BASIC BOOKS IN SCIENCE

– a Series of books that start *at the beginning*

Book 3a

Calculus and differential equations

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Acknowledgements

In a world increasingly driven by information technology and market forces, no educational experiment can expect to make a significant impact without the availability of effective bridges to the 'user community' – the students and their teachers.

In the case of "Basic Books in Science" (for brevity, "the Series"), these bridges have been provided as a result of the enthusiasm and good will of Dr. David Peat (The Pari Center for New Learning), who first offered to host the Series on his website, and of Dr. Jan Visser (The Learning Development Institute), who set up a parallel channel for further development of the project with the use of Distance Learning techniques. The credit for setting up and maintaining the bridgeheads, and for promoting the project in general, must go entirely to them.

Education is a global enterprise with no boundaries and, as such, is sure to meet linguistic difficulties: these will be ameliorated by the provision of translations into some of the world's more widely used languages. We are most grateful to Dr. Angel S. Sanz (Madrid), who has already prepared Spanish versions of the first few books in the Series: these are being posted on the websites indicated as soon as they are ready. This represents a massive step forward: we are now seeking other translators, at first for French and Arabic editions.

The importance of having feedback from user groups, especially those in the Developing World, should not be underestimated. We are grateful for the interest shown by universities in Sub-Saharan Africa (e.g. University of the Western Cape and Kenyatta University), where trainee teachers are making use of the Series; and to the Illinois Mathematics and Science Academy (IMSA) where material from the Series is being used in teaching groups of refugee children from many parts of the world.

All who have contributed to the Series in any way are warmly thanked: they have given freely of their time and energy 'for the love of Science'. Paperback copies of the books in the Series will soon be available, but this will not jeopardize their free downloading from the Web.

Pisa 13 September 2010 Roy McWeeny (Series Editor)

BASIC BOOKS IN SCIENCE

About this Series

All human progress depends on **education**: to get it we need books and schools. Science Education is of key importance.

Unfortunately, books and schools are not always easy to find. But nowadays all the world's knowledge should be freely available to everyone – through the Internet that connects all the world's computers.

The aim of the Series is to bring basic knowledge in all areas of science within the reach of everyone. Every Book will cover in some depth a clearly defined area, starting from the very beginning and leading up to university level, and will be available on the Internet *at no cost to the reader*. To obtain a copy it should be enough to make a single visit to any library or public office with a personal computer and a telephone line. Each book will serve as one of the 'building blocks' out of which Science is built; and together they will form a 'give-away' science library.

About this book

This book, like the others in the Series, is written in simple English – the language most widely used in science and technology. It builds on the foundations laid in Book 1 (Number and symbols) and in Book 2 (Space) and deals with the mathematics we need in describing the *relationships* among the quantities we measure in Physics and the Physical Sciences in general. This leads us into the study of relationships and change, the starting point for Mathematical Analysis and the Calculus – which are needed in all branches of Science.

The present volume is essentially a supplement to Book 3, placing more emphasis on Mathematics as a human activity and on the people who made it – in the course of many centuries and in many parts of the world. Some topics are also taken to a more advanced level, with the addition of Problems and Solutions.

Contents

Chapter 1

Historical background

No single culture can claim to have produced modern science. Science (defined as organized knowledge) has been built up gradually over a long period of time, and gifts from many peoples have merged to form the vast system of verifyable scientific knowledge that is now the common heritage of humanity.

Before starting our discussion of calculus and differential equations, it is interesting to spend a few moments looking at the roots of mathematics, to which many cultures have contributed.

Mesopotamia (present-day Iraq)

Some of the most important early steps in the evolution of human culture were taken in Mesopotamia, the region that we now call Iraq. The Mesopotamians (i.e., the ancient Iraqis) not only invented an early form of writing, but they also contributed importantly to the foundations of mathematics, physics, astronomy and medicine.

In Mesopotamia (which in Greek means "between the rivers"), the settled agricultural people of the Tigris and Euphraties valleys evolved a form of writing. Among the earliest Mesopotamian writings are a set of clay tablets found at Tepe Yahya in southern Iran, the site of an ancient Elamite trading community halfway between Mesopotamia and India.

The Elamite trade supplied the Sumarian civilization of Mesopotamia with silver, copper, tin, lead, precious gems, horses, timber, obsidian, alabaster and soapstone. The practical Sumerians and Elamites probably invented writing as a means of keeping accounts.

The tablets found at Tepe Yahya are inscribed in proto-Elamite, and radio-carbon dating of organic remains associated with the tablets shows them to be from about 3,600 B.C.. The inscriptions on these tablets were made by pressing the blunt and sharp ends of a stylus into soft clay. Similar tablets have been found at the Sumarian city of Susa at the head of the Tigris River.

In about 3,100 B.C. the cuneiform script was developed, and later Mesopotamian tablets are written in cuneiform, which is a phonetic script where the symbols stand for syllables.

Mesopotamian science

Both the mathematics and astronomy of the Mesopotamians were startlingly advanced. Their number system was positional, like ours, and was based on six and sixty. We can still see traces of it in our present method of measuring angles in degrees and minutes, and also in our method of measuring time in hours, minutes and seconds.

The Mesopotamians were acquainted with square roots and cube roots, and they could solve quadratic equations. They also were aware of exponential and logarithmic relationships[1]. They seemed to value mathematics for its own sake, for the sake of enjoyment and recreation, as much as for its practical applications. On the whole, their algebra was more advanced than their geometry. They knew some of the properties of triangles and circles, but did not prove them in a systematic way.

Egypt: books and geometry

The ancient Egyptians were the first to make books. As early as 4,000 B.C., they began to make books in the form of scrolls by cutting papyrus reeds into thin strips and pasting them into sheets of double thickness. The sheets were glued together end to end, so that they formed a long roll. The rolls were sometimes very long indeed. For example, one roll, which is now in the British Museum, is 17 inches wide and 135 feet long.

The periodic flooding of the Nile meant that each year the land had to be surveyed and boundary lines redrawn. Thus the flooding of the Nile, with its surveying problems, together with the engineering problems of pyramid building, led the Egyptians to develop the science of geometry (which in Greek means "earth measurement").

An ancient Egyptian papyrus book on mathematics was found in the nineteenth century and is now in the British Museum. It was copied by the scribe Ahmose in c. 1,650 B.C., but the mathematical knowledge which it contains is probably much older. The papyrus is entitled *"Directions for*

[1]For a discussion of exponentials and logarithms, see Chapter 3.

Attaining Knowledge of All Dark Things", and it deals with simple equations, fractions, and methods for calculating areas, volumes, etc..

The Egyptians knew, for example, that a triangle whose sides are three units, four units, and five units long is a right triangle[2]. They knew many special right triangles of this kind, and they knew that in these special cases the sum of the areas of the squares formed on the two short sides is equal to the area of the square formed on the longest side. However, there is no evidence that they knew that the relationship holds for every right triangle. It was left to Pythagoras to discover and prove this great theorem in its full generality.

Thales of Miletus

It is known that the Greeks arrived in the Aegean region in three waves. The first to come were the Ionians. Next came the Achaeans, and finally the Dorians. Warfare between the Achaeans and the Ionians weakened both groups, and finally they both were conquered by the Dorians. This conquest by the semi-primitive Dorians was probably the event which produced a dark age in Greek culture between 1,075 B.C. and 850 B.C. During this dark age the art of writing was lost to the Greeks, and the level of artistic and cultural achievement deteriorated.

However, beginning in about 850 B.C., there was a rebirth of Greek culture. This cultural renaissance began in Ionia on the west coast of present-day Turkey, where the Greeks were in close contact with the Mesopotamian civilization. Probably the Homeric epics were written in Miletus, a city on the coast of Asia Minor, in about 700 B.C.. The first three philosophers of the Greek world, Thales, Anaximander and Anaximenes, were also natives of Miletus.

Thales was born in 624 B.C. and died in 546 B.C.. The later Greeks considered him to have been the founder of almost every branch of knowledge. Whenever the wise men of ancient times were listed, Thales was invariably mentioned first. However, most of the achievements for which the Greeks admired Thales were probably not invented by him. He is supposed to have been born of a Phoenecean mother, and to have travelled extensively in Egypt and Mesopotamia, and he probably picked up most of his knowledge of science from these ancient civilizations.

[2]In a right triangle, one of the angles is a 90 degree angle (sometimes called a "right angle"). In other words, two of the sides of a right triangle are perpendicular to each other.

Thales brought Egyptian geometry to Greece, and he also made some original contributions to this field. He changed geometry from a set of *ad hoc* rules into an abstract and deductive science. He was the first to think of geometry as dealing not with real lines of finite thickness and imperfect straightness, but with lines of infinitesimal thickness and perfect straightness. (Echoes of this point of view are found in Plato's philosophy).

Thales had a student named Anaximander (610 B.C. - 546 B.C.) who also helped to bring Egyptian and Mesopotamian science to Greece. He imported the sundial from Egypt, and he was the first to try to draw a map of the entire world. He pictured the sky as a sphere, with the earth floating in space at its center. The sphere of the sky rotated once each day about an axis passing through the polar star. Anaximander knew that the surface of the earth is curved. He deduced this from the fact that as one travels northward, some stars disappear below the southern horizon, while others appear in the north. However, Anaximander thought that a north-south curvature was sufficient. He imagined the earth to be cylindrical rather than spherical in shape. The idea of a spherical earth had to wait for Pythagoras.

Pythagoras

Pythagoras, a student of Anaximander, first became famous as a leader and reformer of the Orphic religion. He was born on the island of Samos, near the Asian mainland, and like other early Ionian philosophers, he is said to have travelled extensively in Egypt and Mesopotamia. In 529 B.C., he left Samos for Croton, a large Greek colony in southern Italy. When he arrived in Croton, his reputation had preceded him, and a great crowd of people came out of the city to meet him. After Pythagoras had spoken to this crowd, six hundred of them left their homes to join the Pythagorean brotherhood without even saying goodbye to their families.

For a period of about twenty years, the Pythagoreans gained political power in Croton, and they also had political influence in the other Greek colonies of the western Mediterranean. However, when Pythagoras was an old man, the brotherhood which he founded fell from power, their temples at Croton were burned, and Pythagoras himself moved to Metapontion, another Greek city in southern Italy. Although it was never again politically influential, the Pythagorean brotherhood survived for more than a hundred years.

The Pythagorean brotherhood admitted women on equal terms, and all its members held their property in common. Even the scientific discoveries of the brotherhood were considered to have been made in common by all its members.

Pythagorean harmony

The Pythagoreans practiced medicine, and also a form of psychotherapy. According to Aristoxenius, a philosopher who studied under the Pythagoreans, "They used medicine to purge the body, and music to purge the soul". Music was of great importance to the Pythagoreans, as it was also to the original followers of Dionysos and Orpheus.

Both in music and in medicine, the concept of harmony was very important. Here Pythagoras made a remarkable discovery which united music and mathematics. He discovered that the harmonics which are pleasing to the human ear can be produced by dividing a lyre string into lengths which are expressible as simple ratios of whole numbers. For example, if we divide the string in half by clamping it at the center, (keeping the tension constant), the pitch of its note rises by an octave. If the length is reduced to 2/3 of the basic length, then the note is raised from the fundamental tone by the musical interval which we call a major fifth, and so on. The discovery that harmonious musical tones could be related by rational numbers made the Pythagoreans think that rational numbers[3] are the key to understanding nature, and this belief became a part of their religion.

Having discovered that musical harmonics are governed by mathematics, Pythagoras fitted this discovery into the framework of Orphism. According to the Orphic religion, the soul may be reincarnated in a succession of bodies. In a similar way (according to Pythagoras), the "soul" of the music is the mathematical structure of its harmony, and the "body" through which it is expressed is the gross physical instrument. Just as the soul can be reincarnated in many bodies, the mathematical idea of the music can be expressed through many particular instruments; and just as the soul is immortal, the idea of the music exists eternally, although the instruments through which it is expressed may decay.

In distinguishing very clearly between mathematical ideas and their physical expression, Pythagoras was building on the earlier work of Thales, who thought of geometry as dealing with dimensionless points and lines of perfect straightness, rather than with real physical objects. The teachings of Pythagoras and his followers served in turn as an inspiration for Plato's idealistic philosophy.

Having found mathematical harmony in the world of sound, and having searched for it in astronomy, Pythagoras tried to find mathematical relationships in the visual world. Among other things, he discovered the five possible regular polyhedra. However, his greatest contribution to geometry is the famous Pythagorean theorem, which is considered to be the most important

[3] i.e., numbers that can be expressed as the ratio of two integers

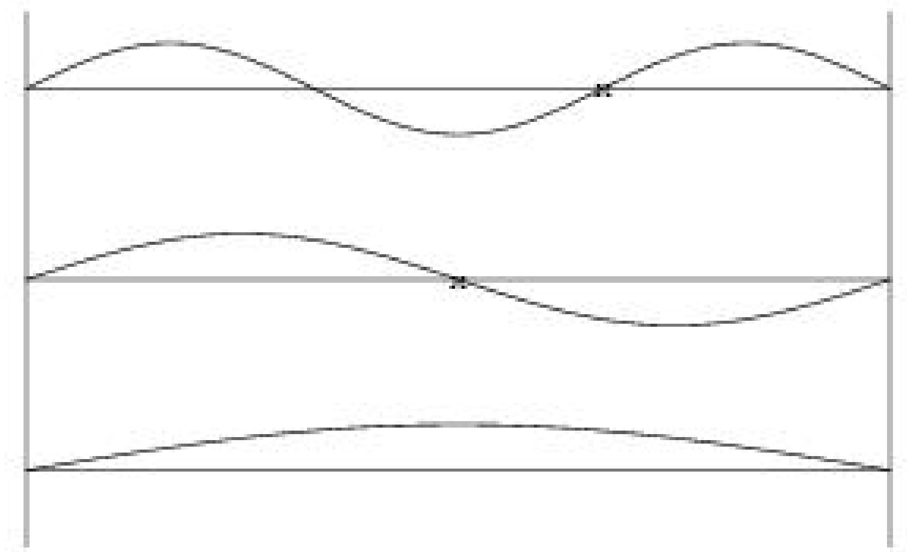

Figure 1.1: *Pythagoras (569 B.C. - 475 B.C.) discovered that the musical harmonics that are pleasing to the human ear can be produced by clamping a lyre string of constant tension at points that are related by rational numbers. In the figure the octave and the major fifth above the octave correspond to the ratios 1/2 and 1/3.*

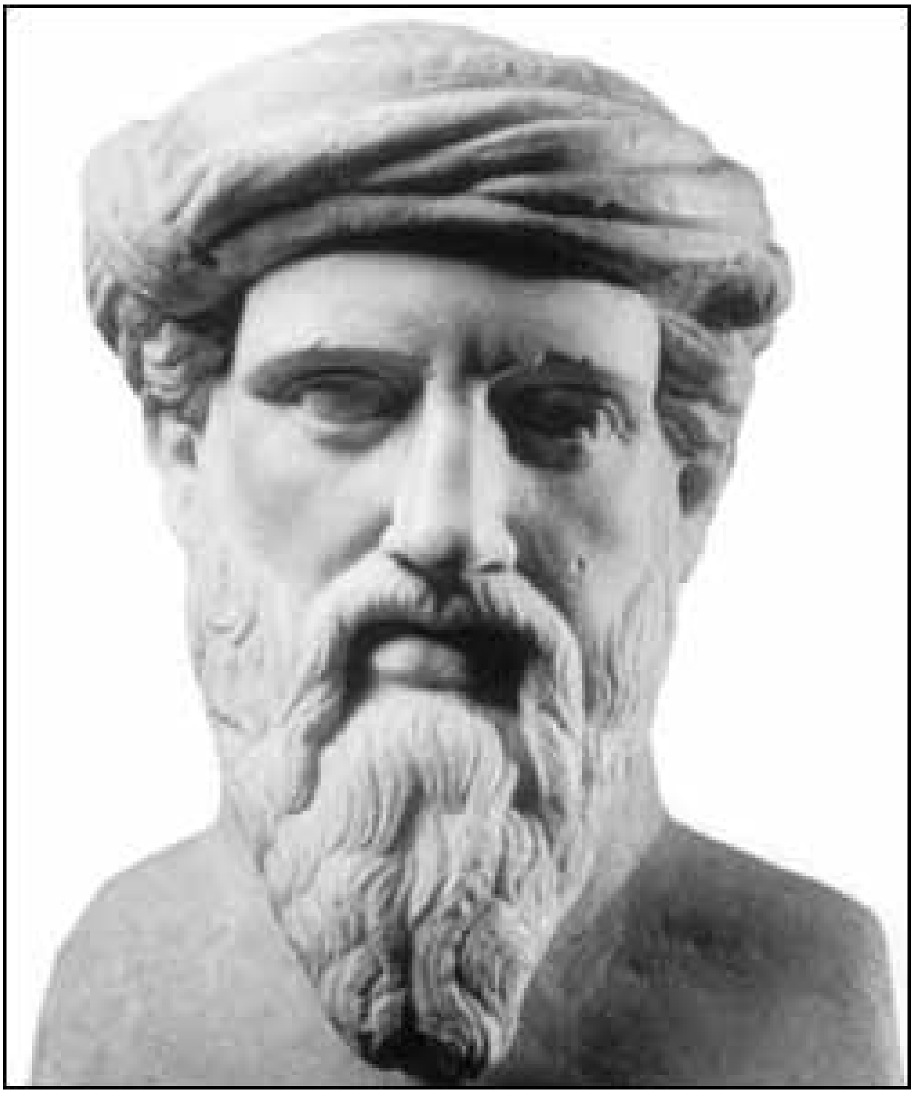

Figure 1.2: *Pythagoras founded a brotherhood that lasted about a hundred years and greatly influenced the development of mathematics and science. The Pythagorean theorem, which he discovered, is considered to be the most important single theorem in mathematics.*

single theorem in the whole of mathematics.

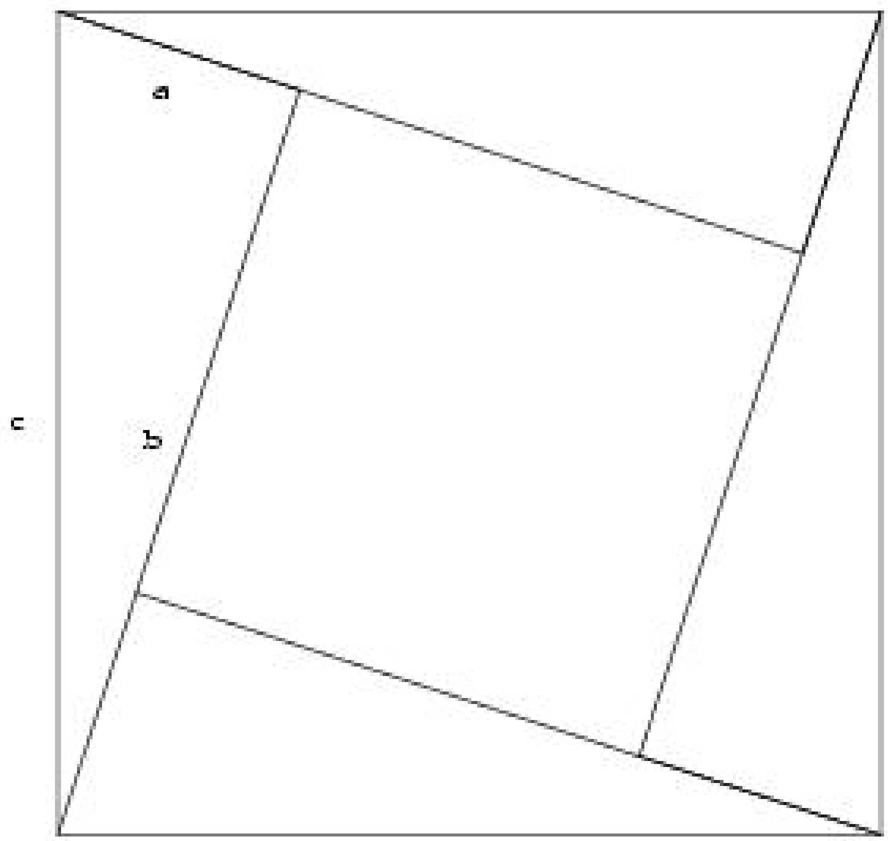

Figure 1.3: *This figure can be used to prove the famous theorem of Pythagoras concerning squares constructed on the sides of a right triangle (i.e. a triangle where two of the sides are perpendicular to each other). It shows a right triangle whose sides, in order of increasing length, are* a, b *and* c. *Four identical copies of this triangle, with total area* $2ab$, *are inscribed inside a square constructed on the long side. The remaining area inside the large square is* $(b-a)^2 = a^2 - 2ab + b^2$ *and therefore the total area of the large square is* $c^2 = a^2 + b^2$.

The Mesopotamians and the Egyptians knew that for many special right triangles, the sum of the squares formed on the two shorter sides is equal to the square formed on the long side. For example, Egyptian surveyors used a triangle with sides of lengths 3, 4 and 5 units. They knew that between the two shorter sides, a right angle is formed, and that for this particular right triangle, the sum of the squares of the two shorter sides is equal to the square of the longer side. Pythagoras proved that this relationship holds for every right triangle.

In exploring the consequences of his great theorem, Pythagoras and his followers discovered that the square root of 2 is an irrational number. (In other words, it cannot be expressed as the ratio of two integers.) The discovery of irrationals upset them so much that they abandoned algebra. They concentrated entirely on geometry, and for the next two thousand years geometrical ideas dominated science and philosophy.

The classical Greek geometers, most of whom were Pythagoreans, discovered many geometrical theorems. They believed that the contemplation of eternal geometrical truths was a way of finding release from the suffering of human existence, and geometry was a part of their religion. There were certain rules that had to be followed in geometrical constructions: only a compass and a straight ruler could be used. The theorems of the geometers of classical Greece were collected and put into a logical order by Euclid, who lived in Alexandria, the capital city of Egypt founded by Alexander of Macedon.

Alexandria

Alexander of Macedon's brief conquest of the entire known world had the effect of blending the ancient cultures of Greece, Persia, India and Egypt, and producing a world culture. The era associated with this culture is usually called the Hellenistic Era (323 B.C. - 146 B.C.). Although the Hellenistic culture was a mixture of all the great cultures of the ancient world, it had a decidedly Greek flavor, and during this period the language of educated people throughout the known world was Greek.

Nowhere was the cosmopolitan character of the Hellenistic Era more apparent than at Alexandria in Egypt. No city in history has ever boasted a greater variety of people. Ideally located at the crossroads of world trading routes, Alexandria became the capital of the world - not the political capital, but the cultural and intellectual capital.

Miletus in its prime had a population of 25,000; Athens in the age of Pericles had about 100,000 people; but Alexandria was the first city in history to reach a population of over a million!

Strangers arriving in Alexandria were impressed by the marvels of the city - machines which sprinkled holy water automatically when a five-drachma coin was inserted, water-driven organs, guns powered by compressed air, and even moving statues, powered by water or steam!

For scholars, the chief marvels of Alexandria were the great library and the Museum established by Ptolemy I, one of Alexander's generals. Credit for making Alexandria the intellectual capital of the world must go to Ptolemy

I and his successors (all of whom were named Ptolemy except the last of the line, the famous queen, Cleopatra). Realizing the importance of the schools which had been founded by Pythagoras, Plato and Aristotle, Ptolemy I established a school at Alexandria. This school was called the Museum, because it was dedicated to the muses.

Near to the Museum, Ptolemy built a great library for the preservation of important manuscripts. The collection of manuscripts which Aristotle had built up at the Lyceum in Athens became the nucleus of this great library. The library at Alexandria was open to the general public, and at its height it was said to contain 750,000 volumes. Besides preserving important manuscripts, the library became a center for copying and distributing books.

One of the first scholars to be called to the newly-established Museum was Euclid. He was born in 325 B.C. and was probably educated at Plato's Academy in Athens. While in Alexandria, Euclid wrote the most successful text-book of all time, the *Elements of Geometry*. The theorems in this splendid book were not, for the most part, originated by Euclid. They were the work of many generations of classical Greek geometers. Euclid's contribution was to take the theorems of the classical period and to arrange them in an order which is so logical and elegant that it almost defies improvement. One of Euclid's great merits is that he reduces the number of axioms to a minimum, and he does not conceal the doubiousness of certain axioms.

Euclid's axiom concerning parallel lines has an interesting history: This axiom states that "Through a given point not on a given line, one and only one line can be drawn parallel to a given line". At first, mathematicians doubted that it was necessary to have such an axiom. They suspected that it could be proved by means of Euclid's other more simple axioms. After much thought, however, they decided that the axiom is indeed one of the necessary foundations of classical geometry. They then began to wonder whether there could be another kind of geometry where the postulate concerning parallels is discarded. These ideas were developed in the 18th and 19th centuries by Lobachevski, Bolyai, Gauss and Riemann, and in the 20th century by Levi-Civita. In 1915, the mathematical theory of non-Euclidian geometry finally became the basis for Einstein's general theory of relativity.

Besides classical geometry, Euclid's book also contains some topics in number theory. For example, he discusses irrational numbers, and he proves that the number of primes is infinite. He also discusses geometrical optics.

Euclid's *Elements* has gone through more than 1,000 editions since the invention of printing - more than any other book, with the exception of the Bible. Its influence has been immense. For more than two thousand years, Euclid's *Elements of Geometry* has served as a model for rational thought.

One of the Pythagorean mottos was: "A diagram and a step, not a di-

Figure 1.4: *Euclid (325 B.C. - 265 B.C.) was probably educated at Plato's Academy in Athens, but he later worked at the Museum in Alexandria. Euclid arranged the theorems of the classical Greek geometers in an order so logical and elegant that it can hardly be improved. His "Elements of Geometry" proved to be the most successful textbook of all time.*

agram and a penny". Euclid, who belonged to the Pythagorean tradition, once rebuked a student who asked what profit could be gained from a knowledge of geometry. Euclid called a slave and said (pointing at the student): "He wants to profit from geometry. Give him a penny." The student was then dismissed from Euclid's school.

The Greeks of the classical age could afford to ignore practical matters, since their ordinary work was performed for them by slaves. It is unfortunate that the craftsmen and metallurgists of ancient Greece were slaves, while the philosophers were gentlemen who refused to get their hands dirty. An unbridgeable social gap separated the philosophers from the craftsmen; and the empirical knowledge of chemistry and physics, which the craftsmen had gained over the centuries, was never incorporated into Greek philosophy.

However in Alexandria the attitude in general was much more practical, and the scholars at the Museum regarded geometry and other branches of mathematics as tools to be used in navigation and engineering.

Eratosthenes (276 B.C. - 196 B.C.), the director of the library at Alexandria, was probably the most cultured man of the Hellenistic Era. His interests and abilities were universal. He was an excellent historian, in fact the first historian who ever attempted to set up an accurate chronology of events. He was also a literary critic, and he wrote a treatise on Greek comedy. He made many contributions to mathematics, including a study of prime numbers and a method for generating primes called the "sieve of Eratosthenes".

Figure 1.5: *Eratosthenes (276 B.C. - 196 B.C.) was the director of the great library at Alexandria in Egypt. He made an astonishingly precise measurement of the radius of the earth. This measurement showed that the earth's surface was much larger than the area of the known world, and Eratosthenes correctly concluded that most of the earth's surface is covered by water. He believed that it would be possible to reach India by sailing westward from Spain.*

As a geographer, Eratosthenes made a map of the world which, at that time, was the most accurate that had ever been made. The positions of various places on Eratosthenes' map were calculated from astronomical observations. The latitude was calculated by measuring the angle of the polar star above the horizon, while the longitude probably was calculated from the apparent local time of lunar eclipses.

As an astronomer, Eratosthenes made an extremely accurate measurement of the angle between the axis of the earth and the plane of the sun's apparent motion; and he also prepared a map of the sky which included the positions of 675 stars.

Eratosthenes' greatest achievement however, was an astonishingly precise measurement of the radius of the earth. The value which he gave for the radius was within 50 miles of what we now consider to be the correct value! To make this remarkable measurement, Eratosthenes of course assumed that the earth is spherical, and he also assumed that the sun is so far away from the earth that rays of light from the sun, falling on the earth, are almost parallel. He knew that directly south of Alexandria there was a city called

Seyne, where at noon on a midsummer day, the sun stands straight overhead. Given these facts, all he had to do to find the radius of the earth was to measure the distance between Alexandria and Seyne. Then, at noon on a midsummer day, he measured the angle which the sun makes with the vertical at Alexandria. From these two values, he calculated the circumference of the earth to be a little over 25,000 miles. This was so much larger than the size of the known world that Eratosthenes concluded (correctly) that most of the earth's surface must be covered with water; and he stated that "If it were not for the vast extent of the Atlantic, one might sail from Spain to India along the same parallel."

The Hellenistic astronomers not only measured the size of the earth - they also measured the sizes of the sun and the moon, and their distances from the earth. Among the astronomers who worked on this problem was Aristarchus (c. 320 B.C. - c. 250 B.C.). Like Pythagoras, he was born on the island of Samos, and he may have studied in Athens under Strato. However, he was soon drawn to Alexandria, where the most exciting scientific work of the time was being done.

Aristarchus calculated the size of the moon by noticing the shape of the shadow of the earth thrown on the face of the moon during a solar eclipse. From the shape of the earth's shadow, he concluded that the diameter of the moon is about a third the diameter of the earth. (This is approximately correct).

From the diameter of the moon and the angle between its opposite edges when it is seen from the earth, Aristarchus could calculate the distance of the moon from the earth. Next he compared the distance from the earth to the moon with the distance from the earth to the sun. To do this, he waited for a moment when the moon was exactly half-illuminated. Then the earth, moon and sun formed a right triangle, with the moon at the corner corresponding to the right angle. Aristarchus, standing on the earth, could measure the angle between the moon and the sun. He already knew the distance from the earth to the moon, so now he knew two angles and one side of the right triangle. This was enough to allow him to calculate the other sides, one of which was the sun-earth distance. His value for this distance was not very accurate, because small errors in measuring the angles were magnified in the calculation.

Aristarchus concluded that the sun is about twenty times as distant from the earth as the moon, whereas in fact it is about four hundred times as distant. Still, even the underestimated distance which Aristarchus found convinced him that the sun is enormous! He calculated that the sun has about seven times the diameter of the earth, and three hundred and fifty times the earth's volume. Actually, the sun's diameter is more than a hundred times the diameter of the earth, and its volume exceeds the earth's volume by a factor of more than a million!

Even his underestimated value for the size of the sun was enough to convince Aristarchus that the sun does not move around the earth. It seemed ridiculous to him to imagine the enormous sun circulating in an orbit around the tiny earth. Therefore he proposed a model of the solar system in which the earth and all the planets move in orbits around the sun, which remains motionless at the center; and he proposed the idea that the earth spins about its axis once every day.

Although it was the tremendous size of the sun which suggested this model to Aristarchus, he soon realized that the heliocentric model had many calculational advantages: For example, it made the occasional retrograde motion of certain planets much easier to explain. Unfortunately, he did not work out detailed table for predicting the positions of the planets. If he had done so, the advantages of the heliocentric model would have been so obvious that it might have been universally adopted almost two thousand years before the time of Copernicus, and the history of science might have been very different.

The model of the solar system on which the Hellenistic astronomers finally agreed was not that of Aristarchus but an alternative (and inferior) earth-centered model developed by Hipparchus (c. 190 B.C. - c. 120 B.C.). Although his model of the solar system was inferior to that of Aristarchus, Hipparchus made many important contributions to astronomy and mathematics. For example, he was the first person to calculate and publish tables of trigonometric functions.

- **Problem 1.1**: Calculate $[cos(a)]^2+[sin(a)]^2$ for all of the angles shown in Table 1.1. How is the result related to Pythagoras' theorem concerning the squares of the sides of right triangles?

- **Problem 1.2**: The total of all three angles inside any triangle is π (or 180 degrees). What will the angles be at the corners of a triangle where all three sides have equal length (an equilateral triangle)? How is this result related to the fact that when t is $\pi/6$ (30 degrees), $sin(t) = 1/2$?

