Manifold Theory

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CHAPTER 1

Manifolds

1.1. Smooth Manifolds

A manifold is a topological space, M, with a *maximal atlas* or a *maximal smooth structure*.

There are two virtually identical definitions. The standard definition is as follows:

DEFINITION 1.1.1. There is an *atlas* $\mathscr{A}$ consisting of maps $x_\alpha : U_\alpha \to \mathbb{R}^{n_\alpha}$ such that

(1) U_α is an open covering of M.
(2) x_α is a homeomorphism onto its image.
(3) The *transition functions* $x_\alpha \circ x_\beta^{-1} : x_\beta \left(U_\alpha \cap U_\beta\right) \to x_\alpha \left(U_\alpha \cap U_\beta\right)$ are diffeomorphisms.

In condition (3) it suffices to show that the transition functions are smooth since $x_\beta \circ x_\alpha^{-1} : x_\alpha \left(U_\alpha \cap U_\beta\right) \to x_\beta \left(U_\alpha \cap U_\beta\right)$ is an inverse.

The second definition is a compromise between the first and a more sheaf theoretic approach. It is, however, essentially the definition of a submanifold of Euclidean space where parametrizations are given as local graphs.

DEFINITION 1.1.2. A *smooth structure* is a collection $\mathscr{D}$ consisting of continuous functions whose domains are open subsets of M with the property that: For each $p \in M$, there is an open neighborhood $U \ni p$ and functions $x^i \in \mathscr{D}$, $i = 1, ..., n$ such that

(1) The domains of x^i contain U.
(2) The map $x = \left(x^1, ..., x^n\right) : U \to \mathbb{R}^n$ is a homeomorphism onto its image $V \subset \mathbb{R}^n$.
(3) For each $f : O \to \mathbb{R}$ in $\mathscr{D}$ there is a smooth function $F : x(U \cap O) \to \mathbb{R}$ such that $f = F \circ x$ on $U \cap O$.

The map in (2) in both definitions is called a *chart* or *coordinate system* on U. The topology of M is recovered by these maps. Observe that in condition (3), $F = f \circ x^{-1}$, but it is usually possible to find F without having to invert x. F is called the *coordinate representation of* f and is normally also denoted by f.

Note that it is very easy to see that these two definitions are equivalent. Both have advantages. The first in certain proofs. The latter is generally easier to work with when showing that a concrete space is a manifold and is also often easier to work with when it comes to defining foundational concepts.

DEFINITION 1.1.3. A continuous function $f : O \to \mathbb{R}$ is said to be smooth wrt $\mathscr{D}$ if $\mathscr{D} \cup \{f\}$ is also a smooth structure. In other words we only need to check that condition (3) still holds when we add f to our collection $\mathscr{D}$. We can more generally define what it means for f to be C^k for any k with smooth being C^∞ and continuous C^0. We shall generally only use smooth or continuous functions.

The space of all smooth functions is a maximal smooth structure. We use the notation $C^k(M)$ for the space of C^k functions defined on all of M and $\mathfrak{C}^k(M)$ for the space of $f: O \to \mathbb{R}$ where $O \subset M$ is open and f is C^k.

It is often the case that all the functions in a $\mathscr{D}$ have domain M. In fact it is possible to always select the smooth structure such that this is the case. We shall also show that it is possible to always use a finite collection $\mathscr{D}$.

A manifold of dimension n or an n-manifold is a manifold such that coordinate charts always use n functions.

PROPOSITION 1.1.4. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets that are diffeomorphic, then $m = n$.*

PROOF. The differential of the diffeomorphism is forced to be a linear isomorphism. This shows that $m = n$. □

COROLLARY 1.1.5. *A connected manifold is an n-manifold for some integer n.*

PROOF. It is not possible to have coordinates around a point into Euclidean spaces of different dimensions. Let $A^n \subset M$ be the set of points that have coordinates using n functions. This is clearly an open set. Moreover if $p_i \to p$ and $p_i \in A^n$ then we see that if p has a chart that uses m functions then p_i will also have this property showing that $m = n$. □

1.2. Examples

If we start with $M \subset \mathbb{R}^k$ as a subset of Euclidean space, then we should obviously use the induced topology and the ambient coordinate functions $x^i|_M : M \to \mathbb{R}$ as the potential differentiable structure $\mathscr{D}$. Depending on what subset we start with this might or might not work. Even when it doesn't there might be other obvious ways that could make it work. For example, we might start with a subset which has corners, such as a triangle. While the obvious choice of a differentiable structure will not work we note that the subset is homeomorphic to a circle, which does have a valid differentiable structure. This structure will be carried over to the triangle via the homeomorphism. This is a rather subtle point and begs the very difficult question: Does every topological manifold carry a smooth structure? The answer is yes in dimensions 1, 2, and 3, but no in dimension 4 and higher. There are also subsets where the induced topology won't make the space even locally homeomorphic to Euclidean space. A figure eight 8 is a good example. But again there is an interesting bijective continuous map $\mathbb{R} \to 8$. It "starts" at the crossing, wraps around in the figure 8 and then ends at the crossing on the opposite side. However, as the interval was open every point on 8 only gets covered once in this process. This map is clearly also continuous. However, it is not a homeomorphism onto its image. Thus we see again that an even more subtle game can be played where we refine the topology of a given subset and thus have the possibility of making it a manifold.

1.2.1. Spheres. The n-sphere is defined as

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \right\}$$

Thus we have $n+1$ natural coordinate functions. On any hemisphere $O_i^{\pm} = \left\{ x \in S^n \mid \pm x^i > 0 \right\}$ we use the coordinate system that comes from using the n functions x^j where $j \neq i$ and the remaining coordinate function is given as a smooth expression:

$$\pm x^i = \sqrt{1 - \sum_{j \neq i} \left(x^j\right)^2}$$

A somewhat different atlas of charts is given by stereographic projection from the points $\pm e_i$, where e_i are the usual basis vectors. The map is geometrically given by drawing a line through a point $z \in \left\{z \in \mathbb{R}^{n+1} \mid z \perp e_i\right\}$ and $\pm e_i$ and then checking where it intersects the sphere. The equator where $x^i = 0$ stays fixed, while the hemisphere closest to $\pm e_i$ is mapped outside this equatorial band, and the hemisphere farthest from $\pm e_i$ is mapped inside the band, finally the map is not defined at $\pm e_i$. The map from the sphere to the subspace is given by the formula:

$$z = \frac{1}{1 \mp x^i}\left(x \mp e_i\right) \pm e_i$$

and the inverse

$$x = \frac{\pm 2}{1 + |z|^2}\left(z \mp e_i\right) \pm e_i$$

Any two of these maps suffice to create an atlas. But one must check that the transition functions are also smooth. One generally takes the ones coming from opposite points, say e_{n+1} and $-e_{n+1}$. In this case the transition is an inversion in the equatorial band and is given by

$$z \mapsto \frac{z}{|z|^2}$$

1.2.2. Projective Spaces. The n-dimensional (real) projective space $\mathbb{RP}^n$ is defined as the space of lines or more properly 1-dimensional subspaces of $\mathbb{R}^{n+1}$. First let us dispel the myth that this is not easily seen to be a subset of some Euclidean space. A subspace $M \subset \mathbb{R}^{n+1}$ is uniquely identified with the orthogonal projection $\mathrm{proj}_M : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ whose image is $M = \mathrm{proj}_M\left(\mathbb{R}^{n+1}\right)$. Orthogonal projections are characterized as idempotent self-adjoint linear maps, i.e., in this case matrices $E \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{R})$ such that $E^2 = E$ and $E^* = E$. Thus it is clear that $\mathbb{RP}^n \subset \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{R})$. We can be more specific. If

$$x = \begin{bmatrix} x^0 \\ x^1 \\ \\ \\ x^n \end{bmatrix} \in \mathbb{R}^{n+1} - \{0\},$$

then the matrix that describes the orthogonal projection onto $\mathrm{span}\{x\}$ is given by

$$\begin{aligned} E_x &= \frac{1}{|x|^2}\begin{bmatrix} x^0x^0 & x^0x^1 & & x^0x^n \\ x^1x^0 & x^1x^1 & & x^1x^n \\ & & & \\ & & & \\ x^nx^0 & x^nx^1 & & x^nx^n \end{bmatrix} \\ &= \frac{1}{|x|^2}xx^*. \end{aligned}$$

Clearly $E_x^* = E_x$ and as $x^*x = |x|^2$ we have $E_x^2 = E_x$ and $E_x x = x$. Thus E_x is the orthogonal projection onto $\mathrm{span}\{x\}$. Finally note that $E_x = E_y$ if and only if $x = \lambda y$, $\lambda \neq 0$. With that in mind we obtain a natural differentiable system by using the coordinate functions

$$f^{ij}(E_x) = \frac{x^i x^j}{|x|^2}.$$

If we fix j and consider the $n+1$ functions f^{ij}, then we have the relationship

$$f^{jj} = \left(f^{jj}\right)^2 + \sum_{i \neq j} \left(f^{ij}\right)^2.$$

This describes a sphere of radius $\frac{1}{2}$ centered at the point where $f^{ij} = 0$ for $i \neq j$ and $f^{jj} = \frac{1}{2}$. The origin on this sphere corresponds to all points where $x^j = 0$. But any other point on the sphere corresponds to a unique element of $O_j = \left\{E_x : x^j \neq 0\right\}$. This means that around any given point in O_j we can use n of the functions f^{ij} as a coordinate chart. The remaining function is then expressed smoothly in terms of the other coordinate functions. This still leaves us with the other functions f^{kl}, but they satisfy

$$f^{kl} = \frac{f^{kj} f^{lj}}{f^{jj}}$$

and so on the given neighborhood in O_j they are also smoothly expressed in terms of our chosen coordinate functions. The more efficient collection of functions f^{ij}, $i \leq j$ yield the *Veronese map*

$$\mathbb{RP}^n \to \mathbb{R}^{\frac{(n+2)(n+1)}{2}}.$$

A more convenient differentiable system can be constructed using *homogeneous coordinates* on $\mathbb{RP}^n$. These are written $\left[x^0 : x^1 : \cdots : x^n\right]$ and represent the equivalence class of non-zero vectors that are multiples of x. The idea is that all elements in the equivalence class have the same ratios $x^i : x^j = \frac{x^i}{x^j}$ on O_j. We can then define a differentiable system by using the functions

$$f_j^i\left(\left[x^0 : x^1 : \cdots : x^n\right]\right) = \frac{x^i}{x^j} = \frac{f^{ij}}{f^{jj}}.$$

These have domain O_j and are smoothly expressed in terms of the coordinate functions we already considered. Conversely note that on $O_i \cap O_j$ the old coordinates are also expressed smoothly in terms of the new functions:

$$f^{ij} = \left(\sum_k f_i^k f_j^k\right)^{-1}.$$

On O_j we can use f_j^i, $i \neq j$ as a coordinate chart. The other coordinate functions f_l^k can easily be expressed as smooth combinations by noting that on $O_l \cap O_j$ we have

$$f_l^k = \frac{f_j^k}{f_j^l}.$$

Thus using the obvious coordinate functions works, but it is often desirable to use a different collection of functions for a differentiable system.

1.2.3. Matrix Spaces. Define $\mathrm{Mat}_{n\times m}^k$ as the matrices with n rows, m columns, and rank k.

The special case where $k = n = m$ is denoted Gl_n and is known as the general linear group. It evidently consists of the nonsingular $n \times n$ matrices and is an open subset of all the $n \times n$ matrices. As such it is obviously a manifold of dimension n^2.

Back to general case. As $\mathrm{Mat}_{n\times m}^k$ is a subspace of a Euclidean space we immediately suspect that the entries will suffice as a differentiable system. The trick is to discover how many of them are needed to create a coordinate system. To that end, assume that we look at the matrices of rank k where the first k rows and the first k columns are linearly independent. If such a matrix is written in block form

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix},$$

then we know that $B = YA$, $Y \in \mathrm{Mat}_{(n-k)\times k}$, $C = AX$, $X \in \mathrm{Mat}_{k\times(m-k)}$, and $D = YAX$. Thus those matrices are uniquely represented by the invertible matrix A and the two general matrices X,Y. Next observe that $Y = BA^{-1}$, $X = A^{-1}C$. Thus we can use the $nm - (n-k)(m-k)$ entries that correspond to A,B,C as a coordinate chart on this set. The remaining entries corresponding to D are then smooth functions of these coordinates as $D = BA^{-1}C$.

More generally we define the sets $O_{i_1,\dots,i_k,j_1,\dots,j_k} \subset \mathrm{Mat}^k_{n\times m}$ as the rank k matrices where the rows indexed by $i_1,\dots,i_k$ and columns by $j_1,\dots,j_k$ are linearly independent. On these sets all entries that lie in the corresponding rows and columns are used as coordinates and the remaining entries are smoothly expressed in terms of these using the above expression with the necessary index modifications.

When $m = n$ we can add other conditions such as having constant determinant, being skew or self-adjoint, orthogonal, unitary and much more. A particularly nasty situation is the *Grassmannian* of k-planes in $\mathbb{R}^n$. These are as indicated the k-dimensional subspaces of $\mathbb{R}^n$. When $k = 1$ we then return to the projective spaces. As such they are represented as the subset

$$\mathrm{Gr}_k(\mathbb{R}^n) = \left\{ E \in \mathrm{Mat}^k_{n\times n} \mid E^2 = E \text{ and } E^* = E \right\}.$$

If $X \in \mathrm{Mat}^k_{n\times k}$, then

$$E_X = X(X^*X)^{-1}X^* \in \mathrm{Gr}_k(\mathbb{R}^n).$$

Moreover, $E_X = E_Y$ if and only if $X = YA$ where $A \in Gl_k$. The question now is if we learned anything else useful from constructing coordinates on projective space. We define sets $O_{i_1,\dots,i_k} \subset \mathrm{Gr}_k(\mathbb{R}^n)$ with the property that the rows of E corresponding to the indices $i_1,\dots,i_k$ are linearly independent. As E is self-adjoint the corresponding columns are also linearly independent. If $E = E_X$, then $O_{i_1,\dots,i_k}$ corresponds the $X \in \mathrm{Mat}^k_{n\times k}$ where the rows indexed by $i_1,\dots,i_k$ are linearly independent. We can then consider the matrix $A_X \in Gl_k$ which consists of those rows from X. Then the remaining rows in XA_X^{-1} parametrize $E_X = E_{XA_X^{-1}}$. To see this more explicitly assume that the first k rows are linearly independent. Then we can use

$$X = \begin{bmatrix} I_k \\ Z \end{bmatrix}, Z \in \mathrm{Mat}_{(n-k)\times k}$$

and

$$E = \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} I_k + Z^*Z & (I_k + Z^*Z)Z^* \\ Z(I_k + Z^*Z) & Z(I_k + Z^*Z)Z^* \end{bmatrix}.$$

Thus we should use the functions coming from the entries of BA^{-1} as our coordinates on $O_{1,\dots,k}$. With that choice we clearly get that the entries are smoothly expressed in terms of these coordinates. But that is not really what we wish to check. However, the types of coordinate functions we are considering are in turn smoothly related to the entries of E so in a somewhat backward way we have worked everything out without having done any hard work.

1.2.4. Tangent Spaces to Spheres. The last example for now is somewhat different in nature and can easily be generalized to manifolds that come from subsets of Euclidean space where standard coordinate functions give a differentiable system.

We'll consider the set of vectors tangent to a sphere. By tangent to the sphere we mean that they are velocity vectors for curves in the sphere. If $c : I \to S^n$, then $|c|^2 = 1$ and consequently $\dot{c} \cdot c = (\dot{c}|c) = 1$. Thus the velocity is always perpendicular to the base vector. This means that we are considering the set

$$TS^n \simeq \left\{ (x,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } (x|v) = 0 \right\}$$

Conversely we see that for $(x,v) \in TS^n$ the curve

$$c(t) = x\cos t + v\sin t$$

is a curve on the sphere that has velocity v at the base point x. Now suppose that we are considering the points $x \in O_j^{\pm}$ with $\pm x^j > 0$. We know that on this set we can use x^i, $i \neq j$ as coordinates. It seems plausible that we could similarly use v^i, $i \neq j$ for the vector component. We already know that we can write x^j as a smooth function of x^i, $i \neq j$. So we now have to write v^j as a smooth function of v^i and x^i. The equation $(x|v) = 0$ tells us that

$$v^j = -\frac{\sum_{i \neq j} x^i v^i}{x^j}$$

so this is certainly possible.

This also helps us in the general case where we might be considering tangent vectors to a general M. For simplicity assume that $x^{n+1} = F\left(x^1, \ldots, x^n\right)$. If c is a curve, then we also have $c^{n+1}(t) = F\left(c^1(t), \ldots, c^n(t)\right)$. Thus

$$\dot{c}^{n+1}(t) = \frac{\partial F}{\partial x^i} \dot{c}^i(t).$$

This means that for the tangent vectors

$$v^{n+1} = \frac{\partial F}{\partial x^i} v^i.$$

Thus we have again written v^{n+1} as a smooth function of our chosen coordinates given that x^{n+1} is already written as a smooth function of $x^1, \ldots, x^n$.

This argument is general enough that we can use it to create a differentiable structure for similarly defined tangent spaces TM for $M^m \subset \mathbb{R}^n$ where we used the n-coordinate functions from $\mathbb{R}^n$ to generate the differentiable structure on M. The only difference is that we'll have $n - m$ functions to describe $n - m$ coordinates on any given set where we've used a specific set of m coordinates as a chart. For instance

$$x^j = F^j\left(x^1, \ldots, x^m\right),\ j > m$$

yields

$$v^j = \sum_{i=1}^{m} \frac{\partial F^j}{\partial x^i} v^i,\ j > m.$$

1.3. Topological Properties of Manifolds

The goal is to show that we can construct partitions of unity on smooth manifolds. This means that we have to start by showing that the space is paracompact. The simplest topological assumptions for this to work is that the space is second countable (there is a countable basis for the topology) and Hausdorff (points can be separated by disjoint open sets). For a manifold, as defined above, this means that the topology will henceforth be assumed to be second countable and Hausdorff. The Hausdorff property is essential for much that we do, but it will also seem as if we rarely use it explicitly. Two essential properties come from the Hausdorff axiom. First, that limits of sequences are uniquely defined. Second, that compact subsets are closed sets and thus have complements that are open.

Checking that the topology is second countable generally follows by checking that the space can be covered by countably many coordinate charts. Clearly open subsets of $\mathbb{R}^n$ are second countable. So this means that the space is a countable union of open sets that are all second countable and thus itself second countable.

Checking that it is Hausdorff is generally also easy. Either two points will lie the same chart in which case they can easily be separated. Otherwise they'll never lie in the same chart and one must then check that there are small charts around the points whose domains don't intersect.

We now proceed to the constructions that are directly related to what we shall later use.

THEOREM 1.3.1. *A smooth manifold has a compact exhaustion and is paracompact.*

PROOF. A *compact exhaustion* is an increasing countable collection of compact sets $K_1 \subset K_2 \subset \cdots$ such that $M = \cup K_i$ and $K_i \subset \mathrm{int} K_{i+1}$ for all i. The crucial ingredients for finding such an exhaustion is second countability and local compactness.

First we show that open sets O in $\mathbb{R}^n$ have this property. Around each $p \in O$ select an open neighborhood U_p such that the closure is compact and $\overline{U}_p \subset O$. Since O is second countable (or just Lindelöf) we can select a countable collection U_{p_i} that covers O. Define $K_1 = \overline{U}_{p_1}$ and given K_i let $K_{i+1} = \overline{U}_{p_1} \cup \cdots \cup \overline{U}_{p_n}$ where $p_1, ..., p_n$ are chosen so that $n \geq i$ and $K_i \subset U_{p_1} \cup \cdots \cup U_{p_n}$.

By definition M is a countable union of open sets that have exhaustions, i.e., there are compact sets $K_{i,j}$ where for fixed j, $K_{i,j}$, $i = 1,2,3...$ are an exhaustion of O_j, and O_j is an open covering. The desired exhaustion is then given by $K_i = \cup_{j \leq i} K_{i,j}$.

To show that the space is paracompact consider the compact "annuli" $C_i = K_i - \mathrm{int} K_{i-1}$ and note that $C_i \cap C_j = \emptyset$ when $|i-j| > 1$. Extend this to a covering of open sets $U_i = \mathrm{int} K_{i+1} - K_{i-1} \supset C_i$ and note that $U_i \cap U_j = \emptyset$ when $|i-j| > 4$. In other words these are locally finite covers. Given an open cover B_α we can consider the refinement $B_\alpha \cap U_i$. For fixed i we can then extract a finite collection of $B_\alpha \cap U_i$ that cover the compact set V_i. This leads to a locally finite refinement of the original cover. □

Another fundamental lemma we need is a smooth version of Urysohn's lemma.

LEMMA 1.3.2. (Smooth Urysohn Lemma) *If M is a smooth manifold and $C_0, C_1 \subset M$ are disjoint closed sets, then there exists a smooth function $f : M \to [0,1]$ such that $C_0 = f^{-1}(0)$ and $C_1 = f^{-1}(1)$.*

PROOF. First we claim that for each open set $O \subset M$ there is a smooth function $f : M \to [0,\infty)$ such that $M - O = f^{-1}(0)$.

We start by proving this in Euclidean space. First note that for any open cube

$$O = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

there is a bump function $\mathbb{R}^n \to [0,\infty)$ that is positive on O and vanishes on the complement. Next write a general open set O as a union of open cubes such that for all $p \in \mathbb{R}^n$ there is a neighborhood U that intersects only finitely many open cubes. Using bump functions on each of the cubes we can then add them up to get a function that is positive only on O.

Next note that if $U \subset M$ is open and the closure is contained in a chart $\bar{U} \subset V$, where $x : V \to O \subset \mathbb{R}^n$, then this construction gives us a function that is positive on U and vanishes on $V - \bar{U}$. If we extend this function to vanish on $M - V$ we obtain a smooth function.

More generally we can find a locally finite cover of M consisting of open U_α, where $\bar{U}_\alpha \subset V_\alpha$ and V_α is the domain for a chart (MIGHT WANT TO PROVE THIS). For a fixed open set $O \subset M$ consider the nonempty intersections $U_\alpha \cap O$ and construct a function as just explained on each of them. Then add all of these functions to obtain a smooth function on M that is positive on O and vanishes on $M - O$.

Finally, the Urysohn function is constructed by selecting $f_i : M \to [0, \infty)$ such that $f_i^{-1}(0) = C_i$ and defining

$$f(x) = \frac{f_0(x)}{f_0(x) + f_1(x)}.$$

This function is well-defined as $C_0 \cap C_1 = \emptyset$ and is the desired Urysohn function. □

We can now easily construct the partitions of unity we need.

LEMMA 1.3.3. *Let M be a smooth manifold. Any countable locally finite covering U_α of open sets has partition of unity subordinate to this covering, i.e., there are smooth functions $\phi_\alpha : M \to [0,1]$ such that $\phi_\alpha^{-1}(0) = M - U_\alpha$ and $1 = \sum_\alpha \phi_\alpha$.*

PROOF. The previous result gives us functions $\lambda_\alpha : M \to [0,1]$ such that $\lambda_\alpha^{-1}(0) = M - U_\alpha$. As the cover is locally finite the sum $\sum_\alpha \lambda_\alpha$ is well-defined. Moreover it is always positive as U_α cover M. We can then define

$$\phi_\alpha = \frac{\lambda_\alpha}{\sum_\alpha \lambda_\alpha}$$

□

PROPOSITION 1.3.4. *If $U \subset M$ is an open set in a smooth manifold and $f : U \to \mathbb{R}^n$ is smooth, then λf defines a smooth function on M if $\lambda : M \to \mathbb{R}$ is smooth and vanishes on $M - U$.*

PROOF. Clearly λf is smooth away from the boundary of U. On the boundary λ and all it derivatives vanish so the product rule shows that λf is also smooth there. □

Finally, we can use this to show

PROPOSITION 1.3.5. *A smooth manifold admits a proper smooth function.*

PROOF. Select a compact exhaustion $K_1 \subset K_2 \subset \cdots$, where each K_i is compact, $K_i \subset \operatorname{int} K_{i+1}$, and $M = \bigcup K_i$. Choose Urysohn functions $\phi_i : M \to [0,1]$ such that $\phi_i(K_{i-1}) = 0$ and $\phi_i(M - \operatorname{int} K_i) = 1$. Then consider $\rho = \sum \phi_i$. □

We finish by mentioning three interesting results that help us understand when topological spaces are metrizable and when metric spaces have compact exhaustions. It should also be mentioned that if we use the topology on $\mathbb{R}$ generated by the half open intervals $[a,b)$ then we obtain a paracompact space that is separable but not second countable and not locally compact (51 in [Steen & Seebach]).

THEOREM 1.3.6. *A connected locally compact metric space has a compact exhaustion.*

PROOF. Assume (M,d) is the metric space. For each $x \in M$ let

$$r(x) = \sup\left\{ r \mid \overline{B(x,r)} \text{ is compact} \right\}.$$

If $r(x) = \infty$ for some x we are finished. Otherwise $r(x)$ is a continuous function, in fact

$$|r(x) - r(y)| \leq d(x,y)$$

since

$$r(y) \leq d(x,y) + r(x)$$

and

$$r(x) \leq d(x,y) + r(y)$$

We now claim that for a fixed compact set C the set $C^{\#} = \left\{ x \in M \mid \exists z \in C : d(x,z) \leq \frac{1}{2} r(z) \right\}$ is also compact and contains C in its interior. The latter statement is obvious since $B\left(x, \frac{1}{2} r(x)\right) \subset C^{\#}$ for all $x \in C$. Next select a sequence $x_i \in C^{\#}$ and select $z_i \in C$ such that $d(x_i, z_i) \leq \frac{1}{2} r(z_i)$. Since C is compact we can after passing to a subsequence assume that $z_i \to z \in C$ and that $d(z, z_i) < \frac{1}{4} r(z)$ for all i. Then $d(z, x_i) \leq d(z, z_i) + d(z_i, x_i) < \frac{1}{4} r(z) + \frac{1}{2} r(z_i)$. Continuity of $r(z_i)$ then shows that $x_i \in B\left(z, \frac{3}{4} r(z)\right)$ for large i. As $\overline{B\left(z, \frac{3}{4} r(z)\right)}$ is compact we can then extract a convergent subsequence of x_i.

Finally consider the compact sets $K_{i+1} = K_i^{\#}$ where K_1 is any non-empty compact set. We claim that $\cup_i K_i$ is both open and closed. The set is open since $B\left(x, \frac{1}{2} r(x)\right) \subset K_i^{\#} = K_{i+1}$ for any $x \in K_i$. To see that the set is closed select a convergent sequence $x_n \in \cup_i K_i$ and let x be the limit point. We have $r(x_n) \to r(x)$ and $d(x_i, x) \to 0$. So it follows that for large n we have $x \in B\left(x_n, \frac{1}{2} r(x_n)\right)$ showing that $x \in K_i^{\#}$ if $x_n \in K_i$. So the fact that M is connected shows that it has a compact exhaustion. □

COROLLARY 1.3.7. *A second countable locally compact metric space has a compact exhaustion and is paracompact.*

PROOF. There are at most countably many connected components and each of these has a compact exhaustion. We can then proceed as above. □

THEOREM 1.3.8 (Baire Category Theorem). *A Hausdorff space that is locally compact satisfies: A countable union of closed sets without interiors has no interior.*

PROOF. Let $C_i \subset M$ be a countable collection of closed sets with no interior points. Select an open set $V_0 \subset X$. Then $V_0 - C_1$ is a nonempty open set as C_1 has no interior points. As M is locally compact we can find an open set V_1 such that $\bar{V}_1 \subset V_0 - C_1$ is compact. Similarly we can find open sets V_i such that $\bar{V}_i \subset V_{i-1} - C_i \subset V_{i-1}$ is compact. By compactness $\bigcap_{i=1}^{\infty} \bar{V}_i$ is nonempty and we also have $\bigcap_{i=1}^{\infty} \bar{V}_i \subset V_0 - \bigcup_{i=1}^{\infty} C_i$. In particular, $V_0 - \bigcup_{i=1}^{\infty} C_i$ is nonempty for any open set V_0. This shows that $\bigcup_{i=1}^{\infty} C_i$ has no interior points. □

EXAMPLE 1.3.9. The set of rationals $\mathbb{Q} \subset \mathbb{R}$ forms a metrizable space that does not admit a complete metric nor is it locally compact.

The Urysohn metrization theorem asserts that a second countable normal Hausdorff space is metrizable. The proof of this result is remarkably simple.

THEOREM 1.3.10. *A second countable normal Hausdorff space is metrizable. Moreover, if the space admits a compact exhaustion, then it is metrizable with a complete metric.*

PROOF. We shall only use that the space is completely regular. In fact Tychonoff's Lemma shows that a regular Lindelöf space is normal. So it suffices to assume that the space is second countable and regular. There are second countable Hausdorff spaces that are not regular (79 in [Steen & Seebach]). Note that such spaces can't be locally compact.

The key is to use that the Hilbert cube: $\times_{i=1}^{\infty} I_i$ where $I_i = [0,1]$ is a metric space with distance

$$d\left(\left(x_i\right),\left(y_i\right)\right) = \sum_i 2^{-i}\left|x_i - y_i\right|.$$

The goal is then simply to show that our space is homeomorphic to a subset in the Hilbert cube.

Choose a countable collection of closed sets $\mathscr{C}$ such that their complements generate the topology of M. Enumerate the all pairs $(C_i, F_i) \in \mathscr{C} \times \mathscr{C}$ with $C_i \subset \mathrm{int}F_i$, and for each such pair select a function $\phi_i : M \to [0,1]$ such that $\phi_i(C_i) = 0$ and $\phi_i(M - \mathrm{int}F_i) = 1$. Then we obtain a map $\Phi : M \to \times_{i=1}^{\infty} I_i$ by $\Phi(x) = \times_{i=1}^{\infty}\phi_i(x)$.

This map is injective since distinct points can be separated by open sets whose complements are in $\mathscr{C}$. Next we show that for each $C \in \mathscr{C}$ the image $\Phi(C)$ is closed. Consider a sequence $c_n \in C$ such that $\Phi(c_n) \to \Phi(x)$. Note that for any fixed index we then have $\phi_i(c_n) \to \phi_i(x)$. If $x \notin C$, then we can find a pair (C_i, F_i) where $x \in M - \mathrm{int}F_i$. Therefore, $\phi_i(c_n) = 0$ and $\phi_i(x) = 1$, which is impossible. Thus $x \in C$ and $\Phi(x) \in \Phi(C)$. This shows that the map is a homeomorphism onto its image.

An explicit metric on M can given by

$$d(x,y) = \sum_i 2^{-i}\left|\phi_i(x) - \phi_i(y)\right|.$$

In case the space also has a compact exhaustion we can find a proper function $\rho : M \to [0,\infty)$ and use the proper map: $(\rho, \Phi) : M \to [0,\infty) \times_{i=1}^{\infty} I_i$. In this way the metric has the property that bounded closed sets are compact. In particular, Cauchy sequences have accumulations points and are consequently convergent. □

These topological properties of manifolds lead us to a very general principle that will be used later.

Consider a class $\mathscr{M}^n$ manifolds with the following properties:

(1) Every $M \in \mathscr{M}$ is σ-compact and has dimension n.
(2) $\mathbb{R}^n \in \mathscr{M}^n$.
(3) If $M \in \mathscr{M}^n$ and $U \subset M$ is open, then $U \in \mathscr{M}^n$.
(4) If $M \in \mathscr{M}^n$ and M is diffeomorphic to N, then $N \in \mathscr{M}^n$.

This can for example be the class of all n-manifolds or all oriented n-manifolds or simply all open subsets of a manifold. The key property to be extracted from σ-compactness is that each manifold has a proper functon $\rho : M \to [0,\infty)$.

The goal is to consider the validity of a statement $P(M)$ for all $M \in \mathscr{M}^n$. We will assume that the statement only depends on the diffeomorphism type of the manifold.

THEOREM 1.3.11. *The statement $P(M)$ is true for all manifolds in $\mathscr{M}^n$ provided the following conditions hold:*

(1) *$P(\mathbb{R}^n)$ is true.*
(2) *If $A, B \subset M \in \mathscr{M}^n$ are open and $P(A), P(B), P(A \cap B)$ are true, then $P(A \cup B)$ is true.*
(3) *If $A_i \subset M \in \mathscr{M}^n$ form a countable collection of pairwise disjoint open sets such that $P(A_i)$ are true, then $P\left(\bigcup A_i\right)$ is true.*

PROOF. We start by showing that $P(U)$ is true for all open sets $U \subset \mathbb{R}^n$. Observe first that any open box $(a_1,b_1) \times \cdots \times (a_n,b_n)$ is diffeomorphic to $\mathbb{R}^n$ and that the intersection of two boxes is either empty or a box. Consider next an open subset of $\mathbb{R}^n$ that is a finite union of open boxes. The claim follows for such sets by induction on the number of boxes. To see this, assume it holds for any union of k or fewer open boxes and consider $k+1$ open boxes B_i. Then the statement holds for $B_1 \cup \cdots \cup B_k$, B_{k+1}, and the intersection as it is a union of k or fewer boxes:

$$(B_1 \cup \cdots \cup B_k) \cap B_{k+1} = (B_1 \cap B_{k+1}) \cup \cdots \cup (B_k \cap B_{k+1}).$$

This in turn shows that we can prove the theorem for all open sets in $\mathbb{R}^n$. Fix an open set $U \subset \mathbb{R}^n$ and a proper function $\rho : U \to [0,\infty)$. Now cover each compact set $\rho^{-1}[i,i+1] \subset U_i$ by an open set U_i that is a finite union of open boxes, where $U_i \cap U_j = \emptyset$ when $|i-j| \geq 2$. Thus the theorem holds for $\bigcup U_{2i}$, $\bigcup U_{2i+1}$. It also holds for the intersection $(\bigcup U_{2i}) \cap (\bigcup U_{2i+1}) = \bigcup (U_j \cap U_{j+1})$ as $U_i \cap U_{i+1} \cap U_j \cap U_{j+1} = \emptyset$ when $i \neq j$. Consequently, the statement holds for the entire union.

Having come this far we use the exact same strategy to prove the statement for an $M \in \mathscr{M}^n$ by considering the class of all open subsets $U \subset M$ and replacing the first statement with:

(1) $P(U)$ is true for all open $U \subset M$ that are diffeomorphic to an open subset of $\mathbb{R}^n$, i.e., all charts $U \subset M$.

Using induction this shows that the statement is true for any open subset of M that is a finite union of charts. Next write $M = \bigcup U_i$ where each U_i is a finite union of charts and $U_i \cap U_j = \emptyset$ when $|i-j| \geq 2$. This means the theorem holds for $\bigcup U_{2i}$, $\bigcup U_{2i+1}$, and $(\bigcup U_{2i}) \cap (\bigcup U_{2i+1})$ and consequently for the entire union. □

1.4. Smooth Maps

1.4.1. Smooth Maps. A map $F : M \to N$ between spaces has a natural dual or pull back that takes functions defined on subsets of N to functions defined on subsets of M. Specifically if $f : A \subset N \to \mathbb{R}$ then $F^*(f) = f \circ F : F^{-1}(A) \subset M \to \mathbb{R}$. Here it could happen that $F^{-1}(A) = \emptyset$. Note that if F is continuous then its pull back will map continuous functions on open subsets of N to continuous functions on open subsets of M. Conversely, if N is normal, and the pull back takes continuous functions to continuous functions, then it will be continuous. To see this fix $O \subset N$ that is open and select a continuous function $\lambda : N \to [0,\infty)$ such that $\lambda^{-1}(0,\infty) = O$. Then $(\lambda \circ F)^{-1}(0,\infty) = F^{-1}(O)$ and is in particular open as we assumed that $\lambda \circ F$ was continuous.

DEFINITION 1.4.1. A map $F : M \to N$ is said to be *smooth* if F^* takes smooth functions to smooth functions, i.e., $F^*(\mathfrak{C}^\infty(N)) \subset \mathfrak{C}^\infty(M)$.

PROPOSITION 1.4.2. *Let $F : M \to N$ be continuous then the following conditions are equivalent:*

(1) *F is smooth.*
(2) *If $\mathscr{D}$ is a differentiable structure on N, then $F^*(\mathscr{D}) \subset \mathfrak{C}^\infty(M)$.*
(3) *$F^*(C^\infty(N)) \subset C^\infty(M)$.*
(4) *If $x_\alpha : U_\alpha \to \mathbb{R}^m$ is an atlas for M and $y_\beta : V_\beta \to \mathbb{R}^n$ an atlas for N, then the coordinate representations $y_\alpha \circ F \circ x_\beta^{-1}$ are smooth when- and where-ever they are defined.*

1.4.2. Maps of Maximal Rank.

DEFINITION 1.4.3. The *rank* of a smooth map at $p \in M$ is denoted $\mathrm{rank}_p F$ and is defined as the rank of the differential $D\left(y \circ F \circ x^{-1}\right)$ at $x(p)$. This definition is independent of the coordinate systems we choose due to the chain rule and the fact that the transition functions have nonsingular differentials at all points.

PROPOSITION 1.4.4. *If $F : M \to N$ and $G : N \to O$ are smooth maps, then*

$$\mathrm{rank}_p (G \circ F) \leq \min \left\{\mathrm{rank}_p F, \mathrm{rank}_{F(p)} G\right\}.$$

PROOF. Using coordinates x around $p \in M$, y around $F(p) \in N$, and z around $G(F(p)) \in O$ we can consider the composition

$$z \circ G \circ F \circ x^{-1} = \left(z \circ G \circ y^{-1}\right) \circ \left(y \circ F \circ x^{-1}\right)$$

The chain rule then implies

$$D\left(z \circ G \circ F \circ x^{-1}\right)|_p = D\left(z \circ G \circ y^{-1}\right)|_{y \circ F(p)} \circ D\left(y \circ F \circ x^{-1}\right)|_{x(p)}$$

This reduces the claim to the corresponding result for linear maps. □

DEFINITION 1.4.5. We say that F is a *diffeomorphism* if it is a bijection and both F and F^{-1} are smooth.

PROPOSITION 1.4.6. *Let $y : U \to \mathbb{R}^m$ be smooth where $U \subset M$ is an open subset. If* $\mathrm{rank}_p y = \dim M = m$*, then y is a chart on a neighborhood of p. Moreover, if* $\mathrm{rank}_p y = m <$ $\dim M$*, then it is possible to select coordinate functions $y^{m+1}, ..., y^n$ such that $y^1, ..., y^n$ form coordinates around p.*

PROOF. This follows from the inverse function theorem. Select a chart $x : V \to \mathbb{R}^m$ on a neighborhood of p and consider the smooth map $y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^m$. By the definition of rank the map has nonsingular differential at $x(p)$ and must therefore be a diffeomorphism from a neighborhood around $x(p)$ to its image. This shows in turn that y is a diffeomorphism on some neighborhood of p onto its image.

For the second claim select an arbitrary coordinate system $z^1, ..., z^n$ around p. Then the map $\left(y \circ z^{-1}, z^1, ..., z^n\right)$ has a differential at $z(p)$ that looks like

$$\begin{bmatrix} D\left(y \circ z^{-1}\right) \\ I_n \end{bmatrix}$$

where I_n is the identity matrix and $D\left(y \circ z^{-1}\right)$ has linearly independent rows. We can then use the replacement procedure to eliminate m of the bottom n rows so as to get a nonsingular $n \times n$ matrix. Assuming after possibly rearranging indices that the remaining rows are the last $n - m$ rows we see that $\left(y \circ z^{-1}, z^{m+1}, ..., z^n\right)$ has rank n at p and thus forms a coordinate system around p. □

DEFINITION 1.4.7. We say that F is an *immersion* if $\mathrm{rank}_p F = \dim M$ for every $p \in M$.

