

Noncommutative analysis,  
Multivariable spectral theory for operators in  
Hilbert space,  
Probability, and Unitary Representations

Palle Jorgensen, and Feng Tian

“‘The Journal of Functional Analysis’ is dedicated to the broadening of the horizons of functional analysis. Accordingly, it encourages original research papers of high quality from all branches of science, provided the core and flavor are of a functional analytic character and the paper is in accordance with contemporary mathematical standards.”

— From the cover page of ‘The Journal of Functional Analysis’ (written in 1967)

Dedicated to the memory of William B. Arveson (see [MJD<sup>+</sup>15])  
(22 November 1934 – 15 November 2011).

“What goes around has come around, and today quantum information theory (QIT) has led us back into a finite-dimensional context. Completely positive maps on matrix algebras are the objects that are dual to quantum channels; in fact, the study of quantum channels reduces to the study of completely positive maps of matrix algebras that preserve the unit. This is an area that is still undergoing vigorous development in efforts to understand entanglement, entropy, and channel-capacity in QIT.”

— William B. Arveson (written around 2009.)

# Foreword

by **Wayne Polyzou, Professor of Physics, University of Iowa**

Progress in science, engineering and mathematics comes fast and it often requires a significant effort to keep up with the advances in other fields that impact applications. Functional analysis (especially operators in Hilbert space, unitary representations of Lie groups, and spectral theory) is one discipline that impacts my physics research. Bringing my students up to speed with the subject facilitates their ability to efficiently perform research, however the typical curriculum in functional analysis courses is not directed to practitioners whose primary objective is applications. This is also reflected in the many excellent available texts on the subject, which primarily focus on the mathematics, and are directed at students aspiring to a career in mathematics.

I have been fortunate to have Palle Jorgensen as a colleague. He participates in a weekly joint mathematical physics seminar that is attended by faculty and students from both departments. It provides a forum to address questions related to the role of mathematics in physics research. Professor Jorgensen has a healthy appreciation of applications of functional analysis; in these seminars he has been at the center of discussions on a diverse range of applications involving wavelets, reflection positivity, path integrals, entanglement, financial mathematics, and algebraic field theory.

A number of the mathematically inclined students in my department have benefited from taking the functional analysis course taught by Professor Jorgensen. These students are motivated to enroll in his class because the course material includes a significant discussion of applications of functional analysis to subjects that interest them.

This book is based on the course that Professor Jorgensen teaches on functional analysis. It fills in a gap that is not addressed by the many excellent available texts on functional analysis, by using applications to motivate basic results in functional analysis. The way that it uses applications makes the material more accessible to students; particularly for students who will eventually find careers in related disciplines. The book also points to additional reference material for students who are motivated to learn more about a specific topic.

W. Polyzou

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Over the decades, Functional Analysis, and the theory of operators in Hilbert space, have been enriched and inspired on account of demands from neighboring fields, within mathematics, *harmonic analysis* (wavelets and signal processing), *numerical analysis* (finite element methods, discretization), *PDEs* (diffusion equations, scattering theory), *representation theory*; *iterated function systems* (fractals, Julia sets, chaotic dynamical systems), *ergodic theory*, *operator algebras*, and many more. And neighboring areas, *probability/statistics* (for example stochastic processes, Itô and Malliavin calculus), *physics* (representation of Lie groups, quantum field theory), and *spectral theory* for Schrödinger operators.

The book is based on a course sequence (two-semester 313-314) taught, over the years at the University of Iowa, by the first-named author. The students in the course made up a mix: some advanced undergraduates, but most of them, first or second year graduate students (from math, as well as some from physics and stats.)

We have subsequently expanded the course notes taken by the second-named author: we completed several topics from the original notes, and we added a number of others, so that the book is now self-contained, and it covers a unified theme; and yet it stresses a multitude of applications. And it offers flexibility for users.

A glance at the table of contents makes it clear that our aim, and choice of topics, is different from that of more traditional Functional Analysis courses. This is deliberate. For example, in our choice of topics, we placed emphasis on the use of Hilbert space techniques which are then again used in our treatment of central themes of applied functional analysis.

We have also strived for a more accessible book, and yet aimed squarely at applications; — we have been serious about motivation: Rather than beginning with the four big theorems in Functional Analysis, our point of departure is an initial choice of topics from applications. And we have aimed for flexibility of use; acknowledging that students and instructors will invariably have a host of diverse goals in teaching beginning analysis courses. And students come to the course with a varied background. Indeed, over the years we found that students have come to the Functional Analysis sequence from other and different areas of math, and even from other departments; and so we have presented the material in a way that minimizes the need for prerequisites. We also found that well motivated students are easily able to fill in what is needed from measure theory, or from a facility with the four big theorems of Functional Analysis. And we found that the approach “learn-by-using” has a comparative advantage.

### **Analysis of Continuous Systems vs Discrete (Networks and Graphs)**

A new theme here, going beyond traditional books in the subject, is applications of functional and harmonic analysis to “*large networks*,” so to discrete problems. More precisely, we study infinite network models. Such models can often be represented as follows: By a pair of sets,  $V$  (vertices), and  $E$ , (edges).

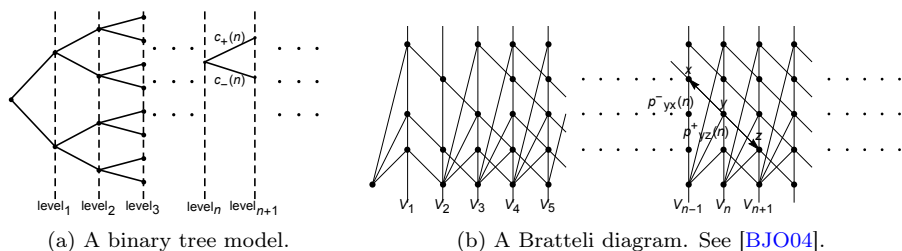


Figure 1: Examples of infinite weighted network; for details, see Chapter 10.

In addition, one specifies a positive function  $c$  defined on the edge set  $E$ . (In electrical network models,  $c$  represents conductance.) There are then two associated operators  $\Delta$  and  $P$ , each depending on the triple  $(V, E, c)$ . (See Figure 1.) Both operators represent actions (i.e., operations) on appropriate spaces of functions, more precisely functions defined on the infinite vertex set  $V$ . For the networks of interest to us, the vertex set  $V$  will be infinite, reflecting statistical and stochastic properties; and it will have additional geometric and ergodic theoretic properties. We are therefore faced with a variety of choices of infinite-dimensional function spaces. Many questions are of spectral theoretic flavor, and as a result, the useful choices of function spaces will be Hilbert spaces.

But even restricting to Hilbert spaces, there are at least three natural (and useful) candidates: (i) the plain  $l^2$  sequence space, so an  $l^2$ -space of functions on  $V$ , (ii) a suitably weighted  $l^2$ -space, and finally (iii), an energy Hilbert space  $\mathcal{H}_E$ . (The latter is an abstraction of more classical notions of Dirichlet spaces.) Which one of the three to use depends on the particular operator considered, and also on the questions asked.

In *infinite network models*, both the Laplacian  $\Delta$ , and the Markov operator  $P$ , will have infinite by infinite matrix representations. Each of these infinite by infinite matrices will have the following property: it will have non-zero entries localized only in finite bands containing the infinite matrix-diagonal (i.e., they are infinite banded matrices.) See Section 1.5 and Figure 1.14. Thus, the standard algebraic matrix operations will be well defined.

Functional analytic and spectral theoretic tools now enter as follows: In passing to appropriate Hilbert spaces, we arrive at various classes of Hilbert space-operators. In the present setting, the operators in question will be Hermitian, some unbounded, and some bounded. The Laplacian  $\Delta$  will typically be an unbounded operator, albeit semibounded. When  $\Delta$  is realized in the energy Hilbert space  $\mathcal{H}_E$ , we must introduce boundary value considerations in order to get selfadjoint extensions. By contrast, for the Markov operator  $P$ , there is a weighted  $l^2$ -space such that  $P$  is a bounded, selfadjoint operator. Moreover, its spectrum is then contained in the finite interval  $[-1, 1]$ . In all of the operator realizations by selfadjoint operators,  $\Delta$  or  $P$ , the corresponding spectra may be continuous, or may have a mix of spectral types, continuous (singular or Lebesgue), and discrete.

For the operator theory, and graph Laplacians, for infinite network models, we refer to Chapter 10.

**An Apology** The central themes in our book are as follows: (i) Operators in Hilbert Space with emphasis on unbounded operators and non-commutativity, (ii) Multivariable Spectral Theory, (iii) Noncommutative Analysis, (iv) Probability with emphasis on Gaussian processes, and (v) Unitary Representations. But more importantly, it is our goal to stress the mutual interconnection between these five themes, or in fact central areas of modern analysis. And these are interrelationships which in the literature, at least up to now, have usually not been thought of as especially related. But here we stress, and elaborate in detail, on how a number of key theorems in anyone of these five areas crucially impact advances in the others. Nonetheless, for readers expecting a rehash of the standard list of topics from books in Functional Analysis of past generations, therefore perhaps an apology is in order.

The number of topics making up Functional Analysis is vast; and when applications are added, the size and diversity are daunting. A glance at the many books out there (see the partial list in Appendix A) will give readers an idea of the vast scope. It is by necessity that we have made choices; and that readers will in all likelihood have favorite topics not covered here. And there are probably surprises too; – things we cover here that are not typically included in standard Functional Analysis books. We apologize to readers who had expected a different table of contents. But we hope our choices are justified in our discussion in Part I below.

Our glaring omissions among the big classical areas of Functional Analysis include more technical aspects of the theory of Banach spaces. Even in our consideration of  $L^p$  spaces we have favored  $p = 1, 2$ , or  $\infty$ . Although we have included some fundamentals from Banach space theory, in this, we made a selection of only a few topics which are of direct relevance to the concrete applications that we do include. As for our bias in the choice of  $L^p$  spaces, we can excuse this in part by the familiar availability of interpolation theorems (the interpolation refers to values of  $p$ ), starting with the *Riesz-Thorin theorem* and related; see e.g., [Kru07, HMU80]. Moreover, there is a host of books out there dealing with the exciting and deep areas of Banach space theory, both new and classical; and we refer readers to [JLS96, Joh88] for a sample.

**Emphasis:** In our applications, such as to physics, and statistics, we have concentrated on those analysis tools that are directly needed for the goal at hand. To fit the material into a single volume, we have been forced to omit a number of classical areas of functional analysis, and to concentrate on those that serve the applications we have selected. And in particular we have omitted a number of proofs, or reduced our discussion of some proofs to a few hints, or to exercises. We feel this is justified as there are many great books out there (see Appendix A) which contain complete proofs of the big theorems in functional analysis; for example, the books by W. Rudin, or by P. Lax.

**Note on Presentation and Exercises** In presenting our results, we have aimed for a reader-friendly account: We found it helpful to include worked examples in order to illustrate abstract ideas. Main theorems hold in various degrees of generality, but when appropriate, we have not chosen to present details in their highest level of generality. Rather, we typically give the result in a setting where the idea is more transparent, and easier to grasp. But we do include comments about the more general versions; sketching them in rough outline. The more general versions will typically be easier for readers to follow, and to appreciate, after ideas have been fleshed out in simpler contexts. We have made a second choice in order to make it easier for students to grasp both ideas and the technical details: We have included a lot of worked examples. At the end of each of these examples, we then outline how details (from the example in question) serve to illustrate one or more features in the general theorems elsewhere in the book. Finally, we have made generous use of both tables and figures. These are listed with page-references at the end of the book.

We shall be using some terminology from neighboring areas. And in order to help readers absorb that, we have included in Appendix B a summary, with cited references, of some key notions from *quantum theory*, *signal processing*, *stochastic processes*, *unitary representations*, and from *wavelet theory*.

Our selection of Exercises varies in level of difficulty, and they vary in purpose as well. Some are really easy, aimed mainly for the benefit of beginners; getting used a definition, or a concept. Others are more traditional homework Exercises that can be assigned in a standard course. And yet others are quite demanding.

We believe the material covered here is accessible to beginning graduate students. In our choice of topics and presentation, we have aimed for a user friendly book which hopefully will also be of interest to both pure and applied mathematicians, as well as to students and scientists, in anyone of a number of neighboring areas. In our presentation, we have stressed applications, and also interconnections between disparate areas.

**Reader's guide to References.** In the Reference list, and in citations, we have included both books and research papers. For the various themes, we have aimed at citing both original sources, as well as timely papers; but we also cite brand new research. As for the latter, i.e., the cited papers in the References dealing with recent research (relating to the present topics), we mention a few, followed by citations:

- **Spectral theory:** [AH13, CJK<sup>+</sup>12, HdSS12, Hel13, JP13b, JPT12a, JPT14b].
- **The theory of frames, including Kadison-Singer:** [Wea03, Cas13, Cas14, CFMT11, KOPT13, MSS15, SWZ11]. (The paper [MSS15] is the solution to K-S.)
- **Stochastic processes and applications:** [SS11a, AJ12, AJS14, Jør14].
- **Analysis of infinite networks:** [RAKK05, BKS13, AJSV13, JP13a, JT15b].



- **Representations of groups and algebras:** [CM06, CM13, DHL09, JÓ00, JP12, JPS05, Boc08, DJ08, JLH06].
- **Quantization and quantum information:** [OH13, ARR13, CJK<sup>+</sup>12, CM07, Fan10, Maa10, OR07].

# Preface

*There are already many books in Functional Analysis, so why another?*

The main reason is that we feel there is a need: in the teaching at the beginning graduate level; more flexibility, more options for students and instructors in pursuing new directions. And aiming for a book which will help students with primary interests elsewhere to acquire a facility with tools of a functional analytic flavor, say in spectral theory for operators in Hilbert space, in commutative and non-commutative harmonic analysis, in PDE, in numerical analysis, in stochastic processes, or in physics.

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# Abbreviations

$\aleph_0$	aleph-sub 0, cardinality of $\mathbb{N}$
$\Im\{z\}$	the imaginary part of $z \in \mathbb{C}$
$\mathcal{O}_N$	the Cuntz-algebra, i.e., generators $\{s_i\}_{i=1}^N$ and relations, $s_i^*s_j = \delta_{i,j}\mathbb{1}$ , and $\sum_1^N s_i s_i^* = \mathbb{1}$ .
$\Re\{z\}$	the real part of $z \in \mathbb{C}$
$NR_T$	numerical range of a given operator $T$
BMO	bounded mean oscillation
conv	convex hull
CP	completely positive map
ext	set of extreme-points
i.i.d	independent identically distributed (system of random variables)
ind	induced representation
irrep	irreducible representation
KS	Kadison-Singer
ODE	ordinary differential equation
ONB	orthonormal basis (in Hilbert space)
PDE	partial differential equation; examples: the heat equation, diffusion equation, the wave equation, the Laplace equation, the Schrödinger equation.
PDO	partial differential operator
Proj	projection
PVM	projection valued measure (the condition $P(A \cap B) = P(A)P(B)$ is part of the definition)



$\text{Rep}(\mathfrak{A}, \mathcal{H})$	representations of an algebra $\mathfrak{A}$ acting on a Hilbert space $\mathcal{H}$
$\text{Rep}(G, \mathcal{H})$	representations of a group $G$ acting on a Hilbert space $\mathcal{H}$
RKHS	reproducing kernel Hilbert space
SDE	stochastic differential equation
span	linear span
supp	support of a function, a measure, or a distribution

# Notation

## *Operators in Hilbert space*

$A = A^*$ .....	selfadjoint
$A \subset A^*$ .....	symmetric (also called Hermitian)
$A \subset -A^*$ .....	skew-symmetric
$AA^* = A^*A$ .....	normal
$UU^* = U^*U = I$ ....	unitary
$I$ , or $I_{\mathcal{H}}$ .....	identity operator in a given Hilbert space $\mathcal{H}$ , i.e., $I(v) = v, \forall v \in \mathcal{H}$
$P = P^* = P^2$ .....	projection
$\mathcal{G}(A)$ .....	graph of operator
$\langle \cdot, \cdot \rangle$ .....	inner product of a given Hilbert space $\mathcal{H}$ , i.e., $\langle v, w \rangle = \langle v, w \rangle$ for $v, w \in \mathcal{H}$ ; linear in the second variable.
$ v\rangle\langle w $ .....	Dirac ket-bra vectors for rank-one operator
$\wedge$ .....	lattice operation “minimum” applied to projections
$\vee$ .....	lattice operation “maximum” applied to projections
$\cap$ .....	set-theoretic intersection
$\cup$ .....	set-theoretic union
ess sup .....	essential supremum
$l^p$ .....	sequence space, $l^p$ -summable
$L^p(\mu)$ .....	$L^p$ -integrable functions on a $\mu$ measure space
span .....	all linear combination of a specified subset
$\overline{\text{span}}$ .....	closure of span

$E^*$ .....	the dual of a given normed space $E$ ( $E^*$ is a Banach space)
$E^{**}$ .....	double-dual
$(\cdot)'$ .....	commutant of a set of operators
$(\cdot)''$ .....	double-commutant
$\widehat{\cdot}$ .....	Fourier transform, or Gelfand transform
$\chi_E$ .....	indicator function of a set $E$
$\delta$ .....	Dirac delta "function"
$*$ .....	convolution
$\text{sp}$ .....	(spec) spectrum
$\text{res}$ .....	resolvent set
$\mathcal{B}(X)$ .....	Borel sets, i.e., the sigma-algebra generated by the open sets in a topological space $X$
$\mathcal{B}(\mathcal{H})$ .....	all bounded linear operators $\mathcal{H} \rightarrow \mathcal{H}$
$\mathcal{FR}(\mathcal{H})$ .....	all finite-rank operators in $\mathcal{H}$
$\text{tr}$ (trace) .....	the trace functional
$\mathcal{T}_1(\mathcal{H})$ .....	all trace class operators in $\mathcal{B}(\mathcal{H})$
$\text{Proj}(\mathcal{H})$ .....	the lattice of all orthogonal projections $P$ in a fixed Hilbert space, i.e., $P = P^2 = P^*$ .
$\text{dom}(A)$ .....	the domain of some linear operator $A$
$\text{ran}(A)$ (or $R(A)$ ) ..	the range of $A$
$\text{Ker}(A)$ .....	the kernel of $A$
$M_\varphi$ .....	the operator of multiplication by some function acting in some $L^2(\mu)$ , or a multiplier in some RKHS.
$T_\varphi = P_+ M_\varphi P_+$ .....	Toeplitz-operator with symbol $\varphi$
$\mu \circ T^{-1}$ .....	transformation of measure, i.e., $(\mu \circ T^{-1})(\Delta) = \mu(T^{-1}(\Delta))$ , $\Delta \in \text{sigma-algebra}$ , $T^{-1}(\Delta) = \{x : Tx \in \Delta\}$ .
$\int^\oplus$ .....	direct integral decomposition
$\oplus$ .....	orthogonal sum
$\otimes$ .....	tensor product

$P, Q$  ..... notation used for pairs of projections, but also for the momentum and position operators from quantum mechanics

$\mathbb{P}$  ..... some probability measure

$\mathbb{E} = \mathbb{E}_{\mathbb{P}}$  ..... expectation  $\mathbb{E}_{\mathbb{P}}(X) = \int_{\Omega} X d\mathbb{P}$

$G$  ..... Lie group

$\mathfrak{g}$  ..... Lie algebra

$\mathfrak{g} \xrightarrow{\text{exp}} G$  ..... exponential mapping

$\mathcal{U}$  ..... representation of some Lie group  $G$

$d\mathcal{U}$  ..... representation of the Lie algebra  $\mathfrak{g}$  corresponding to  $G$ ; the derived representation.

$C^*$ -algebra ..... an algebra  $\mathfrak{A}$  with involution  $\mathfrak{A} \ni a \rightarrow a^* \in \mathfrak{A}$ ,  $a^{**} = a$ ,  $(ab)^* = b^*a^*$ ,  $a, b \in \mathfrak{A}$ ; and norm  $\|\cdot\|$ , such that  $(\mathfrak{A}, \|\cdot\|)$  is complete; and  $\|ab\| \leq \|a\| \|b\|$ ,  $a, b \in \mathfrak{A}$ , holds, as well as  $\|a^*a\| = \|a\|^2$ ,  $a \in \mathfrak{A}$ .

$W^*$ -algebra ..... (also called *von Neumann algebra*, or a *ring of operators*)  
 A  $W^*$ -algebra is a  $C^*$ -algebra  $\mathfrak{A}$  which has the following additional property: There is a Banach space  $(X_*, \|\cdot\|_*)$  such that  $\mathfrak{A}$ , with its  $C^*$ -norm (i.e.,  $\|a^*a\| = \|a\|^2$ ,  $a \in \mathfrak{A}$ ), is the dual,  $\mathfrak{A} = (X_*)^*$ . When such a Banach space  $X_*$  exists, it is called a *pre-dual*. (This characterization of  $W^*$ -algebra is due to Sakai [Sak71].)

**Operations on subspaces of Hilbert spaces  $\mathcal{H}$**

$\mathcal{T} \subset \mathcal{H}$  ..... some subspace in  $\mathcal{H}$

$\mathcal{T}^\perp$  ..... ortho-complement

$$\mathcal{T}^\perp = \{h \in \mathcal{H} : \langle h, s \rangle = 0, \forall s \in \mathcal{T}\} = \mathcal{H} \ominus \mathcal{T}$$

$$\mathcal{T}^{\perp\perp} = \overline{\text{span}} \mathcal{T}$$

**Normal or not! It depends:**

- An operator  $T$  (bounded or not) is normal iff (Def.)

$$T^*T = TT^*$$

- A *state*  $s$  on a  $*$ -algebra  $\mathfrak{A}$  is normal if it allows a representation  $(\mathcal{H}, \rho)$  where  $\mathcal{H}$  is a Hilbert space, and  $\rho$  is a positive trace-class operator in  $\mathcal{H}$  such that  $\text{trace}(\rho) = 1$ , and

$$s(A) = \text{trace}(\rho A), \quad \forall A \in \mathfrak{A}.$$

- A *random variable*  $X$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to be normal iff (Def.) its distribution is normal, i.e.,  $\exists m \in \mathbb{R}, \sigma > 0$  such that

$$\mathbb{P}(\{\omega \in \Omega \mid a \leq X(\omega) \leq b\}) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx.$$

## Part I

# Introduction and Motivation

Below we outline the main areas covered inside the book. We offer some tips for the reader, and conclude with a list of applications.

## 0.1 Motivation

More traditional books on Functional Analysis, and operators in Hilbert space, tend to postpone applications till after all the big theorems from the theory have been covered. The purpose of the present book is to give students a tour of a selection of applications. We aim to do this by first offering a crash course in theory topics tailor made for the purpose (part II). In order to stress the interplay between theory and applications (part III) we have emphasized the traffic in both directions. We believe that the multitude of new applications makes Functional Analysis both a powerful, versatile, and timeless tool in mathematics.

A glance at existing books in Functional Analysis and related areas (see list of reviews in the Appendix A) shows that related books so far already display a rich variety, even if they may have the same title “Functional Analysis” or “Functional Analysis with a subtitle, or a narrowing of the focus.”

Still *the aims, and the contents* of these other books go in a different directions than ours. One thing they have in common is an emphasis on the Four Big Theorems in Functional Analysis, The *Hahn-Banach Theorem*, The *Open Mapping Theorem*, The *Uniform Boundedness Principle*, The *Closed Range Theorem*, and duality principles.

By contrast, we do as follows; rather we select a list of topics and applications that acquire a degree of elegance when presented in a functional analytic setting. There are several reasons for this different approach, the main ones are as follows:

- (i) The subject is ever changing on account of demands from neighboring fields;
- (ii) Students come to our graduate functional analysis course with a diversity of backgrounds, and a “one-size fits all” approach is not practical;
- (iii) Well-motivated students can pick up on their own what is needed from the Four Big Theorems;
- (iv) Concentrating on the Four Big Theorems leaves too little time for a variety of neighboring areas, both within mathematics, and in neighboring sciences.
- (v) Also the more traditional approach, beginning with the Four Big Theorems is already in many existing books (see the Appendix A).

A glance at the *Table of Contents* will reflect our aim: beginning with tools from Hilbert space in Chapters 1 & 2, but motivated by quantum physics; a preview of the Spectral Theorem in Chapter 3; some basic tools from the theory of operator

algebras in Chapter 4, with an emphasis on the Gelfand-Naimark-Segal (GNS) construction; and stressing the many links between properties of states, and the parallel properties of the representations, and the operator algebras they generate.

In Chapter 5, and motivated by physics and harmonic analysis, we discuss dilation theory. This is the general phenomenon (pioneered by Stinespring and Arveson) of studying problems in an initial Hilbert space by passing to an enlarged Hilbert space.

Chapter 6 (Brownian motion), while different from the others, still fits perfectly, and inviting application of the tools already discussed in the first four chapters. The applications we cover in Chapter 7 are primarily to representations of groups and algebras. Chapter 8 is an application of theorems from Chapters 3-4 to the problem named after Kadison and Singer, now abbreviated the KS-problem. It is 50 years old, is motivated by Dirac's formulation of quantum physics (observables, states, and measurements); and it was solved only a year ago (as of present).

The last three chapters are, 9: *selfadjoint extensions*, 10: *graph-Laplacian*, and 11: *reproducing kernel Hilbert spaces* (RKHSs), and they are somewhat more technical, but they are critical for a host of the questions inside the book. Some readers may be familiar with this material already. If not, a quick reading of Chapters 2, 9, and 10 may be useful. Similarly, in the appendix, to help students orient themselves, we give a birds-eye view of, in the order of 20 books out there, all of which cover an approach to Functional Analysis, and its many applications.

## 0.2 Key Themes in the Book: A bird's eye pre-view

While each of our central themes has found book presentations, the particular interconnection and applications that are the focus of the present book, have not previously been explored in a textbook form. To the extent they are in the literature at all, it will be in the form of research papers.

### Operators in Hilbert Space

The notion of a *Hilbert space*, is one of the most successful axiomatic constructions in modern analysis. It was John von Neumann who coined the term Hilbert space. While, historically, the concept originated with problems from partial differential equations (PDE), potential theory, quantum physics, and ergodic theory, it has since found a host of other applications involving the part of functional analysis dealing with infinite-dimensional function spaces; such areas as: the study of unitary representations of groups (for example symmetry groups from physics), complex function theory (Hardy spaces of holomorphic functions), applications to probability, to stochastic processes, to signal processing, to thermodynamics (heat transfer, ergodic theory.) One reason the von



Neumann-Hilbert axioms have proved especially successful is their versatility in dealing with optimization problems arising in the study of infinite-dimensional function spaces. This is so despite the fact that the Hilbert space axioms themselves are formulated in the abstract, independently of the particular context where they are applied. Specifically, the axioms entail a given vector space  $\mathcal{H}$ , equipped with an inner product (part of the axiom system), which in turn induces a norm. The last axiom is that  $\mathcal{H}$  must be complete with respect to this norm.

With this one then proceeds to devise a host of coordinate systems, orthonormal bases (ONB). A noteworthy family of ONBs of more recent vintage are wavelet bases.

Among more recent areas of application, we mention machine learning; a sub-area of artificial intelligence. In its current version, machine learning models are formulated in the setting of reproducing kernel Hilbert spaces (RKHS); see Chapter 11 below, and [SZ07]. Indeed, in modern machine learning theory, the RKHSs play a critical role in the construction of optimization algorithms. A second use of RKHSs is in the solution of maximum-likelihood problems from probability theory.

As for the study of linear transformations (operators), our present dual emphasis will be unbounded operators, and non-commutativity. Specifically, we study systems of densely defined linear operators. A key motivation for this emphasis is again quantum mechanics: Indeed quantum mechanical observables (momentum, position, energy, etc) correspond to non-commuting selfadjoint unbounded operators in Hilbert space.

The first two Hilbert spaces most students encounter are  $l^2$  and  $L^2(\mathbb{R})$ :

$l^2$ : sequences  $(x_n)_{n=1}^\infty$  such that

$$\|x\|_{l^2}^2 = \sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

$L^2(\mathbb{R})$ : measurable functions on  $\mathbb{R}$  such that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

These two examples serve to illustrate the axiom system for Hilbert space which we shall study in Section 1.4 below.

The norms for  $l^2$  and for  $L^2(\mathbb{R})$  come from associated inner products  $\langle \cdot, \cdot \rangle$ , for example,  $\langle x, y \rangle_{l^2} = \sum_{n=1}^{\infty} \bar{x}_n y_n$ ,  $\forall x, y \in l^2$ . The system of vectors  $\delta_1, \delta_2, \dots$  in  $l^2$ , given by

$$\delta_k(n) = \delta_{k,n} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

satisfies  $\langle \delta_k, \delta_l \rangle_{l^2} = \delta_{k,l}$  (the orthonormality property); and, for all  $x = (x_n)_1^\infty \in l^2$ , we have

$$\lim_{N \rightarrow \infty} \left\| x - \sum_{k=1}^N x_k \delta_k \right\|_{l^2}^2 = \lim_{N \rightarrow \infty} \sum_{k=N+1}^{\infty} |x_k|^2 = 0;$$

the second property (called “total”) that orthonormal bases (ONBs) have.

One naturally wonders “what are analogous ONBs for the second mentioned Hilbert space  $L^2(\mathbb{R})$ ?” And we shall turn to this question also in Chapter 1 below: There are two classes, (i) *special functions*, of which the best known are the Hermite functions (Section 1.4); and (ii) *wavelet-bases* (Sections 1.4 and 5.5).

The chapters of special relevance to these topics are: Chapters 1, 2, 8, 9, and 11. Sections of special relevance include 1.6, 1.7, 2.3, 3.4, 9.1, 9.2, and 9.4.

## Multivariable Spectral Theory

In this setting we are dealing with more than one operator at a time. A host of applications naturally present themselves, again with the same applications as mentioned in sect 0.2 above. From in the late nineteen thirties we have the study of selfadjoint algebras in the works of Murray-von Neumann and of Gelfand-Naimark. Later, this was followed up with systematic studies of non-selfadjoint algebras; e.g., the work of Kadison-Singer. Other studies of multivariate operator theory emphasize analogues of analyticity, both in the commutative as well as in non-commutative settings. It has had remarkable successes, including applications in other areas of mathematics such as complex and algebraic geometry, and non-commutative geometry. In the multivariable case, some researchers consider either  $n$ -tuples of operators, or representations of algebras with generators and relations; while others have adopted the language of Hilbert modules; for example, modules over algebras of holomorphic functions, polynomials or entire functions depending on the given number  $n$  (commuting) complex variables.

Historically, the first important multivariable problem in operator theory was perhaps the relations of Heisenberg for a pair of linear operators  $P$  and  $Q$  with dense domain  $\mathcal{D}$  in a fixed Hilbert space  $\mathcal{H}$ . The relations require that

$$PQf - QPf = -if \tag{1}$$

holds for all  $f \in \mathcal{D}$ .

By now (1) is well understood, but there are many subtle points; all of which make important connections to what we call multivariable spectral theory; for example (1) does not have solutions for bounded operators in  $\mathcal{H}$ .

We shall also consider a variety of multivariable systems of bounded operators; – in this case, it is usually in the setting of non-normal operators (so in particular non-selfadjoint), for example for: (i) finite sets of bounded commuting operators in a fixed Hilbert space  $\mathcal{H}$ ; (ii) algebras  $\mathfrak{A}$  of operators on  $\mathcal{H}$  such that the pair  $(\mathfrak{A}, \mathcal{H})$  forms a module; and (iii) finite systems of isometries

in some Hilbert space  $\mathcal{H}$ ; and (iv) sets of isometries subject to the added condition that the ranges forms a system of orthogonal subspaces of  $\mathcal{H}$  with sum equal to  $\mathcal{H}$ . The relations on sets of isometries described in (iv) are called the Cuntz relations (see sect 4.1) and [Cun77, BJ02, BJO04]. The Cuntz relations correspond to representations of a  $C^*$ -algebra, called the Cuntz-algebra. It has many applications, some of which will be studied, see e.g., sect 4.9. A good reference for (i) is [Arv98].

The chapters of special relevance to these topics are: Chapters 5, 7, and 9. Sections of special relevance include 5.2, 5.3, 7.4, 7.5, 7.8, and 9.4.

## Noncommutative Analysis

The above multivariable settings are part of a wider theme: *noncommutative analysis*, a field which extends (classical commutative) Fourier analysis. This began with the study of locally compact groups from physics, and their unitary representations. The case of compact groups encompasses the Peter-Weyl theorem from the 1920s, but needs from number theory (mathematics), and from relativistic quantum physics, have dictated extensions to non-compact (and non-commutative) groups, typically Lie groups.

A more recent area of noncommutative analysis is the study of *free probability*, which we shall only touch on tangentially inside the book. It is an exciting and new, rapidly growing, research direction; with new advances in theory as well as in applications. Fortunately, there are already nice and accessible book treatments, see e.g., [Spe11] and the sources cited there. In free probability, we study systems of non-commutative random variables. As stochastic processes, they are not Gaussian. Rather the notion of free independence dictates the semicircle-law (not the Gaussian distribution). The rigorous study of free probability entails such operator algebraic notions as free products. We emphasize that the important new notion of free independence is dictated by non-commutativity, and that it generalizes the more familiar notion of independence which was used previously in probability. The subject was initiated by Dan Voiculescu in the 1980ties. Its applications up to now include: random matrix theory, representations of symmetric groups, large deviations of stochastic processes, and quantum information theory.

We use the term “noncommutative analysis” more broadly than the related one, “noncommutative geometry.” The latter owes much to the pioneering work of Alain Connes, see e.g., [Con07]. In broad outline, it covers the role von Neumann algebra theory plays in noncommutative considerations in geometry and in quantum physics (the Standard Model); in noncommutative metric theory and spaces, noncommutativity in topology, spectral triples, differential geometry, cyclic cohomology, cyclic homology, K-theory, and M-theory. In more detail, noncommutative geometry (NCG) is concerned with a geometric approach to the construction of spaces that are locally presented via noncommutative algebras of operators. This is the framework of, what in physics, is referred to as “local quantum field theory.” The prime applications of NCG are to particle physics where A. Connes has developed a noncommutative standard model.

Some of the other successes of NCG include extensions of known topological invariants to formal duals of noncommutative operator algebras. Via a Connes-Chern character map, this has led to the discovery of a new homology theory of noncommutative operator algebras; and to a new non-commutative theory of characteristic classes; and to generalizations of the classical index theorems.

The Standard Model of particle physics deals with the electromagnetic, weak, and strong nuclear interactions, and with classifications of all the known subatomic particles, the "theory of almost everything." It received a boost in the mid-1970s after an experimental confirmation of the existence of quarks; and later of the tau neutrino, and the Higgs boson (2013).

Helpful references here are [Arv76, BD91, BR79, DJ08, Gli61, JM84, Jor94, Jor11, Pow75, Tak79].

The chapters of special relevance to these topics are: Chapters 4, 5, and 7. Sections of special relevance include 4.7, 4.8, 5.3, 5.5, 7.4, 7.8, and 7.11.

## Probability

Probability theory originated with the need for quantification of uncertainty, as it arises for example in quantum physics, and in financial markets. In the 1930ties, Kolmogorov's formulated precise mathematical axioms of probability space  $\Omega$ , sample points, events as specified subsets, in a prescribed sigma-algebra  $\mathcal{F}$  of subsets of  $\Omega$ , and a probability measure, defined on  $\mathcal{F}$ .

Our present focus will be a subclass of stochastic processes, the Gaussian processes, especially those which are derived from stochastic integration defined relative to Brownian motion.

Brownian motion is the simplest of the continuous-time stochastic (meaning probabilistic) processes. It is a limit of simpler stochastic processes going by the name random walks; a fact which reflects the universality of the normal distribution, the Gaussians.

It is not an accident that we have focused on problems from quantum physics and from probability. With some over simplification, it is fair to say that Hilbert's 6th problem asked for a mathematical rigorous treatment of these two areas. In 1900, when Hilbert formulated his 23 problems, these two areas did not yet have mathematically rigorous foundations.

The topic from probability that shall concern us the most is that of Brownian motion. In a nutshell, a Brownian motion may be thought as this way: There is a probability measure  $\mathbb{P}$  on a sigma-algebra of subsets of the continuous functions  $\omega$  on  $\mathbb{R}$  such that

$$B_t(\omega) = \omega(t), \quad t \in \mathbb{R}, \omega \in C(\mathbb{R})$$

satisfy a number of axioms of which we mention here only that for each  $t \in \mathbb{R}$ ,  $B_t$  has a Gaussian distribution relative to  $\mathbb{P}$  such that

$$\int_{C(\mathbb{R})} |B_t(\omega) - B_s(\omega)|^2 d\mathbb{P}(\omega) = |t - s|, \quad s, t \in \mathbb{R},$$

and

$$\int_{C(\mathbb{R})} B_t(\omega) d\mathbb{P}(\omega) = 0, \quad \forall t \in \mathbb{R}.$$

Helpful references here are [AJ12, AJS14, GJ60, It04, It06, Nel67, Par82, Sla03].

The chapters of special relevance to these topics are: Chapters 6, and 11. Sections of special relevance include 6.2, 11.1-11.4, and Figures 6.1, 6.2, 6.3, 11.2, and 11.3.

## Unitary Representations

An early motivation (see also Section 0.2 above) is work of J. von Neumann and I.E. Segal. They showed that, if  $G$  is a locally compact unimodular group such that the associated von Neumann group algebra is of type I, then the regular representation of  $G$ , acting on the Hilbert space  $L^2(G)$  relative to Haar measure, as a unitary representation, is a direct integral of irreducible unitary representations (“irreps” for short.) This leads to a notion of a unitary dual for  $G$ , defined as the set of equivalence classes (under unitary equivalence) of such representations, the “irreps.”

But for general locally compact groups, including countable discrete groups, the von Neumann group algebra typically is not of type I and the regular translation-representation of  $G$  cannot be expressed in terms of building blocks of “irreps.”

The applications of our present results on unitary representations will include those discussed above, so in particular, applications to quantum physics, and to probability, especially to Gaussian stochastic processes.

The chapters of special relevance to these topics are: Chapters 2, 4, and especially 7. Sections of special relevance include 2.1, 2.2, 4.2, 4.4, 4.6, 7.2, 7.7, and 7.8.

## 0.3 Note on Cited Books and Papers

For readers looking for references on the foundations, our suggestions are as follows: Operators in Hilbert space: [Arv76, Arv72]. Quantum mechanics: [OR07, Wei03, GG02], and [Pol02, PK88, CP82]. Non-commutative functional analysis and algebras of operators: [BJKR84, BR81b, BR79]. Unitary representations of groups: [Mac92, Mac85, Mac52].

In our use of citations we adopted the following dual approach. Inside the chapters, as the material is developed, we include citations to key sources that we rely on; – but this is done sparingly so as not to interrupt the narrative too much.

To remedy sparse citations inside chapters, and, in order to help the reader orient herself in the literature, each of the 11 chapters concludes with a little bibliographical section, summarizing papers and books of special relevance to

	Subject	Example
A	analysis	$f(x) - f(0) = \int_0^x f'(y) dy$
B	dynamical systems	functions on fractals, Cantor set, etc.
C	PDE	Sobolev spaces
D	numerical analysis	discretization
E	measures / probability theory	probability space $(\Omega, \mathcal{F}, \mathbb{P})$
F	quantum theory	Hilbert spaces of quantum states

Table 1: Examples of Linear Spaces: Banach spaces, Banach algebras, Hilbert spaces  $\mathcal{H}$ , linear operators act in  $\mathcal{H}$ .

the topic inside the text. Thus there is a separate list of citations which concludes each chapter. Readers who do not find a particular citation inside the chapter itself will likely be able to locate it from the *end-of-chapter-list*.

## 0.4 Reader Guide

Below we explain chapter by chapter how the six areas in Table 1 are covered.

- Ch 1: Areas A, E, F.
- Ch 2: Areas A, C, F.
- Ch 3: Areas B, C, E, F.
- Ch 4: Areas A, B, E, F.
- Ch 5: Areas A, E, F.
- Ch 6: Area E.
- Ch 7: Areas D, F.
- Ch 8: Areas E, F.
- Ch 9: Areas B, C, D.
- Ch 10: Areas A, F.
- Ch 11: Areas A, B, C, D, E.

In more detail, the six areas in Table 1 may be fleshed out as follows:

Examples of subjects within *area A* include measure theory, transforms, construction of bases, Fourier series, Fourier transforms, wavelets, and wavelet transforms, as well as a host of operations in analysis.

Subjects from *area B* include solutions to ordinary differential equations (ODEs), and the output of iteration schemes, such as the Newton iteration algorithm. Also included are ergodic theory; and the study of fractals, including harmonic analysis on fractals.

*Area C* encompasses the study of the three types of linear PDEs, elliptic, parabolic and hyperbolic. Sample questions: weak solutions, *a priori* estimates, diffusion equations, and scattering theory.

*Area D* encompasses discretization, algorithms (Newton etc), estimation of error terms, approximation (for example wavelet approximation, and the associated algorithms.)

*Area E* encompasses probability theory, stochastic processes (including Brownian motion), and path-space integration.

Finally *area F* includes the theory of unbounded operators in Hilbert space, the three versions of the Spectral Theorem, as well as representations of Lie groups, and of algebras generated by the commutation relations coming from physics.

## 0.5 A Word About the Exercises

All the chapters have exercises. The topics in the last three chapters are more specialized, and exercises seem less natural there. The purpose of the exercises is to improve and facilitate the use of the book in courses; – to help students and instructors. There is a total of 147 exercises. To help with classroom use, we have listed them in the back, numbering chapter-by-chapter. Each exercise is given a name identification. Here is a sample: Exercise Exercise 1.65 (Lax-Milgram), 1.79 (The Haar wavelet), 2.18 (the resolvent identity), 3.68 (Powers-Størmer), 4.66 (time-reflection), 4.112 (extreme measures), 7.81 (multiplicity), 7.86 (a formula from Peter-Weyl), and so on; . . . 11.6 (Szegő-kernel).

The degree of difficulty of the exercises varies from one to the next, some are relatively easy; for example, serving to give the reader a chance to practice definitions or new concepts; – and some are quite difficult. But of the exercises all interact naturally with the topics developed in the various chapters. This is why we have integrated them into the development of the topics, chapter for chapter. And this is also why some chapters have many exercises, such as Chapter 1 with a total of 38 exercises; – Chapter 3 has 12 exercises; and Chapter 4 has 44 exercises in all. In all of the chapters, we have mixed and interspaced the placement of exercises with the central themes: some exercises supplement examples, and some theorems, within each chapter.

There are two lists after the Appendices, a *List of Exercises*, and a *list of all the figures*. The second should help readers with cross-references; and the first with use of the exercises in course-assignments.

*The Appendices* themselves serve to aid readers navigate the book-literature. Appendix A includes *telegraphic reviews*, and Appendix C is a collection of *biographical sketches* of the pioneers in the subject.

## 0.6 List of Applications

Part of the discussion below will make use of terminology from neighboring areas, such as physics, engineering, and statistics. For readers who might be encountering this for the first time, we have a terminology section in the back. It is a section in the Appendix B, called “Terminology from neighboring areas.” The Appendix also includes other lists: a list of telegraphic reviews of related books; biographical sketches; diagrams with lines illustrating interconnections between disparate areas; a list of figures, and a list of all the Exercises inside the text. Each Exercise is given a descriptive name.

Each of the chapters is illustrated with examples and applications. A recurrent theme is the important notion of *positive definite functions*, and their realization in Hilbert space. Applications to *Wiener measure* and path-space are included in Chapters 1, 6, and 11.

More applications, starting in Chapter 1 are: (i) a variance formula for the Haar-wavelet basis in  $L^2(0, 1)$ ; (ii) a formula for perturbations of diagonal operators; and (iii) the  $\infty \times \infty$  matrix representation of *Heisenberg’s commutation relations* in the case of the canonical pair, momentum and position operators from quantum mechanics.

The case of Heisenberg’s commutation relations motivates the need for a systematic study of *unbounded operators* in Hilbert space. This is started in Chapter 2, and resumed then systematically in Chapters 3 (the *Spectral Theorem*), and 9 (the theory of *extensions of symmetric operators with dense domain*; – von Neumann indices, and All That). Both our study of selfadjoint operators and normal operators, and their spectral theory, throughout the book is motivated by the axioms from quantum theory: observables, states, measurements, and the uncertainty principle. Our systematic treatment of *projection valued measures*, and *quantum states*, in Chapter 3 is a case in point.

This goes for our theme in Chapter 4 as well, the *Gelfand-Naimark-Segal* (GNS) representation. *Quantum states* must be realized in Hilbert space, but what is the relevant Hilbert space, when a *quantum observable* is prepared in a state? To answer this we must realize the observables as selfadjoint operators affiliated with a suitable  $C^*$ -algebra, say  $\mathfrak{A}$ , or von Neumann algebra. States on these algebras then become positive linear functionals. The GNS construction is a device for constructing representation in Hilbert space for every state, defined as a *positive linear functional* on  $\mathfrak{A}$ . In this construction, the pure states are matched up with irreducible representations.

In Section 4.4, our application is to the subject of “*reflection-positivity*” from quantum physics. This notion came up first in a renormalization question in physics: “How to realize observables in relativistic quantum field theory (RQFT)?”

The material in Chapter 5 has applications to signal processing; – to the construction of sub-band filters, and filter banks. These applications are discussed in Chapter 5; – included as one of the applications of a certain family of representations of the Cuntz relations. Other applications of these representations include wavelet filters.



## 0.7 Groups and Physics

“The importance of group theory was emphasized very recently when some physicists using group theory predicted the existence of a particle that had never been observed before, and described the properties it should have. Later experiments proved that this particle really exists and has those properties.”

— Irving Adler

Recall that in RQFT, the symmetry group is the Poincaré group, but its physical representations are often illusive. Starting with papers by Osterwalder-Schrader in the 1970ties (see e.g., [OS75, GJ87, JÓ00]), it was suggested to instead begin with representations of the Euclidian group, and then to get to the Poincaré group through the back door, via an analytic continuation (a *c*-dual group construction), and a *renormalization*. This led to a systematic study of renormalizations for the Hilbert space of quantum states. The “*c*-dual” here refers to an *analytic continuation* which links the two groups. This in turn is accomplished with the use of a certain reflection, and a corresponding change in the inner product. In a simplified summary, the construction is as follows: Starting with the inner product in the initial Hilbert, say  $\mathcal{H}$ , and a unitary representation admitting a reflection  $\mathcal{J}$ , we then pass to a certain invariant subspace of  $\mathcal{H}$ , and use  $\mathcal{J}$  in the definition of the new inner product. The result is a physical energy operator (dual of time) with the correct positive spectrum for the relativistic problem, hence “*reflection-positivity*.” The invariant subspace refers to invariance only in a positive time direction. All of this is presented in Section 4.4, and illustrated with an example.

Chapter 5 deals with the same theme; only there the states are *operator valued*. From the theory of Stinespring and Arveson we know that there is then a different positivity notion, *complete positivity* (CP).

Among the Hilbert spaces we encounter are  $L^2$  spaces of random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The case of Brownian motion is studied in Chapter 6, and again in Chapter 11.

In Chapter 7, we introduce families of *unitary representations* of groups, and *\**-representations of algebras; each one motivated by an application from physics, or from signal-processing. We are stressing examples as opposed to general theory.

Chapter 8 is devoted to the *Kadison-Singer* problem (KS). It is a problem from operator algebras, but originating with Dirac’s presentation of quantum mechanics. By choosing a suitable orthonormal basis (ONB) we may take for Hilbert space the sequence  $l^2(\mathbb{N})$  space, square-summable sequences. Dirac was interested in the algebra  $\mathcal{B}(l^2(\mathbb{N}))$  of all bounded operators in  $l^2(\mathbb{N})$ . But with the  $\infty \times \infty$  matrix representation for elements in  $\mathcal{B}(l^2(\mathbb{N}))$ , we can talk about the *maximal abelian subalgebra*  $\mathcal{D}$  of all diagonal operators in  $\mathcal{B}(l^2(\mathbb{N}))$ . Note  $\mathcal{D}$  is just a copy of  $l^\infty(\mathbb{N})$ . The Dirac-KS question is this: “Does every pure

state on  $\mathcal{D}$  have a *unique* pure-state extension to  $\mathcal{B}(l^2(\mathbb{N}))$ ?”

The problem was solved in the affirmative; just a year ago (see [MSS15]), and we sketch the framework for the KS problem. However the details of the solution are far beyond the scope of our book.

The application in Chapter 10 is to potential theory of *infinite networks*; mathematically infinite graphs  $G = (V, E)$ ,  $V$  the specified set of vertices, and  $E$  the edges. Our emphasis is electrical networks, and the functions include energy, conductance, resistance, voltage, and current. The main operator here is the so called *graph Laplacian*.

The new applications in Chapter 11 include *scattering theory*, *learning theory* (as it is used in machine learning and in pattern recognition.)

## Part II

# Topics from Functional Analysis and Operators in Hilbert Space: a selection

# Chapter 1

## Elementary Facts

“... the [quantum mechanical] observables are operators on a Hilbert space. The algebra of operators on a Hilbert space is noncommutative. It is this noncommutativity of operators on a Hilbert space that provides a precise formulation of [Heisenberg’s] uncertainty principle: There are operator solutions to equations like  $pq - qp = 1$ . This equation has no commutative counterpart. In fact, it has no solution in operators  $p, q$  acting on a finite dimensional space. So if you’re interested in the dynamics of quantum theory, you must work with operators rather than functions and, more precisely, operators on infinite dimensional spaces.”

— William B. Arveson (1934-2011. The quote is from 2009.)

I received an early copy of Heisenberg’s first work a little before publication and I studied it for a while and within a week or two I saw that the noncommutation was really the dominant characteristic of Heisenberg’s new theory. It was really more important than Heisenberg’s idea of building up the theory in terms of quantities closely connected with experimental results. So I was led to concentrate on the idea of noncommutation and to see how the ordinary dynamics which people had been using until then should be modified to include it.

— P. A. M. Dirac

Problems worthy  
of attack  
prove their worth  
by hitting back.

— Piet Hein

Below we outline some *basic concepts, ideas, and examples* which will be studied

inside the book itself. While they represent only a sample, and we favor the setting of Hilbert space, the details below still tie in nicely with diverse tools and techniques not directly related to Hilbert space.

The discussion below concentrates on topics connected to Hilbert space, but we will also have occasion to use some other basic facts from functional analysis; e.g., duality and Hahn-Banach. We have collected those, in a condensed form, in an Appendix at the end of the chapter.

From linear algebra we know precisely what square matrices  $M$  can be diagonalized; the *normal matrices*, i.e.,  $M^*M = MM^*$ . More precisely, a matrix is normal if and only if it is conjugate to a diagonal matrix. More general square matrices don't diagonalize, but they admit a Jordan form.

In the infinite dimensional case, – while infinite matrices are useful, the axiomatic setting of *Hilbert space* and *linear operators* has proved more successful than an infinite matrix formulation; and, following von Neumann and Stone, we will make precise the notion of normal operators. Because of applications, the case of *unbounded operators* is essential. In separate chapters, we will prepare the ground for this.

The *Spectral Theorem* (see [Sto90, Yos95, Nel69, RS75, DS88c]) states that a linear operator  $T$  (in Hilbert space) is *normal*, i.e.,  $T^*T = TT^*$ , if and only if it is unitarily equivalent to a *multiplication operator* in some  $L^2$  space, i.e., multiplication by a measurable function, and the function may be unbounded. The implied Hilbert space  $L^2$  is with respect to some measure space, which of course will depend on the normal operator  $T$ , given at the outset. Hence the classification of *normal operators* is equivalent to the classification of *measure spaces*; – a technically quite subtle problem.

There is a second (and equivalent) version of the Spectral Theorem, one based on *projection valued measures* (PVMs), and we will present this as well. It is a powerful tool in the theory of unitary representations of locally compact groups (see Chapter 7 below), and in a host of areas of pure and applied mathematics.

It is natural to ask whether there is an analogue of the finite-dimensional Jordan form; i.e., extending from finite to the infinite dimensional case. The short answer is “no,” although there are partial results. They are beyond the scope of this book.

In our first two chapters below we prepare the ground for the statement and proof of the *Spectral Theorem*, but we hasten to add that there are several versions. In the bounded case, for *compact selfadjoint operators* (Section 3.5), the analogue to the spectral theorem from linear algebra is closest, i.e., eigenvalues and eigenvectors. Going beyond this will entail an understanding of *continuous spectrum* (Section 3.4), and of *multiplicity theory* in the measure theoretic category (Section 4.11).

With a few exceptions, we will assume that all of the Hilbert spaces considered are separable; i.e., that their orthonormal bases (ONBs) are countable. The exceptions to this will include the  $L^2$ -space of the Bohr completion of the reals  $\mathbb{R}$ . See Exercise 4.60.

## 1.1 A Sample of Topics

“Too many people write papers that are very abstract and at the end they may give some examples. It should be the other way around. You should start with understanding the interesting examples and build up to explain what the general phenomena are.”

— Sir Michael Atiyah

Classical functional analysis is roughly divided into two branches, each with a long list of subbranches:

- study of function spaces (Banach space, Hilbert space)
- applications in physics, statistics, and to engineering

Within pure mathematics, it is manifested in the list below:

- representation theory of groups and algebras, among a long list of diverse topics

We will consider three classes of algebraic objects of direct functional analytic relevance: (i) generators and relations; (ii) algebras, and (iii) groups.

In the case of (i), we illustrate the ideas with the *canonical commutation relation*

$$PQ - QP = -iI, \quad i = \sqrt{-1}. \quad (1.1)$$

The objective is to build a Hilbert space such that the symbols  $P$  and  $Q$  are represented by unbounded essentially selfadjoint operators (see [RS75, Nel69, vN32a, DS88c]), each defined on a common dense domain in some Hilbert space, and with the operators satisfying (1.1) on this domain. (See technical points inside the present book, and in the cited references.)

In class (ii), we consider both  $C^*$ -algebras and von Neumann algebras (also called  $W^*$ -algebras); and in case (iii), our focus is on *unitary representations* of the group  $G$  under consideration. The group may be abelian or non-abelian, continuous or discrete, locally compact or not. Our present focus will be the case when  $G$  is a *Lie group*. In this case, we will study its representations with the use of the corresponding *Lie algebra*.

- $C^*$ -algebras, von Neumann algebras

We will be considering  $C^*$ -, and  $W^*$ -algebras axiomatically. In doing this we use the theorem by S. Sakai to the effect that the  $W^*$ -algebras consist of the subset of the  $C^*$ -algebras that are the dual of a Banach space. If the  $W^*$ -algebra is given, the Banach space is called the pre-dual. Representations will be studied with the use of states, and we stress the theorem of Gelfand, Naimark, and Segal (GNS) linking states with cyclic representations.

- wavelets theory

A wavelet is a special basis for a suitable  $L^2$ -space which is given by generators and relations, plus self-similarity. Our approach to wavelets will be a mix of functional analysis and harmonic analysis, and we will stress a correspondence between a family of representations of a particular  $C^*$ -algebra, called the Cuntz-algebra, on one side and wavelets on the other.

- harmonic analysis

Our approach to harmonic analysis will be general, – encompassing anyone of a set of direct sum (or integral) decompositions. Further our presentation will rely on representations.

- analytic number theory

Our notions from analytic number theory will be those that connect to groups, and representations; such as the study of automorphic forms, and of properties of generalized zeta-functions; see e.g., [CM07, CM06, OPS88, LPS88].

Our brief bird’s eye view of the topics above is only preliminary, only hints; and most questions will be addressed in more detail inside the book.

As for references, the literature on the commutation relations (1.1) is extensive, and we refer to [Sza04, Nel59a, Fug82, Pou73].

Some of the questions regarding the commutation relations involve the subtle difference between (1.1) itself vs its group version, – often referred to as the *Weyl relations*, or the integrated form. As for the other themes mentioned above, operator algebras, math physics, wavelets and harmonic analysis, the reader will find selected references to these themes at the end of this chapter.

A glance at the table of contents makes it clear that we venture into a few topics at the cross roads of mathematics and physics; and a disclaimer is in order. In the 1930s, David Hilbert encouraged the pioneers in quantum physics to axiomatize the theory that was taking shape then with the initial papers by Heisenberg, Schrödinger, Dirac. Others like J. von Neumann joined into this program. These endeavors were partly in response to *Hilbert’s Sixth Problem* [Wig76];

“Give a mathematical treatment of the axioms of physics”

in his famous list of 23 problems announced in 1900 [Hil02]. At this time, quantum physics barely existed. Max Planck’s hypothesis on discreteness of atomic energy-measurements is usually dated a little after the turn of the Century.

**Quantum mechanics** is a first quantized quantum theory that supersedes classical mechanics at the atomic and subatomic levels. It is a fundamental branch of physics that provides the underlying mathematical framework for many fields of physics and chemistry. The term “quantum mechanics” is sometimes used in a more general sense, to mean quantum physics.

With hindsight, we know that there are considerable limitations to the use of axioms in physics. While a number of important questions in quantum physics have mathematical formulations, others depend on physical intuition. Hence in our discussion of questions at the cross-roads of mathematics and physics, we will resort to hand-waiving.

“For those who are not shocked when they first come across quantum theory cannot possibly have understood it.”

Niels Bohr, — quoted in W. Heisenberg, *Physics and Beyond* (1971).

## 1.2 Duality

The “functional” in the name “Functional Analysis” derives from the abstract notion of a *linear functional*: Let  $E$  be a vector space over a field  $\mathbb{F}$  (we shall take  $\mathbb{F} = \mathbb{R}$ , or  $\mathbb{C}$  below.)

**Definition 1.1.** A function  $\varphi : E \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) is said to be a *linear functional*, if we require

$$\varphi(u + \lambda v) = \varphi(u) + \lambda\varphi(v), \quad \forall \lambda \in \mathbb{R}, \forall u, v \in E.$$

If  $E$  comes with a topology (for example from a norm, or from a system of seminorms), we will consider continuous linear functionals. Occasionally, continuity will be implicit.

**Definition 1.2.** The set of all continuous linear functionals is denoted  $E^*$ , and it is called the *dual space*. (In many examples there is a natural identification of  $E^*$  as illustrated in Table 1.1.)

**Definition 1.3.** If  $E$  is a normed vector space, and if it is complete in the given norm, we say that  $E$  is a *Banach space*.

**Lemma 1.4.** Let  $E$  be a normed space with dual  $E^*$ . For  $\varphi \in E^*$ , set

$$\|\varphi\|_* := \sup_{\|x\|=1} |\varphi(x)|.$$

Then  $(E^*, \|\cdot\|_*)$  is a Banach space.

*Proof.* An exercise. □

Given a Banach space  $E$ , there are typically three steps involved in the discovery of an explicit form for the *dual Banach space*  $E^*$ . Table 1.1 illustrates this in two examples, but there are many more to follow; – for example, the case when  $E =$  the Hardy space  $\mathbb{H}_1$ , or  $E =$  the trace-class operators on a Hilbert space.

Moreover, the same idea based on a duality-pairing applies *mutatis mutandis*, to other topological vector spaces as well, for example, to those from Schwartz’ theory of distributions.

The three steps are as follows:

**Step 1.** Given  $E$ , then first come up with a second Banach space  $F$  as a candidate for the dual Banach space  $E^*$ . (Note that  $E^*$  is so far, *a priori*, only an abstraction.)



**Step 2.** Set up a bilinear and non-degenerate pairing, say  $p$ , between the two Banach spaces  $E$  and  $F$ , and check that  $p(\cdot, \cdot)$  is continuous on  $E \times F$ . Rescale such that

$$|p(x, y)| \leq \|x\|_E \|y\|_F, \quad \forall x \in E, y \in F.$$

This way, via  $p$ , we design a linear and isometric embedding of  $F$  into  $E^*$ .

**Step 3.** Verify that the embedding from step 2 is “onto”  $E^*$ . If “yes” we say that  $F$  “is” the dual Banach space. (Example, the dual of  $\mathbb{H}_1$  is BMO [Fef71].)

Examples of Banach spaces include: (i)  $l^p$ ,  $1 \leq p \leq \infty$ , and (ii)  $L^p(\mu)$  where  $\mu$  is a positive measure on some given measure space; details below.

**Example 1.5.**  $l^p$ : all  $p$ -summable sequences.

A sequence  $x = (x_k)_{k \in \mathbb{N}}$  is in  $l^p$  iff  $\sum_{k \in \mathbb{N}} |x_k|^p < \infty$ , and then

$$\|x\|_p := \left( \sum_{k \in \mathbb{N}} |x_k|^p \right)^{\frac{1}{p}}.$$

**Example 1.6.**  $L^p$ : all  $p$ -integrable function with respect to some fixed measure  $\mu$ .

Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be monotone increasing, i.e.,  $x \leq y \implies F(x) \leq F(y)$ ; then there is a Borel measure  $\mu$  on  $\mathbb{R}$  (see [Rud87]) such that  $\mu((x, y]) = F(y) - F(x)$ ; and  $\int \varphi d\mu$  will be the limit of the Stieltjes sums:

$$\sum_i \varphi(x_i) (F(x_{i+1}) - F(x_i)), \quad \text{where } x_1 < x_2 < \dots < x_n.$$

We say that  $\varphi \in L^p(\mu)$  iff  $\int_{\mathbb{R}} |\varphi|^p d\mu$  is well defined and finite; then

$$\|\varphi\|_p = \left( \int_{\mathbb{R}} |\varphi(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

By Stieltjes integral,  $\int |\varphi|^p d\mu = \int |\varphi|^p dF$ . Here, we give the definition of  $L^p(\mu)$  in the case where  $\mu = dF$ , but it applies more generally.

For completeness of  $l^p$  and of  $L^p(\mu)$ , see [Rud87].

*Remark 1.7.* At the foundation of analysis of  $L^p$ -spaces (including  $l^p$  for the case of counting-measure) is Hölder’s inequality; see e.g., [Rud87, ch 3]. Recall conjugate pairs  $p, q \in [1, \infty)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , or equivalently  $p + q = pq$ ; see Figure 1.1.

We present Hölder’s inequality without proof: Fix a measure space  $(X, \mathcal{F}, \mu)$ . If  $p, q$  are conjugate,  $1 < p < \infty$ , then for measurable functions  $f, g$  we have:

$$\left| \int_X fg d\mu \right| \leq \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}} \left( \int_X |g|^q d\mu \right)^{\frac{1}{q}}. \quad (1.2)$$

$E$	$E^*$	how?
$l^p, 1 \leq p < \infty$ with $l^p$ -norm	$l^q, \frac{1}{p} + \frac{1}{q} = 1$	$x = (x_i) \in l^p, y = (y_i) \in l^q,$ $\varphi_y(x) = \sum_i x_i y_i$
$C(I), I = [0, 1]$ with max-norm	signed Borel measures $\mu$ on $I$ , of bounded variation	$\varphi_\mu(f) = \int_0^1 f(x) d\mu(x),$ $\forall f \in C(I).$
$C^\infty(\mathbb{R}),$ system of seminorms	$\mathcal{E}'$ all Schwartz distributions $D$ on $\mathbb{R}$ of compact support	$\varphi_D(f) = D$ applied to $f,$ $f \in C^\infty(\mathbb{R}).$

Table 1.1: Examples of dual spaces.

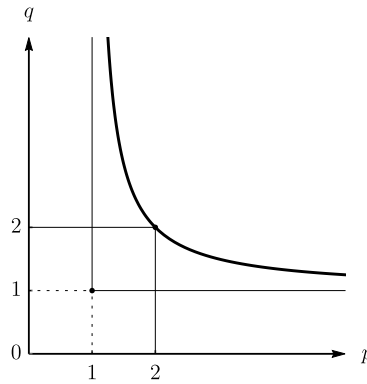


Figure 1.1: Dual exponents for the  $L^p$  spaces,  $\frac{1}{p} + \frac{1}{q} = 1$ .

If  $p = 1, q = \infty$ , and we have:

$$\left| \int_X fg d\mu \right| \leq \left( \int_X |f| d\mu \right) \text{ess sup}_{x \in X} |g(x)|; \tag{1.3}$$

where

$$\|g\|_\infty := \text{ess sup } |g|$$

denotes essential supremum, i.e., neglecting sets of  $\mu$ -measure zero.

The following result is basic in the subject.

**Theorem 1.8** (Hahn-Banach). *Let  $E$  be a Banach space, and let  $x \in E \setminus \{0\}$ , then there is a  $\varphi \in E^*$  such that  $\varphi(x) = \|x\|$ , and  $\|\varphi\|_{E^*} = 1$ .*

*Remark 1.9.* In Examples 1.5 and 1.6 above, i.e.,  $l^p$  and  $L^p(\mu)$ , it is possible to identify the needed elements in  $E^*$ . But the power of Theorem 1.8 is that it yields existence for all Banach spaces, i.e., when  $E$  is given only by the axioms from Definitions 1.1-1.2.

**Definition 1.10.** The weak-\* topology on  $E^*$  is the weakest topology which makes all the linear functionals

$$E^* \ni l \longrightarrow l(x) \in \mathbb{C}$$

continuous, as  $x$  ranges over  $E$ .

**Exercise 1.11** (weak-\* neighborhoods). Show that the neighborhoods of 0 in  $E^*$  have a basis of open sets  $\mathcal{N}$  indexed as follows:

Let  $\epsilon \in \mathbb{R}_+$ ,  $n \in \mathbb{N}$ , and  $x_1, \dots, x_n \in E$ , and set

$$\mathcal{N}_{\epsilon, x_1, \dots, x_n} := \{l \in E^* : |l(x_i)| < \epsilon, i = 1, \dots, n\}.$$

Terminology. The subsets of  $E^*$  in Exercise 1.11 are often called cylinder sets. They form a basis for the weak-\* topology. They also generate a sigma algebra of subsets of  $E^*$ , often called the cylinder sigma algebra. We will be using it in Sections 6.1 (pg. 213), 6.2 (pg. 216), and 11.1 (pg. 337) below.

**Exercise 1.12** (weak-\* vs norm). Let  $1 < p \leq \infty$  be fixed. Set  $l^p = l^p(\mathbb{N})$ , and show that  $\{x \in l^p : \|x\|_{l^p} \leq 1\}$  is weak-\* compact, but not norm-compact.

Hint: By weak-\*, we refer to  $l^p = (l^q)^*$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Exercise 1.13** (Be careful with weak-\* limits.). Settings as in the previous exercise, but now with  $p = 2$ . Let  $\{e_k\}_{k \in \mathbb{N}}$  be the standard ONB in  $l^2$ , i.e.,

$$e_k(i) = \delta_{i,k}, \forall i, k \in \mathbb{N}. \quad (1.4)$$

Show that 0 in  $l^2$  is a weak \*-limit of the sequence  $\{e_k\}_{k \in \mathbb{N}}$ . Conclude that  $\{x \in l^2 : \|x\|_2 = 1\}$  is not weak-\* closed.