- **Problem 1.3**: Give an argument explaining the values of $sin(t)$ and $cos(t)$ when t is $\pi/4$ (45 degrees).

- **Problem 1.4**: How can the minus signs in Table 1.1 be interpreted?

- **Problem 1.5**: Extend Table 1.1 by calculating values of $sin(t)$, $cos(t)$ and $tan(t)$ when $t = 7\pi/6$ and $t = 5\pi/4$.

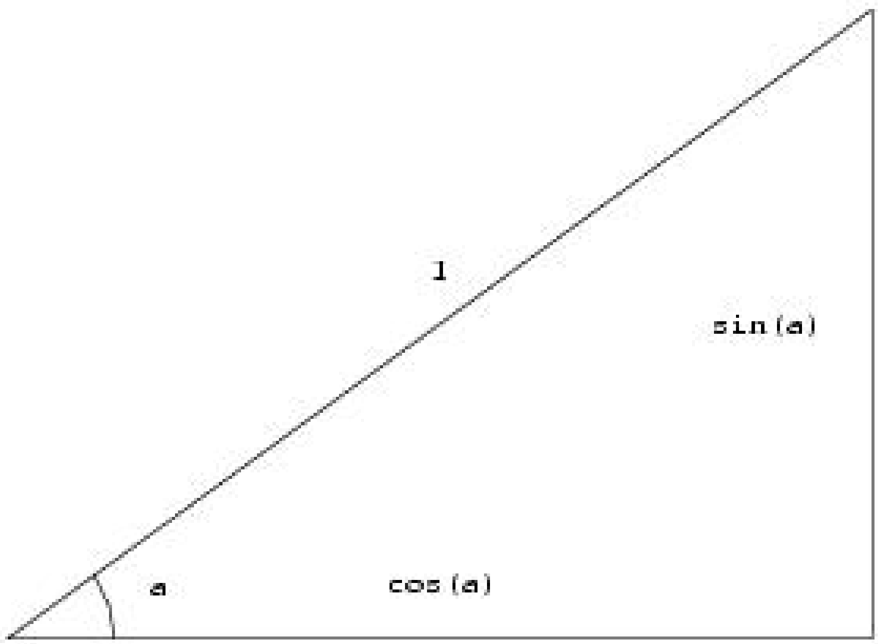

Figure 1.6: *This figure illustrates the definitions of the trigonometric functions sin(a) and cos(a) which were first tabulated by the the Egyptian astronomer Hipparchus. It shows a right triangle whose longest side has a length equal to 1. One of the small angles is called a. The length of the side opposite to this angle is then called sin(a), while the length of the remaining side is called cos(a).*

Table 1.1: This table shows the trigonometric functions $sin(t)$, $cos(t)$ and $tan(t)$ as functions of the angle t, where $tan(t) \equiv sin(t)/cos(t)$. The angles are expressed both in degrees and in radians. (1 radian = $180/\pi$ degrees). Tables like this were first made by the Egyptian astronomer Hipparchus.

t (degrees)	t (radians)	$sin(t)$	$cos(t)$	$tan(t)$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	∞
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
180	π	0	-1	0

Archimedes

Archimedes was the greatest mathematician of the Hellenistic Era. In fact, together with Newton and Gauss, he is considered to be one of the greatest mathematicians of all time.

Archimedes was born in Syracuse in Sicily in 287 B.C.. He was the son of an astronomer, and he was also a close relative of Hieron II, the king of Syracuse. Like most scientists of his time, Archimedes was educated at the Museum in Alexandria, but unlike most, he did not stay in Alexandria. He returned to Syracuse, probably because of his kinship with Hieron II. Being a wealthy aristocrat, Archimedes had no need for the patronage of the Ptolemys.

Unlike Euclid, Archimedes did not belong to the tradition of the classical Pythagorens for whom geometry was a part of religion. He was more in tune with the spirit of busy, commercial Alexandria, where mathematics was regarded as a practical tool to be used in navigation and architecture. In his book *On Method*, Archimedes even confesses to cutting out figures from papyrus and weighing them as a means of obtaining intuition about areas and centers of gravity. Of course, having done this, he then derived the areas and centers of gravity by more rigorous methods.

One of Archimedes' great contributions to mathematics was his development of methods for finding the areas of plane figures bounded by curves, as well as methods for finding the areas and volumes of solid figures bounded by curved surfaces. To do this, he employed the "doctrine of limits". For example, to find the area of a circle, he began by inscribing a square inside the circle. The area of the square was a first approximation to the area of the circle. Next, he inscribed a regular octagon and calculated its area, which was a closer approximation to the area of the circle. This was followed by a figure with 16 sides, and then 32 sides, and so on. Each increase in the number of sides brought him closer to the true area of the circle.

Archimedes also circumscribed polygons about the circle, and thus he obtained an upper limit for the area, as well as a lower limit. The true area was trapped between the two limits. In this way, Archimedes showed that the value of pi lies between 223/71 and 220/70.

Sometimes Archimedes' use of the doctrine of limits led to exact results. For example, he was able to show that the ratio between the volume of a sphere inscribed in a cylinder to the volume of the cylinder is 2/3, and that the area of the sphere is 2/3 the area of the cylinder. He was so pleased with this result that he asked that a sphere and a cylinder be engraved on his tomb, together with the ratio, 2/3.

Another problem which Archimedes was able to solve exactly was the

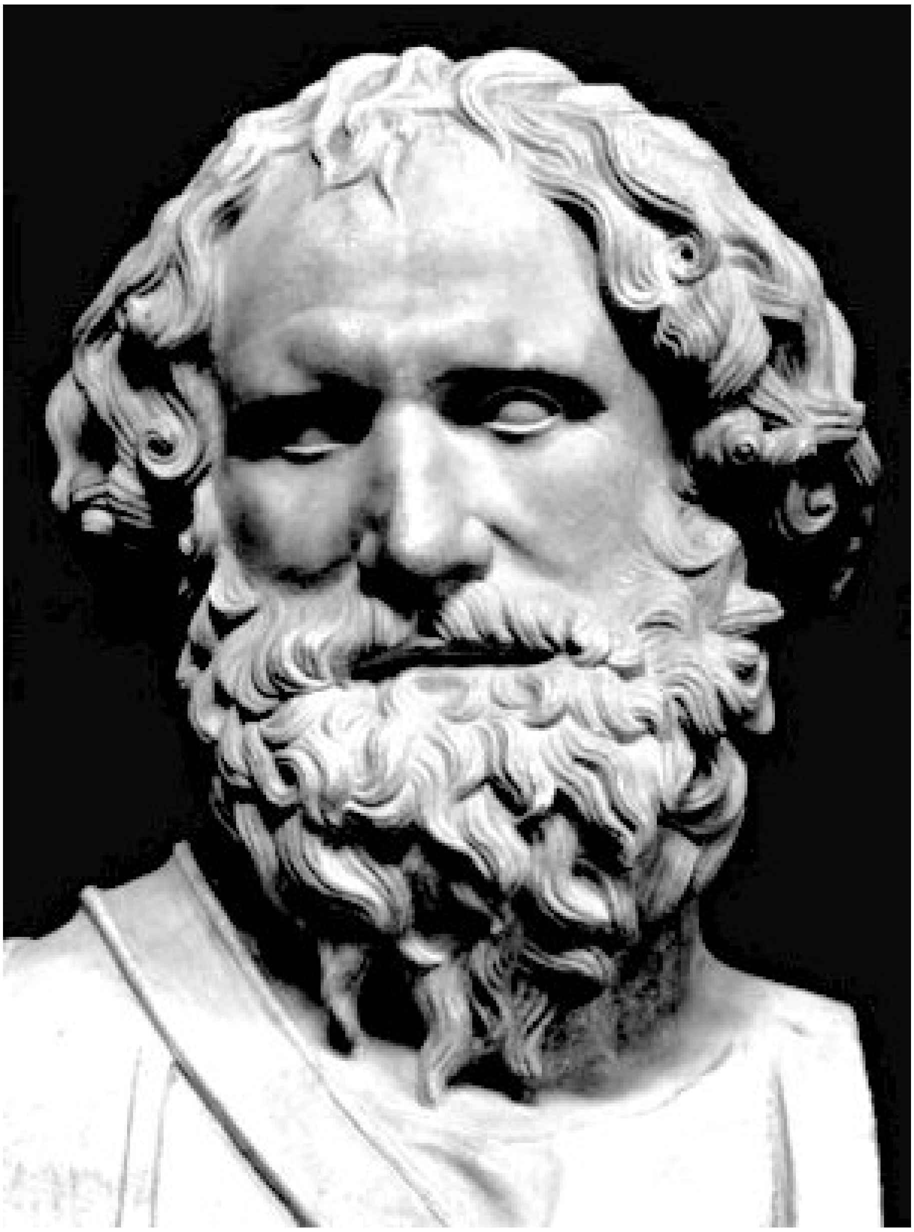

Figure 1.7: *A statue of Archimedes (287 B.C. -212 B.C.). Together with Newton and Gauss, he is considered to be one of the three greatest mathematicians of all time.*

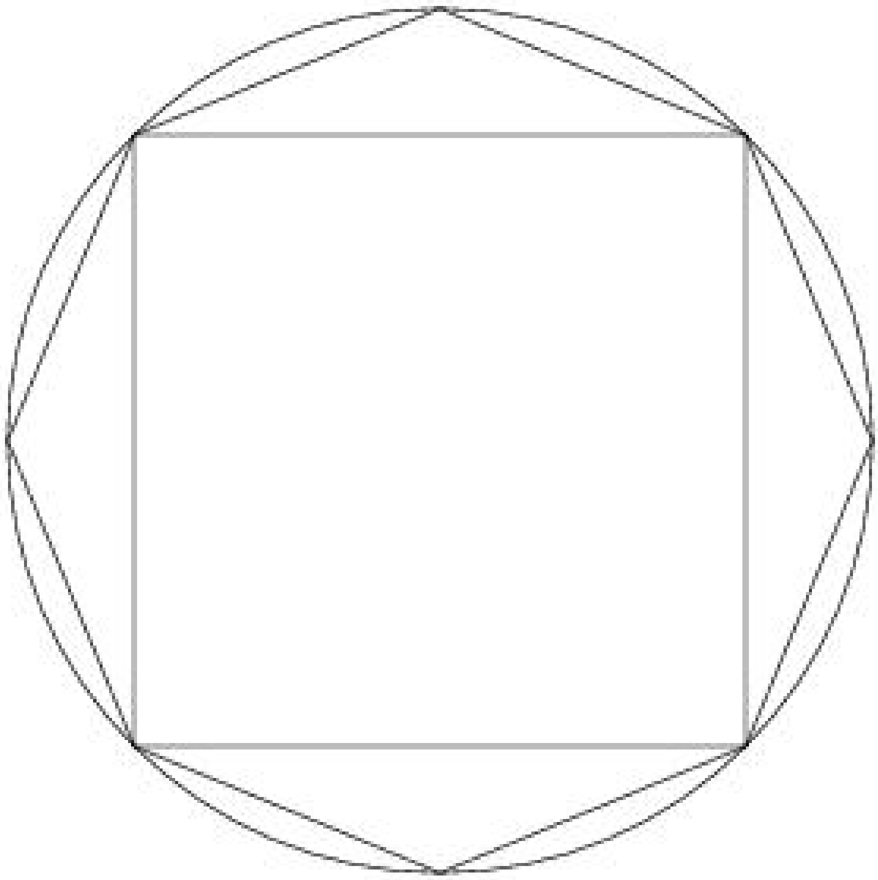

Figure 1.8: *This figure illustrates one of the ways in which Archimedes used his doctrine of limits to calculate the area of a circle. He first inscribed a square within the circle, then an octagon, then a figure with 16 sides, and so on. As the number of sides became very large, the area of these figures (which he could calculate) approached the true area of the circle.*

problem of calculating the area of a plane figure bounded by a parabola. In his book *On method*, Archimedes says that it was his habit to begin working on a problem by thinking of a plane figure as being composed of a very large number of narrow strips, or, in the case of a solid, he thought of it as being built up from a very large number of slices. This is exactly the approach which is used in integral calculus.

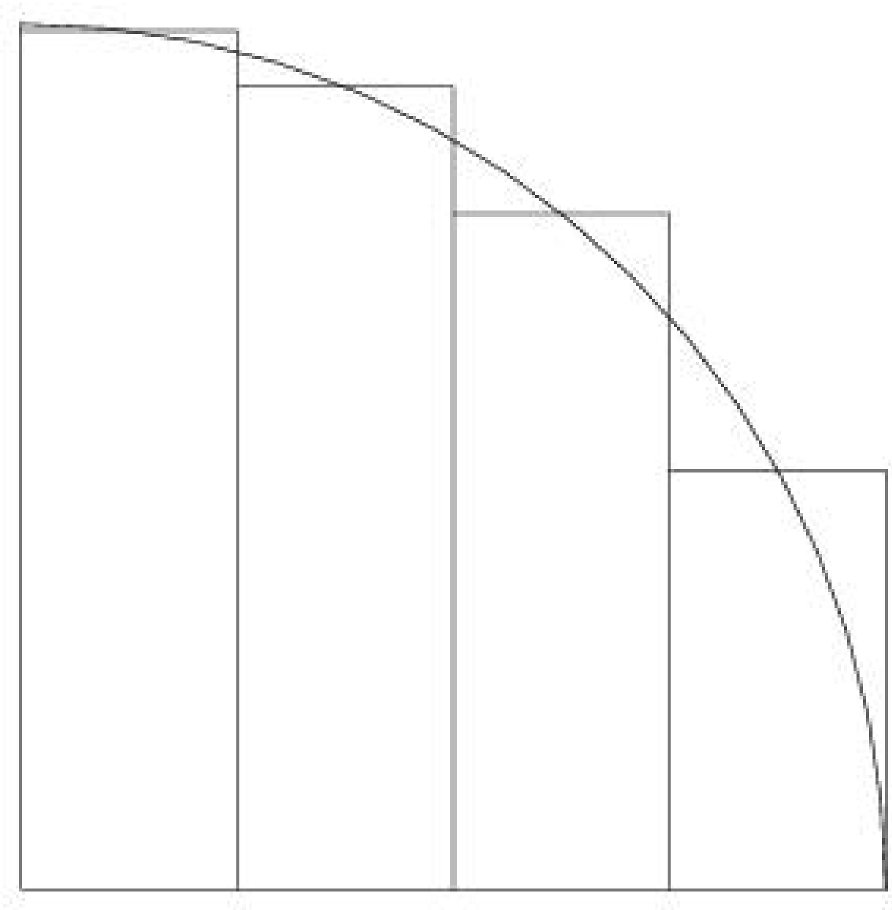

Figure 1.9: *Here we see another way in which Archimedes used his doctrine of limits. He could calculate the areas of figures bounded by curves by dividing up these areas into a large number of narrow strips. As the number of strips became very large, their total area approached the true area of the figure.*

Archimedes must really be credited with the invention of both differential and integral calculus. He used what amounts to integral calculus to find the volumes and areas not only of spheres, cylinders and cones, but also of spherical segments, spheroids, hyperboloids and paraboloids of revolution; and his method for constructing tangents anticipates differential calculus.

Unfortunately, Archimedes was unable to transmit his invention of the calculus to the other mathematicians of his time. The difficulty was that there was not yet any such thing as algebraic geometry. The Pythagoreans had never recovered from the shock of discovering irrational numbers, and they had therefore abandoned algebra in favor of geometry. The union of algebra and geometry, and the development of a calculus which even non-geniuses could use, had to wait for Descartes, Fermat, Newton and Leibniz.

- **Problem 1.6**: In Figure 1.8, a square is inscribed in a circle. If the radius of the circle is r, What is the length of a side of the square?
- **Problem 1.7**: In Figure 1.8, an octagon is also inscribed in the circle. Use the Pythagorean theorem to find the length of a side of the octagon. What is the total length of all eight sides of the octagon?
- **Problem 1.8**: What is the area of the octagon in Figure 1.8?
- **Problem 1.9**: If the circumference of a circle is given by $2\pi r$, and if the area of a circle is given by πr^2, use the results of Problems 1.7 and 1.8 to find a lower limit to the value of π.

Civilizations of the East

After the fall of Rome in the 5th century A.D., Europe became a culturally backward area. However, the great civilizations of Asia and the Middle East continued to flourish, and it was through contact with these civilizations that science was reborn in the west.

During the dark ages of Europe, a particularly high level of civilization existed in China. Paper was invented in China at the end of the first century A.D. facilitated this project, and it greatly stimulated scholarship and literature.

It was during the T'ang period (618 A.D. - 906 A.D) that the Chinese made an invention of immense importance to the cultural evolution of mankind. This was the invention of printing. Together with writing, printing is one of the key inventions which form the basis of human cultural evolution.

The Chinese had for a long time followed the custom of brushing engraved official seals with ink and using them to stamp documents. The type of ink which they used was made from lamp-black, water and binder. In fact, it was what we now call "India ink". However, in spite of its name, India ink is a Chinese invention, which later spread to India, and from there to Europe.

We mentioned that paper of the type which we now use was invented in China in the first century A.D.. Thus, the Buddhist monks of China had all the elements which they needed to make printing practical: They had good ink, cheap, smooth paper, and the tradition of stamping documents with ink-covered engraved seals. The first block prints which they produced date from the 8th century A.D.. They were made by carving a block of wood the size of a printed page so that raised characters remained, brushing ink onto the block, and pressing this onto a sheet of paper.

The Chinese made some early experiments with movable type, but movable type never became popular in China, because the Chinese written language contains 10,000 characters. However, printing with movable type was highly successful in Korea as early as the 15th century A.D..

An "information explosion" occurred in the west following the introduction of printing with movable type, but this never occurred in China. It is ironical that although both paper and printing were invented by the Chinese, the full effect of these immensely important inventions bypassed China and instead revolutionized the west.

In Indian mathematics, algebra and trigonometry were especially highly developed during the dark ages of Europe. For example, the astronomer Brahmagupta (598 A.D. - 660 A.D.) applied algebraic methods to astronomical problems. The notation for zero and the decimal system were invented, perhaps independently, in China and in India. These mathematical techniques were later transmitted to Europe by the Arabs.

Figure 1.10: *One of a series of astronomical observatories built near Jaipur, India, by the astronomer-ruler Sawai Jai Singh (1688-1743), who revived ancient Indian astronomical traditions. Jai Singh also made use of the work of Nasir al-Din al-Tusi (1201-1274) and Ulugh Beg (1394-1449).*

Indian mining and metallurgy were also highly developed. The Europeans of the middle ages prized fine laminated steel from Damascus; but it was not in Damascus that the technique of making steel originated. The Arabs learned steelmaking from the Persians, and Persia learned it from India.

After the burning of the great library at Alexandria and the destruction of Hellenistic civilization, most of the books of the classical Greek and Hellenistic philosophers were lost. However, a few of these books survived and were translated from Greek, first into Syriac, then into Arabic and finally from Arabic into Latin. By this roundabout route, fragments from the wreck of the classical Greek and Hellenistic civilizations drifted back into the consciousness of the west.

We mentioned that the Roman empire was ended in the 5th century A.D. by attacks of barbaric Germanic tribes from northern Europe. However, by that time, the Roman empire had split into two halves. The eastern half, with its capital at Byzantium (Constantinople), survived until 1453, when the last emperor was killed vainly defending the walls of his city against the Turks.

The Byzantine empire included many Syriac-speaking subjects; and in fact, beginning in the 3rd century A.D., Syriac replaced Greek as the major language of western Asia. In the 5th century A.D., there was a split in the Christian church of Byzantium;and the Nestorian church, separated from the official Byzantine church. The Nestorians were bitterly persecuted by the Byzantines, and therefore they migrated, first to Mesopotamia, and later to south-west Persia. (Some Nestorians migrated as far as China.)

During the early part of the middle ages, the Nestorian capital at Gondisapur was a great center of intellectual activity. The works of Plato, Aristotle, Hippocrates, Euclid, Archimedes, Ptolemy, Hero and Galen were translated into Syriac by Nestorian scholars, who had brought these books with them from Byzantium.

Among the most distinguished of the Nestorian translators were the members of a family called Bukht-Yishu (meaning "Jesus hath delivered"), which produced seven generations of outstanding scholars. Members of this family were fluent not only in Greek and Syriac, but also in Arabic and Persian.

In the 7th century A.D., the Islamic religion suddenly emerged as a conquering and proselytizing force. Inspired by the teachings of Mohammad (570 A.D. - 632 A.D.), the Arabs and their converts rapidly conquered western Asia, northern Africa, and Spain. During the initial stages of the conquest, the Islamic religion inspired a fanaticism in its followers which was often hostile to learning. However, this initial fanaticism quickly changed to an appreciation of the ancient cultures of the conquered territories; and during the middle ages, the Islamic world reached a very high level of culture and civilization.

Thus, while the century from 750 to 850 was primarily a period of translation from Greek to Syriac, the century from 850 to 950 was a period of translation from Syriac to Arabic. It was during this latter century that Yuhanna Ibn Masawiah (a member of the Bukht-Yishu family, and medical advisor to Caliph Harun al-Rashid) produced many important translations into Arabic.

The skill of the physicians of the Bukht-Yishu family convinced the Caliphs of the value of Greek learning; and in this way the family played an extremely important role in the preservation of the world's cultural heritage. Caliph al-Mamun, the son of Harun al-Rashid, established at Baghdad a library and a school for translation, and soon Baghdad replaced Gondisapur as a center of learning.

The English word "chemistry" is derived from the Arabic words *"al-chimia"*, which mean "the changing". The earliest alchemical writer in Arabic was Jabir (760-815), a friend of Harun al-Rashid. Much of his writing deals with the occult, but mixed with this is a certain amount of real chemical knowledge. For example, in his *Book of Properties*, Jabir gives the following recipe for making what we now call lead hydroxycarbonate (white lead), which is used in painting and pottery glazes:

"Take a pound of litharge, powder it well and heat it gently with four pounds of vinegar until the latter is reduced to half its original volume. The take a pound of soda and heat it with four pounds of fresh water until the volume of the latter is halved. Filter the two solutions until they are quite clear, and then gradually add the solution of soda to that of the litharge. A white substance is formed, which settles to the bottom. Pour off the supernatant water, and leave the residue to dry. It will become a salt as white as snow."

Another important alchemical writer was Rahzes (c. 860 - c. 950). He was born in the ancient city of Ray, near Teheran, and his name means "the man from Ray". Rhazes studied medicine in Baghdad, and he became chief physician at the hospital there. He wrote the first accurate descriptions of smallpox and measles, and his medical writings include methods for setting broken bones with casts made from plaster of Paris. Rahzes was the first person to classify substances into vegetable, animal and mineral. The word *"al-kali"*, which appears in his writings, means "the calcined" in Arabic. It is the source of our word "alkali", as well as of the symbol K for potassium.

The greatest physician of the middle ages, Avicinna, (Abu-Ali al Hussain Ibn Abdullah Ibn Sina, 980-1037), was also a Persian, like Rahzes. More than a hundred books are attributed to him. They were translated into Latin in the 12th century, and they were among the most important medical books used in Europe until the time of Harvey. Avicinina also wrote on alchemy, and he is important for having denied the possibility of transmutation of elements.

In mathematics, one of the most outstanding Arabic writers was al-Khwarizmi (c. 780 - c. 850). The title of his book, *Ilm al-jabr wa'd muqabalah*, is the source of the English word "algebra". In Arabic *al-jabr* means "the equating". Al-Khwarizmi's name has also become an English word, "algorism", the old word for arithmetic. Al-Khwarizmi drew from both Greek and Hindu sources, and through his writings the decimal system and the use of zero were transmitted to the west.

One of the outstanding Arabic physicists was al-Hazen (965-1038). He made the mistake of claiming to be able to construct a machine which could regulate the flooding of the Nile. This claim won him a position in the service of the Egyptian Caliph, al-Hakim. However, as al-Hazen observed Caliph al-Hakim in action, he began to realize that if he did not construct his machine *immediately*, he was likely to pay with his life! This led al-Hazen to the rather desperate measure of pretending to be insane, a ruse which he kept up for many years. Meanwhile he did excellent work in optics, and in this field he went far beyond anything done by the Greeks.

Al-Hazen studied the reflection of light by the atmosphere, an effect which makes the stars appear displaced from their true positions when they are near the horizon; and he calculated the height of the atmospheric layer above the earth to be about ten miles. He also studied the rainbow, the halo, and the reflection of light from spherical and parabolic mirrors. In his book, *On the Burning Sphere*, he shows a deep understanding of the properties of convex lenses. Al-Hazen also used a dark room with a pin-hole opening to study the image of the sun during an eclipse. This is the first mention of the *camera obscura*, and it is perhaps correct to attribute the invention of the *camera obscura* to al-Hazen.

Another Islamic philosopher who had great influence on western thought was Averröes, who lived in Spain from 1126 to 1198. His writings took the form of thoughtful commentaries on the works of Aristotle. He shocked both his Moslem and his Christian readers by maintaining that the world was not created at a definite instant, but that it instead evolved over a long period of time, and is still evolving.

Like Aristotle, Averröes seems to have been groping towards the ideas of evolution which were later developed in geology by Steno, Hutton and Lyell and in biology by Darwin and Wallace. Much of the scholastic philosophy which developed at the University of Paris during the 13th century was aimed at refuting the doctrines of Averröes; but nevertheless, his ideas survived and helped to shape the modern picture of the world.

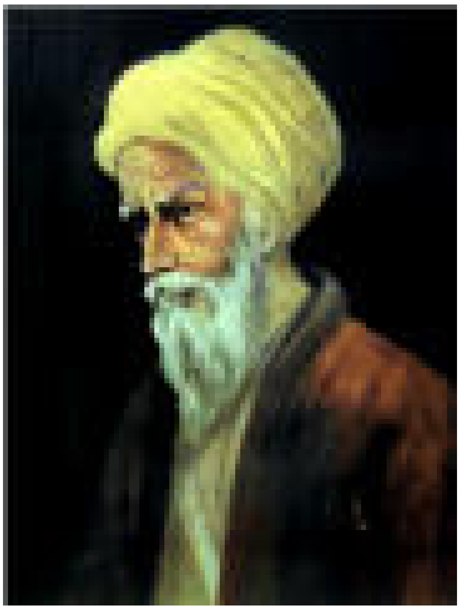

Figure 1.11: *Al-Hazen (965-1038) did important work in many branches of physics, especially in optics. He studied the laws of refraction and is credited with the invention of the camera obscura.*

East-west contacts

Towards the end of the middle ages, Europe began to be influenced by the advanced Islamic civilization. European scholars were anxious to learn, but there was an "iron curtain" of religious intolerance which made travel in the Islamic countries difficult and dangerous for Christians. However, in the 12th century, parts of Spain, including the cities of Córdoba and Toledo, were reconquered by the Christians. These cities had been Islamic cultural centers, and many Moslem scholars, together with their manuscripts, remained in the cities when they passed into the hands of the Christians. Thus Córdoba and Toledo became centers for the exchange of ideas between east and west; and it was these cities that many of the books of the classical Greek and Hellenistic philosophers were translated from Arabic into Latin.

During the Mongol period (1279-1328), direct contact between Europe and China was possible because the Mongols controlled the entire route across central Asia; and during this period Europe received from China three revolutionary inventions: printing, gunpowder and the magnetic compass.

Another bridge between east and west was established by the crusades. In 1099, taking advantage of political divisions in the Moslem world, the Christians conquered Jerusalem and Palestine, which they held until 1187.

Figure 1.12: *Ulugh Beg (1394-1449), a grandson of Tamurlane, became the ruler of Samarkand at the age of 16. He established an institution of higher learning there and built an astronomical observatory. Ulugh Beg's tables of trigonometric functions were accurate to at least 7 figures, and they were tabulated at intervals of 1 degree.*

This was the first of a series of crusades, the last of which took place in 1270. European armies, returning from the Middle East, brought with them a taste for the luxurious spices, jewelry, leatherwork and fine steel weapons of the orient, and their control of Mediterranean sea routes made trade with the east both safe and profitable. Most of the profit from this trade went to a few cities, particularly to Venice and Florence.

The prosperity of these cities, and their close contact with east, gave rise to the Italian Renaissance, a revival of interest in the art, science and literature of the ancient world. This heritage that had been preserved and enriched by the eastern civilizations, and during the 13th-15th centuries it was rediscovered with enthusiasm by the west. In Italy the Renaissance produced such figures as Leonardo da Vinci, Michelangelo and Galileo Galilei. Copernicus spent ten years studying in Italy, where he absorbed the ideas that led him to rediscover and develop his sun-centered model of the solar system, a model that had first been put forward in Egypt by Aristarchus, many centuries earlier. As the Renaissance moved Northward, it produced many important artists, writers and scientists, for example Rembrandt, Dürer, Shakespeare, Erasmus and Descartes. We shall see in the next section that Descartes reunited algebra and geometry, two disciplines that had been separated ever since their combination had led the Pythagoreans to discover irrational numbers. (This discovery that horrified them to such an extent that they abandoned algebra.) By reuniting algebra and geometry, Descartes paved the way for the rediscovery of differential and integral calculus, two fields that had been lost since the time of Archimedes.

Descartes

Until the night of November 10, 1619, algebra and geometry were separate disciplines. On that autumn evening, the troops of the Elector of Bavaria were celebrating the Feast of Saint Martin at the village of Neuberg in Bohemia. With them was a young Frenchman named René Descartes (1596-1659), who had enlisted in the army of the Elector in order to escape from Parisian society. During that night, Descartes had a series of dreams which, as he said later, filled him with enthusiasm, converted him to a life of philosophy, and put him in possession of a wonderful key with which to unlock the secrets of nature.

The program of natural philosophy on which Descartes embarked as a result of his dreams led him to the discovery of analytic geometry, the combination of algebra and geometry. Essentially, Descartes' method amounted to labeling each point in a plane with two numbers, f and t. These numbers represented the distance between the point and two perpendicular fixed

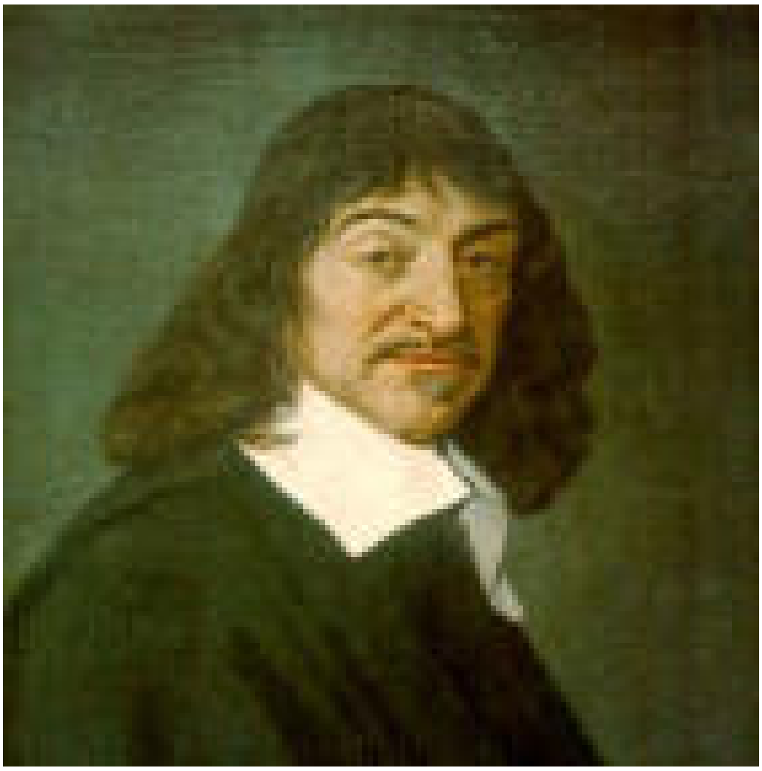

Figure 1.13: *René Descartes (1596-1650) reunited algebra and geometry, which had been separated ever since the Pythagoreans abandoned algebra after their shocking discovery of irrational numbers, a discovery so contrary to their religion that they kept it secret and renounced algebra. Descartes' algebraic geometry paved the way for the rediscovery of calculus by Fermat, Newton, and Leibniz. Cartesian coordinates are named after him.*

lines, (the coordinate axes). Then every algebraic equation relating f and t generated a curve in the plane.

