial, it follows from Theorem 5.2.1, *The identity theorem*, that all coefficients are zero, thus $a_1 = 0$ and

(62) $(n+2)a_{n+2} - 2a_n = 0, \quad n \in \mathbb{N}_0, \qquad \text{thus} \qquad a_{n+2} = \dfrac{2}{n+2}\,a_n, \quad n \in \mathbb{N}_0.$

From $a_1 = 0$ and (62) follow by induction that $a_{2n+1} = 0$ for every $n \in \mathbb{N}_0$. When the index is even, $n = 2m$, then we get from (62) that

$$(2m+2)a_{2m+2} - 2a_{2m} = 0, \qquad \text{thus} \qquad (m+1)a_{2(m+1)} = a_{2\cdot m}, \quad m \in \mathbb{N}.$$

We multiply the latter formula by $m! \neq 0$, and then we get by recursion,

$$(m+1)!\,a_{2(m+1)} = m!\,a_{2\cdot m} = \cdots = 1!\,a_{2\cdot 1} = 0!\,a_0 = a_0 = f(0) = 1.$$

Solving with respect to a_{2m} we get

$$a_{2m} = \frac{1}{m!}, \qquad \text{hence} \qquad a_{2n} = \frac{1}{n!} \quad \text{and} \quad a_{2n+1} = 0 \quad \text{for } n \in \mathbb{N}_0.$$

Therefore, the *formal* power series solution becomes

$$f(x) = \sum_{n=0}^{+\infty} a_{2n}\,x^{2n} = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(x^2\right)^2 = \exp\left(x^2\right),$$

where we at last recognize the power series expansion of the exponential, so the formal solution is also the correct solution. $\Diamond$

5.3 Taylor's formula

We shall here prove Taylor's formula, first in general, and then restrict ourselves to polynomials.

Theorem 5.3.1 Taylor's formula. *Let $f \in C^{n+1}(I)$ be an $n+1$ times continuously differentiable function on an open real interval, and let $a \in I$ be a given point. Then, for $x \in I$,*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t.$$

Proof. We shall here give a direct proof. If x is considered as a constant, then $t-a$ is an integral of 1, and it follows by a series of successive partial integrations that

$$\begin{aligned} f(x) &= f(a) + \int_a^x 1 \cdot f'(t)\,\mathrm{d}t = f(a) + \left[\frac{t-x}{1!} \cdot f'(t)\right]_a^x - \int_a^x \frac{t-x}{1!} f''(t)\,\mathrm{d}t \\ &= f(a) + \frac{f'(a)}{1!}(x-a) - \left[\frac{(t-x)^2}{2!} \cdot f''(t)\right]_a^x + \int_a^x \frac{(t-x)^2}{2!} f^{(3)}(t)\,\mathrm{d}t \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \left[\frac{(t-x)^3}{3!} f^{(3)}(t)\right]_a^x - \int_a^x \frac{(t-x)^3}{3!} f^{(4)}(t)\,\mathrm{d}t \\ &= \cdots \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t, \end{aligned}$$

and the theorem is proved. □

Theorem 5.3.1 holds in particular for polynomials $P(x)$ in the real variable $x \in \mathbb{R}$. We get an even better result, because if $P(x)$ has degree n, then clearly $P^{(n+1)}(t) \equiv 0$, so the error term (the final integral) disappears, and we have

$$(63) \quad P(x)P(a) + \frac{P'(a)}{1!}(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \cdots + \frac{P^{(n)}(a)}{n!}(x-a)^n, \qquad \text{for all } x \in \mathbb{R},$$

without the error term.

By the *identity theorem*, Theorem 1.1.1, the coefficients of a polynomial are unique, so (63) also holds, when $x \in \mathbb{R}$ is replaced by $z \in \mathbb{C}$.

Finally, we leave as an exercise to the reader to prove (63), when $a \in \mathbb{C}$ is a complex constant.

5.4 Weierstraß's approximation theorem

It is the author's experience that this very important theorem is not too well-known in general. It is, however, due to this theorem in many cases possible to work only with approximating polynomials instead of with more general continuous functions, so it plays indeed a very important role in Mathematics.

Theorem 5.4.1 Weierstraß's approximation theorem. *Let f be a continuous function on the bounded closed real interval $[a, b]$. Then there exists a sequence of polynomials $\{P_n\}$, which converges* uniformly *on $[a, b]$ towards f.*

That $P_n \to f$ uniformly on $[a, b]$ for $n \to +\infty$ means that to every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$, such that

$$|f(x) - P_n(x)| < \varepsilon \qquad \text{for all } n \geq n_0 \text{ and all } x \in [a, b],$$

i.e. the graphs of P_n lie eventually in an ε-tube around the graph of the continuous function.

Remark 5.4.1 We shall here give the customary proof, which for given $\varepsilon > 0$ and corresponding $n_0 \in \mathbb{N}$ *explicitly* defines the approximating *Bernstein polynomials*, so the proof of Theorem 5.4.1 is actually constructive. $\Diamond$

Proof. Clearly, we can approximate the real and the imaginary parts of f separately, so without loss of generality we may assume that f is a real function.

Then notice that the linear transformation $g : [0,1] \to [a,b]$, given by

$$g(t) = a + (b-a)t, \qquad t \in [0,1],$$

makes $f \circ g$ a continuous function of $[0,1]$, so we may assume from the beginning that $[a,b] = [0,1]$.

