RIEMANNIAN SUBMANIFOLDS: A SURVEY

BANG-YEN CHEN

Contents

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1. Introduction

Problems in submanifold theory have been studied since the invention of calculus and it was started with differential geometry of plane curves. Owing to his studies of how to draw tangents to smooth plane curves, P. Fermat (1601–1665) is regarded as a pioneer in this field. Since his time, differential geometry of plane curves, dealing with curvature, circles of curvature, evolutes, envelopes, etc., has been developed as an important part of calculus. Also, the field has been expanded to analogous studies of space curves and surfaces, especially of lines of curvatures, geodesics on surfaces, and ruled surfaces.

Some historians date the beginning even before the invention of calculus. Already around 1350, the French bishop Nicole Oresme (1323–1382) proposed to assign 0-curvature to the straight lines and curvature $\frac{1}{r}$ to the circles of radius r. Along the lines of previous work by J. Kepler (1571–1630), R. Descartes (1596–1650) and C. Huygens (1629-1695) in 1671, I. Newton (1642–1727) succeeded in defining and computing the curvature $\kappa(t)$ at each point of a plane curve using the ideas of osculating circles and intersection of neighborhood normals.

The first major contributor to the subject was L. Euler (1707–1783). In 1736 Euler introduced the arc length and the radius of curvature and so began the study of intrinsic differential geometry of submanifolds.

Concerning space curves, G. Monge (1747–1818) obtained in 1770 the expression for the curvature $\kappa(t)$ of a space curve $\gamma = \gamma(t)$. The expression for the torsion $\tau(t)$ was first obtained by M. A. Lancret in 1806. The works of A. L. Cauchy (1789–1857) in 1826, F. Frenet (1816–1900) in 1847 and J. A. Serret (1819–1885) in 1851, resulted in the well-known Frenet-Serret formulas which give all the successive derivatives of a curve. The fundamental theorems or congruence theorems for curves were obtained by L. S. V. Aoust (1814–1885) in 1876.

C. F. Gauss (1777–1855) established the theory of surfaces by introducing the concepts of the geometry of surfaces (*Disquisitiones circa superficie curvas,* 1827). Since then the subject has come to occupy a very firm position in mathematics. The influence of differential geometry of curves and surfaces exerted upon branches of mathematics, dynamics, physics, and engineering has been profound. For instance, the study of geodesics is a topic deeply related to dynamics, calculus of variations, and topology; also the study of minimal surfaces is intimately related to the theory of functions of a complex variable, calculus of variations, and topology. Weierstrass and Schwarz established its relationship with the theory of functions. Among others, J.

L. Lagrange (1736–1813), K. Weierstrass (1815–1897), H. A. Schwarz (1843–1921), J. Douglas (1897–1965), T. Radó (1897–1965), S. S. Chern (1911–), and R. Osserman (1926–) are those who made major contributions on this subject.

Belgian physicist J. A. Plateau (1801–1883) showed experimentally that minimal surfaces can be realized as soap films by dipping wire in the form of a closed space curve into a soap solution (around 1850). The Plateau problem, that is, the problem of proving mathematically the existence of a minimal surface with prescribed boundary curve, was solved by T. Radó (1895–1965) in 1930, and independently by J. Douglas (1897–1965) in 1931.

Before Gauss, geometers viewed a surface as being made of infinitely many curves, whereas Gauss viewed the surface as an entity in itself. Influenced by Gauss' geometry on a surface in Euclidean 3-space, B. Riemann (1826–1866) introduced in 1854 Riemannian geometry. Riemannian geometry includes Euclidean and non-Euclidean geometries as special cases, and it is important for the great influence it exerted on geometric and physical ideas of the twentieth century.

Using the concept of the intrinsic Riemannian structure on the surface, one can compute the curvature of a surface in two different ways. One is to compute the principal curvatures and the other is done intrinsically using the induced Riemannian metric on the surface. The Theorema Egregium of Gauss provides a direct relationship between the intrinsic and the extrinsic geometries of surfaces.

Motivated by the theory of mechanics, G. Darboux (1824–1917) unified the theory of curves and surfaces with his concept of a moving frame. This is the beginning of modern submanifold theory which in turn gave valuable insight into the field.

Since the celebrated embedding theorem of J. F. Nash (1928–) allows geometers to view each Riemannian manifold as a submanifold in a Euclidean space, the problem of discovering simple sharp relationships between intrinsic and extrinsic invariants of a submanifold is one of the most fundamental problems in submanifold theory. Many beautiful results in submanifold theory, including the Gauss-Bonnet theorem and isoperimetric inequalities, are results in this respect. In the modern theory of submanifolds, the study of relations between local and global properties has also attracted the interest of many geometers. This view was emphasized by W. Blaschke (1885–1962), who worked on the differential geometry of ovals and ovaloids. The study of rigidity of ovaloids by S. Cohn-Vossen (1902–1936) belongs in this category.

An important class of Riemannian manifolds was discovered by J. A. Schouten (1883–1971), D. van Dantzig (1900–1959), and E. Kähler (1906–) around 1929–1932. This class of manifolds, called Kähler manifolds today, includes the projective algebraic manifolds. The study of complex submanifolds of a Kähler manifold from differential geometric points of view was initiated by E. Calabi (1923–) in the early 1950's. Besides complex submanifolds, there are some other important classes of submanifolds of a Kähler manifold determined by the behavior of the tangent bundle of the submanifold under the action of the almost complex structure of the ambient manifold. These classes of submanifolds have many interesting properties and many important results have been discovered in the last quarter of this century from this standpoint.

Submanifold theory is a very active vast research field which in turn plays an important role in the development of modern differential geometry in this century. This branch of differential geometry is still so far from being exhausted; only a small portion of an exceedingly fruitful field has been cultivated, much more remains to be discovered in the coming centuries.

2. Nash's embedding theorem and some related results

Throughout this article manifolds are assumed to be connected, of class C^∞, and without boundary, unless mentioned otherwise.

H. Whitney (1907–1989) proved in 1936 that an n-manifold can always be immersed in the Euclidean $2n$-space E^{2n}, and can always be embedded in E^{2n+1} as a closed set.

An immersion f (or an embedding) of a Riemannian manifold (M, g) into another Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be isometric if it satisfies the condition $f^*\tilde{g} = g$. In this case, M is called a Riemannian submanifold (or simply a submanifold) of $\tilde{M}$.

We shall identify the image $f(M)$ with M when there is no danger of confusion.

One of fundamental problems in submanifold theory is the problem of isometric immersibility. The earliest publication on isometric embedding appeared in 1873 by L. Schläfli (1814–1895).

The problem of isometric immersion (or embedding) admits an obvious analytic interpretation; namely, if $g_{ij}(x)$, $x = (x_1, \ldots, x_n)$, are the components of the metric tensor g in local coordinates $x_1, \ldots, x_n$ on a Riemannian n-manifold M, and $y = (y_1, \ldots, y_m)$ are the standard Euclidean coordinates in E^m, then the condition for an isometric immersion in E^m is

$$\sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i} = g_{jk}(x),$$

that is, we have a system of $\frac{1}{2}n(n+1)$ nonlinear partial differential equations in m unknown functions. If $m = \frac{1}{2}n(n+1)$, then this system is definite and so we would like to have a solution. Schläfli asserted that any Riemannian n-manifold can be isometrically embedded in Euclidean space of dimension $\frac{1}{2}n(n+1)$. Apparently it is appropriate to assume that he had in mind of analytic metrics and local analytic embeddings. This was later called Schläfli's conjecture.

2.1. **Cartan-Janet's theorem.** In 1926 M. Janet (1888–1984) published a proof of Schläfli's conjecture which states that a real analytic Riemannian n-manifold M can be locally isometrically embedded into any real analytic Riemannian manifold of dimension $\frac{1}{2}n(n+1)$. In 1927 É. Cartan (1869–1951) revised Janet's paper with the same title; while Janet wrote the problem in the form of a system of partial differential equations which he investigated using rather complicated methods, Cartan applied his own theory of Pfaffian systems in involution. Both Janet's and Cartan's proofs contained obscurities. In 1931 C. Burstin got rid of them. This result of Cartan-Janet implies that every Einstein n-manifold ($n \geq 3$) can be locally isometrically embedded in $E^{n(n+1)/2}$.

The Cartan-Janet theorem is dimensionwise the best possible, that is, there exist real analytic Riemannian n-manifolds which do not possess smooth local isometric embeddings in any Euclidean space of dimension strictly less than $\frac{1}{2}n(n+1)$. Not every Riemannian n-manifold can be isometrically immersed in E^m with $m \leq \frac{1}{2}n(n+1)$. For instance, not every Riemannian 2-manifold can be isometrically immersed in E^3.

Cartan-Janet's theorem implies that an analytic Riemannian 3-manifold can be locally isometrically embedded into E^6. For Riemannian smooth 3-manifolds, R. L. Bryant, P. A. Griffiths and D. Yang (1983) proved the following:

Let M be a smooth Riemannian 3-manifold and let $x \in M$ be such that the Einstein tensor at x is neither zero nor a perfect square $L^2 = \sum \ell_i \ell_j dx^i dx^j$, $L = \sum \ell_i dx^i \in T_x^* M$. Then there exists a neighborhood of x which can be smoothly isometrically embedded into E^6.

G. Nakamura and Y. Maeda (1986) improved the result to the following: Let M be a smooth Riemannian 3-manifold and let $x \in M$ be a point such that the curvature tensor $R(x)$ at x does not vanish, where $R(x)$ is considered as a symmetric linear operator acting on the space of 2-forms. Then there exists a local smooth isometric embedding of a neighborhood of x into E^6.

H. Jacobowitz and J. D. Moore (1973) proved that, for real analytic Riemannian manifolds M and $\tilde{M}$ of dimensions n and N respectively with $n \geq 2$, $N \geq \frac{1}{2}n(n+1) - 1$, if x_0 is a point of M, then there exists an open neighborhood U of $x_0 \in M$ which can be conformally embedded in $\tilde{M}$.

2.2. **Nash's embedding theorem.** A global isometric embedding theorem was proved by J. F. Nash (1956) which states as follows.

Theorem 2.1 *Every compact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean N-space E^N with $N = \frac{1}{2}n(3n+11)$. Every non-compact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean m-space E^m with $m = \frac{1}{2}n(n+1)(3n+11)$.*

R. E. Greene (1970) improved Nash's result and proved that every non-compact Riemannian n-manifold can be isometrically embedded in Euclidean N-space with $N = 2(2n+1)(3n+7)$. Furthermore, Greene (1970) and M. L. Gromov and V. A. Rokhlin (1970) proved independently that a local isometric embedding from Riemannian n-manifold into $E^{\frac{1}{2}n(n+1)+n}$ always exist. Gromov and Rokhlin (1970) also proved that a compact Riemannian

n-manifold of class C^r ($r = \infty$ or ω) can be isometrically C^r-embedded in $E^{\frac{1}{2}n(n+1)+3n+5}$.

Nash (1954) proved that if a manifold M admits a C^1-embedding in E^m, where $m \geq n+2$, then it admits an isometric C^1-embedding in E^m. N. H. Kuiper (1920–1994) improved this result in 1955 by showing that it is true when $m \geq n+1$. M. L. Gromov showed that every Riemannian n-manifold can be C^1 isometrically immersed into E^{2n} (cf. [Gromov 1986]).

It is known that a Hermitian symmetric space G/K of compact type can be equivariantly and isometrically embedded into E^m with $m = \dim G$ [Lichnerowicz 1958]. This result has been extended to almost all symmetric spaces of compact type [Nagano 1965; Kobayashi 1968]. In 1976 J. D. Moore proved that every compact Riemannian homogeneous manifold admits an equivariant isometric embedding in some Euclidean space.

2.3. **Isometric immersions with the smallest possible codimension.** According to Nash's embedding theorem, every Riemannian manifold can be isometrically embedded in a Euclidean space of sufficiently large dimension, it is thus natural to look for a Euclidean space of smallest possible dimension in which a Riemannian manifold can be isometrically embedded.

D. Hilbert (1862–1943) proved in 1901 that a complete surface of constant negative curvature cannot be C^4-isometrically immersed in Euclidean 3-space.

A result of S. S. Chern and N. H. Kuiper (1952) states that a compact Riemannian n-manifold with non-positive sectional curvature cannot be isometrically immersed in E^{2n-1}. A generalization by B. O'Neill (1924–) states that if N is a complete simply-connected Riemannian $(2n-1)$-manifold with sectional curvature $K_N \leq 0$, then a compact Riemannian n-manifold M with sectional curvature $K_M \leq K_N$ cannot be isometrically immersed in N [O'Neill 1960].

T. Otsuki (1917–) proved in 1954 that a Riemannian n-manifold of constant negative sectional curvature cannot be isometrically immersed in E^{2n-2} even locally. He also proved that a Riemannian n-manifold cannot be isometrically immersed in a Riemannian $(2n-2)$-manifold if $K_M < K_N$.

Using the purely algebraic theory of "flat bilinear forms", J. D. Moore (1977) proved that if a compact Riemannian n-manifold M of constant curvature 1 admits an isometric immersion in E^N with $N \leq \frac{3}{2}n$, then M is simply-connected, hence isometric to the unit n-sphere. A real analytic version of this result was obtained by W. Henke (1976) in the special case where $n \geq 4$ and $N = n+2$.

2.4. **Isometric immersions with prescribed Gaussian or Gauss-Kronecker curvature.** A 1915 problem of H. Weyl (1885–1955) is that whether a Riemannian 2-manifold of positive Gaussian curvature that is diffeomorphic to a sphere can be realized as a smooth ovaloid in E^3? Weyl himself suggested an incomplete solution of this problem for analytic surfaces. In fact, he solved the problem in the case of analytic metrics sufficiently close to the metric of a sphere.

A complete solution of Weyl's problem for analytic case was given by H. Lewy (1904–1988) in 1938. L. Nirenberg (1925–) proved in 1953 that given a C^∞-smooth Riemannian metric g on a topological 2-sphere S^2 with Gaussian curvature $K > 0$, there exists a C^∞-smooth global isometric embedding of (S^2, g) into E^3.

A local immersibility for C^k-smooth metric with $K \geq 0$ and $k \geq 10$ in the form of a C^{k-6}-smooth convex surface was proved in 1985 by C. S. Lin. In 1995 J. Hong and C. Zuily extended Nirenberg's global result to the case $K \geq 0$. J. Hong (1997) established isometric embedding in E^3 of complete noncompact nonnegatively curved surfaces.

C. S. Lin (1986) considered the problem of isometric embedding of two-dimensional metrics of curvature that changes sign and proved the following: Let the curvature K of a Riemannian 2-manifold be equal to zero at the point P, but the gradient of the curvature ∇K be nonzero. If the metric of the manifold belongs to the class C^s, $s \geq 6$, then a neighborhood of P admits an isometric embedding of class C^{s-3} in E^3. W. Greub and D. Socolescu (1994) claimed that the condition $\nabla K \neq 0$ in Lin's result can be removed.

N. V. Efimov (1964) proved that a complete surface with Gaussian curvature $K \leq -c^2$, c a positive constant, does not admit an isometric immersion in E^3. C. Baikoussis and T. Koufogiorgos (1980) showed that a complete surface with curvature $-\infty < -a^2 \leq K \leq 0$ in E^3 is unbounded.

B. Smyth and F. Xavier (1987) proved that if a complete Riemannian n-manifold M with negative Ricci curvature is immersed as a hypersurface in a Euclidean space, then the upper bound of the Ricci curvature of M is equal to zero if $n = 3$, or if $n > 3$ and the sectional curvatures of M do not take all real values.

For an oriented hypersurface M in E^{n+1}, the determinant of the shape operator of M with respect to the unit outward normal is called the Gauss-Kronecker curvature of the hypersurface.

Given a smooth positive real function F on E^{n+1}, the problem to find a closed convex hypersurface M in E^{n+1} with Gauss-Kronecker curvature $K = F$ have been studied by various geometers. For instance, sufficient conditions were found under which this problem can be solved, either by topological methods [Oliker 1984; Caffarelli-Nirenberg-Spruck 1986] or by geometric variational approach [Oliker 1986; Tao 1991; X. J. Wang 1996].

2.5. **Isometric immersions with prescribed mean curvature.** Given a closed $(n-1)$-dimensional submanifold Γ in a Riemannian manifold N, the problem of finding an oriented n-dimensional submanifold M with a prescribed mean curvature vector and with Γ as its boundary has been investigated by many mathematicians.

The first necessary conditions for parametric surfaces were given by E. Heinz in 1969 for surfaces of constant mean curvature in Euclidean 3-space with a prescribed rectifiable boundary. R. Gulliver gave in 1983 a necessary condition on the magnitude of the mean curvature vector field for there to exist an oriented submanifold of a Riemannian manifold M having prescribed mean curvature vector and a given closed submanifold as boundary.

I. J. Bakelman and B. E. Kantor (1974), A. Treibergs and W. S. Wei (1983), A. Treibergs (1985), and K. Tso (1989) established the existence of closed convex hypersurfaces in a Euclidean space with prescribed mean curvature.

3. Fundamental theorems, basic notions and results

3.1. **Fundamental equations.** Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion. Denote by ∇ and $\tilde{\nabla}$ the metric connections of M and $\tilde{M}$, respectively. For vector fields X and Y tangent to M, the tangential component of $\tilde{\nabla}_X Y$ is equal to $\nabla_X Y$.

Let

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y. \tag{3.1}$$

The h is a normal-bundle-valued symmetric $(0, 2)$ tensor field on M, which is called the second fundamental form of the submanifold (or of the immersion). Formula (3.1) is known as the Gauss formula [Gauss 1827].

For a normal vector ξ at a point $x \in M$, we put

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi). \tag{3.2}$$

Then A_ξ is a symmetric linear transformation on the tangent space $T_x M$ of M at x, which is called the shape operator (or the Weingarten map) in the direction of ξ. The eigenvalues of A_ξ are called the principal curvatures in the direction of ξ.

The metric connection on the normal bundle $T^\perp M$ induced from the metric connection of $\tilde{M}$ is called the normal connection of M (or of f).

Let D denote covariant differentiation with respect to the normal connection. For a tangent vector field X and a normal vector field ξ on M, we have

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{3.3}$$

where $-A_\xi X$ is the tangential component of $\tilde{\nabla}_X \xi$. (3.3) is known as the Weingarten formula, named after the 1861 paper of J. Weingarten (1836–1910).

Let $R, \tilde{R}$ and R^D denote the Riemannian curvature tensors of $\nabla, \tilde{\nabla}$ and D, respectively. Then the integrability condition for (3.1) and (3.3) implies

$$\begin{aligned} \tilde{R}(X, Y)Z = R(X, Y)Z &+ A_{h(X,Z)} Y - A_{h(Y,Z)} X \\ &+ (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \end{aligned} \tag{3.4}$$

for tangent vector fields X, Y, Z of M, where $\bar{\nabla}$ is the covariant differentiation with respect to the connection in $TM \oplus T^\perp M$. The tangential and normal components of (3.4) yield the following equation of Gauss

$$\langle R(X, Y)Z, W \rangle = \langle\ \tilde{R}(X, Y)Z, W\ \rangle + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{3.5}$$

and the equation of Codazzi

$$(R(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{3.6}$$

where X, Y, Z, W are tangent vectors of M, $(R(X,Y)Z)^{\perp}$ is the normal component of $R(X,Y)Z$, and $\langle\ ,\rangle$ is the inner product.

Similarly, for normal vector fields ξ and η, the relation

$$\langle\ \tilde{R}(X,Y)\xi,\eta\ \rangle = \left\langle R^D(X,Y)\xi,\eta\right\rangle - \langle [A_\xi, A_\eta]X, Y\rangle \tag{3.7}$$

holds, which is called the equation of Ricci.

Equations (3.1), (3.3), (3.5), (3.6) and (3.7) are called the fundamental equations of the isometric immersion $f : M \to \tilde{M}$.

As a special case, suppose the ambient space $\tilde{M}$ is a Riemannian manifold of constant sectional curvature c. Then the equations of Gauss, Codazzi and Ricci reduce respectively to

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle$$

$$+c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\}, \tag{3.8}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{3.9}$$

$$\left\langle R^D(X,Y)\xi,\eta\right\rangle = \langle [A_\xi, A_\eta]X, Y\rangle\,. \tag{3.10}$$

Formulas (3.8) and (3.9) for surfaces in E^3 were given in principal, though not explicitly, in [Gauss 1827]. The formulas can be found in a 1860 paper by D. Codazzi (1824–1875) in his answer to a “concours” of the Paris Academy (printed in the Mémoires présenté à l’Académie in 1880; also in [Codazzi 1868]). These formulas at that time already published by G. Mainardi (1800–1879) in [Mainardi 1856]. The fundamental importance of these formulas was fully recognized by O. Bonnet (1819–1892) in [Bonnet 1867]. The equations of Gauss and Codazzi for general submanifolds were first given by A. Voss in 1880. The equation (3.10) of Ricci was first given by G. Ricci (1853–1925) in 1888.

3.2. **Fundamental theorems.** The fundamental theorems of submanifolds are the following.

Existence Theorem *Let (M,g) be a simply-connected Riemannian n-manifold and suppose there is a given m-dimensional Riemannian vector bundle $\nu(M)$ over M with curvature tensor R^D and a $\nu(M)$-valued symmetric $(0,2)$ tensor h on M. For a cross section ξ of $\nu(M)$, define A_ξ by $g(A_\xi X, Y) = \langle h(X,Y),\xi\rangle$, where $\langle\ ,\ \rangle$ is the fiber metric of $\nu(M)$. If they satisfy* (3.8), (3.9) *and* (3.10)*, then M can be isometrically immersed in an $(n+m)$-dimensional* *complete simply-connected Riemannian manifold $R^{n+m}(c)$ of constant curvature c in such way that $\nu(M)$ is the normal bundle and h is the second fundamental form.*

Uniqueness Theorem*Let $f, f' : M \to R^m(c)$ be two isometric immersions of a Riemannian n-manifold M into a complete simply-connected Riemannian m-manifold of constant curvature c with normal bundles ν and ν' equipped with their canonical bundle metrics, connections and second fundamental forms, respectively. Suppose there is an isometry $\phi : M \to M$ such that ϕ can be covered by a bundle map $\bar{\phi} : \nu \to \nu'$ which preserves the bundle metrics, the connections and the second fundamental forms. Then there is an isometry Φ of R^m such that $\Phi \circ f = f'$.*

The first to give a proof of the fundamental theorems was O. Bonnet (cf. [Bonnet 1867]).

3.3. **Basic notions.** Let M be an n-dimensional Riemannian submanifold of a Riemannian manifold $\tilde{M}$. A point $x \in M$ is called a geodesic point if the second fundamental form h vanishes at x. The submanifold is said to be totally geodesic if every point of M is a geodesic point. A Riemannian submanifold M is a totally geodesic submanifold of $\tilde{M}$ if and only if every geodesic of M is a geodesic of $\tilde{M}$.

Let M be a submanifold of $\tilde{M}$ and let $e_1, \ldots, e_n$ be an orthonormal basis of T_xM. Then the mean curvature vector H at x is defined by

$$H = \frac{1}{n} \sum_{j=1}^{n} h(e_j, e_j).$$

The length of H is called the mean curvature which is denoted by H. M is called a minimal submanifold of $\tilde{M}$ if the mean curvature vector field vanishes identically.

A point $x \in M$ is called an umbilical point if $h = g \otimes H$ at x, that is, the shape operator A_ξ is proportional to the identity transformation for all $\xi \in T_x^\perp M$. The submanifold is said to be totally umbilical if every point of the submanifold is an umbilical point.

A point $x \in M$ is called an isotropic point if $|h(X,X)|/|X|^2$ does not depend on the nonzero vector $X \in T_xM$. A submanifold M is called an isotropic submanifold if every point of M is an isotropic point. The submanifold M is called constant isotropic if $|h(X,X)|/|X|^2$ is also independent of the point $x \in M$. It is clear that umbilical points are isotropic points.

The index of relative nullity at $x \in M$ of a submanifold M in $\tilde{M}$ is defined by

$$\mu(x) = \dim \cap_{\xi \in T^\perp M} \ker A_\xi.$$

If we denote by $N_0(x)$ the null space of the linear mapping $\xi \to A_\xi$, then the orthogonal complement of $N_0(x)$ in the normal space $T_x^\perp M$ is called the first normal space at x.

A normal vector field ξ of M in $\tilde{M}$ is said to be parallel in the normal bundle if $D_X \xi = 0$ for any vector X tangent to M. A submanifold M is said to have parallel mean curvature vector if the mean curvature vector field of M is a parallel normal vector field.

A submanifold M in a Riemannian manifold is called a parallel submanifold if its second fundamental form h is parallel, that is, $\bar{\nabla} h = 0$, identically.

A Riemannian submanifold M is said to have flat normal connection if the curvature tensor R^D of the normal connection D vanishes at each point $x \in M$.

For a Riemannian n-manifold M, we denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_x M$. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_x M$. The Ricci curvature Ric and the scalar curvature ρ of M at x are defined respectively by

$$Ric(X,Y) = \sum_{j=1}^{n} \langle R(e_j, X)Y, e_j \rangle \,, \tag{3.11}$$

$$\rho = \sum_{i \neq j} K(e_i \wedge e_j), \tag{3.12}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of the 2-plane section spanned by e_i and e_j.

In general if L is an r-plane section in $T_x M$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L \subset T_x M$, then the scalar curvature $\rho(L)$ of L is defined by

$$\rho(L) = \sum_{i \neq j} K(e_i \wedge e_j), \quad 1 \leq i, j \leq r. \tag{3.13}$$

3.4. **A general inequality.** Let M be a Riemannian n-manifold. For an integer $k \geq 0$, denote by $\mathcal{S}(n,k)$ the finite set consisting of k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of (unordered) k-tuples with $k \geq 0$ for a fixed positive integer n.

The cardinal number $\#\mathcal{S}(n)$ of $\mathcal{S}(n)$ is equal to $p(n) - 1$, where $p(n)$ denotes the number of partition of n which increases quite rapidly with n. For instance, for

$$n = 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, 20, \ldots, 50, \ldots, 100, \ldots, 200,$$

the cardinal number $\#\mathcal{S}(n)$ are given by

$$1, 2, 4, 6, 10, 14, 21, 29, 41, \ldots, 626, \ldots, 204225, \ldots, 190569291, \ldots, 3972999029387,$$

respectively. The asymptotic behavior of $\#\mathcal{S}(n)$ is given by

$$\#\mathcal{S}(n) \sim \frac{1}{4n\sqrt{3}} \exp\left[\pi\sqrt{2n/3}\right] \quad \text{as } n \to \infty.$$

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ B. Y. Chen (1996f,1997d) introduced a Riemannian invariant $\delta(n_1, \ldots, n_k)$ by

$$\delta(n_1, \ldots, n_k)(x) = \frac{1}{2}\left(\rho(x) - \inf\{\rho(L_1) + \cdots + \rho(L_k)\}\right), \tag{3.14}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j$, $j = 1, \ldots, k$. We put $\delta(\emptyset) = \frac{\rho}{2}$.

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, let

$$a(n_1, \ldots, n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1). \tag{3.15}$$

$$b(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}. \tag{3.16}$$

When $k = 0$, the left hand sides of (3.15) and (3.16) are denoted respectively by $a(\emptyset)$, $b(\emptyset)$.

For any n-dimensional submanifold M of a real space form $R^m(c)$ and for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, regardless of dimension and codimension, there is a sharp general inequality between the invariant $\delta(n_1, \ldots, n_k)$ and the squared mean curvature H^2 given by [Chen 1996c,1996f]:

$$\delta(n_1, \ldots, n_k) \le b(n_1, \ldots, n_k)H^2 + a(n_1, \ldots, n_k)c. \tag{3.17}$$

The equality case of inequality (3.17) holds at a point $x \in M$ if and only if, there exists an orthonormal basis $e_1, \ldots, e_m$ at x, such that the shape operators of M in $R^m(c)$ at x take the following form:

$$A_r = \begin{pmatrix} A_1^r & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & A_k^r & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \mu_r & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \mu_r \end{pmatrix}, \quad r = n+1, \ldots, m, \tag{3.18}$$

where A_j^r are symmetric $n_j \times n_j$ submatrices which satisfy

$$\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \mu_r.$$

Inequality (3.17) has many interesting applications. For instance, regardless of codimension, it implies that the squared mean curvature of every isometric immersion of $S^n(1)$ (respectively, of $S^k(1) \times E^{n-k}$ or of $S^k(1) \times S^{n-k}(1)$) into a Euclidean space must satisfy

(3.19)
$$H^2 \geq 1 \quad \left(\text{respectively, } H^2 \geq \left(\frac{k}{n}\right)^2 \text{ or } H^2 \geq \left(\frac{k}{n}\right)^2 + \left(\frac{n-k}{n}\right)^2\right),$$

with the equality holding if and only if it is a standard embedding.

The equality case of (3.17) with $k = 0$ holds at a point $x \in M$ when and only when x is an umbilical point. In general, there exist ample examples of submanifolds which satisfy the equality case of (3.17) with $k > 0$.

Inequality (3.17) also provides the following sharp estimate of the first nonzero eigenvalue λ_1 of the Laplacian Δ on each compact irreducible homogeneous Riemannian n-manifold M [Chen 1996f, Chen 1997d]:

$$\lambda_1 \geq n\hat{\Delta}_0, \tag{3.20}$$

where

$$\hat{\Delta}_0 = \max\left\{\frac{\delta(n_1, \ldots, n_k)}{b(n_1, \ldots, n_k)} : (n_1, \ldots, n_k) \in \mathcal{S}(n)\right\}. \tag{3.21}$$

Clearly, the invariant $\hat{\Delta}_0$ is constant on a homogeneous Riemannian manifold, since each δ-invariant $\delta(n_1, \ldots, n_k)$ is constant on such a space.

The estimate of λ_1 given in (3.20) improves a well-known result of T. Nagano (1930–) who proved in 1961 that $\lambda_1 \geq \frac{\rho}{n-1}$ for each compact irreducible homogeneous Riemannian n-manifold M, with the equality holding if and only if M is a Riemannian n-sphere. We remark that $\rho/(n-1)$ is nothing but $\delta(\emptyset)/b(\emptyset)$.

Inequality (3.17) for $\delta(2)$ was first proved in [Chen 1993]. The equality case of (3.17) for $\delta(2)$ have been investigated by D. E. Blair, J. Bolton, B. Y. Chen, M. Dajczer, F. Defever, R. Deszcz, F. Dillen, L. A. Florit, C. S. Houh, I. Mihai, M. Petrovic, C. Scharlach, L. Verstraelen, L. Vrancken, L. M. Woodward, J. Yang, and others.

Further applications of (3.17) can be found in §5.3.1, §5.4.1, and §16.7.

3.5. **Product immersions.** Suppose that $M_1, \ldots, M_k$ are Riemannian manifolds and that

$$f : M_1 \times \cdots \times M_k \to E^N$$

is an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into Euclidean N-space. J. D. Moore (1971) proved that if the second fundamental form h of f has the property that $h(X, Y) = 0$, when X is tangent to M_i and Y is tangent to M_j for $i \neq j$, then f is a product immersion, that is, there exist isometric immersions $f_i : M_i \to E^{m_i}$, $1 \le i \le k$ such that

$$f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$$

when $x_i \in M_i$ for $1 \le i \le k$.

Let $M_0, \cdots, M_k$ be Riemannian manifolds, $M = M_0 \times \cdots \times M_k$ their product, and $\pi_i : M \to M_i$ the canonical projection. If $\rho_1, \cdots, \rho_k : M_0 \to \mathbf{R}_+$ are positive-valued functions, then

$$\langle X, Y \rangle := \langle \pi_{0*}X, \pi_{0*}Y \rangle + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 \langle \pi_{i*}X, \pi_{i*}Y \rangle$$

defines a Riemannian metric on M, called a warped product metric. M endowed with this metric is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$.

A warped product immersion is defined as follows: Let $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and let $f_i : N_i \to M_i$, $i = 0, \cdots, k$, be isometric immersions, and define $\sigma_i := \rho_i \circ f_0 : N_0 \to \mathbf{R}_+$ for $i = 1, \cdots, k$. Then the map

$$f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \to M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$$

given by $f(x_0, \cdots, x_k) := (f_0(x_0), f_1(x_1), \cdots, f_k(x_k))$ is an isometric immersion, which is called a warped product immersion.

S. Nölker (1996) extended Moore's result to the following.

Let $f : N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k \to R^N(c)$ be an isometric immersion into a Riemannian manifold of constant curvature c. If h is the second fundamental form of f and $h(X_i, X_j) = 0$, for all vector fields X_i and X_j, tangent to N_i and N_j respectively, with $i \neq j$, then, locally, f is a warped product immersion.

3.6. **A relationship between k-Ricci tensor and shape operator.** Let M be a Riemannian n-manifold and L^k is a k-plane section of T_xM^n, $x \in M$. For each unit vector X in L^k, we choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L^k such that $e_1 = X$. Define the Ricci curvature Ric_{L^k} of L^k at X by

$$Ric_{L^k}(X) = K_{12} + \cdots + K_{1k}, \tag{3.22}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We call $Ric_{L^k}(X)$ a k-Ricci curvature of M at X relative to L^k. Clearly, the n-th Ricci curvature is nothing but the Ricci curvature in the usual sense and second Ricci curvature coincides with the sectional curvature.

For each integer k, $2 \leq k \leq n$, let θ_k denote the Riemannian invariant defined on M by

$$\theta_k(x) = \left(\frac{1}{k-1}\right) \inf_{L^k, X} Ric_{L^k}(X), \quad X \in T_x M, \tag{3.23}$$

where L^k runs over all k-plane sections in $T_x M$ and X runs over all unit vectors in L^k.

The following results provide a sharp relationship between the k-Ricci curvature and the shape operator for an arbitrary submanifold in a real space form, regardless of codimension [Chen 1996e,1998c].

Let $f : M \to R^m(c)$ be an isometric immersion of a Riemannian n-manifold M into a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have

(1) if $\theta_k(x) \neq c$, then the shape operator in the direction of the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}(\theta_k(x) - c)I \quad \text{at } x, \tag{3.24}$$

where I denotes the identity map of $T_x M^n$;

Inequality (3.24) means that $A_H - \frac{n-1}{n}(\theta_k(x) - c)I$ is positive-definite.

(2) if $\theta_k(x) = c$, then $A_H \geq 0$ at x;

(3) a unit vector $X \in T_x M$ satisfies $A_H X = \frac{n-1}{n}(\theta_k(x) - c)X$ if and only if $\theta_k(x) = c$ and X lies in the relative null space at x;

(4) $A_H \equiv \frac{n-1}{n}(\theta_k - c)I$ at x if and only if x is a totally geodesic point.

The estimate of the eigenvalues of A_H given above is sharp.

In particular, the result implies the following:

(i) If there is an integer k, $2 \leq k \leq n$, such that $\theta_k(x) > c$ (respectively, $\theta_k(x) \geq c$) for a Riemannian n-manifold M at a point $x \in M$, then, for any isometric immersion of M into $R^m(c)$, every eigenvalue of the shape operator A_H is greater than $\frac{n-1}{n}$ (respectively, ≥ 0), regardless of codimension.

(ii) If M is a compact hypersurface of E^{n+1} with $\theta_k \geq 0$ (respectively, with $\theta_k > 0$) for a fixed k, $2 \leq k \leq n$, then M is embedded as a convex (respectively, strictly convex) hypersurface in E^{n+1}. In particular, if M has constant scalar curvature, then M is a hypersphere of E^{n+1}; according to a result of W. Süss (1929) which states that the only compact convex hypersurfaces with constant scalar curvature in Euclidean space are hyperspheres.

Statement (ii) implies

(ii)$'$ If M is a compact hypersurface of E^{n+1} with nonnegative Ricci curvature (respectively, with positive Ricci curvature), then M is embedded as a convex (respectively, strictly convex) hypersurface in E^{n+1}. In particular, if M has constant scalar curvature, then M is a hypersphere of E^{n+1}.

3.7. **Completeness of curvature surfaces.** Let M be a hypersurface in a Euclidean space, $N_0(x)$ the null space of the second fundamental form of M at $x \in M$, k the minimal value of the dimensions of the vector spaces $N_0(x)$ on M, and U the open subset of M on which this minimum occurs. Then U is generated by k-dimensional totally geodesic submanifolds along which the normal space of M is constant. Moreover, if M is complete, then these generating submanifolds of U are also complete [Chern-Lashof 1957]. This result was generalized in 1971 by D. Ferus to submanifolds of higher codimension in real space forms.

For an arbitrary principal curvature function λ of an isometric immersion $f : M \to R^m(c)$ of a Riemannian n-manifold M into a Riemannian m-manifold of constant sectional curvature c, a similar result was obtained by T. Otsuki (1970) and H. Reckziegel (1976,1979).

Let $T^{\perp}M$ denote the normal bundle and $T^*_{\perp}M$ its dual bundle. A 1-form $\mu \in T^*_{\perp}M$ at $x \in M$ is called a principal curvature of f at x if the vector space

$$\mathcal{E}(\mu) = \{X \in T_xM \, : \, A_\xi X = \mu(\xi) \cdot X \text{ for all } \xi \in T_x^{\perp}M\}$$

is at least 1-dimensional.

Suppose that there is given a continuous principal curvature function λ of f, that is, a continuous section of the bundle $T^*_{\perp}M$ with $\dim \mathcal{E}(\lambda_x) \geq 1$ for all $x \in M$, and let U be any open subset of M on which the function $x \mapsto \dim \mathcal{E}(\lambda_x)$ is constant, say $\dim \mathcal{E}(\lambda_x) = k$ for all $x \in U$.

H. Reckziegel (1979) obtained the following.

(i) The principal curvature form λ is C^∞-differentiable on U;

(ii) The vector spaces $\mathcal{E}(\lambda_x)$, $x \in U$, form a vector subbundle of $\mathcal{E}$ of the tangent bundle $TM|_U$;

(iii) If L denotes the foliation obtained by integrating $\mathcal{E}$, and $i : L \to M$ its inclusion, then all leaves of L are k-dimensional totally umbilical submanifolds of M and $f \circ i : L \to R^m(c)$ is a totally umbilical immersion into $R^m(c)$;

(iv) If λ is covariant constant along $\mathcal{E}$, that is, if

$$(\nabla_X \lambda)(\xi) := X \cdot \lambda(\xi) - \lambda(D_X \xi) = 0$$

for all $X \in \Gamma(\mathcal{E})$ and $\xi \in \Gamma(T^{\perp}M|_U)$, if furthermore $\gamma : J \to L$ is a geodesic of L with $\delta := \sup J < \infty$, and if $q := \lim_{t \to \delta} \gamma(t)$ exists in M, then also $\dim \mathcal{E}(\lambda_q) = k$;

(v) If, in particular, U is the subset of M on which the function $x \mapsto \dim \mathcal{E}(\lambda_x)$ is minimal (this subset is open, because $x \mapsto \dim \mathcal{E}(\lambda_x)$ is upper-semicontinuous), λ is covariant constant along $\mathcal{E}$, and M is complete, then all the leaves of L are also complete spaces.

If $k \geq 2$, then λ is always covariant constant along $\mathcal{E}$.

The leaves of L are called the curvature surfaces of f in U corresponding to λ.

There exist submanifolds of codimension ≥ 2 without any principal curvature. For instance, the Veronese isometric embedding of $RP^2\left(\sqrt{3}\right)$ in $S^4(1)$ mentioned in §5.4.5 has no principal curvature in the above sense.

4. Rigidity and reduction theorems

4.1. **Rigidity.** An isometric immersion $f : M \to \tilde{M}$ is called rigid if it is unique up to an isometry of $\tilde{M}$, that is, if $f' : M \to \tilde{M}$ is another isometric immersion, then there is an isometry ϕ of $\tilde{M}$ such that $f' = \phi \circ f$. If $f : M \to \tilde{M}$ is rigid, then every isometry of M can be extended to an isometry of $\tilde{M}$.

F. Minding (1806–1885) conjectured in 1839 that a standard sphere a E^3 is rigid. This conjecture was proved by H. Liebmann (1874–1939) in 1899. In 1929, S. Cohn-Vossen (1902–1936) proved that a closed convex surface in Euclidean 3-space is rigid. A. V. Pogorelov (1919–) proved in 1951 that the requirement of smoothness can be removed, by proving that any closed convex surface, that is, the boundary of a bounded convex body, is uniquely determined up to a rigid motion by its metric.

If $f : M \to E^{n+1}$ is a hypersurface of Euclidean $(n+1)$-space, then at each point $x \in M$, the type number of f at x, denoted by $t(x)$, is defined to be the rank of the shape operator of f at x.

A classical result of R. Beez (1876) states that if M is an orientable Riemannian n-manifold and f is an isometric immersion of M into E^{n+1} such that the type number of f is ≥ 3 at every point of M, then f is rigid.

R. Sacksteder (1962) obtained a number of rigidity theorems for hypersurfaces. Among them he proved that a complete convex hypersurface in E^{n+1}, $n \geq 3$, is rigid if its type number is at least 3 at one point.

D. Ferus (1970) proved that if M is a complete Riemannian n-manifold with $n \geq 5$ and if $f : M \to S^{n+1}$ is an isometric immersion whose type number is everywhere ≥ 2, then f is rigid.

C. Harle (1971) proved that if M is a Riemannian n-manifold, $n \geq 4$, with constant scalar curvature $\rho \neq \frac{1}{2}n(n-1)c$ and $c \neq 0$, then every isometric immersion of M in a complete simply-connected Riemannian space form $R^{n+1}(c)$ is rigid. Y. Matsuyama (1976) showed that a hypersurface with nonzero constant mean curvature in a real hyperbolic $(n+1)$-space H^{n+1} with $n \geq 3$ is rigid.

Given a system $\{A_1, \ldots, A_m\}$ of symmetric endomorphisms of a vector space V that are linearly independent. The type number of the system is defined to be the largest integer t for which there are t vectors $v_1, \ldots, v_t$ in V such that the mt vectors $A_r(v_i)$, $1 \leq r \leq m$, $1 \leq i \leq t$, are linearly independent. When $m = 1$, the type number of the system of one single endomorphism A is just equal to the rank of A.

A generalization of R. Beez's result to higher codimension was obtained in 1939 by C. Allendoerfer (1911–1974). Allendoerfer's result states as follows:

Let f and f' be two isometric embeddings of a Riemannian n-manifold M into E^{n+m}. Assume that, for a neighborhood U of a point $x_0 \in M$, we have (1) the dimensions of the first normal space at $x \in U$ for both f and f' are equal to a constant, say k, and (2) the type number of f is at least 3 at each point $x \in M$. Then there exists an isometry ϕ of E^{n+m} such that $f' = \phi \circ f$ on a neighborhood of x_0.

The standard n-sphere S^n of constant curvature one is rigid in E^{n+1} when $n \geq 2$. However, it is not rigid in E^{n+2}, since one can construct an infinite-dimensional family of compositions of isometric immersions $S^n \to E^{n+1} \to E^{n+2}$. J. D. Moore (1996) proved that this is essentially the only way in which rigidity fails when $n \geq 3$. Moreover, in this case he proved that any isometric immersion of S^n into E^{n+2} is homotopic through isometric immersions to a standard embedding into a hyperplane.

E. Berger, R. Bryant and P. Griffiths (1983) proved the following: Consider a local isometric embedding of a Riemannian n-manifold M into E^{n+r}. Assume the embedding is "general" in the sense that the second fundamental form lies in a certain Zariski open subset of all such forms. Then, if $r \leq n$ and $n \geq 8$, or $r \leq 3$ and $n = 4$, or $r \leq 4$ and $n = 5, 6$, or $r \leq 6$ and $n = 7, 8$, then the embedding is unique up to a rigid motion. This rigidity is not a consequence of algebraic properties of the Gauss equation, but depends rather on the properties of the prolonged Gauss equations involving the higher covariant derivatives of the curvature tensor.

By applying inequality (3.17), Chen (1996f) established some rigidity theorems for isometric immersions from some homogeneous Riemannian n-manifolds into Euclidean space E^{n+k}, regardless of codimension k, under the assumption that the immersions have the smallest possible squared mean curvature.

4.2. **A reduction theorem.** Let M be an n-dimensional submanifold of a Riemannian m-manifold N and E a subbundle of the normal bundle $T^{\perp}M$. Then E is said to be parallel in the normal bundle if, for each section ξ of E and each vector X tangent to M, we have $D_X\xi \in E$.

The reduction theorem of J. Erbacher (1971) states as follows:

Let M be an n-dimensional submanifold of a complete simply-connected Riemannian m-manifold $R^m(c)$ of constant curvature c. If there exists normal subbundle E of rank ℓ which is parallel in the normal bundle and the first normal space N_x^1, spanned by $\{h(X,Y) : X, Y \in T_xM\}$, is contained in E_x for each $x \in M$, then M is contained in an $(n+\ell)$-dimensional totally geodesic submanifold of $R^m(c)$.

5. Minimal submanifolds

The theory of minimal submanifolds is closely related with the the theory of calculus of variations. According to historians, it is not quite certain when L. Euler (1707–1783) began his study of the calculus of variations. C. Carathéodory (1873–1950) believed that it occurred during his period in Basel with John Bernoulli (1667–1748). Euler considered in 1732 and 1736 problems more or less arising out the isoperimetric problems of James Bernoulli (1654–1705); and even as early as the end of 1728 or early of 1729. In effect, Euler in 1744, following John Bernoulli, examined the equation of end-curves that cut a family of geodesics so that they have equal length. In his famous 1744 book, L. Euler gave the first systematic treatment of the calculus of variations for curves. In this book he gave a general procedure for writing down the so-called Euler differential equation or the first necessary condition, and to discuss the principle of least action.

The history of the theory of minimal surfaces goes back to J. L. Lagrange (1736–1813) who studied minimal surfaces in Euclidean 3-space. In his famous memoir, "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies" which appeared in 1760–1761, Lagrange developed his algorithm for the calculus of variations; an algorithm which is also applicable in higher dimensions and which leads to what is known today as the Euler-Lagrange differential equation. Lagrange communicated his method in his first letter, dated August 12, 1755 when he was only nineteen, to L. Euler who applauded his results (cf. [Euler 1755]). The basic idea of Lagrange ushered in a new epoch in the calculus of variations. After seeing Lagrange's work, Euler dropped his own method, espoused that of Lagrange, and renamed the subject the calculus of variations.

In his famous memoir Lagrange also discovered the minimal surface equation:

$$(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy} = 0.$$

for a surface defined by $z = f(x, y)$ for (x, y) in a domain of E^2 as the equation for a critical point of the area functional. It was J. Meusnier (1754–1793) in 1776 who gave a geometrical interpretation of this equation as meaning that the surface has vanishing mean curvature function.

Before Lagrange, L. Euler had found in 1744 that a catenoid is a minimal surface; the earliest nontrivial minimal surface discovered which remained the only known nontrivial minimal surface for over twenty years, until J. Meusnier found in 1766 that a right helicoid is a minimal surface. It took almost ninety years until H. F. Scherk (1798–1885) discovered further minimal surfaces. Scherk found in 1834 that the surface defined by

$$z = \log(\cos y) - \log(\cos x)$$

is a minimal surface, which is known today as Scherk's surface.

In 1842 E. Catalan (1814–1894) proved that the helicoid is the only ruled minimal surface in E^3. O. Bonnet (1819–1892) proved in 1860 that the catenoid is the only minimal surface of revolution.

The theory of minimal surfaces experienced a rapid development throughout of the nineteenth century. The major achievements of this period were presented in detail in the 1903 book of L. Bianchi (1856–1928) and the 1887 book of J. G. Darboux (1847–1917). A detailed account of more recent results was given in the 1989 book of J. C. C. Nitsche (1926–1996).

5.1. **First and second variational formulas.** Let $f : M \to \tilde{M}$ be an immersion of a compact n-dimensional manifold M (with or without boundary ∂M) into a Riemannian m-manifold $\tilde{M}$. Let $\{f_t\}$ be a one-parameter family of immersions of $M \to \tilde{M}$ with the property that $f_0 = f$. Assume the map $F : M \times [0,1] \to \tilde{M}$ defined by $F(p,t) = f_t(p)$ is differentiable (we further assume $f_t = 0$ on ∂M when $\partial M \neq \emptyset$). $\{f_t\}$ is called a variation of f.

A variation of f induces a vector field in $\tilde{M}$ defined along the image of M under f, called the variational vector field. We shall denote this field by ζ and it is constructed as follows:

Let $\partial/\partial t$ be the standard vector field in $M \times [0,1]$. We set $\zeta(p) = F_*(\frac{\partial}{\partial t}(p,0))$. Then ζ gives rise to cross-sections ζ^T and ζ^N in TM and $T^\perp M$, respectively. If we have $\zeta^T = 0$, then $\{f_t\}$ is called a normal variation of f.

For a given normal vector field ξ on M, $\exp t\xi$ defines a normal variation $\{f_t\}$ induced from ξ. We denote by V_t the volume of M under f_t with respect to the induced metric.

The first variational formula is given by:

$$V'(\xi) := \frac{dV_t}{dt}|_{t=0} = -n \int_M \langle \xi, H \rangle \, dV_0. \tag{5.1}$$

Hence, the immersion f is minimal if and only if $\frac{dV_t}{dt}|_{t=0} = 0$ for all variations of f. Thus, a minimal submanifold gives an extremal of the volume integral, though not necessarily of the least volume.

For a minimal submanifold M of a Riemannian manifold $\tilde{M}$, the second variational formula is given by

$$V''(\xi) := \frac{d^2 V_t}{dt^2}|_{t=0} = \int_M \{||D\xi||^2 - \bar{S}(\xi,\xi) - ||A_\xi||^2\} dV, \tag{5.2}$$

where $\bar{S}(\xi,\eta)$ is defined by

$$\bar{S}(\xi,\eta)=\sum_{i=1}^{n}\tilde{R}(\xi,e_i,e_i,\eta), \tag{5.3}$$

$e_1,\ldots,e_n$ is a local orthonormal frame of TM, and $\tilde{R}$ is the Riemann curvature tensor of the ambient manifold $\tilde{M}$.

Applying Stokes' theorem to the integral of the first term of (5.2), we have

$$V''(\xi)=\int_M \langle \mathcal{J}\xi,\xi\rangle\, dV, \tag{5.4}$$

in which $\mathcal{J}$ is a self-adjoint strongly elliptic linear differential operator of the second order acting on the space of sections of the normal bundle, given by

$$\mathcal{J}=-\Delta^D-\hat{A}-\hat{S}, \tag{5.5}$$

where $\langle\, \hat{A}\xi,\,\eta\rangle=\mathrm{trace}\,(A_\xi A_\eta)$, $\langle\, \hat{S}\xi,\,\eta\rangle=\bar{S}(\xi,\eta)$ and Δ^D is the Laplacian operator associated with the normal connection.

5.2. **Jacobi operator, index, nullity and Killing nullity.** The differential operator $\mathcal{J}$ defined by (5.5) is called the Jacobi operator of the minimal immersion $f:M\to\tilde{M}$. The Jacobi operator has discrete eigenvalues $\lambda_1<\lambda_2<\ldots\nearrow\infty$. We put

$$E_\lambda=\{\xi\in\Gamma(T^\perp M)\,:\,\mathcal{J}(\xi)=\lambda\xi\,\}.$$

A domain D, of a minimal submanifold M, with compact closure is called stable if the second variation of the induced volume of D is positive for all variations that leave the boundary ∂D of D fixed. The minimal submanifold $f:M\to\tilde{M}$ is said to be stable if every such domain D of M is stable.

The number $i(f):=\sum_{\lambda<0}\dim(E_\lambda)$ is called the index of f which measures how far the minimal submanifold is from being stable.

A vector field ξ in E_0 is called a Jacobi field. The number $n(f):=\dim E_0$ is called the nullity of f.

Define a subspace P of $\Gamma(T^\perp M)$ by

$$P=\{\xi^N\,:\,\xi \text{ is a Killing vector field on } \tilde{M}\},$$

where ξ^N denotes the component of ξ normal to M. Then $P\subset E_0$. The dimension of P is called the Killing nullity of f, which is denoted by $n_k(f)$.

5.3. **Minimal submanifolds of Euclidean space.** An immersion $f:M\to E^m$ can be viewed as a E^m-valued function. In this case, Beltrami's formula relates H to f by

$$\Delta f=-nH, \tag{5.6}$$

where Δ is the Laplacian of M with respect to the induced Riemannian metric which is defined by

$$\Delta f = -\mathrm{div}\,(\mathrm{grad}\, f).$$

Beltrami's formula implies that each coordinate function of f is a harmonic function. Hence, there exists no compact minimal submanifold without boundary in Euclidean space.

This nonexistence result also follows from the following fact: If M is a compact submanifold (without boundary) in a Euclidean space, there always exits a point $x \in M$ such that the shape operator of M at x is positive-definite with respect to some unit normal vector at x. The point x can be chosen to be the farthest point on M from any fixed point in the Euclidean space.

It follows from the equation of Gauss and Moore's lemma that a minimal isometric immersion of a Riemannian product into a Euclidean space is a product of minimal immersions [Ejiri 1979b].

E. F. Beckenbach (1906–1982) and T. Radó proved in 1933 that if the Gaussian curvature of a surface is ≤ 0, then, for any simply-connected domain D on S, one has $L^2 \geq 4\pi A$, where L is the arclength of the boundary of D and A is the area of D. In particular, for any immersed simply-connected minimal surface M in E^m with boundary C, one has the following isoperimetric inequality:

$$L^2 \geq 4\pi A,$$

where L is the arclength of $C = \partial M$ and A the area of M.

When one drops simple connectivity, the isoperimetric inequality does not hold for general domains on surfaces satisfying $K \leq 0$. For instance, on a circular cylinder in E^3, the length of each boundary circle is $2\pi r$ and the area is $2\pi rh$; thus the area can be made arbitrarily large.

The isoperimetric inequality $L^2 \geq 4\pi A$ holds for domains D lying on minimal surfaces in E^m in the following cases [Osserman-Schiffer 1974; Osserman 1978]:

(1) the boundary of D consists of a single rectifiable Jordan curve;

(2) D is doubly-connected and is bounded by two rectifiable curves;

(3) D is bounded by a finite number of rectifiable curves lying on a sphere centered at a point of D;

(4) D is bounded by a finite number of rectifiable Jordan curves, and minimizes area among all surfaces with the same boundary.

L. P. Jorge and F. Xavier (1979) proved that there are no bounded complete minimal surfaces in E^3 with bounded Gaussian curvature. N. Nadirashvili (1996) showed that there exists an immersed complete bounded minimal surface in E^3 with negative Gaussian curvature.

5.3.1. Obstructions to minimal isometric immersions

If M is a minimal surface in E^3 with the induced metric g, then the Gauss curvature K of M is ≤ 0. Thus, $\sqrt{-K}g$ defines a new metric on points where $K \neq 0$.

G. Ricci (1853–1925) proved in 1894 that a given metric g on a plane domain D arises locally as the metric tensor of a minimal surface in R^3 if and only if the Gauss curvature K of (D, g) is everywhere nonpositive and the corresponding Gauss curvature $\bar{K}$ of $\sqrt{-K}g$ vanishes at each point where $K \neq 0$.

Let g be the metric tensor of a minimal surface M in E^m. If g satisfies Ricci's condition, then g corresponds locally to the metric tensor of a minimal surface $\hat{M}$ in E^3. H. Lawson (1971) proved that, in this case, either M lies in E^3 and belongs to a specific one-parameter family of surfaces associated to $\hat{M}$, or else M lies in E^6 and belongs to a specific two-parameter family of surfaces obtained from $\hat{M}$, none of which lie in any E^5.

Lawson's result implies that the Ricci condition is an intrinsic condition which completely characterizes minimal surfaces lying in E^3 among all minimal surfaces in E^4 or E^5; and also the set of all minimal surfaces in E^m isometric to a given minimal surface in E^3 consists of a specific two-parameter family of surfaces lying in E^6.

For a minimal submanifold in Euclidean space in general, the equation of Gauss implies that the Ricci tensor of a minimal submanifold M of a Euclidean space satisfies

$$Ric(X, X) = -\sum_{i=1}^{n} |h(X, e_i)|^2 \leq 0, \tag{5.7}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal local frame field on M. Thus, the Ricci tensor of a minimal submanifold M of a Euclidean space is negative semi-definite and, moreover, the minimal submanifold is totally geodesic if and only if its scalar curvature vanishes identically.

For minimal hypersurfaces in E^{n+1}, consider the metric $\hat{g} = -\rho g$ at the point where the scalar curvature ρ with respect to the induced metric g is negative. J. L. Barbosa and M. do Carmo (1978) proved that the scalar curvature $\hat{\rho}$ of this new metric $\hat{g}$ must satisfy the inequality $\hat{\rho} \leq 2(n-1) - 1$.

Besides the above necessary conditions, inequality (3.17) provides many further sharp necessary conditions for a Riemannian n-manifold to admit a minimal isometric immersion in a Euclidean space, regardless of codimension.

In fact, inequality (3.17) of Chen implies that, for a given Riemannian n-manifold M, if there is a k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ such that the δ-invariant

$$\delta(n_1, \dots, n_k) > 0 \quad \text{at some point } x \in M, \tag{5.8}$$

then M admits no isometric minimal immersion in Euclidean space for any arbitrary codimension.

In particular, if a Riemannian manifold M satisfies $\delta(2) > 0$ at some point $x \in M$ (or equivalently, $\rho(x) > 2(\inf K)(x)$ at some point x), then M admits no isometric minimal immersion in Euclidean space, regardless of codimension.

There exist ample examples of Riemannian manifolds with $Ric \leq 0$ which satisfy condition (5.8).

5.3.2. Branched minimal surfaces

A branch point of a harmonic map $f : M \to E^m$ is a point $x \in M$ at which the differential $(f_*)_x$ is zero. A harmonic map $f : M \to E^m$ of a Riemann surface M is called a branched (or generalized) minimal immersion if it is conformal except at the branch points, and the image $f(M)$ is called a branched minimal surface.

Branched minimal surfaces have the following three basic properties:

(1) **Convex hull property**. Every branched minimal surface with boundary in E^m lies in the "convex hull" of its boundary curve, that is, the smallest closed convex set containing the boundary.

(2) **Minimal principle**. If M_1 and M_2 are two branched minimal surfaces in E^3 such that for a point $x \in M_1 \cap M_2$, the surface M_1 lies locally on one side of M_2 near x, then M_1 and M_2 coincide near x.

(3) **Reflection principle**. If the boundary curve of a branched minimal surface contains a straight line L, then the surface can be analytically continued as a branched minimal surface by reflection across L.

Based on reflection principle, H. Lewy (1904–1988) proved in 1951 the following: Let Γ be an analytic Jordan curve in E^m and $f : M \to E^m$ a branched minimal immersion with boundary Γ. Then f is analytic up to the boundary, that is, $f(M)$ is contained in the interior of a larger branched minimal surface.

S. Hildebrandt (1969) obtained the smooth version of H. Lewy's theorem: If $f : M \to E^m$ is a branched minimal immersion with smooth boundary curve, then f is smooth up to the boundary.

5.3.3. Plateau's problem

The famous problem of Plateau states that given a Jordan curve Γ in E^3 (or in E^m) find a surface of least area which has Γ as its boundary. Plateau's problem was investigated extensively in the second half of the nineteenth century by E. Betti (1823–1892), O. Bonnet (1819–1892), G. Darboux (1842–1917), A. Enneper (1830–1883), É. L. Mathieu (1835–1890), H. Poincaré (1854–1912), B. Riemann (1826–1866), K. Weierstrass (1815–1897), and others. Plateau's problem was finally solved independently by J. Douglas and T. Radó around 1930.

The solution to Plateau's problem given by Douglas and Radó is a branched minimal surface. More precisely, they proved that if Γ is a rectifiable Jordan curve in E^m and $D = \{(x, y) \in E^2 : x^2 + y^2 < 1\}$, then there exists a continuous map $f : \bar{D} \to E^m$ from the closure of D into E^m such that

(a) $f|_{\partial D}$ maps homeomorphically onto Γ,

(b) $f|_D$ is a harmonic map and almost conformal, that is $\langle f_x, f_y \rangle = 0$ and $|f_x| = |f_y|$ in D with $|df| > 0$ except at isolated branch points, and

(c) the induced area of f is the least among the family of piecewise smooth surfaces with Γ as their boundary.

The map f given above is called the classical solution or the Douglas-Radó solution to Plateau's problem for Γ. The resulting surface M is a branched minimal disk.

A branched minimal disk M bounded by a smooth curve Γ in E^3 satisfies the following formula of Gauss-Bonnet-Sasaki-Nitsche:

$$1 + \sum (m_\alpha - 1) + \sum M_\beta + \frac{1}{2\pi} \int_M |K| dA \leq \frac{1}{2\pi} \kappa(\Gamma),$$

where $m_\alpha - 1$ denote the orders of the interior branch points, $2M_\beta$ the orders of the boundary branch points which must be even, K the Gaussian curvature, and $\kappa(\Gamma)$ the total curvature of Γ.

R. Osserman (1970) proved that every classical solution to Plateau's problem in E^3 is free of branch points in its interior. Thus, it is a regular immersion.

If Γ is a curve in E^3 which is real analytic or a smooth curve with total curvature $\kappa(\Gamma) < 4\pi$, then the minimal disk of least area with Γ as its boundary has no boundary branch points [Gulliver-Leslie 1973].

A Douglas-Radó solution is not necessarily an embedding. In fact, if Γ is knotted in E^3, then every solution must have self-intersections. However, immersed minimal disks of least area in E^3 which can self-intersect only in their interiors are embeddings. Also, there exists an unknotted Jordan curve which never bounds an embedded minimal disk. Meeks and Yau (1982) proved that if the Jordan curve lying entirely on the boundary of a convex body, then a Douglas-Radó solution is embedded (see also [Almgren-Simon 1979; Grüter-Jost 1986]).

In general a Douglas-Radó solution to Plateau's problem does not have the uniqueness property. However, Radó (1930) provided a sufficient condition on the boundary curve for which the solution is always unique. More precisely, he showed that if a Jordan curve Γ in E^m admits a one-to-one orthogonal projection onto a convex curve in a plane E^2 in E^m, then the classical solution for Γ is free of branch points and can be expressed as the graph over this plane. Furthermore, when $n = 3$, the solution is unique.

J. C. C. Nitsche (1989) proved that if Γ is an analytic Jordan curve in E^3 with total curvature $\kappa(\Gamma) \leq 4\pi$ or a smooth curve with $\kappa(\Gamma) < 4\pi$, then Γ bounds a unique immersed minimal disk.

A. J. Tromba (1977) proved the following: (1) Any rectifiable Jordan curve Γ "sufficiently close" to a plane curve has a unique simply-connected minimal surface spanning it, and (2) If $\mathcal{F}$ denotes the set of embeddings f of S^1 into E^3 with the property that for the curve $f(S^1)$ every simply connected minimal surface spanning $f(S^1)$ is free from branch points, then there is an $\mathcal{F}$ of embeddings for which the number of minimal surfaces spanning the image is finite.

F. Tomi (1986) showed that an analytic Jordan curve in E^3 bounds only finitely many minimal disks and Nitsche proved that a smooth Jordan curve Γ with total curvature $\leq 6\pi$ also bounds only finitely many minimal disks. On the other hand, P. Lévy and R. Courant constructed an example of rectifiable Jordan curve that is smooth with the exception of one point and it bounds uncountably many minimal discs.

F. Morgan (1978) and A. J. Tromba (1977) proved that, generically, there are at most finitely many minimal surfaces with a given boundary. More precisely, in the space $\mathcal{A}$ of all smooth Jordan curves in E^m with suitable topology, there exists an open and dense subset $\mathcal{B}$ such that for any Γ in $\mathcal{B}$, there exists a unique area-minimizing minimal disk.

H. Iseri (1996) studied the Plateau's problem in which the boundary curves may have self-intersections and provided a condition which guarantees that the minimal surfaces they generated will not be degenerate.

The method was carried further in 1948 by C. B. Morrey (1907–1984) for Plateau's problem in a complete Riemannian manifold which is metrically well behaved at infinity, and includes the class of compact or homogeneous Riemannian manifolds.

Morrey's setting of the generalized Plateau's problem is as follows: Suppose a homotopically trivial rectifiable Jordan curve Γ is given in a Riemannian m-manifold N. Let D denote a disk in E^2. Find a mapping $f : \bar{D} \to N$ such that (i) f maps ∂D homeomorphically onto Γ and (ii) the induced area of f is the least among the class of piecewise smooth surfaces in N bounded by Γ satisfying (i).

Morrey gave a solution under the assumption that N is homogeneously regular, that is, there exist $0 < k < K$ such that, for any point $y \in N$, there is a local coordinate system (U, Ψ) around y for which $\Psi(U) = \{x \in E^m : ||x|| < 1\}$ and the Riemannian metric $g = \sum g_{ij} dx^i dx^j$ satisfies

$$k \sum v_i^2 \leq \sum g_{ij} v_i v_j \leq K \sum v_i^2$$

for any x and $v = (v_1, \ldots, v_m) \in E^m$.

C. B. Morrey (1907–1984) proved in 1948 that if N is a homogeneously regular Riemannian manifold and if Γ is a homotopically trivial rectifiable Jordan curve in N, then there exists a branched minimal immersion $f : D \to M$ with least area bounded by Γ such that $f|_{\partial D}$ maps homeomorphically onto Γ. If Γ is smooth, then so is the solution f up to the boundary. Furthermore, when $m = 3$, the solution f is an immersion in its interior. If N and Γ are real analytic and $m = 3$, then the solution f is an immersion up to the boundary.

M. Ji and G. Y. Wang (1993) investigated disk type minimal surfaces spanned by a given Jordan curve Γ in N and proved the following:

(1) Each smooth Jordan curve Γ in S^n bounds at least two minimal surfaces, sometimes infinitely many ones; and

(2) Let N be a compact oriented Riemannian manifold embedded in a Euclidean m-dimensional space. Suppose that the compact manifold N admits no minimal sphere. If there are two strictly stable minimal disks bounded by a Jordan curve Γ, then there exists another minimal surface bounded by Γ.