Hint: By Parseval, we have, for all  $x \in l^2$ ,

$$\|x\|_2^2 = \sum_{k \in \mathbb{N}} |\langle e_k, x \rangle_2|^2,$$

so  $\lim_{k \rightarrow \infty} \langle e_k, x \rangle_2 = 0$ .

## Duality and Measures

**Definition 1.14.** Let  $E_i$ ,  $i = 1, 2$ , be Banach spaces, and let  $T : E_1 \rightarrow E_2$  be a linear mapping. We say that  $T$  is bounded (continuous) iff (Def.)  $\exists C < \infty$ , such that

$$\|Tx\|_2 \leq C \|x\|_1, \forall x \in E_1. \quad (1.5)$$

**Definition 1.15.** Define  $T^* : E_2^* \rightarrow E_1^*$  by

$$(T^* \varphi_2)(x) = \varphi_2(Tx), \quad \forall x \in E_1, \quad \forall \varphi_2 \in E_2^*. \quad (1.6)$$

We shall adopt the following equivalent notation:

$$\langle T^* \varphi_2, x \rangle = \langle \varphi_2, Tx \rangle, \quad \forall x \in E_1, \quad \forall \varphi_2 \in E_2^*. \quad (1.7)$$

(Here  $E^*$  denotes “dual Banach space.”) It is immediate that (1.5) implies

$$\|T^* \varphi_2\|_* \leq C \|\varphi_2\|_*, \quad \forall \varphi_2 \in E_2^*. \quad (1.8)$$

**Application.** Let  $\Omega_k, k = 1, 2$ , be compact spaces, and let  $\Psi : \Omega_2 \rightarrow \Omega_1$ , be a continuous function. Set

$$Tf = f \circ \Psi, \quad \forall f \in C(\Omega_1). \quad (1.9)$$

Recall the dual Banach spaces:

$$C(\Omega_k)^* = \text{the respective signed measures on } \Omega_k \quad (1.10)$$

of bounded variation,  $k = 1, 2$ ;

$$\|\mu\|_* = |\mu|(\Omega) \quad (= \text{variation of } \mu) \quad (1.11)$$

$$= \sup \sum_i |\mu(E_i)|, \quad (1.12)$$

where  $\{E_i \mid E_i \in \mathcal{B}(\Omega)\}$  in (1.12) runs over all partitions of  $\Omega$ .

**Exercise 1.16** (Transformation of measures). Apply (1.7)-(1.8) to show that

$$(T^* \mu_2)(E) = \mu_2(\Psi^{-1}(E)), \quad \forall E \in \mathcal{B}(\Omega_1),$$

or stated equivalently

$$\int_{\Omega_1} f d\mu(T^* \mu_2) = \int_{\Omega_2} (f \circ \Psi) d\mu_2, \quad \forall f \in C(\Omega_1), \quad \forall \mu_2 \in C(\Omega_2)^*.$$

See Figures 1.3 and 1.4 below.

*Remark 1.17.* We shall make use of the following special case of *pull-back of measures*. It underlies the notion of “the distribution of a *random variable* (math lingo, a measurable function)” from statistics. See Figures 1.2 and 1.3. We shall make use of it below, both in the case of a single random variable, or an indexed family (called a *stochastic process*.)

**Definition 1.18.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a *probability space*:

$\Omega$ : a set, called “the sample space”.

$\mathcal{F}$ : a sigma-algebra of subsets of  $\Omega$ . Elements in  $\mathcal{F}$  are called events.

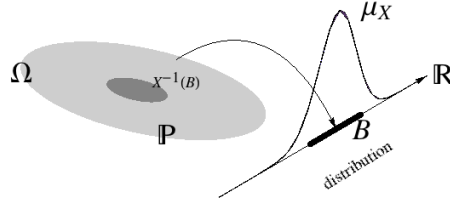


Figure 1.2: A measurement  $X$ ;  $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ . A random variable and its distribution.

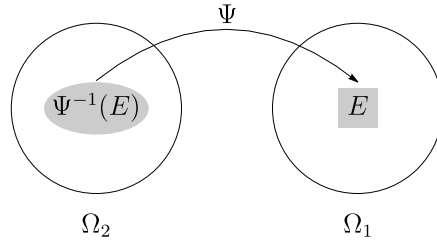


Figure 1.3:  $\Psi^{-1}(E) = \{\omega \in \Omega_2 : \Psi(\omega) \in E\}$ , pull-back.

$\mathbb{P}$ : a probability measure defined on  $\mathcal{F}$ , so  $\mathbb{P}$  is positive, sigma-additive, and  $\mathbb{P}(\Omega) = 1$ .

We say a function  $X : \Omega \rightarrow \mathbb{R}$  is a *random variable* iff (Def.) the following implication holds:

$$B \in \mathcal{B}(\mathbb{R}) \implies X^{-1}(B) \in \mathcal{F}; \text{ see Fig 1.2.} \tag{1.13}$$

So if  $X$  is a fixed random variable, there is an induced measure  $\mu_X$  on  $\mathbb{R}$ , a positive Borel measure. It is the pull-back via  $X$ , i.e.,

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)), \forall B \in \mathcal{B}(\mathbb{R}). \tag{1.14}$$

If  $\mu_X$  is Gaussian, see Section 1.7, we say that  $X$  is a *Gaussian random variable*. If  $\mu_X$  is uniform, we say that  $X$  is uniformly distributed; and similarly for the other probability distributions on  $\mathbb{R}$ ; see Table 6.1 in Section 2.1 below.

### Other Spaces in Duality

Below we consider three spaces of functions on  $\mathbb{R}$ , and their duals. These are basics of the L. Schwartz' theory of distributions:

$\mathcal{D} := C_c^\infty(\mathbb{R}) =$  all  $C^\infty$ -functions on  $\mathbb{R}$  having compact support;

$\mathcal{S} := \mathcal{S}(\mathbb{R}) =$  all  $C^\infty$ -functions on  $\mathbb{R}$  such that  $x^k f^{(n)} \in L^2(\mathbb{R})$  for all  $k, n \in \mathbb{N}$ ;

$\longrightarrow$	compact spaces	$\Omega_2 \xrightarrow{\Psi} \Omega_1$
$\longleftarrow$	Banach spaces	$C(\Omega_2) \xleftarrow{T} C(\Omega_1)$
$\longrightarrow$	duals: measures	$\mathcal{M}(\Omega_2) \xrightarrow{T^*} \mathcal{M}(\Omega_1)$

Figure 1.4: Contra-variance (from point transformations, to transformation of functions, to transformation of measures).

$\mathcal{E} := C^\infty(\mathbb{R}) =$  all  $C^\infty$ -functions on  $\mathbb{R}$  (without support restriction).

Each of the three spaces of test functions  $\mathcal{D}$ ,  $\mathcal{S}$ , and  $\mathcal{E}$  have countable families of seminorms, turning them into topological vector spaces (TVS). The two,  $\mathcal{S}$  and  $\mathcal{E}$  are Fréchet spaces, while  $\mathcal{D}$  is an inductive limit of Fréchet spaces (abbreviated LF.)

For  $\mathcal{S}$ , the seminorms are the max of the absolute value of the above listed functions, so indexed by  $k$  and  $n$ . For the other two,  $\mathcal{E}$ , and  $\mathcal{D}$ , the seminorms are indexed by a number  $n$  of derivatives, and by compact intervals, say  $[-k, k]$ . For each  $n$  and  $k$ , we max  $|f^{(n)}(x)|$  over  $[-k, k]$ . As TVSs, these three spaces in turn are the building blocks of Schwartz' theory of distributions, see [Sch57] and [Trê06a]. In each case, the dual space will be defined with reference to the respective topologies. See details below.

Clearly,

$$\mathcal{D} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}; \tag{1.15}$$

but all of the three spaces come with a natural system of seminorms turning them into topological vector spaces, and we have the continuous inclusions  $\mathcal{D} \hookrightarrow \mathcal{S}$ , and  $\mathcal{S} \hookrightarrow \mathcal{E}$ .

Hence for the duals, we have

$$\mathcal{E}' \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{D}'; \text{ where} \tag{1.16}$$

$\mathcal{E}' =$  the space of all compactly supported distributions on  $\mathbb{R}$ ;

$\mathcal{S}' =$  the space of all tempered distributions on  $\mathbb{R}$ ; and

$\mathcal{D}' =$  all distributions on  $\mathbb{R}$ .

**Exercise 1.19** (Gelfand triple).

- Using Table 1.1, show that  $L^2(\mathbb{R})$  is contained in  $\mathcal{S}'$  (= tempered distributions.)
- Using self-duality of  $L^2$ , i.e.,  $(L^2)^* \simeq L^2$  (by Riesz), make precise the following double inclusions:

$$\mathcal{S} \hookrightarrow L^2 \hookrightarrow \mathcal{S}' \tag{1.17}$$

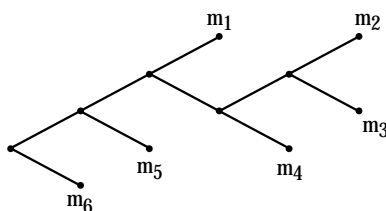


Figure 1.5: Finite Tree (natural order on the set of vertices). Examples of maximal elements:  $m_1, m_2, \dots$

where each inclusion mapping in (1.17) is continuous with respect to the respective topologies; the Fréchet topology on  $\mathcal{S}$ , the norm-topology on  $L^2$ , and the weak-\* (dual) topology on  $\mathcal{S}'$ . (The system (1.17) is an example of a *Gelfand triple*, see Section 9.5.)

### 1.3 Transfinite Induction (Zorn and All That)

Let  $(X, \leq)$  be a *partially ordered set*. By partial ordering, we mean a binary relation “ $\leq$ ” on the set  $X$ , such that (i)  $x \leq x$ ; (ii)  $x \leq y$  and  $y \leq x$  implies  $x = y$ ; and (iii)  $x \leq y$  and  $y \leq z$  implies  $x \leq z$ .

A subset  $C \subset X$  is said to be a *chain*, or *totally ordered*, if  $x, y \in C$  implies that either  $x \leq y$  or  $y \leq x$ . Zorn’s lemma says that if every chain has a majorant then there exists a maximal element in  $X$ .

**Theorem 1.20 (Zorn).** *Let  $(X, \leq)$  be a partially ordered set. If every chain  $C$  in  $X$  has a majorant (upper bound), then there exists an element  $m$  in  $X$  so that  $x \geq m$  implies  $x = m$ .*

An illuminating example of a partially ordered set is the binary tree model (Figs 1.5-1.6). Another example is when  $X$  is a family of subsets of a given set, partially ordered by inclusion.

Zorn’s lemma lies at the foundation of set theory. It is in fact an axiom and is equivalent to the axiom of choice, and to Hausdorff’s maximality principle.

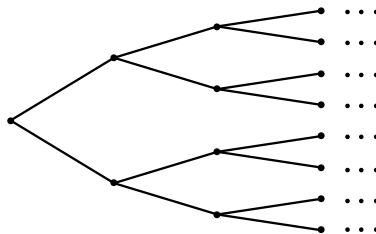
**Theorem 1.21 (Hausdorff Maximality Principle).** *Let  $(X, \leq)$  be a partially ordered set, then there exists a maximal totally ordered subset  $L$  in  $X$ .*

The axiom of choice is equivalent to the following statement on infinite products, which itself is extensively used in functional analysis.

**Theorem 1.22 (axiom of choice).** *Let  $A_\alpha$  be a family of nonempty sets indexed by  $\alpha \in I$ . Then the infinite Cartesian product*

$$\Omega = \prod_{\alpha \in I} A_\alpha = \{ \omega : I \rightarrow \cup_{\alpha \in I} A_\alpha \mid \omega(\alpha) \in A_\alpha \}$$

*is nonempty.*



All finite words in the alphabet  $\{0, 1\}$ , continued indefinitely.

Figure 1.6: Infinite Tree (no maximal element!)

The point of using the axiom of choice is that, if the index set is uncountable, there is no way to verify whether  $(x_\alpha)$  is in  $\Omega$ , or not. It is just impossible to check for each  $\alpha$  that  $x_\alpha$  is contained in  $A_\alpha$ .

*Remark 1.23.* A more dramatic consequence of the axiom of choice is the mind-boggling *Banach-Tarski paradox*; see e.g., [MT13]. It states: For the solid ball  $B$  in 3-dimensional space, there exists a decomposition of  $B$  into a finite number of disjoint subsets, which can then in turn be put back together again, but in a different way which will yield two identical copies of the original ball  $B$ ; – stated informally as: "A pea can be chopped up and reassembled into the Sun." The axiom of choice allows for the construction of nonmeasurable sets, i.e., sets that do not have a volume, and that for their construction would require performing an uncountably infinite number of choices.

In case the set is countable, we simply apply the down to earth standard induction. Note that the standard mathematical induction is equivalent to the Peano's axiom: *Every nonempty subset of the set of natural number has a unique smallest element.* The power of transfinite induction is that it applies to uncountable sets as well.

In applications, the key of using the transfinite induction is to cook up, in a clear way, a partially ordered set, so that the maximal element turns out to be the object to be constructed.

Examples include *Hahn-Banach extension theorem*, *Krein-Milman's theorem* on compact convex set, existence of orthonormal bases in Hilbert space, *Tychonoff's theorem* on infinite Cartesian product of compact spaces (follows immediately from the axiom of choice.)

**Theorem 1.24** (Tychonoff). *Let  $A_\alpha$  be a family of compact sets indexed by  $\alpha \in I$ . Then the infinite Cartesian product  $\prod_\alpha A_\alpha$  is compact with respect to the product topology.*

We will apply transfinite induction (Zorn's lemma) to show that every infinite dimensional Hilbert space has an orthonormal basis (ONB).



## 1.4 Basics of Hilbert Space Theory

Key to functional analysis is the idea of *normed vector spaces*. The interesting ones are infinite-dimensional. To use them effectively in the solution of problems, we must be able to take limits, hence the assumption of completeness. A complete normed linear space is called a *Banach space*. But for applications in physics, statistics, and in engineering it often happens that the norm comes from an *inner product*; – this is the case of *Hilbert space*. With an inner product, one is typically able to get much more precise results, than in the less structured case of Banach space. (Many Banach spaces are not Hilbert spaces.)

The more interesting Hilbert spaces typically arise in concrete applications as infinite-dimensional spaces of function. And as such, they have proved indispensable tools in the study of partial differential equations (PDE), in quantum mechanics, in Fourier analysis, in signal processing, in representations of groups, and in ergodic theory. The term Hilbert space was originally coined by John von Neumann, who identified the axioms that now underlie these diverse applied areas. Examples include spaces of *square-integrable functions* (e.g., the  $L^2$  random variables of a probability space), *Sobolev spaces*, Hilbert spaces of Schwartz distributions, and *Hardy spaces* of holomorphic functions; – to mention just a few.

One reason for their success is that geometric intuition from finite dimensions carries over: e.g., the Pythagorean Theorem, the parallelogram law; and, for optimization problems, the important notion of “*orthogonal projection*.” And the idea (from linear algebra) of diagonalizing a normal matrix; – the *spectral theorem*.

Linear mappings (transformations) between Hilbert spaces are called linear operators, or simply “operators.” They include partial differential operators (PDOs), and many others.

**Definition 1.25.** Let  $X$  be a vector space over  $\mathbb{C}$ .

A norm on  $X$  is a mapping  $\|\cdot\| : X \rightarrow \mathbb{C}$  such that

- $\|cx\| = |c| \|x\|$ ,  $c \in \mathbb{C}$ ,  $x \in X$ ;
- $\|x\| \geq 0$ ;  $\|x\| = 0$  implies  $x = 0$ , for all  $x \in X$ ;
- $\|x + y\| \leq \|x\| + \|y\|$ , for all  $x, y \in X$ .

**Definition 1.26.** Let  $(X, \|\cdot\|)$  be a normed space.  $X$  is called a *Banach space* if it is complete with respect to the induced metric

$$d(x, y) := \|x - y\|, \quad x, y \in X.$$

**Definition 1.27.** Let  $X$  be vector space over  $\mathbb{C}$ . An inner product on  $X$  is a function  $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$  so that for all  $x, y \in \mathcal{H}$ , and  $c \in \mathbb{C}$ , we have

- $\langle x, \cdot \rangle : X \rightarrow \mathbb{C}$  is linear (linearity)
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$  (conjugation)

- $\langle x, x \rangle \geq 0$ ; and  $\langle x, x \rangle = 0$  implies  $x = 0$  (positivity)

*Remark 1.28.* The abstract formulation of Hilbert space was invented by von Neumann in 1925. It fits precisely with the axioms of quantum mechanics (spectral lines, etc.) A few years before von Neumann’s formulation, Max Born had translated Heisenberg’s quantum mechanics into modern mathematics. In 1924, in a break-through paper, Heisenberg had invented quantum mechanics, but he had not been precise about the mathematics. His use of “matrices” was highly intuitive. It was only in the subsequent years, with the axiomatic language of Hilbert space, that the group of physicists and mathematicians around Hilbert in Göttingen were able to give the theory the form it now has in modern textbooks.

**Lemma 1.29** (Cauchy-Schwarz). <sup>1</sup>Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space, then

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle, \quad \forall x, y \in X. \quad (1.18)$$

*Proof.* By the positivity axiom in the definition of an inner product, we see that

$$\sum_{i,j=1}^2 \overline{c_i} c_j \langle x_i, x_j \rangle = \left\langle \sum_{i=1}^2 c_i x_i, \sum_{j=1}^2 c_j x_j \right\rangle \geq 0, \quad \forall c_1, c_2 \in \mathbb{C};$$

i.e., the matrix

$$\begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle \end{bmatrix}$$

is positive definite. Hence the above matrix has nonnegative determinant, and (1.18) follows.  $\square$

**Corollary 1.30.** Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space, then

$$\|x\| := \sqrt{\langle x, x \rangle}, \quad x \in X \quad (1.19)$$

defines a norm.

*Proof.* It suffices to check the triangle inequality (Definition 1.27). For all  $x, y \in X$ , we have (with the use of Lemma 1.29):

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \|x\|^2 + \|y\|^2 + 2\Re\{\langle x, y \rangle\} \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \quad (\text{by (1.18)}) \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

<sup>1</sup>Hermann Amandus Schwarz (1843 - 1921), German mathematician, contemporary of Weierstrass, and known for his work in complex analysis. He is the one in many theorems in books on analytic functions. We will often refer to (1.18) as simply “Schwarz”. The abbreviation is useful because we use it a lot.

There are other two “Schwartz” (with a “t”):

Laurent Schwartz (1915 - 2002), French mathematician, Fields Medal in 1950 for his work of distribution theory.

Jack Schwartz (1930 - 2009), American mathematician, author of the famous book “Linear Operators”.

and the corollary follows.  $\square$

**Definition 1.31.** An inner product space  $(X, \langle \cdot, \cdot \rangle)$  is called a *Hilbert space* if  $X$  is complete with respect to the metric

$$d(x, y) = \|x - y\|, \quad x, y \in X;$$

where the RHS is given by (1.19).

**Exercise 1.32** (Hilbert completion). Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner-product space (Definition 1.27), and let  $\mathcal{H}$  be its *metric completion* with respect to the norm in (1.19). Show that  $\langle \cdot, \cdot \rangle$  on  $X \times X$  extends by limit to a sesquilinear form  $\langle \cdot, \cdot \rangle^\sim$  on  $\mathcal{H} \times \mathcal{H}$ ; and that  $\mathcal{H}$  with  $\langle \cdot, \cdot \rangle^\sim$  is a Hilbert space.

**Exercise 1.33** ( $L^2$  of a measure-space). Let  $(M, \mathcal{B}, \mu)$  be as follows:

$M$  : locally compact Hausdorff space;

$\mathcal{B}$  : the Borel sigma-algebra, i.e., generated by the open subsets of  $M$ ;

$\mu$  : a fixed positive measure defined on  $\mathcal{B}$ .

Let  $\mathcal{F} := \text{span} \{ \chi_E \mid E \in \mathcal{B} \}$ , and on linear combinations, set

$$\left\| \sum_{i: \text{finite}} c_i \chi_{E_i} \right\|_{\mathcal{H}}^2 = \sum_i |c_i|^2 \mu(E_i) \quad (1.20)$$

where  $E_i \in \mathcal{B}$ , and  $E_i \cap E_j = \emptyset$  ( $i \neq j$ ), are assumed.

Show that the Hilbert-completion of  $\mathcal{F}$  with respect to (1.20) agrees with the standard definitions [Rud87, Par82] of the  $L^2(\mu)$ -space.

*Remark 1.34.* An extremely useful method to build Hilbert spaces is the GNS construction. For details, see Chapter 4.

The idea is to start with a positive definite function  $\varphi : X \times X \rightarrow \mathbb{C}$ , defined on an arbitrary set  $X$ . We say  $\varphi$  is *positive definite*, if for all  $n \in \mathbb{N}$ ,

$$\sum_{i, j=1}^n \overline{c_i} c_j \varphi(x_i, x_j) \geq 0 \quad (1.21)$$

for all system of coefficients  $c_1, \dots, c_n \in \mathbb{C}$ , and all  $x_1, \dots, x_n \in X$ .

Given  $\varphi$ , set

$$H_0 := \left\{ \sum_{\text{finite}} c_x \delta_x : x \in X, c_x \in \mathbb{C} \right\} = \text{span}_{\mathbb{C}} \{ \delta_x : x \in X \},$$

and define a sesquilinear form on  $H_0$  by

$$\left\langle \sum c_x \delta_x, \sum d_y \delta_y \right\rangle_{\varphi} := \sum \overline{c_x} d_y \varphi(x, y).$$

Note that

$$\left\| \sum c_x \delta_x \right\|_\varphi^2 := \left\langle \sum c_x \delta_x, \sum c_x \delta_x \right\rangle_\varphi = \sum_{x,y} \overline{c_x} c_y \varphi(x,y) \geq 0$$

by assumption. (All summations are finite.)

However,  $\langle \cdot, \cdot \rangle_\varphi$  is in general not an inner product since the strict positivity axiom may not be satisfied. Hence one has to pass to a quotient space by letting

$$N = \left\{ f \in H_0 \mid \langle f, f \rangle_\varphi = 0 \right\},$$

and set  $\mathcal{H} :=$  completion of the quotient space  $H_0/N$  with respect to  $\|\cdot\|_\varphi$ . (The fact that  $N$  is really a subspace follows from (1.18).)  $\mathcal{H}$  is a Hilbert space.

**Corollary 1.35.** *Let  $X$  be a set, and let  $\varphi : X \times X \rightarrow \mathbb{C}$  be a function. Then  $\varphi$  is positive definite if and only if there is a Hilbert space  $\mathcal{H} = \mathcal{H}_\varphi$ , and a function  $\Phi : X \rightarrow \mathcal{H}$  such that*

$$\varphi(x,y) = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}} \quad (1.22)$$

for all  $(x,y) \in X \times X$ , where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  denotes the inner product in  $\mathcal{H}$ .

Given a solution  $\Phi$  satisfying (1.22), then we say that  $\mathcal{H}$  is minimal if

$$\mathcal{H} = \overline{\text{span}} \{ \Phi(x) : x \in X \}. \quad (1.23)$$

Given two minimal solutions,  $\Phi_i : X \rightarrow \mathcal{H}_i$ ,  $i = 1, 2$  (both satisfying (1.22)); then there is a unitary isomorphism  $\mathcal{U} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that

$$\mathcal{U}\Phi_1(x) = \Phi_2(x)\mathcal{U}, \quad \forall x \in X. \quad (1.24)$$

*Proof.* These conclusions follow from Remark 1.34, and the definitions. (The missing details are left as an exercise to the student.)  $\square$

*Remark 1.36.* It is possible to be more explicit about choice of the pair  $(\Phi, \mathcal{H})$  in Corollary 1.35, where  $\varphi : X \times X \rightarrow \mathbb{C}$  is a given positive definite function. We may in fact choose  $\mathcal{H}$  to be  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  where  $\mathbb{P} = \mathbb{P}_\varphi$  depends on  $\varphi$ , and  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space.

**Example 1.37** (Wiener-measure). In Remark 1.36, take  $X = [0, \infty) = \mathbb{R}_+ \cup \{0\}$ , and set

$$\varphi(s,t) = s \wedge t = \min(s,t),$$

see Figure 1.7. In this case, we may then take  $\Omega = C(\mathbb{R}) =$  all continuous functions on  $\mathbb{R}$ , and  $\Phi_t(\omega) := \omega(t)$ ,  $t \in [0, \infty)$ ,  $\omega \in C(\mathbb{R})$ .

Further, the sigma-algebra in  $C(\mathbb{R})$ ,  $\mathcal{F} := \text{Cyl}$ , is generated by cylinder-sets, and  $\mathbb{P}$  is the *Wiener-measure*; and  $\Phi$  on  $L^2(C(\mathbb{R}), \text{Cyl}, \mathbb{P})$  is the standard *Brownian motion*, i.e.,  $\Phi : [0, \infty) \rightarrow L^2(C(\mathbb{R}), \mathbb{P})$  is a Gaussian process with

$$\mathbb{E}_\mathbb{P}(\Phi(s)\Phi(t)) = \int_{C(\mathbb{R})} \Phi_s(\omega)\Phi_t(\omega) d\mathbb{P}(\omega) = s \wedge t.$$

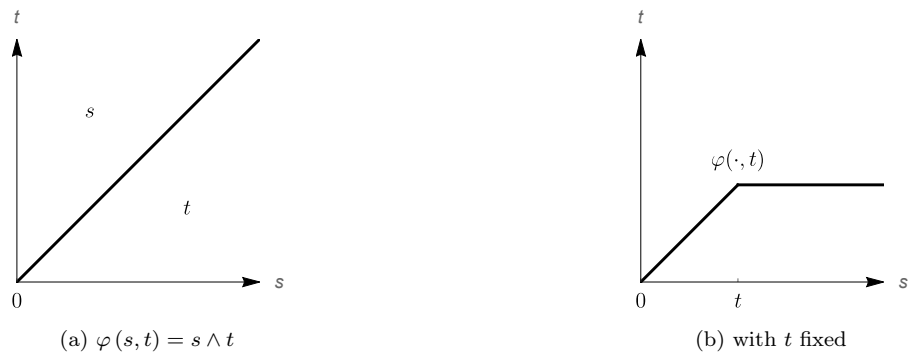


Figure 1.7: Covariance function of Brownian motion.

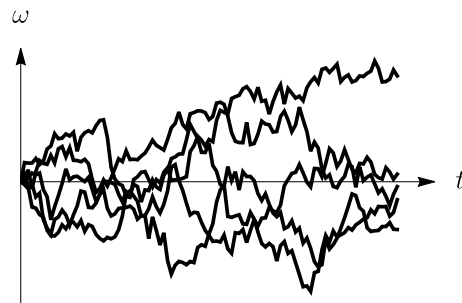


Figure 1.8: A set of Brownian sample-paths generated by a Monte-Carlo computer simulation.

The process  $\{\Phi_t\}$  is called the Brownian motion; its properties include that each  $\Phi_t$  is a Gaussian random variable. We refer to Chapter 6 for full details. Figure 1.8 shows a set of sample path of the standard Brownian motion.

**Exercise 1.38** (Product of two positive definite functions). Let  $\varphi$  and  $\psi$  be positive definite functions  $X \times X \rightarrow \mathbb{C}$  (see (1.21)) and set

$$\xi(x, y) = \varphi(x, y) \psi(x, y), \quad \forall (x, y) \in X \times X.$$

Show that  $\xi = \varphi \cdot \psi$  is again positive definite.

Hint: Use Remark 1.34 and the fact that every positive  $n \times n$  matrix  $B$  has the form  $B = A^*A$ . Fix  $n$ , and  $x_1, \dots, x_n \in X$ . Apply this to  $B_{ij} := \varphi(x_i, x_j)$ .

We resume the discussion of stochastic processes in Chapter 11 below.

*Remark 1.39.* We see from the proof of Lemma 1.29 that the Cauchy-Schwarz inequality holds for all positive definite functions.

**Definition 1.40.** Let  $\mathcal{H}_i$ ,  $i = 1, 2$  be two Hilbert spaces.

A linear operator  $J : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is said to be an *isometry* iff (Def.)

$$\|Jx\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}, \quad \forall x \in \mathcal{H}_1.$$

Note that  $J$  is not assumed “onto.”

**Exercise 1.41** (An Itô-isometry). Let  $T$  be a locally compact Hausdorff space; and let  $\mu$  be a positive Borel measure, i.e., consider the measure space  $(T, \mathcal{B}(T), \mu)$ , where  $\mathcal{B}(T)$  is the Borel sigma-algebra.

1. Show that there is a measure space  $(\Omega, \mathcal{F}, \mathbb{P}(\mu))$  depending on  $\mu$ ; and a function (Gaussian process)

$$\Phi : \mathcal{B}(T) \longrightarrow L^2(\Omega, \mathbb{P}) \quad (1.25)$$

such that every  $\Phi_A$ , for  $A \in \mathcal{B}(T)$ , is a Gaussian random variable, such that  $\mathbb{E}(\Phi_A) = 0$ , and

$$\mathbb{E}(\Phi_A \Phi_B) = \mu(A \cap B) \quad (1.26)$$

holds for all  $A, B \in \mathcal{B}(T)$ . The expectation  $\mathbb{E}$  in (1.26) is with respect to  $\mathbb{P} = \mathbb{P}(\mu)$ .

2. Show that there is an isometry  $J = J_{(It\bar{o})}$  from  $L^2(T, \mu)$  into  $L^2(\Omega, \mathbb{P})$  such that

$$\mathbb{E} \left( \left| \int_T f(t) d\Phi_t \right|^2 \right) = \int_T |f|^2 d\mu \quad (1.27)$$

where the expression  $\int_T f(t) d\Phi_t$  on the LHS in (1.27) is the  $L^2$ -limit of finite sums (simple functions):

$$\sum_i c_i \Phi_{A_i}, \quad (1.28)$$

$c_i \in \mathbb{R}$ , finite indexing; and  $A_i \in \mathcal{B}(T)$ ,  $A_i \cap A_j = \emptyset$ ,  $i \neq j$  (i.e., disjointness.)

Because of (1.27), we can set

$$Jf = \int_T f(t) d\Phi_t \in L^2(\Omega, \mathbb{P}(\mu)). \quad (1.29)$$

Hint:

1. This is an application of Corollary 1.35 (Section 1.4), applied to  $X = \mathcal{B}(T)$ . Note that

$$\mathcal{B}(T) \times \mathcal{B}(T) : (A, B) \mapsto \mu(A \cap B) \quad (1.30)$$

is positive definite. Hence, the existence of the Gaussian process

$$(\Omega, \mathcal{F}, \mathbb{P}, \Phi)$$

subject to the conditions in part (1) of the exercise, follows from Corollary 1.35.

2. Consider the simple functions in (1.28), and use (1.26). Then derive the following:

$$\mathbb{E} \left( \left| \sum_i c_i \Phi_{A_i} \right|^2 \right) = \sum_i |c_i|^2 \mu(A_i) \quad (1.31)$$

Hence, the *Itô-isometry*, defined initially only on simple functions, is isometric. Justify the extension by limits to all of  $L^2(T, \mu)$ . In your last step, taking the limit over partitions, make use of the conclusion from Exercise 1.33.

*Remark 1.42.* The construction in the exercise is an example of a *stochastic process*, and a *stochastic integral*. Both subjects are resumed in Chapters 6 and 11 below.

## Orthonormal Bases

**Definition 1.43.** Let  $\mathcal{H}$  be a Hilbert space. A family of vectors  $\{u_\alpha\}$  in  $\mathcal{H}$  is said to be an orthonormal basis if

1.  $\langle u_\alpha, u_\beta \rangle_{\mathcal{H}} = \delta_{\alpha\beta}$  and
2.  $\overline{\text{span}} \{u_\alpha\} = \mathcal{H}$ . (Here “*span*” means “closure of the linear span.”)

We are now ready to prove the existence of *orthonormal bases* for any Hilbert space. The key idea is to cook up a partially ordered set satisfying all the requirements for transfinite induction, so that each maximal element turns out to be an orthonormal basis (ONB). Notice that all we have at hands are the

abstract axioms of a Hilbert space, and nothing else. Everything will be developed out of these axioms. A separate issue is *constructive* ONBs, for example, wavelets or orthogonal polynomials.

There is a new theory which generalizes the notion of ONB; called “frame”, and it is discussed in Chapter 8 below, along with some more applications.

**Theorem 1.44.** *Every Hilbert space  $\mathcal{H}$  has an orthonormal basis.*

To start out, we need the following lemmas.

**Lemma 1.45.** *Let  $\mathcal{H}$  be a Hilbert space and  $S \subset \mathcal{H}$ . Then the following are equivalent:*

1.  $x \perp S$  implies  $x = 0$
2.  $\overline{\text{span}\{S\}} = \mathcal{H}$

*Proof.* Now we prove Theorem 1.44. If  $\mathcal{H} = \{0\}$  then the proof is done. Otherwise, let  $u_1 \in \mathcal{H}$ . If  $\|u_1\| \neq 1$ , it can be normalized by  $u_1/\|u_1\|$ . Hence we may assume  $\|u_1\| = 1$ . If  $\text{span}\{u_1\} = \mathcal{H}$  the proof finishes again, otherwise there exists  $u_2 \notin \text{span}\{u_1\}$ . By Lemma 1.45, we may assume  $\|u_2\| = 1$  and  $u_1 \perp u_2$ . It follows that there exists a collection  $S$  of orthonormal vectors in  $\mathcal{H}$ .

Let  $\mathbb{P}(S)$  be the set of all orthonormal sets partially ordered by set inclusion. Let  $C \subset \mathbb{P}(S)$  be any chain and let  $M := \bigcup_{E \in C} E$ .  $M$  is clearly a majorant of  $C$ .

In fact,  $M$  is in the partially ordered system. For if  $x, y \in M$ , there exist  $E_x$  and  $E_y$  in  $C$  so that  $x \in E_x$  and  $y \in E_y$ . Since  $C$  is a chain, we may assume  $E_x \leq E_y$ . Hence  $x, y \in E_y$ , and so  $x \perp y$ . This shows that  $M$  is in the partially ordered system and a majorant.

By Zorn’s lemma, there exists a maximal element  $m \in \mathbb{P}(S)$ . It remains to show that the closed span of  $m$  is  $\mathcal{H}$ . Suppose this is false, then by Lemma 1.45 there exists a vector  $x \in \mathcal{H}$  so that  $x \perp \overline{\text{span}\{m\}}$ .

Since  $m \cup \{x\} \geq m$ , and  $m$  is assumed maximal, it follows that  $x \in m$ . This implies  $x \perp x$ . Therefore  $x = 0$ , by the positivity axiom of the definition of Hilbert space.  $\square$

**Corollary 1.46.** *Let  $\mathcal{H}$  be a Hilbert space, then  $\mathcal{H}$  is isomorphic to the  $l^2$  space of the index set of an ONB of  $\mathcal{H}$ . Specifically, given an ONB  $\{u_\alpha\}_{\alpha \in J}$  in  $\mathcal{H}$ , where  $J$  is some index set, then*

$$v = \sum_{\alpha \in J} \langle u_\alpha, v \rangle u_\alpha, \text{ and} \quad (1.32)$$

$$\|v\|^2 = \sum_{\alpha \in J} |\langle u_\alpha, v \rangle|^2, \quad \forall v \in \mathcal{H}. \quad (1.33)$$

Moreover,

$$\langle u, v \rangle = \sum_{\alpha \in J} \langle u, u_\alpha \rangle \langle u_\alpha, v \rangle, \quad \forall u, v \in \mathcal{H}. \quad (1.34)$$



In Dirac's notation (see Section 1.5), (1.32)-(1.34) can be written in the following operator identity:

$$I_{\mathcal{H}} = \sum_{\alpha \in J} |u_{\alpha}\rangle\langle u_{\alpha}|. \quad (1.35)$$

(Note: eq. (1.33) is called the Parseval identity.)

*Proof.* Set  $\mathcal{H}_0 := \text{span}\{u_{\alpha}\}$ . Then, for all  $v \in \mathcal{H}_0$ , we have

$$v = \sum_{\text{finite}} \langle u_{\alpha}, v \rangle u_{\alpha}$$

and

$$\|v\|^2 = \sum_{\text{finite}} |\langle u_{\alpha}, v \rangle|^2.$$

Thus, the map

$$\mathcal{H}_0 \ni v \mapsto \widehat{v} := (\langle u_{\alpha}, v \rangle) \in C_c(J) \quad (1.36)$$

is an isometric isomorphism; where  $C_c$  denotes all the  $l^2$ -sequences indexed by  $J$ , vanishing outside some finite subset of  $J$ .

Since  $\mathcal{H}_0$  is dense in  $\mathcal{H}$ , and  $C_c(J)$  is dense in  $l^2(J)$ , it follows that (1.36) extends to a unitary operator from  $\mathcal{H}$  onto  $l^2(J)$ , see Exercise 1.47. Thus, (1.32)-(1.33) hold.

Using the *polarization identity* (Lemma 3.61) in both  $\mathcal{H}$  and  $l^2(J)$ , we conclude that

$$\begin{aligned} \langle u, v \rangle_{\mathcal{H}} &= \frac{1}{4} \sum_{k=0}^3 i^k \|v + i^k u\|_{\mathcal{H}}^2 \\ &\stackrel{(1.36)}{=} \frac{1}{4} \sum_{k=0}^3 i^k \|\widehat{v} + i^k \widehat{u}\|_{l^2(J)}^2 \\ &= \langle \widehat{u}, \widehat{v} \rangle_{l^2(J)} \\ &= \sum_{j \in J} \langle u, u_{\alpha} \rangle_{\mathcal{H}} \langle u_{\alpha}, v \rangle_{\mathcal{H}}, \quad \forall u, v \in \mathcal{H}, \end{aligned}$$

which is the assertion in (1.34).  $\square$

**Exercise 1.47** (Fischer). Fix an ONB  $\{u_{\alpha}\}_{\alpha \in J}$  as in Corollary 1.46, and set

$$Tv = (\langle u_{\alpha}, v \rangle_{\mathcal{H}})_{\alpha \in J}.$$

Then show that  $T : \mathcal{H} \rightarrow l^2(J)$  is a unitary isomorphism of  $\mathcal{H}$  onto  $l^2(J)$ .

*Remark 1.48.* The correspondence  $\mathcal{H} \longleftrightarrow l^2$  (index set of an ONB) is *functorial*, and an *isomorphism*. Hence, there seems to be just one Hilbert space. But this is misleading, because numerous interesting realizations of an abstract Hilbert space come in when we make a choice of the ONB. The question as to which Hilbert space to use is equivalent to a good choice of an ONB; in  $L^2(\mathbb{R})$ , for example, a wavelet ONB.

**Definition 1.49.** A Hilbert space  $\mathcal{H}$  is said to be *separable* iff (Def.) it has an ONB with cardinality of  $\mathbb{N}$ , (this cardinal is denoted  $\aleph_0$ ).

Many theorems stated first in the separable case also carry over to non-separable; but in the more general cases, there are both surprises, and, in some cases, substantial technical (set-theoretic) complications.

As a result, we shall make the blanket assumption that our Hilbert spaces are separable, (unless stated otherwise.)

**Exercise 1.50** (ONBs and cardinality). Let  $\mathcal{H}$  be a Hilbert space, not necessarily assumed separable, and let  $\{u_\alpha\}_{\alpha \in A}$  and  $\{v_\beta\}_{\beta \in B}$  be two ONBs for  $\mathcal{H}$ .

Show that  $A$  and  $B$  have the same cardinality, i.e., that there is a set-theoretic bijection of  $A$  onto  $B$ .

**Definition 1.51.**

1. Let  $A$  be a set, and  $p : A \rightarrow \mathbb{R}_+$  a function on  $A$ . We say that the sum  $\sum_{\alpha \in A} p(\alpha)$  is well-defined and finite iff (Def.)

$$\sup_{F \subset A, F \text{ finite}} \sum_{\alpha \in F} p(\alpha) < \infty;$$

and we set  $\sum_{\alpha \in A} p(\alpha)$  equal to this supremum.

2. Let  $A$  be a set. By  $l^2(A)$  we mean the set of functions  $f : A \rightarrow \mathbb{C}$ , such that

$$\sum_{\alpha \in A} |f(\alpha)|^2 < \infty.$$

**Exercise 1.52** ( $l^2(A)$ ). Let  $A$  be a set (general; not necessarily countable) then show that  $l^2(A)$  is a Hilbert space.

*Hint:* For  $f, g \in l^2(A)$ , introduce the inner product  $\sum_{\alpha \in A} \overline{f(\alpha)}g(\alpha)$ , by using Cauchy-Schwarz for every finite subset of  $A$ .

**Exercise 1.53** (A functor from sets to Hilbert space). Let  $A$  and  $B$  be sets, and let  $\psi : A \rightarrow B$  be a bijective function, then show that there is an induced unitary isomorphism of  $l^2(A)$  onto  $l^2(B)$ .

**Example 1.54** (Wavelets). Suppose  $\mathcal{H}$  is separable (i.e., having a countable ONB), for instance let  $\mathcal{H} = L^2(\mathbb{R})$ . Then

$$\mathcal{H} \cong l^2(\mathbb{N}) \cong l^2(\mathbb{N} \times \mathbb{N}),$$

and it follows that potentially we could choose a doubly indexed basis

$$\{\psi_{j,k} : j, k \in \mathbb{N}\}$$

for  $L^2(\mathbb{R})$ . It turns out that this is precisely the setting of wavelet basis! What's even better is that in the  $l^2$  space, there are all kinds of diagonalized operators,

which correspond to selfadjoint (or normal) operators in  $L^2$ . Among these operators in  $L^2$ , we single out the following two:

$$\text{scaling: } f(x) \xrightarrow{U_j} 2^{j/2} f(2^j x) \quad (1.37)$$

$$\text{translation: } f(x) \xrightarrow{V_k} f(x - k) \quad (1.38)$$

for all  $j, k \in \mathbb{Z}$ . However,  $U_j$  and  $V_k$  are NOT diagonalized *simultaneously* though. See below for details!

*Remark 1.55.* The two unitary actions  $U_j$  and  $V_k$ ,  $j, k \in \mathbb{Z}$ , in (1.37) and (1.38) satisfy the following important commutation relation:

$$V_k U_j = U_j V_{2^j k}; \quad (1.39)$$

or equivalent:

$$U_j^{-1} V_k U_j = V_{2^j k}. \quad (1.40)$$

Verify details!

**Definition 1.56.** We say a rational number is a *dyadic fraction* or *dyadic rational* if it has the form of  $\frac{a}{2^b}$ , where  $a \in \mathbb{Z}$ , and  $b \in \mathbb{N}$ .

In the language of groups, the pair in (1.39) & (1.40) forms a representation of a *semidirect product*; or, equivalently, of the discrete dyadic  $ax + b$  group (see Section 7.5 for more details): The latter group consists of all  $2 \times 2$  matrices

$$\begin{bmatrix} 2^j & \frac{k}{2^l} \\ 0 & 1 \end{bmatrix}; \quad j, k \in \mathbb{Z}, l \in \mathbb{N}.$$

This group is often referred to as one of the *Baumslag–Solitar groups*; see e.g., [Dud14, DJ08].

## Bounded Operators in Hilbert Space

**Definition 1.57.** A bounded operator in a Hilbert space  $\mathcal{H}$  is a linear mapping  $T : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$\|T\| := \sup \{ \|Tx\|_{\mathcal{H}} : \|x\|_{\mathcal{H}} \leq 1 \} < \infty.$$

We denote by  $\mathcal{B}(\mathcal{H})$  the algebra of all bounded operators in  $\mathcal{H}$ .

Setting  $(ST)(v) = S(T(v))$ ,  $v \in \mathcal{H}$ ,  $S, T \in \mathcal{B}(\mathcal{H})$ , we have

$$\|ST\| \leq \|S\| \|T\|.$$

**Lemma 1.58 (Riesz).** *There is a bijection  $\mathcal{H} \ni h \mapsto l_h$  between  $\mathcal{H}$  and the space of all bounded linear functionals on  $\mathcal{H}$ , where*

$$\begin{aligned} l_h(x) &:= \langle h, x \rangle, \quad \forall x \in \mathcal{H}, \text{ and} \\ \|l_h\| &:= \sup \{ |l(x)| : \|x\|_{\mathcal{H}} \leq 1 \} < \infty. \end{aligned}$$

Moreover,  $\|l_h\| = \|h\|$ .

**Corollary 1.59.** For all  $T \in \mathcal{B}(\mathcal{H})$ , there exists a unique operator  $T^* \in \mathcal{B}(\mathcal{H})$ , called the adjoint of  $T$ , such that

$$\langle x, Ty \rangle = \langle T^*x, y \rangle, \quad \forall x, y \in \mathcal{H};$$

and  $\|T^*\| = \|T\|$ .

*Proof.* Let  $T \in \mathcal{B}(\mathcal{H})$ , then it follows from the Cauchy-Schwarz inequality that

$$|\langle x, Ty \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|.$$

Hence the mapping  $y \mapsto \langle x, Ty \rangle$  is a bounded linear functional on  $\mathcal{H}$ . By Riesz's theorem, there exists a unique  $h_x \in \mathcal{H}$ , such that  $\langle x, Ty \rangle = \langle h_x, y \rangle$ , for all  $y \in \mathcal{H}$ . Set  $T^*x := h_x$ . One checks that  $T^*$  linear, bounded, and in fact  $\|T^*\| = \|T\|$ .  $\square$

**Exercise 1.60** (The  $C^*$  property). Let  $T \in \mathcal{B}(\mathcal{H})$ , then prove

$$\|T^*T\| = \|T\|^2. \quad (1.41)$$

**Definition 1.61.** Let  $T \in \mathcal{B}(\mathcal{H})$ . Then,

- $T$  is *normal* if  $TT^* = T^*T$
- $T$  is *selfadjoint* if  $T = T^*$
- $T$  is *unitary* if  $T^*T = TT^* = I_{\mathcal{H}}$  (= the identity operator)
- $T$  is a (selfadjoint) *projection* if  $T = T^* = T^2$

For  $T \in \mathcal{B}(\mathcal{H})$ , we may write

$$\begin{aligned} R &= \frac{1}{2}(T + T^*) \\ S &= \frac{1}{2i}(T - T^*) \end{aligned}$$

then both  $R$  and  $S$  are selfadjoint, and

$$T = R + iS.$$

This is similar the to decomposition of a complex number into its real and imaginary parts. Notice also that  $T$  is normal if and only if  $R$  and  $S$  commute. (Prove this!) Thus the study of a family of commuting normal operators is equivalent to the study of a family of commuting selfadjoint operators.

**Exercise 1.62** (The group of all unitary operators). Let  $\mathcal{H}$  be a fixed Hilbert space, and denote by  $G_{\mathcal{H}}$  the unitary operators in  $\mathcal{H}$  (see Definition 1.61).

1. Show that  $G_{\mathcal{H}}$  is a group, and that  $T^{-1} = T^*$  for all  $T \in G_{\mathcal{H}}$ .

2. Let  $\{u_\alpha\}_{\alpha \in J}$  be an ONB in  $\mathcal{H}$ , and let  $v_\alpha := T(u_\alpha)$ ,  $\alpha \in J$ ; then show that  $\{v_\alpha\}_{\alpha \in J}$  is also an ONB.
3. Show that, for any pair of ONBs  $\{u_\alpha\}_J, \{w_\alpha\}_J$  with the same index set  $J$ , there is then a unique  $T \in G_{\mathcal{H}}$  such that  $w_\alpha = T(u_\alpha)$ ,  $\alpha \in J$ . We say that  $G_{\mathcal{H}}$  acts transitively on the set of all ONBs in  $\mathcal{H}$ .

**Theorem 1.63.** *Let  $\mathcal{H}$  be a Hilbert space. There is a one-to-one correspondence between selfadjoint projections and closed subspaces of  $\mathcal{H}$  (Figure 1.10),*

$$[\text{Closed subspace } \mathcal{M} \subset \mathcal{H}] \longleftrightarrow \text{Projections.}$$

*Proof.* Let  $P$  be a selfadjoint projection in  $\mathcal{H}$ , i.e.,  $P^2 = P = P^*$ . Then

$$\mathcal{M} = P\mathcal{H} = \{x \in \mathcal{H} : Px = x\}$$

is a closed subspace in  $\mathcal{H}$ . Let  $P^\perp := I - P$  be the complement of  $P$ , so that

$$P^\perp \mathcal{H} = \{x \in \mathcal{H} : P^\perp x = x\} = \{x \in \mathcal{H} : Px = 0\}.$$

Since  $PP^\perp = P(1 - P) = P - P^2 = P - P = 0$ , we have  $P\mathcal{H} \perp P^\perp \mathcal{H}$ .

Conversely, let  $\mathcal{W} \subsetneq \mathcal{H}$  be a closed subspace. Note the following “parallelogram law” holds:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall x, y \in \mathcal{H}; \quad (1.42)$$

see Figure 1.9 for an illustration.

Let  $x \in \mathcal{H} \setminus \mathcal{W}$ , and set

$$d := \inf_{w \in \mathcal{W}} \|x - w\|.$$

The key step in the proof is showing that the infimum is attained; see Figure 1.10.

By definition, there exists a sequence  $\{w_n\}$  in  $\mathcal{W}$  so that  $\|w_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ . Applying (1.42) to  $x - w_n$  and  $x - w_m$ , we get

$$\begin{aligned} & \| (x - w_n) + (x - w_m) \|^2 + \| (x - w_n) - (x - w_m) \|^2 \\ &= 2 \left( \|x - w_n\|^2 + \|x - w_m\|^2 \right); \end{aligned}$$

which simplifies to

$$\begin{aligned} \|w_n - w_m\|^2 &= 2 \left( \|x - w_n\|^2 + \|x - w_m\|^2 \right) - 4 \left\| x - \frac{w_n + w_m}{2} \right\|^2 \\ &\leq 2 \left( \|x - w_n\|^2 + \|x - w_m\|^2 \right) - 4d. \end{aligned} \quad (1.43)$$

Notice here all we require is  $\frac{1}{2}(w_n + w_m) \in \mathcal{W}$ , hence the argument carries over if we simply assume  $\mathcal{W}$  is a closed convex subset in  $\mathcal{H}$ . We conclude from (1.43)

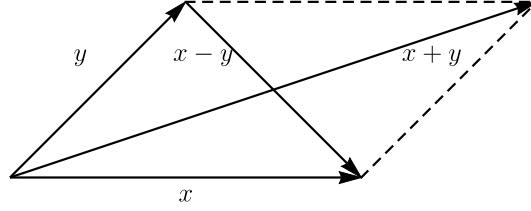


Figure 1.9: The parallelogram law.

that  $\|w_n - w_m\| \rightarrow 0$ , and so  $\{w_n\}$  is a Cauchy sequence. Since  $\mathcal{H}$  is complete, there is a unique limit,

$$Px := \lim_{n \rightarrow \infty} w_n \in \mathcal{W} \quad (1.44)$$

and

$$d = \|x - Px\| \left( = \inf_{w \in \mathcal{W}} \|x - w\| \right). \quad (1.45)$$

See Figure 1.10.

Set  $P^\perp x := x - Px$ . We proceed to verify that  $P^\perp x \in \mathcal{W}^\perp$ . By the minimizing property in (1.45), we have

$$\begin{aligned} \|P^\perp x\|^2 &\leq \|P^\perp x + tw\|^2 \\ &= \|P^\perp x\|^2 + |t|^2 \|w\|^2 + t \langle P^\perp x, w \rangle + \bar{t} \langle w, P^\perp x \rangle \end{aligned} \quad (1.46)$$

for all  $t \in \mathbb{C}$ , and all  $w \in \mathcal{W}$ . Assuming  $w \neq 0$  (the non-trivial case), and setting

$$t = -\frac{\langle w, P^\perp x \rangle}{\|w\|^2}$$

in (1.46), it follows that

$$0 \leq -\frac{|\langle w, P^\perp x \rangle|^2}{\|w\|^2} \implies \langle w, P^\perp x \rangle = 0, \quad \forall w \in \mathcal{W}.$$

This shows that  $P^\perp x \in \mathcal{W}^\perp$ , for all  $x \in \mathcal{H}$ .

For uniqueness, suppose  $P_1$  and  $P_2$  both have the stated properties, then for all  $x \in \mathcal{H}$ , we have

$$x = P_1 x + P_1^\perp x = P_2 x + P_2^\perp x; \text{ i.e.,}$$

$$P_1 x - P_2 x = P_2^\perp x - P_1^\perp x \in \mathcal{W} \cap \mathcal{W}^\perp = \{0\}$$

thus,  $P_1 x = P_2 x, \forall x \in \mathcal{H}$ .

We leave the rest to the reader. See, e.g., [Rud73], [Nel69, p.62].  $\square$

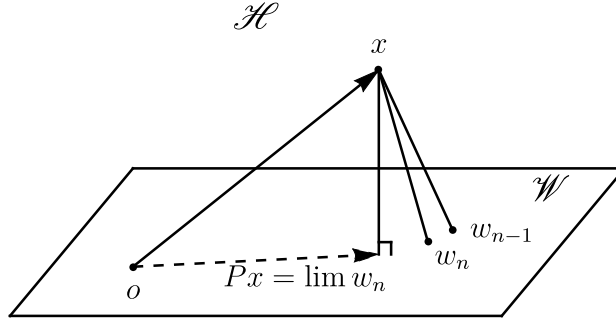


Figure 1.10:  $\|x - Px\| = \inf \{\|x - w\| : w \in \mathcal{W}\}$ . Projection from optimization in Hilbert space.

**Exercise 1.64** (Riesz). As a corollary to Theorem 1.63, prove the following version of Riesz' theorem. Let  $\mathcal{H}$  be a fixed Hilbert space:

For every  $l \in \mathcal{H}^*$ , show that there is a unique  $h (= h_l) \in \mathcal{H}$  such that

$$l(f) = \langle h, f \rangle, \quad \forall f \in \mathcal{H}. \quad (1.47)$$

**Exercise 1.65** (Lax-Milgram [Lax02]). Let  $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  be sesquilinear, and suppose there is a finite constant  $c$  such that

$$|B(h, k)| \leq c \|h\| \|k\|;$$

and  $b > 0$  such that

$$|B(h, h)| \geq b \|h\|^2, \quad \forall h, k \in \mathcal{H}.$$

Then prove that, for every  $h \in \mathcal{H}$ , there is a unique  $k (= k_h) \in \mathcal{H}$  such that

$$\langle h, f \rangle = B(k_h, f), \quad \forall f \in \mathcal{H}. \quad (1.48)$$

*Remark 1.66.* In view of Riesz, Lax-Milgram is an assertion about  $\mathcal{H}^*$ . The Lax-Milgram lemma was proved with view to solving elliptic PDEs, but in Chapter 8 we give an application to frame expansion.

### The Gram-Schmidt Process

Every Hilbert space has an ONB, but it does not mean in practice it is easy to select one that works well for a particular problem. The Gram-Schmidt orthogonalization process was developed a little earlier than von Neumann's formulation of abstract Hilbert space. It is an important tool to get an orthonormal set out of a set of linearly independent vectors.

**Lemma 1.67** (Gram-Schmidt). *Let  $\{u_n\}$  be a sequence of linearly independent vectors in  $\mathcal{H}$ , then there exists a sequence  $\{v_n\}$  of unit vectors so that  $\langle v_n, v_k \rangle = \delta_{n,k}$  and*

$$\text{span} \{u_k\}_{k=1}^n = \text{span} \{v_k\}_{k=1}^n$$

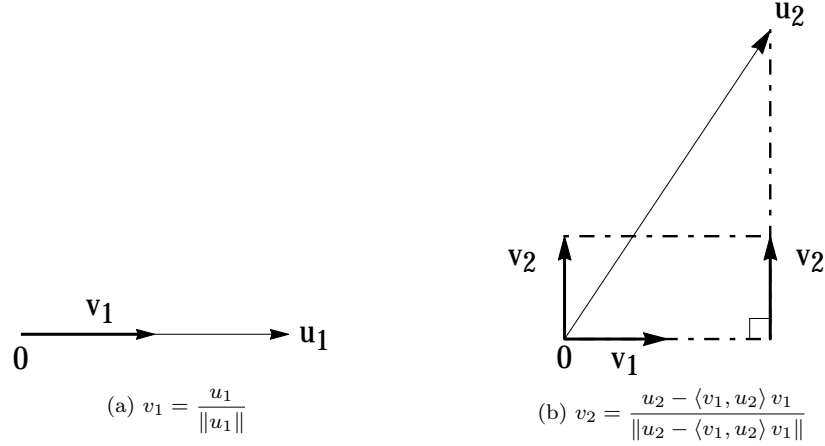


Figure 1.11: The first two steps in G-S.

for all  $n$ ; and therefore,

$$\overline{\text{span}} \{u_k\} = \overline{\text{span}} \{v_k\}.$$

*Proof.* Given  $\{u_n\}$  as in the statement of the lemma, we set

$$\begin{aligned} v_1 &= \frac{u_1}{\|u_1\|}. \\ v_2 &= \frac{u_2 - \langle v_1, u_2 \rangle v_1}{\|u_2 - \langle v_1, u_2 \rangle v_1\|}, \dots \end{aligned}$$

The inductive step: Suppose we have constructed the orthonormal set  $F_n := \{v_1, \dots, v_n\}$ , and let  $P_{F_n}$  be the projection on  $F_n$ . For the induction step, we set

$$v_{n+1} := \frac{u_{n+1} - P_{F_n} u_{n+1}}{\|u_{n+1} - P_{F_n} u_{n+1}\|}, \quad n = 1, 2, \dots \tag{1.49}$$

See Figure 1.11. Note the LHS in (1.49) a unit vector, and orthogonal to  $P_{F_n} \mathcal{H}$ .

The formula for  $P_{F_n}$ , the projection onto the span of  $F_n$ , is

$$P_{F_n} = \sum_{k=1}^n |v_k\rangle \langle v_k|.$$

□

*Remark 1.68.* If  $\mathcal{H}$  is *non-separable*, the standard induction does not work, and the transfinite induction is needed.



**Example 1.69** (Legendre (see Table 1.2)). Let  $\mathcal{H} = L^2(-1, 1)$ . The polynomials  $\{1, x, x^2, \dots\}$  are linearly independent in  $\mathcal{H}$ , for if

$$\sum_{k=1}^n c_k x^k = 0$$

then as an analytic function, the left-hand-side must be identically zero. By Stone-Weierstrass' theorem,  $\text{span}\{1, x, x^2, \dots\}$  is dense in  $C([-1, 1])$  under the  $\|\cdot\|_\infty$  norm. Since  $\|\cdot\|_{L^2} \leq \|\cdot\|_\infty$ , it follows that  $\text{span}\{1, x, x^2, \dots\}$  is also dense in  $\mathcal{H}$ .

By the Gram-Schmidt process, we get a sequence  $\{V_n\}_{n=1}^\infty$  of finite dimensional subspaces in  $\mathcal{H}$ , where  $V_n$  has an orthonormal basis  $\{h_0, \dots, h_n\}$ , so that

$$\begin{aligned} V_n &= \text{span}\{1, x, \dots, x^n\} \\ &= \text{span}\{h_0, h_1, \dots, h_n\}. \end{aligned}$$

Details: Set  $h_0 = \mathbb{1} = \text{constant function}$ , and

$$h_{n+1} := \frac{x^{n+1} - P_n x^{n+1}}{\|x^{n+1} - P_n x^{n+1}\|}, \quad n \in \mathbb{N}.$$

Then the set  $\{h_n : n \in \mathbb{N} \cup \{0\}\}$  is an ONB in  $\mathcal{H}$ . These are the Legendre polynomials, see Table 1.2.

The two important families of orthogonal polynomials on  $(-1, 1)$ , are in Table 1.2 below.

**Definition 1.70.** Let  $\{P_n(x)\}_{n \in \{0\} \cup \mathbb{N}}$  be a sequence of polynomials. We say that the expansion

$$G_P(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n$$

is the corresponding *generating function*.

**Exercise 1.71** (Generating functions), see Table 1.2. Show that the generating functions for the three cases of orthogonal polynomials, *Legendre* (pg.45), *Chebyshev* (pg.45), and *Hermite* (pg.45) are as follows:

$$\begin{aligned} G_L(x, t) &= \frac{1}{\sqrt{1 - 2xt + t^2}}, \\ G_C(x, t) &= \frac{1 - xt}{1 - 2xt + t^2}, \text{ and} \\ G_H(x, t) &= \sum_{\substack{n=0 \\ \text{(modified)}}}^{\infty} P_n^{(H)}(x) \frac{t^n}{n!} = \exp(2xt - t^2). \end{aligned}$$

See [Akh65].

Name	Hilbert space	List
Legendre	$L^2(-1, 1)$ $\ f\ _L^2 = \int_{-1}^1  f(x) ^2 dx, -1 \leq x \leq 1$  orthogonal relation $\int_{-1}^1 P_n(x) P_k(x) dx = \delta_{n,k} \frac{2}{2n+1}$	$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$ $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ $\vdots$
Chebyshev	$L^2\left((-1, 1); \frac{dx}{\sqrt{1-x^2}}\right)$ $\ f\ _C^2 = \int_{-1}^1  f(x) ^2 \frac{dx}{\sqrt{1-x^2}}$ $= \int_0^\pi  f(\theta) ^2 d\theta, x \in [-1, 1]$  orthogonal relation $\int_{-1}^1 \frac{P_n(x) P_k(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & n \neq k \\ \frac{\pi}{2} & n = k \neq 0 \\ \pi & n = k = 0 \end{cases}$	$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$ $P_n(\cos \theta) = \cos(n\theta)$ $P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = 2x^2 - 1$ $P_3(x) = 4x^3 - 3x$ $\vdots$
Hermite	$L^2(\mathbb{R}, e^{-x^2} dx)$ $\ f\ _H^2 = \int_{-\infty}^\infty  f(x) ^2 e^{-x^2} dx, x \in \mathbb{R}$  orthogonal relation $\int_{-\infty}^\infty P_n(x) P_m(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{n,m}$	(physics version) $P_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$ $P_0(x) = 1$ $P_1(x) = 2x$ $p_2(x) = 4x^2 - 2$ $P_3(x) = 8x^3 - 12x$ $\vdots$

Table 1.2: ORTHOGONAL POLYNOMIALS. Legendre, Chebyshev, Hermite

**Exercise 1.72** (Recursive identities). Verify the following recursive identities for the three classes of orthogonal polynomials:

*Legendre:*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

*Chebyshev:*

$$\begin{aligned} P_{n+1}(x) &= 2xP_n(x) - P_{n-1}(x), \\ 2P_m(x)P_n(x) &= P_{m+n}(x) + P_{m-n}(x) \end{aligned}$$

*Hermite:*

$$\begin{aligned} P_{n+1}(x) &= 2xP_n(x) - P'_n(x) \text{ (derivative)} \\ &= 2xP_n(x) - 2nP_{n-1}(x), \\ P_n(x+y) &= 2^{-\frac{n}{2}} \sum_{k=0}^n \binom{n}{k} P_{n-k}(x\sqrt{2})P_k(y\sqrt{2}). \end{aligned}$$

**Exercise 1.73** (Legendre, Chebyshev, Hermite, and Jacobi). see Table 1.2. Find the three  $\infty \times \infty$  Jacobi matrices  $J$  associated with the three systems of polynomials in Table 1.2.

Hint: Before writing down the three respective matrices  $J$ , you must first normalize the polynomials with respect to the respective Hilbert norms. See also [Sho36].

We shall return to the Hermite polynomials, and a corresponding system, the Hermite functions, in Examples 3.10 and 3.11, where they are used in a detailed analysis of the canonical commutation relations, see also Section 1.1 above; as well as the corresponding harmonic oscillator Hamiltonian  $H$ . With the use of raising and lowering operators, we show that the Hermite functions are eigenfunctions for  $H$ , and we derive the spectrum for  $H$  this way.

**Exercise 1.74** (The orthogonality rules). Verify the orthogonality rules contained in Table 1.2.

Hint: You can use direct computations, a clever system of recursions, or Fourier transform (generating function).

**Exercise 1.75** ( $M_x$  in Jacobi form). Let  $J \subset \mathbb{R}$  be an interval (finite, or infinite), and let  $\{p_n(x)\}_{n=0}^\infty$  be a system of polynomial functions on  $J$ ; then show that there is a positive Borel measure  $\mu$  on  $J$ , with infinite support, but moments of all orders, such that  $\{p_n(x)\}_{n=0}^\infty$  is an ONB in  $L^2(J, \mu)$ , if and only if, the multiplication operator  $M_x$  has an  $\infty \times \infty$  matrix representation, with  $\alpha_n \in \mathbb{R}$ ,  $\beta_n \in \mathbb{C}$ , and

$$\begin{aligned} \beta_1 p_1(x) &= (x - \alpha_0) p_0(x) \\ \beta_2 p_2(x) &= (x - \alpha_1) p_1(x) - \overline{\beta_1} p_0(x) \\ &\vdots \\ \beta_{n+1} p_{n+1}(x) &= (x - \alpha_n) p_n(x) - \overline{\beta_n} p_{n-1}(x); \end{aligned}$$



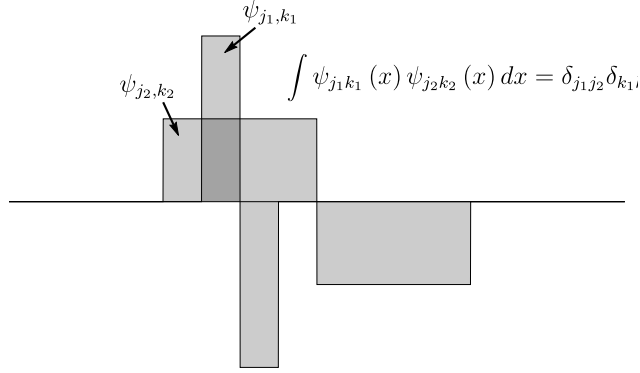


Figure 1.13: Haar wavelet. Scaling properties, resolutions.

*Proof.* Fix  $k$ , if  $j_1 \neq j_2$ , then  $\psi_{j_1, k}$  and  $\psi_{j_2, k}$  are orthogonal since their supports are nested. For fixed  $j$ , and  $k_1 \neq k_2$ , then  $\psi_{j, k_1}$  and  $\psi_{j, k_2}$  have disjoint supports, and so they are also orthogonal (see Figure 1.13).  $\square$

**Exercise 1.79** (The Haar wavelet, and multiplication by  $t$ ). Let  $M := M_t : L^2[0, 1] \rightarrow L^2[0, 1]$  be the standard multiplication operator in  $L^2$  of the unit-interval. Now compute the  $\infty \times \infty$  matrix of  $M$  relative to the orthogonal Haar wavelet basis in Example 1.78.