PROPOSITION 1.4.8. *For a smooth map $F : M \to N$ the following conditions are equivalent:*

(1) *F is an immersion.*
(2) *For each $p \in M$ there are charts $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that*

$$y \circ F \circ x^{-1}\left(x^1, ..., x^m\right) = \left(x^1, ..., x^m, 0, ..., 0\right)$$

(3) *If $\mathscr{D}$ is a differentiable structure on N then $F^*(\mathscr{D})$ is a differentiable structure on M.*

PROOF. It is obvious that 2 implies 1. For 1 implies 2. Select coordinates $z : U \to \mathbb{R}^m$ around p and $\tilde{x} : V \to \mathbb{R}^n$ around $F(p) \in N$. The composition $\tilde{x} \circ F \circ z^{-1}$ has rank m at $z(p)$. After possibly reordering the indices for the $\tilde{x}$-coordinates we can assume that $\left(\tilde{x}^1, \ldots, \tilde{x}^m\right) \circ F \circ z^{-1}$ also has rank m at $z(p)$. But this means that it is a diffeomorphism on some neighborhood around $z(p)$. Consequently $x = \left(\tilde{x}^1, \ldots, \tilde{x}^m\right) \circ F$ is a chart around p. Consider the functions

$$\begin{aligned} y^i &= \tilde{x}^i, \, i = 1, \ldots, m, \\ y^i &= \tilde{x}^i - \tilde{x}^i \circ F \circ x^{-1}\left(\tilde{x}^1, \ldots, \tilde{x}^m\right), \, i > m. \end{aligned}$$

These are defined on a neighborhood of $F(p)$ and when $i > m$ we have

$$\tilde{x}^i \circ F - \tilde{x}^i \circ F \circ x^{-1}\left(x^1 \circ F, \ldots, x^m \circ F\right) = 0.$$

So it remains to check that they are coordinates at $F(p)$. After composing these functions with $\tilde{x}^{-1}$ the differential will have a lower triangular block form

$$\begin{bmatrix} I_m & 0 \\ * & I_{n-m} \end{bmatrix}$$

where the diagonal entries are the identity matrices on first m and last $n-m$ coordinate subspaces. This shows that they will form coordinates on some neighborhood of $F(p)$.

As 1 and 2 are equivalent we can now use the proof that 1 implies 2 to show that if 1 or 2 hold then 3 also holds.

Conversely assume that 3 holds. Select coordinates $z^i = y^i \circ F$ around p where $y^i \in \mathscr{D}$. The chart z has rank m at p, so it follows that the corresponding smooth map y must have rank at least m at $F(p)$. However, the rank can't be greater than m as it maps into $\mathbb{R}^m$. We can now add $n-m$ coordinate functions z^i from some other coordinate system around $F(p)$ so as to get a map $\left(y^1, \ldots, y^m, z^{m+1}, \ldots, z^n\right)$ that has rank n at $F(p)$. These coordinate choices show that 1 holds. □

COROLLARY 1.4.9. *A smooth map $F : M \to N$ is an immersion iff for any smooth map $G : L \to M$ and $o \in L$ we have*

$$\operatorname{rank}_o F \circ G = \operatorname{rank}_o G.$$

DEFINITION 1.4.10. We say that F is an *embedding* if it is an immersion, injective, and $F : M \to F(M)$ is a homeomorphism, where the image is endowed with the induced topology.

PROPOSITION 1.4.11. *For a smooth map $F : M \to N$ the following conditions are equivalent:*

(1) *F is an embedding.*
(2) *$F^*\left(\mathfrak{C}^\infty(N)\right) = \mathfrak{C}^\infty(M)$, i.e., F^* is surjective on smooth functions.*

PROOF. Start by assuming that 2 holds. Given $p, q \in M$ select $f \in \mathfrak{C}^\infty(M)$ such that $f(p) \neq f(q)$. Then find $g \in \mathfrak{C}^\infty(N)$ such that $f = g \circ F$. Then $g(F(p)) \neq g(F(q))$ showing that F is injective. To see that the topology of M agrees with the induced topology on $F(M)$ select an open set $O \in M$ and $\lambda : M \to [0, \infty)$ such that $\lambda^{-1}(0, \infty) = O$. Select $\mu : U \subset N \to \mathbb{R}$ such that $\lambda = \mu \circ F$. Note that $F(M) \subset U$ as λ is defined on all of M. Thus

$$\mu^{-1}(0, \infty) \cap F(M) = F\left(\lambda^{-1}(0, \infty)\right) = F(O)$$

and $F(O)$ is open in $F(M)$. Finally select coordinates x around $p \in M$ and write $x^i = y^i \circ F$ for smooth functions on some neighborhood of $F(p)$. The composition $y \circ F \circ x^{-1}$ has rank m at $x(p)$. So the map $F \circ x^{-1}$ must have rank at least m at $x(p)$. However, the rank can't exceed m so this shows that $\operatorname{rank}_p F = m$ and in turn that F is an immersion.

Conversely assume that F is an embedding and $f : O \subset M \to \mathbb{R}$ a smooth function. Using that F is an immersion we can for each $p \in M$ select charts $x_p : O_p \to \mathbb{R}^m$ around p and $y_p : U_p \to \mathbb{R}^n$ around $F(p)$ such that $y_p^j|_{F(O_p) \cap U_p} = 0$ for $j > m$. Since F is an embedding $U_p \cap F(O_p) \subset F(M)$ is open. This means that we can assume that U_p is chosen so that $F(O_p) = U_p \cap F(M)$. Now select a locally finite subcover $\underline{U_\alpha}$ of $F(M)$ from the cover U_p and let $O_\alpha = F^{-1}(U_\alpha)$. On each U_α define g_α such that $g_\alpha \circ y_\alpha^{-1}\left(a^1, .., a^n\right) = f \circ x_\alpha^{-1}\left(a^1, \ldots, a^m\right)$. We can then define $g = \sum_\alpha \mu_\alpha g_\alpha$, where μ_α is a partition of unity for U_α. This gives us a function on the open set $\cup U_\alpha$. Since F is injective it follows that $g \circ F = f$. □

COROLLARY 1.4.12. *If $F : M \to N$ is an embedding such that $F(M) \subset N$ is closed, then $F^*(C^\infty(N)) = C^\infty(M)$.*

PROOF. The only additional item to worry about is that the function g we just constructed cannot be extended to N and still remain fixed on $F(M)$. When the image is a closed subset this is easily done by finding a smooth Urysohn function ν that is 1 on $F(M)$ and vanishes on $N - U$. The function νg is then a smooth function on N that can be used instead of g. □

DEFINITION 1.4.13. A subset $S \subset M$ is a submanifold if it admits a topology such that the restriction of the differentiable structure on M to S is a differentiable structure. The dimension of the structure on S will generally be less than that of M unless S is an open subset with the induced topology. Note that the topology on S can be different from the induced topology, but it has to be finer as we require all smooth functions on M to be smooth on S. In this way we see that a submanifold is in fact the image of an injective immersion.

DEFINITION 1.4.14. We say that F is a *submersion* if $\operatorname{rank}_p F = \dim N$ for all $p \in M$.

PROPOSITION 1.4.15. *For a smooth map $F : M \to N$ the following conditions are equivalent:*

(1) *F is a submersion.*
(2) *For each $p \in M$ there are charts $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that*
$$y \circ F \circ x^{-1}\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^n\right).$$
(3) *For each $f \in \mathfrak{C}^\infty(N)$ and $p \in M$ we have that* $\operatorname{rank}_p(f \circ F) = \operatorname{rank}_{F(p)}(f)$.

PROOF. Assume that 1 holds and select a chart y around $F(p)$. Then $y \circ F$ has rank n at p. We can then supplement with $m - n$ coordinate functions x^i from any coordinate system around p such that $x^1 = y^1 \circ F, \ldots, x^n = y^n \circ F, x^{n+1}, \ldots, x^m$ are coordinates around p. This yields the desired coordinates.

Clearly 2 implies 3.

If we assume that 3 holds and that we have a chart y around $F(p)$. Then we can consider smooth functions $f = \sum \alpha_i y^i$, where $\alpha_i \in \mathbb{R}$. These have rank 1 at $F(p)$ unless $\alpha^1 = \cdots = \alpha^n = 0$. If we choose coordinates x around p, then $D\left(f \circ F \circ x^{-1}\right)|_{x^{-1}(p)}) =$

$\sum \alpha_i D\left(y^i \circ F \circ x^{-1}\right)|_{x^{-1}(p)}$. So it follows that $D\left(y^i \circ F \circ x^{-1}\right)|_{x^{-1}(p)}$ are linearly independent, which in turn implies that $y \circ F \circ x^{-1}$ has rank n at $x^{-1}(p)$. □

COROLLARY 1.4.16. *A smooth map $F : M \to N$ is a submersion iff for any smooth map $G : N \to O$ and $p \in M$ we have*

$$\text{rank}_p G \circ F = \text{rank}_{F(p)} G$$

Finally we have a few useful properties.

PROPOSITION 1.4.17. *Let $F : M^m \to N^n$ be a smooth map.*

(1) *If F is proper, then it is closed.*
(2) *If F is a submersion, then it is open.*
(3) *If F is a proper submersion and N is connected then it is surjective.*

PROOF. 1. Let $C \subset M$ be a closed set and assume $F(x_i) \to y$, where $x_i \in C$. The set $\{y, F(x_i)\}$ is compact. Thus the preimage is also compact. This implies that $\{x_i\}$ has an accumulation point. If we assume that $x_{i_j} \to x \in C$, then continuity shows that $F\left(x_{i_j}\right) \to F(x)$. Thus $y = F(x) \in F(C)$.
2. Consequence of local coordinate representation of F.
3. Follows directly from the two other properties. □

COROLLARY 1.4.18. *Let $F : M \to N$ be a submersion. If $f : O \subset F(N) \to \mathbb{R}$ is a function on an open set such that $f \circ F$ is smooth, then f is smooth.*

PROOF. Smoothness is clearly a local property so we can confine ourselves to functions that are defined on the coordinate systems guaranteed from 2 in the above characterization of submersions. But then the claim is obvious. □

1.4.3. Regular and Critical Points. We say that F is *non-singular* on M if it is both a submersion and an immersion. This is evidently equivalent to saying that it is locally a diffeomorphism.

A point $p \in M$ is called a *regular point* if $\text{rank}_p F = \dim N$, otherwise it is a *critical point*. A point $q \in N$ is called a *regular value* if $F^{-1}(q)$ is empty or only contains regular points, otherwise it is a *critical value*.

Note that if $p \in M$ is a regular point for $F : M \to N$, then there is a neighborhood $p \in U \subset M$ such that q is a regular value for $F|_U : U \to N$.

THEOREM 1.4.19 (*The Regular Value Theorem*). *If $q \in N$ is a regular value for a smooth function $F : M^m \to N^n$, then $F^{-1}(p)$ is empty or a properly embedded submanifold of M of dimension $m - n$.*

PROOF. Note that the preimage is closed so it follows that its intersections with compact sets is compact. We shall also use the induced topology and show that it is a submanifold with respect to that topology. We claim that $\mathfrak{C}^\infty(M)$ restricts to a differential system on the preimage.

If we select coordinates y^i, $i = 1, ..., n$ around $q \in N$, then the functions $y^i \circ F$ are part of a coordinate system x^i around any point $p \in F^{-1}(q)$. This means that we can find a neighborhood $p \in U$ such that $U \cap F^{-1}(q) = \left\{x \in U \mid y^i(F(x)) = y^i(F(q))\right\}$, i.e., $x^i = y^i \circ F$ are constant on the preimage. Given $f \in \mathfrak{C}^\infty(M)$ defined around p we have that $f = F\left(x^1, ..., x^m\right)$. Now on $U \cap F^{-1}(q)$ the first n coordinates are constant so it follows that $f|_{U \cap F^{-1}(q)} = F\left(x^1(p), ..., x^n(p), x^{n+1}, ..., x^m\right)$. Thus the restriction can be written as a smooth function of the last $m - n$ coordinates. Finally we note that these last $m - n$

coordinates also define the desired chart on $U \cap F^{-1}(q)$ as they are injective and yield a homeomorphism on to the image. □

To complement this we next prove.

THEOREM 1.4.20 (Brown, 1935, A.P. Morse, 1939 and Sard, 1942). *The set of regular values for a smooth function $F : M^m \to N^n$ is a countable intersection of open dense sets and in particular dense. Moreover, the set of critical values has measure 0.*

PROOF. We prove Brown's original statement: the set of critical values has no interior points. The proof we give is fairly standard and is very close to Brown's original proof. The same proof is easily adapted to prove Sard's measure zero version, but this particular statement is in fact rarely used. A.P. Morse proved the measure theoretic result when the target space is $\mathbb{R}$.

Note that the set of critical points is closed but its image need not be closed. However, the set of critical points is a countable union of compact sets and thus the image is also a countable union of compact sets. This means that we rely on the Baire category theorem: a set that is the countable union of closed sets with empty interiors also has empty interior. Thus we only need to show that there are no interior points in the set of critical values that come from critical points in a compact set. Further note that it suffices to prove the theorem for the restriction of F to any open covering of M.

To clarify the meaning of measure 0 and prove Sard's theorem in the case where it is most used, we make some simple observations.

Consider a map $F : O \subset \mathbb{R}^n \to \mathbb{R}^n$. When F is locally Lipschitz, then it maps sets of measure zero to sets of measure zero. Moreover, any differentiable map that has bounded derivative on compact sets is locally Lipschitz. Thus C^1 diffeomorphisms preserve sets of measure zero. This shows that the notion of sets of measure zero is well-defined in a smooth manifold. Now consider $F : M^m \to N^n$, where $m < n$ and construct $\bar{F} : M \times \mathbb{R}^{n-m} \to N$, by $\bar{F}(x,z) = F(x)$. Then $F(M) = \bar{F}(M \times \{0\})$ has measure zero as $M \times \{0\} \subset M \times \mathbb{R}^{n-m}$ has measure zero.

In the general case the proof uses induction on m. For $m = 0$ the claim is trivial as M is forced to be a countable set with the discrete topology. As mentioned above, it suffices to prove it for maps $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$, where U is open. For such a map let C_0 be the set of critical points and define $C_k \subset C_0$ as the set of critical points where all derivatives of order $\leq k$ vanish. Note that all of these sets are closed.

First we show that $F(C_k)$ has no interior points when $k \geq m/n$: Fix a compact set K. Taylor's theorem shows that we can select $r > 0$ and $C > 0$ such that for any $x \in B(p,r)$ with $p \in C_k \cap K$ we have

$$|F(p) - F(x)| \leq C |p-x|^{k+1}.$$

Now cover $C_k \cap K$ by finitely many cubes I_i^ε of side length $\varepsilon < r$, then $F(I_i^\varepsilon)$ lies in a cube J_i^ε of side length $\leq C(m,n)\,\varepsilon^{k+1}$ for a constant $C(m,n)$ that depends on C, m, and n. Thus

$$\begin{aligned} |J_i^\varepsilon| &\leq (C(m,n))^n \varepsilon^{n(k+1)} \\ &= (C(m,n))^n \varepsilon^{n(k+1)-m} |I_i^\varepsilon|. \end{aligned}$$

Since $C_k \cap K$ is compact we can assume that $\sum |I_i^\varepsilon|$ remains bounded as $\varepsilon \to 0$. Thus $\sum |J_i^\varepsilon|$ will converge to 0 since $n(k+1) > m$. This shows that $F(C_k \cap K)$ does not contain any interior points as it could otherwise not be covered by cubes whose total volume is arbitrarily small.

Next we show that $F(C_k - C_{k+1})$ has no interior points for $k > 0$: Denote by ∂^k some specific partial derivative of order k. Thus $\left(\partial^k F\right)(p) = 0$ for $p \in C_k - C_{k+1}$ but some partial derivative $\frac{\partial \partial^k F}{\partial x^j}(p) \neq 0$. Without loss of generality we can assume that $\frac{\partial \partial^k F^1}{\partial x^j}(p) \neq 0$. This means that near p the set where $\partial^k F^1 = 0$ will be a submanifold of dimension $m-1$. Since p is critical for F it'll also be a critical point for the restriction of F to any submanifold. By induction hypothesis the image of such a set has no interior points. Thus for any fixed compact set K the set $K \cap (C_k - C_{k+1})$ can be divided into a finite collection of sets whose images have no interior points.

Finally we show that $F(C_0 - C_1)$ has no interior points: Note that when $n = 1$ it follows that $C_0 = C_1$ so there is nothing to prove in this case. Assume that $p \in C_0 - C_1$ is a point where $\frac{\partial F^i}{\partial x^j} \neq 0$. After rearranging the coordinates in $\mathbb{R}^m$ and $\mathbb{R}^n$ we can assume that $\frac{\partial F^1}{\partial x^1} \neq 0$. In particular, the set $L = \left\{x \mid F^1(x) = F^1(p)\right\}$ is a submanifold of dimension $m-1$ in a neighborhood of p. Let $G = \left(F^2, ..., F^n\right) : L \to \mathbb{R}^{n-1}$. Now observe that if $F(p)$ is an interior point in $F(C_0 - C_1)$, then $G(p)$ is an interior point for $G(L \cap (C_0 - C_1))$. This, however, contradicts our induction hypothesis since all the points in $L \cap (C_0 - C_1)$ are critical for G. (For the measure zero statement, this last part requires a precursor to the Tonelli/Fubini theorem or Cavalieri's principle: A set has measure zero if its intersection with all parallel hyperplanes has measure zero in the hyperplanes.)

Putting these three statements together implies that the set of critical values has no interior points. □

1.4.4. Covering Maps.

LEMMA 1.4.21. *Let $F : M^m \to N^m$ be a smooth proper map. If $y \in N$ is a regular value, then there exists a neighborhood V around y such that $F^{-1}(V) = \bigcup_{k=1}^n U_k$ where U_k are mutually disjoint and $F : U_k \to V$ is a diffeomorphism.*

PROOF. First use that F is proper to show that $F^{-1}(y) = \{x_1, \ldots, x_n\}$ is a finite set. Next use that y is regular to find mutually disjoint neighborhoods W_k around x_k such that $F : W_k \to F(W_k)$ is a diffeomorphism. If the desired V does not exist, then we can find a sequence $z_i \in M - \bigcup_{k=1}^n W_k$ such that $F(z_i) \to y$. Using again that F is proper it follows that (z_i) must have an accumulation point z. Continuity of F then shows that $z \in F^{-1}(y)$. This in turn shows that infinitely many z_i must lie in $\bigcup_{k=1}^n W_k$, a contradiction. □

DEFINITION 1.4.22. A smooth map $\pi : \bar{N} \to N$ is called a covering map if each point in N is evenly covered, i.e., for every $y \in N$ there is a neighborhood V around y such that $\pi^{-1}(V) = \bigcup U_i$ where $\pi : U_i \to V$ is a diffeomorphism and the sets U_i are pairwise disjoint.

COROLLARY 1.4.23. *If $F : M \to N$ is a proper non-singular map with N connected, then F is a covering map.*

The key property for covering maps is the unique path lifting property. A lift of a map $F : M \to N$ into the base of a covering map $\pi : \bar{N} \to N$ is a map $\bar{F} : M \to \bar{N}$ such that $\pi \circ \bar{F} = F$. If $\bar{F}(x_0) = \pi(y_0)$, then we say that the lift goes through y_0.

PROPOSITION 1.4.24. *If M is connected, $x_0 \in M$, and $y_0 \in \bar{N}$ such that $F(x_0) = \pi(y_0)$, then there is at most one lift $\bar{F}$ such that $\bar{F}(x_0) = y_0$.*

PROOF. Assume that we have two lifts F_1 and F_2 with this property and let $A = \{x \in M \mid F_1(x) = F_2(x)\}$. Clearly A is non-empty and closed. The covering maps property shows that A is open. So when M is connected $A = M$. □

THEOREM 1.4.25. *If M is connected and simply connected, then any $F : M \to N$ has lift through each point in $\pi^{-1}(F(x_0))$.*

PROOF. Cover N by connected open sets V_α that are evenly covered by disjoint sets in $\bar{N}$.

Next suppose that M is covered by a string of connected sets U_i, $i = 0,1,2...$ such that $F(U_i) \subset V_{\alpha_i}$. We can then lift F on each of the sets U_i to go through a given point in $\pi^{-1}(F(U_i))$. If we further have the property that $U_k \cap \left(\bigcup_{i=0}^{k-1} U_i\right)$ is non-empty and connected for, then we can use the uniqueness of liftings to successively define $F|_{U_k}$ given that it is defined on $\bigcup_{i=0}^{k-1} U_i$. Note that the sets U_i need not be open.

Unfortunately not a lot of manifolds admit such covers. Clearly $\mathbb{R}^k$ does as it can be covered by coordinate cubes. Also any interval, disc, and square has this property. However, the circle S^1 cannot be covered by such a string of sets. On the other hand spheres S^n, $n > 1$ do have this property. We will use the property for the interval and square.

We can now show that if we have a map $G : M_0 \to M$, where M_0 has the desired covering property, then $F \circ G$ can be lifted. Given two curves $c_i : [0,1] \to M$ where $c_i(0) = x_0$ and $c_i(1) = x \in M$, where $i = 0,1$, we invoke simple connectivity of M to find a homotopy $H : [0,1]^2 \to M$ where $H(s,0) = x_0$, $H(s,1) = x$, and $H(i,t) = c_i(t)$. We can then find a lift of $F \circ H$ such that $\overline{F \circ H}(s,0) = y_0$. The unique path lifting property then guarantees that $\overline{F \circ H}(s,1)$ is constant, and, in particular, that the lift of F at $x \in M$ does not depend on the path connecting it to x_0. This gives us a well-defined lift of F that is smooth when composed with any curve that starts at x_0. It is now easy to check that the lift of F is continuous and smooth using uniqueness of lifts. □

COROLLARY 1.4.26. *If $\pi : \bar{N} \to N$ is a covering map and $F : M \to N$ is a map such that for every closed curve $c : S^1 \to M$ the map $F \circ c$ has a lift that passes through each point in $\pi^{-1}(F \circ c(t_0))$ for a fixed $t_0 \in S^1$, then F has a lift through each point in $\pi^{-1}(F(x_0))$.*

PROOF. This proof is almost identical to the above proof. The one difference is that the curves are no longer necessarily homotopic to each other. However, the fact that lifts of closed curves in M are assumed to become closed shows that the construction is independent of the paths we choose. □

COROLLARY 1.4.27. *If $F_0 : M_0 \to N$ and $F_1 : M_1 \to N$ are coverings where all manifolds are connected and $M_{0,1}$ are both simply connected, then M_0 and M_1 are diffeomorphic.*

PROOF. This is an immediate consequence of the lifting property of each of the covering maps to the other covering space. □

COROLLARY 1.4.28. (Hadamard) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a proper non-singular map, then F is a diffeomorphism.*

1.5. Tangent Spaces

1.5.1. Motivation. Let us start by selecting a countable differentiable system $\{f^i\}$, $i = 1,2,...$ of functions $f^i : M \to \mathbb{R}$. To find such a system we invoke paracompactness, partitions of unity, extensions of smooth function etc from above.

Tangent vectors are supposed to be tangents or velocities to curves on the manifold. These vectors have as such no place to live unless we know that the manifold is in Euclidean space. In the general case we can use the countable collection f^i of smooth functions coming from a differential structure to measure the coordinates of the velocities by calculating the derivatives

$$\frac{d\left(f^{i}\circ c\right)}{dt}$$

for a smooth curve $c:I\to M$. Thus a tangent vector $v\in TM$ looks like a countable collection v^{i} of its coordinates. However, around any given point we know that there will be n coordinate functions, say $f^{1},...,f^{n}$, that yield a chart and then other smooth functions F^{j}, $j>n$ such that $f^{j}=F^{j}\left(f^{1},...,f^{n}\right)$. Thus we also have the relations

$$v^{j}=\sum_{i=1}^{n}\frac{\partial F^{j}}{\partial x^{i}}v^{i}.$$

In other words the n coordinates $v^{1},...,v^{n}$ determine the rest of the coordinate components of v. Note that at a fixed point p, the tangent vectors $v\in T_{p}M$ form an n-dimensional vector space, which is an n-dimensional subspace of a fixed infinite dimensional vector space. Moreover, this tangent space is well-defined as the set of vectors tangent to curves going through p and is thus not dependent on the chosen coordinates. However, the coordinates help us select a basis for this vector space and thus to create suitable coordinates that yield a differentiable structure on TM.

As it stands the definition does depend on our initial choice of a differentiable system. To get around this we could simply use the entire space of smooth functions $C^{\infty}\left(M\right)$ to get around this. This is more or less what we shall do below.

1.5.2. Abstract Derivations. The space of all smooth functions $\mathfrak{C}^{\infty}\left(M\right)$ is not a vector space as we can't add functions that have different domains especially if these domains do not even intersect. If we fix $p\in M$, then we can consider the subset $\mathfrak{C}_{p}\left(M\right)\subset\mathfrak{C}^{\infty}\left(M\right)$ of smooth functions whose domain contains p. Thus any two functions in $\mathfrak{C}_{p}\left(M\right)$ can now be added in a meaningful way by adding them on the intersection of their domains and then noting that this is again an open set containing p. Thus we get a nice and very large vector space of smooth functions defined on some neighborhood of p. To get a logically meaningful theory this space is often modified by considering instead equivalence classes of function in $\mathfrak{C}_{p}\left(M\right)$, the relation being that two functions that are equal on some neighborhood of p are considered equivalent. This quotient space is denoted $\mathfrak{F}_{p}\left(M\right)$ and the elements are called *germs* of functions at p. This is not unlike the idea that the space of L^{2} functions is really supposed to be a quotient space where we divide out by the subspace of functions that vanish almost everywhere.

Now consider a curve $c:I\to M$ with $c\left(t_{0}\right)=p$. The goal is to make sense of the velocity of c at t_{0}. If $f\in\mathfrak{C}_{p}\left(M\right)$, then $f\circ c$ measures how c changes with respect to f. If f had been a coordinate function this would be the corresponding coordinate component of c in a chart. Similarly the derivative $\frac{d}{dt}\left(f\circ c\right)$ measures the change in velocity with respect to f, i.e., what should be the f-component of the velocity.

DEFINITION 1.5.1. The velocity $\dot{c}\left(t_{0}\right)$ of c at t_{0} is the map

$$\begin{array}{rcl}\mathfrak{C}_{p}\left(M\right) & \to & \mathbb{R}\\ f & \mapsto & \frac{d}{dt}\left(f\circ c\right)\left(t_{0}\right).\end{array}$$

Thus $\dot{c}\left(t_{0}\right)$ is implicitly defined by specifying its directional derivatives

$$D_{\dot{c}\left(t_{0}\right)}f=\frac{d}{dt}\left(f\circ c\right)\left(t_{0}\right)$$

for all smooth functions defined on a neighborhood of $p=c\left(t_{0}\right)$.

DEFINITION 1.5.2. A derivation at p or on $\mathfrak{C}_p(M)$ is a linear map $D : \mathfrak{C}_p(M) \to \mathbb{R}$ that is also satisfies the product rule for differentiation at p:

$$D(fg) = D(f)g(p) + f(p)D(g).$$

There is an alternate way of defining derivations as linear functions on $\mathfrak{C}_p(M)$. Let $\mathfrak{C}_p^0(M) \subset \mathfrak{C}_p(M)$ be the maximal ideal of functions that vanish at p and $\left(\mathfrak{C}_p^0(M)\right)^2 \subset \mathfrak{C}_p^0(M)$ the ideal generated by products of elements in $\mathfrak{C}_p^0(M)$.

LEMMA 1.5.3. *The derivations at p are isomorphic to the subspace of linear maps on* $\mathfrak{C}_p^0(M)$ *that vanish on* $\left(\mathfrak{C}_p^0(M)\right)^2$.

PROOF. If D is a derivation, then the derivation property shows that it vanishes on $\left(\mathfrak{C}_p^0(M)\right)^2$. Furthermore, it also vanishes on constant functions as linearity and the derivation property implies

$$D(c) = cD(1) = cD(1\cdot 1) = c(D(1)+D(1))$$

Conversely, any linear map D on $\mathfrak{C}_p^0(M)$ that vanishes on $\left(\mathfrak{C}_p^0(M)\right)^2$ defines a unique linear map on $\mathfrak{C}_p(M)$ by defining it to vanish on constant functions. If $f,g \in \mathfrak{C}_p(M)$, then we have

$$\begin{aligned} 0 &= D((f-f(p))(g-f(p))) \\ &= D(fg) - f(p)Dg - g(p)Df + D(f(p)g(p)) \\ &= D(fg) - f(p)Dg - g(p)Df \end{aligned}$$

showing that it is a derivation. □

Next we show that derivations exist.

PROPOSITION 1.5.4. *The map* $f \mapsto \frac{d}{dt}(f\circ c)(t_0)$ *is a derivation on* $\mathfrak{C}_p(M)$.

PROOF. That it is linear in f is obvious from the fact that differentiation is linear. The derivation property follows from the product rule for differentiation:

$$\frac{d}{dt}((fg)\circ c)(t_0) = \left(\frac{d}{dt}(f\circ c)(t_0)\right)(g\circ c)(t_0) + (f\circ c)(t_0)\frac{d}{dt}(g\circ c)(t_0).$$

□

DEFINITION 1.5.5. The tangent space T_pM for M at p is the vector space of derivations on $\mathfrak{C}_p(M)$.

PROPOSITION 1.5.6. *If* $p \in U \subset M$, *where U is open, then* $T_pU = T_pM$.

PROOF. We already saw that derivations must vanish on constant function. Next consider a function f that vanishes on a neighborhood of p. We can then find $\lambda : M \to \mathbb{R}$ that is 1 on a neighborhood of p and $\lambda = 0$ on the complement of the region where f vanishes. Thus $\lambda f = 0$ on M and

$$0 = D(\lambda f) = D(\lambda)f(p) + \lambda(p)D(f) = D(f).$$

This in turns shows that if two functions f,g agree on a neighborhood of p, then $D(f) = D(g)$. This means that a derivation D on $\mathfrak{C}_p(M)$ restricts to a derivation on $\mathfrak{C}_p(U)$ and conversely that any derivation on $\mathfrak{C}_p(U)$ also defines a derivation on $\mathfrak{C}_p(M)$. This proves the claim. □

We are now ready to prove that there are no more derivations than one would expect.

LEMMA 1.5.7. *The natural map $\mathbb{R}^n \to T_0\mathbb{R}^n$ that maps v to $D_v f = \left(\frac{df}{dt}\right)(tv)|_{t=0}$ is an isomorphism.*

PROOF. The map is clearly linear and as

$$D_v x^i = v^i$$

it follows that its kernel is trivial. Thus we need to show that it is surjective. This claim depends crucially on the fact that derivations are defined on C^∞ functions. The key observation is that we have a Taylor formula

$$f(x) = f(0) + x^i f_i(x)$$

where f_i are also smooth and $f_i(0) = \frac{\partial f}{\partial x^i}(0)$. These functions are defined by

$$f_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)\,dt$$

and the result follows from the fundamental theorem of calculus and the chain rule

$$\frac{d}{dt}(f(tx)) = x^i \frac{\partial f}{\partial x^i}(tx).$$

Now select an abstract derivation $D \in T_0\mathbb{R}^n$ and observe that

$$D(f) = D(f(0)) + D\left(x^i\right) f_i(0) + 0D(f_i) = \frac{\partial f}{\partial x^i}(0) D\left(x^i\right)$$

So if we define a vector $v = \left(D\left(x^1\right), \dots, D(x^n)\right)$, then in fact

$$D(f) = D_v(f).$$

□

REMARK 1.5.8. The space of linear maps on $C^k(\mathbb{R}^n)$, $1 \le k < \infty$ that satisfy the product rule

$$D(fg) = D(f)g(0) + f(0)D(g)$$

is infinite dimensional! Note that it suffices to show this for $n = 1$. Next observe that if $Z \subset C^k(\mathbb{R})$ is the subset of functions that vanish at 0, then we merely need to show that Z/Z^2 is infinite dimensional. To see this first note that if f is C^0 and $g \in Z$ then fg is differentiable with derivative $f(0)g'(0)$ at 0. This in turn implies that functions in Z^2 are not only C^k but also have derivatives of order $k+1$ at 0. However, there is a vast class of functions in Z that do not have derivatives of order $k+1$ at 0.

1.5.3. Concrete Derivations. To avoid the issue of crucially using C^∞ functions we give an alternate definition of the tangent space that obviously gives the above definition.

DEFINITION 1.5.9. T_pM is the space of derivations that are constructed from the derivations coming from curves that pass through p.

Without the above result it is not obvious that this is a vector space so a little more work is needed.

PROPOSITION 1.5.10. *Let $x^1, \dots, x^n$ be coordinates on a neighborhood of p, then two curves c_i passing through p at $t = 0$ define the same derivations if and only if for all $i = 1, \dots, n$*

$$\frac{d\left(x^i \circ c_1\right)}{dt}(0) = \frac{d\left(x^i \circ c_2\right)}{dt}(0).$$

PROOF. The necessity is obvious. Conversely note that any $f \in \mathfrak{C}_p(M)$ can be expressed smoothly as $f = F\left(x^1, \ldots, x^n\right)$ on some neighborhood of p. Thus

$$\begin{aligned}
\frac{d(f \circ c_1)}{dt}(0) &= \frac{d\left(F\left(x^1 \circ c_1, \ldots, x^n \circ c_1\right)\right)}{dt}(0) \\
&= \frac{\partial F}{\partial x^i} \frac{d\left(x^i \circ c_1\right)}{dt}(0) \\
&= \frac{\partial F}{\partial x^i} \frac{d\left(x^i \circ c_2\right)}{dt}(0) \\
&= \frac{d(f \circ c_2)}{dt}(0).
\end{aligned}$$

□

PROPOSITION 1.5.11. *The subset of derivations on $\mathfrak{C}_p(M)$ that come from curves through p form a subspace.*

PROOF. First note that for a curve c through p we have

$$\alpha \frac{d(f \circ c)}{dt}(0) = \frac{d(f \circ c)(\alpha t)}{dt}(0).$$

So scalar multiplication preserves this subset.

Next assume that we have two curves c_i and select a coordinate system x^i around p. Define

$$c = x^{-1}\left(x^1 \circ c_1 + x^1 \circ c_2, \ldots, x^n \circ c_1 + x^n \circ c_2\right)$$

where x^{-1} is the inverse of the chart map $x : U \to V \subset \mathbb{R}^n$. Then

$$x^i \circ c = x^i \circ c_1 + x^i \circ c_2$$

and

$$\frac{d(f \circ c)}{dt}(0) = \frac{d(f \circ c_1)}{dt}(0) + \frac{d(f \circ c_2)}{dt}(0).$$

Showing that addition of such derivations also remain in this subset. □

DEFINITION 1.5.12. The velocity of a curve $c : I \to M$ at t_0 is denoted by $\dot{c}(t_0) \in T_{c(t_0)}M$ and is the derivation corresponding to the map:

$$f \mapsto \frac{d(f \circ c)}{dt}(t_0).$$

As any vector $v \in T_pM$ can be written as $v = \dot{c}(t_0)$ we can also define the directional derivative of f by

$$D_v f = \frac{d(f \circ c)}{dt}(t_0).$$

1.5.4. Local Coordinate Formulas, Differentials, and the Tangent Bundle. Finally let us use coordinates to specify a basis for the tangent space. Fix $p \in M$ and a coordinate system x^i around p. For any $f \in \mathfrak{C}_p(M)$ write $f = F\left(x^1, \ldots, x^n\right)$ and define

$$\frac{\partial f}{\partial x^i} = \frac{\partial F}{\partial x^i}.$$

The map $f \mapsto \frac{\partial f}{\partial x^i}(p)$ is a derivation on $\mathfrak{C}_p(M)$. We denote it by $\frac{\partial}{\partial x^i}|_p$. These tangent vectors in fact form a basis as we saw that

$$D(f) = D\left(x^i\right) \frac{\partial f}{\partial x^i}|_p$$

i.e.,

$$D = v^i \frac{\partial}{\partial x^i}|_p$$

where the components v^i are uniquely determined. Moreover, as

$$\frac{d(f \circ c)}{dt}(0) = \frac{\partial f}{\partial x^i}|_p \frac{d\left(x^i \circ c\right)}{dt}(0)$$

we also get this as a natural basis if we stick to curves.

DEFINITION 1.5.13. The cotangent space T_p^*M to M at $p \in M$ is the vector space of linear functions on T_pM. Alternately this can also be defined as the quotient space $\mathfrak{C}_p^0(M) / \left(\mathfrak{C}_p^0(M)\right)^2$ without even referring to tangent vectors.

Using coordinates we obtain a natural dual basis dx^i satisfying

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

In particular we see that

$$dx^i(v) = dx^i\left(v^j \frac{\partial}{\partial x^j}\right) = v^i$$

calculates the i^{th} coordinate of a vector.

We also obtain a natural set of transformation laws when we have another coordinate system y^i around p:

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

and

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}.$$

Here the matrices $\left[\frac{\partial y^i}{\partial x^j}\right]$ and $\left[\frac{\partial x^j}{\partial y^i}\right]$ have entries that are functions on the common domain of the the charts and they are inverses of each other. These are also the natural transformation laws for a change of basis as well as the change of the dual basis.

The differential d also has a coordinate free definition. Let $f \in \mathfrak{C}_p(M)$, then we can define $df \in T_p^*M$ by

$$df(v) = D_v f = \frac{d(f \circ c)}{dt}(0)$$

if c is a curve with $\dot{c}(0) = v$. In coordinates we already know that

$$df(v) = \frac{\partial f}{\partial x^i} v^i$$

so in fact

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

This shows that our definition of dx^i is consistent with the more abstract definition and that the transformation law for switching coordinates is simply just the law of how to write a vector or co-vector out in components with respect to a basis.

It now becomes very simple to define a differentiable structure on the tangent bundle TM. This space is the disjoint union of the tangent spaces T_pM where $p \in M$. There is also a natural base point projection $p : TM \to M$ that takes a vector in T_pM to its base point p. Starting with a differential system $\left\{f^i\right\}$ for M, we obtain a differentiable system $\left\{f^i \circ p, df^i\right\}$ for TM. Moreover when $f^1, .., f^n$ form a chart on $U \subset M$, then

$f^1 \circ p, ..., f^n \circ p, df^1, ..., df^n$ form a chart on TU. This takes us full circle back to our preliminary definition of tangent vectors.

IMPORTANT: The isomorphism between T_pM and $\mathbb{R}^n$ depends on a choice of coordinates and is not canonically defined. We just saw that in a coordinate system we have a natural identification

$$TU \to U \times \mathbb{R}^n$$

which for fixed $p \in U$ yields a linear isomorphism

$$T_pU \to \{p\} \times \mathbb{R}^n \simeq \mathbb{R}^n.$$

However, this does not mean that TM has a natural map to $M \times \mathbb{R}^n$ that is a linear isomorphism when restricted to tangent spaces. Manifolds that admit such maps are called *parallelizable*. Euclidean space is parallelizable as are all matrix groups. But as we shall see S^2 is not parallelizable.

1.5.5. Derivatives of Maps. Given a smooth function $F : M \to N$ we obtain a derivative or differential $DF|_p : T_pM \to T_{F(p)}N$. If we let $D = v = \dot{c}(0) \in T_pM$ represent a tangent vector, then

$$DF|_p(D) = D \circ F^*,$$

$$D_{DF|_p(v)} f = D_v f \circ F,$$

$$DF|_p(v) = \frac{d(F(c(t)))}{dt}|_{t=0}.$$

When using coordinates around $p \in M$ we can also create the partial derivatives

$$\frac{\partial F}{\partial x^i} \in TN$$

as the velocities of the x^i-curves for $F \circ x^{-1}$ where the other coordinates are kept constant, in fact

$$\frac{\partial F}{\partial x^i}|_p = DF\left(\frac{\partial}{\partial x^i}|_p\right).$$

Note that $\frac{\partial F}{\partial x^i}$ is a function from (a subset of) M to TN which at $p \in M$ is mapped to $T_{F(p)}N$.

These partial derivatives represent the columns in a matrix representation for DF since

$$DF(v) = DF\left(\frac{\partial}{\partial x^i} v^i\right) = DF\left(\frac{\partial}{\partial x^i}\right) v^i = \frac{\partial F}{\partial x^i} v^i.$$

If we also have coordinates at $F(p)$ in N, then we have

$$DF(v) = \frac{\partial F}{\partial x^i} v^i = \frac{\partial\left(y^j \circ F\right)}{\partial x^i} v^i \frac{\partial}{\partial y^j}.$$

So the matrix representation for DF is precisely the matrix of partial derivatives

$$[DF] = \left[\frac{\partial\left(y^j \circ F\right)}{\partial x^i}\right] = \left[\frac{\partial\left(y^j \circ F \circ x^{-1}\right)}{\partial x^i}\right].$$

We can now reformulate what it means for a smooth function to be an immersion or submersion.

DEFINITION 1.5.14. The smooth function $F : M \to N$ is an immersion if $DF|_p$ is injective for all $p \in M$. It is a submersion if $DF|_p$ is surjective for all $p \in M$.

REMARK 1.5.15. When we consider a map $F : M \to \mathbb{R}^k$, then we also have a differential

$$dF = \begin{bmatrix} dF^1 \\ \vdots \\ dF^k \end{bmatrix} : TM \to \mathbb{R}^k.$$

The identification $I : \mathbb{R}^k \times \mathbb{R}^k \to T\mathbb{R}^k$ defined by $I(p,v) = \frac{d}{dt}(p+tv)|_{t=0}$ shows that $DF = I(F, dF)$.