Given two Jordan curves Γ_1, Γ_2 in E^3, does $\Gamma = \Gamma_1 \cup \Gamma_2$ bound a minimal annulus? This is called the Douglas-Plateau problem which is a generalization of the original Plateau problem [Douglas 1931b].

In many cases the answers to the Douglas-Plateau problem are negative. One example is that of two coaxial unit circles C_1 and C_2. If the distance between their centers is large, then $C_1 \cup C_2$ cannot bound a minimal annulus.

When Γ_1 and Γ_2 are smooth convex planar Jordan curves lying in parallel but different planes, the Douglas-Plateau problem has a satisfactory answer.

In fact, [Meeks-White 1991] proved the following: If Γ_1 and Γ_2 are smooth convex planar Jordan curves lying in parallel but different planes, then exactly one of the following three cases occurs:

(a) There are exactly two minimal annuli bounded by $\Gamma = \Gamma_1 \cup \Gamma_2$, one is stable and one is unstable;

(b) There is a unique minimal annulus M bounded by Γ; it is almost stable in the sense that the first eigenvalue of the Jacobi operator of M is zero;

(c) There are no minimal annuli bounded by Γ.

If M is a minimal annulus bounded by $\Gamma = \Gamma_1 \cup \Gamma_2$, then the symmetry group of M is the same as the symmetry group of Γ.

R. Hardt and L. Simon (1979) proved that if Γ is the union of any finite collection of disjoint smooth Jordan curves in E^3, then there exists a compact embedded minimal surface with boundary Γ which is smooth up to the boundary.

In 1985 J. Jost proved the existence of a minimal surface of finite prescribed genus and connectivity spanning a configuration of oriented Jordan curves in a homogeneously regular manifold in the sense of Morrey, under the condition that the area infimum over combinations of surfaces of this topological type is strictly less than the one over combinations of surfaces of lower genus or connectivity.

Although the same problem as Jost's was treated in classical papers by J. Douglas, R. Courant, and M. Shiffman in the 1930s, there was some doubt about the validity of their proofs. Namely, in order to get the compactness of a minimizing sequence, Douglas compactified the moduli space of surfaces of the topological type considered, but he did not show that the boundary of this compactification consists of surfaces of lower genus or connectivity. Courant provided a complete proof of the case of higher connectivity but genus zero, his considerations about the case of higher genus were pointed out by A. J. Tromba to be too vague and not detailed enough to be accepted as a correct proof. Shiffman assumed a priori a condition which is equivalent to the compactness of a minimizing sequence and therefore could only prove a weaker statement.

In 1993 F. Bernatzki treated the same Plateau-Douglas problem as Jost's for nonorientable surfaces, using a method similar to Jost's.

5.3.4. Weierstrass' representation formula

In 1866 Weierstrass gave a general formula to express a simply-connected minimal surface in terms of a complex analytic function f and a meromorphic function g with certain properties. His formula allows one to construct a great variety of minimal surfaces by choosing these functions.

Weierstrass' representation formula states as follows: Every simply-connected (branched) minimal surface in E^3 is represented in the form:

$$x_k(z) = \mathrm{Re}\left\{\int_0^z \phi_k(\zeta)d\zeta\right\} + c_k, \quad k = 1, 2, 3, \tag{5.9}$$

where $\phi_1 = \frac{1}{2}f(1-g^2)$, $\phi_2 = \frac{1}{2}\sqrt{-1}f(1+g^2)$, $\phi_3 = fg$, and c_k are constants. Here $g(z)$ is a meromorphic function on D (= the unit disc or the entire complex plane), $f(z)$ is an analytic function on D satisfying the property that at each point z, where g(z) has a pole of order m and $f(z)$ has a zero of order $2m$.

For instance, Enneper's surface with coordinates

$$(x_1, x_2, x_3) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2), \quad (u, v) \in E^2 \tag{5.10}$$

is obtained from the Weierstrass formula by setting $f = 1$ and $g(z) = z$. The Richmond surface, the catenoid, and helicoid are the minimal surfaces obtained from the Weierstrass formula by setting $(f, g) = (z^2, z^{-2}), (\frac{1}{2}z^{-2}, z)$, and $(-ie^{-z}, e^z)$, respectively.

A minimal surface described by (f, g) via Weierstrass's representation formula has an associated one-parameter family of minimal surfaces given respectively by $(e^{it}f, g)$. Two surfaces of the family described by t_0 and t_1 are called adjoint by O. Bonnet (1853a,1853b) if $t_1 - t_0 = \frac{\pi}{2}$. The catenoid and the helicoid are a pair of adjoint minimal surfaces for a suitable choice of constants.

All the surfaces of an associated family are locally isometric. Conversely, H. A. Schwarz (1890) proved that if a simply-connected minimal surface S_1 is isometric to a simply-connected minimal surface S, then S_1 is congruent to an associate minimal surface of S.

5.3.5. Bernstein's problem

Let $x_1, \ldots, x_n, z$ be standard coordinates in E^{n+1}. Consider minimal hypersurfaces which can be represented by an equation of the form

$$z = z(x_1, \ldots, x_n)$$

for all x_i, that is, which have a one-to-one projection onto a hyperplane.

The answer to Bernstein's problem is known to be affirmative in the following cases:

(a) $n = 2$ [Bernstein 1914];

(b) $n = 3$ [de Giorgi 1965];
(c) $n = 4$ [Almgren 1966]; and
(d) $n = 5, 6, 7$ [Simons 1968].

And the answer is negative for $n \geq 8$ [Bombieri-de Giorgi-Giusti 1969].

In general, when a bounded domain D in E^n and a continuous function ϕ on its boundary ∂D are given, the problem of finding a minimal hypersurface M defined by the graph of a real-valued function f on $\bar{D}$, the closure of D, with $f|_{\partial D} = \phi$ gives rise to a typical Dirichlet problem. The basic questions are those of existence, uniqueness, and regularity of solutions. These problems were studied by T. Radó for $n = 2$ and later by L. Bers, R. Finn, H. Jenkins, J. Serrin, R. Osserman, and others.

5.3.6. Periodic minimal surfaces and minimal surfaces with many symmetries

A minimal surface in E^3 is called periodic if it is invariant under a group G of isometries that acts freely on E^3.

The Gauss-Bonnet theorem implies that if M_g is a minimal surface of genus g in a 3-tours T^3, then its Gauss map $G : M_g \to S^2$ represents M_g as a $(g-1)$ conformal branched covering of S^2. Thus, a surface of genus 2 is never periodic, and a minimal surface of genus g in a 3-torus T^3 has $4(g-1)$ zeros of Gaussian curvature, counted with multiplicities (cf. [Meeks 1993]).

In 1867 H. A. Schwarz established a procedure for generating periodic minimal surfaces in E^3 using octahedral or tetrahedral symmetry. A. H. Schoen (1970) constructed infinitely periodic minimal surfaces in E^3 without self-intersections. Among them is a surface containing no straight lines built out of an infinite number of congruent curvilinear hexagons whose sides form a family of curves which are almost, but not exactly, circular helices. A. H. Schoen (1970) also described some triply periodic minimal surfaces in E^3. H. Karcher (1989) gave a geometric description of the construction of Schoen's surfaces, by solving a conjugate Plateau problem for a polygonal contour. Karcher showed that the solution of an analogous Plateau problem in S^3 provides a deformation of the simpler of these minimal surfaces into constant mean curvature surfaces. The spherical polygon can be obtained from the Euclidean contour of the conjugate minimal surface by considering the straight lines in E^3 as integral curves of parallel vector fields and the contour in S^3 as formed by integral curves of right-invariant vector fields.

In 1978 T. Nagano and B. Smyth gave a construction procedure to construct periodic minimal surfaces in E^m or m-tori with symmetry corresponding to a Weyl group of rank m.

A surface is said to have finite topology if it is homeomorphic to a closed surface with a finite number of points removed. W. H. Meeks and H. Rosenberg (1993) proved that a properly embedded minimal surface in a 3-torus T^3 has finite total curvature if and only if it has finite topology. They also proved that the plane and the helicoid are the only properly embedded simply-connected minimal surfaces in E^3 with infinite symmetry group.

D. Hoffman, F. Wei and H. Karcher (1993) constructed a complete embedded singly periodic minimal surface in E^3 that is asymptotic to the helicoid, has infinite genus and whose quotient by translations has genus one. The quotient of the helicoid by translations has genus zero and the helicoid itself is simply-connected. Using the techniques of N. Kapouleas, M. Traizet (1996) constructed simply periodic minimal surfaces in Euclidean 3-space by glueing together Scherk surfaces. F. J. López, M. Ritoré and F. Wei (1997) found all the properly immersed minimal tori with two parallel embedded planar ends.

W. Fischer and E. Koch (1996) classified triply periodic minimal surfaces containing straight lines, by using their associated crystallographic groups or, more precisely, by group-subgroup pairs with index 2. C. Frohman (1990) proved that if F and F' are triply periodic minimal surfaces in E^3, then there is a homeomorphism $h : E^3 \to E^3$ such that $h(F) = h(F')$.

M. Callahan, D. Hoffman and W. Meeks (1990) proved that a properly embedded minimal surface with more than one end and with infinite symmetry group is either the catenoid or has an infinite number of flat ends and is invariant under a screw motion. They also established the existence of a family of complete embedded minimal surfaces $M_{k,\theta}$ invariant under a rotation of order $k+1$ and a screw motion of angle 2θ about the same axis, where $k > 0$ is any integer and θ is any angle with $|\theta| < \pi/(k+1)$. In 1993 Callahan, Hoffman and Karcher gave an explicit construction of these surfaces using generalized Weierstrass representation; generalized in the sense of using the logarithm derivative of the Gauss map rather than the Gauss map itself as in the usual Weierstrass representation.

Further results on periodic minimal surfaces were obtained by Nagano and Smyth (1975,1976,1978,1980), Meeks and Rosenberg (1989,1993), Meeks (1990,1993), F. Wei (1992), J. Hass, J. T. Pitts and J. H. Rubinstein (1993), and others.

5.3.7. Ruled minimal submanifolds

In 1835 P. Scherk tried unsuccessfully to determine all ruled minimal surfaces in E^3, that is, those minimal surfaces which contain a straight line through each point of the surfaces. The problem was finally solved by E.

Catalan in 1842, who proved that the helicoid is the only nonplanar ruled minimal surface in E^3.

An n-dimensional submanifold M^n of a Riemannian manifold $\tilde{M}$ is called ruled if M^n is foliated by $(n-1)$-dimensional totally geodesic submanifolds of $\tilde{M}$.

Ü. Lumiste (1958) showed that an n-dimensional minimal ruled submanifold of Euclidean space is either

(a) generated by an $(n-1)$-dimensional affine subspace P under a screw motion in E^{2n+1} such that the axis cuts P orthogonally, or

(b) generated by an $(n-1)$-dimensional affine subspace P under a rotation in E^{2n} around a point in P, or

(c) a cylinder on a submanifold of the type (i) or (ii).

Analytically any ruled minimal submanifold therefore can be given by

$$\begin{aligned} X(s, t_1, \ldots, t_{n-1}) = (&t_1 \cos(a_1 s), t_1 \sin(a_1 s), \ldots, \\ &t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \ldots, t_{n-1}, bs) \end{aligned}$$

where $a_1, \ldots, a_k$ and b are real numbers.

A submanifold with this kind of parameterization is called a generalized helicoid.

If $b \neq 0$ (respectively, $b = 0$), then this gives a cylinder on a submanifold of the type (i) (respectively, of the type (ii)). A ruled submanifold of the type (ii) is a cone on a minimal ruled submanifold of some hypersphere of the Euclidean space.

J. M. Barbosa, M. Dacjzer and L. P. Jorge (1984) proved that any minimal ruled submanifold is generated by an affine subspace P under a one-parameter subgroup A of rigid motions of the Euclidean space such that P is orthogonal to the orbits of A. Then they showed that the resulting submanifold (at least if it is minimal) has the same parameterization. They also extended their result to ruled submanifolds of real space forms.

Complete ruled minimal hypersurfaces of Euclidean space were classified by D. E. Blair and J. R. Vanstone (1978); and the classification of general ruled minimal hypersurfaces of Euclidean space were done by G. Aumann (1981).

5.3.8. Minimal immersions of Kähler manifolds

The theory of minimal surfaces in Euclidean space profits substantially from the study of the underlying complex structure. Thus, it is natural to study minimal immersions $f : M^{2n} \to E^{2n+p}$ of Kähler manifolds into Euclidean space.

Kähler manifolds which are isometrically immersed into Euclidean space as real hypersurfaces are called real Kähler hypersurfaces. Such hypersurfaces have been studied by T. Takahashi (1972), P. Ryan (1973), K. Abe (1974), M. Dajczer and D. Gromoll (1985), M. Dajczer and L. Rodriguez (1986,1991), H. Furuhata (1994), and others. For instance, K. Abe (1974) proved that if M^{2n} is a complete real Kähler hypersurface of Euclidean $(2n+1)$-space, then M^{2n} is a product of a Riemann surface and $C^{n-1} = E^{2n-2}$, provided either

(a) M^{2n} has nonnegative scalar curvature, or

(b) M^{2n} has strictly negative scalar curvature, or

(c) the immersion is real analytic.

Minimal real Kähler hypersurfaces are abundant and have been classified by M. Dajczer and D. Gromoll (1985). It turns out that none of them is complete unless $M^{2n} = M^2 \times E^{2n-2}$ and $f = f_1 \times \mathrm{id}$ splits, where $f_1 : M^2 \to E^3$ is a complete minimal surface.

An isometric immersion $f : M^{2n} \to E^{2n+2}$ is said to be complex ruled if M^{2n} is a Kähler manifold and admits a continuous codimension two foliation such that any leaf is a Kähler submanifold of M^{2n} and whose image under f is an affine subspace of E^{2n+2}. f is called completely complex ruled if in addition the leaves are all complete Euclidean E^{2n-2} spaces.

We say that the scalar curvature ρ of a complete manifold has subquadratic growth along geodesics if its growth along any geodesic is less than any quadratic polynomial in the parameter.

M. Dajczer and L. Rodriguez (1991) investigated minimal immersions of complete Kähler manifolds of codimension two in Euclidean space and obtained the following:

Let $f : M^{2n} \to E^{2n+2}$, $n \geq 2$, be a minimal immersion of a complete Kähler manifold. Then one of the following occurs:

(i) f is a holomorphic;

(ii) f is completely complex ruled;

(iii) $M^{2n} = M^4 \times E^{2n-4}$ and $f = f_1 \times \mathrm{id}$.

Moreover,

(a) if the scalar curvature of M^{2n} has subquadratic growth along geodesics, then f is of type (i) or (iii), and

(b) if the index of relative nullity of f is $\geq 2n-4$ everywhere, then f is of type (i) or (ii).

M. Dajczer and L. Rodriguez (1986) also investigated minimal immersions of Kähler manifolds into Euclidean space of higher codimension and obtained the following:

(1) Let $f : M^{2n} \to E^{2n+p}$ be an isometric immersion of a Kähler manifold. If the type number of f is ≥ 3 everywhere, then f is holomorphic.

(2) Let $f : M^{2n} \to C^{n+1}$ be a holomorphic isometric immersion of a Kähler manifold. If $g : M^{2n} \to E^{2n+p}$ is a minimal isometric immersion, then g is congruent to f in E^{2n+p}.

(3) Let $f : M^{2n} \to C^{n+q}$ be a full (or substantial) isometric immersion of a Kähler manifold with type number ≥ 3 everywhere. If $g : M^{2n} \to E^{2n+p}$ is a minimal isometric immersion, then g is congruent to f in E^{2n+p}.

(4) Let $f : M^{2n} \to R^{2n+p}(c)$ be a minimal isometric immersion of a Kähler manifold into a Riemannian manifold of constant curvature c.

(4.1) If $c < 0$, then $n = 1$.

(4.2) If $c = 0$, then f is circular, that is, the second fundamental form of f satisfies $h(X, JY) = h(JX, Y)$ for all $X, Y \in TM$.

(4.3) If $c > 0$, then the Ricci curvature $Ric_M \leq nc$, with equality implying that the second fundamental form is parallel.

M. Dajczer and D. Gromoll (1995) proved that if $f : M^{2n} \to E^{2n+2}$, $n \geq 3$ is a minimal isometric immersion of a complete Kähler manifold and if f is irreducible and not holomorphic, then M^{2n} contains an open dense subset M^* on which f is completely holomorphic ruled. Furthermore, along any holomorphic section, f has a "Weierstrass type representation".

Dajczer and Gromoll (1985) also proved that if an isometric minimal immersion f of a simply-connected Kähler manifold into a Euclidean space is not holomorphic, then there is a one-parameter family, called the associated family, of non-congruent isometric minimal immersions with the same Gauss map. Moreover, the immersion can always be made the real part of a holomorphic isometric immersion, called the holomorphic representative of f.

H. Furuhata (1994) gave a parametrization of the set of isometric minimal immersions of a simply-connected Kähler manifold into a Euclidean space by a set of certain complex matrices, which is described in terms of a full isometric holomorphic immersion of the Kähler manifold into a complex Euclidean space. Furuhata's result is an extension of a result of E. Calabi (1968) on minimal surfaces.

5.4. **Minimal submanifolds of spheres.** A submanifold M of a Euclidean m-space is contained in a hypersphere as a minimal submanifold if and only if it is a pseudo-umbilical submanifold with nonzero parallel mean curvature vector [Yano-Chen 1971]. A submanifold of codimension two in a Euclidean space is contained in a hypersphere as a minimal submanifold if and only if it is a pseudo-umbilical submanifold with nonzero constant mean curvature [Chen 1971].

By a pseudo-umbilical submanifold we mean a submanifold of a Riemannian manifold whose shape operator in the direction of the mean curvature vector is proportional to the identity transformation.

The geometry of minimal submanifolds of a sphere S^n takes a quite different course than in the Euclidean case, because there do exist many compact minimal submanifolds in spheres.

5.4.1. Necessary conditions

Similar to the Euclidean case, there are some necessary conditions for a Riemannian n-manifold to admit an isometric minimal immersion in the unit m-sphere S^m. In fact, the equation of Gauss implies that the Ricci tensor of a minimal submanifold in S^m satisfies $Ric \leq (n-1)g$.

Inequality (3.17) of Chen provides many further necessary conditions. In fact, (3.17) implies that, regardless of codimension, if a Riemannian n-manifold admits an isometric minimal immersion in a unit sphere, it must satisfies

$$\delta(n_1, \ldots, n_k) \leq \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1) \tag{5.11}$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

Since the center of gravity of a compact minimal submanifold of S^m is exactly the center of the S^m, where S^m is viewed as an ordinary hypersphere in E^{m+1}, there exists no compact minimal submanifold in S^m which is contained in an open hemisphere of S^m.

5.4.2. Takahashi's theorem

A fundamental result of T. Takahashi (1933–) obtained in 1966 states that an isometric immersion f of a Riemannian n-manifold M in the unit m-sphere S^m, viewed as a vector-valued function in E^{m+1}, is minimal if and only if $\Delta f = nf$.

An immediate application of Takahashi's theorem is that any compact n-dimensional homogeneous Riemannian manifold whose linear isotropy group is irreducible can be minimally isometrically immersed into the m-sphere of curvature λ/n, corresponding to any nonzero eigenvalue λ, where $m+1$ is the dimension of the corresponding eigenspace.

5.4.3. Minimal isometric immersions of spheres into spheres

M. do Carmo and N. R. Wallach (1971) proved that if an n-sphere of constant curvature c is minimally isometrically immersed into the unit m-sphere, but not in any great hypersphere, then, for each non-negative integer k, we have

$$c = \frac{n}{k(n+k-1)}, \quad m \leq (n+2k-1)\left(\frac{n+k-2)!}{k!(n-1)!}\right) - 1.$$

The immersion is rigid if $n = 2$ [E. Calabi, 1967] or $k \leq 3$.

For general n, do Carmo and Wallach showed that the space of minimal isometric immersions from $S^n(1)$ into $S^m(r)$ can be parametrized by a compact convex body in some finite-dimensional vector space. The immersions corresponding to interior points of this convex body all have images that are embedded spheres or embedded real projective spaces.

A spherical space form is a compact manifold of positive constant sectional curvature. D. DeTurck and W. Ziller (1992) proved that every homogeneous spherical space form admits a minimal isometric embedding into some sphere. C. M. Escher (1996) gave a necessary condition for the existence of a minimal embedding of nonhomogeneous 3-dimensional spherical space forms. In particular, she showed that the lens space $L(5, 2)$ cannot be minimally embedded into any sphere.

G. Toth (1997) provided a general method that associates to set of spherical minimal immersions from S^n a spherical minimal immersion from S^{n+1}. In particular, he proved the following: Let $n \geq 3$ and $p \geq 4$. Given full spherical minimal immersions $f_i : S^n \to S^{m_i}$, $i = 1, \ldots, p$, there exists a full spherical minimal immersion $\tilde{f} : S^{n+1} \to S^N$, where $N = \sum_{i=1}^{p}(m_i + 1)$.

5.4.4. Minimal surfaces in spheres

In contrast to Euclidean case, there exist many compact minimal submanifolds in spheres. In fact, H. B. Lawson (1970) proved the following:

(1) any compact surface of any genus, except the real projective plane, can be minimally immersed in S^3;

(2) there exist minimal immersions of every surface of negative Euler characteristic into S^5 such that none of the images lies in a totally geodesic S^4; and

(3) there is a countable family of minimal immersions of the torus into S^4 where none of the images lies in a totally geodesic S^3.

Many further examples of minimal surfaces in S^3 have also been constructed in [Karcher-Pinkall-Sterling 1988, Pitts-Rubinstein 1988].

F. J. Almgren (1933–1977) proved in 1966 that the only minimal immersion from S^2 into S^3 is the totally geodesic one. H. I. Choi and R. Schoen

(1985) showed that the space of embedded compact minimal surfaces of any fixed genus in S^3 is compact in the C^k topology, $k \geq 1$.

E. Calabi (1968) described in principle minimal immersions of S^2 into S^n. He also proved in 1967 that, for a minimal full immersion of S^2 into a $(m-1)$-sphere $S^{m-1}(r)$ of radius r, m is an odd integer and the area of the immersed S^2 is an integral multiple of $2\pi r^2$, at least $\frac{1}{2}\pi r^2(m^2-1)$. S. S. Chern (1970) provided a general construction method of minimal spheres in the unit 4-sphere using their directrix curves.

A holomorphic curve $\Xi : S^2 \to CP^{2m}$ is called totally isotropic if any of its local representations ξ in homogeneous coordinates satisfies $(\xi,\xi) = (\xi',\xi') = \cdots = (\xi^{(m-1)},\xi^{(m-1)}) = 0$, where the upper indices stand for derivatives and $(\ ,\ )$ denotes the canonical symmetric product in C^{2m+1}. J. L. Barbosa (1975) established a one-to-one correspondence between the set of all full generalized minimal immersions $f : S^2 \to S^{2m}(1)$ and the set of all linearly full totally isotropic curves $\Xi : S^2 \to CP^{2m}$, where CP^{2m} denotes the $2m$-dimensional complex projective space of constant holomorphic curvature 4, with such immersions corresponding to their directrices. It is then natural to define the degree of the minimal immersion f as the degree of its directrix curve. By considering a very particular local expression for the directrix curve, Barbosa obtained a set of minimal immersions such that, for any multiple of 4π greater than or equal to $2\pi m(m+1)$, there is one having that value as its area. He also showed that the group $SO(2m+1,\mathbf{C})$ of all complex matrices A satisfying $\det(A) = 1$ and $AA^T = I$ acts on the space of totally isotropic curves. Identifying the minimal immersions that are isometric, he found that each orbit of this action is diffeomorphic to $SO(2m+1,\mathbf{C})/SO(2m+1,\mathbf{R})$. Barbosa also showed that the space of minimal immersion of degree $2m$ consists of exactly one such orbit. X. X. Li (1995) extended Barbosa's result to the case of minimal immersions of degree $2m+2$. By solving the totally isotropic conditions, he concludes that the set of full minimal immersions $f : S^2 \to S^{2m}(1)$ of degree $2m+2$ is, modulo isometries, diffeomorphic to a disjoint union of $m-1$ such orbits.

N. Ejiri (1986a) investigated equivariant minimal immersions of S^2 into S^{2m} and proved the following:

(a) There are no full minimal immersions from the real projective 2-plane RP^2 into $S^{2(2n-1)}$;

(b) The minimal cone of a full minimal immersion of S^2 into S^{2m} is stable;

(c) If $f : S^2 \to S^{2m}$ is a full minimal immersion whose minimal cone has a parallel calibration, then $m = 3$ and f is holomorphic in the near Kähler S^6; and

(d) Circle bundles of S^2 of positive even Chern number (≤ 4) can be minimally immersed in the near Kähler S^6.

There do exist many minimal isometric immersions from E^2 into S^m. A description of such minimal immersions has been obtained by K. Kenmotsu in 1976.

A. Ros (1995) proved that any 2-equator in a 3-sphere divides each embedded compact minimal surface into two connected pieces, and closed regions in the sphere with mean convex boundary containing a null-homologous great circle are the intersection of two closed half-spheres. As an application of these results Ros proved that the normal surface of an embedded minimal torus is also embedded. He also showed that the Clifford torus is the only embedded minimal torus in S^3 that is symmetric with respect to four pairwise orthogonal hyperplanes in E^4. In his 1997 doctoral thesis at Universidad Federal do Ceará, F. A. Amaral claimed that the Clifford torus is the only embedded minimal torus in S^3.

Y. Kitagawa (1995) proved that if $f : M \to S^3$ is an isometric embedding of a flat torus M into S^3, then the image $f(M)$ is invariant under the antipodal map of S^3. He also showed that there exist an immersed flat torus M in S^3 whose image is not invariant under the antipodal map of S^3. H. Hashimoto and K. Sekigawa (1995) showed that a complete minimal surface in S^4 with nonnegative Gaussian curvature is either superminimal or congruent to the Clifford torus.

H. Gauchman (1986) showed that if the second fundamental form h of a compact minimal submanifold in a unit sphere satisfies $|h(u,u)|^2 \leq 1/3$ for any unit tangent vector at any point, then it is totally geodesic. P. F. Leung (1993) proved that if an n-dimensional ($n \geq 3$) compact oriented submanifold (not necessarily minimal) in a unit sphere satisfies the same condition as Gauchman's, then M is homeomorphic to a sphere when $n > 3$ and M is homotopic to a sphere when $n = 3$. In 1997 C. Y. Xia extended Leung's result to the following:

Let M be an n-dimensional ($n \geq 4$) compact simply-connected submanifold isometrically immersed in a δ-pinched ($\delta > \frac{1}{4}$) Riemannian manifold. If the second fundamental form of M satisfies $|h(u,u)|^2 < \frac{4}{9}(\delta - \frac{1}{4})$ for any unit tangent vector at any point, then M is homeomorphic to an n-sphere.

5.4.5. Simons' theorem

J. Simons (1968) proved that if the squared length of the second fundamental form, denoted by S, of a compact n-dimensional minimal submanifold

M of the unit $(n+p)$-sphere S^{n+p} satisfies

$$S \leq \frac{n}{2-\frac{1}{p}}, \tag{5.12}$$

then either M is totally geodesic or $S = \frac{n}{2-\frac{1}{p}}$. If the second case occurs, then either

(a) M is a generalized Clifford torus:

$$S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right),$$

which is the standard product embedding of the product of two spheres of radius $\sqrt{k/n}$ and $\sqrt{(n-k)/n}$, respectively, or

(b) M is a Veronese surface in S^4.

A. M. Li and J. M. Li (1992) showed that if M is a compact n-dimensional minimal submanifold of S^{n+p} with $p \geq 2$ and $S \leq 2n/3$, then M is either a totally geodesic submanifold or a Veronese surface in S^4.

The Veronese surface in S^4 is defined as follows: Let (x, y, z) be the natural coordinate system of E^3 and $(u_1, \ldots, u_5)$ that of E^5. The mapping defined by

(5.13)
$$u_1 = \frac{yz}{\sqrt{3}}, \quad u_2 = \frac{xz}{\sqrt{3}}, \quad u_3 = \frac{xy}{\sqrt{3}}, \quad u_4 = \frac{x^2-y^2}{2\sqrt{3}}, \quad u_5 = \frac{1}{6}(x^2+y^2-2z^2)$$

gives rise to an isometric immersion of $S^2(\sqrt{3})$ into S^4. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point. Thus, the mapping defines an embedding of the real projective plane RP^2 into S^4. This embedding of $RP^2(\sqrt{3})$ into S^4 is called the Veronese surface.

5.4.6. Chern-do Carmo-Kobayashi's theorem and related results

S. S. Chern, M. do Carmo and S. Kobayashi (1970) proved that the open pieces of the generalized Clifford torus and the Veronese surface are the only minimal submanifolds of S^{n+p} with $S = \frac{n}{2-\frac{1}{p}}$.

A large number of examples of minimal hypersurfaces in S^{n+1} were constructed by W. Y. Hsiang (1967), using Lie group methods. For example, he showed that, for each $n \geq 4$, there exist infinitely many, mutually incongruent minimal embeddings of $S^1 \times S^{n-2}$ (respectively, $S^2 \times S^{n-3}$) into $S^n(1)$.

Hsiang also considered the problem of finding algebraic minimal cones, obtained by setting a homogeneous polynomial equal to zero. For quadratic polynomials, they are

$$q(x_1^2 + \cdots + x_{p+1}^2) - p(x_{p+1}^2 + \cdots + x_{p+q+2}^2) = 0,$$

$$p \geq 1, \quad q \geq 1, \quad p+q+2=n, \tag{5.14}$$

whose intersection with S^{n+1} are the generalized Clifford tori. Hsiang showed that these are in fact the only algebraic minimal cones of degree 2.

S. S. Chern conjectured that for a compact minimal hypersurface with constant scalar curvature in S^{n+1} the values S are discrete. C. K. Peng and C. L. Terng (1983) proved that if M is a compact minimal hypersurface of S^{n+1} with constant scalar curvature, then there exists a constant $\epsilon(n) > 1/(12n)$ such that if $n \leq S \leq n+\epsilon(n)$, then $S = n$, so that M is a generalized Clifford torus. Furthermore, they showed that if $n = 3$ and $S > 3$, then $S \geq 6$; this bound is sharp, since the principal curvatures of the Cartan minimal isoparametric hypersurface $SO(3)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ in S^4 are given by $\sqrt{3}, 0, -\sqrt{3}$. Peng and Terng's result still holds if the 3-dimensional minimal submanifold is assumed to be complete [Cheng 1990]. Peng and Terng conjectured that the third value of S should be $2n$, since there exist Cartan's isoparametric minimal hypersurfaces in S^{n+1} satisfying $S = 2n$.

H. C. Yang and Q. M. Cheng (1997) proved that, for a compact minimal hypersurface M with constant scalar curvature in S^{n+1}, if $S > n > 3$, then $S > n+\frac{1}{3}n$. In particular, if the shape operator A_ξ of M in S^{n+1} with respect to a unit normal vector ξ satisfying trace $(A_\xi^3) =$ constant, then $S \geq n + \frac{2}{3}n$.

Q. M. Wang (1988) constructed examples of compact noncongruent minimal hypersurfaces in odd-dimensional spheres which have the same constant scalar curvature. Thus, the compact minimal hypersurfaces with given constant scalar curvature in a sphere are not necessary unique.

It is still an open problem to determine whether $S \geq 2n$ for a compact minimal hypersurface M with constant scalar curvature in S^{n+1} with $S > n > 3$.

For an n-dimensional compact minimal manifold M in S^{n+p} with $p \geq 2$, C. Xia (1991) proved the following:

(1) If n is even and $S \leq n(3n-2)/(5n-4)$, then M is either totally geodesic or a Veronese surface in S^4;

(2) If n is odd and $S \leq n(3n-5)/(5n-9)$, then

(2-i) when $n > 5$, M is totally geodesic in S^{n+p};

(2-ii) when $n = 5$, M is either totally geodesic or homeomorphic to S^5 and $S = 25/8$ on M; and

(2-iii) when $n = 3$, S is identically equal to 0 or 2; in the latter case M is diffeomorphic to S^3 or RP^3.

T. Itoh (1978) proved that if $f : M \to S^{n+p}$ is a minimal full isometric immersion of a compact orientable Riemannian n-manifold into S^{n+p} and the sectional curvature K of M satisfies $K \geq n/2(n+1)$, then either M

is totally geodesic or M is of constant sectional curvature $n/2(n+1)$ and the immersion is given by the second standard immersion of an n-sphere of sectional curvature $n/2(n+1)$.

N. Ejiri (1979a) showed that if the Ricci tensor of an n-dimensional ($n \geq 4$) compact minimal submanifold of S^{n+p} satisfies $Ric \geq (n-2)g$, then M is totally geodesic, or $n = 2m$ and M is $S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2}) \subset S^{n+1} \subset S^{n+p}$ embedded in a standard way, or M is a 2-dimensional complex projective space CP^2 of constant holomorphic sectional curvature $\frac{4}{3}$ which is isometrically immersed in a totally geodesic S^7 via Hermitian harmonic functions of degree one.

G. Chen and X. Zou (1995) showed that if the sectional curvature is $\geq \frac{1}{2} - \frac{1}{3p}$, then either M is totally geodesic or the Veronese surface in S^4.

5.4.7. Otsuki's theorem and Otsuki's equation

In 1970, T. Otsuki proved the following.

Let M be a complete minimal hypersurface of S^{n+1} with two principal curvatures. If their multiplicities k and $n-k$ are ≥ 2, then M is the generalized Clifford torus $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$. If one of the multiplicities is one, then M is a hypersurface of S^{n+1} in $E^{n+2} = E^n \times E^2$ whose orthogonal projection into E^2 is a curve of which the support function $x(t)$ is a solution of the following nonlinear differential equation:

$$nx(1-x^2)x''(t) + x'(t)^2 + (1-x^2)(nx^2-1) = 0. \tag{5.15}$$

Furthermore, there are countably many compact minimal hypersurfaces immersed but not embedded in S^{n+1}. Only $S^{n-1}(\sqrt{(n-1)/n}) \times S^1(\sqrt{1/n})$ is minimally embedded in S^{n+1}, which corresponds to the trivial solution $x(t) = 1/\sqrt{n}$ of Otsuki's equation (5.15).

Applying Otsuki's result Q. M. Cheng (1996) proved that if M is a compact minimal hypersurface of $S^{n+1}(1)$ with two distinct principal curvatures such that

$$n \leq S \leq n + \frac{2n^2(n+4)}{3(n+2)^2}, \tag{5.16}$$

then $S = n$ and hence M is a generalized Clifford torus.

Otsuki's result was extended by L. P. Jorge and F. Mercuri (1984) to submanifolds of higher codimension: If $f\ M^n \to S^{n+p}\,(n \geq 3, p > 1)$ is a full minimal immersion such that the shape operator A_ξ in any normal direction ξ has at most two distinct eigenvalues, then M^n is an open subset of a projective space over the complex, quaternion or Cayley numbers, and f is a standard embedding with parallel second fundamental form.

5.5. **Minimal submanifolds in hyperbolic space.** N. Ejiri (1979b) proved that every minimal submanifold in a hyperbolic space is irreducible as a Riemannian manifold. Chen (1972) showed that there exists no minimal surface of constant Gaussian curvature in H^3 except the totally geodesic one.

On the other hand, M. do Carmo and M. Dajczer (1983) constructed many minimal rotation hypersurfaces in hyperbolic space, in particular, in H^3. They also proved that complete minimal rotation surfaces of H^3 are embedded.

X. Li-Jost (1994) studied Plateau type problem in hyperbolic space and proved that if Γ is a closed Jordan curve of class $C^{3,\alpha}$ in H^3 with total curvature $\leq 4\pi$, then there exists precisely one minimal surface of disk type, free of branch points, spanning Γ.

Let γ be a geodesic in H^3, $\{\psi_t\}$ the translation along γ, and $\{\varphi_t\}$ the one-parameter subgroup of isometries of H^3 whose orbits are circles centered on γ. Given any $\alpha \in \mathbf{R}$, $\lambda = \{\lambda_t\} = \{\psi_t \circ \varphi_{\alpha t}\}$ is a one-parameter subgroup of isometries of H^3, which is called a helicoidal group of isometries with angular pitch α. Any surface in H^3 which is λ-invariant is called a helicoidal surface.

J. B. Ripoll (1989) proved the following: Let $\alpha \in \mathbf{R}$, $|\alpha| < 1$. Then there exists a one-parameter family Σ of complete simply-connected minimal helicoidal surfaces in H^3 with angular pitch α which foliates H^3. Furthermore, any complete helicoidal minimal surface in H^3 with angular pitch $|\alpha| < 1$ is congruent to an element of Σ.

G. de Oliveira Filho (1993) considered complete minimal immersions in hyperbolic space and proved the following.

(1) If $M^n \to H^m$ is a complete minimal immersion and $\int_M S^{n/2} dV < \infty$, then M is properly immersed and is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary. Furthermore, the immersion $M^n \to H^m$ extends to a continuous map $\overline{M} \to \overline{H}^n$, where $\overline{H}^n$ is the compactification of H^n.

(2) If $M^2 \to H^m$ is a complete minimal immersion with $\int_M S dV < \infty$, then M is conformally equivalent to a compact surface $\overline{M}$ with a finite number of disks removed and the index of the Jacobi operator is finite. Furthermore, the asymptotic boundary $\partial_\infty M$ is a Lipschitz curve.

K. Polthier (1991) constructed complete embedded minimal surfaces in H^3 having the symmetry of a regular tessellation by Coxeter orthoschemes and proved that there exist complete minimal surfaces in H^3 with the symmetry of tessellations given by (a) all compact and noncompact Platonic polyhedra; (b) all Coxeter orthoschemes (p, q, r) with $q \in \{3, 4, \cdots, 1000\}$ and small p and r; (c) all "self-dual" Coxeter orthoschemes (p, q, r) with $p = r$.

Recently, M. Kokubu (1997) established the Weierstrass type representation for minimal surfaces in hyperbolic space.

For stable minimal submanifolds in hyperbolic space, do Carmo and Dajczer (1983) proved that there exists an infinite family of simply connected stable complete minimal surfaces in hyperbolic space H^3 that are not totally geodesic. Furthermore, let $f : M \to H^3$ be an isometric immersion of M into H^3. If D is a simply-connected domain in M with compact closure $\bar{D}$ and piecewise smooth boundary ∂D, Barbosa and do Carmo (1981) proved that if

$$\int_{\bar{D}} \left(|K| + \frac{2}{3} \right) dV < 2\pi,$$

then D is stable. Also, Ripoll's result implies that any complete helicoidal minimal surface in H^3 with angular pitch $|\alpha| < 1$ is globally stable.

Let $M^{p-1} \to S^{n-1}(\infty)$, $p = n - 1 \leq 6$, be an immersed compact submanifold in the $(n-1)$-sphere at infinity of H^n. M. T. Anderson (1982) proved that there exists a complete embedded absolutely area-minimizing submanifold asymptotic to M^{p-1} at infinity. In particular, there are lots of embedded complete minimal submanifolds in case $p = n - 1 \leq 6$.

5.6. **Gauss map of minimal surfaces.** The Gauss map $G : M \to S^2$ of a surface $f : M \to E^3$ is a map from the surface M to the unit sphere S^2 given by $G(x) = \xi(x)$, where $\xi(x)$ is the unit normal of M at x. Since $\xi(x)$ is a unit vector in E^3, one may represent it as a point in S^2.

O. Bonnet (1860) proved that the Gauss map of a minimal surface in E^3 is conformal. Conversely, E. B. Christoffel proved in 1867 that if the Gauss map of a surface in E^3 is conformal, then it is either a minimal surface or a round sphere.

For a surface $f : M \to E^m$, $m \geq 3$, the Gauss map G is defined to be the map which assigns to each point $x \in M$ the oriented tangent space $f_*(T_xM) \subset E^m$. The Gauss map G can be considered as a map from M into the Grassmann manifold $G^R(2, m-2) = SO(m)/SO(2) \times SO(m-2)$ of oriented 2-planes in E^m, which in turn can be identified with the complex quadric $Q_{m-2}(C)$:

$$Q_{m-2}(C) = \{(z_1, z_2, \ldots, z_m) \in CP^{m-1} : \sum z_j^2 = 0\} \tag{5.17}$$

in the complex projective space CP^{m-1} in a natural way. The Gauss map of an n-dimensional submanifold in E^m is a map from M into $G^R(n, m-n)$ defined in a similar way.

The complex projective space admits a unique Kähler metric with constant holomorphic sectional curvature 2. The induced metric on Q_{m-2} defines a metric $\hat{g}$ on the Grassmannian $G^R(2, m-2)$ under the identification, satisfying

$$G^*(\hat{g}) = -Kg \tag{5.18}$$

for any minimal surface M in E^m, where g is the metric on M and K the Gaussian curvature of M. Thus, the Gauss map G is conformal for a minimal surface $f : M \to E^m$. S. S. Chern (1965) showed that an immersion $f : M \to E^m$ is minimal if and only if the Gauss map G of f is antiholomorphic.

Since the area $\hat{A}(G(M))$ of the Gauss image $G(M)$ is related with the total curvature of M by

$$\hat{A}(G(M)) = -\int_M K dA, \tag{5.19}$$

for the minimal surface, one is able to translate statements about the total curvature of a minimal surfaces in E^m into corresponding statements about the area of holomorphic curves in CP^{m-1}.

5.6.1. Chern-Osserman's theorem

S. S. Chern and R. Osserman (1967) proved the following fundamental results:

If $f : M \to E^m$ is a complete orientable minimal surface with finite total curvature $\int_M K dA = -\pi C < \infty$, then

(1) M is conformally a compact Riemann surface $\bar{M}$ with finite number, say r, of points deleted;

(2) C is an even integer and satisfies

$$C \geq 2(r - \chi) = 4g + 4r - 4,$$

where χ is the Euler characteristic and g is the genus of M (= the genus of $\bar{M}$);

(3) if $f(M)$ does not lie in any proper affine subspace of E^m, then

$$C \geq 4g + r + m - 3 \geq 4g + m - 2 \geq m - 2;$$

(4) if $f(M)$ is simply-connected and nondegenerate, that is, $G(M)$ does not lie in a hyperplane of CP^{m-1}, then $C \geq 2n - 2$ and this inequality is sharp;

(5) when $m = 3$, C is a multiple of 4, with the minimum value 4 attained only by Enneper's surface and the catenoid;

(6) the Gauss map G of f extends to a map of $\bar{M}$ whose Gauss image $G(\bar{M})$ is an algebraic curve in CP^{m-1} lying in Q_{m-2}; the total curvature of $f(M)$ is equal to the area of $G(\bar{M})$ in absolute value, counting multiplicity;

(7) $G(M)$ intersects a fixed number of times, say n (counting multiplicity), every hyperplane in CP^{m-1} except for those hyperplanes containing any of

the finite number of points of $G(\bar{M} - M)$; the total curvature of $f(M)$ equals $-2n\pi$.

(8) Enneper's surface and the catenoid are the only two complete minimal surfaces in E^3 whose Gauss map is one-to-one.

For a complete oriented (not necessary minimal) surface M in E^m, B. White (1987) proved that if $\int_M SdA$ is finite, S the squared length of the second fundamental form, then the total curvature, $\int_M KdA$, is an integral multiple of 2π, or of 4π in case $m = 3$.

5.6.2. Value distribution of Gauss map of complete minimal surfaces

The Gauss map of Scherk's surface in Euclidean 3-space omits exactly 4 points of S^2. F. Xavier (1981) proved that the Gauss map of any complete nonflat minimal surface in E^3 can omit at most 6 points of S^2. F. López and A. Ros (1987, unpublished) gave a 1-point improvement by showing that the Gauss map of any complete nonflat minimal surface in E^3 can omit at most 5 points of S^2. Finally, H. Fujimoto (1988) proved that the Gauss map of any complete nonflat minimal surface in E^3 can omit at most 4 points of S^2. Clearly, Fujimoto's estimate is sharp.

For an orientable complete minimal surface M in E^3 with finite total curvature, a theorem of A. Huber (1957) implies that M is conformally equivalent to a compact Riemann surface punctured at a finite number of points; thus there is a closed Riemann surface M_k of genus k and a finite number of points $Q_1, \ldots, Q_r$ on M_k such that M is conformally $M = M_k - \{Q_1, \ldots, Q_r\}$ [Osserman 1969b]. R. Osserman (1964) extended this result to complete surfaces of finite total curvature in E^3 with nonpositive Gaussian curvature.

For a complete minimal surface $f : M \to E^3$ of finite total curvature, the Gauss map G of f can be extended to a meromorphic function $G : M_k \to S^2$.

The total curvature of the catenoid is -4π and its Gauss map misses 2 values. R. Osserman (1961) proved that if the Gauss map of a complete minimal surface of finite total curvature in E^3 omits more than 3 values, then it is a plane. One important consequence of this is a sharpening of Fujimoto's result: If the Gauss map of a complete nonplanar minimal surface in E^3 omits 4 points on S^2, then every other point of S^2 must be covered infinitely often; and hence the total curvature of the minimal surface must be infinite.

There is no known example of a complete minimal surface of finite total curvature whose Gauss map misses 3 values. Osserman (1964) proved that if the Gauss map of a complete minimal surface of finite total curvature in

E^3 misses 3 values, then the genus of the minimal surface is at least one and the total curvature is less than or equal to -12π; A. Weitsman and F. Xavier proved in 1987 that the total curvature is less than or equal to -16π; and Y. Fang proved in 1993 that the total curvature must be at most -20π, and the degree of the Gauss map is at least five.

In embedded case, the Gauss map a complete minimal surface in E^3 with finite total curvature cannot omit more than 2 values, since the limit normal direction at each end belongs to a certain pair of antipodal points [Jorge-Meeks 1983]. In particular, if the minimal surface is embedded or the minimal surface has parallel embedded ends, then it has at least two catenoid type ends [Fang 1993].

In 1990 X. Mo and R. Osserman showed that if the Gauss map of a complete minimal surface in E^3 takes on 5 distinct values only a finite number of times, then the minimal surface has finite total curvature. Mo and Osserman's result is sharp, since there is an embedded complete minimal surface, due to Scherk, in E^3 whose Gauss map misses four points and takes any other points infinitely many times. Mo and Osserman (1990) also proved that the Gauss map of a nonplanar complete minimal surface in E^3 of infinite total curvature takes on every value infinitely often, with the possible exception of four points.

Since the complex quadric surface Q_2 is holomorphically isometric to the product of two spheres of radii $1/\sqrt{2}$, the Gauss map of a surface M in E^4 is thus described by a pair of maps $G_j : M \to S_j$, $j = 1, 2$.

M. Pinl (1953) showed that for a given minimal surface in E^4, the maps G_1 and G_2 defined above are both conformal.

W. Blaschke (1949) proved the following: Let M be a compact surface immersed in E^4 and let χ be its Euler characteristic. Denote by A_j the algebraic area of the image of M under the map G_j, $j = 1, 2$. Then

$$A_1 + A_2 = 4\pi\chi. \tag{5.20}$$

S. S. Chern (1965) proved that if M is a complete minimal surface in E^4 and if the image of M under each of the maps G_1, G_2 omits a neighborhood of some point, then it is a plane.

X. Mo and R. Osserman (1990) proved that if each of the factors G_j of the Gauss map of a complete nonflat minimal surface in E^4 omits 4 distinct points, then each of the G_j must cover every other point infinitely often. If one of the G_j is constant, then the other must cover every point infinitely often with at most 3 exceptions.

For a complete minimal surface $f : M \to E^m$ with $m \geq 3$, H. Fujimoto (1990) proved that G can omit at most $m(m + 1)/2$ hyperplanes in general position if the Gauss map G of f is nondegenerate, that is, $G(M)$ is not contained in any hyperplane in CP^{m-1}. For arbitrary odd number m, the number $m(m+1)/2$ is sharp (cf. [Fujimoto 1993, §5.5]).

M. Ru (1991) improved Fujimoto's result to the following: If the Gauss map G of f omits more than $m(m+1)/2$ hyperplanes in CP^{m-1}, located in general position, then the minimal surface must be a plane.

Recently, R. Osserman and M. Ru (1997) extended the above result to the following: Let $f : M \to E^m$ be a minimal surface immersed in E^m. Suppose that its Gauss map G omits more than $m(m+1)/2$ hyperplanes in CP^{m-1}, located in general position. Then there exists a constant C, depending on the set of omitted hyperplanes, but not on the surface, such that $|K(x)|^{\frac{1}{2}}d(x) \leq C$, where $K(x)$ is the Gaussian curvature of M at x and $d(x)$ is the geodesic distance from x to the boundary of M.

Related to the above results are some results for minimal surfaces defined on the complex plane $\mathbf{C}$ which are given by P. Hall. Consider a minimal surface $x : \mathbf{C} \to E^m$ and the reduced representation $F = (f_1, \ldots, f_m)$ of its Gauss map $G : \mathbf{C} \to CP^{m-1}$. A direction $v = (v_1, \ldots, v_m) \in E^m$ is called a normal to M at $p \in M$ if it is orthogonal to T_pM, that is, $\sum_{i=1}^m v_i f_i(p) = 0$.

P. Hall (1989,1991) proved the following:

(1) If the normals to a minimal surface $x : \mathbf{C} \to E^m$ omits m directions in general position, then $x : \mathbf{C} \to E^m$ has a holomorphic factor, namely, there is an orthogonal decomposition $E^m = E^2 \oplus E^{m-2}$ such that the projection of x into the first factor is holomorphic or antiholomorphic with respect to an orthogonal almost complex structure on E^2;

(2) If the normals to a minimal surface $x : \mathbf{C} \to E^4$ omit four directions in general position, then x is holomorphic in some orthogonal almost complex structure. Moreover, it they omit five directions in general position, then x is a plane.

(3) If the normals to a minimal surface $x : \mathbf{C} \to E^m$ omits $n + k\,(k \geq 0)$ directions in general position, then the dimension d of the linear subspace of CP^{m-1} generated by the image of the Gauss map and the dimension a of the affine subspace of E^m generated by the image of x satisfy

$$1 \leq d \leq \frac{m-3}{k+1}, \quad d + 3 \leq a \leq \min(n - kd, 2d + 2).$$

5.7. **Complete minimal submanifolds in Euclidean space with finite total curvature.** Let M_k be a compact surface of genus k, $Q_1, \cdots, Q_r$ be r points of M_k, and

$$f : M = M_k - \{Q_1, \cdots, Q_r\} \to E^n$$

be a complete minimal immersion in Euclidean n-space E^n. If $D_j \subset M_k$ is a topological disk centered at Q_j, $j = 1, \cdots, r$, $Q_i \notin D_j$, $i \neq j$, then $E_j = f(D_j \cap M)$ is an end of the immersion f, and f is called a complete minimal immersion in E^n, of genus k and with r ends.

A surface is said to have finite topology if it is homeomorphic to a compact Riemann surface from which a finite number of points have been removed. A complete immersed minimal surface of finite total curvature in E^3 has finite topology. In fact, it is conformal to a punctured compact Riemann surface.

The helicoid is a complete embedded simply-connected minimal surface of genus zero and with one end. Since it is periodic and nonflat, its total curvature is infinite. This example shows that finite topology does not imply finite total curvature.

T. Klotz and L. Sario (1965) proved that there exist complete minimal surfaces in E^3 of arbitrary genus with any finite number of ends.

The Gauss map $G : M \to S^2$ of a complete minimal surface M of finite total curvature in E^3 can be extended to M_k such that the extension $\tilde{G} : M_k \to S^2$ is a holomorphic function. Moreover, the total curvature of M is $-4\pi \deg G$, where $\deg G$ is the degree of G [Osserman 1969b].

5.7.1. Jorge-Meeks' formula and its generalization

Let $f : M \to E^m$ be a complete minimal surface with finite total curvature. Assume M is conformally $M = M_k - \{Q_1, \ldots, Q_r\}$, $n \geq 1$, where M_k is a closed Riemann surface of genus k. Each Q_j corresponds to an end E_j of M. For each end E_j of M the immersed circles $\Gamma_t^j = \frac{1}{t}(E_j \cap S^{m-1}(t))$ converge smoothly to closed geodesics γ^j on $S^{m-1}(1)$ with multiplicity I_j, where $S^{m-1}(t)$ is the sphere centered at$(0,0,0)$ with radius t.

L. P. Jorge and W. H. Meeks (1983) proved that if M is a complete minimal surface of finite total curvature in E^m with r ends, then

$$\int_M K\, dA = 2\pi\Big(\chi(M) - \sum_{j=1}^{r} I_j\Big) \leq 2\pi(\chi(M) - r), \tag{5.21}$$

where $\chi(M) = 2(1-k) - r$ is the Euler characteristic of M. Furthermore, if m equals 3, then

$$\int_M K\, dA = 2\pi(\chi(M) - r) \tag{5.22}$$

if and only if all the ends of M are embedded.

For a branched complete minimal surface M in E^3 Y. Fang (1996a) extended Jorge-Meeks' formula to the following:

$$\int_M K\,dA = 2\pi\Big(\chi(M) - \sum_{i=1}^{r}(J_i - 1) + \sum_{i=1}^{N} K_j\Big), \tag{5.23}$$

where $J_i\,(1 \le i \le r)$ and $K_j\,(1 \le j \le N)$ denote respectively the order of the ends E_i and of the branch points q_j of M.

5.7.2. Topology of complete minimal surfaces

5.7.2.1. Ends of complete minimal surfaces in E^3

Let M be a complete minimal surface of finite total curvature in E^3. Suppose the ends of M are embedded. Then after a suitable rotation of the coordinates, each end of M can be written as

$$z = a\log(x^2 + y^2) + b + r^{-2}(cx + dy) + O\left(r^{-2}\right) \tag{5.24}$$

for suitable constants a, b, c and d, where $r^2 = x^2 + y^2$. An end E_j of M is called a flat (or planar) end, if $a = 0$ at E_j. Otherwise, E_j is called a catenoid end. An end E_j of a complete minimal surface M in E^3 is said to be of Enneper type if its multiplicity (or its winding number) is 3. The complete minimal surface M is said to be of flat type if all of its ends are flat ends.

An end of a complete minimal surface M in E^3 is called annular, if it is homeomorphic to a punctured disk.

Geometrically, all the topological ends of a complete minimal surface of finite total curvature in E^3 are conformally equivalent to a punctured disk, and there is a well-defined limit tangent plane at each end. Outside of a sufficiently large compact set, such an end is multisheeted graph over the limit tangent plane, and if the end is embedded, it is asymptotic to either a plane or a half-catenoid [Schoen 1983].

Y. Fang (1996b) proved that if all of the ends of a complete minimal surface of finite total curvature in E^3 are embedded, then either it is of flat type (that is, no catenoid type ends) or it has at least two catenoid type ends.

An end of a complete embedded minimal surface in E^3 is called a Nitsche end if it is fibered by embedded Jordan curves in parallel planes. Meeks and Rosenberg (1993b) proved that if a complete embedded minimal surface of finite topology has more than one end, then any end of infinite total curvature is a Nitsche end.

L. P. Jorge and W. H. Meeks (1983) have shown the following topological uniqueness result: Suppose M_1 and M_2 are complete embedded minimal surfaces in E^3 with finite total curvature. If M_1 and M_2 are diffeomorphic with two topological ends, then there is an orientation preserving diffeomorphism $\phi : E^3 \to E^3$ with $\phi(M_1) = M_2$.

For an embedded complete minimal surface in E^3 with finite total curvature and with k ends, Jorge and Meeks (1983) proved the following:

(1) If M has an odd number of ends, then M disconnects E^3 into two regions diffeomorphic to the interior of a solid $(g + (k-1)/2)$-holed torus where g is the genus of the associated compact surface $\hat{M}$; and

(2) If M has an even number of ends and $\hat{M}$ has genus g, then M disconnects E^3 into N_1 diffeomorphic to the interior of a $(g + k/2)$-holed solid torus and into N_2 diffeomorphic to the interior of a $(g + (k-2)/2)$-holed solid torus.

5.7.3. Properly embedded complete minimal surfaces

A mapping $f : M \to N$ between two topological spaces is called proper if, for any compact set $C \subset N$, $f^{-1}(C)$ is also compact. R. Osserman proved that every complete minimal immersion $f : M \to E^3$ of finite type is proper.

Until 1982 the only known examples of properly embedded minimal surfaces in E^3 of finite total curvature were the plane, the catenoid, and the helicoid. The total curvature of the catenoid is -4π whose Gauss map has degree one. The catenoid has genus zero and two embedded ends. The helicoid is locally isometric to the catenoid and it has infinite total curvature. The Gauss map of the helicoid has an essential singularity.

Hoffman and Meeks (1989) proved that on a properly embedded minimal surface in E^3, at most two distinct annular ends can have infinite total curvature. Thus, all other ends have finite total curvature and are therefore geometrically well behaved, that is, asymptotic to the plane or a half-catenoid. Fang and Meeks (1991) showed that if a properly embedded minimal surface in E^3 with two annular ends having infinite total curvature, then these ends lie in disjoint closed halfspaces and all other annular ends are flat ends parallel to the boundary of the halfspaces.

P. Collin (1997) proved that if a properly embedded minimal surface in E^3 has at least two ends, then it has finite topology if and only if it has finite total curvature.

5.7.4. Half-space theorem

D. A. Hoffman and W. H. Meeks (1990b) showed that a nonplanar proper, possible branched, complete minimal surface in E^3 is not contained in a half-space. This result implies that every nonplanar embedded complete minimal surface of finite total curvature has at least two annular ends.

5.7.5. Complete minimal surfaces of genus zero

F. J. López and A. Ros (1991) showed that the plane and the catenoid are the only embedded complete minimal surfaces of finite total curvature and genus zero in E^3. L. P. Jorge and W. H. Meeks (1983) proved that the only complete finite total curvature minimal embedding of $S^2 - \{Q_1, \cdots, Q_r\} \to E^3$ for $1 \leq r \leq 5$ are the plane ($r = 1$) and the catenoid ($r = 2$). The cases $r = 3, 4$ or 5 do not occur.

K. Yang (1994) showed that, for any finite subset Σ of S^2, one can conformally immerse $S^2 - \Sigma$ into E^3 as a complete minimal surface of finite total curvature.

R. Miyaoka and K. Sato (1994) classified all complete minimal surfaces in E^3 of genus zero and two ends.

C. J. Costa (1993) constructed several examples of complete immersed minimal surfaces of finite topology and infinite total curvature. The first of these examples is a one-parameter family perturbation of the catenoid. Each of the perturbation surfaces has a catenoid-type end and an end of infinite total curvature. The second is a one-parameter family of complete minimal surfaces of genus one and three ends. Among the three ends, one is flat, one is catenoid-type, and the third is an end of infinite total curvature. This second family is a perturbation of Costa's embedded minimal surface of genus one and three ends. The surfaces in these two families are not embedded.

W. Rossman (1995) classified all genus zero catenoid-ended complete minimal surfaces with at most $2n + 1$ ends and high symmetry.

5.7.6. Complete minimal surfaces with one or two ends

Planes are the only complete minimal surface of finite total curvature in E^3 with one end. R. Schoen (1983) proved that catenoids are the only complete embedded minimal surfaces in E^3 of finite total curvatures with two ends. C. Costa (1989) classified complete minimal surfaces of finite total curvature in E^3 with genus one and three ends. Hoffman and Meeks (1989,1990a) classified complete minimal surfaces of finite total curvature in E^3 with three ends and a symmetry group of order at least $4(k + 1)$.

K. Sato (1996) showed the existence of complete immersed minimal surfaces of higher genus in E^3 with finite total curvature and one Enneper-type end.

D. Hoffman, F. Wei and H. Karcher (1993) constructed a properly embedded complete minimal surface of infinite total curvature with genus one and one end that is asymptotic to the end of a helicoid; a genus one helicoid. The surface contains two lines, one vertical and corresponding to the axis of the helicoid, the other horizontal crossing the axial line. Rotation about these lines generates the full isometry group, which is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

5.7.7. Complete minimal surfaces of higher genus

R. Miyaoka and K. Sato (1994) constructed examples of complete minimal surfaces in E^3 of genus k with r ends for $k = 0$, $r = 3$ and for $k = 1$, $r \geq 3$, via the method of generating higher genus algebraic curves through taking branched coverings of the Riemann sphere. Using Weierstrass $\mathfrak{p}$ functions on $M_1 - \{4 \text{ points}\}$, they have constructed two series of examples. As a consequence Miyaoka and Sato have shown that there exist complete minimal surfaces of finite total curvature in E^3, missed 2 values, for $M_k - \{r \text{ points}\}$ with (1) $r \geq 2$ when $k = 0$, (2) $r \geq 3$ when $k = 1$, or (3) $r \geq 4$ when $k \geq 2$.

For every positive integer k, D. Hoffman and W. Meeks constructed in 1990 an infinite family of examples of properly embedded minimal surfaces of genus k with three ends in E^3. The total curvature is $-4\pi(k+2)$. E. C. Thayer (1995) discovered a family of complete minimal surfaces with arbitrary even genus. Recently, For every $k \geq 2$, Hoffman and Meeks discovered a one parameter family, $M_{k,x}$, $x \geq 1$, of embedded minimal surfaces of genus $k-1$ and finite total curvature. The surfaces $M_{k,x}$, $x > 1$ have all three ends of catenoid type and a symmetry group generated by k vertical planes of reflectional symmetry.

In 1995 W. Rossman constructed examples of complete minimal surfaces in E^3 of finite total curvature with catenoid-type ends, of genus zero; and also of higher genus. Rossman's examples include minimal surfaces with symmetry group $D_n \times \mathbf{Z}_2$ (dihedral symmetry) and Platonic symmetry, where D_n is the dihedral group.

For a closed Riemann surface M_k of genus k, a positive integer r is called a puncture number for M_k if M_k can be conformally immersed in E^3 as a complete finite total curvature minimal surface with exactly r punctures. The set of all puncture numbers for M_k is denoted by $P(M_k)$. K. Yang (1994) proved that given any M_k its puncture set $P(M_k)$ always contains the set $\{r \in \mathbf{Z} : r \geq 4k\}$.

J. Pérez and A. Ros (1996) showed that the moduli space of nondegenerate, properly embedded minimal surfaces in E^3 with finite total curvature is a real analytic $(r+3)$-dimensional manifold if the fixed number of ends is r.

5.7.8. Minimal annuli of finite total curvature

The catenoid is topologically an annulus, that is, it is homeomorphic to a punctured disk. It follows from Jorge-Meeks' formula that the catenoid is the only embedded complete minimal annulus in E^3 with finite total curvature.

P. Collin (1997) proved that a properly embedded complete minimal annulus in E^3 with at least two ends has finite total curvature.

5.7.9. Riemann's minimal surfaces

The catenoid is a rotational surface, hence is foliated by circles in parallel planes. In 1867 B. Riemann found a one-parameter family of complete embedded singly-periodic minimal surfaces foliated by circles and lines in parallel planes. Each minimal annulus in this one-parameter family is contained in a slab and foliated by circles, and its boundary is a pair of parallel straight lines. Rotating repeatly about these boundary straight lines gives a one-parameter family of singly periodic minimal surfaces. These surfaces known today as Riemann's minimal surfaces. Riemann's minimal surfaces were characterized by Riemann (1892) as the only minimal surfaces fibered by circles in parallel planes besides the catenoid.

A. Enneper (1869) proved that a minimal surface fibered by circular arcs was an open part of a Riemann's minimal surface or an open part of the catenoid.

M. Shiffman (1956) proved that a minimal annulus spanning two circles in parallel planes was foliated by circles in parallel planes and hence a part of Riemann's examples or a part of the catenoid. Hoffman, Karcher and Rosenberg (1991) showed that an embedded minimal annulus with boundary of two parallel lines on parallel planes and lying between the planes extended by Schwarz reflection to a Riemann's minimal surface.

É. Toubiana (1992) characterized Riemann's minimal surfaces as the only properly embedded minimal annuli between a pair of parallel planes bounded by any pair of lines. He also generalized Riemann's examples to produce a countable family of immersed minimal annuli between a pair of parallel planes bounded by a pair of parallel lines. These surfaces are then extended, via the reflection principle, to produce complete immersed minimal surfaces.

In 1993 P. Romon proved that a properly embedded annulus with one flat end, lying between two parallel planes and bounded by two parallel lines in the planes, is a part of a Riemann example.

J. Pérez (1995) proved that a properly embedded minimal torus in E^3/T (T is the group generated by a nontrivial translation in E^3) with two planar type ends is a Riemann's minimal surface provided that it is symmetric with respect to a plane. A. Douady and R. Douady (1995) showed that Riemann examples are the only singly-periodic with translational symmetries minimal surfaces of genus one with planar ends and a symmetry with respect to a plane. In 1994 Y. Fang proved that a properly embedded minimal annulus in a slab with boundary consisting of two circles or planes must be part of a Riemann's minimal surface. Y. Fang and F. Wei (1998) showed that a properly embedded minimal annulus with a planar end and boundary consisting of circles or lines in parallel planes is a part of a Riemann example. F. J. López, M. Ritoré and F. Wei (1997) characterized Riemann's minimal surfaces as the only properly embedded minimal tori with two planar ends in E^3/T, where T is the group generated by a nontrivial translation in E^3. Using numerical methods, F. Wei (1995) constructed a properly embedded minimal surface of genus two and two planar ends in E^3/T by adding handles to the Riemann examples.

5.7.10. Examples and classification of complete minimal surfaces of finite total curvature in E^3

Clearly, planes in E^3 are embedded complete minimal surfaces with zero total curvature. There are only two complete minimal surfaces in E^3 whose total curvature is -4π. These are the catenoid and the Enneper surface; the only embedded one is the catenoid. Also, it is known that the only complete embedded minimal surfaces with total curvature $\geq -8\pi$ in E^3 are the plane and the catenoid with total curvature 0 and -4π respectively.