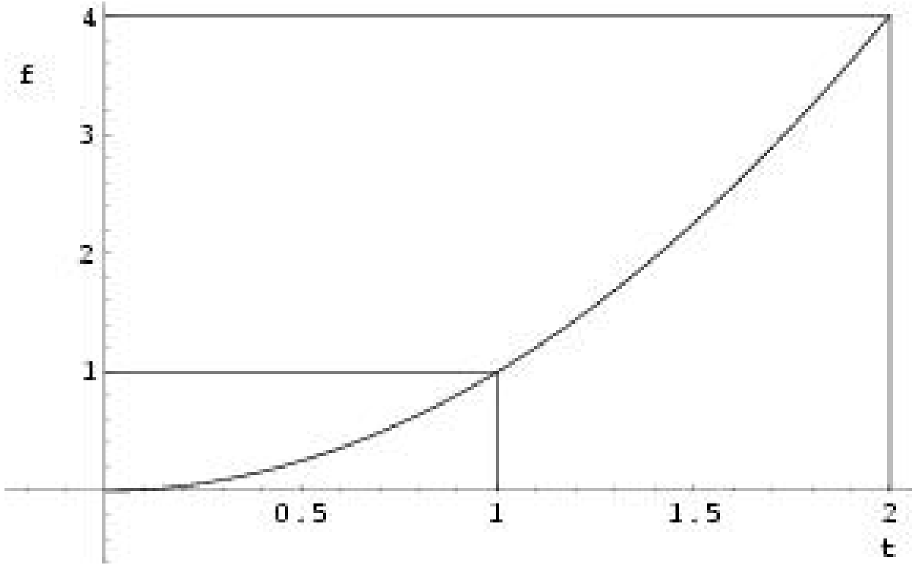

Figure 1.14: *This figure shows the parabola $f = t^2$ plotted using the method of Descartes. Values of f are measured on the vertical axis, while values of t are measured along the horizontal axis. The curve tells us the value of f corresponding to every value of t. For example, when $t = 1$, $f = 1$, while when $t = 2$, $f = 4$. If we want to know the value of $f = t^2$ corresponding to a particular value of t, we go vertically up to the curve from the horizontal axis, and then horizontally left from the curve to the vertical axis.*

Descartes realized the power of using algebra to generate and study geometrical figures; and he developed his method in an important book, which was among the books that Newton studied at Cambridge. Descartes' pioneering work in analytic geometry paved the way for the invention of differential and integral calculus by Fermat, Newton and Leibniz. (Besides taking some steps towards the invention of calculus, the great French mathematician, Pierre de Fermat (1601-1665), also discovered analytic geometry independently, but he did not publish this work.)

- **Problem 1.10**: Looking at the curve $f = t^2$ shown in Figure 1.14, we can see that when $t = 1$, $f = 1$. Suppose that we increase t by an amount $\Delta t = .01$. Then f will increase by an amount Δf. What is the ratio $\Delta f/\Delta t$?

- **Problem 1.11**: Repeat Problem 1.10 for $\Delta t = .0001$ and $\Delta t = .000001$. Does the ratio $\Delta f/\Delta t$ seem to be approaching a limiting value as Δt becomes smaller and smaller? How is this ratio related to the slope of the curve?

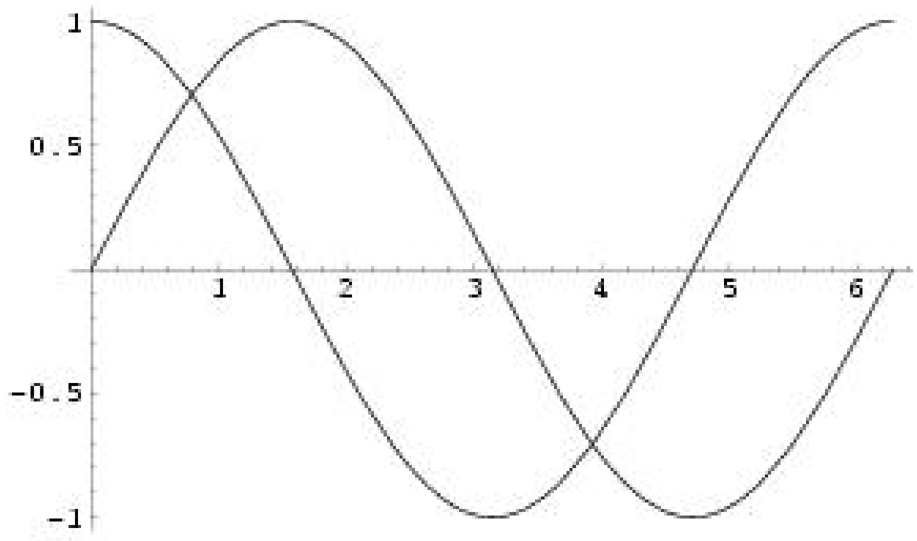

Figure 1.15: *This figure shows the trigonometric functions $f = sin(t)$ and $f = cos(t)$ plotted as functions of t using the method of Descartes. The functions were first tabulated by the Egyptian astronomer Hipparchus. The function $sin(t)$ is zero at $t = 0$, and increases to 1 at $t = \pi/2$. The function $cos(t)$ has the value 1 at $t = 0$, and falls to zero at $t = \pi/2$.* ($\pi = 3.1415927...$)

Descartes did important work in optics, physiology and philosophy. In philosophy, he is the author of the famous phrase *"Cogito, ergo sum"*, "I think; therefore I exist", which is the starting point for his theory of knowledge. He resolved to doubt everything which it was possible to doubt; and finally he was reduced to knowledge of his own existence as the only real certainty.

René Descartes died tragically through the combination of two evils which he had always tried to avoid: cold weather and early rising. Even as a student, he spent a large portion of his time in bed. He was able to indulge in this taste for a womblike existence because his father had left him some estates in Brittany. Descartes sold these estates and invested the money, from which he obtained an ample income. He never married, and he succeeded in avoiding responsibilities of every kind.

Descartes might have been able to live happily in this way to a ripe old age if only he had been able to resist a flattering invitation sent to him by Queen Christina of Sweden. Christina, the intellectual and strong-willed daughter of King Gustav Adolf, was determined to bring culture to Sweden, much to the disgust of the Swedish noblemen, who considered that money from the royal treasury ought to be spent exclusively on guns and fortifications. Unfortunately for Descartes, he had become so famous that Queen Christina wished to take lessons in philosophy from him; and she sent a warship to fetch him from Holland, where he was staying. Descartes, unable to resist this flattering attention from a royal patron, left his sanctuary in Holland and sailed to the frozen north.

The only time Christina could spare for her lessons was at five o'clock in the morning, three times a week. Poor Descartes was forced to get up in the utter darkness of the bitterly cold Swedish winter nights to give Christina her lessons in a draughty castle library; but his strength was by no means equal to that of the queen, and before the winter was over he had died of pneumonia.

Chapter 2

Differential calculus

Newton

On Christmas day in 1642 (the year in which Galileo died), a recently widowed woman named Hannah Newton gave birth to a premature baby at the manor house of Woolsthorpe, a small village in Lincolnshire, England. Her baby was so small that, as she said later, "he could have been put into a quart mug", and he was not expected to live.

When Isaac Newton was four years old, his mother married again and went to live with her new husband, leaving the boy to be cared for by his grandmother. This may have caused Newton to become more solemn and introverted than he might otherwise have been. One of his childhood friends remembered him as "a sober, silent, thinking lad, scarce known to play with the other boys at their silly amusements".

As a boy, Newton was fond of making mechanical models, but at first he showed no special brilliance as a scholar. He showed even less interest in running the family farm, however; and a relative (who was a fellow of Trinity College) recommended that he be sent to grammar school to prepare for Cambridge University.

When Newton arrived at Cambridge, he found a substitute father in the famous mathematician Isaac Barrow, who was his tutor. Under Barrow's guidance, and while still a student, Newton showed his mathematical genius by extending the binomial theorem, which had previously been studied by Pascal and Wallis.

To understand Newton's work on the binomial theorem, we can begin by thinking of what happens when we multiply the quantity $a + b$ by itself, as in equation (2.1):

$$
\begin{array}{r}
a+b \\
\times\ a+b \\
\hline
ab+b^2 \\
a^2+ab \\
\hline
a^2+2ab+b^2
\end{array}
\tag{2.1}
$$

The result is $a^2 + 2ab + b^2$. Now suppose that we continue the process and multiply $a^2 + 2ab + b^2$ by $a + b$, as in equation (2.2):

$$
\begin{array}{r}
a^2+2ab+b^2 \\
\times\ \ a+b \\
\hline
a^2b+2ab^2+b^3 \\
a^3+2a^2b+ab^2 \\
\hline
a^3+3a^2b+3ab^2+b^3
\end{array}
\tag{2.2}
$$

The result of this second multiplication is $a^3 + 3a^2b + 3ab^2 + b^3$, which can also be written as $(a + b)^3$. Continuing in this way we can obtain higher powers of $a + b$:

$$
\begin{aligned}
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
\vdots \;&\vdots\; \vdots
\end{aligned}
\tag{2.3}
$$

and so on. In general, an integral power of $a+b$ can be expressed in the form

$$
(a+b)^n = a^n + \frac{n}{1!}\,a^{n-1}b + \frac{n(n-1)}{2!}\,a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}\,a^{n-3}b^3 + ... + b^n
\tag{2.4}
$$

where

$$
\begin{aligned}
0! &\equiv 1 \\
1! &\equiv 1 = 1 \\
2! &\equiv 2\times 1 = 2 \\
3! &\equiv 3\times 2\times 1 = 6 \\
4! &\equiv 4\times 3\times 2\times 1 = 24 \\
\vdots \;&\vdots\; \vdots
\end{aligned}
\tag{2.5}
$$

and so on. An integer n followed by an exclamation mark stands for the product $n! \equiv n(n-1)(n-2)...1$, and one refers to such a product as "n

factorial", as was mentioned in Book 1. From the definition of $n!$, it follows that

$$n = \frac{n!}{(n-1)!}, \qquad n(n-1) = \frac{n!}{(n-2)!} \qquad ... \tag{2.6}$$

so that we can rewrite equation (2.4) can be rewritten in the form

$$(a+b)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} a^{n-j} b^j \tag{2.7}$$

where $\sum_{j=0}^{n}$ means "sum the expression over all integral values of j, starting at $j = 0$ and ending at $j = n$". The coefficients

$$\left(\begin{array}{c} n \\ j \end{array} \right) \equiv \frac{n!}{j!(n-j)!} \tag{2.8}$$

are called "binomial coefficients". Using this notation, we can alternatively express equation (2.7) in the form

$$(a+b)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) a^{n-j} b^j \tag{2.9}$$

Equation (2.7) is the famous binomial theorem. It can be proved by assuming that it holds for some value of n. One can then show that it holds for $n+1$. Since the binomial theorem obviously holds for $n = 1$, it must hold for all positive integral values of n.

- **Problem 2.1**: Calculate the values of 5!, 6! and 7!.
- **Problem 2.2**: Write expressions for $(a+b)^5$ and $(a+b)^6$ in powers of a and b.
- **Problem 2.3**: What is the value of the binomial coefficient $\left(\begin{array}{c} 8 \\ 5 \end{array} \right)$?

Newton exhibited his genius by asking himself what happens when n is not a positive integer. What if it is a negative integer or a fraction? What then? After studying this question, Newton concluded that the series then contains an infinite number of terms. He found that an infinite series of the form

$$(a+b)^p = a^p + p\, a^{p-1} b + \frac{p(p-1)}{2!}\, a^{p-2} b^2 + \frac{p(p-1)(p-2)}{3!}\, a^{p-3} b^3 + ... \tag{2.10}$$

where p is not a positive integer, converges to a finite value provided that b is sufficiently small compared with a.

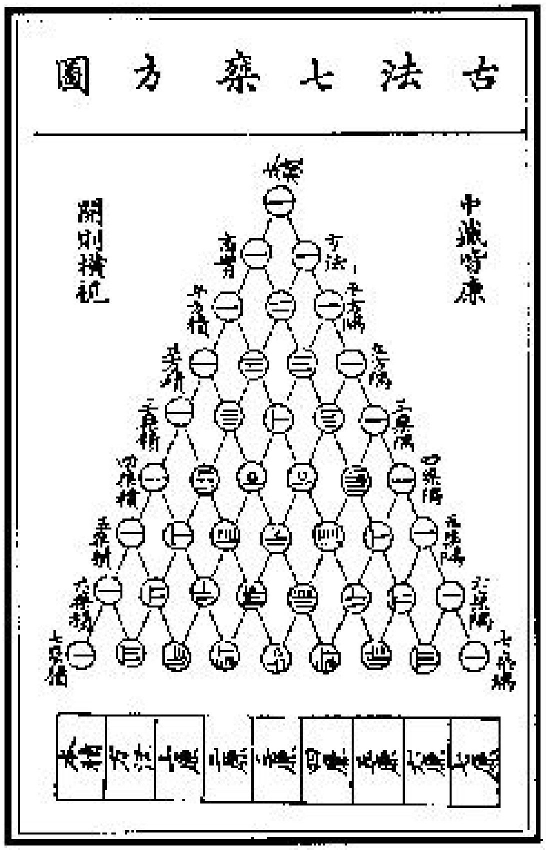

FIGURE 10

Figure 2.1: *Newton's work on binomial coefficients was forshadowed by that of the French mathematician Blaise Pascal (1623-1662), inventor of "Pascal's triangle". However, Pascal was in turn preceded by the Persian mathematician-poet Omar Khayyám (1048-1131) and by the Chinese mathematician Yanghui, who lived 500 years before Pascal. In the figure we see the Yanghui triangle. The binomial coefficients in each successive row are obtained by adding together coefficients in the previous row. The number above and slightly to the left is added to the number above and slightly to the right, and the sum forms the new coefficient.*

- **Problem 2.4**: Use equation (2.10) to make a series expansion of $\sqrt{1+x} \equiv (1+x)^{1/2}$ in powers of x. Evaluate the sum of the first five terms in the series when $x = .1$. Square the result and compare it to 1.1.

- **Problem 2.5**: Try evaluating the the first 5 terms of series of Problem 2.5 when $x = 2$. Does the series converge to a particular number as more and more terms are added?

In 1665, Cambridge University was closed because of an outbreak of the plague, and Newton returned for two years to the family farm at Woolsthorpe. He was then twenty-three years old. During the two years of isolation, Newton developed the binomial theorem into the beginnings of differential calculus. He imagined Δt to be an extremely small increase in the value of a variable t. For example, t might represent time, in which case Δt would represent an infinitesimal increase in time - a tiny fraction of a split-second. Newton realized that the series

$$(t+\Delta t)^p = t^p + p\ t^{p-1}\Delta t + \frac{p(p-1)}{2!}\ t^{p-2}\Delta t^2 + ... \tag{2.11}$$

could then be represented to a very good approximation by its first two terms, and in the limit $\Delta t \to 0$, he obtained the result

$$\begin{matrix}\text{limit}\\ \Delta t \to 0\end{matrix} \left[\frac{(t+\Delta t)^p - t^p}{\Delta t}\right] = p\ t^{p-1} \tag{2.12}$$

Newton then asked himself how much any function $f(t)$ changes when t increases by an infinitesimally small amount. He called the change in the function df and the infinitesimal increase in t he called dt. Newton concluded that the ratio df/dt would be given by

$$\frac{df}{dt} \equiv \begin{matrix}\text{limit}\\ \Delta t \to 0\end{matrix} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right] \tag{2.13}$$

Thus, in the particular case where $f(t) = t^p$ he found

$$\text{if } f = t^p, \text{ then } \frac{df}{dt} = p\ t^{p-1} \tag{2.14}$$

If we substitute various values of p into this relationship, we obtain a variety of relationships, for example:

$$\text{if } f = t^0 = 1, \text{ then } \frac{df}{dt} = 0 \tag{2.15}$$

$$\text{if } f = t^1 = t, \text{ then } \frac{df}{dt} = 1 \tag{2.16}$$

$$\text{if } f = t^2, \text{ then } \frac{df}{dt} = 2t \tag{2.17}$$

$$\text{if } f = t^3, \text{ then } \frac{df}{dt} = 3t^2 \tag{2.18}$$

$$\text{if } f = t^{1.5}, \text{ then } \frac{df}{dt} = 1.5t^{.5} \tag{2.19}$$

$$\text{if } f = t^{-1}, \text{ then } \frac{df}{dt} = -t^{-2} \tag{2.20}$$

and so on.

- **Problem 2.6**: Calculate $\frac{df}{dt}$ when $f(t) = \frac{1}{t^3}$
- **Problem 2.7**: Calculate $\frac{df}{dt}$ when $f(t) = (at)^4$ where a is a constant.
- **Problem 2.8**: Calculate $\frac{df}{dt}$ when $f(t) = 1 + t$.

$\frac{d}{dt}$ can be thought of as an operator which one can apply to a function $f(t)$. Today we call this operation "differentiation", and df/dt is called the function's "derivative".

Equations (2.13)-(2.20) all have geometrical interpretations: For example, the curve $f = t^2$ of equation (2.17) is shown in Figure 2.2. Suppose that we draw a tangent to the curve at some point t, as is shown in the figure. We can then construct a small right triangle whose long side is the tangent line, and whose other sides are respectively horizontal and vertical. If the horizontal side of the triangle has length Δt, then in the limit where Δt becomes infinitesimally small, the vertical side will have length $f(t + \Delta t) - f(t)$, and in this limit, the ratio of the two sides will be equal to the derivative, df/dt.

We have considered the particular case of a parabola, but a similar argument would hold for any well-behaved function. The derivative of a function can be interpreted as the slope (at a particular point t) of a curve representing the function. Differential calculus is the branch of mathematics that deals with differentiation, with slopes, with tangents, and with rates of change.

If we differentiate the sum of two functions, we obtain

$$\frac{d}{dt}[f(t) + g(t)] \equiv \begin{matrix}\text{limit}\\ \Delta t \to 0\end{matrix} \left[\frac{f(t+\Delta t) - f(t) + g(t+\Delta t) - g(t)}{\Delta t}\right] \tag{2.21}$$

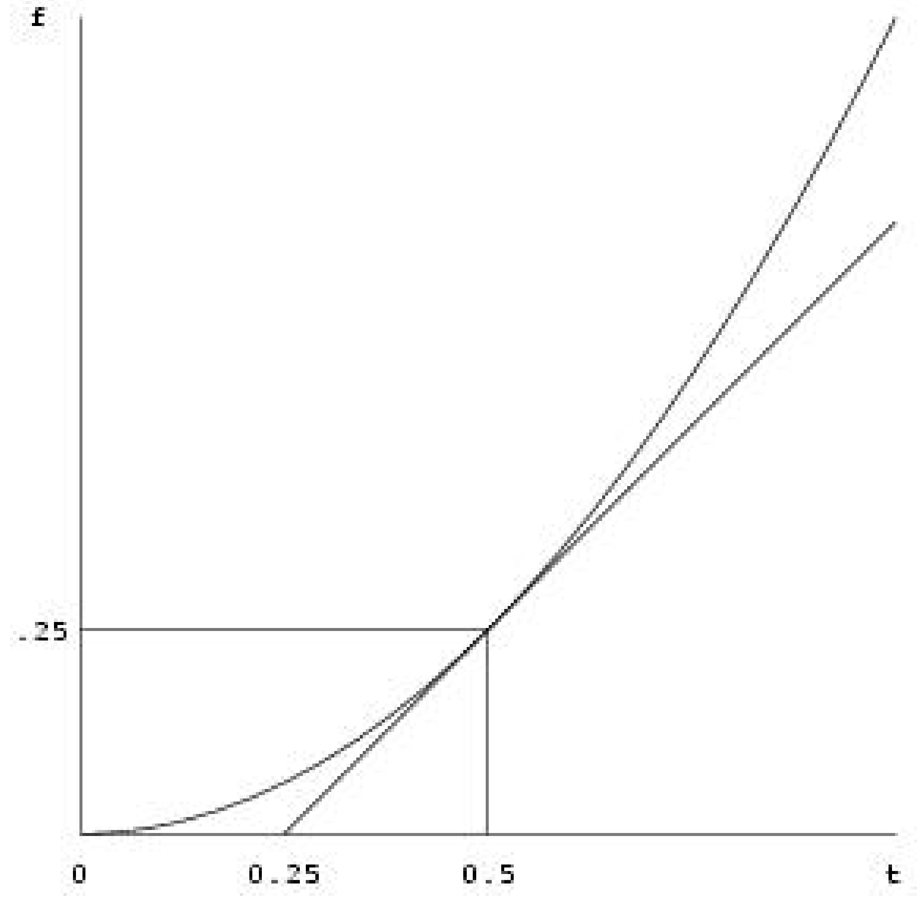

Figure 2.2: *This figure shows a plot of the parabola* $f = t^2$. *A line drawn tangent to the curve at some point* t *will have the same slope as the curve at that point, and the slope of the tangent line is given by the derivative,* $df/dt = 2t$, *(equations (2.13) and (2.17)). In the illustration,* $t{=}.5$, *and the slope of the curve at that point is* $df/dt = 2 \times .5 = 1$.

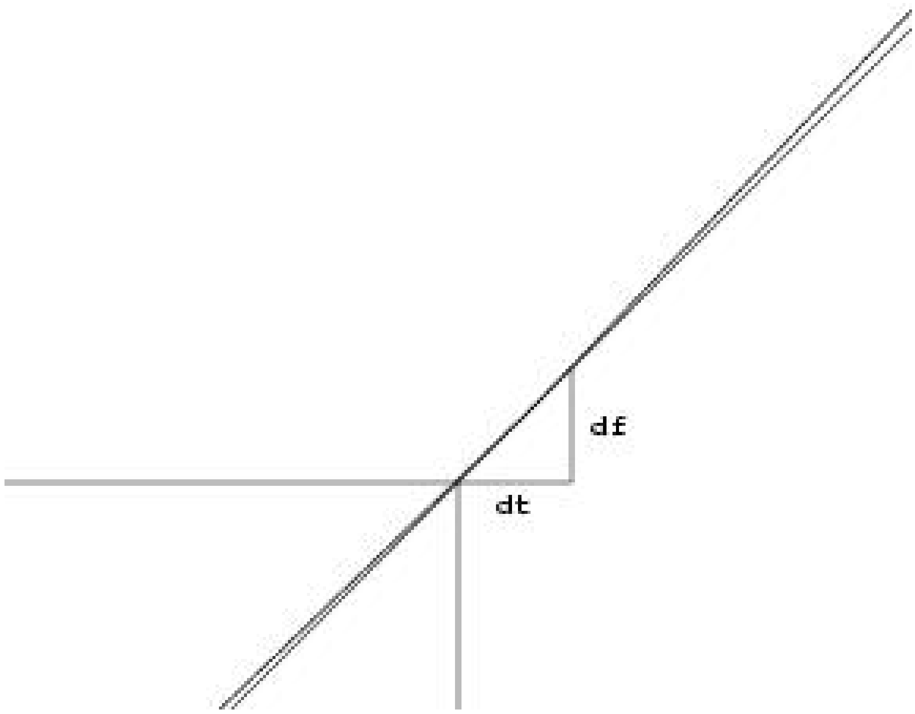

Figure 2.3: *This figure shows a magnified view of the point of contact between the parabola* $f = t^2$ *of the previous figure and the tangent line. A small triangle is drawn whose horizontal side represents an infinitesimal change in* t *while the vertical side represents the resulting change in* f*. The slope of the curve at that point is given by* df/dt*.*

and using equation (2.13), we can rewrite this in the form:

$$\frac{d}{dt}[f+g] = \frac{df}{dt} + \frac{dg}{dt} \tag{2.22}$$

For example

$$\text{if } f+g = t+t^2, \text{ then } \frac{d}{dt}[f+g] = 1+2t \tag{2.23}$$

Differentiating the product of two functions yields

$$\frac{d}{dt}[f(t)g(t)] \equiv \begin{array}{c}\text{limit}\\ \Delta t \to 0\end{array} \left[\frac{f(t+\Delta t)g(t+\Delta t) - f(t)g(t)}{\Delta t}\right] \tag{2.24}$$

which can be rewritten in the form

$$\frac{d}{dt}[fg] = f\frac{dg}{dt} + g\frac{df}{dt} \tag{2.25}$$

Now suppose that $g(t) = a$ where a is a constant, i.e. independent of t. Then from (2.25) we find that

$$\text{if } a = \text{constant}, \text{ then } \frac{d}{dt}[af] = a\frac{df}{dt} \tag{2.26}$$

Combining (2.26) with (2.16)-(2.18) we obtain

$$\frac{d}{dt}\left[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...\right] = a_1 + 2a_2 t + 3a_3 t^2 + ... \tag{2.27}$$

Differentiating a function gives us a new function, but this new function can also be differentiated, and this process will yield another function, which today is called the "second derivative". In modern notation, the new function obtained by differentiating $f(t)$ twice with respect to t is represented by the symbol $\frac{d^2 f}{dt^2}$:

$$\frac{d^2 f}{dt^2} \equiv \frac{d}{dt}\left[\frac{df}{dt}\right] \tag{2.28}$$

For example,

$$\frac{d^2}{dt^2}\left[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...\right] = 2a_2 + 6a_3 t + 12a_4 t^2 + ... \tag{2.29}$$

We can continue and take the third derivative:

$$\frac{d^3}{dt^3}\left[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...\right] = 6a_3 + 24a_4 t + 60a_5 t^2 + ... \tag{2.30}$$

Continuing to differentiate, we obtain in general

$$\text{if } f = \sum_{n=0}^{\infty} a_n t^n, \text{ then } \left[\frac{d^n f}{dt^n}\right]_{t=0} = n! a_n \tag{2.31}$$

Dividing (2.31) by $n!$, we obtain

$$\text{if } f = \sum_{n=0}^{\infty} a_n t^n, \text{ then } a_n = \frac{1}{n!}\left[\frac{d^n f}{dt^n}\right]_{t=0} \tag{2.32}$$

- **Problem 2.9**: Calculate $\frac{d^2 f}{dt^2}$ when $f(t) = t^{1/2}$.
- **Problem 2.10**: Suppose that $f(t) = t^3$. Use equation (2.32) to calculate the expansion coefficients a_n and show that the expansion is consistent with the original definition of the function.

We have used modern notation to go through the reasoning that Newton used to develop his binomial theorem into differential calculus. The quantities that we today call "derivatives", he called "fluxions", i.e. flowing quantities, perhaps because he associated them with a water clock that he had made as a boy - a water-filled jar with a hole in the bottom. If $f(t)$ represents the volume of water in the jar as a function of time, then df/dt represents the rate at which water is flowing out through the hole.

Newton also applied his "method of fluxions" to mechanics. From the three laws of planetary motion discovered by the German astronomer Kepler, Newton had deduced that the force with which the sun attracts a planet must fall off as the square of the distance between the planet and the sun. With great boldness, he guessed that this force is *universal*, and that every object in the universe attracts every other object with a gravitational force that is directly proportional to the product of the two masses, and inversely proportional to the square of the distance between them.

Newton also guessed correctly that in attracting an object outside its surface, the earth acts as though its mass were concentrated at its center. However, he could not construct the proof of this theorem, since it depended on integral calculus, which did not exist in 1666. (Newton himself perfected integral calculus later in his life.)

Referring to the year 1666, Newton wrote later: "I began to think of gravity extending to the orb of the moon; and having found out how to estimate the force with which a globe revolving within a sphere presses the surface of the sphere, from Kepler's rule of the periodical times of the planets being in a sesquialternate proportion of their distances from the centres of

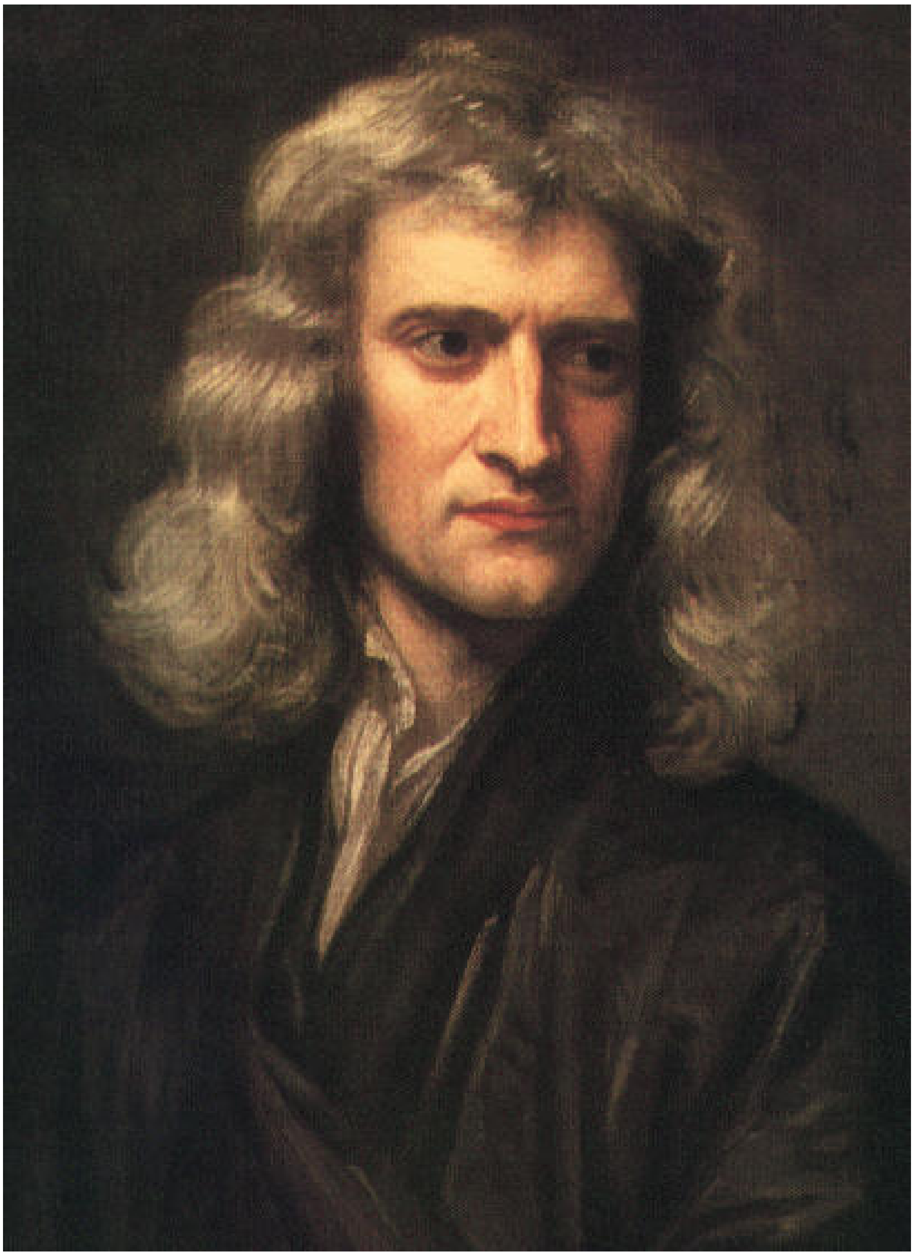

Figure 2.4: *Sir Isaac Newton (1642-1727) became an intellectual hero during his own lifetime, and his work was an inspiration to all of the philosophers of the Enlightenment. Newton is generally considered to have been the greatest physicist of all time.*

their orbs, I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of the distances from the centres about which they revolve; and thereby compared the force requisite to keep the moon in her orb with the force of gravity at the surface of the earth, and found them to answer pretty nearly."