1.5.6. Vector Fields. A v*ector field* is a smooth map (called a *section*) $X : M \to TM$ such that $X|_p \in T_pM$. We use $X|_p$ instead of $X(p)$ as $X|_p$ is a derivation that can also be evaluated on function. In fact we note that we obtain a derivation

$$D_X : C^\infty(M) \to C^\infty(M)$$

by defining

$$(D_X f)(p) = D_{X|_p} f.$$

Conversely any such derivation corresponds to a vector field in the same way that tangent vectors correspond to derivations at a point.

In local coordinates we obtain

$$X = D_X\left(x^i\right)\frac{\partial}{\partial x^i}.$$

Given two vector fields X and Y we can construct their *Lie bracket* $[X,Y]$. Implicitly as a derivation

$$D_{[X,Y]} = D_X D_Y - D_Y D_X = [D_X, D_Y].$$

This clearly defines a linear map and is a derivation as

$$\begin{aligned} D_{[X,Y]}(fg) &= D_X(gD_Yf + fD_Yg) - D_Y(gD_Xf + fD_Xg) \\ &= D_XgD_Yf + D_XfD_Yg + gD_XD_Yf + fD_XD_Yg \\ &\quad -D_YgD_Xf + -D_YfD_Xg - gD_YD_Xf - fD_YD_Xg \\ &= g[D_X, D_Y]f + f[D_X, D_Y]g. \end{aligned}$$

In local coordinates this is conveniently calculated by ignoring second order partial derivatives:

$$\begin{aligned} \left[X^i\frac{\partial}{\partial x^i}, Y^j\frac{\partial}{\partial x^j}\right] &= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i} \\ &\quad +X^iY^j\frac{\partial^2}{\partial x^i\partial x^j} - Y^jX^i\frac{\partial^2}{\partial x^j\partial x^i} \\ &= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i} \\ &= \left(X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}. \end{aligned}$$

Since tangent vectors are also velocities to curves it would be convenient if vector fields had a similar interpretation. A curve $c(t)$ such that

$$\dot{c}(t) = X|_{c(t)}$$

is called an *integral curve* for X. Given an initial value $p \in M$, there is in fact a unique integral curve $c(t)$ such that $c(0) = p$ and it is defined on some maximal interval I that contains 0 as an interior point.

In local coordinates we can write $x^i \circ c(t) = x^i(t)$ and $X = v^i \frac{\partial}{\partial x^i}$. The condition that c is an integral curve then comes down to

$$\dot{c}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} = v^i \frac{\partial}{\partial x^i}$$

or more precisely

$$\frac{dx^i}{dt}(t) = v^i(c(t)).$$

This is a first order ODE and as such will have a unique solution given an initial value.

To get a maximal interval for an integral curve we have to use the local uniqueness of solutions and patch them together through a covering of coordinate charts.

We state the main theorem on integral curves that will be used again and again.

THEOREM 1.5.16. *Let X be a vector field on a manifold M. For each $p \in M$ there is a unique integral curve $c_p(t) : I_p \to M$ where $c_p(0) = p$, $\dot{c}_p(t) = X_{c_p(t)}$ for all $t \in I_p$, and I_p is the maximal open interval for any curve satisfying these two properties. Moreover, the map $(t,p) \mapsto c_p(t)$ is defined on an open subset of $\mathbb{R} \times M$ and is smooth. Finally, for given $p \in M$ the interval I_p either contains $[0,\infty)$ or $c_p(t)$ is not contained in a compact set as $t \to \infty$.*

PROOF. The first part is simply existence and uniqueness of solutions to ODEs. The second part is that such solutions depend smoothly on initial data. This is far more subtle to prove. The last statement is a basic compactness argument. □

We use the general notation that $\Phi_X^t(p) = c_p(t)$ is the flow corresponding to a vector field X, i.e.

$$\frac{d}{dt}\Phi_X^t = X|_{\Phi_X^t} = X \circ \Phi_X^t$$

Let $F : M^m \to N^n$ be a smooth map between manifolds. If X is a vector field on M and Y a vector field on N, then we say that X and Y are *F-related* provided $DF(X|_p) = Y|_{F(p)}$, or in other words $DF(X) = Y \circ F$. Given that tangent vectors are defined as derivations we note that it is equivalent to say that for all $f \in C^\infty(N)$ we have $(D_Y f) \circ F = D_X((f \circ F))$. In particular, when X_i are F-related to Y_i for $i = 1,2$, it follows that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

We can also relate this concept to the integral curves for the vector fields.

PROPOSITION 1.5.17. *X and Y are F-related iff $F \circ \Phi_X^t = \Phi_Y^t \circ F$ whenever both sides are defined.*

PROOF. Assuming that $F \circ \Phi_X^t = \Phi_Y^t \circ F$ we have

$$\begin{aligned} DF(X) &= DF\left(\frac{d}{dt}|_{t=0}\Phi_X^t\right) \\ &= \frac{d}{dt}|_{t=0}\left(F \circ \Phi_X^t\right) \\ &= \frac{d}{dt}|_{t=0}\left(\Phi_Y^t \circ F\right) \\ &= Y \circ \Phi_Y^0 \circ F \\ &= Y \circ F. \end{aligned}$$

Conversely $DF(X) = Y \circ F$ implies that

$$\begin{aligned} \frac{d}{dt}\left(F \circ \Phi_X^t\right) &= DF\left(\frac{d}{dt}\Phi_X^t\right) \\ &= DF\left(X|_{\Phi_X^t}\right) \\ &= Y|_{F\circ\Phi_X^t}. \end{aligned}$$

This shows that $t \mapsto F \circ \Phi_X^t$ is an integral curve for Y. At $t = 0$ it agrees with the integral curve $t \mapsto \Phi_Y^t \circ F$ so by uniqueness we obtain $F \circ \Phi_X^t = \Phi_Y^t \circ F$. □

1.5.7. Proper Submersions. In case F is a submersion it is possible to construct vector fields in M that are F-related to a given vector field in N.

PROPOSITION 1.5.18. *Assume that F is a submersion. Given a vector field Y in N, there are vector fields X in M that are F-related to Y.*

PROOF. First we do a local construction of X. Since F is a submersion proposition 1.4.15 shows that for each $p \in M$ there are charts $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that

$$y \circ F \circ x^{-1}\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^n\right).$$

This relationship evidently implies that $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial x^i}$ are F-related for $i = 1, \ldots, n$. Thus, if we write $Y = Y^i \frac{\partial}{\partial y^i}$, then we can simply define $X = \sum_{i=1}^n Y^i \circ F \frac{\partial}{\partial x^i}$. This gives the local construction.

For the global construction assume that we have a covering U_α, vector fields X_α on U_α that are F-related to Y, and a partition of unity λ_α subordinate to U_α. Then simply define $X = \sum \lambda_\alpha X_\alpha$ and note that

$$\begin{aligned} DF(X) &= DF\left(\sum \lambda_\alpha X_\alpha\right) \\ &= \sum \lambda_\alpha DF(X_\alpha) \\ &= \sum \lambda_\alpha Y \circ F \\ &= Y \circ F. \end{aligned}$$

□

Finally we can say something about the maximal domains of definition for the flows of F-related vector fields given F is proper.

PROPOSITION 1.5.19. *Assume that F is proper and that X and Y are F-related vector fields. If $F(p) = q$ and $\Phi_Y^t(q)$ is defined on $[0,b)$, then $\Phi_X^t(p)$ is also defined on $[0,b)$. In other words the relation $F \circ \Phi_X^t = \Phi_Y^t \circ F$ holds for as long as the RHS is defined.*

PROOF. Assume $\Phi_X^t(p)$ is defined on $[0,a)$. If $a < b$, then the set

$$\begin{aligned} K &= \left\{x \in M \mid F(x) = \Phi_Y^t(p) \text{ for some } t \in [0,a]\right\} \\ &= F^{-1}\left(\left\{\Phi_Y^t(p) \mid t \in [0,a]\right\}\right) \end{aligned}$$

is compact in M since F is proper. The integral curve $t \mapsto \Phi_X^t(q)$ lies in K since $F(\Phi_X^t(p)) = \Phi_Y^t(q)$. It is now a general result that maximally defined integral curves are either defined for all time or leave every compact set. In particular, $[0,a)$ is not the maximal interval on which $t \mapsto \Phi_X^t(p)$ is defined. □

These relatively simple properties lead to some very general and tricky results.

A *fibration* $F : M \to N$ is a smooth map which is locally trivial in the sense that for every $p \in N$ there is a neighborhood U of p such that $F^{-1}(U)$ is diffeomorphic to $U \times F^{-1}(p)$. This diffeomorphism must commute with the natural maps of these sets on to U. In other words $(x,y) \in U \times F^{-1}(p)$ must be mapped to a point in $F^{-1}(x)$. Note that it is easy to destroy the fibration property by simply deleting a point in M. Note also that in this context fibrations are necessarily submersions.

Special cases of fibrations are covering maps and vector bundles. The Hopf fibration $S^3 \to S^2 = \mathbb{P}^1$ is a more non trivial example of a fibration, which we shall study further below. Tubular neighborhoods are also examples of fibrations.

THEOREM 1.5.20 (Ehresman). *If $F : M \to N$ is a proper submersion, then it is a fibration.*

PROOF. As far as N is concerned this is a local result. In N we simply select a set U that is diffeomorphic to $\mathbb{R}^n$ and claim that $F^{-1}(U) \approx U \times F^{-1}(0)$. Thus we just need to prove the theorem in case $N = \mathbb{R}^n$, or more generally a coordinate box around the origin.

Next select vector fields $X_1, \ldots, X_n$ in M that are F-related to the coordinate vector fields $\partial_1, \ldots, \partial_n$. Our smooth map $G : \mathbb{R}^n \times F^{-1}(0) \to M$ is then defined by $G\left(t^1, \ldots, t^n, x\right) = \Phi_{X_1}^{t^1} \circ \cdots \circ \Phi_{X_n}^{t^n}(x)$. The inverse to this map is $G^{-1}(z) = \left(F(z), \Phi_{X_n}^{-t^n} \circ \cdots \circ \Phi_{X_1}^{-t^1}(z)\right)$, where $F(z) = \left(t^1, \ldots, t^n\right)$. □

The theorem also unifies several different results.

COROLLARY 1.5.21 (Basic Lemma in Morse Theory). *Let $F : M \to \mathbb{R}$ be a proper map. If F is regular on $(a,b) \subset \mathbb{R}$, then $F^{-1}(a,b) \simeq F^{-1}(c) \times (a,b)$ where $c \in (a,b)$.*

COROLLARY 1.5.22 (Reeb). *Let M be a closed manifold that admits a map with two critical points, then M is homeomorphic to a sphere. (This is a bit easier to show if we also assume that the critical points are nondegenerate.)*

Finally we can extend the fibration theorem to the case when M has boundary.

THEOREM 1.5.23. *Assume that M is a manifold with boundary and that N is a manifold without boundary, if $F : M \to N$ is proper and a submersion on M as well as on ∂M, then it is a fibration.*

PROOF. The proof is identical and reduced to the case when $N = \mathbb{R}^n$. The assumptions allow us to construct the lifted vector fields so that they are tangent to ∂M. The flows will then stay in ∂M or int$M$ for all time if they start there. □

REMARK 1.5.24. This theorem is sometimes useful when we have a submersion whose fibers are not compact. It is then occasionally possible to add a boundary to M so as to make the map proper. A good example is a tubular neighborhood around a closed submanifold $S \subset U$. By possibly making U smaller we can assume that it is a compact manifold with boundary such that the fibers of $U \to S$ are closed discs rather than open discs.

EXAMPLE 1.5.25. Consider the the projection $\mathbb{R}^2 \to \mathbb{R}$ onto the first axis. This is clearly a submersion and a trivial bundle. The standard vector field ∂_x on $\mathbb{R}$ can be lifted to the related field $\partial_x + y^2 \partial_y$ on $\mathbb{R}^2$. However, the integral curves for this lifted field are not complete as they are given by $\left(t + t_0, \frac{x_0}{1 - x_0(t + t_0)}\right)$ and diverge as t approaches $\frac{1}{x_0} - t_0$. In particular, neither the above proportion or theorem 1.5.20 can be made to work when the submersion isn't proper even though the submersion is a trivial fibration.

REMARK 1.5.26. There is also a very interesting converse problem: If M is a manifold and $\sim$ an equivalence relation on M when is $M/\sim$ a manifold and $M \to M/\sim$ a submersion? Clearly the equivalence classes must form a foliation and the leaves/equivalence classes be closed subsets of M. Also their normal bundles have to be trivial as preimages of regular values have trivial normal bundle.

The most basic and still very nontrivial case is that of a Lie group G and a subgroup H. The equivalence classes are the cosets gH in G and the quotient space is G/H. When H is dense in G the quotient topology is not even Hausdorff. However one can prove that if H is closed in G , so that the equivalence classes are all closed embedded submanifolds, then the quotient is a manifold and the quotient map a submersion.

A nasty example is $\mathbb{R}^2 - \{0\}$ with the equivalence relation being that two points are equivalent if they have the same x-coordinate and lie in the same component of the corresponding vertical line. This means that the above general assumptions are not sufficient as all equivalence classes are closed embedded submanifolds with trivial normal bundles. The quotient space is the line with double origin and so is not Hausdorff!

The key to getting a Hausdorff quotient is to assume that the graph of the equivalence relation

$$\{(x,y) \mid x \sim y\} \subset M \times M$$

is closed.

1.6. Embeddings

1.6.1. Embeddings into Euclidean Space. The goal is to show that any manifold is a proper submanifold of Euclidean space. This requires most importantly that we can find a way to reduce the dimension of the ambient Euclidean space into which the manifold can be embedded.

THEOREM 1.6.1 (Whitney Embedding, Dimension Reduction). *If $F : M^m \to \mathbb{R}^n$ is an injective immersion, then there is also an injective immersion $M^m \to \mathbb{R}^{2m+1}$. Moreover, if one of the coordinate functions of F is proper, then we can keep this property. In particular, when M is compact we obtain an embedding.*

PROOF. For each $v \in \mathbb{R}^n - \{0\}$ consider the orthogonal projection onto the orthogonal complement

$$f_v(x) = x - \frac{(x|v)\, v}{|v|^2}.$$

The image is an $(n-1)$-dimensional subspace. So if we can show that $f_v \circ F$ is an injective immersion, then the ambient dimension has been reduced by 1.

Note that $f_v \circ F(x) = f_v \circ F(y)$ iff $F(x) - F(y)$ is proportional to v. Similarly $d(f_v \circ F)(w) = 0$ iff $dF(w)$ is proportional to v.

As long as $2m+1 < n$ Sard's theorem implies that the union of the two images

$$\begin{aligned} H &: M \times M \times \mathbb{R} \to \mathbb{R}^n \\ h(x,y,t) &= t(F(x) - F(y)) \end{aligned}$$

$$\begin{aligned} G &: TM \to \mathbb{R}^n \\ G(w) &= dF(w) \end{aligned}$$

has dense complement. Therefore, we can select $v\in\mathbb{R}^n-(H(M\times M\times\mathbb{R})\cup G(TM))$.

Assuming $f_v\circ F(x)=f_v\circ F(y)$, we have $F(x)-F(y)=sv$. If $s=0$ this shows that $F(x)=F(y)$ and hence $x=y$. Otherwise $s\neq 0$ showing that $s^{-1}(F(x)-F(y))=v$ and hence that $v\in H(M\times M\times\mathbb{R})$.

Assuming $d(f_v\circ F)(w)=0$ we get that $dF(w)=sv$. If $s=0$, then $dF(w)=0$ and $w=0$. Otherwise $dF\left(s^{-1}w\right)=v$ showing that $v\in G(TM)$.

Note that the v we selected could be taken from $O-(H(M\times M\times\mathbb{R})\cup G(TM))$, where $O\subset\mathbb{R}^n$ is any open subset. This gives us a bit of extra information. While we can't get the ultimate map $M^m\to\mathbb{R}^{2m+1}$ to target a specific $(2m+1)$-dimensional subspace of $\mathbb{R}^n$, we can map it into a subspace arbitrarily close to a fixed subspace of dimension $2m+1$. To be specific simply assume that $\mathbb{R}^{2m+1}\subset\mathbb{R}^n$ consists of the first $2m+1$ coordinates. By selecting $v\in(-\varepsilon,\varepsilon)^{2m+1}\times(1-\varepsilon,1+\varepsilon)^{n-2m-1}$ we see that f_v changes the first coordinates with an error that is small.

This can be used to obtain proper maps $f_v\circ F$. When the first coordinate for F is proper, then $f_v\circ F$ is also proper provided v is not proportional to e_1. This means that we merely have to select $v\in\{|v|<2\mid(v\mid e_1)<\varepsilon\}$ to obtain a proper injective submersion. □

REMARK 1.6.2. Note also that if F starts out only being an immersion, then we can find an immersion into $\mathbb{R}^{2m}$. This is because $G(TM)\subset\mathbb{R}^n$ has measure zero as long as $n>2m$.

LEMMA 1.6.3. *If $A,B\subset M^m$ are open sets that both admit embeddings into $\mathbb{R}^{2m+1}$, then the union $A\cup B$ also admits an embedding into $\mathbb{R}^{2m+1}$.*

PROOF. Select a partition of unity $\lambda_A,\lambda_B:A\cup B\to[0,1]$, i.e., $\operatorname{supp}\lambda_A\subset A$, $\operatorname{supp}\lambda_B\subset B$, and $\lambda_A+\lambda_B=1$. Further, choose embeddings $F_A:A\to\mathbb{R}^{2m+1}$ and $F_B:B\to\mathbb{R}^{2m+1}$. Note multiplying these embeddings with our bump functions we obtain well-defined maps $\lambda_AF_A,\lambda_BF_B:A\cup B\to\mathbb{R}^{2m+1}$. This gives us a map

$$\begin{aligned}F &: A\cup B\to\mathbb{R}^{2m+1}\times\mathbb{R}^{2m+1}\times\mathbb{R}\times\mathbb{R},\\ F(x) &= (\lambda_A(x)F_A(x),\lambda_B(x)F_B(x),\lambda_A(x),\lambda_B(x)),\end{aligned}$$

which we claim is an injective immersion.

If $F(x)=F(y)$, then $\lambda_{A,B}(x)=\lambda_{A,B}(y)$. If, e.g., $\lambda_B(x)>0$ then $F_B(x)=F_B(y)$. This shows that $x=y$ as F_B is an injection.

If $dF(v)=0$ for $v\in T_pM$, then $d\lambda_{A,B}(v)=0$. So if, e.g., $\lambda_A(p)>0$, then by the product rule:

$$d(\lambda_AF_A)|_p=(d\lambda_A)|_pF_A(p)+\lambda_A(p)dF_A|_p=\lambda_A(p)dF_A|_p$$

and consequently

$$dF_A|_p(v)=0$$

showing that $v=0$.

If, in addition, we select a proper function $\rho:A\cup B\to[0,\infty)$, then we obtain a proper injective immersion

$$(\rho,F):A\cup B\to\mathbb{R}\times\mathbb{R}^{2m+1}\times\mathbb{R}^{2m+1}\times\mathbb{R}\times\mathbb{R}$$

and consequently an embedding. The dimension reduction result above then gives us a (proper) embedding into $\mathbb{R}^{2m+1}$. □

THEOREM 1.6.4 (Whitney Embedding, Final Version). *An m-dimensional manifold M admits a proper embedding into $\mathbb{R}^{2m+1}$.*

PROOF. We only need to check the hypotheses in theorem 1.3.11. Clearly the statement is invariant under diffeomorphisms and holds for $\mathbb{R}^m$. Condition (2) was established in the previous lemma. Condition (3) is almost trivial. Given embeddings $F_i : A_i \to \mathbb{R}^{2m+1}$, where $A_i \subset M$ are open and pairwise disjoint we can construct new embeddings $G_i : A_i \to \left(i, i+\frac{1}{2}\right)^{2m+1}$ with disjoint images. This yields an embedding $G : \bigcup_i A_i \to \mathbb{R}^{2m+1}$.

This shows that any m-manifold has an embedding into $\mathbb{R}^{2m+1}$. To obtain a proper embedding we select a proper function $\rho : M \to [0,\infty)$ and use the dimension reduction result on the proper embedding $(\rho, F) : M \to \mathbb{R}\times\mathbb{R}^{2m+1}$. □

1.6.2. Extending Embeddings.

LEMMA 1.6.5. *If $F : M \to N$ is an immersion that is an embedding when restricted to the embedded submanifold $S \subset M$, then F is an embedding on a neighborhood of S.*

PROOF. We only do the case where $\dim M = \dim N$. It is a bit easier and also the only case we actually need.

By assumption F is an open mapping as it is a local diffeomorphism. Thus it suffices to show that it is injective on a neighborhood of S. If it is not injective on any neighborhood, then we can find sequences x_i and y_i that approach S with $F(x_i) = F(y_i)$. If both sequences have accumulation points, then those points will lie in S and we can, by passing to subsequences, assume that they converge to points x and y in S. Then $F(x) = F(y)$ so $x = y$ and $x_i = y_i$ for large i as they lie in a neighborhood of $x = y$ where F is injective. If one or both of these sequences have no accumulation points, then it is possible to find a neighborhood of S that doesn't contain the sequence. This shows that we don't have to worry about the sequence. □

LEMMA 1.6.6. *If $M \subset \mathbb{R}^n$ is an embedded submanifold, then some neighborhood of the normal bundle of M in $\mathbb{R}^n$ is diffeomorphic to a neighborhood of M in $\mathbb{R}^n$.*

PROOF. The normal bundle is defined as

$$\nu(M\subset\mathbb{R}^n) = \left\{(v,p)\in T_p\mathbb{R}^n\times M \mid v\perp T_pM\right\}.$$

There is a natural map

$$\begin{array}{rcl}\nu(M\subset\mathbb{R}^n) & \to & \mathbb{R}^n, \\ (v,p) & \mapsto & v+p.\end{array}$$

One checks easily that this is a local diffeomorphism on some neighborhood of the zero section M and that it is clearly an embedding when restricted to the zero section. The previous lemma then shows that it is a diffeomorphism on a neighborhood of the zero section. □

THEOREM 1.6.7. *If $M \subset N$ is an embedded submanifold, then some neighborhood of the normal bundle of M in N is diffeomorphic to a neighborhood of M in N.*

PROOF. Any subbundle of $TN|_M$ that is transverse to TM is a normal bundle. It is easy to see that all such bundles are isomorphic. One specific choice comes from embedding $N \subset \mathbb{R}^n$ and then defining

$$\nu(M\subset N) = \left\{(v,p)\in T_pN\times M \mid v\perp T_pM\right\}.$$

We don't immediately get a map $\nu(M \subset N) \to N$. What we do, is to select a neighborhood $N \subset U \subset \mathbb{R}^n$ as in the previous lemma. The projection $\pi : U \to N$ that takes $w + q \in U$ to

$q \in N$ is a submersion deformation retraction. We then select a neighborhood $M \subset V \subset \nu(M \subset N)$ such that $v + p \in U$ if $(v, p) \in V$. Now we get a map

$$\begin{array}{rcl} V & \to & N, \\ (v,p) & \mapsto & \pi(v+p). \end{array}$$

that is a local diffeomorphism near the zero section and an embedding on the zero section.

□

1.7. Lie Groups

1.7.1. General Properties. A Lie group is a smooth manifold with a group structure that is also smooth, i.e., We have a manifold G with an associative multiplication $G \times G \to G$ that is smooth and inverse operation $G \to G$ that is smooth. A Lie group homomorphism is a homomorphism between Lie groups that is also smooth.

A Lie group is homogeneous in a canonical way as left translation by group elements: $L_g(x) = g \cdot x$ maps the identity element e to g. Conseqeuntly, $L_{gh^{-1}}$ maps h to g. Since left translation is a diffeomorphim it can be used to calculate the differential of Lie group homomorphisms if we know their differentials at the identity. Let $\phi : G_1 \to G_2$ be a Lie group homomorphism, then the homorphism property implies that

$$\phi \circ L_g = L_{\phi(g)} \circ \phi$$

and by the chain rule we obtain

$$D\phi \circ DL_g = DL_{\phi(g)} \circ D\phi$$

showing that

$$D\phi|_g = DL_{\phi(g)} \circ D\phi|_e \circ DL_{g^{-1}}$$

In particular, ϕ will be an immersion (or submersion) precisely when $D\phi|_e$ is injective (or surjective.)

THEOREM 1.7.1. *A surjective Lie group homomorphism with a differential that is bijective is a covering map. Moreover, when G is connected the kernel is central and in particular Abelian.*

PROOF. Consider a surjective Lie group homomorphism $\phi : G \to H$ whose differential is bijective. The kernel $\ker\phi$ is by definition the pre-image of the identity and by the regular value theorem a closed 0-dimensional submanifold of G. Thus we can select a neighborhood U around $e \in G$ that has compact closure and $\bar{U} \cap \ker\phi = \{e\}$ and that is mapped diffeomorphically to $\phi(U)$. It follows from continuity of the group multiplication and that inversion is a diffeormorphism that there is neighborhood around $e \in G$ such that $V^2 \subset U$ and $V^{-1} = V$ i.e., if $a, b \in V$ then $a \cdot b \in U$ and $a^{-1} \in V$. We claim that if $g, h \in \ker\phi$ and $g \cdot V \cap h \cdot V \neq \emptyset$, then $g = h$. In fact, if $g \cdot v_1 = h \cdot v_2$, then $g^{-1} \cdot h = v_2 \cdot v_1^{-1} \in U \cap \ker\phi$, which implies that $g^{-1} \cdot h = e$. In this way we have found disjoint open sets $g \cdot V$ for $g \in \ker\phi$ that are mapped diffeomorphically to $\phi(V)$. We claim that additionally $\phi^{-1}(\phi(V)) = \bigcup_{g \in \ker\phi} g \cdot V$. To see this let $\phi(x) = \phi(y)$ with $y \in V$. Then $g = xy^{-1} \in \ker\phi$ and $x \in gV$.

This shows that a neighborhood of $e \in H$ is evenly covered. Using left translations we can then show that all points in H are evenly covered.

Finally assume that G is connected. For a fixed $g \in G$ consider conjugation $x \to gxg^{-1}$. We say that x is central if it commutes with all elements in G and this comes down to checking that x is fixed by all conjugations. Now the kernel is already a normal subgroup of G and thus preserved by all conjugations. Consider a path $g(t)$ from $e \in G$ to $g \in G$, then for fixed x we obtain a path $g(t)\cdot x\cdot(g(t))^{-1}$. When $x\in\ker\phi$ this path is necessarily in $\ker\phi$ and starts at x. However, $\ker\phi$ is discrete and so the path must be constant. This shows that any $x\in\ker\phi$ commutes with all elements in G. □

There is also a converse to this result

THEOREM 1.7.2. *Let $f:\bar{G}\to G$ be a covering map, with $\bar{G}$ connected. If G is a Lie group, then $\bar{G}$ has a Lie group structure that makes f a homomorphism. Moreover, the fundamental group of a connected Lie group is Abelian.*

PROOF. The most important and simplest case is when $\bar{G}$ is simply connected. In that case we can simply use the unique lifting property to lift the map $\bar{G}\times\bar{G}\to G$ to a product structure on $\bar{G}$. Simmilarly we obtain the inverse structure. We then have to use the uniquness of lifts to establish associativity as we would otherwise obtain to different lifts for multiplying three elements $\bar{G}\times\bar{G}\times\bar{G}\to G$.

The general case now uses that the kernel of f becomes Abelian. This allows us after having developed the theory of covering spaces to conclude that any connected cover of G is a Galois cover, i.e., if $\tilde{G}\to G$ is the universal cover, then there is a covering map $\tilde{G}\to\bar{G}$ such that $\tilde{G}\to G$ is factored via $\tilde{G}\to\bar{G}\to G$. The group structure on $\bar{G}$ then comes from lifting $\bar{G}\times\bar{G}\to G$ to $\tilde{G}$ and then mapping it down to $\bar{G}$.

Finally, covering space theory shows that the fundamental group is also a group of deck transformations on the universal cover. Specifically the collection of all lifts of the projection $\tilde{G}\to G$. Composition and inverses of these lifts are simply new lifts and so they form a group. This is the fundamental group. However, left translation by elements in the kernel of $\tilde{G}\to G$ are lifts of the projection $\tilde{G}\to G$. The group structure on the kernel is preserved as composition of left translations so the deck transformations form an Abelian group. □

1.7.2. Matrix Groups. The most obvious examples of Lie groups are matrix groups starting with the general linear groups

$$Gl(n,\mathbb{R})\subset\mathrm{Mat}_{n\times n}(\mathbb{R}),$$
$$Gl(n,\mathbb{C})\subset\mathrm{Mat}_{n\times n}(\mathbb{C}).$$

These are open subsets of the vector space of $n\times n$ matrices and and the group operations are explicitly given in terms of multiplication and division of numbers. The determinant map $\det:\mathrm{Mat}_{n\times n}(\mathbb{F})\to\mathbb{F}$ is multiplicative and smooth, and the general linear group is in fact the open subset of matrices with non-zero determinant.

The derivative of the determinant is important to calculate. The determinant function is multi-linear in the columns of the matrix. So if we denote the identity matrix by I, then it follows that

$$\det(I+tX)=1+t(\mathrm{tr}X)+o(t)$$

and for $A\in Gl$ that

$$\det(A+tX)=\det A\left(1+t\left(\mathrm{tr}\left(A^{-1}X\right)\right)+o(t)\right).$$

In particular, all non-zero values in $\mathbb{F}-\{0\}$ are regular values of det. This gives us the special linear groups $Sl(n,\mathbb{F})$ of matrices with $\det=1$. The tangent space T_ISl is given as the kernel of the differential and is thus the space of traceless matrices:

$$T_ISl=\{X\in\mathrm{Mat}_{n\times n}\mid\mathrm{tr}X=0\}.$$

Using that the operation of taking adjoints $A\to A^*$ is smooth we obtain a smooth map $F:\mathrm{Mat}_{n\times n}(\mathbb{F})\to\mathrm{Sym}_n(\mathbb{F})$ defined by $A\to AA^*$ where $\mathrm{Sym}_n(\mathbb{F})$ denotes the real vector space of self-adjoint operators (symmetric or Hermitian depending on the field.) Note that the image of this map consists of the seft-adjoint matrices that are nonnegative definite, i.e., have nonnegative eigenvalues. The differential of this map at the identity can be found using

$$(I+tX)(I+tX^*) = I+t(X+X^*)+o(t)$$

to be

$$X+X^*.$$

This is clearly surjective since it is simply multiplication by 2 when restricted to $\mathrm{Sym}_n(\mathbb{F})$. More generally the differential at an invertible $A \in Gl$ is given by

$$XA^* + AX^*$$

which is also surjective as it is a bijection when retricted to the real subspace $\left\{X\left(A^{-1}\right)^* \mid X \in \mathrm{Sym}_n(\mathbb{F})\right\}$. Thus we obtain a submersion to the space of positive definite self-adjoint matrices:

$$F : Gl(n,\mathbb{F}) \to \mathrm{Sym}_n^+(\mathbb{F}).$$

Note that $\mathrm{Sym}_n^+(\mathbb{F}) \subset \mathrm{Sym}_n(\mathbb{F})$ is an open convex subset of a real vector space and diffeomorphic to a Euclidean space. Finally we observe that this submersion is also proper as $A_kA_k^* \to \infty$ when $A_k \to \infty$. In particular, we can use Ehresmann's theorem to conclude that $Gl(n,\mathbb{F})$ is diffeomorphic to $\mathrm{Sym}_n^+(\mathbb{F}) \times F^{-1}(I)$. The fiber over the identity is identifield with the orthogonal group:

$$O(n) = \{O \in Gl(n,\mathbb{R}) \mid OO^* = I\}$$

or the unitary group

$$U(n) = \{U \in Gl(n,\mathbb{C}) \mid UU^* = I\}$$

and are both compact Lie groups. We note that left translates $L_AF^{-1}(I) = A \cdot F^{-1}(I)$ are diffeomorphic to each other and $A \cdot F^{-1}(I) \subset F^{-1}(A)$. Thus fibers are precisely the lefttranslates of the orthogonal or unitary groups. This is the content of the polar decomposition for invertible matrices.

The tangent spaces to the orthogonal and unitary groups are given as the kernel of the differential of the map $A \to AA^*$ and are thus given by the skew-adjoint matrices

$$T_IO(n) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid X^* = -X\},$$

$$T_IU(n) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid X^* = -X\}.$$

These two families of groups can be intersected with the special linear groups to obtain the special orthogonal groups $SO(n) = O(n) \cap Sl(n,\mathbb{R})$ and the special unitary groups $SU(n) = U(n) \cap Sl(n,\mathbb{C})$. It is not immediately clear that these new groups have well-defined smooth structures as the intersections are not transverse. However, it follows from the canonical forms of orthogonal matrices that $SO(n)$ is the connected component of $O(n)$ that contains I. The other component consists of the orthogonal matrices with $\det = -1$. For the unitary group we obtain a Lie group homomorphism $\det : U(n) \to S^1 \subset \mathbb{C}$ where all values are regular values.

The tangent spaces are the traceless skew-adjoint matrices. In the real case skew-adjoint matrices are skew-symmeteric and thus automatically traceless, this conforms with $SO(n) \subset O(n)$ being open. In the complex case, the skew-adjoint matrices have purely imaginary entries on the diagonal so the additional assumption that they be traceless reduces the real dimension by 1, this conforms with 1 being a regular value of $\det : U(n) \to S^1$.

The matrix exponential map $\exp : \mathrm{Mat}_{n\times n}(\mathbb{F}) \to Gl(n,\mathbb{F})$ is defined using the usual power series expansion. The relationship

$$\det\exp(A) = \exp(\mathrm{tr}A)$$

shows that it image is in the general linear group and in case $\mathbb{F} = \mathbb{R}$ that it maps into the matrices with positive determinant.

It also commutes with the operation of taking adjoints $\exp A^* = (\exp A)^*$. This also shows that we obtain the following restrictions

$$\begin{aligned} \exp : T_I O(n) &\to SO(n), \\ \exp : T_I U(n) &\to U(n), \\ \exp : T_I SU(n) &\to SU(n), \end{aligned}$$

as well as

$$\exp : \mathrm{Sym}_n(\mathbb{F}) = T_I \mathrm{Sym}_n^+(\mathbb{F}) \to \mathrm{Sym}_n^+(\mathbb{F}).$$

These maps are all surjective. In all cases this uses that a matrix in the target can be conjugated to a nice canonical form: O^*CO where C is diagonal in the last three cases and has a block diagonal form in the first case that consists of 2×2 rotations and diagonal entries that are ± 1. In the unitary case the diagonal entries are of the form $e^{i\theta}$. Thus $C = \exp(iD)$, where D is a real diagonal matrix, and $O^*CO = O^*\exp(iD)\,O$. Similarly, in the last case C is a diagonal matric with positive entries and $C = \exp(D)$ for a unique diagonal matrix D with real entries. The first case is the most intricate. First observe that rotations do come from skew-symmetric matrices:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \exp\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}.$$

This also takes care of pairs of eigenvalues of the same sign as they correspond to rotations where $\theta = 0$ or π. Since elements in $SO(n)$ have determinant 1 we can always ensure that the real eigenvalues get paired up except when n is odd, in which case the remaining eigenvalues is 1.

The polar decomposition diffeomorphism $Gl(n,\mathbb{C}) \cong \mathrm{Sym}_n^+(\mathbb{R}) \times U(n)$ now tells us that $Gl(n,\mathbb{C})$ is connected. Similarly, $Gl^+(n,\mathbb{R}) \simeq \mathrm{Sym}_n^+(\mathbb{R}) \times SO(n)$ is connected. As the elements of $O(n)$ with determinant -1 are diffeomorphic to $SO(n)$ via multiplication by any reflection in a coordinate hyperplane it follows that $Gl(n,\mathbb{R})$ has presicely two connected components.

Finally, the exponential map also satisfies the law of exponents $\exp(A+B) = \exp A \exp B$ when A, B commute.

1.7.3. Low Dimensional Groups and Spheres. There are several interesting connections between low dimensional Lie groups and low dimensional spheres.

First we note that rotations in the plane are also complex multiplication by numbers on the unit circle$S^1 \subset \mathbb{C}$ so:

$$SO(2) = U(1) = S^1.$$

The 3-sphere can be thought of as the unit sphere $S^3 \subset \mathbb{C}^2$ and thus $S^3 = \left\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2\right\}$. However

$$SU(2) = U(2) \cap Sl(2,\mathbb{C}) = \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \in U(2) \mid z\bar{z} + w\bar{w} = 1 \right\}$$

so we have:

$$SU(2) = S^3.$$

Next we note that

$$\begin{aligned} SO(3) &= \left\{ \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \mid e_i \cdot e_j = \delta_{ij}, \det \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = 1 \right\} \\ &= \left\{ \begin{bmatrix} e_1 & e_2 & e_1 \times e_2 \end{bmatrix} \mid e_1 \cdot e_2 = 0, |e_1| = |e_2| = 1 \right\} \\ &= US^2 \end{aligned}$$

where $US^2 = \{(p,v) \mid |p| = |v| = 1, p \cdot v = 0\}$ is the set of unit tangent vectors.

There is a another important identification for this space

$$SO(3) = \mathbb{R}P^3.$$

This comes from exhibiting a homomorphism $SU(2) \to SO(3)$ whose kernel is $\{\pm I\}$. This shows that via the identification $SU(2) = S^3$ the preimages are precisely antipodal points. The specifics of the construction take a bit of work and will also lead us to quaternions. First make the identification

$$\mathbb{C}^2 = \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \mid (z,w) \in \mathbb{C}^2 \right\}.$$

On the right hand side we obtain a collection of matrices that is closed under addition and multiplication by real scalars. Since $\mathbb{C}$ is a commutative algebra the right hand side is also closed under multiplication. Thus it forms an algebra over $\mathbb{R}$. It is also a division algebra as non-zero elements have $\det = |z|^2 + |w|^2 > 0$ and thus have inverses. This is the algebra of quaternions also denoted $\mathbb{H}$. Any $A \in SU(2)$ acts by conjugation on this algebra by

$$A \cdot X = AXA^*.$$

In fact $X \mapsto AXA^*$ is an orthogonal transformation when we use the natural real inner product structure where

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ k = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

form an orthonormal basis. Note that these matrices each have Euclidean norm $\sqrt{2}$. So the inner product is scaled to make them have norm 1. The last matrix is defined so that we obtain

$$\begin{aligned} ij &= k = -ji, \\ jk &= i = -kj, \\ ki &= j = -ik, \\ i^2 = j^2 &= k^2 = -1. \end{aligned}$$

In fact conjugation fixes 1 so it also fixes the orthogonal complement spanned by i, j, k. Thus we obtain a homomorphism $SU(2) \to SO(3)$ by letting $A \in SU(2)$ act by conjuagtion on $\operatorname{span}_{\mathbb{R}} \{i, j, k\}$. The kernel of this map consists of matrices $A \in SU(2)$ that commute with all elements in $\mathbb{H}$ since $AX = XA$. This shows that such A must be homotheties and consequently the only possibilities are $\pm I = \pm 1$. It is also not hard to check that $SU(2) \to SO(3)$ is a submersion by calculating the differential at the identity. Thus the image is both open and closed and all of $SO(3)$. This shows that $SO(3) = \mathbb{R}P^3$.

From all of this we can derive the “Hairy Ball Theorem”:

THEOREM 1.7.3. *Every vector field on S^2 vanishes somewhere.*

PROOF. The proof is by contradiction. If we have a non-zero vector field, then we also have a unit vector field $p \mapsto (p, v(p)) \in US^2$. This gives us a diffeomorphism $SO(3) \to S^2 \times S^1 \subset S^2 \times \mathbb{R}^2$ by mapping each $[p, e_2, p \times e_2] \in SO(3)$ to

$$(p, e_2 \cdot v(p), (p \times e_2) \cdot v(p)).$$

This contradicts that $SO(3) = \mathbb{R}P^3$ as $S^2 \times S^1$ has a non-compact simply connected cover $S^2 \times \mathbb{R}$. □

1.8. Projective Space

Given a vector space V we define $\mathbb{P}(V)$ as the space of 1-dimensional subspaces or lines through the origin. It is called the projective space of V. In the special case were $V = \mathbb{F}^{n+1}$ we use the notation $\mathbb{P}\left(\mathbb{F}^{n+1}\right) = \mathbb{F}\mathbb{P}^n = \mathbb{P}^n$. This is a bit confusing in terms of notation. The point is that $\mathbb{P}^n$ is an n-dimensional space as we shall see below.