In 1981 W. H. Meeks showed that if M is diffeomorphic to a real projective plane minus two points, then it does not admit a complete minimal immersion into E^3 with total curvature -6π. A complete Möbius strip in E^3 with total curvature -6π was constructed by Meeks (1975). M. Barbosa and A. G. Colares (1986) showed that, up to rigid motions of E^3, there exists a unique complete minimal immersion of the Möbius strip into E^3 with total curvature -6π.

Osserman, Jorge and Meeks proved that if M is a complete minimal surface in E^3 with total curvature greater than -8π, then, up to a projective transformation of E^3, M is the plane, the catenoid, the Enneper surface, or Meeks' minimal Möbius strip.

By adjoining a handle on Enneper's surface, C. C. Chen and F. Gackstatter (1982) constructed a complete minimal surface of total curvature -8π in E^3; which was characterized by D. Bloss (1989) and F. J. López (1992) as the only complete minimal once punctured torus in E^3 with total curvature -8π.

It follows from a formula of L. Jorge and W. Meeks (1983) that when the total curvature of M is -8π, the genus of the underlying Riemann surface has to be either 0 or 1. Moreover, if the genus is 1, the number of punctures (or ends) has to be 1; and if the genus is 0, the number of punctures can be 1, 2 or 3. The genus zero surfaces were classified rather easily using the Weierstrass representation.

In 1992 F. J. López classified orientable complete minimal surfaces in E^3 with total curvature -8π. In 1993 he gave an example of a once-punctured minimal Klein bottle with total curvature -8π, and proved in 1996 that this minimal Klein bottle is the only complete nonorientable minimal surface in E^3 with total curvature -8π.

M. E. G. G. Oliveira (1984) constructed an example of a nonorientable complete minimal surface of genus one with two ends and total curvature -10π in E^3. S. P. Zhang (1989) observed that there is only one minimal two-punctured projective plane in E^3 of total curvature -10π and such that the branch number of the Gauss map at the ends is greater than or equal to three.

C. C. Chen and F. Gackstatter (1982) constructed a complete minimal surface of genus two with total curvature -12π and one end in E^3. A complete minimal surface of genus one with three ends was discovered by C. J. Costa in 1984 which satisfies the following two properties: (a) the total curvature is -12π, and (b) the ends are embedded. Hoffman and Meeks (1985) showed that Costa's minimal surface is properly embedded. They also showed that it contains two straight lines meeting at right angles, it is composed of eight congruent pieces in different octants, each of which is a function graph, and the entire surface is invariant under a dihedral group of 3-space rigid motions.

C. J. Costa (1991) classified orientable complete minimal surfaces in E^3 with total curvature -12π, assuming that they are embedded. F. J. López, F. Martin and D. Rodriguez (1997) proved that the genus two Chen-Gackstatter example is the unique complete orientable minimal surface of genus two in E^3 with total curvature -12π and eight symmetries.

N. Do Espirito-Santo (1994) showed the existence of a complete minimal surface of genus 3 with total curvature -16π and one Enneper-type end. F. F. Abi-Khuzam (1995) constructed a one-parameter family of complete minimal surfaces of genus one with total curvature -16π and having four embedded planar ends.

Costa also constructed an example of complete minimal surface of genus one with two ends and total curvature -20π in E^3.

Complete Möbius strips in E^3 with total curvature $-2\pi n$, for any odd integer $n \geq 5$, were constructed by Oliveira in 1984. In particular, this implies that there exist complete Möbius strips with total curvature $-10\pi, -14\pi$ or -18π in E^3. In 1993 de Oliveira and Toubiana constructed, for any integer $n \geq 3$, an example of complete minimal Klein bottles in E^3 with total curvature $-2\pi(2n+3)$.

In 1989 H. Karcher obtained a generalization of Chen-Gackstatter surface by increasing the genus and the order of the symmetry group. For each $k \geq 1$ he proved that there exists a complete orientable minimal surface of genus k with one end, total curvature $-4\pi(2k-2)$, and $4k+4$ symmetries. R. Kusner (1987) constructed a family of immersed projective planes with $k\,(k \geq 3)$ embedded flat ends and total curvature $-4\pi(2k-1)$. In 1996 A. Ros proved that if M is an embedded complete minimal surface of genus $k > 0$ with finite total curvature, then the symmetry group of M has at most $4(k+1)$ elements, and it has $4k+4$ elements if and only if M is the Hoffman-Meeks surface M_k (1990).

F. Martin (1995,1997) discovered a family of complete non-orientable highly symmetrical complete minimal surfaces with arbitrary topology and one end and provided characterizations of such minimal surfaces. F. Martin and D. Rodriguez (1997) classified complete minimal surfaces of total curvature $-4\pi(3k-3)$ with $4k$ symmetries and one end in E^3, for k not a multiple of 3.

Jorge and Meeks constructed in 1983 complete minimal surfaces of genus zero in E^3 with total curvature $-4\pi r$ with r embedded ends. In 1993 É. Toubiana proved that there exist nonorientable minimal surfaces of genus k with two ends and total curvature $-10(k+1)\pi$.

The new examples of complete embedded minimal surfaces of finite total curvature were discovered by using the global version of the Enneper-Riemann-Weierstrass representation, which is essentially due to Osserman; The method involves knowledge of the compact Riemann surface structure of the minimal surface as well as its Gauss map and other geometric-analytic data.

5.7.11. Maximum principle at infinity

The maximum principle at infinity for minimal surfaces in E^3 was first studied by R. Langevin and H. Rosenberg (1988), who proved that the distance between two disjoint embedded complete minimal surfaces in E^3 with finite total curvature and compact boundaries must be greater than zero, that is, the surfaces cannot touch each other at infinity. W. Meeks and Rosenberg (1990) extended their result to the following:

Let M_1 and M_2 be disjoint, properly immersed minimal surfaces with nonempty compact boundaries in a complete flat 3-manifold. Then

$$\operatorname{dist}(M_1, M_2) = \min(\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(M_1, \partial M_2)). \tag{5.25}$$

M. Soret (1995) studied the maximum principle at infinity for minimal surfaces with noncompact boundaries and proved that if M_1 and M_2 are disjoint properly embedded minimal surfaces with bounded curvature in a complete flat 3-manifold and one of the surfaces is of parabolic type, then

$$\operatorname{dist}(M_1, M_2) = \min(\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(M_1, \partial M_2)). \tag{5.26}$$

Consequently, if M_1 and M_2 are disjoint, properly embedded stable minimal surfaces with noncompact boundaries in a complete flat 3-manifold, then

$$\operatorname{dist}(M_1, M_2) = \min(\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(M_1, \partial M_2)). \tag{5.27}$$

In particular, if the boundary of one surface, say M_1, is empty then

$$\operatorname{dist}(M_1, M_2) = \operatorname{dist}(M_1, \partial M_2). \tag{5.28}$$

5.7.12. Further results on complete minimal surfaces in E^3 with finite total curvature

H. I. Choi, W. H. Meeks, and B. White (1990) proved that any intrinsic local symmetry of the minimal surface in E^3 with finite total curvature can be extended to a rigid motion of E^3. Y. Xu (1995) observed that this property yields the identity of the intrinsic and exterior symmetry groups for the minimal surfaces with embedded catenoid ends. As a consequence he proved that, for any closed subgroup $G \subset SO(3)$ different from $SO(2)$, there exists a genus zero complete minimal surface whose symmetry group is G. The proof relies on the fact that if G is the symmetry group of the minimal surface, then there exists an appropriate Möbius transformation γ which is conjugate to G by the Weierstrass representation. To construct the corresponding examples, Xu described all γ-invariant polynomials which generate the Gauss map of symmetric minimal surfaces.

5.7.13. Complete minimal surfaces in E^m, $m \geq 4$ with finite total curvature

A complete minimal surface in E^m is said to have quadratic area growth if

$$\operatorname{Area}(M \cap B(R)) \leq C_0 R^2 \tag{5.29}$$

for all $R > 0$, where C_0 is a constant and $B(R)$ is a ball of radius R in E^m centered at 0. According to the fundamental result of Chern-Osserman (1967) if a complete minimal surface in E^m has finite total curvature, it is of quadratic area growth and has finite topological type. Conversely, Q. Chen (1997) proved that if a complete minimal surface in E^m has finite topological type and is of quadratic area growth, then it has finite total curvature; the result is false if one drops the assumption of finite topological type, since the surface $\sin z = \sinh x \sinh y$, a Scherk surface in E^3, has infinite genus and quadratic area growth.

For a complete oriented minimal surface M of finite type in E^4, S. Nayatani (1990a) showed that if M has finite total curvature and degenerate Gauss map, then M is of finite total curvature or a holomorphic curve with respect to some orthogonal almost complex structure on E^4.

For complete minimal surfaces of E^m with $m \geq 4$, C. C. Chen (1979) proved that if a complete minimal surface in E^m has total curvature -2π, then it lies in an affine 4-space $E^4 \subset E^m$, and with respect to a suitable complex structure on E^4, M is a holomorphic curve in C^2. C. C. Chen (1980) also proved that if a complete minimal surface in E^m has total curvature -4π, it must be either simply-connected or doubly-connected. In the former case, it lies in some affine 6-space $E^6 \subset E^m$, and in the latter case, in some $E^5 \subset E^m$.

D. Hoffman and R. Osserman (1980) gave complete description of complete minimal surfaces in Euclidean space with total curvature -4π. In particular, they showed that the dimensions 5 and 6 given by Chen are sharp. It turns out that doubly-connected surfaces are all a kind of "skew catenoid" generated by a one-parameter family of ellipses.

The Chern-Osserman theorem implies that the total curvature of a complete orientable minimal surface M in E^m is a negative integer multiple of 4π. Osserman showed that if the total curvature is -4π, then M must be either Enneper's surface or the catenoid.

5.7.14. Complete minimal submanifolds with finite total scalar curvature

Let $f : M \to E^m$ be a minimally immersed submanifold of E^m. The total scalar curvature of f is defined to be $\int_M S^{n/2} dV$, where S is the squared length of the second fundamental form. This integral is called total scalar curvature because, for minimal submanifolds in a Euclidean space, the scalar curvature is equal to $-S$.

M. Anderson (1984, 1985) studied n-dimensional complete minimal submanifolds of dimension $n \geq 2$ with finite total scalar curvature in a Euclidean space and proved the following:

(1) A complete minimal submanifold M of E^m with finite total scalar curvature is conformally diffeomorphic to a compact Riemannian manifold minus a finite number of points, thus M has only finitely many ends. Moreover, each of finite topological type;

(2) Let M be an n-dimensional complete minimal submanifold of E^m. If $n \geq 3$ and M has finite total scalar curvature and one end, then M is an n-plane;

(3) If a complete minimal submanifold M of E^m has finite total scalar curvature, then each end of M has a unique n-plane as its tangent cone at infinity;

(4) If a complete minimal submanifold M of E^m has finite total scalar curvature, then M is properly immersed, that is, the inverse image of any compact set is compact.

H. Moore (1996) also investigated complete minimal submanifolds of dimension ≥ 3 with finite total scalar curvature. She obtained the following results:

(5) Let M be a complete minimal hypersurface of E^{n+1} with $n \geq 3$. If M has finite total scalar curvature, then M lies between two parallel n-planes in E^{n+1};

(6) Let M be an n-dimensional complete minimal submanifold of E^m with $n \geq 3$. If M has finite total scalar curvature and it has two ends, then either M is the union of two n-planes or M is connected and embedded;

(7) Let M be an n-dimensional complete minimal submanifold of E^m with $n \geq 3$ and $n > m/2$. If M has finite total scalar curvature and it has two ends, then M lies between two parallel n-planes in some affine $(n+1)$-subspace $E^{n+1} \subset E^m$; and

(8) Let M be an n-dimensional complete nonplanar minimal submanifold of E^m with $n \geq 3$ and $n > m/2$. If M has finite total scalar curvature and it has two ends, then M is a catenoid.

J. Tysk (1989) proved that a complete minimal hypersurface M in E^{n+1} has finite index if and only if M has finite total scalar curvature for $n = 3, 4, 5, 6$, provided that the volume growth of M is bounded by a constant times r^n, where r is the Euclidean distance function. Tysk also showed that the result is not valid in E^9 and in higher-dimensional Euclidean spaces.

5.8. **Complete minimal surfaces in E^3 lying between two parallel planes.** In 1980 Jorge and Xavier exhibited a nontrivial example of a complete minimal surface which lies between two parallel planes in E^3. Rosenberg and Toubiana constructed in 1987 a complete minimal surface of the topological type of a cylinder in E^3 which lies between two parallel planes; this surface intersects every parallel plane transversally. Hoffman and Meeks (1990b) proved that there does not exist a properly immersed minimal surface in E^3 that is contained between two parallel planes; this follows from their result that a nonplanar proper minimal surface M in E^3 is not contained in a half-space. In 1992 F. F. de Brito constructed a large family of complete minimal surfaces which lie between two parallel planes in E^3.

For each positive integer k and each integer N, $1 \leq N \leq 4$, C. J. Costa and P. A. Q. Simoes constructed in 1996 an example of complete minimal surface of genus k and N ends in a slab of E^3. More precisely, they showed that there is a complete minimal immersion $f_{k,N} : M_{k,N} \to E^3$ with infinite total curvature such that:

(a) $M_{k,1}$ and $M_{k,2}$ are respectively a compact Riemann surface of genus k minus one disk and two disks,

(b) $M_{k,j+2}$, $j = 1, 2$ are respectively $M_{k,j}$ punctured at two points, and

(c) $f_{k,N}(M_{k,N})$ lies between two parallel planes of E^3 and $f_{k,3}$, $f_{k,4}$ have two embedded planar ends.

5.9. **The geometry of Gauss image.** For a minimal surface M in E^m let $\hat{M} = G(M)$ denote the Gauss image of M under its Gauss map. At all nonsingular points of $\hat{M}$, we have a well-defined Gaussian curvature $\hat{K}$. It follows from the normalization of the metric on CP^{m-1} that $\hat{K} \leq 2$.

D. Hoffman and R. Osserman (1980) proved that the Gaussian curvature $\hat{K}$ of the Gauss image of a minimal surface M in E^m is equal to 2 everywhere if and only if M lies in some affine $E^4 \subset E^m$ and is a holomorphic curve in C^2 with respect to a suitable orthogonal almost complex structure on E^4. Moreover, the Gaussian curvature of the Gauss image of a minimal surface M in E^m is equal to 1 everywhere if and only if M is locally isometric to a minimal surface in E^3.

B. Y. Chen and S. Yamaguchi (1983) proved that a submanifold M of a Euclidean m-space has totally geodesic Gauss image if and only if the second fundamental form h of M in E^m satisfies

$$(\bar{\nabla}_X h)(Y, Z) = h(\nabla^G_X Y, Z) - h(\nabla_X Y, Z) \tag{5.30}$$

for vector fields X, Y, Z tangent to M, where ∇^G denotes the Levi-Civita connection of the Gauss image with respect to the metric induced from the

Gauss map G. By applying this necessary and sufficient condition, Chen and Yamaguchi (1983) proved that the Gauss image of a minimal surface in E^m is totally geodesic in $G^R(2, m-2)$ if and only if either M lies in an affine $E^3 \subset E^m$ or M is a complex curve lying fully in C^2, where C^2 is an affine $E^4 \subset E^m$ endowed with some orthogonal almost complex structure. If the second case occurs, the Gaussian curvature $\hat{K}$ of the Gauss image is 2.

Chen and Yamaguchi (1983) also completely classified surfaces in Euclidean space with totally geodesic Gauss image:

Let M be a surface in E^m whose Gauss image is regular. If the Gauss image $G(M)$ of M is totally geodesic in $G^R(2, m-2)$, then M is one of the following surfaces:

(1) a surface in an affine $E^3 \subset E^m$;

(2) a surface in E^m with parallel second fundamental form, that is, a parallel surface;

(3) a surface in an affine 4-space $E^4 \subset E^m$ which is locally the Riemannian product of two plane curves of nonzero curvature;

(4) a complex curve lying fully in C^2, where C^2 denotes an affine $E^4 \subset E^m$ endowed with some orthogonal almost complex structure.

Conversely, surfaces of type (1), (2), (3) and (4) have totally geodesic Gauss image.

Yu. A. Nikolaevskii (1993) extended Chen-Yamaguchi's result to the following (see, also [Chen-Yamaguchi 1984]):

Let M be an n-dimensional submanifold in E^m whose Gauss image is regular. Then the Gauss image $G(M)$ of M is totally geodesic in $G^R(n, m-n)$ if and only if M is the product of submanifolds, each of the factors is either

(a) a real hypersurface, or

(b) a submanifold with parallel second fundamental form, or

(c) a complex hypersurface.

Chen and Yamaguchi (1984) proved that a submanifold M of E^m is locally the product of real hypersurfaces if and only if the Gauss image is totally geodesic and the normal connection is flat.

5.10. **Stability and index of minimal submanifolds.** .

5.10.1. Stability and λ_1

If $f : M \to E^m$ is a minimal submanifold and ξ is a normal vector field on M, then $f + t\xi$ gives rise to a normal variation $F : (-\epsilon, \epsilon) \times M \to E^m$ for some sufficiently small $\epsilon > 0$.

The second variational formula for the volume functional is given by

$$\frac{d^2V_t}{dt^2}|_{t=0} = \int_M \left(||D\xi||^2 - ||A_\xi||^2\right) dV_0. \tag{5.31}$$

In particular, if M is a minimal surface in E^3, (5.25) reduces to

$$\frac{d^2V_t}{dt^2}|_{t=0} = \int_M \left(|\nabla\phi|^2 + 2K\phi^2\right) dA, \tag{5.32}$$

where $\xi = \phi e_3$ and e_3 is a unit normal vector field of M. A minimal surface M in E^3 is called stable if the second variation is positive for all variations on any bounded domain D in M.

Therefore, a minimal surface M in E^3 is stable if and only if

$$\int_D \left(|\nabla\phi|^2 + 2K\phi^2\right) dA > 0 \tag{5.33}$$

for any smooth function ϕ with compact support on M.

It is convenient to rewrite (5.33) using a new metric $\hat{g} = -Kg$, where g is the metric of M. Then we have

$$d\hat{A} = -K dA \tag{5.34}$$

and

$$|\nabla\phi|^2 = -K|\hat{\nabla}\phi|^2, \tag{5.35}$$

where $\hat{\nabla}$ denotes the gradient in the new metric. We can rewrite (5.33) as

$$\int_D |\hat{\nabla}\phi|^2 d\hat{A} > 2\int_D \phi^2 d\hat{A}. \tag{5.36}$$

The ratio

$$Q(\phi) = \frac{\int_D |\nabla\phi|^2 d\hat{A}}{\int_D \phi^2 dA} \tag{5.37}$$

is called the Rayleigh quotient of D, and the quantity

$$\lambda_1(D) = \inf Q(\phi) \tag{5.38}$$

represents the first eigenvalue of the problem

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{5.39}$$

The "inf" in (5.38) may be taken over all piecewise smooth functions in $\bar{D}$ that vanish on the boundary, where Δ in (5.39) is the Laplacian with respect to a given metric on D. If D has reasonably smooth boundary, then (5.39) has a solution ϕ_1 corresponding to the eigenvalue λ_1, and the "inf" in (5.38) is actually attended when $\phi = \phi_1$.

From these it follows that the stability condition (5.27) is simply the condition:

$$\lambda_1(D) > 2. \tag{5.40}$$

Since for a minimal surface in E^3 the metric $\hat{g}$ is nothing but the pullback under the Gauss map of the metric on the unit sphere S^2, thus we have the following [Barbosa-do Carmo 1976]:

Let D be a relatively compact domain on a minimal surface M in E^3. Suppose that the Gauss map G of the minimal surface maps D one-to-one onto a domain $\hat{D}$ on the unit sphere. If $\lambda_1(\hat{D}) < 2$, then D cannot be area-minimizing with respect to its boundary.

Since $\lambda_1(D_1) = 2$ for a hemisphere D_1 on the unit sphere, this result implies in particular a well-known result of H. A. Schwarz:

If the Gauss map G of a minimal surface M in E^3 maps a relatively compact domain D of a minimal surface M in E^3 one-to-one onto a domain containing a hemisphere, then D cannot be area-minimizing.

H. A. Schwarz also obtained in 1885 a sufficient condition for a domain D in a minimal surface to be stable; namely, suppose a minimal surface M in E^3 has one-to-one Gauss map $G : M \to S^2$, then a relatively compact domain $D \subset M$ is stable if $G(D)$ is contained in a hemisphere of S^2.

Schwarz's result was generalized by J. L. Barbosa and M. do Carmo (1976) to the following: If the area $A(G(D))$ of the Gauss image $G(D)$ is less than 2π, then D is stable.

For a minimal surface M in Euclidean m-space, J. L. Barbosa and M. do Carmo (1980a) proved that if $D \subset M$ is simply-connected and that $\int_M |K| dV < \frac{4}{3}\pi$, then D is stable.

J. Peetre (1959) obtained the following: Let D be a domain on the unit sphere S^2 and $\tilde{D}$ a geodesic disc on the sphere having the same area as D. Then $\lambda_1(D) \geq \lambda_1(\tilde{D})$.

As an analogue to Bernstein's theorem, M. do Carmo and C. K. Peng (1979), and independently by Fischer-Colbrie and Schoen (1980), proved that planes are the only stable complete minimal surfaces in E^3.

H. Mori (1977) studied minimal surfaces in 3-sphere and proved the following:

Let D be a relatively compact domain on a minimal surface M of a unit 3-sphere S^3. Suppose that $\sup_D K = K_0 < 1$ and

$$\int_D (1 - K) dA < \frac{1}{54\pi} \cdot \frac{1 - K_0}{2 - K_0}. \tag{5.41}$$

Then D is stable.

Barbosa and do Carmo (1980a) studied the stability of minimal surfaces in 3-sphere and in hyperbolic 3-space and improved Mori's result to the following:

(1) Let $f : M \to S^3$ be a minimal immersion of a surface M into the unit 3-sphere. Assume that $D \subset M$ is simply-connected and that

$$\int_D (2 - K) dV < 2\pi, \tag{5.42}$$

then D is stable.

Furthermore, the result is sharp in the following sense: given $\delta > 0$ there exists a minimal immersion $f : M \to S^3$ and an unstable domain $D_\delta \subset M$ such that

$$\int_{D_\delta} (2 - K) dV = 2\pi + \delta, \tag{5.43}$$

and

(2) Let $f : M \to H^3$ be a minimal immersion of a surface M into the unit hyperbolic 3-space with constant curvature -1. Assume that $D \subset M$ is simply-connected and that

$$\int_D |K| dV < 2\pi, \tag{5.44}$$

then D is stable.

Barbosa and do Carmo (1980b) also considered stability for minimal immersions in higher dimensional real space form and obtained the following:

Given a minimal surface M of a hypersphere of radius r in E^m, let D be a simply-connected relatively compact domain in M. If

$$\int_D \left(\frac{2}{r^2} - K \right) dA < \frac{2n - 6}{2n - 7} \pi, \tag{5.45}$$

then D is stable.

D. Hoffman and R. Osserman (1982) were able to prove the stability of D under a weaker condition:

$$\int_D \left(\frac{2}{r^2} - K \right) dA < \frac{4}{3} \pi. \tag{5.46}$$

5.10.2. Indices of minimal submanifolds

The index of every compact minimal surface in a Riemannian manifold is always finite. For complete minimal surfaces in E^3, D. Fischer-Colbrie (1985) obtained a direct relationship between index and total curvature. She proved that the index of a complete minimal surface of finite total curvature in E^3 is equal to the index of its Gauss map; thus the index of a complete minimal surface in E^3 is finite if and only if its total curvature is finite.

Since a nonplanar periodic minimal surface in E^3 has infinite total curvature, they have infinite index; hence, the index of a complete Scherk surface is infinite. The helicoid have infinite index as well [Tuzhilin 1992].

J. Tysk (1987) showed that for complete minimal surfaces M in E^3 one has $i_M \leq 7.68183d$, where d is the degree of the Gauss map of M. The number 7.68183 is not optimal, since a catenoid has index one and $d = 1$. It is not known whether the optimal value is 1.

S. Nayatani (1993) related the upper and lower bounds for the index with the degree of the Gauss map and the genus of the minimal surface.

The indices of the catenoid and the Enneper surface are both equal to one. This follows immediately from the fact that the extended Gauss map of these genus one surfaces is a conformal diffeomorphism to the sphere. Osserman (1964) proved that the catenoid and the Enneper surface were the only complete minimal surfaces satisfying this property. S. Montiel and A. Ros (1990) showed that the catenoid and the Enneper surface are the only complete minimal surface in E^3 with index one.

S. Y. Cheng and J. Tysk (1988) showed that if M is a complete orientable minimal surface in E^3 with embedded ends which is neither a plane nor a catenoid, then the index of M is at least 2. F. J. López and A. Ros (1989) showed that the catenoid and the Enneper surface are in fact the only complete orientable minimal surfaces in E^3 with index one.

M. Ritoré and A. Ros (1996) studied the structure of the space of compact index one minimal surfaces embedded in flat 3-tori and obtained the following:

Let M be a complete noncompact orientable index one minimal surface properly embedded in the quotient of E^3 by a discrete subgroup Γ of translations. Then one of the following must occurs:

(i) M is a catenoid in E^3;

(ii) M is a Scherk surface with genus zero and four ends in E^3/Γ;

(iii) M is a Scherk surface with genus zero and four ends in $T^2 \times \mathbf{R}$;

(iv) M is a helicoid with total curvature -4π in E^3/Γ.

If M is a complete oriented minimal surface of genus zero in E^3 which is not the plane, the catenoid, or the Enneper's surface, the index of M is at least 3 [Nayatani 1990b].

Montiel and Ros (1990) and N. Ejiri and M. Kotani (1993) proved that a generic complete orientable finitely branched minimal surface of genus zero in E^3 with finite total curvature $4d\pi$ has index $2d-1$ and nullity 3. Here "generic" means that the Gauss map of M belongs to the complement of an analytic subvariety of the space of such maps.

Ejiri and Kotani (1993) also defined the notion of (nonembedded) flat ends for finitely branched complete minimal surface in E^3 and proved the following:

(a) A complete orientable finitely branched minimal surface in E^3 with finite total curvature has nullity ≥ 4 if and only if its Gauss map can be the Gauss map of a complete finitely branched flat-ended minimal surface in E^3;

(b) The index and the nullity of a complete orientable finitely branched minimal surface of genus zero in E^3 with total curvature -8π both equal to 3.

Let M be a complete submanifold of arbitrary codimension in a Riemannian manifold N, and φ a smooth vector field on N. The horizon of M with respect to φ, denoted by $H(M;\varphi)$, is the set of all points of M at which φ is a tangent vector of M. A connected subset D of M is called visible with respect to φ if D is disjoint from $H(M;\varphi)$. The number of components of $M - H(M;\varphi)$ is called the vision number of M with respect to φ and is denoted by $\nu(M;\varphi)$.

Let φ_n, φ_l and φ_p, respectively, be the variation vector fields in E^3 associated with a 1-parameter family of translations τ_t^n in the direction of a unit vector n, a 1-parameter family of rotations p_t^l around a straight line l, and a 1-parameter family of homothetic expansions μ_t^p with center p.

J. Choe (1990) proved the following:

(a) For any unit vector n in E^3 and any minimal surface M in E^3 of finite total curvature, orientable or nonorientable, $i_M \geq \nu(M;\varphi_n) - 1$;

(b) Let M be a complete minimal surface in E^3 of finite total curvature. If each end of M is embedded and the normal vectors at the points of M at infinity are all parallel to a line l, then $i_M \geq \nu(M;\varphi) - 1$; and

(c) Let M be a complete minimal submanifold in a real space form $R^m(c)$ and φ a Killing vector field on $R^m(c)$. If M is compact, then $i_M \geq \nu(M;\varphi) - 1$, and otherwise $i_M \geq \overline{\nu}(M;\varphi)$, where $\overline{\nu}(M;\varphi)$ is the number of bounded components of $M - H(M;\varphi)$.

Using these Choe showed the following:

(i) The index of the Jorge-Meeks minimal surface with k ends is at least $2k-3$;

(ii) The index of the Hoffman-Meeks minimal surface of genus g [Hoffman-Meeks 1990a] is at least $2g+1$;

(iii) The index of Lawson's minimal surface $\xi_{m,k}$ of genus mk in S^3 [Lawson 1970] is at least $\max(2m+1, 2k+1)$;

(iv) The index of the minimal hypersurface $S^p(\sqrt{p/(p+q)}) \times S^q(\sqrt{q/(p+q)})$ in S^{p+q+1} is at least 3;

(v) The index of any complete immersed nonorientable minimal surface in E^3 of finite total curvature which is conformally equivalent to a projective plane with finite punctures is at least 2;

(vi) The plane, Enneper's surface, and the catenoid are the only three complete immersed orientable minimal surfaces of genus zero and index less than three in E^3;

(vii) The index of Chen-Gackstatter surface is 3.

S. Nayatani (1993) showed that the index of a Hoffman-Meeks minimal surface in E^3 with 3-ends of genus k is $2k+3$ for $k \leq 37$. He also proved that if a complete oriented minimal surface in E^m has finite total curvature, then it has finite index [Nayatani 1990a].

Let M be an n-dimensional complete minimal submanifold of E^m. Then there exists a constant $C_{n,m}$ depending only on the dimensions n, m such that the index of M is less than or equal to $C_{n,m}$ times the total scalar curvature, that is

$$i_M \leq C_{n,m} \int_M S^{n/2} dV, \tag{5.47}$$

which was proved by P. Bérard and G. Besson (1990), S. Y. Cheng and J. Tysk (1988) and N. Ejiri (1991).

In 1994 Cheng and Tysk proved that there exist constants C_m depending only on m such that the index of a branched complete minimal surface in E^m satisfies

$$i_M \leq C_m \int_M (-K) dA, \tag{5.48}$$

where K is the Gaussian curvature of M.

For complete oriented minimal surfaces in E^4, S. Y. Cheng and J. Tysk (1988) proved that the index is less than or equal to $12.72\left(\frac{1}{2\pi}\int(-KdA)\right)$.

For a minimal hypersurface M of E^{n+1} with $n \geq 3$, J. Tysk (1989) showed that the index of M satisfies

$$i_M \leq \omega_n^{-1} \left(\frac{\sqrt{e}(n-1)2^{2n+3}}{n-2}\right)^n \int_M S^{n/2} dV, \tag{5.49}$$

where ω_n is the volume of the unit ball in E^n.

The index of a great 2-sphere in S^3 is one. For a compact orientable minimal surface M in S^3 which is not totally geodesic, F. Urbano (1990) proved that the index of M is at least 5, and equal to 5 if and only if M is the Clifford torus.

N. Ejiri (1983) showed that if M is a closed minimal surface of genus zero, fully immersed in S^{2n} with $n \geq 2$, then the index of M is greater than or equal to $2(n(n+2)-3)$.

In 1994 S. Y. Cheng and J. Tysk proved that if M is a bounded immersed complete minimal surface in S^m, possibly with boundary, then

$$i_M \leq C_m \left(2 \operatorname{Area}(M) - \int_M K dA \right), \tag{5.50}$$

where C_m is a constant depending only on m.

A. A. Tuzhilin (1992) proved that the indices for all the hyperbolic catenoids and the parabolic catenoids in the hyperbolic 3-space H^3 are zero. He also showed that the indices for the spherical catenoids in H^3 do not exceed one. A result of do Carmo and Dajczer (1983) implies the existence of unstable catenoids in the one-parameter family of spherical catenoids. Tuzhilin showed that the indices for these catenoids equal one.

5.10.3. Stability of minimal submanifolds

M. Ross (1992) proved that every complete nonorientable minimal surface in E^3 of finite total curvature is unstable. By a refined analysis of moduli and Teichmuller spaces, J. Jost and M. Struwe (1990) succeed in applying saddle-point methods to prove the existence of unstable complete minimal surfaces of prescribed genus.

Let $L \subset E^3$ be a discrete lattice with rank $L = 1$ or 2 and $f : M \to E^3/L$ be a complete and connected minimal immersion. M. Ross and C. Schoen (1994) proved that if f is stable and M has finite genus, then $f(M)$ is a quotient of the plane, the helicoid or a Scherk surface.

For complete stable minimal surfaces in a given Riemannian manifold, D. Fischer-Colbrie and R. Schoen (1980) proved the following:

(1) Stable complete orientable minimal surfaces in complete orientable 3-manifolds with non-negative Ricci curvature are totally geodesic;

(2) Let N be a complete Riemannian 3-manifold with nonnegative scalar curvature ρ and let M be a complete stable orientable minimal surface in N. Then

(2.1) if M is compact, then either M is conformally equivalent to the Riemann sphere S^2 or else it is a totally geodesic flat torus. Furthermore, if $\rho > 0$, the latter case cannot occur, and

(2.2) if M is not compact, then M is conformally equivalent to the complex plane or a cylinder.

H. Lawson and W. Y. Hsiang gave a complete classification of equivariant stable cones of codimension one in E^{n+1}.

B. Palmer (1991) proved that if M is a complete orientable minimal hypersurface of E^{n+1} and if there exists a codimension one cycle C in M which does not separate M, then M is stable.

M. do Carmo and C. K. Peng (1980) showed that if an oriented stable complete minimal hypersurface M in E^{n+1} satisfies $\int_M SdV < \infty$, then M is a hyperplane. Using Simons' result, P. Bérard (1991) proved that if an oriented stable complete minimal hypersurface M in E^{n+1} has finite total scalar curvature, that is, $\int_M S^{n/2}dV < \infty$, then M is a hyperplane for $n \leq 5$. The same result was proved recently by Y. B. Shen and X. H. Zhu (1998) for $n > 5$, using the interior curvature estimate and Gromov's compactness theorem.

There are results on stability of higher dimensional minimal submanifolds in real space forms obtained by various geometers. For instance, B. H. Lawson and J. Simons (1973) proved that there does not exist stable compact minimal submanifold in S^m.

S. P. Wang and S. W. Wei (1983) showed that, for any dimension $n \geq 2$, there is a non-totally geodesic complete absolute area-minimizing hypersurface in a hyperbolic $(n+1)$-space H^{n+1}.

Further results on indices and stabilities of minimal submanifolds can be found in §11.3, §11.4 and §16.11.

6. Submanifolds of finite type

The study of order and submanifolds of finite type began in the 1970s through Chen's attempts to find the best possible estimate of the total mean curvature of an isometric immersion of a compact manifold in Euclidean space and to find a notion of "degree" for submanifolds in Euclidean space.

The main objects in algebraic geometry are algebraic varieties. One can define the degree of an algebraic variety by its algebraic structure. On the other hand, although every Riemannian manifold can be realized as a submanifold in Euclidean space according to Nash's embedding theorem, one lacks the notion of the degree of a submanifold in Euclidean space. Inspired by this observation, the notions of order and submanifolds of finite type were introduced in [Chen 1979a,1984b].

6.1. **Spectral resolution.** Let (M, g) be a compact Riemannian n-manifold. Then the eigenvalues of the Laplacian Δ form a discrete infinite sequence: $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \nearrow \infty$. Let

$$V_k = \{f \in C^\infty(M) : \Delta f = \lambda_k f\}$$

be the eigenspace of Δ associated with eigenvalue λ_k. Then each V_k is finite-dimensional. Define an inner product $(\; , \;)$ on $C^\infty(M)$ by $(f, h) = \int_M fh\, dV$. Then $\sum_{k=0}^\infty V_k$ is dense in $C^\infty(M)$ (in L^2–sense). If we denote by $\hat{\oplus} V_k$ the completion of $\sum V_k$, we have $C^\infty(M) = \hat{\oplus}_k V_k$.

For each function $f \in C^\infty(M)$, let f_t denote the projection of f onto the subspace V_t. We have the spectral resolution (or decomposition): $f = \sum_{t=0}^\infty f_t$ (in L^2-sense).

Because V_0 is 1-dimensional, there is a positive integer $p \geq 1$ such that $f_p \neq 0$ and $f - f_0 = \sum_{t \geq p} f_t$, where $f_0 \in V_0$ is a constant. If there are infinite many f_t's which are nonzero, put $q = +\infty$; otherwise, there is an integer $q \geq p$ such that $f_q \neq 0$ and $f - f_0 = \sum_{t=p}^q f_t$.

If $x : M \to E^m$ is an isometric immersion of a compact Riemannian n-manifold M into E^m (or, more generally, into a pseudo-Euclidean space), for each coordinate function x_A we have $x_A = (x_A)_0 + \sum_{t=p_A}^{q_A} (x_A)_t$. We put

$$p = \inf_A \{p_A\} \quad \text{and} \quad q = \sup_A \{q_A\}, \tag{6.1}$$

where A ranges over all A such $x_A - (x_A)_0 \neq 0$.

Both p and q are well-defined geometric invariants such that p is a positive integer and q is either $+\infty$ or an integer $\geq p$. Consequently, we have the spectral decomposition of x in vector form:

$$x = x_0 + \sum_{t=p}^{q} x_t, \tag{6.2}$$

which is called the spectral resolution (or decomposition) of the immersion x.

6.2. **Order and type of immersions.** For a compact manifold M, the set $T(f) = \{t \in \mathbf{Z} : f_t \neq \text{constant}\}$ of a function f on M is called the order of f. The smallest element in $T(f)$ is called the lower order of f and the supremum of $T(f)$ is called the upper order of f.

A function f is said to be of finite type if $T(f)$ is a finite set, that is, if its spectral resolution contains only finitely many non-zero terms. Otherwise f is said to be of infinite type.

Let $x : M \to E^m$ be an isometric immersion of a compact Riemannian n-manifold M into E^m (or, more generally, a pseudo-Euclidean space). Put

$$T(x) = \{t \in \mathbf{Z} : x_t \neq \text{constant map}\}. \tag{6.3}$$

The immersion x or the submanifold M is said to be of k-type if $T(x)$ contains exactly k elements. Similarly one can define the lower order and the upper order of the immersion. The immersion x is said to be of finite type if its upper order q is finite; and the immersion is said to be of infinite type if its upper order is $+\infty$. The constant vector x_0 in the spectral resolution is the center of mass of M in E^m.

One cannot make the spectral resolution of a function on a non-compact Riemannian manifold in general. However, it remains possible to define the notion of a function or an immersion of finite type and the related notions of order and type.

For example, a function f is said to be of finite type if it is a finite sum of eigenfunctions of the Laplacian and an immersion x of a non-compact manifold is said to be of finite type if it admits a finite spectral resolution $x = \sum_{t=p}^{q} x_t$ for some natural numbers p and q; otherwise, the immersion is said to be of infinite type. A k-type immersion is said to be of null k-type if the component x_0 in the spectral resolution is non-constant.

A result of [Takahashi 1966] can now be rephrased by saying that 1-type submanifolds of E^m are precisely those which are minimal in E^m or minimal in some hypersphere of E^m. In that regard, submanifolds of finite type provide vast generalization of minimal submanifolds.

Let $x : M \to E^m$ be a k-type isometric immersion whose spectral resolution is given by

$$x = c + x_1 + \ldots + x_k, \Delta x_i = \lambda_i x_i, \qquad \lambda_1 < \ldots < \lambda_k,$$

where c is a constant vector in E^m and $x_1, \ldots, x_k$ are non-constant eigenmaps of the Laplacian. For each $i \in \{1, \ldots, k\}$ we put

$$E_i = \mathrm{Span}\{x_i(u) : u \in M\}.$$

Then each E_i is a linear subspace of E^m. The immersion is said to be linearly independent if the k subspaces $E_1, \ldots, E_k$ are linearly independent, that is, the dimension of the subspace spanned by vectors in $E_1 \cup \ldots \cup E_k$ is equal to $\dim E_1 + \ldots + \dim E_k$. The immersion is said to be orthogonal if the k subspaces $E_1, \ldots, E_k$ defined above are mutually orthogonal in E^m [Chen 1991a].

Let $x : M \to E^m$ be an isometric immersion of finite type. B. Y. Chen and M. Petrovic (1991) proved the following:

(a) The immersion x is linearly independent if and only if it satisfies Dillen-Pas-Verstraelen's condition, that is, it satisfies $\Delta x = Ax + B$ for some $m \times m$ matrix A and some vector $B \in E^m$;

(b) The immersion x is orthogonal if and only if it satisfies $\Delta x = Ax + B$ for some symmetric $m \times m$ matrix A and some vector $B \in E^m$.

Linearly independent submanifolds, equivalently submanifolds satisfying condition $\Delta x = Ax + B$, have also been studied by C. Baikoussis, D. E. Blair, F. Defever, F. Dillen, O. J. Garay, T. Hasanis, J. Pas, M. Petrovic, T. Vlachos, L. Verstraelen, and others.

If $x : M \to E^m$ is an isometric immersion of null k-type whose spectral resolution satisfies $\Delta x_j = 0$, then the immersion is called weakly linearly independent if the $k-1$ subspaces $E_1, \ldots, E_{j-1}, E_{j+1}, \ldots, E_k$ are linearly independent; and the immersion is called weakly orthogonal if $E_1, \ldots, E_{j-1}, E_{j+1}, \ldots, E_k$ are mutually orthogonal.

The mean curvature vector of a submanifold of non-null finite type in E^m satisfies $\Delta H = AH$ for some $m \times m$ matrix A if and only if M is linearly independent. On the other hand, if M is a submanifold of null finite type E^m, then M satisfies $\Delta H = AH$ for some $m \times m$ matrix A if and only if M is weakly linearly independent.

For surfaces in E^3 Chen proved in 1994 the following results:

(1) Minimal surfaces and open parts of circular cylinders are the only ruled surfaces satisfying $\Delta H = AH$ for some 3×3 matrix A;

(2) Minimal surfaces and open parts of circular cylinders are the only finite type surfaces satisfying the condition $\Delta H = AH$ for some 3×3 singular matrix A;

(3) Open parts of circular cylinders are the only tubes satisfying the condition $\Delta H = AH$ for some 3×3 matrix A.

See [Chen 1996d] for the details and for related results.

6.3. **Equivariant submanifolds as minimal submanifolds in their adjoint hyperquadrics.** Let $f : M \to E^m$ be a nonminimal linearly independent isometric immersion and let A denote the $m \times m$ matrix associated with the immersion f defined in §6.2. Then, for any point $u \in M$, the equation

$$\langle Au, u \rangle := \sum_{i,j}^{m} a_{ij} u_i u_j = c_u,$$

with $c_u = \langle Ax, x \rangle (u)$ defines a quadric Q_u of E^m, where $u = (u_1, \ldots, u_m)$ be a Euclidean coordinate system on E^m. The hyperquadric defined above is called the adjoint hyperquadric at u. If $f(M)$ is contained in an adjoint hyperquadric Q_u for some point $u \in M$, then all of the adjoint hyperquadrics $\{Q_u : u \in M\}$ coincide, which give rise to a common adjoint hyperquadric, denoted by Q. This common hyperquadric Q is called the adjoint hyperquadric of the immersion.

Suppose $f : M \to E^m$ is a linearly independent isometric immersion of a compact Riemannian manifold into E^m. Then M is immersed into its adjoint hyperquadric by the immersion f if and only if the immersion is spherical, that is, $f(M)$ is contained in a hypersphere of E^m.

A nonminimal linearly independent isometric immersion $f : M \to E^m$ of a Riemannian manifold is orthogonal if and only if M is immersed as a minimal submanifold of its adjoint hyperquadric by the immersion f.

Although an equivariant isometric immersion of a compact homogeneous Riemannian manifold into Euclidean m-space is of finite type, it is not necessary a minimal submanifold of any hypersphere of the Euclidean m-space in general. However, we have the following general result of Chen (1991a):

If $f : M \to E^m$ is an equivariant isometric immersion of a compact homogeneous Riemannian manifold into Euclidean m-space, then M is isometrically immersed as a minimal submanifold in its adjoint hyperquadric by the immersion f.

6.4. **Submanifolds of finite type.** Although the class of submanifolds of finite type is huge, it consists of "nice" submanifolds of Euclidean spaces. For example, all minimal submanifolds of Euclidean space and all minimal submanifolds of hyperspheres are of 1-type and vice versa. Also, all parallel submanifolds of Euclidean space and all compact homogeneous Riemannian manifolds equivariantly immersed in Euclidean space are of finite type.

Given a natural number k, there do exist infinitely many non-equivalent k-type submanifolds of codimension 2 in Euclidean space. The simplest examples of such codimension two k-type submanifolds of Euclidean space are the Riemannian products of the $(n-1)$-dimensional Euclidean space E^{n-1} with any $(k-1)$-type closed curves in E^3.

Also, according to a result of C. Baikoussis, F. Defever, T. Koufogiorgos and L. Verstraelen (1995), for any natural number k, there exist k-type isometric immersions of flat tori in E^6 which are not product immersions.

6.4.1. Minimal polynomial criterion

Compact finite type submanifolds are characterized by the minimal polynomial criterion which establishes the existence of a polynomial P of the least degree for which $P(\Delta)H = 0$, where H is the mean curvature vector of the submanifold and deg $P = k$ for a k-type immersion [Chen 1984b].

For general submanifolds Chen and M. Petrovic (1991) proved the following: Let $f : M \to E^m$ be an isometric immersion. Then f is of finite type if and only if there exists a vector $c \in E^m$ and a polynomial $P(t)$ with simple roots such that $P(\Delta)(x-c) = 0$. Furthermore, in this case, the type number of f is $\leq \deg P$.

6.4.2. A variational minimal principle

Just like minimal submanifolds, finite type submanifolds are characterized by a variational minimal principle in a natural way; namely as critical points of directional deformations [Chen-Dillen-Verstraelen-Vrancken 1993].

Let $f : M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m. Associated with each E^m-valued vector field ξ defined on M, there is a variation ϕ_t, defined by

$$\phi_t(p) := f(p) + t\xi(p), \quad p \in M, \quad t \in (-\epsilon, \epsilon), \tag{6.4}$$

where ϵ is a sufficiently small positive number.

Let $\mathcal{D}$ denote the class of all variations acting on the submanifold M and let $\mathcal{E}$ denote a nonempty subclass of $\mathcal{D}$. A compact submanifold M of E^m is said to satisfy the variational minimal principle in the class $\mathcal{E}$ if M is a critical point of the volume functional for all variations in $\mathcal{E}$.

Directional deformations were introduced by K. Voss in 1956. Directional deformations are defined as follows : let c be a fixed vector in E^m and let ϕ be a smooth function defined on the submanifold M. Then we have a variation given by

$$\phi_t^{\phi c}(p) := f(p) + t\phi(p)c, \quad p \in M \quad t \in (-\epsilon, \epsilon). \tag{6.5}$$

Such a variation is called a directional deformation in the direction c. For each natural number $q \in \mathbf{N}$, define $\mathcal{C}_q$ to be the class of all directional deformations given by smooth functions ϕ in $\sum_{i \geq q} V_i$.

Chen, Dillen, Verstraelen and Vrancken (1993) proved the following:

(1) There are no compact submanifolds in E^m which satisfy the variational minimal principle in the classes $\mathcal{C}_0$ and $\mathcal{C}_1$.

(2) A compact submanifold M of E^m is of finite type if and only if it satisfies the variational minimal principle in the class $\mathcal{C}_q$ for some $q \geq 2$.

6.4.3. Diagonal immersions

Let $y_i : M \to E^{n_i}$, $i = 1, \ldots, k$, be k isometric immersions of a Riemannian manifold M into E^{n_i}, respectively. For any k real numbers $c_1, \ldots, c_k$ with $c_1^2 + \ldots + c_k^2 = 1$, the immersion

$$f = (c_1 y_1, \ldots, c_k y_k) : M \to E^{n_1 + \cdots + n_k} \tag{6.6}$$

is also an isometric immersion, which is called a diagonal immersion of $y_1, \ldots, y_k$. If $y_1, \ldots, y_k$ are of finite type, then each diagonal immersion of $y_1, \ldots, y_k$ is also of finite type.

6.4.4. Curves of finite type

A closed curve in E^m is of finite type if and only if the Fourier series expansion of each coordinate function of the curve has only finite nonzero terms.

The only curves of finite type in E^2 are open portions of circles or lines, hence plane curves of finite type are of 1-type. In contrast with plane curves, there exist infinitely many non-equivalent curves of k-type in E^3 for each $k \in \{2, 3, 4, \cdots\}$.

Closed curves of finite type in a Euclidean space are rational curves. Furthermore, a closed curve of finite type in E^3 is of 1-type if and only if it lies in a 2-sphere [Chen-Deprez-Dillen-Verstraelen-Vrancken 1990].

6.4.5. Finite type submanifolds in Euclidean space

B. Y. Chen, J. Deprez and P. Verheyen (1987) proved that an isometric immersion of a symmetric space M of compact type into a Euclidean space is of finite type if and only if the immersion maps all geodesics of M into curves of finite type.

In 1988, Chen proved that a surface in E^3 is of null 2-type if and only if it is an open portion of a circular cylinder. Also, Chen and H. S. Lue (1988) proved the following:

(a) a 2-type submanifold M in a Euclidean m-space with parallel mean curvature vector is either spherical or null;

(b) every 2-type hypersurface of constant mean curvature in Euclidean space is of null 2-type;

(b) every compact 2-type hypersurface of a Euclidean space has non-constant mean curvature.

Null 2-type hypersurfaces and open portions of hyperspheres are the only hypersurfaces of Euclidean space with nonzero constant mean curvature and constant scalar curvature (cf. [Chen 1996d]). T. Hasanis and T. Vlachos (1995a) proved that null 2-type hypersurfaces of E^4 have nonzero mean curvature and constant scalar curvature. They also showed that a 3-type surface in E^3 has non-constant mean curvature [Hasanis-Vlachos 1995a].

Chen proved in 1987 that a tube in E^3 is of finite type if and only if it is an open portion of a circular cylinder. O. J. Garay (1988b) showed that open portions of hyperplanes are the only cones of finite type in E^{n+1}.

A ruled surface in E^3 is of finite type if and only if it is open portion of a plane, a circular cylinder or a helicoid. In particular, a flat surface in E^3 is of finite type if and only if it is an open portion of a plane or a circular cylinder [Chen-Dillen-Verstraelen-Vrancken 1990]. F. Dillen (1992) considered finite type ruled submanifolds of Euclidean space and proved that a ruled submanifold of Euclidean space is of finite type if and only if it is a part of a cylinder on a curve of finite type or an open portion of a generalized helicoid.

Chen and Dillen (1990a) proved that open portions of spheres and circular cylinders are the only quadrics of finite type in E^3. Further, Chen, Dillen and H. Z. Song (1992) showed that a quadric hypersurface M of E^{n+1} is of finite type if and only if it is one of the following hypersurfaces:

(1) a hypersphere;

(2) a minimal algebraic cone $C_{k,n-k-1}$, $0 < k < n-1$, over an $(n-1)$-dimensional generalized Clifford torus (defined by (5.14));

(3) a spherical hypercylinder $E^k \times S^{n-k}$, $0 < k < n$;

(4) the standard product embedding of the product of a linear subspace E^ℓ and one of the algebraic cones $C_{k,n-\ell-k-1}$ with $0 < k < n-\ell-1$.

F. Defever, R. Deszcz and L. Verstraelen (1993/4) proved that all compact and noncompact cyclides of Dupin are of infinite type.

A hypersurface M of E^{n+1} is called a translation hypersurface if it is a non-parametric hypersurface of the form:

$$x_{n+1} = P_1(x_1) + \cdots + P_n(x_n),$$

where each P_i is a function of one variable. If each function P_i is a polynomial, the hypersurface is called a polynomial translation hypersurface.

F. Dillen, L. Verstraelen, L. Vrancken and G. Zafindratafa (1995) proved that a polynomial translation hypersurface of a Euclidean space is of finite type if and only if it is a hyperplane.

A surface in E^3 is called a surface of revolution if it is generated by a curve C on a plane π when π is rotated around a straight line L in π. By choosing π to be the xz-plane and line L to be the z-axis, the surface of revolution can be parameterized by

$$x(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

A surface of revolution is said to be of polynomial kind if $f(u)$ and $g(u)$ are polynomial functions in u; and it is said to be of rational kind if g is a rational function in f, that is g is the quotient of two polynomial functions in f.

For finite type surfaces of revolution, Chen and S. Ishikawa (1993) proved the following.

(1) A surface of revolution of polynomial kind is of finite type if and only if either it is an open portion of a plane or it is an open portion of a circular cylinder;

(2) A surface of revolution of rational kind is of finite type if and only if it is an open portion of a plane.

T. Hasanis and T. Vlachos (1993) proved that a surface of revolution with constant mean curvature in E^3 is of finite type if and only if it is an open portion of a plane, of a sphere, or of a circular cylinder. J. Arroyo, O. J. Garay and J. J. Menc´ia (1998) proved that the only finite type surfaces in E^3 obtained by revolving an ellipse around a suitable axis are the round spheres.

A spiral surface is a surface in E^3 generated by rotating a plane curve C about an axis A contained in the plane of the curve C and simultaneously transforming C homothetically relative to a point of A.

C. Baikoussis and L. Verstraelen (1995) proved that a spiral surface is of finite type if and only if it is a minimal surface.

It was conjectured that round spheres are the only compact surfaces of finite type in E^3 [Chen 1987]. All of the results mentioned above support the conjecture.

In 1988, Chen proved that if $f : M \to E^m$ is an isometric immersion of a Riemannian n-manifold into E^m, then the mean curvature vector H of f is an eigenvector of the Laplacian on M, that is, $\Delta H = \lambda H$ for some $\lambda \in \mathbf{R}$, if and only if M is one of the following submanifolds:

(a) 1-type submanifold;

(b) a null 2-type submanifold;

(c) a biharmonic submanifold, that is, a submanifold satisfies $\Delta H = 0$.

I. Dimitrić proved in his doctoral thesis (Michigan State University 1989) that open portions of straight lines are the only biharmonic curves of a Euclidean space. Chen proved in 1985 that minimal surfaces are the only biharmonic surfaces in E^3. I. Dimitrić (1989) extended Chen's result to the following: Minimal hypersurfaces are the only biharmonic hypersurfaces of a Euclidean space with at most two distinct principal curvatures.

T. Hasanis and T. Vlachos (1995b) showed that minimal hypersurfaces are the only biharmonic hypersurfaces of Euclidean 4-space. An alternative proof was given in [Defever 1996].

It was conjectured by Chen that minimal submanifolds are the only biharmonic submanifolds in Euclidean spaces.

The conjecture was also proved to be true if the biharmonic submanifold is one of the following submanifolds:

(1) a spherical submanifold [Chen 1991];

(2) a submanifold of finite type [Dimitrić 1989];

(3) a pseudo-umbilical submanifold of dimension $\neq 4$ [Dimitrić 1989].

The conjecture is false if the ambient space is replaced by a pseudo-Euclidean m-space with $m \geq 4$ [Chen-Ishikawa 1991].

6.4.5. Finite type submanifolds in sphere

Standard 2-spheres in S^3 and products of plane circles are the only finite type compact surfaces with constant Gauss curvature in S^3 [Chen-Dillen 1990b].

Every hypersurface with constant mean curvature and constant scalar curvature in S^{n+1} is either totally umbilical or of 2-type. Consequently, every isoparametric hypersurface in S^{n+1} is either of 1-type or of 2-type [Chen 1984b]. Furthermore, every spherical 2-type hypersurface has constant mean curvature and constant scalar curvature [Hasanis-Vlachos 1991].

2-type surfaces in S^3 are open portions of the standard product embedding of the product of two circles [Hasanis-Vlachos 1991].

Chen (1996d) proved that a compact hypersurface of $S^4(1) \subset E^5$ is of 2-type if and only if it is one of the following hypersurfaces:

(1) $S^1(a) \times S^2(b) \subset S^4(1) \subset E^5$ with $a^2 + b^2 = 1$ and $(a, b) \neq (\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}})$ embedded in the standard way;

(2) a tubular hypersurface of constant radius $r \neq \frac{\pi}{2}$ about the Veronese surface of constant curvature $\frac{1}{3}$ in $S^4(1)$.

For higher dimensional spherical hypersurfaces, Chen (1991b,1984b) proved the following:

(a) Let M be a hypersurface of a unit hypersphere $S^{n+1}(1)$ of E^{n+2} with at most two distinct principal curvatures. Then M is of 2-type if and only if M is an open portion of the product of two spheres $S^k(a) \times S^{n-k}(b)$, $1 \leq k \leq n-1$, such that $a^2 + b^2 = 1$, $(a, b) \neq \left(\sqrt{k/n}, \sqrt{(n-k)/n}\right)$.

(b) Let M be a conformally flat hypersurface of a unit hypersphere $S^{n+1}(1)$ of E^{n+2}. Then M is of 2-type if and only if M is an open portion of the product of two spheres $S^k(a) \times S^{n-k}(b)$ for some k, $1 \leq k \leq n-1$, such that $a^2 + b^2 = 1$, $(a, b) \neq \left(\sqrt{k/n}, \sqrt{(n-k)/n}\right)$.

(c) If M is a 2-type Dupin hypersurface of a hypersphere S^{n+1} with three principal curvatures, then M is an isoparametric hypersurface.

Chen and S. J. Li (1991) proved that every 3-type hypersurface of a sphere has nonconstant mean curvature.

A compact spherical submanifold $M \subset S^{m-1} \subset E^m$ is called mass-symmetric if the center of the mass of M coincides with the center of the hypersphere S^{m-1}. Similarly, a non-compact submanifold $f : M \to S_c^{m-1}(r)$ is called mass-symmetric if its position function admits a spectral resolution of the form:

$$x = x_{t_p} + \cdots + x_{t_q}, \quad \Delta x_{t_j} = \lambda_{t_j} x_{t_j}.$$

Regardless of codimension, every mass-symmetric 2-type submanifold of a hypersphere has constant squared mean curvature, which is determined by the order of the submanifolds [Chen 1984b].

Although every compact 2-type surface in S^3 is mass-symmetric [Barros-Garay 1987], there do not exist mass-symmetric 2-type surfaces which lie fully in S^4 [Barros-Chen 1987a].

M. Kotani (1990) studied mass-symmetric 2-type immersions of a topological 2-sphere into a hypersphere of E^m. She proved that such an immersion is the diagonal immersion of two 1-type immersions. Y. Miyata (1988) classified mass-symmetric 2-type spherical immersions of surfaces of constant curvature (see, also [Garay 1988a]).

6.4.6. Finite type submanifolds in hyperbolic space

Let E_s^m denote the m-dimensional pseudo-Euclidean space with index s endowed with the standard flat metric given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2, \tag{6.7}$$

where $(x_1, \ldots, x_m)$ is a standard coordinate system of E_s^m.

For a number $r > 0$, we denote by $S_s^{m-1}(r)$ the pseudo-Riemannian sphere and by $H_{s-1}^{m-1}(-r)$ the pseudo-hyperbolic space defined respectively by

$$S_s^{m-1}(r) = \{u \in E_s^m : \langle u, u\rangle = r^2\}, \tag{6.8}$$

$$H_{s-1}^{m-1}(-r) = \{u \in E_s^m : \langle u, u\rangle = -r^2\}, \tag{6.9}$$

where $\langle\ ,\ \rangle$ denotes the indefinite inner product on the pseudo-Euclidean space.

Denote by H^{m-1} the hyperbolic space which is embedded standardly in the Minkowski space-time E_1^m by

$$H^{m-1} = \{u \in L^m : \langle u, u\rangle = -1 \text{ and } t > 0\}, \tag{6.10}$$

where $L^m = E_1^m$ and $t = x_1$ is the first coordinate of E_1^m.

Let $f : M \to E_s^m$ be an isometric immersion of a pseudo-Riemannian manifold M into the pseudo-Euclidean m-space with index s. Then M is of 1-type if and only if either

(1) M is a minimal submanifold of E_s^m; or

(2) up to translations, M is a minimal submanifold of a pseudo-Riemannian sphere $S_s^{m-1}(r),\ r > 0$; or

(3) up to translations, M is a minimal submanifold of a pseudo-hyperbolic space $H_{s-1}^{m-1}(-r),\ r > 0$.

For 2- and 3-type hypersurfaces in hyperbolic space, the following are known (cf. [Chen 1996d]).

If M is a hypersurface of the hyperbolic space H^{n+1}, embedded standardly in E_1^{n+2}, then

(a) if M has constant mean curvature and constant scalar curvature, M is either of 1-type or of 2-type;

(b) every 2-type hypersurface of the hyperbolic space H^{n+1} has constant mean curvature and constant scalar curvature.

Furthermore, we have

(c) there do not exist compact 2-type hypersurfaces in the hyperbolic space;

(d) there do not exist null 2-type hypersurfaces in the hyperbolic space;

(e) there do not exist 3-type hypersurfaces with constant mean curvature in the hyperbolic space;

(f) if M is a 2-type hypersurface with at most two distinct principal curvatures in the hyperbolic space, then up to rigid motions of H^{n+1}, M is an open portion of $M_{k,r}^n$ for positive integer $k, 2 \le k \le n$ and for some $r > 0$, where $M^n_{k,r}$ is defined by

$$M^n_{k,r} = \{(t, x_2, \dots, x_{n+2}) : t^2 - x_2^2 - \dots - x_k^2 = 1 + r^2, \\ x_{k+1}^2 + \dots + x_{n+2}^2 = r^2\}. \tag{6.11}$$

(g) if M is a 2-type surface in the hyperbolic 3-space H^3, then it is a flat surface and, up to rigid motions of H^3, M is an open portion of $M^2_{2,r}$ for some $r > 0$.

6.4.7. Finite type immersions of irreducible homogeneous spaces

Every finite type isometric immersion of a compact irreducible homogeneous Riemannian manifold is a screw diagonal immersion, that is, it is the composition of a diagonal immersion followed by a linear map [Chen 1996d, Deprez 1988].

J. Deprez (1988) and T. Takahashi (1988) proved that every equivariant isometric immersion of a compact irreducible homogeneous Riemannian manifold in Euclidean space is the diagonal immersion of some standard 1-type isometric immersions.

7. Isometric immersions between real space forms

Historically the study of surfaces of negative constant curvature in E^3 was closely related with the problem of interpretation of non-Euclidean geometry. During 1839-1840 F. Minding investigated properties of surfaces of constant negative Gaussian curvature in E^3, he discovered the so called helical surfaces of constant curvature. Minding showed that surfaces of revolutions of constant negative curvature in E^3 can be divided into three types; one of them is known as the pseudo-sphere; the surface obtained by rotating the asymptote the so-called curve of pursuit. In 1868 E. Beltrami (1835–1900) discovered a close connection between hyperbolic geometry and the pseudo-sphere.

In 1901 D. Hilbert proved that any complete immersed surface in E^3 with constant positive Gaussian curvature is a round sphere. The analytic case of this result was given by H. Liebmann in 1900. A result of D. Hilbert and E. Holmgren states that there do not exist complete immersed surfaces in E^3 with constant negative Gaussian curvature in E^3. A. W. Pogorelov (1956a,1956b) and P. Hartman and L. Nirenberg (1959) showed that a complete flat surface immersed in E^3 is a generalized cylinder.

Isometric immersions of a Riemannian n-manifold $R^n(c)$ with constant curvature c into another Riemannian $(n+p)$-manifold $R^{n+p}(\bar{c})$ of constant curvature $\bar{c}$ were studied by É. Cartan (1919). He proved, for example, that if $c < \bar{c}$, then the existence of an isometric immersion implies $p \geq n-1$.

Cartan's study was confined to local phenomena of isometric immersions, which are generally complicated. However, sometimes, definite results can be obtained under a global assumption of completeness or compactness.

7.1. **Case: $c = \bar{c}$.** The following global results for $c = \bar{c}$ and $p = 1$ have been known.

7.1.1. Euclidean case: $(c = \bar{c} = 0)$

If a complete flat Riemannian n-manifold $R^n(0)$ is isometrically immersed in a Euclidean $(n+1)$-space E^{n+1}, then $R^n(0)$ is a hypercylinder over a plane curve, that is, $R^n(0) = E^{n-1} \times C$, where E^{n-1} is a Euclidean $(n-1)$-subspace of E^{n+1} and C is a curve lying in a Euclidean plane perpendicular to E^{n-1}. This result was proved in [Hartman-Nirenberg 1959]. **7.1.2. Spherical case: $(c = \bar{c} = 1)$**

If the unit n-sphere S^n is isometrically immersed in S^{n+1}, then S^n is embedded as a great n-sphere in S^{n+1}. This result is a special case of a theorem in [O'Neill-Stiel 1963].

7.1.3. Hyperbolic case: $(c = \bar{c} = -1)$

In this case the situation is more complicated. K. Nomizu (1924–) exhibited three distinct types of isometric immersions of hyperbolic plane $H^2(-1)$ into the hyperbolic 3-space $H^3(-1)$ which are not totally geodesic [Nomizu 1973a.

If an isometric immersion $f : H^n(-1) \to H^{n+1}(-1)$ has no umbilical points, the second fundamental form has nullity $n-1$ at every point. In this case, the relative nullity distribution is integrable and each leaf is an $(n-1)$-dimensional complete totally geodesic submanifold.

In general, a k-dimensional foliation on a Riemannian manifold is called totally geodesic if each leaf is a k-dimensional complete totally geodesic submanifold. D. Ferus (1973) has showed how to obtain all $(n-1)$-dimensional totally geodesic foliations on the hyperbolic space $H^n(-1)$ and that every totally geodesic foliation on $H^n(-1)$ is the nullity distribution of a suitable isometric immersion of $H^n(-1)$ into $H^{n+1}(-1)$ with umbilical points.

Recently, K. Abe, H. Mori and H. Takahashi (1997) parametrized the space of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ by a family of properly chosen (at most) countable n-tuples of real-valued functions.

Concerning the problem of describing isometric immersions of the hyperbolic plane $H^2(-1)$ into the hyperbolic 3-space $H^3(-1)$, by using the stereographic projection of the upper sheet of the hyperboloid $x^2+y^2-z^2=-1$ from the origin on the plane $z=1$, it is possible to parameterize $H^2(-1)$ by points $\xi=(\xi_1,\xi_2)\in D$ of the open unit disc. In this way, the problem of determining isometric immersions $f\colon H^2(-1)\to H^3(-1)$ is reduced to solving a degenerate Monge-Ampere equation on the unit disc: $\det(\partial^2 u/\partial\xi_i\partial\xi_j)=0$, $\xi\in D$, where each isometric immersion corresponds to a solution $u(\xi_1,\xi_2)$ of such an equation.