"All this was in the plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded mathematics and philosophy more than at any time since."

Galileo had studied the motion of projectiles, and Newton was able to build on this work by thinking of the moon as a sort of projectile, dropping towards the earth, but at the same time moving rapidly to the side. The combination of these two motions gives the moon its nearly-circular path.

To see how Newton made this calculation, we can let x, y and z represent the Cartesian position coordinates of a body (for example the moon, or an apple). These are functions of time, and if we assume that the functions can be represented by polynomials in t [1], we can make use of (2.32) and write

$$x(t) = x_0 + t\left[\frac{dx}{dt}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2x}{dt^2}\right]_{t=0} + ... \tag{2.33}$$

$$y(t) = y_0 + t\left[\frac{dy}{dt}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2y}{dt^2}\right]_{t=0} + ... \tag{2.34}$$

and

$$z(t) = z_0 + t\left[\frac{dz}{dt}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2z}{dt^2}\right]_{t=0} + ... \tag{2.35}$$

The three Cartesian coordinates of a particle can be thought of as forming the three components of a vector which we can call $\mathbf{r}$. (A vector is a physical or mathematical quantity that has a direction as well as a size. For example the velocity of an object is a vector, since it has a direction as well as a magnitude.)

$$\mathbf{r} \equiv \{x, y, z\} \tag{2.36}$$

The force acting on an object has components in the directions of the three Cartesian coordinates, and thus the force can also be thought of as a vector:

$$\mathbf{F} \equiv \{F_x, F_y, F_z\} \tag{2.37}$$

[1] A polynomial in the variable t is a sum of powers of t multiplied by constant coefficients, like the sum shown in equation (2.32). The assumption that the moon's orbit can be represented as a polynomial in t is only valid for extremely small values of t, since the force acting on the moon is not constant but changes direction as the time t increases.

(We use bold-face type here to denote vectors). In addition to guessing the universal law of gravitation, Newton also postulated that the second derivative of the position vector of a body with respect to time (i.e. its acceleration) is directly proportional to the force acting on it, the constant of proportionality being the inverse of the body's mass:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}}{m} \tag{2.38}$$

Equation (2.38) is Newton's famous third law of motion. It is a vector equation, and its meaning is that each component of the vector on the left side is equal to the corresponding component of the vector on the right. In other words,

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{F_x}{m} \\ \frac{d^2y}{dt^2} &= \frac{F_y}{m} \\ \frac{d^2z}{dt^2} &= \frac{F_z}{m} \end{aligned} \tag{2.39}$$

Suppose now that the body is an apple, falling to the ground because of the earth's gravitational attraction. If z represents the vertical height of the apple above the earth's surface, while x and y measure its horizontal position on the surface, and if $-mg$ is the force of gravity acting on the apple, then we can write:

$$\mathbf{F} = \{0, 0, -mg\} \tag{2.40}$$

Combining (2.38) and (2.40), we have

$$\left[\frac{d^2\mathbf{r}}{dt^2}\right]_{t=0} \equiv \{0, 0, -g\} \tag{2.41}$$

The constant g which appears in equation (2.41) is the acceleration due to the earth's gravity acting on an object near to its surface, and it has the value

$$g = 32.174\frac{\text{feet}}{\text{sec.}^2} = 9.8066\frac{\text{meters}}{\text{sec.}^2} \tag{2.42}$$

(Newton used the English units, feet and miles. 1 meter = 3.28084 feet. 1 mile = 5280 feet.) Notice that the mass m has now disappeared! The force of gravity in Newton's theory is directly proportional to a body's mass, but the acceleration produced by a force in inversely proportional to it, and therefore the mass cancels out of the equation for gravitational acceleration[2].

To make the problem of the falling apple a little more complicated, let us suppose that a small boy has climbed the tree and that instead of just dropping the apple, he throws it out horizontally with velocity

$$\left[\frac{d\mathbf{r}}{dt}\right]_{t=0} \equiv \{v_x, 0, 0\} \tag{2.43}$$

Then substituting the initial velocity and acceleration of the apple into equations (2.33)-(2.35) and letting $x_0 = y_0 = 0$, we obtain

$$\begin{aligned} x &= v_x t \\ y &= 0 \\ z &= z_0 - g\frac{t^2}{2} \end{aligned} \tag{2.44}$$

We can use the first of these equations to express t in terms of x and rewrite the equation for z in the form:

$$z = z_0 - g\frac{x^2}{2v_x^2} \tag{2.45}$$

Thus we see that if it is thrown out horizontally from the tree, the apple will fall to the ground following a parabolic trajectory. Equations (2.44) and (2.45) describe the motions of projectiles and falling bodies. These were already well known to Galileo, who was the first to study such motions experimentally.

[2]Many years later, Albert Einstein noticed that Newton had used mass in these two different ways, as gravitational mass and as inertial mass, which by a coincidence were the same; and he set out to construct a theory of motion and gravitation where the two would *have* to be the same. The starting point of Einstein's general theory of relativity is the postulate that no local experiment whatever can distinguish between gravitation and acceleration. Thus, in Einstein's theory, an observer inside a closed box cannot tell whether the box is being accelerated or whether it is in a gravitational field. This led Einstein to the conclusion that a ray of light must be very slightly bent when it propagates in a strong gravitational field because such bending would be noticed by an observer looking at a ray of light propagating within an accelerated box. When the bending of light in a gravitational field was actually observed in 1918, Einstein became famous not only to other scientists, but also to ordinary newspaper readers. He was invited to meet the Archbishop of Canterbury as well as Charlie Chaplin and US President Herbert Hoover. While standing with Chaplin amid a huge cheering crowd, Einstein asked, "What does it all mean?" "Nothing!" answered Chaplin. Einstein agreed with him.

- **Problem 2.11**: Use equation (2.44), where $g = 32$ feet/second^2, to calculate how long a stone will take to fall from the top of a tower that is 64 feet high (neglecting air resistance).

- **Problem 2.12**: Suppose that instead of being merely dropped, the stone in Problem 2.11 is thrown horizontally from the top of the same tower with velocity $v_x = 16$ feet/second. Use equation (2.45) to calculate how far from the base of the tower it will land (again neglecting air resistance).

Newton boldly postulated that the laws of motion and gravitation that can be observed here on earth extend throughout the universe. To him it seemed that the moon resembles an apple thrown to the side by a small boy sitting in the apple tree. The moon falls towards the earth, but at the same time it moves to the side with the constant velocity v_x. The combination of these two motions gives the moon its nearly-circular orbit. Of course, after it has moved a little, the force of gravitation comes from a different direction, and therefore the moon does not follow a parabolic orbit but an approximately circular one. However, if we consider only a very short period of time, the circle and parabola fit closely together, as is illustrated in Figure 2.5.

If we take the origin of our coordinate system to be the center of the earth, then $z_0 = R_m$ where R_m is the radius of the moon's orbit, and the trajectory of the moon through a very short interval of time is given by

$$z = R_m - g' \frac{x^2}{2v_x^2} \tag{2.46}$$

We use g' instead of g in equation (2.46) because the moon is much more distant from the earth's center than the apple is, and the moon's gravitational acceleration is much less than the apple's. Building on Kepler's laws of planetary motion, Newton postulated that the force of gravity exerted by the earth falls off as the reciprocal of the square of the distance from the earth's center. Thus g and g' are related by

$$g' = g\left(\frac{R_e}{R_m}\right)^2 = 32.174\frac{\text{feet}}{\text{sec.}^2}\left(\frac{3963 \text{ miles}}{238600 \text{ miles}}\right)^2 = .0089\frac{\text{feet}}{\text{sec.}^2} \tag{2.47}$$

$$z = \sqrt{R_m^2 - x^2} \approx R_m - \frac{x^2}{2R_m} = R_m - g'\frac{x^2}{2v_x^2} \tag{2.48}$$

$$v_x = \frac{2\pi R_m}{\tau} = 3356\frac{\text{feet}}{\text{sec.}} \tag{2.49}$$

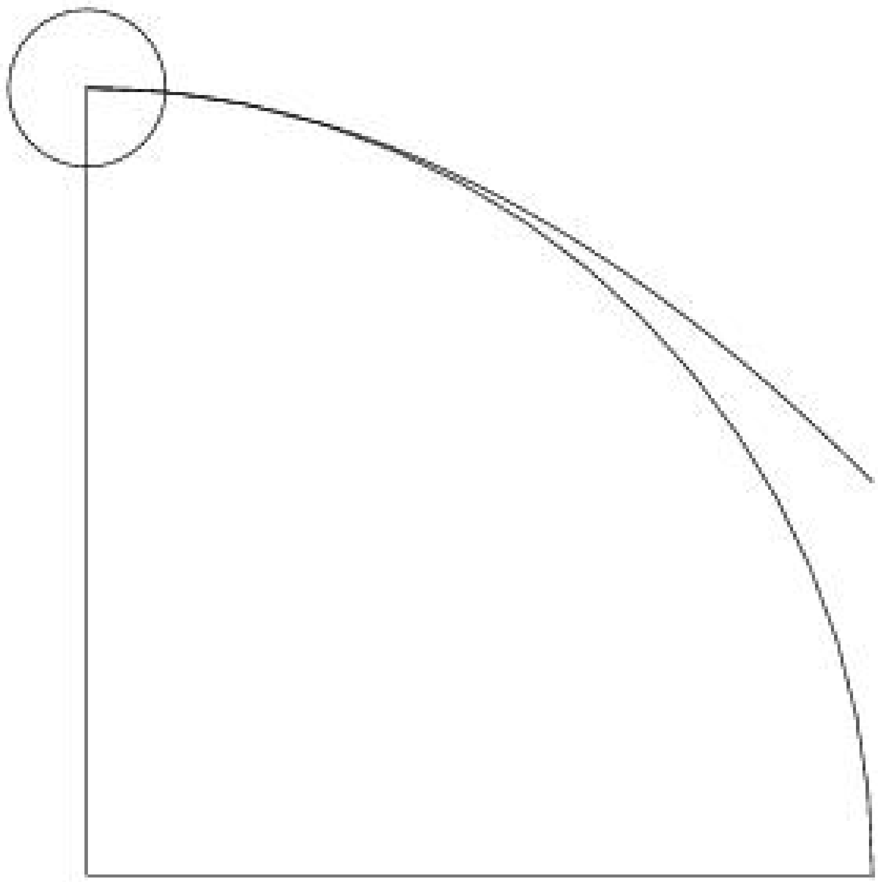

Figure 2.5: *The orbit of the moon is approximately circular in shape. During a very short interval of time, the moon can be thought of as being similar to an object moving horizontally, and at the same time being accelerated in a vertical direction by the force of gravity. The parabolic trajectory of such an object is approximately the same as a circle during that short interval of time, as is shown in the figure.*

$$g' = \frac{v_x^2}{R_m} = .0089 \frac{\text{feet}}{\text{sec.}^2} \tag{2.50}$$

In this way, Newton "compared the force necessary to keep the moon in her orb with the force of gravity on the earth's surface, and found them to answer pretty nearly."

Newton was not satisfied with this incomplete triumph, and he did not show his calculations to anyone. He not only kept his ideas on gravitation to himself, (probably because of the missing proof), but he also refrained for many years from publishing his work on the calculus. By the time Newton published, the calculus had been invented independently by the great German mathematician and philosopher, Gottfried Wilhelm Leibniz (1646-1716); and the result was a bitter quarrel over priority. However, Newton did publish his experiments in optics, and these alone were enough to make him famous.

Chapter 3

Integral calculus

In 1669, Newton's teacher, Isaac Barrow, generously resigned his post as Lucasian Professor of Mathematics so that Newton could have it. Thus, at the age of 27, Newton became the head of the mathematics department at Cambridge. He was required to give eight lectures a year, but the rest of his time was free for research.

Newton worked at this time on developing what he called "the method of inverse fluxions". Today we call his method "integral calculus". What did Newton mean by "inverse fluxions"? By "fluxions" he meant differentials, so we must think of an operation that is the reverse of differentiation.

In Chapter 2, we discussed how to find the differential of a function $f(t)$. Suppose that we know from our experience with differentiation that (for example)

$$\text{if and only if } f = t^p + C, \text{ then } \frac{df}{dt} = p\ t^{p-1} \tag{3.1}$$

where C is a constant. Then we also know that

$$\text{if } \frac{df}{dt} = p\ t^{p-1}, \text{ then } f = t^p + C \tag{3.2}$$

In equation (3.2), we know that C is a constant, but we do not know its value. Knowledge of the derivative df/dt allows us to determine the original function $f(t)$ from which it was derived up to an additive constant that must be determined in some other way. The operation of going backwards from the differential of a function to the function itself is called "integration", and the unknown constant C is called the "constant of integration". If we replace p by $p+1$, it follows from (3.2) that

$$\text{if } \frac{df}{dt} = t^p, \text{ then } f = \frac{t^{p+1}}{p+1} + C \qquad (p \neq -1) \tag{3.3}$$

(We have to exclude $p = -1$ in (3.3) to avoid dividing by zero.) It is customary to write this relationship in the form

$$\int dt\ t^p = \frac{t^{p+1}}{p+1} + C \qquad (p \neq -1) \tag{3.4}$$

Once again the constant of integration, C, is unknown and must be determined in some other way. When $p = 1$, equation (3.3) becomes

$$\text{if } \frac{df}{dt} = t, \text{ then } f = \frac{t^2}{2} + C \tag{3.5}$$

while (3.4) takes on the form

$$\int dt\ t = \frac{t^2}{2} + C \tag{3.6}$$

Equations (3.4) and (3.6) are called “indefinite integrals” - indefinite because the constant of integration is unknown. One also speaks of “definite integrals”, where knowledge of the derivative df/dt is used to find $f(t_2) - f(t_1)$. If the variable t represents time, then $f(t_2) - f(t_1)$ would represent the difference between the function $f(t)$ evaluated at the time $t = t_2$ minus the same function evaluated at the time $t = t_1$. For example,

$$\text{if } \frac{df}{dt} = t \qquad \text{then } f(t_2) - f(t_1) = \frac{t_2^2}{2} - \frac{t_1^2}{2} \tag{3.7}$$

This relationship is written in the form

$$\int_{t_1}^{t_2} dt\ t = \frac{t_2^2}{2} - \frac{t_1^2}{2} \tag{3.8}$$

The integration is said to be taken between the lower limit $t = t_1$ and the upper limit, $t = t_2$. The more general indefinite integral shown in equation (3.4) has a corresponding definite integral of the form:

$$\int_{t_1}^{t_2} dt\ t^p = \frac{t_2^{p+1}}{p+1} - \frac{t_1^{p+1}}{p+1} \qquad (p \neq -1) \tag{3.9}$$

When $p = 0$, this becomes

$$\int_{t_1}^{t_2} dt = t_2 - t_1 \tag{3.10}$$

The reason why integrals taken between two limits are called “definite integrals” is that the unknown constant of integration C has cancelled out so no information is missing when we go from the differential of a function to the function itself.

- **Problem 3.1**: Calculate the indefinite integral $\int dt\ t^4$.

- **Problem 3.2**: Calculate the definite integral $\int_1^2 dt\ t^4$.

- **Problem 3.3**: If $\frac{df}{dt} = t^{1/2}$, what is the form of the function f?

In Chapter 1, we mentioned that Archimedes invented integral calculus and used it to determine the areas of figures bounded by curves. To see how he did this and how Newton, many centuries later, did the same thing, let us begin by multiplying both sides of equation (3.10) by a constant v. This gives us

$$v \int_{t_1}^{t_2} dt = v(t_2 - t_1) \tag{3.11}$$

Equation (3.11) has both a geometrical interpretation and a physical meaning. Figure 3.1 shows a rectangle with height v and a base whose length is $t_2 - t_1$. The area of such a rectangle is $v(t_2 - t_1)$. Now suppose that the rectangle is divided up into a number of small strips, each having a width

$$\Delta t = \frac{t_2 - t_1}{N} \tag{3.12}$$

as is shown in Figure 3.2, where we have let N=5. The total area of the rectangle will be the sum of the areas of the strips.

$$\text{area} = N(v\Delta t) = N\left(\frac{v(t_2 - t_1)}{N}\right) = v(t_2 - t_1) \tag{3.13}$$

Obviously the sum of the areas of the small rectangular strips is independent of how many of them we use to divide up the area of the rectangle, and this is reflected in the fact that N cancels out in equation (3.13), giving an N-independent answer for the total area.

What about the physical meaning of of equation (3.11)? If we imagine an object moving with constant velocity v, then $v\Delta t$ represents the distance it will move in the small but finite interval of time Δt, while vdt can be imagined informally to be the distance moved in an infinitesimal time interval dt. Summing up the small distances moved in small intervals, we obtain the total distance moved in the interval between the initial time t_1 and a later time t_2. Equation (3.11) tells us that this total distance will be $v(t_2 - t_1)$. Alternatively v might represent the constant rate of flow from the water-clock that Isaac Newton made as a boy. In that case, $v(t_2 - t_1)$ would represent all of the water lost in the time interval $t_2 - t_1$.

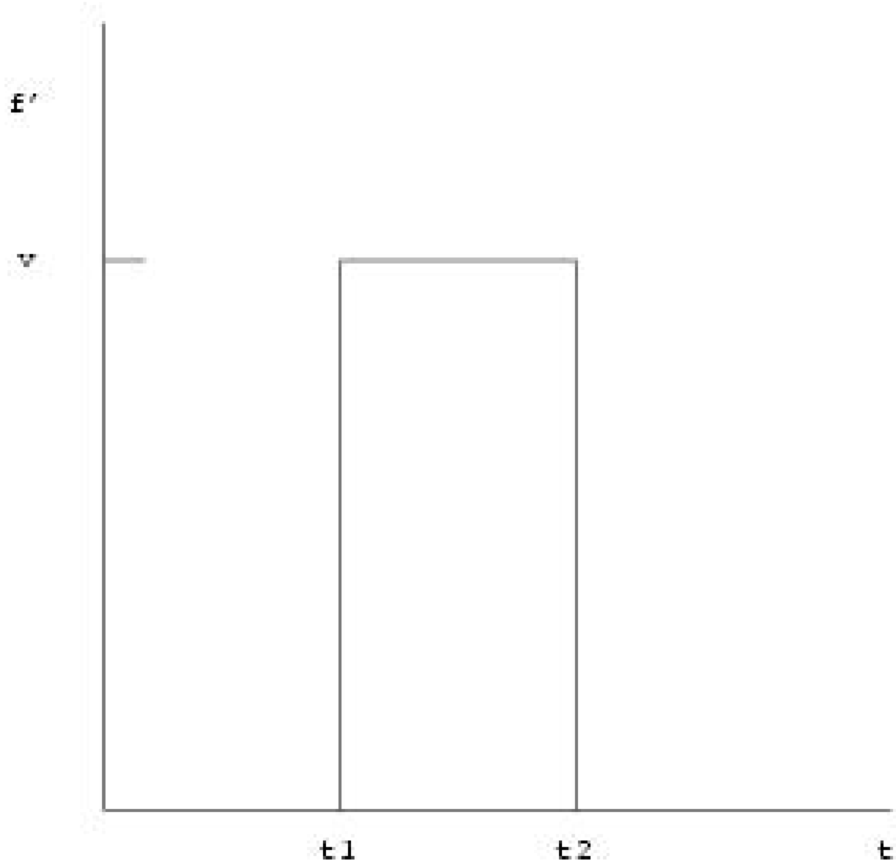

Figure 3.1: *This figure shows a rectangle with height v and base $t_2 - t_1$. The area of the figure is $v(t_2 - t_1)$. If v represents the constant velocity of an object, then the area of the rectangle represents distance that the object moves between the times t_1 and t_2.*

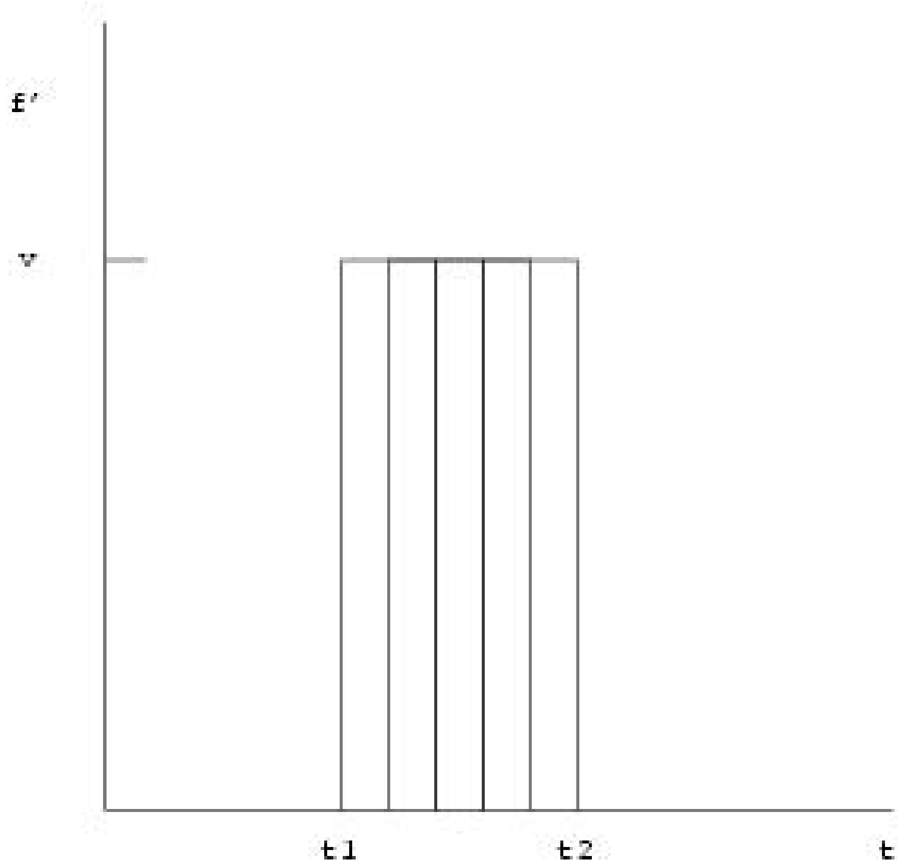

Figure 3.2: *We now divide the large rectangle of Figure 3.1 into five small rectangular strips, each with area* $v\Delta t = v(t_2 - t_1)/5$. *When we add together the areas of the small strips, we get the same answer for the total area of the rectangle. Physically,* $v\Delta t$ *can represent the distance that an object with constant velocity* v *moves in a small interval of time* Δt.

- **Problem 3.4**: Suppose that a man is walking at an average speed of 3 kilometers per hour. How far, on the average, will he walk in 1 second? How is this question related to equation (3.10) and Figure 3.1?

- **Problem 3.5**: As a boy, Isaac Newton constructed a water clock. It was a large container with a small hole in the bottom, and the water ran out through the hole at a constant rate. Let us suppose that its volume was four quarts and that it took 24 hours to go from full to empty. How fast did the water run out through the hole? If we apply the idea of functions and differentials to this problem, what does $f(t)$ represent? What does df/dt represent? What did the word "fluxion" mean to Newton?

What we have done here seems a bit like cracking a peanut with a sledgehammer. Why have we used such a heavy piece of mathematical hardware to crack a problem that we could have solved in 30 seconds in our heads? However, if the reader will be patient with the first two simple examples, which we have included for the sake of clarity, we will soon go on to problems involving figures bounded by curves, and these cannot be solved without the help of integral calculus.

In the next simple example, we multiply both sides of equation (3.8) with the constant a. This will give us

$$a\int_{t_1}^{t_2} dt\ t = a\left(\frac{t_2^2}{2} - \frac{t_1^2}{2}\right) \tag{3.14}$$

If we let $t_1 = 0$ we have

$$a\int_{0}^{t_2} dt\ t = a\left(\frac{t_2^2}{2}\right) \tag{3.15}$$

Like (3.11), this equation has a both a geometrical interpretation and a physical one. The geometrical interpretation is shown in Figures 3.3 and 3.4. In Figure 3.3, we see the straight line

$$f'(t) = at \tag{3.16}$$

where f' is plotted as a function of t. The area under the straight line between $t = 0$ and $t = t_2$ is triangular in shape, and is given by $(at_2)(t_2/2)$, i.e. by the height of the triangle multiplied by half the length of its base. If we divide the area under the line $f' = at$ into N thin strips as is shown in Figure 3.4, and if we sum the area of the strips and let $N \to \infty$ then we will obtain the area under the line. In Figure 3.4, the error is represented by the areas of the five small triangles above the line $f' = at$. When we increase N, the number of these small triangles increases, but the total error decreases because the area of each triangle is proportional to $1/N^2$.

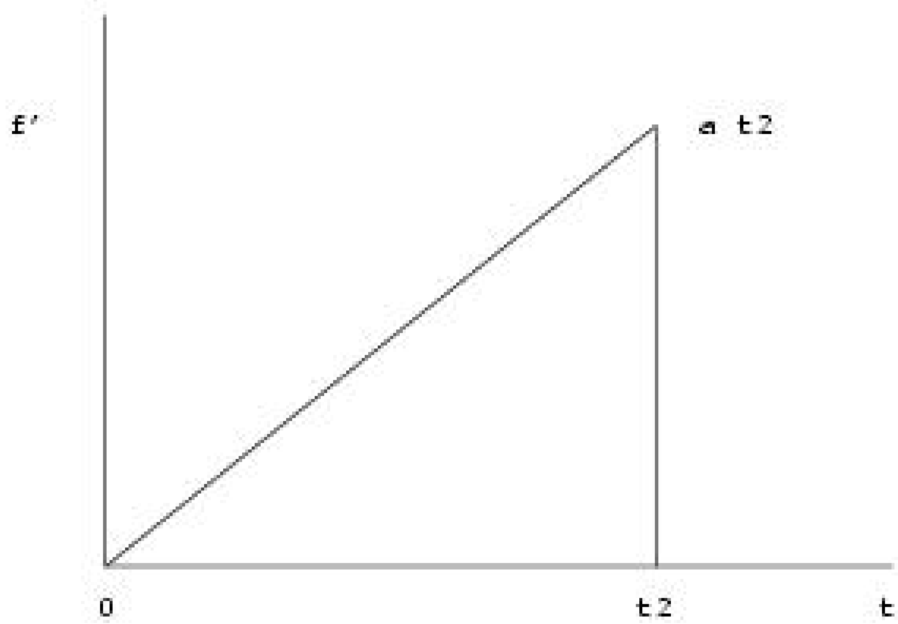

Figure 3.3: *This figure illustrates the geometrical interpretation of equation (3.15). The area under the straight line* $v = at$ *between the points* $t = 0$ *and* $t = t_2$ *is given by* $at_2^2/2$, *i.e., the height of the triangle, multiplied by half the length of the base. Physically, the area of the triangle can represent the distance moved by an object with constant acceleration* a. *It's velocity is then given by* $v = at$, *and the distance travelled is proportional to the square of the elapsed time. Galileo found this law experimentally for falling bodies with constant gravitational acceleration. He observed that the distance travelled by a falling body is proportional to the square of the elapsed time.*

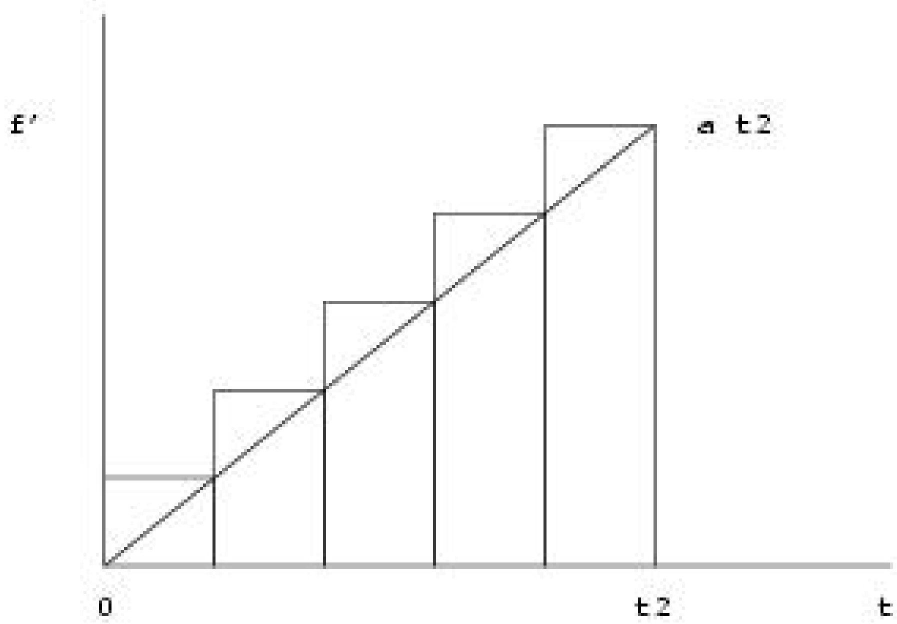

Figure 3.4: *We now divide the triangle of Figure 3.3 into N small rectangular strips. (In the figure, $N = 5$.) The area of the triangle is approximated by the sum of the areas of the small strips. If we increase the number of strips, N, the approximation will become more exact. The area of each of the narrow strips can represent physically the approximate distance that an object with constant acceleration a travels during the interval of time Δt. This distance changes with time because acceleration changes the velocity of the object.*

- **Problem 3.6**: What are the heights of each of the five narrow strips shown in Figure 3.4? What are the areas of each of the strips? What is the sum of their areas?

- **Problem 3.7**: In Chapter 1, Figure 1.9 shows the method which Archimedes used to calculate the area of a circle by dividing it into a number of narrow strips and then letting the strips become more and more narrow and numerous. In the figure, four strips are shown. If the radius of the circle has length $r = 1$, what is the area of each strip? What is their total area?

Equation (3.15) has a physical meaning as well as a geometrical interpretation. Let us think of an object acted on by a constant force, for example the force of gravity. Then according to Newton's laws of motion discussed in Chapter 2, the acceleration of the body will be constant, and its velocity will increase linearly with time according to the rule $v = at$. Thus Figure 3.3 can be thought of as a plot of the velocity of the object as a function of time. In Figure 3.4, the area of each strip represents approximately the distance traveled in the small interval of time Δt. Of course we must remember that the velocity is constantly changing.

The area under the line $v = at$ between the times $t = 0$ and $t = t_2$ represents the total distance travelled by a body when it is acted on by a constant force. We see from our construction that it is proportional to the square of the elapsed time. This is exactly the law of falling bodies that was discovered experimentally by the great Italian physicist, Galileo Galilei, and later explained theoretically by Isaac Newton.

- **Problem 3.8**: If $f(t)$ represents the distance traveled by an object moving in a straight line, what does $\frac{df}{dt}$ represent? What does $\frac{d^2 f}{dt^2}$ represent?

- **Problem 3.9**: Suppose that an object has a constant acceleration a in a particular direction. Express the velocity as an indefinite integral and find an expression for the velocity of the object as a function of time. What is the physical interpretation of the constant of integration? Integrate again to find the distance travelled as a function of time. What is the interpretation of the second constant of integration?

- **Problem 3.10**: Repeat Problem 3.9 for the case where $a = wt$ where w is a constant. In other words, repeat the problem for the case where the acceleration increases linearly with time.