*Remark 1.80.* Even though  $M_t$  has continuous spectrum  $[0, 1]$ , uniform multiplicity, it is of interest to study the diagonal part in an  $\infty \times \infty$  matrix representation of  $M_t$ . Indeed, in the wavelet ONB in  $L^2(0, 1)$  we get the following  $\infty \times \infty$  matrix representation

$$(M_t)_{(j_1, k_1)(j_2, k_2)} = \int_0^1 \psi_{j_1, k_1}(t) t \psi_{j_2, k_2}(t) dt.$$

The diagonal part  $D$  consists of the sequence

$$D(jk) = \int_0^1 t (\psi_{j, k}(t))^2 dt.$$

Anderson's theorem [And79b] states that  $M_t - D \in \mathcal{K}$  (the compact operators in  $L^2(0, 1)$ .) Indeed, Anderson computed the variance

$$V_{jk} = \int_0^1 t^2 \psi_{j, k}^2(t) dt - \left( \int_0^1 t \psi_{j, k}^2(t) dt \right)^2 = \frac{1}{12} 2^{-2j} \quad (1.53)$$

for all  $j \in \mathbb{N} \cup \{0\}$ , and all  $k \in \{0, 1, 2, \dots, 2^j - 1\}$ .

Generally: It is known that if  $A$  is a selfadjoint operator acting in a separable Hilbert space, then  $A = D + K$ , where  $D$  is a diagonal operator, and  $K$  is a

compact perturbation [VN35]. (In fact, there is even a representation  $A = D + K$ , where  $K$  is a Hilbert-Schmidt operator. See Section 1.5 and Chapter 3.)

Note that the multiplication operator  $M_t$  in Exercise 1.79 is bounded and selfadjoint in  $L^2(0, 1)$ . Different ONBs will yield different diagonal representations  $D$ . But the wavelet basis is of special interest.

For more details on compact perturbation of linear operators in Hilbert space, we refer to [And74].

The conclusion from Anderson is of special interests as the function  $t$  on  $[0, 1]$  is as “nice” as can be, while the functions from the wavelet ONB (1.52) are wiggly, and in fact get increasingly more wiggly as the scaling degree  $j$  in the wavelet ONB tends to infinity. The scaling degree  $j$  is log to the base 2 of the frequency applied to the mother wavelet function (1.51). The conclusion from Anderson is that the variance numbers (1.53) fall off as the inverse square of the frequency.

*Remark 1.81.* It is of interest to ask the analogous questions for other functions than  $t$ , and for other wavelet bases, other than the Haar wavelet basis.

**Exercise 1.82** (A duality). Let  $z$  be a complex number, and  $P$  be a selfadjoint projection. Show that  $U(z) = zP + (I - P)$  is unitary if and only if  $|z| = 1$ .

Hint:  $U(z)U(z)^* = U(z)^*U(z) = |z|^2 P + (I - P)$ , so

$$U(z) \text{ is unitary} \iff |z| = 1.$$

## 1.5 Dirac’s Notation

“There is a great satisfaction in building good tools for other people to use.”

— Freeman Dyson

P.A.M. Dirac was very efficient with notation, and he introduced the “bra-ket” vectors [Dir35, Dir47].

This Dirac formalism has proved extraordinarily efficient, and it is widely used in physics. It deserves to be better known in the math community.

**Definition 1.83.** Let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . We denote by “bra” for vectors  $\langle x|$  and “ket” for vectors  $|y\rangle$ , for  $x, y \in \mathcal{H}$ .

With Dirac’s notation, our first observation is the following lemma.

**Lemma 1.84.** Let  $v \in \mathcal{H}$  be a unit vector. The operator  $x \mapsto \langle v, x \rangle v$  can be written as  $P_v = |v\rangle\langle v|$ , i.e., a “ket-bra” vector. And  $P_v$  is a rank-one selfadjoint projection.

*Proof.* First, we see that

$$P_v^2 = (|v\rangle\langle v|)(|v\rangle\langle v|) = |v\rangle\langle v, v\rangle\langle v| = |v\rangle\langle v| = P_v.$$

Also, if  $x, y \in \mathcal{H}$  then

$$\langle x, P_v y \rangle = \langle x, v \rangle \langle v, y \rangle = \langle \overline{\langle x, v \rangle} v, y \rangle = \langle \langle v, x \rangle v, y \rangle = \langle P_v x, y \rangle$$

so  $P_v = P_v^*$ .  $\square$

**Corollary 1.85.** *Let  $F = \text{span} \{v_i\}$  with  $\{v_i\}$  a finite set of orthonormal vectors in  $\mathcal{H}$ , then*

$$P_F := \sum_{v_i \in F} |v_i\rangle\langle v_i|$$

*is the selfadjoint projection onto  $F$ .*

*Proof.* Indeed, we have

$$\begin{aligned} P_F^2 &= \sum_{v_i, v_j \in F} (|v_i\rangle\langle v_i|)(|v_j\rangle\langle v_j|) = \sum_{v_i \in F} |v_i\rangle\langle v_i| = P_F \\ P_F^* &= \sum_{v_i \in F} (|v_i\rangle\langle v_i|)^* = \sum_{v_i \in F} |v_i\rangle\langle v_i| = P_F, \end{aligned}$$

and we have

$$\begin{aligned} P_F w = w &\iff \sum_F |\langle v_i, w \rangle|^2 = \|w\|^2 \\ &\iff w \in F. \end{aligned} \tag{1.54}$$

Since we may take the limit in (1.54), it follows that the Corollary also holds if  $F$  is infinite-dimensional, i.e.,  $F = \overline{\text{span}} \{v_i\}$ , closure.  $\square$

*Remark 1.86.* More generally, any rank-one operator can be written in Dirac notation as

$$|u\rangle\langle v| : \mathcal{H} \ni x \mapsto \langle v, x \rangle u \in \mathcal{H}.$$

With the bra-ket notation, it is easy to verify that the set of rank-one operators forms an algebra, which easily follows from the fact that

$$(|v_1\rangle\langle v_2|)(|v_3\rangle\langle v_4|) = \langle v_2, v_3 \rangle |v_1\rangle\langle v_4|.$$

The moment that an orthonormal basis is selected, the algebra of operators on  $\mathcal{H}$  will be translated to the algebra of matrices (infinite). See Lemma 1.90.

**Exercise 1.87** (Finite-rank reduction). Let  $\mathcal{H}$  be a Hilbert space. For all  $x, y \in \mathcal{H}$ , let  $|x\rangle\langle y|$  denote the corresponding (Dirac) rank-1 operator.

1. Let  $A, B \in \mathcal{B}(\mathcal{H})$ . Verify that

$$A|x\rangle\langle y| = |Ax\rangle\langle y|, \text{ and} \quad (1.55)$$

$$|x\rangle\langle y|B = |x\rangle\langle B^*y|. \quad (1.56)$$

In particular,  $\mathcal{FR}(\mathcal{H})$  is a two-sided ideal in  $\mathcal{B}(\mathcal{H})$ .

2. For all  $x, y \in \mathcal{H}$ , set

$$w_{x,y}(A) := \langle x, Ay \rangle, \forall A \in \mathcal{B}(\mathcal{H}).$$

For  $\|x\| = 1$ , set  $w_x(A) := \langle x, Ax \rangle$ , i.e., a *pure state* on  $\mathcal{B}(\mathcal{H})$ .  
Let  $\{x_i\}_{i=1}^n \subset \mathcal{H}$ ,  $\|x_i\| = 1$ , and set  $T$  by

$$T = \sum_i |x_i\rangle\langle x_i|. \quad (1.57)$$

Show that then

$$TAT = \sum_i \sum_j w_{x_i, x_j}(A) \underbrace{|x_i\rangle\langle x_j|}_{\text{Dirac-rank-1}}. \quad (1.58)$$

In particular, if  $n = 1$ , and  $T = |x_1\rangle\langle x_1|$ , we have:

$$TAT = w_{x_1}(A)T. \quad (1.59)$$

3. Use part (1), and Theorem 1.95 below, to give a quick proof that the compact operators form an ideal in  $\mathcal{B}(\mathcal{H})$ .

**Exercise 1.88** (Numerical Range and Toeplitz-Hausdorff). The set

$$\{w_x(A) : \|x\| = 1\} \subset \mathbb{C}$$

is called the *numerical range* of  $A$ ,  $NR_A$ . Show that  $NR_A$  is convex.

Hint: Difficult! It is called the *Toeplitz-Hausdorff theorem*; see e.g., [Hal67, Hal64]. (There are few assertions that are true for all bounded operators. The Toeplitz-Hausdorff theorem is one on a short list.)

**Lemma 1.89.** Let  $\{u_\alpha\}_{\alpha \in J}$  be an ONB in  $\mathcal{H}$ , then we may write

$$I_{\mathcal{H}} = \sum_{\alpha \in J} |u_\alpha\rangle\langle u_\alpha|.$$

*Proof.* This is equivalent to the decomposition

$$v = \sum_{\alpha \in J} \langle u_\alpha, v \rangle u_\alpha, \forall v \in \mathcal{H}.$$

□



A selection of ONB makes a representation of the algebra of operators acting on  $\mathcal{H}$  by infinite matrices. We check that, using Dirac's notation, the algebra of operators really becomes the algebra of infinite matrices.

For  $A \in \mathcal{B}(\mathcal{H})$ , and  $\{u_i\}$  an ONB, set

$$(M_A)_{i,j} = \langle u_i, Au_j \rangle_{\mathcal{H}}.$$

Most of the operators we use in the math physics problems are unbounded, so it is a big deal that the conclusion about matrix product is valid for unbounded operators subject to the condition that the chosen ONB is in the domain of such operators.

**Lemma 1.90** (matrix product). *Assume some ONB  $\{u_i\}_{i \in J}$  satisfies  $u_i \in \text{dom}(A^*) \cap \text{dom}(B)$ ; then  $M_{AB} = M_A M_B$ , i.e.,  $(M_{AB})_{ij} = \sum_k (M_A)_{ik} (M_B)_{kj}$ .*

*Proof.* By  $AB$  we mean the operator given by

$$(AB)(u) = A(B(u)).$$

Pick an ONB  $\{u_i\}$  in  $\mathcal{H}$ , and the two operators as stated. We denote by  $M_A = A_{ij} := \langle u_i, Au_j \rangle$  the matrix of  $A$  under the ONB. We compute  $\langle u_i, ABu_j \rangle$ .

$$\begin{aligned} (M_A M_B)_{ij} &= \sum_k A_{ik} B_{kj} = \sum_k \langle u_i, Au_k \rangle \langle u_k, Bu_j \rangle \\ &= \sum_k \langle A^* u_i, u_k \rangle \langle u_k, Bu_j \rangle \\ &= \langle A^* u_i, Bu_j \rangle \quad [\text{by Parseval}] \\ &= \langle u_i, ABu_j \rangle \\ &= (M_{AB})_{ij} \end{aligned}$$

where we used that  $I = \sum |u_i\rangle\langle u_i|$ . □

**Exercise 1.91** (Matrix product of  $\infty \times \infty$  banded matrices). Consider two linear operators  $A$  and  $B$  both defined on a dense subspace  $\mathcal{D}$  in a fixed Hilbert space  $\mathcal{H}$ . Suppose  $\mathcal{D}$  contains an ONB  $\{e_i\}_{i \in \mathbb{N}}$ , and that the corresponding matrices  $M_A$  and  $M_B$  with respect to  $\{e_i\}$  are both banded. Then show that the matrix-product

$$M_{AB} = M_A M_B \tag{1.60}$$

is well defined, and is again banded. See Figure 1.14.

Hint: Use Lemma 1.90, and the equation

$$(M_{AB})_{i,j} = \langle e_i, AB e_j \rangle = \langle A^* e_i, B e_j \rangle,$$

and note that  $\mathcal{D} \subset \text{dom}(A^*)$ .

*Remark 1.92.* Here are two *open questions* regarding *banded operators/matrices*.

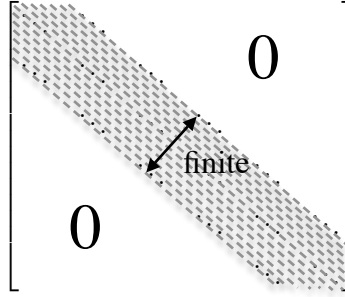


Figure 1.14:  $\infty \times \infty$  banded matrix. Supported on a band around the diagonal.

1. Fix a separable Hilbert space  $\mathcal{H}$ , is there an intrinsic geometric characterization of the linear operators  $A$  (with dense domain) in  $\mathcal{H}$  which admit an ONB  $\{e_i\}_{i \in \mathbb{N}}$  such that the matrix

$$(M_A)_{i,j} := \langle e_i, Ae_j \rangle \tag{1.61}$$

is banded?

2. Given an ONB  $\{e_i\}_{i \in \mathbb{N}}$ , what is the  $*$ -algebra  $\mathfrak{A}$  of unbounded operators with dense domain

$$\mathcal{D} = \text{span} \{e_i\} \tag{1.62}$$

such that every  $A \in \mathfrak{A}$  is banded with respect to  $\{e_i\}_{i \in \mathbb{N}}$ ?

**Exercise 1.93** (A transform). Let  $\mathcal{H}$  be a Hilbert space, and  $A$  a set which indexes a fixed ONB  $\{v_\alpha\}_{\alpha \in A}$ . Now define  $T : \mathcal{H} \rightarrow l^2(A)$ , by

$$(Th)(\alpha) := (\langle v_\alpha, h \rangle), \quad \forall \alpha \in A, h \in \mathcal{H}.$$

Show that  $T$  is unitary, and onto  $l^2(A)$ , i.e.,

$$\begin{aligned} TT^* &= I_{l^2(A)}, \text{ and} \\ T^*T &= I_{\mathcal{H}}. \end{aligned}$$

$T$  is called the analysis transformation, and  $T^*$  the synthesis transformation.

### Three Norm-Completions

Let  $\mathcal{H}$  be a fixed Hilbert space, infinite-dimensional in the discussion below.

Let

$$\begin{aligned} \mathcal{FR}(\mathcal{H}) &= \{\text{all finite-rank operators } \mathcal{H} \longrightarrow \mathcal{H}\} \\ &= \text{span} \{|v\rangle\langle w| : v, w \in \mathcal{H}\} \end{aligned}$$

where  $|v\rangle\langle w|$  denotes the Dirac ket-bra-operator.

**Definition 1.94.** On  $\mathcal{F}R(\mathcal{H})$  we introduce the following three norms: the *uniform norm* (UN), the *trace-norm* (TN), and the *Hilbert-Schmidt* norm as follows:

- (UN) For all  $T \in \mathcal{F}R(\mathcal{H})$ , set

$$\|T\|_{UN} := \sup \{\|Tv\| : \|v\| = 1\}.$$

- (TN) Set

$$\|T\|_{TN} := \text{trace} \left( \sqrt{T^*T} \right);$$

- (HSN) Set

$$\|T\|_{HSN} = (\text{trace}(T^*T))^{\frac{1}{2}}.$$

**Theorem 1.95.** *The completion of  $\mathcal{F}R(\mathcal{H})$  with respect to  $\|\cdot\|_{UN}$ ,  $\|\cdot\|_{TN}$ , and  $\|\cdot\|_{HSN}$  are respectively the compact operators, the trace-class operators, and the Hilbert-Schmidt operators. (See Definition 1.94 above.)*

We will prove, as a consequence of the Spectral Theorem (Chapter 3), that the  $\|\cdot\|_{UN}$ -completion agrees with the usual definition of the compact operators.

*Remark 1.96.* Note that Theorem 1.95 is for Hilbert space, and it is natural to ask “what carries over to Banach space?” Not everything by a theorem of Per Enflo [Enf73]. In detail: The assertion in the first part of Theorem 1.95 (Hilbert space) is that every compact operator is the norm limit of finite-rank operators; referring to the uniform norm (UN) in Definition 1.94. But there are Banach spaces where this is false; – although the easy implication is true, i.e., that operators in the norm closure of finite-rank operators are compact.

**Definition 1.97.** Let  $T \in \mathcal{B}(\mathcal{H})$ , then we say that  $T$  is compact iff (Def.)  $T(\mathcal{H}_1)$  is relatively compact in  $\mathcal{H}$ , where  $\mathcal{H}_1 := \{v \in \mathcal{H} \mid \|v\| \leq 1\}$ .

A similar remark applies to the other two Banach spaces of operators. The Hilbert-Schmidt operators forms a Hilbert space.

**Exercise 1.98** (The identity-operator in infinite dimension). If  $\dim \mathcal{H} = \infty$ , show that the identity operator  $I_{\mathcal{H}}$  is *not* compact.

Hint: Use an ONB.

**Definition 1.99.** The following is useful in working through the arguments above.

**Lemma 1.100.** *Let  $v, w \in \mathcal{H}$ , then*

$$\text{trace}(|v\rangle\langle w|) = \langle v, w \rangle_{\mathcal{H}}.$$

*Proof.* Introduce an ONB  $\{u_{\alpha}\}$  in  $\mathcal{H}$ , and compute:

$$\begin{aligned} \text{trace}(|v\rangle\langle w|) &= \sum_{\alpha} \langle u_{\alpha}, |v\rangle\langle w| u_{\alpha} \rangle_{\mathcal{H}} \\ &= \sum_{\alpha} \langle u_{\alpha}, v \rangle_{\mathcal{H}} \langle w, u_{\alpha} \rangle_{\mathcal{H}} \\ &= \langle v, w \rangle_{\mathcal{H}} \end{aligned}$$

where we used Parseval in the last step of the computation.  $\square$

**Exercise 1.101** (Comparing norms). Let  $T \in \mathcal{F}R(\mathcal{H})$ , and let the three norms be as in Definition 1.94. Then show that

$$\|T\|_{UN} \leq \|T\|_{HSN} \leq \|T\|_{TN}; \quad (1.63)$$

and conclude the following contractive inclusions:

$$\begin{aligned} \{\text{Trace-class operators}\} &\subset \{\text{Hilbert-Schmidt operators}\} \\ &\subset \{\text{compact operators}\}. \end{aligned}$$

*Remark 1.102.* In the literature, the following notation is often used for the three norms in Definition 1.94:

$$\begin{cases} \|T\|_{UN} = \|T\|_{\infty} \\ \|T\|_{TN} = \|T\|_1 \\ \|T\|_{HSN} = \|T\|_2 \end{cases} \quad (1.64)$$

Note that if  $T$  is a diagonal operator,  $T = \sum_k x_k |u_k\rangle\langle u_k|$  in some ONB  $\{u_k\}$ , then the respective norms are  $\|x\|_{\infty}$ ,  $\|x\|_1$ , and  $\|x\|_2$ .

With the notation in (1.64), the inequalities (1.63) now take the form

$$\|T\|_{\infty} \leq \|T\|_2 \leq \|T\|_1, \quad T \in \mathcal{F}R(\mathcal{H}).$$

**Exercise 1.103** (Matrix entries in infinite dimensions). Let  $\mathcal{H}$  be a separable Hilbert space. Pick an ONB  $\{e_j\}_{j \in J}$ , and set  $E_{ij} := |e_i\rangle\langle e_j|$ ,  $(i, j) \in J^2$ .

1. Show that this is an ONB in  $\mathcal{H}S(\mathcal{H})$  (= Hilbert Schmidt operators) with respect to the inner product

$$\langle A, B \rangle_{\mathcal{H}S} := \text{trace}(A^*B), \quad A, B \in \mathcal{H}S(\mathcal{H}). \quad (1.65)$$

2. Show that the corresponding orthogonal expansion for  $A \in \mathcal{H}S(\mathcal{H})$  is

$$A = \sum_{(i,j) \in J^2} \langle e_i, Ae_j \rangle_{\mathcal{H}} E_{ij}. \quad (1.66)$$

**Exercise 1.104** (The three steps). Let  $\mathcal{H}$  be a separable Hilbert space, and let

1.  $\mathcal{B}(\mathcal{H})$ : all bounded operators in  $\mathcal{H}$ ; with the uniform norm.
2.  $\mathcal{T}_1(\mathcal{H})$ : all trace-class operators, with the trace-norm; see Definition 1.94.

Use the three steps from Section 1.2 to show that

$$(\mathcal{T}_1(\mathcal{H}))^* = \mathcal{B}(\mathcal{H});$$

i.e., that  $\mathcal{B}(\mathcal{H})$  is the dual Banach space where the respective norms are specified as in (1)-(2). (For more about this duality, see also Theorem 4.55.)

## Connection to Quantum Mechanics

One of the powerful applications of the theory of operators in Hilbert space, and more generally of functional analysis, is in quantum mechanics (QM) [Pol02, PK88, CP82]. Even the very formulation of the central questions in QM entails the mathematics of *unbounded selfadjoint operators*, of projection valued measures (PVM), and *unitary one-parameter groups* of operators. The latter usually abbreviated to “unitary one-parameter groups.”

By contrast to what holds for the more familiar case of bounded operators, we stress that for unbounded selfadjoint operators, mathematical precision necessitates making a sharp distinction between the following three notions: *selfadjoint*, *essentially selfadjoint*, and *formally selfadjoint* (also called Hermitian, or symmetric). See [RS75, Nel69, vN32a, DS88c]. We define them below; see especially the appendix at the end of Chapter 2. One reason for this distinction is that quantum mechanical observables, momentum  $P$ , position  $Q$ , energy etc, become selfadjoint operators in the axiomatic language of QM. What makes it even more subtle is that these operators are both unbounded and non-commuting (take the case of  $P$  and  $Q$  which was pair from Heisenberg’s pioneering paper on uncertainty.) Another subtle point entails the relationship between selfadjoint operators, projection-valued measures, and unitary one-parameter groups (as used in the dynamical description of states in QM, i.e., describing the solution of the wave equation of Schrödinger.) Unitary one-parameter groups are also used in the study of other partial differential equations, especially hyperbolic PDEs.

The discussion which follows below will make reference to this setting from QM, and it serves as motivation. However the more systematic mathematical presentation of selfadjoint operators, projection-valued measures, and unitary one-parameter groups will be postponed to later in the book. We first need to develop a number of technical tools. However we have included an outline of the bigger picture in the appendix (Stone’s Theorem), to the present chapter. Stone’s theorem shows that the following three notions, (i) selfadjoint operator, (ii) projection-valued measure, and (iii) unitary one-parameter group, are incarnations of one and the same; i.e., when one of the three is known, any one of the other two can be computed from it.

We emphasize that there is a host of other applications of this, for example to harmonic analysis, to statistics, and to PDE. These will also play an important role in later chapters.

Much of the motivation for the axiomatic approach to the theory of linear operators in Hilbert space dates back to the early days of quantum mechanics (Planck, Heisenberg, and Schrödinger), but in the form suggested by J. von Neumann. (von Neumann’s formulation is the one now adopted by most books on functional analysis.) Here we will be brief, as a systematic and historical discussion is far beyond our present scope. Suffice it to mention here that what is known as the “matrix-mechanics” of Heisenberg takes the form infinite by infinite matrices with entries representing, in turn, *transition probabilities*, where “transition” refers to “jumps” between energy levels. See (1.68)-(1.69) below,

and Exercises 1.106-1.107. By contrast to matrices, in Schrödinger's wave mechanics, the Hilbert space represents wave solutions to Schrödinger's equation. Now this entails the study of one-parameter groups of unitary operators in  $\mathcal{H}$ . In modern language, with the two settings we get the dichotomy between the case when the Hilbert space is an  $l^2$  space (i.e., a  $l^2$ -sequence space), vs the case of Schrödinger when  $\mathcal{H}$  is an  $L^2$ -space of functions on phase-space.

In both cases, the observables are represented by families of selfadjoint operators in the respective Hilbert spaces. For the purpose here, we pick the pair of selfadjoint operators representing momentum (denoted  $P$ ) and position (denoted  $Q$ ). In one degree of freedom, we only need a single pair. The canonical commutation relation is

$$PQ - QP = -iI; \text{ or } PQ - QP = -i\hbar I$$

where  $\hbar = \frac{h}{2\pi}$  is Planck's constant, and  $i = \sqrt{-1}$ .

A few years after the pioneering work of Heisenberg and Schrödinger, J. von Neumann and M. Stone proved that the two approaches are *unitarily equivalent*, hence they produce the same "measurements." In modern lingo, the notion of measurement take the form of projection valued measures, which in turn are the key ingredient in the modern formulation of the spectral theorem for selfadjoint, or normal, linear operators in Hilbert space. (See [Sto90, Yos95, Nel69, RS75, DS88c].) Because of dictates from physics, the "interesting" operators, such as  $P$  and  $Q$  are *unbounded*.

The first point we will discuss about the pair of operators  $P$  and  $Q$  is non-commutativity. As is typical in mathematical physics, non-commuting operators will satisfy conditions on the resulting commutators. In the case of  $P$  and  $Q$ , the commutation relation is called the canonical commutation relation; see below. For reference, see [Dir47, Hei69, vN31, vN32c].

Quantum mechanics was born during the years from 1900 to 1933. It was created to explain phenomena in black body radiation, hydrogen atom, where a discrete pattern occurs in the frequencies of waves in the radiation. The radiation energy turns out to be  $E = \nu\hbar$ , with  $\hbar$  being the Plank's constant, and  $\nu$  is frequency. Classical mechanics runs into trouble.

During the years of 1925 and 1926, Heisenberg found a way to represent the energy  $E$  as a matrix (spectrum = energy levels), so that the matrix entries  $\langle v_j, E v_i \rangle$  represent transition probability for transitions from energy level  $i$  to energy level  $j$ . (See Figure 1.17 below.) A fundamental relation in quantum mechanics is the commutation relation satisfied by the momentum operator  $P$  and the position operator  $Q$ , where

$$PQ - QP = -iI, \quad i = \sqrt{-1}. \quad (1.67)$$

Heisenberg represented the operators  $P, Q$  by infinite matrices, although his solution to (1.67) is not really matrices, and not finite matrices.

**Lemma 1.105.** *Eq. (1.67) has no solutions for finite matrices, in fact, not even for bounded operators.*

*Proof.* The reason is that for matrices, there is a trace operation where

$$\text{trace}(AB) = \text{trace}(BA).$$

This implies the trace on the left-hand-side is zero, while the trace on the RHS is not.  $\square$

This shows that there is no finite dimensional solution to the commutation relation above, and one is forced to work with infinite dimensional Hilbert space and operators on it. Notice also that  $P, Q$  do not commute, and the above commutation relation leads to the uncertainty principle (Hilbert, Max Born, von Neumann worked out the mathematics). It states that the statistical variance  $\Delta P$  and  $\Delta Q$  satisfy  $\Delta P \Delta Q \geq \hbar/2$ . We will come back to this later in Exercise 3.57.

We will show that non-commutativity always yields “uncertainty.”

However, Heisenberg [vN31, Hei69] found his “matrix” solutions by tri-diagonal  $\infty \times \infty$  matrices, where

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \quad (1.68)$$

and

$$Q = \frac{1}{i\sqrt{2}} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & \sqrt{2} & & \\ & -\sqrt{2} & 0 & \sqrt{3} & \\ & & -\sqrt{3} & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \quad (1.69)$$

the complex  $i$  in front of  $Q$  is to make it selfadjoint.

**Exercise 1.106** (The canonical commutation relation). Using matrix multiplication for  $\infty \times \infty$  matrices verify directly that the two matrices  $P$  and  $Q$  satisfy  $PQ - QP = -iI$ , where  $I$  is the identity matrix in  $l^2(\mathbb{N}_0)$ , i.e.,  $(I)_{ij} = \delta_{ij}$ . Hint: use the rules in Lemma 1.90.

**Exercise 1.107** (Raising and lowering operators (non-commutative complex variables)). Set

$$A_{\mp} := P \pm iQ; \quad (1.70)$$

and show that the matrix representation for these operators is as follows:

$$A_- = \sqrt{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & \sqrt{2} & 0 & 0 & \cdots \\ \vdots & \vdots & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 & \sqrt{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

and

$$A_+ = \sqrt{2} \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots & \cdots \\ \vdots & 0 & \sqrt{3} & 0 & 0 & \cdots \\ \vdots & \vdots & 0 & \sqrt{4} & \ddots & \ddots \end{bmatrix}.$$

In other words, the raising operator  $A_+$  is a sub-banded matrix, while the lowering operator  $A_-$  is a super-banded matrix. Both  $A_+$  and  $A_-$  has 0s down the diagonal.

Further, show that

$$A_- A_+ = 2 \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & 0 & \cdots & \cdots & \cdots \\ \vdots & 0 & 3 & 0 & \cdots & \cdots \\ \vdots & \vdots & 0 & 4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

i.e., a diagonal matrix, with the numbers  $\mathbb{N}$  down the diagonal inside  $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ ; and

$$A_+ A_- = 2 \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots & \cdots \\ \vdots & 0 & 2 & 0 & \cdots & \cdots \\ \vdots & \vdots & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 & 4 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix};$$

so that

$$\frac{1}{2} [A_-, A_+] = I.$$

*Remark 1.108* (Raising and lowering in an ONB). In the canonical ONB  $\{e_n\}_{n=0}^\infty$  in  $l^2(\mathbb{N}_0)$ , we have the following representations of the two operators  $A_\pm$  (see



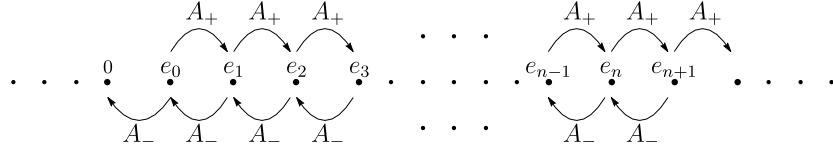


Figure 1.15: The raising and lowering operators  $A_{\pm}$ . The lowering operator  $A_-$  kills  $e_0$ .

(1.70)):

$$\begin{aligned} A_+ e_n &= \sqrt{2} \sqrt{n+1} e_{n+1}; \quad n = 0, 1, 2, \dots, \text{ and} \\ A_- e_n &= \sqrt{2} \sqrt{n} e_{n-1}; \quad n = 1, 2, \dots, \\ A_- e_0 &= 0, \text{ see Fig 1.15.} \end{aligned}$$

The vector  $e_0$  is called the *ground state*, or *vacuum vector*.

*Remark 1.109.* The conclusion of the discussion above is that the Heisenberg commutation relation (1.67) for pairs of selfadjoint operators has two realizations, one in  $L^2(\mathbb{R})$ , and the other in  $l^2(\mathbb{N})$ .

In the first one we have

$$\begin{aligned} (Pf)(x) &= \frac{1}{i} \frac{d}{dx} f \\ (Qf)(x) &= xf(x) \end{aligned}$$

for all  $f \in \mathcal{S} \subset L^2(\mathbb{R})$ , where  $\mathcal{S}$  denotes the Schwartz test-function subspace in  $L^2(\mathbb{R})$ .

The second realization is by  $\infty \times \infty$  matrices, and it is given in detail above. In Section 7.5 we shall return to the first realization.

The *Stone-von Neumann uniqueness theorem* (see [vN32b, vN31]) implies the two solutions are unitarily equivalent; see Chapter 7.

**Exercise 1.110** (Infinite banded matrices). Give an example of two sequences

$$d_1, d_2, \dots \in \mathbb{R}, \quad a_1, a_2, \dots \in \mathbb{C}$$

such that the corresponding Hermitian symmetric  $\infty \times \infty$  tri-diagonal (banded) matrix  $A$  in Figure 1.16 satisfies  $A \subset A^*$ , but  $\overline{A} \neq A^*$ , i.e.,  $A$  is *not* essentially selfadjoint when realized as a Hermitian operator in  $l^2$ .

*Remark 1.111* (Matrices vs operators). Every bounded linear operator (and many unbounded operators too) in separable Hilbert space (and, in particular, in  $l^2$ ) can be realized as a well-defined *infinite “square” matrix*. In  $l^2$  we pick the canonical ONB, but in a general Hilbert space, a choice of ONB must be made. We saw that most rules for finite matrices carry over the case of infinite matrices; sums, products, and adjoints.



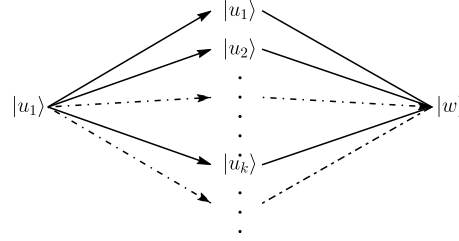


Figure 1.17: Transition of quantum-states

*Remark 1.113.* Note that the Hilbert matrix  $H$  is not banded; in fact every entry in  $H$  is positive. Nonetheless, it follows from an application of Corollary 1.122 (Section 1.7) that the Hilbert matrix  $H$  in Exercise 1.112 is equivalent to a banded matrix  $J$ ; and there is a choice of  $J$  to be tri-diagonal; a Jacobi matrix; see Figure 1.16. Since  $H$  yields a bounded selfadjoint operator in  $l^2$ , it follows from Corollary 1.122 that  $J$  is in fact a bounded Jacobi-matrix with the same norm as  $H$ .

### Probabilistic Interpretation of Parseval in Hilbert Space

**Case 1.** Let  $\mathcal{H}$  be a complex Hilbert space, and let  $\{u_k\}_{k \in \mathbb{N}}$  be an ONB, then Parseval’s formula reads:

$$\langle v, w \rangle_{\mathcal{H}} = \sum_{k \in \mathbb{N}} \langle v, u_k \rangle \langle u_k, w \rangle, \quad \forall v, w \in \mathcal{H}. \quad (1.71)$$

Translating this into a statement about “transition probabilities” for quantum states,  $v, w \in \mathcal{H}$ , with  $\|v\|_{\mathcal{H}} = \|w\|_{\mathcal{H}} = 1$ , we get

$$\text{Prob}(v \rightarrow w) = \sum_{k \in \mathbb{N}} \text{Prob}(v \rightarrow u_k) \text{Prob}(u_k \rightarrow w). \quad (1.72)$$

See Figure 1.17. The states  $v$  and  $w$  are said to be *uncorrelated* iff (Def.) they are orthogonal.

Fix a state  $w \in \mathcal{H}$ , then

$$\|w\|^2 = \sum_{k \in \mathbb{N}} |\langle u_k, w \rangle|^2 = 1.$$

The numbers  $|\langle u_k, w \rangle|^2$  represent a probability distribution over the index set, where  $|\langle u_k, w \rangle|^2$  is the probability that the quantum system is in the state  $|u_k\rangle$ .

Disclaimer: The notation “transition-probability” in (1.72) and Figure 1.17 is a stretch since the inner products  $\langle v, u_k \rangle$  are not positive. Nonetheless, it is justified by

$$\sum_k |\langle v, u_k \rangle|^2 = 1$$

when  $v \in \mathcal{H}$  (is a state vector).

**Case 2.** If  $P : \mathcal{B}(\mathbb{R}) \rightarrow \text{Proj}(\mathcal{H})$  is a projection valued measure (Appendix 2.A), we get the analogous assertions, but with integration, as opposed to summation. In this case eq. (1.71) holds the following form:

$$\langle v, w \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle v, P(d\lambda) w \rangle_{\mathcal{H}}; \quad (1.73)$$

and for  $v = w$ , it reads:

$$\|v\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} \|P(d\lambda) v\|_{\mathcal{H}}^2. \quad (1.74)$$

Recall the other axioms of  $P(\cdot)$  are:

1.  $P(A) = P(A)^* = P(A)^2, \forall A \in \mathcal{B}(\mathbb{R})$ .
2.  $P(\cdot)$  is countably additive on the Borel subsets of  $\mathbb{R}$ , i.e.,

$$\sum_j P(A_j) = P(\cup_j A_j)$$

where  $A_j \in \mathcal{B}(\mathbb{R}), A_i \cap A_j = \emptyset, i \neq j$ .

3.  $P(A \cap B) = P(A)P(B), \forall A, B \in \mathcal{B}(\mathbb{R})$ .

## 1.6 The Lattice Structure of Projections

A lattice is a partially ordered set in which every two elements have a supremum (also called a least upper bound or join) and an infimum (also called a greatest lower bound or meet).

The purpose of the discussion below is twofold; one to identify two cases: (i) the (easy) lattice of subsets of a fixed total set; and (ii) the lattice of projections in a fixed Hilbert space. Secondly we point out how non-commutativity of projections makes the comparison of (i) and (ii) subtle; even though there are some intriguing correspondences; see Table 1.4 for illustration.

Notation: In this section we will denote projections  $P, Q$ , etc.

von Neumann invented the notion of abstract Hilbert space in 1928 as shown in one of the earliest papers.<sup>2</sup> His work was greatly motivated by quantum mechanics. In order to express quantum mechanics logic operations, he created lattices of projections, so that everything we do in set theory with set operation has a counterpart in the operations of projections. See Table 1.4.

For example, if  $P$  and  $Q$  are two projections in  $\mathcal{B}(\mathcal{H})$ , then

$$P\mathcal{H} \subset Q\mathcal{H} \quad (1.75)$$

$$\Downarrow$$

$$P = PQ \quad (1.76)$$

$$\Downarrow$$

$$P \leq Q. \quad (1.77)$$

<sup>2</sup>Earlier authors, Schmidt and Hilbert, worked with infinite bases, and  $\infty \times \infty$  matrices.

SETS	CHAR	PROJ	DEF
$A \cap B$	$\chi_A \chi_B$	$P \wedge Q$	$P\mathcal{H} \cap Q\mathcal{H}$
$A \cup B$	$\chi_{A \cup B}$	$P \vee Q$	$\overline{\text{span}}\{P\mathcal{H} \cup Q\mathcal{H}\}$
$A \subset B$	$\chi_A \chi_B = \chi_A$	$P \leq Q$	$P\mathcal{H} \subset Q\mathcal{H}$
$A_1 \subset A_2 \subset \dots$	$\chi_{A_i} \chi_{A_{i+1}} = \chi_{A_i}$	$P_1 \leq P_2 \leq \dots$	$P_i\mathcal{H} \subset P_{i+1}\mathcal{H}$
$\bigcup_{k=1}^{\infty} A_k$	$\chi_{\bigcup_k A_k}$	$\bigvee_{k=1}^{\infty} P_k$	$\overline{\text{span}}\{\bigcup_{k=1}^{\infty} P_k\mathcal{H}\}$
$\bigcap_{k=1}^{\infty} A_k$	$\chi_{\bigcap_k A_k}$	$\bigwedge_{k=1}^{\infty} P_k$	$\bigcap_{k=1}^{\infty} P_k\mathcal{H}$
$A \times B$	$(\chi_{A \times X})(\chi_{X \times B})$	$P \otimes Q$	$P \otimes Q \in \text{proj}(\mathcal{H} \otimes \mathcal{H})$

Table 1.4: Lattice of projections in Hilbert space.

This is similar to the following equivalence relation in set theory

$$A \subset B \text{ (containment of sets)} \quad (1.78)$$

$$\Updownarrow$$

$$A = A \cap B. \quad (1.79)$$

In general, product and sum of projections are not projections. But if  $P\mathcal{H} \subset Q\mathcal{H}$  then the product  $PQ$  is in fact a projection. Taking adjoint in (1.76) yields

$$P^* = (PQ)^* = Q^*P^* = QP.$$

It follows that  $PQ = QP = P$ , i.e., *containment of subspaces implies the corresponding projections commute*.

Two decades before von Neumann developed his Hilbert space theory, Lebesgue developed his integration theory [Leb05] which extends the classical Riemann integral. The monotone sequence of sets  $A_1 \subset A_2 \subset \dots$  in Lebesgue's integration theory also has a counterpart in the theory of Hilbert space.

**Lemma 1.114.** *Let  $P_1$  and  $P_2$  be orthogonal projections acting on  $\mathcal{H}$ , then*

$$P_1 \leq P_2 \iff \|P_1x\| \leq \|P_2x\|, \forall x \in \mathcal{H} \quad (1.80)$$

$$\Updownarrow$$

$$P_1 = P_1P_2 = P_2P_1 \quad (1.81)$$

(see Table 1.4.)

*Proof.* Indeed, for all  $x \in \mathcal{H}$ , we have

$$\|P_1x\|^2 = \langle P_1x, P_1x \rangle = \langle x, P_1x \rangle = \langle x, P_2P_1x \rangle \leq \|P_1P_2x\|^2 \leq \|P_2x\|^2.$$

□

**Theorem 1.115.** *For every monotonically increasing sequence of projections*

$$P_1 \leq P_2 \leq \cdots,$$

and setting

$$P := \vee P_k = \lim_k P_k,$$

then  $P$  defines a projection, the limit.

*Proof.* The assumption  $P_1 \leq P_2 \leq \cdots$  implies that  $\{\|P_kx\|\}_{k=1}^\infty$ ,  $x \in \mathcal{H}$ , is a monotone increasing sequence in  $\mathbb{R}$ , and the sequence is bounded by  $\|x\|$ , since  $\|P_kx\| \leq \|x\|$ , for all  $k \in \mathbb{N}$ . Therefore the sequence  $\{P_k\}_{k=1}^\infty$  converges to  $P \in \mathcal{B}(\mathcal{H})$  (strongly), and  $P$  really defines a selfadjoint projection. (We use “ $\leq$ ” to denote the lattice operation on projection.) Note the convergence refers to the strong operator topology, i.e., for all  $x \in \mathcal{H}$ , there exists a vector, which we denote by  $Px$ , so that  $\lim_k \|P_kx - Px\| = 0$ . □

The examples in Section 1.4 using Gram-Schmidt process can now be formulated in the lattice of projections.

Recall (Lemma 1.67) that for a linearly independent subset  $\{u_k\} \subset \mathcal{H}$ , the Gram-Schmidt process yields an orthonormal set  $\{v_k\} \subset \mathcal{H}$ , with  $v_1 := u_1/\|u_1\|$ , and

$$v_{n+1} := \frac{u_{n+1} - P_n u_{n+1}}{\|u_{n+1} - P_n u_{n+1}\|}, \quad n = 1, 2, \dots;$$

where  $P_n$  is the orthogonal projection on the  $n$ -dimensional subspace

$$V_n := \text{span}\{v_1, \dots, v_n\}.$$

See Figure 1.18.

Note that

$$\begin{aligned} V_n \subset V_{n+1} &\rightarrow \bigcup_n V_n \sim P_n \leq P_{n+1} \rightarrow P \\ P_n^\perp &\geq P_{n+1}^\perp \rightarrow P^\perp. \end{aligned}$$

Assume  $\bigcup_n V_n$  is dense in  $\mathcal{H}$ , then  $P = I$  and  $P^\perp = 0$ . In lattice notations, we may write

$$\begin{aligned} \vee P_n &= \sup P_n = I \\ \wedge P_n^\perp &= \inf P_n^\perp = 0. \end{aligned}$$

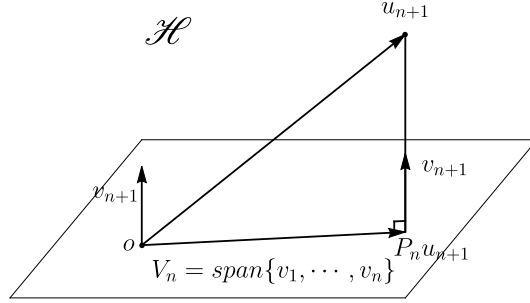


Figure 1.18: Gram-Schmidt:  $V_n \rightarrow V_{n+1}$

**Lemma 1.116.** *Let  $P, Q \in Proj(\mathcal{H})$ , then*

$$P + Q \in Proj(\mathcal{H}) \iff PQ = QP = 0, \text{ i.e., } P \perp Q.$$

*Proof.* Notice that

$$(P + Q)^2 = P + Q + PQ + QP \tag{1.82}$$

and so

$$(P + Q)^2 = P + Q \tag{1.83}$$

$$\Updownarrow$$

$$PQ + QP = 0. \tag{1.84}$$

Suppose  $PQ = QP = 0$  then

$$(P + Q)^2 = P + Q = (P + Q)^*, \text{ i.e., } P + Q \in Proj(\mathcal{H}).$$

Conversely, if  $P + Q \in Proj(\mathcal{H})$ , then  $(P + Q)^2 = P + Q \implies PQ + QP = 0$  by (1.84). Also,  $(PQ)^* = Q^*P^* = QP$ , combining with (1.84) yields

$$(PQ)^* = QP = -PQ. \tag{1.85}$$

Then,

$$(PQ)^2 = P(QP)Q \stackrel{(1.85)}{=} -P(PQ)Q = -PQ$$

which implies  $PQ(I + PQ) = 0$ . Hence,

$$PQ = 0 \quad \text{or} \quad PQ = I. \tag{1.86}$$

But by (1.85),  $PQ$  is skew-adjoint, it follows that  $PQ = 0$ , and so  $QP = 0$ .  $\square$

*Remark 1.117.* Eq. (1.82) is analogous to the following identity for characteristic functions:

$$\chi_A + \chi_B = \chi_{A \cup B} - \chi_{A \cap B}$$

Therefore,  $(\chi_A + \chi_B)^2 = \chi_A + \chi_B$  iff  $\chi_{A \cap B} = 0$ , i.e., iff  $A \cap B = \emptyset$ .

The set of projections in a Hilbert space  $\mathcal{H}$  is partially ordered according to the corresponding closed subspaces partially ordered by inclusion. Since containment implies commuting, the chain of projections

$$P_1 \leq P_2 \leq \dots$$

is a family of commuting selfadjoint operators. By the spectral theorem (Chapter 3),  $\{P_i\}$  may be simultaneously diagonalized, so that  $P_i$  is unitarily equivalent to the operator of multiplication by  $\chi_{E_i}$  on the Hilbert space  $L^2(X, \mu)$ , where  $X$  is compact and Hausdorff. Therefore the lattice structure of projections in  $\mathcal{H}$  is precisely the lattice structure of  $\chi_E$ , or equivalently, the lattice structure of measurable sets in  $X$ .

**Lemma 1.118.** *Consider  $L^2(X, \mu)$ . The following are equivalent.*

1.  $E \subset F$ ;
2.  $\chi_E \chi_F = \chi_F \chi_E = \chi_E$ ;
3.  $\|\chi_E f\| \leq \|\chi_F f\|$ , for any  $f \in L^2$ ;
4.  $\chi_E \leq \chi_F$ , in the sense that

$$\langle f, \chi_E f \rangle \leq \langle f, \chi_F f \rangle, \quad \forall f \in L^2(X).$$

*Proof.* The proof is trivial. Note that

$$\begin{aligned} \langle f, \chi_E f \rangle &= \int \chi_E |f|^2 d\mu \\ \|\chi_E f\|^2 &= \int |\chi_E f|^2 d\mu = \int \chi_E |f|^2 d\mu \end{aligned}$$

where we used that fact that

$$\chi_E = \overline{\chi_E} = \chi_E^2.$$

□

What makes  $\text{Proj}(\mathcal{H})$  intriguing is the non-commutativity. For example, if  $P, Q \in \text{Proj}(\mathcal{H})$  are given, it does not follow (in general) that  $P + Q \in \text{Proj}(\mathcal{H})$ ; nor that  $PQP \in \text{Proj}(\mathcal{H})$ . These two conclusions only hold if it is further assumed that  $P$  and  $Q$  commute; see Lemmas 1.116 and 1.119.

**Lemma 1.119.** *Let  $P, Q \in \text{Proj}(\mathcal{H})$ ; then the following conditions are equivalent:*

1.  $PQP \in \text{Proj}(\mathcal{H})$ ;
2.  $PQ = QP$ .



*Proof.* First note that the operator  $A = PQ - QP$  is skew-symmetric, i.e.,  $A^* = -A$ , and so its spectrum is contained in the imaginary line  $i\mathbb{R}$ .

The implication (2) $\Rightarrow$ (1) above is immediate so assume (1), i.e., that

$$(PQP)^2 = PQP.$$

And using this, one checks by a direct computation that  $A^3 = 0$ . But with  $A^* = -A$ , and the spectral theorem, we therefore conclude that  $A = 0$ , in other words, (2) holds.  $\square$

## 1.7 Multiplication Operators

**Exercise 1.120** (Multiplication operators). Let  $(X, \mathcal{F}, \mu)$  be a measure space, assume  $\mu$  is  $\sigma$ -finite. Let  $L^2(\mu)$  be the corresponding Hilbert space. Let  $\varphi$  be a locally integrable function on  $X$ , and set

$$M_\varphi f := \varphi f,$$

pointwise product, defined for

$$f \in \text{dom}(M_\varphi) = \{f \in L^2(\mu) : \varphi f \in L^2(\mu)\}.$$

$M_\varphi$  is called a *multiplication operator*.

1. Show that  $M_\varphi$  is normal.
2. Show that  $M_\varphi$  is selfadjoint iff  $\varphi$  is  $\mu$ -a.e. real-valued.
3. Show that  $M_\varphi$  is bounded in  $L^2(\mu)$  iff  $\varphi \in L^\infty(\mu)$ ; and, in this case,

$$\|M_\varphi\|_{UN} = \|\varphi\|_{L^\infty(\mu)}. \quad (1.87)$$

4. Show that if  $\varphi \in L^\infty(\mu)$ , then  $\text{dom}(M_\varphi) = L^2(\mu)$ .
5. Discuss the converse.

**Exercise 1.121** (Moment theory [Akh65]). Let  $\mu$  be a positive Borel measure on  $\mathbb{R}$  such that

$$\int_{\mathbb{R}} x^{2n} d\mu(x) < \infty \quad (1.88)$$

for all  $n \in \mathbb{N}$ , i.e.,  $\mu$  has finite moments of all orders. Let  $\varphi(x) = x$ , and  $M = M_\varphi$  the corresponding multiplication operator in  $L^2(\mu) = L^2(\mathbb{R}, \mathcal{B}, \mu)$ , i.e.,

$$(Mf)(x) = (Qf)(x) = xf(x), \quad (1.89)$$

for all  $f \in L^2(\mu)$  such that  $xf \in L^2(\mu)$ .

- Using Gram-Schmidt (Lemma 1.67), show that  $M$  has a matrix representation by an  $\infty \times \infty$  tri-diagonal (banded) matrix as in Figure 1.16.

Akhiezer calls these infinite banded matrices *Jacobi matrices*. They define formally selfadjoint (alias symmetric) operators in  $l^2$ ; unbounded of course. And these operators can only attain von Neumann indices  $(0, 0)$  or  $(1, 1)$ . Both are possible.

- Work out a recursive formula for the two sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(d_n)_{n \in \mathbb{N}}$  in the expression for  $M$  by the matrix of Figure 1.16 in terms of the moments  $(s_n)_{n \in \mathbb{N} \cup \{0\}}$ :

$$s_n := \int_{\mathbb{R}} x^n d\mu(x).$$

- Same question as in (2) but for the special case when  $\mu$  is

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

i.e., the  $N(0, 1)$  Gaussian measure on  $\mathbb{R}$ .

Hint: Show first that the moments  $\{s_k\}_{k \in \{0\} \cup \mathbb{N}}$  of  $\mu_{N(0,1)}$  are as follows (the Gaussian moments):

$$\begin{aligned} s_{2n+1} &= 0, \text{ and} \\ s_{2n} &= \frac{(2n)!}{2^n \cdot n!} = (2n-1)!! (= (2n-1)(2n-3)\cdots 5 \cdot 3). \end{aligned}$$

- Give necessary and sufficient conditions for *essential selfadjointness* of the associated Jacobi matrix as an operator in  $L^2(\mu)$ , expressed in terms  $\mu$  and of the moments  $(s_n)$  in (3).

**Corollary 1.122.** *Let  $A$  be a selfadjoint operator in a separable Hilbert space, and suppose there is a cyclic vector  $u_0$ ,  $\|u_0\| = 1$ , such that*

$$u_0 \in \bigcap_{k \in \mathbb{N}} \text{dom}(A^k). \quad (1.90)$$

*Then there is an ONB  $\{e_i\}_{i \in \mathbb{N}}$  in  $\mathcal{H}$ , contained in  $\text{dom}(A)$  such that the corresponding  $\infty \times \infty$  matrix*

$$(M_A)_{i,j} = \langle e_i, A e_j \rangle, \quad i, j \in \mathbb{N} \quad (1.91)$$

*is banded; what is more, it is a Jacobi-matrix, see Exercise 1.121.*

*Proof.* Using the Spectral Theorem, we conclude that there is a measure  $\mu_0$  on  $\mathbb{R}$  and a unitary transform  $W : L^2(\mathbb{R}, \mu_0) \rightarrow \mathcal{H}$  such that  $\square$

- $Wf = f(A)u_0, \forall f \in L^2(\mu_0);$

- (a)  $WM_t = AW$ , where  $M_t$  denotes multiplication by  $t$  in  $L^2(\mu_0)$ ; and  
 (b)  $\langle u_0, f(A)u_0 \rangle = \int_{\mathbb{R}} f(t) d\mu_0(t)$ ,  $\forall f \in L^2(\mu_0)$ .

We refer the reader to Chapter 3 for more details.

*Proof.* Now apply Gram-Schmidt to the monomials  $\{t^k\}_{k \in \{0\} \cup \mathbb{N}}$ , to get orthogonal polynomials  $\{p_k(t)\}$  such that

$$\text{span}_{k \leq n} \{p_k(t)\} = \text{span}_{k \leq n} \{t^k\}$$

holds for all  $n \in \mathbb{N}$ .

Set

$$e_k := p_k(A)u_0, \quad k \in \{0\} \cup \mathbb{N}, \quad (1.92)$$

$e_0 = u_0$ , and this is then the desired ONB. To see this, use the conclusion from Exercise 1.121, together with the following:

$$\begin{aligned} \langle e_j, e_k \rangle &= \langle p_j(A)u_0, p_k(A)u_0 \rangle \\ &= \langle u_0, p_j(A)p_k(A)u_0 \rangle \\ &= \langle u_0, (p_j p_k)(A)u_0 \rangle \\ &= \int_{\mathbb{R}} p_j(t)p_k(t) d\mu_0(t) \\ &\stackrel{\text{by (iii)}}{=} \delta_{j,k} \quad (\text{by Gram-Schmidt.}) \end{aligned}$$

□

Historical Note.

In [GIS90], Lax relates an account of von Neumann and F. Rellich speaking in Hilbert's seminar, in Göttingen (around 1930). When they came to "selfadjoint operator in Hilbert space," Erhard Schmidt (of Gram-Schmidt) would interrupt: "Please, young man, say *infinite matrix*."

Ironically, von Neumann invented numerical methods for "large" matrices toward the end of his career [GvN51].

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [Tay86, Arv72, Ban93, BR79, Con90, DM85, DS88c, Lax02, RS75, RSN90, Rud73, Rud87, AJS14, AJLM13, AJL13, AJ12, ARR13, BM13, Hid80, It06, Jør14, KL14a, BJ02, Jor06, CW14, Gro64, Joh88, KF75, JM80, Hel13, KW12, Con07].

## 1.A Hahn-Banach Theorems

*Version 1.* Let  $S$  be a subspace of a real vector space  $X$ . Let  $l : S \rightarrow \mathbb{R}$  be a linear functional, and let  $p : X \rightarrow \mathbb{R}$  satisfy

$$p(x+y) \leq p(x) + p(y), \quad x, y \in X; \quad (1.93)$$

$$p(tx) = tp(x), \quad t \in \mathbb{R}_+, x \in X. \quad (1.94)$$

**Theorem 1.123** (HB1). *Let  $X, S, p$ , and  $l$  be as above, and assume:*

$$l(x) \leq p(x), \quad x \in S. \quad (1.95)$$

*Then there is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$ , extending  $l$  on the subspace, and satisfying*

$$\tilde{l}(x) \leq p(x), \quad \forall x \in X. \quad (1.96)$$

Hint: Introduce a partially ordered set (p.o.s.)  $(T, m)$  where  $S \subset T \subset X$ ,  $T$  is a subspace,  $m : T \rightarrow \mathbb{R}$  is a linear functional extending  $(S, l)$  and satisfying

$$m(x) \leq p(x), \quad \forall x \in T. \quad (1.97)$$

Define the order  $(T, m) \leq (T', m')$  to mean that  $T \subseteq T'$  and  $m'$  agrees with  $m$  on  $T$ . (Both satisfying (1.97).) Apply Zorn's lemma to this p.o.s., and show that every maximal element must be a solution to (1.96).

## 1.B Banach-Limit

Consider  $X := l_{\mathbb{R}}^{\infty}(\mathbb{N}) =$  all bounded real sequences  $x = (x_1, x_2, \dots)$ , and the shift  $\sigma : l^{\infty} \rightarrow l^{\infty}$ , defined by

$$\sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Consider the subspace  $S \subset X$  consisting of all convergent sequences, and for  $x \in S$ , set  $l(x) = \lim_{k \rightarrow \infty} x_k$ , i.e., it holds that, for  $\forall \varepsilon \in \mathbb{R}_+$ ,  $\exists n$  such that

$$|x_k - l(x)| < \varepsilon \quad \text{for } \forall k \geq n.$$

**Theorem 1.124** (Banach). *There is a linear functional, called LIM :  $X \rightarrow \mathbb{R}$ , (a Banach limit) having the following properties:*

(i) *LIM is an extension of  $l$  on  $S$ .*

(ii)

$$\liminf_k x_k \leq LIM(x) \leq \limsup_k x_k \quad \text{and}$$

(iii)

$$LIM \circ \sigma = LIM,$$

*i.e., LIM is shift-invariant.*

*Proof.* This is an application of (HB1) but with a modification; we set

$$q(x) = \limsup x_k, \quad \text{and}$$

$$p(x) = \inf_n \frac{1}{n+1} \sum_{k=0}^n q(\sigma^k(x)), \quad \forall x \in X,$$

and we note that  $l(x) \leq p(x)$ ,  $\forall x \in S$ . The rest of the proof follows that of HB1 *mutatis mutandis*.  $\square$

*Remark 1.125.* Note that *LIM* is *not* unique.

It follows by the theorem above that *LIM* is in  $(l^\infty(\mathbb{N}))^*$ , and that it is *not* represented by any  $y \in l^1(\mathbb{N})$ , see Table 1.1. As a result, we have

$$(l^\infty(\mathbb{N}))^* \supsetneq l^1(\mathbb{N}); \quad (1.98)$$

i.e., the dual  $(l^\infty)^*$  is (much) bigger than  $l^1$ .

**Theorem 1.126** (HB2). *Let  $X$  be a normed space,  $S \subset X$  a closed subspace,  $l : S \rightarrow \mathbb{R}$  a linear functional such that*

$$|l(x)| \leq \|x\|, \quad \forall x \in S. \quad (1.99)$$

*Then there is a  $\tilde{l} \in X^*$  such that*

$$|\tilde{l}(x)| \leq \|x\|, \quad \forall x \in X,$$

*$\tilde{l}$  extending  $l$  from  $S$ , and*

$$\|\tilde{l}\|_{X^*} = \|l\|_{S^*}. \quad (1.100)$$

**Theorem 1.127** (HB3-Separation). *Let  $X$  be a real vector space. Assume  $X$  is equipped with a topology making the two vector-operations continuous. Let  $K \neq \emptyset$  be an open convex subset of  $X$ . Let  $y \in X \setminus K$  (in the complement).*

*Then there is a linear functional  $l : X \rightarrow \mathbb{R}$  such that*

$$l(x) < l(y), \quad \forall x \in K. \quad (1.101)$$

*(In fact, there exists  $c \in \mathbb{R}$  such that  $l(x) < c$ ,  $\forall x \in K$ , and  $l(y) = c$ .)*

Hint: Assume (by translation) that  $0 \in K$ ; and set

$$p_K(x) := \inf \left\{ a : a \in \mathbb{R}_+, \frac{x}{a} \in K \right\}; \quad (1.102)$$

and then apply version 2 to  $p_K$ , which can be shown to be sub-additive. For the separation property, see Figure 1.19.

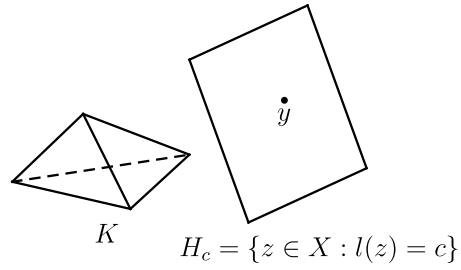


Figure 1.19: Separation of  $K$  and  $y$  by the hyperplane  $H_c$ .

**Theorem 1.128** (HB4). *Let  $\mathfrak{A}$  be a  $C^*$ -algebra and let  $\mathfrak{B} \subset \mathfrak{A}$  be a  $*$ -subalgebra. Let  $l : \mathfrak{B} \rightarrow \mathbb{C}$  satisfy*

$$l(b^*b) \geq 0, \quad \forall b \in \mathfrak{B}, \quad \text{and} \quad \|l\| = 1 \text{ (positivity),}$$

*then there is a positive linear functional  $\tilde{l} : \mathfrak{A} \rightarrow \mathbb{C}$ , such that*

1.  $\tilde{l}$  extends  $l$  on  $\mathfrak{B}$ ;
2.  $\tilde{l}(a^*a) \geq 0, \forall a \in \mathfrak{A}$ ; and
3.  $\|\tilde{l}\| = 1$ .

*Remark 1.129.* This version is due to M. Krein, but its proof uses the same ideas which we sketched above in versions 1-2.

## Chapter 2

# Unbounded Operators in Hilbert Space

We were [initially] entirely in Heisenberg's footsteps. He had the idea that one should take matrices, although he did not know that his dynamical quantities were matrices.... And when one had such a programme of formulating everything in matrix language, it takes some effort to get rid of matrices. Though it seemed quite natural for me to represent perturbation theory in the algebraic way, this was not a particularly new way.

— Max Born

...practical methods of applying quantum mechanics should be developed, which can lead to an explanation of the main features of complex atomic systems without too much computation.

— Paul Adrien Maurice Dirac

“...Mathematics ... are not only part of a special science, but are also closely connected with our general culture ... , a bridge to the Arts and Sciences, and the seemingly so non-exact sciences ...Our purpose is to help build such a bridge. Not for the sake of history but for the genesis of problems, facts and proofs, ... By going back to the roots of these conceptions, back through the dust of times past, the scars of long use would disappear, and they would be reborn to us as creatures full of life.”

— Otto Toeplitz, 1926

Quantum physics is one of the sources of problems in Functional Analysis, in particular the study of operators in Hilbert space. In the dictionary translating between Quantum physics and operators in Hilbert space we already saw

that “quantum observables” are “selfadjoint operators.” (See also Chapter 8, and especially Figure 8.1.)

As noted, even in a finite number of degrees of freedom, the relevant operators such as momentum, position, and energy are unbounded. Most Functional Analysis books stress the bounded case, and below we identify questions and theorems related to key issues for unbounded linear operators. (See, e.g., Appendix 2.A.)

In this chapter, we review the basic theory of unbounded operators in Hilbert space. For general notions, we refer to [DS88b, DS88c].

## 2.1 Domain, Graph, and Adjoints

Among the classes of operators in Hilbert space, the family of selfadjoint linear operators is crucially important for a host of applications, e.g., to mathematical physics, and to the study of partial differential equations (PDE). For a study of each of the three classes of linear PDOs, elliptic, hyperbolic, and parabolic, the Spectral Theorem (see [Sto90, Yos95, Nel69, RS75, DS88c]) for associated unbounded selfadjoint operators is a “workhorse.”

Every selfadjoint operator is densely defined, is closed, and it is necessarily (Hermitian) symmetric. For unbounded operators, the converse fails; although it does hold for bounded operators. It follows that selfadjointness is a much more restricting property than the related three properties. Moreover we will see that the distinction (between “symmetric” and selfadjoint) lies at the heart of key issues from applications. We will further see that symmetric operators with dense domain are automatically closable; but they may, or may not, have selfadjoint extensions; – again an issue of importance in physics.

The Spectral Theorem holds for selfadjoint operators, and for normal operators. But the case of normal operators reduces to the Spectral Theorem for two commuting selfadjoint operators.

On account of Stone’s theorem (Appendix 2.A) for one-parameter unitary groups we know that the class of selfadjoint operators coincides precisely with the infinitesimal generators of strongly continuous one-parameter groups of unitary operators acting on Hilbert space; – hence applications to the Schrödinger equation, and to wave equations.

Let  $\mathcal{H}$  be a complex Hilbert space. An operator  $A$  is a linear mapping whose domain  $dom(A)$  and range  $ran(A)$  are subspaces in  $\mathcal{H}$ . The kernel  $ker(A)$  of  $A$  consists of all  $a \in dom(A)$  such that  $Aa = 0$ . The operator  $A$  is uniquely determined by its graph

$$\mathcal{G}(A) = \{(a, Aa) : a \in dom(A)\}. \quad (2.1)$$

(Here the parentheses are used to denote an ordered pair, rather than an inner product.) Thus,  $\mathcal{G}(A)$  is a subspace in  $\mathcal{H} \oplus \mathcal{H}$  equipped with the inherited



graph inner product

$$\langle a, b \rangle_A = \langle a, b \rangle + \langle Aa, Ab \rangle, \text{ and} \quad (2.2)$$

$$\|a\|_A^2 = \langle a, a \rangle_A, \forall a, b \in \text{dom}(A) \quad (2.3)$$

In general, a subspace  $K \subset \mathcal{H} \oplus \mathcal{H}$  is the graph of an operator if and only if  $(0, a) \in K$  implies  $a = 0$ .

Given two operators  $A$  and  $B$ , we say  $B$  is an extension of  $A$ , denoted by  $B \supset A$ , if  $\mathcal{G}(B) \supset \mathcal{G}(A)$  in  $\mathcal{H} \oplus \mathcal{H}$ . The operator  $A$  is *closable* if  $\overline{\mathcal{G}(A)}$  is the graph of an operator  $\bar{A}$ , namely, the closure of  $A$ . We say  $A$  is *closed* if  $A = \bar{A}$ .

Let  $A$  be a closed operator. A dense subspace  $K \subset \mathcal{H}$  is called a *core* of  $A$ , if the closure of the restriction  $A|_K$  is equal to  $A$ .

Let  $G$  be the group of all  $3 \times 3$  real matrices  $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$  with Lie algebra  $\mathfrak{g}$

consisting of matrices  $\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$ ,  $(a, b, c) \in \mathbb{R}^3$ . Let  $\mathcal{U}$  be a strongly continuous

unitary representation of  $G$  acting on a Hilbert space  $\mathcal{H}$ . Then for every  $X \in \mathfrak{g}$ ,  $t \mapsto \mathcal{U}(e^{tX})$  defines a strongly continuous one-parameter group; and hence its infinitesimal generator, denoted  $d\mathcal{U}(X)$  is a skew-adjoint operator with dense domain in  $\mathcal{H}$ .