Using a special family of solutions of the Monge-Ampere equation, Z. J. Hu and G. S. Zhao (1997a) gave a way of constructing many incongruent examples of immersions $f\colon H^2(-1)\to H^3(-1)$ with or without an umbilic set and with bounded or unbounded principal curvature.

7.2. **Case:** $c\neq\bar{c}$**.** For $c\neq\bar{c}$ a classical result of K. Liebmann and D. Hilbert states that a complete Riemannian 2-manifold of constant negative curvature cannot be isometrically immersed in Euclidean 3-space. L. Bianchi proved in 1896 that there exist infinitely many isometric immersions from a complete flat surface into S^3 (cf. [Bianchi 1903]). Ju. A. Volkov and S. M. Vladimirova in 1971 and S. Sasaki (1912–1987) in 1973 showed that an isometric immersion of a complete flat surface into a complete simply-connected $H^3(-1)$ is either a horosphere or a set of points at a fixed distance from a geodesic.

An isometric immersion of a Riemannian n-manifold $R^n(c)$ with $n \geq 3$ into a Riemannian $(n+1)$-manifold $R^{n+1}(\bar{c})$ with $c > \bar{c}$ is totally umbilical. If $p \leq n-1$, Ferus (1975) proved that every isometric immersion of a complete Riemannian n-manifold $R^n(c)$ into a complete Riemannian $(n+p)$-manifold $R^{n+p}(c)$ with the same constant curvature is totally geodesic.

D. Blanusa (1955) proved that a hyperbolic n-space $H^n(-1)$ can be isometrically embedded into E^{6n-5}. On the other hand, J. D. Moore (1972) proved that if $p \leq n-1$, then there do not exist isometric immersions from a complete Riemannian n-manifold $R^n(c)$ of constant curvature c into a complete simply-connected Riemannian manifold $R^{n+p}(\bar{c})$ with $\bar{c} > c > 0$.

By applying Morse theory, Moore (1977) proved that if a compact Riemannian n-manifold $R^n(1)$ of constant curvature 1 admits an isometric immersion in E^N with $N \leq \frac{3}{2}n$, then $R^n(1)$ is simply-connected, hence isometric to $S^n(1)$.

Flat surfaces in E^4 with flat normal connection were classified in [Dajczer-Tojeiro 1995a].

D. Ferus and F. Pedit (1996) gave a method for finding local isometric immersions between real space forms by integrable systems techniques. The idea is that, given an isometric immersion between real space forms of nonzero different curvatures, the structural equations can be rewritten as a zero curvature equation involving an auxiliary (spectral) parameter, that is, as the flatness equation for a 1-form with values in a loop algebra. The isometric immersion thus generates a one-parameter family of isometric immersions with flat normal bundle. A large class of solutions to the flatness equation can then be found by integrating certain commuting vector fields on a loop algebra.

The isometric immersions so constructed are real-analytic and depend on the same number of functions as predicted by Cartan-Kähler theory, though not all isometric immersions are real-analytic.

8. Parallel submanifolds

The first fundamental form, that is, the metric tensor, of a submanifold of a Riemannian submanifold is automatically parallel, thus, $\nabla g \equiv 0$ with respect to the Riemannian connection ∇ on the tangent bundle TM. A Riemannian submanifold is said to be parallel if its second fundamental form h is parallel, that is $\bar{\nabla} h \equiv 0$ with respect to the connection $\bar{\nabla}$ on $TM \oplus T^{\perp}M$.

8.1. **Parallel submanifolds in Euclidean space.** The first result on parallel submanifolds was given by V. F. Kagan in 1948 who showed that the class of parallel surfaces in E^3 consists of open parts of planes, round spheres, and circular cylinders $S^1 \times E^1$. U. Simon and A. Weinstein (1969) determined parallel hypersurfaces of Euclidean $(n+1)$-space. A general classification theorem of parallel submanifolds in Euclidean space was obtained by D. Ferus in 1974.

An affine subspace of E^m or a symmetric R-space $M \subset E^m$, which is minimally embedded in a hypersphere of E^m as described in [Takeuchi-Kobayashi 1965] is a parallel submanifold of E^m. The class of symmetric R-spaces includes:

(a) all Hermitian symmetric spaces of compact type,

(b) Grassmann manifolds $O(p+q)/O(p) \times O(q), Sp(p+q)/Sp(p) \times Sp(q)$,

(c) the classical groups $SO(m)$, $U(m)$, $Sp(m)$,

(d) $U(2m)/Sp(m)$, $U(m)/O(m)$,

(e) $(SO(p+1) \times SO(q+1))/S(O(p) \times O(q))$, where $S(O(p) \times O(q))$ is the subgroup of $SO(p+1) \times SO(q+1)$ consisting of matrices of the form

$$\begin{pmatrix} \epsilon & 0 & & \\ 0 & A & & \\ & & \epsilon & 0 \\ & & 0 & B \end{pmatrix}, \quad \epsilon = \pm 1, \quad A \in O(p), \quad B \in O(q),$$

(f) the Cayley projective plane $\mathcal{O}P^2$, and

(g) the three exceptional spaces $E_6/Spin(10) \times T, E_7/E_6 \times T$, and E_6/F_4.

D. Ferus (1974) proved that essentially these submanifolds exhaust all parallel submanifolds of E^m in the following sense: A complete full parallel submanifold of the Euclidean m-space E^m is congruent to

"(1)" $M = E^{m_0} \times M_1 \times \cdots \times M_s \subset E^{m_0} \times E^{m_1} \times \cdots \times E^{m_s} = E^m$, $s \geq 0$, or to

"(2)" $M = M_1 \times \cdots \times M_s \subset E^{m_1} \times \cdots \times E^{m_s} = E^m$, $s \geq 1$, where each $M_i \subset E^{m_i}$ is an irreducible symmetric R-space.

Notice that in case (1) M is not contained in any hypersphere of E^m, but in case (2) M is contained in a hypersphere of E^m.

For an n-dimensional submanifold $f : M \to E^m$, for each point $x \in M$ and each unit tangent vector X at x, the vector $f_*(X)$ and the normal space $T_x^\perp$ determine an $(m-n+1)$-dimensional subspace $E(x, X)$ of E^m. The intersection of $f(M)$ and $E(x, X)$ defines a curve γ in a neighborhood of $f(x)$, which is called the normal section of f at x in the direction X. A point p on a plane curve is called a vertex if its curvature function $\kappa(s)$ has a critical point at p.

Parallel submanifolds of E^m are characterized by the following simple geometric property: normal sections of M at each point $x \in M$ are plane curves with x as one of its vertices [Chen 1981a].

A submanifold $f : M \to E^m$ is said to be extrinsic symmetric if, for each $x \in M$, there is an isometry ϕ of M into itself such that $\phi(x) = x$ and $f \circ \phi = \sigma_x \circ f$, where σ_x denotes the reflection at the normal space $T_x^\perp M$ at x, that is the motion of E^m which fixes the space through $f(x)$ normal to $f_*(T_x M)$ and reflects $f(x) + f_*(T_x M)$ at $f(x)$. The submanifold $f : M \to E^m$ is said to be extrinsic locally symmetric, if each point $x \in M$ has a neighborhood U and an isometry ϕ of U into itself, such that $\phi(x) = x$ and $f \circ \phi = \sigma_x \circ f$ on U. In other words, a submanifold M of E^m is extrinsic locally symmetric if each point $x \in M$ has a neighborhood which is invariant under the reflection of E^m with respect to the normal space at x.

D. Ferus (1980) proved that extrinsic locally symmetric submanifolds of E^m have parallel second fundamental form and vice versa.

A canonical connection on a Riemannian manifold (M, g) is defined as any metric connection ∇^c on M such that the difference tensor $\hat{D}$ between ∇^c and the Levi-Civita connection ∇ is ∇^c-parallel.

An embedded submanifold M of E^m is said to be an extrinsic homogeneous submanifold with constant principal curvatures if, for any given $x, y \in M$ and a given piecewise differentiable curve γ from x to y, there exists an isometry φ of E^m satisfying (1) $\phi(M) = M$, (2) $\phi(x) = y$, and (3) $\phi_{*x|T_x^\perp M} : T_x^\perp M \to T_y^\perp M$ coincides with $\hat{D}$-parallel transport along γ.

C. Olmos and C. Sánchez (1991) extended Ferus' result and obtained the following: Let M be a connected compact Riemannian submanifold fully in E^m, and let h be its second fundamental form. Then the following three statements are equivalent:

(i) M admits a canonical connection ∇^c such that $\nabla^c h = 0$,

(ii) M is an extrinsic homogeneous submanifold with constant principal curvatures,

(iii) M is an orbit of an s-representation, that is, of an isotropy representation of a semisimple Riemannian symmetric space.

The notion of extrinsic k-symmetric submanifold of E^m was introduced and classified for odd k in [Sánchez 1985]. Furthermore, Sánchez (1992) proved that the extrinsic k-symmetric submanifolds are essentially characterized by the property of having parallel second fundamental form with respect to the canonical connection of k-symmetric space. Thus, the above result implies that every extrinsic k-symmetric submanifold of a Euclidean space is an orbit of an s-representation.

8.2. **Parallel submanifolds in spheres.** Regarding the unit $(m-1)$-sphere S^{m-1} as an ordinary hypersphere of E^m, a submanifold $M \subset S^{m-1}$ is parallel if and only if $M \subset S^{m-1} \subset E^m$ is a parallel submanifold of E^m.

Consequently, Ferus' result implies that M is a parallel submanifold of S^{m-1} if and only if M is obtained by a submanifold of type (2).

8.3. **Parallel submanifolds in hyperbolic spaces.** Parallel submanifolds of hyperbolic spaces were classified in 1981 by M. Takeuchi (1921–) which is given as follows: For each $c < 0$, let $H^m(c)$ denote the hyperbolic m-space defined by

$$H^m(c) = \{(x_0, \ldots, x_m) \in E^{m+1} : -x_0^2 + x_1^2 + \cdots + x_m^2 = 1/c, x_0 > 0\}.$$

Assume M is a parallel submanifold of $H^m(\bar{c})$, $\bar{c} < 0$. Then

(1) if M is not contained in any complete totally geodesic hypersurface of $H^m(\bar{c})$, then M is congruent to the product

$$H^{m_0}(c_0) \times M_1 \times \cdots \times M_s \subset H^{m_0}(c_0) \times S^{m-m_0-1}(c') \subset H^{m_0}(\bar{c})$$

with $c_0 < 0$, $c' > 0, 1/c_0 + 1/c' = 1/\bar{c}$, $s \geq 0$, where $M_1 \times \cdots \times M_s \subset S^{m-m_0-1}(c')$ is a parallel submanifold as described in Ferus' result; and

(2) if M is contained in a complete totally geodesic hypersurface N of $H^m(\bar{c})$, then N is either isometric to an $(m-1)$-sphere, or to a Euclidean $(m-1)$-space, or to a hyperbolic $(m-1)$-space. Hence, such parallel submanifolds reduce to the parallel submanifolds described before.

8.4. **Parallel submanifolds in complex projective and complex hyperbolic spaces.** A parallel submanifold M of a Riemannian manifold $\tilde{M}$ is curvature-invariant, that is, for each point $x \in M$ and $X, Y \in T_xM$, we have

$$\tilde{R}(X,Y)T_xM \subset T_xM,$$

where $\tilde{R}$ is the curvature tensor of $\tilde{M}$. Thus, according to a result of [Chen-Ogiue 1974b], parallel submanifolds of complex projective and complex hyperbolic spaces are either parallel Kähler submanifolds or parallel totally real submanifolds.

Complete parallel Kähler submanifolds of complex projective spaces and of complex hyperbolic spaces have been completely classified in [Nakagawa-Takagi 1976] and in [Kon 1974], respectively (see §15.9 for details).

[Naitoh 1981] showed that the classification of complete totally real parallel submanifolds in complex projective spaces is reduced to that of certain cubic forms of n-variables and [Naitoh-Takeuchi 1982] classified these submanifolds by the theory of symmetric bounded domains of tube type.

The complete classification of parallel submanifolds in complex projective spaces and in complex hyperbolic spaces was given in [Naitoh 1983].

8.5. **Parallel submanifolds in quaternionic projective spaces.** Parallel submanifolds of a quaternionic projective m-space or its non-compact dual were classified in [Tsukada 1985b]. Such submanifolds are parallel totally real submanifolds in a totally real totally geodesic submanifold RP^m, or parallel totally real submanifolds in a totally complex totally geodesic submanifold CP^m, or parallel complex submanifolds in a totally complex totally geodesic submanifold CP^m, or parallel totally complex submanifolds in a totally geodesic quaternionic submanifold HP^k whose dimension is twice the dimension of the parallel submanifold.

8.6. **Parallel submanifolds in the Cayley plane.** Parallel submanifolds of the Cayley plane $\mathcal{O}P^2$ are contained either in a totally geodesic quaternion projective plane HP^2 as parallel submanifolds or in a totally geodesic 8-sphere as parallel submanifolds [Tsukada 1985c].

All parallel submanifolds in E^m are of finite type. Furthermore, if a compact symmetric space N of rank one is regarded as a submanifold of a Euclidean space via its first standard embedding, then a parallel submanifold of N is also of finite type [Chen 1996d].

9. Standard immersions and submanifolds with simple geodesics

9.1. **Standard immersions.** Let $M = G/K$ be a compact irreducible homogeneous Riemannian manifold. For each positive eigenvalue λ of the Laplacian on M, we denote by m_λ the multiplicity of the eigenvalue λ. Let $\phi_1, \ldots, \phi_{m_\lambda}$ be an orthonormal basis of the eigenspace of the Laplacian with eigenvalue λ. Define a map $f_\lambda : M \to E^{m_\lambda}$ by

$$f_\lambda(u) = \frac{c_\lambda}{m_\lambda^2}(\phi_1(u), \ldots, \phi_{m_\lambda}(u)), \tag{7.1}$$

where c_λ is a positive number. The map f_λ defines an isometric minimal immersion of M into $S_0^{m_\lambda - 1}(1)$ for some suitable constant $c_\lambda > 0$.

According to a result of T. Takahashi (1966) each such f_λ is an isometric minimal immersion of M into a hypersphere of E^{m_λ}.

If λ_i is the i-th positive eigenvalue of Laplacian of M, then the immersion $\psi_i = f_{\lambda_i}$ is called the i-th standard immersion of $M = G/K$.

Every full isometric minimal immersion of a Riemannian n-sphere into a hypersphere of a Euclidean space is a standard immersion if either $n = 2$ or $n \geq 3$ and the order of the immersion is either $\{1\}, \{2\}$ or $\{3\}$.

Not every full isometric minimal immersion of a Riemannian n-sphere into a hypersphere is a standard immersion. For instance, N. Ejiri (1981) constructed a full minimal isometric immersion of $S^3\left(\frac{1}{16}\right)$ into $S^6(1)$ of order $\{6\}$, which is not a standard immersion. An explicit construction was given by K. Mashimo (1985) and by F. Dillen, L. Verstraelen and L. Vrancken (1990), who also showed that the immersion is a 24-fold cover onto its image. The image of this minimal immersion in $S^6(1)$ was identified in [DeTurck-Ziller 1992] as S^3/T^*, where T^* is the binary tetrahedral group of order 24.

According to a result of Moore (1972), the minimum number m for which $S^3(c)$ can admit a non-totally geodesic isometric minimal immersion into S^m is 6. D. DeTurck and W. Ziller (1992) showed that every non-totally geodesic $SU(2)$-equivariant minimal isometric immersion of $S^3(\frac{1}{16})$ into $S^6(1)$ is congruent to the immersion mentioned above.

9.2. **Submanifolds with planar geodesics.** A surface in E^3 whose geodesics are all planar curves is open portion of a plane or sphere. S. L. Hong (1973) was the first to ask for all submanifolds of Euclidean space whose geodesics are plane curves. He showed that if $f : M \to E^m$ is an isometric immersion which is not totally geodesic and such that for each geodesic γ in M, $f \circ \gamma$ is a plane curve in E^m, then $f \circ \gamma$ is a plane circle.

Let $f : M \to R^m(c)$ be an isometric immersion of a Riemannian manifold into a complete simply-connected real space form of constant curvature c. If the image of each geodesic of M is contained in a 2-dimensional totally geodesic submanifold of $R^m(c)$, then f is either a totally geodesic immersion, a totally umbilical immersion or a minimal immersion of a compact symmetric space of rank one by harmonic functions of degree 2.

The later case occurs only when $c > 0$ and in this case the immersions are the first standard embeddings of the real, complex and quaternionic projective spaces or the Cayley plane [Hong 1973, Little 1976, Sakamoto 1977].

9.3. **Submanifolds with pointwise planar normal sections.** Let M be an n-dimensional submanifold of a Euclidean m-space E^m. For a point x in M and a unit vector X tangent to M at x, the vector X and the normal space $T_x^\perp M$ to M at x determine an $(m-n+1)$-dimensional affine subspace $E(x,X)$ of E^m through x. The intersection of M and $E(x,X)$ gives rise to a curve γ in a neighborhood of x which is called the normal section of M at x in the direction X.

A submanifold M of E^m is said to have planar normal sections if each normal section $\gamma_X(s)$ at x of M in E^m is a planar curve where s is an arclength parametrization of γ_X; thus the first three derivatives $\{\gamma_X'(s), \gamma_X''(s), \gamma_X'''(s)\}$ of $\gamma(s)$ are linearly dependent as vectors in E^m. Hypersurface of Euclidean space and Euclidean submanifolds with planar geodesics are examples of submanifolds with planar normal sections. Conversely, B. Y. Chen (1983a) proved that if a surface M of E^m has planar normal sections, then either it lies locally in an affine 3-space E^3 of E^m or it has planar geodesics. If the later case occurs, M is an open portion of a Veronese surface in an affine 5-space E^5 of E^m.

A submanifold M of E^m is said to have pointwise planar normal sections if, for each normal section γ at x, $x \in M$, the three vectors $\{\gamma'(0), \gamma''(0), \gamma'''(0)\}$ at x are linearly dependent. Clearly, every hypersurface of E^{n+1} has planar normal sections, and hence has pointwise planar normal sections. A submanifold M of E^m is called spherical if M is contained in a hypersphere of E^m.

In 1982 B. Y. Chen proved that a spherical submanifold of a Euclidean space has pointwise planar normal sections if and only if it has parallel second fundamental form. K. Arslan and A. West (1996) showed that if an n-dimensional submanifold M of a Euclidean m-space E^m has pointwise planar normal sections and does not have parallel second fundamental form, then locally it must lies in an affine $(n+1)$-space E^{n+1} of E^m as a hypersurface, that is, for each point $x \in M$, there exists a neighborhood U of x such that U is contained in an affine $(n+1)$-space E^{n+1} of E^m.

W. Dal Lago, A. Garc´ia, and C. U. Sánchez (1994) studied the set $\mathcal{X}[M]$ of pointwise planar normal sections on the natural embedding of a flag manifold M and proved that it is a real algebraic submanifold of the real projective n-space RP^n with $n = \dim M$. They also computed the Euler characteristic of $\chi[M]$ and its complexification $\chi_c[M]$ and showed that the Euler characteristics of $\chi[M]$ and of $\chi_c[M]$ depend only on the dimension of M and not on the nature of M itself.

9.4. **Submanifolds with geodesic normal sections and helical immersions.** A submanifold $f : M \to E^m$ is said to have geodesic normal sections if, for each point $x \in M$ and each unit tangent vector X at x, the image of the geodesic γ_X with $\gamma_X'(0) = X$ is the normal section of f at x in the direction X.

Submanifolds in Euclidean space with planar geodesics also have geodesic normal sections.

Chen and Verheyen (1981) asked for all submanifolds of Euclidean space with geodesic normal sections. They proved that a submanifold M of E^m has geodesic normal sections if and only if all normal sections of M, considered as curves in E^m, have the same constant first curvature κ_1; also if and only if every curve γ of M which is a normal section of M at $\gamma(0)$ in the direction $\gamma'(0)$ remains a normal section of M at $\gamma(s)$ in the direction $\gamma'(s)$, for all s in the domain of γ.

In particular, Chen and Verheyen's result implies that if a compact symmetric space, isometrically immersed in Euclidean space, has geodesic normal sections, then it is of rank one.

Chen and Verheyen also proved the following results:

(1) If M is a submanifold of E^m all of whose geodesics are 3-planar, that is, each geodesic lies in some 3-plane, then M has geodesic normal sections if and only if it is isotropic; and

(2) if M is a submanifold of Euclidean space all of whose geodesics are 4-planar, then M has geodesic normal sections if and only if it is constant isotropic.

Recall that a submanifold M of a Riemannian manifold $\tilde{M}$ is called isotropic if, for each point $x \in M$ and each unit vector $X \in T_xM$, the length $|h(X,X)|$ of $h(X,X)$ depends only on x and not on the unit vector X. In other words, each geodesic of M emanating from x, considered as a curve in $\tilde{M}$, has the same first curvature κ_1 at x. In particular, if the length of $h(X,X)$ is also independent of the point $x \in M$, then M is called constant isotropic.

Chen and Verheyen (1981) showed that submanifolds in Euclidean space with geodesic normal sections are constant isotropic.

Let M be a compact Riemannian manifold. It has a unique kernel of the heat equation: $K : M \times M \times \mathbf{R}_+^* \to \mathbf{R}$. If there exists a function $\Psi : \mathbf{R}_+ \times \mathbf{R}_+^* \to \mathbf{R}$ such that $K(u,v,t) = \Psi(d(u,v),t)$ for every $u, v \in M$ and $r \in \mathbf{R}_+^*$, then M is called strongly harmonic.

Related to submanifolds with planar geodesics and to submanifolds with geodesic normal sections is the notion of helical immersions. An isometric immersion $f : M \to E^m$ is called a helical immersion if each geodesic γ of M is mapped to a curve with constant Frenet curvatures, that is, to a W-curve, which are independent of the chosen geodesic.

A. Besse (1978) constructed helical immersions of strongly harmonic manifolds into a unit sphere. Conversely, K. Sakamoto (1982) proved that if a complete Riemannian manifold admits a helical minimal immersion into a hypersphere of E^m, then M is a strongly harmonic manifold.

Y. Hong (1986) proved that every helical immersion of a compact homogeneous Riemannian manifold into Euclidean space is spherical. Hong also proved that every helical immersion of a compact rank one symmetric space into a Euclidean space is a diagonal immersion of some 1-type standard isometric immersions.

Chen and Verheyen (1984) showed that a helical submanifold of Euclidean space is a submanifold with geodesic normal sections. Conversely, P. Verheyen (1985) proved that every submanifold of Euclidean space with geodesic normal sections is a helical submanifold.

Helical submanifolds were further investigated by K. Sakamoto, B. Y. Chen, P. Verheyen, Y. Hong, C.-S. Houh, K. Mashimo, K. Tsukada, H. Nakagawa, and others.

9.5. **Submanifolds whose geodesics are generic W-curves.** A W-curve $\gamma : \mathbf{R} \supset I \to E^N$ is said to be of rank r, if for all $t \in I$ the derivatives $\gamma'(t), \dots, \gamma^{(r)}(t)$ are linearly independent and the derivatives $\gamma'(t), \dots, \gamma^{(r+1)}(t)$ are linearly dependent. Let $\gamma : \mathbf{R} \supset I \to E^N$ be a W-curve of infinite length, parametrized by arc length. If the image $\gamma(I)$ is bounded, then the rank of γ is even, say $r = 2k$. There are positive constants $a_1, \dots, a_k$, unique up to order, corresponding positive constants $r_1, \dots, r_k$ and orthonormal vectors

$e_1, \ldots, e_{2k}$ in E^N such that

$$\gamma(t) = c + \sum_{i=1}^{k} r_i(e_{2i-1} \sin a_i t + e_{2i} \cos a_i t),$$

where c is a constant vector.

The rank of unbounded W-curves is odd and the expression of $\gamma(t)$ contains an additional term linear in t. A W-curve γ is called a generic W-curve if the a_i are independent over the rationals, that is, if the closure of $\gamma(\mathbf{R})$ is a standard torus $S^1(r_1) \times \cdots \times S^1(r_k)$ up to a motion.

D. Ferus and S. Schirrmacher (1982) proved that if $f : M \to E^m$ is an isometric immersion of a compact Riemannian manifold into E^m, then f is extrinsic symmetric if and only if, for almost every geodesic γ in M, the image $f(\gamma)$ is a generic W-curve.

The above result is false if the condition on $f(\gamma)$ were replaced by the condition: for each geodesic γ in M, the image $f(\gamma)$ is a W-curve.

For a compact Riemannian 2-manifold M, D. Ferus and S. Schirrmacher (1982) proved that if $f : M \to E^4$ is an isometric immersion such that, for every geodesic γ in M, the image $f(\gamma)$ is a W-curve, then either one of the following holds:

(1) if M contains a non-periodic geodesic, f covers (up to a motion) a standard torus $S^1(r_1) \times S^2(r_2) \subset E^4$, or

(2) if all geodesics in M are periodic, f is (up to a motion) an isometry onto a Euclidean 2-sphere $S^2(r) \subset E^3 \subset E^4$.

Y. H. Kim and E. K. Lee (1993) proved the following: Let M be a complete surface in E^4. If there is a point $x \in M$ such that every geodesic through x, considered as a curve in E^4, is a W-curve, then M is an affine 2-space, a round sphere or a circular cylinder in an affine 3-space, a product of two plane circles, or a Blaschke surface at a point $o \in E^4$, diffeomorphic to a real projective plane, and up to a motion, it is immersed in E^4 by

$$\left(\frac{1}{\kappa} \sin \kappa s \cos \theta, \frac{1}{\kappa} \sin \kappa s \sin \theta, \frac{1}{\kappa}(1 - \cos \kappa s) \cos 2\theta, \frac{1}{\kappa}(1 - \cos \kappa s) \sin 2\theta\right),$$

where κ is the Frenet curvature of geodesic through o. The converse is also true.

9.6. **Symmetric spaces in Euclidean space with simple geodesics.** Submanifolds in Euclidean space with finite type geodesics were studied by Chen, Deprez and Verheyen (1987). They proved the following results:

(a) An isometric immersion of a compact symmetric space M into a Euclidean space is of finite type if and only if each geodesic of M is mapped into curves of finite type;

(b) if $f : S^n \to E^m$ is an isometric immersion of a unit n-sphere into E^m, then the immersion maps all geodesics of S^n into 1-type or 2-type curves if and only if the immersion is of finite type with order $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}$ or $\{2, 4\}$;

(c) if $f : FP^n \to E^m$ is an isometric immersion of a real, complex, or quaternionic projective space, or the Cayley plane into E^m, then the immersion maps all geodesics of FP^n into 1-type or 2-type curves if and only if the immersion is of finite type with order $\{1\}, \{2\}$ or $\{1, 2\}$;

(d) a finite type isometric immersion of a unit n-sphere in E^m of order $\{1, 2\}$ is a diagonal immersion of the first and the second standard immersions of the n-sphere;

(e) a finite type isometric immersion of a unit n-sphere in E^m of order $\{1, 3\}$ for which all geodesics are mapped to W-curves is a diagonal immersion of the first and the third standard immersions of the n-sphere; and

(f) there exist finite type isometric immersions of the unit 2-sphere of order $\{1, 3\}$ or of order $\{2, 4\}$ which are not diagonal immersions.

Chen, Deprez and Verheyen (1987) also studied an isometric immersion $f : M \to E^m$ which satisfies the condition: there is a point $x_0 \in M$ such that every geodesic through x_0 is mapped to a circle.

They proved the following results:

(g) Let $f : S^n \to E^m$ be an isometric embedding. If there exists a point $x_0 \in S^n$ such that f maps all geodesics of S^n through x_0 to 1-type curves, then the embedding is the first standard embedding of S^n into a totally geodesic E^{n+1};

(h) Let $f : FP^n \to E^m$ be a finite type isometric immersion. If there exists a point $x_0 \in FP^n$ such that the immersion maps all geodesics of FP^n through x_0 to 1-type curves, then the immersion is the first standard embedding of FP^n; and

(i) up to motions, the set of isometric immersions of a projective n-space $FP^n\,(F = R, C$ or $H)$ into Euclidean m-space which map all geodesics through a fixed point $x_o \in FP^n$ to circles is in one-to-one correspondence with the set of isometric immersions of FP^{n-1} into S^{m-dn-1}, where d is 1, 2 or 4, according to the field F is real, complex or quaternion.

10. Hypersurfaces of real space forms

Complete simply-connected Riemannian n-manifolds of constant curvature are frame homogeneous, that is, for any pair of points x and y and any orthonormal frames u at x and v at y there is an isometry ϕ such that $\phi(x) = y$ and ϕ_* maps u onto v. Such Riemannian manifolds are Euclidean n-space E^n, Riemannian n-spheres and real hyperbolic n-spaces.

Consider an isometrically immersed orientable hypersurface M in a complete simply-connected real space form $R^{n+1}(c)$ of constant curvature c with a unit normal vector field ξ. We simply denote the shape operator A_ξ at ξ by A.

Let

$$\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$$

denote the n eigenvalues of A at each point x of M. Then each κ_i, $(1 \leq i \leq n)$, is a continuous function on M, and is called a principal curvature of M.

For each $x \in M$ and each $\kappa \in \{\kappa_1, \dots, \kappa_n\}$, we define a subspace $\mathcal{D}_x(\kappa)$ of T_xM by

$$\mathcal{D}_x(\kappa) = \{X \in T_xM : AX = \kappa(x)X\}. \tag{10.1}$$

Let $\mathcal{D}(\kappa)$ assign each point $x \in M$ the subspace $\mathcal{D}_x(\kappa)$.

The following basic results are well-known: If $\dim \mathcal{D}_x(\kappa)$ is constant on M, say m, then

(1) κ is a differentiable function on M;

(2) $\mathcal{D}(\kappa)$ is a differentiable distribution on M;

(3) $\mathcal{D}(\kappa)$ is completely integrable, called the principal foliation;

(4) If $m \geq 2$, then κ is constant along each leaf of $\mathcal{D}(\kappa)$;

(5) If κ is constant along a leaf L of $\mathcal{D}(\kappa)$, then L is locally an m-sphere of $R^{n+1}(c)$, where an m-sphere means a hypersphere of an $(m+1)$-dimensional totally geodesic submanifold of $R^{n+1}(c)$.

Suppose that a continuous principal curvature κ has constant multiplicity m on an open subset $U \subset M$. Then κ and its principal foliation $\mathcal{D}(\kappa)$ are smooth on U. The leaves of this principal foliation are the curvature surfaces corresponding to κ on U. The principal curvature κ is constant along each of its curvature surfaces in U if and only if these curvature surfaces are open subsets of m-dimensional Euclidean spheres or planes. The focal map f_κ corresponding to κ is the map which maps $x \in M$ to the focal point $f_\kappa(x)$ corresponding to κ, that is,

$$f_\mu(x) = f(x) + \frac{1}{\mu(x)}\xi(x),$$

where ξ is a unit normal vector field. The principal curvature κ is constant along each of its curvature surfaces in U if and only if the focal map f_μ factors through an immersion of the $(n-1-m)$-dimensional space of leaves $M/\mathcal{D}(\kappa)$ into E^n.

10.1. **Einstein hypersurfaces.** A Riemannian manifold is said to be Einstein if the Ricci tensor is a constant multiple of the metric tensor, that is $Ric = \rho g$, where ρ is a constant.

A. Fialkow (1938) classified Einstein hypersurfaces in real space forms.

Let M^n, $n > 2$, be an Einstein hypersurface in $R^{n+1}(\bar{c})$. Then

(a) if $\rho > (n-1)\bar{c}$, then M is totally umbilical, hence, M is also a real space form;

(b) if $\rho = (n-1)\bar{c}$, then the type number, $t(x) = \operatorname{rank} A_x$, is ≤ 1 for all $x \in M$ and M is of constant curvature $\bar{c}$; and

(c) if $\rho < (n-1)\bar{c}$, then $\bar{c} > 0$, $\rho = (n-2)\bar{c}$, and M is locally a standard product embedding of $S^p\left(\left(\frac{n-2}{p-1}\right)\bar{c}\right) \times S^{n-p}\left(\left(\frac{n-2}{n-p-1}\right)\bar{c}\right)$, where $1 < p < n-1$.

In particular, complete Einstein hypersurfaces of E^{n+1} are hyperspheres, hypercylinders over complete plane curves and hyperplanes; and complete Einstein hypersurfaces in S^{n+1} are the small hyperspheres, the great hyperspheres and certain standard product embedding of products of spheres.

P. Ryan (1971) studied hypersurfaces of real space forms with parallel Ricci tensor and proved that if a hypersurface M of dimension > 2 in a real space form of constant curvature c is not of constant curvature c and if it has parallel Ricci tensor, then M has at most two distinct principal curvatures. Furthermore, if $c \neq 0$, then both principal curvatures are constant.

10.2. **Homogeneous hypersurfaces.** Let M be a homogeneous Riemannian n-manifold isometrically immersed into an $(n+1)$-dimensional complete simply-connected real space form $R^{n+1}(c)$. Then

(1) if $c = 0$, then M is isometric to the hypercylinder $S^k \times E^{n-k}$ [Kobayashi 1958, Nagano-Takahashi 1960];

(2) if $c > 0$, then M is represented as an orbit of a linear isotropy group of a Riemannian symmetric space of rank 2; and M is isometric to E^2 or else is given as an orbit of a subgroup of the isometry group of $R^{n+1}(c)$ [Hsiang-Lawson 1971]; and

(3) if $c < 0$, then M is isometric to a standard product embedding of E^n, $S^k \times H^{n-k}$, or a 3-dimension group manifold

$$B = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbf{R}$$

with the metric $ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2$ [Takahashi 1971].

Each of the hypersurfaces above except E^2 in (2) and B in (3) is given as an orbit of a certain subgroup of the isometry group of $R^{n+1}(c)$.

10.3. **Isoparametric hypersurfaces.** The history of isoparametric hypersurfaces can be traced back to 1918 of the work of E. Laura (1918) and of C. Somigliana (1918) on geometric optics. T. Levi-Civita (1873–1941) and B. Segre (1903–1977) studied such hypersurfaces of Euclidean space during the period of 1924–1938. A major progress on isoparametric hypersurfaces was made by É. Cartan during the period of 1938–1940.

A hypersurface M of a Riemannian manifold $\tilde{M}$ is called isoparametric if M is locally defined as the level set of a function ϕ on (an open set of) $\tilde{M}$ with property: $d\phi \wedge d||d\phi||^2 = 0$ and $d\phi \wedge d(\Delta\phi) = 0$.

A hypersurface M of a complete simply-connected Riemannian manifold $R^{n+1}(c)$ of constant curvature c is isoparametric if and only if M has constant principal curvatures. Each isoparametric hypersurface of $R^{n+1}(c)$ determines a unique complete embedded isoparametric hypersurface in $R^{n+1}(c)$. Thus, every open piece of an isoparametric hypersurface can be extended to a unique complete isoparametric hypersurface.

10.3.1. Isoparametric family of hypersurfaces

Let $f : M \to R^{n+1}(c)$ be a hypersurface and ξ is a unit normal vector field of f. For each $t > 0$, let $f_t(x)$, $x \in M$, be the point of $R^{n+1}(c)$ on the geodesic from $f(x)$ starting in the direction ξ at x which has geodesic distance t from $f(x)$. In the Euclidean case ($c = 0$), we have $f_t(x) = f(x) + t\xi_x$. In the spherical case ($c = 1$), we have $f_t(x) = (\cos t)f(x) + (\sin t)\xi_x$ by considering S^{n+1} as the unit sphere in E^{n+2}. In all cases, f_t is an immersion of M in $R^{n+1}(c)$ for sufficiently small t.

In $R^{n+1}(c)$ an isoparametric family of hypersurfaces is a family of hypersurfaces $f_t : M \to R^{n+1}(c)$ obtained from a hypersurface $f : M \to R^{n+1}(c)$ with constant principal curvatures.

10.3.2. Cartan's basic identity

The starting point of Cartan's work on isoparametric hypersurfaces is the following basic identity concerning all the distinct constant principal curvatures $a_1, \dots, a_g$ with their respective multiplicities $\nu_1, \dots, \nu_g$.

For $g \geq 2$, Cartan's basic identity is the following: For each i, $1 \leq i \leq g$, we have

$$\sum_{j \neq i} \nu_j \frac{c + a_i a_j}{a_i - a_j} = 0, \quad 1 \leq j \leq g. \tag{10.2}$$

Cartan's basic identity holds for all c, positive, negative and zero.

K. Nomizu (1975) observed that the focal set of an isoparametric hypersurface in a real space form admits a submanifold structure locally, and that the mean curvature vector of this submanifold at $f_t(x)$ can always be expressed as a multiple of ξ_x. Moreover, if $f_t(x)$ is a focal point corresponding to a principal curvature κ, then the multiple is the left hand side of (10.2). Thus, Cartan's identity is equivalent to the minimality of the focal varieties.

10.3.3. Isoparametric hypersurfaces in Euclidean space

In the Euclidean case ($c = 0$), Cartan's identity implies $g \leq 2$. If $g = 2$, one of the principal curvatures must be 0.

Isoparametric hypersurfaces of a Euclidean $(n+1)$-space E^{n+1} are locally hyperspheres, hyperplanes or a standard product embedding of $S^k \times E^{n-k}$.

This result was proved in 1937 by T. Levi-Civita for $n = 2$ and in 1938 by B. Segre for arbitrary n.

10.3.4. Isoparametric hypersurfaces in hyperbolic space

Cartan's basic identity also yields $g \leq 2$ in the case $c < 0$. In the hyperbolic space $H^{n+1}(-1)$, É. Cartan proved that the number of the distinct principal curvatures of an isoparametric hypersurface is one or two. An isoparametric hypersurface in hyperbolic space is totally umbilical or locally a standard product embedding of $S^k \times H^{n-k}$.

10.3.5. Isoparametric hypersurfaces in spheres

Cartan's basic identity does not restrict g if $c > 0$. In fact, let θ be any number such that $\sin \theta \neq 0$ and let g be any positive integer. Set

$$\lambda_k = \cot \left(\theta + \frac{k-1}{g} \pi \right)$$

for $k = 1, \dots, g$. Then such a collection of λ_k with equal multiplicities satisfies (10.2).

One major part of Cartan's work on isoparametric hypersurfaces is to give an algebraic method of finding an isoparametric family of hypersurfaces in S^{n+1} with g distinct principal curvatures with the same multiplicity. His result is that such a family is defined by

$$M_t = \{x \in S^{n+1} : \phi(x) = \cos gt\},$$

where ϕ is a harmonic homogeneous polynomial of degree g on E^{n+2} satisfying

$$||d\phi||^2 = g^2(x_1^2 + \cdots + x_{n+2}^2)^{g-1},$$

where $(x_1, \cdots, x_{n+2})$ are the Euclidean coordinates on E^{n+2}.

In S^{n+1} an isoparametric hypersurface with $g = 2$ is locally a standard product embedding of the product of two spheres with appropriate radii.

Cartan (1939a) proved that if $g = 3$, then $n = 3, 6, 12$, or 24; and the multiplicities are the same in each case. Moreover, the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of a projective plane FP^2 into S^{3m+1}, where F is the division algebra of reals, complex numbers, quaternions, or Cayley numbers for the (common) multiplicity $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m. These isoparametric hypersurfaces with three principal curvatures are known as Cartan hypersurfaces. If $g = 4$ and if the multiplicities are equal, then $n = 4$ or 8.

For each of the above cases, Cartan gave an example of a hypersurface with constant principal curvatures. All the compact isoparametric hypersurfaces in S^{n+1} given by Cartan are homogeneous; indeed, each of them is the orbit of a certain point by an appropriate closed subgroup of the isometry group of S^{n+1}.

H. F. Münzner(1980,1981) showed that a parallel family of isoparametric hypersurfaces in S^n always consists of the level sets in S^n of a homogeneous polynomial defined on E^{n+1}.

R. Takagi and T. Takahashi (1972) determined all the orbit hypersurfaces in S^{n+1}. They showed that the number of distinct principal curvatures of a homogeneous isoparametric hypersurfaces in a sphere is 1, 2, 3, 4, or 6, and they listed the possible multiplicities. The list of homogeneous isoparametric hypersurfaces of Takagi and Takahashi contains five different classes of orbit hypersurfaces in S^{n+1} each with four distinct principal curvatures not of the same multiplicity. It also contains an example with $g = 6$ and $m_1 = m_2 = 1$, and one with $g = 6$ and $m_1 = m_2 = 2$. H. Ozeki and M. Takeuchi (1975,1976) produced two infinite series of isoparametric families which are not on the list of Takagi and Takahashi and are, therefore, inhomogeneous. These examples all have four distinct principal curvatures.

H. F. Münzner (1980,1981) proved that isoparametric hypersurfaces in S^{n+1} with g distinct principal curvatures exist only when $g = 1, 2, 3, 4$, or 6.

Through a geometric study of the focal submanifolds and their second fundamental forms, Münzner showed that if M is an isoparametric hypersurface with principal curvatures $\cot\theta_i$, $0 < \theta_1 < \cdots < \theta_g < \pi$, with multiplicities m_i, then

$$\theta_k = \theta_1 + \frac{k-1}{g}\pi, \quad 1 \le k \le g,$$

and the multiplicities satisfy

$$m_i = m_{i+2} \quad \text{(subscripts mod } g\text{)}.$$

As a consequence, if g is odd, then all of the multiplicities must be equal; and if g is even, then $m_1 = m_3 = \cdots = m_{g-1}$ and $m_2 = m_4 = \cdots = m_g$.

Moreover, H. F. Münzner (1981) proved that if a hypersurface M in S^{n+1} splits S^{n+1} into two disks bundles D_1 and D_2 over compact manifolds with fibers of dimensions $m_1 + 1$ and $m_2 + 1$ respectively, then $\dim H^*(M; \mathbf{Z}_2) = 2h$, where h is $1, 2, 3, 4$ or 6. In the case that M is an isoparametric hypersurface, M splits S^{n+1} into two disk bundles such that the numbers m_1 and m_2 coincide with the multiplicities of the principal curvatures of M.

Although isoparametric hypersurfaces with four principal curvatures have not been completely classified, there is a large class of examples due to D. Ferus, H. Karcher and Münzner (1981). In fact, Ferus, Karcher and Münzner constructed isoparametric hypersurfaces with four distinct principal curvatures using representations of Clifford algebras, which include all known examples, except two. They are able to show geometrically that many of their examples are not homogeneous. The isoparametric hypersurfaces which belong to the Clifford series discovered by Ferus, Karcher and Münzner are the regular level sets of an isoparametric function on S^{2k-1} determined by an orthogonal representation of the Clifford algebra C_{m-1} on E^k.

J. Dorfmeister and E. Neher (1983) extended the work of Ferus, Karcher and Münzner to a more algebraic setting involving triple systems. The topology of isoparametric hypersurfaces of the Clifford examples have been studied by Q. M. Wang (1988). Among others he showed that there exist noncongruent hypersurfaces in two different isoparametric families which are diffeomorphic.

S. Stolz (1997) proved that the only possible triples (g, m_1, m_2) with $g = 4$ are exactly those that appear either in the homogeneous examples or those appear in the Clifford examples of Ferus, Karcher and Münzner.

Münzner (1981) proved that $m_1 = m_2$ always hold in the case $g = 6$. U. Abresch (1983) showed that the common multiplicity of m_1 and m_2 must be 1 or 2. Münzner's computation also yields that the focal submanifolds must be minimal.

J. Dorfmeister and E. Neher (1983) proved that every compact isoparametric hypersurface in the sphere with $g = 6$ and $m_1 = m_2 = 1$ must be homogeneous. In particular, an isoparametric hypersurface in S^7 with $g = 6$ is an orbit of the isotropy action of the symmetric space $G_2/SO(4)$, where G_2 is the automorphism group of the Cayley algebra. A geometric study of such isoparametric hypersurfaces in S^7 was made in [Miyaoka 1993]. In fact, Miyaoka has shown that a homogeneous isoparametric hypersurface M in S^7 can be obtained as the inverse image under the Hopf fibration $\psi : S^7 \to S^4$ of an isoparametric hypersurface with three principal curvatures of multiplicity one in S^4. She also showed that the two focal submanifolds of M are not congruent, even though they are lifts under ψ^{-1} of congruent Veronese surfaces in S^4. Thus, these focal submanifolds are two noncongruent minimal taut homogeneous embeddings of $RP^2 \times S^3$ in S^7. Fang (1995) studied the topology of isoparametric hypersurfaces with six principal curvatures. Peng and Hou (1989) gave explicit forms for the isoparametric polynomials of degree six for the homogeneous isoparametric hypersurfaces with $g = 6$. Recently, R. Miyaoka (1998) proves that all isoparametric hypersurfaces in the sphere with $g = 6$ are homogeneous.

A smooth immersion $f : M \to E^m$ from a compact manifold M into a Euclidean m-space is called taut if every nondegenerate Euclidean squared distance function has the minimum number of critical points. T. E. Cecil and P. Ryan (1979a) showed that all isoparametric hypersurfaces and their focal submanifolds in the spheres are taut, and every isoparametric hypersurface in a sphere is totally focal, that is, every squared distance function of the hypersurfaces is either nondegenerate or has only degenerate critical points. S. Carter and A. West (1982) proved that every totally focal hypersurface in the sphere is isoparametric.

R. Miyaoka (1982) proved that if a complete hypersurface in S^{n+1} has constant mean curvature and three non-simple principal curvatures, then it is isoparametric. B. Y. Chen (1984b) proved that an isoparametric hypersurface in the sphere is either of 1-type or of 2-type. S. Chang (1993) showed that a compact hypersurface in S^4 with constant scalar curvature and constant mean curvature is isoparametric.

M. Kotani (1985) proved that if M is an n-dimensional compact homogeneous minimal hypersurface in a unit sphere with g distinct principal curvatures, then the first nonzero eigenvalue λ_1 of the Laplacian on M is n unless $g = 4$. B. Solomon (1990a,1990b) determined the spectrum of the Laplacian operator on isoparametric minimal hypersurfaces of spheres with $g = 3$. All such hypersurfaces are algebraic and homogeneous. Solomon

(1992) also studied the spectrum of the Laplacian on quartic isoparametric hypersurfaces in the unit sphere. These are hypersurfaces with $g = 4$. J. H. Eschenburg and V. Schroeder (1991) investigated the behavior of the Tits metric on isoparametric hypersurfaces. B. Wu (1994) showed that for each n there are only finitely many diffeomorphism classes of compact isoparametric hypersurfaces of S^{n+1} with four distinct principal curvatures. J. K. Peng and Z. Z. Tang (1996) derived an explicit formula for the Brouwer degree of the gradient map of an isoparametric function f in terms of the multiplicities of the principal curvatures of the isoparametric hypersurface defined by f. They applied this formula to determine the Brouwer degree in a variety of examples.

10.4. **Dupin hypersurfaces.** .

10.4.1. Cyclide of Dupin

C. Dupin (1784–1873) defined in 1922 a cyclide to be a surface M in E^3 which is the envelope of the family of spheres tangent to three fixed spheres. This was shown to be equivalent to requiring that both sheets of the focal set degenerated into curves. The cyclides are equivalently characterized by requiring that the lines of curvatures in both families be arcs of circles or straight lines. Thus, one can obtain three obvious examples: a torus of revolution, a circular cylinder and a circular cone. It turns out that all cyclides can be obtained from these three by inversions in a sphere in E^3.

The cyclides were studied by many prominent mathematicians of the last century including A. Cayley (1821–1895), J. G. Darboux (1842–1917), F. Klein (1849–1925), J. Liouville (1809–1882) and J. C. Maxwell (1931–1879). A detailed treatment of the cyclides can be found in the book [Fladt-Baur 1975]. A recent survey on Dupin hypersurfaces was given in [Cecil 1997].

10.4.2. Proper Dupin hypersurfaces

The study of Dupin hypersurfaces was initiated by T. E. Cecil and P. J. Ryan in 1978. Let M be a hypersurface in a complete simply-connected real space form $R^{n+1}(c)$. A submanifold S of M is called a curvature surface if, at each point $x \in S$, the tangent space T_xS is a principal space, that is, it is an eigenspace of the shape operator.

If a principal curvature κ has constant multiplicity m on some open set $U \subset M$, then the corresponding distribution of principal spaces is a foliation of rank m, and the leaves of this principal foliation are curvature surfaces. Furthermore, if the multiplicity m of κ is greater than one, then κ is constant along each of these curvature surfaces.

A hypersurface M is called a Dupin hypersurface if, along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface is called proper if each principal curvature has constant multiplicity on M, that is, the number of distinct principal curvatures is constant.

In E^3 the only proper Dupin hypersurfaces are spheres, planes, and the cyclides of Dupin. There exist many examples of Dupin hypersurfaces which are not proper, for instance, a tube M of sufficiently small constant radius r in E^4 over a torus of revolution $T^2 \subset E^3 \subset E^4$, since there are only two distinct principal curvatures on the set $T^2 \times \pm\{r\}$ but three distinct principal curvatures elsewhere on M.

G. Thorbergsson (1983a) proved that the possible number g of distinct principal curvatures of a compact embedded proper Dupin hypersurface in E^{n+1} is 1, 2, 3, 4 or 6; the same as for an isoparametric hypersurface. Thorbergsson's result implies that compact Dupin hypersurfaces in a sphere satisfy the same periodicity $m_i = m_{i+2}$ (subscripts mod g) for the multiplicities of the principal curvatures, just like isoparametric hypersurfaces.

S. Stolz (1997) proved that the possible multiplicities m_1, m_2 of compact proper Dupin hypersurfaces are exactly the same as in the isoparametric case.

10.4.3. Local construction of Dupin hypersurfaces

U. Pinkall (1985a) gave four local constructions for obtaining a proper Dupin hypersurface with $g + 1$ distinct principal curvatures from a lower dimensional proper Dupin hypersurface with g distinct principal curvatures. Using these Pinkall proved that there exists a proper Dupin hypersurface in Euclidean space with an arbitrary number of distinct principal curvatures with any given multiplicities.

Pinkall's construction is done by using the following basic constructions:

Start with a Dupin hypersurface W^{n-1} in E^n and then consider E^n as the linear subspace $E^n \times \{0\}$ in E^{n+1}. Then the following constructions yield a Dupin hypersurface M in E^{n+1}:

(1) Let M be the cylinder $W^{n-1} \times E^1$ in E^{n+1};

(2) Let M be the hypersurface in E^{n+1} obtained by rotating W^{n-1} around an axis $E^{n-1} \subset E^n$;

(3) Project W^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset E^{n+1}$. Let M be the cone over V^{n-1} in E^{n+1};

(4) Let M be a tube in E^{n+1} around W^{n-1}.

These constructions give rise to a new principal curvature of multiplicity one which is constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of W^{n-1}, and the Dupin property is preserved for these principal curvatures. These construction can be easily be generalized to produce a new principal curvature of multiplicity m by considering E^n as a subset of $E^n \times E^m$ rather than $E^n \times E^1$. These constructions only yield a compact proper Dupin hypersurface if the original manifold W^{n-1} is itself a sphere. Otherwise, the number of distinct principal curvatures is not constant on a compact manifold obtained in this way.

A Dupin hypersurface which is obtained as the result of one of the four constructions is said to be reducible. A proper Dupin hypersurface which does not contain any reducible open subset is called locally irreducible.

10.4.4. Dupin hypersurfaces with 2, 3 or 4 distinct principal curvatures

Let M be a complete proper Dupin hypersurface in E^{n+1} with two distinct principal curvatures. If one of the principal curvatures is identically zero, M is a standard product embedding of $S^k(r) \times E^{n-k}$, where $S^k(r)$ is a round sphere in a Euclidean subspace E^{k+1} orthogonal to E^{n-k}. Otherwise, T. E. Cecil and P. Ryan (1985) proved that a compact cyclide M of characteristic $(k, n-k)$ embedded in S^{n+1} must be Möbius equivalent to a standard product embedding of two spheres $S^k(r) \times S^{n-k}(s) \subset S^{n+1}(t)$, $r^2+s^2 = t^2$. The proof of Cecil and Ryan used the compactness assumption in an essential way, whereas the classification of Dupin surfaces in E^3 obtained in the nineteenth century does not need such an assumption. Using Lie sphere geometry of S. Lie (1842–1899), Pinkall (1985a) proved that every cyclide of Dupin is contained in a unique compact connected cyclide, and any two cyclides of the same characteristic are locally Lie equivalent.

If the pole of the projection does not lie on $S^k(r) \times S^{n-k}(s)$, M is called a ring cyclide. Otherwise, M is noncompact and is called a parabolic ring cyclide. In both cases, the two sheets of the focal set are a pair of focal conics.

Complete proper Dupin hypersurfaces embedded in E^{n+1} with two distinct principal curvatures were completely classified by Cecil and Ryan (1985). They proved that if such a hypersurface is compact, it is a ring cyclide; if such a hypersurface is noncompact and one of the two distinct principal curvatures is zero identically, then it is a standard product embedding of $S^k \times E^{n-k}$ or a parabolic ring cyclide.

Pinkall (1985b) gave local classification of Dupin hypersurfaces with three distinct principal curvatures in E^4 up to Lie-equivalence, by using the method of moving frames. Niebergall (1991) proved that every proper Dupin hypersurface in E^5 with three distinct principal curvatures at each point is reducible.

T. E. Cecil and G. R. Jensen (1998) proved that if a proper Dupin hypersurface in E^n with three distinct principal curvatures does not contain a reducible open subset, then it is equivalent by a Lie sphere transformation to an isoparametric hypersurface in a sphere S^n.

A necessary condition on a Dupin hypersurface with at least 4 distinct principal curvature to be Lie equivalent to a piece of an isoparametric hypersurface is the constancy of the Lie curvatures. Niebergall (1992) gave a local classification of proper Dupin hypersurfaces in E^5 with four distinct principal curvatures which are Lie equivalent to isoparametric hypersurfaces. He showed that if the Lie curvature of the hypersurface M is constant and a condition on certain half-invariants are satisfied, then M is Lie equivalent to an isoparametric hypersurface. Cecil and Jensen (1997) proved that the condition on the half-invariants can be removed. In 1989, R. Miyaoka gave necessary and sufficient conditions for a compact embedded Dupin hypersurface with four or six principal curvatures to be Lie equivalent to an isoparametric hypersurface. She showed that a compact proper Dupin hypersurface embedded in E^{n+1} is Lie equivalent to an isoparametric hypersurface if it has constant Lie curvatures and it satisfies certain global conditions regarding the intersections of leaves of its various principal foliations.

10.4.5. Dupin hypersurfaces and Lie sphere transformations

The classes of Dupin and proper Dupin hypersurfaces in S^{n+1} are invariant under conformal transformations of S^{n+1} and under stereographic projection from S^{n+1} to E^{n+1} (cf. [Cecil-Ryan 1985]). U. Pinkall (1985a) proved that they are invariant under parallel transformations; and thus under the group of Lie sphere transformations. Hence, Dupin hypersurfaces are most naturally studied in the Lie geometric framework.

A Lie sphere transformation is a projective transformation of RP^{n+2} which takes the Lie hyperquadric Q^{n+1} into itself, where Q^{n+1} in RP^{n+2} is defined by

$$-x_0^2 + x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = 0$$

in terms of homogeneous coordinates.

In terms of the geometry of E^n a Lie sphere transformation preserves the family of oriented spheres and planes. The group of Lie sphere transformations is isomorphic to $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations of E_2^{n+3}. The group of Lie sphere transformations contains the group of conformal transformations of S^{n+1} as a proper subgroup.

A proper Dupin hypersurface with one principal curvature at each point is, of course, totally umbilical and must, therefore, be an open subset of a great or small sphere.

Cecil and Ryan (1978) showed that a compact proper Dupin hypersurface in S^{n+1} with two distinct principal curvatures must be a ring cyclide, that is, the image under a conformal transformation of S^{n+1} of a standard product of two spheres. Pinkall (1985) was able to drop the assumption of compactness and showed that any proper Dupin hypersurface in S^{n+1} with two distinct principal curvatures is the image under a Lie sphere transformation of an open subset of a standard product embedding of the product of two spheres.

It was conjectured by T. E. Cecil and P. Ryan that every compact proper Dupin hypersurface in S^{n+1} is Lie equivalent to an isoparametric hypersurface. As noted above, this is true for g equal to 1 or 2.

R. Miyaoka (1984a) proved that a compact embedded proper Dupin hypersurface in a real space form with three principal curvatures is Lie equivalent to an isoparametric hypersurface.

Cecil and Ryan's conjecture has been shown to be false by U. Pinkall and G. Thorbergsson (1989a) and also independently by R. Miyaoka and T. Ozawa around 1988 who have produced different counterexamples to the conjecture.

For a proper Dupin hypersurface with $g = 4$, one can order the principal curvatures so that $\nu_1 < \nu_2 < \nu_3 < \nu_4$, the Lie curvature Ψ is defined to be the cross ratio:

$$\Psi = \frac{(\nu_4 - \nu_3)(\nu_1 - \nu_2)}{(\nu_4 - \nu_2)(\nu_1 - \nu_3)}. \tag{10.3}$$

The examples of Miyaoka and Ozawa involve the Hopf fibration of S^7 over S^4. Let $E^8 = H \times H$, where H is the division ring of quaternions. The Hopf fibering of the unit sphere S^7 in E^8 over the unit sphere S^4 in $E^5 = H \times E^1$ is given by

$$\psi(u, v) = (2u\bar{v}, |u|^2 - |v|^2), \quad u, v \in H. \tag{10.4}$$

Miyaoka and Ozawa showed that if M^3 is a compact proper Dupin hypersurface embedded in S^4, then $\psi^{-1}(M^3)$ is a proper Dupin hypersurface embedded in S^7. Furthermore, if M^3 has g distinct principal curvatures, then $\psi^{-1}(M^3)$ has $2g$ distinct principal curvatures. If M^3 is not isoparametric, then the Lie curvature of $\psi^{-1}(M^3)$ is not constant, and so $\psi^{-1}(M^3)$ is not Lie equivalent to an isoparametric hypersurface.

The examples of Pinkall and Thorbergsson are given as follows:

Let $E^{2n+2} = E^{n+1} \times E^{n+1}$ and let S^{2n+1} denote the unit sphere in E^{2n+2}. The Stiefel manifold V of orthogonal 2-frames in E^{n+1} of length $1/\sqrt{2}$ is given by

$$V = \{(u, v) \in E^{2n+2} | u \cdot v = 0, |u| = |v| = 1/\sqrt{2}\}.$$

The submanifold V lies in S^{2n+1} with codimension 2, so V has dimension $2n - 1$. Let α and β be positive real numbers satisfying $\alpha^2 + \beta^2 = 1$, and let $T_{\alpha,\beta}$ be the linear transformation of E^{2n+2} defined by

$$T_{\alpha,\beta}(u, v) = \sqrt{2}(\alpha u, \beta v).$$

Then the image $W^{\alpha,\beta} = T_{\alpha,\beta}V$ is contained in S^{2n+1} and it is proper Dupin with $g = 4$. On can show that the Lie curvature is not constant on $W^{\alpha,\beta}$ if $\alpha \neq 1/\sqrt{2}$. Thus $W^{\alpha,\beta}$ is not Lie equivalent to an isoparametric hypersurface if $\alpha \neq 1/\sqrt{2}$.

If M^3 is an isoparametric hypersurface with k principal curvatures, the inverse image $\psi^{-1}(M^3)$ is an isoparametric hypersurface with $2k$ principal curvatures in S^7. When $k = 3$, M^3 must be a tube over a Veronese surface, and so the unique family of isoparametric hypersurfaces with $g = 6$ in S^7 has a precise geometric characterization in terms of ψ. Miyaoka (1993) showed that in this case $\psi^{-1}(M^3)$ is homeomorphic to $M^3 \times S^3$ and it has a foliation whose leaves are isoparametric hypersurfaces with three principal curvatures of multiplicity one. The two focal submanifolds of $\psi^{-1}(M^3)$ are obtained from the two focal submanifolds of M^3 via ψ^{-1}. However, although the two focal submanifolds of M^3 are Veronese surfaces which are congruent in S^4, the two focal submanifolds of $\psi^{-1}(M^3)$ are not congruent in S^7. Thus, they are two noncongruent minimal taut homogeneous embeddings of $RP^2 \times S^3$ into S^7.

Pinkall and Thorbergsson (1989) introduced the Möbius curvature which can distinguish among the Lie equivalent parallel hypersurfaces in a family of isoparametric hypersurfaces. C. P. Wang (1992) applied the method of moving frames to determine a complete set of Möbius invariants for surfaces in E^3 without umbilic points and for hypersurfaces in E^4 with three distinct principal curvatures at each point. He then applied this result to derive a local classification of Dupin hypersurfaces in E^4 with three principal curvatures up to Möbius transformation.

Ferapontov (1995a,1995b) studied the relationship between Dupin and isoparametric hypersurfaces and Hamiltonian systems of hydrodynamic type.

10.4.6. Tubes as Dupin hypersurfaces

M. Takeuchi (1991) studied Dupin hypersurfaces in real space forms which are tubes around symmetric submanifolds in $R^{n+1}(c)$ and proved the following: Let M be a non-totally geodesic symmetric submanifold of $R^{n+1}(c)$ of codimension > 1. Then the ε-tube $T_\varepsilon(M)$ around M is a proper Dupin hypersurface if and only if either

(i) M is a complete extrinsic sphere of $R^{n+1}(c)$ of codimension > 1; or

(ii) M is one of the following symmetric submanifolds of the n-dimensional sphere S^n:

(ii-a) the projective plane $FP^2 \subset S^{3d+1}$, $d = \dim_R F$, over $F = R, C$, the quaternions H, or octonions $\mathcal{O}$;

(ii-b) the complex quadric $Q_3 \subset S^9$;

(ii-c) the Lie quadric $Q^{m+1} \subset S^{2m+1}$, $m \geq 2$;

(ii-d) the unitary symplectic group $Sp(2) \subset S^{15}$.

In case (i), $T_\varepsilon(M)$ is a Dupin cyclide, that is, a proper Dupin hypersurface with two distinct principal curvatures, but it is not isoparametric. In case (ii), $T_\varepsilon(M)$ is a homogeneous isoparametric hypersurface with three or four distinct principal curvatures, and it is irreducible in the sense of Pinkall.

10.4.7. Topology of Dupin hypersurfaces

G. Thorbergsson (1983a) proved the following:

(1) a compact embedded proper Dupin hypersurface M in $R^{n+1}(c)$ satisfies $\dim H^*(M; \mathbf{Z}_2) = 2g$, where g is the number of distinct principal curvatures; and

(2) a compact embedded proper Dupin hypersurface M divides S^{n+1} into two ball bundles.

By using (2), Thorbergsson showed that $g = 1, 2, 3, 4$ or 6.

The integral homology, fundamental group and rational homotopy type of a compact Dupin hypersurface in the sphere were determined in [Grove-Halperin 1987].

In particular, K. Grove and S. Halperin proved that if M is a compact Dupin hypersurface of S^{n+1}, then

(1) there are two integers k, ℓ (possible equal) such that each principal curvature has multiplicity k or ℓ;

(2) the integral homology of M determines k, ℓ, and the number g of principal curvatures. Conversely, g, k and ℓ determine the fundamental group, integral homology, and rational homotopy type of M;

(3) the integers g, k, ℓ satisfy the following restrictions:

(3-a) if $k \neq \ell$, then $g = 2$ or 4, and k and ℓ are each the multiplicity of $g/2$ principal curvatures;

(3-b) if $g = 3$, then $k = 1, 2, 4$ or 8,

(3-c) if $g = 4$ and $k = \ell$, then $k = 1$ or 2; furthermore, if $g = 4$ and $k > \ell \geq 2$, then $k + \ell$ is odd,

(3-d) if $g = 6$, then $k = 1$ or 2.

It is immediate from the above results that $n = \frac{1}{2}(k + \ell)g$. Thus, $g = 1$ if and only if $k + \ell > n$; $g = 2$ if and only if $k + \ell = n$; and $g = 3, 4$ or 6 if and only if $k + \ell < n$.

10.4.8. Dupin hypersurfaces of T_1S^{n+1}

Let T_1S^{n+1} denote the unit tangent bundle of S^{n+1}. Consider T_1S^{n+1} as the $(2n+1)$-dimensional submanifold of $S^{n+1} \times S^{n+1} \subset E^{n+2} \times E^{n+2}$ given by

$$T_1S^n = \{(x, \xi) : \ |x| = 1, \, |\xi| = 1, \, \langle x, \xi \rangle = 0\}.$$

Then T_1S^{n+1} admits a canonical contact 1-form ω induced from the canonical almost complex structure on $C^{n+2} = E^{n+2} \times E^{n+2}$; thus, $\omega \wedge (d\omega)^n \neq 0$ everywhere. The contact structure gives rise to a codimension one distribution on T_1S^{n+1} which has integrable submanifolds of dimension n, but none of higher codimension. An n-dimensional integrable submanifold of T_1S^{n+1} is called a Legendre submanifold.

Pinkall (1985a) investigated the Legendre submanifold of T_1S^{n+1} that he called Lie geometric hypersurfaces of the sphere S^{n+1}. Each oriented hypersurface M of S^{n+1} gives rise to a Legendre submanifold L_M of T_1S^{n+1} by associating to M the set L_M of oriented unit normal vectors along M.

The image of a Legendre submanifold is called a wavefront. Pinkall showed how the basic theory of hypersurfaces can be extended to wavefronts. A contact transformation is a diffeomorphism F of T_1S^{n+1} which satisfies the property $dF(\ker \omega) = \ker \omega$. In particular, if F leaves the class of Legendre submanifolds that come from lifting an oriented hypersurface of S^{n+1} invariant, then F is called a Lie sphere transformation. The class of transformations of S^{n+1} that map spheres to spheres are called Möbius transformations; they are exactly the conformal automorphisms of S^{n+1}, according to a result of J. Liouville.