The two simple examples given here follow a pattern: In each example,

$$\int_{t_1}^{t_2} dt\ f'(t) = f(t_2) - f(t_1) \tag{3.17}$$

was interpreted as the area under the curve represented by $f'(t)$ between vertical lines drawn at $t = t_1$ and $t = t_2$, the lower boundary of the figure being the horizontal axis. This is in fact the general geometrical interpretation of the definite integral of a function of a single variable where f' is the first derivative of f, i.e.,

$$f'(t) \equiv \begin{array}{c} \text{limit} \\ \Delta t \to 0 \end{array} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right] \equiv \frac{df}{dt} \tag{3.18}$$

We can see this if we consider the sum of the areas of N strips of width Δt and height $f'(t_1 + j\Delta t)$:

$$S = \sum_{j=0}^{N-1} \Delta t f'(t_1 + j\Delta t) \tag{3.19}$$

Here

$$\Delta t \equiv \frac{t_2 - t_1}{N} \tag{3.20}$$

and $f'(t)$ is is defined by (3.18). If N is sufficiently large, but still finite, so that Δt is extremely small but still finite, and if $f(t)$ is a smooth continuous function, we have

$$f'(t) \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{3.21}$$

so that

$$S \approx \sum_{j=0}^{N-1} \left[f(t_1 + j\Delta t + \Delta t) - f(t_1 + j\Delta t)\right] \tag{3.22}$$

Writing out the terms in this sum yields

$$\begin{aligned} S \ \approx\ & f(t_1+\Delta t) - f(t_1) + f(t_1 + 2\Delta t) - f(t_1 + \Delta t) \\ & + f(t_1 + 3\Delta t) - f(t_1 + 2\Delta t) + ... \\ & + f(t_1 + N\Delta t) - f(t_1 + N\Delta t - \Delta t) \end{aligned} \tag{3.23}$$

We can notice a cancellation between the 1st and 4th terms of this sum, between the 3rd and 6th terms, and so on. In fact, all of the terms cancel out except the 2nd term and the next to last one. Therefore we can write

$$S \approx f(t_2) - f(t_1) \tag{3.24}$$

where we have used the fact that $t_2 = t_1 + N\Delta t$. As N becomes larger and larger, the approximation in equation (3.24) becomes progressively more accurate, provided that the function $f(t)$ is smooth and continuous. This establishes the general geometrical interpretation of a definite integral, since as N becomes larger, S more and more closely approximates the area enclosed by the curve $f'(t)$, the horizontal axis and the vertical lines $t = t_1$ and $t = t_2$.

Newton realized that the operation of intergration (finding "inverse fluxions") is equivalent to dividing the area under a curve into N narrow rectangular strips and adding together the areas of the strips in the limit where $N \to \infty$. He introduced the symbol $\int$ for this operation. The symbol is, in fact, an old-fashioned S, standing for "Summa", the Latin word for sum.

As Newton showed, integrals can be used to find the areas of figures bounded by curves. For example, suppose that we let $p = 2$ in equation (3.9). Then it will reduce to

$$\int_{t_1}^{t_2} dt\ t^2 = \frac{t_2^3}{3} - \frac{t_1^3}{3} \tag{3.25}$$

The right-hand side of equation (3.25) represents the area under the curve

$$f'(t) = t^2 \tag{3.26}$$

between vertical lines drawn at $t = t_1$ and $t = t_2$, as shown in Figure 3.5.

After inventing differential and integral calculus, Isaac Newton used it to solve many of the problems that had been worrying him in his earlier work on motion and gravitation. For example, he was able to show that when the gravitational force of the earth acts on an object outside its surface, the result is the same as it would be if all the mass of the earth were concentrated at its center. However, he did not publish any of this work until many years later.

Meanwhile, the problems of gravitation and planetary motion were increasingly discussed by the members of the Royal Society. In January, 1684, three members of the Society were gathered in a London coffee house. One of them was Robert Hooke (1635-1703), author of *Micrographia* and Professor of Geometry at Gresham College, a brilliant but irritable man. He had begun his career as Robert Boyle's assistant, and had gone on to do important work in many fields of science. Hooke claimed that he could calculate the motion of the planets by assuming that they were attracted to the sun by a force which diminished as the square of the distance.

Listening to Hooke were Sir Christopher Wren (1632-1723), the designer of St. Paul's Cathedral, and the young astronomer, Edmund Halley (1656-1742). Wren challenged Hooke to produce his calculations; and he offered to

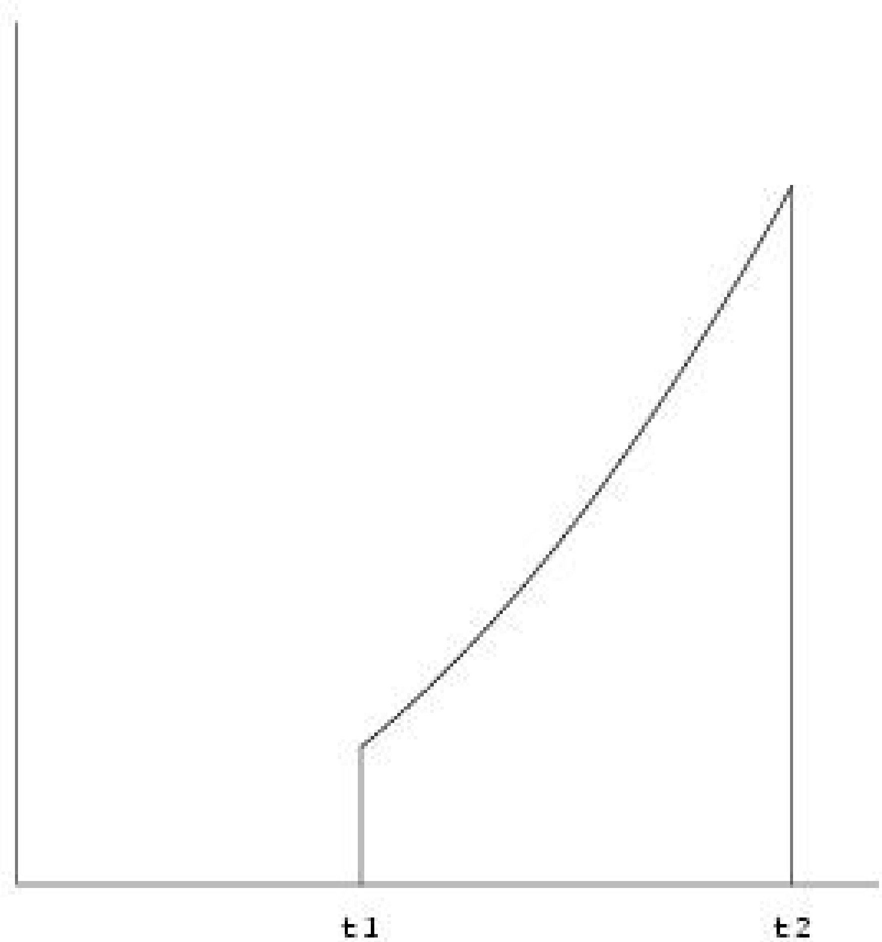

Figure 3.5: *Equation (3.25) tells us how to find the area under the parabola $f'(t) = t^2$ between vertical lines drawn at $t = t_1$ and $t = t_2$. The other boundary of the calculated area is the horizontal axis.*

present Hooke with a book worth 40 shillings if he could prove his inverse square force law by means of rigorous mathematics. Hooke tried for several months, but he was unable to win Wren's reward.

In August, 1684, Halley made a journey to Cambridge to talk with Newton, who was rumored to know very much more about the motions of the planets than he had revealed in his published papers. According to an almost-contemporary account, what happened then was the following:

"Without mentioning his own speculations, or those of Hooke and Wren, he (Halley) at once indicated the object of his visit by asking Newton what would be the curve described by the planets on the supposition that gravity diminished as the square of the distance. Newton immediately answered: an Ellipse. Struck with joy and amazement, Halley asked how he knew it? 'Why', replied he, 'I have calculated it'; and being asked for the calculation, he could not find it, but promised to send it to him."

Newton soon reconstructed the calculation and sent it to Halley; and Halley, filled with enthusiasm and admiration, urged Newton to write out in detail all of his work on motion and gravitation. Spurred on by Halley's encouragement and enthusiasm, Newton began to put his research in order. He returned to the problems which had occupied him during the plague years, and now his progress was rapid because he had invented integral calculus. Newton also had available an improved value for the radius of the earth, measured by the French astronomer Jean Picard (1620-1682). This time, when he approached the problem of gravitation, everything fell into place.

By the autumn of 1684, Newton was ready to give a series of lectures on dynamics, and he sent the notes for these lectures to Halley in the form of a small booklet entitled *On the Motion of Bodies*. Halley persuaded Newton to develop these notes into a larger book, and with great tact and patience he struggled to keep a controversy from developing between Newton, who was neurotically sensitive, and Hooke, who was claiming his share of recognition in very loud tones, hinting that Newton was guilty of plagiarism.

Although Newton was undoubtedly the greatest physicist of all time, he had his shortcomings as a human being; and he reacted by striking out from his book every single reference to Robert Hooke. The Royal Society at first offered to pay for the publication costs of Newton's book, but because a fight between Newton and Hooke seemed possible, the Society discretely backed out. Halley then generously offered to pay the publication costs himself, and in 1686 Newton's great book was printed. It is entitled *Philosophae Naturalis Principia Mathematica*, (The Mathematical Principles of Natural Philosophy), and it is divided into three sections.

The first book sets down the general principles of mechanics. In it, Newton states his three laws of motion, and he also discusses differential and integral calculus (both invented by himself).

In the second book, Newton applies these methods to systems of particles and to hydrodynamics. For example, he calculates the velocity of sound in air from the compressibility and density of air; and he treats a great variety of other problems, such as the problem of calculating how a body moves when its motion is slowed by a resisting medium, such as air or water.

The third book is entitled *The System of the World*. In this book, Newton sets out to derive the entire behavior of the solar system from his three laws of motion and from his law of universal gravitation. From these, he not only derives all three of Kepler's laws, but he also calculates the periods of the planets and the periods of their moons; and he explains such details as the flattened, non-spherical shape of the earth, and the slow precession of its axis about a fixed axis in space. Newton also calculated the irregular motion of the moon resulting from the combined attractions of the earth and the sun; and he determined the mass of the moon from the behavior of the tides.

Newton's *Principia* is generally considered to be the greatest scientific work of all time. To present a unified theory explaining such a wide variety of phenomena with so few assumptions was a magnificent and unprecedented achievement; and Newton's contemporaries immediately recognized the importance of what he had done.

The great Dutch physicist, Christian Huygens (1629-1695), inventor of the pendulum clock and the wave theory of light, travelled to England with the express purpose of meeting Newton. Voltaire, who for reasons of personal safety was forced to spend three years in England, used the time to study Newton's *Principia*; and when he returned to France, he persuaded his mistress, Madame du Chatelet, to translate the *Principia* into French; and Alexander Pope, expressing the general opinion of his contemporaries, wrote a famous couplet, which he hoped would be carved on Newton's tombstone:

Nature and Nature's law lay hid in night.
God said: 'Let Newton be!', and all was light!

The Newtonian synthesis was the first great achievement of a new epoch in human thought, an epoch which came to be known as the "Age of Reason" or the "Enlightenment". We might ask just what it was in Newton's work that so much impressed the intellectuals of the 18th century. The answer is that in the Newtonian system of the world, the entire evolution of the solar system is determined by the laws of motion and by the positions and velocities of the planets and their moons at a given instant of time. Knowing these, it is possible to predict all of the future and to deduce all of the past.

The Newtonian system of the world is like an enormous clock which has to run on in a predictable way once it is started. In this picture of the world, comets and eclipses are no longer objects of fear and superstition. They too are part of the majestic clockwork of the universe. The Newtonian laws are simple and mathematical in form; they have complete generality; and they are unalterable. In this picture, although there are no miracles or exceptions to natural law, nature itself, in its beautiful works, can be regarded as miraculous.

Newton's contemporaries knew that there were other laws of nature to be discovered besides those of motion and gravitation; but they had no doubt that, given time, all of the laws of nature would be discovered. The climate of intellectual optimism was such that many people thought that these discoveries would be made in a few generations, or at most in a few centuries.

Huygens and Leibniz

Meanwhile, on the continent, mathematics and physics had been developing rapidly, stimulated by the writings of René Descartes. One of the most distinguished followers of Descartes was the Dutch physicist, Christian Huygens (1629-1695).

Huygens was the son of an important official in the Dutch government. After studying mathematics at the University of Leiden, he published the first formal book ever written about probability. However, he soon was diverted from pure mathematics by a growing interest in physics.

In 1655, while working on improvements to the telescope together with his brother and the Dutch philosopher Benedict Spinoza, Huygens invented an improved method for grinding lenses. He used his new method to construct a twenty-three foot telescope, and with this instrument he made a number of astronomical discoveries, including a satellite of Saturn, the rings of Saturn, the markings on the surface of Mars and the Orion Nebula.

Huygens was the first person to estimate numerically the distance to a star. By assuming the star Sirius to be exactly as luminous as the sun, he calculated the distance to Sirius, and found it to be 2.5 trillion miles. In fact, Sirius is more luminous than the sun, and its true distance is twenty times Huygens' estimate.

Another of Huygens' important inventions is the pendulum clock. Improving on Galileo's studies, he showed that for a pendulum swinging in a circular arc, the period is not precisely independent of the amplitude of the swing. Huygens then invented a pendulum with a modified arc, not quite circular, for which the swing was exactly isochronous. He used this improved pendulum to regulate the turning of cog wheels, driven by a falling weight;

and thus he invented the pendulum clock, almost exactly as we know it today.

Among the friends of Christian Huygens was the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646-1716). Leibniz was a man of universal and spectacular ability. In addition to being a mathematician and philosopher, he was also a lawyer, historian and diplomat. He invented the doctrine of balance of power, attempted to unify the Catholic and Protestant churches, founded academies of science in Berlin and St. Petersberg, invented combinatorial analysis, introduced determinants into mathematics, independently invented the calculus, invented a calculating machine which could multiply and divide as well as adding and subtracting, acted as advisor to Peter the Great and originated the theory that "this is the best of all possible worlds" (later mercilessly satirized by Voltaire in *Candide*).

Leibniz learned mathematics from Christian Huygens, whom he met while travelling as an emissary of the Elector of Mainz. Since Huygens too was a man of very wide interests, he found the versatile Leibniz congenial, and gladly agreed to give him lessons. Leibniz continued to correspond with Huygens and to receive encouragement from him until the end of the older man's life.

In 1673, Leibniz visited England, where he was elected to membership by the Royal Society. During the same year, he began his work on calculus, which he completed and published in 1684. Newton's invention of differential and integral calculus had been made much earlier than the independent work of Leibniz, but Newton did not publish his discoveries until 1687. This set the stage for a bitter quarrel over priority between the admirers of Newton and those of Leibniz. The quarrel was unfortunate for everyone concerned, especially for Leibniz himself. He had taken a position in the service of the Elector of Hanover, which he held for forty years. However, in 1714, the Elector was called to the throne of England as George I. Leibniz wanted to accompany the Elector to England, but was left behind, mainly because of the quarrel with the followers of Newton. Leibniz died two years later, neglected and forgotten, with only his secretary attending the funeral.

The Bernoullis and Euler

Among the followers of Leibniz was an extraordinary family of mathematicians called Bernoulli. They were descended from a wealthy merchant family in Basle, Switzerland. The head of the family, Nicolas Bernoulli the Elder, tried to force his three sons, James (1654-1705), Nicolas II (1662-1716) and John (1667-1748) to follow him in carrying on the family business. However, the eldest son, James, had taught himself the Leibnizian form of calculus, and instead became Professor of Mathematics at the University of Basle. His motto was *"Invicto patre sidera verso"* ("Against my father's will, I study the stars").

Nicolas II and John soon caught their brother's enthusiasm, and they learned calculus from him. John became Professor of Mathematics in Grönigen and Nicolas II joined the faculty of the newly-formed Academy of St. Petersberg. John Bernoulli had three sons, Nicolas III (1695-1726), Daniel (1700-1782) and John II (1710-1790), all of whom made notable contributions to mathematics and physics. In fact, the family of Nicolas Bernoulli the Elder produced a total of nine famous mathematicians in three generations!

Daniel Bernoulli's brilliance made him stand out even among the other members of his gifted family. He became professor of mathematics at the Academy of Sciences in St. Petersberg when he was twenty-five. After eight Russian winters however, he returned to his native Basle. Since the chair in mathematics was already occupied by his father, he was given a vacant chair, first in anatomy, then in botany, and finally in physics. In spite of the variety of his titles, however, Daniel's main work was in applied mathematics, and he has been called the father of mathematical physics.

One of the good friends of Daniel Bernoulli and his brothers was a young man named Leonhard Euler (1707-1783). He came to their house once a week to take private lessons from their father, John Bernoulli. Euler was destined to become the most prolific mathematician in history, and the Bernoullis were quick to recognize his great ability. They persuaded Euler's father not to force him into a theological career, but instead to allow him to go with Nicolas III and Daniel to work at the Academy in St. Petersberg.

Euler married the daughter of a Swiss painter and settled down to a life of quiet work, producing a large family and an unparalleled output of papers. A recent edition of Euler's works contains 70 quatro volumes of published research and 14 volumes of manuscripts and letters. His books and papers are mainly devoted to algebra, the theory of numbers, analysis, mechanics, optics, the calculus of variations (invented by Euler), geometry, trigonometry and astronomy; but they also include contributions to shipbuilding science, architecture, philosophy and musical theory!

Euler achieved this enormous output by means of a calm and happy disposition, an extraordinary memory and remarkable powers of concentration, which allowed him to work even in the midst of the noise of his large family. His friend Thiébault described Euler as sitting "..with a cat on his shoulder and a child on his knee - that was how he wrote his immortal works".

In 1771, Euler became totally blind. Nevertheless, aided by his sons and his devoted scientific assistants, he continued to produce work of fundamental importance. It was his habit to make calculations with chalk on a board for the benefit of his assistants, although he himself could not see what he was writing. Appropriately, Euler was making such computations on the day of his death. On September 18, 1783, Euler gave a mathematics lesson to one of his grandchildren, and made some calculations on the motions of balloons. He then spent the afternoon discussing the newly-discovered planet Uranus with two of his assistants. At five o'clock, he suffered a cerebral hemorrhage, lost consciousness, and died soon afterwards. As one of his biographers put it, "The chalk fell from his hand; Euler ceased to calculate, and to live".

In the eighteenth century it was customary for the French Academy of Sciences to propose a mathematical topic each year, and to award a prize for the best paper dealing with the problem. Leonhard Euler and Daniel Bernoulli each won the Paris prize more than ten times, and they share the distinction of being the only men ever to do so. John Bernoulli is said to have thrown his son out of the house for winning the Paris prize in a year when he himself had competed for it.

Euler and the Bernoullis did more than anyone else to develop the Leibnizian form of calculus into a workable tool and to spread it throughout Europe. They applied it to a great variety of problems, from the shape of ships' sails to the kinetic theory of gasses.

Logarithms, exponentials and Euler's identity

To understand the problems on which the Bernoulli's and Euler worked, we will need to know how to differentiate and integrate the trigonometric functions $sin(t)$ and $cos(t)$, whose definitions are illustrated in Figure 1.6 of Chapter 1. In Figure 1.6, a right triangle is inscribed in a circle of unit radius, with one corner touching the circle, another corner at the center of the circle, and the third corner a distance called $cos(t)$ from the center along the horizontal axis. The length of the vertical side of the right triangle is called $sin(t)$, where t is the angle at the center of the circle.

Now imagine that the angle t is increased by a small amount Δt. Both the slightly changed triangle and the original one are shown in Figure 3.8.

Figure 3.6: *Daniel Bernoulli (1700-1782) is sometimes called the "father of mathematical physics" because of the far-reaching importance of his work with partial differential equations.*

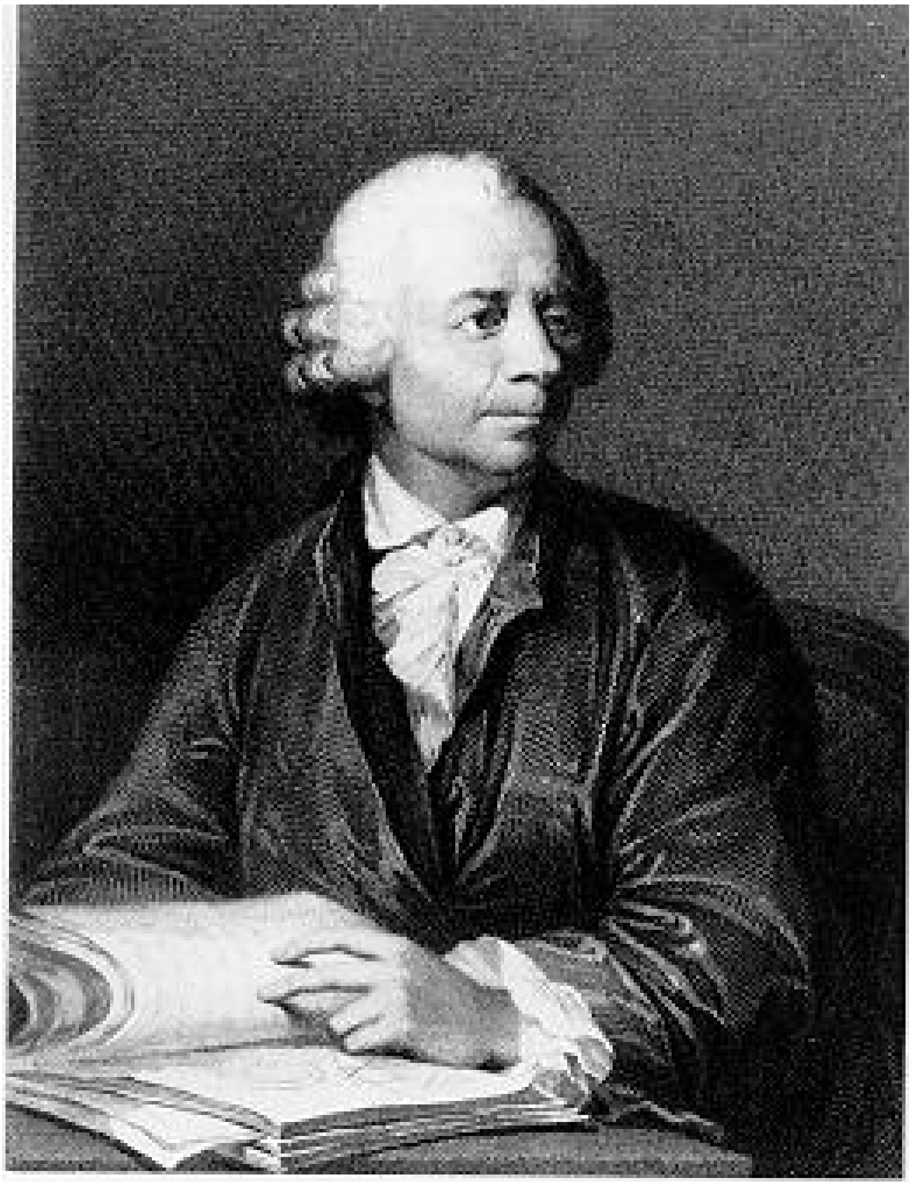

Figure 3.7: *Leonhard Euler (1707-1783) was the most prolific mathematician in history. His memory and his powers of concentration were amazing. Many of his important results were obtained during the last period of his life, when he was totally blind.*

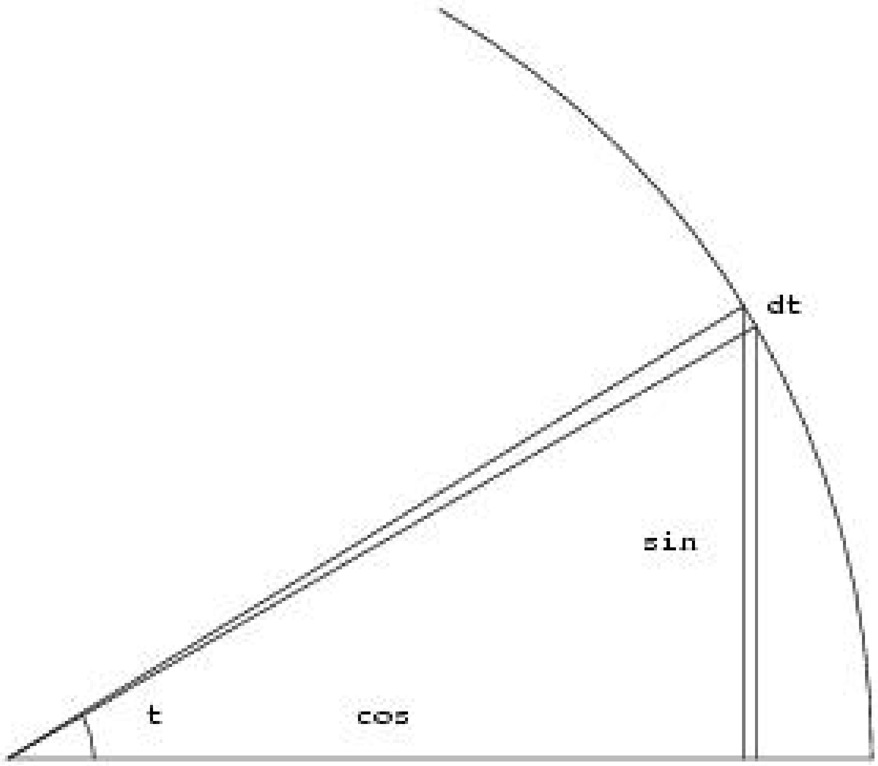

Figure 3.8: *This figure shows a circle of unit radius, inside which a right triangle is drawn with one corner touching the circle, another corner at the center of the circle, and the third corner on the horizontal axis. If the angle t at the center of the circle is slightly changed, the vertical side of the triangle becomes a little longer, and the horizontal side a little shorter.*

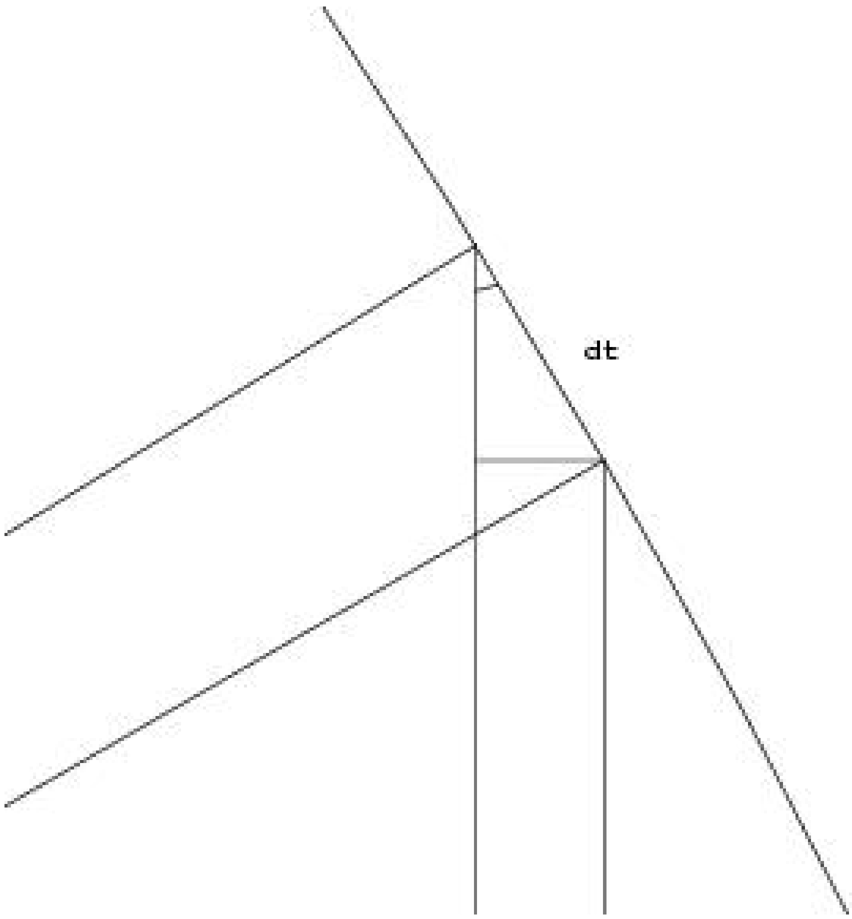

Figure 3.9: *This figure shows a magnified view of a portion of the previous figure. A small triangle is drawn, whose angles are the same as the angles of the previous triangle. It follows that if the central angle changes by an amount dt, the length of the vertical side will change by* $cos(t)dt$*, while the horizontal side will change by* $-sin(t)dt$*. These results are used in equations (3.27) and (3.28).*

As can be seen from this figure, the vertical side of the triangle has been increased by a small amount. In the limit where Δt becomes extremely small, considerations of geometry allow us to calculate by how much the vertical side of the right triangle has been increased. In that limit small arc of the circle joining the corner of the original triangle with the corner of the slightly altered one approaches a straight line of length Δt. Figure 3.9 shows a magnified view of this portion of Figure 3.8. From elementary geometry it is possible to show that the angle between the small arc and the vertical side of the new right triangle will approach t as Δt approaches zero. If we add a small horizontal line, as shown in Figure 3.9, we will obtain a tiny right triangle similar to our original triangle. For any two similar triangles the ratios of corresponding sides are equal. Therefore

$$\frac{d\ sin(t)}{dt} \equiv \begin{array}{c}\text{limit}\\ \Delta t \to 0\end{array} \left[\frac{sin(t+\Delta t) - sin(t)}{\Delta t}\right] = cos(t) \tag{3.27}$$

and

$$\frac{d\ cos(t)}{dt} \equiv \begin{array}{c}\text{limit}\\ \Delta t \to 0\end{array} \left[\frac{cos(t+\Delta t) - cos(t)}{\Delta t}\right] = -sin(t) \tag{3.28}$$

Equations (3.27) and (3.28) tell us how to differentiate $sin(t)$ and $cos(t)$ with respect to t. We also know, from the definitions of these functions, that

$$sin(0) = 0 \qquad \text{and} \qquad cos(0) = 1 \tag{3.29}$$

We are now in a position to use equation (2.32) to derive series representations of $sin(t)$ and $cos(t)$ in terms of powers of the variable t. If we let

$$sin(t) = \sum_{n=0}^{\infty} a_n t^n \tag{3.30}$$

then we know from equations (2.32), (3.27), (3.28) and (3.29) that

$$a_0 = \frac{1}{0!}\left[\frac{d^0 sin(t)}{dt^0}\right]_{t=0} = sin(0) = 0 \tag{3.31}$$

(where we have used the fact that $0! \equiv 1$) while

$$a_1 = \frac{1}{1!}\left[\frac{dsin(t)}{dt}\right]_{t=0} = cos(0) = 1 \tag{3.32}$$

and

$$a_2 = \frac{1}{2!}\left[\frac{d^2 sin(t)}{dt^2}\right]_{t=0} = sin(0) = 0 \tag{3.33}$$

and so on. Continuing in this way we obtain the series:

$$sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ... \tag{3.34}$$

and similarly,

$$cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ... \tag{3.35}$$

These series representations of $sin(t)$ and $cos(t)$ were known to Leonhard Euler. He was also familiar with another series that had been studied previously by the mathematician John Napier (1550-1617), Lord of Merchiston Castle near Edinburgh, Scotland:

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... \tag{3.36}$$

When he evaluated this series numerically for various values of t, Lord Napier noticed that

$$f(t)^2 = f(2t) \tag{3.37}$$

while

$$f(t)^3 = f(3t) \tag{3.38}$$

and in general

$$f(t)^n = f(nt) \tag{3.39}$$

Because of the property shown in equations (3.37)-(3.39), Napier thought of the series as representing some number e raised to the power t:

$$f = \sum_{n=0}^{\infty} \frac{t^n}{n!} \equiv e^t \tag{3.40}$$

since

$$(e^t)^2 = e^{2t} \qquad (e^t)^3 = e^{3t} \tag{3.41}$$

and so on. By evaluating the series at $t = 0$, Napier was able to find the value of the mysterious number e:

$$f(0) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = 2.718281828459045235... \equiv e \tag{3.42}$$

and this number is called the “Napierian base” in his honor. Napier also invented the concept of logarithms, which are closely related to equation (3.40). If we make a plot of Napier’s exponential function e^t, we can use the plot to find the values of t that must be substituted into the series (3.40) to give a particular result $f = e^t = a$. Napier called this particular value of t

the "logarithm of a". If the abbreviation "ln" is used to denote it, then we can write

$$a = e^{\ln(a)} \tag{3.43}$$

$$b = e^{\ln(b)} \tag{3.44}$$

and so on. Napier used his invention of logarithms to reduce the effort required to perform a multiplication numerically. He noticed that

$$ab = e^{\ln(a)} \times e^{\ln(b)} = e^{\ln(a)+\ln(b)} \tag{3.45}$$

so that

$$\ln(ab) = \ln(a) + \ln(b) \tag{3.46}$$

Similarly

$$\ln(\frac{b}{a}) = \ln(b) - \ln(a) \tag{3.47}$$

With the help of these relationships, Napier showed that tables of logarithms can be used to reduce the work involved in multiplication and division.