One can similarly develop a theory of the space of subspaces of any given dimension. The space of k-dimensional subspaces is denoted $G_k(V)$ and is called the Grassmannian.

1.8.1. Basic Geometry of Projective Spaces. The space of operators or endomorphisms on V is denoted $\text{End}(V)$ and the invertible operators or automorphisms by $\text{Aut}(V)$. When $V = \mathbb{F}^n$ these are represented by matrices $\text{End}(\mathbb{F}^n) = \text{Mat}(\mathbb{F})$ and $\text{Aut}(\mathbb{F}^n) = Gl_n(\mathbb{F})$. Since invertible operators map lines to lines we see that $\text{aut}(V)$ acts in a natural way on $\mathbb{P}(V)$. In fact this action is homogeneous, i.e., if we have $p, q \in \mathbb{P}(V)$, then there is an operator $A \in \text{aut}(V)$ such that $A(p) = q$. Moreover, as any two bases in V can be mapped to each other by invertible operators it follows that any collection of k independent lines $p_1, \ldots, p_k$, i.e., $p_1 + \cdots + p_k = p_1 \oplus \cdots \oplus p_k$ can be mapped to any collection of k independent lines $q_1, \ldots, q_k$. This means that the action of $\text{Aut}(V)$ on $\mathbb{P}(V)$ is k-point homogeneous for all $k \leq \dim(V)$. Note that this action is not effective, i.e., some transformations act trivially on $\mathbb{P}(V)$. Specifically, the maps that act trivially are precisely the homotheties $A = \lambda 1_V$.

Since an endomorphism might have a kernel it is not true that it maps lines to lines, however, if we have $A \in \text{end}(V)$, then we do get a map $A : \mathbb{P}(V) - \mathbb{P}(\ker A) \to \mathbb{P}(V)$ defined on lines that are not in the kernel of A.

Let us now assume that V is an inner product space with an inner product $\langle v, w \rangle$ that can be real or complex. The key observation in relation to subspaces is that they are completely characterized by the orthogonal projections onto the subspaces. Thus the space of k-dimensional subspaces is the same as the space of orthogonal projections of rank k. It is convenient to know that an endomorphism $E \in \text{End}(V)$ is an orthogonal projection iff it is a projection, $E^2 = E$ that is self-adjoint, $E^* = E$. In the case of a one dimensional subspace $p \in \mathbb{P}(V)$ spanned by a unit vector $v \in V$, the orthogonal projection is given by

$$\text{proj}_p(x) = \langle x, v \rangle v.$$

Clearly we get the same formula for all unit vectors in p. Note that the formula is quadratic in v. This yields a map $\mathbb{P}(V) \to \text{End}(V)$. This gives $\mathbb{P}(V)$ a natural topology and even a metric. One can also easily see that $\mathbb{P}(V)$ is compact.

The angle between lines in V gives a natural metric on $\mathbb{P}(V)$. Automorphisms clearly do not preserve angles between lines and so are not necessarily isometries. However if we restrict attention to unitary or orthogonal transformations $U \subset \text{Aut}(V)$, then we know that they preserve inner products of vectors. Therefore, they must also preserve angles between lines. Thus U acts by isometries on $\mathbb{P}(V)$. This action is again homogeneous so $\mathbb{P}(V)$ looks the same everywhere.

1.8.2. Coordinates in more Detail. We are now ready to coordinatize $\mathbb{P}(V)$. Select $p \in \mathbb{P}(V)$ and consider the set of lines $\mathbb{P}(V) - \mathbb{P}(p^{\perp})$ that are not perpendicular to p. This is clearly an open set in $\mathbb{P}(V)$ and we claim that there is a coordinate map G_p : $\text{Hom}(p, p^{\perp}) \to \mathbb{P}(V) - \mathbb{P}(p^{\perp})$. To construct this map decompose $V \simeq p \oplus p^{\perp}$ and note that any line not in $p^{\perp}$ is a graph over p given by a unique homomorphism in $(p, p^{\perp})$. The next thing to check is that G_p is a homeomorphism onto its image and is differentiable as a map into $\text{End}(V)$. Neither fact is hard to verify. Finally observe that $\text{Hom}(p, p^{\perp})$ is a vector space of dimension $\dim V - 1$. In this way $\mathbb{P}(V)$ becomes a manifold of dimension $\dim V - 1$.

In case we are considering $\mathbb{P}^n$ we can construct a more explicit coordinate map. First we introduce homogenous coordinates: select $z = (z^0, ..., z^n) \in \mathbb{F}^{n+1} - \{0\}$ denote the line by $[z^0 : \cdots : z^n] \in \mathbb{P}^n$, thus $[z^0 : \cdots : z^n] = [w^0 : \cdots : w^n]$ iff and only if z and w are proportional and hence generate the same line. If we let $p = [1 : 0 : \cdots : 0]$, then $\mathbb{F}^n \to \mathbb{P}^n$ is simply $G_p(z^1, ..., z^n) = [1 : z^1 : \cdots : z^n]$.

Keeping in mind that p is the only line perpendicular to all lines in $p^{\perp}$ we see that $\mathbb{P}^n - p$ can be represented by

$$\mathbb{P}^n - p = \left\{ [z : z^1 : \cdots : z^n] \mid (z^1, ..., z^n) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F} \right\}.$$

Here the subset

$$\mathbb{P}\left(p^{\perp}\right) = \left\{ [0 : z^1 : \cdots : z^n] \mid (z^1, ..., z^n) \in \mathbb{F}^n - \{0\} \right\}$$

can be identified with $\mathbb{P}^{n-1}$. Using the projection

$$\begin{aligned} R_0 &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \\ \ker(R_0) &= p \end{aligned}$$

we get a retract $R_0 : \mathbb{P}^n - p \to \mathbb{P}^{n-1}$, whose fibers are diffeomorphic to $\mathbb{F}$. Using the transformations

$$R_t = \begin{bmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

we see that R_0 is in fact a deformation retraction.

Finally we check the projective spaces in low dimensions. When $\dim V = 1$, $\mathbb{P}(V)$ is just a point and that point is in fact V it self. Thus $\mathbb{P}(V) = \{V\}$. When $\dim V = 2$, we note that for each $p \in \mathbb{P}(V)$ the orthogonal complement $p^{\perp}$ is again a one dimensional subspace and therefore an element of $\mathbb{P}(V)$. This gives us an involution $p \to p^{\perp}$ on $\mathbb{P}(V)$ just like the antipodal map on the sphere. In fact

$$\begin{aligned} \mathbb{P}(V) &= (\mathbb{P}(V) - \{p\}) \cup \left(\mathbb{P}(V) - \left\{p^{\perp}\right\}\right), \\ \mathbb{P}(V) - \{p\} &\simeq \mathbb{F} \simeq \mathbb{P}(V) - \left\{p^{\perp}\right\}, \\ \mathbb{F} - \{0\} &\simeq (\mathbb{P}(V) - \{p\}) \cap \left(\mathbb{P}(V) - \left\{p^{\perp}\right\}\right). \end{aligned}$$

Thus $\mathbb{P}(V)$ is simply a one point compactification of $\mathbb{F}$. In particular, we have that $\mathbb{RP}^1 \simeq S^1$ and $\mathbb{CP}^1 \simeq S^2$, (you need to convince your self that this is a diffeomorphism.) Since the geometry doesn't allow for distances larger than $\frac{\pi}{2}$ it is natural to suppose that these projective "lines" are spheres of radius $\frac{1}{2}$ in $\mathbb{F}^2$. This is in fact true.

1.8.3. Bundles. Define the tautological or canonical line bundle

$$\tau(\mathbb{P}^n) = \left\{(p,v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} \mid v \in p\right\}.$$

This is a natural subbundle of the trivial vector bundle $\mathbb{P}^n \times \mathbb{F}^{n+1}$ and therefore has a natural orthogonal complement

$$\tau^{\perp}(\mathbb{P}^n) \simeq \left\{(p,v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} \mid p \perp v\right\}$$

Note that in the complex case we are using Hermitian orthogonality. These are related to the tangent bundle in an interesting fashion

$$T\mathbb{P}^n \simeq \mathrm{Hom}\left(\tau(\mathbb{P}^n), \tau^{\perp}(\mathbb{P}^n)\right)$$

This identity comes from our coordinatization around a point $p \in \mathbb{P}^n$. We should check that these bundle are locally trivial, i.e., fibrations over $\mathbb{P}^n$. This is quite easy, for each $p \in \mathbb{P}^n$ we use the coordinate neighborhood around p and show that the bundles are trivial over these neighborhoods.

Note that the fibrations $\tau(\mathbb{P}^n) \to \mathbb{P}^n$ and $\mathbb{F}^{n+1} - \{0\} \to \mathbb{P}^n$ are suspiciously similar. The latter has fibers $p - \{0\}$ where the former has p. This means that the latter fibration can be identified with the nonzero vectors in $\tau(\mathbb{P}^n)$. This means that the missing 0 in $\mathbb{F}^{n+1} - \{0\}$ is replaced by the zero section in $\tau(\mathbb{P}^n)$ in order to create a larger bundle. This process is called a *blow up* of the origin in $\mathbb{F}^{n+1}$. Essentially we have a map $\tau(\mathbb{P}^n) \to \mathbb{F}^{n+1}$ that maps the zero section to 0 and is a bijection outside that. We can use $\mathbb{F}^{n+1} - \{0\} \to \mathbb{P}^n$ to create a new fibration by restricting it to the unit sphere $S \subset \mathbb{F}^{n+1} - \{0\}$.

The conjugate to the tautological bundle can also be seen internally in $\mathbb{P}^{n+1}$ as the map

$$\mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$$

When $p = [1:0:\cdots:0]$ this fibration was given by

$$\left[z : z^0 : \cdots : z^n\right] \to \left[z^0 : \cdots : z^n\right].$$

This looks like a vector bundle if we use fiberwise addition and scalar multiplication on z.

The equivalence is obtained by mapping

$$\mathbb{P}^{n+1} - \{[1:0:\cdots:0]\} \to \tau(\mathbb{P}^n),$$

$$\left[z : z^0 : \cdots : z^n\right] \to \left(\left[z^0 : \cdots : z^n\right], \bar{z}\frac{(z^0,\ldots,z^n)}{|(z^0,\ldots,z^n)|^2}\right)$$

It is necessary to conjugate z to get a well-defined map. This is why the identification is only conjugate linear. The conjugate to the tautological bundle can also be identified with the dual bundle $\hom(\tau(\mathbb{P}^n), \mathbb{C})$ via the natural inner product structure coming from $\tau(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{F}^{n+1}$. The relevant linear functional corresponding to $\left[z : z^0 : \cdots : z^n\right]$ is given by

$$v \to \left\langle v, \bar{z}\frac{(z^0,\ldots,z^n)}{|(z^0,\ldots,z^n)|^2}\right\rangle$$

This functional appears to be defined on all of $\mathbb{F}^{n+1}$, but as it vanishes on the orthogonal complement to $\left(z^0,...,z^n\right)$ we only need to consider the restriction to span $\left\{\left(z^0,...,z^n\right)\right\} = \left[z^0 : \cdots : z^n\right]$.

Finally we prove that these bundles are not trivial. In fact, we show that there can't be any smooth sections $F : \mathbb{P}^n \to S \subset \mathbb{F}^{n+1} - \{0\}$ such that $F(p) \in p$ for all p, i.e., it is not possible to find a smooth (or continuous) choice of basis for all 1-dimensional subspaces. Should such a map exist it would evidently be a lift of the identity on $\mathbb{P}^n$ to a map $\mathbb{P}^n \to S$. In case $\mathbb{F} = \mathbb{R}$, the map $S \to \mathbb{RP}^n$ is a nontrivial two fold covering map. So it is not possible to find $\mathbb{RP}^n \to S$ as a lift of the identity. In case $\mathbb{F} = \mathbb{C}$ the unit sphere S has larger dimension than $\mathbb{CP}^n$ so Sard's theorem tells us that $\mathbb{CP}^n \to S$ isn't onto. But then it is homotopic to a constant, thus showing that the identity $\mathbb{CP}^n \to \mathbb{CP}^n$ is homotopic to the constant map. We shall see below that this is not possible.

In effect, we proved that a fibration of a sphere $S \to B$ is nontrivial if either $\pi_1(B) \neq \{1\}$ or $\dim B < \dim S$.

1.8.4. Lefschetz Numbers. Finally we are going to study Lefschetz numbers for linear maps on projective spaces. The first general observation is that a map $A \in \mathrm{Aut}(V)$ has a fixed point $p \in \mathbb{P}(V)$ iff p is an invariant one dimensional subspace for A. In other words fixed points for A on $\mathbb{P}(V)$ correspond to eigenvectors, but without information about eigenvalues.

We start with the complex case as it is a bit simpler. The claim is that any $A \in \mathrm{Aut}(V)$ with distinct eigenvalues is a Lefschetz map on $\mathbb{P}(V)$ with $L(A) = \dim V$. Since such maps are diagonalizable we can restrict attention to $V = \mathbb{C}^{n+1}$ and the diagonal matrix

$$A = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

By symmetry we need only study the fixed point $p = [1 : 0 : \cdots : 0]$. Note that the eigenvalues are assumed to be distinct and none of then vanish. To check the action of A on a neighborhood of p we use the coordinates introduced above $\left[1 : z^1 : \cdots : z^n\right]$. We see that

$$\begin{aligned} A\left[1 : z^1 : \cdots : z^n\right] &= \left[\lambda_0 1 : \lambda_1 z^1 : \cdots : \lambda_n z^n\right] \\ &= \left[1 : \frac{\lambda_1}{\lambda_0} z^1 : \cdots : \frac{\lambda_n}{\lambda_0} z^n\right]. \end{aligned}$$

This is already (complex) linear in these coordinates so the differential at p must be represented by the complex $n \times n$ matrix

$$DA|_p = \begin{bmatrix} \frac{\lambda_1}{\lambda_0} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n}{\lambda_0} \end{bmatrix}.$$

As the eigenvalues are all distinct 1 is not an eigenvalue of this matrix, showing that A really is a Lefschetz map. Next we need to check the differential of $\det\left(I - DA|_p\right)$. Note that in [**Guillemin-Pollack**] the authors use the sign of $\det\left(DA|_p - I\right)$, but this is not consistent with Lefschetz' formula for the Lefschetz number as we shall see below. Since $Gl_n(\mathbb{C})$ is connected it must lie in $Gl_{2n}^{+}(\mathbb{R})$ as a real matrix, i.e., complex matrices always have positive determinant when viewed as real matrices. Since $DA|_p$ is complex it must follow that $\det\left(I - DA|_p\right) > 0$. So all local Lefschetz numbers are 1. This shows that $L(A) = n + 1$.

Since $Gl_{n+1}(\mathbb{C})$ is connected any linear map is homotopic to a linear Lefschetz map and must therefore also have Lefschetz number $n+1$.

In particular, we have shown that all invertible complex linear maps must have eigenvectors. Note that this fact is obvious for maps that are not invertible. This could be one of the most convoluted ways of proving the Fundamental Theorem of Algebra. We used the fact that $Gl_n(\mathbb{C})$ is connected. This in turn follows from the polar decomposition of matrices, which in turn follows from the Spectral Theorem. Finally we observe that the Spectral Theorem can be proven without invoking the Fundamental Theorem of Algebra.

The alternate observation that the above Lefschetz maps are dense in $Gl_n(\mathbb{C})$ is also quite useful in many situations.

The real projective spaces can be analyzed in a similar way but we need to consider the parity of the dimension as well as the sign of the determinant of the linear map.

For $A \in GL^+_{2n+2}(\mathbb{R})$ we might not have any eigenvectors whatsoever as A could be $n+1$ rotations. Since $GL^+_{2n+2}(\mathbb{R})$ is connected this means that $L(A)=0$ on $\mathbb{RP}^{2n+1}$ if $A \in GL^+_{2n+2}(\mathbb{R})$. When $A \in GL^-_{2n+2}(\mathbb{R})$ it must have at least two eigenvalues of opposite sign. Since $GL^-_{2n+2}(\mathbb{R})$ is connected we just need to check what happens for a specific

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & & & & & \\ 0 & -1 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & R \end{bmatrix} \end{aligned}$$

We have two fixed points

$$\begin{aligned} p &= [1:0:\cdots:0], \\ q &= [0:1:\cdots:0]. \end{aligned}$$

For p we can quickly guess that

$$DA_p = \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}.$$

This matrix doesn't have 1 as an eigenvalue and

$$\begin{aligned} \det\left(I - \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}\right) &= \det \begin{bmatrix} 2 & 0 \\ 0 & I-R \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & & & & & \\ & 1 & 1 & & & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & -1 & 1 \end{bmatrix} \\ &= 2^{n+1}. \end{aligned}$$

So we see that the determinant is positive. For q we use the coordinates $\left[z^0 : 1 : z^2 : \cdots : z^n\right]$ and easily see that the differential is

$$\begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$$

which also doesn't have 1 as an eigenvalue and again gives us positive determinant for $I - DA_q$. This shows that $L(A) = 2$ if $A \in GL^-_{2n+2}(\mathbb{R})$.

In case $A \in Gl_{2n+1}(\mathbb{R})$ it is only possible to compute the Lefschetz number mod 2 as $\mathbb{RP}^{2n}$ isn't orientable. We can select

$$A^{\pm} = \begin{bmatrix} \pm 1 & 0 \\ 0 & R \end{bmatrix} \in GL^{\pm}_{2n+1}(\mathbb{R})$$

with R as above. In either case we have only one fixed point and it is a Lefschetz fixed point since $DA^{\pm}_p = \pm R$. Thus $L(A^{\pm}) = 1$ and all $A \in G(2n+1, \mathbb{R})$ have $L(A) = 1$.

CHAPTER 2

Basic Tensor Analysis

2.1. Lie Derivatives and Its Uses

Let X be a vector field and $\Phi^t = \Phi^t_X$ the corresponding locally defined flow on a smooth manifold M. Thus $\Phi^t(p)$ is defined for small t and the curve $t \mapsto \Phi^t(p)$ is the integral curve for X that goes through p at $t = 0$. The Lie derivative of a tensor in the direction of X is defined as the first order term in a suitable Taylor expansion of the tensor when it is moved by the flow of X.

2.1.1. Definitions and Properties. Let us start with a function $f : M \to \mathbb{R}$. Then

$$f\left(\Phi^t(p)\right) = f(p) + t(L_X f)(p) + o(t),$$

where the Lie derivative $L_X f$ is just the directional derivative $D_X f = df(X)$. We can also write this as

$$\begin{aligned} f \circ \Phi^t &= f + tL_X f + o(t), \\ L_X f &= D_X f = df(X). \end{aligned}$$

When we have a vector field Y things get a little more complicated. We wish to consider $Y|_{\Phi^t}$, but this can't be directly compared to Y as the vectors live in different tangent spaces. Thus we look at the curve $t \to D\Phi^{-t}\left(Y|_{\Phi^t(p)}\right)$ that lies in T_pM. We can expand for t near 0 to get

$$D\Phi^{-t}\left(Y|_{\Phi^t(p)}\right) = Y|_p + t(L_X Y)|_p + o(t)$$

for some vector $(L_X Y)|_p \in T_pM$. If we compare this with proposition 1.5.17, then we see that $L_X Y$ measures how far Y is from being Φ^t related to itself for small t. This is made more precise in the next result.

PROPOSITION 2.1.1. *Consider two vector fields X, Y on M. The following are equivalent:*

(1) $\Phi^t_X \circ \Phi^s_Y = \Phi^s_Y \circ \Phi^t_X$,
(2) $D\Phi^t_X(Y) = Y \circ \Phi^t_X$, *i.e., Y is Φ^t_X-related to itself,*
(3) $L_X Y = 0$ *on M.*

PROOF. The fact that (1) and (2) are equivalent follows from proposition 1.5.17. The fact that (2) implies (3) follows from

$$L_X Y = \lim_{t \to 0} \frac{Y|_{\Phi^t_X} - D\Phi^t_X(Y)}{t}.$$

Conversely, consider the curve $c(t) = D\Phi_X^{-t}\left(Y|_{\Phi_X^t(p)}\right) \in T_pM$. Its velocity at t_0 is calculated by considering the difference:

$$\begin{aligned}
D\Phi_X^{-t}\left(Y|_{\Phi_X^t(p)}\right) - D\Phi_X^{-t_0}\left(Y|_{\Phi_X^{t_0}(p)}\right) &= D\Phi_X^{-t_0}\left(D\Phi^{-(t-t_0)}\left(Y|_{\Phi_X^{t-t_0}(\Phi_X^{t_0}(p))}\right)\right) - D\Phi_X^{-t_0}\left(Y|_{\Phi_X^{t_0}(p)}\right) \\
&= D\Phi_X^{-t_0}\left(D\Phi^{-(t-t_0)}\left(Y|_{\Phi_X^{t-t_0}(\Phi_X^{t_0}(p))}\right) - Y|_{\Phi_X^{t_0}(p)}\right) \\
&= D\Phi_X^{-t_0}\left((t-t_0) L_X Y|_{\Phi_X^{t_0}(p)} + o(t-t_0)\right) \\
&= o(t-t_0).
\end{aligned}$$

Showing that the curve is constant and consequently that (2) holds provided the Lie bracket vanishes. □

This Lie derivative of a vector field is in fact the Lie bracket.

PROPOSITION 2.1.2. *For vector fields X, Y on M we have*

$$L_X Y = [X, Y].$$

PROOF. We see that the Lie derivative satisfies

$$D\Phi^{-t}(Y|_{\Phi^t}) = Y + tL_X Y + o(t)$$

or equivalently

$$Y|_{\Phi^t} = D\Phi^t(Y) + tD\Phi^t(L_X Y) + o(t).$$

It is therefore natural to consider the directional derivative of a function f in the direction of $Y|_{\Phi^t} - D\Phi^t(Y)$.

$$\begin{aligned}
D_{\left(Y|_{\Phi^t} - D\Phi^t(Y)\right)} f &= D_{Y|_{\Phi^t}} f - D_{D\Phi^t(Y)} f \\
&= (D_Y f) \circ \Phi^t - D_Y\left(f \circ \Phi^t\right) \\
&= D_Y f + tD_X D_Y f + o(t) \\
&\quad - D_Y(f + tD_X f + o(t)) \\
&= t(D_X D_Y f - D_Y D_X f) + o(t) \\
&= tD_{[X,Y]} f + o(t).
\end{aligned}$$

This shows that

$$\begin{aligned}
L_X Y &= \lim_{t \to 0} \frac{Y|_{\Phi^t} - D\Phi^t(Y)}{t} \\
&= [X, Y].
\end{aligned}$$

□

We are now ready to define the Lie derivative of a $(0, p)$-tensor T and also give an algebraic formula for this derivative. We define

$$\left(\Phi^t\right)^* T = T + t(L_X T) + o(t)$$

or more precisely

$$\begin{aligned}
\left(\left(\Phi^t\right)^* T\right)(Y_1, ..., Y_p) &= T\left(D\Phi^t(Y_1), ..., D\Phi^t(Y_p)\right) \\
&= T(Y_1, ..., Y_p) + t(L_X T)(Y_1, ..., Y_p) + o(t).
\end{aligned}$$

PROPOSITION 2.1.3. *If X is a vector field and T a $(0, p)$-tensor on M, then*

$$(L_X T)(Y_1, ..., Y_p) = D_X(T(Y_1, ..., Y_p)) - \sum_{i=1}^{p} T(Y_1, ..., L_X Y_i, ..., Y_p)$$

PROOF. We restrict attention to the case where $p=1$. The general case is similar but requires more notation. Using that

$$Y|_{\Phi^t} = D\Phi^t(Y) + tD\Phi^t(L_X Y) + o(t)$$

we get

$$\begin{aligned}\left(\left(\Phi^t\right)^* T\right)(Y) &= T\left(D\Phi^t(Y)\right)\\ &= T\left(Y|_{\Phi^t} - tD\Phi^t(L_X Y)\right) + o(t)\\ &= T(Y)\circ\Phi^t - tT\left(D\Phi^t(L_X Y)\right) + o(t)\\ &= T(Y) + tD_X(T(Y)) - tT\left(D\Phi^t(L_X Y)\right) + o(t).\end{aligned}$$

Thus

$$\begin{aligned}(L_X T)(Y) &= \lim_{t\to 0}\frac{\left((\Phi^t)^* T\right)(Y) - T(Y)}{t}\\ &= \lim_{t\to 0}\left(D_X(T(Y)) - T\left(D\Phi^t(L_X Y)\right)\right)\\ &= D_X(T(Y)) - T(L_X Y).\end{aligned}$$

□

Finally we have that Lie derivatives satisfy all possible product rules. From the above propositions this is already obvious when multiplying functions with vector fields or $(0,p)$-tensors. However, it is less clear when multiplying tensors.

PROPOSITION 2.1.4. *Let T_1 and T_2 be $(0,p_i)$-tensors, then*

$$L_X(T_1\cdot T_2) = (L_X T_1)\cdot T_2 + T_1\cdot(L_X T_2).$$

PROOF. Recall that for 1-forms and more general $(0,p)$-tensors we define the product as

$$T_1\cdot T_2(X_1,\dots,X_{p_1},Y_1,\dots,Y_{p_2}) = T_1(X_1,\dots,X_{p_1})\cdot T_2(Y_1,\dots,Y_{p_2}).$$

The proposition is then a simple consequence of the previous proposition and the product rule for derivatives of functions. □

PROPOSITION 2.1.5. *Let T be a $(0,p)$-tensor and $f: M\to\mathbb{R}$ a function, then*

$$L_{fX}T(Y_1,\dots,Y_p) = fL_X T(Y_1,\dots,Y_p) + df(Y_i)\sum_{i=1}^{p} T(Y_1,\dots,X,\dots,Y_p).$$

PROOF. We have that

$$\begin{aligned}L_{fX}T(Y_1,\dots,Y_p) &= D_{fX}(T(Y_1,\dots,Y_p)) - \sum_{i=1}^{p} T\left(Y_1,\dots,L_{fX}Y_i,\dots,Y_p\right)\\ &= fD_X(T(Y_1,\dots,Y_p)) - \sum_{i=1}^{p} T(Y_1,\dots,[fX,Y_i],\dots,Y_p)\\ &= fD_X(T(Y_1,\dots,Y_p)) - f\sum_{i=1}^{p} T(Y_1,\dots,[X,Y_i],\dots,Y_p)\\ &\quad + df(Y_i)\sum_{i=1}^{p} T(Y_1,\dots,X,\dots,Y_p)\end{aligned}$$

□

The case where $X|_p = 0$ is of special interest when computing Lie derivatives. We note that $\Phi^t(p) = p$ for all t. Thus $D\Phi^t : T_pM \to T_pM$ and

$$\begin{aligned} L_X Y|_p &= \lim_{t\to 0} \frac{D\Phi^{-t}(Y|_p) - Y|_p}{t} \\ &= \frac{d}{dt}\left(D\Phi^{-t}\right)|_{t=0}(Y|_p). \end{aligned}$$

This shows that $L_X = \frac{d}{dt}\left(D\Phi^{-t}\right)|_{t=0}$ when $X|_p = 0$. From this we see that if θ is a 1-form, then $L_X\theta = -\theta \circ L_X$ at points p where $X|_p = 0$.

Before moving on to some applications of Lie derivatives we introduce the concept of interior product, it is simply evaluation of a vector field in the first argument of a tensor:

$$i_X T(X_1, ..., X_k) = T(X, X_1, ..., X_k)$$

We list 4 general properties of Lie derivatives.

$$\begin{aligned} L_{[X,Y]} &= L_X L_Y - L_Y L_X, \\ L_X(fT) &= L_X(f)T + fL_X T, \\ L_X[Y,Z] &= [L_X Y, Z] + [Y, L_X Z], \\ L_X(i_Y T) &= i_{L_X Y} T + i_Y(L_X T). \end{aligned}$$

2.1.2. Lie Groups. Lie derivatives also come in handy when working with Lie groups. For a Lie group G we have the inner automorphism $\mathrm{Ad}_h : x \to hxh^{-1}$ and its differential at $x = e$ denoted by the same letters

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}.$$

LEMMA 2.1.6. *The differential of $h \to \mathrm{Ad}_h$ is given by $U \to \mathrm{ad}_U(X) = [U, X]$*

PROOF. If we write $\mathrm{Ad}_h(x) = R_{h^{-1}}L_h(x)$, then its differential at $x = e$ is given by $\mathrm{Ad}_h = DR_{h^{-1}}DL_h$. Now let Φ^t be the flow for U. Then $\Phi^t(g) = g\Phi^t(e) = L_g(\Phi^t(e))$ as both curves go through g at $t = 0$ and have U as tangent everywhere since U is a left-invariant vector field. This also shows that $D\Phi^t = DR_{\Phi^t(e)}$. Thus

$$\begin{aligned} \mathrm{ad}_U(X)|_e &= \frac{d}{dt} DR_{\Phi^{-t}(e)} DL_{\Phi^t(e)}(X|_e)|_{t=0} \\ &= \frac{d}{dt} DR_{\Phi^{-t}(e)}\left(X|_{\Phi^t(e)}\right)|_{t=0} \\ &= \frac{d}{dt} D\Phi^{-t}\left(X|_{\Phi^t(e)}\right)|_{t=0} \\ &= L_U X = [U, X]. \end{aligned}$$

□

This is used in the next Lemma.

LEMMA 2.1.7. *Let $G = Gl(V)$ be the Lie group of invertible matrices on V. The Lie bracket structure on the Lie algebra $\mathfrak{gl}(V)$ of left invariant vector fields on $Gl(V)$ is given by commutation of linear maps. i.e., if $X, Y \in T_I Gl(V)$, then*

$$[X, Y]|_I = XY - YX.$$

PROOF. Since $x \mapsto hxh^{-1}$ is a linear map on the space $\hom(V,V)$ we see that $\mathrm{Ad}_h(X) = hXh^{-1}$. The flow of U is given by $\Phi^t(g) = g(I + tU + o(t))$ so we have

$$\begin{aligned}
[U,X] &= \frac{d}{dt}\left(\Phi^t(I)X\Phi^{-t}(I)\right)|_{t=0} \\
&= \frac{d}{dt}\left((I+tU+o(t))X(I-tU+o(t))\right)|_{t=0} \\
&= \frac{d}{dt}\left(X+tUX-tXU+o(t)\right)|_{t=0} \\
&= UX - XU.
\end{aligned}$$

□

2.1.3. The Hessian. Lie derivatives are also useful for defining Hessians of functions.

We start with a Riemannian manifold (M^m, g). The Riemannian structure immediately identifies vector fields with 1-froms. If X is a vector field, then the corresponding 1-form is denoted ω_X and is defined by

$$\omega_X(v) = g(X,v).$$

In local coordinates this looks like

$$\begin{aligned}
X &= a^i\partial_i, \\
\omega_X &= g_{ij}a^i dx^j.
\end{aligned}$$

This also tells us that the inverse operation in local coordinates looks like

$$\begin{aligned}
\phi &= a_j dx^j \\
&= \delta_j^k a_k dx^j \\
&= g_{ji}g^{ik}a_k dx^j \\
&= g_{ij}\left(g^{ik}a_k\right)dx^j
\end{aligned}$$

so the corresponding vector field is $X = g^{ik}a_k\partial_i$. If we introduce an inner product on 1-forms that makes this correspondence an isometry

$$g(\omega_X, \omega_Y) = g(X,Y).$$

Then we see that

$$\begin{aligned}
g\left(dx^i, dx^j\right) &= g\left(g^{ik}\partial_k, g^{jl}\partial_l\right) \\
&= g^{ik}g^{jl}g_{kl} \\
&= \delta_l^i g^{jl} \\
&= g^{ji} = g^{ij}.
\end{aligned}$$

Thus the inverse matrix to g_{ij}, the inner product of coordinate vector fields, is simply the inner product of the coordinate 1-forms.

With all this behind us we define the gradient $\mathrm{grad}f$ of a function f as the vector field corresponding to df, i.e.,

$$\begin{aligned}
df(v) &= g(\mathrm{grad}f, v), \\
\omega_{\mathrm{grad}f} &= df, \\
\mathrm{grad}f &= g^{ij}\partial_i f\partial_j.
\end{aligned}$$

This correspondence is a bit easier to calculate in orthonormal frames $E_1,\ldots,E_m$, i.e., $g(E_i,E_j)=\delta_{ij}$, such a frame can always be constructed from a general frame using the Gram-Schmidt procedure. We also have a dual frame $\phi^1,\ldots,\phi^m$ of 1-forms, i.e., $\phi^i(E_j)=\delta^i_j$. First we observe that

$$\phi^i(X)=g(X,E_i)$$

thus

$$\begin{aligned} X &= a^iE_i=\phi^i(X)E_i=g(X,E_i)E_i \\ \omega_X &= \delta_{ij}a^i\phi^j=a^i\phi^i=g(X,E_i)\phi^i \end{aligned}$$

In other words the coefficients don't change. The gradient of a function looks like

$$\begin{aligned} df &= a_i\phi^i=(D_{E_i}f)\phi^i, \\ \mathrm{grad}f &= g(\mathrm{grad}f,E_i)E_i=(D_{E_i}f)E_i. \end{aligned}$$

In Euclidean space we know that the usual Cartesian coordinates ∂_i also form an orthonormal frame and hence the differentials dx^i yield the dual frame of 1-forms. This makes it particularly simple to calculate in $\mathbb{R}^n$. One other manifold with the property is the torus T^n. In this case we don't have global coordinates, but the coordinates vector fields and differentials are defined globally. This is precisely what we are used to in vector calculus, where the vector field $X=P\partial_x+Q\partial_y+R\,\partial_z$ corresponds to the 1-form $\omega_X=Pdx+Qdy+Rdz$ and the gradient is given by $\partial_xf\partial_x+\partial_yf\partial_y+\partial_zf\partial_z$.

Having defined the gradient of a function the next goal is to define the Hessian of F. This is a bilinear form, like the metric, $\mathrm{Hess}f(X,Y)$ that measures the second order change of f. It is defined as the Lie derivative of the metric in the direction of the gradient. Thus it seems to measure how the metric changes as we move along the flow of the gradient

$$\mathrm{Hess}f(X,Y)=\frac{1}{2}\left(L_{\mathrm{grad}f}g\right)(X,Y)$$

We will calculate this in local coordinates to check that it makes some sort of sense:

$$\begin{aligned}
\mathrm{Hess}f(\partial_i,\partial_j) &= \frac{1}{2}\left(L_{\mathrm{grad}f}g\right)(\partial_i,\partial_j) \\
&= \frac{1}{2}L_{\mathrm{grad}f}g_{ij} - \frac{1}{2}g\left(L_{\mathrm{grad}f}\partial_i,\partial_j\right) - \frac{1}{2}g\left(\partial_i,L_{\mathrm{grad}f}\partial_j\right) \\
&= \frac{1}{2}L_{\mathrm{grad}f}g_{ij} - \frac{1}{2}g\left([\mathrm{grad}f,\partial_i],\partial_j\right) - \frac{1}{2}g\left(\partial_i,[\mathrm{grad}f,\partial_j]\right) \\
&= \frac{1}{2}L_{g^{kl}\partial_l f\partial_k}g_{ij} - \frac{1}{2}g\left(\left[g^{kl}\partial_l f\partial_k,\partial_i\right],\partial_j\right) - \frac{1}{2}g\left(\partial_i,\left[g^{kl}\partial_l f\partial_k,\partial_j\right]\right) \\
&= \frac{1}{2}g^{kl}\partial_l f\partial_k(g_{ij}) + \frac{1}{2}g\left(\partial_i\left(g^{kl}\partial_l f\right)\partial_k,\partial_j\right) + \frac{1}{2}g\left(\partial_i,\partial_j\left(g^{kl}\partial_l f\right)\partial_k\right) \\
&= \frac{1}{2}g^{kl}\partial_l f\partial_k(g_{ij}) + \frac{1}{2}\partial_i\left(g^{kl}\partial_l f\right)g_{kj} + \frac{1}{2}\partial_j\left(g^{kl}\partial_l f\right)g_{ik} \\
&= \frac{1}{2}g^{kl}\partial_k(g_{ij})\partial_l f + \frac{1}{2}\partial_i\left(g^{kl}\right)g_{kj}\partial_l f + \frac{1}{2}\partial_j\left(g^{kl}\right)g_{ik}\partial_l f \\
&\quad + \frac{1}{2}g^{kl}\partial_i(\partial_l f)g_{kj} + \frac{1}{2}g^{kl}\partial_j(\partial_l f)g_{ik} \\
&= \frac{1}{2}g^{kl}\partial_k(g_{ij})\partial_l f - \frac{1}{2}g^{kl}\partial_i(g_{kj})\partial_l f - \frac{1}{2}g^{kl}\partial_j(g_{ik})\partial_l f \\
&\quad + \frac{1}{2}\delta_j^l\partial_i\partial_l f + \frac{1}{2}\delta_i^l(\partial_j\partial_l f) \\
&= \frac{1}{2}g^{kl}\left(\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ik}\right)\partial_l f + \partial_i\partial_j f.
\end{aligned}$$

So if the metric coefficients are constant, as in Euclidean space, or we are at a critical point, this gives us the old fashioned Hessian.

It is worth pointing out that these more general definitions and formulas are useful even in Euclidean space. The minute we switch to some more general coordinates, such as polar, cylindrical, spherical etc, the metric coefficients are no longer all constant. Thus the above formulas are our only way of calculating the gradient and Hessian in such general coordinates. We also have the following interesting result that is often used in Morse theory.

LEMMA 2.1.8. *If a function $f : M \to \mathbb{R}$ has a critical point at p then the Hessian of f at p does not depend on the metric.*

PROOF. Assume that $X = \nabla f$ and $X|_p = 0$. Next select coordinates x^i around p such that the metric coefficients satisfy $g_{ij}|_p = \delta_{ij}$. Then we see that

$$\begin{aligned}
L_X\left(g_{ij}dx^i dx^j\right)|_p &= L_X(g_{ij})|_p + \delta_{ij}L_X\left(dx^i\right)dx^j + \delta_{ij}dx^i L_X\left(dx^j\right) \\
&= \delta_{ij}L_X\left(dx^i\right)dx^j + \delta_{ij}dx^i L_X\left(dx^j\right) \\
&= L_X\left(\delta_{ij}dx^i dx^j\right)|_p.
\end{aligned}$$

Thus $\mathrm{Hess}f|_p$ is the same if we compute it using g and the Euclidean metric in the fixed coordinate system. □

2.2. Operations on Forms

2.2.1. General Properties. Given p 1-forms $\omega_i \in \Omega^1(M)$ on a manifold M we define

$$(\omega_1 \wedge \cdots \wedge \omega_p)(v_1,...,v_p) = \det([\omega_i(v_j)])$$

where $[\omega_i(v_j)]$ is the matrix with entries $\omega_i(v_j)$. We can then extend the wedge product to all forms using linearity and associativity. This gives the *wedge product* operation

$$\begin{aligned}\Omega^p(M)\times\Omega^q(M) &\to \Omega^{p+q}(M),\\ (\omega,\psi) &\to \omega\wedge\psi.\end{aligned}$$

This operation is bilinear and antisymmetric in the sense that:

$$\omega\wedge\psi=(-1)^{pq}\,\psi\wedge\omega.$$

The wedge product of a function and a form is simply standard multiplication.