The classification of Lie sphere transformations is done by mapping the space of oriented spheres of S^{n+1} onto the Lie quadric, that is, the quadric of type $(n+2, 2)$ in RP^{n+3}. Then the Lie sphere transformations correspond to the projective transformations of RP^{n+3} which leave the Lie quadric invariant.

Pinkall pointed out that, for each Legendre submanifold L of T_1S^{n+1}, the concept of a principal direction can be defined at a point $p \in L$ as a direction in which L has higher order contact at p with the lift of a hypersphere, called an osculating sphere, to T_1S^{n+1}. Define the principal radii of L at p to be the radii of the corresponding osculating spheres. The tangent space T_pL then decomposes into $E_1 \oplus \cdots \oplus E_g$, where each E_i is a maximal subspace which consists of principal directions. The multiplicity of a principal radius is nothing but the dimension of the corresponding space E_i.

Let L be a Legendre submanifold of T_1S^{n+1} and let S be a submanifold of L such that T_pS is one of the spaces $E_1, \ldots, E_g$ in the decomposition. S is called a curvature surface of L according to Pinkall (1985a). When L is a Legendre submanifold of T_1S^{n+1} such that a continuous principal radius function is constant along its corresponding curvature surface, then L is called a Dupin hypersurface [Pinkall 1985a].

In [Pinkall 1985a] a proper Dupin hypersurface is defined as a Dupin hypersurface L which satisfied the property that the multiplicities of the principal radii at each point $p \in L$ are independent of the point p. Pinkall showed that the class of proper Dupin hypersurfaces is invariant under the Lie sphere transformations.

As a generalization of the classical cyclides in E^3, Dupin hypersurfaces in T_1S^{n+1} with two distinct principal radii at each point are called cyclides of Dupin. Pinkall classified the cyclides of Dupin in T_1S^{n+1} and proved that they are Lie equivalent to the Lagrangian submanifold of T_1S^{n+1} obtained by the standard product embedding of $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, where k and $n-k$ are the multiplicities of the principal curvatures of the cyclide of Dupin.

Pinkall (1985b) proved that a Dupin hypersurface with three distinct principal curvatures in T_1S^4 is either reducible or Lie equivalent to a piece of Cartan's isoparametric hypersurface in S^4 with three distinct principal curvatures. T. E. Cecil and G. R. Jensen (1998) extended Pinkall's result and showed that if a proper Dupin hypersurface in T_1S^{n+1} contains no reducible open subset, then it is Lie equivalent to a piece of an isoparametric hypersurface with three principal curvatures.

10.5. **Hypersurfaces with constant mean curvature.** Surfaces of constant mean curvature occur naturally in physics. S. D. Poisson (1781–1840) showed in 1828 that if a surface in E^3 is the interface between two media in equilibrium, then the mean curvature H of the surface is constant and equal to $H = k(p_1 - p_2)$, where p_1 and p_2 denote the pressures in the media and $\lambda = 1/k$ is called the coefficient of surface tension. Furthermore, a hypersurface with constant mean curvature is a solution to a variational problem; namely, with respect to any volume-preserving normal variation of a domain D in a Euclidean space, the mean curvature of $M = \partial D$, the boundary of

D, is constant if and only if the area of M is critical, that is, it satisfies $A'(0) = 0$.

10.5.1. Hopf's problem and Wente's tori

H. Hopf (1894–1971) proved in 1951 that any immersion of a surface, topologically a sphere, with constant mean curvature in E^3 must be a round sphere. A. D. Alexandrov (1912–) showed in 1958 that the only compact embedded surfaces in E^3 of constant mean curvature are round spheres. That left open the possibility of immersed constant mean curvature surfaces of higher genus.

Surprisingly, H. C. Wente constructed in 1984 examples of tori of constant mean curvature in E^3. Wente's examples solved the long-standing problem of Hopf: Is a compact constant mean curvature immersed surface in E^3 necessarily a round sphere?

Wente's work inspired a string of further research on compact tori of constant mean curvature [Abresch 1987, Spruck 1986, Pinkall-Sterling 1989]. For instance, U. Abresch classified all constant mean curvature tori having one family of planar curvature lines.

N. Kapouleas (1991) showed that there also exist compact constant mean curvature surfaces of every genus ≥ 3 in E^3. Also Kapouleas (1991) provided a general construction method for complete surfaces of constant mean curvature in E^3.

W. Y. Hsiang (1982b,1982c) constructed infinitely many noncongruent immersions of topological n-sphere S^n into E^{n+1} with constant mean curvature for each $n \geq 3$.

10.5.2. Delaunay's surfaces and generalizations

Delaunay's surfaces in E^3 introduced by C. E. Delaunay (1916–1872) are surfaces of revolution of constant mean curvature. N. J. Korevaar, R. Kusner and B. Solomon (1989) proved that, besides the circular cylinder, Delaunay surfaces are the only doubly-connected surfaces properly embedded in E^3 with nonzero constant mean curvature.

W. Y. Hsiang and W. C. Yu (1981) and W. Y. Hsiang (1982a,1982c) constructed one-parameter family of hypersurfaces of revolution, symmetric under $O(n-1)$, having constant mean curvature in the n-sphere $S^n(1)$, in the Euclidean n-space E^n, or in the hyperbolic n-space $H^n(-1)$.

10.5.3. Hypersurfaces with $K \leq 0$ or with $K \geq 0$

T. Klotz and R. Osserman (1966) proved that a complete surface of nonzero constant mean curvature is a circular cylinder if its Gauss curvature

K is ≤ 0. B. Smyth and K. Nomizu (1969) showed that the only compact hypersurfaces of constant mean curvature in E^{n+1} with non-negative sectional curvatures are round n-spheres.

B. Smyth and K. Nomizu (1969) also proved that totally umbilical hypersurfaces and a standard product embedding of the product of two spheres are the only compact hypersurfaces of constant mean curvature in S^{n+1} with nonnegative sectional curvature.

Let M be a compact hypersurface of constant mean curvature in S^{n+1}. Denote by $B = \sqrt{S}$ the norm of the second fundamental form of M in S^{n+1}. Z. H. Hou (1997) proved the following:

(1) If $B < 2\sqrt{n-1}$, M is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+B)}$.

(2) If $B = 2\sqrt{n-1}$, M is either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where

$$r_0^2 = \frac{n}{n+2\sqrt{n-1}}, \quad r^2 = \frac{1}{1+\sqrt{n-1}}, \quad s^2 = \frac{\sqrt{n-1}}{1+\sqrt{n-1}}.$$

10.5.4. Hypersurfaces of constant mean curvature in hyperbolic spaces

In the hyperbolic 3-space H^3 of sectional curvature -1, the behavior of a surface of constant mean curvature H depends on the value of H. If $|H| < 1$, the area of the surface grows exponentially. In the case where $|H| > 1$, the surface can be compact, like geodesic spheres. In the border case $|H| = 1$, there exist examples such that the area grows polynomially, and it is known that some properties similar to those of minimal surfaces in Euclidean space hold.

N. Korevaar, R. Kusner, W. H. Meeks and B. Solomon (1992) studied constant mean curvature surfaces in H^3 and proved the following:

Let M be a complete properly embedded surface in H^3 with constant mean curvature greater than that of a horosphere. Then

(1) M is not homeomorphic to a closed surface punctured in one point.

(2) If M is homeomorphic to a closed surface punctured in two points, then M is Delaunay, that is, M is a constant mean curvature surface of revolution.

(3) If M is homeomorphic to a closed surface punctured in three points, then M remains at a bounded distance from a geodesic plane of reflective symmetry and each half of M determined by the geodesic plane is a graph over this plane with respect to the distance function to the plane.

Moreover, they showed that the annular ends of M must exponentially converge to Delaunay surfaces, which are constant mean curvature surfaces of revolution in H^3.

For hypersurfaces of constant mean curvature in H^{n+1}, M. do Carmo and H. B. Lawson (1983) proved the following:

Let M be a complete hypersurface properly embedded in H^{n+1} with constant mean curvature, and let $\partial_\infty M \subset S^n(\infty)$ be its asymptotic boundary. Then

(a) if M is compact, it is a sphere; if $\partial_\infty M$ consists of exactly one point, then M is a horosphere; and

(b) if $\partial_\infty M$ is a sphere and M separates poles, then M is a hypersphere.

As a consequence, if M is a hypersurface of constant mean curvature in H^{n+1}, admitting a one-to-one projection onto a geodesic hyperplane, then M is a hyperplane. This result can be restated as follows: A nonparametric entire hypersurface, that is, the graph of a function f defined in some H^n, of constant mean curvature is a hypersphere, which is a close analogue of the Bernstein theorem.

10.5.5. Weierstrass type representation for surfaces with constant mean curvature

K. Kenmotsu (1979) established an integral representation formula for arbitrary surfaces in E^3 with nonvanishing mean curvature H which describes the surface as a branched conformal immersion in terms of its mean curvature and its Gauss map. Specifically, let ψ denote the complex-valued function on the surface obtained by composing the Gauss map with stereographic projection. Then, letting $f = -\psi_z/H(1+\psi^2)$, one has

$$x_1 = \mathrm{Re}\int(1-\psi^2)f dz, \quad x_2 = \mathrm{Re}\int i(1+\psi^2)f dz, \quad x_3 = 2\mathrm{Re}\int \psi f dz.$$

Thus, given a real function H and a complex function ψ on a simply-connected domain D, a necessary and sufficient condition that there exists a map $x : D \to E^3$ defining a branched surface in isothermal coordinates having H as mean curvature and ψ as the Gauss map is that the differential relations above between H and ψ be satisfied. As a special case, if H is a nonzero constant, then the resultant equation for ψ is the precise condition for ψ to define a harmonic map into the unit sphere. By virtue of this representation formula, if a harmonic map ψ from a Riemann surface Σ into S^2 is given, then one can construct a branched immersion of a constant mean curvature surface whose Gauss map is ψ.

J. Dorfmeister, F. Pedit and H. Wu (1997) showed that every constant mean curvature immersion $\Phi : D \to E^3$, D the whole complex plane or the open unit disk in $\mathbf{C}$, can be produced from a meromorphic matrix valued

1-form

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f(z) \\ g(z) & 0 \end{pmatrix} dz,$$

$\lambda \in S^1$, the so called meromorphic potential. Here f and g are meromorphic functions of $z \in D$ and $f(z)g(z) = E(z)$, where $E(z)dz^2$ is, up to a constant factor, the Hopf differential of the surface. J. Dorfmeister, and G. Haak (1997) gave an explicit characterization of the zero and pole order of the meromorphic functions for the branched constant mean curvature surface to be a smooth immersion.

R. Bryant (1987b) showed that there also exists a Weierstrass type representation for surfaces of constant mean curvature $H = c > 0$ in $H^3(-c^2)$. In particular, any constant mean curvature one surface in $H^3(-1)$ can be constructed from an $\mathfrak{sl}(2, \mathbf{C})$-valued holomorphic 1-form satisfying some conditions (or equivalently a pair of a meromorphic function and a holomorphic 1-form) on a Riemann surface.

Bryant's representation can be stated as follows.

We identify each point (t, x_2, x_3, x_4) of L^4 with a 2×2 Hermitian matrix

$$\begin{pmatrix} t + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & t - x_4 \end{pmatrix} \in \text{Herm}\,(2). \tag{10.5}$$

Then $H^3(-1)$ is identified with

$$\begin{aligned} H^3(-1) &= \{X \in \text{Herm}\,(2) : \det(X) = 1,\ \text{trace}\,(X) > 0\} \\ &= \{a \cdot a^* : a \in SL(2, C)\}, \end{aligned} \tag{10.6}$$

where $a^* = \bar{a}^T$. Under this identification, each element a of the group $PSL(2, C) := SL(2, C)/\{\pm 1\}$ acts isometrically on $H^3(-1) \ni X \mapsto a \cdot X \cdot a^*$.

Bryant proved the following result.

Let M be a simply-connected Riemann surface and $z_0 \in M$ a fixed point. Take a meromorphic function ψ and a holomorphic 1-form ω on M such that $ds^2 := (1 + |\psi|^2)^2 \omega \cdot \bar{\omega}$ is positive definite on M. Then there exists a unique holomorphic immersion $F : M \to PSL(2, C)$ that satisfies

(1) $F(z_0) = \pm\,\text{id.}$;

(2) $F^{-1} \cdot dF = \begin{pmatrix} \psi & -\psi^2 \\ 1 & -\psi \end{pmatrix} \omega$;

(3) $f = F \cdot F^* : M \to H^3(-1)$ is a conformal immersion with constant man curvature 1 whose first fundamental form is ds^2.

Conversely, any conformal constant mean curvature 1 immersions in $H^3(-1)$ are obtained as above.

M. Umehara and K. Yamada (1996) showed that Bryant's representation formula for surfaces of constant mean curvature c in $H^3(-c^2)$ can be deformed to the Weierstrass representation formula as c tends to 0.

R. Aiyama and K. Akutagawa (1997a) gave representation formulas for surfaces of constant mean curvature H in the hyperbolic 3-space $H^3(-c^2)$ with $H > c > 0$. R. Aiyama and K. Akutagawa (1997b) also gave representation formulas for surfaces of constant mean curvature in the 3-sphere $S^3(c^2)$. Their formulas show that every such surface in $H^3(-c^2)$ or in $S^3(c^2)$ can be represented locally by a harmonic map to the unit 2-sphere.

Further results on surfaces of constant mean curvature in H^3 were obtained by M. Umehara and K. Yamada (1992,1993,1996,1997a, 1997b).

10.5.6. Stability of surfaces with constant mean curvature

Since a compact constant mean curvature surface in E^3 is a critical point of the area functional with respect to volume-preserving normal variations, one can define the stability of such surfaces: A compact constant mean curvature surface in E^3 is called stable if $A''(0) > 0$ with respect to the class of volume-preserving normal variations.

M. do Carmo and A. Da Silveira (1990) proved that the index of $\Delta - 2K$ is finite if and only if the total curvature is finite for a complete surface of constant mean curvature one in the hyperbolic 3-space H^3.

J. A. Barbosa and M. do Carmo (1984) proved that the spheres are the only compact stable hypersurfaces of constant mean curvature in E^{n+1}. This result was generalized for closed constant mean hypersurfaces in S^{n+1} and H^{n+1} by Barbosa, do Carmo and J. Eschenburg in 1988. For surfaces this result was extended by H. Mori (1983), B. Palmer (1986) and F. J. López and A. Ros (1989) to complete surfaces, where the stability assumption applied to every compact subdomain and the surface is assumed to have nonzero constant mean curvature.

Also A. Da Silveira (1987) studied complete noncompact surfaces which are immersed as stable constant mean curvature surfaces in E^3 or in the hyperbolic space H^3. In the case of E^3 he proved that the immersion is a plane. For H^3 he showed, under the condition of nonnegative mean curvature H, that for $H \geq 1$ only horospheres can occur, while for $H < 1$ there exists a one-parameter family of stable nonumbilic embeddings. His theorems are generalizations and improvements of previous results by Barbosa, do Carmo and Eschenburg, Fischer-Colbrie, Peng, and Schoen. His notion of stability is slightly weaker than the previous one for minimal immersions. Examples show that the two definitions in general do not agree. For three-dimensional simply-connected complete Riemannian manifolds with positive constant sectional curvature he proved that there exist no complete and noncompact stable immersions with constant mean curvature.

For hypersurfaces in E^{n+1} H. P. Luo (1996) proved that if a complete noncompact stable hypersurface has nonnegative Ricci curvature, it is minimal.

10.6. **Hypersurfaces with constant higher order mean curvature.** The r-th mean curvature H_r of a hypersurface M is defined as the elementary symmetric polynomial of degree r in the principal curvatures $\kappa_1, \ldots, \kappa_n$ of M, that is,

$$H_r = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}. \tag{10.7}$$

For a hypersurface in E^{n+1}, H_1, H_2 and H_n are the mean curvature, the scalar curvature, and the Gauss-Kronecker curvature, respectively (up to suitable constants).

For a compact oriented hypersurface $f : M \to E^{n+1}$, the r-th mean curvatures are related by the following formulas of H. Minkowski (1903):

$$\binom{n}{r-1} \int_M H_{r-1} dV = -\binom{n}{r} \int_M \langle f, \xi \rangle \, H_r dV, \quad 1 \leq r \leq n, \tag{10.8}$$

where ξ is a unit normal vector field of M in E^{n+1}.

A. Ros (1987) and, independently, N. J. Korevaar (1988) proved that, for any $r, 1 \leq r \leq n$, the round sphere is the only compact hypersurface with constant r-th mean curvature H_r embedded in E^{n+1}.

For a compact hypersurface M embedded in hyperbolic space, N. J. Korevaar (1988) and Montiel and Ros (1991) proved that if any of the higher order mean curvatures is constant, then M must be a geodesic hypersphere. The same is true if M lies in a hemisphere of S^{n+1}.

The above results are not true in general, since all the isoparametric hypersurfaces of S^{n+1} have all mean curvatures constant.

For a compact immersed hypersurface $f : M \to R^{n+1}(c)$ in a complete simply-connected real space form $R^{n+1}(c)$, let $F_r(H_1, H_2, \ldots, H_r)$ be the function defined inductively by

$$F_0 = 1, \quad F_1 = H_1, \quad F_r = H_r + \frac{(n-r+1)c}{r-1} F_{r-2}, \quad 2 \leq r \leq n-1. \tag{10.9}$$

A variation of f is a differentiable map $X : I \times M \to R^{n+1}(c)$ such that $X_0 = f$ and, for each $t \in I$, $X_t(x) = X(t, x)$, $x \in M$, is an immersion.

The balance of volume is defined to be the function $V : I \to \mathbf{R}$ given by $V(t) = \int_{[0,t] \times M} F^*(d\bar{V})$, where $d\bar{V}$ denotes the volume element of $R^{n+1}(c)$. In Euclidean case, it measures the balance of the volume of the enclosed domain from the time 0 to time t. So, in this case, $V \equiv 0$ means that the volume of the domain bounded by the hypersurface is kept constant while the time changes.

A variation of f is said to be volume-preserving if $V(t) \equiv 0$.

Put

$$A_r = \int_M F_r(H_1, H_2, \dots, H_r) dV. \tag{10.10}$$

In the class of volume-preserving variations of $f : M \to R^{n+1}(c)$, the first variational formula of f is given by

$$A'_r(\phi) = \int_M \{-(r+1)H_{r+1} + \kappa\}\phi dV, \tag{10.11}$$

where κ stands for a constant and ϕ is the normal projection of the variation vector field ξ. Thus, immersions with constant $(r+1)$-th mean curvature arise as critical points for the variational problem of minimizing A_r, keeping the balance of volume zero (cf. [Barbosa-Colares 1997, Reilly 1973]).

An immersion $f : M \to R^{n+1}(c)$ with constant $(r+1)$-th mean curvature is said to be r-stable if its second variation $A''_r(\phi)$ is > 0, for any compact support function $\phi : M \to \mathbf{R}$ that satisfies $\int_M \phi dV = 0$.

H. Alencar, M. do Carmo and H. Rosenberg (1993) proved that hyperspheres are the only r-stable immersed compact orientable hypersurfaces in Euclidean space.

J. L. Barbosa and A. G. Colares (1997) showed that geodesic hyperspheres are the only r-stable immersed compact orientable hypersurfaces in an open hemisphere of S^{n+1} or in the hyperbolic space H^{n+1}. When $r = 1$, this is due to [Alencar-do Carmo-Colares 1993].

10.7. **Harmonic spaces and Lichnerowicz conjecture.** A Riemannian manifold M is called a harmonic space if all sufficiently small geodesic hyperspheres have constant mean curvature.

Let $(x_1, \dots, x_n)$ denote a system of Cartesian coordinates of Euclidean n-space E^n centered at a point 0, and Δ denote the Laplacian of E^n. Then it is well-known that the Laplace equation $\Delta\phi = 0$ admits a nice solution given by

$$\phi(x) = \begin{cases} r^{2-n} & \text{if} \quad n > 2, \\ \ln r & \text{if} \quad n = 2, \end{cases}$$

where r is the distance of the point x from the origin 0. This implies that the Laplace equation has a solution which is constant on each hypersphere centered at 0. Clearly, this is true for any arbitrary center in E^n.

In his 1930 doctoral thesis at Oxford University, H. S. Ruse (1905–1974) made an attempt to solve Laplace's equation on a general Riemannian manifold and to find a solution which depends only on the geodesic distance. He realized later that it was implicitly assumed in his thesis that such a solution always exists and this is not the case. Stimulated by this incident, E. C. Copson (1901–1980) and Ruse started in 1939 the study of the class of Riemannian manifolds which admit such a solution. This is the beginning of the study of harmonic manifolds.

In 1944 A. Lichnerowicz (1915–) showed that a harmonic manifold of dimension ≤ 4 is either a flat space or a rank one locally symmetric space. From this one conjectures that the same conclusion holds true without the dimension hypotheses; which is known as Lichnerowicz's conjecture.

Lichnerowicz's conjecture has been proved by Z. I. Szabö (1990) for compact harmonic manifolds with finite fundamental groups.

A Riemannian manifold of negative curvature is said to be asymptotically harmonic if the mean curvatures of the geodesic horospheres are constant. P. Foulon and F. Labourie (1992) proved that if M is a compact (C^∞-) negatively curved asymptotically harmonic manifold, then the geodesic flow of M is C^∞ conjugate to that of a rank one locally symmetric space. On the other hand, G. Besson, G. Courtois and S. Gallot (1995) proved that a Riemannian manifold whose geodesic flow is C^1 conjugate to that of a compact locally symmetric manifold N is isometric to N. Thus, a compact negatively curved asymptotically harmonic Riemannian manifold is locally symmetric; this in particular proves Lichnerowicz's conjecture for the compact negatively curved case.

On the other hand, E. Damek and F. Ricci showed in 1992 that there exist noncompact counterexamples to the conjecture; namely, there exists a class of harmonic homogeneous simply connected manifolds of negative curvature which are not symmetric.

Damek and Ricci's examples are given as follows: Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with inner product $\langle\ ,\ \rangle$, such that if $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{o} = \mathfrak{z}^\perp$, the map $J_Z : \mathfrak{o} \to \mathfrak{o}$ defined by $\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle$ satisfies $J_Z^2 = -|Z|^2$ for all X, $Y \in \mathfrak{o}$ and $Z \in \mathfrak{z}$. The connected and simply connected Lie group N generated by this algebra $\mathfrak{n}$ is classically referred to as the Heisenberg group. Let $\mathfrak{n}$ be solvably extended to $\mathfrak{s} = \mathfrak{o} \oplus \mathfrak{z} \oplus \mathbf{R}T$ by adding the rule $[T, X+Y] = X/2 + Z$, and denote by $S = NA$ ($A = \exp_S(\mathbf{R}T)$), the corresponding connected and simply connected Lie group. By the use of the admissible invariant metric, S is then made into a Riemannian manifold. Effecting on S a suitable Cayley transform one can introduce the normal coordinates (r, w) on the ball

$$B = \{(X, Y, t) \in \mathfrak{o} \oplus \mathfrak{z} \oplus \mathbf{R}T,\ r^2 = |X|^2 + |Y|^2 + |t|^2 < 1\}$$

around the identity element $\mathfrak{e} = (0, 0, 1)$, with respect to which the volume element on B is computed as

$$2^{m+k}\left(\cosh\left(\frac{\rho}{2}\right)\right)^k \left(\sinh\left(\frac{\rho}{2}\right)\right)^{m+k} d\rho\, d\sigma(w),$$

where

$$m = \dim \mathfrak{z}, \quad k = \dim \mathfrak{o}, \quad \rho = \log\left(\frac{1+r}{1-r}\right)$$

and $d\sigma(w)$ denotes the surface element of the sphere S^{m+k}. Thus S turns out to be a harmonic space. Note that m $(= \dim \mathfrak{z})$ is quite arbitrary.

Consider the symmetric space $M = G/K$ of noncompact type and let $G = NAK$ be the Iwasawa decomposition. Then the map $s \to sK$ is an isometry of $S = NA$ onto G/K, and it is known that if M is a symmetric space of rank one, the dimension k of the center $\mathfrak{z}$ of N equals $1, 3$ or 7 only. This leads to infinitely many harmonic spaces S that are not rank-one symmetric.

11. Totally geodesic submanifolds

The notion of totally geodesic submanifolds was introduced in 1901 by J. Hadamard (1865–1963). Hadamard defined (totally) geodesic submanifolds of a Riemannian manifold as submanifolds such that each geodesic of them is a geodesic of the ambient space. This condition is equivalent to the vanishing on the second fundamental form on the submanifolds. 1-dimensional totally geodesic submanifolds are nothing but geodesics. Totally geodesic submanifolds are the simplest and the most fundamental submanifolds of Riemannian manifolds.

It is easy to show that every connected component of the fixed point set of an isometry on a Riemannian manifold is a totally geodesic submanifold.

Totally geodesic submanifolds of a Euclidean space are affine subspaces and totally geodesic submanifolds of a Riemannian sphere are the greatest spheres.

It is much more difficult to classify totally geodesic submanifolds of a Riemannian manifold in general.

11.1. **Cartan's theorem.** Let M be a Riemannian n-manifold with $n \geq 3$. For a vector v in the tangent space T_pM at $p \in M$, denote by γ_v the geodesic through p whose tangent vector at p is v. Denote by $R_v(t)$ the $(1,3)$-tensor on T_pM obtained by the parallel translation of the curvature tensor at $\gamma_v(t)$ along the geodesic γ_v. Also define a (1,2)-tensor $r_v(t)$ on T_pM by

$$r_v(t)(x,y) = R_v(t)(v,x)y, \quad x, y \in T_pM.$$

The following result of É. Cartan provides necessary and sufficient conditions for the existence of totally geodesic submanifolds in Riemannian manifolds in general.

Let V be a subspace of the tangent space T_pM of a Riemannian manifold M at a point p. Then the following three conditions are equivalent.

(1) There is a totally geodesic submanifold of M through p whose tangent space at p is V.

(2) There is a positive number ϵ such that for any unit vector $v \in V$ and any $t \in (-\epsilon, \epsilon)$, $R_v(t)(x,y)z \in V$ for any $x, y, z \in V$.

(3) There is a positive number ϵ such that for any unit vector $v \in V$ and any $t \in (-\epsilon, \epsilon)$, $r_v(t)(x,y) \in V$ for any $x, y \in V$.

11.2. **Totally geodesic submanifolds of symmetric spaces.** The class of Riemannian manifolds with parallel Riemannian curvature tensor, that is, $\nabla R = 0$, was first introduced independently by P. A. Shirokov (1895–1944) in 1925 and by H. Levy in 1926. This class is known today as the class of locally symmetric Riemannian spaces. È. Cartan noticed in 1926 that irreducible spaces of this type are separated into ten large classes each of which depends on one or two arbitrary integers, and in addition there exist twelve special classes corresponding to the exceptional simple groups. Based on this, Cartan created his theory of symmetric Riemannian spaces in his famous papers "Sur une classe remarquable d'espaces de Riemann" [Cartan 1926/7].

An isometry s of a Riemannian manifold M is called involutive if its iterate $s^2 = s \circ s$ is the identity map. A Riemannian manifold M is called a symmetric space if, for each point $p \in M$, there exists an involutive isometry s_p of M such that p is an isolated fixed point of s_p. The s_p is called the (point) symmetry of M at the point p.

Denote by G_M, or simply by G, the closure of the group of isometries generated by $\{s_p : p \in M\}$ in the compact-open topology. Then G is a Lie group which acts transitively on the symmetric space; hence the typical isotropy subgroup H, say at o, is compact and $M = G/H$.

Every complete totally geodesic submanifold of a symmetric space is a symmetric space. For a symmetric space M, the dimension of a maximal flat totally geodesic submanifold of M is a well-defined natural number which is called the rank of M, denoted by $rk(M)$.

It follows from the equation of Gauss that $rk(B) \leq rk(M)$ for each totally geodesic submanifold B of a symmetric space M.

11.2.1. Canonical decomposition and Cartan's criterion

If $M = G/H$ is a symmetric space and o is a point in M, then the map

$$\sigma : G \to G$$

defined by $\sigma(g) = s_o g s_o$ is an involutive automorphism of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Then σ gives rise to an involutive automorphism of $\mathfrak{g}$, also denoted by σ. $\mathfrak{h}$ is the eigenspace of σ with eigenvalue 1.

Let $\mathfrak{m}$ denote the eigenspace of σ on $\mathfrak{g}$ with eigenvalue -1. One has the decomposition:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

which is called the Cartan decomposition or the canonical decomposition of $\mathfrak{g}$ with respect to σ.

The subspace $\mathfrak{m}$ can be identified with the tangent space of the symmetric space M at o in a natural way. A linear subspace $\mathcal{L}$ of $\mathfrak{m}$ is called a Lie triple system if it satisfies $[[\mathcal{L}, \mathcal{L}], \mathcal{L}] \subset \mathcal{L}$.

The following result of É. Cartan provides a simple relationship between totally geodesic submanifolds and Lie triple systems of a symmetric space: Let M be a symmetric space. Then a subspace $\mathcal{L}$ of $\mathfrak{m}$ forms a Lie triple system if and only if $\mathcal{L}$ is the tangent space of a totally geodesic submanifold of M through o.

11.2.2. Totally geodesic submanifolds of rank one symmetric spaces

Applying Cartan's criterion, J. A. Wolf completely classified in 1963 totally geodesic submanifolds in rank one symmetric spaces and obtained the following:

(1) The maximal totally geodesic submanifolds of the real projective m-space RP^m are RP^{m-1};

(2) The maximal totally geodesic submanifolds of the complex projective m-space CP^m are RP^m and CP^{m-1};

(3) The maximal totally geodesic submanifolds of the quaternionic projective m-space HP^m are HP^{m-1} and CP^m; and

(4) The maximal totally geodesic submanifolds of the Cayley plane $\mathcal{O}P^2$ are HP^2 and $\mathcal{O}P^1 = S^8$.

11.2.3. Totally geodesic submanifolds of complex quadric

Applying Cartan's criterion, Chen and H. S. Lue (1975a) classified totally geodesic surfaces in the complex quadric: $Q_m = SO(m+2)/SO(2) \times SO(m)$, $m > 1$. The complete classification of totally geodesic submanifolds of Q_m was obtained by Chen and T. Nagano in 1977. More precisely, they proved the following.

(1) If B is a maximal totally geodesic submanifold of Q_m, B is one of the following three spaces:

(1-a) Q_{m-1}, embedded as a Kähler submanifold;

(1-b) a local Riemannian product, $(S^p \times S^q)/\{\pm \mathrm{id}\}$, of two spheres S^p and S^q, $p + q = m$, of the same radius, embedded as a Lagrangian submanifold; and

(1-c) the complex projective space CP^n with $2n = m$, embedded as a Kähler submanifold.

(2) If B is a non-maximal totally geodesic submanifold of Q_m, M is either contained in Q_{m-1} in an appropriate position in Q_m, or the real projective space RP^n with $2n = m$.

Chen and Nagano (1977) also proved the following:

(3) Each homology group $H_k(Q_m; \mathbf{Z})$, $k < 2m$, is spanned by the classes of totally geodesic submanifolds of Q_m;

(4) The cohomology ring $H^*(Q_m; \mathbf{Z})$ is generated by the Poincaré duals of totally geodesic submanifolds of Q_m; and

(4) There is a maximal totally geodesic submanifold M of Q_m such that the differentiable manifold Q_m is the union of the normal bundles to M and to its focal manifold with the nonzero vectors identified in some way.

11.2.4. Totally geodesic submanifolds of compact Lie groups

Totally geodesic submanifolds of compact Lie groups equipped with biinvariant metrics have been determined in [Chen-Nagano 1978].

Let M be a compact Lie group with a biinvariant metric. Then the local isomorphism classes of totally geodesic submanifolds of M are those of symmetric space $B = G_B/H_B$ such that G_B are subgroup of $G_M = M \times M$.

11.2.5. (M_+, M_-)-method

In general, it is quite difficult to classify totally geodesic submanifolds of a given symmetric space with rank ≥ 2 by classifying the Lie triple systems via Cartan's criterion. For this reason a new approach to compact symmetric spaces was introduced by Chen and Nagano [Chen-Nagano 1978, Chen 1987a]. Using their method, totally geodesic submanifolds in compact symmetric spaces were systematically investigated.

The method of Chen and Nagano works as follows: A pair of points $\{o, p\}$ in a compact symmetric space M is called an antipodal pair if there exists a smooth closed geodesic γ in M such that p is the midpoint of γ from o. For each pair $\{o, p\}$ of antipodal points in a compact symmetric space $M = G/H$, they introduced a pair of orthogonal totally geodesic submanifolds $M_+^o(p), M_-^o(p)$ through p such that

$$\dim M_+^o(p) + \dim M_-^o(p) = \dim M, \quad rk(M_-^o(p)) = rk(M), \quad M_+^o(p) = H(p).$$

The totally geodesic submanifolds M_+'s and M_-'s are called polars and meridians of M, respectively.

A compact symmetric space M is globally determined by its polars and meridians. In fact, two compact symmetric spaces M and N are isometric if and only if some pair $(M_+(p), M_-(p))$ of M is isometric to some pair $(N_+(q), N_-(q))$ of N pairwise [Chen-Nagano 1978, Nagano 1992].

If B is a complete totally geodesic submanifold of a compact symmetric space M, then, for any pair $(B_+(p), B_-(p))$ of B, there is a pair $(M_+(q), M_-(q))$ of M such that $B_+(p)$ and $B_-(p)$ are totally geodesic submanifolds of $M_+(q)$ and $M_-(q)$, respectively. Since the same argument applies to the totally geodesic submanifold $B_+(p) \subset M_+(q)$ and to the totally geodesic submanifold $B_-(p) \subset M_-(q)$, one obtains strings of conditions.

Also, given a pair of antipodal points $\{o, p\}$ in a compact symmetric space M, one obtains an ordered pair $(M^o_+(p), M^o_-(p))$ as above. Two pairs $(M^o_+(p), M^o_-(p))$ and $(M^{o'}_+(p'), M^{o'}_-(p'))$ are called equivalent if there is an isometry on M which carries one to the other. Let $P(M)$ denote the corresponding moduli space. Then $P(M)$ is a finite set which is a global Riemannian invariant of M.

In general, one has $\#P(M) \le 2^{rk(M)} - 1$. A compact irreducible symmetric space satisfying $\#P(M) = 2^{rk(M)} - 1$ if and only if it is a rank one symmetric space.

Every isometric totally geodesic embedding $f : B \to M$ of a compact symmetric space into another induced a pairwise totally geodesic immersion $P(f) : P(B) \to P(M)$. In particular, if B and M have the same rank, then $P(f)$ is surjective, hence, $\#P(B) \ge \#P(M)$; this provides us a useful obstruction to totally geodesic embeddings as well.

In particular, Chen and Nagano's results imply the following:

(1) Any compact symmetric space M of dimension ≥ 2 admits a totally geodesic submanifold B satisfying $\frac{1}{2}\dim M \le \dim B < \dim M$;

(2) Spheres and hyperbolic spaces are the only simply-connected irreducible symmetric spaces admitting a totally geodesic hypersurface.

The result (2) was extended by K. Tojo (1997a) to the following: Let G be a compact simple Lie group and K a closed subgroup of G. If the normal homogeneous space $M = G/K$ contains a totally geodesic hypersurface, then M is a space with constant sectional curvature.

Further information on polars and meridians and on their applications to both geometry and topology can be found in [Chen 1987; Chen-Nagano 1978; Nagano 1988; Nagano 1992; Nagano-Sumi 1989; Burns 1992,1993; Peterson 1987; Burns-Clancy 1994].

There is a duality between totally geodesic submanifolds of symmetric spaces of compact type and of their non-compact duals.

For the investigation of totally geodesic submanifolds in some symmetric spaces of non-compact type with rank ≥ 2, see also [Berger 1957].

11.2.6. The 2-number $\#_2 M$

The notion of 2-number was introduced by Chen and Nagano in 1982. The notion of 2-number can also be applied to determine totally geodesic embeddings in symmetric spaces.

For a compact symmetric space M, the 2-number, denoted by $\#_2 M$, is defined as the maximal possible cardinality $\#_2 A_2$ of a subset A_2 of M such that the point symmetry s_x fixes every point of A_2 for every $x \in A_2$.

The 2-number $\#_2 M$ is finite. The definition is equivalent to saying that $\#_2 M$ is a maximal possible cardinality $\# A_2$ of a subset A_2 of M such that, for every pair of points x and y of A_2, there exists a closed geodesic of M on which x and y are antipodal to each other. Thus, the invariant can also be defined on any Riemannian manifold.

The geometric invariant $\#_2 M$ is an obstruction to the existence of a totally geodesic embedding $f : B \to M$, since the existence of f clearly implies the inequality $\#_2 B \leq \#_2 M$. For example, while the complex Grassmann manifold $G^C(2,2)$ of the 2-dimensional complex subspaces of the complex vector space C^4 is obviously embedded into $G^C(3,3)$ as a totally geodesic submanifold, the "bottom space" $G^C(2,2)^*$ obtained by identifying every member of $G^C(2,2)$ with its orthogonal complement in C^4, however, cannot be totally geodesically embedded into $G^C(3,3)^*$, simply because $\#_2 G^C(2,2)^* = 15 > 12 = \#_2 G^C(3,3)^*$.

The 2-number is not an obstruction to a topological embedding; for instance, the real projective space RP^n can be topologically embedded in a sufficiently high dimensional sphere, but the 2-number $\#_2 RP^n = n + 1 > 2$ simply prohibits a totally geodesic embedding of RP^n into any sphere whose 2-number is 2, regardless of dimension.

The invariant $\#_2 M$ has certain bearings on the topology of M in other aspects; for instance, Chen and Nagano proved that $\#_2 M$ is equal to the Euler number $\mathcal{X}(M)$ of M, if M is a semisimple Hermitian symmetric space. And in general they proved that the inequality $\#_2 M \geq \mathcal{X}(M)$ holds for any compact symmetric space M (cf. [Chen-Nagano 1988] for details).

M. Takeuchi (1989) proved that $\#_2 M = \dim H(M; \mathbf{Z}_2)$ for any symmetric R-space. This formula is actually correct for every compact symmetric space which have been checked.

For a group manifold G, Chen and Nagano showed that $\#_2 G = 2^{r_2}$, where r_2 is the 2-rank of G, which by definition is the maximal possible rank of the elementary 2-subgroup $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ of G. An immediate application of this fact is that the algebraic notion of 2-rank of a compact Lie group G, studied by Borel and Serre (1953), can be determined by computing the 2-number $\#_2 G$ of the group manifold G via the theory of submanifolds.

For the determination of $\#_2 M$ of compact symmetric spaces and of group manifolds, and the relationship between $\#_2 M$ and (M_+, M_-)'s, see [Chen-Nagano 1988].

The notion of 2-number was extended in 1993 to k-number for compact k-symmetric spaces by C. U. Sánchez. She also obtained a formula for k-number similar to Takeuchi's for flag manifolds.

11.3. **Stability of totally geodesic submanifolds.** A minimal submanifold N of a Riemannian manifold M is called stable if its second variation for the volume functional of M is positive for every variation of N in M. It is an interesting and important problem to find all stable minimal submanifolds in each symmetric space, in particular, to determine all stable totally geodesic submanifolds.

11.3.1. Stability of submanifolds in compact rank one symmetric spaces

Stability of compact totally geodesic submanifolds in compact rank one symmetric spaces have been completely determined in [Simons 1968, Lawson-Simons 1973, Ohnita 1986a]:

(1) Compact totally geodesic submanifolds of S^m are unstable.

(2) Compact totally geodesic submanifolds of RP^m are stable.

(3) A compact totally geodesic submanifold of CP^m is stable if and only if it is a complex projective subspace.

(4) A compact totally geodesic submanifold of HP^m is stable if and only if it is a quaternionic projective subspace.

(5) A compact totally geodesic submanifold of the Cayley plane $\mathcal{O}P^2$ is stable if and only if it is a Cayley projective line $\mathcal{O}P^1 = S^8$ of $\mathcal{O}P^2$.

11.3.2. An algorithm for determining the stability of totally geodesic submanifolds in symmetric spaces

There is a general algorithm, discovered by B. Y. Chen, P. F. Leung and T. Nagano in 1980, for determining the stability of totally geodesic submanifolds in compact symmetric spaces.

Let N be a totally geodesic submanifold of a compact symmetric space M. There is a finitely covering group G_N of the connected isometry group G_N^o of N such that G_N is a subgroup of the connected isometry group G_M of M which leaves N invariant, provided that G_N^o is semisimple. Let $\mathcal{P}$ denote the orthogonal complement of the Lie algebra $\mathfrak{g}_N$ in the Lie algebra $\mathfrak{g}_M$ with respect to the biinvariant inner product on $\mathfrak{g}_M$ which is compatible with the metric of M. Every member of $\mathfrak{g}_M$ is thought of as a Killing vector field because of the action of G_M on M.

Let $\hat{P}$ denote the space of the vector fields corresponding to the member of $\mathcal{P}$ restricted to the submanifold N. Then, to every member of $\mathcal{P}$ there corresponds a unique (but not canonical) vector field $v \in \hat{P}$ which is a normal vector field and hence $\hat{P}$ is a G_N-invariant subspace of the space $\Gamma(T^{\perp}N)$ of the sections of the normal bundle to N. Moreover, $\hat{P}$ is homomorphic with $\mathcal{P}$ as a G_N-module. The group G_N acts on $\Gamma(T^{\perp}N)$ and hence on the differential operators: $\Gamma(T^\perp N) \to \Gamma(T^\perp N)$. G_N leaves L fixed, since L is defined with N and the metric of M only. Therefore, each eigenspace of L is left invariant by G_N.

Let V be one of its G_N-invariant irreducible subspaces. One has a representation $\rho : G_N \to GL(V)$. Denote by $c(V)$ or $c(\rho)$ the eigenvalue of the corresponding Casimir operator. To define $c(V)$ one fixes an orthonormal basis (e_λ) for $\mathfrak{g}_N$ and consider the linear endomorphism C or C_V of V defined by

$$C = -\sum \rho(e_\lambda)^2. \tag{11.1}$$

Then C is $c(V)I_V$, where I_V is the identity map on V. The Casimir operator is given by $C_V = -\sum [e_\lambda, [e_\lambda, V]\,]$ for every member v of V (after extending to a neighborhood of N).

A compact totally geodesic submanifold N $(= G_N/K_N)$ of a compact symmetric space $M\,(= G_M/K_M)$ is stable as a minimal submanifold if and only if one has $c(V) \geq c(P')$ for the eigenvalue of the Casimir operator of every simple G_N-module V which shares as a K_N-module some simple K_N-submodule of the K_N-module $T_o^\perp N$ in common with some simple G_N-submodule P' of $\hat{P}$.

Roughly speaking, the algorithm says that N is stable if and only if $c(V) \geq c(\mathcal{P})$ for every G_N-invariant irreducible space V (cf. [Chen 1990]).

Applying their algorithm, Chen, Leung and Nagano obtained in 1980 the following results:

(1) A compact subgroup N of a compact Lie group M with a biinvariant metric is stable if N has the same rank as M and M has nontrivial center.

(2) Every meridian M_- of a compact group manifold M is stable if M has nontrivial center.

(3) Let $G^R(p,q) = SO(p+q)/SO(p) \times SO(q)$ be a real Grassmann manifold isometrically immersed in a complex Grassmann manifold $G^C(p,q)$ as a totally real totally geodesic submanifold in a natural way. Then $G^R(p,q)$ is unstable in $G^C(p,q)$.

When $p = 1$, statement (3) reduces to a result of Lawson and Simons (1973).

Applying the algorithm, M. Takeuchi (1984) completely determined the stability of totally geodesic Lagrangian submanifolds of compact Hermitian symmetric spaces. He proved that if M is a compact Hermitian symmetric space and B a compact Lagrangian totally geodesic submanifold of M, then B is stable if and only if B is simply-connected.

K. Mashimo and H. Tasaki (1990b) applied the same algorithm to determine the stability of maximal tori of compact Lie groups and obtained the following:

(1) Let G be a connected closed subgroup of maximal rank in a compact Lie group U equipped with biinvariant metric. If a maximal torus of U is stable, then G is also stable.

(2) Let U be a compact connected simple Lie group and T be a maximal torus. Then T is unstable if and only if U is isomorphic to

$$SU(r+1), \quad Spin(5), \quad Spin(7), \quad Sp(r) \quad \text{or} \quad G_2.$$

Further results concerning the stability of certain subgroups of compact Lie groups equipped with biinvariant metrics can also be found in [Fomenko 1972, Thi 1977, Brothers 1986, Mashimo-Tasaki 1990a].

The stabilities of all the M_+'s (polars) and the M_-'s (meridians) of a compact irreducible symmetric space M were determined by M. S. Tanaka (1995). In particular, she proved that all polars and meridians of a compact Hermitian symmetric space are stable.

Let G be a compact connected Lie group, σ an automorphism of G and $K = \{k \in G\colon \sigma(k) = k\}$. A mapping $\Sigma\colon G \to G$, $g \mapsto g\sigma g^{-1}$, induces the Cartan embedding of G/K into G in a natural way. If M is a compact simple Lie group G, then the G_+'s are images of Cartan embeddings and the G_-'s are the sets of fixed points of involutive automorphisms.

K. Mashimo (1992) proved that, if G is simple and σ is involutive, the image $\Sigma(G/K)$ is unstable only if either G/K is a Hermitian symmetric space or the pair (G, K) is one of the four cases:

$$(SU(n), SO(n)) \;\; (n \geq 3), \quad (SU(4m{+}2)/\{\pm I\}, SO(4m{+}2)/\{\pm I\}) \;\; (m \geq 1),$$

$$(Spin(n), (Spin(n-3) \times Spin(3))/\mathbf{Z}_2) \;\; (n \geq 7), \quad (G_2, SO(4)).$$

11.3.3. Ohnita's formulas

Y. Ohnita (1987) improved the above algorithm to include the formulas for the index, the nullity and the Killing nullity of a compact totally geodesic submanifold in a compact symmetric space.

Let $f : N \to M$ be a compact totally geodesic submanifold of a compact Riemannian symmetric space. Then $f : N \to M$ is expressed as follows: There are compact symmetric pairs (G, K) and (U, L) with $N = G/K$, $M = U/L$ so that $f : N \to M$ is given by $uK \mapsto \rho(u)L$, where $\rho : G \to U$ is an analytic homomorphism with $\rho(K) \subset L$ and the injective differential $\rho : \mathfrak{g} \to \mathfrak{u}$ which satisfies $\rho(\mathfrak{m}) \subset \mathfrak{p}$. Here $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ are the canonical decompositions of the Lie algebras u and g, respectively.

Let $\mathfrak{m}^\perp$ denote the orthogonal complement of $\rho(\mathfrak{m})$ with $\mathfrak{p}$ relative to the $\mathrm{ad}(U)$-invariant inner product $(\ ,\ )$ on $\mathfrak{u}$ such that $(\ ,\ )$ induces the metric of M. Let $\mathfrak{k}^\perp$ be the orthogonal complement of $\rho(\mathfrak{k})$ in $\mathfrak{l}$. Put $\mathfrak{g}^\perp = \mathfrak{k}^\perp + \mathfrak{m}^\perp$. Then $\mathfrak{g}^\perp$ is the orthogonal complement of $\rho(\mathfrak{g})$ in $\mathfrak{u}$ relative to $(\ ,\ )$, and $\mathfrak{g}^\perp$ is $\mathrm{ad}_\rho(G)$-invariant.

Let θ be the involutive automorphism of the symmetric pair (U, L). Choose an orthogonal decomposition $\mathfrak{g}^\perp = \mathfrak{g}_1^\perp \oplus \cdots \oplus \mathfrak{g}_t^\perp$ such that each $\mathfrak{g}_i^\perp$ is an irreducible $\mathrm{ad}_\rho(G)$-invariant subspace with $\theta(\mathfrak{g}_i^\perp) = \mathfrak{g}_i^\perp$. Then, by Schur's lemma, the Casimir operator C of the representation of G on each $\mathfrak{g}_i^\perp$ is $a_i I$ for some $a_i \in \mathbf{C}$.

Put $\mathfrak{m}_i^\perp = \mathfrak{m} \cap \mathfrak{g}_i^\perp$ and let $D(G)$ denote the set of all equivalent classes of finite dimensional irreducible complex representations of G. For each $\lambda \in D(G)$, $(\rho_\lambda, V_\lambda)$ is a fixed representation of λ.

For each $\lambda \in D(G)$, we assign a map A_λ from $V_\lambda \otimes \mathrm{Hom}_K(V_\lambda, W)$ to $C^\infty(G, W)_K$ by the rule $A_\lambda(v \otimes L)(u) = L(\rho_\lambda(u^{-1})v)$. Here $\mathrm{Hom}_K(V_\lambda, W)$ denotes the space of all linear maps L of V_λ into W so that $\sigma(k)\cdot L = L\cdot\rho_\lambda(k)$ for all $k \in K$.

Y. Ohnita's formulas for the index $i(f)$, the nullity $n(f)$, and the Killing nullity $n_k(f)$ are given respectively by

$i(f) = \sum_{i=1}^t \sum_{\lambda\in D(G), a_\lambda < a_i} m(\lambda) d_\lambda$,

$n(f) = \sum_{i=1}^t \sum_{\lambda\in D(G), a_\lambda = a_i} m(\lambda) d_\lambda$,

$n_k(f) = \sum_{i=1, \mathfrak{m}_i^\perp \neq \{0\}}^t \dim \mathfrak{g}_i^\perp$,

where $m(\lambda) = \dim \mathrm{Hom}_K(V_\lambda, (\mathfrak{m}_i^\perp)^C)$ and d_λ denotes the dimension of the representation λ.

By applying his formulas, Ohnita determined the indices, the nullities and the Killing nullities for all totally geodesic submanifolds in compact rank one symmetric spaces and Helgason spheres in all compact irreducible symmetric spaces.

Q. Zhao applied Ohnita's formulas in [Zhao 1996] to determine the indices, the nullities and the Killing nullities of maximal totally geodesic submanifolds of the complex quadric Q_m.

11.4. **Helgason's spheres.** Let M be a compact irreducible symmetric space and let κ denote the maximum of the sectional curvatures of M. By a theorem of É. Cartan, the same dimensional maximal totally geodesic flat submanifolds of M, that is, the maximal tori, are all conjugate under the largest connected Lie group $I_0(M)$ of isometries.

S. Helgason (1927–) proved in 1966 an analogous result for the maximum curvature κ as follows:

(1) A compact irreducible symmetric space M contains totally geodesic submanifolds of maximum constant curvature κ.

(2) Any two such totally geodesic submanifolds of the same dimension are conjugate under $I_0(M)$.

(3) The maximal dimension of such submanifolds is $1+m(\bar{\delta})$, where $m(\bar{\delta})$ is the multiplicity of the highest restricted root. Also $\kappa = ||\bar{\delta}||^2$, where $|| \ ||$ denotes length.

(4) If M is a simply-connected compact irreducible symmetric space, then the closed geodesics in M of minimal length are permuted transitively by $I_0(M)$.

A maximal dimensional totally geodesic sphere with maximal possible sectional curvature κ in a compact irreducible symmetric space is known as a Helgason's sphere.

The stability of Helgason's spheres was determined by Y. Ohnita in 1987. He proved that every Helgason's sphere in a compact irreducible symmetric space is stable as totally geodesic submanifolds.

The Helgason sphere in a compact simple Lie group is the compact simple 3-dimensional Lie subgroup associated with the highest root. In 1977 C. T. Dao showed that Helgason's spheres in some compact classical Lie groups are homologically volume minimizing. H. Tasaki proved in 1985 that the Helgason spheres in any compact simple Lie groups are homologically volume minimizing using the canonical 3-form $\langle [X,Y], Z\rangle$ divided by the length of the highest root as a calibration and by root systems.

Le Khong Van showed in 1993 the volume minimizing property of Helgason spheres in all compact simply-connected irreducible symmetric spaces. In 1995 H. Tasaki obtained estimates of the volume of Helgason's spheres and of cut loci using the generalized Poincaré formula proved by R. Howard (1993) and established the inequality of volume minimizing of the Helgason spheres. He then proved that under certain suitable conditions a k-dimensional Helgason sphere of a compact symmetric space is volume minimizing in the class of submanifolds of dimension k whose inclusion maps are not null homotopic. Tasaki's conditions hold automatically for compact symmetric spaces of rank one, compact Hermitian symmetric spaces, and quaternionic Grassmann manifolds.

The maximal dimension of totally geodesic submanifolds of positive constant sectional curvature in an arbitrary compact irreducible symmetric space have been completely determined in [Chen-Nagano 1978, Nagano-Sumi

1991] by applying the (M_+, M_-)-method. Such dimensions play an important role in the study of totally umbilical submanifolds, in particular, in the study of extrinsic spheres, in locally symmetric spaces.

11.5. **Frankel's theorem.** T. J. Frankel (1961) proved that if M is a complete Riemannian manifold of positive sectional curvature and V and W are two compact totally geodesic submanifolds with $\dim V + \dim W \geq \dim M$, then V and W have a nonempty intersection. Applying this result, Frankel showed in 1966 that if M is a complete Riemannian manifold of strictly positive sectional curvature and if V is a compact totally geodesic submanifold of M with $\dim V \geq \frac{1}{2} \dim M$, then the homomorphism of fundamental groups: $\pi_1(V) \to \pi_1(M)$ is surjective.

In 1961 Frankel also showed that if M is a complete Kähler manifold of positive sectional curvature, then any two compact Kähler submanifolds must intersect if their dimension sum is at least that of M. S. I. Goldberg and S. Kobayashi (1967) extended this Frankel's result to complete Kähler manifolds of positive bisectional curvature.

Frankel's theorems were extended in 1996 by K. Kenmotsu and C. Xia to complete Riemannian manifolds which has positive k-Ricci curvature or to Kähler manifold with partially positive bisectional curvature. For instance, they proved that if M is a complete Riemannian manifold with nonnegative k-Ricci curvature and V and W are complete totally geodesic submanifolds satisfying (a) V and W are immersed as closed subsets, (b) one of V and W is compact, (3) M has positive k-Ricci curvature either at all points of V or at all points of W, and (4) $\dim V + \dim W \geq \dim M + k - 1$, then V and W must intersect.

In 1966 Frankel also proved that two compact minimal hypersurfaces of a compact Riemannian manifold with positive Ricci curvature must intersect.

12. Totally umbilical submanifolds

A submanifold N of a Riemannian manifold M is called totally umbilical if its second fundamental form h is proportional to its first fundamental form g, that is, $h(X,Y) = g(X,Y)H$ for vectors X, Y tangent to N, where H is the mean curvature vector. Total umbilicity is a conformal invariant in the sense that if N is a totally umbilical submanifold of a Riemannian manifold M, then N is also a totally umbilical submanifold of M endowed with another Riemannian metric which is conformal equivalent to the original Riemannian metric on M. Each connected component of the fixed point set of a conformal transformation on a Riemannian manifold is a totally umbilical submanifold. From Riemannian geometric point of views, totally umbilical submanifolds are the simplest submanifolds next to totally geodesic ones.

A totally umbilical submanifold N in a Riemannian manifold M is called an extrinsic sphere if its mean curvature vector field is a nonzero parallel normal vector field. 1-dimensional extrinsic spheres in Riemannian manifolds are called circles.

Hyperspheres in Euclidean space are the most well-known examples of totally umbilical submanifolds and also of extrinsic spheres.

Since every curve in a Riemannian manifold is totally umbilical, we shall only consider totally umbilical submanifolds of dimension ≥ 2.

Totally umbilical surfaces in E^3 are open parts of planes and round spheres. This result was first proved by J. Meusnier in 1785 who showed that open parts of planes and spheres are the only surfaces in E^3 satisfying the property that the curvature of the plane sections through each point of the surface are equal.

In 1954 J. A. Schouten proved that every totally umbilical submanifold of dimension ≥ 4 in a conformally flat space is conformally flat.

An n-dimensional submanifold M of E^m satisfies $S \geq nH^2$, with the equality holding identically if and only if M is totally umbilical, where S and H^2 denote the squared norm of the second fundamental form and the squared mean curvature function.

12.1. Totally umbilical submanifolds of real space forms.

Totally umbilical submanifolds in real space forms have been completely classified. Totally umbilical submanifolds of dimension ≥ 2 in real space forms are either totally geodesic submanifolds or extrinsic spheres. An n-dimensional non-totally geodesic, totally umbilical submanifold of a Euclidean m-space E^m is an ordinary hypersphere which is contained in an affine $(n+1)$-subspace of E^m.

An n-dimensional totally umbilical submanifold of a Riemannian m-sphere S^m or of a hyperbolic m-space H^m is contained in an $(n+1)$-dimensional totally geodesic submanifold as a totally umbilical hypersurface.

An n-dimensional non-totally geodesic totally umbilical submanifold of a real projective m-space RP^m is contained in a totally geodesic RP^{n+1} of RP^m. Such a submanifold is obtained from a totally umbilical submanifold of a Riemannian m-sphere via the two-fold Riemannian covering map $\pi : S^m \to RP^m$.

12.2. **Totally umbilical submanifolds of complex space forms.** Totally umbilical submanifolds in other rank one symmetric spaces are also known. Totally umbilical submanifolds in CP^m and in its non-compact dual are classified in [Chen-Ogiue 1974c].

Let N be an n-dimensional, $(n \geq 2)$, totally umbilical submanifold of a real $2m$-dimensional Kähler manifold M of constant holomorphic sectional curvature $4c$, $c \neq 0$. Then N is one of the following submanifolds:

(1) a complex space form isometrically immersed in M as a totally geodesic complex submanifold;

(2) a real space form isometrically immersed in M as a totally real, totally geodesic submanifold;

(3) a real space form isometrically immersed in an $(n+1)$-dimensional totally real totally geodesic submanifold of M as an extrinsic hypersphere.

12.3. **Totally umbilical submanifolds of quaternionic space forms.** Totally umbilical submanifolds in a quaternionic projective space HP^m and in its non-compact dual have been classified in [Chen 1978].

Let N be an n-dimensional, $(n \geq 4)$, totally umbilical submanifold of a real $4m$-dimensional quaternionic space form M of constant quaternionic sectional curvature $4c$, $c \neq 0$. Then N is one of the following submanifolds:

(1) a quaternionic space form isometrically immersed in M as a totally geodesic quaternionic submanifold;

(2) a complex space form isometrically immersed in M as a totally geodesic, totally complex submanifold;

(3) a real space form isometrically immersed in M as a totally real, totally geodesic submanifold;

(4) a real space form isometrically immersed in an $(n+1)$-dimensional totally real totally geodesic submanifold of M as an extrinsic hypersphere.

12.4. **Totally umbilical submanifolds of the Cayley plane.** Totally umbilical submanifolds of the Cayley plane $\mathcal{O}P^2$ and of its noncompact dual have also been classified [Chen 1977b, Nikolaevskij 1994]:

(1) The maximum dimension of totally umbilical submanifolds in the Cayley plane $\mathcal{O}P^2$ is 8.

(2) A maximal totally umbilical submanifold of $\mathcal{O}P^2$ is one of the following:

(2-a) a totally geodesic quaternionic projective plane HP^2;

(2-b) a totally geodesic Cayley line $\mathcal{O}P^1 = S^8$;

(2-b) an extrinsic hypersurface of a totally geodesic Cayley line $\mathcal{O}P^1 = S^8$; or

(2-c) a non-totally geodesic totally umbilical submanifold of a totally geodesic quaternionic projective plane HP^2.

The corresponding result also holds for totally umbilical submanifolds in the non-compact dual of the Cayley plane.

12.5. **Totally umbilical submanifolds in complex quadric.** Yu. A. Nikolaevskij (1991) classified totally umbilical submanifolds in complex quadric $Q_m = SO(m+2)/SO(m) \times SO(2)$.

An n-dimensional ($n \geq 3$) totally umbilical submanifold N of the complex quadric Q_m is one of the following:

(1) a totally geodesic submanifold;

(2) an extrinsic sphere;

(3) a totally umbilical submanifold with nonzero and nonparallel mean curvature vector.

The last case can be described in the following two ways:

(3-a) N is an umbilical hypersurface of non-constant mean curvature lying in the totally geodesic $S^p \times S^1 \in Q_m$; or

(3-b) N is a diagonal of the product of two small spheres lying in the totally geodesic $S^{l+1} \times S^{l+1} \in Q_m$; moreover, the mean and sectional curvatures of N are both constant.

12.6. **Totally umbilical submanifolds of locally symmetric spaces.** T. Miyazawa and G. Chuman (1972) studied totally umbilical submanifolds of locally symmetric spaces and obtained the following.

(1) A totally umbilical submanifold of a locally symmetric space is locally symmetric if and only if its mean curvature is constant.

(2) Let N be a totally umbilical submanifold of dimension ≥ 4 in a locally symmetric space. If the mean curvature is nowhere zero, then M is conformally flat.

A Riemannian manifold is called reducible if it is locally the Riemannian product of two Riemannian manifolds of positive dimensions.

The following results of Chen (1981a) determines reducible totally umbilical submanifolds of locally symmetric spaces:

(1) Every reducible totally umbilical submanifold of a locally symmetric space has constant mean curvature.

(2) A reducible totally umbilical submanifold of a locally symmetric space M is one of the following locally symmetric spaces:

(2.1) A totally geodesic submanifold,

(2.2) a locally Riemannian product of a curve and a Riemannian space form of constant curvature,

(2.3) a locally Riemannian product of two Riemannian space forms of constant sectional curvatures c and $-c$, $c \neq 0$, respectively.

Irreducible totally umbilical submanifolds in locally symmetric spaces do not have constant mean curvature in general. For irreducible totally umbilical submanifolds with constant mean curvature, we have the following [Chen 1980a]: If N is an n-dimensional ($n \geq 2$) irreducible totally umbilical submanifold with constant mean curvature in a locally symmetric space M, then N is either a totally geodesic submanifold or a real space form. Furthermore, N is either a totally geodesic submanifold or an extrinsic sphere, unless $\dim N < \frac{1}{2} \dim M$.

Chen and P. Verheyen (1983) studied totally umbilical submanifolds in locally Hermitian symmetric spaces and obtained the following: Let N be an n-dimensional ($n \geq 4$) totally umbilical submanifold of a locally Hermitian symmetric space M. If $n > \dim_C M$, then either

(1) N is a totally geodesic submanifold, or

(2) $rk(M) > n$ and N is an extrinsic sphere in a maximal flat totally geodesic submanifold of M.

This result implies, in particular, that if N is an n-dimensional ($n \geq 4$) totally umbilical submanifold in a locally Hermitian symmetric space M of compact or non-compact type and if $n > \dim_C M$, then N is totally geodesic.

For totally umbilical submanifolds in symmetric spaces, Yu. A. Nikolaevskij (1994) proved that an n-dimensional ($n \geq 3$) totally umbilical submanifold in a globally symmetric space M is either totally geodesic or totally umbilical and complete in a totally geodesic submanifold of M which is isometric to the product of some Riemannian space forms. In particular, he proved that if N is an n-dimensional totally umbilical submanifold in an irreducible symmetric space of compact type, then it is either totally geodesic or totally umbilical in the totally geodesic product $\bar{M}$ of flat torus and spheres.

12.7. **Extrinsic spheres in locally symmetric spaces.** Extrinsic spheres in Riemannian manifolds can be characterized as follows: Let N be an n-dimensional ($n \geq 2$) submanifold of a Riemannian manifold M. If, for some

$r > 0$, every circle of radius r in N is a circle in M, then N is an extrinsic sphere in M. Conversely, if N is an extrinsic sphere in M, then every circle in N is a circle in M [Nomizu-Yano 1974].

Extrinsic spheres in locally symmetric spaces were completely classified by Chen (1977a) as follows:

Let N be an n-dimensional ($n \geq 2$) extrinsic sphere of a locally symmetric space M. Then N is an extrinsic hypersphere of an $(n+1)$-dimensional totally geodesic submanifold of M with constant sectional curvature.

Conversely, every extrinsic sphere of dimension ≥ 2 in a locally symmetric space is obtained in such way.

The above result implies, in particular, that

(1) every extrinsic sphere in a locally symmetric space is a real space form, and

(2) real space forms are the only locally symmetric spaces of dimension ≥ 3 which admit an extrinsic hypersphere.

An extrinsic sphere in a Riemannian manifold is not necessary a Riemannian sphere in general. In contrast, for extrinsic spheres in Kähler manifolds, we have the following result of [Chen 1976a].

Every complete simply-connected even-dimensional extrinsic sphere in a Kähler manifold is isometric to a Riemannian sphere if it has flat normal connection.

This result is false if the extrinsic sphere is odd-dimensional. In fact, there exist many complete odd-dimensional simply-connected extrinsic spheres with flat normal connection in Kähler manifolds which are not Riemannian spheres, even not homotopy spheres [Chen 1981a].

In 1984 S. Yamaguchi, H. Nemoto and N. Kawabata showed that if a complete, connected and simply-connected extrinsic sphere in a Kähler manifold is not isometric to an ordinary sphere, then it is homothetic to either a Sasakian manifold or a totally real submanifold.

Extrinsic spheres in locally conformally Kähler manifolds were treated in [Dragomir-Ornea 1998].

12.8. **Totally umbilical hypersurfaces.** O. Kowalski (1972) proved that every totally umbilical hypersurface of an Einstein manifold of dimension ≥ 3 is either a totally geodesic hypersurface or an extrinsic hypersphere.

Not every Riemannian manifold admits a totally umbilical hypersurface. The following result of Chen (1981a) determined all locally symmetric spaces which admit a non-totally geodesic totally umbilical hypersurface:

A locally symmetric space admits a non-totally geodesic, totally umbilical hypersurface if and only if, locally, it is one of the following spaces:

(1) a real space form;

(2) a Riemannian product of a line and a Riemannian space form;

(3) a Riemannian product of two real space forms of constant curvatures c and $-c$, respectively.

Thus, if an irreducible locally symmetric space admits a totally umbilical hypersurface, then it is a real space form.