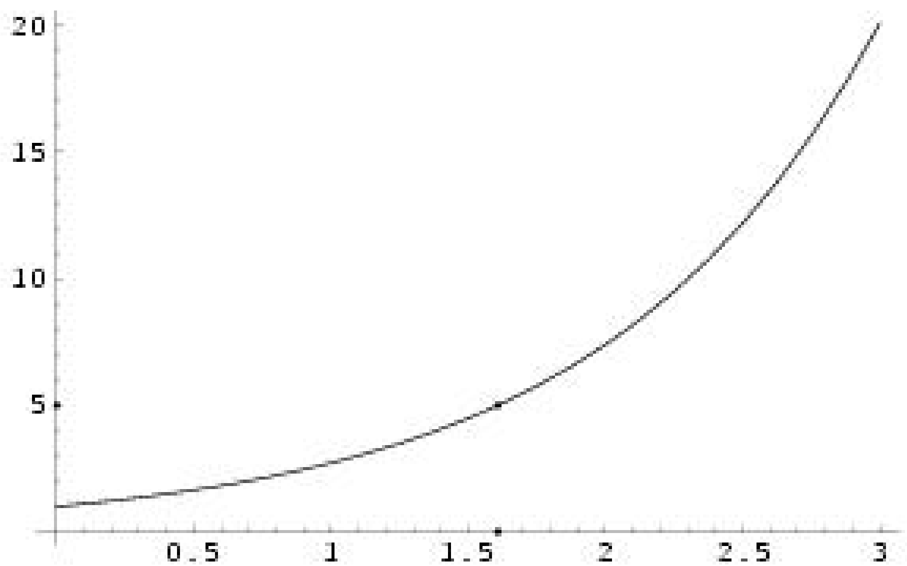

Figure 3.10: *This figure shows the exponential function* e^t *studied by Napier. If* $e^t = a$, *then* $t \equiv \ln a$. *In the figure,* $\ln 5$ *is marked with a dot on the* t *axis.*

- **Problem 3.11**: Use the series of equations (3.34) and (3.35) to evaluate $sin(1)$ and $cos(1)$. What is the value of $[sin(1)]^2 + [cos(1)]^2$? Why is this value nearly equal to 1? Is $[sin(t)]^2 + [cos(t)]^2$ equal to 1 for every value of t?

- **Problem 3.12**: Evaluate the first five terms in the series for the Napierian base e shown in equation (3.42). How close is the sum of these terms to the value of e given in the equation? Do you think that e is a rational number? (A rational number is a number that can be expressed as the ratio of two integers.)

- **Problem 3.13**: Use the series in equation (3.36) to evaluate e^2 up to five terms. How close is the value of $(e^1)^2$ to e^2?

- **Problem 3.14**: Calculate e^3 and e^4 and use these results, together with the results of Problem 3.13, to make a small table of logarithms. Try using this table, together with equations (3.46) and(3.47), to perform multiplications and divisions.

Building on Napier's work, Leonhard Euler studied the series

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + ... \tag{3.48}$$

where

$$i \equiv \sqrt{-1} \tag{3.49}$$

Since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on, equation (3.48) can also be written in the form:

$$e^{it} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + ... \tag{3.50}$$

Comparing this result with the series expansions for $cos(t)$ and $sin(t)$, Euler was able to write down his famous identity:

$$e^{it} = \cos(t) + i\sin(t) \tag{3.51}$$

Replacing i by $-i$, he found that

$$e^{-it} = \cos(t) - i\sin(t) \tag{3.52}$$

Then by adding these two equations and by subtracting them he obtained two related identities:

$$\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right) \tag{3.53}$$

and

$$\sin(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right) \tag{3.54}$$

Euler's identities make it easy to derive relationships between trigonometric functions. For example, if we square equation (3.54), we obtain

$$[\sin(t)]^2 = \left[\frac{1}{2i}\left(e^{it} - e^{-it}\right)\right]^2 = -\frac{1}{4}\left[e^{2it} + e^{-2it} - 2\right] \tag{3.55}$$

But from (3.53) it follows that this can be rewritten in the form

$$\sin^2(t) \equiv [\sin(t)]^2 = \frac{1}{2}[1 - \cos(2t)] \tag{3.56}$$

Euler then generalized the relationships (3.54) and (3.55) to define two new functions

$$\cosh(t) \equiv \frac{1}{2}\left(e^t + e^{-t}\right) \tag{3.57}$$

which he called the "hyperbolic cosine", and

$$\sinh(t) \equiv \frac{1}{2}\left(e^t - e^{-t}\right) \tag{3.58}$$

which he called the "hyperbolic sine" (or "sinus hyperbolicus"). Euler was able to show, using the calculus of variations, which he helped to invent, that the equilibrium configuration of a chain hanging between two fixed supports is described by a hyperbolic cosene. Equations (3.57) and (3.58) can be used to derive many relationships between the hyperbolic functions. For example, one can show that

$$\sinh^2(t) \equiv \frac{1}{2}[1 + \cosh(2t)] \tag{3.59}$$

- **Problem 3.15**: Use Euler's identities (3.51) and (3.52) together with equations (3.27) and (3.28) to evaluate $\frac{d}{dt}\left[e^{it}\right]$.

- **Problem 3.16**: Compare the result of Problem 3.15 with the result of differentiating the series of equation (3.50) term by term.

- **Problem 3.17**: Evaluate the indefinite integral $\int dt\ e^{it}$.

- **Problem 3.18**: Use equations (3.53) and (3.54) to evaluate $[cos(t)]^2 + [sin(t)]^2$.

- **Problem 3.19**: Use equations (3.57) and (3.58) to evaluate $[cosh(t)]^2 - [sinh(t)]^2$.

Equations (3.27) and (3.28) can be used to find the indefinite integrals of $sin(t)$ and $cos(t)$:

$$\int dt\ cos(t) = sin(t) + C \tag{3.60}$$

and

$$\int dt\ sin(t) = -cos(t) + C \tag{3.61}$$

To end this chapter on integral calculus let us return to the story of Archimedes, whose calculations showed that the ratio of the volume of a sphere to the volume of a cylinder circumscribed around it is exactly 2/3. He was so pleased with this result that he wished it to be carved onto his tombstone. Can we use integral calculus to follow in the steps of Archimedes? To do so, we must first find the area of a circle of radius r. We do this by calculating the following definite integral, whose meaning is shown in Figure 3.11.

$$\frac{r^2}{2}\int_0^{2\pi} dt = \pi r^2 = \text{area of a circle} \tag{3.62}$$

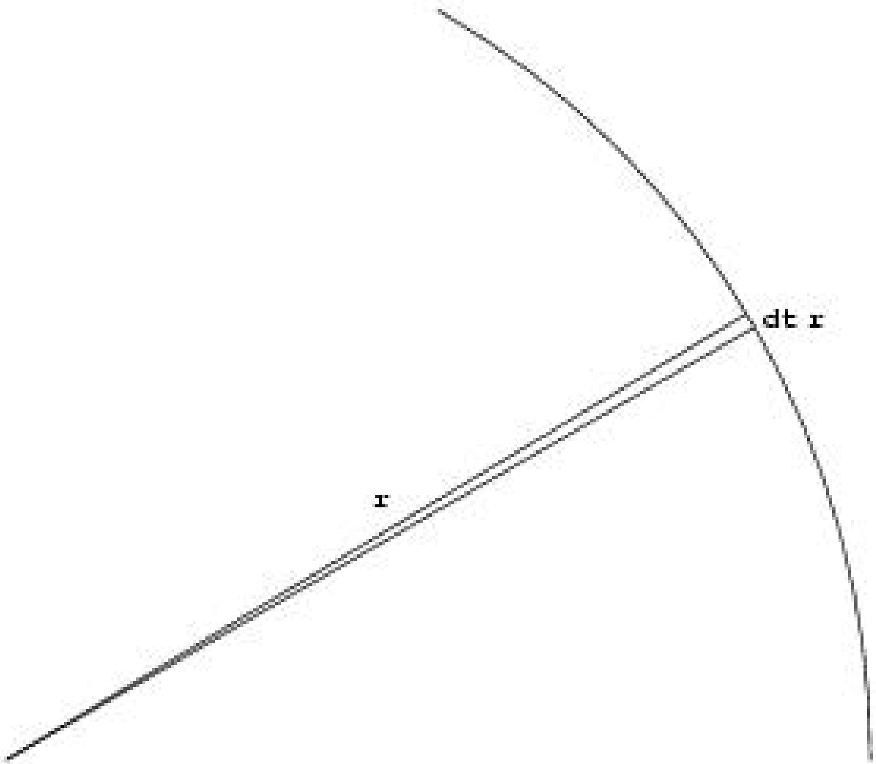

Figure 3.11: *This figure shows the geometrical interpretation of equation (3.62). The extremely narrow triangle shown in the figure has height* r*, base* $r\ dt$*, and area* $r^2dt/2$*. The integral represents the sum of all these small area contributions, and the result is the total area of the circle.*

Having found the area of a circle, we can easily find the volume of a cylinder of height h which has the circle as its base.

$$\text{volume of a cylinder } = \pi r^2 h \tag{3.63}$$

Thus the volume of the cylinder will be $\pi r^2 h$, but in the particular case where $h = 2r$, it will be $2\pi r^3$.

The definite integral

$$\rho^2 \int_0^{2\pi} ds \int_0^{\pi} dt \ \sin(t) = 4\pi\rho^2 \tag{3.64}$$

can be interpreted as the area of the surface of a sphere of radius ρ. To see that this is the case, we must imagine a globe representing the earth. On the surface of the globe are drawn lines representing latitude t and longitude s. Thus the line $t = \pi/2$ represents the equator, while $t = 0$ and $t = \pi$ respectively represent the north and south poles. The angle s (longitude) indicates how far west we are from the Greenwich Meridian. For example, if we are on the equator, $s = 0$ places us somewhere in Africa, while $s = \pi$ is in the Pacific. If we let $s = 2\pi$, we are back again in Africa. Figure 3.12 shows on the globe a small element of area which is approximately rectangular in shape. One of the sides has length $\rho\Delta s$, while the other has length $\rho sin(t)\Delta t$. Thus the area of the rectangle will be

$$\Delta A = \rho^2 \ \Delta s \ sin(t)\Delta t \tag{3.65}$$

The integral shown in equation (3.64) can be interpreted as the result we get from adding together all the small elements of area in the limit where both Δs and Δt become infinitesimally small. Thus (3.64) tells us that

$$\text{the surface of a sphere of radius } \rho = 4\pi\rho^2 \tag{3.66}$$

The definite integral

$$4\pi \int_0^r d\rho \ \rho^2 = \frac{4\pi r^3}{3} = \text{volume of a sphere of radius } r \tag{3.67}$$

on the left of equation (3.67) can be interpreted as the volume of a sphere of radius r, since the operation of integration can be interpreted as adding together many small volume elements

$$\Delta V = 4\pi \ \Delta\rho \ \rho^2 \tag{3.68}$$

in the limit where ΔV becomes infinitesimally small.

Thus, finally we obtain Archimedes famous result

$$\frac{\text{volume of a sphere}}{\text{volume of the circumscribed cylinder}} = \frac{2}{3} \tag{3.69}$$

During the centuries that separated Archimedes from Newton, the methods by which he obtained this result were lost, but through the work of Descartes, Newton, Leibniz, the Bernoulli's, Euler and many others, both differential and integral calculus were rediscovered and turned into practical tools that form part of the foundation of the modern world.

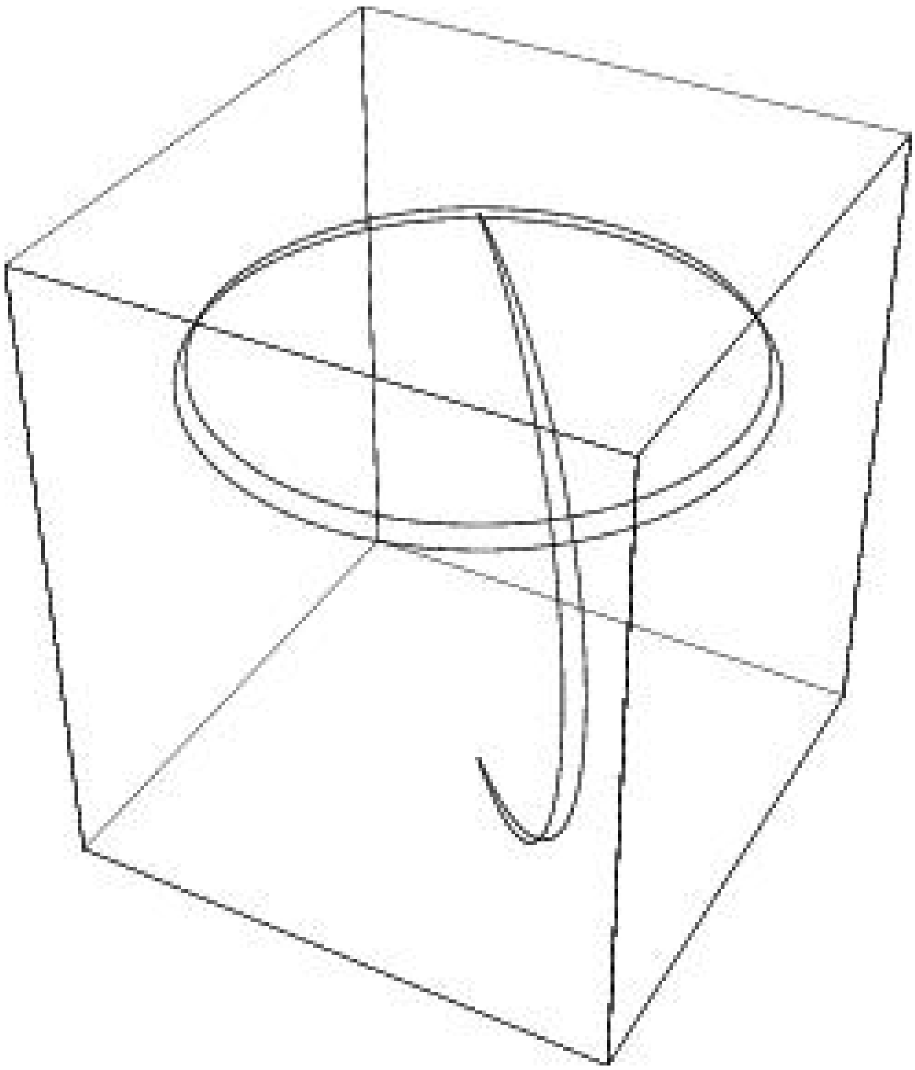

Figure 3.12: *On a globe representing the earth, we can let the angle t represent latitude while s represents longitude. A small approximately rectangular area is shown on the globe. The sides of this small rectangle are* $\rho\Delta s$ *and* $\rho sin(t)\Delta t$*, where* ρ *is the radius of the globe, while* Δt *and* Δs *are small changes in the two angles. By letting these changes become infinitesimal and integrating over the two angles, we obtain the total area of the globe, as is shown in equation (3.64).*

Chapter 4

Differential equations

Linear ordinary differential equations; rates of growth and decay

Leonhard Euler and all the members of the Bernoulli family were very much interested in differential equations, i.e., in equations relating the differentials of functions to the functions themselves. The simplest example of this type of relationship is the equation

$$\frac{df}{dt} = kf \tag{4.1}$$

where k is some constant. Equation (4.1) states that the rate of change of some function $f(t)$ is proportional to the function itself. This equation might (for example) describe the rate of growth of money that we have put into the bank, where k is the interest rate. It might also describe the increase or decrease of a population, where k represents the difference between the birth rate and the death rate. In both cases, the rate of change of f is proportional to the amount of f present at a given time. We can try to solve equation (4.1) by assuming that the solution can be represented by a series of the form

$$f = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... \tag{4.2}$$

where the a_n's are constants that we have to determine. Then the first derivative of the function f with respect to t will be given by

$$\frac{df}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1} = a_1 + 2a_2 t + 3a_3 t^2 + ... \tag{4.3}$$

Substituting equations (4.2) and (4.3) into (4.1), we obtain:

$$a_1 + 2a_2 t + 3a_3 t^3 + ... = ka_0 + ka_1 t + ka_2 t^2 + ... \tag{4.4}$$

In order for (4.4) to hold for all values of t, we need the following relationships between the constant coefficients a_n:

$$
\begin{aligned}
a_1 &= ka_0 \\
2a_2 &= ka_1 \\
3a_3 &= ka_2 \\
\vdots \quad & \vdots \quad \vdots \\
na_n &= ka_{n-1}
\end{aligned}
\tag{4.5}
$$

This set of equations can be solved to give all of the higher coefficients in terms of a_0:

$$
\begin{aligned}
a_1 &= \frac{k^2}{1!}a_0 \\
a_2 &= \frac{k^2}{2!}a_0 \\
a_3 &= \frac{k^3}{3!}a_0 \\
\vdots \quad & \vdots \quad \vdots \\
a_n &= \frac{k^n}{n!}a_0
\end{aligned}
\tag{4.6}
$$

Substituting these values of the coefficients back into (4.2) and remembering Napier's series (3.40) we obtain

$$f = a_0\left(1 + kt + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + ...\right) = a_0e^{kt} \tag{4.7}$$

From equation (4.7) we can see that e^{kt} multiplied by a constant a_0 will satisfy the differential equation (4.1). It follows that

$$\frac{d}{dt}\ e^{kt} = ke^{kt} \tag{4.8}$$

In other words, when we differentiate e^{kt} with respect to t, we obtain the same function again, multiplied by k. Using the relationships discussed in Chapter 3, we can also see that

$$\int dt\ e^{kt} = \frac{e^{kt}}{k} + C \tag{4.9}$$

The constant a_0 that appears in equation (eq:4.1g) is analogous to a constant of integration. It has to be chosen to satisfy other conditions imposed on the solution besides the differential equation. In general, such conditions are called "boundary conditions". We gave two examples of how equation (4.1) might be interpreted: $f(t)$ might represent the growth of money deposited in a bank at interest rate k. In that case, a_0 would represent the amount of money at the initial time, $t = 0$. On the other hand, if $f(t)$ represents a biological population changing as a function of time, where the constant k is the difference between the birth rate and the death rate, then a_0 represents the population at $t = 0$.

- **Problem 4.1**: Use equation (4.9) and Euler's identities (3.53) and (3.54) to show that
$$\int dt\ cos(\omega t) = \frac{1}{\omega} sin(\omega t) + C'$$
and that
$$\int dt\ sin(\omega t) = -\frac{1}{\omega} cos(\omega t) + C'$$
where C' is a constant.

- **Problem 4.2**: If (on the average) 0.1% of the soup bowls that a cafeteria owns are broken every day, write a differential equation that describes the average decrease in the number of soup bowls as a function of time. Suppose that the cafeteria decides to replace the bowls after half are gone. How long will it be before they have to replace them? Use the fact that $\ln(2) = 0.693$.

- **Problem 4.3**: Suppose that the population of a country increases on the average by 2% each year. If it continues to increase at this rate, by what factor will it have increased in a century? By how much in two centuries? By how much in three centuries?

Equation (eq:4.1a) is called a "first-order ordinary differential equation" - first-order because it involves only the function itself and its first derivative with no higher derivatives appearing; ordinary because it involves only one variable, t. We will now go on to discuss an example of a second-order ordinary differential equation, where we will see that there are two constants that must be determined by the boundary conditions of the problem.

The harmonic oscillator

As an example of a second-order ordinary differential equation, let us consider the relationship
$$\frac{d^2 f}{dt^2} = -\omega_0^2 f \tag{4.10}$$

which is sometimes called the "harmonic oscillator equation". We can solve this equation in two different ways. The first way is to make use of Euler's identities, (3.53) and (3.54), together with equation (4.8). If we let $k = i\omega_0$ and if we express $sin(\omega_0 t)$ and $cos(\omega_0 t)$ in terms of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ we obtain

$$\frac{d}{dt}sin(\omega_0 t) = \frac{d}{dt}\left[\frac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)\right] = \frac{1}{2i}\left(i\omega_0 e^{i\omega_0 t} + i\omega_0 e^{-i\omega_0 t}\right) = \omega_0 cos(\omega_0 t) \tag{4.11}$$

and

$$\frac{d}{dt}cos(\omega_0 t) = \frac{d}{dt}\left[\frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)\right] = \frac{1}{2}\left(i\omega_0 e^{i\omega_0 t} - i\omega_0 e^{-i\omega_0 t}\right) = -\omega_0 sin(\omega_0 t) \tag{4.12}$$

These equations are closely similar to (3.27) and (3.27), except for the factor ω_0. If we differentiate (4.11) and (4.12) a second time with respect to t, we obtain:

$$\frac{d^2}{dt^2}sin(\omega_0 t) = \omega_0 \frac{d}{dt}cos(\omega_0 t) = -\omega_0^2 sin(\omega_0 t) \tag{4.13}$$

and

$$\frac{d^2}{dt^2}cos(\omega_0 t) = -\omega_0 \frac{d}{dt}sin(\omega_0 t) = -\omega_0^2 cos(\omega_0 t) \tag{4.14}$$

Looking at equations (4.13) and (4.14), and comparing them with (4.10), we can see that both $sin(\omega_0 t)$ and $cos(\omega_0 t)$ are solutions to the harmonic oscillator equation, (4.10). It follows that if a_1 and a_0 are constants,

$$f(t) = a_1 sin(\omega_0 t) + a_0 cos(\omega_0 t) \tag{4.15}$$

must also be a solution. Here we can see that the solution of a second-order ordinary differential equation contains two constants analogous to the constants of integration that we encountered when evaluating indefinite integrals. These constants cannot be found from the differential equation itself. They are determined by the boundary conditions of the problem.

The second way of solving the harmonic oscillator equation is to assume that the solution $f(t)$ can be expanded in a series of the form shown in equation (4.2). The first derivative of f will then be given by (4.3), and if we differentiate a second time, we obtain:

$$\frac{d^2 f}{dt^2} = 2\ a_2 + 6\ a_3 t + 12\ a_4 t^2 + 20\ a_5 t^3 + ... \tag{4.16}$$

Multiplying f by $-\omega_0^2$ gives

$$-\omega_0^2 f = -\omega_0^2 a_0 - \omega_0^2 a_1 t - \omega_0^2 a_2 t^2 - \omega_0^2 a_3 t^3 - .... \tag{4.17}$$

Thus the harmonic oscillator equation requires that

$$2\ a_2 + 6\ a_3 t + 12\ a_4 t^2 + 20\ a_5 t^3 + ... = -\omega_0^2 a_0 - \omega_0^2 a_1 t - \omega_0^2 a_2 t^2 - \omega_0^2 a_3 t^3 - .... \tag{4.18}$$

The requirement that (4.18) must hold for all values of t gives us a set of equations relating the higher even coefficients to a_0:

$$\begin{array}{rcl} a_2 & = & -\frac{\omega_0^2}{2!} a_0 \\ a_4 & = & +\frac{\omega_0^4}{4!} a_0 \\ a_6 & = & -\frac{\omega_0^6}{6!} a_0 \\ \vdots & \vdots & \vdots \end{array} \tag{4.19}$$

and another set of equations relating the higher odd coefficients to a_1:

$$\begin{array}{rcl} a_3 & = & -\frac{\omega_0^3}{3!} a_1 \\ a_5 & = & +\frac{\omega_0^5}{5!} a_1 \\ a_7 & = & -\frac{\omega_0^7}{7!} a_1 \\ \vdots & \vdots & \vdots \end{array} \tag{4.20}$$

Thus the solution can be written in the form

$$f = a_1 \left(\omega_0 t - \frac{(\omega_0 t)^3}{3!} + \frac{(\omega_0 t)^5}{5!} - ... \right) + a_0 \left(1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - ... \right) \tag{4.21}$$

where the constants a_1 and a_0 must be determined from the boundary conditions of the problem. Comparing these series with the series in equations (3.34) and (3.35), we can rewrite (4.21) in the form

$$f(t) = a_1 sin(\omega_0 t) + a_0 cos(\omega_0 t) \tag{4.22}$$

which is exactly the same as our previous result.

- **Problem 4.4**: The solution to the harmonic oscillator equation shown in equation (4.22) contains two constants of integration, a_0 and a_1. If the initial conditions require that

  $$f(0) = 1 \qquad \left[\frac{df}{dt}\right]_{t=0} = 0$$

  what are the values of the constants a_0 and a_1?

What happens when friction is added?

If we want to make the harmonic oscillator equation a little more complicated, we can add a term proportional to df/dt:

$$\frac{d^2f}{dt^2} + a\frac{df}{dt} + \omega_0^2 f = 0 \tag{4.23}$$

Equation (4.23) is called the "damped harmonic oscillator equation". Our original harmonic oscillator equation, (4.10) might (for example) represent the motion of a frictionless pendulum, while in (4.23) the effects of friction are included. In order to solve the differential equation of a damped harmonic oscillator, let us assume that it is possible to write the solution in the form

$$f = e^{kt} \tag{4.24}$$

where k is a constant that may have both real and imaginary parts. (An imaginary number is a number that is proportional to i, where $i \equiv \sqrt{-1}$.) If we cannot find a solution of this form, we will have to think of some other trial function, but let us begin by examining $f = e^{kt}$ to see whether this will work. From (4.8) we have

$$\frac{df}{dt} = kf \qquad \frac{d^2f}{dt^2} = k^2 f \tag{4.25}$$

Substituting (4.24) and (4.25) into the damped harmonic oscillator equation, we have

$$\left(k^2 + ak + \omega_0^2\right) f = 0 \tag{4.26}$$

Since f is in general not zero, the quantity in brackets must vanish. We said that we would allow k to have both real and imaginary parts. To make this explicit, we write

$$k = u + iv \tag{4.27}$$

where u and v are real numbers and $i \equiv \sqrt{-1}$. Substituting this into the requirement

$$k^2 + ak + \omega_0^2 = 0 \tag{4.28}$$

yields

$$(u + iv)^2 + a(u + iv) + \omega_0^2 = 0 \tag{4.29}$$

or

$$u^2 + 2iuv - v^2 + au + iav + \omega_0^2 = 0 \tag{4.30}$$

The imaginary part of (4.30) must be separately equal to zero, and therefore

$$2u + a = 0 \qquad u = \frac{a}{2} \tag{4.31}$$

The real part of (4.30) must also vanish, which gives us the relationship

$$\frac{a^2}{4} - v^2 + \omega_0^2 = 0 \tag{4.32}$$

where we have used the fact that $u = a/2$. Solving (4.32) for v, we obtain:

$$v = \pm\sqrt{\omega_0^2 + \frac{a^2}{4}} \tag{4.33}$$

The positive value of the square root in equation (4.33) gives us one solution to the damped harmonic oscillator equation, and the negative square root yields another independent solution. The most general solution thus has the form:

$$f = A_1 e^{k_+ t} + A_2 e^{k_- t} = e^{-at/2}\left[A_1 e^{i\omega' t} + A_2 e^{-i\omega' t}\right] \tag{4.34}$$

where A_1 and A_2 are constants that must be determined from the boundary conditions, and where

$$\omega' \equiv \sqrt{\omega_0^2 + \frac{a^2}{4}} \tag{4.35}$$

Using Euler's identities, we can rewrite the general solution in the form.

$$f(t) = e^{-at/2}\left[a_1 sin(\omega' t) + a_0 cos(\omega' t)\right] \tag{4.36}$$

Figure 4.1 shows the solution $f(t)$ in equation (4.36) for the case where $a_0 = 1$ and $a_1 = 0$ and $a = \omega_0/10$.

- **Problem 4.5**: Repeat Problem 4.4 for the damped harmonic oscillator transient solution shown in equation (4.36).

What happens if we add a driving force?

If we want to make our damped harmonic oscillator equation *still* more complicated[1], we can add a term representing an external driving force. For example, if the damped harmonic oscillator in question is a playground swing in which a small girl is sitting, the driving force might be her brother, who occasionally pushes the swing. Or if the damped harmonic oscillator represents a musical instrument, the driving force comes from the efforts of the musician. Both equations (4.10) and (4.23) are said to be "homogeneous" differential equations. This means that they only contain terms proportional to f and to its derivatives. However, when we add a term representing an external driving force, we obtain what is called an "inhomogeneous" differential

[1]Do I hear someone saying "No! No! Help!"?

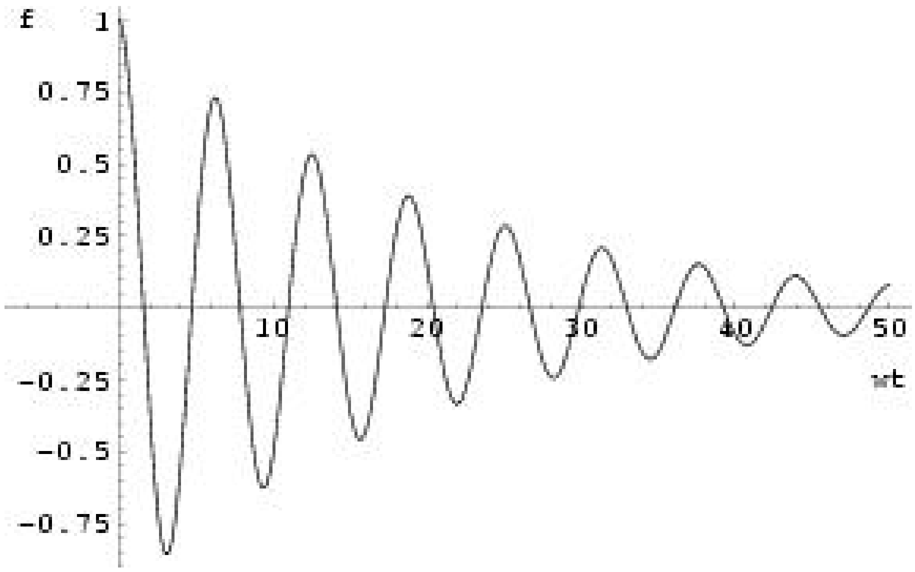

Figure 4.1: *This figure illustrates the behavior of a damped harmonic oscillator as a function of time. The figure shows the solution in equation (4.36) for the case where $a_0 = 1$ and $a_1 = 0$ and $a = \omega_0/10$. Because of damping, the oscillations gradually disappear, and for this reason they are called "transients".*

equation. For example, if the driving force has the form $cos(\omega t)$ the inhomogeneous differential equation for the driven, damped harmonic oscillator has the form:

$$\frac{d^2 f}{dt^2} + a\frac{df}{dt} + \omega_0^2 f = cos(\omega t) \tag{4.37}$$

The most general solution to an inhomogeneous differential equation is the sum of a solution to the corresponding homogeneous equation, plus a particular solution to the inhomogeneous equation:

$$f = f_{homogeneous} + f_{particular} \tag{4.38}$$

In our case, the homogeneous equation corresponding to (4.37) is equation (4.23). If we rewrite (4.37) in the form

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + \omega_0^2\right) f = cos(\omega t) \tag{4.39}$$

we can see that the differential operator in round brackets, acting on a solution to the homogeneous equation, (4.23), will give zero. Suppose that we can manage, somehow or other, to find a function such that the same operator acting on it gives $cos(\omega t)$. We call this function a particular solution to the inhomogeneous equation. Thus the most general solution to (4.39) will have the form shown in equation (4.38), because when the operator in round brackets acts on $f_{homogeneous}$ the result is zero, but when the same operator acts on $f_{particular}$, it yields $cos(\omega t)$. We already have a solution to the homogeneous equation, namely the function shown in (4.34). The particular solution to the inhomogeneous equation can be found by assuming that it has the form

$$f_{particular} = A(\omega)e^{i\omega t} + B(\omega)e^{-i\omega t} \tag{4.40}$$

Substituting (4.40) into (4.39), and making use of Euler's identities, we obtain the requirements

$$A(\omega)\left(-\omega^2 + ia\omega + \omega_0^2\right) = \frac{1}{2} \tag{4.41}$$

and

$$B(\omega)\left(-\omega^2 - ia\omega + \omega_0^2\right) = \frac{1}{2} \tag{4.42}$$

so that $f_{particular}$ must have the form

$$f_{particular} = \frac{e^{i\omega t}}{2\left(-\omega^2 + ia\omega + \omega_0^2\right)} + \frac{e^{-i\omega t}}{2\left(-\omega^2 - ia\omega + \omega_0^2\right)} \tag{4.43}$$

Notice that the particular solution of the inhomogeneous differential equation does not contain any constants analogous to constants of integration.