The exterior derivative of a form is defined by

$$\begin{aligned}d\omega(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i L_{X_i}\left(\omega\left(X_0,\ldots,\widehat{X}_i,\ldots,X_k\right)\right)\\ &\quad -\sum_{i<j}(-1)^i\,\omega\left(X_0,\ldots,\widehat{X}_i,\ldots,L_{X_i}X_j,\ldots,X_k\right)\\ &= \sum_{i=0}^{k}(-1)^i L_{X_i}\left(\omega\left(X_0,\ldots,\widehat{X}_i,\ldots,X_k\right)\right)\\ &\quad +\sum_{i<j}(-1)^{i+j}\,\omega\left(L_{X_i}X_j,X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_k\right)\\ &= \frac{1}{2}\sum_{i=0}^{k}(-1)^i\left(\begin{array}{c}(L_{X_i}\omega)\left(X_0,\ldots,\widehat{X}_i,\ldots,X_k\right)\\ +L_{X_i}\left(\omega\left(X_0,\ldots,\widehat{X}_i,\ldots,X_k\right)\right)\end{array}\right)\end{aligned}$$

Lie derivatives, interior products, wedge products and exterior derivatives on forms are related as follows:

$$\begin{aligned}d(\omega\wedge\psi) &= (d\omega)\wedge\psi+(-1)^p\,\omega\wedge(d\psi),\\ i_X(\omega\wedge\psi) &= (i_X\omega)\wedge\psi+(-1)^p\,\omega\wedge(i_X\psi),\\ L_X(\omega\wedge\psi) &= (L_X\omega)\wedge\psi+\omega\wedge(L_X\psi),\end{aligned}$$

and the composition properties

$$\begin{aligned}d\circ d &= 0,\\ i_X\circ i_X &= 0,\\ L_X &= d\circ i_X+i_X\circ d,\\ L_X\circ d &= d\circ L_X,\\ i_X\circ L_X &= L_X\circ i_X.\end{aligned}$$

The third property $L_X=d\circ i_X+i_X\circ d$ is also known a H. Cartan's formula (son of the geometer E. Cartan). It is behind the definition of exterior derivative we gave above in the form

$$i_{X_0}\circ d=L_{X_0}-d\circ i_{X_0}.$$

2.2.2. The Volume Form. We are now ready to explain how forms are used to unify some standard concepts from differential vector calculus. We shall work on a Riemannian manifold (M,g) and use orthonormal frames $E_1,\ldots,E_m$ as well as the dual frame $\phi^1,\ldots,\phi^m$ of 1-forms.

The local volume form is defined as:

$$d\mathrm{vol} = d\mathrm{vol}_g = \phi^1 \wedge \cdots \wedge \phi^m.$$

We see that if $\psi^1, ..., \psi^m$ is another collection of 1-forms coming from an orthonormal frame $F_1, ..., F_m$, then

$$\begin{aligned}
\psi^1 \wedge \cdots \wedge \psi^m (E_1, ..., E_m) &= \det\left(\psi^i (E_j)\right) \\
&= \det\left(g\left(F_i, E_j\right)\right) \\
&= \pm 1.
\end{aligned}$$

The sign depends on whether or not the two frames define the same orientation. In case M is oriented and we only use positively oriented frames we will get a globally defined volume form. Next we calculate the local volume form in local coordinates assuming that the frame and the coordinates are both positively oriented:

$$\begin{aligned}
d\mathrm{vol}\left(\partial_1, ...\partial_m\right) &= \det\left(\phi^i (\partial_j)\right) \\
&= \det\left(g\left(E_i, \partial_j\right)\right).
\end{aligned}$$

As E_i hasn't been eliminated we have to work a little harder. To this end we note that

$$\begin{aligned}
\det\left(g\left(\partial_i, \partial_j\right)\right) &= \det\left(g\left(g\left(\partial_i, E_k\right) E_k, g\left(\partial_j, E_l\right) E_l\right)\right) \\
&= \det\left(g\left(\partial_i, E_k\right) g\left(\partial_j, E_l\right) \delta_{kl}\right) \\
&= \det\left(g\left(\partial_i, E_k\right) g\left(\partial_j, E_k\right)\right) \\
&= \det\left(g\left(\partial_i, E_k\right)\right) \det\left(g\left(\partial_j, E_k\right)\right) \\
&= \left(\det\left(g\left(E_i, \partial_j\right)\right)\right)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
d\mathrm{vol}\left(\partial_1, ...\partial_m\right) &= \sqrt{\det g_{ij}}, \\
d\mathrm{vol} &= \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^m.
\end{aligned}$$

2.2.3. Divergence. The divergence of a vector field is defined as the change in the volume form as we flow along the vector field. Note the similarity with the Hessian.

$$L_X d\mathrm{vol} = \mathrm{div}(X)\, d\mathrm{vol}$$

In coordinates using that $X = a^i\partial_i$ we get

$$\begin{aligned}
L_X d\mathrm{vol} &= L_X\left(\sqrt{\det g_{kl}}dx^1\wedge\cdots\wedge dx^m\right)\\
&= L_X\left(\sqrt{\det g_{kl}}\right)dx^1\wedge\cdots\wedge dx^m\\
&\quad+\sqrt{\det g_{kl}}\sum_i dx^1\wedge\cdots\wedge L_X\left(dx^i\right)\wedge\cdots\wedge dx^m\\
&= a^i\partial_i\left(\sqrt{\det g_{kl}}\right)dx^1\wedge\cdots\wedge dx^m\\
&\quad+\sqrt{\det g_{kl}}\sum_i dx^1\wedge\cdots\wedge d\left(L_X x^i\right)\wedge\cdots\wedge dx^m\\
&= a^i\partial_i\left(\sqrt{\det g_{kl}}\right)dx^1\wedge\cdots\wedge dx^m\\
&\quad+\sqrt{\det g_{kl}}\sum_i dx^1\wedge\cdots\wedge d\left(a^i\right)\wedge\cdots\wedge dx^m\\
&= a^i\partial_i\left(\sqrt{\det g_{kl}}\right)dx^1\wedge\cdots\wedge dx^m\\
&\quad+\sqrt{\det g_{kl}}\sum_i dx^1\wedge\cdots\wedge\left(\partial_j a^i dx^j\right)\wedge\cdots\wedge dx^m\\
&\quad a^i\partial_i\left(\sqrt{\det g_{kl}}\right)dx^1\wedge\cdots\wedge dx^m\\
&\quad+\sqrt{\det g_{kl}}\sum_i dx^1\wedge\cdots\wedge\left(\partial_i a^i dx^i\right)\wedge\cdots\wedge dx^m\\
&= \left(a^i\partial_i\left(\sqrt{\det g_{kl}}\right)+\sqrt{\det g_{kl}}\partial_i a^i\right)dx^1\wedge\cdots\wedge dx^m\\
&= \frac{\partial_i\left(a^i\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}}\sqrt{\det g_{kl}}dx^1\wedge\cdots\wedge dx^m\\
&= \frac{\partial_i\left(a^i\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}}d\mathrm{vol}
\end{aligned}$$

We see again that in case the metric coefficients are constant we get the familiar divergence from vector calculus.

H. Cartan's formula for the Lie derivative of forms gives us a different way of finding the divergence

$$\begin{aligned}
\operatorname{div}(X)\,d\mathrm{vol} &= L_X d\mathrm{vol}\\
&= d i_X\left(d\mathrm{vol}\right)+i_X d\left(d\mathrm{vol}\right)\\
&= d i_X\left(d\mathrm{vol}\right),
\end{aligned}$$

in particular $\operatorname{div}(X)\,d\mathrm{vol}$ is always exact.

This formula suggests that we should study the correspondence that takes a vector field X to the $(n-1)$-form $i_X\left(d\mathrm{vol}\right)$. Using the orthonormal frame this correspondence is

$$\begin{aligned}
i_X\left(d\mathrm{vol}\right) &= i_{g(X,E_j)E_j}\left(\phi^1\wedge\cdots\wedge\phi^m\right)\\
&= g\left(X,E_j\right)i_{E_j}\left(\phi^1\wedge\cdots\wedge\phi^m\right)\\
&= \sum(-1)^{j+1}g\left(X,E_j\right)\phi^1\wedge\cdots\wedge\widehat{\phi^j}\wedge\cdots\wedge\phi^m
\end{aligned}$$

while in coordinates

$$\begin{aligned} i_X(d\mathrm{vol}) &= i_{a^j\partial_j}\left(\sqrt{\det g_{kl}}\,dx^1\wedge\cdots\wedge dx^m\right)\\ &= \sqrt{\det g_{kl}}\sum a^j i_{\partial_j}\left(dx^1\wedge\cdots\wedge dx^m\right)\\ &= \sqrt{\det g_{kl}}\sum(-1)^{j+1}a^j dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^m \end{aligned}$$

If we compute $di_X(d\mathrm{vol})$ using this formula we quickly get back our coordinate formula for $\mathrm{div}(X)$.

In vector calculus this gives us the correspondence

$$\begin{aligned} i_{(P\partial_x+Q\partial_y+R\partial_z)}dx\wedge dy\wedge dz &= Pi_{\partial_x}dx\wedge dy\wedge dz\\ &\quad +Qi_{\partial_y}dx\wedge dy\wedge dz\\ &\quad +Ri_{\partial_z}dx\wedge dy\wedge dz\\ &= Pdy\wedge dz-Qdx\wedge dz+Rdx\wedge dy\\ &= Pdy\wedge dz+Qdz\wedge dx+Rdx\wedge dy \end{aligned}$$

If we compose the grad and div operations we get the Laplacian:

$$\mathrm{div}(\mathrm{grad}f)=\Delta f$$

For this to make sense we should check that it is the "trace" of the Hessian. This is most easily done using an orthonormal frame E_i. On one hand the trace of the Hessian is:

$$\begin{aligned} \sum_i \mathrm{Hess}f(E_i,E_i) &= \sum_i\frac{1}{2}\left(L_{\mathrm{grad}f}g\right)(E_i,E_i)\\ &= \sum_i\frac{1}{2}L_{\mathrm{grad}f}(g(E_i,E_i))-\frac{1}{2}g\left(L_{\mathrm{grad}f}E_i,E_i\right)-\frac{1}{2}g\left(E_i,L_{\mathrm{grad}f}E_i\right)\\ &= -\sum_i g\left(L_{\mathrm{grad}f}E_i,E_i\right). \end{aligned}$$

While the divergence is calculated as

$$\begin{aligned} \mathrm{div}(\mathrm{grad}f) &= \mathrm{div}(\mathrm{grad}f)\,d\mathrm{vol}(E_1,...,E_m)\\ &= \left(L_{\mathrm{grad}f}\phi^1\wedge\cdots\wedge\phi^m\right)(E_1,...,E_m)\\ &= \sum\left(\phi^1\wedge\cdots\wedge L_{\mathrm{grad}f}\phi^i\wedge\cdots\wedge\phi^m\right)(E_1,...,E_m)\\ &= \sum\left(L_{\mathrm{grad}f}\phi^i\right)(E_i)\\ &= \sum L_{\mathrm{grad}f}\left(\phi^i(E_i)\right)-\phi^i\left(L_{\mathrm{grad}f}E_i\right)\\ &= -\sum\phi^i\left(L_{\mathrm{grad}f}E_i\right)\\ &= -\sum g\left(L_{\mathrm{grad}f}E_i,E_i\right). \end{aligned}$$

2.2.4. Curl. While the gradient and divergence operations work on any Riemannian manifold, the curl operator is specific to oriented 3 dimensional manifolds. It uses the above two correspondences between vector fields and 1-forms as well as 2-forms:

$$d(\omega_X)=i_{\mathrm{curl}X}(d\mathrm{vol})$$

If $X=P\partial_x+Q\partial_y+R\partial_z$ and we are on $\mathbb{R}^3$ we can easily see that

$$\mathrm{curl}X=(\partial_yR-\partial_zQ)\,\partial_x+(\partial_zP-\partial_xR)\,\partial_y+(\partial_xQ-\partial_yP)\,\partial_z$$

Taken together these three operators are defined as follows:

$$\begin{aligned} \omega_{\mathrm{grad}f} &= df, \\ i_{\mathrm{curl}X}(d\mathrm{vol}) &= d(\omega_X), \\ \mathrm{div}(X)\,d\mathrm{vol} &= di_X(d\mathrm{vol}). \end{aligned}$$

Using that $d \circ d = 0$ on all forms we obtain the classical vector analysis formulas

$$\begin{aligned} \mathrm{curl}(\mathrm{grad}f) &= 0, \\ \mathrm{div}(\mathrm{curl}X) &= 0, \end{aligned}$$

from

$$\begin{aligned} i_{\mathrm{curl}(\mathrm{grad}f)}(d\mathrm{vol}) &= d\left(\omega_{\mathrm{grad}f}\right) = ddf, \\ \mathrm{div}(\mathrm{curl}X)\,d\mathrm{vol} &= di_{\mathrm{curl}X}(d\mathrm{vol}) = dd\omega_X. \end{aligned}$$

2.3. Orientability

Recall that two ordered bases of a finite dimensional vector space are said to represent the same orientation if the transition matrix from one to the other is of positive determinant. This evidently defines an equivalence relation with exactly two equivalence classes. A choice of such an equivalence class is called an orientation for the vector space.

Given a smooth manifold each tangent space has two choices for an orientation. Thus we obtain a two fold covering map $O_M \to M$, where the preimage of each $p \in M$ consists of the two orientations for T_pM. A connected manifold is said to be *orientable* if the orientation covering is disconnected. For a disconnected manifold, we simply require that each connected component be connected. A choice of sheet in the covering will correspond to a choice of an orientation for each tangent space.

To see that O_M really is a covering just note that if we have a chart $\left(x^1, x^2, ..., x^n\right) : U \subset M \to \mathbb{R}^n$, where U is connected, then we have two choices of orientations over U, namely, the class determined by the framing $(\partial_1, \partial_2, ..., \partial_n)$ and by the framing $(-\partial_1, \partial_2, ..., \partial_n)$. Thus U is covered by two sets each diffeomorphic to U and parametrized by these two different choices of orientation. Observe that this tells us that $\mathbb{R}^n$ is orientable and has a canonical orientation given by the standard Cartesian coordinate frame $(\partial_1, \partial_2, ..., \partial_n)$.

Note that since simply connected manifolds only have trivial covering spaces they must all be orientable. Thus S^n, $n > 1$ is always orientable.

An other important observation is that the orientation covering O_M is an orientable manifold since it is locally the same as M and an orientation at each tangent space has been picked for us.

THEOREM 2.3.1. *The following conditions for a connected n-manifold M are equivalent.*

1. M is orientable.
2. Orientation is preserved moving along loops.
3. M admits an atlas where the Jacobians of all the transitions functions are positive.
4. M admits a nowhere vanishing n-form.

PROOF. $1 \Leftrightarrow 2$: The unique path lifting property for the covering $O_M \to M$ tells us that orientation is preserved along loops if and only if O_M is disconnected.

$1 \Rightarrow 3$: Pick an orientation. Take any atlas (U_α, F_α) of M where U_α is connected. As in our description of O_M from above we see that either each F_α corresponds to the chosen orientation, otherwise change the sign of the first component of F_α. In this way we get an atlas where each chart corresponds to the chosen orientation. Then it is easily checked that the transition functions $F_\alpha \circ F_\beta^{-1}$ have positive Jacobian as they preserve the canonical orientation of $\mathbb{R}^n$.

$3 \Rightarrow 4$: Choose a locally finite partition of unity (λ_α) subordinate to an atlas (U_α, F_α) where the transition functions have positive Jacobians. On each U_α we have the nowhere vanishing form $\omega_\alpha = dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n$. Now note that if we are in an overlap $U_\alpha \cap U_\beta$ then

$$\begin{aligned} dx_\alpha^1 \wedge \ldots \wedge dx_\alpha^n \left(\frac{\partial}{\partial x_\beta^1}, \ldots, \frac{\partial}{\partial x_\beta^n} \right) &= \det \left(dx_\alpha^i \left(\frac{\partial}{\partial x_\beta^j} \right) \right) \\ &= \det \left(D \left(F_\alpha \circ F_\beta^{-1} \right) \right) \\ &> 0. \end{aligned}$$

Thus the globally defined form $\omega = \sum \lambda_\alpha \omega_\alpha$ is always nonnegative when evaluated on $\left(\frac{\partial}{\partial x_\beta^1}, \ldots, \frac{\partial}{\partial x_\beta^n} \right)$. What is more, at least one term must be positive according to the definition of partition of unity.

$4 \Rightarrow 1$: Pick a nowhere vanishing n-form ω. Then define two sets $O_\pm$ according to whether ω is positive or negative when evaluated on a basis. This yields two disjoint open sets in O_M which cover all of M. □

With this result behind us we can try to determine which manifolds are orientable and which are not. Conditions 3 and 4 are often good ways of establishing orientability. To establish non-orientability is a little more tricky. However, if we suspect a manifold to be non-orientable then 1 tells us that there must be a non-trivial 2-fold covering map $\pi : \hat{M} \to M$, where $\hat{M}$ is oriented and the two given orientations at points over $p \in M$ are mapped to different orientations in M via $D\pi$. A different way of recording this information is to note that for a two fold covering $\pi : \hat{M} \to M$ there is only one nontrivial deck transformation $A : \hat{M} \to \hat{M}$ with the properties: $A(x) \neq x$, $A \circ A = id_M$, and $\pi \circ IA = \pi$. With this is mind we can show

PROPOSITION 2.3.2. *Let $\pi : \hat{M} \to M$ be a non-trivial 2-fold covering and $\hat{M}$ an oriented manifold. Then M is orientable if and only if A preserves the orientation on $\hat{M}$.*

PROOF. First suppose A preserves the orientation of $\hat{M}$. Then given a choice of orientation $e_1, \ldots, e_n \in T_x\hat{M}$ we can declare $D\pi(e_1), \ldots, D\pi(e_n) \in T_{\pi(x)}M$ to be an orientation at $\pi(x)$. This is consistent as $DA(e_1), \ldots, DA(e_n) \in T_{I(x)}\hat{M}$ is mapped to $D\pi(e_1), \ldots, D\pi(e_n)$ as well (using $\pi \circ A = \pi$) and also represents the given orientation on $\hat{M}$ since A was assumed to preserve this orientation.

Suppose conversely that M is orientable and choose an orientation for M. Since we assume that both $\hat{M}$ and M are connected the projection $\pi : \hat{M} \to M$, being nonsingular everywhere, must always preserve or reverse the orientation. We can without loss of generality assume that the orientation is preserved. Then we just use $\pi \circ A = \pi$ as in the first part of the proof to see that A must preserve the orientation on $\hat{M}$. □

We can now use these results to check some concrete manifolds for orientability.

We already know that $S^n, n > 1$ are orientable, but what about S^1? One way of checking that this space is orientable is to note that the tangent bundle is trivial and thus a uniform choice of orientation is possible. This clearly generalizes to Lie groups and other parallelizable manifolds. Another method is to find a nowhere vanishing form. This can be done on all spheres S^n by considering the n-form

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$

on $\mathbb{R}^{n+1}$. This form is a generalization of the 1-form $xdy - ydx$, which is $\pm$ the angular form in the plane. Note that if $X = x^i \partial_i$ denotes the radial vector field, then we have (see also the section below on the classical integral theorems)

$$i_X \left(dx^1 \wedge \cdots \wedge dx^{n+1} \right) = \omega.$$

From this it is clear that if $v_2, ..., v_n$ form a basis for a tangent space to the sphere, then

$$\begin{aligned} \omega\left(v_2, ..., v_n\right) &= dx^1 \wedge \cdots \wedge dx^{n+1} \left(X, v_2, ..., v_{n+1}\right) \\ &\neq 0. \end{aligned}$$

Thus we have found a nonvanishing n-form on all spheres regardless of whether or not they are parallelizable or simply connected. As another exercise people might want to use one of the several coordinate atlases known for the spheres to show that they are orientable.

Recall that $\mathbb{R}P^n$ has S^n as a natural double covering with the antipodal map as a natural deck transformation. Now this deck transformation preserves the radial field $X = x^i \partial_i$ and thus its restriction to S^n preserves or reverses orientation according to what it does on $\mathbb{R}^{n+1}$. On the ambient Euclidean space the map is linear and therefore preserves the orientation iff its determinant is positive. This happens iff $n+1$ is even. Thus we see that $\mathbb{R}P^n$ is orientable iff n is odd.

Using the double covering lemma show that the Klein bottle and the Möbius band are non-orientable.

Manifolds with boundary are defined like manifolds, but modeled on open sets in $L^n = \left\{ x \in \mathbb{R}^n \mid x^1 \leq 0 \right\}$. The boundary ∂M is then the set of points that correspond to elements in $\partial L^n = \left\{ x \in \mathbb{R}^n \mid x^1 = 0 \right\}$. It is not hard to prove that if $F : M \to \mathbb{R}$ has $a \in \mathbb{R}$ as a regular value then $F^{-1}(-\infty, a]$ is a manifold with boundary. If M is oriented then the boundary is oriented in such a way that if we add the outward pointing normal to the boundary as the first basis vector then we get a positively oriented basis for M. Thus $\partial_2, ..., \partial_n$ is the positive orientation for ∂L^n since ∂_1 points away from L^n and $\partial_1, \partial_2, ..., \partial_n$ is the usual positive orientation for L^n.

2.4. Integration of Forms

We shall assume that M is an oriented n-manifold. Thus, M comes with a covering of charts $\varphi_\alpha = \left(x^1_\alpha, \ldots, x^n_\alpha\right) : U_\alpha \longleftrightarrow B(0,1) \subset \mathbb{R}^n$ such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ preserve the usual orientation on Euclidean space, i.e., $\det\left(D\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)\right) > 0$. In addition, we shall also assume that a partition of unity with respect to this covering is given. In other words, we have smooth functions $\phi_\alpha : M \to [0,1]$ such that $\phi_\alpha = 0$ on $M - U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. For the last condition to make sense, it is obviously necessary that the covering be also locally finite.

Given an n-form ω on M we wish to define:

$$\int_M \omega.$$

When M is not compact, it might be necessary to assume that the form has compact support, i.e., it vanishes outside some compact subset of M.

In each chart we can write

$$\omega = f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

Using the partition of unity, we then obtain

$$\begin{aligned} \omega &= \sum_\alpha \phi_\alpha \omega \\ &= \sum_\alpha \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \end{aligned}$$

where each of the forms $\phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ has compact support in U_α. Since U_α is identified with $\bar{U}_\alpha \subset \mathbb{R}^n$, we simply declare that

$$\int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \int_{\bar{U}_\alpha} \phi_\alpha f_\alpha dx^1 \cdots dx^n.$$

Here the right-hand side is simply the integral of the function $\phi_\alpha f_\alpha$ viewed as a function on $\bar{U}_\alpha$. Then we define

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

whenever this sum converges. Using the standard change of variables formula for integration on Euclidean space, we see that indeed this definition is independent of the choice of coordinates.

With these definitions behind us, we can now state and prove Stokes' theorem for manifolds with boundary.

THEOREM 2.4.1. *For any $\omega \in \Omega^{n-1}(M)$ with compact support we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

PROOF. If we use the trick

$$d\omega = \sum_\alpha d(\phi_\alpha \omega),$$

then we see that it suffices to prove the theorem in the case $M = L^n$ and ω has compact support. In that case we can write

$$\omega = \sum_{i=1}^n f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

The differential of ω is now easily computed:

$$\begin{aligned} d\omega &= \sum_{i=1}^n (df_i) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n \left(\frac{\partial f_i}{\partial x^i} \right) dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus,

$$\begin{aligned}\int_{L^n} d\omega &= \int_{L^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{L^n} \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int \left(\int \left(\frac{\partial f_i}{\partial x^i} \right) dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n.\end{aligned}$$

The fundamental theorem of calculus tells us that

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\frac{\partial f_i}{\partial x^i} \right) dx^i &= 0, \text{ for } i > 1, \\ \int_{-\infty}^{0} \left(\frac{\partial f_1}{\partial x^1} \right) dx^1 &= f_1 \left(0, x^2, \ldots, x^n \right).\end{aligned}$$

Thus

$$\int_{L^n} d\omega = \int_{\partial L^n} f_1 \left(0, x^2, \ldots, x^n \right) dx^2 \wedge \cdots \wedge dx^n.$$

Since $dx^1 = 0$ on ∂L^n it follows that

$$\omega|_{\partial L^n} = f_1 dx^2 \wedge \cdots \wedge dx^n.$$

This proves the theorem. □

We get a very nice corollary out of Stokes' theorem.

THEOREM. (Brouwer) *Let M be a connected compact manifold with nonempty boundary. Then there is no retract $r : M \to \partial M$.*

PROOF. Note that if ∂M is not connected such a retract clearly can't exists so we need only worry about having connected boundary.

If M is oriented and ω is a volume form on ∂M, then we have

$$\begin{aligned}0 &< \int_{\partial M} \omega \\ &= \int_{\partial M} r^* \omega \\ &= \int_M d\left(r^* \omega \right) \\ &= \int_M r^* d\omega \\ &= 0.\end{aligned}$$

If M is not orientable, then we lift the situation to the orientation cover and obtain a contradiction there. □

We shall briefly discuss how the classical integral theorems of Green, Gauss, and Stokes follow from the general version of Stokes' theorem presented above.

Green's theorem in the plane is quite simple.

THEOREM 2.4.2. (Green's Theorem) *Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. If $X = P\partial_x + Q\partial_y$ is a vector field defined on a region containing Ω then*

$$\int_{\Omega} \left(\partial_x Q - \partial_y P \right) dx dy = \int_{\partial\Omega} P dx + Q dy.$$

PROOF. Note that the integral on the right-hand side is a line integral, which can also be interpreted as the integral of the 1-form $\omega = Pdx^1 + Qdx^2$ on the 1-manifold $\partial\Omega$. With this in mind we just need to observe that $d\omega = (\partial_1 Q - \partial_2 P)\, dx^1 \wedge dx^2$ in order to establish the theorem. □

Gauss' Theorem is quite a bit more complicated, but we did some of the ground work when we defined the divergence above. The context is a connected, compact, oriented Riemannian manifold M with boundary, but the example to keep in mind is a domain $M \subset \mathbb{R}^n$ with smooth boundary

THEOREM 2.4.3. (The divergence theorem or Gauss' theorem) *Let X be a vector field defined on M and N the outward pointing unit normal field to ∂M, then*

$$\int_M (\mathrm{div} X)\, d\mathrm{vol}_g = \int_{\partial M} g(X,N)\, d\mathrm{vol}_{g|_{\partial M}}$$

PROOF. We know that

$$\mathrm{div} X d\mathrm{vol}_g = d\left(i_X\left(d\mathrm{vol}_g\right)\right).$$

So by Stokes' theorem it suffices to show that

$$i_X\left(d\mathrm{vol}_g\right)|_{\partial M} = g(X,N)\, d\mathrm{vol}_{\mathrm{g}|_{\partial \mathrm{M}}}$$

The orientation on $T_p\partial M$ is so that $v_2,...,v_n$ is a positively oriented basis for $T_p\partial M$ iff $N, v_2,...,v_n$ is a positively oriented basis for T_pM. Therefore, the natural volume form for ∂M denoted $d\mathrm{vol}_{g|_{\partial M}}$ is given by $i_N\left(d\mathrm{vol}_g\right)$. If $v_2,...,v_n \in T_p\partial M$ is a basis, then

$$\begin{aligned} i_X\left(d\mathrm{vol}_g\right)|_{\partial M}(v_2,...,v_n) &= d\mathrm{vol}_g(X, v_2,...,v_n) \\ &= d\mathrm{vol}_g(g(X,N)N, v_2,...,v_n) \\ &= g(X,N)\, d\mathrm{vol}_g(N, v_2,...,v_n) \\ &= g(X,N)\, i_N\left(d\mathrm{vol}_g\right) \\ &= g(X,N)\, d\mathrm{vol}_{g|_{\partial M}} \end{aligned}$$

where we used that $X - g(X,N)X$, the component of X tangent $T_p\partial M$, is a linear combination of $v_2,...,v_n$ and therefore doesn't contribute to the form. □

Stokes' Theorem is specific to 3 dimensions. Classically it holds for an oriented surface $S \subset \mathbb{R}^3$ with smooth boundary but can be formulated for oriented surfaces in oriented Riemannian 3-manifolds.

THEOREM 2.4.4. (Stokes' theorem) *Let $S \subset M^3$ be an oriented surface with boundary ∂S. If X is a vector field defined on a region containing S and N is the unit normal field to S, then*

$$\int_S g(\mathrm{curl} X, N)\, d\mathrm{vol}_{g|_S} = \int_{\partial S} \omega_X.$$

PROOF. Recall that ω_X is the 1-form defined by

$$\omega_X(v) = g(X,v).$$

This form is related to $\mathrm{curl} X$ by

$$d(\omega_X) = i_{\mathrm{curl} X}\left(d\mathrm{vol}_g\right).$$

So Stokes' Theorem tells us that

$$\int_{\partial S} \omega_X = \int_S i_{\mathrm{curl} X}\left(d\mathrm{vol}_g\right).$$

The integral on the right-hand side can now be understood in a manner completely analogous to our discussion of $i_X(d\mathrm{vol}_g)|_{\partial M}$ in the Divergence Theorem. We note that N is chosen perpendicular to T_pS in such a way that $N, v_2, v_3 \in T_pM$ is positively oriented iff $v_2, v_3 \in T_pS$ is positively oriented. Thus we have again that

$$d\mathrm{vol}_{g|_S} = i_N d\mathrm{vol}_g$$

and consequently

$$i_{\mathrm{curl}X}(d\mathrm{vol}_g) = g(\mathrm{curl}X, N)\, d\mathrm{vol}_{g|_S}$$

□

2.5. Frobenius

In the section we prove the theorem of Frobenius for vector fields and relate it to equivalent versions for forms and differential equations that involve Lie derivatives.

CHAPTER 3

Basic Cohomology Theory

3.1. De Rham Cohomology

Throughout we let M be an n-manifold. Using that $d \circ d = 0$, we trivially get that the exact forms

$$B^p(M) = d\left(\Omega^{p-1}(M)\right)$$

are a subset of the closed forms

$$Z^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0\}.$$

The de Rham cohomology is then defined as

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}.$$

Given a closed form ψ, we let $[\psi]$ denote the corresponding cohomology class.

The first simple property comes from the fact that any function with zero differential must be locally constant. On a connected manifold we therefore have

$$H^0(M) = \mathbb{R}.$$

Given a smooth map $F : M \to N$, we get an induced map in cohomology:

$$\begin{array}{rcl} H^p(N) & \to & H^p(M), \\ F^*([\psi]) & = & [F^*\psi]. \end{array}$$

This definition is independent of the choice of ψ, since the pullback F^* commutes with d.

The two key results that are needed for a deeper understanding of de Rham cohomology are the Meyer-Vietoris sequence and homotopy invariance of the pull back map.

LEMMA 3.1.1. (The Mayer-Vietoris Sequence) *If $M = A \cup B$ for open sets $A, B \subset M$, then there is a long exact sequence*

$$\cdots \to H^p(M) \to H^p(A) \oplus H^p(B) \to H^p(A \cap B) \to H^{p+1}(M) \to \cdots.$$

PROOF. The proof is given in outline, as it is exactly the same as the corresponding proof in algebraic topology. We start by defining a short exact sequence

$$0 \to \Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B) \to \Omega^p(A \cap B) \to 0.$$

The map $\Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B)$ is simply restriction $\omega \mapsto (\omega|_A, \omega|_B)$. The second is given by $(\omega, \psi) \mapsto (\omega|_{A\cap B} - \psi|_{A\cap B})$. With these definitions it is clear that $\Omega^p(M) \to \Omega^p(A) \oplus \Omega^p(B)$ is injective and that the sequence is exact at $\Omega^p(A) \oplus \Omega^p(B)$. It is a bit less obvious why $\Omega^p(A) \oplus \Omega^p(B) \to \Omega^p(A \cap B)$ is surjective. To see this select a partition of unity λ_A, λ_B with respect to the covering A, B. Given $\omega \in \Omega^p(A \cap B)$ we see that $\lambda_A\omega$ defines a form on B, while $\lambda_B\omega$ defines a form on A. Then $(\lambda_B\omega, -\lambda_A\omega) \mapsto \omega$.

These maps induce maps in cohomology

$$H^p(M) \to H^p(A) \oplus H^p(B) \to H^p(A \cap B)$$

such that this sequence is exact. The connecting homomorphisms

$$\delta : H^p(A \cap B) \to H^{p+1}(M)$$

are constructed using the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^{p+1}(M) & \to & \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) & \to & \Omega^{p+1}(A \cap B) & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & \Omega^{p}(M) & \to & \Omega^{p}(A) \oplus \Omega^{p}(B) & \to & \Omega^{p}(A \cap B) & \to & 0 \end{array}$$

If we take a form $\omega \in \Omega^p(A \cap B)$, then $(\lambda_B \omega, -\lambda_A \omega) \in \Omega^p(A) \oplus \Omega^p(B)$ is mapped onto ω. If $d\omega = 0$, then

$$\begin{aligned} d(\lambda_B \omega, -\lambda_A \omega) &= (d\lambda_B \wedge \omega, -d\lambda_A \wedge \omega) \\ &\in \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) \end{aligned}$$

vanishes when mapped to $\Omega^{p+1}(A \cap B)$. So we get a well-defined form

$$\begin{aligned} \delta\omega &= \begin{cases} d\lambda_B \wedge \omega & \text{on } A \\ -d\lambda_A \wedge \omega & \text{on } B \end{cases} \\ &\in \Omega^{p+1}(M). \end{aligned}$$

It is easy to see that this defines a map in cohomology that makes the Meyer-Vietoris sequence exact.

The construction here is fairly concrete, but it is a very general homological construction. □

The first part of the Meyer-Vietoris sequence

$$0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to H^1(M)$$

is particularly simple since we know what the zero dimensional cohomology is. In case $A \cap B$ is connected it must be a short exact sequence

$$0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to 0$$

so the Meyer-Vietoris sequence for higher dimensional cohomology starts with

$$0 \to H^1(M) \to H^1(A) \oplus H^1(B) \to \cdots$$

To study what happens when we have homotopic maps between manifolds we have to figure out how forms on the product $[0,1] \times M$ relate to forms on M.

On the product $[0,1] \times M$ we have the vector field ∂_t tangent to the first factor as well as the corresponding one form dt. In local coordinates forms on $[0,1] \times M$ can be written

$$\omega = a_I dx^I + b_J dt \wedge dx^J$$

if we use summation convention and multi index notation

$$\begin{aligned} a_I &= a_{i \cdots i_k}, \\ dx^I &= dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

For each form the dt factor can be integrated out as follows

$$\mathscr{I}(\omega) = \int_0^1 \omega = \int_0^1 b_J dt \wedge dx^J = \left(\int_0^1 b_J dt \right) dx^J$$

Thus giving a map

$$\Omega^{k+1}([0,1]\times M)\to\Omega^{k}(M)$$

To see that this is well-defined note that it can be expressed as

$$\mathscr{I}(\omega)=\int_0^1 dt\wedge i_{\partial_t}\omega$$

since

$$i_{\partial_t}(\omega)=b_J dx^J.$$

LEMMA 3.1.2. *Let* $j_t : M\to[0,1]\times M$ *be the map* $j_t(x)=(t,x)$, *then*

$$\mathscr{I}(d\omega)+d\mathscr{I}(\omega)=j_1^*(\omega)-j_0^*(\omega)$$

PROOF. The key is to prove that

$$\mathscr{I}(d\omega)+d\mathscr{I}(\omega)=\int_0^1 dt\wedge L_{\partial_t}\omega$$

Given this we see that the right hand side is

$$\begin{aligned}\int_0^1 dt\wedge L_{\partial_t}\omega &= \int_0^1 dt\wedge L_{\partial_t}\left(a_I dx^I+b_J dt\wedge dx^J\right)\\ &= \int_0^1 dt\wedge\left(\partial_t a_I dx^I+\partial_t b_J dt\wedge dx^J\right)\\ &= \int_0^1 dt\wedge(\partial_t a_I)\,dx^I\\ &= \left(\int_0^1 dt\,\partial_t a_I\right)dx^I\\ &= (a_I(1,x)-a_I(0,x))\,dx^I\\ &= j_1^*(\omega)-j_0^*(\omega)\end{aligned}$$

The first formula follows by noting that

$$\begin{aligned}\mathscr{I}(d\omega)+d\mathscr{I}(\omega) &= \int_0^1 dt\wedge i_{\partial_t}d\omega+d\left(\int_0^1 dt\wedge i_{\partial_t}\omega\right)\\ &= \int_0^1 dt\wedge i_{\partial_t}d\omega+\int_0^1 dt\wedge d i_{\partial_t}\omega\\ &= \int_0^1 dt\wedge\left(i_{\partial_t}d\omega+d i_{\partial_t}\omega\right)\\ &= \int_0^1 dt\wedge\left(L_{\partial_t}\omega\right)\end{aligned}$$

The one tricky move here is the identity

$$d\left(\int_0^1 dt\wedge i_{\partial_t}\omega\right)=\int_0^1 dt\wedge d i_{\partial_t}\omega$$

On the left hand side it is clear what d does, but on the right hand side we are computing d of a form on the product. However, as we are wedging with dt this does not become an issue. Specifically, if d is exterior differentiation on $[0,1]\times M$ and d_x exterior differentiation on M, then

$$\begin{aligned} d_x\left(\int_0^1 dt\wedge i_{\partial_t}\omega\right) &= d_x\left(\int_0^1 b_J dt\right)\wedge dx^J \\ &= \sum_i \frac{\partial \int_0^1 b_J dt}{\partial x^i}\wedge dx^i\wedge dx^J \\ &= \sum_i \int_0^1 \frac{\partial b_J}{\partial x^i} dt\wedge dx^i\wedge dx^J \\ &= \left(\int_0^1 dt\wedge\left(\sum_i \frac{\partial b_J}{\partial x^i}dx^i\right)\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge (d_x b_J)\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge (db_J - \partial_t b_J dt)\right)\wedge dx^J \\ &= \left(\int_0^1 dt\wedge db_J\right)\wedge dx^J \\ &= \int_0^1 dt\wedge d i_{\partial_t}\omega \end{aligned}$$

□

We can now establish homotopy invariance.

PROPOSITION 3.1.3. *If $F_0, F_1 : M\to N$ are smoothly homotopic, then they induce the same maps on de Rham cohomology.*

PROOF. Assume we have a homotopy $H : [0,1]\times M\to N$, such that $F_0 = H\circ j_0$ and $F_1 = H\circ j_1$, then

$$\begin{aligned} F_1^*(\omega) - F_0^*(\omega) &= (H\circ j_1)^*(\omega) - (H\circ j_0)^*(\omega) \\ &= j_1^*(H^*(\omega)) - j_0^*(H^*(\omega)) \\ &= d\mathscr{I}(H^*(\omega)) + \mathscr{I}(H^*(d\omega)) \end{aligned}$$

So if $\omega\in\Omega^k(N)$ is closed, then we have shown that the difference

$$F_1^*(\omega) - F_0^*(\omega)\in\Omega^k(M)$$

is exact. Thus the two forms $F_1^*(\omega)$ and $F_0^*(\omega)$ must lie in the same de Rham cohomology class. □

COROLLARY 3.1.4. *If two manifolds, possibly of different dimension, are homotopy equivalent, then they have the same de Rham cohomology.*

PROOF. This follows from having maps $F : M\to N$ and $G : N\to M$ such that $F\circ G$ and $G\circ F$ are homotopic to the identity maps. □

LEMMA 3.1.5. (The Poincaré Lemma) *The cohomology of a contractible manifold M is*

$$\begin{aligned} H^0(M) &= \mathbb{R}, \\ H^p(M) &= \{0\} \text{ for } p>0. \end{aligned}$$

In particular, convex sets in $\mathbb{R}^n$ have trivial de Rham cohomology.