It follows from the above result that non-totally geodesic, totally umbilical hypersurfaces of a locally symmetric space are given locally by the fixed point sets of some conformal mappings. More precisely, if N is a non-totally geodesic totally umbilical hypersurface of a locally symmetric space M, then, for each point $x \in N$, there is a neighborhood U of x in M and a conformal mapping ϕ of U into M such that $U \cap N$ lies in the fixed point set of ϕ.

K. Tojo (1997b) studied Riemannian homogeneous spaces which admit extrinsic hyperspheres and proved that if a natural reductive homogeneous Riemannian manifold admits an extrinsic hypersphere, then it must be a real space form.

K. Tsukada (1996) proved that if $M = G/H$ is a Riemannian homogeneous space such that the identity component of H acts irreducibly on the tangent space and if $\dim M \geq 3$, then M admits no totally umbilical hypersurfaces unless M is a real space form.

13. Conformally flat submanifolds

A Riemannian n-manifold M is called conformally flat if, at each point $x \in M$, there is a neighborhood of x in M which is conformal to the Euclidean n-space. An immersed submanifold $f : M \to E^m$ is called a conformally flat submanifold if the submanifold is conformally flat with respect to the induced metric.

Since every Riemannian 2-manifold is conformally flat, due to the existence of local isothermal coordinate system, we only consider conformally flat manifolds of dimension greater than or equal to 3.

According to a well-known result of H. Weyl (1918), a Riemannian manifold of dimension $n \geq 4$ is conformally flat if and only if its Weyl conformal curvature tensor W vanishes identically. The Weyl conformal curvature tensor W vanishes identically for $n = 3$.

Define a tensor L of type $(0, 2)$ on a Riemannian n-manifold by

$$L(X,Y) = -\left(\frac{1}{n-2}\right) Ric(X,Y) + \left(\frac{\rho}{2(n-1)(n-2)}\right) g(X,Y), \tag{13.1}$$

where Ric is the Ricci tensor and $\rho =$ trace Ric. Let D be the (0,3)-tensor defined by

$$D(X,Y,Z) = (\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z). \tag{13.2}$$

H. Weyl (1918) proved that the tensor D vanishes identically for a conformally flat manifold of dimension $n \geq 4$; and a Riemannian 3-manifold is conformally flat if and only if D vanishes identically..

N. H. Kuiper (1949) proved that a compact simply-connected conformally flat manifold of dimension ≥ 2 is conformally equivalent to S^n.

13.1. **Conformally flat hypersurfaces.** .

13.1.1. Quasi-umbilicity of conformally flat hypersurfaces

The study of nonflat conformally flat hypersurfaces of dimension $n \geq 4$ was initiated by É. Cartan around 1918. He proved that a hypersurface of dimension ≥ 4 in Euclidean space is conformally flat if and only if it is quasi-umbilical, that is, it has a principal curvature with multiplicity $\geq n-1$.

The Codazzi equation implies that every quasi-umbilical hypersurface of dimension 3 in a conformally flat 4-manifold is conformally flat. The converse of this is not true in general. In fact, G. M. Lancaster (1973) showed that there exist conformally flat hypersurfaces in E^4 which have three distinct principal curvatures; hence there exist conformally flat hypersurfaces in E^4 which are not quasi-umbilical.

13.1.2. Canal hypersurfaces

A hypersurface of Euclidean space is called a canal hypersurface if it is the envelope of one-parameter family of hyperspheres. É. Cartan (1918) proved that a canal hypersurface of dimension $n \geq 4$ in Euclidean space is conformally flat.

13.1.3. Conformally flat hypersurfaces as loci of spheres

A conformally flat hypersurface of dimension ≥ 4 in a real space form $R^{n+1}(c)$ is a locus of $(n-1)$-spheres, in the sense that it is given by smooth gluing of some n-dimensional submanifolds of M (possibly with boundary) such that each of the submanifolds is foliated by totally umbilical $(n-1)$-submanifolds of $R^{n+1}(c)$ (cf. [Chen 1973b]).

D. E. Blair (1975) proved that the generalized catenoid and the hyperplanes are the only conformally flat minimal hypersurfaces in E^{n+1} with $n \geq 4$.

13.1.4. Intrinsic properties of conformally flat hypersurfaces

An intrinsic characterization of conformally flat manifolds admitting isometric immersions in real space forms as hypersurfaces was given by Chen and Yano.

A conformally flat manifold M of dimension $n \geq 4$ is called special if there exist three functions α, β and γ on M such that the tensor L defined by (13.1) takes the form:

$$L = -\frac{1}{2}\left(k + \alpha^2\right) g - \alpha\beta\, \omega \otimes \omega, \tag{13.3}$$

for some constant k, where ω is a unit 1-form satisfying $d\alpha = \gamma\omega$ on the open subset $U = \{x \in M : \beta(x) \neq 0\}$, where $L = -\frac{1}{2}\left(k + \alpha^2\right) g$ on $\{x \in M : \beta(x) = 0\}$.

For a special conformally flat space M, we define a real number i_M by

$$\begin{aligned} i_M = \sup\{k \in \mathbf{R} : L = -\frac{k+\alpha^2}{2} g - \alpha\beta\, \omega \otimes \omega \\ \text{for some functions } \alpha,\ \beta \text{ on } M\}, \end{aligned}$$

which is called the index of the special conformally flat manifold.

Chen and Yano proved the following (cf. [Chen 1973b]):

(1) Every conformally flat hypersurface of dimension $n \geq 4$ in a real space form is special.

(2) Conversely, every simply-connected special conformally flat manifold of dimension $n \geq 4$ with index i_M can be isometrically immersed in every real space form of curvature $k < i_M$ as a hypersurface and it cannot be isometrically immersed in any real space form of curvature $k > i_M$ as a hypersurface.

In 1982, M. do Carmo and M. Dajczer showed that conformally flat manifolds are the only Riemannian n-manifolds, $n \geq 4$, which can be isometrically immersed as a hypersurface in two real space forms of different curvatures.

U. Pinkall (1988) proved that every compact conformally flat hypersurface in E^{n+1}, $n \geq 4$ is conformally equivalent to a classical Schottky manifold. Pinkall's result improves a result of [do Carmo-Dajczer-Mercuri 1985].

It is not known whether every classical Schottky manifold admits a conformal immersion into E^{n+1}.

13.1.5. Taut conformally flat hypersurfaces

Let $f : M \to E^m$ be an immersion and $p \in E^m$. Denote the function

$$x \in M \to |f(x) - p|^2$$

by L_p. Suppose $\phi : M \to \mathbf{R}$ is a Morse function on manifold M. If for all real r, $M_r = \phi^{-1}(-\infty, r]$ is compact, then the Morse inequality $\mu_k \geq \beta_k$ holds, where μ_k is the number of critical points of index k which ϕ has on M_r, and β_k is the k-th Betti number of M_r over any field F.

The function ϕ is called a T-function if there is a field F such that the Morse inequality is an equality for all r and k. An immersion $f : M \to E^m$ is said to be taut if every function of the form L_p, $p \in E^m$, is a T-function.

T. E. Cecil and P. Ryan (1980) proved that a complete conformally flat hypersurface of E^{n+1}, $n \geq 4$, is taut if and only if it is one of the following:

(a) a hyperplane or a round sphere;

(b) a cylinder over a circle or round $(n-1)$-sphere;

(c) a ring cyclide (diffeomorphic to $S^1 \times S^{n-1}$);

(d) a parabolic cyclide (diffeomorphic to $S^1 \times S^{n-1}$ with a point removed).

The proofs of all of the above results rely on Cartan's condition of quasi-umbilicity on conformally flat hypersurfaces.

13.2. **Conformally flat submanifolds.** .

13.2.1. Totally quasi-umbilical submanifolds

An n-dimensional submanifold M of a Riemannian $(n+p)$-manifold N is called quasiumbilical (respectively, umbilical) with respect to a normal vector field ξ if the shape operator A_ξ has an eigenvalue with multiplicity $\geq n-1$ (respectively, multiplicity n). In this case, ξ is called a quasiumbilical (respectively, umbilical) normal section of M.

An n-dimensional submanifold M of a Riemannian $(n+p)$-manifold is called totally quasiumbilical if there exist p mutually orthogonal quasiumbilical normal sections on M.

A result of Chen and Yano (1972) states that a totally quasiumbilical submanifold of dimension ≥ 4 in a conformally flat manifold is conformally flat.

The property of being totally quasiumbilicity is a conformal invariant, that is, the property remains under every conformal change of the metric of the ambient space.

For conformally flat submanifolds of higher codimension, Chen and L. Verstraelen (1977) proved that an n-dimensional ($n \geq 4$) conformally flat submanifold M with flat normal connection in a conformally flat $(n+p)$-manifold is totally quasiumbilical if $p < n-2$.

Ü. Lumiste and M. Väljas (1989) showed that Chen-Verstraelen's result is sharp in the sense that there exists a conformally flat submanifold M of dimension $n \geq 4$ with flat normal connection in a conformally flat $(2n-2)$-manifold which is not totally quasiumbilical.

J. D. Moore and J. M. Morvan (1978) showed that an n-dimensional conformally flat submanifold M in E^{n+p} is totally quasiumbilical if $n > 7$ and $p < 4$.

A relationship between the notion of quasiumbilicity and focal points of conformally flat submanifolds was given in [Morvan-Zafindratafa 1986].

13.2.2. Submanifolds admitting an umbilical normal section

A normal vector field ξ of a submanifold is said to be parallel if $D_X\xi = 0$ for each tangent vector X of M, where D denotes the normal connection. The length of a parallel normal vector field is constant. A normal vector field ξ of M is called nonparallel if, for each $x \in M$, there is a tangent vector $X \in T_xM$ such that $D_X\xi \neq 0$.

A submanifold $f : M \to E^{n+p}$ of a Euclidean space admits an umbilical parallel unit normal section if and only if f is spherical, that is, $f(M)$ is contained in a hypersphere of E^{n+p}. On the other hand, if a submanifold M of codimension 2 in a Euclidean space admits an umbilical nonparallel unit normal section ξ, then M must be quasiumbilical with respect to each unit normal vector field perpendicular to ξ. Thus, M is a totally quasiumbilical submanifold. In particular, if $\dim M \geq 4$, then the submanifold is conformally flat [Chen-Yano 1973].

13.2.3. Normal curvature tensor as a conformal invariant

Another general conformal property of submanifolds is that the normal curvature tensor of a submanifold of a Riemannian manifold is a conformal invariant [Chen 1974].

If M is a submanifold of a Riemannian manifold N endowed with Riemannian metric $\tilde{g}$ and if $\tilde{g}^* = e^{2\rho}\tilde{g}$ is a conformal change of the metric $\tilde{g}$, then the normal curvature tensor R^D of M in $(N, \tilde{g})$ and the normal curvature tensor R^{*D} of M in $(N, \tilde{g}^*)$ satisfy $R^{*D}(X,Y) = R^D(X,Y)$ for any vectors X, Y tangent to M.

13.2.4. CW-decomposition of a conformally flat submanifold

Applying Morse theory, J. D. Moore (1977) proved that a compact n-dimensional conformally flat submanifold of E^{n+p} possesses a CW-decomposition with no cells of dimension k with $p < k < n - p$.

13.2.5. A sufficient condition for a locus of spheres to be conformally flat

Chen (1973b) proved that a locus M^n of $(n-1)$-spheres $(n \geq 4)$ in an $(n+p)$-dimensional real space form $R^{n+p}(c)$ with arbitrary p is conformally flat if the unit normal vector field of each leaf in M^n is a parallel vector field in the normal bundle of the leaf in $R^{n+p}(c)$.

13.2.6. Conformally flat manifolds and hypersurfaces of light cone

A simply-connected Riemannian manifold of dimension $n \geq 3$ is conformally flat if and only if it can be isometrically immersed as a hypersurface of the light cone

$$V^{n+1} = \{X \in L^{n+2} : \langle X, X \rangle = 0, X \neq 0\},$$

where $\langle\ ,\rangle$ is the semi-definite metric on V^{n+1} induced from the standard Lorentzian metric on the flat $(n+2)$-dimensional Lorentzian space L^{n+2} [Brinkmann 1923, Asperti-Dajczer 1989].

13.2.7. Conformally flat submanifolds with constant index of conformal nullity

When $f : M^n \to E^{n+p}$, $p \leq n-3$, is a conformally flat submanifold, at each point $x \in M$, there is an umbilical subspace $\mathcal{U}(x) \subset T_xM$ with $\dim \mathcal{U}(x) \geq n - p$. Hence, there is a unit vector $\eta \in T_x^{\perp}M$ and a nonnegative number λ so that the second fundamental form satisfies $h(Z,X) = \lambda \langle Z, X \rangle \eta$, for each $Z \in \mathcal{U}(x)$ and each $X \in T_xM$.

The umbilical distribution $\mathcal{U}$ is integrable on any open subset where the dimension of $\mathcal{U}(x)$ is constant, which is denoted by $\nu_f^c(x)$. $\nu_f^c(x)$ is called the index of conformal nullity.

The leaves of the umbilical distribution are extrinsic spheres in M, hence they are totally umbilical submanifolds with parallel mean curvature vector.

We say that an isometric immersion $F : N^{n+1} \to \tilde{N}^{n+p}$ extends an isometric immersion $f : M^n \to \tilde{N}^{n+p}$ when there exists an isometric embedding of M^n into N^{n+1} such that $F|_M = f$.

M. Dajczer and L. A. Florit (1996) proved the following: Let $f : M^n \to E^{n+p}$, $n \geq 5$, $p \leq n-3$, be a simply-connected conformally flat submanifold without flat points. If f has constant index of conformal nullity ν_f^c, say ℓ, then there exist an extension $F : N^{n+1} \to E^{n+p}$ of f and an isometric immersion $G : N^{n+1} \to L^{n+2}$ so that $M^n = G(N^{n+1}) \cap V^{n+1}$. Moreover, F and G carry a common $(\ell+1)$-dimensional relative nullity foliation.

A conformally flat submanifold $f : M^n \to E^{n+2}$, $n \geq 5$, is called generic when its umbilical direction $\eta \in T^{\perp}M$ possesses everywhere a nonzero principal curvature λ of multiplicity $n-2$. An immersion $f : M^n \to E^{n+2}$ is called a composition if there exist an open subset $U \subset E^{n+1}$ and isometric immersions $\tilde{f} : M^n \to U$ and $H : U \to E^{n+2}$ such that $f = H \circ \tilde{f}$.

Dajczer and Florit (1996) proved that any conformally flat submanifold $f : M^n \to E^{n+2}$, $n \geq 5$, without flat points is locally along an open dense subset either generic or a composition.

13.2.8. A non-immersion theorem

H. Rademacher (1988) proved that if there exists a constant c such that the Ricci curvature of a compact conformally flat n-manifold M, $n \geq 4$, satisfies

$$-c^2 \leq \mathrm{Ric}(X) \leq -\left(\frac{n-1}{2n-3}\right) c^2$$

for any unit vector X tangent to M, then M cannot be conformally immersed in S^{2n-2}.

14. Submanifolds with parallel mean curvature vector

A submanifold of a Riemannian manifold is said to have parallel mean curvature vector if the mean curvature vector field H is parallel as a section of the normal bundle. Trivially, every minimal submanifold of a Riemannian manifold has parallel mean curvature vector. A minimal submanifold of a hypersphere in Euclidean space has nonzero parallel mean curvature vector when it is considered as a submanifold of the ambient Euclidean space. Furthermore, a hypersurface of a Riemannian manifold has parallel mean curvature vector if and only if it has constant mean curvature. By introducing the notion of twisted product, Chen (1981a) showed that every Riemannian manifold can be embedded in some twisted product Riemannian manifold as a submanifold with nonzero parallel mean curvature vector. H. Reckziegel (1974) proved if M is a compact submanifold of a manifold N such that $TN|_M$ has a metric γ and η is a nonzero normal vector field of constant length in $(TM)^\perp$, then γ can be extended to a Riemannian metric on N such that M is an extrinsic sphere with parallel mean curvature vector η.

It was S. S. Chern who first suggested in the mid 1960s that the notion of parallel mean curvature vector as the natural extension of constant mean curvature for hypersurfaces.

14.1. **Gauss map and mean curvature vector.** For an isometric immersion $f : M \to E^m$ of an oriented n-dimensional Riemannian manifold into Euclidean m-space, the Gauss map

$$G : M \to G^R(m-n, m)$$

of f is a smooth map which carries a point $x \in M$ into the oriented $(m-n)$-plane in E^m, which is obtained from the parallel translation of the normal space of M at x in E^m, where $G^R(m-n,m)$ denotes the Grassmannian manifold consisting of oriented $(m-n)$-planes in E^m.

E. A. Ruh and J. Vilms (1970) characterized submanifolds of Euclidean space with parallel mean curvature vector as follows: A submanifold M of a Euclidean m-space E^m has parallel mean curvature vector if and only if its Gauss map G is harmonic.

14.2. **Riemann sphere with parallel mean curvature vector.** One-dimensional submanifolds with parallel mean curvature vector are nothing but geodesics and circles.

D. Ferus (1971b) and E. A. Ruh (1971) determined completely closed surfaces of genus zero with parallel mean curvature vector in Euclidean space:

Let M be a closed oriented surface of genus zero in E^m. If M has parallel mean curvature vector, then M is contained in a hypersphere of E^m as a minimal surface.

14.3. **Surfaces with parallel mean curvature vector.** Surfaces in E^4 with parallel mean curvature vector were classified by D. Hoffman in his doctoral thesis (Stanford University 1971).

The complete classification of surfaces in Euclidean n-space, $n \geq 4$, with parallel mean curvature vector was obtained by Chen (1972) and, independently by Yau (1974).

A surface M of Euclidean m-space E^m has parallel mean curvature vector if and only if it is one of the following surfaces:

(1)a minimal surface of E^m;

(2) a minimal surface of a hypersphere of E^m;

(3) a surface of E^3 with constant mean curvature;

(4) a surface of constant mean curvature lying in a hypersphere of an affine 4-subspace of E^m.

Similar results hold for surfaces with parallel mean curvature vector in spheres and in real hyperbolic spaces as well (cf. [Chen 1973b]).

For surfaces of constant Gaussian curvature, Chen and G. D. Ludden (1972) and D. Hoffman (1973) proved that minimal surfaces of a small hypersphere, open pieces of the product of two plane circles, and open pieces of a circular cylinder are the only non-minimal surfaces in Euclidean space with parallel mean curvature vector and with constant Gaussian curvature.

For a compact surface M with positive constant Gaussian curvature, K. Enomoto (1985) proved that if $f : M \to E^m$ is an isometric embedding with constant mean curvature and flat normal connection, then $f(M)$ is a round sphere in an affine 3-subspace of E^m. Enomoto also proved that if $f : M \to E^{n+2}$ is an isometric embedding of a compact Riemannian n-manifold, $n \geq 4$, with positive constant sectional curvature and with constant mean curvature, then $f(M)$ is a round n-sphere in a hyperplane of E^{n+2}.

For codimension two using the method of equivariant differential geometry, W. T. Hsiang, W. Y. Hsiang and I. Sterling (1985) proved the following:

(a) There exist infinitely many codimension two embeddings of distinct knot types of S^{4k+1} into $S^{4k+3}(1)$ with parallel mean curvature vector of arbitrarily small constant length.

(b) There exist infinitely many codimension two embeddings of distinct knot types of the Kervaire exotic sphere Σ_0^{4k+1} into $S^{4k+3}(1)$ with parallel mean curvature vector having length of arbitrarily small constant value.

(c) There exist infinitely many constant mean curvature embeddings of $(4k-1)$-dimensional generalized lens spaces into $S^{4k+1}(1)$.

It remains as an open problem to completely classify submanifolds of dimension ≥ 3 with parallel mean curvature vector in real space forms.

14.4. **Surfaces with parallel normalized mean curvature vector.** Chen (1980b) defined a submanifold in a Riemannian manifold to have parallel normalized mean curvature vector field if there exists a unit parallel vector field ξ which is parallel to the mean curvature vector field H, that is, $H = \alpha\xi$ for some unit parallel normal vector field ξ.

Submanifolds with nonzero parallel mean curvature vector field also have parallel normalized mean curvature vector field. The condition to have parallel normalized mean curvature vector field is much weaker than the condition to have parallel mean curvature vector field. For instance, every hypersurface in a Riemannian manifold always has parallel normalized mean curvature vector field.

For surfaces with parallel normalized mean curvature vector field, we have the following results from [Chen 1980b]:

(1) Let M be a Riemann sphere immersed in a Euclidean m-space E^m. Then M has parallel normalized mean curvature vector field if and only if either

(1-a) M is immersed in a hypersphere of E^m as a minimal surface, or

(1-b) M is immersed in an affine 3-subspace of E^m.

(2) A surface M of class C^ω in a Euclidean m-space E^m has parallel normalized mean curvature vector field if and only if M is one of the following surfaces:

(2-a) a minimal surface of a hyperplane of E^m,

(2-b) a surface in an affine 4-subspace of E^m with parallel normalized mean curvature vector.

Every surface in a Euclidean 3-space has parallel normalized mean curvature vector field. Moreover, there exist abundant examples of surfaces which lie fully in a Euclidean 4-space with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field.

14.5. **Submanifolds satisfying additional conditions.** It is a classical theorem of Liebmann that the only closed convex surfaces in Euclidean 3-space having constant mean curvature are round spheres. B. Smyth (1973) extended Liebmann's result to the following: Let M be a compact n-dimensional submanifold with nonnegative sectional curvature in Euclidean m-space. If M has parallel mean curvature vector, then M is a product submanifold

$M_1 \times \cdots \times M_k$, where each M_i is a minimal submanifold in a hypersphere of an affine subspace of E^m.

Further, K. Yano (1912–1993) and S. Ishihara (1922–) in 1971 and J. Erbacher in 1972 extended Liebmann's result to the following: Let M be an n-dimensional submanifold in Euclidean m-space with nonnegative sectional curvature. Suppose that the mean curvature vector is parallel in the normal bundle and the normal connection is flat. If M is either compact or has constant scalar curvature, then M is the standard product immersion of the product $S^{n_1}(r_1) \times \cdots \times S^{n_k}(r_k)$ of some spheres.

Recently, Y. Zheng (1997) proved the following: Let M be a compact orientable submanifold with constant scalar curvature and with nonnegative sectional curvature immersed in a real space form of constant sectional curvature c. Suppose that M has flat normal connection. If the normalized scalar curvature of M is greater than c, then M is either totally umbilical or locally the Riemannian product of several totally umbilical constantly curved submanifolds.

For complete submanifolds M^n of dimension ≥ 3 in Euclidean space, Y. B. Shen (1985) proved the following: Let M^n $(n \geq 3)$ be a complete submanifold in the Euclidean space E^m with parallel mean curvature vector. If the squared mean curvature H^2 and the squared length S of the second fundamental form of M satisfies

$$(n-1)S \leq n^2 H^2,$$

then M^n is an n-plane, an n-sphere S^n, or a circular cylinder $S^{n-1} \times E^1$. This extended some results of [Chen-Okumura 1973, Okumura 1973].

G. Chen and X. Zou (1995) studied compact submanifolds of spheres with parallel mean curvature vector and proved the following:

Let M be an n-dimensional compact submanifold with nonzero parallel mean curvature vector in the unit $(n+p)$-sphere. Then

(1) M is totally geodesic, if one of the following two conditions hold:

$$S \leq \min \left\{ \frac{2}{3} n, \frac{2n}{1+\sqrt{\frac{n}{2}}} \right\}, \quad p \geq 2 \quad \text{and} \quad n \neq 8;$$

$$S \leq \min \left\{ \frac{n}{2-\frac{1}{p-1}}, \frac{2n}{1+\sqrt{\frac{n}{2}}} \right\}, \quad p \geq 1 \quad \text{and} \quad (n,p) \neq (8,3);$$

(2) M is totally umbilical, if $2 \leq n \leq 7$, $p \geq 2$, and $S \leq \frac{2}{3} n$.

14.6. **Homogeneous submanifolds with parallel mean curvature vector.** C. Olmos (1994,1995) studied homogeneous submanifolds of Euclidean space and proved the following.

(a) If M is a compact homogeneous submanifold of a Euclidean space with parallel mean curvature vector which is not minimal in a sphere, then M is an orbit of the isotropy representation of a simple symmetric space;

(b) A homogeneous irreducible submanifold of Euclidean space with parallel mean curvature vector is either minimal, or minimal in a sphere, or an orbit of the isotropy representation of a simple Riemannian symmetric space.

15. Kähler submanifolds of Kähler manifolds

According to the behavior of the tangent bundle of a submanifold with respect to the action of the almost complex structure J of the ambient manifold, there are several typical classes of submanifolds, namely, Kähler submanifolds, totally real submanifolds, CR-submanifolds and slant submanifolds.

In this section, the dimensions of complex manifolds always mean the complex dimensions, unless mentioned otherwise.

The theory of submanifolds of a Kähler manifold began as a separate area of study in the last century with the investigation of algebraic curves and algebraic surfaces in classical algebraic geometry. The study of complex submanifolds of Kähler manifolds from differential geometrical points of view (that is, with emphasis on the Riemannian metric) was initiated by E. Calabi in the early of 1950's.

15.1. **Basic properties of Kähler submanifolds.** A submanifold of a complex manifold is called a complex submanifold if each of its tangent spaces is invariant under the almost complex structure of the ambient manifold. A complex submanifold of a Kähler manifold is itself a Kähler manifold with respect to its induced metric. By a Kähler submanifold we mean a complex submanifold with the induced Kähler structure.

It was proved by Calabi (1953) that Kähler submanifolds of Kähler manifolds always have rigidity. Thus, for any two full Kähler immersions f and f' of the same Kähler manifold M into CP^m and into CP^N, respectively, we have $m = N$ and, moreover, there exists a unique holomorphic isometry Ψ of CP^m onto itself such that $\Psi \circ f = f'$.

The second fundamental form of a Kähler submanifold M of a Kähler manifold with the almost complex structure J satisfies

$$h(JX, Y) = h(X, JY) = Jh(X, Y),$$

for X, Y tangent to M. From this it follows that Kähler submanifolds of Kähler manifolds are always minimal.

Compact Kähler submanifolds of Kähler manifolds are also stable and have the property of being absolutely volume minimizing inside the homology class. Moreover, a compact Kähler submanifold M of a Kähler manifold $\tilde{M}$ can never be homologous to zero, that is, there exists no submanifold M' of $\tilde{M}$ such that M is the boundary of M' [Wirtinger 1936].

A Kähler submanifold of a Kähler manifold is said to be of degree k if the pure part of the $(k-1)$-st covariant derivative of the second fundamental form is identically zero but the pure part of the $(k-2)$-nd one is not identically zero. In particular, degree 1 is nothing but totally geodesic and degree 2 is equivalent to parallel second fundamental form but not totally geodesic.

Let $CP^m(c)$ denote the complex projective m-space equipped with the Fubini-Study metric of constant holomorphic sectional curvature c and M be an n-dimensional Kähler submanifold of $CP^m(c)$.

Denote by g, K, H, Ric, ρ and h the metric tensor, the sectional curvature, the holomorphic sectional curvature, the Ricci tensor, the scalar curvature and the second fundamental form of M, where $\rho = \sum_{i \neq j} K(e_i \wedge e_j)$ and $\{e_1, \ldots, e_n\}$ is an orthonormal frame on M.

It follows from the equation of Gauss that an n-dimensional Kähler submanifold of a Kähler m-manifold $\tilde{M}^m(c)$ of constant holomorphic sectional curvature c satisfies the following curvature properties in general:

(1) $H \leq c$, with equality holding identically if and only if M is a totally geodesic Kähler submanifold.

(2) $Ric \leq \frac{c}{2}(n+1)g$, with equality holding identically if and only if M is a totally geodesic Kähler submanifold.

(3) $\rho \leq n(n+1)c$, with equality holding identically if and only if M is a totally geodesic Kähler submanifold.

Let $z_0, z_1, \ldots, z_{n+1}$ be a homogeneous coordinate system of $CP^{n+1}(c)$ and Q_n be the complex quadric hypersurface of $CP^{n+1}(c)$ defined by

$$Q_n = \{(z_0, z_1, \ldots, z_{n+1}) \in CP^{n+1}(c) : \sum z_i^2 = 0\}.$$

Then Q_n is complex analytically isometric to the compact Hermitian symmetric space $SO(n+2)/SO(n) \times SO(2)$.

With respect to the induced Kähler metric g, Q_n satisfies the following:

(1) $0 \leq K \leq c$ for $n \geq 2$ and $K = \frac{c}{2}$ for $n = 1$.

(2) $\frac{c}{2} \leq H \leq c$ for $n \geq 2$ and $H = \frac{c}{2}$ for $n = 1$.

(3) $Ric = \frac{n}{2}cg$.

(4) $\rho = n^2 c$.

P. B. Kronheimer and T. S. Mrowka (1994) proved the Thom conjecture concerning the genus of embedded surfaces in CP^2, namely, they proved that if C is a smooth holomorphic curve in CP^2, and C' is a smoothly embedded oriented 2-manifold representing the same homology class as C, then the genus of C' satisfies $g(C') \geq g(C)$. The proof of this result uses Seiberg-Witten's invariant on 4-manifolds.

15.2. **Complex space forms and Chern classes.** A Kähler manifold is called a complex space form if it has constant holomorphic sectional curvature. The universal covering of a complete complex space form $\tilde{M}^n(c)$ is the complex projective n-space $CP^n(c)$, the complex Euclidean n-space C^n, or the complex hyperbolic space $CH^n(c)$, according to $c > 0$, $c = 0$, or $c < 0$.

Complex space forms can be characterized in terms of the first and the second Chern classes. In fact, B. Y. Chen and K. Ogiue (1975) proved the following sharp inequality between the first and the second Chern numbers for Einstein-Kähler manifolds. They also applied their inequality to characterize complex space forms:

Let M be an n-dimensional compact Einstein-Kähler manifold. Then the first and the second Chern classes of M satisfy

$$\varepsilon^n \int_M 2(n+1)c_1^{n-2}c_2 \geq \varepsilon^n \int_M nc_1^n \quad (\varepsilon \text{ the sign of } \rho), \tag{15.1}$$

with the equality holding if and only if M is either a complex space form or a Ricci-flat Kähler manifold. Chen-Ogiue's inequality was extended by M. Lübke in 1982 to Einstein-Hermitian vector bundle over compact Kähler manifolds in the sense of S. Kobayashi (1932–) (cf. [Kobayashi 1987]).

In this respect, we mention that T. Aubin (1976) proved that if M is a compact Kähler manifold with $c_1 < 0$ (that is, c_1 is represented by a negative definite real $(1,1)$-form), then there exists a unique Einstein-Kähler metric on M whose Kähler form is cohomologous to the Kähler form of the initial given metric. Consequently, by combining these two results, it follows that every compact Kähler manifold with $c_1 < 0$ satisfies inequality (15.1), with the equality holding if and only if M is covered by the complex hyperbolic n-space.

S. T. Yau (1977) proved that if M is a compact Kähler manifold with $c_1 = 0$, then it admits a Ricci-flat Kähler metric.

15.3. **Kähler immersions of complex space forms in complex space forms.** M. Umehara (1987a) studied Kähler immersions between complex space forms and obtained the following.

(1) A Kähler submanifold of a complex Euclidean space cannot be a Kähler submanifold of any complex hyperbolic space;

(2) A Kähler submanifold of a complex Euclidean space cannot be a Kähler submanifold of any complex projective space, and

(3) A Kähler submanifold of a complex hyperbolic space cannot be a Kähler submanifold of any complex projective space.

Kähler immersions of complex space forms in complex space forms are completely classified by H. Nakagawa and K. Ogiue in 1976 as follows.

Let $M^n(c)$ be an n-dimensional complex space form isometrically immersed in an m-dimensional complex space form $\tilde{M}^m(\bar{c})$ as a Kähler submanifold such that the immersion is full. Then

(1) if $\bar{c} \leq 0$, then the immersion is totally geodesic;

(2) if $\bar{c} > 0$, then $\bar{c} = \mu c$ and $m = \binom{n+\mu}{\mu} - 1$ for some positive integer μ. Moreover, either the immersion is totally geodesic or locally the immersion is given by one of the Veronese embeddings.

This result is due to Calabi (1953) when both $M^n(c)$ and $\tilde{M}^m(\bar{c})$ are complete simply-connected complex space forms.

An immersion $f\, M \to \tilde{M}$ between Riemannian manifolds is called proper d-planar geodesic if every geodesic in M is mapped into a d-dimensional totally geodesic subspace of $\tilde{M}$, but not into any $d-1$-dimensional totally geodesic subspace of $\tilde{M}$.

J. S. Pak and K. Sakamoto (1986,1988) proved that and $f : M^n \to CP^m$ a proper d-planar geodesic Kähler immersion from a Kähler manifold into CP^m, d odd or $d \in \{2, 4\}$, then f is equivalent to the d-th Veronese embedding of CP^n into CP^m.

15.4. **Einstein-Kähler submanifolds and Kähler submanifolds $\tilde{M}$ satisfying $Ric(X,Y) = \widetilde{Ric}(X,Y)$.** For complex hypersurfaces of complex space forms we have the following:

(1) Let M be a Kähler hypersurface of an $(n+1)$-dimensional complex space form $\tilde{M}^{n+1}(c)$. If $n \geq 2$ and M is Einstein, then either M is totally geodesic or $Ric = \frac{n}{2}cg$.

The latter case occurs only when $c > 0$. Moreover, the immersion is rigid [Smyth 1968, Chern 1967].

(2) Let M be a compact Kähler hypersurface embedded in CP^{n+1}. If M has constant scalar curvature, then M is either totally geodesic in CP^{n+1} or holomorphically isometric to Q_n in CP^{n+1}. [Kobayashi 1967a].

(3) Let M be a Kähler hypersurface of an $(n+1)$-dimensional complex space form $\tilde{M}^{n+1}(c)$. If the Ricci tensor of M is parallel, then M is an Einstein space [Takahashi 1967].

J. Hano (1975) proved that, besides linear subspaces, Q_n is the only Einstein-Kähler submanifold of a complex projective space which is a complete intersection.

M. Umehara (1987b) proved that every Einstein-Kähler submanifold of C^m or CH^m is totally geodesic.

B. Smyth (1968) proved that the normal connection of a Kähler hypersurface M^n in a Kähler manifold $\tilde{M}^{n+1}$ is flat if and only if the Ricci tensors of M^n and $\tilde{M}^{n+1}$ satisfy $Ric(X,Y) = \widetilde{Ric}(X,Y)$ for X, Y tangent to M^n.

For general Kähler submanifolds Chen and Lue (1975b) proved the following.

Let M be a compact Kähler submanifold of a compact Kähler manifold $\tilde{M}$. Then

(1) if the normal connection is flat, the Ricci tensors of M and $\tilde{M}$ satisfy $Ric(X,Y) = \widetilde{Ric}(X,Y)$ for X,Y tangent to M;

(2) if $Ric(X,Y) = \widetilde{Ric}(X,Y)$ for any X,Y tangent to M, then the first Chern class of the normal bundle is trivial, that is, $c_1(T^{\perp}M) = 0$;

(3) if $\tilde{M}$ is flat, then the first Chern class of the normal bundle is trivial if and only if the normal connection is flat.

15.5. **Ogiue's conjectures and curvature pinching.** An n-dimensional complex projective space of constant holomorphic sectional curvature c can be analytically isometrically embedded into an $\left(\binom{n+\mu}{\mu} - 1\right)$-dimensional complex projective space of constant holomorphic sectional curvature μc. Such an embedding is given by all homogeneous monomials of degree μ in homogeneous coordinates, which is called the μ-th Veronese embedding of $CP^n(c)$. The degree of the μ-th Veronese embedding is μ.

The Veronese embeddings were characterized by A. Ros (1986) in terms of holomorphic sectional curvature in the following theorem: If a compact n-dimensional Kähler submanifold M immersed in $CP^m(c)$ satisfies

$$\frac{c}{\mu+1} < H \leq \frac{c}{\mu},$$

then $M = CP^n(\frac{c}{\mu})$ and the immersion is given by the μ-th Veronese embedding.

Kähler submanifolds of degree ≤ 2 are characterized by Ros (1985b) as follows: If a compact Kähler submanifold M immersed in $CP^m(c)$ satisfies $H \geq 1/2$, then degree ≤ 2; and, moreover, the Kähler submanifold is congruent to one of the following six Kähler manifolds:

$$CP^n(c), \;\; CP^n\left(\frac{c}{2}\right), \; Q_n = SO(n+2)/SO(n) \times SO(2),$$

$$SU(r+2)/S(U(r) \times U(2)), \; r \geq 3, \;\; SO(10)/U(5),$$

$$E_6/Spin(10) \times SO(2).$$

A. Ros and L. Verstraelen (1984) and Liao (1988) characterized the second Veronese embedding in terms of sectional curvature in the following theorem.

If a compact n-dimensional ($n \geq 2$) Kähler submanifold M immersed in $CP^m(c)$ satisfies $K \geq 1/8$, then either M is totally geodesic or $M = CP^n(\frac{c}{2})$ and the immersion is the second Veronese embedding.

A. Ros (1986) characterized all Veronese embeddings by curvature pinching as follows: If a compact n-dimensional, $n \leq 2$, Kähler submanifold M

immersed in $CP^m(c)$ satisfies

$$\frac{c}{4(\mu+1)} \leq K \leq \frac{c}{\mu},$$

then either $M = CP(\frac{c}{\mu})$ and the immersion is given by the μ-th Veronese embedding or $M = CP^n(\frac{c}{\mu+1})$ and the immersion is given by the $(\mu+1)$-st Veronese embedding.

J. H. Cheng (1981) and R. J. Liao (1988) characterized complex quadric hypersurface in terms of its scalar curvature:

If a compact n-dimensional Kähler submanifold M immersed in $CP^m(c)$ satisfies $\rho \geq n^2$, then either M is totally geodesic or $M = Q_n$.

The above results gave affirmative answers to some of Ogiue's conjectures.

Totally geodesic Kähler submanifolds of $CP^m(c)$ are characterized in terms of Ricci curvature by K. Ogiue (1972a): If a compact n-dimensional Kähler submanifold M immersed in $CP^m(c)$ satisfies $Ric > \frac{n}{2}c$, then M is totally geodesic.

Chen and Ogiue (1974a) proved that if an n-dimensional Kähler submanifold M of $CP^m(c)$ satisfies $Ric = \frac{n}{2}cg$, then M is an open piece of Q_n which is embedded in some totally geodesic $CP^{n+1}(c)$ in $CP^m(c)$.

For compact Kähler submanifolds with positive sectional curvature, Y. B. Shen (1995) proved the following.

Let M^n be an n-dimensional, $n \geq 2$, compact Kähler submanifold immersed in CP^{n+p} with $p < n$. Then M^n has nonnegative sectional curvature if and only if M^n is one of the following:

(1) a totally geodesic Kähler submanifold in CP^{n+p}; or

(2) an embedded submanifold congruent to the standard full embedding of

(2-a) the complex quadric Q_n as a hypersurface; or

(2-b) of $CP^{n-1} \times CP^1$ with codimension $n-1$; or

(2-c) of $U(5)/U(3) \times U(2)$ of codimension 3; or

(2-d) of $U(6)/U(4) \times U(2)$ with codimension 6; or

(2-e) of $SO(10)/U(5)$ with codimension 5, or

(2-f) of $E_6/Spin(10) \times T$ with codimension 10.

F. Zheng (1996) studied Kähler submanifolds of complex Euclidean spaces and proved that if M is an n-dimensional Kähler submanifold with nonpositive sectional curvature in C^{n+r} with $r \leq n$, then

(a) if $r < n$, M is ruled, that is, there exists a holomorphic foliation $\mathcal{F}$ on an open dense subset U of M with each leaf of $\mathcal{F}$ totally geodesic in C^{n+r};

(b) if $r = n$, then either M is ruled, or M is locally holomorphically isometric to the product of n complex plane curves.

The dimension bound is sharp, as there exist examples of negatively curved submanifolds M^n in C^{2n+1} which are not ruled or product manifolds.

15.6. **Segre embedding.** Let $(z_0^i, \ldots, z_{N_i}^i)$ $(1 \le i \le s)$ denote the homogeneous coordinates of CP^{N_i}. Define a map:

$$S_{N_1 \cdots N_s} : CP^{N_1} \times \cdots \times CP^{N_s} \to CP^N, \quad N = \prod_{i=1}^{s}(N_i + 1) - 1,$$

which maps a point $((z_0^1, \ldots, z_{N_1}^1), \ldots, (z_0^s, \ldots, z_{N_s}^s))$ of the product Kähler manifold $CP^{N_1} \times \cdots \times CP^{N_s}$ to the point $(z_{i_1}^1 \cdots z_{i_j}^s)_{1 \le i_1 \le N_1, \ldots, 1 \le i_s \le N_s}$ in CP^N. The map $S_{N_1 \cdots N_s}$ is a Kähler embedding which is called the Segre embedding [Segre 1891].

Concerning Segre embedding, B. Y. Chen (1981b) and B. Y. Chen and W. E. Kuan (1985) proved the following:

Let $M_1, \ldots, M_s$ be Kähler manifolds of dimensions $N_1, \ldots,$ N_s, respectively. Then every Kähler immersion

$$f : M_1 \times \cdots \times M_s \to CP^N, \quad N = \prod_{i=1}^{s}(N_i + 1) - 1,$$

of $M_1 \times \cdots \times M_s$ into CP^N is locally the Segre embedding, that is, $M_1, \ldots, M_s$ are open portions of $CP^{N_1}, \ldots, CP^{N_s}$, respectively, and moreover, the Kähler immersion f is congruent to the Segre embedding.

This theorem was proved in [Nakagawa-Takagi 1976] under two additional assumptions; namely, $s = 2$ and the Kähler immersion f has parallel second fundamental form.

15.7. **Parallel Kähler submanifolds.** Ogiue (1972b) studied complex space forms in complex space forms with parallel second fundamental form and proved the following: Let $M^n(c)$ be a complex space form analytically isometrically immersed in another complex space form $M^m(\bar{c})$. If the second fundamental form of the immersion is parallel, then either the immersion is totally geodesic or $\bar{c} > 0$ and the immersion is given by the second Veronese embedding.

Complete Kähler submanifolds in a complex projective space with parallel second fundamental form were completely classified by H. Nakagawa and R. Tagaki in 1976. Their result states as follows.

Let M be a complete Kähler submanifold embedded in $CP^m(c)$. If M is irreducible and has parallel second fundamental form, then M is congruent to one of the following six kinds of Kähler submanifolds:

$$CP^n(c),\ \ CP^n\left(\frac{c}{2}\right),\ Q_n = SO(n+2)/SO(n)\times SO(2),$$

$$SU(r+2)/S(U(r)\times U(2)),\ r\geq 3,\ \ SO(10)/U(5), E_6/Spin(10)\times SO(2).$$

If M is reducible and has parallel second fundamental form, then M is congruent to $CP^{n_1}\times CP^{n_2}$ with $n=n_1+n_2$ and the embedding is given by the Segre embedding.

K. Tsukada (1985a) studied parallel Kähler submanifolds of Hermitian symmetric spaces and obtained the following: Let $f: M\to \tilde{M}$ be a Kähler immersion of a complete Kähler manifold M into a simply-connected Hermitian symmetric space $\tilde{M}$. If f has parallel second fundamental form, then M is the direct product of a complex Euclidean space and semisimple Hermitian symmetric spaces. Moreover, $f=f_2\circ f_1$, where f_1 is a direct product of identity maps and (not totally geodesic) parallel Kähler embeddings into complex projective spaces, and f_2 is a totally geodesic Kähler embedding.

15.8. **Symmetric and homogeneous Kähler submanifolds.** Suppose $f_i: M_i\to CP^{N_i},\ i=1,\dots,s$, are full Kähler embeddings of irreducible Hermitian symmetric spaces of compact type and $N=\prod_{i=1}^s(N_i+1)-1$. Then the composition

$$S_{N_1\cdots N_s}\circ(f_1\times\cdots\times f_s): M_1\times\cdots\times M_s\to CP^N$$

is a full Kähler embedding, which is called the tensor product of $f_1,\dots,f_s$.

H. Nakagawa and R. Tagaki (1976) and R. Tagaki and M. Takeuchi (1977) had obtained a close relation between the degree and the rank of a symmetric Kähler submanifold in complex projective space; namely, they proved the following.

Let $f_i: M_i\to CP^{N_i},\ i=1,\dots,s$, are p_i-th full Kähler embeddings of irreducible Hermitian symmetric spaces of compact type. Then the degree of the tensor product of $f_1,\dots,f_s$ is given by $\sum_{i=1}^s r_ip_i$, where $r_i=rk(M_i)$.

M. Takeuchi (1978) studied Kähler immersions of homogeneous Kähler manifolds and proved the following:

Let $f: M\to CP^N$ be a Kähler immersion of a globally homogeneous Kähler manifold M. Then

(1) M is compact and simply-connected;

(2) f is an embedding; and

(3) M is the orbit in CP^N of the highest weight in an irreducible unitary representation of a compact semisimple Lie group.

A different characterization of homogeneous Kähler submanifolds in CP^N was given by S. Console and A. Fino (1996).

H. Nakagawa and R. Tagaki (1976) proved that there do not exist Kähler immersions from locally symmetric Hermitian manifolds into complex Euclidean spaces and complex hyperbolic spaces except the totally geodesic ones.

15.9. **Relative nullity of Kähler submanifolds and reduction theorem.** For a submanifold M in a Riemannian manifold, the subspace

$$N_p = \{X \in T_pM : h(X, Y) = 0, \text{ for all } Y \in T_pM\}, \quad p \in M$$

is called the relative nullity space of M at p. The dimension $\mu(p)$ of N_p is called the relative nullity of M at p. The subset U of M where $\mu(p)$ assumes the minimum, say μ, is open in M, and μ is called the index of relative nullity.

K. Abe (1973) studied the index of relative nullity of Kähler submanifolds and obtained the following:

(1) Let M be an n-dimensional Kähler submanifold of a complex projective m-space CP^m. If M is complete, then the index of relative nullity is either 0 or $2n$. In particular, if $\mu > 0$, then $M = CP^n$ and it is embedded as a totally geodesic submanifold.

(2) Let M be an n-dimensional complete Kähler submanifold of the complex Euclidean m-space C^m. If the relative nullity is greater than or equal to $n-1$, then M is $(n-1)$-cylindrical.

Let M be an n-dimensional Kähler submanifold of an $(n+p)$-dimensional complex space form $\tilde{M}^{n+p}(c)$. A subbundle E of the normal bundle $T^{\perp}M$ is called holomorphic if E is invariant under the action of the almost complex structure J of $\tilde{M}^{n+p}(c)$. For a holomorphic subbundle E of $T^{\perp}M$, let

$$\nu_E(x) = \dim_C\{X \in T_xM : A_\xi X = 0 \text{ for all } \xi \in E_x\}.$$

Put $\nu_E = \mathrm{Min}_{x\in M}\nu_E(x)$, which is called the index of relative nullity with respect to E.

Chen and Ogiue (1973a) proved the following: Let M be an n-dimensional Kähler submanifold of a complex space form $\tilde{M}^{n+p}(c)$. If there exists an r-dimensional parallel normal subbundle E of the normal bundle such that $\nu_E(x) \equiv 0$, then M is contained in an $(n+r)$-dimensional totally geodesic submanifold of $\tilde{M}^{n+p}(c)$.

This result implies the following result of Chen and Ogiue (1973a) and T. E. Cecil (1974): If M is an n-dimensional Kähler submanifold of a complete complex space form $\tilde{M}^{n+p}(c)$ such that the first normal space, Im h, defines an r-dimensional parallel subbundle of the normal bundle, then M is contained in an $(n+r)$-dimensional totally geodesic submanifold of $\tilde{M}^{n+p}$.

16. Totally real and Lagrangian submanifolds of Kähler manifolds

The study of totally real submanifolds of a Kähler manifold from differential geometric points of views was initiated in the early 1970's. A totally real submanifold M of a Kähler manifold $\tilde{M}$ is a submanifold such that the almost complex structure J of the ambient manifold $\tilde{M}$ carries each tangent space of M into the corresponding normal space of M, that is, $J(T_pM) \subset T_p^{\perp}M$ for any point $p \in M$. In other words, M is a totally real submanifold if and only if, for any nonzero vector X tangent to M at any point $p \in M$, the angle between JX and the tangent plane T_pM is equal to $\frac{\pi}{2}$, identically. A totally real submanifold M of a Kähler manifold $\tilde{M}$ is called Lagrangian if $\dim_R M = \dim_C \tilde{M}$.

1-dimensional submanifolds, that is, real curves, in a Kähler manifold are always totally real. For this reason, we only consider totally real submanifolds of dimension ≥ 2.

A submanifold M of dimension ≥ 2 in a non-flat complex space form $\tilde{M}$ is curvature invariant, that is, the Riemann curvature tensor $\tilde{R}$ of $\tilde{M}$ satisfies $\tilde{R}(X,Y)TM \subset TM$ for X, Y tangent to M, if and only if M is either a Kähler submanifold or a totally real submanifold [Chen-Ogiue 1974b].

For a Lagrangian submanifold M of a Kähler manifold $(\tilde{M}, g, J)$, the tangent bundle TM and the normal bundle $T^{\perp}M$ are isomorphic via the almost complex structure J of the ambient manifold. In particular, this implies that the Lagrangian submanifold has flat normal connection if and only if the submanifold is a flat Riemannian manifold.

Let h denote the second fundamental form of the Lagrangian submanifold in $\tilde{M}$ and let $\alpha = Jh$. Another important property of Lagrangian submanifolds is that $g(\alpha(X,Y), JZ)$ is totally symmetric, that is, we have

$$g(\alpha(X,Y), JZ) = g(\alpha(Y,Z), JX) = g(\alpha(Z,X), JY) \tag{16.1}$$

for any vectors X, Y, Z tangent to M.

A result of M. L. Gromov (1985) implies that every compact embedded Lagrangian submanifold of C^n is not simply-connected (see [Sikorav 1986] for a complete proof of this fact). This result is not true when the compact Lagrangian submanifolds were immersed but not embedded.

16.1. **Basic properties of Lagrangian submanifolds.** A general Kähler manifold may not have any minimal Lagrangian submanifold. Also, the only minimal Lagrangian immersion of a topological 2-sphere into CP^2 is the totally geodesic one. In contrast, minimal Lagrangian submanifolds in an Einstein-Kähler manifold exist in abundance, at least locally (cf. [Bryant 1987]).

For surfaces in E^4 Chen and J. M. Morvan (1987) proved that an orientable minimal surface M in E^4 is Lagrangian with respect to an orthogonal almost complex structure on E^4 if and only if it is holomorphic with respect to some orthogonal almost complex structure on E^4.

A simply-connected Riemannian 2-manifold (M, g) with Gaussian curvature K less than a constant c admits a Lagrangian isometric minimal immersion into a complete simply-connected complex space form $\tilde{M}^2(4c)$ if and only if it satisfies the following differential equation [Chen 1997c; Chen-Dillen-Verstraelen-Vrancken 1995b]:

$$\Delta \ln(c - K) = 6K, \tag{16.2}$$

where Δ is the Laplacian on M associated with the metric g.

The intrinsic and extrinsic structures of Lagrangian minimal surfaces in complete simply-connected complex space forms were determined in [Chen 1997c] as follows:

Let $f : M \to \tilde{M}^2(4c)$ be a minimal Lagrangian surface without totally geodesic points. Then, with respect to a suitable coordinate system $\{x, y\}$, we have

(1) the metric tensor of M takes the form of

$$g = E\left(dx^2 + dy^2\right) \tag{16.3}$$

for some positive function E satisfying

$$\Delta_0(\ln E) = 4E^{-2} - 2cE, \tag{16.4}$$

where $\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and

(2) the second fundamental form of L is given by

$$h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -\frac{1}{E} J\left(\frac{\partial}{\partial x}\right), \quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{E} J\left(\frac{\partial}{\partial y}\right),$$

$$h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{E} J\left(\frac{\partial}{\partial x}\right). \tag{16.5}$$

Conversely, if E is a positive function defined on a simply-connected domain U of E^2 satisfying (16.4) and $g = E(dx^2 + dy^2)$ is the metric tensor on U, then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique minimal Lagrangian isometric immersion of (U, g) into a complete simply-connected complex space form $\tilde{M}^2(4c)$ whose second fundamental form is given by (16.5).

R. Harvey and H. B. Lawson (1982) studied the so-called special Lagrangian submanifolds in C^n, which are calibrated by the n-form $Re(dz_1 \wedge \cdots \wedge dz_n)$.

Being calibrated implies volume minimizing in the same homology class. So, in particular, the special Lagrangian submanifolds are oriented minimal Lagrangian submanifolds. In fact, they proved that a special Lagrangian submanifold M (with boundary ∂M) in C^n is volume minimizing in the class of all submanifolds N of C^n satisfying $[M] = [N] \in H_n^c(C^n; \mathbf{R})$ with $\partial M = \partial N$. Harvey and Lawson (1982) constructed many examples of special Lagrangian submanifolds in C^n.

Using the idea of calibrations, one can show that every Lagrangian minimal submanifold in an Einstein-Kähler manifold $\tilde{M}$ with $c_1(\tilde{M}) = 0$ is volume minimizing. It is false for the case $c_1 = \lambda\omega$ with $\lambda > 0$, where ω is the canonical symplectic form on $\tilde{M}$. It is unknown for the case $c_1 = \lambda\omega$ with $\lambda < 0$ (cf. [Bryant 1987c]). J. G. Wolfson (1989) showed that a compact minimal surface without complex tangent points in an Einstein-Kähler surface with $c_1 < 0$ is Lagrangian.

Y. I. Lee (1994) studied embedded surfaces which represent a second homology class of an Einstein-Kähler surface. She obtained the following:

Let N be an Einstein-Kähler surface with $c_1(N) < 0$. Suppose $[A] \in H_2(N, \mathbf{Z})$ and there exists an embedded surface without complex tangent points of genus r which represents $[A]$. Then every connected embedded minimal surface in $[A]$ has genus at least r. Moreover, the equality occurs if and only if the embedded minimal surface is Lagrangian.

Notice that by using an adjunction inequality for positive classes obtained by P. B. Kronheimer and T. S. Mrowka (1994), the minimality condition in her result can be dropped; namely, under the same hypothesis, one can conclude that every connected embedded surface in [A] has genus at least r. Moreover, equality occurs if and only if the embedded surface is Lagrangian.

Recently, Y. I. Lee also obtained the following result:

Let (N, g) be an Einstein-Kähler surface with $c_1 < 0$. If an integral homology class $[A] \in H_2(N, \mathbf{Z})$ can be represented by a union of Lagrangian branched minimal surfaces with respect to g, then, for any other Einstein-Kähler metric g' on N which can be connected to g via a family of Einstein-Kähler metrics on N, $[A]$ can also be represented by a union of branched minimal surfaces with respect to g'.

16.2. **A vanishing theorem and its applications.** For compact Lagrangian submanifolds in Einstein-Kähler manifolds, there is the following vanishing theorem [Chen 1998a]:

Let M be a compact manifold with finite fundamental group $\pi_1(M)$ or vanishing first Betti number $\beta_1(M)$. Then every Lagrangian immersion from M into any Einstein-Kähler manifold must have some minimal points.

This vanishing theorem has the following interesting geometrical consequences:

(1) There do not exist Lagrangian isometric immersions from a compact Riemannian n-manifold with positive Ricci curvature into any flat Kähler n-manifold or into any complex hyperbolic n-space;

(2) Every Lagrangian isometric immersion of constant mean curvature from a compact Riemannian manifold with positive Ricci curvature into any Einstein-Kähler manifold is a minimal immersion; and

(3) Every Lagrangian isometric immersion of constant mean curvature from a spherical space form into a complex projective n-space CP^n is a totally geodesic immersion.

This vanishing theorem is sharp in the following sense:

(a) The conditions on β_1 and π_1 given in the vanishing theorem cannot be removed, since the standard Lagrangian embedding of $T^n = S^1 \times \cdots \times S^1$ into $C^1 \times \cdots \times C^1 = C^n$ is a Lagrangian embedding with nonzero constant mean curvature; and

(b) "Lagrangian immersion" in the theorem cannot be replaced by the weaker condition "totally real immersion", since S^n has both trivial first Betti number and trivial fundamental group; and the standard totally real embedding of S^n in $E^{n+1} \subset C^{n+1}$ is a totally real submanifold with nonzero constant mean curvature.

16.3. **The Hopf lift of Lagrangian submanifolds of nonflat complex space forms.** There is a general method for constructing Lagrangian submanifolds both in complex projective spaces and in complex hyperbolic spaces.

Let $S^{2n+1}(c)$ be the hypersphere of C^{n+1} with constant sectional curvature c centered at the origin. We consider the Hopf fibration

$$\pi\colon\ S^{2n+1}(c) \to CP^n(4c). \tag{16.6}$$

Then π is a Riemannian submersion, meaning that π_*, restricted to the horizontal space, is an isometry. Note that given $z \in S^{2n+1}(c)$, the horizontal space at z is the orthogonal complement of iz with respect to the metric induced on $S^{2n+1}(c)$ from the usual Hermitian Euclidean metric on C^{n+1}. Moreover, given a horizontal vector x, then ix is again horizontal (and tangent to the sphere) and $\pi_*(ix) = J(\pi_*(x))$, where J is the almost complex structure on $CP^n(4c)$.

The main result of H. Reckziegel (1985) is the following: Let $g\colon\ M \to CP^n(4c)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau\colon\ \widehat{M} \to M$, and a horizontal isometric immersion $f\colon\ \widehat{M} \to S^{2n+1}(c)$ such that $g(\tau) = \pi(f)$. Hence every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a horizontal immersion of the same Riemannian manifold.

Conversely, let $f\colon\ \widehat{M} \to S^{2n+1}(c)$ be a horizontal isometric immersion. Then $g = \pi(f)\colon\ M \to CP^n(4c)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π.

In the complex hyperbolic case, we consider the complex number $(n+1)$-space C_1^{n+1} endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j \tag{16.7}$$

Put

$$H_1^{2n+1}(c) = \left\{ z = (z_1, z_2, \ldots, z_{n+1})\colon\ g_0(z,z) = \tfrac{1}{c} < 0 \right\}. \tag{16.8}$$

$H_1^{2n+1}(c)$ is known as the anti-de Sitter space-time.

Let

$$T'_z = \{u \in C_1^{n+1}\colon\ \langle u, z\rangle = 0\}, \quad H_1^1 = \{\lambda \in \mathbf{C}\colon\ \lambda\bar{\lambda} = 1\},$$

where $\langle\ ,\ \rangle$ denotes the Hermitian inner product on C_1^{n+1} whose real part is g_0. Then we have an H_1^1-action on $H_1^{2n+1}(c)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(c)$, the vector iz is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $g_0(iz, iz) = \frac{1}{c}$. Note that the orbit is given by $x_t = (\cos t + i \sin t)z$ and $\frac{dx_t}{dt} = iz_t$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2n+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(4c)$ with constant holomorphic sectional curvature $4c$, with the complex structure J induced from the canonical complex structure J on C_1^{n+1} via the following totally geodesic fibration:

$$\pi\colon\ H_1^{2n+1}(c) \to CH^n(4c). \tag{16.9}$$

Just as in the case of complex projective spaces, let $g\colon\ M \to CH^n(4c)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau\colon\ \widehat{M} \to M$, and a horizontal isometric immersion $f\colon\ \widehat{M} \to H_1^{2n+1}(c)$ such that $g(\tau) = \pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a horizontal immersion.

Conversely, let $f : \widehat{M} \to H_1^{2n+1}(c)$ be a horizontal isometric immersion. Then $g = \pi(f)\colon\ M \to CH^n(4c)$ is again an isometric immersion, which is Lagrangian.

Similarly, under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π.

This construction method have been used by various authors. For instance, M. Dajczer and R. Tojeiro (1995b) applied this technique to show that the projection of the Hopf fibration provides a one-to-one correspondence between the set of totally real flat submanifolds in complex projective space and the set of symmetric flat submanifolds in Euclidean sphere with the same codimension, where a flat submanifold $F : M_0^{n+1} \to S^{2n+1}$ is called symmetric if its corresponding principal coordinates generate a solution of the symmetric generalized wave equation.

16.4. **Totally real minimal submanifolds of complex space forms.** The equation of Gauss implies that an n-dimensional totally real minimal submanifold of a complex space form $\tilde{M}^n(4c)$ satisfies the following properties:

(1) $Ric \leq (n-1)cg$, with equality holding if and only if it is totally geodesic.

(2) $\rho \leq n(n-1)c$, with equality holding if and only if it is totally geodesic.

B. Y. Chen and C. S. Houh (1979) proved that if the sectional curvature of an n-dimensional compact totally real submanifold M of $CP^m(4c)$ satisfies $K \geq (n-1)c/(2n-1)$, then either M is totally geodesic or $n = 2$, $m \geq 3$, and M is of constant Gaussian curvature $c/3$. In both cases, the immersion is rigid.

K. Kenmotsu (1985) gave a canonical description of all totally real isometric minimal immersions of E^2 into CP^m as follows.

Let Ω be a simply-connected domain in the Euclidean plane E^2 with metric $g = 2|dz|^2$ where $z = x + \sqrt{-1}y$ is the standard complex coordinate on E^2, and let $\psi : \Omega \to CP^m$ be a totally real isometric minimal immersion. Then, up to a holomorphic isometry of CP^m, ψ is given by

$$\psi(z) = [r_0 e^{\mu_0 z - \bar{\mu}_0 \bar{z}}, \ldots, r_m e^{\mu_m z - \bar{\mu}_m \bar{z}}], \tag{16.10}$$

where $r_0, \dots, r_m$ are non-negative real numbers and $\mu_0, \dots, \mu_m$ are complex numbers of unit modulus satisfying the following three conditions:

$$\sum_{j=0}^{m} r_j^2 = 1, \quad \sum_{j=0}^{m} r_j^2 \mu_j = 0, \quad \sum_{j=0}^{m} r_j^2 \mu_j^2 = 0. \tag{16.11}$$

Furthermore, ψ is linearly full, that is, $\psi(\Omega)$ does not lie in any linear hyperplane of CP^m, if and only if $r_0, \dots, r_m$ are strictly positive and $\mu_0, \dots, \mu_m$ are distinct, in which case the complex numbers $r_0\mu_0, \dots, r_m\mu_m$ are uniquely determined up to permutations.

Conversely, any map ψ defined by (16.10), where $r_0, \dots, r_m, \mu_0, \dots, \mu_m$ satisfy (16.11), is a totally real isometric minimal immersion of Ω into CP^m.

16.5. **Lagrangian real space form in complex space form.** The simplest examples of Lagrangian (or more generally totally real) submanifolds of complex space forms are totally geodesic Lagrangian submanifolds. A totally geodesic Lagrangian submanifold M of a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$ is a real space form of constant curvature c.

The real projective n-space $RP^n(1)$ (respectively, the real hyperbolic n-space $H^n(-1)$) can be isometrically embedded in $CP^n(4)$ (respectively, in complex hyperbolic space $CH^n(-4)$) as a Lagrangian totally geodesic submanifold.

Non-totally geodesic Lagrangian isometric immersions from real space forms of constant curvature c into a complex space form $\tilde{M}^n(4c)$ were determined by Chen, Dillen, Verstraelen, and Vrancken (1998). Associated with each twisted product decomposition of a real space form, they introduced a canonical 1-form, called the twistor form of the twisted product decomposition. Their result says that if the twistor form of a twisted product decomposition of a simply-connected real space form of constant curvature c is twisted closed, then, up to motions, it admits a unique "adapted" Lagrangian isometric immersion into the complex space form $\tilde{M}^n(4c)$.

Conversely, if $L : M^n(c) \to \tilde{M}^n(4c)$ is a non-totally geodesic Lagrangian isometric immersion of a real space form $M^n(c)$ of constant sectional curvature c into a complex space form $\tilde{M}^n(4c)$, then $M^n(c)$ admits an appropriate twisted product decomposition with twisted closed twistor form and, moreover, the Lagrangian immersion L is given by the corresponding adapted Lagrangian isometric immersion of the twisted product.

Chen and Ogiue (1974b) proved that a Lagrangian minimal submanifold of constant sectional curvature c in a complex space form $\tilde{M}^n(4\tilde{c})$ is either totally geodesic or $c \leq 0$. N. Ejiri (1982) proved that the only Lagrangian minimal submanifolds of constant sectional curvature $c \leq 0$ in a complex space form are the flat ones. Ejiri's result extends the corresponding result of Chen and Ogiue (1974b) for $n = 2$ to $n \geq 2$.

A submanifold M of a Riemannian manifold is called a Chen submanifold if

$$\sum_{i,j} \langle\, h(e_i, e_j), H \,\rangle\, h(e_i, e_j) \tag{16.12}$$

is parallel to the mean curvature vector H, where $\{e_i\}$ is an orthonormal frame of the submanifold M (for general properties of Chen submanifolds, cf. [Gheysens-Verheyen-Verstraelen 1981, Rouxel 1994]).

M. Kotani (1986) studied Lagrangian Chen submanifolds of constant curvature in complex space forms and obtained the following:

If M is a Lagrangian Chen submanifold with constant sectional curvature in a complex space form $\tilde{M}^n(4\tilde{c})$ with $c < \tilde{c}$, then either M is minimal, or locally, $M = I \times \tilde{L}^{n-1}$ with metric $g = dt^2 + f(t)\tilde{g}$, where I is an open interval, $(\tilde{L}^{n-1}, \tilde{g})$ is the following submanifold in $\tilde{M}^n(4\tilde{c})$:

$$\begin{array}{ccc} \tilde{L}^{n-1} & \subset S^{2n-1} & \subset \tilde{M}^n(4\tilde{c}) \\ \pi\downarrow & \downarrow\pi & \\ L^{n-1} & \subset CP^{n-1}, & \end{array} \tag{16.13}$$

S^{2n-1} is a geodesic hypersphere in $\tilde{M}^n(4\tilde{c})$, and $\tilde{L}$ is the horizontal lift of a Lagrangian minimal flat torus L^{n-1} in CP^{n-1}.