For a more systematic account of the interplay between the Lie algebra and the corresponding Lie group, as it relates to representations, see also Chapter 7 below, especially Exercise 7.23 for the present setting.

In the example below, we apply this to  $\mathcal{H} = L^2(\mathbb{R})$ , and the unitary representation  $\mathcal{U}$  of  $G$  is defined as follows: For  $f \in \mathcal{H} = L^2(\mathbb{R})$ , set

$$(\mathcal{U}(g)f)(x) = e^{i(c+bx)}f(x+a), \quad x \in \mathbb{R};$$

called the *Schrödinger representation*. Differentiating in the three directions in the Lie algebra, we get

$$(d\mathcal{U}(X_1)f)(x) = f'(x) = \frac{d}{dx}f, \quad (2.4)$$

$$(d\mathcal{U}(X_2)f)(x) = ix f(x), \text{ and} \quad (2.5)$$

$$(d\mathcal{U}(X_3)f)(x) = i f(x). \quad (2.6)$$

The first two operators in (2.4)-(2.5) are often written as follows:

$$d\mathcal{U}(X_1) = iP$$

where  $P$  is the *momentum operator* of a single quantum mechanical particle (wave function); and

$$d\mathcal{U}(X_2) = iQ$$

where  $Q$  is the corresponding *position operator* (in a single degree of freedom.) These two operators may be realized as follows:

*Remark 2.1.* It would appear that the function  $f_\lambda(x) = e^{i\lambda x}$ ,  $\lambda$  fixed is an eigenfunction for  $P = \frac{1}{i} \frac{d}{dx}$ . For all  $\lambda$ ,

$$Pf_\lambda = \lambda f_\lambda$$

holds pointwise, but “spectrum” depends on the ambient Hilbert space  $\mathcal{H}$ , in this case  $\mathcal{H} = L^2(\mathbb{R})$ ; and  $f_\lambda \notin L^2(\mathbb{R})$ , so  $\lambda$  is not an eigenvalue. Nonetheless, if we allow an intervals for the  $\lambda$  variable, e.g.,  $a < \lambda < b$ , with  $a$  and  $b$  being finite, then

$$F_{a,b}(x) = \int_a^b e^{i\lambda x} d\lambda = \frac{e^{ibx} - e^{iax}}{ix}$$

is in  $L^2(\mathbb{R})$ ; and hence  $P$  has *continuous* spectrum. The functions  $F_{a,b}(\cdot)$  are examples of wave-packets in quantum mechanics.

**Example 2.2.** The two operators  $d/dx$  and  $M_x$  in QM, are acting on  $L^2(\mathbb{R})$  with dense domain = the *Schwartz space*  $\mathcal{S}$ .

An alternative way to get a dense common domain, a way that works for all representations, is to use *Gårding space*, or  $C^\infty$ -vectors.

Let  $u \in \mathcal{H}$  and define

$$u_\varphi := \int_G \varphi(g) \mathcal{U}_g u dg$$

where  $\varphi \in C_c^\infty$ , and  $\mathcal{U} \in \text{Rep}(G, \mathcal{H})$ . Let  $\varphi_\epsilon$  be an approximation of identity. Then for functions on  $G$ ,  $\varphi_\epsilon \star \psi \rightarrow \psi$  as  $\epsilon \rightarrow 0$ ; and for  $C^\infty$  vectors,  $u_{\varphi_\epsilon} \rightarrow u$ , as  $\epsilon \rightarrow 0$  in  $\mathcal{H}$ , i.e., in the  $\|\cdot\|_{\mathcal{H}}$  norm.

The set  $\{u_\varphi\}$  is dense in  $\mathcal{H}$ . It is called the Gårding space, or  $C^\infty$  vectors, or Schwartz space. Notice that not only  $u_\varphi$  is dense in  $\mathcal{H}$ , their derivatives are also dense in  $\mathcal{H}$ .

Differentiating  $\mathcal{U}_g$ , we then get a Lie algebra representation

$$\rho(X) := \left. \frac{d}{dt} \right|_{t=0} \mathcal{U}(e^{tX}) = d\mathcal{U}(X).$$

**Lemma 2.3.**  $\|u_{\varphi_\epsilon} - u\| \rightarrow 0$ , as  $\epsilon \rightarrow 0$ .

*Proof.* Since  $u - u_{\varphi_\epsilon} = \int \varphi_\epsilon(g)(u - \mathcal{U}_g u) dg$ , we have

$$\begin{aligned} \|u_{\varphi_\epsilon} - u\| &= \left\| \int_G \varphi_\epsilon(g)(u - \mathcal{U}_g u) dg \right\| \\ &\leq \int_G \varphi_\epsilon(g) \|u - \mathcal{U}_g u\| dg \end{aligned}$$

where the integration on  $G$  is with respect to Haar measure, and where we used the fact that  $\int_G \varphi_\epsilon = 1$ . Notice that we always assume the representations are norm continuous in the  $g$  variable, otherwise it is almost impossible to get anything interesting. i.e., assume  $\mathcal{U}$  being strongly continuous. So for all  $\delta > 0$ , there is a neighborhood  $\mathcal{O}$  of  $e \in G$  so that  $\|u - \mathcal{U}_g u\| < \delta$  for all  $g \in \mathcal{O}$ . Choose  $\epsilon_\delta$  so that  $\varphi_\epsilon$  is supported in  $\mathcal{O}$  for all  $\epsilon < \epsilon_\delta$ . Then the statement is proved. □

**Corollary 2.4.** *For all  $X \in \mathfrak{g}$ ,  $d\mathcal{U}(X)$  is essentially skew-adjoint on the Gårding domain.*

*Proof.* Let  $\mathcal{U} \in \text{Rep}(G, \mathcal{H})$  be a unitary strongly continuous representation, where  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ . Let  $X \in \mathfrak{g}$ ; we claim that  $d\mathcal{U}(X)$  is essentially skew-adjoint (see [RS75, Nel69, vN32a, DS88c]) on the Gårding space, i.e., the span of vectors

$$v_\varphi = \int_G \varphi(g) \mathcal{U}_g v \, dg \tag{2.7}$$

where  $\varphi \in C_c^\infty(G)$ .

Hence we must show that

$$\ker(d\mathcal{U}(X)^* \pm I) = 0. \tag{2.8}$$

The argument is the same in both cases of (2.8). Hence we must show that, if  $w \in \mathcal{H}$  satisfies

$$\langle d\mathcal{U}(X) v_\varphi - v_\varphi, w \rangle = 0 \tag{2.9}$$

for all  $v_\varphi$  in (2.7), then  $w = 0$ . Now if we view  $X$  as an invariant vector field on  $G$ , then (2.9) states that the continuous function

$$f_w(g) := \mathcal{U}(g) w \tag{2.10}$$

is a weak solution to the ODE

$$X f_w = f_w;$$

equivalently

$$f_{w,X}(t) := \mathcal{U}(\exp(tX)) w \tag{2.11}$$

satisfies

$$\frac{d}{dt} f_{w,X}(t) = f_{w,X}(t), \quad t \in \mathbb{R}; \tag{2.12}$$

and so

$$f_{w,X}(t) = \text{const} \cdot e^t, \quad t \in \mathbb{R}. \tag{2.13}$$

But, since  $\mathcal{U}$  is unitary,  $f_{w,X}$  in (2.11) is bounded; so the constant in (2.13) is zero. Hence  $f_{w,X}(t) \equiv 0$ . But  $f_{w,X}(0) = w$ , and so  $w = 0$ .  $\square$

Now, let  $A$  be an arbitrary linear operator in a Hilbert space  $\mathcal{H}$  with  $\text{dom}(A)$  dense in  $\mathcal{H}$ .

**Theorem 2.5.** *The following are equivalent.*

1.  $A = \bar{A}$ .
2.  $\mathcal{G}(A) = \overline{\mathcal{G}(A)}$ .
3.  $\text{dom}(A)$  is a Hilbert space with respect to the graph inner product  $\langle \cdot, \cdot \rangle_A$ .

4. If  $\{(a_n, Aa_n)\}_{n=1}^\infty$  is a sequence in  $\mathcal{G}(A)$ , and  $(a_n, Aa_n) \rightarrow (a, b)$  as  $n \rightarrow \infty$ , then  $(a, b) \in \mathcal{G}(A)$ . In particular,  $b = Aa$ . (The round braces  $(\cdot, \cdot)$  mean “pair-of vectors.”)

*Proof.* All follow from definitions. □

Let  $X$  be a vector space over  $\mathbb{C}$ . Suppose there are two norms defined on  $X$ , such that

$$\|\cdot\|_1 \leq \|\cdot\|_2. \tag{2.14}$$

Let  $\overline{X}_i$  be the completion of  $X$  with respect to  $\|\cdot\|_i$ ,  $i = 1, 2$ . The ordering (2.14) implies the identify map

$$\varphi : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$$

is continuous, hence it has a unique continuous extension  $\tilde{\varphi}$  to  $\overline{X}_2$ ; and (2.14) passes to the closure  $\overline{X}_2$ . If  $\tilde{\varphi}$  is injective,  $\overline{X}_2$  is embedded into  $\overline{X}_1$  as a dense subspace. In that case,  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are said to be *topologically consistent*.

**Lemma 2.6.**  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are topologically equivalent if and only if

$$\left\{ \begin{array}{l} \{x_n\} \subset X \text{ is Cauchy under } \|\cdot\|_2 \\ \text{(hence Cauchy under } \|\cdot\|_1) \\ \|x_n\|_1 \rightarrow 0 \end{array} \right\} \implies \|x_n\|_2 \rightarrow 0.$$

*Proof.* Note  $\tilde{\varphi}$  is linear, and

$$\ker \tilde{\varphi} = \left\{ x \in \overline{X}_2 \mid \exists (x_n) \subset X, \|x_n - x\|_2 \rightarrow 0, \tilde{\varphi}(x) = 0 \right. \\ \left. \text{(note } \tilde{\varphi}(x) = \lim_n \varphi(x_n) = \lim_n x_n \text{ in } \overline{X}_1) \right\}.$$

The lemma follows from this. □

**Lemma 2.7.** The graph norm of  $A$  is topologically equivalent to  $\|\cdot\| + \|A\cdot\|$ .

*Proof.* This follows from the estimate

$$\frac{1}{2} (\|x\| + \|Ax\|)^2 \leq \|x\|^2 + \|Ax\|^2 \leq (\|x\| + \|Ax\|)^2, \quad \forall x \in \text{dom}(A).$$

□

**Theorem 2.8.** An operator  $A$  is closable if and only if  $\|\cdot\|$  and  $\|\cdot\|_A$  are topologically equivalent. (When they are, the completion of  $\text{dom}(A)$  with respect to  $\|\cdot\|_A$  is identified as a subspace of  $\mathcal{H}$ .)

*Proof.* First, assume  $A$  is closable. Let  $\{x_n\}$  be a sequence in  $\text{dom}(A)$ . Suppose  $\{x_n\}$  is a Cauchy sequence with respect to  $\|\cdot\|_A$ , and  $\|x_n\| \rightarrow 0$ . We need to show  $\{x_n\}$  converges to 0 under the  $A$ -norm, i.e.,  $\|x_n\|_A \rightarrow 0$ . Since  $\{(x_n, Ax_n)\} \subset \mathcal{G}(A)$ , and  $A$  is closable, it follows that  $(x_n, Ax_n) \rightarrow (0, 0) \in \mathcal{G}(A)$ . Therefore,  $\|Ax_n\| \rightarrow 0$ , and (see Lemma 2.7)

$$\|x_n\|_A = \|x_n\| + \|Ax_n\| \rightarrow 0.$$

Conversely, assume  $\|\cdot\|$  and  $\|\cdot\|_A$  are topologically consistent. Let  $\{x_n\} \subset \text{dom}(A)$ , such that

$$(x_n, Ax_n) \rightarrow (0, b) \text{ in } \mathcal{H} \oplus \mathcal{H}. \quad (2.15)$$

We proceed to show that  $b = 0$ , which implies that  $A$  is closable.

By (2.15),  $\{x_n\} \subset \text{dom}(A)$  is a Cauchy sequence with respect to the  $\|\cdot\|_A$ -norm, and  $\|x_n\| \rightarrow 0$ . Since the two norms are topologically consistent, then  $\|x_n\|_A \rightarrow 0$  and so  $\|Ax_n\| \rightarrow 0$ . We conclude that  $b = 0$ .  $\square$

**Corollary 2.9.** *An operator  $A$  with dense domain is closable if and only if its adjoint  $A^*$  has dense domain.*

We will focus on unbounded operators. In the sequel, we will consider densely defined Hermitian (symmetric) operators. Such operators are necessarily closable.

The following result is usually applied to operators whose inverses are bounded.

**Proposition 2.10.** *Let  $A$  be a bounded operator with domain  $\text{dom}(A)$ , and acting in  $\mathcal{H}$ . Then  $\text{dom}(A)$  is closed in  $\|\cdot\|_A$  if and only if it is closed in  $\|\cdot\|$ . (That is, for bounded operators,  $\|\cdot\|$  and  $\|\cdot\|_A$  are topologically equivalent. )*

*Proof.* This is the result of the following estimate:

$$\|x\| \leq \|x\|_A = \|x\| + \|Ax\| \leq (1 + \|A\|) \|x\|, \quad \forall x \in \text{dom}(A).$$

$\square$

**Corollary 2.11.** *If  $A$  is a closed operator in  $\mathcal{H}$  and  $A^{-1}$  is bounded, then  $\text{ran}(A)$  is closed in both  $\|\cdot\|$  and  $\|\cdot\|_{A^{-1}}$ .*

*Proof.* Note the  $\mathcal{G}(A)$  is closed iff  $\mathcal{G}(A^{-1})$  is closed; and  $\text{ran}(A) = \text{dom}(A^{-1})$ . Now, apply Proposition 2.10 to  $A^{-1}$ .  $\square$

Let  $A$  be an operator in a Hilbert space  $\mathcal{H}$ . The set  $\mathcal{G}(A)^\perp$  consists of  $(-b^*, b)$  such that  $(-b^*, b) \perp \mathcal{G}(A)$  in  $\mathcal{H} \oplus \mathcal{H}$ .

**Proposition 2.12.** *The following are equivalent.*

1.  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ .
2.  $(b, 0) \perp \mathcal{G}(A) \implies b = 0$ .
3. If  $(b, -b^*) \perp \mathcal{G}(A)$ , the map  $b \mapsto b^*$  is well-defined.

*Proof.* Let  $a \in \mathcal{D}(A)$ , and  $b, b^* \in \mathcal{H}$ ; then

$$(-b^*, b) \perp (a, Aa) \text{ in } \mathcal{H} \oplus \mathcal{H} \iff \langle b^*, a \rangle = \langle b, Aa \rangle$$

and the desired results follow from this.  $\square$

If any of the conditions is satisfied,  $A^* : b \mapsto b^*$  defines an operator, called the adjoint of  $A$ , such that

$$\langle b, Aa \rangle = \langle A^*b, a \rangle \tag{2.16}$$

for all  $a \in \mathcal{D}(A)$ .  $\mathcal{G}(A)^\perp$  is the inverted graph of  $A^*$ . The adjoints are only defined for operators with dense domains in  $\mathcal{H}$ .

**Example 2.13.**  $A = d/dx$  on  $L^2[0, 1]$  with dense domain

$$\mathcal{D} = \{f \in C^1 \mid f(0) = f(1) = 0\}.$$

Integration by parts shows that  $A \subset -A^*$ .

For unbounded operators,  $(AB)^* = B^*A^*$  does not hold in general. The situation is better if one of them is bounded.

**Theorem 2.14** ([Rud90, Theorem 13.2]). *If  $S, T, ST$  are densely defined operators then  $(ST)^* \supset T^*S^*$ . If, in addition,  $S$  is bounded then  $(ST)^* = T^*S^*$ .*

The next theorem follows directly from the definition of the adjoint operators.

**Theorem 2.15.** *If  $A$  is densely defined then  $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{K}(A^*)$ .*

Finally, we recall some definitions.

**Definition 2.16.** Let  $A$  be a linear operator acting in  $\mathcal{H}$ .  $A$  is said to be

- selfadjoint if  $A = A^*$ .
- essentially selfadjoint if  $\overline{A} = A^*$ .
- normal if  $A^*A = AA^*$ .
- regular if  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ , and closed in  $\|\cdot\|_A$ .

**Definition 2.17.** Let  $A$  be a linear operator on a Hilbert space  $\mathcal{H}$ . The resolvent  $R(A)$  is defined as

$$R(A) = \left\{ \lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ exists} \right\} \text{ (the resolvent set)}$$

and the *spectrum* of  $A$  is the complement of  $R(A)$ , and it is denoted by  $sp(A)$  or  $\sigma(A)$ .

**Exercise 2.18** (The resolvent identity). Let  $A$  be a linear operator in a Hilbert space  $\mathcal{H}$ , and, for  $\lambda_i \in R(A)$ ,  $i = 1, 2$  consider two operators  $(\lambda_i - A)^{-1}$ . Show that

$$(\lambda_1 - A)^{-1} - (\lambda_2 - A)^{-1} = (\lambda_2 - \lambda_1)(\lambda_1 - A)^{-1}(\lambda_2 - A)^{-1}.$$

This formula is called the resolvent identity.

## 2.2 Characteristic Matrix

The method of characteristic matrix was developed by M.H. Stone's [Sto51]. It is extremely useful in operator theory, but has long been overlooked in the literature. We recall some of its applications in normal operators.

If  $\mathcal{H}$  is a fixed Hilbert space, and  $A$  a given liner operator, then its graph

$$\mathcal{G}(A) = \left\{ \begin{bmatrix} u \\ Au \end{bmatrix} : u \in \text{dom}(A) \right\}$$

is a linear subspace in  $\mathcal{H} \oplus \mathcal{H}$ , represented as column vectors  $\begin{bmatrix} u \\ v \end{bmatrix}$ ,  $u, v \in \mathcal{H}$ .

In the case where  $A$  is assumed closed, we now compute the projection onto  $\overline{\mathcal{G}(A)}$  = the  $\mathcal{H} \oplus \mathcal{H}$ -closure of the graph.

Let  $A$  be an operator in a Hilbert space  $\mathcal{H}$ . Let  $P = (P_{ij})$  be the projection from  $\mathcal{H} \oplus \mathcal{H}$  onto  $\overline{\mathcal{G}(A)}$ . The  $2 \times 2$  operator matrix  $(P_{ij})$  of bounded operators in  $\mathcal{H}$  is called the characteristic matrix of  $A$ .

Since  $P^2 = P^* = P$ , the following identities hold

$$P_{ij}^* = P_{ji} \quad (2.17)$$

$$\sum_k P_{ik} P_{kj} = P_{ij} \quad (2.18)$$

In particular,  $P_{11}$  and  $P_{22}$  are selfadjoint.

**Theorem 2.19.** *Let  $P = (P_{ij})$  be the projection from  $\mathcal{H} \oplus \mathcal{H}$  onto a closed subspace  $\mathcal{K}$ . The following are equivalent.*

1.  $\mathcal{K}$  is the graph of an operator.

2.

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \implies a = 0.$$

3.

$$\left( P_{12}a = 0, P_{22}a = a \right) \implies a = 0.$$

If any of these conditions is satisfied, let  $A$  be the operator with  $\mathcal{G}(A) = \mathcal{K}$ , then for all  $a, b \in \mathcal{H}$ ,

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} P_{11}a + P_{12}b \\ P_{21}a + P_{22}b \end{bmatrix} \in \mathcal{G}(A);$$

i.e.,

$$A : (P_{11}a + P_{12}b) \mapsto P_{21}a + P_{22}b. \quad (2.19)$$

In particular,

$$AP_{11} = P_{21} \quad (2.20)$$

$$AP_{12} = P_{22} \quad (2.21)$$

*Proof.* Let  $v := (a, b) \in \mathcal{H} \oplus \mathcal{H}$ . Then  $v \in \mathcal{H}$  if and only if  $Pv = v$ ; and the theorem follows from this.  $\square$

The next theorem describes the adjoint operators.

**Theorem 2.20.** *Let  $A$  be an operator with characteristic matrix  $P = (P_{ij})$ . The following are equivalent.*

1.  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ .
2.  $\begin{bmatrix} b \\ 0 \end{bmatrix} \perp \mathcal{G}(A) = 0 \implies b = 0$ .
3. If  $\begin{bmatrix} -b^* \\ b \end{bmatrix} \in \mathcal{G}(A)^\perp$ , the map  $A^* : b \mapsto b^*$  is a well-defined operator.
4.  $\begin{bmatrix} 1 - P_{11} & -P_{12} \\ -P_{21} & 1 - P_{22} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \implies b = 0$ .
5.  $(P_{11}b = 0, P_{21}b = 0) \implies b = 0$ .

If any of the above conditions is satisfied, then

$$\begin{bmatrix} 1 - P_{11} & -P_{12} \\ -P_{21} & 1 - P_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1 - P_{11})a - P_{12}b \\ (1 - P_{22})b - P_{21}a \end{bmatrix} \in \mathcal{G}(A)^\perp$$

that is,

$$A^* : P_{21}a - (1 - P_{22})b \mapsto (1 - P_{11})a - P_{12}b. \quad (2.22)$$

In particular,

$$A^*P_{21} = 1 - P_{11} \quad (2.23)$$

$$A^*(1 - P_{22}) = P_{12}. \quad (2.24)$$

*Proof.* (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) is a restatement of Proposition 2.12. Note the projection from  $\mathcal{H} \oplus \mathcal{H}$  on  $\mathcal{G}(A)^\perp = \overline{\mathcal{G}(A)}^\perp$  is

$$1 - P = \begin{bmatrix} 1 - P_{11} & -P_{12} \\ -P_{21} & 1 - P_{22} \end{bmatrix}$$

and so

$$\begin{bmatrix} b \\ 0 \end{bmatrix} \perp \mathcal{G}(A) \iff (1 - P) \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Therefore, (2)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5). Finally, (2.22)-(2.24) follow from the definition of  $A^*$ .  $\square$



**Theorem 2.21.** *Let  $A$  be a regular operator (i.e., densely defined, closed) with characteristic matrix  $P = (P_{ij})$ .*

1. *The matrix entries  $P_{ij}$  are given by*

$$\begin{aligned} P_{11} &= (1 + A^*A)^{-1} & P_{12} &= A^*(1 + AA^*)^{-1} \\ P_{21} &= A(1 + A^*A)^{-1} & P_{22} &= AA^*(1 + AA^*)^{-1} \end{aligned} \quad (2.25)$$

2.  $1 - P_{22} = (1 + AA^*)^{-1}$ .

3.  $1 + A^*A$ ,  $1 + AA^*$  are selfadjoint operators.

4. *The following containments hold*

$$A^*(1 + AA^*)^{-1} \supset (1 + A^*A)^{-1}A^* \quad (2.26)$$

$$A(1 + A^*A)^{-1} \supset (1 + AA^*)^{-1}A \quad (2.27)$$

*Proof.* By (2.20) and (2.23), we have

$$\left[ \begin{array}{l} AP_{11} = P_{21} \\ A^*P_{21} = 1 - P_{11} \end{array} \right] \implies A^*AP_{11} = 1 - P_{11}, \text{ i.e., } (1 + A^*A)P_{11} = 1.$$

That is,  $1 + A^*A$  is a Hermitian extension of  $P_{11}^{-1}$ . By (2.17),  $P_{11}$  is selfadjoint and so is  $P_{11}^{-1}$ . Therefore,  $1 + A^*A = P_{11}^{-1}$ , or

$$P_{11} = (1 + A^*A)^{-1}.$$

By (2.20),

$$P_{21} = AP_{11} = A(1 + A^*A)^{-1}.$$

Similarly, by (2.21) and (2.24), we have

$$\left[ \begin{array}{l} AP_{12} = P_{22} \\ A^*(1 - P_{22}) = P_{12} \end{array} \right] \implies AA^*(1 - P_{22}) = P_{22}, \text{ i.e.,}$$

$$(1 + AA^*)(1 - P_{22}) = 1.$$

This means  $1 + AA^* \supset (1 - P_{22})^{-1}$  is a Hermitian extension of the selfadjoint operator  $(1 - P_{22})^{-1}$  (note  $P_{22}$  is selfadjoint), hence  $1 + AA^*$  is selfadjoint, and

$$1 - P_{22} = (1 + AA^*)^{-1}.$$

By (2.24),

$$P_{12} = A^*(1 - P_{22}) = A^*(1 + AA^*)^{-1}.$$

By (2.21),

$$P_{22} = AP_{12} = AA^*(1 + AA^*)^{-1}$$

We have proved (1), (2) and (3).

Finally,

$$P_{12} = P_{21}^* = (AP_{11})^* \supset P_{11}A^*$$

yields (2.26); and

$$P_{21} = P_{12}^* = (A^*(1 - P_{22}))^* \supset (1 - P_{22})A$$

gives (2.27).  $\square$

**Exercise 2.22** ( $A^{**} = \overline{A}$ ). Let  $A$  be a *regular* operator in a Hilbert space (i.e., we assume that  $A$  has dense domain and is closable.) Then show that

$$A^{**} = \overline{A} \tag{2.28}$$

where  $\overline{A}$  denotes the *closure* of  $A$ ; i.e.,  $\mathcal{G}(\overline{A}) = \overline{\mathcal{G}(A)}$ .

Hint: Establish the desired identity (2.28) by justifying the following steps:  
Set  $\chi : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ ,

$$\chi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}^2.$$

Then

$$\begin{aligned} \mathcal{G}(A^{**}) &= (\chi \mathcal{G}(A^*))^\perp \\ &= \left( \chi (\chi \mathcal{G}(A))^\perp \right)^\perp \\ &= (\chi^2 \mathcal{G}(A))^{\perp\perp} \\ &= (\mathcal{G}(A))^{\perp\perp} = \overline{\mathcal{G}(A)} = \mathcal{G}(\overline{A}). \end{aligned}$$

## Commutants

Let  $A, B$  be operators in a Hilbert space  $\mathcal{H}$ , and suppose  $B$  is bounded. The operator  $B$  is said to commute (strongly) with  $A$  if  $BA \subset AB$ .

**Lemma 2.23.** *Assume that  $\overline{A}$  exists. Then  $B$  commutes with  $A$  if and only if  $B$  commutes with  $\overline{A}$ .*

*Proof.* Suppose  $BA \subset AB$ , and we check that  $B\overline{A} \subset \overline{AB}$ . The converse is trivial. For  $(a, \overline{A}a) \in \mathcal{G}(\overline{A})$ , choose a sequence  $(a_n, Aa_n) \in \mathcal{G}(A)$  such that  $(a_n, Aa_n) \rightarrow (a, \overline{A}a)$ . By assumption,  $(Ba_n, ABa_n) = (Ba_n, BAa_n) \in \mathcal{G}(A)$ . Thus,

$$(Ba_n, ABa_n) \rightarrow (Ba, B\overline{A}a) \in \mathcal{G}(\overline{A}).$$

That is,  $Ba \in \mathcal{D}(\overline{A})$  and  $\overline{A}Ba = B\overline{A}a$ .  $\square$

**Lemma 2.24.** *Let  $A$  be a closed operator with characteristic matrix  $P = (P_{ij})$ . Let  $B$  be a bounded operator, and*

$$Q_B := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

1.  $B$  commutes with  $A \Leftrightarrow B$  leaves  $\mathcal{G}(A)$  invariant  $\Leftrightarrow Q_B P = P Q_B P$ .
2.  $B$  commutes with  $P_{ij} \Leftrightarrow Q_B P = P Q_B \Leftrightarrow Q_{B^*} P = P Q_{B^*} \Leftrightarrow B^*$  commutes with  $P_{ij}$ .
3. If  $B, B^*$  commute with  $A$ , then  $B, B^*$  commute with  $P_{ij}$ .

*Proof.* Obvious. □

A closed operator is said to be *affiliated* with a von Neumann algebra  $\mathfrak{M}$  if it commutes with every unitary operator in  $\mathfrak{M}'$ . By [KR97a, Thm 4.1.7], every operator in  $\mathfrak{M}'$  can be written as a finite linear combination of unitary operators in  $\mathfrak{M}'$ . Thus,  $A$  is affiliated with  $\mathfrak{M}$  if and only if  $A$  commutes with every operator in  $\mathfrak{M}'$ .

*Remark 2.25.* Let  $\mathfrak{M}$  be a von Neumann algebra. Let  $x \in \mathfrak{M}$  such that  $\|x\| \leq 1$  and  $x = x^*$ . Set  $y := x + i\sqrt{1-x^2}$ . Then,  $y^*y = yy^* = x^2 + 1 - x^2 = 1$ , i.e.,  $y$  is unitary. Also,  $x = (y + y^*)/2$ .

**Theorem 2.26.** *Let  $A$  be a closed operator with characteristic matrix  $P = (P_{ij})$ . Let  $\mathfrak{M}$  be a von Neumann algebra, and*

$$Q_B := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, B \in \mathfrak{M}'.$$

*The following are equivalent:*

1.  $A$  is affiliated with  $\mathfrak{M}$ .
2.  $PQ_B = Q_B P$ , for all  $B \in \mathfrak{M}'$ .
3.  $P_{ij} \in \mathfrak{M}$ .
4. If  $\mathcal{D}(A)$  is dense, then  $A^*$  is affiliated with  $\mathfrak{M}$ .

*Proof.* Notice that  $\mathfrak{M}$  is selfadjoint. The equivalence of 1, 2, 3 is a direct consequence of Lemma 2.24.

$P^\perp := 1 - P$  is the projection onto the inverted graph of  $A^*$ , should the latter exist.  $PQ_B = Q_B P$  if and only if  $P^\perp Q_B = Q_B P^\perp$ . Thus, 1 is equivalent to 4. □

## 2.3 Normal Operators

Theorem 2.27 below concerning operators of the form  $A^*A$  is an application of Stone's characteristic matrix.

**Theorem 2.27** (von Neumann). *If  $A$  is a regular operator in a Hilbert space  $\mathcal{H}$ , then*

1.  $A^*A$  is selfadjoint;

2.  $\mathcal{D}(A^*A)$  is a core of  $A$ , i.e.,

$$\overline{A|_{\mathcal{D}(A^*A)}} = A;$$

3. In particular,  $\mathcal{D}(A^*A)$  is dense in  $\mathcal{H}$ .

*Proof.* By Theorem 2.21,  $A^*A = P_{11}^{-1} - 1$ . Since  $P_{11}$  is selfadjoint, so is  $P_{11}^{-1}$ . Thus,  $AA^*$  is selfadjoint.

Suppose  $(a, Aa) \in \mathcal{G}(A)$  such that

$$(a, Aa) \perp \mathcal{G}(A|_{\mathcal{D}(A^*A)}); \text{ i.e.,}$$

$$\langle a, b \rangle + \langle Aa, Ab \rangle = \langle a, (1 + A^*A)b \rangle = 0, \forall b \in \mathcal{D}(A^*A).$$

Since  $1 + A^*A = P_{11}^{-1}$ , and  $P_{11}$  is a bounded operator, then

$$\mathcal{R}(1 + A^*A) = \mathcal{D}(P_{11}) = \mathcal{H}.$$

It follows that  $a \perp \mathcal{H}$ , and so  $a = 0$ . □

**Theorem 2.28** (von Neumann). *Let  $A$  be a regular operator in a Hilbert space  $\mathcal{H}$ . Then  $A$  is normal if and only if  $\mathcal{D}(A) = \mathcal{D}(A^*)$  and  $\|Aa\| = \|A^*a\|$ , for all  $a \in \mathcal{D}(A)$ .*

*Proof.* Suppose  $A$  is normal. Then for all  $a \in \mathcal{D}(A^*A) (= \mathcal{D}(AA^*))$ , we have

$$\|Aa\|^2 = \langle Aa, Aa \rangle = \langle a, A^*Aa \rangle = \langle a, AA^*a \rangle = \langle Aa, A^*a \rangle = \|A^*a\|^2;$$

i.e.,  $\|Aa\| = \|A^*a\|$ , for all  $a \in \mathcal{D}(A^*A)$ . It follows that

$$\mathcal{D}\left(\overline{A|_{\mathcal{D}(A^*A)}}\right) = \mathcal{D}\left(\overline{A^*|_{\mathcal{D}(AA^*)}}\right).$$

By Theorem 2.27,  $\mathcal{D}(A) = \mathcal{D}\left(\overline{A|_{\mathcal{D}(A^*A)}}\right)$  and  $\mathcal{D}(A^*) = \mathcal{D}\left(\overline{A^*|_{\mathcal{D}(AA^*)}}\right)$ . Therefore,  $\mathcal{D}(A) = \mathcal{D}(A^*)$  and  $\|Aa\| = \|A^*a\|$ , for all  $a \in \mathcal{D}(A)$ .

Conversely, the map  $Aa \mapsto A^*a$ ,  $a \in \mathcal{D}(A)$ , extends uniquely to a partial isometry  $V$  with initial space  $\overline{\mathcal{R}(A)}$  and final space  $\overline{\mathcal{R}(A^*)}$ , such that  $A^* = VA$ . By Theorem 2.14,  $A = A^*V^*$ . Then  $A^*A = A^*(V^*V)A = (A^*V^*)(VA) = AA^*$ . Thus,  $A$  is normal. □

The following theorem is due to M.H. Stone.

**Theorem 2.29.** *Let  $A$  be a regular operator in a Hilbert space  $\mathcal{H}$ . Let  $P = (P_{ij})$  be the characteristic matrix of  $A$ . The following are equivalent.*

1.  $A$  is normal.
2.  $P_{ij}$  are mutually commuting.
3.  $A$  is affiliated with an abelian von Neumann algebra.

*Remark 2.30.* For the equivalence of 1 and 2, we refer to the original paper of Stone. The most interesting part is  $1 \Leftrightarrow 3$ . The idea of characteristic matrix gives rise to an elegant proof without reference to the spectral theorem.

*Proof of Theorem 2.29.* Assuming  $1 \Leftrightarrow 2$ , we prove that  $1 \Leftrightarrow 3$ .

Suppose  $A$  is normal, i.e.  $P_{ij}$  are mutually commuting. Then  $A$  is affiliated with the abelian von Neumann algebra  $\{P_{ij}\}''$ . For if  $B \in \{P_{ij}\}'$ , then  $B$  commutes  $P_{ij}$ , and so  $B$  commutes with  $A$  by Lemma 2.24.

Conversely, if  $A$  is affiliated with an abelian von Neumann algebra  $\mathfrak{M}$ , then by Theorem 2.26,  $P_{ij} \in \mathfrak{M}$ . This shows that  $P_{ij}$  are mutually commuting, and  $A$  is normal.  $\square$

## 2.4 Polar Decomposition

We show that the intuition behind the familiar polar decomposition (or polar factorization) for complex numbers carries over remarkably well to operators in Hilbert space. Indeed (Theorem 2.32) the operators that admit a polar decomposition are precisely the regular operators, meaning closable and with dense domain.

Let  $A$  be a regular operator in a Hilbert space  $\mathcal{H}$ . By Theorem 2.27,  $A^*A$  is a positive selfadjoint operator and it has a unique positive square root  $|A| := \sqrt{A^*A}$ .

### Theorem 2.31.

1.  $|A| := \sqrt{A^*A}$  is the unique positive selfadjoint operator  $T$  satisfying  $\mathcal{D}(T) = \mathcal{D}(A)$ , and  $\|Ta\| = \|Aa\|$  for all  $a \in \mathcal{D}(A)$ .
2.  $\ker(|A|) = \ker(A)$ ,  $\overline{\mathcal{R}(|A|)} = \overline{\mathcal{R}(A^*)}$ .

*Proof.* Suppose  $T = \sqrt{A^*A}$ , i.e.  $T^*T = A^*A$ . Let  $\mathcal{D} := \mathcal{D}(T^*T) = \mathcal{D}(A^*A)$ . By Theorem 2.27,  $\mathcal{D}$  is a core of both  $T$  and  $A$ . Moreover,  $\|Ta\| = \|Aa\|$ , for all  $a \in \mathcal{D}$ . We conclude from this norm identity that  $\mathcal{D}(T) = \mathcal{D}(A)$  and  $\|Ta\| = \|Aa\|$ , for all  $a \in \mathcal{D}(A)$ .

Conversely, suppose  $T$  has the desired properties. For all  $a \in \mathcal{D}(A) = \mathcal{D}(T)$ , and  $b \in \mathcal{D}(A^*A)$ ,

$$\langle Tb, Ta \rangle = \langle Ab, Aa \rangle = \langle A^*Ab, a \rangle$$

This implies that  $Tb \in \mathcal{D}(T^*) = \mathcal{D}(T)$ ,  $T^2b = A^*Ab$ , for all  $b \in \mathcal{D}(A^*A)$ . That is,  $T^2$  is a selfadjoint extension of  $A^*A$ . Since  $A^*A$  is selfadjoint,  $T^2 = A^*A$ .

The second part follows from Theorem 2.15.  $\square$

Consequently, the map  $|A|a \mapsto Aa$  extends to a unique partial isometry  $V$  with initial space  $\overline{\mathcal{R}(A^*)}$  and final space  $\overline{\mathcal{R}(A)}$  (the overbar means “norm-closure”), such that

$$A = V|A|. \tag{2.29}$$

Equation (2.29) is called the *polar decomposition* of  $A$ . It is clear that such decomposition is unique.

We have proved:

**Theorem 2.32.** *Let  $A$ ,  $V$  and  $|A|$  be as described; then*

$$A = V |A|.$$

Taking adjoints in (2.29) yields  $A^* = |A| V^*$ , so that

$$AA^* = VA^*AV^* \tag{2.30}$$

Restrict  $AA^*$  to  $\overline{\mathcal{R}(A)}$ , and restrict  $A^*A$  restricted to  $\overline{\mathcal{R}(A^*)}$ . Then the two restrictions are unitarily equivalent. It follows that  $A^*A$ ,  $AA^*$  have the same spectrum, aside from possibly the point 0.

By (2.30),  $|A^*| = V |A| V^* = VA^*$ , where  $|A^*| = \sqrt{AA^*}$ . Apply  $V^*$  on both sides gives

$$A^* = V^* |A^*|. \tag{2.31}$$

By uniqueness, (2.31) is the polar decomposition of  $A^*$ .

**Theorem 2.33.**  *$A$  is affiliated with a von Neumann algebra  $\mathfrak{M}$  if and only if  $|A|$  is affiliated with  $\mathfrak{M}$  and  $V \in \mathfrak{M}$ .*

*Proof.* Let  $U$  be a unitary operator in  $\mathfrak{M}'$ . The operator  $UAU^*$  has polar decomposition

$$UAU^* = (UVU^*)(U|A|U^*).$$

By uniqueness,  $A = UAU^*$  if and only if  $V = UVU^*$ ,  $|A| = U|A|U^*$ . Since  $U$  is arbitrary, we conclude that  $V \in \mathfrak{M}$ , and  $A$  is affiliated with  $\mathfrak{M}$ .  $\square$

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest:

Of these, refs [vN32a] and [DS88c] are especially central. A more comprehensive list is: [BR81b, DS88c, Jor08, Kat95, KR97b, Sto51, Sto90, Wei03, Yos95, JL01, Die75, Emc00, Jor88, Jor94, RS75, Akh65, BN00, BR79, Con90, dBR66, FL28, Fri80, GJ87, JM84, Kre46, Nel69, vN32a, Hel13].

## 2.A Stone's Theorem

The gist of the result (Theorem 2.36) is as follows: Given a fixed Hilbert space, there is then a 1-1 correspondence between any two in pairs from the following three: (i) strongly continuous unitary one-parameter groups  $\mathcal{U}(t)$ ; (ii) selfadjoint operators  $H$  (generally unbounded) with dense domain; and (iii) projection valued measures  $P(\cdot)$ , abbreviated PVM. Starting with  $\mathcal{U}(t)$ , we say that the corresponding selfadjoint operator  $H$  is its generator, and then the PVM  $P(\cdot)$  will be from the Spectral Theorem applied to  $H$ .

**Definition 2.34** (Projection valued measure (PVM)). Let  $\mathcal{B}(\mathbb{R})$  be the Borel sigma algebra of subsets of  $\mathbb{R}$ . Let  $\mathcal{H}$  be a Hilbert space. A function  $P : \mathcal{B}(\mathbb{R}) \rightarrow \text{Proj}(\mathcal{H})$  is called a projection valued measure (PVM) iff (Def),  $P(\emptyset) = 0$ ;  $P(\mathbb{R}) = I_{\mathcal{H}}$ ; and for all  $(E_i)_{i=1}^{\infty}$  such that  $E_i \cap E_j = \emptyset$  ( $i \neq j$ ), we have:

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i) \quad (2.32)$$

**Definition 2.35.** A unitary one-parameter group is a function:

$$\mathcal{U} : \mathbb{R} \longrightarrow (\text{unitary operators in } \mathcal{H})$$

such that:

$$\mathcal{U}(s+t) = \mathcal{U}(s)\mathcal{U}(t), \quad \forall s, t \in \mathbb{R}; \quad (2.33)$$

and for  $\forall h \in \mathcal{H}$ ,

$$\lim_{t \rightarrow 0} \mathcal{U}(t)h = h \text{ (strong continuity)}. \quad (2.34)$$

**Theorem 2.36** (Stone's Theorem [Lax02, RS75, Rud73]). *There is a sequence of bijective correspondences between (1)-(3) below, i.e., (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1):*

1. PVMs  $P(\cdot)$ ;
2. unitary one-parameter groups  $\mathcal{U}$ ; and
3. selfadjoint operators  $H$  with dense domain in  $\mathcal{H}$ .

The correspondence is given explicitly as follows:

(1)  $\Rightarrow$  (2): Given  $P$ , a PVM, set

$$\mathcal{U}(t) = \int_{\mathbb{R}} e^{i\lambda t} P(d\lambda) \quad (2.35)$$

where the integral on the RHS in (2.35) is the limit of finite sums of

$$\sum_k e^{i\lambda_k t} P(E_k), \quad t \in \mathbb{R}; \quad (2.36)$$

$E_i \cap E_j = \emptyset$  ( $i \neq j$ ),  $\bigcup_k E_k = \mathbb{R}$ .

(2)  $\Rightarrow$  (3): Given  $\{\mathcal{U}(t)\}_{t \in \mathbb{R}}$ , set

$$\text{dom}(H) = \left\{ f \in \mathcal{H}, \text{ s.t. } \lim_{t \rightarrow 0_+} \frac{1}{it} (\mathcal{U}(t)f - f) \text{ exists} \right\}$$

and

$$iHf = \lim_{t \rightarrow 0_+} \frac{\mathcal{U}(t)f - f}{t}, \quad f \in \text{dom}(H), \quad (2.37)$$

then  $H^* = H$ .

(3)  $\Rightarrow$  (1): Given a selfadjoint operator  $H$  with dense domain in  $\mathcal{H}$ ; then by the spectral theorem (Section 2.3) there is a unique PVM,  $P(\cdot)$  such that

$$H = \int_{\mathbb{R}} \lambda P(d\lambda); \quad \text{and} \quad (2.38)$$

$$\text{dom}(H) = \left\{ f \in \mathcal{H}; \text{ s.t. } \int_{\mathbb{R}} \lambda^2 \|P(d\lambda) f\|^2 < \infty \right\}. \quad (2.39)$$

*Remark 2.37.* We state Stone's theorem already now even though the proof details will require a number of technical tools to be developed systematically only in Chapters 3 and 4 below.

*Remark 2.38.* Note that the selfadjointness condition on  $H$  in (3) in Theorem 2.36 is stronger than merely Hermitian symmetry, i.e., the condition

$$\langle Hu, v \rangle = \langle u, Hv \rangle \quad (2.40)$$

for all pairs of vectors  $u$  and  $v \in \text{dom}(H)$ . We shall discuss this important issue in much detail in Part 4 of the book, both in connection with the theory, and its applications. The applications are in physics, statistics, and infinite networks.

Here we limit ourselves to comments and some definitions; a full discussion will follow in part 4 below.

**Observations.** Introducing the adjoint operator  $H^*$ , we note that (2.40) is equivalent to

$$H \subset H^*, \text{ or} \quad (2.41)$$

$$\mathcal{G}(H) \subset \mathcal{G}(H^*), \quad (2.42)$$

where  $\mathcal{G}$  denotes the graph of the respective operators and where (2.41) & (2.42) mean that  $\text{dom}(H) \subset \text{dom}(H^*)$ , and  $Hu = H^*u$  for  $\forall u \in \text{dom}(H)$ .

If (2.41) holds, then it may, or may not, have selfadjoint extensions.

We introduce the two indices  $d_{\pm}$  (deficiency-indices)

$$d_{\pm} = \dim(H^* \pm iI). \quad (2.43)$$

The following will be proved in part 4:

**Theorem 2.39.** (i) Suppose  $H \subset H^*$ , then  $H$  has selfadjoint extensions iff  $d_+ = d_-$ .

(ii) If  $H$  has selfadjoint extensions, say  $K$  (i.e.,  $K^* = K$ ), so  $H \subset K$ , then it follows that

$$H \subset K \subset H^*. \quad (2.44)$$

So, if there are selfadjoint extensions, they lie between  $H$  and  $H^*$ .

**Definition 2.40.** If  $H \subset H^*$ , and if the closure  $\overline{H} = H^{**}$  is selfadjoint, we say that  $H$  is essentially selfadjoint.



## Chapter 3

# The Spectral Theorem

As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.

— Albert Einstein

**A large part of mathematics which becomes useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so.** By and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and the moment when it is useful; and that this lapse of time can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful.

— John von Neumann

“The spectral theorem together with the multiplicity theory is one of the pearls of mathematics.”

— M. Reed and B. Simon [RS75]

Most Functional Analysis books, when covering the Spectral Theorem, stress the bounded case. Because of dictates from applications (especially quantum physics), below we stress questions directly related to key-issues for unbounded linear operators. These themes will be taken up again in Chapters 9 and 10. In a number of applications, some operator from physics may only be “formally selfadjoint” also called Hermitian; and in such cases, one asks for selfadjoint extensions (if any), Chapter 9. Chapter 10 is a particular case in point, arising in the study of infinite graphs.

### 3.1 An Overview

von Neumann's spectral theorem (see [Sto90, Yos95, Nel69, RS75, DS88c]) states that an operator  $A$  acting in a Hilbert space  $\mathcal{H}$  is normal if and only if there exists a projection-valued measure on  $\mathbb{C}$  so that

$$A = \int_{sp(A)} z P_A(dz) \quad (3.1)$$

i.e.,  $A$  is represented as an integral against the projection-valued measure  $P_A$  over its spectrum.

In quantum mechanics, an *observable* is represented by a selfadjoint operator. Functions of observables are again observables. This is reflected in the spectral theorem as the functional calculus, where we may define

$$\varphi(A) = \int_{sp(A)} \varphi(z) P_A(dz) \quad (3.2)$$

using the spectral representation of  $A$ .

When  $P$  is a selfadjoint projection,  $\langle f, Pf \rangle_{\mathcal{H}} = \|Pf\|_{\mathcal{H}}^2$  is a real number and it represents the expected value of the observable  $P$  prepared in the state  $f$ , unit vector in  $\mathcal{H}$ . Hence, in view of (3.2),  $\|P_A(\cdot) f\|_{\mathcal{H}}^2$  is a Borel probability measure on  $sp(A)$ , and

$$\langle f, \varphi(A) f \rangle_{\mathcal{H}} = \int_{sp(A)} \varphi(z) \|P(dz) f\|_{\mathcal{H}}^2 \quad (3.3)$$

is the expected value of the observable  $\varphi(A)$ .

*Remark 3.1.* Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be measurable and let  $A = A^*$  be given; then, for every  $f \in \mathcal{H} \setminus \{0\}$ , set  $d\mu_f^{(A)}(\lambda) := \|P_A(d\lambda) f\|_{\mathcal{H}}^2 \in \mathcal{M}_+(\mathbb{R})$  (the finite positive Borel measures on  $\mathbb{R}$ .) Then the transformation formula (3.3) takes the following equivalent form:

$$d\mu_f^{(\varphi(A))} = d\mu_f^{(A)} \circ \varphi^{-1}, \text{ i.e.,} \quad (3.4)$$

$$d\mu_f^{(\varphi(A))}(\Delta) = d\mu_f^{(A)}(\varphi^{-1}(\Delta)), \forall \Delta \in \mathcal{B}(\mathbb{R}), \quad (3.5)$$

where  $\varphi^{-1}(\Delta) = \{x : \varphi(x) \in \Delta\}$ .

**Corollary 3.2.** *Let  $A = A^*$ , and  $f \in \mathcal{H} \setminus \{0\}$  be given, and let  $\mu_f$  and  $\varphi$  be as in Remark 3.1, then  $\varphi(A)^* = \varphi(A)$ , and*

$$f \in \text{dom}(\varphi(A)) \iff \varphi \in L^2(\mathbb{R}, \mu_f),$$

where “dom” is short for “domain.”

*Proof.* This is immediate from (3.3)-(3.5). Indeed, setting

$$d\mu_f(\lambda) := \|P(d\lambda) f\|_{\mathcal{H}}^2,$$

we get

$$\int_{\mathbb{R}} |\varphi(\lambda)|^2 d\mu_f(\lambda) = \|\varphi(A)f\|_{\mathcal{H}}^2.$$

□

*Remark 3.3.* The standard diagonalization of Hermitian matrices in linear algebra is a special case of the spectral theorem. Recall that any Hermitian matrix  $A$  can be decomposed as  $A = \sum_k \lambda_k P_k$ , where  $\lambda_k$ 's are the eigenvalues of  $A$  and  $P_k$ 's are the selfadjoint projections onto the eigenspaces associated with  $\lambda_k$ 's. The projection-valued measure in this case can be written as  $P(E) = \sum_{\lambda_k \in E} P_k$ , for all  $E \in \mathcal{B}(\mathbb{R})$ ; i.e., the counting measure supported on  $\lambda_k$ 's.

Quantum mechanics is stated using an abstract Hilbert space as the state space. In practice, one has the freedom to choose exactly which Hilbert space to use for a particular problem. Physical measurements remain unchanged when choosing different realizations of a Hilbert space. The concept needed here is unitary equivalence.

**Definition 3.4.** Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $B : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  be operators.  $A$  is said to be unitarily equivalent to  $B$  if there exists a unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $B = UAU^*$ .

Suppose  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a unitary operator,  $P : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is a selfadjoint projection. Then  $UPU^* : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  is a selfadjoint projection on  $\mathcal{H}_2$ . In fact,

$$(UPU^*)(UPU^*) = UPU^*$$

where we used  $UU^* = U^*U = I$ , since  $U$  is unitary. Let  $|f_1\rangle$  be a state in  $\mathcal{H}_1$  and  $|f_2\rangle = |Uf_1\rangle$  be the corresponding state in  $\mathcal{H}_2$ . Then

$$\langle f_2, UPU^*f_2 \rangle_{\mathcal{H}_2} = \langle U^*f_2, PU^*f_2 \rangle_{\mathcal{H}_1} = \langle f_1, Pf_1 \rangle_{\mathcal{H}_1}$$

i.e., the observable  $P$  has the same expectation value. Since every selfadjoint operator is, by the spectral theorem, decomposed into selfadjoint projections, it follows that the expectation value of any observable remains unchanged under unitary transformations.

We will also consider family of selfadjoint operators. Heisenberg's commutation relation  $PQ - QP = -iI$ ,  $i = \sqrt{-1}$ , is an important example of two non-commuting selfadjoint operators.

**Example 3.5.** The classical Fourier transform  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is unitary, so  $\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I_{L^2(\mathbb{R})}$ , and in particular, the Parseval identity

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})}^2 = \|f\|_{L^2(\mathbb{R})}^2$$

holds for all  $f \in L^2(\mathbb{R})$ .

**Example 3.6.** Let  $Q$  and  $P$  be the position and momentum operators in quantum mechanics. That is,  $Q = M_x$  = multiplication by  $x$ , and  $P = -id/dx$  both defined on the Schwartz space  $\mathcal{S}(\mathbb{R})$ —space of rapidly decreasing functions on  $\mathbb{R}$ , which is dense in the Hilbert space  $L^2(\mathbb{R})$ . On  $\mathcal{S}(\mathbb{R})$ , the operators  $P$  and  $Q$  satisfy the canonical commutation relation:  $PQ - QP = -iI_{L^2(\mathbb{R})}$ .

**Example 3.7.** Denote  $\mathcal{F}$  the Fourier transform on  $L^2(\mathbb{R})$  as before. Specifically, setting

$$\begin{aligned} (\mathcal{F}\varphi)(x) &= \widehat{\varphi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\xi) e^{-i\xi x} d\xi, \text{ and} \\ (\mathcal{F}^*\psi)(\xi) &= \psi^\vee(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{i\xi x} dx, \quad \xi \in \mathbb{R}. \end{aligned}$$

Note that  $\mathcal{F}$  is an automorphism in  $\mathcal{S}(\mathbb{R})$ , continuous with respect to the standard l.c. topology. Moreover,

$$(\mathcal{F}^*Q\mathcal{F}\varphi)(\xi) = \mathcal{F}^*(x\widehat{\varphi}(x)) = \frac{1}{i} \frac{d}{d\xi} \varphi(\xi), \quad \forall \varphi \in \mathcal{S}.$$

Therefore,

$$P = \mathcal{F}^*Q\mathcal{F} \tag{3.6}$$

and so  $P$  and  $Q$  are unitarily equivalent.

A multiplication operator version of the spectral theorem is also available. It works especially well in physics. It says that  $A$  is a normal operator in  $\mathcal{H}$  if and only if  $A$  is unitarily equivalent to the operator of multiplication by a measurable function  $f$  on  $L^2(X, \mu)$ , where  $X$  is locally compact and Hausdorff. The two versions are related via a measure transformation.

**Example 3.8.** Eq. (3.6) says that  $P$  is diagonalized by Fourier transform in the following sense.

Let  $\psi$  be any Borel function on  $\mathbb{R}$ , and set  $M_\psi$  = multiplication by  $\psi(x)$  in  $L^2(\mathbb{R})$ , with

$$\begin{aligned} \text{dom}(M_\psi) &= \{f \mid f, \psi f \in L^2(\mathbb{R})\} \\ &= \left\{ f \mid \int_{-\infty}^{\infty} (1 + |\psi(x)|^2) |f(x)|^2 dx < \infty \right\}; \end{aligned} \tag{3.7}$$

then we define, via eq. (3.6),

$$\psi(P) := \mathcal{F}^*\psi(Q)\mathcal{F}.$$

In particular, given any  $\Delta \in \mathcal{B}(\mathbb{R})$ , let  $\psi = \chi_\Delta$  = characteristic function, then

$$E(\Delta) = \mathcal{F}^*M_{\chi_\Delta}\mathcal{F}.$$

One checks directly that  $E(\Delta)^2 = E(\Delta) = E(\Delta)^*$ , so  $E(\Delta)$  is a selfadjoint projection. Indeed,  $E(\cdot)$  is a convolution operator, where

$$(E(\Delta)f)(x) = \int_a^b e^{i\xi x} \widehat{f}(\xi) d\xi = f * (\chi_{[a,b]})^\wedge(x), \quad \forall f \in L^2(\mathbb{R}).$$

Thus,

$$E(\Delta)(L^2(\mathbb{R})) = \left\{ f \in L^2(\mathbb{R}) \mid \text{supp}(\widehat{f}) \subset \Delta \right\};$$

i.e., the space of “band-limited” functions, with the “pass-band” being  $\Delta$ .

**Example 3.9.** Below, it helps to denote the Fourier transformed space (or frequency space) by  $L^2(\widehat{\mathbb{R}})$ . Fix any  $f \in L^2(\mathbb{R})$ ,  $\Delta \in \mathcal{B}(\mathbb{R})$ , then

$$\begin{aligned} \mu_f(\Delta) &:= \|E(\Delta)f\|_{L^2(\mathbb{R})}^2 &= \langle f, E(\Delta)f \rangle_{L^2(\mathbb{R})} \\ & &= \langle \mathcal{F}f, M_{\chi_\Delta} \mathcal{F}f \rangle_{L^2(\widehat{\mathbb{R}})} \\ & &= \int_\Delta |\widehat{f}(x)|^2 dx \end{aligned}$$

which is a Borel measure on  $\mathbb{R}$ , such that

$$\mu_f(\mathbb{R}) = \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|_{L^2(\mathbb{R})}^2$$

by the Parseval identity.

Now, let  $\psi(x) = x$  be the identity function, and note that it is approximated pointwisely by simple functions of the form  $\sum_{i=1}^n c_i \chi_{\Delta_i}$ , where  $\Delta_i \in \mathcal{B}(\mathbb{R})$ , and  $\Delta_i$ 's are mutually disjoint, i.e.,  $x = \lim_{n \rightarrow \infty} \sum_{i=1}^n c_i \chi_{\Delta_i}$ .

Fix  $f \in \text{dom}(M_x)$ , see (3.7), it follows from Lebesgue dominated convergence theorem, that

$$\begin{aligned} \langle f, Pf \rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \overline{\widehat{f}(x)} x \widehat{f}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \overline{\widehat{f}(x)} \left( \sum_{i=1}^n c_i \chi_{\Delta_i}(x) \right) \widehat{f}(x) dx \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n c_i \langle f, E(\Delta_i)f \rangle_{L^2(\mathbb{R})} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n c_i \|E(\Delta_i)f\|_{L^2(\mathbb{R})}^2 \\ &= \int_{-\infty}^{\infty} x \|E(dx)f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

The last step above yields the projection-valued measure (PVM) version of the spectral theorem for  $P$ , where we write

$$P = \int_{-\infty}^{\infty} x dE(x). \quad (3.8)$$

Consequently, we get two versions of the spectral theorem for  $P = \frac{1}{i} \frac{d}{dx} \Big|_{\mathcal{S}(\mathbb{R})}$ :

1. Multiplication operator version, i.e.,  $P \simeq M_x =$  multiplication by  $x$  in  $L^2(\widehat{\mathbb{R}})$ ; and
2. PVM version, as in (3.8).

This example illustrates the main ideas of the spectral theorem of a single selfadjoint operator in Hilbert space. We will develop the general theory in this chapter, and construct both versions of the spectral decomposition.

**Example 3.10.** Applying the Gram-Schmidt process to all polynomials against the measure  $d\mu = e^{-x^2/2} dx$ , one gets orthogonal polynomials  $P_n$  in  $L^2(\mu)$ . These are called the Hermite polynomials, and the associated Hermite functions are given by

$$h_n := e^{-x^2/2} P_n = e^{-x^2} \left( \frac{d}{dx} \right)^n e^{x^2/2}.$$

The Hermite functions (after normalization) forms an ONB in  $L^2(\mathbb{R})$ , which transforms  $P$  and  $Q$  to Heisenberg's infinite matrices in (1.68)-(1.69).

**Example 3.11** (The harmonic oscillator Hamiltonian). Let  $P, Q$  be as in the previous example. We consider the quantum Hamiltonian

$$H := \frac{1}{2}(Q^2 + P^2 - 1).$$

It can be shown that

$$Hh_n = nh_n$$

or equivalently,

$$(P^2 + Q^2)h_n = (2n + 1)h_n$$

$n = 0, 1, 2, \dots$ . That is,  $H$  is diagonalized by the Hermite functions.

$H$  is called the energy operator in quantum mechanics. This explains mathematically why the energy levels are discrete (in quanta), being a multiple the Plank's constant  $\hbar$ .

**Example 3.12** (Purely discrete spectrum v.s. purely continuous spectrum). The two operators  $P^2 + Q^2$  and  $P^2 - Q^2$  acting in  $L^2(\mathbb{R})$ ; see Figure 3.1.

*Remark 3.13.* Note that both of the two operators  $H_{\pm} := P^2 \pm Q^2$  in Example 3.12 are essentially selfadjoint as operators in  $L^2(\mathbb{R})$  (see [RS75, Nel69, vN32a, DS88c]), and with common dense domain equal to the Schwartz space. The potential in  $H_-$  is repulsive, see Figure 3.1 (b).

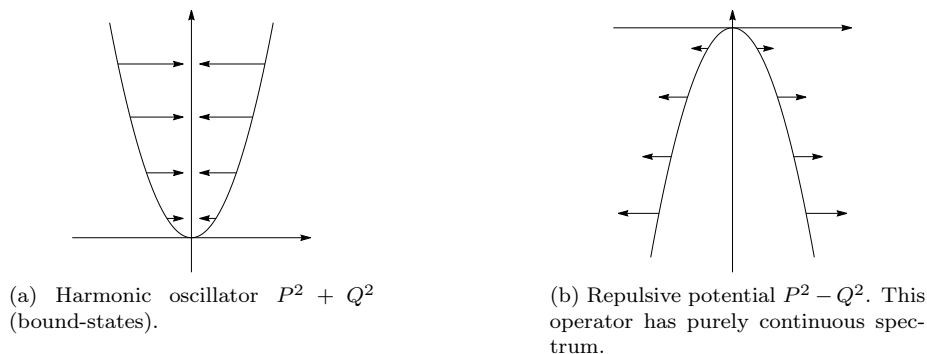


Figure 3.1: Illustration of forces: attractive vs repulsive. The case of “only bound states” (a), vs continuous Lebesgue spectrum (b).

By comparison, the operator  $H_4 := P^2 - Q^4$  is not essentially selfadjoint. (It can be shown that it has deficiency indices  $(2, 2)$ .) The following argument from physics is illuminating: For  $E \in \mathbb{R}_+$ , consider a classical particle  $x(t)$  on the energy surface

$$S_E := \left\{ x(t) : (x'(t))^2 - (x(t))^4 = E \right\}.$$

The travel time to  $\pm\infty$  is finite; in fact, it is

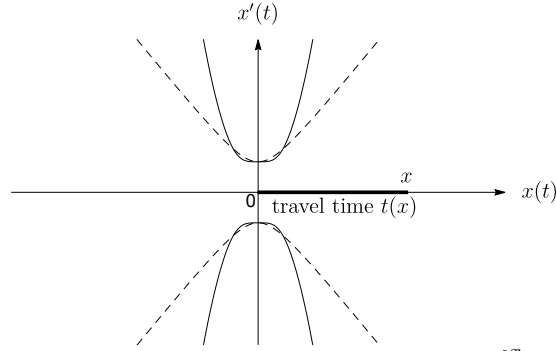
$$t_\infty = \int_0^\infty \frac{dx}{\sqrt{E + x^4}} < \infty.$$

There is a principle from quantum mechanics which implies that the quantum mechanical particle must be assigned conditions at  $\pm\infty$ , which translates into non-zero deficiency indices. (A direct computation, which we omit, yields indices  $(2, 2)$ .)

In the following sections, we present some main ideas of the spectral theorem for single normal operators acting in Hilbert space. Since every normal operator  $N$  can be written as  $N = T_1 + iT_2$ , where  $T_1$  and  $T_2$  are strongly commuting and selfadjoint, the presentation will be focused on selfadjoint operators.

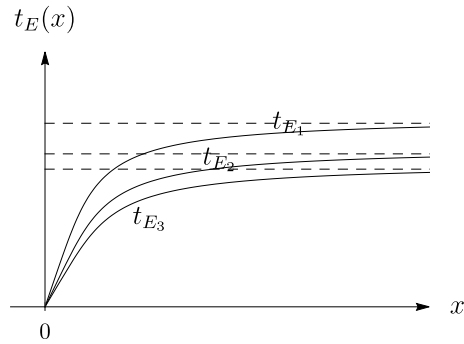
## 3.2 Multiplication Operator Version

Together, the results below serve to give a spectral representation (by multiplication operators) for the most general case: an arbitrary given selfadjoint (or normal) operator with dense domain in a Hilbert space. It applies both to the bounded, and unbounded cases; and it even applies to arbitrary families of strongly commuting selfadjoint operators. Caution: There is a number of subtle points in such representations. Since we aim for realizations up to unitary equivalence, care must be exercised in treating “multiplicity” for the most



A phase-space plot,  $(x, x')$ . Travel time  $t(x) = \int_0^x \frac{ds}{\sqrt{E + s^4}}$ .

Figure 3.2: The energy surface  $S_E$  for the quantum mechanical  $H_4 = P^2 - Q^4$ , with  $P \rightsquigarrow x'(t)$ .



Travel time  $t_E(x) = \int_0^x \frac{ds}{\sqrt{E + s^4}}$ .

Figure 3.3: Fix  $E_1 < E_2 < E_3$ , then  $0 < t_{E_3}(\infty) < t_{E_2}(\infty) < t_{E_1}(\infty)$ .



general spectral types. What we present below may be thought as a modern version of what is often called the *Hahn-Hellinger theory of spectral multiplicity*.

This version of the spectral theory states that every selfadjoint operator  $A$  is unitarily equivalent to the operator of multiplication by a measurable function on some  $L^2$ -space.

**Theorem 3.14.** *Let  $A$  be a linear operator acting in the Hilbert space  $\mathcal{H}$ , then  $A = A^*$  iff there exists a measure space  $(X, \mu)$  and a unitary operator  $U : L^2(X, \mu) \rightarrow \mathcal{H}$  such that*

$$M_\varphi = U^*AU; \quad (3.9)$$

where  $X$  is locally compact and Hausdorff,  $\varphi$  is a real-valued  $\mu$ -measurable function, and

$$M_\varphi f := \varphi f, \forall f \in \text{dom}(M_\varphi), \text{ where} \quad (3.10)$$

$$\text{dom}(M_\varphi) := \{h \in L^2(X, \mu) : \varphi h \in L^2(X, \mu)\}. \quad (3.11)$$

Hence, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{A} & \mathcal{H} \\ \uparrow U & & \uparrow U \\ L^2(X, \mu) & \xrightarrow{M_\varphi} & L^2(X, \mu) \end{array}$$

If  $A \in \mathcal{B}(\mathcal{H})$ , then  $\varphi \in L^\infty(X, \mu)$  and  $\text{dom}(M_\varphi) = \mathcal{H}$ .

We postpone the detailed proof till Section 3.2 below.

**Exercise 3.15** (Multiplication operators, continued). Prove that  $M_\varphi$  in (3.10)-(3.11) is selfadjoint.

**Exercise 3.16** (Continuous spectrum). Let  $M_t : L^2[0, 1] \rightarrow L^2[0, 1]$ ,  $f(t) \mapsto tf(t)$ . Show that  $M_t$  has no eigenvalues in  $L^2[0, 1]$ .

Before giving a proof of Theorem 3.14, we show below that one can go one step further and get that  $A$  is unitarily equivalent to the operator of multiplication by the independent variable on some  $L^2$ -space. This is done by a transformation of the measure  $\mu$  in (3.11).