PROOF. Being contractible is the same as being homotopy equivalent to a point. □

3.2. Examples of Cohomology Groups

For S^n we use that

$$\begin{aligned} S^n &= (S^n - \{p\}) \cup (S^n - \{-p\}), \\ S^n - \{\pm p\} &\simeq \mathbb{R}^n, \\ (S^n - \{p\}) \cap (S^n - \{-p\}) &\simeq \mathbb{R}^n - \{0\}. \end{aligned}$$

Since $\mathbb{R}^n - \{0\}$ deformation retracts onto S^{n-1} this allows us to compute the cohomology of S^n by induction using the Meyer-Vietoris sequence. We start with S^1, which a bit different as the intersection has two components. The Meyer-vietoris sequence starting with $p = 0$ looks like

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1\left(S^1\right) \to 0.$$

Showing that $H^1\left(S^1\right) \simeq \mathbb{R}$. For $n \geq 2$ the intersection is connected so the connecting homomorphism must be an isomorphism

$$H^{p-1}\left(S^{n-1}\right) \to H^p\left(S^n\right)$$

for $p \geq 1$. Thus

$$H^p\left(S^n\right) = \begin{cases} 0, & p \neq 0, n, \\ \mathbb{R}, & p = 0, n. \end{cases}$$

For $\mathbb{P}^n$ we use the decomposition

$$\mathbb{P}^n = \left(\mathbb{P}^n - \mathbb{P}^{n-1}\right) \cup \left(\mathbb{P}^n - p\right),$$

where

$$\begin{aligned} p &= [1 : 0 : \cdots : 0], \\ \mathbb{P}^{n-1} &= \mathbb{P}\left(p^{\perp}\right) = \left\{\left[0 : z^1 : \cdots : z^n\right] \mid \left(z^1, ..., z^n\right) \in \mathbb{F}^n - \{0\}\right\}, \end{aligned}$$

and consequently

$$\begin{aligned} \mathbb{P}^n - p &= \left\{\left[z : z^1 : \cdots : z^n\right] \mid \left(z^1, ..., z^n\right) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F}\right\} \simeq \mathbb{P}^{n-1}, \\ \mathbb{P}^n - \mathbb{P}^{n-1} &= \left\{\left[1 : z^1 : \cdots : z^n\right] \mid \left(z^1, ..., z^n\right) \in \mathbb{F}^n\right\} \simeq \mathbb{F}^n, \\ \left(\mathbb{P}^n - \mathbb{P}^{n-1}\right) \cap \left(\mathbb{P}^n - p\right) &= \left\{\left[1 : z^1 : \cdots : z^n\right] \mid \left(z^1, ..., z^n\right) \in \mathbb{F}^n - \{0\}\right\} \simeq \mathbb{F}^n - \{0\}. \end{aligned}$$

We have already identified $\mathbb{P}^1$ so we don't need to worry about having a disconnected intersection when $\mathbb{F} = \mathbb{R}$ and $n = 1$. Using that $\mathbb{F}^n - \{0\}$ deformation retracts to the unit sphere S of dimension $\dim_{\mathbb{R}} \mathbb{F}^n - 1$ we see that the Meyer-Vietoris sequence reduces to

$$\begin{aligned} 0 &\to H^1\left(\mathbb{P}^n\right) \to H^1\left(\mathbb{P}^{n-1}\right) \to H^1(S) \to \cdots \\ \cdots &\to H^{p-1}(S) \to H^p\left(\mathbb{P}^n\right) \to H^p\left(\mathbb{P}^{n-1}\right) \to H^p(S) \to \cdots \end{aligned}$$

for $p \geq 2$. To get more information we need to specify the scalars and in the real case even distinguish between even and odd n. First assume that $\mathbb{F} = \mathbb{C}$. Then $S = S^{2n-1}$ and $\mathbb{CP}^1 \simeq S^2$. A simple induction then shows that

$$H^p\left(\mathbb{CP}^n\right) = \begin{cases} 0, & p = 1, 3, ..., 2n-1, \\ \mathbb{R}, & p = 0, 2, 4, ..., 2n. \end{cases}.$$

When $\mathbb{F} = \mathbb{R}$, we have $S = S^{n-1}$ and $\mathbb{RP}^1 \simeq S^1$. This shows that $H^p\left(\mathbb{RP}^n\right) = 0$ when $p = 1, ..., n-2$. The remaining cases have to be extracted from the last part of the sequence

$$0 \to H^{n-1}\left(\mathbb{RP}^n\right) \to H^{n-1}\left(\mathbb{RP}^{n-1}\right) \to H^{n-1}\left(S^{n-1}\right) \to H^n\left(\mathbb{RP}^n\right) \to 0$$

where we know that

$$H^{n-1}\left(S^{n-1}\right)=\mathbb{R}.$$

This first of all shows that $H^n\left(\mathbb{RP}^n\right)$ is either 0 or $\mathbb{R}$. Next we observe that the natural map

$$H^k\left(\mathbb{RP}^n\right)\to H^k\left(S^n\right)$$

is always an injection. To see this note that if $\pi : S^n \to \mathbb{RP}^n$ is the natural projection and $A : S^n \to S^n$ the antipodal map then $\pi\circ A=\pi$. So if $\pi^*\omega = d\phi$, then $\pi^*\omega = d\frac{1}{2}\left(\phi + A^*\phi\right)$. But $\frac{1}{2}\left(\phi + A^*\phi\right)$ is invariant under the antipodal map and thus defines a form on projective space. Thus showing that ω is itself exact. Note that this uses that the projection is a local diffeomorphism. This means that we obtain the simpler exact sequence

$$0\to H^{n-1}\left(\mathbb{RP}^{n-1}\right)\to H^{n-1}\left(S^{n-1}\right)\to H^n\left(\mathbb{RP}^n\right)\to 0$$

Form this we conclude that $H^n\left(\mathbb{RP}^n\right)=0$ iff $H^{n-1}\left(\mathbb{RP}^{n-1}\right)=\mathbb{R}$. Given that $H^1\left(\mathbb{RP}^1\right)=\mathbb{R}$ we then obtain the cohomology groups:

$$H^p\left(\mathbb{RP}^{2n}\right)=\begin{cases}0, & p\geq 1,\\ \mathbb{R}, & p=0,\end{cases}$$

$$H^p\left(\mathbb{RP}^{2n+1}\right)=\begin{cases}0, & 2n\geq p\geq 1,\\ \mathbb{R}, & p=0,2n+1.\end{cases}$$

3.3. Poincaré Duality

The last piece of information we need to understand is how the wedge product acts on cohomology. It is easy to see that we have a map

$$\begin{aligned}H^p(M)\times H^q(M) &\to H^{p+q}(M),\\ ([\psi],[\omega]) &\mapsto [\psi\wedge\omega].\end{aligned}$$

We are interested in understanding what happens in case $p+q=n$. This requires a surprising amount of preparatory work. We claim that, if M is a closed connected oriented n-manifold, then

$$\begin{aligned}H^n(M) &\to \mathbb{R},\\ [\omega] &\mapsto \int_M \omega\end{aligned}$$

is a well-defined isomorphism

In order to establish this result it turns out that we also need to work with open manifolds such as Euclidean space. Thus we choose to establish a more general result that depends on introducing a new cohomology theory.

DEFINITION 3.3.1. *Compactly supported cohomology* is defined as follows: Let $\Omega^p_c(M)$ denote the compactly supported p-forms. With this we have the compactly supported exact and closed forms $B^p_c(M)\subset Z^p_c(M)$ (note that $d:\Omega^p_c(M)\to\Omega^{p+1}_c(M)$). Then define

$$H^p_c(M)=\frac{Z^p_c(M)}{B^p_c(M)}.$$

Needless to say, for closed manifolds the two cohomology theories are identical. For connected open manifolds, on the other hand, we have that the closed 0-forms must be zero, as they also have to have compact support. Thus $H^0_c(M)=\{0\}$ if M has no compact connected components.

Note that only proper maps $F:M\to N$ have the property that they map $F^*:\Omega^p_c(N)\to\Omega^p_c(M)$. In particular, if $A\subset M$ is open, we do not have a restriction map $H^p_c(M)\to$

$H_c^p(A)$. Instead, we observe that there is a natural inclusion $\Omega_c^p(A) \to \Omega_c^p(M)$, which induces

$$H_c^p(A) \to H_c^p(M).$$

Thus compactly supported cohomology behaves more like a homology theory.

The above claim can, with our new terminology, be generalized to the claim that

$$\begin{aligned} H_c^n(M) &\to \mathbb{R}, \\ [\omega] &\mapsto \int_M \omega \end{aligned}$$

is an isomorphism for connected oriented n-manifolds.

To start off we establish this result for Euclidean space and then proceed to the even more general result on Poiancaré duality.

LEMMA 3.3.2. *The compactly supported cohomology of Euclidean space is*

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{when } p = n, \\ 0 & \text{when } p \neq n. \end{cases}$$

PROOF. We focus on the case where $p = n$, the other cases will be handled in a similar way.

First observe that for any connected oriented n-manifold, M, the map

$$\begin{aligned} \Omega_c^n(M) &\to \mathbb{R}, \\ \omega &\mapsto \int_M \omega \end{aligned}$$

vanishes on closed forms by Stokes' theorem. Thus it induces a map

$$\begin{aligned} H_c^n(M) &\to \mathbb{R}, \\ [\omega] &\mapsto \int_M \omega. \end{aligned}$$

It is also onto, since any form with the property that it is positive when evaluated on a positively oriented frame is integrated to a positive number.

Case 1: $M = S^n$. We know that $H^n(S^n) = \mathbb{R}$, so $\int : H^n(S^n) \to \mathbb{R}$ must be an isomorphism.

Case 2: $M = \mathbb{R}^n$. We can think of $M = S^n - \{p\}$. Any compactly supported form ω on M therefore yields a form on S^n. Given that $\int_M \omega = 0$, we therefore also get that $\int_{S^n} \omega = 0$. Thus, ω must be exact on S^n. Let $\psi \in \Omega^{n-1}(S^n)$ be chosen such that $d\psi = \omega$. Use again that ω is compactly supported to find an open disc U around p such that ω vanishes on U and $U \cup M = S^n$. Then ψ is clearly closed on U and must by the Poincaré lemma be exact. Thus, we can find $\theta \in \Omega^{n-2}(U)$ with $d\theta = \psi$ on U. This form doesn't necessarily extend to S^n, but we can select a bump function $\lambda : S^n \to [0,1]$ that vanishes on $S^n - U$ and is 1 on some smaller neighborhood $V \subset U$ around p. Now observe that $\psi - d(\lambda\theta)$ is actually defined on all of S^n. It vanishes on V and clearly

$$d(\psi - d(\lambda\theta)) = d\psi = \omega.$$

The case for p-forms proceeds in a similar way using that $H^p(S^n) = 0$ for $0 < p < n$. Finally $H_c^0(M) = 0$ for all non-compact manifolds. □

In order to carry out induction proofs with this cohomology theory, we also need a Meyer-Vietoris sequence:

$$\cdots \to H_c^p(A \cap B) \to H_c^p(A) \oplus H_c^p(B) \to H_c^p(M) \to H_c^{p+1}(A \cap B) \to \cdots.$$

This is established in the same way as before using the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega_c^{p+1}(A\cap B) & \to & \Omega_c^{p+1}(A)\oplus\Omega_c^{p+1}(B) & \to & \Omega_c^{p+1}(M) & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & \Omega_c^{p}(A\cap B) & \to & \Omega_c^{p}(A)\oplus\Omega_c^{p}(B) & \to & \Omega_c^{p}(M) & \to & 0 \end{array}$$

where the horizontal arrows are defined by:

$$\begin{array}{rcl} \Omega_c^p(A\cap B) & \to & \Omega_c^p(A)\oplus\Omega_c^p(B), \\ [\omega] & \mapsto & ([\omega],-[\omega]), \end{array}$$

and

$$\begin{array}{rcl} \Omega_c^p(A)\oplus\Omega_c^p(B) & \to & \Omega_c^p(M), \\ ([\omega_A],[\omega_B]) & \mapsto & [\omega_A+\omega_B]. \end{array}$$

THEOREM 3.3.3 (Poincaré Duality). *Let M be an oriented n-manifold. The pairing*

$$\begin{array}{rl} H^p(M)\times H_c^{n-p}(M) & \to \mathbb{R}, \\ ([\omega],[\psi]) & \mapsto \int_M \omega\wedge\psi \end{array}$$

is well-defined and non-degenerate. In particular, the two cohomology groups $H^p(M)$ and $H_c^{n-p}(M)$ are dual to each other and therefore have the same dimension provided they are finite-dimensional vector spaces.

PROOF. It is easy to see that the pairing is well-defined. Next note that it defines a linear map

$$H^p(M)\to\left(H_c^{n-p}(M)\right)^* = \operatorname{Hom}\left(H_c^{n-p}(M),\mathbb{R}\right).$$

We claim that this map is an isomorphism for all orientable, but not necessarily connected, manifolds. The case when $p=0$ corresponds to the above mentioned results for integrating compactly supported n-forms.

There is also a map

$$H_c^{n-p}(M)\to\left(H^p(M)\right)^*$$

which is an isomorphism when $H_c^{n-p}(M)$ is finite dimensional, but not necessarily otherwise.

We only need to check conditions (1)-(3) in theorem 1.3.11.

1: That $P(\mathbb{R}^n)$ is true follows from the Poincaré lemma and the above calculation of $H_c^p(\mathbb{R}^n)$.

2: In general suppose $A,B\subset M$ are open and our claim is true for A, B, and $A\cap B$. Using that taking duals reverses arrows, we obtain a diagram where the left- and right most columns have been eliminated

$$\begin{array}{ccccccc} \to & H^{p-1}(A\cap B) & \to & H^p(A\cup B) & \to & H^p(A)\oplus H^p(B) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & \left(H_c^{n-p+1}(A\cap B)\right)^* & \to & \left(H_c^{n-p}(A\cup B)\right)^* & \to & \left(H_c^{n-p}(A)\right)^*\oplus\left(H_c^{n-p}(B)\right)^* & \to \end{array}$$

Each square in this diagram is either commutative or anti-commutative (i.e., commutes with a minus sign.) As all vertical arrows, except for the middle one, are assumed to be isomorphisms, we see by a simple diagram chase that the middle arrow is also an isomorphism. More precisely, the five lemma asserts that if we have a commutative diagram:

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

where the two horizontal rows are exact and $A_i \to B_i$ are isomorphisms for $i = 1,2,4,5$, then $A_3 \to B_3$ is an isomorphism.

3: Consider an arbitrary union of pairwise disjoint open sets $\bigcup U_i$. In this case we have

$$\begin{aligned} H^p\left(\bigcup U_i\right) &= \times_i H^p(U_i) \\ H_c^{n-p}\left(\bigcup U_i\right) &= \oplus_i H_c^{n-p}(U_i) \\ \left(H_c^{n-p}\left(\bigcup U_i\right)\right)^* &= \times_i \left(H_c^{n-p}(U_i)\right)^* \end{aligned}$$

so the claim also follows in this case. □

COROLLARY 3.3.4. *If M^n is contractible, then*

$$H_c^p(M) = \begin{cases} \mathbb{R} & \text{when } p = n, \\ 0 & \text{when } p \neq n. \end{cases}$$

COROLLARY 3.3.5. *On a closed oriented n-manifold M we have that $H^p(M)$ and $H^{n-p}(M)$ are isomorphic.*

PROOF. This requires that we know that $H^p(M)$ is finite dimensional for all p. First note that if $O \subset \mathbb{R}^k$ is a finite union of open boxes, then the de Rham cohomology groups are finite dimensional.

This will give the result for $M \subset \mathbb{R}^k$ as we can find a tubular neighborhood $M \subset U \subset \mathbb{R}^k$ and a retract $r : U \to M$, i.e., $r|_M = id_M$. Now cover M by open boxes that lie in U and use compactness of M to find $M \subset O \subset U$ with O being a union of finitely many open boxes. Since $r|_M = id_M$ the retract $r^* : H^p(M) \to H^p(O)$ is an injection so it follows that $H^p(M)$ is finite dimensional. □

Note that $\mathbb{RP}^2$ does not satisfy this duality between H^0 and H^2. In fact we always have

THEOREM 3.3.6. *Let M be a connected n-manifold that is not orientable, then*

$$H_c^n(M) = 0.$$

PROOF. We use the two-fold orientation cover $F : \hat{M} \to M$ and the involution $A : \hat{M} \to \hat{M}$ such that $F = F \circ A$. The fact that M is not orientable means that A is orientation reversing. This implies that pull-back by A changes integrals by a sign:

$$\int_{\hat{M}} \eta = -\int_{\hat{M}} A^*\eta, \ \eta \in \Omega_c^n\left(\hat{M}\right).$$

To prove the theorem select $\omega \in \Omega_c^n(M)$ and consider the pull-back $F^*\omega \in \Omega_c^n\left(\hat{M}\right)$. Since $F = F \circ A$ this form is invariant under pull-back by A

$$\int_{\hat{M}} F^*\omega = \int_{\hat{M}} A^* \circ F^*\omega.$$

On the other hand as A reverses orientation we must also have

$$\int_{\hat{M}} F^*\omega = -\int_{\hat{M}} A^* \circ F^*\omega.$$

Thus

$$\int_{\hat{M}} F^*\omega = 0.$$

This shows that the pull back is exact

$$F^*\omega = d\psi, \ \psi \in \Omega_c^{n-1}\left(\hat{M}\right).$$

The form ψ need not be a pull back of a form on M, but we can average it

$$\bar{\psi} = \frac{1}{2}(\psi + A^*\psi) \in \Omega_c^{n-1}\left(\hat{M}\right)$$

to get a form that is invariant under A

$$\begin{aligned} A^*\bar{\psi} &= \frac{1}{2}(A^*\psi + A^*A^*\psi) \\ &= \frac{1}{2}(A^*\psi + \psi) \\ &= \bar{\psi}. \end{aligned}$$

The differential, however, stays the same

$$\begin{aligned} d\bar{\psi} &= \frac{1}{2}(d\psi + A^*d\psi) \\ &= \frac{1}{2}(F^*\omega + A^*F^*\omega) \\ &= F^*\omega. \end{aligned}$$

Now there is a unique $\phi \in \Omega_c^{n-1}(M)$, such that $F^*\phi = \bar{\psi}$. Moreover $d\phi = \omega$, since F is a local diffeomorphism and

$$\omega = F^*d\phi = dF^*\phi = d\bar{\psi}.$$

□

The last part of this proof yields a more general result:

COROLLARY 3.3.7. *Let $F : M \to N$ be a two-fold covering map, then*

$$F^* : H_c^p(N) \to H_c^p(M)$$

and

$$F^* : H^p(N) \to H^p(M)$$

are injections.

COROLLARY 3.3.8. *Let M be an open connected n-manifold, then*

$$H^n(M) = 0.$$

PROOF. By the previous corollary it suffices to prove this for orientable manifolds. In this case it follows from Poincaré duality that

$$0 \simeq H_c^0(M) \simeq (H^n(M))^*.$$

This proves the claim. □

3.4. Degree Theory

Given the simple nature of the top cohomology class of a manifold, we see that maps between manifolds of the same dimension can act only by multiplication on the top cohomology class. We shall see that this multiplicative factor is in fact an integer, called the *degree* of the map.

To be precise, suppose we have two oriented n-manifolds M and N and also a proper map $F : M \to N$. Then we get a diagram

$$\begin{array}{ccc} H_c^n(N) & \stackrel{F^*}{\to} & H_c^n(M) \\ \downarrow \int & & \downarrow \int \\ \mathbb{R} & \stackrel{d}{\to} & \mathbb{R}. \end{array}$$

Since the vertical arrows are isomorphisms, the induced map F^* yields a unique map $d : \mathbb{R} \to \mathbb{R}$. This map must be multiplication by some number, which we call the degree of F, denoted by $\deg F$. Clearly, the degree is defined by the property

$$\int_M F^*\omega = \deg F \cdot \int_N \omega.$$

From the functorial properties of the induced maps on cohomology we see that

$$\deg(F \circ G) = \deg(F)\deg(G)$$

LEMMA 3.4.1. *If $F : M \to N$ is a diffeomorphism between oriented n-manifolds, then* $\deg F = \pm 1$, *depending on whether F preserves or reverses orientation.*

PROOF. Note that our definition of integration of forms is independent of coordinate changes. It relies only on a choice of orientation. If this choice is changed then the integral changes by a sign. This clearly establishes the lemma. □

THEOREM 3.4.2. *If $F : M \to N$ is a proper map between oriented n-manifolds, then* $\deg F$ *is an integer.*

PROOF. The proof will also give a recipe for computing the degree. First, we must appeal to Sard's theorem. This theorem ensures that we can find $y \in N$ such that for each $x \in F^{-1}(y)$ the differential $DF : T_xM \to T_yN$ is an isomorphism. The inverse function theorem then tells us that F must be a diffeomorphism in a neighborhood of each such x. In particular, the preimage $F^{-1}(y)$ must be a discrete set. As we also assumed the map to be proper, we can conclude that the preimage is finite: $\{x_1, \ldots, x_k\} = F^{-1}(y)$. We can then find a neighborhood U of y in N, and neighborhoods U_i of x_i in M, such that $F : U_i \to U$ is a diffeomorphism for each i. Now select $\omega \in \Omega^n_c(U)$ with $\int \omega = 1$. Then we can write

$$F^*\omega = \sum_{i=1}^{k} F^*\omega|_{U_i},$$

where each $F^*\omega|_{U_i}$ has support in U_i. The above lemma now tells us that

$$\int_{U_i} F^*\omega|_{U_i} = \pm 1.$$

Hence,

$$\begin{aligned} \deg F &= \deg F \cdot \int_N \omega \\ &= \deg F \cdot \int_U \omega \\ &= \int_M F^*\omega \\ &= \sum_{i=1}^{k} \int_{U_i} F^*\omega|_{U_i} \end{aligned}$$

is an integer. □

Note that $\int_{U_i} F^*\omega|_{U_i}$ is ± 1, depending simply on whether F preserves or reverses the orientations at x_i. Thus, the degree simply counts the number of preimages for regular values with sign. In particular, a finite covering map has degree equal to the number of sheets in the covering.

We get several nice results using degree theory. Several of these have other proofs as well using differential topological techniques. Here we emphasize the integration formula for the degree. The key observation is that the degree of a map is a homotopy invariant. However, as we can only compute degrees of proper maps it is important that the homotopies are through proper maps. When working on closed manifolds this is not an issue. But if the manifold is Euclidean space, then all maps are homotopy equivalent, although not necessarily through proper maps.

COROLLARY 3.4.3. *Let $F: M \to N$ be a proper non-singular map of degree ± 1 between oriented connected manifolds, then F is a diffeomorphism.*

PROOF. Since F is non-singular everywhere it either reverses or preserves orientations at all points. If the degree is well defined it follows that it can only be ± 1 if the map is injective. On the other hand the fact that it is proper shows that it is a covering map, thus it must be a diffeomorphism. □

COROLLARY 3.4.4. *The identity map on a closed manifold is not homotopic to a constant map.*

PROOF. The constant map has degree 0 while the identity map has degree 1 on an oriented manifold. In case the manifold isn't oriented we can lift to the orientation cover and still get it to work. □

COROLLARY 3.4.5. *Even dimensional spheres do not admit non-vanishing vector fields.*

PROOF. Let X be a vector field on S^n we can scale it so that it is a unit vector field. If we consider it as a function $X: S^n \to S^n \subset \mathbb{R}^{n+1}$ then it is always perpendicular to its foot point. We can then create a homotopy

$$H(p,t) = p\cos(\pi t) + X_p \sin(\pi t).$$

Since $p \perp X_p$ and both are unit vectors the Pythagorean theorem shows that $H(p,t) \in S^n$ as well. When $t = 0$ the homotopy is the identity, and when $t = 1$ it is the antipodal map. Since the antipodal map reverses orientations on even dimensional spheres it is not possible for the identity map to be homotopic to the antipodal map. □

On an oriented Riemannian manifold (M,g) we always have a canonical volume form denoted by $d\mathrm{vol}_g$. Using this form, we see that the degree of a map between closed Riemannian manifolds $F: (M,g) \to (N,h)$ can be computed as

$$\deg F = \frac{\int_M F^*(d\mathrm{vol}_h)}{\mathrm{vol}(N)}.$$

In case F is locally a Riemannian isometry, we must have that:

$$F^*(d\mathrm{vol}_h) = \pm d\mathrm{vol}_g.$$

Hence,

$$\deg F = \pm \frac{\mathrm{vol}M}{\mathrm{vol}N}.$$

This gives the well-known formula for the relationship between the volumes of Riemannian manifolds that are related by a finite covering map.

On $\mathbb{R}^n - \{0\}$ we have an interesting $(n-1)$-form

$$w = r^{-n} \sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

that is closed. If we restrict this to a sphere of radius ε around the origin we see that

$$\begin{aligned}
\int_{S^{n-1}(\varepsilon)} w &= \varepsilon^{-n}\int_{S^{n-1}(\varepsilon)}\sum_{i=1}^{n}(-1)^{i+1}x^i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n \\
&= \varepsilon^{-n}\int_{\bar{B}(0,\varepsilon)} d\left(\sum_{i=1}^{n}(-1)^{i+1}x^i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\right) \\
&= \varepsilon^{-n}\int_{\bar{B}(0,\varepsilon)} n dx^1\wedge\cdots\wedge dx^n \\
&= n\varepsilon^{-n}\mathrm{vol}\bar{B}(0,\varepsilon) \\
&= n\mathrm{vol}\bar{B}(0,1) \\
&= \mathrm{vol}_{n-1}S^{n-1}(1).
\end{aligned}$$

More generally if $F : M^{n-1}\to\mathbb{R}^n-\{0\}$ is a smooth map then it is clearly homotopic to the map $F_1 : M^{n-1}\to S^{n-1}(1)$ defined by $F_1 = F/|F|$ so we obtain

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1}S^{n-1}(1)}\int_M F^*w &= \frac{1}{\mathrm{vol}_{n-1}S^{n-1}(1)}\int_M F_1^*w \\
&= \deg F_1
\end{aligned}$$

This is called the winding number of F.

3.5. The Künneth-Leray-Hirch Theorem

In this section we shall compute the cohomology of a fibration under certain simplifying assumptions. We assume that we have a submersion-fibration $\pi : E\to M$ where the fibers are diffeomorphic to a manifold N. As an example we might have the product $N\times M\to M$. We shall further assume that the restriction to any fiber $\pi^{-1}(p)\cong N$ is a surjection in cohomology

$$H^*(E)\to N^*\left(\pi^{-1}(p)\right)\to 0,\ \text{for all } p\in M.$$

In the case of a product this obviously holds since the projection $N\times M\to N$ is a right inverse to all the inclusions $N\to N\times\{s\}\subset N\times M$. In general such cohomology classes might not exist, e.g., the fibration $S^3\to S^2$ is a good counter example.

It seems a daunting task to check the condition for all fibers in a general situation. Assuming we know it is true for a specific fiber $N=\pi^{-1}(p)$ we can select a neighborhood U around p such that $\pi^{-1}(U)=N\times U$. As long as U is contractible we see that $\pi^{-1}(U)$ and N are homotopy equivalent and so the restriction to any of the fibers over U will also give a surjection in cohomology. When M is connected a covering of such contractible sets shows that $H^*(E)\to N^*\left(\pi^{-1}(p)\right)$ is a surjection for all $p\in M$. In fact this construction gives us a bit more. First note that for a specific fiber N it is possible to select and subspace $\mathscr{H}^*\in H^*(N)$ that is isomorphic to $H^*(N)$. The construction now shows that $\mathscr{H}^*$ is isomorphic to $N^*\left(\pi^{-1}(p)\right)$ for all $p\in M$ as long as M is connected.

THEOREM 3.5.1 (*Künneth-Leray-Hirch*). *Assume we have $\mathscr{H}^*\subset H^*(E)$ that is isomorphic to $H^*\left(\pi^{-1}(p)\right)$ via restriction for all $p\in M$. If $H^*(N)$ is finite dimensional, then there is an isomorphism:*

$$\bigoplus_{p+q=k} H^q(M)\otimes\mathscr{H}^p\to H^k(E)$$

where the map $H^q(M)\otimes\mathscr{H}^p\to H^{p+q}(E)$ is defined by $\psi\otimes\omega\mapsto\psi\wedge\pi^*(\omega)$.

REMARK 3.5.2. Observe that for any map $E \to M$ the space $H^*(E)$ is naturally a $H^*(M)$ module:

$$H^*(M) \times H^*(E) \to H^*(E)$$

via pull-back $H^*(M) \to H^*(E)$ and wedge product in $H^*(E)$. The statement of the theorem can then be rephrased as giving a condition for when $H^*(E)$ is a free $H^*(M)$-module.

PROOF. Note that for each open $U \subset M$ there is a natural restriction

$$\mathscr{H}^* \subset H^*(E) \to \mathscr{H}^*|_U \subset H^*\left(\pi^{-1}(U)\right).$$

This shows that the assumption of the theorem holds for all of the bundles $\pi^{-1}(U) \to U$, where $U \subset M$ is open. One more important piece of information to check is diffeomorphism invariance. To that end assume that $F : V \to U$ is a diffeomorphism. We can then consider the pull-back bundle

$$F^*\pi^{-1}(U) = \left\{(v,e) \in V \times \pi^{-1}(U) \mid e \in \pi^{-1}(F(v))\right\}.$$

Note that the pull-back is a subbundle of a trivial product bundle over V. It also fits in to a diagram

$$\begin{array}{ccc} F^*\pi^{-1}(U) & \stackrel{\pi_2}{\to} & \pi^{-1}(U) \\ \pi_1 \downarrow & & \pi \downarrow \\ V & \stackrel{F}{\to} & U \end{array}$$

where π_i is the projection onto the i^{th} factor. We can use $\pi_2^*\mathscr{H}^* = \{\pi_2^*\omega \mid \omega \in \mathscr{H}^*\}$ for this pull-back bundle.

With these constructions in mind we can employ the same strategy as in the universal theorem. To that end restrict attention to open subsets $U \subset M$ with the statement $P(U)$ being that for all k the map

$$\bigoplus_{p+q=k} \mathscr{H}^p|_U \otimes H^q(U) \to H^k\left(\pi^{-1}(U)\right)$$

is an isomorphism.

This statement clearly holds for any $U \subset M$ that is contractible and where the bundle is trivial $\pi^{-1}(U) \cong N \times U$. Now any $U \subset M$ that is diffeoemorphic to $\mathbb{R}^n$ has the property that $\pi^{-1}(U)$ is trivial. This follows from the proof of Ehresman's theorem when $\pi : E \to M$ is proper but is fact true for any fibration.

Next assume that the result holds for open sets $U, V, U \cap V \subset M$, then we can use the same strategy as in the proof of theorem 3.3.3 to verify the statement for $U \cup V$.

Finally when the statement holds for pairwise disjoint open sets: $U_i \subset M$, then it will also hold for the union. This depends crucially on $\mathscr{H}^p$ being finite dimensional as tensor products do not, in general, respect infinite products (see example below). Specifically, if we denote dual by $(\mathscr{H}^p)^* = \mathrm{Hom}(\mathscr{H}^p, \mathbb{R})$ we know that

$$\mathrm{Hom}\left((\mathscr{H}^p)^*, V\right) = (\mathscr{H}^p)^{**} \otimes V = \mathscr{H}^p \otimes V.$$

In particular, if $V = \times_i V_i$, then

$$\mathscr{H}^p \otimes (\times_i V_i) = \mathrm{Hom}\left((\mathscr{H}^p)^*, \times_i V_i\right) = \times_i \mathrm{Hom}\left((\mathscr{H}^p)^*, V_i\right) = \times_i (\mathscr{H}^p \otimes V_i).$$

This leads us to the desired isomorphism:

$$\begin{aligned}\bigoplus_{p+q=k}\mathscr{H}^p\otimes H^q\left(\bigcup_i U_i\right) &= \bigoplus_{p+q=k}\mathscr{H}^p\otimes\left(\times_i H^q\left(U_i\right)\right)\\ &= \bigoplus_{p+q=k}\times_i\left(\mathscr{H}^p\otimes H^q\left(U_i\right)\right)\\ &= \times_i\bigoplus_{p+q=k}\left(\mathscr{H}^p\otimes H^q\left(U_i\right)\right)\\ &= \times_i H^k\left(\pi^{-1}\left(U_i\right)\right)\\ &= H^k\left(\pi^{-1}\left(\bigcup_i U_i\right)\right).\end{aligned}$$

□

Künneth's theorem or formula is the above result in the case where the fibration is a product, while the Leray-Hirch theorem or formula is for a fibration of the above type.

EXAMPLE 3.5.3. In case both spaces have infinite dimensional cohomology the result does not necessarily hold. Consider two 0-dimensional manifolds A,B, i.e., they are finite or countable sets. Here $H^0\left(A\times B\right)$ is isomorphic the the space of functions $A\times B\to\mathbb{R}$, while $H^0\left(A\right)\otimes H^0\left(B\right)$ consists of finite sums of elements of the form $f_A\otimes f_B$, where $f_C : C\to\mathbb{R}$. Thus the map $H^0\left(A\right)\otimes H^0\left(B\right)\to H^0\left(A\times B\right)$ is only an isomorphism when A or B is finite. To address the construction in the above proof note that

$$H^0\left(A\right)\otimes H^0\left(B\right)=H^0\left(A\right)\otimes\times_{b\in B}H^0\left(b\right)$$

while

$$\times_{b\in B}H^0\left(A\right)\otimes H^0\left(b\right)=\times_{b\in B}H^0\left(A\right)\otimes\mathbb{R}=\times_{b\in B}H^0\left(A\right)=\times_{a\in A,b\in B}\mathbb{R}=H^0\left(A\times B\right).$$

3.6. Generalized Cohomology

In this section we are going to explain how one can define relative cohomology and also indicate how it can be used to calculate some of the cohomology groups we have seen earlier.

We start with the simplest and most important situation where $S\subset M$ is a closed submanifold of a closed manifold.

PROPOSITION 3.6.1. *If $S\subset M$ is a closed submanifold of a closed manifold, then*

(1) *The restriction map $i^*:\Omega^p\left(M\right)\to\Omega^p\left(S\right)$ is surjective.*
(2) *If $\theta\in\Omega^{p-1}\left(S\right)$ is closed, then there exists $\psi\in\Omega^{p-1}\left(M\right)$ such that $\theta=i^*\psi$ and $d\psi\in\Omega^p_c\left(M-S\right)$.*
(3) *If $\omega\in\Omega^p\left(M\right)$ with $d\omega\in\Omega^{p+1}_c\left(M-S\right)$ and $i^*\omega\in\Omega^p\left(S\right)$ is exact, then there exists $\theta\in\Omega^{p-1}\left(M\right)$ such that $\omega-d\theta\in\Omega^p_c\left(M-S\right)$.*

PROOF. Select a neighborhood $S\subset U\subset M$ that deformation retracts $\pi:U\to S$. Then $i^*:H^p\left(U\right)\to H^p\left(S\right)$ is an isomorphism. We also need a function $\lambda:M\to[0,1]$ that is compactly supported in U and is 1 on a neighborhood of S.

1. Given $\omega\in\Omega^p\left(S\right)$ let $\bar{\omega}=\lambda\pi^*\left(\omega\right)$.

2. This also shows that $d\left(\lambda\pi^*\theta\right)=d\lambda\wedge\pi^*\theta+\lambda d\pi^*\theta$ has compact support in $M-S$.

3. Conversely assume that $\omega\in\Omega^p\left(M\right)$ has $d\omega\in\Omega^{p+1}_c\left(M-S\right)$. By possibly shrinking U we can assume that it is disjoint from the support of $d\omega$. Thus, $d\omega|_U=0$ since $i:S\to U$

is an isomorphism in cohomology and we assume that $i^*\omega$ is exact, it follows that $\omega|_U = d\psi$ for some $\psi \in \Omega^{p-1}(U)$. Define $\theta = \lambda\psi$ and then note that

$$\begin{aligned}\omega - d\theta &= \omega - \lambda d\psi - d\lambda \wedge \psi \\ &= \omega - \lambda\omega|_U - d\lambda \wedge \theta \\ &\in \Omega_c^p(M-S).\end{aligned}$$

□

Part (1) shows that we have a short exact sequence

$$\begin{aligned}0 &\to \Omega^p(M,S) \to \Omega^p(M) \to \Omega^p(S) \to 0, \\ \Omega^p(M,S) &= \ker(i^* : \Omega^p(M) \to \Omega^p(S)).\end{aligned}$$

We claim that (2) and (3) show that the natural inclusion

$$\Omega_c^p(M-S) \to \Omega^p(M,S)$$

induces an isomorphism $H_c^p(M-S) \to H^p(M,S)$.

To show that it is injective consider $\omega \in \Omega_c^p(M-S)$, such that $\omega = d\theta$, where $\theta \in \Omega^{p-1}(M,S)$. We can apply (3) to θ and to find $\psi \in \Omega^{p-2}(M)$ such that $\theta - d\psi \in \Omega_c^{p-1}(M-S)$. This shows that $\omega = d(\theta - d\psi)$ for a form $\theta - d\psi \in \Omega_c^{p-1}(M-S)$.

To show that it is surjective consider $\omega \in \Omega^p(M,S)$ with $d\omega = 0$. By (3) we can find $\theta \in \Omega^{p-1}(M)$ such that $\omega - d\theta \in \Omega_c^p(M-S)$, but we don't know that $\theta \in \Omega^{p-1}(M,S)$. To fix that problem use (2) to find $\psi \in \Omega^{p-1}(M)$ such that $i^*\theta = i^*\psi$ and $d\psi \in \Omega_c^p(M-S)$. Then $\omega - d(\theta - \psi) = (\omega - d\theta) - d\psi \in \Omega_c^p(M-S)$ and $\theta - \psi \in \Omega^{p-1}(M,S)$.

COROLLARY 3.6.2. *Assume $S \subset M$ is a closed submanifold of a closed manifold, then*

$$\to H_c^p(M-S) \to H^p(M) \to H^p(S) \to H_c^{p+1}(M-S) \to$$

is a long exact sequence of cohomology groups.

Good examples are $S^{n-1} \subset S^n$ with $S^n - S^{n-1}$ being two copies of $\mathbb{R}^n$ and $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ where $\mathbb{P}^n - \mathbb{P}^{n-1} \simeq \mathbb{F}^n$. This gives us a slightly different inductive method for computing the cohomology of these spaces. Conversely, given the cohomology groups of those spaces, it computes the compactly supported cohomology of $\mathbb{R}^n$.

It can also be used on manifolds with boundary:

$$\to H_c^p(\mathrm{int}M) \to H^p(M) \to H^p(\partial M) \to H_c^{p+1}(\mathrm{int}M) \to$$

where we can specialize to $M = D^n \subset \mathbb{R}^n$, the closed unit ball. The Poincaré lemma computes the cohomology of D^n so we get that

$$H_c^{p+1}(B^n) \simeq H^p\left(S^{n-1}\right).$$

For general connected compact manifolds with boundary we also get some interesting information.

THEOREM 3.6.3. *If M is a connected compact n-manifold with boundary, then*

$$H^n(M) = 0.$$

PROOF. If M is not orientable, then neither is the interior so $H_c^n(\mathrm{int}M) = 0$ and $H^n(\partial M) = 0$, this shows from the long exact sequence that $H^n(M) = 0$.

If M is oriented, then we know that ∂M is also oriented and that

$$\begin{aligned} H^n(M,\partial M) &= H^n_c(\text{int}M) \simeq \mathbb{R} \\ H^n(\partial M) &= \{0\}, \\ H^{n-1}(\partial M) &\simeq \mathbb{R}^k, \end{aligned}$$

where k is the number of components of ∂M. The connecting homomorphism $H^{n-1}(\partial M) \to H^n_c(\text{int}M)$ can be analyzed from the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^n(M,\partial M) & \to & \Omega^n(M) & \to & \Omega^n(\partial M) & \to & 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d & & \\ 0 & \to & \Omega^{n-1}(M,\partial M) & \to & \Omega^{n-1}(M) & \to & \Omega^{n-1}(\partial M) & \to & 0 \end{array}$$

Evidently any $\omega \in \Omega^{n-1}(\partial M)$ is the restriction of some $\bar{\omega} \in \Omega^{n-1}(M)$, where we can further assume that $d\bar{\omega} \in \Omega^n_c(\text{int}M)$. Stokes' theorem then tells us that

$$\int_M d\bar{\omega} = \int_{\partial M} \bar{\omega} = \int_{\partial M} \omega.$$

This shows that the map $H^{n-1}(\partial M) \to H^n_c(\text{int}M)$ is nontrivial and hence surjective, which in turn implies that $H^n(M) = \{0\}$. □

It is possible to extend the above long exact sequence to the case where M is non-compact by using compactly supported cohomology on M. This gives us the long exact sequence

$$\to H^p_c(M-S) \to H^p_c(M) \to H^p(S) \to H^{p+1}_c(M-S) \to$$

It is even possible to also have S be non-compact if we assume that the embedding is proper and then also use compactly supported cohomology on S

$$\to H^p_c(M-S) \to H^p_c(M) \to H^p_c(S) \to H^{p+1}_c(M-S) \to$$

We can generalize further to a situation where S is simply a compact subset of M. In that case we define the deRham-Cech cohomology groups $\check{H}^p(S)$ using

$$\check{\Omega}^p(S) = \frac{\{\omega \in \Omega^p(M)\}}{\omega_1 \sim \omega_2 \text{ iff } \omega_1 = \omega_2 \text{ on a ngbd of } S},$$

i.e., the elements of $\check{\Omega}^p(S)$ are germs of forms on M at S. We now obtain a short exact sequence

$$0 \to \Omega^p_c(M-S) \to \Omega^p_c(M) \to \check{\Omega}^p(S) \to 0.$$

This in turn gives us a long exact sequence

$$\to H^p_c(M-S) \to H^p_c(M) \to \check{H}^p(S) \to H^{p+1}_c(M-S) \to$$

Finally we can define a more general relative cohomology group. We take a differentiable map $F : S \to M$ between manifolds. It could, e.g., be an embedding of $S \subset M$, but S need not be closed. Define

$$\Omega^p(F) = \Omega^p(M) \oplus \Omega^{p-1}(S)$$

and the differential

$$\begin{aligned} d : \Omega^p(F) &\to \Omega^{p+1}(F) \\ d(\omega,\psi) &= (d\omega, F^*\omega - d\psi) \end{aligned}$$

Note that $d^2 = 0$ so we get a complex and cohomology groups $H^p(F)$. These "forms" fit into a sort exact sequence

$$0 \to \Omega^{p-1}(S) \to \Omega^p(F) \to \Omega^p(M) \to 0,$$

where the maps are just the natural inclusion and projection. When we include the differential we get a large diagram where the left square is anti-commutative and the right one commutative

$$\begin{array}{ccccccccc} 0 & \to & \Omega^p(S) & \to & \Omega^{p+1}(M)\oplus\Omega^p(S) & \to & \Omega^{p+1}(M) & \to & 0 \\ & & \uparrow d & & \uparrow (d, F^*-d) & & \uparrow d & & \\ 0 & \to & \Omega^{p-1}(S) & \to & \Omega^p(M)\oplus\Omega^{p-1}(S) & \to & \Omega^p(M) & \to & 0 \end{array}$$

This still leads us to a long exact sequence

$$\to H^{p-1}(S) \to H^p(F) \to H^p(M) \to H^p(S) \to$$

The connecting homomorphism $H^p(M) \to H^p(S)$ is in fact the pull-back map F^* as can be seen by a simple diagram chase.