A flat torus T^n can be isometrically immersed in CP^n as Lagrangian minimal submanifold with parallel second fundamental form, hence with parallel nonzero mean curvature vector.

M. Dajczer and R. Tojeiro (1995b) proved that a complete flat Lagrangian submanifold in CP^n with constant mean curvature is a flat torus T^n (with parallel second fundamental form).

16.6. **Inequalities for Lagrangian submanifolds.** For any n-dimensional Lagrangian submanifold M in a complex space form $\tilde{M}^n(4c)$ and for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the δ-invariant $\delta(n_1, \ldots, n_k)$ must satisfies following inequality [Chen 1996f]:

$$\delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k)H^2 + a(n_1, \ldots, n_k)c, \tag{16.14}$$

where $\delta(n_1, \ldots, n_k)$, $a(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ are defined by (3.14), (3.15) and (3.16), respectively.

There exist abundant examples of Lagrangian submanifolds in complex space forms which satisfy the equality case of (16.14).

Inequality (16.14) implies that for any Lagrangian submanifold M in a complex space form $\tilde{M}^n(4c)$, one has

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)}H^2 + \tfrac{1}{2}(n+1)(n-2)c. \tag{16.15}$$

Chen, Dillen, Verstraelen and Vrancken (1994) proved that if a Lagrangian submanifold M of $\tilde{M}^n(4c)$ satisfies the equality case of (16.17) identically, then M must be minimal in $\tilde{M}^n(4c)$.

There exist many Lagrangian minimal submanifolds of $\tilde{M}^n(4c)$, $c \in \{-1, 0, 1\}$, which satisfy the equality case of (16.15) identically. However, Chen, Dillen, Verstraelen and Vrancken (1996) proved that if M has constant scalar curvature, then the equality occurs when and only when either M is totally geodesic in $\tilde{M}^n(4c)$ or $n = 3$, $c = 1$ and the immersion is locally congruent to a special non-standard immersion $\psi : S^3 \to CP^3$ of a topological 3-sphere into CP^3 which is called an exotic immersion of S^3. The classification of Lagrangian submanifolds of CP^3 satisfying the equality case of (16.15) was given in [Bolton-Scharlach-Vrancken-Woodward 1998]

Lagrangian submanifolds of the complex hyperbolic n-space CH^n, $n \geq 3$, satisfying the equality case of (16.15) were classified in [Chen-Vrancken 1997c].

We remark that inequality (16.14) holds for an arbitrary n-dimensional submanifold in $CH^n(4c)$ for $c < 0$ as well.

16.7. **Riemannian and topological obstructions to Lagrangian immersions.** M. L. Gromov (1971) proved that a compact n-manifold M admits a Lagrangian immersion into C^n if and only if the complexification of the tangent bundle of M, $TM \otimes \mathbf{C}$, is trivial. Since the tangent bundle of a 3-manifold is always trivial, Gromov's result implies that there does not exist topological obstruction to Lagrangian immersions for compact 3-manifolds.

In contrast, by applying inequality (16.14) and the vanishing theorem mentioned in §16.2, one obtains the following sharp obstructions to isometric Lagrangian immersions of compact Riemannian manifolds into complex space forms [Chen 1996f]:

Let M be a compact Riemannian manifold with finite fundamental group $\pi_1(M)$ or $b_1(M) = 0$. If there exists a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that

$$\delta(n_1, \ldots, n_k) > \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big)c, \tag{16.16}$$

then M admits no Lagrangian isometric immersion into a complex space form of constant holomorphic sectional curvature $4c$.

An immediate important consequence of the above result is the first necessary intrinsic condition for compact Lagrangian submanifolds in C^n; namely, the Ricci curvature of every compact Lagrangian submanifold M in C^n must satisfies $\inf_u Ric(u) \leq 0$, where u runs over all unit tangent vectors of M [Chen 1997d]. For Lagrangian surfaces, this means that the Gaussian curvature of every compact Lagrangian surface M in C^2 must be nonpositive at some points on M. Another immediate consequence is that every compact irreducible symmetric space cannot be isometrically immersed in a complex Euclidean space as a Lagrangian submanifold.

Let $f : E^{n+1} \to C^n$ be the map defined by

$$f(x_0, \dots, x_n) = \frac{1}{1+x_0^2}(x_1, \dots, x_n, x_0x_1, \dots, x_0x_n). \tag{16.17}$$

Then f induces an immersion $w : S^n \to C^n$ of S^n into C^n which has a unique self-intersection point $f(-1, 0, \dots, 0) = f(1, 0, \dots, 0)$.

With respect to the canonical almost complex structure J on C^n, the immersion w is a Lagrangian immersion of S^n into C^n, which is called the Whitney immersion. S^n endowed with the Riemannian metric induced from the Whitney immersion is called a Whitney n-sphere.

The example of the Whitney immersion shows that the condition on the δ-invariants given above is sharp, since S^n $(n \geq 2)$ has trivial fundamental group and trivial first Betti number; moreover, for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the Whitney n-sphere satisfies $\delta(n_1, \dots, n_k) > 0$ except at the unique point of self-intersection.

Also, the assumptions on the finiteness of $\pi_1(M)$ and vanishing of $b_1(M)$ given above are both necessary for $n \geq 3$. This can be seen from the following example:

Let $F : S^1 \to \mathbf{C}$ be the unit circle in the complex plane given by $F(s) = e^{is}$ and let $\iota : S^{n-1} \to E^n$ $(n \geq 3)$ be the unit hypersphere in E^n centered at the origin. Denote by $f : S^1 \times S^{n-1} \to C^n$ the complex extensor defined by $f(s, p) = F(s) \otimes \iota(p)$, $p \in S^{n-1}$. Then f is an isometric Lagrangian immersion of $M =: S^1 \times S^{n-1}$ into C^n which carries each pair $\{(u, p), (-u, -p)\}$ of points in $S^1 \times S^{n-1}$ to a point in C^n (cf. [Chen 1997b]). Clearly, $\pi_1(M) = \mathbf{Z}$ and $b_1(M) = 1$, and moreover, for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the δ-invariant $\delta(n_1, \dots, n_k)$ on M is a positive constant. This example shows that both the conditions on $\pi_1(M)$ and $b_1(M)$ cannot be removed.

16.8. **An inequality between scalar curvature and mean curvature.** Besides inequality (16.14), there is another sharp inequality for Lagrangian submanifolds in complex space forms.

Let $\rho = \sum_{i \neq j} K(e_i \wedge e_j)$ denote the scalar curvature of a Riemannian n-manifold M, where $e_1, \ldots, e_n$ is an orthonormal local frame. The scalar curvature ρ and the squared mean curvature H^2 of a Lagrangian submanifold in complex space form $\tilde{M}^n(4c)$ satisfy the following general sharp inequality:

$$H^2 \geq \frac{n+2}{n^2(n-1)}\rho - \left(\frac{n+2}{n}\right)c. \tag{16.18}$$

Inequality (16.18) with $c = 0$ and $n = 2$ was proved in [Castro-Urbano 1993]. Their proof relies on complex analysis which is not applicable to $n \geq 3$. The general inequality was established in [Borrelli-Chen-Morvan 1995] for $c = 0$ and arbitrary n; and in [Chen 1996b] for $c \neq 0$ and arbitrary n; and independently by Castro and Urbano (1995), for $c \neq 0$ with $n = 2$, also using the method of complex analysis.

If $\tilde{M}^n(4c) = C^n$, the equality of (16.18) holds identically if and only if either the Lagrangian submanifold M is an open portion of a Lagrangian n-plane or, up to dilations, M is an open portion of the Whitney immersion [Borrelli-Chen-Morvan 1995] (see also [Ros-Urbano 1998] for an alternative proof).

Chen (1996b) proved that there exists a one-parameter family of Riemannian n-manifolds, denoted by $P_a^n\,(a > 1)$, which admit Lagrangian isometric immersions into $CP^n(4)$ satisfying the equality case of the inequality (16.18) for $c = 1$; and there are two one-parameter families of Riemannian manifolds, denoted by $C_a^n\,(a > 1), D_a^n\ \ (0 < a < 1)$, and two exceptional n-spaces, denoted by F^n, L^n, which admit Lagrangian isometric immersion into $CH^n(-4)$, satisfying the equality case of the inequality for $c = -1$. Besides the totally geodesic ones, these are the only Lagrangian submanifolds in $CP^n(4)$ and in $CH^n(-4)$ which satisfy the equality case of (16.18) (see also [Castro-Urbano 1995] for the case $n = 2$).

The explicit expressions of those Lagrangian immersions of P_a^n, C_a^n, D_a^n, F^n and L^n satisfying the equality case of (16.18) were completely determined in [Chen-Vrancken 1996].

I. Castro and F. Urbano (1995) showed that a Lagrangian surface in CP^2 satisfies the equality case of (16.14) for $n = 2$ and $c = 1$ if and only if the Lagrangian surface has holomorphic twistor lift.

16.9. **Characterizations of parallel Lagrangian submanifolds.** Compact Lagrangian submanifolds of $CP^n(4c)$ with parallel second fundamental form were completely classified by H. Naitoh in [Naitoh 1981].

There are various pinching results for totally real submanifolds in complex space forms similar to Kähler submanifolds given as follows.

Y. Ohnita (1986b) and F. Urbano (1986) proved the following: Let M be a compact Lagrangian submanifold of a complex space form with parallel mean curvature vector. If M has nonnegative sectional curvature, then the second fundamental form of M is parallel.

By applying the results of Ohnita and Urbano, it follows that compact Lagrangian submanifolds of $CP^n(4)$ with parallel mean curvature vector satisfying $K \geq 0$ must be the products $T \times M_1 \times \cdots \times M_k$, where T is a flat torus with $\dim M \geq k-1$ and each M_i is one of the following:

$$
\begin{gathered}
RP^r(1) \to CP^r(4) \ \ (r \geq 2) \ \ \text{(totally geodesic)}, \\
SU(r)/SO(r) \to CP^{(r-1)(r+2)/2}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
SU(2r)/Sp(r) \to CP^{(r-1)(2r+1)}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
SU(r) \to CP^{r^2-1}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
E_6/F_4 \to CP^{26}(4) \ \ \text{(minimal)}.
\end{gathered}
\tag{16.19}
$$

S. Montiel, A.Ros and F. Urbano (1986) proved that if a compact Lagrangian minimal submanifold of $CP^n(4)$ satisfies $Ric \geq \frac{3}{4}(n-2)g$, then the second fundamental form is parallel.

Combining this theorem with the result of H. Naitoh (1981), it follows that compact Lagrangian minimal submanifold of $CP^n(4)$ satisfying $Ric \geq \frac{3}{4}(n-2)g$ is either one of the Lagrangian minimal submanifolds given in (16.19) or a minimal Lagrangian flat torus in $CP^2(4)$.

Compact Lagrangian minimal submanifolds in $CP^n(4)$ satisfying a pinching of scalar curvature were studied in [Chen-Ogiue 1974b, Shen-Dong-Guo 1995], among others.

For scalar curvature pinching, we have the following: If M is an n-dimensional compact Lagrangian minimal submanifold in $CP^n(4)$ whose scalar curvature ρ satisfies $\rho \geq 3(n-2)n/4$, then M has parallel second fundamental form.

F. Urbano (1989) proved that if M^3 is a 3-dimensional compact Lagrangian submanifold of a complex space form $\tilde{M}^3(4c)$ with nonzero parallel mean curvature vector, then M^3 is flat and has parallel second fundamental form.

For complete Lagrangian submanifolds in C^n with parallel mean curvature vector, F. Urbano (1989) and U. H. Ki and Y. H. Kim (1996) proved the following.

Let M be a complete Lagrangian submanifold embedded in C^n. If M has parallel mean curvature vector, then M is either a minimal submanifold or a product submanifold $M_1 \times \cdots \times M_k$, where each M_i is a Lagrangian submanifold embedded in some C^{n_i} and each M_i is also a minimal submanifold of a hypersphere of C^{n_i}.

J. S. Pak (1978) studied totally real planar geodesic immersions and obtained that if $f : M \to CP^m$ is a totally real isometric immersion such that each geodesic γ of M is mapped into a 2-dimensional totally real totally geodesic submanifold of CP^m, then M is a locally a compact rank one symmetric space and the immersion is rigid.

16.10. **Lagrangian H-umbilical submanifolds and Lagrangian catenoid.** Since there do not exist totally umbilical Lagrangian submanifolds in complex space forms except totally geodesic ones, it is natural to look for the "simplest" Lagrangian submanifolds next to totally geodesic ones. As a result, the following notion of Lagrangian H-umbilical submanifolds was introduced.

A non-totally geodesic Lagrangian submanifold M of a Kähler manifold is called a Lagrangian H-umbilical submanifold if its second fundamental form takes the following simple form:

$$h(e_1,e_1) = \lambda J e_1, \quad h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu J e_1,$$

$$h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \;\; j \neq k, \quad j,k = 2,\ldots,n \tag{16.20}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

Clearly, a non-minimal Lagrangian H-umbilical submanifold satisfies the following two conditions:

(a) JH is an eigenvector of the shape operator A_H and

(b) the restriction of A_H to $(JH)^{\perp}$ is proportional to the identity map.

In fact, Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds satisfying both conditions (a) and (b). In this way Lagrangian H-umbilical submanifolds can be considered as the "simplest" Lagrangian submanifolds in a complex space form next to the totally geodesic ones.

There exist ample examples of Lagrangian H-umbilical submanifolds in a complex space form. For instance, Chen (1997b) showed that every complex extensor of the unit hypersurface of E^n via any unit speed curve in the complex plane gives rise to a Lagrangian H-umbilical submanifold in C^n. Using this method one can constructed ample examples of compact Lagrangian submanifolds in a complex Euclidean space.

Lagrangian H-umbilical submanifolds were classified in [Chen 1997a,1997b] in connections with the notions of Legendre curves and complex extensors. In particular, Chen proved that, except the flat ones, Lagrangian H-umbilical submanifolds of dimension ≥ 3 in a complex Euclidean space are Lagrangian pseudo-sphere ($\lambda = 2\mu$) and complex tensors of the unit hypersphere of a Euclidean space. Moreover, except some exceptional classes, Lagrangian H-umbilical submanifolds in CP^n and in CH^n, $n \geq 3$, are obtained from Legendre curves in S^3 or in H_1^3 via warped products in some natural ways. The intrinsic and extrinsic structures of Lagrangian H-umbilical surfaces in complex space forms were determined in [Chen 1997c]. In [Chen 1998e], a representation formula for flat Lagrangian H-umbilical submanifolds in complex Euclidean spaces was discovered.

By a complex extensor of the unit hypersphere S^{n-1} of E^n, we mean a Lagrangian submanifold of C^n which is given by the tensor product of the unit hypersphere and a unit speed (real) curve in the complex line C^1. The Lagrangian catenoid was constructed by R. Harvey and H. B. Lawson (1982) which is defined by

$$\begin{aligned} M_0 =&\{(x,y) \in C^n = E^n \times E^n : |x|y = |y|x, \\ &\operatorname{Im}(|x| + i|y|)^n = 1,\ |y| < |x| \tan \frac{\pi}{n}\}. \end{aligned} \tag{16.21}$$

Besides being a minimal Lagrangian submanifold of C^n, M_0 is invariant under the diagonal action of $SO(n)$ on $C^n = E^n \times E^n$.

I. Castro and F. Urbano (1997) characterized Lagrangian catenoid as the only minimal nonflat Lagrangian submanifold in C^n which is foliated by pieces of round $(n-1)$-spheres of C^n (up to dilations).

16.11. **Stability of Lagrangian submanifolds.** The stability of minimal Lagrangian submanifolds of a Kähler manifold was first investigated by B. Y. Chen, P. F. Leung and T. Nagano in 1980 (cf. [Chen 1981a]). In particular, they proved that the second variational formula of a compact Lagrangian submanifold M in a Kähler manifold $\tilde{M}$ is given by

$$V''(\xi) = \int_M \left\{ \frac{1}{2}||dX^{\#}||^2 + (\delta X^{\#})^2 - \tilde{S}(X,X) \right\} dV, \tag{16.22}$$

where $JX = \xi$, $X^{\#}$ is the dual 1-form of X on M, δ is the codifferential operator, and $\tilde{S}$ is the Ricci tensor of $\tilde{M}$.

By applying (16.22), Chen, Leung and Nagano proved the following (cf. [Chen 1981a]):

Let $f : M \to \tilde{M}$ be a compact Lagrangian minimal submanifold of a Kaehler manifold $\tilde{M}$.

(1) If $\tilde{M}$ has positive Ricci curvature, then the index of f satisfies $i(f) \geq \beta_1(M)$, where $\beta_1(M)$ denotes the first Betti number of M. In particular, if

the first cohomology group of M is nontrivial, that is, $H^1(M;\mathbf{R}) \neq 0$, then M is unstable;

(2) If $\tilde{M}$ has nonpositive Ricci curvature, then M is stable.

Y. G. Oh (1990) introduced the notion of Hamiltonian deformations in Kähler manifolds. He considered normal variations V along a minimal Lagrangian submanifold M such that the 1-form $\alpha_V = \langle JV, \cdot \rangle$ is exact and call such variations Hamiltonian variations.

A minimal Lagrangian submanifold is called Hamiltonian stable if the second variation is nonnegative in the class of Hamiltonian variations.

Oh (1990) establishes the following Hamiltonian stability criterion on Einstein-Kähler manifolds: Let $\tilde{M}$ be an Einstein-Kähler manifold with $Ric = cg$, where c is a constant. Then a minimal Lagrangian submanifold M is locally Hamiltonian stable if and only if $\lambda_1(M) \geq c$, where $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplacian acting on $C^\infty(M)$.

According to [Lawson-Simons 1973] the Lagrangian totally geodesic $RP^n(1)$ in $CP^n(4)$ is unstable in the usual sense. In contrast, Oh's result implies that the Lagrangian totally geodesic $RP^n(1)$ is Hamiltonian stable in $CP^n(4)$.

I. Castro and F. Urbano (1998) constructed examples of unstable Hamiltonian minimal Lagrangian tori in C^2.

16.12. **Lagrangian immersions and Maslov class.** Let Ω denote the canonical symplectic form on C^n defined by

$$\Omega(X,Y) = \langle JX, Y \rangle . \tag{16.23}$$

Consider the Grassmannian $\mathcal{L}(C^n)$ of all Lagrangian vector subspaces of C^n. $\mathcal{L}(C^n)$ can be identified with the symmetric space $U(n)/O(n)$ in a natural way.

$U(n)/O(n)$ is a bundle over the circle S^1 in C^1 with the projection

$$\det{}^2 : U(n)/O(n) \to S^1,$$

where $\det^2$ is the square of the determinant.

For a Lagrangian submanifold M in C^n, the Gauss map takes the values in $\mathcal{L}(C^n)$ which yields the following sequence:

$$M \to \mathcal{L}(C^n) \cong U(n)/O(n) \to S^1. \tag{16.24}$$

If ds denotes the volume form of S^1, then $m_M := (\det^2 \circ G)^*(ds)$ is a closed 1-form on M. The cohomology class $[m_M] \in H^1(M;\mathbf{Z})$ is called the Maslov class of the Lagrangian submanifold M.

J. M. Morvan (1981) proved that the Maslov form m_M and the mean curvature vector of a Lagrangian submanifold M in C^n are related by

$$m_M(X) = \frac{1}{\pi} \langle\, JH, X \,\rangle\,, \quad X \in TM. \tag{16.25}$$

Hence, a Lagrangian submanifold M in C^n is minimal if and only if $\det^2 \circ G$ is a constant map.

Let ξ be a normal vector field of a Lagrangian submanifold M of a Kähler manifold $\tilde{M}$. Denote by α_ξ the 1-form on M defined by

$$\alpha_\xi(X) = \Omega(\xi, X) = \langle J\xi, X\rangle\,, \quad X \in TM, \tag{16.26}$$

where Ω is the Kähler form of $\tilde{M}$.

Chen and Morvan (1994) introduced the notion of harmonic deformations in Kähler manifolds: A normal vector field ξ of a Lagrangian submanifold M is called harmonic if the 1-form α_ξ associated with ξ is a harmonic 1-form. A normal variation of a Lagrangian submanifold in a Kähler manifold is called harmonic if its variational vector field is harmonic.

A Lagrangian submanifold M of a Kähler manifold is called harmonic minimal if it is a critical point of the volume functional in the class of harmonic variations.

Chen and Morvan (1994) proved that the Maslov class of a Lagrangian submanifold of an Einstein-Kähler manifold vanishes if and only if it is harmonic minimal, thus providing a simple relationship between the calculus of variations and Maslov class. This result implies in particular that a closed curve γ in a Kähler manifold $\tilde{M}$ with $\dim_R \tilde{M} = 2$ is harmonic minimal if and only if it has zero total curvature, that is, $\int_\gamma \kappa(s)ds = 0$; an extension of a result mentioned in [Vaisman 1987].

A Lagrangian submanifold M is said to have conformal Maslov form if JH is a conformal vector filed on M. The Whitney immersion w defined by (16.12) is known to have conformal Maslov form.

A. Ros and F. Urbano (1998) proved that, up to dilations, Whitney's immersion is the only Lagrangian immersion of a compact manifold with zero first betti number and conformal Maslov form.

17. CR-submanifolds of Kähler manifolds

17.1. **Basic properties of CR-submanifolds of Kähler manifolds.** Let M be a submanifold of a Kähler manifold (or more generally, of an almost Hermitian manifold) with almost complex structure J and metric g. At each point $x \in M$, let $\mathcal{D}_x$ denote the maximal holomorphic subspace of the tangent space T_xM, that is, $\mathcal{D}_x = T_xM \cap J(T_xM)$. If the dimension of $\mathcal{D}_x$ is the same for all $x \in M$, we have a holomorphic distribution $\mathcal{D}$ on M. The submanifold M is called a CR-submanifold if there exists on M a holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^\perp$ is totally real, that is, $J(\mathcal{D}_x^\perp) \subset T_x^\perp M$, for all $x \in M$ [Bejancu 1978].

Every real hypersurface of a Hermitian manifold is a CR-submanifold. A CR-submanifold is called proper if it is neither a Kähler submanifold ($\mathcal{D} = TM$) nor a totally real submanifold ($\mathcal{D}^\perp = TM$).

Blair and Chen (1979) proved that a submanifold M of a non-flat complex space form $\tilde{M}$ is a CR-submanifold if and only if the maximal holomorphic subspaces define a holomorphic distribution $\mathcal{D}$ on M such that $\tilde{R}(\mathcal{D}, \mathcal{D}; \mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}$, where $\mathcal{D}^\perp$ denotes the orthogonal distribution of $\mathcal{D}$ in TM, and $\tilde{R}$ is the Riemann curvature tensor of $\tilde{M}$.

In this section, we denote by h the complex rank of the holomorphic distribution $\mathcal{D}$ and by p the real rank of the totally real distribution $\mathcal{D}^\perp$ of a CR-submanifold M so that $\dim M = 2h + p$.

Let N be a differentiable manifold and T_CN be the complexified tangent bundle of N. A CR-structure on N is a complex subbundle $\mathcal{H}$ of T_CN such that $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ and $\mathcal{H}$ is involutive, that is, for complex vector fields U and V in $\mathcal{H}$, $[U, V]$ is also in $\mathcal{H}$. A manifold endowed with a CR-structure is called a CR-manifold [Greenfield 1968].

D. E. Blair and B. Y. Chen (1979) proved that every CR-submanifold of a Hermitian manifold is a CR-manifold.

In 1978 Chen proved the integrability theorem for CR-submanifolds of a Kähler manifold; namely, the totally real distribution $\mathcal{D}^\perp$ of a CR-submanifold of a Kähler manifold is always completely integrable. This theorem implies that every proper CR-submanifold of a Kähler manifold is foliated by totally real submanifolds. By applying this integrability theorem A. Bejancu (1979) proved that a CR-submanifold of a Kähler manifold is mixed totally geodesic if and only if each leaf of the totally real distribution is totally geodesic in the CR-submanifold.

Chen's integrability theorem was extended to CR-submanifolds of various families of Hermitian manifolds by various geometers. For instance, this theorem was extended to CR-submanifolds of locally conformal symplectic manifolds by Blair and Chen. Furthermore, they constructed CR-submanifolds in some Hermitian manifolds with non-integrable totally real distributions [Blair-Chen 1979].

Let M be a CR-submanifold with Riemannian connection ∇ and let $e_1, \ldots, e_{2h}$ be an orthonormal frame field of the holomorphic distribution $\mathcal{D}$. Put $\hat{H} = \text{trace}\, \hat{\sigma}$, where $\hat{\sigma}(X,Y) = (\nabla_X Y)^\perp$ is the component of $\nabla_X Y$ in the totally real distribution. The holomorphic distribution $\mathcal{D}$ is called minimal if $\hat{H} = 0$, identically.

Although the holomorphic distribution is not necessarily integrable in general, Chen (1981c) proved that the holomorphic distribution of a CR-submanifold is always a minimal distribution.

Besides the minimality Chen (1984a) also proved the following properties for the holomorphic distributions:

(1) If M is a compact proper CR-submanifold of a Hermitian symmetric space of non-compact type, then the holomorphic distribution is non-integrable.

(2) Let M be a compact proper CR-submanifold of the complex Euclidean space. If the totally real distribution is a minimal distribution, then the holomorphic distribution is a non-integrable distribution.

A. Bejancu (1978) obtained a necessary and sufficient condition for the integrability of the holomorphic distribution: Let M be a CR-submanifold of a Kähler manifold. Then the holomorphic distribution $\mathcal{D}$ is integrable if and only if the second fundamental form of M satisfies $h(X,JY) = h(Y,JX)$ for any X, Y tangent to M.

Chen (1981c) discovered a canonical cohomology class $c(M) \in H^{2h}(M;\mathbf{R})$ for every compact CR-submanifold M of a Kähler manifold. By applying this cohomology class, he proved the following: Let M be a compact CR-submanifold of a Kähler manifold. If the cohomology group $H^{2k}(M;\mathbf{R}) = \{0\}$ for some integer $k \leq h$, then either the holomorphic distribution $\mathcal{D}$ is not integrable or the totally real distribution $\mathcal{D}^\perp$ is not minimal. Chen's cohomology class was used by S. Dragomir in his study concerning the minimality of Levi distribution (cf. [Dragomir 1995, Dragomir-Ornea 1998]).

A. Ros (1983) proved that if M is an n-dimensional compact minimal CR-submanifold of $CP^m(4)$, then the first nonzero eigenvalue of the Laplacian of M satisfies $\lambda_1(M) \leq 2(n^2 + 4h + p)/n$.

17.2. **Totally umbilical CR-submanifolds.** Bejancu (1980) and Chen (1980c) investigated totally umbilical CR-submanifolds of Kähler manifolds and obtained the following: Let M be a CR-submanifold of a Kähler manifold. If M is totally umbilical, then either

(1) M is totally geodesic, or

(2) M is totally real, or

(3) the totally real distribution is 1-dimensional, that is, $p = 1$.

Totally umbilical CR-submanifolds with 1-dimensional totally real distribution were investigated in [Chen 1980c]. For instance, he proved that every totally umbilical hypersurface of dimension ≥ 5 in a Kähler manifold has constant mean curvature. This result is no longer true if the totally umbilical hypersurface is 3-dimensional.

Totally geodesic CR-submanifolds of a Kähler manifold are classified by Blair and Chen (1979). In particular, they proved that if M is a totally geodesic CR-submanifold of a Kähler manifold, then M is a CR-product.

17.3. **Inequalities for CR-submanifolds.** For CR-submanifolds in complex space forms, there is a sharp relationship between the invariant $\delta_M = \frac{1}{2}\rho - \inf K$ and the squared mean curvature H^2 [Chen 1996a]:

Let M be an n-dimensional CR-submanifold in a complex space form $\tilde{M}^m(4c)$. Then

$$(17.1)\quad \delta_M \leq \begin{cases} \dfrac{n^2(n-2)}{2(n-1)}H^2 + \left\{\dfrac{1}{2}(n+1)(n-2) + 3h\right\}c, & \text{if} \quad c > 0; \\ \dfrac{n^2(n-2)}{2(n-1)}H^2, & \text{if} \quad c = 0; \\ \dfrac{n^2(n-2)}{2(n-1)}H^2 + \dfrac{1}{2}(n+1)(n-2)c, & \text{if} \quad c < 0. \end{cases}$$

There exist many CR-submanifolds in complex space forms which satisfy the equality cases of the above inequalities.

Proper CR-submanifolds of complex hyperbolic spaces satisfying the equality case were completely determined by Chen and Vrancken (1997b) as follows:

Let U be a domain of $\mathbf{C}$ and $\Psi : U \to C^{m-1}$ be a nonconstant holomorphic curve in C^{m-1}. Define $z : E^2 \times U \to C_1^{m+1}$ by

$$z(u,t,w) = \Big(-1 - \frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, -\frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, \Psi(w)\Big)e^{it}.$$

Then $\langle z, z\rangle = -1$ and the image $z(E^2 \times U)$ in H_1^{2m+1} is invariant under the group action of H_1^1. Moreover, away from points where $\Psi'(w) = 0$, the image $\pi(E^2 \times U)$, under the projection $\pi : H_1^{2m+1}(-1) \to CH^m(-4)$, is a proper CR-submanifold of $CH^m(-4)$ which satisfies $\delta_M = \frac{n^2(n-2)}{2(n-1)}H^2 + \frac{1}{2}(n+1)(n-2)c$.

Conversely, up to rigid motions of $CH^m(-4)$, every proper CR-submanifold of $CH^m(-4)$ satisfying the equality is obtained in such way. For a given submanifold M of a Kähler manifold, let $P : TM \to TM$ denote the tangential component of $J : TM \to J(TM)$.

M. Kon (1989) proved the following: Suppose M is a compact orientable n-dimensional minimal CR-submanifold of the complex projective space CP^m. Suppose the Ricci tensor of M satisfies

$$Ric(X,X) \geq (n-1)g(X,X) + 2g(PX,PX). \tag{17.2}$$

Then M is isometric to one of the following:

(1) a real projective space RP^n;

(2) a complex projective space $CP^{n/2}$;

(3) a pseudo-Einstein real hypersurface $\pi(S^{(n+1)/2}(\frac{1}{2}) \times S^{(n+1)/2}(\frac{1}{2}))$, of some

$CP^{(n+1)/2}$ in CP^m, where $\pi : S^{2m+1} \to CP^m$ is the Hopf fibration.

17.4. CR**-products.** The notion of CR-products was introduced in [Chen 1981b]: A CR-submanifold M of a Kähler manifold $\tilde{M}$ is called a CR-product if it is locally a Riemannian product of a Kähler submanifold M^T and a totally real submanifold $N^\perp$ of $\tilde{M}$.

Chen (1981b) showed that a submanifold M of a Kähler manifold is a CR-product if and only if $\nabla P = 0$, that is, P is parallel with respect to the Levi-Civita connection of M, where P is the endomorphism on the tangent bundle TM induced from the almost complex structure J of $\tilde{M}$.

Let $f : M^\perp \to CP^p(4)$ be a Lagrangian submanifold of $CP^p(4)$. Then the composition

$$CP^h \times M^\perp @>i \times f>> CP^h \times CP^p @>S_{hp}>> CP^{h+p+hp} \tag{17.3}$$

is a CR-product in CP^{h+p+hp}, where $i : CP^h \to CP^h$ is the identity map and S_{hp} is the Segre embedding.

A CR-product $M = M^T \times M^\perp$ in CP^m is called a standard CR-product if $m = h+p+hp$ and M^T is a totally geodesic Kähler submanifold of CP^m.

For CR-products in complex space forms, Chen (1981b) proved the following:

(1) A CR-product in a complex hyperbolic space is non-proper, that is, it is either a Kähler submanifold or a totally real submanifold.

(2) A CR-product in complex Euclidean m-space C^m is a product submanifold of a complex linear subspace C^r of C^m and a totally real submanifold in a complex linear subspace C^{m-r} of C^m.

(3) If $M = M^T \times M^\perp$ is a CR-product of $CP^m(4)$, then

(3.1) $m \geq h + p + hp$,

(3.2) the squared length S of the second fundamental form satisfies $S \geq 4hp$,

(3.3) if $m = h + p + hp$, then M is a standard CR-product, and

(3.4) if $S = 4hp$, then $M = M^T \times M^\perp$ is a standard CR-product contained in a totally geodesic Kähler submanifold $CP^{h+p+hp}(4)$ of $CP^m(4)$. Moreover, M^T is an open portion of $CP^h(4)$ and $M^\perp$ is an open portion of $RP^p(1)$.

(4) If M is a minimal CR-product in CP^m, then the scalar curvature ρ of M satisfies

$$\rho \geq 4h^2 + 4h + p^2 - p, \tag{17.4}$$

with the equality holding when and only when $S = 4hp$.

S. Maeda and N. Sato (1983) studied CR-submanifolds M in a complex space form $\tilde{M}^m(4c)$ such that geodesics in M are circles in $\tilde{M}^m(4c)$ and obtained the following: Let M be a CR-submanifold in a complex space form $\tilde{M}^m(4c)$. If geodesics in M are circles in $\tilde{M}^m(4c)$, then M is a CR-product.

17.5. **Cyclic parallel CR-submanifolds.** Concerning the covariant derivative of the second fundamental form of CR-submanifolds of a complex space form $\tilde{M}^m(4c)$, K. Yano and M. Kon (1980) (for $c > 0$) and Chen, G. D. Ludden and S. Montiel (1984) (for $c < 0$) proved the following general inequality.

Let M be a CR-submanifold in a complex space form $\tilde{M}^m(4c)$. Then the squared length of the covariant derivative of the second fundamental form satisfies

$$||\bar{\nabla} h||^2 \geq 4c^2 hp, \tag{17.5}$$

with the equality holding if and only if M a cyclic-parallel CR-submanifold, that is, M satisfies

$$(\bar{\nabla}_X h)(Y, Z) + (\bar{\nabla}_Y h)(Z, X) + (\bar{\nabla}_Z h)(X, Y) = 0 \tag{17.6}$$

for X, Y, Z tangent to M.

Let $H_1^{2m+1}(-1)$ denote the anti-de Sitter space time with constant sectional curvature -1 and let

$$\pi : H_1^{2m+1}(-1) \to CH^m(-4) \tag{17.7}$$

denote the corresponding Hopf fibration. For a submanifold M of $CH^m(-4)$, let $\tilde{M}$ denote the pre-image of M.

B. Y. Chen, G. D. Ludden and S. Montiel (1984) showed that a CR-submanifold M of $CH^m(-4)$ is cyclic-parallel if and only if the preimage

$\tilde{M}$ has parallel second fundamental form in $H_1^{2m+1}(-1)$. Similar result also holds for CR-submanifolds in $CP^m(4)$ [Yano-Kon 1983].

A submanifold of a real space form is cyclic-parallel if and only if it is a parallel submanifold.

A Riemannian manifold M is called a two-point locally homogeneous space if it is either flat or a rank one locally symmetric space. Chen and L. Vanhecke (1981) proved that a Riemannian manifold is a two-point locally homogeneous space if and only if sufficiently small geodesic hypersurfaces of M are cyclic-parallel hypersurfaces.

17.6. **Homogeneous and mixed foliate CR-submanifolds.** Y. Shimizu (1983) constructed homogeneous CR-submanifolds in CP^n which are not CR-products. Shimizu's results state as follows.

Let G/H be an irreducible Hermitian symmetric space of compact type. Denote by $\pi : S^{2n+1}(1) \to CP^n(4)$ the Hopf fibration. For a point $x \in S^{2n+1}$ denote by N the H-orbit of x and $M = \pi(N)$. If the rank of G/H is greater than one and if N has the maximal dimension, then

(1) M is a proper CR-submanifold of CP^n of codimension $rk(G/H) - 1$,
(2) M is not a CR-product,
(3) M has parallel mean curvature vector, and
(4) M has flat normal connection.

A CR-submanifold M in a Kähler manifold is called mixed foliate if its holomorphic distribution $\mathcal{D}$ is integrable and its second fundamental form h satisfies $h(X,Z) = 0$ for X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$.

Mixed foliate CR-submanifolds in complex space forms were completely determined as follows.

(1) A complex projective space admits no mixed foliate proper CR-submanifolds [Bejancu-Kon-Yano 1981].

(2) A CR-submanifold in C^m is mixed foliate if and only if it is a CR-product [Chen 1981b].

(3) A CR-submanifold in a complex hyperbolic space CH^m is mixed foliate if and only if it is either a Kähler submanifold or a totally real submanifold [Chen-Wu 1988].

17.7. **Nullity of CR-submanifolds.** T. Gotoh (1997) investigated the second variational formula of a compact minimal CR-submanifold in a complex projective space and estimated its nullity of the second variations to obtain the following.

Let $f : M \to CP^m$ be an n-dimensional compact minimal CR-submanifold of CP^m.

(1) If n is even, then the nullity of M satisfies

$$n(f) \geq 2\left(\frac{n}{2}+1\right)\left(m-\frac{n}{2}\right), \tag{17.8}$$

with equality holding if and only if M is a totally geodesic Kähler submanifold;

(2) If n is odd and equal to m, then the nullity of M satisfies

$$n(f) \geq \frac{n(n+3)}{2}, \tag{17.9}$$

with equality holding if and only if M is a totally real totally geodesic submanifold;

(3) If n is odd and not equal to m, then the nullity of M satisfies

$$n(f) \geq n+1+2\left(\frac{n+1}{2}+1\right)\left(m-\frac{n+1}{2}\right), \tag{17.10}$$

with equality holding if and only when

$$M = \pi\left(S^1\left(\sqrt{\frac{1}{n+1}}\right) \times S^n\left(\sqrt{\frac{n}{n+1}}\right)\right) \subset CP^{(n+1)/2},$$

where $CP^{(n+1)/2}$ is embedded in CP^m as a totally geodesic Kähler submanifold and $\pi : S^{(n+1)/2+1} \to CP^{(n+1)/2}$ is the Hopf fibration.

18. Slant submanifolds of Kähler manifolds

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Kähler manifold $\tilde{M}$ with almost complex structure J and Kähler metric g. For any vector X tangent to M let PX and FX denote the tangential and the normal components of JX, respectively. Then P is an endomorphism of the tangent bundle TM. For any nonzero vector X tangent to M at a point $p \in M$, the angle $\theta(X)$ between JX and the tangent space T_pM is called the Wirtinger angle of X.

A submanifold M of $\tilde{M}$ is called slant if the Wirtinger angle $\theta(X)$ is constant (which is independent of the choice of $x \in M$ and of $X \in T_xN$). The Wirtinger angle of a slant submanifold is called the slant angle of the slant submanifold [Chen 1990].

Kähler submanifolds and totally real submanifolds are nothing but slant submanifolds with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is called proper if it is neither complex nor totally real. In this sense, both CR-submanifolds and slant submanifolds are generalizations of both Kähler submanifolds and totally real submanifolds.

Slant surfaces in almost Hermitian manifolds do exist extensively. In fact, Chen and Y. Tazawa (1990) proved the following:

Let $f : M \to \tilde{M}$ be an embedding from an oriented surface M into an almost Hermitian manifold $\tilde{M}$ endowed with an almost complex structure J and an almost Hermitian metric g. If f has no complex tangent points, then for any prescribed angle $\theta \in (0, \pi)$, there exists an almost complex structure $\tilde{J}$ on $\tilde{M}$ satisfying the following two conditions:

(i) $(\tilde{M}, g, \tilde{J})$ is an almost Hermitian manifold, and

(ii) f is a θ-slant surface with respect to $(g, \tilde{J})$.

By a complex tangent point of f, we mean a point $x \in M$ such that the tangent space T_xM of M at x is invariant under the action of the almost complex structure J on M.

18.1. **Basic properties of slant submanifolds.** Proper slant submanifolds are even-dimensional, such submanifolds do exist extensively for any even dimension greater than zero (cf. [Chen 1990, Tazawa 1994a, 1994b]).

Slant submanifolds of Kähler manifolds are characterized by a simple condition; namely, $P^2 = \lambda I$ for a fixed real number $\lambda \in [-1, 0]$, where I is the identity map of the tangent bundle of the submanifold.

A proper slant submanifold is called Kählerian slant if the endomorphism P is parallel with respect to the Riemannian connection, that is, $\nabla P = 0$. A Kählerian slant submanifold is a Kähler manifold with respect to the induced metric and the almost complex structure defined by $\tilde{J} = (\sec\theta)P$.

Kähler submanifolds, totally real submanifolds and Kählerian slant submanifolds satisfy the condition: $\nabla P = 0$. In general, let M be a submanifold of a Kähler manifold $\tilde{M}$. Then M satisfies $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times \cdots \times M_k$, where each M_i is a Kähler submanifold, a totally real submanifold or a Kählerian slant submanifold of $\tilde{M}$.

Slant submanifolds have the following topological properties:

(1) If M is a compact $2k$-dimensional proper slant submanifold of a Kähler manifold, then $H^{2i}(M; \mathbf{R}) \neq \{0\}$ for $i = 1, \ldots, k$ [Chen 1990].

(2) Let M be a slant submanifold in a complex Euclidean space. If M is not totally real, then M is non-compact [Chen-Tazawa 1991].

Although there do not exist compact proper slant submanifolds in complex Euclidean spaces, there do exist compact proper slant submanifolds in complex flat tori.

The following result of Chen (1996c) provides a Riemannian obstruction to the isometric slant immersion in a flat Kähler manifold.

Let M be a compact Riemannian n-manifold with finite fundamental group $\pi_1(M)$. If there exists a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that $\delta(n_1, \ldots, n_k) > 0$ on M, then M admits no slant immersion into any flat Kählerian n-manifold.

18.2. **Equivariant slant immersions.** S. Maeda, Y. Ohnita and S. Udagawa (1993) investigated slant immersions between Kähler manifolds and obtained the following.

Let $f : M \to N$ be an isometric immersion of an m-dimensional compact Kähler manifold with Kähler form ω_M into a Kähler manifold with Kähler form ω_N. Assume that the second Betti number $b_2(M) = 1$ and $f^*\omega_N$ is of type $(1, 1)$. Then the following three conditions are equivalent:

(1) f is a slant immersion with slant angle $\cos^{-1}(\frac{|c|}{m})$, for some nonnegative constant c.

(2) $f^*\omega_N = (\frac{c}{m})\omega_M$,

(3) $\mathrm{trace}_g\, f^*\omega_N = \sqrt{-1}c$ is a constant, where g is the Kähler metric on M.

A compact simply-connected homogeneous Kähler manifold is simply called a Kähler C-space. Let $f : M = G/H \to CP^m(4c)$ be an isometric immersion of a compact homogeneous Riemannian manifold into $CP^m(4c)$. The immersion f is called G-equivariant if there is a homomorphism $\rho : G \to SU(m+1)$ such that $f(a \cdot p) = \rho(a)f(p)$ for any $p \in M$ and $a \in G$.

If $f : M \to CP^m(4c)$ is a map of a Kähler manifold M with $H_2(M; \mathbf{Z}) \cong \mathbf{Z}$. Denote by ω_M and $\tilde{\omega}$ the Kähler forms of M and $CP^m(4c)$, respectively. Let S be a positive generator of $H_2(M; \mathbf{Z})$. Define the degree of f by $\deg(f) = \frac{c}{\pi}[f^*\tilde{\omega}](S)$, where $[f^*\tilde{\omega}](S)$ is the evaluation of the cohomology class $[f^*\tilde{\omega}]$ represented by $f^*\tilde{\omega}$ at S.

The following theorem of Maeda, Ohnita and Udagawa provides some nice examples of proper slant submanifolds in complex projective spaces.

Let $f : M = G/H \to CP^m(4c)$ be a G-equivariant isometric immersion of an m-dimensional Kähler C-space into a complex projective space with Kähler form $\tilde{\omega}$. Then $f^*\tilde{\omega}$ is of type $(1,1)$ and $\mathrm{trace}_g\,\tilde{\omega}$ is constant, where g is the Kähler metric on M. Moreover, if $b_2(M) = 1$, then f is a slant immersion with slant angle given by

$$\cos^{-1}\left(|deg\,(f)| \cdot \frac{\pi}{c\,vol(S)}\right),$$

where S is a rational curve of M which represents the generator of $H_2(M; \mathbf{Z})$.

18.3. **Slant surfaces in complex space forms.** Slant submanifolds of dimension two have some special geometric properties. For instance, Chen (1990) proved that a surface in a Kähler manifold is a proper slant surface if and only if it is a Kählerian slant surface. He also showed that there do not exist flat minimal proper slant surfaces in C^2. Also Chen and Tazawa proved in 1997 that there exist no proper slant minimal surfaces in CP^2 and in CH^2.

If the mean curvature of a complete oriented proper slant surface in C^2 is bounded below by a positive number, then topologically it is either a circular cylinder or a 2-plane [Chen-Morvan 1992].

Suppose M is an immersed surface in a Kähler surface $\tilde{M}$ which is neither Kählerian nor Lagrangian. Then M is a proper slant surface of $\tilde{M}$ if and only if the shape operator of M satisfies

$$A_{FX}Y = A_{FY}X \tag{18.1}$$

for vectors X, Y tangent to M.

Applying this special property of the shape operator for slant surfaces, Chen (1995,1998b,1998d) proved that the squared mean curvature and the Gaussian curvature of a proper slant surface in a 2-dimensional complex space form $\tilde{M}^2(4c)$ satisfies

$$H^2 \geq 2K - 2(1 + 3\cos^2\theta)c, \tag{18.2}$$

where θ denotes the slant angle.

There do not exist proper slant surfaces satisfying the equality case of inequality (18.2) for $c > 0$. A proper slant surface in a flat Kähler surface satisfies the equality of inequality (18.2) if and only if it is totally geodesic. Furthermore, a proper slant surface in the complex hyperbolic plane $CH^2(-4)$ satisfying the equality case of inequality (18.2) is a surface of constant Gaussian curvature $-\frac{2}{3}$ with slant angle $\theta = \cos^{-1}(\frac{1}{3})$. Moreover, the immersion of such a slant surface is rigid.

A submanifold N of a pseudo-Riemannian Sasakian manifold $(\tilde{M}, g, \phi, \xi)$ is called contact θ-slant if the structure vector field ξ of $\tilde{M}$ is tangent to N at each point of N and, moreover, for each unit vector X tangent to N and orthogonal to ξ at $p \in N$, the angle $\theta(X)$ between $\phi(X)$ and T_pN is independent of the choice of X and p.

Let $H_1^{2m+1}(-1) \subset C_1^{m+1}$ denote the anti-de Sitter space-time and $\pi\colon\ H_1^{2m+1}(-1) \to CH^m(-4)$ the corresponding totally geodesic fibration (cf. section 16.3). Then every n-dimensional proper θ-slant submanifold M in $CH^m(-4)$ lifts to an $(n+1)$-dimensional proper contact θ-slant submanifold $\pi^{-1}(M)$ in $H_1^{2m+1}(-1)$ via π. Conversely, every proper contact θ-slant submanifold of $H_1^{2m+1}(-1)$ projects to a proper θ-slant submanifold of $CH^m(-4)$ via π. Similar correspondence also holds between proper θ-slant submanifolds of $CP^m(4)$ and proper contact θ-slant submanifolds of the Sasakian unit $(2m+1)$-sphere $S^{2m+1}(1)$.

The contact slant representation of the unique proper slant surface in $CH^2(-4)$ which satisfies the equality case of (18.2) in $H_1^5(-1) \subset C_1^3$ has been determined by Chen and Y. Tazawa in 1997. Up to rigid motions of C_1^3, this contact slant representation is given by

$$\begin{aligned} z(u,v,t) = e^{it}\Big(&1 + \frac{3}{2}\Big(\cosh\sqrt{\frac{2}{3}}v - 1\Big) + \frac{u^2}{6}e^{-\sqrt{\frac{2}{3}}v} - i\frac{u}{\sqrt{6}}(1 + e^{-\sqrt{\frac{2}{3}}v}), \\ &\frac{u}{3}\Big(1 + 2e^{-\sqrt{\frac{2}{3}}v}\Big) + \frac{i}{6\sqrt{6}}e^{-\sqrt{\frac{2}{3}}v}\Big(\Big(e^{\sqrt{\frac{2}{3}}v} - 1\Big)\Big(9e^{\sqrt{\frac{2}{3}}v} - 3\Big) + 2u^2\Big), \\ &\frac{u}{3\sqrt{2}}\Big(1 - e^{-\sqrt{\frac{2}{3}}v}\Big) + \frac{i}{12\sqrt{3}}\Big(6 - 15e^{-\sqrt{\frac{2}{3}}v} + 9e^{\sqrt{\frac{2}{3}}v} + 2e^{-\sqrt{\frac{2}{3}}v}u^2\Big)\Big). \end{aligned} \tag{18.3}$$

In 1990 Chen classified slant surfaces in C^2 with parallel mean curvature vector:

Let M be a slant surface in C^2 with parallel mean curvature vector. Then M is one of the following surfaces:

(1) an open portion of the product surface of two plane circles;

(2) an open portion of a circular cylinder which is contained in a real hyperplane of C^2;

(3) a minimal slant surface.

Cases (1) and (2) occur only when M is a Lagrangian surface of C^2.

J. Yang (1997) showed that a flat proper slant surface with nonzero constant mean curvature in C^2 is an open portion of a helical cylinder and there do not exist proper slant surfaces with nonzero constant mean curvature and nonzero constant Gauss curvature in C^2.

Y. Ohnita (1989) proved that totally geodesic surfaces are the only minimal slant surfaces with constant Gauss curvature in complex hyperbolic spaces. In contrast, Chen and Vrancken (1997a) proved that for each constant θ, $0 < \theta < \frac{\pi}{2}$, there exist complete θ-slant surfaces in the complex hyperbolic plane CH^2 with nonzero constant mean curvature and constant negative Gaussian curvature.

Chen and Vrancken (1997a) also proved the following:

(1) For a given constant θ with $0 < \theta \leq \frac{\pi}{2}$ and a given function λ, there exist infinitely many θ-slant surfaces in C^2 with λ as the prescribed mean curvature function.

(2) For a given constant θ with $0 < \theta \leq \frac{\pi}{2}$ and a given function K, there exist infinitely many θ-slant surfaces in C^2 with K as the prescribed Gaussian curvature function.

Slant surfaces in C^2 were completely classified by Chen and Y. Tazawa (1991) for the following cases:

(1) spherical slant surfaces;

(2) slant surfaces lying in a real hyperplane of C^2; or

(3) slant surfaces whose Gauss map has rank less than two.

For case (1), they proved that a spherical surface in C^2 is proper slant if and only if it is locally a spherical helical cylinder in a hypersphere S^3; for case (2), the surfaces are doubly slant and they are the unions of some open portions of planes, circular cones and the tangent developable surfaces obtained by generalized helices; and for case (3) the slant surfaces are unions of some special flat ruled surfaces.

18.4. **Slant surfaces and almost complex structures.** Let $C^2 = (E^4, J_0)$ be the complex Euclidean plane with the canonical complex structure J_0. Then J_0 is an orientation preserving isomorphism. Denote by $\mathcal{J}$ the set of all almost complex structures on E^4 which are compatible with the inner product $\langle\, ,\, \rangle$, that is, $\mathcal{J}$ consists of all linear endomorphisms J of E^4 such that $J^2 = -I$ and $\langle JX, JY\rangle = \langle X, Y\rangle$ for $X, Y \in E^4$

An orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on E^4 is called a J-basis if $Je_1 = e_2, Je_3 = e_4$. Any two J-bases associated with the same almost complex structure have the same orientation.

With respect to the canonical orientation on E^4 one can divide $\mathcal{J}$ into two disjoint subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ which consist of all positive and all negative J-bases, respectively.

For an immersion ϕ of a Riemann surface M into a Kähler manifold N, the Kähler angle α of ϕ is defined to be the angle between $J\phi_*(\partial/\partial x)$ and $\phi_*(\partial/\partial y)$, where $z = x + \sqrt{-1}y$ is a local complex coordinate on M and J the almost complex structure on N.

The relation between θ and the Kähler angle α for an immersion ϕ of a Riemann surface M into a Kähler manifold N is

$$\theta(X) = \min\{\alpha(T_pM), \pi - \alpha(T_pM)\}$$

for any nonzero vector $X \in T_pM$.

The immersion f of a Riemann surface in N is called holomorphic (respectively, anti-holomorphic) if $\alpha \equiv 0$ (respectively, $\alpha \equiv \pi$).

The following results of B. Y. Chen and Y. Tazawa (1990) determine whether a surfaces in E^4 is slant with respect to some compatible almost complex structure on E^4:

(1) Let $f : N \to E^4$ be a minimal immersion. If there exists a compatible complex structure $\hat{J} \in \mathcal{J}^+$ (respectively, $\hat{J} \in \mathcal{J}^-$) such that the immersion is slant with respect to $\hat{J}$, then

(1-a) for any $\alpha \in [0, \pi]$, there is a compatible complex structure $J_\alpha \in \mathcal{J}^+$ (respectively, $J_\alpha \in \mathcal{J}^-$) such that f is α-slant with respect to the complex structure J_α, and

(1-b) the immersion f is slant with respect to any complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$).

(2) If $f : N \to E^4$ is a non-minimal immersion, then there exist at most two complex structures $\pm J^+ \in \mathcal{J}^+$ and at most two complex structures $\pm J^- \in \mathcal{J}^-$ such that the immersion f is slant with respect to them.

(3) If $f : N \to C^2 = (E^4, J_0)$ is holomorphic, then the immersion f is slant with respect to every complex structure $J \in \mathcal{J}^+$.

(4) If $f : N \to C^2 = (E^4, J_0)$ is anti-holomorphic, then the immersion f is slant with respect to every complex structure $J \in \mathcal{J}^-$.

(5) If $f : N \to E^3$ is a non-totally geodesic minimal immersion, then $f : N \to E^3 \subset E^4$ is not slant with respect to every compatible complex structure on E^4.

18.5. **Slant spheres in complex projective spaces.** For each $k = 0, \ldots, m$, let $\psi_k : S^2 \to CP^m(4)$ be given by

$$\psi_k([z_0, z_1]) = \left[g_{k,0}\left(\frac{z_0}{z_1}\right), \ldots, g_{k,m}\left(\frac{z_0}{z_1}\right)\right], \tag{18.4}$$

where $[z_0, z_1] \in CP^1 = S^2$, and for $j = 0, \ldots, m$, $g_{k,j}(z)$ is given by

$$(18.5) \quad g_{k,j}(z) = \frac{k!}{(1+z\bar{z})^k}\sqrt{\binom{m}{j}}\, z^{j-k}\sum_p (-1)^p \binom{j}{k-p}\binom{m-j}{p}(z\bar{z})^p.$$

It was proved by Bolton, Jensen, Rigoli and Woodward (1988) that each ψ_k is a conformal minimal immersion with constant Gaussian curvature $4(m + 2k(m-k))^{-1}$ and constant Kähler angle α_k given by

$$\tan^2\left(\frac{\alpha_k}{2}\right) = \frac{k(m-k+1)}{(k+1)(m-k)}.$$

Each ψ_k is an embedding unless $m = 2k$, in which case ψ_k is a totally real immersion.

The immersions $\psi_0, \ldots, \psi_m$ defined above are called the Veronese sequence. For Veronese sequence, J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward (1988) proved that

(1) Let $\psi : S^2 \to CP^m(4)$ be a conformal minimal immersion with constant Gaussian curvature and assume that $\psi(S^2)$ is not contained in any hyperplane of $CP^m(4)$. Then, up to a holomorphic isometry of $CP^m(4)$, the immersion ψ is an element of the Veronese sequence.

(2) Let $\psi, \psi' : S^2 \to CP^m(4)$ be conformal minimal immersions. Then ψ, ψ' differ by a holomorphic isometry of $CP^m(4)$ if and only if they have the same Kähler angle and induced metrics at each point.

(3) If $\psi : S^2 \to CP^m(4)$ is a totally real minimal immersion, then ψ is totally geodesic.

For each linearly full minimal immersion $\psi : S^2 \to CP^n$, let $\psi_0, \psi_1, \cdots, \psi_n$ denote the corresponding Veronese sequence with $\psi = \psi_k$ for some $k = 0, 1, \cdots, n$, where ψ_0 is holomorphic, called the directrix of ψ. ψ is called a minimal immersion with position k.

Z. Q. Li (1995) proved the following:

(1) Let ψ: $S^2 \to CP^n$ be a linearly full minimal immersion with position 2. Suppose the Kähler angle α is constant but the Gaussian curvature is not. If $\alpha \neq 0, \pi, \pi/2$ and the directrix ψ_0 of ψ is unramified, then $n \leq 10$ and $\tan^2(\alpha/2) = \frac{3}{4}$.

(2) There are at least three families of totally unramified minimal immersions $\psi : S^2 \to CP^{10}$ such that ψ is neither holomorphic, anti-holomorphic nor totally real, with constant Kähler angle and nonconstant Gaussian curvature. Moreover, ψ is homotopic to the Veronese minimal immersion.

Y. Ohnita (1989) studied minimal surfaces with constant curvature and constant Kähler angle. He obtained the following.

Let M be a minimal surface with constant Gaussian curvature K immersed fully in $CP^m(4)$. If the Kähler angle α of M is constant.

(1) If $K > 0$, then there exists some constant k with $0 \leq k \leq m$ such that

$$K = \frac{4}{2k(m-k)+m}, \quad \cos\alpha = \left(\frac{m-2k}{4}\right) K$$

and M is an open submanifold of an element of Veronese sequence.

(2) If $K = 0$, then M is totally real.

(3) $K < 0$ is impossible.

A minimal surface in CP^n is called superconformal if its harmonic sequence is orthogonally periodic, and it is called pseudo-holomorphic if its harmonic sequence terminates at each end.

M. Sakaki (1996) proved the following:

(1) Any superconformal minimal slant surface in CP^3 is totally real.

(2) Any pseudo-holomorphic minimal slant surface in CP^4 is either holomorphic, anti-holomorphic, totally real or of constant curvature.

19. Submanifolds of the nearly Kähler 6-sphere

It was proved by E. Calabi (1958) that any oriented submanifold M^6 of the hyperplane $\mathrm{Im}\,\mathcal{O}$ of the imaginary octonions carries a $U(3)$-structure (that is, an almost Hermitian structure). For instance, let $S^6 \subset \mathrm{Im}\,\mathcal{O}$ be the sphere of unit imaginary vectors; then the right multiplication by $u \in S^6$ induces a linear transformation $J_u : \mathcal{O} \to \mathcal{O}$ which is orthogonal and satisfies $(J_u)^2 = -I$. The operator J_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane which may be identified with $T_u S^6$. Thus J_u induces an almost complex structure on $T_u S^6$ which is compatible with the inner product induced by the inner product of $\mathcal{O}$ and S^6 has an almost complex structure.

The almost complex structure J on S^6 is a nearly Kähler structure in the sense that the (2,1)-tensor field G on S^6, defined by $G(X,Y) = (\widetilde{\nabla}_X J)(Y)$, is skew-symmetric, where $\widetilde{\nabla}$ denotes the Riemannian connection on S^6.

The group of automorphisms of this nearly Kähler structure is the exceptional simple Lie group G_2 which acts transitively on S^6 as a group of isometries.

A. Gray (1969) proved the following:

(1) every almost complex submanifold of the nearly Kähler S^6 is a minimal submanifold, and

(2) the nearly Kähler S^6 has no 4-dimensional almost complex submanifolds.

19.1. **Almost complex curves.**

Almost complex curves, that is, real 2-dimensional almost complex submanifolds, in the nearly Kähler S^6 have been studied by various authors.

An almost complex curve in S^6 is a non-constant smooth map $f\colon S \to S^6$, from a Riemann surface S, whose differential is complex linear. Such a map is necessarily a weakly conformal harmonic map or, equivalently, a weakly conformal branched minimal immersion.

Almost complex curves have ellipse of curvature a circle, that is, the map $v \mapsto h(v,v)$ describes a circle in the normal space where $v \in UM_p$, UM_p the unit hypersphere of T_pM, and h denotes the second fundamental form. If the map $v \mapsto (\bar{\nabla}_v h)(v,v)$, $v \in UM_p$ also describes a circle, then M^2 is called superminimal. This class of almost complex curves has been investigated by Bryant (1982).

In [Bryant 1982] a Frenet formalism for almost complex curves in S^6 was developed; the first, second and third fundamental forms are defined as holomorphic sections of line bundles over S. In particular, he showed that the third fundamental form III, analogous to the torsion of a space curve, plays a crucial role.

The hypothesis $III \neq 0$ is very restrictive and Bryant proved that if $S = CP^1$, then $III \neq 0$ is impossible. On the contrary, he constructed almost complex curves $f : S \to S^6$ with $III = 0$ for any Riemann surface S such that the ramification divisor of f has arbitrarily large degree. He also proved that if S is compact and $f : S \to S^6$ is an almost complex curve with $III = 0$, then f is algebraic. In particular, it is real analytic.

J. Bolton, L. Vrancken and L. M. Woodward (1994) proved that there are four basic types of almost complex curves in S^6; namely,

(i) linearly full in S^6 and superminimal,
(ii) linearly full in S^6 but not superminimal,
(iii) linearly full in some totally geodesic S^5 in S^6, and
(iv) totally geodesic.

They also provided metric criteria for recognizing when a weakly conformal harmonic map $f : S \to S^6$ is $O(7)$-congruent to an almost complex curve of any one of the four types.

A surface S in S^{2m} is called superminimal if S is the image of a horizontal holomorphic curve in the Hermitian symmetric space $SO(2m+1)/U(m)$ under the Riemannian submersion $\pi : SO(2m+1)/U(m) \to S^{2m}$ induced by the inclusions $U(m) \subset SO(2m) \subset SO(2m+1)$. E. Calabi (1967) proved that all minimal 2-spheres in S^{2m} are superminimal.

R. Bryant (1982) used a twistor construction to obtain all almost complex curves in S^6 of type (I). Bryant's construction involves consideration of the twistor bundle $\pi : Q^5 \to S^6$, where Q^5 denotes the quadric in CP^6 given by

$$Q^5 = \{[a+ib] \in CP^6 : a, b \in E^7,\ |a| = |b| \text{ and } a \perp b\}$$

and $\pi : Q^5 \to S^6$ is the map

$$\pi([a+ib]) = -\frac{a \times b}{|a \times b|}.$$

An alternative description of π may be given in terms of a projection between quotient spaces of G_2 as follows. Recall that the standard action of G_2 on E^7 induces transitive actions on S^6 and Q^5. If $\{e_1, \ldots, e_7\}$ is the standard basis of E^7, then the stabilizer of $e_4 \in S^6$ may be identified with $SU(3)$ and that of $[e_1 + ie_5] \in Q^5$ with $U(2)$. Using these identifications, if $g \in U(2)$, then

$$g(e_4) = -g(e_1 \times e_5) = -e_1 \times e_5 = e_4,$$

so that $U(2) \subset SU(3)$. The above projection π may also be described as the standard projection map $\pi : Q^5 = G_2/U(2) \to G_2/SU(3) = S^6$.

A holomorphic map $g : S \to Q^5$ is said to be superhorizontal if $g \times g_z = 0$, where $z = x+iy$ is a local complex coordinate on S. Such maps are horizontal in the sense that $g(S)$ intersects the fibres orthogonally. Although the map π above is not a Riemannian submersion, if $g : S \to Q^5$ is holomorphic and superhorizontal, the metrics induced on S by g and $\pi \circ g$ are equal.

R. Bryant (1982) showed that there is a one-to-one correspondence between linearly full superhorizontal maps $g : S \to Q^5$ and almost complex curves $f : S \to S^6$ of type (I), where g corresponds to the map $f = \pi \circ g$.

Bryant (1982) also gave a “Weierstrass representation” theorem for almost complex curves in S^6 and proved that every compact Riemann surface admits an infinite number of almost complex maps of type (I) into S^6.

Almost complex curves of genus zero are necessarily of type (I) or (IV), which have been studied by N. Ejiri (1986a) who described all S^1-symmetric examples.

A description of almost complex curves of types (II) was given by Bolton, F. Pedit and Woodward (1995). They showed that almost complex curves of type (II) all arise from solutions of the affine 2-dimensional G_2-Toda equations.

The method of construction of almost complex curves of type (III) uses the Hopf fibration $\pi \colon S^5 \to CP^2$ to obtain these complex curves horizontal lifts of suitable totally real branched minimal immersions into CP^2. In fact, if $f : S \to S^6$ is an almost complex curve of type (III), by applying an element of G_2 if necessary we may assume that $f(S) \subset S^5 = S^6 \cap P$ where P is the hyperplane of E^7 given by $P = \{(x_1, \ldots, x_7) : x_4 = 0\}$. Then the map $\pi : S^5 \to CP^2$ given by

$$\pi(x_1, x_2, x_3, 0, x_5, x_6, x_7) = [x_1 + ix_5, x_2 + ix_6, x_3 + ix_7],$$

is the Hopf fibration and it is shown in [Bolton-Vrancken-Woodward 1994] that f is horizontal and $\pi \circ f : S \to CP^2$ is a totally real non-totally geodesic weakly conformal harmonic map of S into CP^2.

Conversely, if S is simply-connected and if $\psi : S \to CP^2$ is such a map, then among the horizontal lifts of ψ there are exactly three almost complex curves (which are all G_2-congruent).

Using the theory of calibrations, Palmer (1997) gave estimates for the nullity and Morse index of almost complex curves in the nearly Kähler 6-sphere.

19.2. **Minimal surfaces of constant curvature in nearly Kähler 6-sphere.** The minimal surfaces of constant curvature in the 6-sphere have been known for some time.

R. L. Bryant (1985) proved that there are no minimal surfaces in S^n (in particular, in S^6) of constant negative Gaussian curvature. Flat minimal surfaces in S^n were classified by K. Kenmotsu (1976).