## Transformation of Measures

**Definition 3.17.** Let  $\varphi : X \rightarrow Y$  be a measurable function,  $\mathcal{T}_X$  and  $\mathcal{T}_Y$  be the respective sigma-algebras. Fix a measure  $\mu$  on  $\mathcal{T}_X$ , the measure

$$\mu_\varphi := \mu \circ \varphi^{-1} \quad (3.12)$$

defined on  $\mathcal{T}_Y$  is called the transformation of  $\mu$  under  $\varphi$ . Note that

$$\chi_E \circ \varphi(x) = \chi_{\varphi^{-1}(E)}(x) \quad (3.13)$$

for all  $E \in \mathcal{T}_Y$  and  $x \in X$ .

**Lemma 3.18.** For all  $\mathcal{T}$ -measurable function  $f$ ,

$$\int_X f \circ \varphi d\mu = \int_Y f d(\mu \circ \varphi^{-1}). \quad (3.14)$$

(This is a generalization of the substitution formula in calculus.)

*Proof.* For any simple function  $s = \sum c_i \chi_{E_i}$ ,  $E_i \in \mathcal{T}_Y$ , it follows from (3.13) that

$$\begin{aligned} \int_X s \circ \varphi d\mu &= \sum c_i \int_X \chi_{E_i} \circ \varphi d\mu \\ &= \sum c_i \int_X \chi_{\varphi^{-1}(E_i)} d\mu \\ &= \sum c_i \mu(\varphi^{-1}(E_i)) \\ &= \int_Y s d(\mu \circ \varphi^{-1}). \end{aligned}$$

Note all the summations in the above calculation are finite.

Since any measurable function  $f : X \rightarrow Y$  is approximated pointwisely by simple functions, eq. (3.14) follows.  $\square$

*Remark 3.19.* If  $\varphi$  is nasty, even if  $\mu$  is a nice measure (say the Lebesgue measure), the transformation measure  $\mu \circ \varphi^{-1}$  in (3.14) can still be nasty, e.g., it could even be singular.

To simplify the discussion, we consider bounded selfadjoint operators below.

**Corollary 3.20.** Let  $\varphi : X \rightarrow X$  be any measurable function. Then the operator  $Uf := f \circ \varphi$  in  $L^2(X, \mu)$  is isometric iff  $\mu \circ \varphi^{-1} = \mu$ . Moreover,  $M_\varphi U = U M_t$ . In particular,  $U$  is unitary iff  $\varphi$  is invertible.

*Proof.* Follows immediately from Lemma 3.18.  $\square$

**Lemma 3.21.** In Theorem 3.14, assume  $A$  is bounded selfadjoint, so that  $\varphi \in L^\infty(X, \mu)$ , and real-valued. Let  $\mu_\varphi := \mu \circ \varphi^{-1}$  (eq. (3.12)), supported on the essential range of  $\varphi$ . Then the operator  $W : L^2(\mathbb{R}, \mu_\varphi) \rightarrow L^2(X, \mu)$ , by

$$(Wf)(x) = f(\varphi(x)), \quad \forall f \in L^2(\mu_\varphi) \quad (3.15)$$

is isometric, and

$$W M_t = M_\varphi W, \quad (3.16)$$

where  $M_t : L^2(Y, \mu_f) \rightarrow L^2(Y, \mu_f)$ , given by

$$(M_t f)(t) = t f(t); \quad (3.17)$$

i.e., multiplication by the identify function.

*Proof.* For all  $f \in L^2(Y, \mu_\varphi)$ , we have

$$\|f\|_{L^2(Y, \mu_\varphi)}^2 = \int_Y |f|^2 d\mu_\varphi = \int_X |f \circ \varphi|^2 d\mu = \|Wf\|_{L^2(X, \mu)}^2$$

so  $W$  is isometric. Moreover,

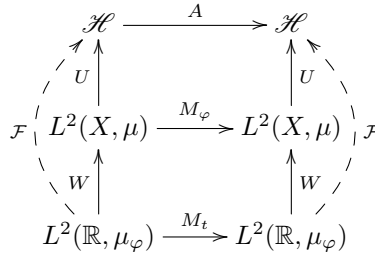
$$\begin{aligned} M_\varphi Wf &= \varphi(x) f(\varphi(x)) \\ WM_t f &= W(tg(t)) = \varphi(x) f(\varphi(x)) \end{aligned}$$

hence (3.16)-(3.17) follows.  $\square$

**Corollary 3.22.** *Let  $\varphi$  be as in Lemma 3.21. Assume  $\varphi$  is invertible, we get that  $W$  in (3.15)-(3.17) is unitary. Set  $\mathcal{F} = UW : L^2(\mathbb{R}, \mu_\varphi) \rightarrow \mathcal{H}$ , then  $\mathcal{F}$  is unitary and*

$$M_t = \mathcal{F}^* A \mathcal{F}; \tag{3.18}$$

*i.e., the following diagram commutes.*



*Remark 3.23.*

1. Eq. (3.18) is a vast extension of *diagonalizing hermitian matrices* in linear algebra, or a generalization of Fourier transform.
2. Given a selfadjoint operator  $A$  in the Hilbert space  $\mathcal{H}$ , what's involved are two algebras: the algebra of measurable functions on  $X$ , treated as multiplication operators, and the algebra of operators generated by  $A$  (with identity). The two algebras are  $*$ -isomorphic. The Spectral Theorem offers two useful tools:
  - (a) Representing the algebra generated by  $A$  by the algebra of functions. In this direction, it helps to understand  $A$ .
  - (b) Representing the algebra of functions by the algebra of operators generated by  $A$ . In this direction, it reveals properties of the function algebra and the underlying space  $X$ .
3. Let  $\mathfrak{A}$  be the algebra of functions. We say that  $\pi$  is a representation of  $\mathfrak{A}$  on the Hilbert space  $\mathcal{H}$ , denoted by  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ , if  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$

is a  $*$ -homomorphism, i.e.,  $\pi(gh) = \pi(g)\pi(h)$ , and  $\pi(\bar{g}) = \pi(g)^*$ , for all  $g, h \in \mathfrak{A}$ . Given  $\mathcal{F}$  as in (3.18), then

$$\pi(\psi) = \mathcal{F}M_\psi\mathcal{F}^* \in \text{Rep}(\mathfrak{A}, \mathcal{H}); \quad (3.19)$$

where the LHS in (3.19) defines the operator

$$\psi(A) := \pi(\psi), \quad \psi \in \mathfrak{A}. \quad (3.20)$$

To see that (3.20) is an algebra isomorphism, one checks that

$$\begin{aligned} (\psi_1\psi_2)(A) &= \mathcal{F}M_{\psi_1\psi_2}\mathcal{F}^* \\ &= \mathcal{F}M_{\psi_1}M_{\psi_2}\mathcal{F}^* \\ &= (\mathcal{F}M_{\psi_1}\mathcal{F}^*)(\mathcal{F}M_{\psi_2}\mathcal{F}^*) \\ &= \psi_1(A)\psi_2(A) \end{aligned}$$

using the fact that

$$M_{\psi_1\psi_2} = M_{\psi_1}M_{\psi_2}$$

i.e., multiplication operators always commute.

4. Eq. (3.19) is called the *spectral representation* of  $A$ . In particular, the spectral theorem of  $A$  implies the following substitution rule

$$\sum c_k x^k \longmapsto \sum c_k A^k$$

is well-defined, and it extends to all bounded measurable functions.

Let  $\varphi$  be as in Lemma 3.21. For the more general case when  $\varphi$  is not necessarily invertible, so  $W$  in (3.16) may not be unitary, we may still diagonalize  $A$ , i.e., get that  $A$  is unitarily equivalent to multiplication by the independent variable in some  $L^2$ -space; but now the corresponding  $L^2$ -space is vector-valued, and we get a direct integral representation. This approach is sketched in the next section.

## Direct Integral Representation

Throughout, we assume all the Hilbert spaces are separable.

The multiplication operator version of the spectral theorem says that  $A = A^*$   $\iff A \simeq M_\varphi$ , where

$$M_\varphi : L^2(X, \mu) \longrightarrow L^2(X, \mu) \quad (3.21)$$

$$f \longmapsto \varphi f. \quad (3.22)$$

Note that  $\varphi$  is real-valued. Moreover,  $A$  is bounded iff  $M_\varphi \in L^\infty(X, \mu)$ . When the Hilbert space  $\mathcal{H}$  is separable, we may further assume that  $\mu$  is finite, or a probability measure.

To further diagonalize  $M_\varphi$  in the case when  $\varphi$  is “nasty”, we will need the following tool from measure theory.

**Definition 3.24.** Let  $X$  be a locally compact and Hausdorff space, and  $\mu$  a Borel probability measure on  $X$ . Let  $\varphi : X \rightarrow Y$  be a measurable function, and set

$$\nu := \mu \circ \varphi^{-1}.$$

A *disintegration* of  $\mu$  with respect to  $\varphi$  is a system of probability measures  $\{\mu_y : y \in Y\}$  on  $X$ , satisfying

1.  $\mu_y(X \setminus \varphi^{-1}(\{y\})) = 0$ ,  $\nu$ -a.e. i.e.,  $\mu_y$  is supported on the “fiber”  $\varphi^{-1}(\{y\})$ .
2. For all Borel set  $E$  in  $X$ , the function  $y \mapsto \mu_y(E)$  is  $\nu$ -measurable, and

$$\mu(E) = \int_Y \mu_y(E) d\nu(y). \quad (3.23)$$

Now, back to the Spectral Theorem.

Let  $M_\varphi$  be as in (3.21)-(3.22), and let  $\nu := \mu \circ \varphi^{-1}$ , i.e., a Borel probability measure on  $\mathbb{R}$ . In fact,  $\nu$  is supported on the essential range of  $\varphi$ .

It is well-known that, in this case, there exists a unique (up to measure zero sets) disintegration of  $\mu$  with respect to  $\varphi$ . See, e.g., [Par82]. Therefore, we get the direct integral decomposition

$$L^2(\mu) \simeq \int^\oplus L^2(\mu_y) d\nu(y) \quad (\text{unitarily equivalent}) \quad (3.24)$$

where  $\{\mu_y : y \in \text{essential range of } \varphi \subset \mathbb{R}\}$  is the system of probability measures as in Definition 3.24.

The RHS in (3.24) is the Hilbert space consisting of measurable cross-sections  $f : \mathbb{R} \rightarrow \bigcup L^2(\mu_y)$ , where  $f(y) \in L^2(\mu_y)$ ,  $\forall y$ , and with the inner product given by

$$\langle f, g \rangle_{L^2(\nu)} := \int_Y \langle f(y), g(y) \rangle_{L^2(\mu_y)} d\nu(y). \quad (3.25)$$

**Exercise 3.25** (Direct integral Hilbert space). Let the setting be as in (3.23)-(3.24); let  $(Y, \mathcal{F}_Y, \nu)$  be a fixed measure space; and let  $\{\mu_y\}_{y \in Y}$  be a field of Borel measures. Show that the space of all functions  $f$  specified as follows (i)-(iii) form a Hilbert space:

- (i)  $f : \mathbb{R} \rightarrow \bigcup_{y \in Y} L^2(\mu_y)$ ;
- (ii)  $y \mapsto \|f(y, \cdot)\|_{L^2(\mu_y)}^2$  is measurable, and in  $L^1(Y, \nu)$ ; with
- (iii)  $\int_Y \|f(y, \cdot)\|_{L^2(\mu_y)}^2 d\nu(y) < \infty$ .

Set

$$\|f\|_{\text{Dir. sum}}^2 = \text{RHS in (iii)},$$

and define the corresponding inner product by the RHS in (3.25).

**Theorem 3.26.** Let  $M_\varphi : L^2(X, \mu) \rightarrow L^2(X, \mu)$  be the multiplication operator in (3.21)-(3.22),  $\nu := \mu \circ \varphi^{-1}$  as before. Then  $M_\varphi$  is unitarily equivalent to multiplication by the independent variable on  $\int^\oplus L^2(\mu_y) d\nu(y)$ .

For details, see, e.g., [Dix81, Seg50].

**Proof of Theorem 3.14**

We try to get the best generalization of diagonalizing Hermitian matrices in finite dimensional linear algebra.

Nelson's idea [Nel69] is to get from selfadjoint operators  $\rightarrow$  cyclic representation of function algebra  $\rightarrow$  measure  $\mu \rightarrow L^2(\mu)$ .

Sketch proof of Theorem 3.14:

1. Start with a single selfadjoint operator  $A$  acting in an abstract Hilbert space  $\mathcal{H}$ . Assume  $A$  is bounded.
2. Fix  $u \in \mathcal{H}$ . The set  $\{f(A)u\}$ , as  $f$  runs through some function algebra, generates a subspace  $\mathcal{H}_u \subset \mathcal{H}$ .  $\mathcal{H}_u$  is called a *cyclic subspace*, and  $u$  the corresponding *cyclic vector*. The function algebra might be taken as the algebra of polynomials, then later it is extended to a much bigger algebra containing polynomials as a dense sub-algebra.
3. Break up  $\mathcal{H}$  into a direct sum of mutually disjoint cyclic subspaces,

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j,$$

with the family of cyclic vectors  $u_j \in \mathcal{H}_j$ .

4. Each  $\mathcal{H}_j$  leaves  $A$  invariant, and the restriction of  $A$  to each  $\mathcal{H}_j$  is unitarily equivalent to  $M_x$  on  $L^2(sp(A), \mu_j)$ , where  $sp(A)$  denotes the spectrum of  $A$ .
5. Piecing together all the cyclic subspace: set

$$X = \bigsqcup_j sp(A), \quad \mu = \bigsqcup_j \mu_j$$

i.e., taking disjoint union as  $u_j$  runs through all the cyclic vectors. When  $\mathcal{H}$  is separable, we get  $\mathcal{H} = \bigoplus_{j \in \mathbb{N}} \mathcal{H}_j$ , and we may set  $\mu := \sum_{j=1}^{\infty} 2^{-j} \mu_j$ .

Details below.

**Lemma 3.27.** *There exists a family of cyclic vector  $\{u_\alpha\}$  such that  $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_{u_\alpha}$ , orthogonal sum of cyclic subspaces.*

*Proof.* An application of Zorn's lemma. See Theorem 4.32.  $\square$

**Lemma 3.28.** *Set  $K := [-\|A\|, \|A\|]$ . For each cyclic vector  $u$ , there exists a Borel measure  $\mu_u$  such that  $\text{supp}(\mu_u) \subset K$ ; and  $\mathcal{H}_{u_\alpha} \simeq L^2(K, \mu_u)$ .*

*Proof.* The map

$$f \mapsto w_u(f) := \langle u, f(A)u \rangle_{\mathcal{H}}$$

is a positive, bounded linear functional on polynomials over  $K$ ; the latter is dense in  $C(K)$  by Stone-Weierstrass theorem. Hence  $w_u$  extends uniquely to

$C(K)$ . ( $w_u$  is a *state* of the  $C^*$ -algebra  $C(K)$ . See Chapter 4.) By Riesz, there exists a unique Borel measure  $\mu_u$  on  $K$ , such that

$$w_u(f) = \langle u, f(A)u \rangle_{\mathcal{H}} = \int_K f d\mu_u. \quad (3.26)$$

Therefore we get  $L^2(K, \mu_u)$ , a Hilbert space containing polynomials as a dense subspace. Let

$$\mathcal{H}_u = \overline{\text{span}}\{f(A)u : f \in \text{polynomials}\}$$

Define  $W : \mathcal{H}_u \rightarrow L^2(K, \mu_u)$ , by

$$W : f(A)u \mapsto f \in L^2(K, \mu_u) \quad (3.27)$$

for polynomials  $f$ , which then extends to  $\mathcal{H}_u$  by density.  $\square$

**Lemma 3.29.** *Let  $W$  be the operator in (3.27), then*

1.  $W$  is an isometric isomorphism; and
2.  $WA = M_t W$ , i.e.,  $W$  intertwines  $A$  and  $M_t$ . Hence  $W$  diagonalizes  $A$ .

*Remark 3.30.*  $WA = M_t W \iff WAW^* = M_t$ . In finite dimension, it is less emphasized that the adjoint  $W^*$  equals the inverse  $W^{-1}$ . For finite dimensional case,  $M_t = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$  where the measure  $\mu = \sum_{\text{finite}} \delta_{\lambda_i}$ , where  $\delta_{\lambda_i}$  is the Dirac measure at the eigenvalue  $\lambda_i$  of  $A$ .

*Proof.* For the first part, let  $f \in L^2(K, \mu_u)$ , then

$$\begin{aligned} \|f\|_{L^2(K, \mu_u)}^2 &= \int_{\mathbb{R}} |f|^2 d\mu_u = \langle u, |f|^2(A)u \rangle_{\mathcal{H}} \\ &= \langle u, \bar{f}(A)f(A)u \rangle_{\mathcal{H}} \\ &= \langle u, f(A)^* f(A)u \rangle_{\mathcal{H}} \\ &= \langle f(A)u, f(A)u \rangle_{\mathcal{H}} \\ &= \|f(A)u\|_{\mathcal{H}}^2. \end{aligned}$$

Notice that strictly speaking,  $f(A^*)^* = \bar{f}(A)$ . Since  $A^* = A$ , we then get

$$f(A)^* = \bar{f}(A).$$

Also,  $f \xrightarrow{\pi} f(A)$  is a  $*$ -representation of the algebra  $C(K)$ ; i.e.,  $\pi$  is a homomorphism, and  $\pi(\bar{f}) = f(A)^*$ .

For the second part, let  $f(t) = a_0 + a_1 t + \dots + a_n t^n$  be any polynomial, then

$$\begin{aligned} W A f(A) &= W A (a_0 + a_1 A + a_2 A^2 + \dots + a_n A^n) \\ &= W (a_0 A + a_1 A^2 + a_2 A^3 + \dots + a_n A^{n+1}) \\ &= a_0 t + a_1 t^2 + a_2 t^3 + \dots + a_n t^{n+1} \\ &= t f(t) \\ &= M_t W f(A) \end{aligned}$$

thus  $WA = M_tW$ . The assertion then follows from standard approximation.

It remains to show that the isometry  $\mathcal{H}_u \xrightarrow{W} L^2(\mu_u)$  in (3.27) maps onto  $L^2(\mu_u)$ . But this follows from (3.26). Indeed if  $f \in L^2(\mu_u)$ , then  $f(A)u \in \mathcal{H}_u$  is well defined by the reasoning above. As a result, for the adjoint operator  $L^2(\mu_u) \xrightarrow{W^*} \mathcal{H}_u$ , we have

$$W^*(f) = f(A)u, \quad \forall f \in L^2(\mu_u). \quad (3.28)$$

□

Finally we piece together all the cyclic subspaces.

**Lemma 3.31.** *There exists a locally compact Hausdorff space  $X$  and a Borel measure  $\mu$ , a unitary operator  $\mathcal{F} : \mathcal{H} \rightarrow L^2(X, \mu)$ , such that*

$$A = \mathcal{F}^* M_\varphi \mathcal{F}$$

where  $\varphi \in L^\infty(\mu)$ .

*Proof.* Recall that we get a family of states  $w_j$ , with the corresponding measures  $\mu_j$ , and Hilbert spaces  $\mathcal{H}_j = L^2(\mu_j)$ . Note that all the  $L^2$ -spaces are on  $K = sp(A)$ . So it's the same underlying set, but with possibly different measures. □

To get a single measure space with  $\mu$ , Nelson [Nel69] suggested taking the disjoint union

$$X := \bigcup_j K \times \{j\}$$

and  $\mu :=$  the disjoint union of  $\mu_j$ 's. The existence of  $\mu$  follows from Riesz. Then we get

$$\mathcal{H} = \bigoplus \mathcal{H}_j \xrightarrow{\mathcal{F}} L^2(X, \mu).$$

*Remark 3.32.* Note the representation of  $L^\infty(X, \mu)$  onto  $\mathcal{H} = \sum^\oplus \mathcal{H}_j$  is highly non unique. There we enter into the multiplicity theory, which starts with breaking up each  $\mathcal{H}_j$  into irreducible components.

*Remark 3.33.* A representation  $\pi \in Rep(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$  is said to be multiplicity free if and only if  $\pi(\mathfrak{A})'$  is abelian. We say  $\pi$  has multiplicity equal to  $n$  if and only if  $(\pi(\mathfrak{A}))' \simeq M_n(\mathbb{C})$ . This notation of multiplicity generalizes the one in finite dimensional linear algebra. See Section 4.11.

**Exercise 3.34** (Multiplicity free). Prove that  $\pi \in Rep(L^\infty(\mu), L^2(\mu))$  is multiplicity free. Conclude that each cyclic representation is multiplicity free (i.e., it is maximal abelian.)

Hint: Suppose  $B \in \mathcal{B}(L^2(\mu))$  commutes with all  $M_\varphi$ ,  $\varphi \in L^2(\mu)$ . Define  $g = B\mathbb{1}$ , where  $\mathbb{1}$  is the constant function. Then, for all  $\psi \in L^2(\mu)$ , we have

$$B\psi = B\psi\mathbb{1} = BM_\psi\mathbb{1} = M_\psi B\mathbb{1} = M_\psi g = \psi g = g\psi = M_g\psi$$

thus  $B = M_g$ .



**Corollary 3.35.**  $T \in \mathcal{B}(\mathcal{H})$  is unitary iff there exists  $\mathcal{F} : \mathcal{H} \rightarrow L^2(X, d\mu)$ , unitary, such that

$$T = \mathcal{F}^* M_z \mathcal{F},$$

where  $|z| \in \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ .

**Exercise 3.36** (The Cayley transform). Finish the proof of Theorem 3.14 for the case when  $A$  is unbounded and selfadjoint.

Hint: Suppose  $A = A^*$ , unbounded. The Cayley transform

$$C_A := (A - i)(A + i)^{-1}$$

is then unitary. See Section 9.2. Apply Corollary 3.35 to  $C_A$ , and convert the result back to  $A$ . See, e.g., [Nel69, Rud73].

### 3.3 Projection-Valued Measure (PVM)

A projection valued measure (PVM)  $P$  satisfies the usual axioms of measures (here Borel measures) but with the main difference:

1.  $P(\Delta)$  is a projection for all  $\Delta \in \mathcal{B}$ , i.e.,  $P(\Delta) = P(\Delta)^* = P(\Delta)^2$ .
2. We assume that  $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$  for all  $\Delta_1, \Delta_2 \in \mathcal{B}$ ; this property is called “orthogonality”.

*Remark 3.37.* The notion of PVM extends the familiar notion of an ONB:

Let  $\mathcal{H}$  be a separable Hilbert space, and suppose  $\{u_k\}_{k \in \mathbb{N}}$  is an ONB in  $\mathcal{H}$ ; then for  $\Delta \in \mathcal{B}(\mathbb{R})$ , set

$$P(\Delta) := \sum_{k \in \Delta} |u_k\rangle\langle u_k|. \quad (3.29)$$

**Exercise 3.38** (A concrete PVM). Show that  $P$  is a PVM.

Note that, under the summation on the RHS in (3.29), we used Dirac’s notation  $|u_k\rangle\langle u_k|$  for the rank-one projection onto  $\mathbb{C}u_k$ . Further note that the summation is over  $k$  from  $\Delta$ ; so it varies as  $\Delta$  varies.

The PVM version of the spectral theorem says that  $A = A^*$  iff

$$A = \int x P_A(dx)$$

where  $P$  is a projection-valued measure defined on the Borel  $\sigma$ -algebra of  $\mathbb{R}$ .

**Definition 3.39.** Let  $\mathcal{B}(\mathbb{C})$  be the Borel  $\sigma$ -algebra of  $\mathbb{C}$ .  $\mathcal{H}$  is a Hilbert space.

$$P : \mathcal{B}(\mathbb{C}) \rightarrow \text{Proj}(\mathcal{H})$$

is a *projection-valued measure* (PVM), if

1.  $P(\emptyset) = 0$ ,  $P(\mathbb{C}) = I$ ,  $P(A)$  is a projection for all  $A \in \mathcal{B}$
2.  $P(A \cap B) = P(A)P(B)$
3.  $P(\cup_k E_k) = \sum P(E_k)$ ,  $E_k \cap E_j = \emptyset$  if  $k \neq j$ . The convergence is in terms of the strong operator topology. By assumption, the sequence of projections  $\sum_{k=1}^N P(E_k)$  is monotone increasing, hence it has a limit, and

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N P(E_k) = P(\cup E_k).$$

The standard Lebesgue integration extends to PVM.

$$\langle \varphi, P(E)\varphi \rangle = \langle P(E)\varphi, P(E)\varphi \rangle = \|P(E)\varphi\|^2 \geq 0$$

since  $P$  is countably additive, the map  $E \mapsto P(E)$  is also countably additive. Therefore, each  $\varphi \in \mathcal{H}$  induces a regular Borel measure  $\mu_\varphi$  on the Borel  $\sigma$ -algebra of  $\mathbb{R}$ .

For a measurable function  $\psi$ ,

$$\begin{aligned} \int \psi d\mu_\varphi &= \int \psi(x) \langle \varphi, P(dx)\varphi \rangle \\ &= \left\langle \varphi, \left( \int \psi P(dx) \right) \varphi \right\rangle \end{aligned}$$

hence we may define

$$\int \psi P(dx)$$

as the operator so that for all  $\varphi \in \mathcal{H}$ ,

$$\left\langle \varphi, \left( \int \psi P(dx) \right) \varphi \right\rangle.$$

*Remark 3.40.*  $P(E) = F\chi_E F^{-1}$  defines a PVM. In fact all PVMs come from this way. In this sense, the  $M_t$  version of the spectral theorem is better, since it implies the PVM version. However, the PVM version facilitates some formulations in quantum mechanics, so physicists usually prefer this version.

*Remark 3.41.* Suppose we start with the PVM version of the spectral theorem. How to prove  $(\psi_1\psi_2)(A) = \psi_1(A)\psi_2(A)$ ? i.e. how to check we do have an algebra isomorphism? Recall in the PVM version,  $\psi(A)$  is defined as the operator so that for all  $\varphi \in \mathcal{H}$ , we have

$$\int \psi d\mu_\varphi = \langle \varphi, \psi(A)\varphi \rangle.$$

As a standard approximation technique, once starts with simple or even step functions. Once it is worked out for simple functions, the extension to any measurable functions is straightforward. Hence let's suppose (WLOG) that the functions are simple.

**Lemma 3.42.** *We have  $\psi_1(A)\psi_2(A) = (\psi_1\psi_2)(A)$ .*

*Proof.* Let

$$\begin{aligned}\psi_1 &= \sum \psi_1(t_i)\chi_{E_i} \\ \psi_2 &= \sum \psi_2(t_j)\chi_{E_j}\end{aligned}$$

then

$$\begin{aligned}\int \psi_1 P(dx) \int \psi_2 P(dx) &= \sum_{i,j} \psi_1(t_i)\psi_2(t_j)P(E_i)P(E_j) \\ &= \sum_i \psi_1(t_i)\psi_2(t_i)P(E_i) \\ &= \int \psi_1\psi_2 P(dx)\end{aligned}$$

where we used the fact that  $P(A)P(B) = 0$  if  $A \cap B = \phi$ .  $\square$

*Remark 3.43.* As we delve into Nelson's lecture notes [Nel69], we notice that on page 69, there is another unitary operator. By piecing these operators together is precisely how we get the spectral theorem. This "piecing" is a vast generalization of Fourier series.

**Lemma 3.44.** *Pick  $\varphi \in \mathcal{H}$ , get the measure  $\mu_\varphi$  where*

$$\mu_\varphi(\cdot) = \|P(\cdot)\varphi\|^2$$

*and we have the Hilbert space  $L^2(\mu_\varphi)$ . Take  $\mathcal{H}_\varphi := \overline{\text{span}}\{\psi(A)\varphi : \psi \in L^2(\mu_\varphi)\}$ . Then the map*

$$\mathcal{H} \ni \psi(A)\varphi \mapsto \psi \in L^2(\mu_\varphi)$$

*is an isometry, and it extends uniquely to a unitary operator from  $\mathcal{H}_\varphi$  to  $L^2(\mu_\varphi)$ .*

*Proof.* We have

$$\begin{aligned}\|\psi(A)\varphi\|^2 &= \langle \psi(A)\varphi, \psi(A)\varphi \rangle \\ &= \langle \varphi, \bar{\psi}(A)\psi(A)\varphi \rangle \\ &= \int_{\mathbb{R}} |\psi(\lambda)|^2 \|P(d\lambda)\varphi\|^2 \\ &= \langle \varphi, |\psi|^2(A)\varphi \rangle \\ &= \int |\psi|^2 d\mu_\varphi.\end{aligned}$$

$\square$

*Remark 3.45.*  $\mathcal{H}_\varphi$  is called the cyclic space generated by  $\varphi$ . Before we can construct  $\mathcal{H}_\varphi$ , we must make sense of  $\psi(A)\varphi$ .

**Lemma 3.46** ([Nel69, p.67]). *Let  $p = a_0 + a_1x + \dots + a_nx^n$  be a polynomial. Then  $\|p(A)u\| \leq \max |p(t)|$ , where  $\|u\| = 1$  i.e.  $u$  is a state.*

*Proof.*  $M := \text{span}\{u, Au, \dots, A^n u\}$  is a finite dimensional subspace in  $\mathcal{H}$  (automatically closed). Let  $E$  be the orthogonal projection onto  $M$ . Then

$$p(A)u = Ep(A)Eu = p(EAE)u.$$

Since  $EAE$  is a Hermitian matrix on  $M$ , we may apply the spectral theorem for finite dimensional space and get

$$EAE = \sum \lambda_k P_{\lambda_k}$$

where  $\lambda_k$ 's are eigenvalues associated with the projections  $P_{\lambda_k}$ . It follows that

$$p(A)u = p\left(\sum \lambda_k P_{\lambda_k}\right)u = \left(\sum p(\lambda_k) P_{\lambda_k}\right)u$$

and

$$\begin{aligned} \|p(A)u\|^2 &= \sum |p(\lambda_k)|^2 \|P_{\lambda_k}u\|^2 \\ &\leq \max |p(t)|^2 \sum \|P_{\lambda_k}u\|^2 \\ &= \max |p(t)|^2 \end{aligned}$$

since

$$\sum \|P_{\lambda_k}u\|^2 = \|u\|^2 = 1.$$

Notice that  $I = \sum P_{\lambda_k}$ . □

*Remark 3.47.* How to extend this? polynomials - continuous functions - measurable functions.  $[-\|A\|, \|A\|] \subset \mathbb{R}$ ,

$$\|EAE\| \leq \|A\|$$

is a uniform estimate for all truncations. Apply Stone-Weierstrass' theorem to the interval  $[-\|A\|, \|A\|]$  we get that any continuous function  $\psi$  is uniformly approximated by polynomials. i.e.  $\psi \sim p_n$ . Thus

$$\|p_n(A)u - p_m(A)u\| \leq \max |p_n - p_m| \|u\| = \|p_n - p_m\|_\infty \rightarrow 0$$

and  $p_n(A)u$  is a Cauchy sequence, hence

$$\lim_n p_n(A)u =: \psi(A)u$$

where we may define the operator  $\psi(A)$  so that  $\psi(A)u$  is the limit of  $p_n(A)u$ .

### 3.4 Convert $M_\varphi$ to a PVM (projection-valued measure)

**Theorem 3.48.** *Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a selfadjoint operator. Then  $A$  is unitarily equivalent to the operator  $M_t$  of multiplication by the independent variable on the Hilbert space  $L^2(\mu)$ . There exists a unique projection-valued measure  $P$  so that*

$$A = \int tP(dt)$$

i.e. for all  $h, k \in \mathcal{H}$ ,

$$\langle k, Ah \rangle_{\mathcal{H}} = \int t \langle k, P(dt)h \rangle_{\mathcal{H}}.$$

*Proof.* The uniqueness part follows from a standard argument. We will only prove the existence of  $P$ .

Let  $\mathcal{F} : L^2(\mu) \rightarrow \mathcal{H}$  be the unitary operator so that  $A = \mathcal{F}M_t\mathcal{F}^*$ . Define

$$P(E) := \mathcal{F}\chi_E\mathcal{F}^*$$

for all  $E$  in the Borel  $\sigma$ -algebra  $\mathfrak{B}$  of  $\mathbb{R}$ . Then  $P(\emptyset) = 0$ ,  $P(\mathbb{R}) = I$ ; and for all  $E_1, E_2 \in \mathfrak{B}$ ,

$$\begin{aligned} P(E_1 \cap E_2) &= \mathcal{F}\chi_{E_1 \cap E_2}\mathcal{F}^{-1} \\ &= \mathcal{F}\chi_{E_1}\chi_{E_2}\mathcal{F}^{-1} \\ &= (\mathcal{F}\chi_{E_1}\mathcal{F}^{-1})(\mathcal{F}\chi_{E_2}\mathcal{F}^{-1}) \\ &= P(E_1)P(E_2). \end{aligned}$$

Suppose  $\{E_k\}$  is a sequence of mutually disjoint elements in  $\mathfrak{B}$ . Let  $h \in \mathcal{H}$  and write  $h = \mathcal{F}\widehat{h}$  for some  $\widehat{h} \in L^2(\mu)$ . Then

$$\begin{aligned} \langle h, P(\cup_k E_k)h \rangle_{\mathcal{H}} &= \langle \mathcal{F}\widehat{h}, P(\cup_k E_k)\mathcal{F}\widehat{h} \rangle_{\mathcal{H}} = \langle \widehat{h}, \mathcal{F}^{-1}P(\cup_k E_k)\mathcal{F}\widehat{h} \rangle_{L^2(\mu)} \\ &= \langle \widehat{h}, \chi_{\cup_k E_k}\widehat{h} \rangle_{L^2(\mu)} = \int_{\cup_k E_k} |\widehat{h}|^2 d\mu \\ &= \sum_k \int_{E_k} |\widehat{h}|^2 d\mu = \sum_k \langle h, P(E_k)h \rangle_{\mathcal{H}}. \end{aligned}$$

Therefore,  $P$  is a projection-valued measure.

For any  $h, k \in \mathcal{H}$ , write  $h = \mathcal{F}\widehat{h}$  and  $k = \mathcal{F}\widehat{k}$ . Then

$$\begin{aligned} \langle k, Ah \rangle_{\mathcal{H}} &= \langle \mathcal{F}\widehat{k}, A\mathcal{F}\widehat{h} \rangle_{\mathcal{H}} = \langle \widehat{k}, \mathcal{F}^*A\mathcal{F}\widehat{h} \rangle_{\mathcal{H}} \\ &= \langle \widehat{k}, M_t\widehat{h} \rangle_{L^2(\mu)} = \int t\widehat{k}(t)\widehat{h}(t)d\mu(t) \\ &= \int t \langle k, P(dt)h \rangle_{\mathcal{H}}. \end{aligned}$$

Thus  $A = \int tP(dt)$ . □

*Remark 3.49.* In fact,  $A$  is in the closed (under norm or strong topology) span of  $\{P(E) : E \in \mathfrak{B}\}$ . Equivalently, since  $M_t = \mathcal{F}^* A \mathcal{F}$ , the function  $f(t) = t$  is in the closed span of the set of characteristic functions; the latter is again a standard approximation in measure theory. It suffices to approximate  $t\chi_{[0,\infty]}$ .

The wonderful idea of Lebesgue is not to partition the domain, as was the case in Riemann integral over  $\mathbb{R}^n$ , but instead the range. Therefore integration over an arbitrary set is made possible. Important examples include analysis on groups.

**Proposition 3.50.** *Let  $f : [0, \infty] \rightarrow \mathbb{R}$ ,  $f(x) = x$ , i.e.  $f = x\chi_{[0,\infty]}$ . Then there exists a sequence of step functions  $s_1 \leq s_2 \leq \dots \leq f(x)$  such that  $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ .*

*Proof.* For  $n \in \mathbb{N}$ , define

$$s_n(x) = \begin{cases} i2^{-n} & x \in [i2^{-n}, (i+1)2^{-n}) \\ n & x \in [n, \infty] \end{cases}$$

where  $0 \leq i \leq n2^{-n} - 1$ . Equivalently,  $s_n$  can be written using characteristic functions as

$$s_n = \sum_{i=0}^{n2^{-n}-1} i2^{-n} \chi_{[i2^{-n}, (i+1)2^{-n})} + n\chi_{[n, \infty]}.$$

Notice that on each interval  $[i2^{-n}, (i+1)2^{-n})$ ,

$$\begin{aligned} s_n(x) &\equiv i2^{-n} \leq x \\ s_n(x) + 2^{-n} &\equiv (i+1)2^{-n} > x \\ s_n(x) &\leq s_{n+1}(x). \end{aligned}$$

Therefore, for all  $n \in \mathbb{N}$  and  $x \in [0, \infty]$ ,

$$x - 2^{-n} < s_n(x) \leq x \tag{3.30}$$

and  $s_n(x) \leq s_{n+1}(x)$ .

It follows from (3.30) that

$$\lim_{n \rightarrow \infty} s_n(x) = f(x)$$

for all  $x \in [0, \infty]$ . □

**Corollary 3.51.** *Let  $f(x) = x\chi_{[0,M]}(x)$ . Then there exists a sequence of step functions  $s_n$  such that  $0 \leq s_1 \leq s_2 \leq \dots \leq f(x)$  and  $s_n \rightarrow f$  uniformly, as  $n \rightarrow \infty$ .*

*Proof.* Define  $s_n$  as in Proposition 3.50. Let  $n > M$ , then by construction

$$f(x) - 2^{-n} < s_n(x) \leq f(x)$$

for all  $s \in [0, M]$ . Hence  $s_n \rightarrow f$  uniformly as  $n \rightarrow \infty$ . □

Proposition 3.50 and its corollary immediately imply the following.

**Corollary 3.52.** *Let  $(X, S, \mu)$  be a measure space. A function (real-valued or complex-valued) is measurable if and only if it is the point-wise limit of a sequence of simple functions. A function is bounded measurable if and only if it is the uniform limit of a sequence of simple functions. Let  $\{s_n\}$  be an approximation sequence of simple functions. Then  $s_n$  can be chosen such that  $|s_n(x)| \leq |f(x)|$  for all  $n = 1, 2, 3, \dots$*

**Theorem 3.53.** *Let  $M_f : L^2(X, S, \mu) \rightarrow L^2(X, S, \mu)$  be the operator of multiplication by  $f$ . Then,*

1. *if  $f \in L^\infty$ ,  $M_f$  is a bounded operator, and  $M_f$  is in the closed span of the set of selfadjoint projections under norm topology.*
2. *if  $f$  is unbounded,  $M_f$  is an unbounded operator.  $M_f$  is in the closed span of the set of selfadjoint projections under the strong operator topology.*

*Proof.* If  $f \in L^\infty$ , then there exists a sequence of simple functions  $s_n$  so that  $s_n \rightarrow f$  uniformly. Hence  $\|f - s_n\|_\infty \rightarrow 0$ , as  $n \rightarrow \infty$ .

Suppose  $f$  is unbounded. By Proposition 3.50 and its corollaries, there exists a sequence of simple functions  $s_n$  such that  $|s_n(x)| \leq |f(x)|$  and  $s_n \rightarrow f$  point-wisely, as  $n \rightarrow \infty$ . Let  $h$  be any element in the domain of  $M_f$ , i.e.

$$\int (|h| + |fh|^2) d\mu < \infty.$$

Then

$$\lim_{n \rightarrow \infty} |(f(x) - s_n(x))h(x)|^2 = 0$$

and

$$|(f(x) - s_n(x))h(x)|^2 \leq \text{const} \cdot |h(x)|^2.$$

Hence by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int |(f(x) - s_n(x))h(x)|^2 d\mu = 0$$

or equivalently,

$$\|(f - s_n)h\|^2 \rightarrow 0$$

as  $n \rightarrow \infty$ . i.e.  $M_{s_n}$  converges to  $M_f$  in the strong operator topology.  $\square$

**Exercise 3.54** (An application to numerical range). Let  $A$  be a bounded normal operator in a separable Hilbert space  $\mathcal{H}$ ; then prove that

$$NR_A \subseteq \overline{\text{conv}}(\text{spec}(A)); \quad (3.31)$$

i.e., that the numerical range of  $A$  is contained in the closed convex hull of the spectrum of  $A$ . (We refer to Exercise 1.88 for details on “numerical range.”)

Hint: Since  $A$  is normal, by the Spectral Theorem, it is represented in a PVM  $P_A(\cdot)$ , i.e., taking valued in  $\text{Proj}(\mathcal{H})$ . For  $x \in \mathcal{H}$ ,  $\|x\| = 1$ , we have

$$w_x(A) = \langle x, Ax \rangle = \int_{\text{spec}(A)} \lambda \|P_A(d\lambda)x\|^2, \text{ and} \quad (3.32)$$

$$\int_{\text{spec}(A)} \|P_A(d\lambda)x\|^2 = \|x\|^2 = 1, \quad (3.33)$$

so  $d\mu_x(\lambda) = \|P_A(d\lambda)x\|^2$  is a regular Borel probability measure on  $\text{spec}(A)$ . Now approximate (3.32) with simple functions on  $\text{spec}(A)$ .

**Quantum States** Let  $\mathcal{H}$  be a Hilbert space (corresponding to some quantum system), and let  $A$  be a selfadjoint operator in  $\mathcal{H}$ , possibly unbounded. Vectors  $f \in \mathcal{H}$  represent quantum states if  $\|f\| = 1$ .

The *mean* of  $A$  in the state  $f$  is

$$\langle f, Af \rangle = \int_{\mathbb{R}} \lambda \|P_A(d\lambda)f\|^2$$

where  $P_A(\cdot)$  denotes the spectral resolution of  $A$ .

The *variance* of  $A$  in the state  $f$  is

$$\begin{aligned} v_f(A) &= \|Af\|^2 - (\langle f, Af \rangle)^2 \\ &= \int_{\mathbb{R}} \lambda^2 \|P_A(d\lambda)f\|^2 - \left( \int_{\mathbb{R}} \lambda \|P_A(d\lambda)f\|^2 \right)^2 \\ &= \int_{\mathbb{R}} (\lambda - \langle f, Af \rangle)^2 \|P_A(d\lambda)f\|^2. \end{aligned} \quad (3.34)$$

**Theorem 3.55** (Uncertainty Principle). *Let  $\mathcal{D}$  be a dense subspace in  $\mathcal{H}$ , and  $A, B$  be two Hermitian operators such that  $A, B : \mathcal{D} \hookrightarrow \mathcal{D}$  (i.e.,  $\mathcal{D}$  is assumed invariant under both  $A$  and  $B$ .)*

Then,

$$\|Ax\| \|Bx\| \geq \frac{1}{2} |\langle x, [A, B]x \rangle|, \quad \forall x \in \mathcal{D}; \quad (3.35)$$

where  $[A, B] := AB - BA$  is the commutator of  $A$  and  $B$ .

In particular, setting

$$\begin{aligned} A_1 &:= A - \langle x, Ax \rangle \\ B_1 &:= B - \langle x, Bx \rangle \end{aligned}$$

then  $A_1, B_1$  are Hermitian, and

$$[A_1, B_1] = [A, B].$$

Therefore,

$$\|A_1x\| \|B_1x\| \geq \frac{1}{2} |\langle x, [A, B]x \rangle|, \quad \forall x \in \mathcal{D}.$$



*Proof.* By the Cauchy-Schwarz inequality (Lemma 1.29), and for  $x \in \mathcal{D}$ , we have

$$\begin{aligned}
 \|Ax\| \|Bx\| &\geq |\langle Ax, Bx \rangle| \\
 &\geq |\Im \{ \langle Ax, Bx \rangle \}| \\
 &= \frac{1}{2} \left| \langle Ax, Bx \rangle - \overline{\langle Ax, Bx \rangle} \right| \\
 &= \frac{1}{2} |\langle Ax, Bx \rangle - \langle Bx, Ax \rangle| \\
 &= \frac{1}{2} |\langle x, [AB - BA]x \rangle|.
 \end{aligned}$$

□

**Corollary 3.56.** *If  $[A, B] = ihI$ ,  $h \in \mathbb{R}_+$ , and  $\|x\| = 1$  (i.e.,  $x$  is a state); then*

$$w_x(A^2)^{\frac{1}{2}} w_x(B^2)^{\frac{1}{2}} \geq \frac{h}{2}. \quad (3.36)$$

**Exercise 3.57** (Heisenberg's uncertainty principle). Let  $\mathcal{H} = L^2(\mathbb{R})$ , and  $f \in L^2(\mathbb{R})$  given, with  $\|f\|^2 = \int |f(x)|^2 dx = 1$ . Suppose  $f \in \text{dom}(P) \cap \text{dom}(Q)$ , where  $P$  and  $Q$  are the momentum and position operators, respectively. (See eq. (2.4)-(2.5) in Section 2.1.)

Show that

$$v_f(P) v_f(Q) \geq \frac{1}{4}. \quad (3.37)$$

Inequality (3.37) is the mathematically precise form of Heisenberg's uncertainty relation, often written in the form

$$\sigma_f(P) \sigma_f(Q) \geq \frac{1}{2}$$

where  $\sigma_f(P) = \sqrt{v_f(P)}$ , and  $\sigma_f(Q) = \sqrt{v_f(Q)}$ .

## 3.5 The Spectral Theorem for Compact Operators

### Preliminaries

The setting for the first of the two Spectral Theorems (direct integral vs representation) we will consider is as following: (Restricting assumptions will be relaxed in subsequent versions!)

Let  $\mathcal{H}$  be a separable (typically infinite dimensional Hilbert space assumed here!) and let  $A \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$  be compact and selfadjoint, i.e.,  $A = A^*$ , and  $A$  is in the  $\|\cdot\|_{UN}$ -closure of  $\mathcal{FR}(\mathcal{H})$ . See Section 1.5.

**Theorem 3.58.** *With  $A$  as above, there is an orthonormal set  $\{u_k\}_{k \in \mathbb{N}}$ , and a sequence  $\{\lambda_k\}_{k \in \mathbb{N}}$  in  $\mathbb{R} \setminus \{0\}$ , such that  $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| \geq |\lambda_{k+1}| \geq \cdots$ ,  $\lim_{k \rightarrow \infty} \lambda_k = 0$ , and*

1.  $Au_k = \lambda_k u_k$ ,  $k \in \mathbb{N}$ ,
2.  $A = \sum_{k=1}^{\infty} \lambda_k |u_k\rangle\langle u_k|$ ,
3.  $\text{spec}(A) = \{\lambda_k\} \cup \{0\}$ ,
4.  $\dim \{v \in \mathcal{H} \mid Av = \lambda_k v\} < \infty$ , for all  $k \in \mathbb{N}$ .

The set  $\{u_k\}$  extends to an ONB, containing an ONB (possibly infinite) for the subspace

$$\ker(A) = \{v \in \mathcal{H} \mid Av = 0\}.$$

**Exercise 3.59** (An eigenspace). Let  $A \in \mathcal{B}(\mathcal{H})$  be compact, and let  $\lambda \in \mathbb{C} \setminus \{0\}$ . Show that the eigenspace

$$\mathcal{E}_\lambda := \text{Ker}(\lambda I - A)$$

is finite-dimensional.

Hint: Assume the contrary  $\dim \mathcal{E}_\lambda = \infty$ . Pick an ONB in  $\mathcal{E}_\lambda$ , say  $\{u_i\}_{i \in \mathbb{N}}$ . Since  $A$  is compact, the sequence  $\{Au_i\}_{i \in \mathbb{N}}$  has a convergent subsequence, say  $\{Au_{i_k}\}$ . But then

$$\|Au_{i_k} - Au_{i_l}\| \xrightarrow{k, l \rightarrow \infty} 0.$$

On the other hand,

$$\|Au_{i_k} - Au_{i_l}\|^2 = |\lambda|^2 \|u_{i_k} - u_{i_l}\|^2 = 2|\lambda|^2;$$

so a contradiction.

**Exercise 3.60** (Attaining the sup). Suppose  $A \in \mathcal{B}(\mathcal{H})$  is compact, and  $A^* = A$ . Suppose further that

$$\lambda = \sup \{\langle x, Ax \rangle : \|x\| = 1\}$$

satisfies  $\lambda > 0$ , strict.

1. Show that  $\|A\| = \lambda$ .
2. Show that, if  $\{x_i\}_{i \in \mathbb{N}}$  satisfies:

$$\|x_i\| = 1, \langle x_i, Ax_i \rangle \xrightarrow{i \rightarrow \infty} \lambda,$$

then  $\exists x \in \mathcal{H}$ ,  $\|x\| = 1$ , and a subsequence  $\{x_{i_k}\}$  such that

$$\|Ax_{i_k} - Ax_{i_l}\| \xrightarrow{k, l \rightarrow \infty} 0, \text{ and}$$

$$\langle x_{i_k} - x, v \rangle \xrightarrow{k \rightarrow \infty} 0, \forall v \in \mathcal{H}.$$

3. Conclude from (1)-(2) that  $Ax = \lambda x$ .
4. Make (1)-(2) the first step in an induction; thus finishing the proof of the Spectral Theorem for compact selfadjoint operators.

We begin with some preliminaries in the preparation for the proof.

**Lemma 3.61** (polarization identity). *Let  $X$  be a set, and  $f : X \times X \rightarrow \mathbb{C}$  a sesquilinear form, conjugate linear in the first variable and linear in the second variable. Then the following polarization identity holds:*

$$f(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k f(y + i^k x, y + i^k x) \quad (3.38)$$

for all  $x, y \in X$ .

*Proof.* A direct computation shows that

$$\begin{aligned} f(y+x, y+x) - f(y-x, y-x) &= 2f(y, x) + 2f(x, y), \text{ and} \\ i(f(y+ix, y+ix) - f(y-ix, y-ix)) &= -2f(y, x) + 2f(x, y) \end{aligned}$$

Adding the above two equations yields the desired result.  $\square$

**Corollary 3.62.** *A bounded operator  $A$  in  $\mathcal{H}$  is selfadjoint if and only if*

$$\langle x, Ax \rangle \in \mathbb{R}, \quad \forall x \in \mathcal{H}.$$

*Proof.* Suppose  $A$  is selfadjoint, i.e.,  $\langle x, Ay \rangle = \langle Ax, y \rangle$ ,  $\forall x, y \in \mathcal{H}$ . Setting  $x = y$ , then

$$\langle x, Ax \rangle = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle} \implies \langle x, Ax \rangle \in \mathbb{R}, \quad \forall x \in \mathcal{H}.$$

Conversely, suppose  $\langle x, Ax \rangle \in \mathbb{R}$ ,  $\forall x \in \mathcal{H}$ . Note that

$$(x, y) \mapsto \langle x, Ay \rangle \quad \text{and} \quad (x, y) \mapsto \langle Ax, y \rangle$$

are both sesquilinear forms defined on  $\mathcal{H}$ . It follows from Lemma 3.61, that

$$\langle x, Ay \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle y + i^k x, A(y + i^k x) \rangle, \text{ and} \quad (3.39)$$

$$\langle Ax, y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle A(y + i^k x), y + i^k x \rangle. \quad (3.40)$$

But by the assumption ( $\langle x, Ax \rangle \in \mathbb{R}$ ,  $x \in \mathcal{H}$ ), the RHS in (3.39) and (3.40) are equal. Therefore, we conclude that  $\langle x, Ay \rangle = \langle Ax, y \rangle$ ,  $\forall x, y \in \mathcal{H}$ , i.e.,  $A$  is selfadjoint.  $\square$

**Theorem 3.63.** Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{C}$ . Let  $f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  be a sesquilinear form, and set

$$M := \sup \{ |f(x, y)| : \|x\| = \|y\| = 1 \} < \infty.$$

Then there exists a unique bounded operator  $A$  in  $\mathcal{H}$ , satisfying

$$f(x, y) = \langle Ax, y \rangle, \quad \forall x, y \in \mathcal{H}; \quad \text{and} \quad (3.41)$$

$$\|A\| = M. \quad (3.42)$$

*Proof.* Given  $x, y \in \mathcal{H}$ , nonzero, we have

$$\left| f\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \right| \leq M \iff |f(x, y)| \leq M \|x\| \|y\|. \quad (3.43)$$

Thus, for each  $x \in \mathcal{H}$ , the map  $y \mapsto f(x, y)$  is a bounded linear functional on  $\mathcal{H}$ . By Riesz's theorem, there exists a unique element  $\xi_x \in \mathcal{H}$ , such that

$$f(x, y) = \langle \xi_x, y \rangle.$$

Set  $\xi_x := Ax$ ,  $x \in \mathcal{H}$ . (The uniqueness part of follows from Riesz.)

Note the map  $x \mapsto Ax$  is linear. For if  $c \in \mathbb{C}$ , then

$$f(x_1 + cx_2, y) = \langle A(x_1 + cx_2), y \rangle, \quad \text{and} \quad (3.44)$$

$$\begin{aligned} f(x_1 + cx_2, y) &= f(x_1, y) + \bar{c}f(x_2, y) \\ &= \langle Ax_1, y \rangle + \bar{c}\langle Ax_2, y \rangle \\ &= \langle Ax_1 + cAx_2, y \rangle; \end{aligned} \quad (3.45)$$

where in (3.45), we used the fact that  $f$  is conjugate linear in the first variable. It follows that  $A(x_1 + cx_2) = Ax_1 + cAx_2$ , i.e.,  $A$  is linear.

Finally,

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \left( \sup_{\|y\|=1} |\langle Ax, y \rangle| \right)$$

and (3.42) follows.  $\square$

**Corollary 3.64.** Any bounded operator  $A$  in  $\mathcal{H}$  is uniquely determined by the corresponding sesquilinear form  $(x, y) \mapsto \langle x, Ay \rangle$ ,  $(x, y) \in \mathcal{H} \times \mathcal{H}$ .

**Corollary 3.65.** For all  $A \in \mathcal{B}(\mathcal{H})$ , we have

$$\|A\| = \sup \{ |\langle x, Ay \rangle| : \|x\| = \|y\| = 1 \}. \quad (3.46)$$

Eq. (3.47) below is the key step in the proof of Theorem 3.58.

**Corollary 3.66.** Let  $A$  be a bounded selfadjoint operator in  $\mathcal{H}$ , then

$$\|A\| = \sup \{ |\langle x, Ax \rangle| : \|x\| = 1 \}. \quad (3.47)$$

*Proof.* Set  $M := \sup \{ |\langle x, Ax \rangle| : \|x\| = 1 \}$ .

For all unit vector  $x$  in  $\mathcal{H}$ , we see that

$$|\langle x, Ax \rangle| \leq \|x\| \|Ax\| \leq \|x\| \|x\| \|A\| = \|A\|;$$

where the first step above uses the Cauchy-Schwarz inequality. Thus,  $M \leq \|A\|$ .

Conversely, by the polarization identity (3.38), we have

$$\begin{aligned} 4 \langle x, Ay \rangle &= \langle A(x+y), x+y \rangle - \langle A(-x+y), -x+y \rangle \\ &\quad + i \langle A(ix+y), ix+y \rangle - i \langle A(-ix+y), -ix+y \rangle. \end{aligned} \quad (3.48)$$

Since  $A$  is selfadjoint, the four inner products on the RHS of (3.48) are all real-valued (Corollary 3.62). Therefore,

$$\Re \{ \langle x, Ay \rangle \} = \frac{1}{4} (\langle A(x+y), x+y \rangle - \langle A(-x+y), -x+y \rangle).$$

Now, there exists a phase factor  $e^{i\theta}$  (depending on  $x, y$ ) such that

$$\begin{aligned} |\langle x, Ay \rangle| &= e^{i\theta} \langle x, Ay \rangle \\ &= |\Re \{ \langle x, Ay \rangle \}| \\ &= \frac{1}{4} |\langle A(x+y), x+y \rangle - \langle A(-x+y), -x+y \rangle| \\ &\leq \frac{1}{4} M (\|x+y\|^2 + \|x-y\|^2) \\ &= \frac{1}{4} \|M\| (2\|x\|^2 + 2\|y\|^2) = M \end{aligned}$$

valid for all unit vectors  $x, y$  in  $\mathcal{H}$ , i.e., with  $\|x\| = \|y\| = 1$ . It follows from this and (3.46) that  $\|A\| \leq M$ .

Therefore, we have

$$M = \|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle|$$

which is the assertion.  $\square$

*Integral operators* with continuous kernel form an important subclass of compact operators.

Setting. Let  $X$  be a compact space,  $\mu$  a finite positive Borel measure on  $X$ , and

$$K : X \times X \longrightarrow \mathbb{C} \quad (3.49)$$

a given function, assumed *continuous* on  $X \times X$ . Define

$$T_K : L^2(\mu) \longrightarrow L^2(\mu) \text{ by}$$

$$(T_K f)(x) = \int_X K(x, y) f(y) d\mu(y), \quad \forall f \in L^2(\mu), \quad \forall x \in X. \quad (3.50)$$

**Exercise 3.67** (An application of Arzelà-Ascoli). Prove that  $T_K$  is a compact operator in  $L^2(\mu)$  subject to the stated assumptions above.

Hint:

Step 1. Show that, for  $\forall x_1, x_2 \in X$ , and  $f \in L^2(\mu)$ , we have:

$$\begin{aligned} & |T_K f(x_1) - T_K f(x_2)| \\ & \leq \sqrt{\mu(X)} \max_{y \in X} |K(x_1, y) - K(x_2, y)| \|f\|_{L^2(\mu)}. \end{aligned}$$

Step 2. Show that

$$|(T_K f)(x)| \leq \sqrt{\mu(X)} \max_{y \in X} |K(x, y)| \|f\|_{L^2(\mu)}.$$

Step 3. Conclude from steps 1-2, and an application of Arzelà-Ascoli's theorem that  $T_K : L^2(\mu) \rightarrow L^2(\mu)$  is a compact operator.

**Exercise 3.68** (Powers-Størmer [PS70]). Let  $\mathcal{H}$  be a Hilbert space, and let  $A$  and  $B$  be positive operators ( $\in \mathcal{B}(\mathcal{H})$ ). Then show that

$$\left\| A^{\frac{1}{2}} - B^{\frac{1}{2}} \right\|_{HS}^2 \leq \|A - B\|_{TR}. \quad (3.51)$$

(Note  $A^{\frac{1}{2}} = \sqrt{A}$  is defined via the Spectral Theorem.)

Hint: Difficult. (The inequality (3.51) is called the Powers-Størmer inequality.) Set  $S = A^{\frac{1}{2}} - B^{\frac{1}{2}}$ , and  $T = A^{\frac{1}{2}} + B^{\frac{1}{2}}$ .

Note that (3.51) is trivial if  $A - B$  is not trace-class, so assume it has finite trace-norm. Then diagonalize  $S$  in an ONB (use the Spectral Theorem), i.e., pick an ONB  $\{f_i\}$  of eigenvectors with eigenvalues  $\{\lambda_i\}$ ,  $Sf_i = \lambda_i f_i$ . Then

$$\begin{aligned} Tr(|A - B|) &= \frac{1}{2} \sum_i \langle f_i, |ST + TS| f_i \rangle \\ &\geq \sum_i |\lambda_i \langle f_i, T f_i \rangle| \\ &\geq \sum_i |\lambda_i|^2. \end{aligned}$$

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [Con90, FL28, Kat95, Kre55, Lax02, LP89, Nel69, RS75, Rud73, Sto51, Sto90, Yos95, HJL<sup>+</sup>13, DHL09, HKLW07, Alp01, CZ07, Fan10, Jor06, AG93, DS88c, Hal67, Jor02, Mac52, VN35, Hel13, MJD<sup>+</sup>15].

Part III

Applications

## Chapter 4

# GNS and Representations

If one finds a difficulty in a calculation which is otherwise quite convincing, one should not push the difficulty away; one should rather try to make it the centre of the whole thing.

— Werner Heisenberg

“Mathematics is a way of thinking in everyday life ...”

— I.M. Gelfand

“First quantization is a mystery; – second quantization is a functor.”

— Edward Nelson

Explanation:

A **category** is an algebraic structure that comprises "objects" linked by morphisms, also called "arrows". A category has two basic properties: one allows us to compose the arrows associatively; and the existence of an identity arrow for each object. A functor is a type of mapping, or transformation, between categories. Functors can be thought of as transformation between categories that transform the rules in the first category into those of the second: objects to objects, arrows to arrows, and diagrams to diagrams. In small categories, functors can be thought of as morphisms.

First quantization is the replacement of classical observables, such as energy or momentum, by operators; and of classical states by "wave functions". Second quantization usually refers to the introduction of field operators when describing quantum many-body systems. In second quantization, one passes from wave functions to an operators; hence the non-commutativity.

Expanding upon the more traditional interpretation, Ed Nelson suggested yet another second quantization functor; it goes from the category of Hilbert space (Hilb) to that of probability space (Prob). In this version, the objects in the category of Hilbert space “Hilb” are Hilbert spaces, and the morphisms are contractive linear operators.



In the category “Prob,” the morphisms are point-transformation of measures.

A state on a  $C^*$ -algebra  $\mathfrak{A}$  is a positive linear (and normalized) functional on  $\mathfrak{A}$ . Given a  $C^*$ -algebra  $\mathfrak{A}$ , then there is a bijective correspondence between states of  $\mathfrak{A}$ , on one side, and cyclic representations of  $\mathfrak{A}$  on the other; it is called the Gelfand-Naimark-Segal construction (abbreviated GNS), and it yields an explicit correspondence between the set of all cyclic  $*$ -representations of  $\mathfrak{A}$ ,  $Rep_{cyc}(\mathfrak{A})$ ; and the states of  $\mathfrak{A}$ ,  $S(\mathfrak{A})$ . It is named for Israel Gelfand, Mark Naimark, and Irving Segal.

The importance of the GNS construction is that it offers answers to a host of questions about representations of algebras, and of groups; the natural question regarding elementary building blocks.

More precisely, in the case of unitary representations of groups, the first questions that present themselves are: “What is the “right” notion of decomposition of a given unitary representation in terms of irreducible unitary representations of  $G$ ?” And “how to compute decompositions?” In this generality, there is not a precise answer. But if  $G$  is assumed locally compact and unimodular, I.E. Segal [Seg50] established precise answers. They entail direct integral theory for representations; see details below. The power of the GNS-construction (states vs representations) is that it allows us to answer a parallel question for states. Indeed, there is a precise notion of “building blocks” for states, they are the pure states (extreme points).

Segal’s insight was twofold: (i) Showing that the pure states correspond to irreducible representations via the GNS correspondence. And (ii), make the precise link between a direct integral decomposition for states, on one side, and on the other, direct integrals for unitary representations. Part (ii) in turn involves Krein-Milman and Choquet theory; see Sections 4.2, 4.8, and Chapter 8.

A corollary of the GNS construction is the Gelfand-Naimark theorem. The latter characterizes  $C^*$ -algebras as precisely the norm-closed  $*$ -algebras arising as  $*$ -subalgebras of  $\mathcal{B}(\mathcal{H})$ , the  $C^*$ -algebra of all bounded operators on a Hilbert space. By extreme-point theory [Phe01], one shows that every  $C^*$ -algebra has sufficiently many pure states (corresponding to irreducible representations under GNS). As a result, the representation of  $\mathfrak{A}$  arising as a direct sum of these corresponding irreducible GNS-representations is faithful.

Since states in quantum physics are vectors (of norm one) in Hilbert space; two question arise: “Where does the Hilbert space come from?” And “What are the algebras of operators from which the selfadjoint observables must be selected?” In a general framework, we offer an answer below, it goes by the name “the Gelfand-Naimark-Segal (GNS) theorem,” which offers a direct correspondence between states and cyclic representations.

Chapters 4 and 5 below form a pair; – to oversimplify, the theme in Chapter 5 is a generalization of that of the present chapter.

One could say that Chapter 4 is about scalar valued “states”; while the “states” in Chapter 5 are operator valued. In both cases, we must specify the appropriate notion of positivity, and this notion in the setting of Chapter 5 is more subtle; – it is called “complete positivity.”

But the goal in both cases is to induce in order to create representation of some given non abelian algebra  $\mathfrak{A}$  coming equipped with a star-involution; for example a  $C^*$ -algebra. The representations, when induced from states, will be  $*$ -representations; i.e., will take the  $*$ -involution in  $\mathfrak{A}$  to “adjoint operator” – where “adjoint” refers to the Hilbert space of the induced representation.

In Chapters 4-5, this notion of induction is developed in detail; and its counterpart for the case of unitary representations of groups is discussed in detail in Chapter 7.

Historically, the two notion of *induction of representations* were used by researchers in parallel universes, for the case of operator algebras (Chapters 4-5), they were pioneered by Gelfand, Naimark, Segal, and brought to fruition by Stinespring and Arveson.

On the other side of the divide, in the study of unitary representations of groups, the names are Harish-Chandra, G.W. Mackey (and more, see cited references in Chapter 7); – and this is the subject of Chapter 7 below. We caution the reader that the theory of unitary representations is a vast subject, and motivated by a number of diverse areas, such as quantum theory, ergodic theory, harmonic analysis, to mention just a few. And the theory of representations of groups, and their induction, is in turn developed by different researchers; and often with different groups  $G$  in mind; – continuous vs discrete; Lie groups vs the more general case of locally compact groups. The case when the group  $G$  is assumed locally compact is attractive because we then always will have left (or right-) Haar measure at our disposal. And there is an associated left-regular representation on the  $L^2$  space of Haar measure. The left-invariance of Haar measure makes this representation unitary. The analogous hold of course for the constructions with right-invariant Haar measure; the two are linked by the *modular function* of  $G$ .

## 4.1 Definitions and Facts: An Overview

Let  $\mathfrak{A}$  be an algebra over  $\mathbb{C}$ , with an involution  $\mathfrak{A} \ni a \mapsto a^* \in \mathfrak{A}$ , and the unit-element  $\mathbf{1}$ . Let  $\mathfrak{A}_+$  denote the set of positive elements in  $\mathfrak{A}$ ; i.e.,

$$\mathfrak{A}_+ = \{b^*b \mid b \in \mathfrak{A}\}.$$

**Definition 4.1.** We say  $\mathfrak{A}$  is a  $C^*$ -algebra if it is complete in a norm  $\|\cdot\|$ , which satisfies:

1.  $\|ab\| \leq \|a\| \|b\|, \forall a, b \in \mathfrak{A}$ ;
2.  $\|\mathbf{1}\| = \mathbf{1}$ ;

$$3. \|b^*b\| = \|b\|^2, \forall b \in \mathfrak{A}.^1$$

**Example 4.2.** Let  $X$  be a compact Hausdorff space, the algebra  $C(X)$  of all continuous function on  $X$  is a  $C^*$ -algebra under the sup-norm.