In case $i : S \subset M$ is an embedding we also use the notation $H^p(M,S) = H^p(i)$. In this case it'd seem that the connecting homomorphism is more naturally defined to be $H^{p-1}(S) \to H^p(M,S)$, but we don't have a short exact sequence

$$0 \to \Omega^p(M) \oplus \Omega^{p-1}(S) \to \Omega^p(M) \to \Omega^p(S) \to 0$$

hence the tricky shift in the groups.

We can easily relate the new relative cohomology to the one defined above. This shows that the relative cohomology, while trickier to define, is ultimately more general and useful.

PROPOSITION 3.6.4. *If $i : S \subset M$ is a closed submanifold of a closed manifold then the natural map*

$$\begin{array}{ccc} \Omega_c^p(M-S) & \to & \Omega^p(M)\oplus\Omega^{p-1}(S) \\ \omega & \to & (\omega, 0) \end{array}$$

defines an isomorphism

$$H_c^p(M-S) \simeq H^p(i).$$

PROOF. Simply observe that we have two long exact sequences

$$\to H^p(i) \to H^p(M) \to H^p(S) \to H^{p+1}(i) \to$$

$$\to H_c^p(M-S) \to H^p(M) \to H^p(S) \to H_c^{p+1}(M-S) \to$$

where two out of three terms are equal. □

Now that we have a fairly general relative cohomology theory we can establish the well-known excision property.

THEOREM 3.6.5. *Assume that a manifold $M = U \cup V$, where U and V are open, then the restriction map*

$$H^p(M,U) \to H^p(V, U\cap V)$$

is an isomorphism.

PROOF. First select a partition of unity λ_U, λ_V relative to U, V.

We start with injectivity. Take a class $[(\omega,\psi)] \in H^p(M,U)$, i.e.,

$$\begin{array}{rcl} d\omega & = & 0, \\ \omega|_U & = & d\psi. \end{array}$$

If the restriction to $(V, U\cap V)$ is exact, then we can find $(\bar{\omega},\bar{\psi})\in\Omega^{p-1}(V)\oplus\Omega^{p-2}(U\cap V)$ such that

$$\begin{aligned}\omega|_V &= d\bar{\omega},\\ \psi|_{U\cap V} &= \bar{\omega}|_{U\cap V}-d\bar{\psi}.\end{aligned}$$

Using that $\bar{\psi}=\lambda_U\bar{\psi}+\lambda_V\bar{\psi}$ we obtain

$$\begin{aligned}(\psi+d(\lambda_V\bar{\psi}))|_{U\cap V} &= (\bar{\omega}-d(\lambda_U\bar{\psi}))|_{U\cap V},\\ \psi+d(\lambda_V\bar{\psi}) &\in \Omega^{p-1}(U),\\ \bar{\omega}-d(\lambda_U\bar{\psi}) &\in \Omega^{p-1}(V).\end{aligned}$$

Thus we have a form $\tilde{\omega}\in\Omega^{p-1}(M)$ defined by $\psi+d(\lambda_V\bar{\psi})$ on U and $\bar{\omega}-d(\lambda_U\bar{\psi})$ on V. Clearly $d\tilde{\omega}=\omega$ and $\psi=\tilde{\omega}|_U-d(\lambda_V\bar{\psi})$ so we have shown that (ω,ψ) is exact.

For surjectivity select $(\bar{\omega},\bar{\psi})\in\Omega^{p}(V)\oplus\Omega^{p-1}(U\cap V)$ that is closed:

$$\begin{aligned}d\bar{\omega} &= 0,\\ \bar{\omega}|_{U\cap V} &= d\bar{\psi}.\end{aligned}$$

Using

$$\begin{aligned}\bar{\omega}|_{U\cap V}-d(\lambda_U\bar{\psi}) &= d(\lambda_V\bar{\psi}),\\ \bar{\omega}-d(\lambda_U\bar{\psi}) &\in \Omega^{p}(V),\\ d(\lambda_V\bar{\psi}) &\in \Omega^{p}(U)\end{aligned}$$

we can define ω as $\bar{\omega}-d(\lambda_U\bar{\psi})$ on V and $d(\lambda_V\bar{\psi})$ on U. Clearly ω is closed and $\omega|_U = d(\lambda_V\bar{\psi})$. Thus we define $\psi=\lambda_V\bar{\psi}$ in order to get a closed form $(\omega,\psi)\in\Omega^{p}(M)\oplus\Omega^{p-1}(U)$. Restricting this form to $\Omega^{p}(V)\oplus\Omega^{p-1}(U\cap V)$ yields $(\bar{\omega}-d(\lambda_U\bar{\psi}),\lambda_V\bar{\psi})$ which is not $(\bar{\omega},\bar{\psi})$. However, the difference is exact:

$$\begin{aligned}(\bar{\omega},\bar{\psi})-(\bar{\omega}-d(\lambda_U\bar{\psi}),\lambda_V\bar{\psi}) &= (d(\lambda_U\bar{\psi}),\lambda_U\bar{\psi})\\ &= d(\lambda_U\bar{\psi},0).\end{aligned}$$

Thus $[(\omega,\psi)]\in H^{p}(M,U)$ is mapped to $[(\bar{\omega},\bar{\psi})]\in H^{p}(V,U\cap V)$. □

CHAPTER 4

Characteristic Classes

4.1. Intersection Theory

4.1.1. The Poincaré Dual. Let $S^k \subset N^n$ be a closed oriented submanifold of an oriented manifold with finite dimensional de Rham cohomology. The codimension is denoted by $m = n-k$. By integrating k-forms on N over S we obtain a linear functional $H^k(N) \to \mathbb{R}$. The Poincaré dual to this functional is an element $\eta_S^N \in H_c^m(N)$ such that

$$\int_S \omega = \int_N \eta_S^N \wedge \omega$$

for all $\omega \in H^k(N)$. We call η_S^N the dual to $S \subset N$. The obvious defect of this definition is that several natural submanifolds might not have nontrivial duals for the simple reason that $H_c^m(N)$ vanishes, e.g., $N = S^n$.

To get a nontrivial dual we observe that $\int_S \omega$ only depends on the values of ω in a neighborhood of S. Thus we can find duals supported in any neighborhood U of S in N, i.e., $\eta_S^U \in H_c^m(U)$. We normally select the neighborhood so that there is a deformation retraction $\pi : U \to S$. In particular,

$$\pi^* : H^k(S) \to H^k(U)$$

is an isomorphism. In case S is connected we also know that integration on $H^k(S)$ defines an isomorphism

$$\int : H^k(S) \to \mathbb{R}.$$

This means that η_S^U is just the Poincaré dual to $1 \in \mathbb{R}$ modulo these isomorphisms. Specifically, if $\omega \in H^k(S)$ is a volume form that integrates to 1, then

$$\int_U \eta_S^U \wedge \pi^* \omega = 1.$$

Our first important observation is that if we change the orientation of S, then integration changes sign on S and hence η_S^U also changes sign. This will become important below.

The dual gives us an interesting isomorphism called the *Thom isomorphism*.

LEMMA 4.1.1. (Thom) Recall that $k+m=n$. *The map*

$$\begin{array}{rcl} H_c^{p-m}(S) & \to & H_c^p(U), \\ \omega & \mapsto & \eta_S^U \wedge \pi^*(\omega) \end{array}$$

is an isomorphism.

PROOF. Using Poincaré duality twice we see that

$$\begin{aligned} H_c^p(U) &\simeq \hom\left(H^{n-p}(U), \mathbb{R}\right) \\ &\simeq \hom\left(H^{n-p}(S), \mathbb{R}\right) \\ &\simeq H_c^{p-m}(S) \end{aligned}$$

Thus it suffices to show that the map

$$\begin{aligned} H_c^{p-m}(S) &\to H_c^p(U) \\ \omega &\mapsto \eta_S^N \wedge \pi^*(\omega) \end{aligned}$$

is injective. When $p = n$ this is clearly the above construction. For $p < n$ select $\tau \in H^{n-p}(S) \simeq H^{n-p}(U)$, then $\omega \wedge \tau \in H^k(S)$ so

$$\begin{aligned} \int_U \eta_S^U \wedge \pi^*(\omega) \wedge \pi^*(\tau) &= \int_U \eta_S^U \wedge \pi^*(\omega \wedge \tau) \\ &= \int_S \omega \wedge \tau. \end{aligned}$$

When $\eta_S^U \wedge \pi^*(\omega)$ is exact, i.e., vanishes in $H_c^p(U)$, then $\eta_S^U \wedge \pi^*(\omega) \wedge \pi^*(\tau)$ is also exact as $d\tau = 0$. In particular, the linear map $\tau \to \int_S \omega \wedge \tau$ is trivial when $\eta_S^U \wedge \pi^*(\omega)$ is trivial in $H_c^p(U)$. Poincaré duality then implies that ω itself is trivial in $H_c^{p-m}(S)$. □

The next goal is to find a characterization of η_S^U when we have a deformation retraction submersion $\pi : U \to S$.

PROPOSITION 4.1.2. *The dual is characterized as a closed form with compact support that integrates to 1 along fibers $\pi^{-1}(p)$ for all $p \in S$.*

PROOF. The characterization requires a choice of orientation for the fibers. It is chosen so that $T_p\pi^{-1}(p) \oplus T_pS$ and T_pN have the same orientation (this is consistent with [**Guillemin-Pollack**], but not with several other texts.) For $\omega \in \Omega^k(S)$ we note that $\pi^*\omega$ is constant on $\pi^{-1}(p)$, $p \in S$. Therefore, if η is a closed compactly supported form that integrates to 1 along all fibers, then

$$\int_U \eta \wedge \pi^*\omega = \int_S \omega$$

as desired.

Conversely we define

$$\begin{aligned} f &: S \to \mathbb{R}, \\ f(p) &= \int_{\pi^{-1}(p)} \eta_S^U \end{aligned}$$

and note that

$$\int_S \omega = \int_U \eta_S^U \wedge \pi^*\omega = \int_S f\omega$$

for all ω. Since the support of ω can be chosen to be in any open subset of S, this shows that $f = 1$ on S. □

In case S is not connected the dual is constructed on each component.

Next we investigate naturality of the dual.

THEOREM 4.1.3. *Let $F : M \to N$ be proper and transverse to S, then for suitable U we have*

$$F^*\left(\eta_S^U\right) = \eta_{F^{-1}(S)}^{F^{-1}(U)}.$$

PROOF. To make sense of $\eta_{F^{-1}(S)}^{F^{-1}(U)}$ we need to choose orientations for $F^{-1}(S)$. This is done as follows. First note that by shrinking U we can assume that $F^{-1}(U)$ deformation retracts onto $F^{-1}(S)$ in such a way that we have a commutative diagram

$$\begin{array}{ccc} F^{-1}(U) & \xrightarrow{F} & U \\ \downarrow \pi & & \downarrow \pi \\ F^{-1}(S) & \xrightarrow{F} & S \end{array}$$

Transversality of F then shows that F restricted to the fibers $F : \pi^{-1}(q) \to \pi^{-1}(F(q))$ is a diffeomorphism. We then select the orientation on $\pi^{-1}(q)$ such that F has degree 1 and then on $T_qF^{-1}(S)$ such that $T_q\pi^{-1}(q) \oplus T_qF^{-1}(S)$ has the orientation of T_qM. In case $F^{-1}(S)$ is a finite collection of points we are simply assigning 1 or -1 to each point depending on whether $\pi^{-1}(q)$ got oriented the same way as M or not. With all of these choices it is now clear that if η_S^U integrates to 1 along fibers then so does the pullback $F^*(\eta_S^U)$, showing that the pullback must represent $\eta_{F^{-1}(S)}^{F^{-1}(U)}$. □

This gives us a new formula for intersection numbers.

COROLLARY 4.1.4. *If* $\dim M + \dim S = \dim N$, *and* $F : M \to N$ *is proper and transverse to* S, *then*

$$I(F,S) = \int_{F^{-1}(U)} F^*(\eta_S^U).$$

The advantage of this formula is that the right-hand side can be calculated even when F isn't transverse to S. And since both sides are invariant under homotopies of F this gives us a more general way of calculating intersection numbers. We shall see how this works in the next section.

Another interesting special case of naturality occurs for submanifolds.

COROLLARY 4.1.5. *Assume* $S_1, S_2 \subset N$ *are compact and transverse and oriented, with suitable orientations on* $S_1 \cap S_2$ *the dual is given by*

$$\eta_{S_1} \wedge \eta_{S_2} = \eta_{S_1 \cap S_2}.$$

4.1.2. The Euler Class. Finally we wish to study to what extent η depends only on its values on the fibers. First we note that if the tubular neighborhood $S \subset U$ is a product neighborhood, i.e. there is a diffeomorphism $F : D \times S \to U$ which is a degree 1 diffeomorphism on fibers: $D \times \{p\} \to \pi^{-1}(p)$ for all $p \in S$, then $\eta_S^{D\times S} = F^*(\eta_S^U)$ can be represented as the volume form on D pulled back to $D \times S$.

To better measure this effect we define the Euler class

$$e_S^U = i^*(\eta_S^U) \in H^m(S)$$

as the restriction of the dual to S. Since duals are natural we quickly get

PROPOSITION 4.1.6. *Let* $F : M \to N$ *be proper and transverse to* S, *then for suitable* U *we have*

$$F^*(e_S^U) = e_{F^{-1}(S)}^{F^{-1}(U)}.$$

COROLLARY 4.1.7. *If* U *is a trivial tubular neighborhood of* S, *then* $e_S^U = 0$.

PROOF. Note that when $U = D \times S$, then the projection to the fiber $F : U \to D$ is proper and transeverse to any point $s \in D$. Clearly $e_{\{s\}}^D = 0$ so the corollary follows from the previous proposition. □

We also see that intersection numbers of maps are carried by the Euler class.

LEMMA 4.1.8. *If* $\dim M + \dim S = \dim N$, *and* $F : M \to N$ *is proper and transeverse to S, then*

$$I(F,S) = \int_{F^{-1}(U)} F^*\left(\pi^*\left(e_S^U\right)\right).$$

PROOF. Assume that $\pi : U \to S$ is a deformation retraction. Then F and $i \circ \pi \circ F$ are homotopy equivalent as maps from $F^{-1}(U)$. This shows that

$$\begin{aligned} I(F,S) &= \int_{F^{-1}(U)} F^*\left(\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (i \circ \pi \circ F)^*\left(\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (\pi \circ F)^*\left(i^*\eta_S^U\right) \\ &= \int_{F^{-1}(U)} (\pi \circ F)^*\left(e_S^U\right) \\ &= \int_{F^{-1}(U)} F^*\left(\pi^*\left(e_S^U\right)\right). \end{aligned}$$

□

This formula makes it clear that this integral really is an intersection number as it must vanish if F doesn't intersect S.

COROLLARY 4.1.9. *If* $m = k$, *then the self intersection number of S with itself is given by*

$$I(S,S) = \int_S e_S^U.$$

PROOF. The left hand side can be calculated by finding a section $F : S \to U$ that is transeverse to S. On the other hand the right hand side

$$\int_S F^*\left(\pi^*\left(e_S^U\right)\right) = \int_S (\pi \circ F)^*\left(e_S^U\right) = \int_S e_S^U$$

for any section. This proves the claim. □

Finally we show that Euler classes vanish if the codimension is odd.

THEOREM 4.1.10. *The Euler class is characterized by*

$$\eta_S^U \wedge \pi^*\left(e_S^U\right) = \eta_S^U \wedge \eta_S^U \in H_c^{2m}(U).$$

In particular $e_S^U = 0$ *if m is odd.*

PROOF. Since $\pi^*\left(e_S^U\right)$ and η_S^U represent the same class in $H^m(U)$ we have that

$$\pi^*\left(e_S^U\right) - \eta_S^U = d\omega.$$

Then

$$\begin{aligned} \eta_S^U \wedge \pi^*\left(e_S^U\right) - \eta_S^U \wedge \eta_S^U &= \eta_S^U \wedge (d\omega) \\ &= d\left(\eta_S^U \wedge \omega\right). \end{aligned}$$

Since $\eta_S^U \wedge \omega$ is compactly supported this shows that $\eta_S^U \wedge \pi^*\left(e_S^U\right) = \eta_S^U \wedge \eta_S^U$.

Moreover, as the map

$$\begin{array}{rcl} H^m(S) & \to & H_c^{2m}(U), \\ e & \mapsto & \eta_S^U \wedge \pi^*(e) \end{array}$$

is injective, it follows that that the relation $\eta_S^U \wedge \pi^*(e) = \eta_S^U \wedge \eta_S^U$ implies that $e = e_S^U$. In particular, $e_S^U = 0$ when $\eta_S^U \wedge \eta_S^U = 0$. This applies to the case when m is odd as

$$\eta_S^U \wedge \eta_S^U = -\eta_S^U \wedge \eta_S^U.$$

□

4.2. The Hopf-Lefschetz Formulas

We are going to relate the Euler characteristic and Lefschetz numbers to the cohomology of the space.

THEOREM 4.2.1. (Hopf-Poincaré) *Let M be a closed oriented n-manifold, then*

$$\chi(M) = I(\Delta,\Delta) = \sum (-1)^p \dim H^p(M).$$

PROOF. If we consider the map

$$\begin{array}{rcl} (id,id) & : & M \to \Delta, \\ (id,id)(x) & = & (x,x), \end{array}$$

then the Euler characteristic can be computed as the intersection number

$$\begin{array}{rcl} \chi(M) & = & I(\Delta,\Delta) \\ & = & I((id,id),\Delta) \\ & = & \int_M (id,id)^* \left(\eta_\Delta^{M\times M}\right). \end{array}$$

Thus we need a formula for the Poincaré dual $\eta_\Delta = \eta_\Delta^{M\times M}$. To find this formula we use Künneth's formula for the cohomology of the product. To this end select a basis ω_i for the cohomology theory $H^*(M)$ and a dual basis τ_i, i.e.,

$$\int_M \omega_i \wedge \tau_j = \delta_{ij},$$

where the integral is assumed to be zero if the form $\omega_i \wedge \tau_j$ doesn't have degree n.

By Künneth's theorem $\pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j)$ is a basis for $H^*(M\times M)$. The dual basis is up to a sign given by $\pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l)$ as we can see by calculating

$$\begin{array}{rl} & \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\ = & (-1)^{\deg\tau_j \deg\tau_k} \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\tau_j) \wedge \pi_2^*(\omega_l) \\ = & (-1)^{\deg\tau_j(\deg\tau_k+\deg\omega_l)} \int_{M\times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \wedge \pi_2^*(\tau_j) \\ = & (-1)^{\deg\tau_j(\deg\tau_k+\deg\omega_l)} \left(\int_M \omega_i \wedge \tau_k\right)\left(\int_M \omega_l \wedge \tau_j\right) \\ = & (-1)^{\deg\tau_j(\deg\tau_k+\deg\omega_l)} \delta_{ik}\delta_{lj} \end{array}$$

Clearly this vanishes unless $i = k$ and $l = j$.

This can be used to compute η_Δ for $\Delta \subset M\times M$. We assume that

$$\eta_\Delta = \sum c_{ij}\pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j).$$

On one hand

$$\begin{aligned}
& \int_{M\times M}\eta_\Delta\wedge\pi_1^*(\tau_k)\wedge\pi_2^*(\omega_l)\\
= & \sum c_{ij}\int_{M\times M}\pi_1^*(\omega_i)\wedge\pi_2^*(\tau_j)\wedge\pi_1^*(\tau_k)\wedge\pi_2^*(\omega_l)\\
= & \sum c_{ij}(-1)^{\deg\tau_j(\deg\tau_k+\deg\omega_l)}\delta_{ki}\delta_{jl}\\
= & c_{kl}(-1)^{\deg\tau_l(\deg\tau_k+\deg\omega_l)}
\end{aligned}$$

On the other hand using that $(id,id):M\to\Delta$ is a map of degree 1 tells us that

$$\begin{aligned}
\int_{M\times M}\eta_\Delta\wedge\pi_1^*(\tau_k)\wedge\pi_2^*(\omega_l) &= \int_\Delta\pi_1^*(\tau_k)\wedge\pi_2^*(\omega_l)\\
&= \int_M(id,id)^*(\pi_1^*(\tau_k)\wedge\pi_2^*(\omega_l))\\
&= \int_M\tau_k\wedge\omega_l\\
&= (-1)^{\deg(\tau_k)\deg(\omega_l)}\delta_{kl}.
\end{aligned}$$

Thus

$$c_{kl}(-1)^{\deg\tau_l(\deg\omega_l+\deg\tau_k)}=(-1)^{\deg\tau_k\deg\omega_l}\delta_{kl}$$

or in other words $c_{kl}=0$ unless $k=l$ and in that case

$$\begin{aligned}
c_{kk} &= (-1)^{\deg\tau_k(2\deg\omega_k+\deg\tau_k)}\\
&= (-1)^{\deg\tau_k\deg\tau_k}\\
&= (-1)^{\deg\tau_k}.
\end{aligned}$$

This yields the formula

$$\eta_\Delta=\sum(-1)^{\deg\tau_i}\pi_1^*(\omega_i)\wedge\pi_2^*(\tau_i).$$

The Euler characteristic can now be computed as follows

$$\begin{aligned}
\chi(M) &= \int_M(id,id)^*\left(\eta_\Delta^{M\times M}\right)\\
&= \int_M(id,id)^*\left(\sum(-1)^{\deg\tau_i}\pi_1^*(\omega_i)\wedge\pi_2^*(\tau_i)\right)\\
&= \sum(-1)^{\deg\tau_i}\int_M\omega_i\wedge\tau_i\\
&= \sum(-1)^{\deg\tau_i}\\
&= \sum(-1)^p\dim H^p(M).
\end{aligned}$$

□

A generalization of this leads us to a similar formula for the Lefschetz number of a map $F:M\to M$.

THEOREM 4.2.2. (Hopf-Lefschetz) *Let* $F:M\to M$, *then*

$$L(F)=I(\operatorname{graph}(F),\Delta)=\sum(-1)^p\operatorname{tr}(F^*:H^p(M)\to H^p(M)).$$

PROOF. This time we use the map $(id,F) : M \to \text{graph}(F)$ sending x to $(x,F(x))$ to compute the Lefschetz number

$$\begin{aligned} I(\text{graph}(F),\Delta) &= \int_M (id,F)^* \eta_\Delta \\ &= \int_M (id,F)^* \left(\sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i)\right) \\ &= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F^* \tau_i \\ &= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F_{ij} \tau_j \\ &= \sum (-1)^{\deg \tau_i} F_{ij} \delta_{ij} \\ &= \sum (-1)^{\deg \tau_i} F_{ii} \\ &= \sum (-1)^p \operatorname{tr}(F^* : H^p(M) \to H^p(M)). \end{aligned}$$

□

The definition $I(\text{graph}(F),\Delta)$ for the Lefschetz number is not consistent with [**Guillemin-Pollack**]. But if we use their definition then the formula we just established would have a sign $(-1)^{\dim M}$ on it. This is a very common confusion in the general literature.

4.3. Examples of Lefschetz Numbers

It is in fact true that $\operatorname{tr}(F^* : H^p(M) \to H^p(M))$ is always an integer, but to see this requires that we know more algebraic topology. In the cases we study here this will be established directly. Two cases where we do know this to be true are when $p = 0$ or $p = \dim M$, in those cases

$$\begin{aligned} \operatorname{tr}\left(F^* : H^0(M) \to H^0(M)\right) &= \#\text{ of components of } M, \\ \operatorname{tr}(F^* : H^n(M) \to H^n(M)) &= \deg F. \end{aligned}$$

4.3.1. Spheres and Real Projective Spaces. The simplicity of the cohomology of spheres and odd dimensional projective spaces now immediately give us the Lefschetz number in terms of the degree.

When $F : S^n \to S^n$ we have $L(F) = 1 + (-1)^n \deg F$. This conforms with our knowledge that any map without fixed points must be homotopic to the antipodal map and therefore have degree $(-1)^{n+1}$.

When $F : \mathbb{RP}^{2n+1} \to \mathbb{RP}^{2n+1}$ we have $L(F) = 1 - \deg(F)$. This also conforms with our feeling for what happens with orthogonal transformations. Namely, if $F \in Gl^+_{2n+2}(\mathbb{R})$ then it doesn't have to have a fixed point as it doesn't have to have an eigenvector, while if $F \in Gl^-_{2n+2}(\mathbb{R})$ there should be at least two fixed points.

The even dimensional version $F : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ is a bit trickier as the manifold isn't orientable and thus our above approach doesn't work. However, as the only nontrivial cohomology group is when $p = 0$ we would expect the mod 2 Lefschetz number to be 1 for all F. When $F \in Gl_{2n+1}(\mathbb{R})$, this is indeed true as such maps have an odd number of real eigenvalues. For general F we can lift it to a map $\tilde{F} : S^{2n} \to S^{2n}$ satisfying the symmetry condition

$$\tilde{F}(-x) = \pm \tilde{F}(x).$$

The sign $\pm$ must be consistent on the entire sphere. If it is $+$ then we have that $\tilde{F} \circ A = \tilde{F}$, where A is the antipodal map. This shows that $\deg \tilde{F} \cdot (-1)^{2n+1} = \deg \tilde{F}$, and hence that

$\deg \tilde{F} = 0$. In particular, $\tilde{F}$ and also F must have a fixed point. If the sign is $-$ and we assume that $\tilde{F}$ doesn't have a fixed point, then the homotopy to the antipodal map

$$H(x,t) = \frac{(1-t)\tilde{F}(x) - tx}{\left|(1-t)\tilde{F}(x) - tx\right|}$$

must also be odd

$$\begin{aligned} H(-x,t) &= \frac{(1-t)\tilde{F}(-x) - t(-x)}{\left|(1-t)\tilde{F}(-x) - t(-x)\right|} \\ &= -\frac{(1-t)\tilde{F}(x) - t(x)}{\left|(1-t)\tilde{F}(x) - t(x)\right|} \\ &= -H(x,t). \end{aligned}$$

This implies that F is homotopic to the identity on $\mathbb{RP}^{2n}$ and thus $L(F) = L(id) = 1$.

4.3.2. Tori. Next let us consider $M = T^n$. The torus is a product of n circles. If we let θ be a generator for $H^1(S^1)$ and $\theta_i = \pi_i^*(\theta)$, where $\pi_i : T^n \to S^1$ is the projection onto the i^{th} factor, then Künneth formula tells us that $H^p(T^n)$ has a basis of the form $\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$, $i_1 < \cdots < i_p$. Thus F^* is entirely determined by knowing what F^* does to θ_i. We write $F^*(\theta_i) = \alpha_{ij}\theta_j$. The action of F^* on the basis $\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$, $i_1 < \cdots < i_p$ is

$$\begin{aligned} F^*\left(\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}\right) &= F^*(\theta_{i_1}) \wedge \cdots \wedge F^*\left(\theta_{i_p}\right) \\ &= \alpha_{i_1 j_1}\theta_{j_1} \wedge \cdots \wedge \alpha_{i_p j_p}\theta_{j_p} \\ &= \left(\alpha_{i_1 j_1} \cdots \alpha_{i_p j_p}\right)\theta_{j_1} \wedge \cdots \wedge \theta_{j_p} \end{aligned}$$

this is zero unless $j_1, ..., j_p$ are distinct. Even then, these indices have to be reordered thus introducing a sign. Note also that there are $p!$ ordered $j_1, ..., j_p$ that when reordered to be increasing are the same. To find the trace we are looking for the "diagonal" entries, i.e., those $j_1, ..., j_p$ that when reordered become $i_1, ..., i_p$. If $S(i_1, ..., i_p)$ denotes the set of permutations of $i_1, ..., i_p$ then we have shown that

$$\operatorname{tr} F^*|_{H^p(T^n)} = \sum_{i_1 < \cdots < i_p} \sum_{\sigma \in S\left(i_1, ..., i_p\right)} \operatorname{sign}(\sigma)\, \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma\left(i_p\right)}.$$

This leads us to the formula

$$L(F) = \sum_{p=0}^{n} (-1)^p \sum_{i_1 < \cdots < i_p} \sum_{\sigma \in S\left(i_1, ..., i_p\right)} \operatorname{sign}(\sigma)\, \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma\left(i_p\right)}.$$

We claim that this can be simplified considerably by making the observation

$$\begin{aligned} \det(\delta_{ij} - \alpha_{ij}) &= \sum_{\sigma \in S(1,...,n)} \operatorname{sign}(\sigma)\left(\delta_{1\sigma(1)} - \alpha_{1\sigma(1)}\right) \cdots \left(\delta_{n\sigma(n)} - \alpha_{n\sigma(n)}\right) \\ &= \sum_{\sigma \in S(1,...,n)} \operatorname{sign}(\sigma)(-1)^p \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma\left(i_p\right)} \delta_{i_{p+1}\sigma\left(i_{p+1}\right)} \cdots \delta_{i_n \sigma(i_n)}, \end{aligned}$$

where in the last sum $\left\{i_1, ..., i_p, i_{p+1}, ..., i_n\right\} = \{1, ..., n\}$. Since the terms vanish unless the permutation fixes $i_{p+1}, ..., i_n$ we have shown that

$$L(F) = \det(\delta_{ij} - \alpha_{ij}).$$

Finally we claim that the $n \times n$ matrix $[\alpha_{ij}]$ has integer entries. To see this first lift F to $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ and think of $T^n = \mathbb{R}^n / \mathbb{Z}^n$ where $\mathbb{Z}^n$ is the usual integer lattice. Let e_i be the canonical basis for $\mathbb{R}^n$ and observe that $e_i \in \mathbb{Z}^n$. The fact that $\tilde{F}$ is a lift of a map in T^n

means that $\tilde{F}(x+e_i)-\tilde{F}(x)\in\mathbb{Z}^n$ for all x and $i=1,...,n$. Since $\tilde{F}$ is continuous we see that

$$\tilde{F}(x+e_i)-\tilde{F}(x)=\tilde{F}(e_i)-\tilde{F}(0)=Ae_i\in\mathbb{Z}^n$$

For some $A=[a_{ij}]\in\mathrm{Mat}_{n\times n}(\mathbb{Z})$. We can then construct a linear homotopy

$$H(x,t)=(1-t)\tilde{F}(x)+t(Ax).$$

Since

$$\begin{aligned} H(x+e_i,t) &= (1-t)\tilde{F}(x+e_i)+tA(x+e_i) \\ &= (1-t)\left(\tilde{F}(x)+Ae_i\right)+t(Ax+Ae_i) \\ &= (1-t)\left(\tilde{F}(x)\right)+t(Ax)+Ae_i \\ &= H(x,t)+Ae_i \end{aligned}$$

we see that this defines a homotopy on T^n as well. Thus showing that F is homotopic to the linear map A on T^n. This means that $F^*=A^*$. Since $A^*(\theta_i)=a_{ji}\theta_j$, we have shown that $[\alpha_{ij}]$ is an integer valued matrix.

4.3.3. Complex Projective Space. The cohomology groups of $\mathbb{P}^n=\mathbb{CP}^n$ vanish in odd dimensions and are one dimensional in even dimensions. The trace formula for the Lefschetz number therefore can't be too complicated. It turns out to be even simpler and completely determined by the action of the map on $H^2(\mathbb{P}^n)$, analogously with what happened on tori. To show this we need to show that any generator $\omega\in H^2(\mathbb{P}^n)$ has the property that $\omega^k\in H^{2k}(\mathbb{P}^n)$ is a generator. We can use induction on n to show this. Fix $\mathbb{P}^{n-1}\subset\mathbb{P}^n$ and recall from section 3.2 that $H^{2k}(\mathbb{P}^n)\to H^{2k}\left(\mathbb{P}^{n-1}\right)$ is an isomorphism for $k\leq n-1$. We can now use the induction hypothesis to claim that $\omega^k|_{\mathbb{P}^{n-1}}\in H^{2k}\left(\mathbb{P}^{n-1}\right)$ are nontrivial for $k\leq n-1$. This in turn shows that $\omega^k\in H^{2k}(\mathbb{P}^n)$ are nontrivial for $k\leq n-1$. Finally, since the duality pairing

$$\begin{aligned} H^2(\mathbb{P}^n)\times H^{2(n-1)}(\mathbb{P}^n) &\to H^{2n}(\mathbb{P}^n), \\ (\omega_1,\omega_2) &\mapsto \omega_1\wedge\omega_2 \end{aligned}$$

is nondegenerate it follows that $\omega^n=\omega\wedge\omega^{n-1}\in H^{2n}(\mathbb{P}^n)$ is a generator.

Such a form can be constructed to have the property that $\int_{\mathbb{P}^1}\omega=1$ for all $\mathbb{P}^1\subset\mathbb{P}^n$. One way of constructing such a form is to note that $U(n+1)$ acts transitively on the space of $\mathbb{P}^1$s in $\mathbb{P}^n$. Specifically, a $\mathbb{P}^1$ corresponds to a complex subspace of dimension 2 in $\mathbb{C}^{n+1}$ and for any two such subspaces there is a unitary transformation that take one into the other. Thus we are finished if we can find a closed 2-form that is invariant under the unitary group and integrates to 1 on just one $\mathbb{P}^1$.

Since $U(n+1)$ is compact we can average any closed 2-form on $\mathbb{P}^n$ to get an invariant closed 2-form

$$\bar{\tau}=\frac{1}{\mathrm{vol}U(n+1)}\int_{U(n+1)}(U^*\tau)\,dU.$$

Next note that $U\mapsto\int_{\mathbb{P}^1}U^*\tau$ is nonnegative, continuous, and positive for $U=I$. In particular, $\int_{\mathbb{P}^1}\bar{\tau}>0$. We can then define $\omega=\frac{1}{\int_{\mathbb{P}^1}\bar{\tau}}\bar{\tau}$. A more explicit form is described at the end of the section.

Now let $F:\mathbb{P}^n\to\mathbb{P}^n$ and define λ by $F^*(\omega)=\lambda\omega$. Then $F^*\left(\omega^k\right)=\lambda^k\omega^k$ and

$$L(F)=1+\lambda+\cdots+\lambda^n.$$

If $\lambda = 1$ this gives us $L(F) = n+1$, which was the answer we got for maps from $Gl_{n+1}(\mathbb{C})$. In particular, the Euler characteristic $\chi(\mathbb{P}^n) = n+1$. When $\lambda \neq 1$, the formula simplifies to

$$L(F) = \frac{1-\lambda^{n+1}}{1-\lambda}.$$

Since λ is real we note that this can't vanish unless $\lambda = -1$ and $n+1$ is even. Thus all maps on $\mathbb{P}^{2n}$ have fixed points, just as on $\mathbb{RP}^{2n}$. On the other hand $\mathbb{P}^{2n+1}$ does admit a map without fixed points, it just can't come from a complex linear map. Instead we just select a real linear map without fixed points that still yields a map on $\mathbb{P}^{2n+1}$

$$I\left(\left[z^0 : z^1 : \cdots\right]\right) = \left[-\bar{z}^1 : \bar{z}^0 : \cdots\right].$$

If I fixes a point then

$$\begin{aligned} -\lambda \bar{z}^1 &= z^0, \\ \lambda \bar{z}^0 &= z^1 \end{aligned}$$

which implies

$$-|\lambda|^2 z^i = z^i$$

for all i. Since this is impossible the map does not have any fixed points.

Finally we should justify why λ is an integer. Let $F_1 = F|_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^n$ and observe that

$$\begin{aligned} \lambda &= \int_{\mathbb{P}^1} F^*(\omega) \\ &= \int_{\mathbb{P}^1} F_1^*(\omega). \end{aligned}$$

We now claim that F_1 is homotopic to a map $\mathbb{P}^1 \to \mathbb{P}^1$. To see this note that $F_1\left(\mathbb{P}^1\right) \subset \mathbb{P}^n$ is compact and has measure 0 by Sard's theorem. Thus we can find $p \notin \operatorname{im}(F_1) \cup \mathbb{P}^1$. This allows us to deformation retract $\mathbb{P}^n - p$ to a $\mathbb{P}^{n-1} \supset \mathbb{P}^1$. This $\mathbb{P}^{n-1}$ might not be perpendicular to p in the usual metric, but one can always select a metric where p and $\mathbb{P}^1$ are perpendicular and then use the $\mathbb{P}^{n-1}$ that is perpendicular to p. Thus $F_1 : \mathbb{P}^1 \to \mathbb{P}^n$ is homotopic to a map $F_2 : \mathbb{P}^1 \to \mathbb{P}^{n-1}$. We can repeat this argument until we get a map $F_n : \mathbb{P}^1 \to \mathbb{P}^1$ homotopic to the original F_1. This shows that

$$\begin{aligned} \lambda &= \int_{\mathbb{P}^1} F_1^*(\omega) \\ &= \int_{\mathbb{P}^1} F_n^*(\omega) \\ &= \deg(F_n) \int_{\mathbb{P}^1} \omega \\ &= \deg(F_n). \end{aligned}$$

EXAMPLE 4.3.1. Finally we give a concrete description of such a form. This description combined with the fact that $\tau(\mathbb{P}^n)$ and $\mathbb{P}^{n+1} - \{p\}$ are isomorphic bundles over $\mathbb{P}^n$ with conjugate structures, i.e., they have opposite orientations but are isomorphic over $\mathbb{R}$, shows that the Euler class $e_{\mathbb{P}^n}^{\tau(\mathbb{P}^n)} \in H^2(\mathbb{P}^n)$ also generates the cohomology of $\mathbb{P}^n$.