What is the physical interpretation of these results? Looking at the particular solution, (4.43), we may be surprised (and perhaps a little disturbed) to see it expressed in terms of $i \equiv \sqrt{-1}$. How can the solution to a physical problem involve imaginary numbers? However, closer examination shows that $f_{particular}$ is real. The first term in $f_{particular}$ is complex, i.e. it has both a real part and an imaginary part. Suppose that we define two real numbers, x and y, in such a way that x is the real part of the first term, while iy is the imaginary part:

$$x + iy \equiv \frac{e^{i\omega t}}{2\left(-\omega^2 + ia\omega + \omega_0^2\right)} \tag{4.44}$$

Then it must be true that

$$x - iy = \frac{e^{-i\omega t}}{2\left(-\omega^2 - ia\omega + \omega_0^2\right)} \tag{4.45}$$

since we can go from (4.44) to (4.45) by replacing i everywhere by $-i$. Adding the two equations, we obtain $2x$ on the left-hand side, which is real, while on the right we obtain $f_{particular}$. Multiplying both the top and bottom of (4.44) by $(\omega_0^2 - \omega^2 - ia\omega)$ and looking at the real part of the result, we obtain

$$f_{particular} = \frac{(\omega_0^2 - \omega^2)cos(\omega t) + a\omega sin(\omega t)}{(\omega_0^2 - \omega^2)^2 + a^2\omega^2} \tag{4.46}$$

In equation (4.46), ω_0 is the natural frequency of the oscillator, while ω is the frequency of the driving force. We can see that when the driving frequency approaches the natural frequency of the oscillator, the amplitude of the induced oscillations will become large. Figure 4.2 shows the factor

$$A(\omega) = \frac{1}{(\omega_0^2 - \omega^2)^2 + a^2\omega^2} \tag{4.47}$$

as a function of the driving frequency ω for several values of the damping constant a. It can be seen that the smaller the damping constant a, the larger the induced oscillations become. The peaking of the amplitude factor is called a "resonance". When the damping is small, the resonance is sharp.

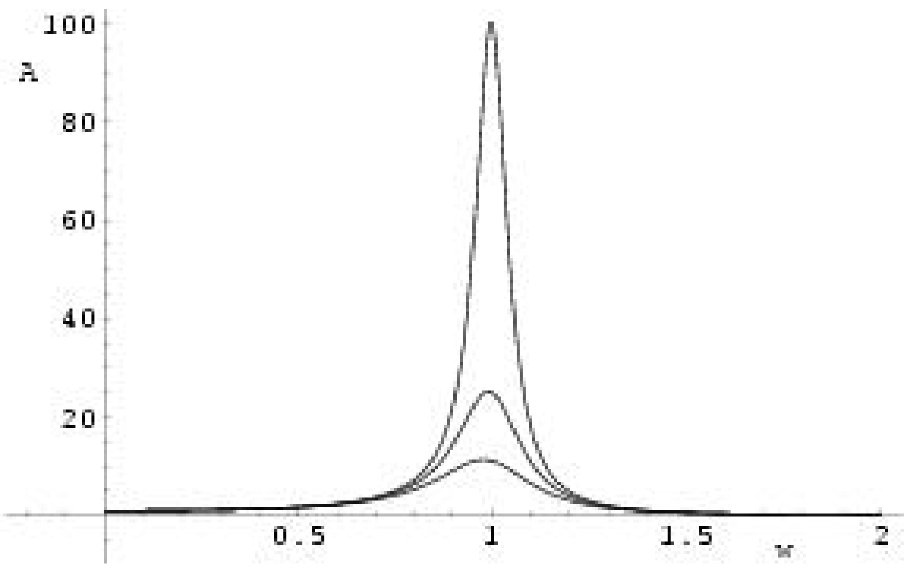

Figure 4.2: *This figure shows the shape of the resonance of a driven damped harmonic oscillator, equation (4.47). The curve $A(\omega)$ in that equation is plotted as a function of the driving frequency ω for various values of the damping constant a. In the graph, $\omega_0 = 1$, and the three curves correspond to $a = .1$, $a = .2$ and $a = .3$. When the damping is small, the resonance is sharp.*

The reader may enjoy trying the following simple experiment, which simulates the behavior of a driven damped harmonic oscillator: Take a small weight and attach it to a rubber band. Hold the rubber band in your hand so that the weight dangles from it. Lift the weight a little with your other hand, and release it. The weight will move up and down with a characteristic frequency which we have called $\omega' \approx \omega_0$ in our discussion. After a little while, the oscillations will become smaller and and they will finally disappear because of friction (damping). These transient oscillations are those shown in equation (4.34). Now move your hand holding the rubber band up and down with some other frequency, ω. Gradually increase ω so that it approaches ω_0. The oscillations of the suspended weight will become large as you pass through the resonance, and smaller again when ω is higher than ω_0. Notice that the phase between the induced oscillations and the driving force changes as you pass through the resonance, as is predicted by equation (4.46).

Partial differentiation; Daniel Bernoulli's wave equation

Having discussed differential equations involving only a single variable (ordinary differential equations), let us now turn to differential equations involving several variables. These are called "partial differential equations". The most important pioneer of this branch of mathematics was Daniel Bernoulli.

In 1727, John Bernoulli in Basle, corresponding with his son Daniel in St. Petersberg, developed an approximate set of equations for the motion of a vibrating string by considering it to be a row of point masses, joined together by weightless springs. Then Daniel boldly passed over to the continuum limit, where the masses became infinitely numerous and small.

The result was Daniel Bernoulli's famous wave equation, which is what we would now call a partial differential equation. But what is a partial differential equation? What is partial differentiation?

Daniel Bernoulli developed his wave equation to describe the motion of a vibrating string, for example a violin string, and in this problem there are two variables: x, which represents the distance along the string, and t, which represents time. The displacement of the string from its equilibrium position is represented by $f(x,t)$. In other words, the displacement is a function of two variables, position and time. To deal with this problem, Daniel Bernoulli defined partial differentials in much the same way that Isaac Newton defined ordinary differentials (equation (2.13)). He introduced the definitions:

$$\frac{\partial f}{\partial x} \equiv \begin{array}{c} \text{limit} \\ \Delta x \to 0 \end{array} \left[\frac{f(x+\Delta x, t) - f(x,t)}{\Delta x} \right] \tag{4.48}$$

and

$$\frac{\partial f}{\partial t} \equiv \begin{matrix} \text{limit} \\ \Delta t \to 0 \end{matrix} \left[\frac{f(x, t + \Delta t) - f(x, t)}{\Delta t} \right] \tag{4.49}$$

We can understand of the partial differentials defined by equations (4.48) and (4.49) by imagining that we are walking in a landscape of hills and valleys. In this landscape, $f(x, t)$ represents the height above sea level, while x represents the north-south position and t the east-west position. If we take an infinitesimal step northward, the change in our height above sea level will be

$$\frac{\partial f}{\partial x}\, dx \tag{4.50}$$

where dx is the length of our northward step, whereas if we take a step eastward, the change will be

$$\frac{\partial f}{\partial t}\, dt \tag{4.51}$$

The rules for partial differentiation are the same as for ordinary differentiation, except that we must add an extra rule: When performing partial differentiation with respect to one variable, all other variables must be regarded as constants. Second partial derivatives are defined similarly. For example, to find

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial x} \right] \tag{4.52}$$

we simply differentiate twice with respect to x, and we remember that during this process, all other variables must be regarded as constant. For, example, when we have two variable, x and t, then t is regarded as constant when we evaluate the second partial derivative with respect to x. Similarly, when we evaluate the second partial derivative with respect to t,

$$\frac{\partial^2 f}{\partial t^2} \equiv \frac{\partial}{\partial t} \left[\frac{\partial f}{\partial t} \right] \tag{4.53}$$

x is regarded as constant. It is also possible to define mixed partial derivatives, and it turns out that in the mixed second partial derivative

$$\frac{\partial^2 f}{\partial x \partial t} \equiv \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial t} \right] = \frac{\partial^2 f}{\partial t \partial x} = \frac{\partial}{\partial t} \left[\frac{\partial f}{\partial x} \right] \tag{4.54}$$

the order of differentiation does not matter.

- **Problem 4.6**: Suppose that

$$(x + iy)^n = u + iv$$

where $n = 3$ and where x, y, u and v all are real, with $i \equiv \sqrt{-1}$. Find u and v and show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations are called the “Cauchy-Riemann equations”.

- **Problem 4.7**: Repeat Problem 4.6 for $n = 1$ and $n = 2$.
- **Problem 4.8**: Show that if u and v satisfy the Cauchy-Riemann equations, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0$$

The second-order differential equation satisfied by both u and v is called the “Laplace equation”.

In the notation that we have been discussing, Daniel Bernoulli's wave equation has the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x,t) = 0 \tag{4.55}$$

where c is a constant. Bernoulli was able to show that in the case of a vibrating string,

$$c = \sqrt{\frac{T}{\mu}} \tag{4.56}$$

where T is the tension in the string and where μ is the mass per unit length. Daniel Bernoulli solved his wave equation by assuming that a solution could be written in the form

$$f(x,t) = \phi(x)\left[cos(\omega t) + a_1 sin(\omega t)\right] \tag{4.57}$$

where the constant a_1 is determined by the initial conditions of the problem. Then, since

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[cos(\omega t) + a_1 sin(\omega t)\right] = -\omega^2\left[cos(\omega t) + a_1 sin(\omega t)\right] \tag{4.58}$$

x-dependent part of the solution had to satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\right)\phi(x) = \left(\frac{d^2}{dx^2} + k^2\right)\phi(x) = 0 \tag{4.59}$$

where

$$k^2 \equiv \frac{\omega^2}{c^2} \tag{4.60}$$

Daniel Bernoulli showed that (4.59) has sinusoidal solutions of the form

$$\phi(x) = A_1 sin(kx) + A_2 cos(kx) \tag{4.61}$$

where the constants A_1 and A_2 as well as the value of k are determined by the boundary conditions. For example, if the vibrating string is clamped at the positions $x = 0$ and $x = L$, then we know that $A_2 = 0$ (since $cos(0) \neq 0$), and that

$$\phi(L) = \sin(kL) = 0 \tag{4.62}$$

The boundary condition shown in equation (4.62) determines the allowed values of k; they must such that kL is an integral multiple of π, and thus the only allowed values are

$$k = \frac{n\pi}{L} \qquad n = 1, 2, 3, 4, ... \tag{4.63}$$

Only positive integers need be considered, because although the negative integers would satisfy the boundary conditions, they do not yield any new independent solutions. Thus Daniel Bernoulli's wave equation, with the boundary conditions $f(0, t) = 0$ and $f(L, t) = 0$, can be satisfied by any function of the form

$$f_n(x, t) = A_n sin(kx) [cos(kct) + a_n sin(kct)] \tag{4.64}$$

where k is an integral multiple of π/L.

Bernoulli realized that the sum of any two solutions to his wave equation is also a solution. This is easy to prove: We know that if $f_n(x, t)$ has the form shown in equation (4.64), then

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) f_n(x, t) = 0 \tag{4.65}$$

Then a function of the form

$$\Phi(x, t) = \sum_n f_n(x, t) \tag{4.66}$$

will also be solution, since

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \Phi(x, t) = \sum_n \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) f_n(x, t) = 0 \tag{4.67}$$

where we have made use of equation (4.65).

Daniel Bernoulli's superposition principle is a mathematical proof of a property of wave motion noticed by Huygens. The fact that many waves can propagate simultaneously through the same medium without interacting was one of the reasons for Huygens' belief that light is wavelike, since he knew that many rays of light from various directions can cross a given space simultaneously without interacting.

The argument between Bernoulli and Euler; Fourier analysis

Leonhard Euler and Daniel Bernoulli were both such great mathematicians and great friends that it is strange to think that there could ever have been a disagreement between them. Nevertheless, a long argument between these two geniuses began as a result of their independent solutions to the wave equation. The argument was by no means sterile, however, and eventually it led to the foundation of a new branch of mathematics - Fourier analysis.

We have just seen Bernoulli's solution to the wave equation. Leonhard Euler also solved it, and in a completely different way. Euler showed that if F and G are any two well-behaved functions of a single variable, then

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) F(x+ct) = 0 \tag{4.68}$$

and

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(x-ct) = 0 \tag{4.69}$$

for example, when

$$F(x+ct) = (x+ct)^2 = x^2 + 2xct + c^2t^2 \tag{4.70}$$

then

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[x^2 + 2xct + c^2t^2\right] = \frac{\partial}{\partial x}\left[2x + 2ct\right] = 2 \tag{4.71}$$

while

$$-\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[x^2 + 2xct + c^2t^2\right] = -\frac{1}{c^2}\frac{\partial}{\partial t}\left[2xc + 2c^2t\right] = -2 \tag{4.72}$$

Adding equations (4.71) and (4.72) we obtain (4.68). Notice that in carrying out the partial differentiations with respect to x, we regard t as a constant, while in (4.72), where we differentiate with respect to t, we hold x constant.

Leonhard Euler was able to show that if F is a function of some variable w, then

$$\frac{\partial}{\partial x}F(w) = \frac{dF}{dw}\frac{\partial w}{\partial x} \qquad \frac{\partial}{\partial t}F(w) = \frac{dF}{dw}\frac{\partial w}{\partial t} \tag{4.73}$$

and using these relationships he was able to prove that equations (4.68) and (4.69) hold in general, no matter what the functions F and G might be.

- **Problem 4.9**: Use the relationships shown in equation (4.73) to show that $F(x+ct)$ satisfies the wave equation, (4.68).

- **Problem 4.10**: Show that $G(x-ct)$ also satisfies the wave equation.

- **Problem 4.11**: Use equation (4.73) to show that if

  $$F(x+iy) = u + iv$$

  where u and v are real functions of x and y, then u and v satisfy the Cauchy-Riemann equations and the Laplace equation.

Meanwhile, Daniel Bernoulli had derived his own solutions to the wave equation, the ones shown in equation (4.64), and he had also shown that if these solutions are added together, with various values of the constants A_n and a_n, the result is also a solution. Euler and Bernoulli wrote letters to each other about their work on the wave equation, and being great mathematicians, they were able draw the logical conclusion that followed from their results: If they were both right, it had to follow that by choosing the constants A_n in the right way it would be possible to construct series such that

$$f(x) = \sum_{n=0}^{\infty} A_n sin(\frac{n\pi x}{L}) \qquad n = 1, 2, 3, ... \tag{4.74}$$

regardless of the form of $f(x)$, the only restriction being that f should be single-valued, continuous and differentiable and that it should obey the boundary conditions $f(x) = 0$ and $f(L) = 0$. Euler found this hard to believe, and to the end of his life he continued to think that there must be something wrong. Euler believed the he himself had found the most general solutions to the wave equation, and that his friend Daniel's set of solutions was somehow incomplete - not sufficiently general.

Fourier analysis

The controversy about the completeness of Bernoulli's solutions was still raging when Jean-Baptiste Fourier (1768-1830) arrived on the scene. Although he began life as the orphaned son of a poor tailor, Fourier later achieved distinction as Professor of Mathematics at Napoleon's École Normale, and he even became a personal friend of the emperor. He followed Napoleon to Egypt, where he helped to set up the Egyptian Institute, and where he made estimates of the ages of the pyramids and other monuments. Napoleon finally appointed Fourier as the Prefect of a district in southern France in the vicinity of Grenoble. Fourier worked hard at this job, supervising (for example) the draining of swamps to eliminate malaria. Nevertheless, he continued his mathematical research, and during his time in Grenoble he composed a monumental study of heat conduction, his *Mémoir sur la Chaleur*. In this work, he made use of a method that later became known as Fourier analysis.

Figure 4.3: *Jean-Baptiste Fourier (1768-1830) founded a branch of mathematics now known as Fourier analysis. Its generalizations have great importance for many branches of theoretical science and engineering.*

The diffusion equation, which governs heat flow, is similar to the wave equation except that it involves only first-order differentiation with respect to time. For the case of heat flow in a metal rod, the equation for the temperature as a function of both position and time has the form

$$\frac{\partial}{\partial t}T(x,t) = C\frac{\partial^2}{\partial x^2}T(x,t) \tag{4.75}$$

where C is a constant and where $T(x,t) + T_0$ is the temperature.

- **Problem 4.12**: Show that functions of the form

  $$T_n(x,t) = A_n e^{-a_n t} sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ...$$

  are solutions to the diffusion equation satisfying the boundary conditions $T_n(0,t) = 0$ and $T_n(L,t) = 0$. What condition must the constants a_n fulfill in order that the diffusion equation should be satisfied? How should the constants A_n be chosen?

Fourier was able to use a slightly modified version of Daniel Bernoulli's methods to find solutions to the diffusion equation, and given the initial temperature distribution, he was able to calculate the temperature distribution at any future time. To do this, he needed to determine the constants A_n in series such as the one shown in equation (4.74). (Today, this type of series is called a Fourier series.) One of the equations that Fourier used to determine these constants had the form

$$\frac{2}{L}\int_0^L dx\ sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 \text{ if } n \neq m \\ \\ 1 \text{ if } n = m \end{cases} \tag{4.76}$$

where both n and m are integers. From equation (4.76) if follows that

$$\begin{aligned} & \frac{2}{L}\int_0^L dx\ sin\left(\frac{n\pi x}{L}\right) f(x) \\ = & \frac{2}{L}\sum_{m=0}^{\infty} A_m \int_0^L dx\ sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) = A_n \end{aligned} \tag{4.77}$$

Fourier was able to substitute the A_n's calculated from (4.77) back into the the series for $f(x)$. For example, suppose that

$$f(x) = \begin{cases} 1 \ \text{ if } \ 0 < x < L/2 \\ \\ 0 \ \text{ if } \ L/2 \leq x < L \end{cases} \tag{4.78}$$

then

$$
\begin{aligned}
A_n &= \frac{2}{L}\int_0^L dx\ sin\left(\frac{n\pi x}{L}\right) f(x) \\
&= \frac{2}{L}\int_0^{L/2} dx\ sin\left(\frac{n\pi x}{L}\right) \\
&= \frac{2}{n\pi}\left\{1 - cos\left(\frac{n\pi}{2}\right)\right\}
\end{aligned}
\tag{4.79}
$$

Figure 4.4 shows the function defined by equation (4.78) compared with the Fourier series for the function carried out to 50 terms. As more and more terms are added to the series, it becomes more and more accurate, but in order to be completely accurate, it would need an infinite number of terms.

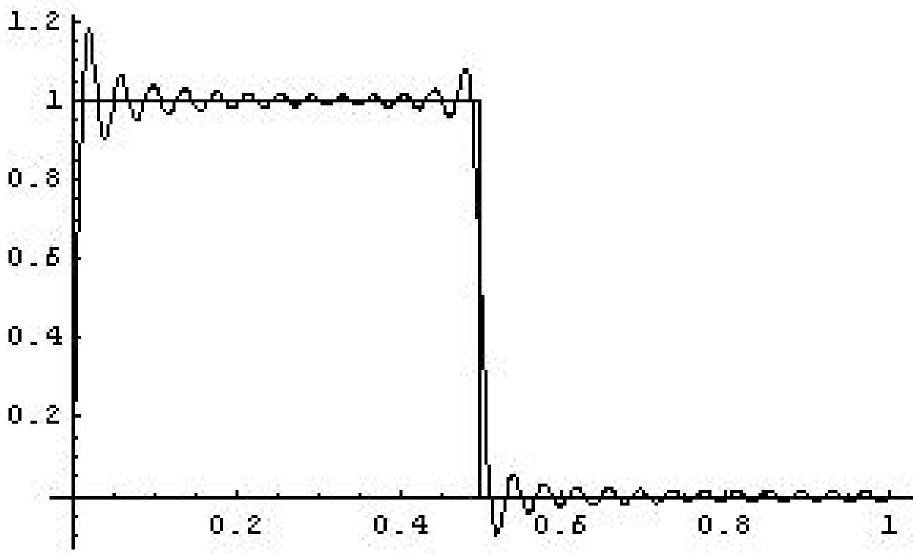

Figure 4.4: *This figure shows the Fourier series representation of the function defined by equation(4.76) compared with the function itself. The slowly convergent series has been truncated after 50 terms, and thus it fails to represent the function with complete accuracy. However, if an infinite number of terms had been included, the Fourier series would be completely accurate. "Square waves" of the kind shown here are sometimes used to test high fidelity electronic amplifiers, because very high frequencies are needed to accurately reproduce the sharp corners of the square wave.*

When Fourier submitted his *Mémoir sur la Chaleur* to the Academy of Sciences in Paris, it was severely criticised and it failed to win the annual prize set by the Academy. The jury consisted of three of the most eminent mathematicians of the period, Joseph-Louis Lagrange (1736-1813), Pierre-Simon Laplace (1749-1827) and Adrien-Marie Legendre (1749-1827). Lagrange, Laplace and Legendre objected that although Fourier's methods worked extremely well in practice, he had not really overcome Euler's objections, i.e. he had not really shown that *every* continuous, single-valued and differentiable function $f(x)$ obeying the boundary conditions $f(0) = 0$ and $f(L) = 0$ can be represented by the series shown in equation (4.74). (This property of the set of functions in the series is called "completeness", and it was not proved until much later.) Undeterred by the criticism, Fourier published his book without any changes. Both parties were right. Fourier was right in believing his set of functions to be complete, and the jury was right in pointing out that he had not proved it. The generalizations of Fourier's methods are extremely powerful, and they form the basis for many branches of theoretical science and engineering.

Modern times

When Pythagoras found the relationship between musical tones and rational numbers through his studies of the harmonics of a vibrating string, his strong intuition told him that he was approaching a deep truth about the nature of the universe. If Pythagoras were alive today, he would rejoice in the discoveries of modern physics, since they have shown that the structure of atoms can be understood in terms of harmonics that are closely analogous to the harmonics of a vibrating string. In 1926, the physicist Erwin Schrödinger wrote down a differential equation that governs the motion of very small particles such as electrons moving in an atom. For an electron moving in a 1-dimensional box of length L, the Schrödinger equation has the form

$$-\frac{1}{2}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \tag{4.80}$$

with the boundary conditions

$$\psi(0) = 0 \qquad \psi(L) = 0 \tag{4.81}$$

Because of the similarity to the equation for a vibrating string, we can immediately write down a solution in the form

$$\psi_n(x) = sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ... \tag{4.82}$$

The quantity E is interpreted as the energy of the electron in the state ψ_n. Substituting (4.82) into (4.80) we obtain a set of energies that are allowed by the Schrödinger equation (4.80) and the boundary conditions (4.81):

$$E_n = \frac{n^2 \pi^2}{2L^2} \qquad n = 1, 2, 3, ... \tag{4.83}$$

In other words, not every energy is allowed. Only certain energies are allowed, each corresponding to a "quantum number" n. Discrete allowed energies of this kind were observed experimentally by atomic spectroscopists at the end of the 19th century, but until the work of Schrödinger and others at the beginning of the 20th century these experimental results were a deep mystery.

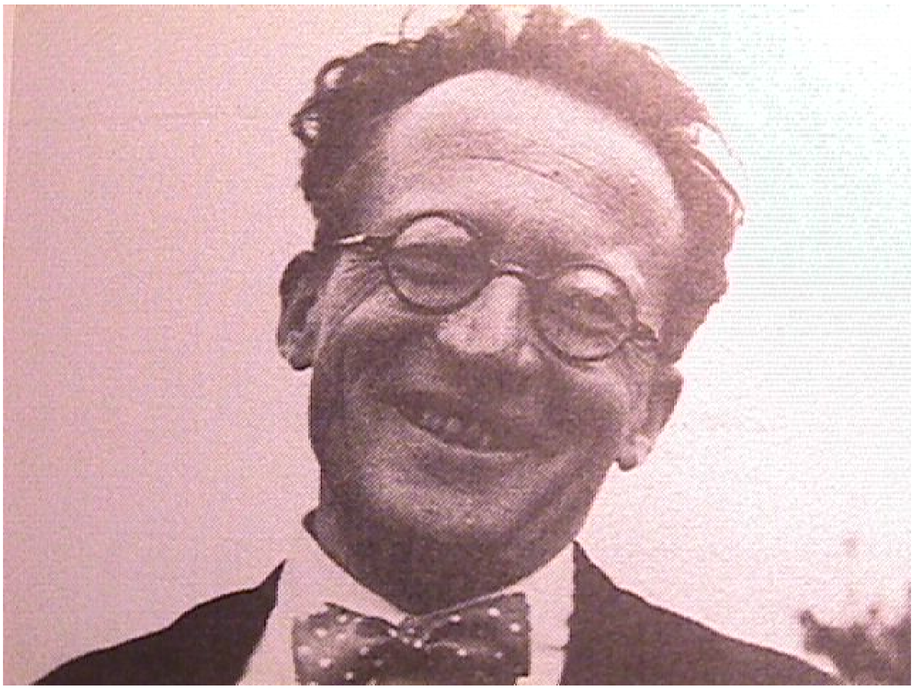

Figure 4.5: *A photograph of Erwin Schrödinger, (1887-1961). His famous wave equation (1926) describes the behavior of very small particles such as electrons. Using the Schrödinger equation, one can analyse in a very exact way the allowed states of atoms. These allowed states are found to be closely analogous to the harmonics of vibrating strings, studied by Pythagoras many centuries earlier.*

How big are the energies E_n? The 1-dimensional Schrödinger equation shown in equation (4.80) is written in special units called "atomic units", which have been found to be especially convenient for calculations on atoms. In atomic units, lengths are measured in "Bohrs" and energies are measured in "Hartrees", (the names having been chosen to honor two of the pioneers of atomic science).

$$1 \text{ Bohr} = .529 \times 10^{-8} \text{ centimeters} \tag{4.84}$$

while

$$1 \text{ Hartree} = 27.1 \text{ electron volts} \tag{4.85}$$

If the length of our box L is a few Bohrs (the approximate size of an atom), equations (4.83) and (4.85) tell us that $E_2 - E_1$ will be a few electron volts. An electron volt is defined as the energy needed to move an electron through a potential of one volt. Energy changes (per electron) of this order of magnitude are observed experimentally for chemical reactions.

By solving the Schrödinger equation for electrons in atoms, it has been possible calculate atomic properties with great precision. In fact, most of the physical and chemical properties of matter can in principle be calculated by solving differential equations, although the calculations are often so complicated that they strain the power of modern computers.

Acknowledgements

I would like to thank my son James for help with the computer techniques used to produce this book, and for his extremely valuable suggestions regarding the mathematical structure of Chapter 3. The book is dedicated to Professor Roy McWeeny, one of the greatest pioneers of quantum chemistry.

Chapter 5

Solutions to the problems

- **Problem 1.1**: Calculate $[cos(a)]^2+[sin(a)]^2$ for all of the angles shown in Table 1.1. How is the result related to Pythagoras' theorem concerning the squares of the sides of right triangles?
  Solution: In all cases, $[cos(a)]^2 + [sin(a)]^2 = 1$. This is because of the definitions of $sin(a)$ and $cos(a)$, shown in Figure 1.6, and because of the Pythagorean theorem. If the length of the long side of the right triangle shown in the figure is 1, then its square is also 1. The sum of the areas of squares constructed on the two shorter sides is $[cos(a)]^2 + [sin(a)]^2$, and by the Pythagorean theorem this sum is equal to the area of a square constructed on the long side, i.e. equal to 1.

- **Problem 1.2**: The total of all three angles inside any triangle is π (or 180 degrees). What will the angles be at the corners of a triangle where all three sides have equal length (an equilateral triangle)? How is this result related to the fact that when t is $\pi/6$ (30 degrees), $sin(t) = 1/2$?: **Solution**: For an equilateral triangle, all three angles are equal (by symmetry), and hence each of them is equal to $\pi/3$ (60 degrees). Now imagine a line from one corner of the equilateral triangle to the midpoint of the opposite side. This line will divide the equilateral triangle into two right triangles, each with angles $\pi/6$, $\pi/3$ and $\pi/2$. For one of these right triangles, the ratio of the shortest side to the longest is 1/2, and this ratio is $sin(\pi/6)$.

- **Problem 1.3**: Give an argument explaining the values of $sin(t)$ and $cos(t)$ when t is $\pi/4$ (45 degrees).
  Solution: For a right triangle where one of the angles is $\pi/4$, the other two angles must be $\pi/4$ and $\pi/2$. Therefore, from symmetry, the two short sides of the triangle are of equal length. Then from the Pythagorean theorem it follows that $sin(\pi/4) = cos(\pi/4) = 1/\sqrt{2}$.

- **Problem 1.4**: How can the minus signs in Table 1.1 be interpreted?
  Solution: The corner of the triangle where the angle a occurs in Figure 1.6 can be thought of as the origin of a Cartesian coordinate system. On the right-hand side of the origin, the horizontal axis is positive, while on the left it is negative. Similarly, the vertical axis is positive above the origin, and negative below it. The short sides of the right triangles used to define trigonometric functions can be thought of as positive or negative according to this system.

- **Problem 1.5**: Extend Table 1.1 by calculating values of $sin(t)$, $cos(t)$ and $tan(t)$ when $t = 7\pi/6$ and $t = 5\pi/4$.
  Solution:

t (degrees)	t (radians)	$sin(t)$	$cos(t)$	$tan(t)$
120	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1

- **Problem 1.6**: In Figure 1.8, a square is inscribed in a circle. If the radius of the circle is r, What is the length of a side of the square?
  Solution: Let d_1 represent the length of a side of the square. A diagonal of the square will have length $2r$, and from the Pythagorean theorem, $d_1^2 + d_1^2 = (2r)^2$. Solving this equation, we have $d_1 = \sqrt{2}\, r = 1.4142\, r$.

- **Problem 1.7**: In Figure 1.8, an octagon is also inscribed in the circle. Use the Pythagorean theorem to find the length of a side of the octagon. What is the total length of all eight sides of the octagon?
  Solution: Let d_2 represent the length of a side of the octagon. Then from the Pythagorean theorem, we know that $d_2^2 = (d_1/2)^2 + (r - d_1/2)^2$. Solving for d_2 we obtain $d_2 = .76537\, r$. The length of all eight sides of the octagon is thus $8d_2 = 6.12293\, r$.

- **Problem 1.8**: What is the area of the octagon in Figure 1.8?
  Solution: The area of the octagon is the area of the square plus the

area of the eight small right triangles that can be constructed to fill out the octagon. The area of the square is $(\sqrt{2}r)^2 = 2r^2$. The area of the eight small triangles is $2d_1(r - d_1/2) = .82843r^2$. Thus the total area of the octagon is 2.82843 r^2.

- **Problem 1.9**: If the circumference of a circle is given by $2\pi r$, and if the area of a circle is given by πr^2, use the results of Problems 1.7 and 1.8 to find a lower limit to the value of π.
  Solution: From Problem 1.7 it follows that π must be larger than 3.06147. From Problem 1.8 we know that π must be larger than 2.82843. Thus Problem 1.7 gives more accurate information about the value of π than Problem 1.8.