**Example 4.3.**  $\mathcal{B}(\mathcal{H})$ : all bounded linear operators on a fixed Hilbert space  $\mathcal{H}$  is a  $C^*$ -algebra.

**Example 4.4.**  $\mathcal{O}_N$ : the Cuntz-algebra,  $N > 1$ ; it is the  $C^*$ -completion of  $N$  generators  $s_1, s_2, \dots, s_N$  satisfying the following relations [Cun77]:

1.  $s_i^*s_j = \delta_{ij}\mathbf{1}$ ;
2.  $\sum_{i=1}^N s_i s_i^* = \mathbf{1}$ .

For the representations of  $\mathcal{O}_N$ , see [Gli60, Gli61, BJO04].

**Definition 4.5.** We denote  $Rep(\mathfrak{A}, \mathcal{H})$  the representations of  $\mathfrak{A}$  acting on some Hilbert space  $\mathcal{H}$ , i.e.,  $\pi \in Rep(\mathfrak{A}, \mathcal{H})$  iff  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a homomorphism of  $*$ -algebras,  $\pi(\mathbf{1}) = I_{\mathcal{H}}$  = the identity operator in  $\mathcal{H}$ ; in particular

$$\langle \pi(b)u, v \rangle_{\mathcal{H}} = \langle u, \pi(b^*)v \rangle_{\mathcal{H}}, \forall b \in \mathfrak{A}, \forall u, v \in \mathcal{H}. \quad (4.1)$$

Let  $S(\mathfrak{A})$  be the states  $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$  on  $\mathfrak{A}$ ; i.e., (axioms)  $\varphi \in \mathfrak{A}^*$  = the dual of  $\mathfrak{A}$ ,  $\varphi(\mathbf{1}) = 1$ , and

$$\varphi(b^*b) \geq 0, \forall b \in \mathfrak{A}. \quad (4.2)$$

**Theorem 4.6** (Gelfand-Naimark-Segal (GNS)). *There is a bijection:*

$$S(\mathfrak{A}) \longleftrightarrow \boxed{\text{cyclic representations, up to unitary equivalence}}$$

as follows:

$\longleftarrow$  (easy direction): Given  $\pi \in Rep(\mathfrak{A}, \mathcal{H})$ ,  $u_0 \in \mathcal{H}$ ,  $\|u_0\| = 1$ , set

$$\varphi(a) = \langle u_0, \pi(a)u_0 \rangle_{\mathcal{H}}, \forall a \in \mathfrak{A}. \quad (4.3)$$

$\longrightarrow$  (non-trivial direction): Given  $\varphi \in S(\mathfrak{A})$ , there is a system  $(\pi, \mathcal{H}, u_0)$  such that (4.3) holds. (Notation, we set  $\pi = \pi_{\varphi}$  to indicate the state  $\varphi$ .)

*Proof.* ( $\longrightarrow$ ) Given  $\varphi \in S(\mathfrak{A})$ , then on  $\mathfrak{A} \times \mathfrak{A}$  consider the sesquilinear form

$$(a, b) \mapsto \varphi(a^*b) \quad (4.4)$$

$$\mathcal{H}_{\varphi} = \{\mathfrak{A} / \{b \in \mathfrak{A} \mid \varphi(b^*b) = 0\}\}^{\sim}$$

where  $\sim$  refers to Hilbert completion in (4.4). Note

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \forall a, b \in \mathfrak{A}.$$

<sup>1</sup>Kadison et al. in 1950's reduced the axioms of  $C^*$ -algebra from about 6 down to just one (3) on the  $C^*$ -norm.

Set  $\Omega = \text{class}(\mathbf{1})$  in  $\mathcal{H}_\varphi$ , and

$$\pi_\varphi(a)(\text{class}(b)) = \text{class}(ab), \quad \forall a, b \in \mathfrak{A}; \quad (\text{Schwarz.})$$

Then it is easy to show that  $(\mathcal{H}_\varphi, \Omega, \pi_\varphi)$  satisfies conclusion (4.3), i.e.,

$$\varphi(a) = \langle \Omega, \pi(a)\Omega \rangle_{\mathcal{H}_\varphi}, \quad \forall a \in \mathfrak{A}.$$

□

**Definition 4.7.** Let  $\varphi \in S(\mathfrak{A})$ , we say it is a *pure state* iff  $\varphi \in \text{ext}S(\mathfrak{A}) :=$  the extreme-points in  $S(\mathfrak{A})$ . (See [Phe01].)

*Remark 4.8* (GNS-correspondence). 1. If  $\mathfrak{A}$  is a  $C^*$ -algebra, then  $S(\mathfrak{A}) (\subset \mathfrak{A}^*)$  is convex and weak \*-compact.

2. Given  $\varphi \in S(\mathfrak{A})$ , and let  $\pi_\varphi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$  be the GNS-representation, see (4.3); then

$$\begin{array}{c} \boxed{\varphi \in \text{ext}S(\mathfrak{A}), \text{ i.e., it is pure}} \\ \Downarrow \\ \boxed{\pi_\varphi \text{ is an irreducible representation}} \end{array}$$

3. If  $\psi \in S(\mathfrak{A})$ ,  $\exists$  a measure  $P_\psi$  on  $\text{ext}(S(\mathfrak{A}))$  such that  $\psi = \int w dP_\psi(w)$ , and then

$$\pi_\psi = \int^\oplus \pi_w dP_\psi(w).$$

**Example 4.9** (Pure states, cases where the full list is known!).  $\varphi \in S(\mathfrak{A})$ :

$\mathfrak{A}$	$\text{ext}S(\mathfrak{A})$
$C(X)$	points $x \in X$ , and $\varphi = \delta_x$ (Dirac mass); $\varphi(f) = f(x)$ , $\forall f \in C(X)$
$\mathcal{B}(\mathcal{H})$	$v \in \mathcal{H}, \ v\  = 1, \varphi = \varphi_v; \varphi_v(A) = \langle v, Av \rangle, \forall A \in \mathcal{B}(\mathcal{H})$
$\mathcal{O}_N$	Partial list: $u = (u_1, \dots, u_N) \in \mathbb{C}^N, \sum_1^N  u_j ^2 = 1, \varphi = \varphi_u$ , specified by $\varphi(s_i s_j^*) = u_i \bar{u}_j, \forall i, j = 1, \dots, N$ ; see 1-2.

Table 4.1: Examples of pure states.

**Exercise 4.10** (Irreducible representations). Using GNS, write down explicitly the irreducible representations of the three  $C^*$ -algebras in Table 4.1 corresponding to the listed pure states.

Hint: In the case of  $C(X)$ , the representations are one-dimensional, but in the other cases, they are infinite-dimensional, i.e.,  $\dim \mathcal{H}_{\pi_\varphi} = \infty$ .

*Remark 4.11.* It is probably impossible to list all pure states of  $\mathcal{O}_N$ ; see [Gli60].

**Exercise 4.12** (Infinite-product measures and representations of  $\mathcal{O}_N$ ). Fix  $N \in \mathbb{N}$ ,  $N > 1$ , and denote the cyclic group of order  $N$ ,

$$\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{0, 1, 2, \dots, N-1\},$$

residue classes mod  $N$ . Let  $(z_i)_{i=0}^{N-1}$  be complex numbers such that  $\sum_i |z_i|^2 = 1$ , and assume  $z_j \neq 0$  for all  $j$ ; see line 3 in Table 4.1.

Let  $p$  be the probability measure on  $\mathbb{Z}_N$  with weights  $p_i = |z_i|^2$ , and let  $\mu = \mu_p$  be the infinite-product measure on  $\Omega_N := \times_{\mathbb{N}} \mathbb{Z}_N = \times_{\mathbb{N}} \{0, 1, \dots, N-1\}$ ,

$$\mu_p := \times_{\mathbb{N}} p = \underbrace{p \times p \times \dots}_{\aleph_0\text{-infinite}}. \quad (4.5)$$

Set  $\Omega_N(j) = \{(x_i) \in \Omega_N; x_1 = j\} = \{j\} \times \Omega_N$ .

1. For all  $x = (x_1, x_2, x_3, \dots)$ , and all  $f \in L^2(\mu)$ , set

$$(S_j f)(x) = \frac{1}{z_j} \chi_{\Omega_N(j)}(x) f(x_2, x_3, \dots).$$

Show that the adjoint operator with respect to  $L^2(\mu)$  is

$$(S_j^* f)(x) = z_j f(j, x_1, x_2, x_3, \dots);$$

and that this system  $\{S_j\}_{j=0}^{N-1}$  defines an irreducible representation of  $\mathcal{O}_N$ , i.e., is in  $Rep_{irr}(\mathcal{O}_N, L^2(\mu))$ .

2. Denote the representation in (1)  $\pi_p^{(N)}$ , and setting  $\mathbb{1}$  to be the constant function in  $L^2(\mu)$ , show that we recover the pure state from line 3 in Table 4.1 corresponding to  $u_j = z_j$ ; i.e., using the formula:

$$\varphi(s_j s_k^*) = \langle \mathbb{1}, S_j S_k^* \mathbb{1} \rangle_{L^2(\mu)} = z_j \overline{z_k}, \quad \forall j, k \in \mathbb{Z}_N.$$

3. Show that  $\pi_p^{(N)}$  is not irreducible when restricted to the abelian subalgebra in  $\mathcal{O}_N$  generated by  $\{S_J S_J^*\}$ , as  $J$  ranges over all finite words in the fixed alphabet  $\mathbb{Z}_N$ .

**Exercise 4.13** (A representation of  $\mathcal{O}_N$ ). What can you say about the representation of  $\mathcal{O}_N$  corresponding to  $(0, z_1, \dots, z_{N-1})$ ,  $\sum |z_j|^2 = 1$ ?

Hint: Modify (1) from Exercise 4.12. (The state  $\varphi(s_j s_k^*) = z_j \overline{z_k}$  then yields  $\varphi(s_j s_0^*) = 0, \forall j \in \mathbb{Z}_N$ .)

## Groups

Case 1. Groups contained in  $\mathcal{B}(\mathcal{H})$  where  $\mathcal{H}$  is a fixed Hilbert space:

**Definition 4.14.** Set

- $\mathcal{B}(\mathcal{H})^{-1}$ : all bounded linear operators in  $\mathcal{H}$  with bounded inverse.
- $\mathcal{B}(\mathcal{H})_{uni}$ : all unitary operators  $u : \mathcal{H} \rightarrow \mathcal{H}$ , i.e.,  $u$  satisfies

$$uu^* = u^*u = I_{\mathcal{H}}.$$

**Definition 4.15.** Fix a group  $G$ , and set:

- $Rep(G, \mathcal{H})$ : all homomorphisms  $\rho \in G \rightarrow \mathcal{B}(\mathcal{H})^{-1}$
- $Rep_{uni}(G, \mathcal{H})$ : all homomorphisms,  $\rho : G \rightarrow \mathcal{B}_{uni}(\mathcal{H})$ , i.e.,

$$\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*, \forall g \in G$$

- $Rep_{cont}(G, \mathcal{H})$ : Elements  $\rho \in Rep(G, \mathcal{H})$  such that  $\forall v \in \mathcal{H}$ ,

$$G \ni g \mapsto \rho(g)v$$

is continuous from  $G$  into  $\mathcal{H}$ ; called *strongly continuous*.

*Remark 4.16.* In the case of  $Rep_{cont}(G, \mathcal{H})$  it is assumed that  $G$  is a continuous group, i.e., is equipped with a topology such that the following two operations are both continuous:

1.  $G \times G \ni (g_1, g_2) \mapsto g_1g_2 \in G$
2.  $G \ni g \mapsto g^{-1} \in G$

**Exercise 4.17** (The regular representation of  $G$ ). Let  $G$  be a locally compact group with  $\mu =$  a left-invariant Haar measure. Set

$$(\rho_L(g)f)(x) := f(g^{-1}x), \quad g, x \in G, \quad f \in L^2(G, \mu).$$

Then show that  $\rho_L$  is a strongly continuous unitary representation of  $G$  acting in  $L^2(G, \mu)$ .

## The Group Algebra

Let  $G$  be a group, and set  $\mathbb{C}[G] :=$  all linear combinations, i.e., finite sums

$$A = \sum_g A_g g \tag{4.6}$$

where  $A_g \in \mathbb{C}$ , and making  $\mathfrak{A} := \mathbb{C}[G]$  into a  $*$ -algebra with the following two operations on finite sums as in (4.6):  $\mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ , given by

$$\left( \sum_{g \in G} A_g g \right) \left( \sum_{h \in G} B_h h \right) := \sum_g \left( \sum_{hk=g} A_h B_k \right) g \quad (4.7)$$

and

$$\left( \sum_{g \in G} A_g g \right)^* := \sum_{g \in G} \overline{A_g} g^{-1}. \quad (4.8)$$

**Lemma 4.18.** *There is a bijection between  $\text{Rep}_{\text{uni}}(G, \mathcal{H})$  and  $\text{Rep}(\mathbb{C}[G], \mathcal{H})$  as follows: If  $\pi \in \text{Rep}_{\text{uni}}(G, \mathcal{H})$ , set  $\tilde{\pi} \in \text{Rep}(\mathbb{C}[G], \mathcal{H})$ :*

$$\tilde{\pi} \left( \sum_{g \in G} A_g g \right) := \sum_{g \in G} A_g \pi(g) \quad (4.9)$$

where the element  $\sum_{g \in G} A_g g$  in (4.9) is a generic element in  $\mathbb{C}[G]$ , see (4.6), i.e., is a finite sum with  $A_g \in \mathbb{C}$ , for all  $g \in G$ .

**Exercise 4.19** (Unitary representations). Fill in the proof details of the assertion in Lemma 4.18.

**Example 4.20.** Let  $G$  be a group, considered as a countable discrete group (the countability is not important). Set  $\mathcal{H} = l^2(G)$ , and

$$\pi(g) \delta_h := \delta_{gh}, \quad \forall g, h \in G. \quad (4.10)$$

**Exercise 4.21** (A proof detail). Show that  $\pi$  in (4.10) is in  $\text{Rep}(G, l^2(G))$ .

**Definition 4.22.** Let  $G$ , and  $\pi \in \text{Rep}(G, l^2(G))$  be as in (4.10), and let  $\tilde{\pi} \in \text{Rep}(\mathbb{C}[G], l^2(G))$  be the corresponding representation of  $\mathbb{C}[G]$ ; see Lemma 4.18. Set

$$C_{\text{red}}^*(G) := \text{the norm closure of } \tilde{\pi}(\mathbb{C}[G]) \subset \mathcal{B}(l^2(G));$$

then  $C_{\text{red}}^*(G)$  is called the reduced  $C^*$ -algebra of the group  $G$ .

**Exercise 4.23** (Reduced  $C^*$ -algebra). Prove that  $C_{\text{red}}^*(G)$  is a  $C^*$ -algebra.

*Remark 4.24.* It is known [Pow75] that  $C_{\text{red}}^*(F_2)$  is simple, where  $F_2$  is the free group on two generators. (“red” short for reduced; it is called the reduced  $C^*$ -algebra on the group.)

## 4.2 The GNS Construction

The GNS construction is a general principle for getting representations from given data in applications, especially in quantum mechanics [Pol02, PK88, CP82]. It was developed independently by I. Gelfand, M. Naimark, and I. Segal around the 1960s, see e.g., [GJ60, Seg50].

**Definition 4.25.** Let  $\mathfrak{A}$  be a  $*$ -algebra with identity. A representation of  $\mathfrak{A}$  is a map  $\pi : \mathfrak{A} \rightarrow B(\mathcal{H}_\pi)$ , where  $\mathcal{H}_\pi$  is a Hilbert space, such that for all  $A, B \in \mathfrak{A}$ ,

1.  $\pi(AB) = \pi(A)\pi(B)$
2.  $\pi(A^*) = \pi(A)^*$

The  $*$  operation (involution) is given on  $\mathfrak{A}$  so that  $A^{**} = A$ ,  $(AB)^* = B^*A^*$ ,  $(\lambda A)^* = \bar{\lambda}A^*$ , for all  $\lambda \in \mathbb{C}$ .

**Example 4.26.** The multiplication version of the spectral theorem of a single selfadjoint operator, say  $A$  acting on  $\mathcal{H}$ , yields a representation of the algebra of  $L^\infty(sp(A))$  (or  $C(sp(A))$ ) as operators on  $\mathcal{H}$ , where

$$L^\infty(sp(A)) \ni f \xrightarrow{\pi} f(A) \in \mathcal{B}(\mathcal{H})$$

such that  $\pi(fg) = \pi(f)\pi(g)$  and  $\pi(\bar{f}) = \pi(f)^*$ .

The general question is given any  $*$ -algebra, where to get such a representation? The answer is given by *states*. One gets representations from algebras vis states. For abelian algebras, the states are Borel measures, so the measures come out as a corollary of representations.

**Definition 4.27.** Let  $\mathfrak{A}$  be a  $*$ -algebra. A state on  $\mathfrak{A}$  is a linear functional  $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$  such that  $\varphi(1_{\mathfrak{A}}) = 1$ , and  $\varphi(A^*A) \geq 0$ , for all  $A \in \mathfrak{A}$ .

**Example 4.28.** Let  $\mathfrak{A} = C(X)$ , i.e.,  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . Note that there is a natural involution  $f \mapsto f^* := \bar{f}$  by complex conjugation. Let  $\mu_\varphi$  be a Borel probability measure on  $X$ , then

$$C(X) \ni f \mapsto \varphi(f) = \int_X f d\mu_\varphi$$

is a state. In fact, in the abelian case, all states are Borel probability measures.

Because of this example, we say that the GNS construction is non-commutative measure theory.

**Example 4.29.** Let  $G$  be a discrete group, and let  $\mathfrak{A} = \mathbb{C}[G]$  be the group-algebra, see Section 4.1.

If we make the assumption (defining  $\varphi$  first on points in  $G$ )

$$\varphi(g) = \begin{cases} 1 & \text{if } g = e \text{ (the unit element in } G) \\ 0 & \text{if } g \in G \setminus \{e\}, \end{cases} \quad (4.11)$$

then the argument from above shows that  $\varphi$  extends to a *linear functional* on  $\mathfrak{A}$ .



**Exercise 4.30** (The trace state on  $\mathbb{C}[G]$ ).

1. Show that  $\varphi$  as defined in (4.11), extended to  $\mathfrak{A} = \mathbb{C}[G]$  is a state, and if  $A = \sum_g A_g g$  (finite sum), then

$$\varphi(A^*A) = \sum_{g \in G} |A_g|^2; \quad (4.12)$$

and moreover (the trace property):

$$\varphi(AB) = \varphi(BA), \quad \forall A, B \in \mathfrak{A}. \quad (4.13)$$

We are aiming at a proof of the GNS theorem (Theorem 4.6), and a way to get more general representations of  $*$ -algebras. Indeed, any representation is built up by the cyclic representations (Definition 4.31), and each cyclic representation is in turn given by a GNS construction.

**Definition 4.31.** A representation  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$  is called *cyclic*, with a *cyclic vector*  $u \in \mathcal{H}$ , if  $\mathcal{H} = \overline{\text{span}} \{ \pi(A)u \mid A \in \mathfrak{A} \}$ .

**Theorem 4.32.** *Given any representation  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ , there exists an index set  $J$ , and closed subspaces  $\mathcal{H}_j \subset \mathcal{H}$  ( $j \in J$ ) such that*

1.  $\mathcal{H}_i \perp \mathcal{H}_j, \forall i \neq j$ ;
2.  $\sum_{j \in J}^{\oplus} \mathcal{H}_j = \mathcal{H}$ ; and
3. *there exists cyclic vectors  $v_j \in \mathcal{H}_j$  such that the restriction of  $\pi$  to  $\mathcal{H}_j$  is cyclic.*

*Remark 4.33.* The proof of 4.32 is very similar to the construction of orthonormal basis (ONB) (use Zorn's lemma!); but here we get a family of mutually orthogonal subspaces.

Of course, if  $\mathcal{H}_j$ 's are all one-dimensional, then it is a decomposition into ONB. Note that not every representation is irreducible, but every representation can be decomposed into direct sum of cyclic representations.

**Exercise 4.34** (Cyclic subspaces). Prove Theorem 4.32. Hint: pick  $v_1 \in \mathcal{H}$ , and let

$$\mathcal{H}_{v_1} := \overline{\text{span}} \{ \pi(A)v_1 : A \in \mathfrak{A} \},$$

i.e., the cyclic subspace generated by  $v_1$ . If  $\mathcal{H}_{v_1} \neq \mathcal{H}$ , then  $\exists v_2 \in \mathcal{H} \setminus \mathcal{H}_{v_1}$ , and the cyclic subspace  $\mathcal{H}_{v_2}$ , so that  $\mathcal{H}_{v_1}$  and  $\mathcal{H}_{v_2}$  are orthogonal. If  $\mathcal{H}_{v_1} \oplus \mathcal{H}_{v_2} \neq \mathcal{H}$ , we then build  $\mathcal{H}_{v_3}$  and so on. Now use transfinite induction or Zorn's lemma to show the family of direct sum of mutually orthogonal cyclic subspaces is total. The final step is exactly the same argument for the existence of an ONB of any Hilbert space.

Now we proceed to prove the Theorem 4.6 (GNS), which is restated below.

**Theorem 4.35** (Gelfand-Naimark-Segal). *There is a bijection between states  $\varphi$  and cyclic representations  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H}, u)$ , with  $\|u\| = 1$ ; where*

$$\varphi(A) = \langle u, \pi(A)u \rangle, \quad \forall A \in \mathfrak{A}. \quad (4.14)$$

Moreover, fix a state  $\varphi$ , the corresponding cyclic representation is unique up to unitary equivalence. Specifically, if  $(\pi_1, \mathcal{H}_1, u_1)$  and  $(\pi_2, \mathcal{H}_2, u_2)$  are two cyclic representations, with cyclic vectors  $u_1, u_2$ , respectively, satisfying

$$\varphi(A) = \langle u_1, \pi_1(A)u_1 \rangle = \langle u_2, \pi_2(A)u_2 \rangle, \quad \forall A \in \mathfrak{A}; \quad (4.15)$$

then

$$W : \pi_1(A)u_1 \mapsto \pi_2(A)u_2, \quad A \in \mathfrak{A} \quad (4.16)$$

extends to a unitary operator from  $\mathcal{H}_1$  onto  $\mathcal{H}_2$ , also denoted by  $W$ , and such that

$$\pi_2 W = W \pi_1, \quad (4.17)$$

i.e.,  $W$  intertwines the two representations.

*Remark 4.36.* For the non-trivial direction, let  $\varphi$  be a given state on  $\mathfrak{A}$ , and we need to construct a cyclic representation  $(\pi, \mathcal{H}_\varphi, u_\varphi)$ . Note that  $\mathfrak{A}$  is an algebra, and it is also a complex vector space. Let us try to turn  $\mathfrak{A}$  into a Hilbert space and see what conditions are needed. There is a homomorphism  $\mathfrak{A} \rightarrow \mathfrak{A}$  which follows from the associative law of  $\mathfrak{A}$  being an algebra, i.e.,  $(AB)C = A(BC)$ . To continue,  $\mathfrak{A}$  should be equipped with an inner product. Using  $\varphi$ , we may set  $\langle A, B \rangle_\varphi := \varphi(A^*B)$ ,  $\forall A, B \in \mathfrak{A}$ . Then  $\langle \cdot, \cdot \rangle_\varphi$  is linear in the second variable, and conjugate linear in the first variable. It also satisfies  $\langle A, A \rangle_\varphi = \varphi(A^*A) \geq 0$ . Therefore we take  $\mathcal{H}_\varphi := [\mathfrak{A}/\{A : \varphi(A^*A) = 0\}]^{cl}$ .

*Proof.* Given a cyclic representation  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H}, u)$ , define  $\varphi$  as in (4.14). Clearly  $\varphi$  is linear, and

$$\begin{aligned} \varphi(A^*A) &= \langle u, \pi(A^*A)u \rangle \\ &= \langle u, \pi(A^*)\pi(A)u \rangle \\ &= \langle \pi(A)u, \pi(A)u \rangle \\ &= \|\pi(A)u\|^2 \geq 0. \end{aligned}$$

Thus  $\varphi$  is a state.

Conversely, fix a state  $\varphi$  on  $\mathfrak{A}$ . Set

$$\mathcal{H}_0 := \left\{ \sum_{i=1}^n c_i A_i \mid c_i \in \mathbb{C}, n \in \mathbb{N} \right\}$$

and define the inner product

$$\left\langle \sum c_i A_i, \sum d_j B_j \right\rangle_\varphi := \sum \sum \bar{c}_i d_j \varphi(A_i^* B_j).$$

Note that, by definition,

$$\left\| \sum c_i A_i \right\|_\varphi^2 = \left\langle \sum c_i A_i, \sum c_i A_i \right\rangle_\varphi = \sum \sum \bar{c}_i c_j \varphi(A_i^* A_j) \geq 0. \quad (4.18)$$

The RHS of (4.18) is positive since  $\varphi$  is a state. Recall that  $\varphi(A^*A) \geq 0$ , for all  $A \in \mathfrak{A}$ , and this implies that for all  $n \in \mathbb{N}$ , the matrix  $(\varphi(A_i^* A_j))_{i,j=1}^n$  is positive definite, hence (4.18) holds.

*Proof of (4.14):* Now, let  $\mathcal{H}_\varphi :=$  completion of  $\mathcal{H}_0$  under  $\langle \cdot, \cdot \rangle_\varphi$  modulo elements  $s$  such that  $\|s\|_\varphi = 0$ . See Lemma 4.37 below.  $\mathcal{H}_\varphi$  is the desired cyclic space, consisting of equivalence classes  $[A]$ ,  $\forall A \in \mathfrak{A}$ . Next, let  $u_\varphi = [1_\mathfrak{A}] =$  equivalence class of the identity element, and set

$$\pi(A) := [A] = [A1_\mathfrak{A}] = [A][1_\mathfrak{A}];$$

then one checks that  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H}_\varphi)$ , and therefore  $\varphi(A) = \langle u_\varphi, \pi(A) u_\varphi \rangle_\varphi$ ,  $\forall A \in \mathfrak{A}$ .

For uniqueness, let  $(\pi_1, \mathcal{H}_1, u_1)$  and  $(\pi_2, \mathcal{H}_2, u_2)$  be as in the statement of the theorem, and let  $W$  be as in (4.16). By (4.15), we have

$$\varphi(A^*A) = \|\pi_2(A) u_2\|^2 = \|\pi_1(A) u_1\|^2$$

so that  $W$  is isometric. But since  $\mathcal{H}_i = \overline{\text{span}} \{ \pi_i(A) u_i : A \in \mathfrak{A} \}$ ,  $i = 1, 2$ , then  $W$  extends by density to a unitary operator from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ .

*Proof of (4.17):* Finally, for all  $A, B \in \mathfrak{A}$ , we have

$$\begin{aligned} W\pi_1(A)(\pi_1(B)u_1) &= W\pi_1(AB)u_1 \\ &= \pi_2(AB)u_2 \\ &= \pi_2(A)(\pi_2(B)u_2) \\ &= \pi_2(A)W\pi_1(B)u_1; \end{aligned}$$

therefore, by the density argument again, we conclude that

$$\pi_2(A) = W\pi_1(A) \quad \forall A \in \mathfrak{A}.$$

This is the intertwining property in (4.17).  $\square$

**Lemma 4.37.**  $\{A \in \mathfrak{A} : \varphi(A^*A) = 0\}$  is a closed two-sided ideal in  $\mathfrak{A}$ .

*Proof.* This follows from the Schwarz inequality. Note that

$$\begin{bmatrix} \varphi(A^*A) & \varphi(A^*B) \\ \varphi(B^*A) & \varphi(B^*B) \end{bmatrix}$$

is a positive definite matrix, and so its determinant is positive, i.e.,

$$|\varphi(A^*B)|^2 \leq \varphi(A^*A)\varphi(B^*B); \quad (4.19)$$

using the fact that  $\varphi(C^*) = \varphi(C)^*$ ,  $\forall C \in \mathfrak{A}$ . The lemma follows from the estimate (4.19).  $\square$

**Example 4.38.** Let  $\mathfrak{A} = C[0, 1]$ . Set  $\varphi : f \mapsto f(0)$ , so that  $\varphi(f^*f) = |f(0)|^2 \geq 0$ . Then,

$$\ker \varphi = \{f \in C[0, 1] \mid f(0) = 0\}$$

and  $C[0, 1] / \ker \varphi$  is one dimensional. The reason is that if  $f \in C[0, 1]$  such that  $f(0) \neq 0$ , then we have  $f(x) \sim f(0)$  since  $f(x) - f(0) \in \ker \varphi$ , where  $f(0)$  represents the constant function  $f(0)$  over  $[0, 1]$ . This shows that  $\varphi$  is a pure state, since the representation has to be irreducible.

**Exercise 4.39** (The GNS construction). Fill in the remaining details in the above proof of the GNS theorem.

Using GNS construction we get the following structure theorem for abstract  $C^*$ -algebras. As a result, all  $C^*$ -algebras are sub-algebras of  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .

**Theorem 4.40** (Gelfand-Naimark). *Every  $C^*$ -algebra (abelian or non-abelian) is isometrically isomorphic to a norm-closed sub-algebra of  $\mathcal{B}(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ .*

*Proof.* Let  $\mathfrak{A}$  be any  $C^*$ -algebra, no Hilbert space  $\mathcal{H}$  is given from outside. Let  $S(\mathfrak{A})$  be the states on  $\mathfrak{A}$ , which is a compact convex subset of the dual space  $\mathfrak{A}^*$ . Here, compactness refers to the weak  $*$ -topology.

We use Hahn-Banach theorem to show that there are plenty of states. Specifically,  $\forall a \in \mathfrak{A}, \exists \varphi \in \mathfrak{A}^*$  such that  $\varphi(a) > 0$ . It is done first on the 1-dimensional subspace

$$tA \mapsto t \in \mathbb{R},$$

and then extends to  $\mathfrak{A}$ . (Note this is also a consequence of Krein–Milman, i.e.,  $S(\mathfrak{A}) = cl(\text{pure states})$ . We will come back to this point later.)

For each state  $\varphi$ , one gets a *cyclic representation*  $(\pi_\varphi, \mathcal{H}_\varphi, u_\varphi)$ . Applying transfinite induction, one concludes that  $\pi := \bigoplus \pi_\varphi$  is a representation on the Hilbert space  $\mathcal{H} := \bigoplus \mathcal{H}_\varphi$ . For details, see e.g., [Rud73].  $\square$

**Theorem 4.41.** *Let  $\mathfrak{A}$  be an abelian  $C^*$ -algebra. Then there is a compact Hausdorff space  $X$ , unique up to homeomorphism, such that  $\mathfrak{A} \cong C(X)$ .*

### 4.3 States, Dual and Pre-dual

Let  $V$  be a *Banach space*, i.e., (recall, Chapter 1):

- $V$  is a vector space over  $\mathbb{C}$ ;
- $\exists$  norm  $\|\cdot\|$
- $V$  is complete with respect to  $\|\cdot\|$

The dual space  $V^*$  consists of linear functionals  $l : V \rightarrow \mathbb{C}$  satisfying

$$\|l\| := \sup_{\|v\|=1} |l(v)| < \infty.$$

These are the continuous linear functionals.

The *Hahn-Banach Theorem* implies that for all  $v \in V$ ,  $\|v\| \neq 0$ , there exists  $l_v \in V^*$ , of norm 1, such that  $l_v(v) = \|v\|$ . Recall the construction is to first define  $l_v$  on the one-dimensional subspace spanned by the vector  $v$ , then use transfinite induction to extend  $l_v$  to all of  $V$ . Notice that  $V^*$  is always complete, even if  $V$  is an incomplete normed space. In other words,  $V^*$  is always a Banach space.

Now  $V$  is embedded into  $V^{**}$  (as we always do this) via the mapping

$$\begin{aligned} V \ni v &\mapsto \psi(v) \in V^{**}, \text{ where} \\ \psi(v)(l) &:= l(v), \forall l \in V^*. \end{aligned} \tag{4.20}$$

Below we give a number of applications:

**Exercise 4.42** (Identification by isometry). Show that  $V \xrightarrow{\psi} V^{**}$  in (4.20) is isometric, i.e.,

$$\|\psi(v)\|_{**} = \|v\|, \forall v \in V.$$

**Example 4.43.** Let  $X$  be a compact Hausdorff space. The algebra  $C(X)$  of all continuous functions on  $X$  with the sup norm, i.e.,  $\|f\|_\infty := \sup_{x \in X} |f(x)|$ , is a Banach space.

**Example 4.44.** The classical  $L^p$  space:  $(l^p)^* = l^q$ ,  $(L^p)^* = L^q$ , for  $1/p + 1/q = 1$  and  $1 \leq p < \infty$ . If  $1 < p < \infty$ , then  $(l^p)^{**} = l^p$ , i.e., these spaces are *reflexive*. For  $p = 1$ , however, we have  $(l^1)^* = l^\infty$ , but  $(l^\infty)^*$  is much bigger than  $l^1$ . Also note that  $(l^p)^* \neq l^q$  except for  $p = q = 2$ . And  $l^p$  is a Hilbert space iff  $p = 2$ .

Let  $B$  be a Banach space and denote by  $B^*$  its dual space.  $B^*$  is a Banach space as well, where the norm is defined by

$$\|f\|_{B^*} = \sup_{\|x\|=1} \{|f(x)|\}.$$

Let  $B_1^* = \{f \in B^* : \|f\| \leq 1\}$  be the unit ball in  $B^*$ .

**Theorem 4.45** (Banach-Alaoglu).  $B_1^*$  is weak \* compact in  $B^*$ .

*Proof.* This is proved by showing  $B_1^*$  is a closed subset in  $\Omega := \prod_{\|x\|=1} \mathbb{C}_1$ , with  $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| \leq 1\}$ ; and  $\Omega$  is given its product topology, and is compact and Hausdorff.  $\square$

As an application, we have

**Corollary 4.46.** Let  $B$  be a separable Banach space. Then every bounded sequence in  $B^*$  has a convergent subsequence in the weak \*-topology.

**Corollary 4.47.** *Every bounded sequence in  $\mathcal{B}(\mathcal{H})$  contains a convergence subsequence in the weak \*-topology.*

We show in Theorem 4.55 that  $\mathcal{B}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})^*$ , where  $\mathcal{T}_1(\mathcal{H}) =$  trace-class operators.

Now we turn to Hilbert space, say  $\mathcal{H}$ :

- $\mathcal{H}$  is a vector space over  $\mathbb{C}$ ;
- it has an inner product  $\langle \cdot, \cdot \rangle$ , and the norm  $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ ;
- $\mathcal{H}$  is complete with respect to  $\|\cdot\|$ ;
- $\mathcal{H}^* = \mathcal{H}$ , i.e.,  $\mathcal{H}$  is reflexive;
- every Hilbert space has an orthonormal basis (by Zorn's lemma)

The identification  $\mathcal{H} = \mathcal{H}^*$  is due to Riesz, and the corresponding map is given by

$$h \mapsto \langle h, \cdot \rangle \in \mathcal{H}^*$$

This can also be seen by noting via an ONB that  $\mathcal{H}$  is unitarily equivalent to  $l^2(A)$ , with some index set  $A$ , and  $l^2(A)$  is reflexive.

The set of all bounded operators  $\mathcal{B}(\mathcal{H})$  on  $\mathcal{H}$  is a Banach space. We ask two questions:

1. What is the dual  $\mathcal{B}(\mathcal{H})^*$ ?
2. Is  $\mathcal{B}(\mathcal{H})$  the dual space of some Banach space?

The first question is extremely difficult and we will discuss that later.

For the present section, we show that

$$\mathcal{B}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})^*$$

where we denote by  $\mathcal{T}_1(\mathcal{H})$  the trace-class operators in  $\mathcal{B}(\mathcal{H})$ . For more details, see Theorem 4.55, and Section 1.5.

Let  $\rho : \mathcal{H} \rightarrow \mathcal{H}$  be a compact selfadjoint operator. Assume  $\rho$  is positive, i.e.,  $\langle x, \rho x \rangle \geq 0$  for all  $x \in \mathcal{H}$ . By the spectral theorem of compact operators, we get the following decomposition

$$\rho = \sum \lambda_k P_k \tag{4.21}$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \rightarrow 0$ , and  $P_k$  is the projection onto the finite dimensional eigenspace of  $\lambda_k$ .

In general, we want to get rid of the assumption that  $\rho \geq 0$ . This is done using the polar decomposition, which we will consider in Section 2.4 even for unbounded operators. It is much easier for bounded operators: If  $A \in \mathcal{B}(\mathcal{H})$ ,  $A^*A$  is positive, selfadjoint, and so by the spectral theorem, we may take  $|A| := \sqrt{A^*A}$ . Then, one checks that

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle = \left\langle \sqrt{A^*A}x, \sqrt{A^*A}x \right\rangle = \||A|x\|^2,$$

thus

$$\|A\| = \||A|\| \quad (4.22)$$

and there is a partial isometry  $V : \text{range}(|A|) \rightarrow \text{range}(A)$ , and the following polar decomposition holds:

$$A = V|A| \quad (4.23)$$

We will come back to this point in Section 2.4 when we consider unbounded operators.

**Corollary 4.48.** *Let  $A \in \mathcal{T}_1(\mathcal{H})$ , then  $A$  has the following decomposition*

$$A = \sum_n \lambda_n |f_n\rangle\langle e_n| \quad (4.24)$$

where  $\{e_n\}$  and  $\{f_n\}$  are ONBs in  $\mathcal{H}$ .

*Proof.* Using the polar decomposition  $A = V|A|$ , we may first diagonalize  $|A|$  with respect to some ONB  $\{e_n\}$  as

$$|A| = \sum_n \lambda_n |e_n\rangle\langle e_n|, \text{ then}$$

$$A = V|A| = \sum_n \lambda_n |Ve_n\rangle\langle e_n| = \sum_n \lambda_n |f_n\rangle\langle e_n|$$

where  $f_n := Ve_n$ . □

With the above discussion, we may work, instead, with compact operators  $A : \mathcal{H} \rightarrow \mathcal{H}$  so that  $A$  is a trace class operator if  $|A|$  (positive, selfadjoint) satisfies condition (4.25).

**Definition 4.49.** Let  $A$  be a compact operator with its polar decomposition  $A = V|A|$ , where  $|A| := \sqrt{A^*A}$ . Let  $\{\lambda_k\}_{k=1}^\infty$  be the eigenvalues of  $|A|$ , and  $P_k$  the corresponding spectral projections, see (4.21). We say  $A$  is a trace class operator, if

$$\|A\|_1 := \text{trace}(|A|) = \sum_n \lambda_n \text{rank}(P_k) < \infty. \quad (4.25)$$

*Caution.* In our consideration of eigenvalue lists, we may of course have multiplicity. But for compact operators, the multiplicity is automatically finite for each non-zero eigenvalue. And if we have sets of associated eigenvectors run through a local ONB in each of the finite-dimensional eigenspaces, then multiplicity is counted this way. But, alternatively, when computing a trace as a sum of eigenvalues, then the term in such a sum must be counted with multiplicity. Or each of the distinct numbers in an eigenvalue list can be multiplied with the respective multiplicity. This will be clear from the context.

We now continue the discussion from Section 1.5 on spaces of operators.

**Definition 4.50.** Let  $A \in \mathcal{T}_1(\mathcal{H})$ , and  $\{e_n\}$  an ONB in  $\mathcal{H}$ . Set

$$\text{trace}(A) := \sum_n \langle e_n, Ae_n \rangle \quad (4.26)$$

Note the RHS in (4.26) is independent of the choice of the ONB. For if  $\{f_n\}$  is another ONB in  $\mathcal{H}$ , using the Parseval identity repeatedly, we have

$$\begin{aligned} \sum \langle f_n, Af_n \rangle &= \sum_n \sum_m \langle f_n, e_m \rangle \langle e_m, Af_n \rangle \\ &= \sum_m \sum_n \langle f_n, e_m \rangle \langle A^* e_m, f_n \rangle \\ &= \sum_m \langle A^* e_m, e_m \rangle = \sum_m \langle e_m, Ae_m \rangle. \end{aligned}$$

**Corollary 4.51.** Let  $A \in \mathcal{T}_1(\mathcal{H})$ , then

$$|\text{trace}(A)| \leq \|A\|_1.$$

Therefore, the RHS in (4.26) is absolutely convergent.

*Proof.* By Corollary 4.48, there exists ONBs  $\{e_n\}$  and  $\{f_n\}$ , and  $A$  has a decomposition as in (4.24). Then,

$$\begin{aligned} |\text{trace}(A)| &\leq \sum_n |\langle e_n, Ae_n \rangle| = \sum_n \lambda_n |\langle e_n, f_n \rangle| \\ &\leq \sum_n \lambda_n \|e_n\| \|f_n\| = \sum_n \lambda_n = \|A\|_1 < \infty. \end{aligned}$$

We have used the fact that  $\text{trace}(A)$  is independent of the choice of an ONBs.  $\square$

**Lemma 4.52.** Let  $\mathcal{T}_1(\mathcal{H})$  be the trace class introduced above. Then,

1.  $\mathcal{T}_1(\mathcal{H})$  is a two-sided ideal in  $\mathcal{B}(\mathcal{H})$ .
2.  $\text{trace}(AB) = \text{trace}(BA)$
3.  $\mathcal{T}_1(\mathcal{H})$  is a Banach space with respect to the trace norm (4.25).

**Exercise 4.53** (A pre-dual). Prove Lemma 4.52.

**Lemma 4.54.** Let  $\rho \in \mathcal{T}_1(\mathcal{H})$ , then the map  $A \mapsto \text{trace}(A\rho)$  is a state on  $\mathcal{B}(\mathcal{H})$ . These are called the normal states.

*Proof.* By Lemma 4.52,  $A\rho \in \mathcal{T}_1(\mathcal{H})$  for all  $A \in \mathcal{B}(\mathcal{H})$ . The map  $A \mapsto \text{trace}(A\rho)$  is in  $\mathcal{B}(\mathcal{H})^*$  means that the pairing  $(A, \rho) \mapsto \text{trace}(A\rho)$  satisfies

$$|\text{trace}(A\rho)| \leq \|A\| \|\rho\|_1.$$



By Corollary 4.51, it suffices to verify, instead, that

$$\|A\rho\|_1 \leq \|A\| \|\rho\|_1.$$

Indeed, if we choose an ONB  $\{e_n\}$  in  $\mathcal{H}$  that diagonalizes  $|\rho|$ , i.e.,

$$|\rho| = \sum \lambda_n |e_n\rangle\langle e_n|, \text{ where } \sum \lambda_n < \infty, \lambda_n > 0, \forall n;$$

then

$$\begin{aligned} \|A\rho\|_1 &= \text{trace} \left( \sqrt{\rho^* A^* A \rho} \right) = \text{trace} \left( \sqrt{\rho^* \rho} \sqrt{A^* A} \right) \\ &= \sum_n \langle e_n, |A| |\rho| e_n \rangle = \sum_n \lambda_n \langle e_n, |A| e_n \rangle \\ &\leq \|A\| \sum_k \lambda_k = \|A\| \|\rho\|_1. \end{aligned}$$

□

**Theorem 4.55.**  $\mathcal{T}_1^*(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ .

*Proof.* Let  $l \in \mathcal{T}_1^*$ . By  $\mathcal{T}_1^* = (\mathcal{T}_1)^*$  we mean the dual Banach, duality with respect to the trace-norm.

How to get an operator  $A$ ? The operator  $A$  must satisfy

$$l(\rho) = \text{trace}(\rho A), \quad \forall \rho \in \mathcal{T}_1.$$

How to pull an operator  $A$  out of the hat? The idea also goes back to Dirac. It is in fact not difficult to find  $A$ . Since  $A$  is determined by its matrix, it suffices to find  $\langle f, Af \rangle$ , the entries in the matrix of  $A$ .

For any  $f_1, f_2 \in \mathcal{H}$ , the rank-one operator  $|f_1\rangle\langle f_2|$  is in  $\mathcal{T}_1$ , hence we know what  $l$  does to it, i.e., we know the numbers  $l(|f_1\rangle\langle f_2|)$ . But since  $l(|f_1\rangle\langle f_2|)$  is linear in  $f_1$ , and conjugate linear in  $f_2$ , by the Riesz theorem for Hilbert space, there exists a unique operator  $A$  such that

$$l(|f_1\rangle\langle f_2|) = \langle f_2, Af_1 \rangle.$$

Now we check that  $l(\rho) = \text{trace}(\rho A)$ . By Corollary 4.48, any  $\rho \in \mathcal{T}_1$  can be written as  $\rho = \sum_n \lambda_n |f_n\rangle\langle e_n|$ , where  $\{e_n\}$  and  $\{f_n\}$  are some ONBs in  $\mathcal{H}$ .

Then,

$$\begin{aligned}
\text{trace}(\rho A) &= \text{trace} \left( \sum_n \lambda_n |f_n\rangle \langle e_n| A \right) \\
&= \text{trace} \left( \sum_n \lambda_n |Af_n\rangle \langle e_n| \right) \\
&= \sum_m \sum_n \lambda_n \langle u_m, Af_n \rangle \langle e_n, u_m \rangle \\
&= \sum_n \lambda_n \left( \sum_m \langle u_m, Af_n \rangle \langle e_n, u_m \rangle \right) \\
&= \sum_n \lambda_n \langle e_n, Af_n \rangle (= l(\rho))
\end{aligned}$$

where  $\{u_n\}$  is an ONB in  $\mathcal{H}$ , and the last step follows from Parseval's identity.  $\square$

*Remark 4.56.* If  $B$  is the dual of a Banach space, then we say that  $B$  has a pre-dual. For example  $l^\infty = (l^1)^*$ , hence  $l^1$  is the pre-dual of  $l^\infty$ .

Another example: Let  $\mathbb{H}_1$  be hardy space of analytic functions on the disk [Rud87].  $(\mathbb{H}_1)^* = \text{BMO}$ , where BMO refers to bounded mean oscillation. It was developed by Charles Fefferman in 1974 who won the fields medal for this theory. See [Fef71]. (Getting hands on a specific dual space is often a big thing.)

**Definition 4.57.** Let  $\mathbb{D}$  be the complex disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Consider functions  $f$  analytic on  $\mathbb{D}$  such that

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})| dt < \infty. \quad (4.27)$$

This is the  $\mathbb{H}_1$ -Hardy space, and the  $\mathbb{H}_1$ -norm is the supremum in (4.27). (The literature on Hardy space is extensive, and we refer to [Rud87] for overview and details.)

**Theorem 4.58** (C. Fefferman).  $\mathbb{H}_1^* = \text{BMO}$ .

*Proof.* We refer to [Fef71].  $\square$

**Definition 4.59.** Let  $f$  be a locally integrable function on  $\mathbb{R}^n$ , and let  $Q$  run through all  $n$ -cubes  $\subset \mathbb{R}^n$ . Set

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy.$$

We say that  $f \in BMO$  iff (Def.)

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty. \quad (4.28)$$

In this case the LHS in (4.28) is the BMO-norm of  $f$ . Moreover, BMO is a Banach space.

**Exercise 4.60** (The Bohr compactification). From abstract harmonic analysis (see [Rud90]), we know that every locally Abelian (l.c.A.) group  $G$  has a Haar measure, unique up to scalar normalization. When  $G$  is a given l.c.A. group, we denote by  $\widehat{G}$  its dual group (of all continuous unitary characters.) The duality theorem for l.c.A. groups  $G$  states the following:

$$G \text{ is compact} \iff \widehat{G} \text{ is discrete.} \quad (4.29)$$

Moreover, in general,  $G \simeq \widehat{\widehat{G}}$  where  $G$  is l.c.A.. Now consider the group  $\mathbb{R}$  (the reals) with addition, but in its *discrete topology*, (usually denoted  $\mathbb{R}_d$ .) The corresponding dual group  $\mathbb{R}_b = (\mathbb{R}_d)^\wedge$  is therefore compact by (4.29). It is called the *Bohr-compactification* of  $\mathbb{R}$ . Let  $d\chi$  denote its Haar measure.

1. Show that  $L^2(G_b, d\chi)$  is a *non-separable* Hilbert space.
2. Find an ONB in  $L^2(G_b, d\chi)$  indexed by  $\mathbb{R}$ .

## 4.4 New Hilbert Spaces From “old”

Below we consider some cases of building new Hilbert spaces from given ones. Only sample cases are fleshed out; and they will be needed in the sequel.

### An Overview:

#### GNS

See Section 4.2.

#### Direct sum $\bigoplus_\alpha \mathcal{H}_\alpha$

- (a) Let  $\mathcal{H}_i$ ,  $i = 1, 2$  be two given Hilbert spaces, then the direct “orthogonal” sum  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  is as follows:

$$\begin{aligned} \mathcal{H} &= \{\text{symbol pairs } h_1 \oplus h_2, h_i \in \mathcal{H}_i, i = 1, 2\}, \text{ and} \\ \|h_1 \oplus h_2\|_{\mathcal{H}}^2 &= \|h_1\|_{\mathcal{H}_1}^2 + \|h_2\|_{\mathcal{H}_2}^2. \end{aligned} \quad (4.30)$$

- (b) Given an indexed family of Hilbert spaces  $\{\mathcal{H}_\alpha\}_{\alpha \in A}$  where  $A$  is a set; then set  $\mathcal{H} := \bigoplus_A \mathcal{H}_\alpha$  to be

$$\left\| \sum_{\alpha \in A}^{\oplus} h_\alpha \right\|_{\mathcal{H}}^2 = \sum_{\alpha \in A} \|h_\alpha\|_{\mathcal{H}_\alpha}^2 < \infty; \quad (4.31)$$

i.e., finiteness of the sum in (4.31) is part of the definition.

**Exercise 4.61** (Unitary operators on a direct Hilbert sum). Let  $\mathcal{H}_i, i = 1, 2$ , be Hilbert spaces, and set  $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ .

- (i) Let  $G_{\mathcal{H}}$ , and  $G_{\mathcal{H}_i}, i = 1, 2$ , be the respective groups of unitary operators. Show that  $G_{\mathcal{H}_1} \times G_{\mathcal{H}_2}$  is a subgroup of  $G_{\mathcal{H}}$ .
- (ii) Let  $L \in \mathcal{B}(\mathcal{H})$ , where  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , and suppose  $L$  commutes with the group  $G_{\mathcal{H}_1} \times G_{\mathcal{H}_2}$  in (i); then show that  $L$  must have the following form

$$L = (\alpha I_{\mathcal{H}_1}) \times (\beta I_{\mathcal{H}_2})$$

where  $\alpha, \beta \in \mathbb{C}$ . (We say that the *commutant* of the group  $G_{\mathcal{H}_1} \times G_{\mathcal{H}_2}$  has this form; it is two-dimensional.)

- (iii) Let  $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ , and  $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ ; show that the block-operator matrix  $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$  defines a unitary operator in  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  if and only if

$$AA^* = I_{\mathcal{H}_1}, \quad A^*A = I_{\mathcal{H}_2},$$

and

$$BB^* = I_{\mathcal{H}_2}, \quad B^*B = I_{\mathcal{H}_1},$$

i.e., the two operators are unitary between the respective Hilbert spaces.

### Hilbert-Schmidt operators (continuing the discussion in 1)

Let  $\mathcal{H}$  be a fixed Hilbert space, and set

$$\mathcal{HS}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T^*T \text{ is trace class}\} \quad (4.32)$$

and set

$$\|T\|_{\mathcal{HS}}^2 := \text{trace}(T^*T); \quad (4.33)$$

similarly if  $S, T \in \mathcal{HS}(\mathcal{H})$ , set

$$\langle S, T \rangle_{\mathcal{HS}} := \text{trace}(S^*T). \quad (4.34)$$

Note that finiteness on the RHS in (4.33) is part of the definition.

### Tensor-product $\mathcal{H}_1 \otimes \mathcal{H}_2$

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Hilbert spaces, and consider finite-rank operators (rank-1 in this case):

$$\begin{aligned} & |h_1\rangle\langle h_2| \text{ (Dirac ket-bra)} \\ |h_1\rangle\langle h_2|(u) &= \langle h_2, u \rangle_{\mathcal{H}_2} h_1, \quad \forall u \in \mathcal{H}_2, \end{aligned} \quad (4.35)$$

so  $T = |h_1\rangle\langle h_2| : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  with the identification

$$h_1 \otimes h_2 \longleftrightarrow |h_1\rangle\langle h_2|. \quad (4.36)$$

Set

$$\|h_1 \otimes h_2\|^2 := \text{trace}(T^*T) = \|h_1\|_{\mathcal{H}_1}^2 \|h_2\|_{\mathcal{H}_2}^2. \quad (4.37)$$

For the Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$  we take the  $\mathcal{H}S$ -completion of the space of finite rank operators spanned by the set in (4.35). The tensor product construction fits with composite system in quantum mechanics.

### Contractive inclusion

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Hilbert spaces, and let  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a contractive linear operator, i.e.,

$$I_{\mathcal{H}_1} - T^*T \geq 0. \quad (4.38)$$

On the subspace

$$\mathcal{R}(T) = \{Th_1 \mid h_1 \in \mathcal{H}_1\} \quad (4.39)$$

(generally not closed in  $\mathcal{H}_2$ ), set

$$\|Th_1\|_{\text{new}} := \|h_1\|, \quad h_1 \in \mathcal{H}_1; \quad (4.40)$$

then with  $\|\cdot\|_{\text{new}}$ ,  $\mathcal{R}(T)$  becomes a Hilbert space.

### Inflation (dilations)

Let  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a contraction, and set

$$\mathcal{U} = \left[ \begin{array}{c|c} T & (I_2 - TT^*)^{\frac{1}{2}} \\ \hline (I_1 - T^*T)^{\frac{1}{2}} & -T^* \end{array} \right] \quad (4.41)$$

The two operators in the off-diagonal slots are called the “defect operators” for the contraction  $T$ . Reason: the pair of defect-operators are  $(0, 0)$  if and only if  $T$  is a unitary isomorphism of  $\mathcal{H}_1$  onto  $\mathcal{H}_2$ .

**Exercise 4.62** (The Julia operator). Show that the matrix-block (4.41) defines a unitary operator  $\mathcal{U}$  in  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  (called the Julia operator); and that  $P_1\mathcal{U}P_1 = T$  where  $P_1$  denotes the projection of  $\mathcal{H}$  onto  $\mathcal{H}_1$ .

**Reflection Positivity (or renormalization)**  $(\mathcal{H}_+/\mathcal{N})^\sim$ 

**New Hilbert space from reflection positivity:** Let  $\mathcal{H}$  be a given Hilbert space,  $\mathcal{H}_+ \subset \mathcal{H}$  a closed subspace, and let  $U, \mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$  be two unitary operators,  $\mathcal{J}$  satisfying the idempotency condition

$$\mathcal{J}^2 = I, \text{ as well as} \quad (4.42)$$

$$\mathcal{J}U\mathcal{J} = U^*, \text{ and} \quad (4.43)$$

$$U\mathcal{H}_+ \subset \mathcal{H}_+; \text{ and} \quad (4.44)$$

finally

$$\langle h_+, \mathcal{J}h_+ \rangle \geq 0, \forall h_+ \in \mathcal{H}_+. \quad (4.45)$$

Note that (4.43) states that  $U$  is unitarily equivalent to its adjoint  $U^*$ .

*Note 4.63.* Set  $P_+ := \text{Proj}\mathcal{H}_+$  (= the projection onto  $\mathcal{H}_+$ ), then (4.45) is equivalent to

$$P_+\mathcal{J}P_+ \geq 0$$

with respect to the usual ordering of operators.

Set

$$\begin{aligned} \mathcal{N} &= \text{Ker}(P_+\mathcal{J}P_+) \\ &= \{h_+ \in \mathcal{H}_+ : \langle h_+, \mathcal{J}h_+ \rangle = 0\}. \end{aligned} \quad (4.46)$$

Set

$$\mathcal{K} = (\mathcal{H}_+/\mathcal{N})^\sim \quad (4.47)$$

where “ $\sim$ ” in (4.47) means Hilbert completion with respect to the sesquilinear form:  $\mathcal{H}_+ \times \mathcal{H}_+ \rightarrow \mathbb{C}$ , given by

$$\langle h_+, h_+ \rangle_{\mathcal{K}} := \langle h_+, \mathcal{J}h_+ \rangle, \quad (4.48)$$

a renormalized inner product.

**Exercise 4.64** (An induced operator). Let the setting be as above. Show that  $\tilde{U} : \mathcal{K} \rightarrow \mathcal{K}$ , given by

$$\tilde{U}(\text{class } h_+) = \text{class}(Uh_+), \quad h_+ \in \mathcal{H}_+ \quad (4.49)$$

where  $\text{class } h_+$  refers to the quotient in (4.47), is selfadjoint and contractive (see Figure 4.1).

*Remark 4.65.* The construction outlined above is called “reflection positivity”; see e.g., [JÓ00, PK88]. It has many applications in physics and in representation theory.

*Proof of the assertions in Figure 4.1.* Denote the “new” inner product in  $\mathcal{K}$  by  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ , and the initial inner product in  $\mathcal{H}$  by  $\langle \cdot, \cdot \rangle$ .

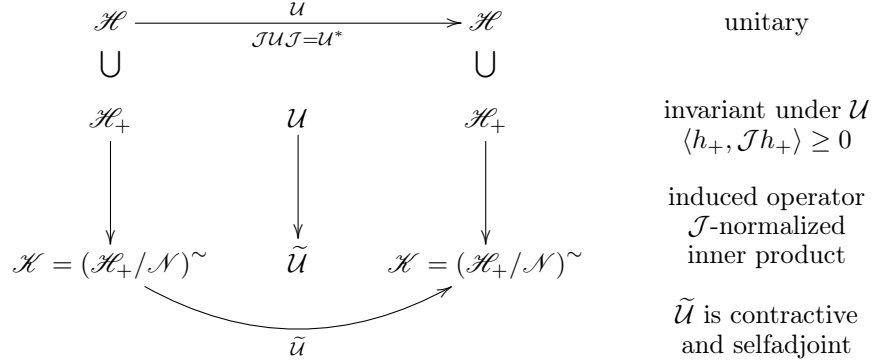


Figure 4.1: Reflection positivity. A unitary operator  $\mathcal{U}$  transforms into a self-adjoint contraction  $\tilde{\mathcal{U}}$ .

$\tilde{\mathcal{U}}$  is symmetric: Let  $x, y \in \mathcal{H}_+$ , then

$$\begin{aligned} \langle x, \tilde{\mathcal{U}}y \rangle_{\mathcal{K}} &= \langle x, \mathcal{J}\mathcal{U}y \rangle = \langle x, \mathcal{U}^* \mathcal{J}y \rangle \\ &= \langle \mathcal{U}x, \mathcal{J}y \rangle = \langle \tilde{\mathcal{U}}x, y \rangle_{\mathcal{K}} \end{aligned}$$

is the desired conclusion.

$\tilde{\mathcal{U}}$  is contractive: Let  $x \in \mathcal{H}_+$ , then

$$\begin{aligned} \|\tilde{\mathcal{U}}x\|_{\mathcal{K}}^2 &= \langle \mathcal{U}x, \mathcal{J}\mathcal{U}x \rangle = \langle \mathcal{U}x, \mathcal{U}^* \mathcal{J}x \rangle \\ &= \langle \mathcal{U}^2 x, \mathcal{J}x \rangle = \langle \mathcal{U}^2 x, x \rangle_{\mathcal{K}} \\ &\leq \|\mathcal{U}^2 x\|_{\mathcal{K}} \cdot \|x\|_{\mathcal{K}} \quad (\text{by Schwarz in } \mathcal{K}) \\ &\leq \|\mathcal{U}^4 x\|_{\mathcal{K}}^{\frac{1}{2}} \cdot \|x\|_{\mathcal{K}}^{1+\frac{1}{2}} \quad (\text{by the first step}) \\ &\leq \|\mathcal{U}^{2^{n+1}} x\|_{\mathcal{K}}^{\frac{1}{2^n}} \cdot \|x\|_{\mathcal{K}}^{1+\frac{1}{2}+\dots+\frac{1}{2^n}}. \quad (\text{by iteration}) \end{aligned}$$

By the spectral-radius formula,

$$\lim_{n \rightarrow \infty} \|\mathcal{U}^{2^n} x\|_{\mathcal{K}}^{\frac{1}{2^n}} = 1;$$

and we get  $\|\tilde{\mathcal{U}}x\|_{\mathcal{K}}^2 \leq \|x\|_{\mathcal{K}}^2$ , which is the desired contractivity.  $\square$

**Exercise 4.66** (Time-reflection). Show that if  $\{\mathcal{U}_t\}_{t \in \mathbb{R}}$  is a unitary one-parameter group in  $\mathcal{H}$  such that

$$\begin{aligned} \mathcal{J}\mathcal{U}_t\mathcal{J} &= \mathcal{U}_{-t}, \quad t \in \mathbb{R}, \text{ and} \\ \mathcal{U}_t\mathcal{H}_+ &\subset \mathcal{H}_+, \quad t \in \mathbb{R}_+, \end{aligned}$$

$$\begin{array}{ccc}
 A & \mathcal{H} \xrightarrow{\mathcal{U}_t=e^{-tA}} \mathcal{H} & A^* = -A \\
 \downarrow & & \\
 L & \mathcal{H} \xrightarrow[\mathcal{U}_t=e^{-tL}]{[\mathcal{S}_t]_{t \in \mathbb{R}_+}} \mathcal{H} & L^* = L, L \geq 0
 \end{array}$$

 Figure 4.2: Transformation of skew-adjoint  $A$  into selfadjoint semibounded  $L$ .

then

$$\mathcal{S}_t = \tilde{\mathcal{U}}_t : \mathcal{H} \rightarrow \mathcal{H}$$

is a selfadjoint contraction semigroup,  $t \in \mathbb{R}_+$ , i.e., there is a selfadjoint generator  $L$  in  $\mathcal{H}$ ,

$$\langle k, Lk \rangle_{\mathcal{H}} \geq 0, \forall k \in \text{dom}(L), \quad (4.50)$$

where

$$\mathcal{S}_t (= \tilde{\mathcal{U}}_t) = e^{-tL}, t \in \mathbb{R}_+ \quad (4.51)$$

and

$$\mathcal{S}_{t_1} \mathcal{S}_{t_2} = \mathcal{S}_{t_1+t_2}, t_1, t_2 \in \mathbb{R}_+. \quad (4.52)$$

**Example 4.67** ([Jor02]). Fix  $0 < \sigma < 1$ , and let  $\mathcal{H}$  ( $= \mathcal{H}_\sigma$ ) be the Hilbert space of all locally integral functions on  $\mathbb{R}$  satisfying

$$\|f\|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} f(y) |x-y|^{\sigma-1} dx dy < \infty. \quad (4.53)$$

Set

$$(\mathcal{U}(t)f)(x) = e^{(\sigma+1)t} f(e^{2t}x), \text{ and} \quad (4.54)$$

$$(\mathcal{J}f)(x) = |x|^{-\sigma-1} f\left(\frac{1}{x}\right). \quad (4.55)$$

Then  $\{\mathcal{U}(t)\}_{t \in \mathbb{R}}$  and  $\mathcal{J}$  satisfy the reflection property, i.e.,

$$\mathcal{J}\mathcal{U}(t)\mathcal{J} = \mathcal{U}(-t), t \in \mathbb{R} \quad (4.56)$$

as operators in  $\mathcal{H}$ ; and  $\{\mathcal{U}(t)\}_{t \in \mathbb{R}}$  is a unitary one-parameter group.

We now turn to the “reflected” version of the Hilbert norm (4.53):

The reflection Hilbert space  $\mathcal{H}$  will be generated by the completion of the space of functions  $f$  supported in  $(-1, 1)$ , that satisfy

$$\|f\|_{\mathcal{H}}^2 = \int_{-1}^1 \int_{-1}^1 \overline{f(x)} f(y) |1-xy|^{\sigma-1} dx dy < \infty. \quad (4.57)$$

We show below that this is a Hilbert space of *distributions*.



The selfadjoint contractive semigroup  $\{\tilde{U}(t)\}_{t \in \mathbb{R}_+}$  acting in  $\mathcal{H}$  is given by the same formula as in (4.54), but now acting in the Hilbert space  $\mathcal{H}$  defined by (4.57). Note  $\tilde{U}(t)$  is only defined for  $t \in \mathbb{R}_+ \cup \{0\}$ .

**Exercise 4.68** (Renormalization).

1. Show that the distributions  $\{\delta_0^{(n)}\}_{n \in \{0\} \cup \mathbb{N}}$  forms an orthogonal and total system in  $\mathcal{H}_\sigma$  from (4.57), for all fixed  $0 < \sigma < 1$ .

2. Show that

$$\left\| \delta_0^{(n)} \right\|_{\mathcal{H}_\sigma}^2 = n! (1 - \sigma) (2 - \sigma) \cdots (n - \sigma). \quad (4.58)$$

The idea of reflection positivity originated in physics. Now, when it is carried out in concrete cases, the initial function spaces change; but, more importantly, the inner product which produces the respective Hilbert spaces of quantum states changes as well.

What is especially intriguing is that before reflection we may have a Hilbert space of functions, but after the time-reflection is turned on, then, in the new inner product, the corresponding completion magically becomes a *Hilbert space of distributions*.

Now this is illustrated already in the simple examples above, Exercise 4.66, and Example 4.67. We include details below to stress the distinction between an abstract Hilbert-norm completion on the one hand, and a concretely realized Hilbert space on the other.

Constructing physical Hilbert spaces entail completions, often a completion of a suitable space of functions. What can happen is that the completion may fail to be a Hilbert space of *functions*, but rather a suitable Hilbert space of distributions.

Recall that a completion, say  $\mathcal{H}$  is defined axiomatically, and the “real” secret is revealed only when the elements in  $\mathcal{H}$  are identified.

To make the idea more clear we illustrate the point by considering functions on the interval  $-1 < x < 1$ .

Let  $C_c^\infty(-1, 1)$  be the  $C^\infty$ -functions with compact supports contained in  $(-1, 1)$ .

A linear functional  $\varphi$  on  $C_c^\infty(-1, 1)$  is said to be a *distribution* if for  $\forall K \subset (-1, 1)$  compact,  $\forall n \in \mathbb{N}$ ,  $\exists C = C_{K,n}$  such that

$$|\varphi(f)| \leq C \sup_{x \in K} \max_{0 \leq j \leq n} \left| \left( \frac{d}{dx} \right)^j f(x) \right|, \quad \forall f \in C_c^\infty(-1, 1). \quad (4.59)$$

Examples of distributions are Dirac “functions”  $\delta_{x_0}$ , and the derivatives  $\left( \frac{d}{dx} \right)^n \delta_{x_0}$ ,  $x_0 \in (-1, 1)$ , are defined by:

$$\left( \left( \frac{d}{dx} \right)^n \delta_{x_0} \right) (f) = (-1)^n f^{(n)}(x_0), \quad f \in C_c^\infty(-1, 1). \quad (4.60)$$

(Note: Distributions are not functions, but in Gelfand's rendition of the theory [GS77] they are called "generalized functions.")