Using the submersion $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ that sends $\left(z^0, ..., z^n\right)$ to $\left[z^0 : \cdots : z^n\right]$ we should be able to construct ω on $\mathbb{C}^{n+1}$. To make the form as nice as possible we want it to be $U(n+1)$ invariant. This is extremely useful as it will force $\int_{\mathbb{P}^1} \omega$ to be the same for all $\mathbb{P}^1 \subset \mathbb{P}^n$. Since ω is closed it will also be exact on $\mathbb{C}^{n+1}$. We use a bit of auxiliary notation to define the desired 2-form ω on $\mathbb{C}^{n+1} - \{0\}$ as well as some complex differentiation notation

$$\begin{aligned} dz^i &= dx^i + \sqrt{-1}dy^i, \\ d\bar{z}^i &= dx^i - \sqrt{-1}dy^i, \\ \frac{\partial f}{\partial z^i} &= \frac{1}{2}\left(\frac{\partial f}{\partial x^i} - \sqrt{-1}\frac{\partial f}{\partial y^i}\right), \\ \frac{\partial f}{\partial \bar{z}^i} &= \frac{1}{2}\left(\frac{\partial f}{\partial x^i} + \sqrt{-1}\frac{\partial f}{\partial y^i}\right) \\ \partial f &= \frac{\partial f}{\partial z^i}dz^i, \\ \bar{\partial} f &= \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i. \end{aligned}$$

The factor $\frac{1}{2}$ and strange signs ensure that the complex differentials work as one would think

$$\begin{aligned} dz^j\left(\frac{\partial}{\partial z^i}\right) &= \frac{\partial z_j}{\partial z^i} = \delta_i^j = \frac{\partial \bar{z}_j}{\partial \bar{z}^i} = d\bar{z}^j\left(\frac{\partial}{\partial \bar{z}^i}\right), \\ dz^j\left(\frac{\partial}{\partial \bar{z}^i}\right) &= 0 = d\bar{z}^j\left(\frac{\partial}{\partial z^i}\right) \end{aligned}$$

More generally we can define $\partial\omega$ and $\bar{\partial}\omega$ for complex valued forms by simply computing ∂ and $\bar{\partial}$ of the coefficient functions just as the local coordinate definition of d, specifically

$$\begin{aligned} \partial\left(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\right) &= \partial f \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}, \\ \bar{\partial}\left(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\right) &= \bar{\partial} f \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}. \end{aligned}$$

With this definition we see that

$$\begin{aligned} d &= \partial + \bar{\partial}, \\ \partial^2 &= \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \end{aligned}$$

and the Cauchy-Riemann equations for holomorphic functions can be stated as

$$\bar{\partial} f = 0.$$

Working on $\mathbb{C}^{n+1} - \{0\}$ define

$$\begin{aligned} \Phi(z) &= \log|z|^2 \\ &= \log\left(z^0\bar{z}^0 + \cdots + z^n\bar{z}^n\right) \end{aligned}$$

and

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi.$$

As $|z|^2$ is invariant under $U(n+1)$ the form ω will also be invariant. If we multiply $z \in \mathbb{C}^{n+1} - \{0\}$ by a nonzero scalar λ then

$$\begin{aligned} \Phi(\lambda z) &= \log\left(|\lambda z|^2\right) = \log|\lambda|^2 + \log|z|^2 \\ &= \log|\lambda|^2 + \Phi(z) \end{aligned}$$

so when taking derivatives the constant $\log|\lambda|^2$ disappears. This shows that the form ω becomes invariant under multiplication by scalars. That said, it is not possible to define Φ

on all of $\mathbb{P}^n$. We give a local coordinate representation below. It is called the potential, or Kähler potential, of ω. Note that the form is exact on $\mathbb{C}^{n+1}-\{0\}$ since

$$\partial\bar{\partial}=\left(\partial+\bar{\partial}\right)\bar{\partial}=d\bar{\partial}.$$

To show that ω is a nontrivial element of $H^2\left(\mathbb{P}^n\right)$ it suffices to show that $\int_{\mathbb{P}^1}\omega\neq 0$. By deleting a point from $\mathbb{P}^1$ we can coordinatize it by $\mathbb{C}$. Specifically we consider

$$\mathbb{P}^1=\left[z^0:z^1:0:\cdots:0\right],$$

and coordinatize $\mathbb{P}^1-\{[0:1:0:\cdots:0]\}$ by $z\mapsto[1:z:0:\cdots:0]$. Then

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1+z\bar{z}\right)\\
&= \frac{\sqrt{-1}}{2\pi}\left(\partial\left(\frac{zd\bar{z}}{1+|z|^2}\right)\right)\\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{\partial\left(zd\bar{z}\right)}{1+|z|^2}-\left(\partial\left(1+|z|^2\right)\right)\wedge\frac{zd\bar{z}}{\left(1+|z|^2\right)^2}\right)\\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{dz\wedge d\bar{z}}{1+|z|^2}-\left(\bar{z}dz\right)\wedge\frac{zd\bar{z}}{\left(1+|z|^2\right)^2}\right)\\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{dz\wedge d\bar{z}}{1+|z|^2}-\frac{|z|^2\,dz\wedge d\bar{z}}{\left(1+|z|^2\right)^2}\right)\\
&= \frac{\sqrt{-1}}{2\pi}\frac{dz\wedge d\bar{z}}{\left(1+|z|^2\right)^2}\\
&= \frac{\sqrt{-1}}{2\pi}\frac{d\left(x+\sqrt{-1}y\right)\wedge d\left(x-\sqrt{-1}y\right)}{\left(1+x^2+y^2\right)^2}\\
&= \frac{\sqrt{-1}}{2\pi}\frac{2\sqrt{-1}dy\wedge dx}{\left(1+x^2+y^2\right)^2}\\
&= \frac{1}{\pi}\frac{dx\wedge dy}{\left(1+x^2+y^2\right)^2}\\
&= \frac{1}{\pi}\frac{rdr\wedge d\theta}{\left(1+r^2\right)^2}
\end{aligned}$$

If we delete the π in the formula this is the volume form for the sphere of radius $\frac{1}{2}$ in stereographic coordinates, or the volume form for that sphere in Riemann's conformally flat model.

$$\begin{aligned}\int_{\mathbb{P}^1}\omega &= \int_{\mathbb{P}^1-\{[0:1:0:\cdots:0]\}}\omega\\ &= \int_{\mathbb{C}}\frac{1}{2\pi\sqrt{-1}}\frac{d\bar{z}\wedge dz}{\left(1+|z|^2\right)^2}\\ &= \int_{\mathbb{R}^2}\frac{1}{\pi}\frac{dx\wedge dy}{(1+x^2+y^2)^2}\\ &= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi}\frac{rdr\wedge d\theta}{(1+r^2)^2}\\ &= \int_0^\infty\frac{2rdr}{(1+r^2)^2}\\ &= 1.\end{aligned}$$

We can more generally calculate ω in the coordinates $z=\left(z^1,\ldots,z^n\right)\in\mathbb{C}^n$ corresponding to points $\left[1:z^1:\cdots:z^n\right]\in\mathbb{P}^n$.

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1+z^1\bar{z}^1+\cdots+z^n\bar{z}^n\right)\\ &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1+|z|^2\right)\\ &= \frac{\sqrt{-1}}{2\pi}\left(\partial\left(\frac{\bar{\partial}|z|^2}{1+|z|^2}\right)\right)\\ &= \frac{\sqrt{-1}}{2\pi}\left(\frac{\partial\bar{\partial}|z|^2}{1+|z|^2}-\frac{\partial|z|^2\wedge\bar{\partial}|z|^2}{\left(1+|z|^2\right)^2}\right)\\ &= \frac{\sqrt{-1}}{2\pi\left(1+|z|^2\right)^2}\left(\left(1+|z|^2\right)\partial\bar{\partial}|z|^2-\partial|z|^2\wedge\bar{\partial}|z|^2\right)\end{aligned}$$

and in coordinates

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1+|z|^2\right)\\ &= \frac{\sqrt{-1}}{2\pi}\frac{\partial^2\log\left(1+|z|^2\right)}{\partial z^i\partial\bar{z}^j}dz^i\wedge d\bar{z}^j\\ &= \frac{\sqrt{-1}}{2\pi}F_{i\bar{j}}dz^i\wedge d\bar{z}^j.\end{aligned}$$

Here the matrix $\left[F_{i\bar{j}}\right]$ is Hermitian and we clain it is positive definite. The entries are given by

$$F_{i\bar{j}}=\frac{\left(1+|z|^2\right)\delta_{ij}-z^j\bar{z}^i}{\left(1+|z|^2\right)^2}.$$

Here the matrix $\left[z^j\bar{z}^i\right]=z\cdot z^*$, where z^* is the adjoint of the column matrix z. In particular, the kernel of $z\cdot z^*$ consists of all the vectors orthogonal to z and z is an eigenvector with eigenvalue $|z|^2$. This also gives the eigenspace decomposition for $\left[F_{i\bar{j}}\right]$. Specifically, $n-1$ eigenvectors with eigenvalue $\frac{1}{1+|z|^2}$ and one with eigenvalue $\frac{1}{\left(1+|z|^2\right)^2}$. Thus $\det\left[F_{i\bar{j}}\right] = \left(1+|z|^2\right)^{-n-1}$.

We can now calculate

$$\begin{aligned}
\omega^n &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(F_{i\bar{j}}dz^i \wedge d\bar{z}^j\right)^n \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(F_{i_1\bar{j}_1}\cdots F_{i_n\bar{j}_n} dz^{i_1}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge dz^{i_n}\wedge d\bar{z}^{j_n}\right)
\end{aligned}$$

Now note that this vanishes unless all of the indices $i_1,...,i_n$, as well as $j_1,...,j_n$, are distinct. After rearraging then we obtain

$$\begin{aligned}
\omega^n &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \operatorname{sign}(i_1,...,i_n)\operatorname{sign}(j_1,...,j_n) F_{i_1\bar{j}_1}\cdots F_{i_n\bar{j}_n} dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n n!\det\left[F_{i\bar{j}}\right] dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n \\
&= \frac{n!}{\pi^n\left(1+|z|^2\right)^{n+1}} dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n
\end{aligned}$$

and

$$\int_{\mathbb{P}^n}\omega^n = \int_{\mathbb{P}^n-\mathbb{P}^{n-1}}\omega^n = \int_{\mathbb{R}^{2n}}\pi n!\left(\frac{1}{\pi\left(1+|z|^2\right)}\right)^{n+1} dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n > 0.$$

This shows that ω^n is a volume form and that $\omega^k \in H^{2k}(\mathbb{P}^n)$ is a generator for all $k = 0,...,n$.

4.4. The Euler Class

We are interested in studying duals and in particular Euler classes in the special case where we have a vector bundle $\pi : E \to M$ and M is thought of a submanifold of E by embedding it into E via the zero section. The total space E is assumed oriented in such a way that a positive orientation for the fibers together with a positive orientation of M gives us the orientation for E. The dimensions are set up so that the fibers of $E \to M$ have dimension m.

The dual $\eta_M^E \in H_c^m(E)$ is in this case usually called the Thom class of the bundle $E \to M$. The embedding $M \subset E$ is proper so by restriction this dual defines a class $e(E) \in H^m(M)$ called the Euler class (note that we only defined duals to closed submanifolds so $H_c(M) = H(M)$.) Since all sections $s : M \to E$ are homotopy equivalent we see that $e(E) = s^*\eta_M$. This immediately proves a very interesting theorem generalizing our earlier result for trivial tubular neighborhoods.

THEOREM 4.4.1. *If a bundle $\pi : E \to M$ has a nowhere vanishing section then $e(E) = 0$.*

PROOF. Let $s : M \to E$ be a section and consider $C\cdot s$ for a large constant C. Then the image of $C\cdot s$ must be disjoint from the compact support of η_M and hence $s^*(\eta_M) = 0$. □

This Euler class is also natural

PROPOSITION 4.4.2. *Let $F : N \to M$ be a map that is covered by a vector bundle map $\bar{F} : E' \to E$, i.e., $\bar{F}$ is a linear orientation preserving isomorphism on fibers. Then*

$$e\left(E'\right) = F^*\left(e\left(E\right)\right).$$

An example is the pull-back vector bundle is defined by

$$F^*\left(E\right) = \left\{\left(p,v\right) \in N \times E : \pi\left(v\right) = F\left(q\right)\right\}.$$

Reversing orientation of fibers changes the sign of η_M^E and hence also of $e\left(E\right)$. Using $F = id$ and $\bar{F}\left(v\right) = -v$ yields an orientation reversing bundle map when k is odd, showing that $e\left(E\right) = 0$. Thus we usually only consider Euler classes for even dimensional bundles.

The Euler class can also be used to detect intersection numbers as we have see before. In case M and the fibers have the same dimension, we can define the intersection number $I\left(s,M\right)$ of a section $s : M \to E$ with the zero section or simply M. The formula is

$$\begin{aligned} I\left(s,M\right) &= \int_M s^*\left(e\left(E\right)\right) \\ &= \int_M e\left(E\right) \end{aligned}$$

since all sections are homotopy equivalent to the zero section.

In the special case of the tangent bundle to an oriented manifold M we already know that the intersection number of a vector field X with the zero section is the Euler characteristic. Thus

$$\chi\left(M\right) = I\left(X,M\right) = \int_M e\left(TM\right)$$

This result was first proven by Hopf and can be used to compute χ using a triangulation. This is explained in [**Guillemin-Pollack**] and [**Spivak**].

The Euler class has other natural properties when we do constructions with vector bundles.

THEOREM 4.4.3. *Given two vector bundles $E \to M$ and $E' \to M$, the Whitney sum has Euler class*

$$e\left(E \oplus E'\right) = e\left(E\right) \wedge e\left(E'\right).$$

PROOF. As we have a better characterization of duals we start with a more general calculation.

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be bundles and consider the product bundle $\pi \times \pi' : E \times E' \to M \times M'$. With this we have the projections $\pi_1 : E \times E' \to E$ and $\pi_2 : E \times E' \to E'$. Restricting to the zero sections gives the projections $\pi_1 : M \times M' \to M$ and $\pi_2 : M \times M' \to M'$. We claim that

$$\eta_{M\times M'} = \left(-1\right)^{n\cdot m'} \pi_1^*\left(\eta_M\right) \wedge \pi_2^*\left(\eta_{M'}\right) \in H_c^{m+m'}\left(E \times E'\right).$$

Note that since the projections are not proper it is not clear that $\pi_1^*\left(\eta_M\right) \wedge \pi_2^*\left(\eta_{M'}\right)$ has compact support. However, the support must be compact when projected to E and E' and thus be compact in $E \times E'$. To see the equality we select volume forms $\omega \in H^n\left(M\right)$ and $\omega' \in H^{n'}\left(M'\right)$ that integrate to 1. Then $\pi_1^*\left(\omega\right) \wedge \pi_2^*\left(\omega'\right)$ is a volume form on $M \times M'$ that integrates to 1. Thus it suffices to compute

$$\begin{aligned}
& \int_{E\times E'} \pi_1^*(\eta_M)\wedge\pi_2^*(\eta_{M'})\wedge(\pi\times\pi')^*\left(\pi_1^*(\omega)\wedge\pi_2^*(\omega')\right) \\
= & \int_{E\times E'} \pi_1^*(\eta_M)\wedge\pi_2^*(\eta_{M'})\wedge\pi_1^*(\pi^*(\omega))\wedge\pi_2^*\left((\pi')^*(\omega')\right) \\
= & (-1)^{n\cdot m'}\int_{E\times E'} \pi_1^*(\eta_M)\wedge\pi_1^*(\pi^*(\omega))\wedge\pi_2^*(\eta_{M'})\wedge\pi_2^*\left((\pi')^*(\omega')\right) \\
= & (-1)^{n\cdot m'}\left(\int_E \eta_M\wedge\pi^*(\omega)\right)\left(\int_{E'}\eta_{M'}\wedge(\pi')^*(\omega')\right) \\
= & (-1)^{n\cdot m'}.
\end{aligned}$$

When we consider Euler classes this gives us

$$e(E\times E') = \pi_1^*(e(E))\wedge\pi_2^*(e(M'))\in H_c^{m+m'}(M\times M').$$

The sign is now irrelevant since $e(M')=0$ if m' is odd.

The Whitney sum $E\oplus E'\to M$ of two bundles over the same space is gotten by taking direct sums of the vector space fibers over points in M. This means that $E\oplus E' = (id,id)^*(E\times E')$ where $(id,id):M\to M\times M$ since

$$(id,id)^*(E\times E') = \{(p,v,v')\in M\times E\times E' : \pi(v)=p=\pi'(v')\} = E\oplus E'.$$

Thus we get the formula

$$e(E\oplus E') = e(E)\wedge e(E').$$

□

This implies

COROLLARY 4.4.4. *If a bundle $\pi:E\to M$ admits an orientable odd dimensional sub-bundle $F\subset E$, then $e(E)=0$.*

PROOF. We have that $E=F\oplus E/F$ or if E carries an inner product structure $E=F\oplus F^{\perp}$. Now orient F and then E/F so that $F\oplus E/F$ and E have compatible orientations. Then $e(E)=e(F)\wedge e(E/F)=0$. □

Note that if there is a nowhere vanishing section, then there is a 1 dimensional orientable subbundle. So this recaptures our earlier vanishing theorem. Conversely any orientable 1 dimensional bundle is trivial and thus yields a nowhere vanishing section.

A meaningful theory of invariants for vector bundles using forms should try to avoid odd dimensional bundles altogether. The simplest way of doing this is to consider vector bundles where the vector spaces are complex and then insist on using only complex and Hermitian constructions. This will be investigated further below.

The trivial bundles $\mathbb{R}^m\oplus M$ all have $e(\mathbb{R}^m\oplus M)=0$. This is because these bundles are all pull-backs of the bundle $\mathbb{R}^m\oplus\{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1}-\{p\}\to\mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n}=\omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n))=-\omega$.

Since $\chi(\mathbb{P}^n)=n+1$ we know that $e(T\mathbb{P}^n)=(n+1)\,\omega^n$.

We go on to describe how the dual and Euler class can be calculated locally. Assume that M is covered by sets U_k such that $E|_{U_k}$ is trivial and that there is a partition of unit λ_k relative to this covering.

First we analyze what the dual restricted to the fibers might look like. For that purpose we assume that the fiber is isometric to $\mathbb{R}^m$. We select a volume form $\psi \in \Omega^{m-1}\left(S^{m-1}\right)$ that integrates to 1 and a bump function $\rho : [0,\infty) \to [-1,0]$ that is -1 on a neighborhood of 0 and has compact support. Then extend ψ to $\mathbb{R}^m - \{0\}$ and consider

$$d(\rho\psi) = d\rho \wedge \psi.$$

Since $d\rho$ vanishes near the origin this is a globally defined form with total integral

$$\begin{aligned}\int_{\mathbb{R}^m} d\rho \wedge \psi &= \int_0^\infty d\rho \int_{S^{m-1}} \psi \\ &= (\rho(\infty) - \rho(0)) \\ &= 1.\end{aligned}$$

Each fiber of E carries such a form. The bump function ρ is defined on all of E by $\rho(v) = \rho(|v|)$, but the "angular" form ψ is not globally defined. As we shall see, the Euler class is the obstruction for ψ to be defined on E. Over each U_k the bundle is trivial so we do get a closed form $\psi_k \in \Omega^{m-1}\left(S\left(E|_{U_k}\right)\right)$ that restricts to the angular form on fibers. As these forms agree on the fibers the difference depends only on the footpoints:

$$\psi_k - \psi_l = \pi^* \phi_{kl},$$

where $\phi_{kl} \in \Omega^{m-1}(U_k \cap U_l)$ are closed. These forms satisfy the cocycle conditions

$$\begin{aligned}\phi_{kl} &= -\phi_{lk}, \\ \phi_{ki} + \phi_{il} &= \phi_{kl}.\end{aligned}$$

Now define

$$\varepsilon_k = \sum_i \lambda_i \phi_{ki} \in \Omega^{m-1}(U_k)$$

and note that the cocycle conditions show that

$$\begin{aligned}\varepsilon_k - \varepsilon_l &= \sum_i \lambda_i \phi_{ki} - \sum_i \lambda_i \phi_{li} \\ &= \sum_i \lambda_i (\phi_{ki} - \phi_{li}) \\ &= \sum_i \lambda_i \phi_{kl} \\ &= \phi_{kl}.\end{aligned}$$

Thus we have a globally defined form $e = d\varepsilon_k$ on M since $d(\varepsilon_k - \varepsilon_l) = d\phi_{kl} = 0$. This will turn out to be the Euler form

$$e = d\left(\sum_i \lambda_i \phi_{ki}\right) = \sum_i d\lambda_i \wedge \phi_{ki}.$$

Next we observe that

$$\pi^*\varepsilon_k - \pi^*\varepsilon_l = \psi_k - \psi_l$$

so

$$\psi = \psi_k - \pi^*\varepsilon_k$$

defines a form on E. This is our global angular form. We now claim that

$$\begin{aligned} \eta &= d(\rho\psi) \\ &= d\rho \wedge \psi + \rho d\psi \\ &= d\rho \wedge \psi - \rho \pi^* d\varepsilon_k \\ &= d\rho \wedge \psi - \rho \pi^* e \end{aligned}$$

is the dual. First we note that it is defined on all of E, is closed, and has compact support. It yields e when restricted to the zero section as $\rho(0) = -1$. Finally when restricted to a fiber we can localize the expression

$$\eta = d\rho \wedge \psi_k - d\rho \wedge \pi^* \varepsilon_k - \rho \pi^* e.$$

But both $\pi^*\varepsilon_k$ and $\pi^* e$ vanish on fibers so η, when restricted to a fiber, is simply the form we constructed above whose integral was 1. This shows that η is the dual to M in E and that e is the Euler class.

We are now going to specialize to complex line bundles with a Hermitian structure on each fiber. Since an oriented Euclidean plane has a canonical complex structure this is the same as studying oriented 2-plane bundles. The complex structure just helps in setting up the formulas.

The angular form is usually denoted $d\theta$ as it is the differential of the locally defined angle. To make sense of this we select a unit length section $s_k : U_k \to S\left(E|_{U_k}\right)$. For $v \in S\left(E|_{U_k}\right)$ the angle can be defined by

$$v = h_k(v)\, s_k = e^{\sqrt{-1}\theta_k} s_k.$$

This shows that the angular form is given by

$$\begin{aligned} d\theta_k &= -\sqrt{-1}\frac{dh_k}{h_k} \\ &= -\sqrt{-1}\, d\log h_k. \end{aligned}$$

Since we want the unit circles to have unit length we normalize this and define

$$\psi_k = -\frac{\sqrt{-1}}{2\pi} d\log h_k.$$

On $U_k \cap U_l$ we have that

$$h_l s_l = v = h_k s_k$$

So

$$(h_l)^{-1} h_k s_k = s_l.$$

But $(h_l)^{-1} h_k$ now only depends on the base point in $U_k \cap U_l$ and not on where v might be in the unit circle. Thus

$$\pi^* g_{kl} = g_{kl} \circ \pi = h_k (h_l)^{-1}$$

where $g_{kl} : U_k \cap U_l \to S^1$ satisfy the cocycle conditions

$$\begin{aligned} (g_{kl})^{-1} &= g_{lk} \\ g_{ki} g_{il} &= g_{kl}. \end{aligned}$$

Taking logarithmic differentials then gives us

$$\begin{aligned}
-\frac{\sqrt{-1}}{2\pi}\pi^*\frac{dg_{kl}}{g_{kl}} &= -\frac{\sqrt{-1}}{2\pi}\pi^* d\log(g_{kl}) \\
&= \left(-\frac{\sqrt{-1}}{2\pi}d\log(h_k)\right) - \left(-\frac{\sqrt{-1}}{2\pi}d\log(h_l)\right) \\
&= \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_k}{h_k}\right) - \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_l}{h_l}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\varepsilon_k &= -\frac{\sqrt{-1}}{2\pi}\sum_i \lambda_i d\log(g_{ki}), \\
\psi &= \left(-\frac{\sqrt{-1}}{2\pi}\frac{dh_k}{h_k}\right) - \pi^*\varepsilon_k
\end{aligned}$$

$$\begin{aligned}
e &= d\varepsilon_k \\
&= d\left(\frac{\sqrt{-1}}{2\pi}\sum_i \lambda_i d\log(g_{ki})\right) \\
&= \frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i \wedge d\log(g_{ki})
\end{aligned}$$

This can be used to prove an important result.

LEMMA 4.4.5. *Let $E \to M$ and $E' \to M$ be complex line bundles, then*

$$\begin{aligned}
e\left(\hom\left(E,E'\right)\right) &= -e(E) + e\left(E'\right), \\
e\left(E \otimes E'\right) &= e(E) + e\left(E'\right).
\end{aligned}$$

PROOF. Note that the sign ensures that the Euler class vanishes when $E = E'$.

Select a covering U_k such that E and E' have unit length sections s_k respectively t_k on U_k. If we define $L_k \in \hom(E,E')$ such that $L_k(s_k) = t_k$, then h_k is a unit length section of $\hom(E,E')$ over U_k. The transitions functions are

$$\begin{aligned}
g_{kl}s_k &= s_l, \\
\bar{g}_{kl}t_k &= t_l.
\end{aligned}$$

For $\hom(E,E')$ we see that

$$\begin{aligned}
L_l(s_k) &= h_k(g_{lk}s_l) \\
&= g_{lk}L_l(s_l) \\
&= g_{lk}t_l \\
&= g_{lk}\bar{g}_{kl}t_k \\
&= (g_{kl})^{-1}\bar{g}_{kl}t_k
\end{aligned}$$

Thus

$$L_l = (g_{kl})^{-1}\bar{g}_{kl}L_k.$$

This shows that

$$\begin{aligned} e\left(\hom\left(E,E'\right)\right) &= -\frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i\wedge d\log\left(\left(g_{ki}\right)^{-1}\bar{g}_{ki}\right) \\ &= \frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i\wedge d\log\left(g_{ki}\right)-\frac{\sqrt{-1}}{2\pi}\sum_i d\lambda_i\wedge d\log\left(\bar{g}_{ki}\right) \\ &= -e\left(E\right)+e\left(E'\right). \end{aligned}$$

The proof is similar for tensor products using

$$\begin{aligned} s_l\otimes t_l &= \left(g_{kl}s_k\right)\otimes\left(\bar{g}_{kl}t_k\right) \\ &= g_{kl}\bar{g}_{kl}\left(s_k\otimes t_k\right). \end{aligned}$$

□

4.5. Characteristic Classes

All vector bundles will be complex and for convenience also have Hermitian structures. Dimensions etc will be complex so a little bit of adjustment is sometimes necessary when we check where classes live. Note that complex bundles are always oriented since $Gl_m\left(\mathbb{C}\right)\subset Gl_{2m}^{+}\left(\mathbb{R}\right)$.

We are looking for a characteristic class $c\left(E\right)\in H^*\left(M\right)$ that can be written as

$$\begin{aligned} c\left(E\right) &= c_0\left(E\right)+c_1\left(E\right)+c_2\left(E\right)+\cdots, \\ c_0\left(E\right) &= 1\in H^0\left(M\right), \\ c_1\left(E\right) &\in H^2\left(M\right), \\ c_2\left(E\right) &\in H^4\left(M\right), \\ &\vdots \\ c_m\left(E\right) &\in H^{2m}\left(M\right), \\ c_l\left(E\right) &= 0, l>m \end{aligned}$$

For a 1 dimensional or line bundle we simply define $c\left(E\right)=1+c_1\left(E\right)=1+e\left(E\right)$. There are two more general properties that these classes should satisfy. First they should be natural in the sense that

$$c\left(E\right)=F^*\left(c\left(E'\right)\right)$$

where $F:M\to M'$ is covered by a complex bundle map $E\to E'$ that is an isomorphism on fibers. Second, they should satisfy the product formula

$$\begin{aligned} c\left(E\oplus E'\right) &= c\left(E\right)\wedge c\left(E'\right) \\ &= \sum_{p=0}^{m+m'}\sum_{i=0}^{p}c_i\left(E\right)\wedge c_{p-i}\left(E'\right) \end{aligned}$$

for Whitney sums.

There are two approaches to defining $c\left(E\right)$. In **[Milnor-Stasheff]** an inductive method is used in conjunction with the Gysin sequence for the unit sphere bundle. As this approach doesn't seem to have any advantage over the one we shall give here we will not present it. The other method is more abstract, clean, and does not use the Hermitian structure. It is analogous to the construction of splitting fields in Galois theory and is due to Grothendieck.

First we need to understand the cohomology of $H^*(\mathbb{P}(E))$. Note that we have a natural fibration $\pi : \mathbb{P}(E) \to M$ and a canonical line bundle $\tau(\mathbb{P}(E))$. The Euler class of the line bundle is for simplicity denoted

$$e = e(\tau(\mathbb{P}(E))) \in H^2(\mathbb{P}(E)).$$

The fibers of $\mathbb{P}(E) \to M$ are $\mathbb{P}^{m-1}$ and we note that the natural inclusion $i : \mathbb{P}^{m-1} \to \mathbb{P}(E)$ is also natural for the tautological bundles

$$i^*(\tau(\mathbb{P}(E))) = \tau\left(\mathbb{P}^{m-1}\right)$$

thus showing that

$$i^*(e) = e\left(\tau\left(\mathbb{P}^{m-1}\right)\right).$$

As $e\left(\tau\left(\mathbb{P}^{m-1}\right)\right)$ generates the cohomology of the fiber we have shown that the Leray-Hirch formula for the cohomology of the fibration $\mathbb{P}(E) \to M$ can be applied. Thus any element $\omega \in H^*(\mathbb{P}(E))$ has an expression of the form

$$\omega = \sum_{i=1}^{m} \pi^*(\omega_i) \wedge e^{m-i}$$

where $\omega_i \in H^*(M)$ are unique. In particular we can write:

$$\begin{aligned} 0 &= (-e)^m + \pi^*(c_1(E)) \wedge (-e)^{m-1} + \cdots + \pi^*(c_{m-1}(E)) \wedge (-e) + \pi^*(c_m(E)) \\ &= \sum_{i=0}^{m} \pi^*(c_i(E)) \wedge (-e)^{m-i} \end{aligned}$$

This means that $H^*(\mathbb{P}(E))$ is an extension of $H^*(M)$ with a unique monic polynomial

$$p_E(t) = t^m + c_1(E)t^{m-1} + \cdots + c_{m-1}(E)t + c_m(E)$$

such that $p_E(-e) = 0$. Moreover, the total Chern class is defined as

$$p_E(1) = c(E) = 1 + c_1(E) + \cdots + c_m(E).$$

The reason for using $-e$ rather than e is that $-e$ restricts to the form ω on the fibers of $\mathbb{P}(E)$.

THEOREM 4.5.1. *Assume that we have vector bundles $E \to M$ and $E' \to M'$ both of rank m, and a smooth map $F : M \to M'$ that is covered by a bundle map that is fiberwise an isomorphism. Then*

$$c(E) = F^*\left(c\left(E'\right)\right).$$

PROOF. We start by selecting a Hermitian structure on E' and then transfer it to E by the bundle map. In that way the bundle map preserves the unit sphere bundles. Better yet, we get a bundle map

$$\pi^*(E) \to \left(\pi'\right)^*\left(E'\right)$$

that also yields a bundle map

$$\tau(\mathbb{P}(E)) \to \tau\left(\mathbb{P}\left(E'\right)\right).$$

Since the Euler classes for these bundles is natural we have

$$F^*\left(e'\right) = e$$

and therefore

$$\begin{aligned} 0 &= F^*\left(\sum_{i=0}^{m} c_i\left(E'\right) \wedge \left(-e'\right)^{m-i}\right) \\ &= \sum_{i=0}^{m} F^* c_i\left(E'\right) \wedge (-e)^{m-i} \end{aligned}$$

Since $c_i(E)$ are uniquely defined by

$$0 = \sum_{i=0}^{m} c_i(E) \wedge (-e)^{m-i}$$

we have shown that

$$c_i(E) = F^* c_i\left(E'\right).$$

□

The trivial bundles $\mathbb{C}^m \oplus M$ all have $c(\mathbb{C}^m \oplus M) = 1$. This is because these bundles are all pull-backs of the bundle $\mathbb{C}^m \oplus \{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n} = \omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n)) = -\omega$.

The Whitney sum formula is established by proving the splitting principle.

THEOREM 4.5.2. *If a bundle $\pi : E \to M$ splits $E = L_1 \oplus \cdots \oplus L_m$ as a direct sum of line bundles, then*

$$c(E) = \prod_{i=1}^{m} (1 + e(L_i)).$$

PROOF. We pull back all classes to E without changing notation. We know that $c(E) = p_E(1)$ so it suffices to identify p_E with the monic polynomial of degree m defioned by $p(t) = \prod_{i=1}^{m}(t + e(L_i))$. To prove this we need to show that

$$p(-e) = \prod_{i=1}^{m} (-e + e(L_i)) = 0.$$

Note that we can identify $-e + e(L_i)$ with the Euler class of $\hom(\tau, L_i)$. With that in mind:

$$\begin{aligned} \prod_{i=1}^{m} (-e + e(L_i)) &= e\left(\bigoplus_{i=1}^{m} \hom(\tau, L_i)\right) \\ &= e(\hom(\tau, L_1 \oplus \cdots \oplus L_m)) \\ &= e(\hom(\tau, E)) \\ &= e\left(\hom\left(\tau, \tau \oplus \tau^{\perp}\right)\right) \\ &= e(\hom(\tau, \tau)) \wedge e\left(\hom\left(\tau, \tau^{\perp}\right)\right) \\ &= 0. \end{aligned}$$

Where the last equality follows from the fact that $\hom(\tau, \tau)$ has the identity map as a nowhere vanishing section. □

The splitting principle can be used to compute $c(T\mathbb{P}^n)$. First note that $T\mathbb{P}^n \simeq \hom\left(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^{\perp}\right)$. Thus

$$\begin{aligned}
T\mathbb{P}^n \oplus \mathbb{C} &= \hom\left(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^{\perp}\right) \oplus \mathbb{C} \\
&= \hom\left(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^{\perp}\right) \oplus \hom(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)) \\
&= \hom\left(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^{\perp} \oplus \tau(\mathbb{P}^n)\right) \\
&= \hom\left(\tau(\mathbb{P}^n), \mathbb{C}^{n+1}\right) \\
&= \hom(\tau(\mathbb{P}^n), \mathbb{C}) \oplus \cdots \oplus \hom(\tau(\mathbb{P}^n), \mathbb{C}).
\end{aligned}$$

Thus

$$\begin{aligned}
c(T\mathbb{P}^n) &= c(T\mathbb{P}^n \oplus \mathbb{C}) \\
&= (1+\omega)^{n+1}.
\end{aligned}$$

This shows that

$$c_i(T\mathbb{P}^n) = \binom{n+1}{i}\omega^i$$

which conforms with

$$e(T\mathbb{P}^n) = c_n(T\mathbb{P}^n) = (n+1)\,\omega^n.$$

We can now finally establish the Whitney sum formula.

THEOREM 4.5.3. *For two vector bundles $E \to M$ and $E' \to M$ we have*

$$c\left(E \oplus E'\right) = c(E) \wedge c\left(E'\right).$$

PROOF. First we repeatedly projectivize so as to create a map $\tilde{N} \to M$ with the property that it is an injection on cohomology and the pull-back of E to $\tilde{N}$ splits as a direct sum of line bundles. Then repeat this procedure on the pull-back of E' to $\tilde{N}$ until we finally get a map $F : N \to M$ such that F^* is an injection on cohomology and both of the bundles split

$$\begin{aligned}
F^*(E) &= L_1 \oplus \cdots \oplus L_m, \\
F^*\left(E'\right) &= K_1 \oplus \cdots \oplus K_{m'}
\end{aligned}$$

The splitting principle together with naturality then implies that

$$\begin{aligned}
F^*\left(c\left(E \oplus E'\right)\right) &= c\left(F^*\left(E \oplus E'\right)\right) \\
&= c(L_1) \wedge \cdots \wedge c(L_m) \wedge c(K_1) \wedge \cdots \wedge c(K_{m'}) \\
&= c(F^*(E)) \wedge c\left(F^*\left(E'\right)\right) \\
&= F^* c(E) \wedge F^* c\left(E'\right) \\
&= F^*\left(c(E) \wedge c\left(E'\right)\right).
\end{aligned}$$

Since F^* is an injection this shows that

$$c\left(E \oplus E'\right) = c(E) \wedge c\left(E'\right).$$

□

4.6. The Gysin Sequence

This sequence allows us to compute the cohomology of certain fibrations where the fibers are spheres. As we saw above, these fibrations are not necessarily among the ones where we can use the Hirch-Leray formula. This sequence uses the Euler class and will recapture the dual, or Thom class, from the Euler class.

We start with an oriented vector bundle $\pi : E \to M$. It is possible to put a smoothly varying inner product structure on the vector spaces of the fibration, using that such bundles are locally trivial and gluing inner products together with a partition of unity on M. The function $E \to \mathbb{R}$ that takes v to $|v|^2$ is then smooth and the only critical value is 0. As such we get a smooth manifold with boundary

$$D(E) = \{v \in E : |v| \leq 1\}$$

called the disc bundle with boundary

$$S(E) = \partial D(E) = \{v \in E : |v| = 1\}$$

being the unit sphere bundle and interior

$$\text{int}D(E) = \{v \in E : |v| < 1\}.$$

Two different inner product structures will yield different disc bundles, but it is easy to see that they are all diffeomorphic to each other. We also note that $\text{int}D(E)$ is diffeomorphic to E, while $D(E)$ is homotopy equivalent to E. This gives us a diagram

$$\begin{array}{ccccccccc} \to & H_c^p(\text{int}D(E)) & \to & H^p(D(E)) & \to & H^p(S(E)) & \to & H_c^{p+1}(\text{int}D(E)) & \to \\ & \downarrow & & \uparrow & & \updownarrow & & \uparrow & \\ \to & H_c^p(E) & \to & H^p(E) & \to & H^p(S(E)) & \dashrightarrow & H_c^{p+1}(E) & \to \end{array}$$

where the vertical arrows are simply pull-backs and all are isomorphims. The connecting homomorphism

$$H^p(S(E)) \to H_c^{p+1}(\text{int}D(E))$$

then yields a map

$$H^p(S(E)) \dashrightarrow H_c^{p+1}(E)$$

that makes the bottom sequence a long exact sequence. Using the Thom isomorphism

$$H^{p-m}(M) \to H_c^p(E)$$

then gives us a new diagram

$$\begin{array}{ccccccccc} \to & H^{p-m}(M) & \xrightarrow{e\wedge} & H^p(M) & \to & H^p(S(E)) & \dashrightarrow & H^{p+1-m}(M) & \to \\ & \downarrow \eta_M \wedge \pi^*(\cdot) & & \updownarrow & & \updownarrow & & \downarrow & \\ \to & H_c^p(E) & \to & H^p(E) & \to & H^p(S(E)) & \to & H_c^{p+1}(E) & \to \end{array}$$

Most of the arrows are pull-backs and the vertical arrows are isomorphisms. The first square is commutative since $\pi^* i^*(\eta_M) = \pi^*(e)$ is represented by η_M in $H^m(E)$. This is simply because the zero section $I : M \to E$ and projection $\pi : E \to M$ are homotopy equivalences. The second square is obviously commutative. Thus we get a map

$$H^p(S(E)) \dashrightarrow H^{p+1-m}(M)$$

making the top sequence exact. This is the Gysin sequence of the sphere bundle of an oriented vector bundle. The connecting homomorphism which lowers the degree by $m-1$ can be constructed explicitly and geometrically by integrating forms on $S(E)$ along the unit spheres, but we won't need this interpretation.

The Gysin sequence also tells us how the Euler class can be used to compute the cohomology of the sphere bundle from M.

To come full circle with the Leray-Hirch Theorem we now assume that $E \to M$ is a complex bundle of complex dimension m and construct the projectivized bundle

$$\mathbb{P}(E) = \left\{ (p, L) \mid L \subset \pi^{-1}(p) \text{ is a 1 dimensional subspace} \right\}$$

This gives us projections

$$S(E) \to \mathbb{P}(E) \to M.$$

There is also a tautological bundle

$$\tau(\mathbb{P}(E)) = \{(p, L, v) \mid v \in L\}.$$

The unit-sphere bundle for τ is naturally identified with $S(E)$ by

$$\begin{array}{ccc} S(E) & \to & S(\tau(\mathbb{P}(E))), \\ (p, v) & \to & (p, \operatorname{span}\{v\}, v). \end{array}$$

This means that $S(E)$ is part of two Gysin sequences. One where M is the base and one where $\mathbb{P}(E)$ is the base. These two sequences can be connected in a very interesting manner.

If we pull back E to $\mathbb{P}(E)$ and let

$$\tau^{\perp} = \left\{ (p, L, w) \mid w \in L^{\perp} \right\}$$

be the orthogonal complement then we have that

$$\pi^*(e(E)) = e(\pi^*(E)) = e(\tau(\mathbb{P}(E))) \wedge e\left(\tau^{\perp}\right) \in H^*(\mathbb{P}(E)).$$

Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & & H^{p-2}(\mathbb{P}(E)) & \overset{e(\tau)\wedge\cdot}{\longrightarrow} & H^p(\mathbb{P}(E)) & & & \\ \searrow & & \nearrow & & & & \searrow & & \nearrow \\ & H^{p-1}(S(E)) & & \uparrow e\left(\tau^{\perp}\right) \wedge \pi^*(\cdot) & & \uparrow \pi^* & & H^p(S(E)) & \\ \nearrow & & \searrow & & & & \nearrow & & \searrow \\ & & & H^{p-2m}(M) & \overset{e(E)\wedge\cdot}{\longrightarrow} & H^p(M) & & & \end{array}$$

What is more we can now show in two ways that

$$\operatorname{span}\left\{1, e, \ldots, e^{m-1}\right\} \otimes H^*(M) \to H^*(\mathbb{P}(E))$$

is an isomorphism. First we can simply use the Leray-Hirch result by noting that the classes $1, e, \ldots, e^{m-1}$ when restricted to the fibers are the usual cohomology classes of the fiber $\mathbb{P}^m$. Or we can use diagram chases on the above diagram.

4.7. Further Study

There are several texts that expand on the material covered here. The book by [**Guillemin-Pollack**] is the basic prerequisite for the material covered in the early chapters. The cohomology aspects we cover here correspond to a simplified version of [**Bott-Tu**]. Another text is the well constructed [**Madsen-Tornehave**], which in addition explains how characteristic classes can be computed using curvature. The comprehensive text [**Spivak**, vol. V] is also worth consulting for many aspects of the theory discussed here. For a more topological approach we recommend [**Milnor-Stasheff**]. Other useful texts are listed in the references.

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