K. Sekigawa (1983) studied almost complex curves of constant curvature in the nearly Kähler 6-sphere and proved that, if the Gaussian curvature K of an almost complex surface M in the nearly Kähler S^6 is constant on M, then $K = 1$, $K = \frac{1}{6}$ or $K = 0$. Moreover, up to G_2-congruence and conformal transformation of the domain, the immersion $f : M \to S^6$ of the almost complex curve is one of the following:

(1) $K = 1$, $M = S^2 = \{(x, y, z) \in E^3 : x^2 + y^2 + z^2 = 1\}$ and

$$f(x, y, z) = (x, y, z, 0, 0, 0, 0),$$

for $(x, y, z) \in S^2$,

(2) $K = \frac{1}{6}$, $M = S^2$, and

$$f(x,y,z) = \frac{1}{2\sqrt{2}}(\sqrt{3}x(-x^2 - y^2 + 4z^2), \sqrt{30}z(x^2 - y^2),$$
$$\sqrt{5}y(3x^2 - y^2), \sqrt{2}z(-3x^2 - 3y^2 + 2z^2),$$
$$-\sqrt{3}y(-x^2 - y^2 + 4z^2), -2\sqrt{30}xyz, \sqrt{5}x(-x^2 + 3y^2)),$$

(3) $K = 0$, $M = C^1$, and

$$f(w) = \frac{1}{\sqrt{6}} \sum_{j=1}^{3} \left(e^{\mu_j w - \overline{\mu_j w}} \nu_j + e^{-\mu_j w + \overline{\mu_j w}} \bar{\nu}_j \right),$$

where

$$\mu_1 = 1,\ \mu_2 = \exp(\frac{2\pi}{3}), \mu_3 = \exp(\frac{4\pi i}{3}), \nu_j = \frac{1}{\sqrt{2}}(e_j + ie_{j+4}),$$

for $j = 1, 2, 3$; and $e_1, \ldots, e_7$ is the standard basis of E^7.

F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken (1987b) studied almost complex surfaces M in the nearly Kähler S^6 and proved that

(1) If $0 \le K \le 1/6$, then K is constant and $K = 0$ or $K = 1/6$.

(2) If $1/6 \le K \le 1$, then K is constant and $K = 1/6$ or $K = 1$.

The condition $K \le 1$ in (2) is not necessary, since it is always satisfied for an almost complex surface. (2) improves an earlier result of Sekigawa (1983).

19.3. **Hopf hypersurfaces and almost complex curves.** Suppose that M is a real hypersurface in the nearly Kähler S^6. Applying the almost complex structure J on S^6 to the normal bundle of M, one obtains a 1-dimensional distribution in M. The 1-dimensional foliation induced by this distribution is called the Hopf foliation, and M is said to be a Hopf hypersurface if this foliation is totally geodesic.

J. Berndt, J. Bolton and L. Woodward (1995) proved that a hypersurface of the nearly Kähler S^6 is a Hopf hypersurface if and only if it is an open part of a tube around an almost complex submanifold of S^6.

As the nearly Kähler six-sphere admits no four-dimensional almost complex submanifolds, this implies that M is a Hopf hypersurface if and only if it is an open part of either a geodesic hypersphere or a tube around an almost complex curve. As a consequence, every Hopf hypersurface of S^6 has exactly 1, 2, or 3 distinct principal curvatures at each point.

In the case where M is umbilical, it is an open subset of a geodesic hypersphere. The Hopf hypersurface M has exactly 2 distinct principal curvatures if and only if M is an open part of a tube around a totally geodesic almost complex curve in the nearly Kähler S^6.

19.4. **Lagrangian submanifolds in nearly Kähler 6-sphere.** .

19.4.1. Ejiri's theorems for Lagrangian submanifolds in S^6

A 3-dimensional submanifold M of the nearly Kähler S^6 is called Lagrangian if the almost complex structure J on the nearly Kähler 6-sphere carries each tangent space T_xM, $x \in M$ onto the corresponding normal space $T_x^\perp M$.

NĖjiri (1981) proved that a Lagrangian submanifold M in S^6 is always minimal and orientable. He also proved that if M has constant sectional curvature, then M is either totally geodesic or has constant curvature 1/16. The first nonhomogeneous examples of Lagrangian submanifolds in the nearly Kähler 6-sphere were described in [Ejiri 1986b].

F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken (1987a) proved that if M is a compact Lagrangian submanifold of S^6 with $K > 1/16$, then M is a totally geodesic submanifold.

19.4.2. Equivariant Lagrangian submanifolds in S^6

K. Mashimo (1985) classified the G_2-equivariant Lagrangian submanifolds M of the nearly Kähler 6-sphere. It turns out that there are five models, and every equivariant Lagrangian submanifold in the nearly Kähler 6-sphere is G_2-congruent to one of the five models.

These five models can be distinguished by the following curvature properties:

(1) M^3 is totally geodesic ($\delta(2) = 2$),
(2) M^3 has constant curvature 1/16 ($\delta(2) = 1/8$),
(3) the curvature of M^3 satisfies $1/16 \leq K \leq 21/16$ ($\delta(2) = 11/8$),
(4) the curvature of M^3 satisfies $-7/3 \leq K \leq 1$ ($\delta(2) = 2$),
(5) the curvature of M^3 satisfies $-1 \leq K \leq 1$ ($\delta(2) = 2$).

F. Dillen, L. Verstraelen and L. Vrancken (1990) characterized models (1), (2) and (3) as the only compact Lagrangian submanifolds in S^6 whose sectional curvatures satisfy $K \geq 1/16$. They also obtained an explicit expression for the Lagrangian submanifold of constant curvature 1/16 in terms of harmonic homogeneous polynomials of degree 6. Using these formulas, it follows that the immersion has degree 24. Further, they also obtained an explicit expression for model (3).

It follows from inequality (3.17) and Ejiri's result that the invariant

$$\delta(2) := \frac{\rho}{2} - \inf K \tag{19.1}$$

always satisfies $\delta(2) \leq 2$, for every Lagrangian submanifold of the nearly Kähler S^6.

The models (1), (4) and (5) satisfy the equality $\delta(2) = 2$ identically. Chen, Dillen, Verstraelen and Vrancken (1995a) proved that these three models are the only Lagrangian submanifolds of the nearly Kähler S^6 with constant scalar curvature that satisfy the equality $\delta(2) = 2$.

Many further examples of Lagrangian submanifolds in the nearly Kähler S^6 satisfying the equality $\delta(2) = 2$ have been constructed in [Chen-Dillen-Verstraelen-Vrancken 1995a, 1995b].

19.4.3. Lagrangian submanifolds in S^6 satisfying $\delta(2) = 2$

A Riemannian n-manifold M whose Ricci tensor has an eigenvalue of multiplicity at least $n - 1$ is called quasi-Einstein. R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken (1997) proved that Lagrangian submanifolds of the nearly Kähler 6-sphere satisfying $\delta(2) = 2$ are quasi-Einstein.

The complete classification of Lagrangian submanifolds in the nearly Kähler 6-sphere satisfying the equality $\delta(2) = 2$ was established by Dillen and Vrancken (1996). More precisely, they proved the following:

(1) Let $\phi : N_1 \to CP^2(4)$ be a holomorphic curve in $CP^2(4)$, PN_1 the circle bundle over N_1 induced by the Hopf fibration $\pi : S^5(1) \to CP^2(4)$,

and ψ the isometric immersion such that the following diagram commutes:

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4). \end{array}$$

Then, there exists a totally geodesic embedding i of S^5 into the nearly Kähler 6-sphere such that the immersion $i \circ \psi : PN_1 \to S^6$ is a 3-dimensional Lagrangian immersion in S^6 satisfying equality $\delta(2) = 2$.

(2) Let $\bar{\phi} : N_2 \to S^6$ be an almost complex curve (with second fundamental form h) without totally geodesic points. Denote by UN_2 the unit tangent bundle over N_2 and define a map

$$\text{19.2} \qquad \bar{\psi} : UN_2 \to S^6 : v \mapsto \bar{\phi}_\star(v) \times \frac{h(v,v)}{\|h(v,v)\|}.$$

Then $\bar{\psi}$ is a (possibly branched) Lagrangian immersion into S^6 satisfying equality $\delta(2) = 2$. Moreover, the immersion is linearly full in S^6.

(3) Let $\bar{\phi} : N_2 \to S^6$ be a (branched) almost complex immersion. Then, SN_2 is a 3-dimensional (possibly branched) Lagrangian submanifold of S^6 satisfying equality $\delta(2) = 2$.

(4) Let $f : M \to S^6$ be a Lagrangian immersion which is not linearly full in S^6. Then M automatically satisfies equality $\delta(2) = 2$ and there exists a totally geodesic S^5, and a holomorphic immersion $\phi : N_1 \to CP^2(4)$ such that f is congruent to ψ, which is obtained from ϕ as in (1).

(5) Let $f : M \to S^6$ be a linearly full Lagrangian immersion of a 3-dimensional manifold satisfying equality $\delta(2) = 2$. Let p be a non totally geodesic point of M. Then there exists a (possibly branched) almost complex curve $\bar{\phi} : N_2 \to S^6$ such that f is locally around p congruent to $\bar{\psi}$, which is obtained from $\bar{\phi}$ as in (3).

Let $f : S \to S^6$ be an almost complex curve without totally geodesic points. Define

$$\text{(19.3)} \qquad F : T_1S \to S^6(1) : v \mapsto \frac{h(v,v)}{||h(v,v)||},$$

where T_1S denotes the unit tangent bundle of S.

L. Vrancken (1997) showed that the following:

(i) F given by (19.2) defines a Lagrangian immersion if and only if f is superminimal, and

(ii) If $\psi : M \to S^6(1)$ be a Lagrangian immersion which admits a unit length Killing vector field whose integral curves are great circles. Then there exist an open dense subset U of M such that each point p of U has a neighborhood V such that $\psi : V \to S^6$ satisfies $\delta(2) = 2$, or $\psi : V \to S^6$ is obtained as in (i).

19.5. **Further results.** L. Vrancken (1988) proved that a locally symmetric Lagrangian submanifold of the nearly Kähler S^6 has constant curvature 1 or 1/16.

H. Li (1996) showed that if the Ricci tensor of a compact Lagrangian submanifold in the nearly Kähler S^6 satisfies $Ric \geq (53/64)g$, then either $Ric = 2g$ or the submanifold is totally geodesic.

F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken (1988) proved that if a totally real surface M in the nearly Kähler S^6 is homeomorphic to a sphere, then M is totally geodesic.

K. Sekigawa studied CR-submanifolds in the nearly Kähler S^6 and proved that there exists no proper CR-product in the nearly Kähler S^6, although there do exist 3-dimensional CR-submanifolds in the nearly Kähler S^6 whose totally real and holomorphic distributions are both integrable.

Y. B. Shen (1998) studied slant minimal surfaces in the nearly Kähler S^6. He proved that if $f : M \to S^6$ is a minimal slant isometric immersion of a complete surface of nonnagative Gauss curvature K in the nearly Kähler S^6 such that f is neither holomorphic nor antiholomorphic, then either $K = 1$ and f is totally geodesic, or $K = 0$ and f is either totally real or superminimal. He also showed that if $f : S^2 \to S^6$ is a minimal slant immersion of a topological 2-sphere which is neither holomorphic nor antiholomorphic, then f is totally geodesic.

20. Axioms of submanifolds

20.1. **Axiom of planes.** Historically the axiom of planes was originally introduced by G. Riemann in postulating the existence of a surface S passing through three given points with the property that every straight line having two points in S is completely contained in this surface.

E. Beltrami (1835–1900) had shown in 1868 that a Riemannian manifold of constant curvature satisfies the axiom of 2-planes and F. Schur (1856–1932) proved in 1886 that the converse is also true. The later result was also obtained by L. Schläfli (1873) in combination with the work of F. Klein (1849–1925).

In his 1928 book, È. Cartan defined the axiom of planes as follows: A Riemannian n-manifold M, $n \geq 3$, is said to satisfy the axiom of k-planes if, for each point $x \in M$ and each k-dimensional subspace T'_x of the tangent space T_xM, there exists a k-dimensional totally geodesic submanifold N containing x such that the tangent space of N at x is T'_x, where k is a fixed integer $2 \leq k < n$.

È. Cartan's result states that real space forms are the only Riemannian manifolds of dimension ≥ 3 which satisfy the axiom of k-planes, for some k with $2 \leq k < n$.

20.2. **Axioms of spheres and of totally umbilical submanifolds.** As a generalization of the axiom of k-planes, D. S. Leung and K. Nomizu (1971) introduced the axiom of k-spheres: for each point $x \in M$ and for each k-dimensional linear subspace T'_x of T_xM, there exists a k-dimensional totally umbilical submanifold N of M containing x with parallel mean curvature vector such that the tangent space of N at x is T'_x.

Leung and Nomizu proved that a Riemannian manifold of dimension $n \geq 3$ satisfies the axiom of k-spheres, $2 \leq k < n$, if and only if it is a real space form.

The proof of Leung and Nomizu's result was based on Codazzi's equation and the following result of É. Cartan: A Riemannian manifold M of dimension > 2 is a real space form if and only if its curvature tensor R satisfies $R(X, Y, Z, X) = 0$ for any orthonormal vector fields X, Y, Z in M.

S. I. Goldberg and E. M. Moskal (1976) observed that the result also holds if totally umbilical submanifolds with parallel mean curvature vector are replaced by submanifolds with parallel second fundamental form. W. Strübing (1979) pointed out that totally umbilical submanifolds with parallel mean curvature vector can further be replaced by submanifolds satisfying Codazzi equation for real space forms.

J. A. Schouten in 1924 proved that a Riemannian manifold of dimension $n \geq 4$ is conformally flat if and only if it satisfies the axiom of totally umbilical submanifolds of dimension k, $3 \leq k < n$.

By the axiom of totally umbilical submanifolds of dimension k, we mean that, for each point $x \in M$ and for each k-dimensional linear subspace T'_x of T_xM, there exists a k-dimensional totally umbilical submanifold N of M containing x such that the tangent space of N at x is T'_x.

K. L. Stellmacher (1951) showed that the same result holds for Riemannian 3-manifold for $k = 2$.

K. Yano and Y. Muto (1941) proved that a Riemannian manifold of dimension ≥ 4 is conformally flat if and only if it satisfies the axiom of totally umbilical surfaces with prescribed mean curvature vector.

D. Van Lindt and L. Verstraelen (1981) proved that a Riemannian manifold of dimension $n \geq 4$ is conformally flat if and only if it satisfies the axiom of conformally flat totally quasiumbilical submanifolds of dimension k, $3 < k < n$. Here by the axiom of conformally flat totally quasiumbilical submanifolds of dimension k we mean that, for each point $x \in M$ and for each k-dimensional linear subspace T'_x of T_xM, there exists a k-dimensional conformally flat totally quasiumbilical submanifold N of M containing x such that the tangent space of N at x is T'_x.

In 1975, Chen and Verstraelen proved that a Riemannian manifold M of dimension $n \geq 4$ is conformally flat if and only if, for each point $x \in M$ and for each k-dimensional ($2 \leq k < n$) linear subspace T'_x of T_xM, there exists a k-dimensional submanifold N which passes through x and which at x tangent to T'_x such that N has flat normal connection and commutative shape operators.

20.3. **Axiom of holomorphic $2k$-planes.** In 1955 K. Yano and Y. Mogi introduced the axiom of holomorphic $2k$-planes on a Kähler manifold $\tilde{M}$ as follows: for each point $x \in \tilde{M}$ and for each holomorphic $2k$-dimensional linear subspace T'_x of T_xM, there exists a $2k$-dimensional totally geodesic submanifold N of $\tilde{M}$ containing x such that the tangent space of N at x is T'_x.

Yano and Mogi proved that a Kähler manifold of real dimension $2n \geq 4$ is a complex space form if and only if it satisfies the axiom of holomorphic $2k$-planes for some k, $1 \leq k < n$. Goldberg and Moskal pointed out that the same result holds if the $2k$-dimensional totally geodesic submanifolds are replaced by $2k$-dimensional totally umbilical submanifolds with parallel mean curvature vector. In 1982, O. Kassabov proved that the same result also holds if the $2k$-dimensional totally geodesic submanifolds are replaced by $2k$-dimensional totally umbilical submanifolds.

20.4. **Axiom of antiholomorphic k-planes.** The axiom of antiholomorphic k-planes was introduced in 1973 by Chen and Ogiue: for each point $x \in \tilde{M}$ and for each totally real k-dimensional linear subspace T'_x of $T_x\tilde{M}$, there exists a k-dimensional totally geodesic submanifold N of $\tilde{M}$ containing x such that the tangent space of N at x is T'_x.

Chen and Ogiue proved that a Kähler manifold of real dimension $2n \geq 4$ is a complex space form if and only if it satisfies the axiom of antiholomorphic k-planes for $2 \leq k \leq n$. The same result was also obtained independently by K. Nomizu (1973b).

M. Harada (1974) pointed out that the same result holds if the k-dimensional totally geodesic submanifolds were replaced by k-dimensional totally umbilical submanifolds with parallel mean curvature vector. Also, S. Yamaguchi and M. Kon (1978) observed that the same result holds if the k-dimensional totally geodesic submanifolds are replaced by k-dimensional totally umbilical totally real submanifolds. O. Kassabov proved that the same result also holds if the k-dimensional totally geodesic submanifolds are further replaced by k-dimensional totally umbilical submanifolds.

D. Van Lindt and L. Verstraelen proved that a Kähler manifold of real dimension $2n > 4$ is a complex space form if and only if, for each point $x \in M$ and for each k-dimensional $(2 \leq k < n)$ totally real linear subspace T'_x of T_xM, there exists a totally real k-dimensional submanifold N which passes through x and which at x tangent to T'_x such that N has commutative shape operators and parallel f-structure in the normal bundle. Here the f-structure is the endomorphism on the normal bundle induced from the almost complex structure on the ambient space.

20.5. **Axioms of coholomorphic spheres.** Chen and Ogiue (1974a) introduced the axiom of coholomorphic $(2k+\ell)$-spheres as follows: for each point $x \in \tilde{M}$ and for each totally real $(2k+\ell)$-dimensional CR-plane section T'_x of $T_x\tilde{M}$, there exists a k-dimensional totally umbilical submanifold N of $\tilde{M}$ containing x such that the tangent space of N at x is T'_x. Chen and Ogiue proved that a Kählerian manifold of real dimension $2n \geq 4$ is locally flat if and only if it satisfies the axiom of coholomorphic $(2k+\ell)$-spheres for some integers k and ℓ such that $1 \leq k, \ell < n$ and $2k + \ell < 2n$.

An almost Hermitian manifold M is called a RK-manifold if its Riemann curvature tensor R and its almost complex structure J satisfies

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW)$$

for X, Y, Z, W tangent to M. L. Vanhecke (1976) studied RK-manifolds satisfying the axiom of coholomorphic $(2k+1)$-spheres for some k and obtained characterization theorems for space forms.

S. Tachibana and S. Kashiwada (1973) proved that every geodesic hypersphere S with unit normal vector field ξ in a complex space form is $J\xi$-quasiumbilical, that is, S is quasiumbilical with respect to ξ and $J\xi$ is a principal direction with multiplicity equal to either one or n, where $n = \dim S$. The later case occurs only when the complex space form is flat.

A hypersurface of a Kähler manifold is said to be $J\xi$-hypercylindric if $J\xi$ is a principal direction and the principal curvatures other than the principal curvature associated with $J\xi$ are zero.

A Kähler manifold $\tilde{M}$ is said to satisfy the axiom of $J\xi$-quasiumbilical hypersurfaces if, for each point $x \in \tilde{M}$ and for each hyperplane H of $T_x\tilde{M}$ with hyperplane normal ξ, there exists a $J\xi$-quasiumbilical hypersurface N containing x such that the tangent space of N at x is H.

B. Y. Chen and L. Verstraelen (1980) proved that a Kähler manifold of real dimension > 4 satisfies the axiom of $J\xi$-quasiumbilical hypersurfaces if and only if it is a complex space form. This improves a result of L. Vanhecke and T. J. Willmore (1977).

D. van Lindt and L. Verstraelen showed that a Kähler manifold of real dimension > 4 is locally flat if and only if, for each point $x \in \tilde{M}$ and for each hyperplane L of $T_x\tilde{M}$ with hyperplane normal ξ, there exists a $J\xi$-hypercylindric hypersurface N containing x such that the tangent space of N at x is L.

20.6. **Submanifolds contain many circles.** An ordinary torus contains exactly four circles through each points. Since each compact cyclide of Dupin in E^3 can be obtained from inversion of a torus of revolution; thus it contains four circles through each point. R. Blum (1980) investigated the cyclide in E^3 defined by

$$(x^2 + y^2 + z^2)^2 - 2ax^2 - 2by^2 - 2cz^2 + d^2 = 0, \tag{20.1}$$

where the real coefficients a, b, c, d satisfy the condition $0 < d < b \le a$, $c < d$. He proved that (a) if $a \neq b$ and $c \neq -d$, there exist 6 circles through each point; (b) if $a = b$, $c \neq -d$ or $a \neq b$, $c = -d$, there exist 5 circles through each point; and (c) if $a = b$ and $c = -d$, there exist 4 circles through each point of the cyclide. The case(c) represents a torus of revolution. On the other hand, N. Takeuchi (1987) showed that a smooth compact surface of genus one in E^3 cannot contain seven circles through each point. Obviously, there exist infinitely many circles which pass through each point of a round sphere in E^3.

In 1984 K. Ogiue and R. Takagi proved that a surface M in E^3 is locally a plane or a sphere if, through each point $p \in M$, there exist two Euclidean circles such that (i) they are contained in M in some neighborhood of p and (ii) they are tangent to each other at p. Condition (i) alone is not sufficient, as it is satisfied by a torus of revolution. Ogiue and Takagi also generalized this to obtain similar characterizations of totally geodesic submanifolds and extrinsic spheres of arbitrary dimension in Riemannian manifolds. In particular, they proved that a 2-dimensional surface M in a Riemannian manifold N is totally geodesic if through each point $p \in M$ there exist three geodesics of N which lie in M in some neighborhood of p.

R. Miyaoka and N. Takeuchi (1992) proved that a complete simply-connected surface in E^3 which contains two transversal circles through each point must be a plane or a sphere.

K. Ogiue and N. Takeuchi (1992) proved that a compact smooth surface of revolution which contains at least two circles through each point is either a sphere or a hulahoop surface, that is, a surface obtained by revolving a circle around a suitable axis. A hulahoop surface has 4, 5, or infinitely many circles through each point. A hulahoop surface, which is neither a sphere nor an ordinary torus, contains exactly 5 circles through each point. Ogiue and Takeuchi also described the concrete geometric construction of a torus in Euclidean 3-space containing five circles through each point.

J. Arroyo, O. J. Garay and J. J. Menc´ia (1998) showed that if a compact surface of revolution in E^3 contains at least two ellipses through each point, then it is an elliptic hulahoop surface, that is, a surface obtained by revolving an ellipse around a suitable axis.

Without making distinction between real and nonreal circles, E. E. Kummer (1810–1893) already observed in 1865 that a general cyclide has the property that there exist 10 circles through each generic point of the cyclide.

21. Total absolute curvature

21.1. **Rotation index and total curvature of a curve.** Let γ be closed smooth curve in the plane. As a point moves along γ, the line through a fixed point O and parallel to the tangent line of γ rotates through an angle $2n\pi$ or rotates n times about O. This integer n is called the rotation index of γ. If γ is a simple closed curve, $n = \pm 1$.

Two curves are said to be regularly homotopic if one can be deformed to the other through a family of closed smooth curves. Because the rotation index is an integer and it varies continuously through the deformation, it must be constant. Therefore, two closed smooth curves have the same rotation index if they are regular homotopic. A theorem of W. C. Graustein (1888–1941) and H. Whitney states that the converse of this is also true; a result suggested by Graustein whose proof was first published in [Whitney 1937]. Hence, the only invariant of a regular homotopy class is the rotation index.

Let $\gamma(s) = (x(s), y(s))$ be a unit-speed smooth closed curve in E^2. Then

$$x''(x) = -\kappa(s)y'(s), \quad y''(x) = \kappa(s)x'(s),$$

where $\kappa = \kappa(s)$ is the curvature of the curve. If $\theta(s)$ denotes the angle between the tangent line and the x-axis, then $d\theta = \kappa(s)ds$. Thus, we have

$$\int_\gamma \kappa(s)ds = 2n\pi, \tag{21.1}$$

where n is the rotation index of γ.

From (21.1), it follows that the total absolute curvature of γ satisfies

$$\int_\gamma |\kappa(s)|ds \geq 2\pi, \tag{21.2}$$

with the equality holding if and only if γ is a convex plane curve.

Inequality (21.2) was generalized to closed curves in E^3 by W. Fenchel (1905–) in 1929, and to closed curves in E^m, $m > 3$, by K. Borsuk (1905–1982) in 1947.

I. Fary (1922–) in 1949 and J. Milnor (1931–) in 1950 proved that if a closed curve γ in E^m satisfies

$$\int_\gamma |\kappa(s)|ds \leq 4\pi,$$

then γ is unknotted.

21.2. **Total absolute curvature of Chern and Lashof.** Let $f : M \to E^m$ be an isometric immersion of a compact Riemannian n-manifold M into E^m. Let $\nu_1(M)$ denote the unit normal bundle of f, and S^{m-1} the unit hypersphere centered at the origin of E^m, and let $T_f : \nu_1(M) \to S^{m-1}$ be the parallel translation.

Denote by ω and Ω the volume elements of $\nu_1(M)$ and of S^{m-1}, respectively. For each $\xi \in \nu_1(M)$, we have

$$T_f^*\Omega = (\det A_\xi)\omega,$$

where A_ξ is the shape operator of f in the direction ξ.

As a generalization of the total absolute curvature for a space curve, S. S. Chern and R. K. Lashof (1957) defined the total absolute curvature of f as follows:

$$\tau(f) = \frac{1}{s_{m-1}} \int_{\nu_1(M)} |T_f^*\Omega| = \frac{1}{s_{m-1}} \int_{\nu_1(M)} |\det A_\xi|\,\omega, \tag{21.3}$$

where s_{m-1} is the volume of the unit $(m-1)$-sphere.

A function ϕ on M which has only nondegenerate critical point is called a Morse function. Since M is assumed to be compact, each Morse function on M has only a finite number of critical points. The Morse number $\gamma(M)$ of M is defined as the least number of critical points of Morse functions on M. The Morse inequalities imply that

$$\gamma(M) \geq \beta(M;F) = \sum_{k=0}^{n} \beta_k(M;F) \tag{21.4}$$

for any field F, where $\beta_k(M;F)$ is the k-th betti number of M over F. For a (smooth) manifold M of dimension greater than five, the Morse number of M is equal to the number of cells in the smallest CW-complex of the same simple homotopy type as M [Sharpe 1989a].

S. S. Chern and R. K. Lashof (1957,1958) proved the following results.

Let $f : M \to E^m$ be an isometric immersion of a compact Riemannian n-manifold M into E^m. Then

(1) $\tau(f) \geq \gamma(M) \geq 2$;

(2) $\tau(f) = 2$ if and only if f is an embedding and $f(M)$ is a convex hypersurface in an affine $(n+1)$-subspace E^{n+1} of E^m;

(3) if $\tau(f) < 3$, then M is homeomorphic to S^n.

If $\tau(f) = 3$, M needs not be homeomorphic to S^n. In fact, the Veronese embedding of the real projective plane into E^5 satisfies $\tau(f) = 3$.

J. Eells and N. H. Kuiper (1962) classified compact manifolds which admit a Morse function with three nondegenerate critical points. They called such manifolds "manifolds like projective spaces", which include the real, complex and quaternionic projective planes and the Cayley plane.

Applying Eells-Kuiper's result, it follows that if an immersion $f : M \to E^m$ of a compact manifold M into E^m satisfies $\tau(f) < 4$, then M is homeomorphic either to the sphere S^n or else to one of the manifolds like projective planes.

R. W. Sharpe (1989b) proved that for manifolds M of dimension greater than five, the best possible lower bound for the total absolute curvature $\tau(f)$ is the Morse number $\gamma(M)$, as the immersion $f : M \to E^m$ varies over all possibilities.

Let M be a compact n-manifold with $n > 5$. Denote by $\tau[i]$ the infimum of the total absolute curvature $\tau(j)$ as j varies over all immersions in the regular homotopy class of the immersion $i : M \to E^m$. Sharpe (1989b) also proved that, for $n > 5$, if $m > n + 1$ or if $m = n + 1$ is odd, then $\tau[i] = \gamma(M)$. If $m = n + 1$ is even, then $\tau[i] = \max\{\gamma(M), 2|d|\}$, where d is the normal degree of i and $\gamma(M)$ is the Morse number of M. Examples are given of codimension-one immersions of odd-dimensional spheres which have arbitrary odd normal degree and which attain the infimum of the total absolute curvature in their regular homotopy class..

R. Langevin and H. Rosenberg (1976) proved that if the total absolute curvature of an embedded surface of genus $g > 1$ is $< 2g+6$, then the surface is unknotted. N. H. Kuiper and W. Meeks (1984) showed that if the total absolute curvature $\tau(f)$ of an embedding torus $f : T \to E^3$ is ≤ 8, then $f(T)$ is unknotted.

Kuiper and Meeks (1984) also proved that, for an embedding of a compact manifold $f : M^n \to E^N$, the total absolute curvature of f satisfies

$$\tau(f) > \beta + 4\sigma_1, \tag{21.5}$$

where β is the sum of the mod 2 Betti numbers and $\beta + \sigma_1$ is the minimal number of generators of the fundamental group of the complement of the image $f(M^n)$.

21.3. **Tight immersions.** An immersion $f : M \to E^m$ is called tight (or minimal total absolute curvature immersion) if $\tau(f) = b(M) := \min_\phi \beta(\phi)$, where $\beta(\phi)$ denotes the number of critical points of a Morse function ϕ on M. This condition is equivalent to requiring that every height function that is Morse have the minimum number of critical points required by the Morse inequalities. Not every compact manifold admits a tight immersion. For instance, N. H. Kuiper (1958) observed that the exotic 7-sphere of J. Milnor admits no immersion with minimal total absolute curvature. This can be seems as follows: Since a manifold M homeomorphic to a sphere admits a function with only two critical points, M satisfies $b(M) = 2$. Thus, if a manifold M is homeomorphic to S^7, then a tight immersion $f : M \to E^m$ would embed M as a convex hypersurface in an $E^8 \subset E^m$, and hence M would be diffeomorphic to the standard 7-sphere.

D. Ferus (1967) proved that every embedding f of an exotic n-sphere $(n \geq 5)$ in E^{n+2} has total absolute curvature $\tau(f) \geq 4$.

S. Kobayashi (1967b) showed that every compact homogeneous Kähler manifold admits a tight embedding. R. Bott and H. Samelson (1958) proved that symmetric R-spaces admit tight immersions. This result was also proved independently by M. Takeuchi and S. Kobayashi (1968). A tight immersion of a symmetric R-space is a minimal immersion into a hypersphere.

In 1960's the theory of tight immersions underwent substantial development and reformulation. Since this notion is a generalization of convexity, N. H. Kuiper called these "convex immersions". T. F. Banchoff (1965) first used tight in conjunction with his study of the two-piece property.

Kuiper (1962) formulated tightness in terms of intersections with half-spaces and injectivity of induced maps on homology and proved that his formulation is equivalent to the minimal total absolute curvature of manifolds which satisfy the condition: the Morse number $\gamma(M)$ of M is equal to the sum $\beta(M;F)$ of the Betti numbers for some field F.

N. H. Kuiper (1961) obtained smooth tight embeddings into E^3 for all orientable surfaces, and smooth tight immersions for all nonorientable surfaces with Euler characteristic less than -1. He also showed that smooth tight immersions of the projective plane and the Klein bottle into E^3 do not exist.

The question of whether there is a smooth tight immersion of the projective plane with an attached handle into E^3 has been open for 30 years. In 1992, F. Haab proved that no such immersion exists.

Kuiper and Meeks (1984) showed that if the genus g of a compact surface M is greater than 2, then there exists a knotted tight embedding in E^3, whereas if $g \leq 2$, there does not exist such an embedding. Pinkall (1986a) showed that if the Euler characteristic $\chi(M)$ of M satisfies $\chi(M) < -9$, then every immersion f of M into E^3 is regularly homotopic to a tight immersion. This is also true if M is orientable with genus $g \geq 4$. On the other hand, there are immersions which are not regularly homotopic to a tight immersion. This is clearly true for any immersion of the projective plane or Klein bottle. Pinkall (1986a) also showed that every tight immersion of the torus T^2 is regularly homotopic to a standard embedding, and thus there are no tight immersions in the nonstandard regular homotopy class of immersed tori.

Kuiper (1961) showed that smooth immersions into E^4 of orientable surfaces exist for every genus ≥ 1. He also showed that for a substantial (that is, not contained in any proper affine subspace) tight immersion of a surface into E^N one must have $N \leq 5$, with equality only for surfaces projectively equivalent to the Veronese surface and thus analytic.

Kuiper (1979) also proved the following:

(1) If $f : M^{2d} \to E^N$ is a tight substantial continuous embedding of a manifold like a projective space, then $N \leq 3d+2$;

(2) Let $f : M^{2d} \to E^{3d+2}$ be a tight smooth substantial embedding of a compact manifold with Morse number $\gamma(M) = 3$. Then M^{2d} is algebraic. Moreover, it is the union of its E^{d+1}-top-sets, smooth d-sphere S^d that are quadratic d-manifolds.

T. F. Banchoff and N. H. Kuiper have produced tight analytic immersions into E^3 of all orientable compact surfaces, while Kuiper has produced tight analytic immersions into E^3 of all nonorientable compact surfaces with even Euler characteristic other than zero.

G. Thorbergsson (1991) proved that an analytic tight immersion of a compact orientable surface into E^4 which is substantial must be a torus. He also showed that the surface is the intersection of two developable ruled hypersurfaces, possibly with singularities, with two-dimensional rulings.

After the spheres, the $(k-1)$-connected $2k$-dimensional compact manifolds have the most simple topology, for their homology groups vanish in all dimensions except 0, k and $2k$. Among these so-called highly connected manifolds, the only ones known to admit tight immersions into some Euclidean space are the connected sums of copies of $S^k \times S^k$, the projective planes, and all surfaces except for the Klein bottle and the projective plane with one handle attached. Kuiper has conjectured that the only $2k$-dimensional, $(k-1)$-connected manifolds with trivial k-th Stiefel-Whitney class that admit tight immersions into E^{2k+l} are homeomorphic to $S^k \times S^k$. (The condition on the k-th Stiefel-Whitney class follows from $(k-1)$-connectedness for $k \neq 1, 2, 4, 8$.) J. J. Hebda (1984) and G. Thorbergsson (1986) were able to construct counterexamples for $l = 1$ and $l = 2$, respectively. Assume the immersion is analytic, R. Niebergall (1994) used top-set techniques to show that Kuiper's conjecture is true for $l \geq 2$.

C. S. Chen (1979) proved that if f is a substantial tight embedding of $S^k \times S^{n-k}$ $(k/(n-k) \neq 2, \frac{1}{2})$ into E^{n+2} whose image lies on an ovoid, then f is projectively equivalent to a product embedding of two ovaloids of dimensions k and $n-k$, respectively.

M. van Gemmeren (1996) generalized tightness properties of immersed compact manifolds to noncompact manifolds with a finite number of ends.

21.4. **Taut immersions.** T. Banchoff initiated the study of taut immersions in 1970 by attempting to find all tight surfaces which lie in a hypersphere of a Euclidean m-space. Via stereographic projection, this problem is equivalent to the study of surfaces in E^m which have the spherical two-piece property.

S. Carter and A. West (1972) extended the spherical two-piece property and defined an immersion of a compact manifold M to be taut if every nondegenerate Euclidean distance function L_p has the minimum number of critical points.

The property of tautness is preserved under Lie sphere transformations [Cecil-Chern 1987]. Furthermore, an embedding $f : M \to E^m$ is taut (or more precisely F-taut) if and only if the embedding $\sigma \circ f : M \to S^m$ has the property that every nondegenerate spherical distance function has $\beta(M;F)$ critical points on M, where $\sigma : E^m \to S^m - \{P\}$ is stereographic projection and $\beta(M;F)$ is the sum of F-Betti numbers of M for any field F.

A spherical distance function $d_p(q) = \cos^{-1}(\ell_p(q))$ is essentially a Euclidean height function $\ell_p(q) = p \cdot q$, for $p, q \in S^m$, which has the same critical points as ℓ_p. Thus, the embedding f is taut if and only if the spherical embedding $\sigma \circ f$ is tight, that is, every nondegenerate height function ℓ_p has $\beta(M;F)$ critical points on M. A tight spherical embedding $F : M \to S^m \subset E^{m+1}$ is taut when regarded as an embedding of M in E^{m+1} (cf. [Cecil-Ryan 1985]).

S. Carter and A. West (1972) observed that if $f : M \to E^{n+1}$ is a compact orientable embedded taut hypersurface, then for sufficiently small $r > 0$ the hypersurface $f_r : M \to E^{n+1}$ defined by $f_r(p) = f(p) + r\xi(p)$ if taut if and only if f is taut, where ξ is a global unit normal vector field of f. Carter and West (1972) also pointed out their idea can be generalized to taut embeddings of higher codimension. Pinkall (1986b) proved that if M is a compact embedded submanifold of dimension $n < m-1$ in S^m, then a tube $T_r(M)$ of sufficiently small radius r over M is taut if and only if M is $\mathbf{Z}_2$-taut.

S. Carter and A. West (1972) also observed that if M is noncompact but the immersion is proper, then the Morse inequalities still hold on compact subset of the form

$$M_r(L_p) = \{x \in M : L_p(x) \leq r\}$$

and the notion of a taut immersion extends to this case.

Carter and West (1972) showed that a taut immersion is tight; and it must be an embedding, since if $p \in E^m$ were a double point then the function L_p would have two absolute minima. Thus, tautness is a stronger condition than tightness. Via stereographic projection and Chern-Lashof's theorem, Banchoff and Carter-West observed that a taut embedding of a sphere in a Euclidean space must be a round sphere, not merely convex.

S. Carter and A. West (1972) classified substantial taut embeddings as follows:

Let $f : M \to E^m$ be a substantial taut embedding of an n-manifold.

(a) If M is compact, then $m \leq \frac{1}{2}n(n+3)$. In particular, if $m = \frac{1}{2}n(n+3)$, then f is a spherical Veronese embedding of a real projective space RP^n;

(b) If M is noncompact, then $m \leq \frac{1}{2}n(n+3) - 1$. In particular, if $m = \frac{1}{2}n(n+3) - 1$, then $f(M)$ is the image under stereographic projection of a Veronese manifold, where the pole of the projection is on the Veronese manifold.

Pinkall (1985a) proved that any taut submanifold in a real space form is Dupin. G. Thorbergsson (1983) proved that a complete embedded proper Dupin hypersurface in E^{n+1} is taut, and thus it must be embedded. Pinkall (1985a) extended this result to compact submanifolds of higher codimension for which the number of distinct principal curvatures is constant on the unit normal bundle. It remains as an open problem whether Dupin implies taut without this assumption.

Taut surfaces in a Euclidean space have been completely classified by Banchoff (1970) and Cecil (1976) as follows:

Let M be a taut surface substantially embedded in a Euclidean m-space.

(c) If M is a compact, then M is a round sphere or a ring cyclide in E^3, a spherical Veronese surface in E^5, or a compact surface in E^4 related to one of these by stereographic projection;

(d) If M is noncompact, then M is a plane, a circular cylinder, a parabolic ring cyclide in E^3, or it is the image in E^4 of a punctured spherical Veronese surface under stereographic projection.

Conversely, all of the surfaces listed in (c) and (d) are taut.

Let $\psi : S^7 \to S^4$ denote the Hopf fibration defined by (10.4) in §10.4.5. R. Miyaoka and T. Ozawa (1988) proved that if a compact submanifold of S^4 is taut, then $\psi^{-1}(M)$ is also taut in S^7.

Pinkall and Thorbergsson (1989b) classified, up to diffeomorphism, compact 3-manifolds which admit taut embeddings into some Euclidean space. They showed that there are seven such manifolds: $S^1 \times S^2$, $S^1 \times RP^2$, the twisted S^2-bundle over S^1, S^3, RP^3, the quaternion space $S^3/\{\pm 1, \pm i, \pm j, \pm k\}$, and the torus T^3. Each has at least one known taut embedding. The taut embeddings of $S^1 \times S^2, S^3$ and the quaternion space have been shown to be unique up to an appropriate equivalence. RP^3 is known to admit taut substantial embeddings of codimension 2 and 5, but it is still unknown whether it admits a taut substantial embedding of codimensions 3 or 4. The possible codimensions for taut substantial embeddings of the other three manifolds have been completely determined. Yet, the complete geometric classification of taut 3-manifolds remains open.

For higher dimensional taut hypersurfaces, Carter and West (1972) proved the following:

(e) Let $f : M \to E^{n+1}$ be a taut embedding of a noncompact n-manifold with $H_k(M; \mathbf{Z}_2) = \mathbf{Z}_2$ for some k, $0 < k < n$, and $H_i(M; \mathbf{Z}_2) = 0$ for $i \neq 0, k$. Then M is diffeomorphic to $S^k \times E^{n-k}$ and f is a standard product embedding;

(f) Let $f : M \to E^{n+q}$ be a substantial taut embedding of a noncompact n-manifold whose $\mathbf{Z}_2$-Betti numbers satisfy $\beta_k(M) = j > 0$ for some k, $\frac{n}{2} < k < n$ and $\beta_i = 0$ for $i \neq 0, k$. Then $q = 1$, $j = 1$ and f embeds M as a standard product $S^k \times E^{n-k}$ in E^{n+1}.

Both the hypothesis $k > \frac{n}{2}$ in statement (f) and the hypothesis that the codimension is one in statement (e) are necessary due to the example of the taut substantial embedding of the Möbius band $M^2 = RP^2 - \{\text{a point}\}$ into E^4 obtained from a Veronese surface V^2 in S^4 by stereographic projection with respect to a pole on V^2.

Cecil and Ryan (1978a) studied taut hypersurfaces of a Euclidean space and proved the following:

(g) Let M be a taut hypersurface in E^{2k+1} such that $H_i(M; \mathbf{Z}_2) = 0$ for $i \neq 0, k, 2k$. Then M is a round hypersphere, a hyperplane, a standard product $S^k \times E^k$, a ring cyclide, or a parabolic cyclide;

(h) A taut compact hypersurface M in E^{n+1} with the same $\mathbf{Z}_2$-homology as $S^k \times S^{n-k}$ is a ring cyclide.

T. Ozawa (1986) showed that the codimension of a taut substantial embedding of the product of two spheres, $S^p \times S^q, p < q$, is either 1 or $q - p + 1$, and that a connected sum of such products of two spheres cannot be tautly embedded into any sphere.

G. Thorbergsson (1983b) generalized statement (h) to higher codimension. He obtained the following:

(i) If M^{2k} is a compact $(k-1)$-connected, but not k-connected, taut submanifold of E^m which does not lie in any hypersphere of E^m, then either (i-1) $m = 2k + 1$ and M^{2k} is a cyclide of Dupin diffeomorphic to $S^k \times S^k$, or

(i-2) $m = 3k + 1$ and M^{2k} is diffeomorphic to one of the projective planes RP^2, CP^2, HP^2 or $\mathcal{O}P^2$.

It is known that the compact focal submanifolds of a taut hypersurface in E^{n+1} need not be taut. For instance, one focal submanifold of a non-round cyclide of Dupin in E^3 is an ellipse, which is tight but not taut. On the other hand, S. Buyske (1989) used Lie sphere-geometric techniques to show that if a hypersurface M in E^{n+1} is Lie equivalent to an isoparametric hypersurface in S^{n+1}, then each compact focal submanifold of M is tight in E^{n+1}.

R. Bott and H. Samelson (1958) proved that the principal orbits of the isotropy representations of symmetric space, known as R-spaces, are taut. M. Takeuchi and S. Kobayashi (1968) also proved the same result independently.

G. Thorbergsson (1988) found some necessary conditions for the existence of a taut embedding of a manifold M. He proved the following:

(j) Suppose that M is tautly embedded with respect to F into E^n and that $i > 0$ is the smallest number such that $H_i(M; F)$ is nontrivial. Then every torsion element in $H_i(M; \mathbf{Z})$ is of order two. In particular, $H_i(M; \mathbf{Z})$ is without torsion if the characteristic of F is not two.

As a corollary, Thorbergsson showed that the only lens space admitting a taut embedding is the real projective space. He also showed that a coset space $M = G/H$ cannot be tautly embedded if one of the following conditions is satisfied:

(j-1) G is simply connected with a torsion element in the fundamental group of order greater than two;

(j-2) G and H are simple and simply-connected and H is a subgroup with index greater than two.

Thorbergsson gave several explicit examples of homogeneous spaces which do not admit taut embeddings because of these two conditions.

C. Olmos (1994) proved that if M is a substantial compact homogeneous submanifold of a Euclidean space, then the following five statements are equivalent:

(0-1) M is taut;

(0-2) M is Dupin;

(0-3) M is a submanifold with constant principal curvatures;

(0-4) M is an orbit of the isotropy representation of a symmetric space;

(O-5) the first normal space of M coincides with the normal space.

T. E. Cecil and S. S. Chern (1987,1989) established a relationship between a given immersion f and its induced Legendre submanifold in T_1S^{n+1} (cf. §10.4.8). They proved the following:

Suppose $L : M^n \to T_1S^{n+1}$ is a Legendre submanifold whose Möbius projection is a taut immersion. If β is a Lie transformation such that the Möbius projection of $\beta \circ L$ is an immersion, then the Möbius projection of $\beta \circ L$ is also taut.

Two hypersurfaces of S^{n+1} are called Lie equivalent if their induced Legendre submanifolds are Lie equivalent, that is, there is a Lie sphere transformation which carries one to the other. The result of Cecil and Chern implies that if two compact hypersurfaces of S^{n+1} are Lie equivalent, then one of the two hypersurfaces is taut if and only if the other is taut.

When M is a compact embedded submanifold in S^m of codimension > 1, it induces a Legendre submanifold defined on the unit normal bundle of M in S^m. As with hypersurfaces two submanifolds of arbitrary codimension in S^m are said to be Lie equivalent if their induced Legendre submanifolds are Lie equivalent. Cecil and Chern's result also implies that if two compact embedded submanifolds of S^m are Lie equivalent, then one is $\mathbf{Z}_2$-taut if and only if the other is $\mathbf{Z}_2$-taut.

Cecil and Chern (1987,1989) also investigated the relationship between Legendre submanifolds and Dupin submanifolds.

J. J. Hebda (1988) investigated possible $\mathbf{Z}_2$-cohomology of some taut submanifolds in spheres. E. Curtin (1994) extended the notion of taut embeddings to manifolds with boundary and obtained some classification results.

Tight and taut submanifolds have also been studied in hyperbolic space. In fact, Cecil and Ryan (1979a,1979b) introduced three classes of distance functions in hyperbolic m-space H^m whose level sets are spheres centered at a point $p \in H^m$, equidistant hypersurfaces from a hyperplane in H^m, and horospheres equidistant from a fixed horosphere. Suppose M is a compact embedded submanifold in H^m and that $\pi : H^m \to D^m$ is stereographic projection of H^m onto a disk $D^m \subset E^m$. They proved that $\pi(M)$ is taut in E^m if and only if every nondegenerate hyperbolic distance function of each of the three types has $\beta(M;F)$ critical points on M.

G. Thorbergsson (1983b) obtained a result similar to statement (i) for noncompact taut submanifolds of E^m and for taut compact $(k-1)$-connected hypersurfaces in real hyperbolic space H^{2k+1}.

22. Total mean curvature

22.1. **Total mean curvature of surfaces in Euclidean 3-space.** It is well-known that the two most important geometric invariants of a surface in a Euclidean m–space are the Gaussian curvature K and the squared mean curvature H^2. According to Gauss' Theorema Egregium, Gaussian curvature is an intrinsic invariant and the integral of the Gaussian curvature over a compact surface gives the famous Gauss–Bonnet formula:

$$\int_M K dV = 2\pi\chi(M), \tag{22.1}$$

where $\chi(M)$ denotes the Euler number of M.

For a compact surface M in E^3, Chern-Lashof's inequality yields the following inequality:

$$\int_M |K| dV \geq 4\pi(1+g), \tag{22.2}$$

where g denotes the genus of M.

G. Thomsen initiated in 1923 the study of total mean curvature:

$$w(f) = \int_M H^2 dV \tag{22.3}$$

of an immersion $f : M \to E^3$ of a surface in E^3. Among others, G. Thomsen studied the first variations of the total mean curvature and showed that the Euler-Lagrange equation of (22.3) is given by

$$\Delta H + 2H(H^2 - K) = 0. \tag{22.4}$$

W. Blaschke proved in his 1923 book that the total mean curvature of a compact surface in E^3 is a conformal invariant. Chen (1973c,1974) extended Blaschke's result to any submanifold of dimension ≥ 2 in an arbitary Riemannian manifold.

Using the inequality $H^2 \geq K$ for a surface M in E^3, T. J. Willmore (1968) observed that combining Gauss-Bonnet's formula and Chern-Lashof's inequality yields the following inequality:

$$w(f) = \int_M H^2 dV \geq 4\pi, \tag{22.5}$$

for an immersion $f : M \to E^3$ of a compact surface M in E^3, with equality holding if and only if M is a round sphere.

The functional $w(f)$ defined by (22.3), initially studied by G. Thomsen, is also known as Willmore's functional and a surface satisfying Thomsen's equation (22.4) is called a stationary surface or a Willmore surface.

An immersed surface $\bar{M}$ in a Euclidean space is said to be conformally equivalent to another immersed surface M in a Euclidean space if $\bar{M}$ can be obtained from M via conformal mappings on Euclidean space.

22.2. **Willmore's conjecture.** Since the total mean curvature of an immersion $f : M \to E^3$ of a compact surface in E^3 is at least 4π, it is a natural question to determine the infinimum of $w(f)$ among all immersions of a compact surface M_g of a given genus g, or among all isometric immersions of a compact Riemannian surface.

T. J. Willmore conjectured that if $f : M \to E^3$ is an immersion of a torus, then $w(f) \geq 2\pi^2$.

Willmore's conjecture have been proved to be true for various classes of immersed tori in E^3 (or more generally, in E^m, $m \geq 3$).

For instance, the following are known.

(1) If $f : M \to E^3$ is a closed tube with fixed radius over a closed curve in E^3, then $w(f) \geq 2\pi^2$, with the equality holding if and only if it is a torus of revolution whose generating circle has radius r and distance $(\sqrt{2}-1)\, r$ from the axis of revolution [Shiohama-Tagaki 1970, Willmore 1971, Langer-Singer 1984].

(2) If $f : M \to E^3$ is a knotted torus, then $w(f) \geq 8\pi$ [Chen 1971,1984b].

(3) If an immersed torus M in a Euclidean space E^m is conformally equivalent to a flat torus in a Euclidean space, then $w(f) \geq 2\pi^2$, with the equality holding if and only if M is a conformal Clifford torus, that is, M is conformally equivalent to a standard square torus in an affine $E^4 \subset E^m$ [Chen 1976b, 1984b].

(4) U. Hertrich-Jeromin and U. Pinkall (1992) proved that the conjecture is true for elliptic tubular tori in E^3. (A special case of this was proved by van de Woestijne and Verstraelen in 1990, cf. [Verstraelen 1990]).

(5) If an immersed tori in E^3 has self-intersections, then $w(f) \geq 8\pi$ [Li-Yau 1982].

(6) Li and Yau (1982) showed that the conjecture is true for certain bounded domain of the moduli space of conformal structure on torus. Montiel and Ros (1985) proved that the conjecture is true for a larger domain in this moduli space.

22.3. **Further results on total mean curvature for surfaces in Euclidean space.** Chen (1973) proved that if $f : M \to E^m$ is a compact surface in E^m which is conformally equivalent to a compact surface in $E^4 \subset E^m$ with nonnegative Gaussian curvature and $w(f) \leq (2+\pi)\pi$, then M is homeomorphic to S^2.

Let $f : M \to E^4$ be an embedding of a compact surface M into E^4. The fundamental group $\pi_1(E^4 - f(M))$ of $E^4 - f(M)$ is called the knot group of f. The minimal number of generators of the knot group of f is called the knot number of f.

P. Wintgen (1978,1979) proved the following:

(a) If $f : M \to E^4$ is an embedding of a compact surface M into E^4, then $w(f) \geq 4\pi\rho$, where ρ is the knot number of f;

(b) If $f : M \to E^4$ is an immersion of a compact oriented surface M into E^4, then

$$w(f) \geq 4\pi(1 + |I_f| - g), \tag{22.6}$$

where I_f is the self-intersection number of f and g is the genus of M.

Li and Yau (1982) showed that if $f : RP^2 \to E^3$ is an immersion of a real projective plane RP^2 into E^3, then $w(f) \geq 12\pi$. R. Bryant (1987a) and, independently, R. Kusner (1987) found explicit immersions of RP^2 in E^3 satisfying $w(f) = 12\pi$.

Let M be an oriented surface immersed in E^4 and $\{X_1, X_2\}$ be an orthonormal oriented frame field of TM. For each point $x \in M$ and each unit tangent vector of M at x, we put $X = (\cos\theta)X_1 + (\sin\theta)X_2$. Then the second fundamental form h of M satisfies

$$h(X,X) = H + (\cos 2\theta)\left(\frac{h(X_1,X_1) - h(X_2,X_2)}{2}\right) + (\sin 2\theta)h(X_1,X_2),$$

which shows that

$$E_x = \{h(X,X) : X \in T_xM,\ |X| = 1\}$$

is an ellipse in the normal space $T_x^\perp M$ centered at H. The ellipse E_x is called the ellipse of curvature at x.

Let $\{e_3, e_4\}$ be an orthonormal oriented frame field of the normal bundle of M. The normal curvature K^D of M in E^4 is defined by

$$K^D = \left\langle R^D(X_1,X_2)e_4, e_3 \right\rangle.$$

I. V. Guadalupe and L. Rodriguez (1983) proved that if $f : M \to E^4$ is an immersion of a compact oriented surface M into E^4 and if the normal curvature K^D of f is everywhere positive, then $w(f) \geq 12\pi$, with the equality holding if and only if the ellipse of curvature of f is always a circle.

For an immersion $f : M \to E^3$ of a compact surface into E^3, W. Kühnel and U. Pinkall (1986) showed the following:

(1) if M is nonorientable with even Euler number, then $w(f) \geq 8\pi$, and

(2) if M has odd Euler number, then $w(f) \geq 12\pi$.

Kühnel and Pinkall (1986) also proved that, for any genus g, there are compact orientable surfaces of genus g immersed in E^3 with $\int H^2 dV \leq 8\pi$.

22.4. **Total mean curvature for arbitrary submanifolds and applications.** According to Nash's embedding theorem, every compact Riemannian n-manifold can be isometrically embedded in $E^{n(3n+11)/2}$. On the other hand, most compact Riemannian n-manifolds cannot be isometrically immersed in E^{n+1} as hypersurfaces. For instance, every compact surface with nonpositive Gaussian curvature cannot be isometrically immersed in E^3.

Chen (1971) proved the following general inequality for compact submanifolds in Euclidean space for arbitrary dimension and arbitrary codimension:

Let M be a compact n-dimensional submanifold of E^m. Then

$$\int_M H^n dV \geq s_n, \tag{22.7}$$

where s_n is the volume of unit n-sphere. The equality sign of (22.7) holds if and only if M is a convex planar curve when $n = 1$; and M is embedded as a hypersphere in an affine $(n+1)$–subspace of E^m when $n > 1$.

If $n = 1$, inequality (22.7) reduces to the well-known Fenchel-Borsuk inequality for closed curves in Euclidean space.

Some geometric applications of inequality (22.7) are the following:

(1) If M is a compact n-dimensional minimal submanifold of the unit m-sphere S^m, then the volume of M satisfies $\mathrm{vol}\,(M) \geq s_n$, with the equality holding if and only if M is a great n-sphere of S^m;

(2) If M is a compact n-dimensional minimal submanifold of RP^m of constant sectional curvature 1, then $\mathrm{vol}\,(M) \geq s_n/2$, with the equality holding if and only if M is a totally geodesic submanifold of RP^m;

(3) if M is a compact n-dimensional minimal submanifold of CP^m of constant holomorphic sectional curvature 4, then $\mathrm{vol}\,(M) \geq s_{n+1}/2\pi$, with the equality holding if and only if $M = CP^k$ which is embedded as a totally geodesic complex submanifold of $CP^m(4)$;

(4) If M is a compact n-dimensional minimal submanifold of HP^m of constant quaternionic sectional curvature 4, then $\mathrm{vol}\,(M) \geq s_{n+3}/2\pi^2$, with the equality holding if and only if $M = HP^k$ which is embedded as a totally geodesic quaternionic submanifold of $HP^m(4)$; and

(5) If M is a compact n-dimensional minimal submanifold of the Cayley plane $\mathcal{O}P^2$ of maximal sectional curvature 4, then $\mathrm{vol}\,(M) \geq s_n/2^n$.

If M is a compact n-dimensional submanifold M with nonnegative scalar curvature in a Euclidean m-space, then the total mean curvature of M satisfying [Chen 1972]:

$$\int_M H^n dV \geq C(n)\beta(M), \tag{22.8}$$

where $C(n)$ is a positive constant depending on n and $\beta(M)$ is the topological invariant: $\beta(M) = \max\{\sum_{i=0}^n \beta_i(M;F) : F \text{ fields}\}$, $\beta_i(M;F)$ the i-th Betti number of M over F.

Related to inequality (22.8), Chen (1976c) proved that the total scalar curvature of an arbitrary compact n-dimensional submanifold M in the Euclidean m-space, regardless of codimension, satisfies

$$\int_M S^{n/2} dV \geq \left(\left(\frac{n}{2}\right)^{n/2} s_n\right)\beta(M). \tag{22.9}$$

The equality sign of (22.9) holds if and only if M is embedded as a hypersphere in an affine $(n+1)$-subspace of E^m.

22.5. **Some related results.** In the 1973 AMS symposium held at Standard University, Chen asked to find the relationship between the total mean curvature and Riemannian invariants of a compact submanifold in a Euclidean space (cf. [Chen 1975]). In the late 1970s Chen obtained a solution to this problem; discovering a sharp relationship between the total mean curvature of a compact submanifold and the order of the immersion (cf. [Chen 1984b]). More precisely, he proved the following:

Let M be an n-dimensional compact submanifold of E^m. Then

$$\left(\frac{\lambda_p}{n}\right)\mathrm{vol}(M) \leq \int_M H^2 dV \leq \left(\frac{\lambda_q}{n}\right)\mathrm{vol}(M), \tag{22.10}$$

where p and q are the lower order and the upper orders of $f : M \to E^m$. Either equality sign in (22.10) holds if and only if M is of 1-type.

Inequality (22.10) improves a result of Reilly (1977) who proved that, for any compact submanifold M in E^m, the total mean curvature of M satisfies

$$\int_M H^2 dV \geq \left(\frac{\lambda_1}{n}\right)\mathrm{vol}(M). \tag{22.11}$$

We also have the following sharp inequalities for the total mean curvature [Chen 1987b, Chen-Jiang 1995]:

Let $f : M \to E^m$ be a compact n-dimensional submanifold of E^m and let c denote the distance from the origin to the center of gravity of M in E^m. Then

(i) if M is contained in a closed ball $\overline{B_0(R)}$ with radius R centered at the origin, then M satisfies

$$\int_M |H|^k dV \geq \frac{\mathrm{vol}(M)}{(R^2 - c^2)^{k/2}}, \quad k = 2, 3, \cdots, n, \tag{22.12}$$

with equality holding for some $k \in \{2, 3, \ldots, n\}$ if and only if M is a minimal submanifold of the hypersphere $S_0^{m-1}(R)$ of radius R centered at the origin;

(ii) if M is contained in $E^m - B_0(r)$, then

$$\left(\frac{\lambda_p}{n}\right)^2 (r^2 - c^2) \leq \frac{1}{\mathrm{vol}(M)} \int_M |H|^2 dV \leq \left(\frac{\lambda_q}{n}\right)^2 (R^2 - c^2) \tag{22.13}$$

where p and q denote the lower and the upper orders of M in E^m. Either equality of (22.13) holds if and only if M is a minimal submanifold of a hypersphere centered at the origin; and

(iii) if $f(M)$ is contained in a unit hypersphere of E^m, then the first and the second nonzero eigenvalues of the Laplacian of M satisfy

$$\int_M |H|^2 dV \geq \frac{1}{n^2}\{n(\lambda_1 + \lambda_2) - \lambda_1\lambda_2\}\mathrm{vol}(M), \tag{22.14}$$

with equality sign of (22.14) holding if and only if either f is of 1-type with order $\{1\}$ or with order $\{2\}$, or f is of 2-type with order $\{1, 2\}$. Some easy applications of (iii) are the following [Chen 1987b]:

(1) If M is an n-dimensional compact minimal submanifold of $RP^m(1)$, then the first and the second nonzero eigenvalues of the Laplacian of M satisfy

$$\frac{m}{2(m+1)}\lambda_1\lambda_2 \geq n(\lambda_1 + \lambda_2 - 2n - 2). \tag{22.15}$$

(2) If M is an n-dimensional compact minimal submanifold of $CP^m(4)$, then

$$\frac{m}{2(m+1)}\lambda_1\lambda_2 \geq n(\lambda_1 + \lambda_2 - 2n - 4), \tag{22.16}$$

with equality holding if and only if M is one of the following compact Hermitian symmetric spaces:

$$CP^k(4), \;\; CP^k(2), \;\; SO(2+k)/SO(2) \times SO(k),$$
$$CP^k(4) \times CP^k(4), \;\; U(2+k)/U(2) \times U(k), \; (k > 2),$$
$$SO(10)/U(5), \;\text{ and }\; E_6/\mathrm{Spin}(10) \times T,$$

with an appropriate metric, where m is given respectively by

$$k, \; \frac{k(k+3)}{2}, \; k+1, \; k(k+2), \; \frac{k(k+3)}{2}, \; 15, \;\text{ and }\; 26.$$

(3) If M is an n-dimensional compact minimal submanifold of $HP^m(4)$, then

$$\frac{m}{2(m+1)}\lambda_1\lambda_2 \geq n(\lambda_1 + \lambda_2 - 2n - 8), \tag{22.17}$$

with equality holding if and only if M is a totally geodesic quaternionic submanifold and $n = 4m$.

For further applications of (i), (ii) and (iii), see [Chen 1996d]. The following conformal property of $\lambda_1 \mathrm{vol}(M)$ as another application of the notion of order was first discovered in [Chen 1979b]:

If a compact Riemannian surface M admits an order $\{1\}$ isometric embedding into E^m, then, for any compact surface $\overline{M} \subset E^m$ which is conformally equivalent to $M \subset E^m$, we have

$$\lambda_1 \mathrm{vol}(M) \geq \bar{\lambda}_1 \mathrm{vol}(\bar{M}). \tag{22.18}$$

Equality sign of (22.18) holds if and only if $\bar{M}$ also admits an isometric embedding of order $\{1\}$.

Some further applications of (22.10) and (22.11) are the following [Chen 1983b, 1984b].

Let M be an n-dimensional compact submanifold of the unit hypersphere S^{m-1} of Euclidean m-space. Denote by p and q the lower order and the upper order of M in E^m. Then

(a) If M is mass-symmetric in S^{m-1}, then $\lambda_1 \leq \lambda_p \leq n$. In particular, $\lambda_p = n$ if and only if M is of 1-type and of order $\{p\}$;

(b) If M is of finite type, then $\lambda_q \geq n$. In particular, $\lambda_q = n$ if and only if M is of 1-type and of order $\{q\}$;

(c) If M is a compact n-dimensional minimal submanifold of $RP^m(1)$, then λ_1 of M satisfies $\lambda_1 \leq 2(n+1)$, with equality holding if and only if M is a totally geodesic $RP^n(1)$ in $RP^m(1)$;

(d) If M is a compact n-dimensional minimal submanifold of $CP^m(4)$, then $\lambda_1 \leq 2(n+2)$, with equality holding if and only if M is holomorphically isometric to a $CP^{\frac{n}{2}}(4)$, and M is embedded as a complex totally geodesic submanifold of $CP^m(4)$.

(e) If M is a compact n-dimensional minimal submanifold of $HP^m(4)$, then $\lambda_1 \leq 2(n+4)$, with equality sign holding if and only if M is a $HP^{\frac{n}{4}}(4)$ and M is a quaternionic totally geodesic submanifold of $HP^m(4)$; and

(f) If M is a compact n-dimensional minimal submanifold of the Cayley plane $\mathcal{O}P^2(4)$, then $\lambda_1 \leq 4n$.

I. Dimitrić (1998) improved the result of statement (f) to $\lambda_1 < 676/15 = 45.0\bar{6}$ for compact minimal hypersurfaces in $\mathcal{O}P^2(4)$.

For compact Kähler submanifolds of $CP^m(4)$, statement (d) was due to A. Ros (1983), and independently by N. Ejiri (1983).

The Euler-Lagrange equation of $w(f) = \int_M H^n dV$ for a compact hypersurface M in E^{n+1} is given by

$$\Delta H^{n-1} + H^{n-1}\left(nH^2 - S\right) = 0, \tag{22.19}$$

where S denotes the squared length of the second fundamental form.

Chen (1973d) proved that hyperspheres in E^{n+1} are the only solutions of (22.19) when n is odd (for compact submanifolds in Euclidean space with higher codimension, see [Chen-Houh 1975]). On the other hand, there do exist many solutions of (22.19) other than hyperspheres for even n. For example, a torus of revolution in E^3 whose generating circle has radius r and distance $(\sqrt{2}-1)\,r$ from the axis of revolution is a solution of (22.19). Furthermore, the stereographic projections of compact minimal surfaces M in S^3 also satisfy (22.19) with $n = 2$.

U. Pinkall (1985c) found examples of compact embedded surfaces in E^3 satisfying (22.19) that are not stereographic projections of compact minimal surfaces in S^3 (see, also [Weiner 1979, Pinkall-Sterling 1987]). Barros and Garay [1998,1999] constructed many submanifolds in spheres which satisfies (22.19).

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