Now equip  $C_c^\infty(-1, 1)$  with the sesquilinear form from (4.57) in Example 4.67, i.e.,

$$\langle f, g \rangle_{\mathcal{K}_\sigma} := \int_{-1}^1 \int_{-1}^1 \overline{f(x)} g(y) |1 - xy|^{\sigma-1} dx dy.$$

**Exercise 4.69** (A Hilbert space of distributions).

1. Show that each of the distributions  $\left(\frac{d}{dx}\right)^n \delta_{x_0}$ ,  $n \in \{0\} \cup \mathbb{N}$ ,  $x_0 \in (-1, 1)$  is in the completion  $\mathcal{K}_\sigma$  with respect to (4.57).
2. Compute the Hilbert norm of  $\left(\frac{d}{dx}\right)^n \delta_{x_0}$  in  $\mathcal{K}_\sigma$ , i.e., find

$$\left\| \left( \frac{d}{dx} \right)^n \delta_{x_0} \right\|_{\mathcal{K}_\sigma} \quad (4.61)$$

for all  $n \in \{0\} \cup \mathbb{N}$ , and  $x_0 \in (-1, 1)$ .

Hint: The answer to (2) (i.e., (4.61)) is as follows:

- $n = 0$ :

$$\|\delta_{x_0}\|_{\mathcal{K}_\sigma}^2 = (1 - x_0^2)^{\sigma-1};$$

- $n = 1$  (one derivative):

$$\|\delta'_{x_0}\|_{\mathcal{K}_\sigma}^2 = (1 - \sigma) (1 - x_0^2)^{\sigma-3} (1 + (1 - \sigma) x_0^2).$$

**Exercise 4.70** (Taylor for distributions). Fix  $0 < \sigma < 1$ , and let  $\mathcal{K}_\sigma$  be the corresponding Hilbert space of distributions. As an identity in  $\mathcal{K}_\sigma$ , establish:

$$\delta_x = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \delta_0^{(n)}, \quad (4.62)$$

valid for all  $x$ ,  $|x| < 1$ .

Historical note.

Laurent Schwartz has developed a systematic study of Hilbert spaces of distributions; see [Sch64b].

## 4.5 A second duality principle: A metric on the set of probability measures

Let  $(X, d)$  be a separable metric space, and denote by  $\mathcal{M}_1(X)$  and  $\mathcal{M}_1(X \times X)$  the corresponding sets of regular probability measures. Let  $\pi_i, i = 1, 2$ , denote the projections:  $\pi_1(x_1, x_2) = x_1, \pi_2(x_1, x_2) = x_2$ , for all  $(x_1, x_2) \in X \times X$ .

For  $\mu \in \mathcal{M}_1(X \times X)$ , set

$$\mu^{\pi_i} := \mu \circ \pi_i^{-1}.$$

For  $P_i \in \mathcal{M}_1(X), i = 1, 2$ , set

$$\mathcal{M}(P_1, P_2) = \{\mu \in \mathcal{M}_1(X \times X) : \mu^{\pi_i} = P_i, i = 1, 2\}.$$

Finally, let  $\text{Lip}_1$  = the Lipchitz functions on  $(X, d)$ , i.e.,  $f \in \text{Lip}_1$  iff (Def.)

$$|f(x) - f(y)| \leq d(x, y), \forall x, y \in X.$$

**Theorem 4.71** (Kantorovich-Rubinstein). *Setting*

$$\text{dist}_W(P_1, P_2) = \inf \left\{ \int_{X \times X} d(x, y) d\mu(x, y) : \mu \in \mathcal{M}(P_1, P_2) \right\}$$

and

$$\text{dist}_K(P_1, P_2) = \sup \left\{ \int_X f d(P_1 - P_2) : f \in \text{Lip}_1 \right\}$$

then

$$\text{dist}_W(P_1, P_2) = \text{dist}_K(P_1, P_2)$$

for all  $P_1, P_2 \in \mathcal{M}_1(X)$ .

*Proof.* We omit the proof here, but refer to [Kan58, KR57, Rüs07].  $\square$

**Exercise 4.72** (A complete metric space). Show that  $\mathcal{M}_1(X)$  is a complete metric space when equipped with the metric  $\text{dist}_K$ .

**Exercise 4.73** (A distance formula). Let  $(X, d)$  be  $\mathbb{R}$  with the usual distance  $d(x, y) = |x - y|$ . For  $P \in \mathcal{M}_1(\mathbb{R})$  set  $F_P(x) = P((-\infty, x])$ . Show that then

$$\text{dist}(P_1, P_2) = \int_{\mathbb{R}} |F_{P_1}(x) - F_{P_2}(x)| dx.$$

Let  $(X, d)$  and  $\mathcal{M}_1(X)$  be as above. Now apply Banach's Fixed point theorem to the complete metric space  $(\mathcal{M}_1(X), \text{dist}_K)$  to get a solution to the following:

**Exercise 4.74** (Iterated function systems). Let  $N \in \mathbb{N}$ , and let  $\varphi_i : X \rightarrow X$ ,  $i = 1, \dots, N$  be a system of strict contractions in  $(X, d)$ . On  $\mathcal{M}_1(X)$ , set

$$T_\mu := \frac{1}{N} \sum_{i=1}^N d\mu \circ \varphi_i^{-1}. \quad (4.63)$$

Recall  $(\mu \circ \varphi_i^{-1})(\Delta) = \mu(\varphi_i^{-1}(\Delta))$ .

1. Show that, if  $c$  is the smallest of the contractivity constant for  $\{\varphi_i\}_{i=1}^N$ , then

$$\text{dist}(T_\mu, T_\nu) \leq c \text{dist}(\mu, \nu), \quad \forall \mu, \nu \in \mathcal{M}_1(X). \quad (4.64)$$

2. Show that there is a unique solution  $\mu_L \in \mathcal{M}_1(X)$  to

$$T\mu_L = \mu_L, \text{ i.e.,} \quad (4.65)$$

$$\int_X f(x) d\mu_L(x) = \frac{1}{N} \sum_{i=1}^N \int_X f(\varphi_i(x)) d\mu_L(x) \quad (4.66)$$

holds for  $\forall f \in C_b(X)$  (= bounded continuous.)

Hint: The desired conclusion in (2), i.e., both existence and uniqueness of  $\mu_L$ , follows from Banach's fixed point theorem: Every strict contraction in a complete metric space has a unique fixed-point.

**Exercise 4.75** (The Middle-Third Cantor-measure). Set  $X = [0, 1]$  = the unit interval with the usual metric, set  $N = 2$ , and

$$\varphi_1(x) = \frac{x}{3}, \quad \varphi_2(x) = \frac{x+2}{3}, \quad (4.67)$$

and let  $\mu_L$  be the corresponding measure, i.e.,

$$\int_0^1 f(x) d\mu_L(x) = \frac{1}{2} \left( \int_0^1 f\left(\frac{x}{3}\right) d\mu_L(x) + \int_0^1 f\left(\frac{x+2}{3}\right) d\mu_L(x) \right). \quad (4.68)$$

Show that  $\mu_L$  is supported on the Middle-Third Cantor set (see Figure 4.3).

*Remark 4.76.* Starting with the Middle-Third Cantor measure  $\mu_L$ ; see (4.68), we get the cumulative distribution function  $F$  defined on the unit interval  $[0, 1]$ ,

$$F(x) = \mu_L([0, x]). \quad (4.69)$$

It follows from Exercise 4.75 that the graph of  $F$  is the Devil's Staircase; see Figure 4.4. Endpoints:  $F(0) = 0$ , and  $F(1) = 1$ . The union  $\mathcal{O}$  of all the omitted open intervals has total length:

$$\frac{1}{3} + \frac{2}{3^2} + \dots = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1,$$

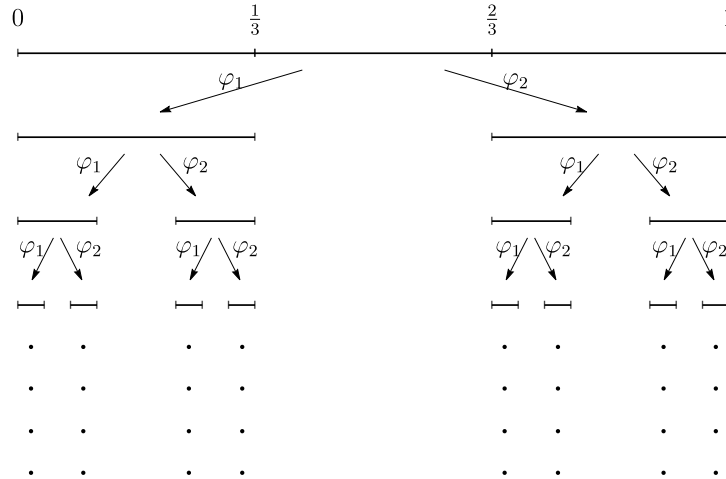


Figure 4.3: The Middle-Third Cantor set as a limit.

and  $F'(x) = 0$  for all  $x \in \mathcal{O}$ .

Using the argument from Exercise 1.12 above, we get

$$\int_0^1 dF = \int_0^1 F'(x) dx = 0.$$

Since  $F(1) - F(0) = 1$ , it would seem that the conclusion in the Fundamental Theorem of Calculus fails. (Explain! See e.g., [Rud87, ch 7].)

**Exercise 4.77** (Straightening out the Devil's staircase). Repeat the construction from the previous exercise, but now with the two functions  $\varphi_1, \varphi_2$  modified as follows:

$$\varphi_1(x) = \frac{x}{2}, \quad \varphi_2(x) = \frac{x+1}{2}; \tag{4.70}$$

compare with (4.67) above.

Then rewrite formula (4.68), and show that the cumulative distribution  $F$  from (4.69) becomes (Figure 4.5)

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 0 & x > 1. \end{cases}$$

Explain this!

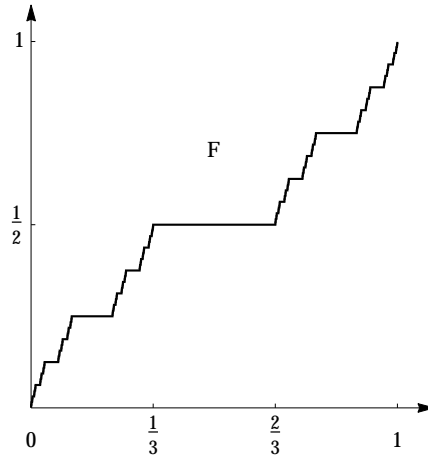


Figure 4.4: The Devil's Staircase.

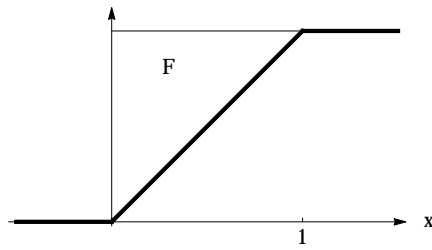


Figure 4.5: Straightening out the Devil's staircase.

## 4.6 Abelian $C^*$ -algebras

Diagonalizing a commuting family of bounded selfadjoint operators may be formulated in the setting of abelian  $C^*$ -algebras. By the structure theorem of Gelfand and Naimark, every abelian  $C^*$ -algebra containing the identity element is isomorphic to the algebra  $C(X)$  of continuous functions on some compact Hausdorff space  $X$ , which is unique up to homeomorphism. The classification of all the representations abelian  $C^*$ -algebras, therefore, amounts to that of  $C(X)$ . This problem can be understood using the idea of  $\sigma$ -measures (square densities). It also leads to the multiplicity theory of selfadjoint operators. The best treatment on this subject can be found in [Nel69].

Here we discuss *Gelfand's theory on abelian  $C^*$ -algebras*. Throughout, we assume all the algebras contain unit element.

**Definition 4.78.**  $\mathfrak{A}$  is *Banach algebra* if it is a complex algebra and a Banach space such that the norm satisfies  $\|ab\| \leq \|a\| \|b\|$ , for all  $a, b \in \mathfrak{A}$ .

Let  $\mathfrak{A}$  be an abelian Banach. Consider the closed ideals in  $\mathfrak{A}$  (since  $\mathfrak{A}$  is normed, so consider closed ideals) ordered by inclusion. By Zorn's lemma, there exists maximal ideals  $M$ , which are closed by maximality. Then  $\mathfrak{A}/M$  is 1-dimensional, i.e.,  $\mathfrak{A}/M = \{tv\}$  for some  $v \in \mathfrak{A}$ , and  $t \in \mathbb{R}$ . Therefore the combined map

$$\varphi : \mathfrak{A} \rightarrow \mathfrak{A}/M \rightarrow \mathbb{C}, \quad a \mapsto a/M \mapsto t_a$$

is a (complex) homomorphism. In particular,  $\mathfrak{A} \ni 1_{\mathfrak{A}} \mapsto v := 1_{\mathfrak{A}}/M \in \mathfrak{A}/M \simeq \mathbb{C}$ , and  $\varphi(1_{\mathfrak{A}}) = 1$ .

Conversely, the kernel of any homomorphism is a maximal ideal in  $\mathfrak{A}$  (since the co-dimension = 1.) Therefore there is a bijection between maximal ideas and homomorphisms.

**Lemma 4.79.** *Let  $\mathfrak{A}$  be an abelian Banach algebra. If  $a \in \mathfrak{A}$ , and  $\|a\| < 1$ , then  $1_{\mathfrak{A}} - a$  is invertible.*

*Proof.* It is easy to verify that  $(1 - a)^{-1} = 1 + a + a^2 + \dots$ , and the RHS is norm convergent.  $\square$

**Corollary 4.80.** *Any homomorphism  $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$  is a contraction.*

*Proof.* Let  $a \in \mathfrak{A}$ ,  $a \neq 0$ . Suppose  $\lambda := \varphi(a)$  such that  $|\lambda| > \|a\|$ . Then  $\|a/\lambda\| < 1$  and so  $1_{\mathfrak{A}} - a/\lambda$  is invertible by Lemma 4.79. Since  $\varphi$  is a homomorphism, it must map invertible element to invertible element, hence  $\varphi(1_{\mathfrak{A}} - a/\lambda) \neq 0$ , i.e.,  $\varphi(a) \neq \lambda$ , which is a contradiction.  $\square$

Let  $X$  be the set of all *maximal ideals*, identified with all homomorphisms in  $\mathfrak{A}_1^*$ , where  $\mathfrak{A}_1^*$  is the unit ball in  $\mathfrak{A}^*$ . Since  $\mathfrak{A}_1^*$  is compact (see Banach-Alaoglu, Theorem 4.45), and  $X$  is closed in it, therefore  $X$  is also compact. Here, compactness refers to the weak\*-topology.

**Definition 4.81.** The Gelfand transform  $\mathcal{F} : \mathfrak{A} \rightarrow C(X)$  is given by

$$\mathcal{F}(a)(\varphi) = \varphi(a), \quad a \in \mathfrak{A}, \varphi \in C(X). \quad (4.71)$$

Hence  $\mathfrak{A}/\ker \mathcal{F}$  is homomorphic to a closed subalgebra of  $C(X)$ . Note  $\ker \mathcal{F} = \{a \in \mathfrak{A} : \varphi(a) = 0, \forall \varphi \in X\}$ . It is called the *radical* of  $\mathfrak{A}$ .

The theory is takes a more pleasant form when  $\mathfrak{A}$  is a  $C^*$ -algebra. So there is an involution, and the norm satisfies the  $C^*$  axiom:  $\|aa^*\| = \|a\|^2$ , for all  $a \in \mathfrak{A}$ .

**Theorem 4.82** (Gelfand). *If  $\mathfrak{A}$  is an abelian  $C^*$ -algebra then the Gelfand transform (4.71) is an isometric  $*$ -isomorphism from  $\mathfrak{A}$  onto  $C(X)$ , where  $X$  is the maximal ideal space of  $\mathfrak{A}$ .*

**Example 4.83.** Consider  $l^1(\mathbb{Z})$ , the convolution algebra:

$$\begin{aligned} (ab)_n &= \sum_k a_k b_{n-k} \\ a_n^* &= \overline{a_{-n}} \\ \|a\| &= \sum_n |a_n| \\ 1_{\mathfrak{A}} &= \delta_0 \text{ (Dirac mass at 0);} \end{aligned} \quad (4.72)$$

the unit-element for the product (4.72) in  $l^1(\mathbb{Z})$ .

To identify  $X$  in practice, we always start with a guess, and usually it turns out to be correct. Since Fourier transform converts convolution to multiplication,

$$l^1(\mathbb{Z}) \ni a \xrightarrow{\varphi_z} \sum a_n z^n$$

is a complex homomorphism. To see  $\varphi_z$  is multiplicative, we have

$$\begin{aligned} \varphi_z(ab) &= \sum (ab)_n z^n \\ &= \sum_{n,k} a_k b_{n-k} z^n \\ &= \sum_k a_k z^k \sum_n b_{n-k} z^{n-k} \\ &= \left( \sum_k a_k z^k \right) \left( \sum_k b_k z^k \right) \\ &= \varphi_z(a) \varphi_z(b). \end{aligned}$$

Thus  $\{z : |z| = 1\}$  is a subspace in the Gelfand space  $X$ . Note that we cannot use  $|z| < 1$  since we are dealing with two-sided  $l^1$  sequence. (If the sequences were truncated, so that  $a_n = 0$  for  $n < 0$  then we allow  $|z| < 1$ .)

$$\varphi_z \text{ is contractive: } |\varphi_z(a)| = \left| \sum a_n z^n \right| \leq \sum_n |a_n| = \|a\|.$$



**Exercise 4.84** (The homomorphism of  $l^1$ ). Prove that every homomorphism of  $l^1(\mathbb{Z})$  is obtained as  $\varphi_z$  for some  $|z| = 1$ . Hence  $X = \mathbb{T}^1 (= \{z \in \mathbb{C} : |z| = 1\})$ .

**Example 4.85.**  $l^\infty(\mathbb{Z})$ , with  $\|a\| = \sup_n |a_n|$ . The Gelfand space in this case is  $X = \beta\mathbb{Z}$ , the Stone-Čech compactification of  $\mathbb{Z}$ , which are the ultra-filters on  $\mathbb{Z}$ .  $\beta\mathbb{Z}$  is much bigger than  $p$ -adic numbers. Pure states on diagonal operators correspond to  $\beta\mathbb{Z}$ . See Chapter 8 for details.

## 4.7 States and Representations

Let  $\mathfrak{A}$  be a  $*$ -algebra, a representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  generates a  $*$ -subalgebra  $\pi(\mathfrak{A})$  in  $\mathcal{B}(\mathcal{H})$ . By taking norm closure, one gets a  $C^*$ -algebra, i.e., a Banach  $*$ -algebra with the axiom  $\|a^*a\| = \|a\|^2$ . On the other hand, by Gelfand and Naimark's theorem, all abstract  $C^*$ -algebras are isometrically isomorphic to closed subalgebras of  $\mathcal{B}(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$  (Theorem 4.40). The construction of  $\mathcal{H}$  comes down to states  $S(\mathfrak{A})$  on  $\mathfrak{A}$  and the GNS construction. Therefore, the GNS construction gives rise to a bijection between states and representations.

Let  $\mathfrak{A}_+$  be the positive elements in  $\mathfrak{A}$ .  $s \in S(\mathfrak{A})$ ,  $s : \mathfrak{A} \rightarrow \mathbb{C}$  and  $s(\mathfrak{A}_+) \subset [0, \infty)$ . For  $C^*$ -algebra, positive elements can be written  $f = (\sqrt{f})^2$  by the spectral theorem. In general, positive elements have the form  $a^*a$ . There is a bijection between states and GNS representations  $Rep(\mathfrak{A}, \mathcal{H})$ , where  $s(A) = \langle \Omega, \pi(A)\Omega \rangle$ .

**Example 4.86.**  $\mathfrak{A} = C(X)$  where  $X$  is a compact Hausdorff space.  $s_\mu$  given by  $s_\mu(a) = \int a d\mu$  is a state. The GNS construction gives  $\mathcal{H} = L^2(\mu)$ ,  $\pi(f)$  is the operator of multiplication by  $f$  on  $L^2(\mu)$ .  $\{\varphi 1 : \varphi \in C(X)\}$  is dense in  $L^2$ , where 1 is the cyclic vector.  $s_\mu(f) = \langle \Omega, \pi(f)\Omega \rangle = \int 1f1d\mu = \int f d\mu$ , which is also seen as the expectation of  $f$  in case  $\mu$  is a probability measure.

We consider decomposition of representations or equivalently states, i.e., breaking up representations corresponds to breaking up states.

The thing that we want to do with representations comes down to the smallest ones, i.e., the irreducible representations. Irreducible representations correspond to pure states which are extreme points in the states (see [Phe01]).

A representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is irreducible, if whenever  $\mathcal{H}$  breaks up into two pieces  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , where  $\mathcal{H}_i$  is invariant under  $\pi(\mathfrak{A})$ , one of them is zero (the other is  $\mathcal{H}$ ). Equivalently, if  $\pi = \pi_1 \oplus \pi_2$ , where  $\pi_i = \pi|_{\mathcal{H}_i}$ , then one of them is zero. This is similar to the decomposition of natural numbers into product of primes. For example,  $6 = 2 \times 3$ , but 2 and 3 are primes and they do not decompose further.

Hilbert spaces are defined up to unitary equivalence. A state  $\varphi$  may have equivalent representations on different Hilbert spaces (but unitarily equivalent), however  $\varphi$  does not see the distinction, and it can only detect equivalent classes of representations.

**Example 4.87.** Let  $\mathfrak{A}$  be a  $*$ -algebra. Given two states  $s_1$  and  $s_2$ , by the GNS construction, we get cyclic vectors  $\xi_i$ , and representations  $\pi_i : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_i)$ , so that  $s_i(A) = \langle \xi_i, \pi_i(A)\xi_i \rangle$ ,  $i = 1, 2$ . Suppose there is a unitary operator  $W : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , such that for all  $A \in \mathfrak{A}$ ,

$$\pi_1(A) = W^* \pi_2(A) W.$$

Then

$$\begin{aligned} \langle \xi_1, \pi_1(A)\xi_1 \rangle_1 &= \langle \xi_1, W^* \pi_2(A) W \xi_1 \rangle_1 \\ &= \langle W \xi_1, \pi_2(A) W \xi_1 \rangle_2 \\ &= \langle \xi_2, \pi_2(A)\xi_2 \rangle_2, \forall A \in \mathfrak{A}; \end{aligned}$$

i.e.,  $s_2(A) = s_1(A)$ . Therefore the same state  $s = s_1 = s_2$  has two distinct (unitarily equivalent) representations.

*Remark 4.88.* A special case of states are measures when the algebra is abelian. Recall that all abelian  $C^*$ -algebras with identity are  $C(X)$ , where  $X$  is the corresponding Gelfand space. Two representations are mutually singular  $\pi_1 \perp \pi_2$ , if and only if the two measures are mutually singular,  $\mu_1 \perp \mu_2$ .

The theorem below is fundamental in representation theory. Recall that if  $M$  is a subset of  $\mathcal{B}(\mathcal{H})$ , the *commutant*  $M'$  consists of  $A \in \mathcal{B}(\mathcal{H})$  that commutes with all elements in  $M$ .

**Theorem 4.89** (Schur). *Let  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  be a representation. The following are equivalent.*

1.  $\pi$  is irreducible.
2. The commutant  $(\pi(\mathfrak{A}))'$  is one-dimensional, i.e.,  $(\pi(\mathfrak{A}))' = cI_{\mathfrak{A}}$ ,  $c \in \mathbb{C}$ .

*Proof.* Suppose  $(\pi(\mathfrak{A}))'$  has more than one dimension. Let  $X \in (\pi(\mathfrak{A}))'$ , then by taking adjoint,  $X^* \in (\pi(\mathfrak{A}))'$ .  $X + X^*$  is selfadjoint, and  $X + X^* \neq cI$  since by hypothesis  $(\pi(\mathfrak{A}))'$  has more than one dimension. Therefore  $X + X^*$  has a non trivial spectral projection  $P(E)$ , i.e.,  $P(E) \notin \{0, I\}$ . Let  $\mathcal{H}_1 = P(E)\mathcal{H}$  and  $\mathcal{H}_2 = (I - P(E))\mathcal{H}$ .  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are both nonzero proper subspaces of  $\mathcal{H}$ . Since  $P(E)$  commutes with  $\pi(A)$ , for all  $A \in \mathfrak{A}$ , it follows that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are both invariant under  $\pi$ .

Conversely, suppose  $(\pi(\mathfrak{A}))'$  is one-dimensional. If  $\pi$  is not irreducible, i.e.,  $\pi = \pi_1 \oplus \pi_2$ , then for

$$P_{\mathcal{H}_1} = \begin{bmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{\mathcal{H}_2} = 1 - P_{\mathcal{H}_1} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}_2} \end{bmatrix}$$

we have

$$P_{\mathcal{H}_i} \pi(A) = \pi(A) P_{\mathcal{H}_i}, \quad i = 1, 2$$

for all  $A \in \mathfrak{A}$ . Hence  $(\pi(\mathfrak{A}))'$  has more than one dimension.  $\square$

**Corollary 4.90.**  $\pi$  is irreducible if and only if the only projections in  $(\pi(\mathfrak{A}))'$  are 0 or  $I$ .

Thus to test invariant subspaces, one only needs to look at projections in the commutant.

**Corollary 4.91.** If  $\mathfrak{A}$  is abelian, then  $\pi$  is irreducible if and only if  $\mathcal{H}$  is one-dimensional.

*Proof.* Obviously, if  $\dim \mathcal{H} = 1$ ,  $\pi$  is irreducible. Conversely, by Theorem 4.89,  $(\pi(\mathfrak{A}))' = cI$ . Since  $\pi(\mathfrak{A})$  is abelian,  $\pi(\mathfrak{A}) \subset (\pi(\mathfrak{A}))'$ . Thus for all  $A \in \mathfrak{A}$ ,  $\pi(A) = c_A I$ , for some constant  $c_A$ .  $\square$

If instead of taking the norm closure, but using the strong operator topology, ones gets a von Neumann algebra. von Neumann showed that the weak closure of  $\mathfrak{A}$  is equal to  $\mathfrak{A}''$ .

**Corollary 4.92.**  $\pi$  is irreducible  $\iff (\pi(\mathfrak{A}))'$  is 1-dimensional  $\iff (\pi(\mathfrak{A}))'' = \mathcal{B}(\mathcal{H})$ .

*Remark 4.93.* In matrix notation, we write  $\pi = \pi_1 \oplus \pi_2$  as

$$\pi(A) = \begin{bmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{bmatrix}.$$

If

$$\begin{bmatrix} X & Y \\ U & V \end{bmatrix} \in (\pi(\mathfrak{A}))'$$

then

$$\begin{aligned} \begin{bmatrix} X & Y \\ U & V \end{bmatrix} \begin{bmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{bmatrix} &= \begin{bmatrix} X\pi_1(A) & Y\pi_2(A) \\ U\pi_1(A) & V\pi_2(A) \end{bmatrix} \\ \begin{bmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{bmatrix} \begin{bmatrix} X & Y \\ U & V \end{bmatrix} &= \begin{bmatrix} \pi_1(A)X & \pi_1(A)Y \\ \pi_2(A)U & \pi_2(A)V \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} X\pi_1(A) &= \pi_1(A)X \\ V\pi_2(A) &= \pi_2(A)V \\ U\pi_1(A) &= \pi_2(A)U \\ Y\pi_2(A) &= \pi_1(A)Y. \end{aligned}$$

Therefore,

$$\begin{aligned} X &\in (\pi_1(\mathfrak{A}))', \quad V \in (\pi_2(\mathfrak{A}))', \text{ and} \\ U, Y &\in \text{int}(\pi_1, \pi_2) = \text{intertwining operators of } \pi_1, \pi_2. \end{aligned}$$

This is illustrated by the diagram below.

$$\begin{array}{ccc}
 \mathcal{H}_1 & \xrightarrow{\pi_1(A)} & \mathcal{H}_1 \\
 \downarrow U & & \downarrow U \\
 \mathcal{H}_2 & \xrightarrow{\pi_2(A)} & \mathcal{H}_2
 \end{array}
 \begin{array}{c}
 \curvearrowright Y \\
 \curvearrowleft Y
 \end{array}$$

We say  $\pi_1$  and  $\pi_2$  are *inequivalent* if and only if  $\text{int}(\pi_1, \pi_2) = 0$ . For  $\pi_1 = \pi_2$ ,  $\pi$  has *multiplicity 2*. Multiplicity  $> 1$  is equivalent to the commutant being non-abelian. In the case  $\pi = \pi_1 \oplus \pi_2$  where  $\pi_1 = \pi_2$ ,  $(\pi(\mathfrak{A}))' \simeq M_2(\mathbb{C})$ .

Schur's lemma addresses all representations. It says that a representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is irreducible if and only if  $(\pi(\mathfrak{A}))'$  is 1-dimensional. When specialize to the GNS representation of a given state  $s$ , this is also equivalent to saying that for all positive linear functional  $t$ ,  $t \leq s \Rightarrow t = \lambda s$  for some  $\lambda \geq 0$ . This latter equivalence is obtained by using a more general result, which relates  $t$  and selfadjoint operators in the commutant  $(\pi(\mathfrak{A}))'$ .

We now turn to characterize the relation between state and its GNS representation, i.e., specialize to the GNS representation. Given a  $*$  algebra  $\mathfrak{A}$ , the states  $S(\mathfrak{A})$  forms a compact convex subset in the unit ball of the dual  $\mathfrak{A}^*$ .

Let  $\mathfrak{A}_+$  be the set of positive elements in  $\mathfrak{A}$ . Given  $s \in S(\mathfrak{A})$ , let  $t$  be a positive linear functional. By  $t \leq s$ , we mean  $t(A) \leq s(A)$  for all  $A \in \mathfrak{A}_+$ . We look for relation between  $t$  and the commutant  $(\pi(\mathfrak{A}))'$ .

**Lemma 4.94** (Schur-Sakai-Nicodym). *Let  $t$  be a positive linear functional, and let  $s$  be a state. There is a bijection between  $t$  such that  $0 \leq t \leq s$ , and selfadjoint operator  $A$  in the commutant with  $0 \leq A \leq I$ . The relation is given by*

$$t(\cdot) = \langle \Omega, \pi(\cdot)A\Omega \rangle$$

*Remark 4.95.* This is an extension of the classical Radon-Nikodym derivative theorem to the non-commutative setting. We may write  $A = dt/ds$ . The notation  $0 \leq A \leq I$  refers to the partial order of selfadjoint operators. It means that for all  $\xi \in \mathcal{H}$ ,  $0 \leq \langle \xi, A\xi \rangle \leq \langle \xi, \xi \rangle$ . See [Sak71, KR97b].

*Proof.* Easy direction, suppose  $A \in (\pi(\mathfrak{A}))'$  and  $0 \leq A \leq I$ . As in many applications, the favorite functions one usually applies to selfadjoint operators is the square root function  $\sqrt{\cdot}$ . So let's take  $\sqrt{A}$ . Since  $A \in (\pi(\mathfrak{A}))'$ , so is  $\sqrt{A}$ . We need to show  $t(a) = \langle \Omega, \pi(a)A\Omega \rangle \leq s(a)$ , for all  $a \geq 0$  in  $\mathfrak{A}$ . Let  $a = b^2$ , then

$$\begin{aligned}
 t(a) &= \langle \Omega, \pi(a)A\Omega \rangle \\
 &= \langle \Omega, \pi(b^2)A\Omega \rangle \\
 &= \langle \Omega, \pi(b)^* \pi(b)A\Omega \rangle \\
 &= \langle \pi(b)\Omega, A\pi(b)\Omega \rangle \\
 &\leq \langle \pi(b)\Omega, \pi(b)\Omega \rangle \\
 &= \langle \Omega, \pi(a)\Omega \rangle \\
 &= s(a).
 \end{aligned}$$

Conversely, suppose  $t \leq s$ . Then for all  $a \geq 0$ ,  $t(a) \leq s(a) = \langle \Omega, \pi(a)\Omega \rangle$ . Again write  $a = b^2$ . It follows that

$$t(b^2) \leq s(b^2) = \langle \Omega, \pi(a)\Omega \rangle = \|\pi(b)\Omega\|^2.$$

By Riesz's theorem, there is a unique  $\eta$ , so that

$$t(a) = \langle \pi(b)\Omega, \eta \rangle.$$

Conversely, let  $a = b^2$ , then

$$t(b^2) \leq s(b^2) = \langle \Omega, \pi(a)\Omega \rangle = \|\pi(b)\Omega\|^2$$

i.e.,  $\pi(b)\Omega \mapsto t(b^2)$  is a bounded quadratic form. Therefore, there exists a unique  $A \geq 0$  such that

$$t(b^2) = \langle \pi(b)\Omega, A\pi(b)\Omega \rangle.$$

It is easy to see that  $0 \leq A \leq I$ . Also,  $A \in (\pi(\mathfrak{A}))'$ , the commutant of  $\pi(\mathfrak{A})$ .  $\square$

**Corollary 4.96.** *Let  $s$  be a state.  $(\pi, \Omega, \mathcal{H})$  is the corresponding GNS construction. The following are equivalent.*

1. For all positive linear functional  $t$ ,  $t \leq s \Rightarrow t = \lambda s$  for some  $\lambda \geq 0$ .
2.  $\pi$  is irreducible.

*Proof.* By Sakai-Nicodym derivative,  $t \leq s$  if and only if there is a selfadjoint operator  $A \in (\pi(\mathfrak{A}))'$  so that

$$t(\cdot) = \langle \Omega, \pi(\cdot)A\Omega \rangle$$

Therefore  $t = \lambda s$  if and only if  $A = \lambda I$ .

Suppose  $t \leq s \Rightarrow t = \lambda s$  for some  $\lambda \geq 0$ . Then  $\pi$  must be irreducible, since otherwise there exists  $A \in (\pi(\mathfrak{A}))'$  with  $A \neq cI$ , hence  $\mathfrak{A} \ni a \mapsto t(a) := \langle \Omega, \pi(a)A\Omega \rangle$  defines a positive linear functional, and  $t \leq s$ , however  $t \neq \lambda s$ . Thus a contradiction to the hypothesis.

Conversely, suppose  $\pi$  is irreducible. Then by Schur's lemma,  $(\pi(\mathfrak{A}))'$  is 1-dimensional. i.e. for all  $A \in (\pi(\mathfrak{A}))'$ ,  $A = \lambda I$  for some  $\lambda$ . Therefore if  $t \leq s$ , by Sakai's theorem,  $t(\cdot) = \langle \Omega, \pi(\cdot)A\Omega \rangle$ . Thus  $t = \lambda s$  for some  $\lambda \geq 0$ .  $\square$

**Definition 4.97.** A state  $s$  is pure if it cannot be broken up into a convex combination of two distinct states. i.e. for all states  $s_1$  and  $s_2$ ,  $s = \lambda s_1 + (1 - \lambda)s_2 \Rightarrow s = s_1$  or  $s = s_2$ .

The main theorem in this section is a corollary to Sakai's theorem.

**Corollary 4.98.** *Let  $s$  be a state.  $(\pi, \Omega, \mathcal{H})$  is the corresponding GNS construction. The following are equivalent.*

1.  $t \leq s \Rightarrow t = \lambda s$  for some  $\lambda \geq 0$ .

2.  $\pi$  is irreducible.

3.  $s$  is a pure state.

*Proof.* By Sakai-Nicodym derivative,  $t \leq s$  if and only if there is a selfadjoint operator  $A \in (\pi(\mathfrak{A}))'$  so that

$$t(a) = \langle \Omega, \pi(a) A \Omega \rangle, \quad \forall a \in \mathfrak{A}.$$

Therefore  $t = \lambda s$  if and only if  $A = \lambda I$ .

We show that (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Leftrightarrow$ (2) Suppose  $t \leq s \Rightarrow t = \lambda s$ , then  $\pi$  must be irreducible, since otherwise there exists  $A \in (\pi(\mathfrak{A}))'$  with  $A \neq cI$ , hence  $t(\cdot) := \langle \Omega, \pi(\cdot) A \Omega \rangle$  defines a positive linear functional with  $t \leq s$ , however  $t \neq \lambda s$ . Conversely, suppose  $\pi$  is irreducible. If  $t \leq s$ , then  $t(\cdot) = \langle \Omega, \pi(\cdot) A \Omega \rangle$  with  $A \in (\pi(\mathfrak{A}))'$ . By Schur's lemma,  $(\pi(\mathfrak{A}))' = \{0, \lambda I\}$ . Therefore,  $A = \lambda I$  and  $t = \lambda s$ .

(1) $\Rightarrow$ (3) Suppose  $t \leq s \Rightarrow t = \lambda s$  for some  $\lambda \geq 0$ . If  $s$  is not pure, then  $s = cs_1 + (1-c)s_2$  where  $s_1, s_2$  are states and  $c \in (0, 1)$ . By hypothesis,  $s_1 \leq s$  implies that  $s_1 = \lambda s$ . It follows that  $s = s_1 = s_2$ .

(3) $\Rightarrow$ (2) Suppose  $\pi$  is not irreducible, i.e. there is a non trivial projection  $P \in (\pi(\mathfrak{A}))'$ . Let  $\Omega = \Omega_1 \oplus \Omega_2$  where  $\Omega_1 = P\Omega$  and  $\Omega_2 = (I - P)\Omega$ . Then

$$\begin{aligned} s(a) &= \langle \Omega, \pi(a) \Omega \rangle \\ &= \langle \Omega_1 \oplus \Omega_2, \pi(a) \Omega_1 \oplus \Omega_2 \rangle \\ &= \langle \Omega_1, \pi(a) \Omega_1 \rangle + \langle \Omega_2, \pi(a) \Omega_2 \rangle \\ &= \|\Omega_1\|^2 \left\langle \frac{\Omega_1}{\|\Omega_1\|}, \pi(a) \frac{\Omega_1}{\|\Omega_1\|} \right\rangle + \|\Omega_2\|^2 \left\langle \frac{\Omega_2}{\|\Omega_2\|}, \pi(a) \frac{\Omega_2}{\|\Omega_2\|} \right\rangle \\ &= \|\Omega_1\|^2 \left\langle \frac{\Omega_1}{\|\Omega_1\|}, \pi(a) \frac{\Omega_1}{\|\Omega_1\|} \right\rangle + (1 - \|\Omega_1\|^2) \left\langle \frac{\Omega_2}{\|\Omega_2\|}, \pi(a) \frac{\Omega_2}{\|\Omega_2\|} \right\rangle \\ &= \lambda s_1(a) + (1 - \lambda) s_2(a). \end{aligned}$$

Hence  $s$  is not a pure state.  $\square$

## Normal States

More general states in physics come from the mixture of particle states, which correspond to composite system. These are called normal states in mathematics.

Let  $\rho \in \mathcal{T}_1(\mathcal{H}) =$  trace class operator, such that  $\rho > 0$  and  $tr(\rho) = 1$ . Define state  $s_\rho(A) := tr(A\rho)$ ,  $A \in \mathcal{B}(\mathcal{H})$ . Since  $\rho$  is compact, by spectral theorem of compact operators,

$$\rho = \sum_k \lambda_k P_k$$

such that  $\lambda_1 > \lambda_2 > \dots \rightarrow 0$ ;  $\sum \lambda_k = 1$  and  $P_k = |\xi_k\rangle\langle \xi_k|$ , i.e., the rank-1 projections. (See Section 4.3.) We have

- $s_\rho(I) = \text{tr}(\rho) = 1$ ; and
- for all  $A \in \mathcal{B}(\mathcal{H})$ ,

$$\begin{aligned}
s_\rho(A) = \text{tr}(A\rho) &= \sum_n \langle u_n, A\rho u_n \rangle \\
&= \sum_n \langle A^* u_n, \rho u_n \rangle \\
&= \sum_n \sum_k \lambda_k \langle A^* u_n, \xi_k \rangle \langle \xi_k, u_n \rangle \\
&= \sum_k \lambda_k \left( \sum_n \langle u_n, A\xi_k \rangle \langle \xi_k, u_n \rangle \right) \\
&= \sum_k \lambda_k \langle \xi_k, A\xi_k \rangle;
\end{aligned}$$

where  $\{u_k\}$  is any ONB in  $\mathcal{H}$ . Hence,

$$s_\rho = \sum_k \lambda_k s_{\xi_k} = \sum_k \lambda_k |\xi_k\rangle\langle \xi_k|$$

i.e.,  $s_\rho$  is a convex combination of pure states  $s_{\xi_k} := |\xi_k\rangle\langle \xi_k|$ .

*Remark 4.99.* Notice that  $\text{tr}(|\xi\rangle\langle \eta|) = \langle \eta, \xi \rangle$ . In fact, take any ONB  $\{e_n\}$  in  $\mathcal{H}$ , then

$$\text{tr}(|\xi\rangle\langle \eta|) = \sum_n \langle e_n, \xi \rangle \langle \eta, e_n \rangle = \langle \eta, \xi \rangle$$

where the last step follows from Parseval identity. (If we drop the condition  $\rho \geq 0$  then we get the duality  $(\mathcal{T}_1\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ . See Theorem 4.55.)

## A Dictionary of OT and QM<sup>2</sup>

- states - unit vectors  $\xi \in \mathcal{H}$ . These are all the pure (normal) states on  $\mathcal{B}(\mathcal{H})$ .
- observable - selfadjoint operators  $A = A^*$
- measurement - spectrum

The spectral theorem was developed by J. von Neumann and later improved by Dirac and others. (See [Sto90, Yos95, Nel69, RS75, DS88c].) A selfadjoint operator  $A$  corresponds to a quantum observable, and result of a quantum measurement can be represented by the spectrum of  $A$ .

- simple eigenvalue:  $A = \lambda |\xi_\lambda\rangle\langle \xi_\lambda|$ ,

$$s_{\xi_\lambda}(A) = \langle \xi_\lambda, A\xi_\lambda \rangle$$

<sup>2</sup>The abbreviation OT is for operator theory, and QM for quantum mechanics.

- compact operator:  $A = \sum_{\lambda} \lambda |\xi_{\lambda}\rangle\langle \xi_{\lambda}|$ , such that  $\{\xi_{\lambda}\}$  is an ONB of  $\mathcal{H}$ . If  $\xi = \sum c_{\lambda} \xi_{\lambda}$  is a unit vector, then

$$s_{\xi}(A) = \sum_{\lambda} \lambda \langle \xi_{\lambda}, A\xi_{\lambda} \rangle$$

where  $\{|c_{\lambda}|^2\}_{\lambda}$  is a probability distribution over the spectrum of  $A$ , and  $s_{\xi}$  is the expectation value of  $A$ .

- more general, allowing continuous spectrum:

$$\begin{aligned} A &= \int \lambda E(d\lambda) \\ A\xi &= \int \lambda E(d\lambda)\xi. \end{aligned}$$

We may write the unit vector  $\xi$  as

$$\xi = \int \overbrace{E(d\lambda)\xi}^{\xi_{\lambda}}$$

so that

$$\|\xi\|^2 = \int \|E(d\lambda)\xi\|^2 = 1$$

It is clear that  $\|E(\cdot)\xi\|^2$  is a probability distribution on spectrum of  $A$ .  $s_{\xi}(A)$  is again seen as the expectation value of  $A$  with respect to  $\|E(\cdot)\xi\|^2$ , since

$$s_{\xi}(A) = \langle \xi, A\xi \rangle = \int \lambda \|E(d\lambda)\xi\|^2.$$

## 4.8 Krein-Milman, Choquet, Decomposition of States

We study some examples of compact convex sets in locally convex topological spaces.<sup>3</sup> Typical examples include the set of positive semi-definite functions, taking values in  $\mathbb{C}$  or  $\mathcal{B}(\mathcal{H})$ .

**Definition 4.100.** A vector space is locally convex if it has a topology which makes the vector space operators continuous, and if the neighborhoods  $\{x + \text{Nbh}_0\}$  have a basis consisting of convex sets.

The context for Krein-Milman is locally convex topological spaces. It is in all functional analysis books. Choquet's theorem however comes later, and it's not found in most books. A good reference is the book by R. Phelps [Phe01]. The proof of Choquet's theorem is not specially illuminating. It uses standard integration theory.

<sup>3</sup>Almost all spaces one works with are locally convex.



**Theorem 4.101** (Krein-Milman). *Let  $K$  be a compact convex set in a locally convex topological space. Then  $K$  is the closed convex hull of its extreme points  $E(K)$ , i.e.,*

$$K = \overline{\text{conv}}(E(K)).$$

*Proof.* (sketch) If  $K \not\supseteq \overline{\text{conv}}(E(K))$ , we get a linear functional  $w$ , such that  $w$  is zero on  $\overline{\text{conv}}(E(K))$  and not zero on  $w \in K \setminus \overline{\text{conv}}(E(K))$ . Extend  $w$  by Hahn-Banach theorem to a linear functional to the whole space, and get a contradiction.  $\square$

*Note 4.102.* The dual of a normed vector space is always a Banach space, so the theorem applies. The convex hull in an infinite dimensional space is not always closed, so close it. A good reference to locally convex topological space is the lovely book by F. Trèves [Trè06b].

A convex combination of points  $(\xi_i)$  in  $K$  takes the form  $v = \sum c_i \xi_i$ , where  $c_i > 0$  and  $\sum c_i = 1$ . Closure refers to taking limit, so we allow all limits of such convex combinations. Such a  $v$  is obviously in  $K$ , since  $K$  was assumed to be convex. The point of the Krein-Milman's theorem is the converse.

The decomposition of states into pure states was developed by Choquet et al; see [Phe01]. The idea goes back to Krein and Choquet.

**Theorem 4.103** (Choquet).  *$K = S(\mathfrak{A})$  is a compact convex set in a locally convex topological space. Let  $E(K)$  be the set of extreme points on  $K$ . Then for all  $p \in K$ , there exists a Borel probability measure  $\mu_p$ , supported on a Borel set  $bE(K) \supset E(K)$ , such that for all affine functions  $f$ , we have*

$$f(p) = \int_{bE(K)} f(\xi) d\mu_p(\xi). \quad (4.73)$$

The expression in Choquet's theorem is a generalization of convex combination. In stead of summation, it is an integral against a measure. Since there are some bizarre cases where the extreme points  $E(K)$  do not form a Borel set, the measure  $\mu_p$  is actually supported on  $bE(K)$ , such that  $\mu_p(bE(K) - E(K)) = 0$ .

Applications of Choquet theory and of Theorem 4.103 are manifold, and we shall discuss some of them in Chapter 7 below. Among them are applications to representations of  $C^*$ -algebras; e.g., the problem of finding "Borel-cross sections" for the set of equivalence classes of representations of a particular  $C^*$ -algebra. Equivalence here means "unitary equivalence." By a theorem of Glimm [Gli60, Gli61], we know that there are infinite simple  $C^*$ -algebras which do not admit such Borel parameterizations. Examples of this case include the Cuntz algebras  $\mathcal{O}_N$ ,  $N > 1$ . Nonetheless we shall study subclasses of representations of  $\mathcal{O}_N$  which correspond to sub-band filters in signal processing, and to pyramid algorithms for wavelet constructions. In Chapter 7 we shall also study representations of the  $C^*$ -algebra of the free group on 2 generators, as well as the  $C^*$ -algebra on two generators  $u$  and  $v$ , subject to the relation  $uvu^{-1} = u^2$ . It is called the Baumslag-Solitar algebra ( $BS_2$ ), after Gilbert Baumslag and Donald Solitar; and it is of great importance in a more a systematic analysis of families

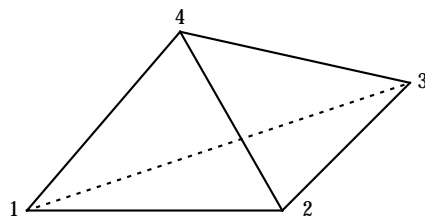


Figure 4.6: A simplex. The four extreme points are marked.

of wavelets. It is the algebra of a Baumslag-Solitar group. In fact, there are indexed families of Baumslag-Solitar groups, given by their respective group presentation. They are examples of two-generator one-relator groups, and they play an important role in combinatorial group theory, and in geometric group theory as (counter) examples and test-cases.

Other examples of uses of Choquet theory in harmonic analysis and representation theory include such decompositions from classical analysis as Fourier transform, Laplace transform, as well as direct integral theory for representations [Sti59, Seg50].

*Note 4.104.*  $\mu_p$  in (4.73) may not be unique. If it is unique,  $K$  is called a simplex. The unit disk has its boundary as extreme points. But representation of points in the interior using points on the boundary is not unique. Therefore the unit disk is not a simplex. A tetrahedron is (Figure 4.6).

**Example 4.105.** Let  $(X, \mathfrak{M}, \mu)$  be a measure space, where  $X$  is compact and Hausdorff. The set of all probability measures  $\mathcal{P}(X)$  is a convex set. To see this, let  $\mu_1, \mu_2 \in \mathcal{P}(X)$  and  $0 \leq t \leq 1$ , then  $t\mu_1 + (1-t)\mu_2$  is a measure on  $X$ , moreover  $(t\mu_1 + (1-t)\mu_2)(X) = t + 1 - t = 1$ , hence  $t\mu_1 + (1-t)\mu_2 \in \mathcal{P}(X)$ . Usually we don't want all probability measures, but a closed subset.

**Example 4.106.** We compute extreme points in the previous example.  $K = \mathcal{P}(X)$  is compact convex in  $C(X)^*$ , which is identified as the set of all measures due to Riesz.  $C(X)^*$  is a Banach space hence is always convex. The importance of being the dual of some Banach space is that the unit ball is always weak\*-compact (Banach-Alaoglu, Theorem 4.45). Note the weak\*-topology is just the cylinder/product topology. The unit ball  $B_1^*$  sits inside the infinite product space (compact, Hausdorff)  $\prod_{v \in B, \|v\|=1} D_1$ , where  $D_1 = \{z \in \mathbb{C} : |z| = 1\}$ . The weak\* topology on  $B_1^*$  is just the restriction of the product topology onto  $B_1^*$ .

**Example 4.107.** *Claim:*  $E(K) = \{\delta_x : x \in X\}$ , where  $\delta_x$  is the Dirac measure supported at  $x \in X$ . By Riesz, to know the measure is to know the linear functional.  $\int f d\delta_x = f(x)$ . Hence we get a family of measures indexed by  $X$ . If  $X = [0, 1]$ , we get a continuous family of measures. To see these really are extreme points, we do the GNS construction on the algebra  $\mathfrak{A} = C(X)$ , with the state  $\mu \in \mathcal{P}(X)$ . The Hilbert space so constructed is simply  $L^2(\mu)$ . It's

clear that  $L^2(\delta_x)$  is 1-dimensional, hence the representation is irreducible. We conclude that  $\delta_x$  is a pure state, for all  $x \in X$ .

There is a bijection between state  $\varphi$  and Borel measure  $\mu := \mu_\varphi$ ,

$$\varphi(a) = \int_X a d\mu_\varphi.$$

In  $C(X)$ ,  $1_{\mathfrak{A}} = \mathbb{1} = \text{constant function}$ . We check that

$$\varphi(1_{\mathfrak{A}}) = \varphi(1) = \int 1 d\mu = \mu(X) = 1$$

since  $\mu \in \mathcal{P}(X)$  is a probability measure. Also, if  $f \geq 0$  then  $f = g^2$ , with  $g := \sqrt{f}$ ; and

$$\varphi(f) = \int g^2 d\mu \geq 0.$$

*Note 4.108.*  $\nu$  is an extreme point in  $\mathcal{P}(X)$  if and only if

$$(\nu \in [\mu_1, \mu_2] = \text{convex hull of } \{\mu_1, \mu_2\}) \implies (\nu = \mu_1 \text{ or } \nu = \mu_2).$$

**Example 4.109.** Let  $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ , and  $S(\mathfrak{A}) = \text{states of } \mathfrak{A}$ . For each  $\xi \in \mathcal{H}$ , the map  $A \mapsto w_\xi(A) := \langle \xi, A\xi \rangle$  is a state, called vector state.

*Claim:*  $E(S) = \text{vector states}$ .

To show this, suppose  $W$  is a subspace of  $\mathcal{H}$  such that  $0 \subsetneq W \subsetneq \mathcal{H}$ , and suppose  $W$  is invariant under the action of  $\mathcal{B}(\mathcal{H})$ . Then  $\exists h \in \mathcal{H}$ ,  $h \perp W$ . Choose  $\xi \in W$ . The wonderful rank-1 operator (due to Dirac)  $T : \xi \mapsto h$  given by  $T := |h\rangle\langle \xi|$ , shows that  $h \in W$  (since  $TW \subset W$  by assumption.) Hence  $h \perp h$  and  $h = 0$ . Therefore  $W = \mathcal{H}$ . We say  $\mathcal{B}(\mathcal{H})$  acts transitively on  $\mathcal{H}$ .

*Note 4.110.* In general, any  $C^*$ -algebra is a closed subalgebra of  $\mathcal{B}(\mathcal{H})$  for some  $\mathcal{H}$  (Theorem 4.40). All the pure states on  $\mathcal{B}(\mathcal{H})$  are vector states.

**Example 4.111.** Let  $\mathfrak{A}$  be a  $*$ -algebra,  $S(\mathfrak{A})$  be the set of states on  $\mathfrak{A}$ .  $w : \mathfrak{A} \rightarrow \mathbb{C}$  is a state on  $\mathfrak{A}$  if  $w(1_{\mathfrak{A}}) = 1$  and  $w(A) \geq 0$ , whenever  $A \geq 0$ . The set of completely positive (CP) maps is a compact convex set. CP maps are generalizations of states (Chapter 5).

**Exercise 4.112** (Extreme measures). Take the two state sample space  $\Omega = \prod_1^\infty \{0, 1\}$  with product topology. Assign probability measure, so that we might favor one outcome than the other. For example, let  $s = x_1 + \dots + x_n$ ,  $P_\theta(C_x) = \theta^s(1 - \theta)^{n-1}$ , i.e.  $s$  heads,  $(n - s)$  tails. Notice that  $P_\theta$  is invariant under permutation of coordinates.  $x_1, x_2, \dots, x_n \mapsto x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)}$ .  $P_\theta$  is a member of the set of all such invariant measures (invariant under permutation)  $P_{inv}(\Omega)$ . Prove that

$$E(P_{inv}(\Omega)) = [0, 1]$$

i.e.,  $P_\theta$  are all the possible extreme points.

*Remark 4.113.* Let  $\sigma : X \rightarrow X$  be a measurable transformation. A (probability) measure  $\mu$  is *ergodic* if

$$[E \in \mathfrak{M}, \sigma E = E] \Rightarrow \mu(E) \in \{0, 1\}.$$

Intuitively, it says that the whole space  $X$  can't be divided non-trivially into parts where  $\mu$  is invariant. The set  $X$  will be mixed up by the transformation  $\sigma$ .

**Exercise 4.114** (Irrational rotation). Let  $\theta > 0$  be a fixed irrational number, and set

$$\sigma_\theta(x) = \theta x \bmod 1 \tag{4.74}$$

i.e., multiplication by  $\theta$  modulo 1. Show that  $\sigma_\theta$  in (4.74) is ergodic in the measure space  $\mathbb{R}/\mathbb{Z} \simeq [0, 1)$  with Lebesgue measure (see Figure 4.7).

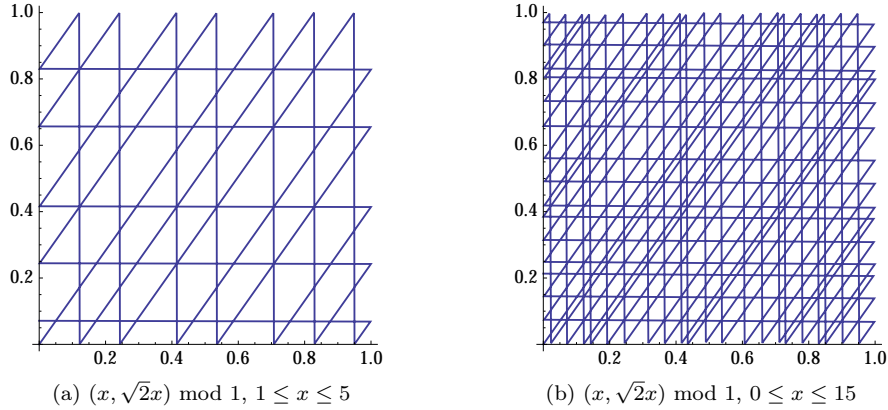


Figure 4.7: Irrational rotation.

### Noncommutative Radon-Nikodym Derivative

Let  $w$  be a state on a  $C^*$ -algebra  $\mathfrak{A}$ , and let  $K$  be an operator in  $\mathfrak{A}_+$ . Set

$$w_K(A) = \frac{w(\sqrt{K}A\sqrt{K})}{w(K)}.$$

Then  $w_K$  is a state, and  $w_K \ll w$ , i.e.,  $w(A) = 0 \Rightarrow w_K(A) = 0$ . We say that  $K = \frac{dw}{dw_K}$  is a noncommutative Radon-Nikodym derivative.

Check:

$$\begin{aligned} w_K(1) &= 1 \\ w_K(A^*A) &= \frac{w(\sqrt{K}A^*A\sqrt{K})}{w(K)} \\ &= \frac{w((A\sqrt{K})^*(A\sqrt{K}))}{w(K)} \geq 0 \end{aligned}$$

The converse holds too [Sak71] and is called the noncommutative Radon-Nikodym theorem.

### Examples of Disintegration

**Example 4.115.**  $L^2(I)$  with Lebesgue measure. Let

$$F_x(t) = \begin{cases} 1 & t \geq x \\ 0 & t < x \end{cases}$$

$F_x$  is a monotone increasing function on  $\mathbb{R}$ , hence by Riesz, we get the corresponding Riemann-Stieltjes measure  $dF_x$ .

$$d\mu = \int^{\oplus} dF_x(t)dx.$$

i.e.

$$\int f d\mu = \int dF_x(f)dx = \int f(x)dx.$$

Equivalently,

$$d\mu = \int \delta_x dx$$

i.e.

$$\int f d\mu = \int \delta_x(f)dx = \int f(x)dx.$$

$\mu$  is a state,  $\delta_x = dF_x(t)$  is a pure state,  $\forall x \in I$ . This is a decomposition of state into direct integral of pure states. See [Sti59, Seg50].

**Example 4.116.**  $\Omega = \prod_{t>0} \bar{\mathbb{R}}$ ,  $\Omega_x = \{w \in \Omega : w(0) = x\}$ . Kolmogorov gives rise to  $P_x$  by conditioning  $\bar{P}$  with respect to “starting at  $x$ ”.

$$P = \int^{\oplus} P_x dx$$

i.e.

$$P() = \int P(\cdot | \text{start at } x) dx.$$

**Example 4.117.** Harmonic function on  $D$

$$h \mapsto h(z) = \int_{\partial\mathbb{D}} \hat{f} d\mu_z$$

Poisson integration.

## 4.9 Examples of $C^*$ -algebras

Let  $\mathcal{H}$  be an infinite-dimensional separable Hilbert space, and let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an isometry; i.e., we have

$$S^*S = I_{\mathcal{H}}. \quad (4.75)$$

We shall be interested in the case when  $S$  is non-unitary, so the projection

$$P_S := SS^*$$

is not  $I_{\mathcal{H}}$ , i.e.,  $P_S \not\leq I_{\mathcal{H}}$ .

**Theorem 4.118** (Wold, see [Wol51, Con90]). *Let  $S : \mathcal{H} \rightarrow \mathcal{H}$  be an isometry. Set*

$$\mathcal{H}_0 := \left\{ x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|S^{*n}x\| = 0 \right\}, \text{ and} \quad (4.76)$$

$$\mathcal{H}_1 := \left\{ x \in \mathcal{H} : \|S^{*n}x\| = \|x\|, \forall n \in \mathbb{N} \right\}. \quad (4.77)$$

1. Then

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad (4.78)$$

where “ $\oplus$ ” in (4.78) refers to orthogonal sum, i.e.,  $\mathcal{H}_0 \perp \mathcal{H}_1$ .

2.  $S|_{\mathcal{H}_0} : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is a shift-operator;

3.  $S|_{\mathcal{H}_1} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is a unitary operator in  $\mathcal{H}_1$ .

**Exercise 4.119** (Wold’s decomposition). Carry out the details in the proof of Wold’s theorem.

In summary, associate to every isometry  $\mathcal{H} \xrightarrow{S} \mathcal{H}$ , there are three subspaces

$$\begin{aligned} \mathcal{H}_{\text{shift}} &= \mathcal{H}_0 \text{ in (4.76)} \\ \mathcal{H}_{\text{unit}} &= \mathcal{H}_1 \text{ in (4.77), and} \\ \mathfrak{h} &= \ker(S^*) \text{ in (4.80), the multiplicity space.} \end{aligned}$$

For the closed subspace  $\mathcal{H}_0$  in the shift-part of the decomposition, it holds that  $\mathcal{H}_0$  is the countable direct sum of  $\mathfrak{h}$  with itself.

**Exercise 4.120** (Substitution by  $z^N$ ). Let  $\mathcal{H} = \mathbb{H}_2 = \mathbb{H}_2(\mathbb{D})$  be the Hardy space of the disk, and let  $N \in \mathbb{N}$ ,  $N > 1$ . Set

$$(Sf)(z) = f(z^N), \quad f \in \mathbb{H}_2, \quad z \in \mathbb{D}. \quad (4.79)$$

Show that the three closed subspaces for this isometry are as follows:

$$\begin{aligned}
 \mathcal{H}_{\text{unit}} &= \text{the constant functions on } \mathbb{D} \\
 &= \mathbb{C}e_0, e_0(z) = z^0 = 1. \\
 \mathcal{H}_{\text{shift}} &= \mathcal{H} \ominus \mathbb{C}e_0 \\
 &= \{f \in \mathbb{H}_2 : f(0) = 0\} \\
 \ker(S^*) &= \overline{\text{span}}\{z^k : N \nmid k \text{ (not divisible by } N)\}, \text{ i.e.,} \\
 &\text{powers of } z^k, k \in (\{0\} \cup \mathbb{N}) \setminus N\mathbb{Z}, \text{ so } k \text{ not} \\
 &\text{divisible by } N.
 \end{aligned}$$

The isometry  $S$  in (4.79) is an example of an *isometry of infinite multiplicity*.

**$C^*$ -algebras generated by isometries.** An important family of non-abelian  $C^*$ -algebras includes those generated by one, or more, isometries:

**Case 1. One Isometry** Because of Wold's decomposition, if a  $C^*$ -algebra  $\mathfrak{A}$  is generated by one isometry, we may "split off" the one generated by the unitary part; and then reduce the study to the case where  $\mathfrak{A}$  is generated by a shift  $S$ .

Introduce  $\mathfrak{h} := \ker(S^*)$ , and

$$\mathfrak{h}^\infty = \bigoplus_{\mathbb{N}} \mathfrak{h} = \{(x, x_2, \dots) : x_i \in \mathfrak{h}\} \quad (4.80)$$

$$\|(x_1, x_2, \dots)\|_{\mathfrak{h}^\infty}^2 := \sum_{i=1}^{\infty} \|x_i\|_{\mathfrak{h}}^2; \quad (4.81)$$

and set

$$S_\infty(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots). \quad (4.82)$$

**Exercise 4.121** (The backwards shift). Show that

$$S_\infty^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots), \text{ and that}$$

$$\|(S_\infty^*)^n x\| \xrightarrow{n \rightarrow \infty} 0.$$

*Remark 4.122.* It would seem like the backwards shift is an overly specialized example. Nonetheless it plays a big role in operator theory, see for example [AD03, MQ14, KLR09], and it is an example of a wider class of operators going by the name "the Cowen-Dougllass class," playing an important role in complex geometry, see [CD78].

**Exercise 4.123** (Infinite multiplicity). For the isometry  $(Sf)(z) = f(z^N)$ ,  $f \in \mathbb{H}_2$ ,  $z \in \mathbb{D}$ , write out the representation (4.80)-(4.82) above.

**Exercise 4.124** (A shift is really a shift). Show that  $S$  and  $S_\infty$  are unitarily equivalent if and only if  $S$  is a shift.

**Exercise 4.125** (Multiplication by  $z$  is a shift in  $\mathbb{H}_2$ ). If  $\dim \mathfrak{h} = 1$  (multiplicity one), show that  $S$  in (4.82) is unitarily equivalent to

$$\left(\tilde{S}f\right)(z) = zf(z), \quad z \in \mathbb{D}, \quad f \in \mathbb{H}_2 = \text{the Hardy space.} \quad (4.83)$$

Hint: By  $\mathbb{H}_2$ , we mean the Hilbert space of all analytic functions  $f$  on the disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  such that

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad \text{and } (a_k) \in l^2. \quad (4.84)$$

We set  $\|f\|_{\mathbb{H}_2} = \|(a_k)\|_{l^2}$ .

**Exercise 4.126** (The two shifts in  $\mathbb{H}_2$ ). Show that the adjoint to the generator  $\tilde{S}$  (from (4.83)) is

$$\left(\tilde{S}^*f\right)(z) = \frac{f(z) - f(0)}{z}, \quad \forall f \in \mathbb{H}_2, \forall z \in \mathbb{D} \setminus \{0\}, \quad (4.85)$$

and

$$\left(\tilde{S}^*f\right)(0) = f'(0), \quad f \in \mathbb{H}_2.$$

Hint: Show that, if  $f, g \in \mathbb{H}_2$ , then the following holds:

$$\left\langle \tilde{S}f, g \right\rangle_{\mathbb{H}_2} = \left\langle f, \tilde{S}^*g \right\rangle_{\mathbb{H}_2} \quad (4.86)$$

where we use formula (4.85) in computing the  $\mathbb{H}_2$ -inner product on the RHS in (4.86).

Compare this with the result from Exercise 4.121.

**Exercise 4.127** (A numerical range). Let  $T := S_{\infty}^*$  be the backward shift (expressed in coordinates) in Exercise 4.121. Since  $TT^* - T^*T$  is the rank-one projection  $|e_1\rangle\langle e_1|$ , of course  $T$  is not normal.