Since f is continuous on the bounded closed interval $[0,1]$, it follows from one of the main theorems of continuous functions that f is *uniformly* continuous on $[0,1]$. This means that given $\varepsilon > 0$ there exists a $\delta > 0$, such that

(64) $$|f(x) - f(y)| < \frac{\varepsilon}{2}, \qquad \text{whenever } x,\, y \in [0,1] \text{ and } |x-y| < \delta.$$

Let in the following ε and δ be given, such that (64) is fulfilled, and define the *Bernstein polynomials* $B_{n,f}(t)$ corresponding to the function f, in the following way,

$$B_{n,f}(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}, \qquad n \in \mathbb{N}.$$

We shall prove that

(65) $$|f(t) - B_{n,f}(t)| \le \varepsilon \qquad \text{for all } t \in [0,1] \text{ and all } n \ge n_0,$$

where

(66) $$n_0 := \frac{1}{\varepsilon \cdot \delta^2} \max_{t \in [0,1]} |f(t)|.$$

We shall need the following

Lemma 5.4.1

$$\sum_{k=0}^{n} \left\{ t - \frac{k}{n} \right\}^2 \binom{n}{k} t^k (1-t)^{n-k} = \frac{t(1-t)}{n} \qquad \textit{for } t \in [0,1].$$

Proof of Lemma 5.4.1. If one is familiar with the variance of the binomial distribution from *Probability Theory*, this is trivial. If not, one proceeds in the following way:

$$\sum_{k=0}^{n}\left\{t-\frac{k}{n}\right\}^2\binom{n}{k}t^k(1-t)^{n-k}=\sum_{k=0}^{n}\left\{t^2-2\cdot\frac{k}{n}+\frac{k^2}{n^2}\right\}^2\binom{n}{k}t^k(1-t)^{n-k}$$

$$=t^2\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}-2t\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}t^k(1-t)^{n-k}$$

$$+\sum_{k=1}^{n}\frac{k-1+1}{n}\cdot\frac{(n-1)!}{(k-1)!(n-k)!}t^k(1-t)^{n-k}$$

$$=t^2\cdot 1^n-2t\cdot t\sum_{k=0}^{n-1}\binom{n-1}{k}t^k\,(1-t)^{n-1-k}+\frac{1}{n}\cdot t\sum_{k=0}^{n-1}\binom{n-1}{k}t^k\,(1-t)^{n-1-k}$$

$$+\frac{n-1}{n}\sum_{k=2}^{n}\frac{(n-2)!}{(k-2)!(n-k)!}t^k\,(1-t)^{n-k}$$

$$=t^2-2t^2\cdot 1^{n-1}+\frac{1}{n}\cdot t\cdot 1^{n-1}+\left\{1-\frac{1}{n}\right\}^2\sum_{k=0}^{n-2}\binom{n-2}{k}t^k\,(1-t)^{n-2-k}$$

$$=t^2-2t^2+\frac{1}{n}t+\left\{1-\frac{1}{n}\right\}t^2\cdot 1^{n-2}=\frac{1}{n}t(1-t),$$

so

$$0\le\sum_{k=0}^{n}\left\{t-\frac{k}{n}\right\}^2\binom{n}{k}t^k\,(1-t)^{n-k}=\frac{t(1-t)}{n}\le\frac{1}{4n}\qquad\text{for } t\in[0,1].\qquad\square$$

Returning to the proof of Weierstraß's approximation theorem, we choose

$$n\ge\frac{1}{\varepsilon\cdot\delta^2}\max_{t\in[0,1]}|f(t)|,$$

and then we get by (64) and Lemma 5.4.1 the following computation,

$$|f(t) - B_{n,f}(t)| = \left| f(t) \cdot \{t + (1-t)\}^n - \sum_{n=0}^{n} f\left(\frac{k}{n}\right) \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k} \right|$$

$$= \left| \sum_{k=0}^{n} \left\{ f(t) - f\left(\frac{k}{n}\right) \right\} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k} \right|$$

$$\leq \left\{ \sum_{\left|t - \frac{k}{n}\right| < \delta} + \sum_{\left|t - \frac{k}{n}\right| \geq \delta} \right\} \left| f(t) - f\left(\frac{k}{n}\right) \right| \cdot \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-k)^{n-k}$$

$$\leq \frac{\varepsilon}{2} \sum_{\left|t - \frac{k}{n}\right| < \delta} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k} + \sum_{\left|t - \frac{k}{n}\right| \geq \delta} 2 \max_{t \in [0,1]} |f(t)| \cdot \frac{\left\{ t - \frac{k}{n} \right\}^2}{\delta^2} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k}$$

$$\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k} + \frac{2}{\delta^2} \max_{t \in [0,1]} |f(t)| \sum_{k=0}^{n} \left\{ t - \frac{k}{n} \right\}^2 \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k}$$

$$= \frac{\varepsilon}{2} + \frac{2}{\delta^2} \max_{t \in [0,1]} |f(t)| \cdot \frac{1}{n} t(1-t)$$

$$\leq \frac{\varepsilon}{2} + \frac{2}{\delta^2} \max_{t \in [0,1]} |f(t)| \cdot \frac{\varepsilon \cdot \delta^2}{\max_{t \in [0,1]} |f(t)|} \cdot \frac{1}{4} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

assuming that f is not identical 0. It is of course trivial, when $f \equiv 0$. □

Index