- **Problem 1.10**: Looking at the curve $f = t^2$ shown in Figure 1.14, we can see that when $t = 1$, $f = 1$. Suppose that we increase t by an amount $\Delta t = .01$. Then f will increase by an amount Δf. What is the ratio $\Delta f/\Delta t$?
  Solution: Since $f(1.01) = (1.01)^2 = 1.0201$ we have

$$\Delta f = f(1.01) - f(1) = .0201 \qquad \frac{\Delta f}{\Delta t} = \frac{.0201}{.01} = 2.01$$

- **Problem 1.11**: Repeat Problem 1.10 for $\Delta t = .0001$ and $\Delta t = .000001$. Does the ratio $\Delta f/\Delta t$ approach a limiting value as Δt becomes smaller and smaller? How is this ratio related to the slope of the curve?
  Solution:

$$\Delta f = f(1.0001) - f(1) = .00020001 \qquad \frac{\Delta f}{\Delta t} = \frac{.000201}{.0001} = 2.0001$$

$$\Delta f = f(1.000001) - f(1) = .000002000001 \qquad \frac{\Delta f}{\Delta t} = 2.000001$$

  The ratio $\Delta f/\Delta t$ seems to be approaching 2 more and more precisely as Δt becomes smaller. This ratio is a measure of the slope of the curve at the point $t = 1$.

- **Problem 2.1**: Calculate the values of 5!, 6! and 7!.
  Solution: $5! = 5 \times 4! = 120$, $6! = 6 \times 5! = 720$, $7! = 7 \times 6! = 5040$.

- **Problem 2.2**: Write expressions for $(a+b)^5$ and $(a+b)^6$ in powers of a and b.
  Solution:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

- **Problem 2.3**: What is the value of the binomial coefficient $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$?
  Solution:
$$\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \frac{8!}{5!(8-5)!} = 56$$

- **Problem 2.4**: Use equation (2.10) to make a series expansion of $\sqrt{1+x} \equiv (1+x)^{1/2}$ in powers of x. Evaluate the sum of the first five terms in the series when $x = .1$. Square the result and compare it to 1.1.
  Solution:
$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{128} + ...$$
$1 + .05 - .00125 + .0000625 - .000004 = 1.04881 \qquad (1.04881)^2 = 1.10000$

- **Problem 2.5**: Try evaluating the the first 5 terms of series of Problem 2.5 when $x = 2$. Does the series converge to a particular number as more and more terms are added?
  Solution:
$$1 + 1 - .5 + .5 - .625 + ... = ?$$
The series does not seem to be converging.

- **Problem 2.6**: Calculate $\frac{df}{dt}$ when $f(t) = \frac{1}{t^3}$
  Solution:
$$\frac{d}{dt}\left[t^{-3}\right] = -3t^{-4} = -\frac{3}{t^4}$$

- **Problem 2.7**: Calculate $\frac{df}{dt}$ when $f(t) = (at)^4$ where a is a constant.
  Solution:
$$\frac{d}{dt}\left[(at)^4\right] = a^4 \frac{d}{dt}\left[t^4\right] = 4a^4t^3$$

- **Problem 2.8**: Calculate $\frac{df}{dt}$ when $f(t) = 1 + t$.
  Solution:
$$\frac{d}{dt}[1+t] = \frac{d}{dt}[1] + \frac{d}{dt}[t] = 0 + 1 = 1$$

- **Problem 2.9**: Calculate $\dfrac{d^2 f}{dt^2}$ when $f(t) = t^{1/2}$.
  Solution:
$$\frac{d^2}{dt^2}\left[t^{1/2}\right] = \frac{d}{dt}\left[\frac{1}{2}t^{-1/2}\right] = -\frac{1}{4}t^{-3/2}$$

- **Problem 2.10**: Suppose that $f(t) = t^3$. Use equation (2.32) to calculate the expansion coefficients a_n and show that the expansion is consistent with the original definition of the function.
  Solution:
$$a_0 = [f]_{t=0} = \left[t^3\right]_{t=0} = 0$$
$$a_1 = \frac{1}{1!}\left[\frac{df}{dt}\right]_{t=0} = \left[3t^2\right]_{t=0} = 0$$
$$a_2 = \frac{1}{2!}\left[\frac{d^2 f}{dt^2}\right]_{t=0} = \frac{1}{2}\left[6t\right]_{t=0} = 0$$
$$a_3 = \frac{1}{3!}\left[\frac{d^3 f}{dt^3}\right]_{t=0} = \frac{1}{6}\left[6\right]_{t=0} = 1$$
$$a_4 = \frac{1}{4!}\left[\frac{d^4 f}{dt^4}\right]_{t=0} = \frac{1}{24}\left[0\right]_{t=0} = 0$$
$$f = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + ... = t^3$$

- **Problem 2.11**: Use equation (2.44), where $g = 32$ feet/second^2, to calculate how long a stone will take to fall from the top of a tower that is 64 feet high (neglecting air resistance).
  Solution:
$$0 = z_0 - \frac{1}{2}gt^2$$
$$t^2 = \frac{2z_0}{g} = \frac{2 \times 64}{32}\text{sec.}^2$$
  It will take 2 seconds for the stone to fall to the bottom of the tower.

- **Problem 2.12**: Suppose that instead of being merely dropped, the stone in Problem 2.11 is thrown horizontally from the top of the same tower with velocity $v_x = 16$ feet/second. Use equation (2.45) to calculate how far from the base of the tower it will land (again neglecting air resistance).
  Solution:
$$0 = z_0 - \frac{gx^2}{2v_x^2}$$

$$x^2 = \frac{2z_0 v_x^2}{g} = \frac{2 \times 64 \times 16 \times 16}{32} \text{feet}^2$$

The stone will fall 32 feet out from the bottom of the tower.

- **Problem 3.1**: Calculate the indefinite integral $\int dt\ t^4$.
  Solution:
  $$\int dt\ t^4 = \frac{t^5}{5} + C$$

- **Problem 3.2**: Calculate the definite integral $\int_1^2 dt\ t^4$.
  Solution:
  $$\int_1^2 dt\ t^4 = \frac{2^5}{5} - \frac{1^5}{5} = \frac{31}{5}$$

- **Problem 3.3**: If $\dfrac{df}{dt} = t^{1/2}$, what is the form of the function f?
  Solution:
  $$f = \int dt\ t^{1/2} = \frac{t^{3/2}}{3/2} + C$$

- **Problem 3.4**: Suppose that a man is walking at an average speed of 3 kilometers per hour. How far, on the average, will he walk in 1 second? How is this question related to equation (3.10) and Figure 3.1?
  Solution:
  $$v = \frac{3000 \text{ m}}{3600 \text{ s}} = 0.83333 \text{ m/s}$$
  $$v(t_2 - t_1) = 0.83333 \text{ meters}$$

  The integral of instantaneous velocity over time between t_1 and t_2 gives the distance traveled in this time interval. When the velocity is constant (v), as in equation (3.10), the distance traveled is $v(t_2 - t_1)$, and this distance is represented by a rectangle in Figure 3.1. If the velocity had been a function of time, the distance traveled would have been given by the definite integral of that function, taken between t_1 and t_2.

- **Problem 3.5**: As a boy, Isaac Newton constructed a water clock. It was a large container with a small hole in the bottom, and the water ran out through the hole at a constant rate. Let us suppose that its volume was four quarts and that it took 24 hours to go from full to empty. How fast did the water run out through the hole? If we apply the idea of functions and differentials to this problem, what does $f(t)$ represent? What does df/dt represent? What did the word "fluxion"

mean to Newton?
Solution:
$$v = -\frac{4 \text{ quarts}}{24 \text{ hours}} = -1.66667 \text{ q/h}$$
In this problem, $f(t)$ represents the amount of water in the container, and df/dt represents the rate of change of that amount, or the rate of flow (flux). Newton used the word "fluxion" to mean the rate of change of some quantity that is a function of time.

- **Problem 3.6**: What are the heights of each of the five narrow strips shown in Figure 3.4? What are the areas of each of the strips? What is the sum of their areas?
Solution: The heights of the strips in Figure 3.4 are given by
$$h_1 = \frac{at_2}{5} \qquad h_2 = \frac{2at_2}{5} \qquad h_3 = \frac{3at_2}{5} \qquad h_4 = \frac{4at_2}{5} \qquad h_5 = \frac{5at_2}{5}$$
Their areas are
$$h_j \Delta t = \frac{h_j t_2}{5}$$
and their total area is
$$\sum_{j=1}^{5} \frac{h_j t_2}{5} = (1+2+3+4+5)\frac{at_2^2}{25} = \frac{3at_2^2}{5}$$
If the number of strips were increased, their total area would more closely approximate the true area, $at_2^2/2$.

- **Problem 3.7**: In Chapter 1, Figure 1.9 shows the method which Archimedes used to calculate the area of a circle by dividing it into a number of narrow strips and then letting the strips become more and more narrow and numerous. In the figure, four strips are shown. If the radius of the circle has length $r = 1$, what is the area of each strip? What is their total area?
Solution: If we let h_j represent the height of the jth strip, then from the Pythagorean theorem we have
$$h_1 = \sqrt{1 - \left(\frac{1}{8}\right)^2} = 0.99216$$
$$h_2 = \sqrt{1 - \left(\frac{3}{8}\right)^2} = 0.92702$$
$$h_3 = \sqrt{1 - \left(\frac{5}{8}\right)^2} = 0.78062$$

$$h_4 = \sqrt{1 - \left(\frac{7}{8}\right)^2} = 0.48412$$

To find the areas of the strips, we divide each of these numbers by 4, so that the areas are 0.24804, 0.23176, 0.19516 and 0.12103. The total area of the four strips is 0.79602. To get the approximate total area of the circle, we must multiply by 4, which yields 3.1841. This can be compared with the value of π that is known from more exact calculations, 3.141592654... . If the number of strips had been increased, we would have obtained a more exact result.

- **Problem 3.8**: If $f(t)$ represents the distance traveled by an object moving in a straight line, what does $\frac{df}{dt}$ represent? What does $\frac{d^2f}{dt^2}$ represent?
  Solution: $\frac{df}{dt}$ represents the velocity at a given time, while $\frac{d^2f}{dt^2}$ represents the object's acceleration.

- **Problem 3.9**: Suppose that an object has a constant acceleration a in a particular direction. Express the velocity as an indefinite integral and find an expression for the velocity of the object as a function of time. What is the physical interpretation of the constant of integration? Integrate again to find the distance travelled as a function of time. What is the interpretation of the second constant of integration?
  Solution:
  $$v(t) = \int a\ dt = at + v_0$$
  The constant of integration, v_0, represents the velocity of the object at the initial time $t = 0$.
  $$x(t) = \int v(t)\ dt = \frac{1}{2}at^2 + v_0t + x_0$$
  The second constant of integration represents the position of the object at the initial time, $t = 0$.

- **Problem 3.10**: Repeat Problem 3.9 for the case where $a = wt$ where w is a constant. In other words, repeat the problem for the case where the acceleration increases linearly with time.
  Solution:
  $$v(t) = \int a(t)\ dt = \int wt\ dt = \frac{1}{2}wt^2 + v_0$$
  $$x(t) = \int v(t)\ dt = \int \left(\frac{1}{2}wt^2 + v_0\right) dt = \frac{1}{6}wt^3 + v_0t + x_0$$
  The constants of integration have the same meaning as in Problem 3.9.

- **Problem 3.11**: Use the series of equations (3.34) and (3.35) to evaluate $sin(1)$ and $cos(1)$. What is the value of $[sin(1)]^2 + [cos(1)]^2$? Why is this value nearly equal to 1? Is $[sin(t)]^2 + [cos(t)]^2$ equal to 1 for every value of t?
  Solution: Taking the first five terms in the series gives
  $$cos(1) \approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540303$$
  and
  $$sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = 0.841471$$
  $(0.540303)^2 + (0.841471)^2 = 1.000000$ because of the definition of $sin(t)$ and $cos(t)$, combined with the Pythagorean theorem. For the same reason, $[sin(t)]^2 + [cos(t)]^2 = 1$ for all values of t.

- **Problem 3.12**: Evaluate the first eight terms in the series for the Napierian base e shown in equation (3.42). How close is the sum of these terms to the value of e given in the equation? Do you think that e is a rational number? (A rational number is a number that can be expressed as the ratio of two integers.)
  Solution: The first five terms give
  $$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = 2.71825$$
  which agrees to 5 figures with the true value, e=2.718281828459045...

- **Problem 3.13**: Use the series in equation (3.36) to evaluate e^2 up to eight terms. How close is the value of $(e^1)^2$ to e^2?
  Solution:
  $$e^2 \approx 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} = 7.38095$$
  We can compare this result with $(2.718281...)^2 = 7.38906$, from which we can see that the series gave us 3-figure accuracy - less than in the previous problem because with a larger argument, the series converges less rapidly.

- **Problem 3.14**: Calculate e^3 and e^4 and use these results, together with the results of Problem 3.13, to make a small table of logarithms. Try using this table, together with equations (3.46) and(3.47), to perform multiplications and divisions.
  Solution: Since $e^1 = 2.718281$, $e^2 = 7.38906$, $e^3 = 20.0855$ and $e^4 = 54.5982$, we can construct the following small table of logarithms:

x	2.718281	7.38906	20.0855	54.5982
$\ln(x)$	1	2	3	4

From this table it follows (for example) that 2.718281×7.38906=20.0855, since ln(2718281) = 1, ln(7.38906) = 2 and 1+2=3. After adding the two logarithms, we look in the table to find the number to which the sum corresponds, in this case 20.0855, and that is the result of our multiplication. Much larger tables, together with interpolation procedures, were used to reduce the work of multiplication and division before the days of electronic calculators.

- **Problem 3.15**: Use Euler's identities (3.51) and (3.52) together with equations (3.27) and (3.28) to evaluate $\frac{d}{dt}\left[e^{it}\right]$.
  Solution:

$$\frac{d}{dt}\left[e^{it}\right] = \frac{d}{dt}\left[cos(t) + i\ sin(t)\right] = -sin(t) + i\ cos(t) = ie^{it}$$

- **Problem 3.16**: Compare the result of Problem 3.15 with the result of differentiating the series of equation (3.50) term by term.
  Solution:

$$\frac{d}{dt}\left[1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + ...\right] = i + \frac{2(i)^2 t}{2!} + \frac{3(i)^3 t^2}{3!} + ... = ie^{it}$$

- **Problem 3.17**: Evaluate the indefinite integral $\int dt\ e^{it}$.
  Solution:

$$\int dt\ e^{it} = \frac{1}{i}e^{it} + C$$

- **Problem 3.18**: Use equations (3.53) and (3.54) to evaluate $[cos(t)]^2 + [sin(t)]^2$.
  Solution:

$$[cos(t)]^2 + [sin(t)]^2 = \frac{1}{4}\left[e^{2it} + e^{-2it} + 2\right] - \frac{1}{4}\left[e^{2it} + e^{-2it} - 2\right] = 1$$

- **Problem 3.19**: Use equations (3.57) and (3.58) to evaluate $[cosh(t)]^2 - [sinh(t)]^2$.
Solution:
$$[cosh(t)]^2 - [sinh(t)]^2 = \frac{1}{4}\left[e^{2t} + e^{-2t} + 2\right] - \frac{1}{4}\left[e^{2t} + e^{-2t} - 2\right] = 1$$

- **Problem 4.1**: Use equation (4.9) and Euler's identities to show that
$$\int dt\ cos(\omega t) = \frac{1}{\omega} sin(\omega t) + C'$$
and that
$$\int dt\ sin(\omega t) = -\frac{1}{\omega} cos(\omega t) + C'$$
where C' is a constant.
Solution:
$$\int dt\ cos(\omega t) = \frac{1}{2}\int dt \left[e^{i\omega t} + e^{-i\omega t}\right] = \frac{1}{2i\omega}\left[e^{i\omega t} - e^{-i\omega t}\right] + C'$$
$$\int dt\ sin(\omega t) = \frac{1}{2i}\int dt \left[e^{i\omega t} - e^{-i\omega t}\right] = -\frac{1}{2\omega}\left[e^{i\omega t} + e^{-i\omega t}\right] + C'$$
Once more making use of Euler's identities, we can identify the terms on the right respectively as $sin(\omega t)/\omega + C'$ and $-cos(\omega t)/\omega + C'$

- **Problem 4.2**: If (on the average) 0.1% of the soup bowls that a cafeteria owns are broken every day, write a differential equation that describes the average decrease in the number of soup bowls as a function of time. Suppose that the cafeteria decides to replace the bowls after half are gone. How long will it be before they have to replace them? Use the fact that $\ln(2) = 0.693$.
Solution: Let $S(t)$ be the number of soup bowls as a function of time, where the time t is measured in days, and let k=.001 (days)$^{-1}$. Then S obeys the first-order ordinary differential equation
$$\frac{dS}{dt} = -kS$$
Solving this equation, we obtain
$$S = S_0 e^{-kt}$$
where S_0 is a constant that represents the number of soup bowls at time $t = 0$. We now let τ represent the time after which half the bowls are gone. (Sometimes this is called the "half-life"). Then
$$e^{-k\tau} = \frac{1}{2} \qquad k\tau = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

from which we have

$$\tau = \frac{\ln(2)}{k} = \frac{0.693}{.001} = 693 \text{ days}$$

- **Problem 4.3**: Suppose that the population of a country increases on the average by 2% each year. If it continues to increase at this rate, by what factor will it have increased in a century? By how much in two centuries? By how much in three centuries?
  Solution: Let $P(t)$ represent the population and let $k = .02$ (years)$^{-1}$. Then P will obey the differential equation

$$\frac{dP}{dt} = kP$$

  which has the solution

$$P = P_0 e^{kt}$$

  where P_0 is a constant that represents the population when $t = 0$. After a century, the population will have increased by a factor $e^2 = 7.389$, after two centuries by a factor $e^4 = 54.60$, and after three centuries by a factor $e^6 = 403.4$.

- **Problem 4.4**: The solution to the harmonic oscillator equation shown in equation (4.22) contains two constants of integration, a_0 and a_1. If the initial conditions require that

$$f(0) = 1 \qquad \left[\frac{df}{dt}\right]_{t=0} = 0$$

  what are the values of the constants a_0 and a_1?
  Solution: Since $sin(0) = 0$ and $cos(0) = 1$, the condition $f(0) = 1$ requires that $a_0 = 1$. Differentiating f with respect to time, we obtain

$$0 = \left[\frac{df}{dt}\right]_{t=0} = \left[\omega_0 a_1 cos(\omega_0 t) - \omega_0 a_0 sin(\omega_0 t)\right]_{t=0} = \omega_0 a_1$$

  Thus the second initial condition requires that $a_1 = 0$.

- **Problem 4.5**: Repeat Problem 4.4 for the damped harmonic oscillator transient solution shown in equation (4.36).
  Solution: The initial condition $f(0) = 1$ requires that $a_0 = 1$. Differentiating the damped harmonic oscillator solution of equation (4.36), we obtain:

$$\frac{df}{dt} = e^{-at/2}\left[\omega' a_1 cos(\omega' t) - \omega' a_0 sin(\omega' t)\right]$$

$$-\frac{a}{2}e^{-at/2}\left[a_1 sin(\omega' t) + a_0 cos(\omega' t)\right]$$

and thus the second initial condition requires that

$$0 = \left[\frac{df}{dt}\right]_{t=0} = \omega' a_1 - \frac{a}{2}a_0$$

from which we have

$$a_1 = \frac{a}{2\omega'}$$

- **Problem 4.6**: Suppose that

$$(x+iy)^n = u + iv$$

where $n = 3$ and where x, y, u and v all are real, with $i \equiv \sqrt{-1}$. Find u and v and show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations are called the "Cauchy-Riemann equations".
Solution:

$$(x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = 6xy = -\frac{\partial u}{\partial y}$$

- **Problem 4.7**: Repeat Problem 4.6 for $n = 1$ and $n = 2$.
Solution: For $n = 1$

$$(x+iy)^1 = x + iy$$

$$u = x \qquad v = y$$

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y}$$

For $n = 2$

$$(x+iy)^2 = x^2 + 2x(iy) + 2(iy)^2 = x^2 - y^2 + i(2xy)$$

$$u = x^2 - y^2 \qquad v = 2xy$$
$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

- **Problem 4.8**: Show that if u and v satisfy the Cauchy-Riemann equations, then
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$
and
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0$$
The second-order differential equation satisfied by both u and v is called the "Laplace equation".
Solution: If u and v satisfy the Cauchy-Riemann equations, then
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
and
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$$
from which it can be seen that u satisfies the Laplace equation. Also
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$$
so that v also satisfies the Laplace equation.

- **Problem 4.9**: Use the relationships shown in equation (4.73) to show that $F(x + ct)$ satisfies the wave equation, (4.68).
Solution: If $w = x + ct$ and
$$\frac{\partial}{\partial x} F(w) = \frac{dF}{dw}\frac{\partial w}{\partial x} \qquad \frac{\partial}{\partial t} F(w) = \frac{dF}{dw}\frac{\partial w}{\partial t}$$
then
$$\frac{\partial}{\partial x} F(w) = \frac{dF}{dw} \qquad \frac{\partial}{\partial t} F(w) = c\frac{dF}{dw} \qquad \frac{\partial}{\partial x} F(w) = \frac{1}{c}\frac{\partial}{\partial t} F(w)$$
from which
$$\frac{\partial^2}{\partial x^2} F(w) = \frac{1}{c}\frac{\partial^2}{\partial x \partial t} F(w) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} F(w)$$

- **Problem 4.10**: Show that $G(x - ct)$ also satisfies the wave equation.
Solution: If $w = x - ct$ and

$$\frac{\partial}{\partial x}G(w) = \frac{dG}{dw}\frac{\partial w}{\partial x} \qquad \frac{\partial}{\partial t}G(w) = \frac{dG}{dw}\frac{\partial w}{\partial t}$$

then

$$\frac{\partial}{\partial x}G(w) = \frac{dG}{dw} \qquad \frac{\partial}{\partial t}G(w) = -c\frac{dG}{dw} \qquad \frac{\partial}{\partial x}G(w) = -\frac{1}{c}\frac{\partial}{\partial t}G(w)$$

from which

$$\frac{\partial^2}{\partial x^2}G(w) = -\frac{1}{c}\frac{\partial^2}{\partial x \partial t}G(w) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(w)$$

- **Problem 4.11**: Use equation (4.73) to show that if

$$F(x + iy) = u + iv$$

where u and v are real functions of x and y, then u and v satisfy the Cauchy-Riemann equations and the Laplace equation.
Solution: If $w = x + iy$ and

$$\frac{\partial}{\partial x}F(w) = \frac{dF}{dw}\frac{\partial w}{\partial x} \qquad \frac{\partial}{\partial y}F(w) = \frac{dF}{dw}\frac{\partial w}{\partial y}$$

then

$$\frac{\partial}{\partial x}F(w) = \frac{dF}{dw} \qquad \frac{\partial}{\partial y}F(w) = i\frac{dF}{dw} \qquad \frac{\partial}{\partial x}F(w) = \frac{1}{i}\frac{\partial}{\partial y}F(w)$$

from which

$$\frac{\partial}{\partial x}(u + iv) = \frac{1}{i}\frac{\partial}{\partial y}(u + iv)$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

so that u and v obey the Cauchy-Riemann equations. As was shown in Problem 4.8, both u and v must then also satisfy the Laplace equation.

- **Problem 4.12**: Show that functions of the form

$$T_n(x,t) = A_n e^{-a_n t} sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ...$$

are solutions to the diffusion equation satisfying the boundary conditions $T_n(0,t) = 0$ and $T_n(L,t) = 0$. What condition must the constants

a_n fulfill in order that the diffusion equation should be satisfied? How should the constants A_n be chosen?
Solution: Substituting $T_n(x,t)$ into the diffusion equation we obtain

$$-a_n T_n = -C\left(\frac{n\pi}{L}\right)^2 T_n$$

so that T_n will be a solution provided that

$$a_n = C\left(\frac{n\pi}{L}\right)^2$$

This solution also satisfies the boundary conditions because

$$sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ..$$

vanishes both when $x = 0$ and when $x = L$. The constants A_n are determined by the initial conditions of the problem.

Appendix A

Tables

The following tables may be useful in practical calculations. Much larger tables are available, for example *Tables of Integrals, Series and Products*, by I.S. Gradshteyn and I.M. Ryshik, Academic Press, New York, or *CRC Standard Mathematical Tables and Formulae, 30th Edition*, by Daniel Zwillinger, published by the Chemical Rubber Company. In using mathematical tables, a student or research worker does not need to be able to rederive the results, since these have been checked and rechecked by generations of mathematicians.

The gamma function, $\Gamma(x)$, defined in Table A4 and tabulated in Table A6, reduces to a factorial for positive integral arguments:

$$\Gamma(n+1) = n! \qquad n = 0, 1, 2, 3, ...$$

and it has the property that

$$\Gamma(x+1) = x\Gamma(x)$$

This function is useful for evaluating definite integrals of the form

$$\int_0^\infty dt\ t^x e^{-at} = \frac{\Gamma(x+1)}{a^{x+1}}$$

Students who have access to the computer program *Mathematica* will greatly enjoy using it. This program can perform both numerical and algebraic operations (for example, it can differentiate or integrate functions, and can make series expansions of them), and all of the common mathematical functions are available on it.

Table A.1: Some fundamental differentials

$$
\begin{aligned}
\frac{d}{dt}\left[t^p\right] &= pt^{p-1} \\
\frac{d}{dt}\left[f+g\right] &= \frac{df}{dt}+\frac{dg}{dt} \\
\frac{d}{dt}\left[fg\right] &= f\frac{dg}{dt}+g\frac{df}{dt} \\
\frac{d}{dt}\left[\frac{f}{g}\right] &= \frac{1}{g^2}\left[g\frac{df}{dt}-f\frac{dg}{dt}\right] \\
\frac{d}{dt}\left[e^{at}\right] &= ae^{at} \\
\frac{d}{dt}\left[ln(t)\right] &= \frac{1}{t} \\
\frac{d}{dt}\left[f(g)\right] &= \frac{df}{dg}\frac{dg}{dt} \\
\frac{d}{dt}\left[af\right] &= a\frac{df}{dt} \\
\frac{d}{dt}\left[\sin(at)\right] &= a\cos(at) \\
\frac{d}{dt}\left[\cos(at)\right] &= -a\sin(at) \\
\frac{d}{dt}\left[\sinh(at)\right] &= a\cosh(at) \\
\frac{d}{dt}\left[\cosh(at)\right] &= a\sinh(at)
\end{aligned}
$$

Table A.2: Differentials of inverse trigonometric functions

$$\begin{aligned}
\frac{d}{dt}\left[\sin^{-1}(t)\right] &= \frac{1}{\sqrt{1-t^2}} \\
\frac{d}{dt}\left[\cos^{-1}(t)\right] &= \frac{-1}{\sqrt{1-t^2}} \\
\frac{d}{dt}\left[\tan^{-1}(t)\right] &= \frac{1}{1+t^2} \\
\frac{d}{dt}\left[\cot^{-1}(t)\right] &= \frac{-1}{1+t^2} \\
\frac{d}{dt}\left[\sinh^{-1}(t)\right] &= \frac{1}{\sqrt{1+t^2}} \\
\frac{d}{dt}\left[\tanh^{-1}(t)\right] &= \frac{1}{1-t^2}
\end{aligned}$$

Table A.3: Some fundamental indefinite integrals

$$
\begin{aligned}
\int dt\ t^p &= \frac{t^{p+1}}{p+1} + C \qquad p \neq -1 \\
\int dt\ t^{-1} &= \ln(t) + C \\
\int dt\ e^{at} &= \frac{e^{at}}{a} + C \\
\int dt\ \cos(at) &= \frac{1}{a}\sin(at) + C \\
\int dt\ \sin(at) &= \frac{-1}{a}\cos(at) + C \\
\int dt\ \cosh(at) &= \frac{1}{a}\sinh(at) + C \\
\int dt\ \sinh(at) &= \frac{1}{a}\cosh(at) + C \\
\int dt\ f\frac{dg}{dt} &= fg - \int dt\ g\frac{df}{dt} + C
\end{aligned}
$$

Table A.4: A few important definite integrals.

$\int_0^{\infty} dt\ t^n e^{-at}$	$=$	$\frac{n!}{a^{n+1}}$	$n = \text{integer},\ a = \text{real},\ a > 0$
$\int_0^{\infty} dt\ t^{2n} e^{-at^2}$	$=$	$\frac{1 \times 3 \times 5...(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$	$n = \text{integer},\ a > 0$
$\int_0^{\infty} dt\ t^p e^{-t}$	$\equiv$	$\Gamma(p+1)$	
$\int_0^{\infty} dt\ \frac{a}{a^2+x^2}$	$=$	$\pm\frac{\pi}{2}$	if $\pm a > 0,\ a = \text{real}$
$\int_0^{\infty} dt\ \frac{t^{p-1}}{1+t}$	$=$	$\frac{\pi}{\sin(p\pi)}$	if $1 > p > 0,\ p = \text{real}$
$\int_0^{\infty} dt\ \frac{\sin^2(t)}{t^2}$	$=$	$\frac{\pi}{2}$	
$\int_0^{\infty} dt\ \frac{\sin(at)}{t}$	$=$	$\frac{\pi}{2}$	if $a > 0$
$\int_0^{\pi} dt\ \sin^2(nt)$	$=$	$\frac{\pi}{2}$	$n = \text{integer}$
$\int_0^{\pi} dt\ \cos^2(nt)$	$=$	$\frac{\pi}{2}$	$n = \text{integer}$
$\int_0^{\pi} dt\ \cos(nt)\cos(mt)$	$=$	0	m and $n = \text{integers},\ n \neq m$
$\int_0^{\pi} dt\ \sin(nt)\sin(mt)$	$=$	0	m and $n = \text{integers},\ n \neq m$
$\int_0^{\pi/n} dt\ \sin(nt)\cos(nt)$	$=$	0	$n = \text{integer}$

Table A.5: Series expansions of functions.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$$

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...$$

$$a^t = 1 + \frac{t\ln(a)}{1!} + \frac{[t\ln(a)]^2}{2!} + \frac{[t\ln(a)]^3}{3!} + ...$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ... \qquad -1 < t \leq 1$$

$$\ln(t) = 2\left[\frac{t-1}{t+1} + \frac{1}{3}\left(\frac{t-1}{t+1}\right)^3 + \frac{1}{5}\left(\frac{t-1}{t+1}\right)^5 + ...\right] \qquad t > 0$$

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ...$$

$$\sin(t) = x - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ...$$

$$\cosh(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + ...$$

$$\sinh(t) = x + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + ...$$

$$\tan^{-1}(t) = x - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + ...$$

Table A.6: The exponential, logarithm and gamma functions. Values of these functions for other values of x can be found by using the relationships $\ln(ab) = \ln(a) + \ln(b)$, $\ln(1/a) = -\ln(a)$, $e^{x+n} = e^n e^x$ and $\Gamma(x+1) = x\Gamma(x)$.

x	$\ln(x)$	e^x	$\Gamma(x)$
0.1	-2.302585	1.105171	9.513507
0.2	-1.609438	1.221403	4.590844
0.3	-1.203973	1.349859	2.991569
0.4	-0.916291	1.491825	2.218159
0.5	-0.693147	1.648721	1.772454
0.6	-0.510826	1.822119	1.489192
0.7	-0.356675	2.013753	1.298055
0.8	-0.223144	2.225541	1.164230
0.9	-0.105361	2.459603	1.068629
1.0	0.000000	2.718282	1.000000
1.1	0.095310	3.004166	0.951351
1.2	0.182322	3.320117	0.918169
1.3	0.262364	3.669297	0.897471
1.4	0.336472	4.055200	0.887264
1.5	0.405465	4.481689	0.886227
1.6	0.470004	4.953032	0.893515
1.7	0.530628	5.473947	0.908639
1.8	0.587787	6.049647	0.931384
1.9	0.641854	6.685894	0.961766
2.0	0.693147	7.389056	1.000000

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