1. Show that  $x_{\lambda} = (1, \lambda, \lambda^2, \lambda^3, \dots)$  satisfies

$$Tx_{\lambda} = \lambda x_{\lambda}, \quad \forall \lambda \in \mathbb{C}. \quad (4.87)$$

2. Since

$$x_{\lambda} \in l^2 \iff |\lambda| < 1, \quad (4.88)$$

conclude that the point-spectrum of  $T$  is  $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ .

3. Combine (1) & (2) in order to conclude that

$$NR_T = \mathbb{D}.$$



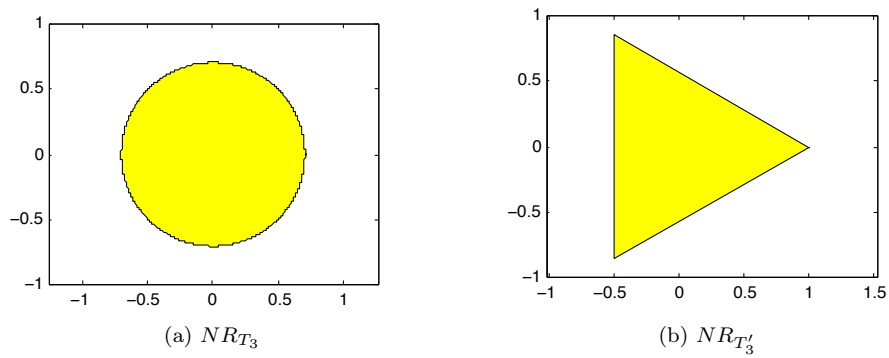


Figure 4.8: The numerical range of  $T_3$  vs  $T'_3$

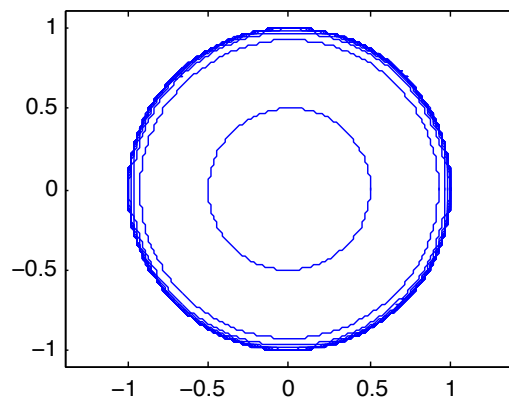


Figure 4.9: The numerical range (NR) of the truncated finite matrices: Expanding truncations of the infinite matrix  $T$  corresponding to the backward shift, and letting the size  $\rightarrow \infty$ :  $T_3, T_4, \dots, T_n, T_{n+1}, \dots$ ; the limit-NR fills the open disk of radius 1.

**Exercise 4.128** (The finite shift). Compare the infinite case above with the analogous matrix case

$$T_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T'_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

A sketch of  $NR_{T_3}$  and  $NR_{T'_3}$  are in Figure 4.8 below. See also Figure 4.9.

**Exercise 4.129** (The Hardy space  $\mathbb{H}_2$ ; a transform). Let  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , and set

$$\tilde{f}(t) = \sum_{k=0}^{\infty} a_k e^{i2\pi kt}, \quad t \in \mathbb{R}. \quad (4.89)$$

Show that

$$f \in \mathbb{H}_2 \iff \tilde{f} \in L^2(\mathbb{T}), \quad \mathbb{T} = \partial\mathbb{D};$$

and that

$$\|\tilde{f}\|_{L^2(\mathbb{T})} = \|f\|_{\mathbb{H}_2} \quad (4.90)$$

holds.

Because of Exercise 4.129, we may identify  $\mathbb{H}_2$  with a closed subspace in  $L^2(\mathbb{T})$ . Let  $P_+$  denote the projection of  $L^2(\mathbb{T})$  onto  $\mathbb{H}_2$ .

**Definition 4.130.** For  $\varphi \in L^\infty(\mathbb{T})$ , set

$$T_\varphi f = P_+(\varphi f), \quad \forall f \in \mathbb{H}_2, \quad (4.91)$$

equivalently,  $T_\varphi = P_+ M_\varphi P_+$ .

The operator  $T_\varphi$  in (4.91) is called a *Toeplitz-operator*; and

$$\mathcal{T} := C^*(\{T_\varphi : \varphi \in L^\infty(\mathbb{T})\}) \quad (4.92)$$

is called the *Toeplitz-algebra*.

**Exercise 4.131** (Multiplicity-one and  $\mathbb{H}_2$ ). Show that there is a short exact sequence (in the category of  $C^*$ -algebras):

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\pi} \mathcal{T}/\mathcal{K} \longrightarrow 0 \quad (4.93)$$

where  $\mathcal{K}$  = the  $C^*$ -algebra of compact operators; and  $\mathcal{T}/\mathcal{K}$  is the quotient; finally

$$\mathcal{T}/\mathcal{K} \simeq L^\infty(\mathbb{T}),$$

realized via the mapping  $T_\varphi \xrightarrow{\pi} \varphi$  (the symbol mapping) in (4.93), i.e.,  $\pi(T_\varphi) := \varphi$ , is assigning the symbol  $\varphi$  to the Toeplitz operator  $T_\varphi$ .

**Exercise 4.132** (The Toeplitz matrices). Suppose  $\varphi \in L^\infty(\mathbb{T})$  has Fourier expansion

$$\varphi(t) = \sum_{n \in \mathbb{Z}} b_n e^{i2\pi nt}, \quad t \in \mathbb{R}. \quad (4.94)$$

Then show that the  $\infty \times \infty$  matrix of the corresponding Toeplitz operator  $T_\varphi$  is as follows w.r.t the standard ONB in  $\mathbb{H}_2$ ,  $\{z^n : n \in \{0\} \cup \mathbb{N}\}$ .

$$\text{Mat}(T_\varphi) = \begin{bmatrix} b_0 & b_{-1} & b_{-2} & b_{-3} & b_{-4} & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1 & b_0 & b_{-1} & b_{-2} & b_{-3} & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_2 & b_1 & b_0 & b_{-1} & b_{-2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_3 & b_2 & b_1 & b_0 & b_{-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_4 & b_3 & b_2 & b_1 & b_0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & b_0 & b_{-1} & b_{-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & b_1 & b_0 & b_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & b_2 & b_1 & b_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (4.95)$$

*Note 4.133.* Matrices of the form given in (4.95) are called *Toeplitz matrices*, i.e., with the banded pattern, constant numbers down the diagonal lines, with  $b_0$  in the main diagonal.

*Remark 4.134.* Note that the mapping  $\varphi \rightarrow T_\varphi$  (Toeplitz),  $L^\infty(\mathbb{T}) \rightarrow \mathcal{T}$  is not a homomorphism of the algebra  $L^\infty(\mathbb{T})$  into  $\mathcal{T} = C^*(\{T_\varphi\})$ . Here we view  $\overline{L^\infty}(\mathbb{T})$  as an abelian  $C^*$ -algebra under pointwise product, i.e.,

$$(\varphi_1 \varphi_2)(t) := \varphi_1(t) \varphi_2(t), \quad \forall t \in \mathbb{R}/\mathbb{Z}.$$

The point of mapping of the *short exact sequence* (lingo from homological algebra):

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow L^\infty(\mathbb{T}) \rightarrow 0 \quad (4.96)$$

is that  $\varphi \rightarrow T_\varphi$  is only a “homomorphism mod  $\mathcal{K}$  (= the compact operators)”, i.e., that we have

$$T_{\varphi_1} T_{\varphi_2} - T_{\varphi_1 \varphi_2} \in \mathcal{K} \quad (4.97)$$

valid for all  $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T})$ .

There is an extensive literature on (4.96) and (4.97), see especially [Dou80].

**Exercise 4.135** (homomorphism mod  $\mathcal{K}$ ). Give a direct proof that the operator on the LHS in (4.97) is a compact operator in  $\mathbb{H}_2$ .

*Remark 4.136.* The subject of Toeplitz operators, and Toeplitz algebras is vast (see e.g., [AZ07]). The more restricted case where the symbol  $\varphi$  of  $T_\varphi = P_+ M_\varphi P_+$  is continuous (i.e.,  $\varphi \in C(S^1)$ ,  $S^1 =$  the circle) is especially rich; starting with *Szegő’s Index Theorem* :

**Definition 4.137.** Let  $X$  and  $Y$  be Banach spaces. A *Fredholm operator* is a bounded linear operator  $T : X \rightarrow Y$ , such that  $\ker(T)$  and  $\ker(T^*)$  are finite-dimensional, and  $\text{ran}(T)$  is closed. The *index* of  $T$  is given as

$$\text{ind}(T) := \dim(\ker(T)) - \dim(\ker(T^*)).$$

(The assumption on the range of  $T$  in the definition is redundant [AA02].)

**Theorem 4.138** (Szegö [BS94]). *If  $\varphi \in C(S^1)$  and  $\varphi$  does not vanish on  $S^1$ , then  $T_\varphi$  is Fredholm, and the index of  $T_\varphi$  computes as follows:*

$$\text{ind}(T_\varphi) = \dim(\ker(T_\varphi)) - \dim(\ker(T_\varphi^*)) = -\#w(\varphi) \quad (4.98)$$

where  $\#w(\varphi)$  in (4.98) is the winding number

$$\#w(\varphi) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\varphi'(e^{i\theta})}{\varphi(e^{i\theta})} d\theta. \quad (4.99)$$

Note:  $\#w(e^{in\theta}) = n$ , for  $n \in \mathbb{Z}$ , and  $\ker(T_\varphi^*) = (\text{ran}(T_\varphi))^\perp$ .

**Case 2. Multiple Isometries** Here we refer to the Cuntz-algebra  $\mathcal{O}_N$  (see [Cun77]), the unique  $C^*$ -algebra  $\mathcal{O}_N$ ,  $N > 1$ , generated by  $\{S_i\}_{i=1}^N$  and the relations

$$S_i^* S_j = \delta_{ij}, \text{ and} \quad (4.100)$$

$$\sum_{i=1}^N S_i S_i^* = \mathbf{1}. \quad (4.101)$$

Cuntz showed ([Cun77]) that this is a simple  $C^*$ -algebra (i.e., no non-trivial closed two-sided ideals), purely infinite.

We shall return to the study of its *representation* in Chapter 7.

**Exercise 4.139** (An element in  $\text{Rep}(\mathcal{O}_N, \mathbb{H}_2)$ ). Fix  $N \in \mathbb{N}$ ,  $N > 1$ , and consider the following operators  $\{S_k\}_{k=0}^{N-1}$  acting in the Hardy space  $\mathbb{H}_2 = \mathbb{H}_2(\mathbb{D})$ :

$$(S_k f)(z) = z^k f(z^N), \quad \forall f \in \mathbb{H}_2, \quad \forall z \in \mathbb{D}, \quad k = 0, 1, \dots, N-1. \quad (4.102)$$

Show that the operators  $(S_k)$  in (4.102) satisfy the  $\mathcal{O}_N$ -relations (4.100)-(4.101), i.e., that

$$S_j^* S_k = \delta_{jk} I_{\mathbb{H}_2}, \text{ and} \\ \sum_{j=0}^{N-1} S_j S_j^* = I_{\mathbb{H}_2};$$

hence a *representation* of  $\mathcal{O}_N$  in  $\mathbb{H}_2$ .

**Exercise 4.140** (The multivariable Toeplitz algebra). For  $k \in \mathbb{N}$ , set  $\mathcal{H}_k = \mathbb{C}^k$  = the  $k$ -dimensional complex Hilbert space with the usual inner product:

$$\langle v, w \rangle = \sum_{j=1}^k \overline{v_j} w_j. \quad (4.103)$$

For  $k = 1$ , pick a normalized basis vector  $\Omega$ . For  $N > 1$ , set

$$\mathcal{F}(\mathcal{H}_N) = \mathcal{H}_1 \oplus \sum_{n=1}^{\infty} \oplus \mathcal{H}_N^{\otimes n}. \quad (4.104)$$

(The letter  $\mathcal{F}$  is for Fock-space.) For  $f \in \mathcal{H}_N$ , set:

$$T_f(\otimes_1^n h_j) = f \otimes (\otimes_1^n h_j), \text{ and} \quad (4.105)$$

$$T_f^*(\otimes_1^n h_j) = \langle f, h_1 \rangle \otimes_2^n h_j, \quad n \in \mathbb{N}. \quad (4.106)$$

And finally, the vacuum rule:

$$T_f^* \Omega = 0. \quad (4.107)$$

1. Show that the following hold:

$$T_f^* T_g = \langle f, g \rangle_N I_{\mathcal{F}(\mathcal{H}_N)}, \quad \forall f, g \in \mathcal{H}_N. \quad (4.108)$$

Define  $T_i$  and  $T_i^*$  from and ONB in  $\mathcal{H}_N$ , we get

$$\sum_{i=1}^N T_i T_i^* = I_{\mathcal{F}(\mathcal{H}_N)} - |\Omega\rangle\langle\Omega|. \quad (4.109)$$

The  $C^*$ -algebra generated by  $\{T_f : f \in \mathcal{H}_N\}$  is called the (multivariable) *Toeplitz algebra*, and is denoted  $\mathcal{T}_N$ .

2. Show, with the use of (4.108)-(4.109), that there is a natural short exact sequence of  $C^*$ -algebras:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_N \longrightarrow \mathcal{O}_N \longrightarrow 0.$$

Compare with (4.96) in Remark 4.134.

## 4.10 Examples of Representations

We consider the Fourier algebra.

1. Discrete case:  $l^1(\mathbb{Z})$  and the Gelfand transform

$$\begin{aligned} (a * b)_n &= \sum_k a_k b_{n-k} \\ (a^*)_n &= \overline{a_{-n}} \\ 1_{\mathfrak{A}} &= \delta_0 \end{aligned}$$

$$a \xrightarrow[\text{Gelfand}]{\mathcal{F}} F(z) := \sum_n a_n z^n$$

We may specialize to  $z = e^{it}$ ,  $t \in \mathbb{R} \bmod 2\pi$ .  $\{F(z)\}$  is an abelian algebra of functions, with multiplication is given by

$$F(z)G(z) = \sum_n (a * b)_n z^n$$

In fact, most abelian algebras can be thought of as function algebras.

Homomorphism:

$$\begin{aligned} (l^1, *) &\xrightarrow{\mathcal{F}} C(\mathbb{T}^1) \\ (a_n) &\mapsto F(z). \end{aligned}$$

If we want to write  $F(z)$  as power series, then we need to drop  $a_n$  for  $n < 0$ . Then  $F(z)$  extends to an analytic function over the unit disk. The representation by the sequence space

$$\{a_0, a_1, \dots\}$$

was suggested by Hardy. We set

$$\|F\|_{\mathbb{H}_2}^2 = \sum_{k=0}^{\infty} |a_k|^2;$$

the natural isometric isomorphism. Rudin has two nice chapters on  $H^2$ , as a Hilbert space, a RKHS. See [Rud87, ch16].

2. Continuous case:  $L^1(\mathbb{R})$

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{\infty} f(s)g(x-s)ds \\ f^*(x) &= \overline{f(-x)} \end{aligned}$$

The algebra  $L^1$  has no identity, but we may always insert one by adding  $\delta_0$ . So  $\delta_0$  is the homomorphism  $f \mapsto f(0)$ ; and  $L^1(\mathbb{R}) \cup \{\delta_0\}$  is again a Banach  $*$ -algebra.

The *Gelfand map* is the classical Fourier transform, i.e.,

$$f \xrightarrow[\text{Gelfand}]{\mathcal{F}} \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

where  $\widehat{f * g} = \hat{f}\hat{g}$ .

*Remark 4.141.*  $C(\mathbb{T}^1)$  is called the  $C^*$ -algebra completion of  $l^1$ .  $L^\infty(X, B, \mu) = L^1(\mu)^*$  is also a  $C^*$ -algebra. It is a  $W^*$ -algebra, or von Neumann algebra (see, e.g., [Sak71]). The  $W^*$  refers to the fact that its topology comes from the weak  $*$ -topology. Recall that  $\mathcal{B}(\mathcal{H})$ , for any Hilbert space, is a von Neumann algebra.

**Example 4.142.** Fix  $\varphi$  and set  $uf = e^{i\theta}f(\theta)$ ,  $vf = f(\theta - \varphi)$ , restrict to  $[0, 2\pi]$ , i.e.,  $2\pi$  periodic functions.

$$\begin{aligned}vuv^{-1} &= e^{i\varphi}u \\vu &= e^{i\varphi}uv\end{aligned}$$

$u, v$  generate a noncommutative  $C^*$ -algebra. See [EN12, Boc08].

**Example 4.143** (Quantum Mechanics). Consider the canonical commutation relation

$$[p, q] = -iI, \quad i = \sqrt{-1},$$

where  $[x, y] := xy - yx$  denotes the commutator of  $x$  and  $y$ .

The two symbols  $p, q$  generate an algebra, but they can not be represented by bounded operators. But we may apply bounded functions to them and get a  $C^*$ -algebra.

**Exercise 4.144** (No bounded solutions to the canonical commutation relations). Show that  $p, q$  and not be represented by bounded operators. Hint: take the trace.

**Example 4.145.** Let  $\mathcal{H}$  be an infinite dimensional Hilbert space, then  $\mathcal{H}$  is isometrically isomorphic to a proper subspace of itself. For example, let  $\{e_n\}$  be an ONB.  $\mathcal{H}_1 = \overline{\text{span}}\{e_{2n}\}$ ,  $\mathcal{H}_2 = \overline{\text{span}}\{e_{2n+1}\}$ . Let

$$\begin{aligned}V_1(e_n) &= e_{2n} \\V_2(e_n) &= e_{2n+1}\end{aligned}$$

then we get two isometries. Also,

$$\begin{aligned}V_1V_1^* + V_2V_2^* &= I \\V_i^*V_i &= I \\V_iV_i^* &= P_i\end{aligned}$$

where  $P_i$  is a selfadjoint *projection*,  $i = 1, 2$  onto the respective  $\mathcal{H}_i$ . This is the Cuntz algebra  $\mathcal{O}_2$ . More general  $\mathcal{O}_N$ ,  $N > 2$ .

Cuntz (in 1977) showed that this is a *simple*  $C^*$ -algebra, i.e., it does not have non-trivial closed two-sided ideals. For studies of its representations, see, e.g., [Gli60, Gli61, BJO04].

## 4.11 Beginning of Multiplicity Theory

The main question here is how to break up a representation into smaller ones. The smallest are the irreducible representations, and the next would be the multiplicity free representations.

Let  $\mathfrak{A}$  be an algebra.

- *commutative*: e.g., function algebras
- *non-commutative*: e.g., matrix algebra, algebras generated by representation of non-abelian groups

Smallest representation:

- *irreducible*:  $\pi \in \text{Rep}_{\text{irr}}(\mathfrak{A}, \mathcal{H})$ , where the commutant  $\pi(\mathfrak{A})'$  is 1-dimensional. This is the starting point of further analysis.
- *multiplicity free*: Let  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ . We may assume  $\pi$  is cyclic, since otherwise  $\pi$  can be decomposed into a direct sum of cyclic representations, i.e.,  $\pi = \oplus \pi_{\text{cyc}}$ ; see Theorem 4.32. Then,

$$\pi \text{ is multiplicity free} \iff \pi(\mathfrak{A})' \text{ is abelian.}$$

Fix a Hilbert space  $\mathcal{H}$ , and let  $\mathfrak{C}$  be a  $*$ -algebra in  $\mathcal{B}(\mathcal{H})$ . The commutant  $\mathfrak{C}'$  is given by

$$\mathfrak{C}' = \{X \in \mathcal{B}(\mathcal{H}) : XC = CX, \forall C \in \mathfrak{C}\}.$$

The commutant  $\mathfrak{C}'$  is also a  $*$ -algebra, and

$$\mathfrak{C} \text{ is abelian} \iff \mathfrak{C} \subset \mathfrak{C}'.$$

Note that  $\mathfrak{C} \subset \mathfrak{C}''$  (double-commutant.)

**Theorem 4.146** (von Neumann). *If  $M$  is a von Neumann algebra, then  $M = M''$ .*

*Proof.* See, e.g., [BR79, KR97a]. □

**Definition 4.147.** Let  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ . We say that  $\pi$  has *multiplicity*  $n$ ,  $n \in \{0\} \cup \mathbb{N}$ , if  $\pi(\mathfrak{A})' \simeq M_n(\mathbb{C})$ , i.e., the commutant  $\pi(\mathfrak{A})'$  is  $*$ -isomorphic to the algebra of all  $n \times n$  complex matrices.  $\pi$  is said to be *multiplicity-free* if  $\pi(\mathfrak{A})' \simeq \mathbb{C}I_{\mathcal{H}}$ .

**Example 4.148.** Let

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 2 \end{bmatrix}.$$



Let  $C \in M_3(\mathbb{C})$ , then  $AC = CA$  if and only if  $C$  has the form

$$C = \begin{bmatrix} a & b & \\ c & d & \\ & & 1 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$$

where  $B \in M_2(\mathbb{C})$ .

Let  $A$  be a linear operator (not necessarily bounded) acting in the Hilbert space  $\mathcal{H}$ . By the Spectral Theorem (Chapter 3), we have  $A = A^*$  if and only if

$$A = \int_{sp(A)} \lambda P_A(d\lambda);$$

where  $P_A$  is the corresponding projection-valued measure (PVM).

**Example 4.149.** The simplest example of a PVM is when  $\mathcal{H} = L^2(X, \mu)$ , for some compact Hausdorff space  $X$ , and  $P(\omega) := \chi_\omega$ , for all Borel subsets  $\omega$  in  $X$ . Indeed, the Spectral Theorem states that all PVMs come this way.

**Example 4.150.** Let  $A$  be compact and selfadjoint. We may further assume that  $A$  is positive,  $A \geq 0$ , in the usual order of Hermitian operators (i.e.,  $\langle x, Ax \rangle \geq 0, \forall x \in \mathcal{H}$ .) Then by Theorem 3.58,  $A$  has the decomposition

$$A = \sum_{n=1}^{\infty} \lambda_n P_n \quad (4.110)$$

where  $\lambda_n$ 's are the eigenvalues of  $A$ , such that  $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \rightarrow 0$ ; and  $P_n$ 's are the selfadjoint projections onto the (finite dimensional) eigenspace of  $\lambda_n$ . In this case, the projection-valued measure  $P_A$  is supported on  $\mathbb{N}$ , and  $P_A(\{n\}) = P_n, \forall n \in \mathbb{N}$ .

In (4.110), we may arrange the eigenvalues as follows:

$$\overbrace{\lambda_1 = \dots = \lambda_1}^{s_1} > \overbrace{\lambda_2 = \dots = \lambda_2}^{s_2} > \dots > \overbrace{\lambda_n = \dots = \lambda_n}^{s_n} > \dots \rightarrow 0. \quad (4.111)$$

We say that  $\lambda_i$  has multiplicity  $s_i$ , i.e., the dimension of the eigenspace of  $\lambda_i$ . Note that

$$\dim \mathcal{H} = \sum_{i=1}^{\infty} s_i.$$

Question: What does  $A$  look like if it is represented as the operator of multiplication by the independent variable?

**Example 4.151.** Let  $s_1, s_2, \dots$  be a sequence in  $\mathbb{N}$ , set

$$E_k = \{x_1^{(k)}, \dots, x_{s_k}^{(k)}\} \subset \mathbb{C}, \text{ and } E = \bigcup_{k=1}^{\infty} E_k.$$

Let  $\mathcal{H} = l^2(E)$ , and

$$f := \sum_{k=1}^{\infty} \lambda_k \chi_{E_k}, \text{ s.t. } \lambda_1 > \lambda_2 > \cdots > \lambda_n \rightarrow 0.$$

Let We represent  $A$  as the operator  $M_f$  of multiplication by  $f$  on  $L^2(X, \mu)$ . Let  $E_k = \{x_{k,1}, \dots, x_{k,s_k}\} \subset X$ , and let  $\mathcal{H}_k = \text{span}\{\chi_{\{x_{k,j}\}} : j \in \{1, 2, \dots, s_k\}\}$ . Let Notice that  $\chi_{E_k}$  is a rank  $s_1$  projection.  $M_f$  is compact if and only if it is of the given form.

**Example 4.152.** Follow the previous example, we represent  $A$  as the operator  $M_t$  of multiplication by the independent variable on some Hilbert space  $L^2(\mu_f)$ . For simplicity, let  $\lambda > 0$  and

$$f = \lambda \chi_{\{x_1, x_2\}} = \lambda \chi_{\{x_1\}} + \lambda \chi_{\{x_2\}}$$

i.e.  $f$  is compact since it is  $\lambda$  times a rank-2 projection;  $f$  is positive since  $\lambda > 0$ . The eigenspace of  $\lambda$  has two dimension,

$$M_f \chi_{\{x_i\}} = \lambda \chi_{\{x_i\}}, \quad i = 1, 2.$$

Define  $\mu_f(\cdot) = \mu \circ f^{-1}(\cdot)$ , then

$$\mu_f = \mu(\{x_1\})\delta_\lambda \oplus \mu(\{x_2\})\delta_\lambda \oplus \text{cont. sp } \delta_0$$

and

$$L^2(\mu_f) = L^2(\mu(\{x_1\})\delta_\lambda) \oplus L^2(\mu(\{x_2\})\delta_\lambda) \oplus L^2(\text{cont. sp } \delta_0).$$

Define  $U : L^2(\mu) \rightarrow L^2(\mu_f)$  by

$$(Ug) = g \circ f^{-1}.$$

$U$  is unitary, and the following diagram commute:

$$\begin{array}{ccc} L^2(X, \mu) & \xrightarrow{M_f} & L^2(X, \mu) \\ \downarrow U & & \downarrow U \\ L^2(\mathbb{R}, \mu_f) & \xrightarrow{M_t} & L^2(\mathbb{R}, \mu_f) \end{array}$$

To check  $U$  preserves the  $L^2$ -norm,

$$\begin{aligned} \|Ug\|^2 &= \int \|g \circ f^{-1}(\{x\})\|^2 d\mu_f \\ &= \|g \circ f^{-1}(\{\lambda\})\|^2 + \|g \circ f^{-1}(\{0\})\|^2 \\ &= |g(x_1)|^2 \mu(\{x_1\}) + |g(x_2)|^2 \mu(\{x_2\}) + \int_{X \setminus \{x_1, x_2\}} |g(x)|^2 d\mu \\ &= \int_X |g(x)|^2 d\mu \end{aligned}$$

To see  $U$  diagonalizes  $M_f$ ,

$$\begin{aligned} M_t U g &= \lambda g(x_1) \oplus \lambda g(x_2) \oplus 0g(t)\chi_{X \setminus \{x_1, x_2\}} \\ &= \lambda g(x_1) \oplus \lambda g(x_2) \oplus 0 \\ U M_f g &= U(\lambda g(x)\chi_{\{x_1, x_2\}}) \\ &= \lambda g(x_1) \oplus \lambda g(x_2) \oplus 0 \end{aligned}$$

Thus

$$M_t U = U M_f.$$

*Remark 4.153.* Notice that  $f$  should really be written as

$$f = \lambda\chi_{\{x_1, x_2\}} = \lambda\chi_{\{x_1\}} + \lambda\chi_{\{x_2\}} + 0\chi_{X \setminus \{x_1, x_2\}}$$

since 0 is also an eigenvalue of  $M_f$ , and the corresponding eigenspace is the kernel of  $M_f$ .

**Example 4.154.** diagonalize  $M_f$  on  $L^2(\mu)$  where  $f = \chi_{[0,1]}$  and  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ .

**Example 4.155.** diagonalize  $M_f$  on  $L^2(\mu)$  where

$$f(x) = \begin{cases} 2x & x \in [0, 1/2] \\ 2 - 2x & x \in [1/2, 1] \end{cases}$$

and  $\mu$  is the Lebesgue measure on  $[0, 1]$ .

*Remark 4.156.* see direct integral and disintegration of measures.

In general, let  $A$  be a selfadjoint operator acting on  $\mathcal{H}$ . Then there exists a second Hilbert space  $K$ , a measure  $\nu$  on  $\mathbb{R}$ , and unitary transformation  $F : \mathcal{H} \rightarrow L_K^2(\mathbb{R}, \nu)$  such that

$$M_t F = F A$$

for measurable function  $\varphi : \mathbb{R} \rightarrow K$ ,

$$\|\varphi\|_{L_K^2(\nu)} = \int \|\varphi(t)\|_K^2 d\nu(t) < \infty.$$

Examples that do have multiplicities in finite dimensional linear algebra:

**Example 4.157.** 2-d,  $\lambda I, \{\lambda I\}' = M_2(\mathbb{C})$  which is not abelian. Hence  $\text{mult}(\lambda) = 2$ .

**Example 4.158.** 3-d,

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 I & \\ & \lambda_2 \end{bmatrix}$$

where  $\lambda_1 \neq \lambda_2$ . The commutant is

$$\begin{bmatrix} B & \\ & b \end{bmatrix}$$

where  $B \in M_2(\mathbb{C})$ , and  $b \in \mathbb{C}$ . Therefore the commutant is isomorphic to  $M_2(\mathbb{C})$ , and multiplicity is equal to 2.

**Example 4.159.** The example of  $M_\varphi$  with repetition.

$$M_\varphi \oplus M_\varphi : L^2(\mu) \oplus L^2(\mu) \rightarrow L^2(\mu) \oplus L^2(\mu)$$

$$\begin{bmatrix} M_\varphi & \\ & M_\varphi \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \varphi f_1 \\ \varphi f_2 \end{bmatrix}$$

the commutant in this case is isomorphic to  $M_2(\mathbb{C})$ . If we introduce the tensor product, then the representation space is also written as  $L^2(\mu) \otimes V_2$ , the multiplication operator is amplified to  $M_\varphi \otimes I$ , whose commutant is represented as  $I \otimes V_2$ . Hence it's clear that the commutant is isomorphic to  $M_2(\mathbb{C})$ . To check

$$\begin{aligned} (\varphi \otimes I)(I \otimes B) &= \varphi \otimes B \\ (I \otimes B)(\varphi \otimes I) &= \varphi \otimes B. \end{aligned}$$

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [Arv76, BR79, Mac85, BD91, Dou80, Cob67, BJ97b, BJ97a, GJ87, AD03, Alp01, BJ02, Cun77, Con90, Dix81, Gli61, KR97a, KR97b, Seg50, Sak71, Tay86, MJD<sup>+</sup>15, Hal13, Hal15].

## Chapter 5

# Completely Positive Maps

“Completely positive maps on von Neumann algebras or between  $C^*$ -algebras have fascinated me since my days as a graduate student.”

— William B. Arveson

“... the development of mathematics is not something one can predict, and it would be foolish to try. One reason we love doing mathematics is that we don't know what lies ahead that future research will uncover.”

— Alain Connes

The study of completely positive maps dates back five decades, but because of a recent observation of Arveson (see e.g., [Arv09a, Arv09c, Arv09b]), they have acquired a brand new set of applications; applications to quantum information theory (QIT). In this framework, one studies completely positive maps on matrix algebras. They turn out to be the objects that are dual to quantum channels. Even more: Arveson proved the converse: that the study of quantum channels reduces to the study of unital completely positive maps of matrix algebras. This work is part of QIT, and it is still ongoing, with view to the study of entanglement, entropy and channel-capacity.

In the last chapter we studied two question from the use of algebras of operators in quantum physics: “Where does the Hilbert space come from?” And “What are the algebras of operators from which the selfadjoint observables must be selected?” An answer is given in “the Gelfand-Naimark-Segal (GNS) theorem;” a direct correspondence between states and cyclic representations. But states are scalar valued positive definite functions on  $*$ -algebras. For a host of applications, one must instead consider operator valued “states.” For this a different notion of positivity is needed, “*complete positivity*.”

The GNS construction gives a bijection between states and cyclic representations. An extension to the GNS construction is Stinespring's completely positive

maps. It appeared in an early paper by Stinespring in 1955 [Sti55]. Arveson in 1970's greatly extended Stinespring's result using tensor product [Arv72]. He showed that completely positive maps are the key in multivariable operator theory, and in noncommutative dynamics.

## 5.1 Motivation

Let  $\mathfrak{A}$  be a  $*$ -algebra with identity. Recall that a functional  $w : \mathfrak{A} \rightarrow \mathbb{C}$  is a *state* if  $w(1_{\mathfrak{A}}) = 1$ ,  $w(A^*A) \geq 0$ . If  $\mathfrak{A}$  was a  $C^*$ -algebra,  $A \geq 0 \Leftrightarrow sp(A) \geq 0$ , hence we may take  $B = \sqrt{A}$  and  $A = B^*B$ .

Given a state  $w$ , the GNS construction gives a Hilbert space  $\mathcal{H}$ , a cyclic vector  $\Omega \in \mathcal{H}$ , and a representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ , such that

$$w(A) = \langle \Omega, \pi(A)\Omega \rangle$$

$$\mathcal{H} = \overline{\text{span}}\{\pi(A)\Omega : A \in \mathfrak{A}\}.$$

Moreover, the Hilbert space is unique up to unitary equivalence.

Stinespring modified the GNS construction as follows: Instead of a state  $w : \mathfrak{A} \rightarrow \mathbb{C}$ , he considered a positive map  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ , i.e.,  $\varphi$  maps positive elements in  $\mathfrak{A}$  to positive operators in  $\mathcal{B}(\mathcal{H})$ .  $\varphi$  is a natural extension of  $w$ , since  $\mathbb{C}$  can be seen as a 1-dimensional Hilbert space, and  $w$  is a positive map  $w : \mathfrak{A} \rightarrow \mathcal{B}(\mathbb{C})$ . He further realized that  $\varphi$  being a positive map is not enough to produce a Hilbert space and a representation. It turns out that the condition to put on  $\varphi$  is *complete positivity*:

**Definition 5.1.** Let  $\mathfrak{A}$  be a  $*$ -algebra. A map  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is *completely positive*, if for all  $n \in \mathbb{N}$ ,

$$\varphi \otimes I_{M_n} : \mathfrak{A} \otimes M_n \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \quad (5.1)$$

maps positive elements in  $\mathfrak{A} \otimes M_n$  to positive operators in  $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ .  $\varphi$  is called a completely positive map, or a CP map. (CP maps are developed primarily for nonabelian algebras.)

The algebra  $M_n$  of  $n \times n$  matrices can be seen as an  $n^2$ -dimensional Hilbert space with an ONB given by the matrix units  $\{e_{ij}\}_{i,j=1}^n$ . It is also a  $*$ -algebra generated by  $\{e_{ij}\}_{i,j=1}^n$  such that

$$e_{ij}e_{kl} = \begin{cases} e_{il} & j = k \\ 0 & j \neq k \end{cases}$$

Members of  $\mathfrak{A} \otimes M_n$  are of the form

$$\sum_{i,j} A_{ij} \otimes e_{ij}.$$

In other words,  $\mathfrak{A} \otimes M_n$  consists of precisely the  $\mathfrak{A}$ -valued  $n \times n$  matrices. Similarly, members of  $\mathcal{H} \otimes \mathbb{C}^n$  are the  $n$ -tuple column vectors with  $\mathcal{H}$ -valued entries.

Let  $I_{M_n} : M_n \rightarrow \mathcal{B}(\mathbb{C}^n)$  be the identity representation of  $M_n$  onto  $\mathcal{B}(\mathbb{C}^n)$ . Then,

$$\varphi \otimes I_{M_n} : \mathfrak{A} \otimes M_n \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n) (= \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)) \quad (5.2)$$

$$\varphi \otimes I_{M_n} \left( \sum_{i,j} A_{ij} \otimes e_{ij} \right) = \sum_{i,j} \varphi(A_{ij}) \otimes e_{ij}. \quad (5.3)$$

Note the RHS in (5.3) is an  $n \times n$  matrix with  $\mathcal{B}(\mathcal{H})$ -valued entries.

*Remark 5.2.* The algebra  $\mathcal{B}(\mathbb{C}^n)$  of all bounded operators on  $\mathbb{C}^n$  is generated by the rank-one operators, i.e.,

$$I_{M_n}(e_{ij}) = |e_i\rangle\langle e_j|. \quad (5.4)$$

Hence the  $e_{ij}$  on the LHS of (5.3) is seen as an element in the algebra  $M_n(\mathbb{C})$ , i.e.,  $n \times n$  complex matrices; while on the RHS of (5.3),  $e_{ij}$  is treated as the rank one operator  $|e_i\rangle\langle e_j| \in \mathcal{B}(\mathbb{C}^n)$ . Using Dirac's notation, when we look at  $e_{ij}$  as operators, we may write

$$e_{i,j}(e_k) = |e_i\rangle\langle e_j|e_k\rangle = \begin{cases} |e_i\rangle & j = k \\ 0 & j \neq k \end{cases}$$

$$e_{i,j}e_{kl} = |e_i\rangle\langle e_j|e_k\rangle\langle e_l| = \begin{cases} |e_i\rangle\langle e_l| & j = k \\ 0 & j \neq k \end{cases}$$

This also shows that  $I_{M_n}$  is in fact an algebra isomorphism.

The CP condition in (5.1) is illustrated in the following diagram.

$$\otimes \begin{cases} \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}) : & A \mapsto \varphi(A) \\ M_n \rightarrow M_n : & x \mapsto I_{M_n}(X) = X \text{ (identity representation of } M_n) \end{cases}$$

It is saying that if  $\sum_{i,j} A_{ij} \otimes e_{ij}$  is a positive element in the algebra  $\mathfrak{A} \otimes M_n$ , then the  $n \times n$   $\mathcal{B}(\mathcal{H})$ -valued matrix  $\sum_{i,j} \varphi(A_{ij}) \otimes e_{ij}$  is a positive operator acting on the Hilbert space  $\mathcal{H} \otimes \mathbb{C}^n$ .

Specifically, take any  $v = \sum_{k=1}^n v_k \otimes e_k$  in  $\mathcal{H} \otimes \mathbb{C}^n$ , we must have

$$\begin{aligned}
& \left\langle \sum_l v_l \otimes e_l, \left( \sum_{i,j} \varphi(A_{ij}) \otimes e_{ij} \right) \left( \sum_k v_k \otimes e_k \right) \right\rangle \\
&= \left\langle \sum_l v_l \otimes e_l, \sum_{i,j,k} \varphi(A_{ij}) v_k \otimes e_{ij}(e_k) \right\rangle \\
&= \left\langle \sum_l v_l \otimes e_l, \sum_{i,j} \varphi(A_{ij}) v_j \otimes e_i \right\rangle \\
&= \sum_{i,j,l} \langle v_l, \varphi(A_{ij}) v_j \rangle \langle e_l, e_i \rangle \\
&= \sum_{i,j} \langle v_i, \varphi(A_{ij}) v_j \rangle \geq 0.
\end{aligned} \tag{5.5}$$

Using matrix notation, the CP condition is formulated as:

For all  $n \in \mathbb{N}$ , and all  $v \in \mathcal{H} \otimes \mathbb{C}^n$ , i.e.,

$$v = \sum_{k=1}^n v_k \otimes e_k = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

we have

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \varphi(A_{11}) & \varphi(A_{12}) & \cdots & \varphi(A_{1n}) \\ \varphi(A_{21}) & \varphi(A_{22}) & \cdots & \varphi(A_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(A_{n1}) & \varphi(A_{n2}) & \cdots & \varphi(A_{nn}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \geq 0. \tag{5.6}$$

## 5.2 CP v.s. GNS

The GNS construction can be reformulated as a special case of the Stinespring's theorem [Sti55].

Let  $\mathfrak{A}$  be a  $*$ -algebra, given a state  $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$ , there exists a triple  $(\mathcal{H}, \Omega, \pi)$ , all depending on  $\varphi$ , such that

$$\varphi(A) = \langle \Omega, \pi(A) \Omega \rangle_{\mathcal{H}}$$

where

$$\begin{aligned}
\Omega &= \pi(1_{\mathfrak{A}}) \in \mathcal{H} \\
\mathcal{H} &= \overline{\text{span}}\{\pi(A)\Omega : A \in \mathfrak{A}\}.
\end{aligned}$$

The 1-dimensional Hilbert space  $\mathbb{C}$  is thought of being embedded into  $\mathcal{H}$  (possibly infinite dimensional) via

$$\mathbb{C} \ni t \xrightarrow{V} t\Omega \in \mathbb{C}\Omega \tag{5.7}$$



where  $\mathbb{C}\Omega$  = the one-dimensional subspace in  $\mathcal{H}$  generated by the unit cyclic vector  $\Omega$ .

**Lemma 5.3.** *The map  $V$  in (5.7) is an isometry, such that  $V^*V = I_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ , and*

$$VV^* : \mathcal{H} \rightarrow \mathbb{C}\Omega \quad (5.8)$$

is the projection from  $\mathcal{H}$  onto the 1-d subspace  $\mathbb{C}\Omega$  in  $\mathcal{H}$ .

Moreover,

$$\varphi(A) = V^*\pi(A)V, \quad \forall A \in \mathfrak{A}. \quad (5.9)$$

*Proof.* Let  $t \in \mathbb{C}$ , then  $\|Vt\|_{\mathcal{H}} = \|t\Omega\|_{\mathcal{H}} = |t|$ , and so  $V$  is an isometry.

For all  $\xi \in \mathcal{H}$ , we have

$$\langle \xi, Vt \rangle_{\mathcal{H}} = \langle V^*\xi, t \rangle_{\mathbb{C}} = t\overline{V^*\xi}.$$

By setting  $t = 1$ , we get

$$V^*\xi = \overline{\langle \xi, V1 \rangle_{\mathcal{H}}} = \overline{\langle \xi, \Omega \rangle_{\mathcal{H}}} = \langle \Omega, \xi \rangle_{\mathcal{H}} \iff V^* = \langle \Omega, \cdot \rangle_{\mathcal{H}}.$$

Therefore,

$$V^*Vt = V^*(t\Omega) = \langle \Omega, t\Omega \rangle_{\mathcal{H}} = t, \quad \forall t \in \mathbb{C} \iff V^*V = I_{\mathbb{C}}$$

$$VV^*\xi = V(\langle \Omega, \xi \rangle_{\mathcal{H}}) = \langle \Omega, \xi \rangle_{\mathcal{H}} \Omega, \quad \forall \xi \in \mathcal{H} \iff VV^* = |\Omega\rangle\langle \Omega|.$$

It follows that

$$\begin{aligned} \varphi(A) &= \langle \Omega, \pi(A)\Omega \rangle_{\mathcal{H}} \\ &= \langle V1, \pi(A)V1 \rangle_{\mathcal{H}} \\ &= \langle 1, V^*\pi(A)V1 \rangle_{\mathbb{C}} \\ &= V^*\pi(A)V, \quad \forall A \in \mathfrak{A} \end{aligned}$$

which is the assertion in (5.9).  $\square$

In other words,  $\Omega \mapsto \pi(A)\Omega$  sends the unit vector  $\Omega$  from the 1-dimensional subspace  $\mathbb{C}\Omega$  to the vector  $\pi(A)\Omega \in \mathcal{H}$ , and  $\langle \Omega, \pi(A)\Omega \rangle_{\mathcal{H}}$  cuts off the resulting vector  $\pi(A)\Omega$  and only preserves the component corresponding to the 1-d subspace  $\mathbb{C}\Omega$ . Notice that the unit vector  $\Omega$  is obtained from embedding the constant  $1 \in \mathbb{C}$  via the map  $V$ , i.e.,  $\Omega = V1$ . In matrix notation, if we identify  $\mathbb{C}$  with its image  $\mathbb{C}\Omega$  in  $\mathcal{H}$ , then  $\varphi(A)$  is put into a matrix corner:

$$\pi(A) = \begin{bmatrix} \varphi(A) & * \\ * & * \end{bmatrix}$$

so that when acting on vectors,

$$\varphi(A) = \begin{bmatrix} \Omega & 0 \end{bmatrix} \begin{bmatrix} \varphi(A) & * \\ * & * \end{bmatrix} \begin{bmatrix} \Omega \\ 0 \end{bmatrix}.$$

Equivalently

$$\varphi(A) = P_1\pi(A) : P_1\mathcal{H} \rightarrow \mathbb{C};$$

where  $P_1 := VV^* = |\Omega\rangle\langle\Omega| = \text{rank-1 projection on } \mathbb{C}\Omega$ .

Stinespring's construction is a generalization of the above formulation: Let  $\mathfrak{A}$  be a  $*$ -algebra, given a CP map  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ , there exists a Hilbert space  $\mathcal{H}$  ( $= \mathcal{H}_\varphi$ ), an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ , and a representation  $\pi (= \pi_\varphi) : \mathfrak{A} \rightarrow \mathcal{K}$ , such that

$$\varphi(A) = V^*\pi(A)V, \forall A \in \mathfrak{A}.$$

Notice that this construction starts with a possibly infinite dimensional Hilbert space  $\mathcal{H}$  (instead of the 1-dimensional Hilbert space  $\mathbb{C}$ ), the map  $V$  embeds  $\mathcal{H}$  into a bigger Hilbert space  $\mathcal{K}$ . If  $\mathcal{H}$  is identified with its image in  $\mathcal{K}$ , then  $\pi(A)$  is put into a matrix corner,

$$\begin{bmatrix} \pi(A) & * \\ * & * \end{bmatrix}$$

so that when acting on vectors,

$$\varphi(A)\xi = \begin{bmatrix} V\xi & 0 \end{bmatrix} \begin{bmatrix} \pi(A) & * \\ * & * \end{bmatrix} \begin{bmatrix} V\xi \\ 0 \end{bmatrix}.$$

This can be formulated alternatively:

For every CP map  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ , there is a dilated Hilbert space  $\mathcal{K} (= \mathcal{H}_\varphi) \supset \mathcal{H}$ , a representation  $\pi (= \pi_\varphi) : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{K})$ , such that

$$\varphi(A) = P_{\mathcal{H}}\pi(A)$$

i.e.,  $\pi(A)$  can be put into a matrix corner.  $\mathcal{K}$  is chosen as minimal in the sense that

$$\mathcal{K} = \overline{\text{span}}\{\pi(A)(Vh) : A \in \mathfrak{A}, h \in \mathcal{H}\}.$$

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{V} & \mathcal{K} \\ \varphi(A) \downarrow & & \downarrow \pi(A) \\ \mathcal{H} & \xrightarrow{V} & \mathcal{K} \end{array}$$

*Note 5.4.* The containment  $\mathcal{H} \subset \mathcal{K}$  comes after the identification of  $\mathcal{H}$  with its image in  $\mathcal{K}$  under the isometric embedding  $V$ . We write  $\varphi(A) = P_{\mathcal{H}}\pi(A)$ , as opposed to  $\varphi(A) = P_{\mathcal{H}}\pi(A)P_{\mathcal{H}}$ , since  $\varphi(A)$  only acts on the subspace  $\mathcal{H}$ .

### 5.3 Stinespring's Theorem

**Theorem 5.5** (Stinespring [Sti55]). *Let  $\mathfrak{A}$  be a  $*$ -algebra. The following are equivalent:*

1.  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a completely positive map, and  $\varphi(1_{\mathfrak{A}}) = I_{\mathcal{H}}$ .

2. There exists a Hilbert space  $\mathcal{K}$ , an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ , and a representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{K})$  such that

$$\varphi(A) = V^* \pi(A) V, \quad \forall A \in \mathfrak{A}. \quad (5.10)$$

3. If the dilated Hilbert space  $\mathcal{K}$  is taken to be minimum, then it is unique up to unitary equivalence. Specifically, if there are two systems  $(V_i, \mathcal{K}_i, \pi_i)$ ,  $i = 1, 2$ , satisfying

$$\varphi(A) = V_i^* \pi_i(A) V_i \quad (5.11)$$

$$\mathcal{K}_i = \overline{\text{span}}\{\pi_i(A) V h : A \in \mathfrak{A}, h \in \mathcal{H}\} \quad (5.12)$$

then there exists a unitary operator  $W : \mathcal{K}_1 \rightarrow \mathcal{K}_2$  so that

$$W \pi_1 = \pi_2 W \quad (5.13)$$

*Proof.*

(Part (3), uniqueness) Let  $(V_i, \mathcal{K}_i, \pi_i)$ ,  $i = 1, 2$ , be as in the statement satisfying (5.11)-(5.12). Define

$$W \pi_1(A) V h = \pi_2(A) V h$$

then  $W$  is an isometry, since

$$\begin{aligned} \|\pi_i(A) V h\|_{\mathcal{K}}^2 &= \langle \pi_i(A) V h, \pi_i(A) V h \rangle_{\mathcal{K}} \\ &= \langle h, V^* \pi_i(A^* A) V h \rangle_{\mathcal{H}} \\ &= \langle h, \varphi(A^* A) h \rangle_{\mathcal{H}}. \end{aligned}$$

Hence  $W$  extends uniquely to a unitary operator  $W : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ . To see that  $W$  intertwines  $\pi_1, \pi_2$ , notice that a typical vector in  $\mathcal{K}_i$  is  $\pi_i(A) V h$ , and

$$\begin{aligned} W \pi_1(B) \pi_1(A) V h &= W \pi_1(BA) V h \\ &= \pi_2(BA) V h \\ &= \pi_2(B) \pi_2(A) V h \\ &= \pi_2(B) W \pi_1(A) V h. \end{aligned}$$

Since such vectors are dense in the respective dilated space, we conclude that  $W \pi_1 = \pi_2 W$ , so (5.13) holds.  $\square$

*Note 5.6.*  $\|\pi_1(B) \pi_1(A) V h\|^2 = \langle h, V^* \pi_1(A^* B^* B A) V h \rangle$ . Fix  $A \in \mathcal{B}(\mathcal{H})$ , the map  $B \mapsto A^* B A$  is an automorphism on  $\mathcal{B}(\mathcal{H})$ .

*Proof.* (2)  $\implies$  (1)

Now suppose  $\varphi(A) = V^* \pi(A) V$ , and we verify it is completely positive.

Since positive elements in  $\mathfrak{A} \otimes M_n$  are sums of the operator matrix

$$\sum_{i,j} A_i^* A_j \otimes e_{ij} = \begin{bmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \end{bmatrix} \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

it suffices to show that

$$\varphi \otimes I_{M_n} \left( \sum_{i,j} A_i^* A_j \otimes e_{ij} \right) = \sum_{i,j} \varphi(A_i^* A_j) \otimes e_{ij}$$

is a positive operator in  $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ , i.e., need to show that for all  $v \in \mathcal{H} \otimes \mathbb{C}^n$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \varphi(A_1^* A_1) & \varphi(A_1^* A_2) & \cdots & \varphi(A_1^* A_n) \\ \varphi(A_2^* A_1) & \varphi(A_2^* A_2) & \cdots & \varphi(A_2^* A_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(A_n^* A_1) & \varphi(A_n^* A_2) & \cdots & \varphi(A_n^* A_n) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \geq 0. \quad (5.14)$$

This is true, since

$$\begin{aligned} \text{RHS}_{(5.14)} &= \sum_{i,j} \langle v_i, \varphi(A_i^* A_j) v_j \rangle_{\mathcal{H}} \\ &= \sum_{i,j} \langle v_i, V^* \pi(A_i^* A_j) V v_j \rangle_{\mathcal{H}} \\ &= \sum_{i,j} \langle \pi(A_i) V v_i, \pi(A_j) V v_j \rangle_{\mathcal{H}} \\ &= \left\| \sum_i \pi(A_i) V v_i \right\|_{\mathcal{H}}^2 \geq 0. \end{aligned}$$

(1)  $\implies$  (2)

Given a completely positive map  $\varphi$ , we construct  $\mathcal{K} (= \mathcal{K}_\varphi)$ ,  $V (= V_\varphi)$  and  $\pi (= \pi_\varphi)$ . Recall that  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a CP map means that for all  $n \in \mathbb{N}$ ,

$$\varphi \otimes I_{M_n} : \mathfrak{A} \otimes M_n \rightarrow \mathcal{B}(\mathcal{H} \otimes M_n)$$

is positive, and

$$\varphi \otimes I_{M_n} (1_{\mathfrak{A}} \otimes I_{M_n}) = I_{\mathcal{H}} \otimes I_{M_n}.$$

The condition on the identity element can be stated using matrix notation as

$$\begin{bmatrix} \varphi & 0 & \cdots & 0 \\ 0 & \varphi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi \end{bmatrix} \begin{bmatrix} 1_{\mathfrak{A}} & 0 & \cdots & 0 \\ 0 & 1_{\mathfrak{A}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{\mathfrak{A}} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & 0 & \cdots & 0 \\ 0 & I_{\mathcal{H}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{\mathcal{H}} \end{bmatrix}.$$

Let  $K_0$  be the algebraic tensor product  $\mathfrak{A} \otimes \mathcal{H}$ , i.e.,

$$K_0 = \text{span} \left\{ \sum_{\text{finite}} A_i \otimes \xi_i : A_i \in \mathfrak{A}, \xi_i \in \mathcal{H} \right\}.$$

Define a sesquilinear form  $\langle \cdot, \cdot \rangle_\varphi : K_0 \times K_0 \rightarrow \mathbb{C}$ , by

$$\left\langle \sum_{i=1}^n A_i \otimes \xi_i, \sum_{j=1}^n B_j \otimes \eta_j \right\rangle_\varphi := \sum_{i,j} \langle \xi_i, \varphi(A_i^* B_j) \eta_j \rangle_{\mathcal{H}}. \quad (5.15)$$

By the CP condition (5.1), we have

$$\left\langle \sum_{i=1}^n A_i \xi_i, \sum_{j=1}^n A_j \xi_j \right\rangle_\varphi = \sum_{i,j} \langle \xi_i, \varphi(A_i^* A_j) \xi_j \rangle_{\mathcal{H}} \geq 0.$$

Let  $N := \{v \in K_0 : \langle v, v \rangle_\varphi = 0\}$ . Since the Schwarz inequality holds for any sesquilinear form, it follows that

$$N = \{v \in K_0 : \langle s, v \rangle_\varphi = 0, \forall s \in K_0\}.$$

Thus  $N$  is a closed subspace in  $K_0$ . Let  $\mathcal{K} (= \mathcal{K}_\varphi)$  be the Hilbert space by completing  $K_0/N$  with respect to

$$\|\cdot\|_{\mathcal{K}} := \langle \cdot, \cdot \rangle_\varphi^{1/2}.$$

Let  $V : \mathcal{H} \rightarrow K_0$ , by

$$V\xi := 1_{\mathfrak{A}} \otimes \xi, \forall \xi \in \mathcal{H}.$$

Then,

$$\begin{aligned} \|V\xi\|_\varphi^2 &= \langle 1_{\mathfrak{A}} \otimes \xi, 1_{\mathfrak{A}} \otimes \xi \rangle_\varphi \\ &= \langle \xi, \varphi(1_{\mathfrak{A}}^* 1_{\mathfrak{A}}) \xi \rangle_{\mathcal{H}} \\ &= \langle \xi, \xi \rangle_{\mathcal{H}} = \|\xi\|_{\mathcal{H}}^2 \end{aligned}$$

i.e.,  $V$  is isometric, and so  $\mathcal{H} \xrightarrow{V} K_0$  is an isometric embedding.

**Claim.**

- (i)  $V^*V = I_{\mathcal{H}}$ ;
- (ii)  $VV^*$  = projection from  $K_0$  on the subspace  $1_{\mathfrak{A}} \otimes \mathcal{H}$ .

Indeed, for any  $A \otimes \eta \in K_0$ , we have

$$\begin{aligned} \langle A \otimes \eta, V\xi \rangle_\varphi &= \langle A \otimes \eta, 1_{\mathfrak{A}} \otimes \xi \rangle_\varphi \\ &= \langle \eta, \varphi(A^*) \xi \rangle_{\mathcal{H}} \\ &= \langle \varphi(A^*)^* \eta, \xi \rangle_{\mathcal{H}} \end{aligned}$$

which implies that

$$V^*(A \otimes \eta) = \varphi(A^*)^*\eta.$$

It follows that

$$V^*V\xi = V^*(1_{\mathfrak{A}} \otimes \xi) = \varphi(1_{\mathfrak{A}}^*)^*\xi = \xi, \quad \forall \xi \in \mathcal{H}$$

i.e.,  $V^*V = I_{\mathcal{H}}$ . Moreover, for any  $A \otimes \eta \in K_0$ ,

$$VV^*(A \otimes \eta) = V(\varphi(A^*)^*\eta) = 1_{\mathfrak{A}} \otimes \varphi(A^*)^*\eta.$$

This proves the claim. It is clear that the properties of  $V$  pass to the dilated space  $\mathcal{K} (= \mathcal{K}_\varphi) = cl_\varphi(K_0/N)$ .

To finish the proof of the theorem, define  $\pi (= \pi_\varphi)$  as follows: Set

$$\pi(A) \left( \sum_j B_j \otimes \eta_j \right) := \sum_j AB_j \otimes \eta_j, \quad \forall A \in \mathfrak{A}$$

and extend it to  $\mathcal{K}$ .

For all  $\xi, \eta \in \mathcal{H}$ , then,

$$\begin{aligned} \langle \xi, V^*\pi(A)V\eta \rangle_{\mathcal{H}} &= \langle V\xi, \pi(A)V\eta \rangle_{\mathcal{K}} \\ &= \langle 1_{\mathfrak{A}} \otimes \xi, \pi(A)1_{\mathfrak{A}} \otimes \eta \rangle_{\mathcal{K}} \\ &= \langle 1_{\mathfrak{A}} \otimes \xi, A \otimes \eta \rangle_{\mathcal{K}} \\ &= \langle \xi, \varphi(1_{\mathfrak{A}}^*A)\eta \rangle_{\mathcal{H}} \\ &= \langle \xi, \varphi(A)\eta \rangle_{\mathcal{H}}. \end{aligned}$$

We conclude that  $\varphi(A) = V^*\pi(A)V$ , for all  $A \in \mathfrak{A}$ .  $\square$

### Application of Stinespring's Theorem to Representations of $\mathcal{O}_N$

**Corollary 5.7.** *Let  $N \in \mathbb{N}$ ,  $N > 1$ , and let  $A_i \in \mathcal{B}(\mathcal{H})$ ,  $1 \leq i \leq N$ , be a system of operators in a Hilbert space  $\mathcal{H}$  such that*

$$\sum_{i=1}^N A_i^* A_i = I_{\mathcal{H}}; \quad (5.16)$$

*then there is a second Hilbert space  $\mathcal{K}$ , and an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$ , and a representation  $\pi \in \text{Rep}(\mathcal{O}_N, \mathcal{K})$  such that*

$$V^*\pi(s_i)V = A_i^*, \quad 1 \leq i \leq N, \quad (5.17)$$

*where  $\{s_i\}_{i=1}^N$  are generators for  $\mathcal{O}_N$ .*

*Proof.* Given  $\mathcal{O}_N$  with generators  $\{s_i\}_{i=1}^N$ , then set

$$\varphi(s_i s_j^*) = A_i^* A_j$$

using (5.16), it is easy to see that  $\varphi$  is completely positive.

Now let  $(\pi, \mathcal{H})$  be the pair obtained from Theorem 5.5 (Stinespring); then as a block-matrix of operators, we have as follows

$$\pi(s_i)^* = \begin{bmatrix} A_i & * \\ \mathbf{0} & * \end{bmatrix} \quad (5.18)$$

relative to the splitting

$$\mathcal{H} = V\mathcal{H} \oplus (\mathcal{H} \ominus V\mathcal{H}), \quad (5.19)$$

and so  $V^* \pi(s_i)^* V = A_i$ , which is equivalent to (5.17).  $\square$

## 5.4 Comments

In Stinespring's theorem, the dilated space comes from a general principle (using positive definite functions) when building Hilbert spaces out of the given data. We illustrate this point with a few familiar examples.

**Example 5.8.** In linear algebra, there is a bijection between inner product structures on  $\mathbb{C}^n$  and positive-definite  $n \times n$  matrices. Specifically,  $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$  is an inner product if and only if there exists a positive definite matrix  $A$  such that

$$\langle v, w \rangle_A = v^* A w$$

for all  $v, w \in \mathbb{C}^n$ . We think of  $\mathbb{C}^n$  as  $\mathbb{C}$ -valued functions on  $\{1, 2, \dots, n\}$ , then  $\langle \cdot, \cdot \rangle_A$  is an inner product built on the function space.

This is then extended to infinite dimensional space.

**Example 5.9.** If  $F$  is a positive definite function on  $\mathbb{R}$ , then on  $K_0 = \text{span}\{\delta_x : x \in \mathbb{R}\}$ ,  $F$  defines a sesquilinear form  $\langle \cdot, \cdot \rangle_F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ , where

$$\left\langle \sum_i c_i \delta_{x_i}, \sum_j d_j \delta_{x_j} \right\rangle_F := \sum_{i,j} \bar{c}_i d_j F(x_i, x_j), \text{ and}$$

$$\left\| \sum_i c_i \delta_{x_i} \right\|_F^2 := \left\langle \sum_i c_i \delta_{x_i}, \sum_j c_j \delta_{x_j} \right\rangle_F = \sum_{i,j} \bar{c}_i c_j F(x_i, x_j) \geq 0.$$

Let  $N = \{v \in K_0 : \langle v, v \rangle = 0\}$ , then  $N$  is a closed subspace in  $K_0$ . We get a Hilbert space:  $\mathcal{H} := \text{cl}_F(K_0/N)$  = the completion of  $K_0/N$  with respect to  $\|\cdot\|_F$ .

What if the index set is not  $\{1, 2, \dots, n\}$  or  $\mathbb{R}$ , but a  $*$ -algebra?

**Example 5.10.**  $C(X)$ ,  $X$  compact Hausdorff. It is a  $C^*$ -algebra, where  $\|f\| := \sup_x |f(x)|$ . By Riesz's theorem, there is a bijection between positive states (linear functionals) on  $C(X)$  and Borel probability measures on  $X$ .

Let  $\mathfrak{B}(X)$  be the Borel sigma-algebra on  $X$ , which is also an abelian algebra: The associative multiplication is defined as  $AB := A \cap B$ . The identity element is just  $X$ .

Let  $\mu$  be a probability measure, then  $\mu(A \cap B) \geq 0$ , for all  $A, B \in \mathfrak{B}(X)$ , and  $\mu(X) = 1$ . Hence  $\mu$  is a state. As before, we apply the GNS construction. Set

$$K_0 = \text{span}\{\delta_A : A \in \mathfrak{M}\} = \text{span}\{\chi_A : A \in \mathfrak{M}\}$$

Note the index set here is  $\mathfrak{B}(X)$ , and  $\sum_i c_i \delta_{A_i} = \sum_i c_i \chi_{A_i}$ , i.e., these are precisely the simple functions. Define

$$\left\langle \sum_i c_i \chi_{A_i}, \sum_j d_j \chi_{B_j} \right\rangle := \sum_{i,j} \bar{c}_i d_j \mu(A_i \cap B_j)$$

which is positive definite, since

$$\left\langle \sum_i c_i \chi_{A_i}, \sum_i c_i \chi_{A_i} \right\rangle = \sum_{i,j} \bar{c}_i c_j \mu(A_i \cap A_j) = \sum_i |c_i|^2 \mu(A_i) \geq 0.$$

Here,  $N = \{v \in K_0 : \langle v, v \rangle = 0\} = \mu$ -measure zero sets, and

$$\mathcal{H} = \text{cl}_\mu(K_0/N) = L^2(\mu).$$

**Example 5.11 (GNS).** Let  $\mathfrak{A}$  be a  $*$ -algebra. The set of  $\mathbb{C}$ -valued functions on  $\mathfrak{A}$  is  $\mathfrak{A} \otimes \mathbb{C}$ , i.e., functions of the form

$$\left\{ \sum_i A_i \otimes c_i = \sum_i c_i \delta_{A_i} \right\} \quad (5.20)$$

with finite summation over  $i$ . Note that  $\mathbb{C}$  is naturally embedded into  $\mathfrak{A} \otimes \mathbb{C}$  as  $1_{\mathfrak{A}} \otimes \mathbb{C}$  (i.e.,  $c \mapsto c \delta_{1_{\mathfrak{A}}}$ ), and the latter is a 1-dimensional subspace. In order to build a Hilbert space out of (5.20), one needs a positive definite function. A state  $\varphi$  on  $\mathfrak{A}$  does exactly the job. The sesquilinear form is given by

$$\left\langle \sum_i c_i \delta_{A_i}, \sum_j d_j \delta_{B_j} \right\rangle_\varphi := \sum_{i,j} \bar{c}_i d_j \varphi(A_i^* B_j)$$

so that

$$\left\| \sum_i c_i \delta_{A_i} \right\|_\varphi^2 = \sum_{i,j} \bar{c}_i c_j \varphi(A_i^* A_j) \geq 0.$$

Finally, let  $\mathcal{H}_\varphi = \text{Hilbert completion of } \mathfrak{A} \otimes \mathbb{C} / \ker \varphi$ . Define  $\pi(A) \delta_B := \delta_{BA}$ , so a "shift" in the index variable, and extend to  $\mathcal{H}_\varphi$ .



In Stinespring's construction,  $\mathfrak{A} \otimes \mathbb{C}$  is replaced by  $\mathfrak{A} \otimes \mathcal{H}$ , i.e., in stead of working with  $\mathbb{C}$ -valued functions on  $\mathfrak{A}$ , one looks at  $\mathcal{H}$ -valued functions. Hence we are looking at functions of the form

$$\left\{ \sum_i A_i \otimes \xi_i = \sum_i \xi_i \delta_{A_i} \right\}$$

with finite summation over  $i$ .  $\mathcal{H}$  is embedded into  $\mathfrak{A} \otimes \mathcal{H}$  as  $1_{\mathfrak{A}} \otimes \mathcal{H}$ , by  $\mathcal{H} \ni 1_{\mathfrak{A}} \otimes \xi = \xi \delta_{1_{\mathfrak{A}}}$ .  $1_{\mathfrak{A}} \otimes \mathcal{H}$  is in general infinite dimensional, or we say that the function  $\xi \delta_{1_{\mathfrak{A}}}$  at  $1_{\mathfrak{A}}$  has infinite multiplicity. If  $\mathcal{H}$  is separable, we are actually attaching an  $l^2$  sequence at every point  $A \in \mathfrak{A}$ .

How to build a Hilbert space out of these  $\mathcal{H}$ -valued functions? The question depends on the choice of a quadratic form. If  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is positive, i.e.,  $\varphi$  maps positive elements in  $\mathfrak{A}$  to positive operators on  $\mathcal{H}$ , then quadratic form

$$\langle A \otimes \xi, B \otimes \eta \rangle_{\varphi} := \langle \xi, \varphi(A^* B) \eta \rangle_{\mathcal{H}}$$

is indeed positive definite. But when extend linearly, one is in trouble. For

$$\begin{aligned} & \left\langle \sum_i A_i \otimes \xi_i, \sum_j B_j \otimes \eta_j \right\rangle_{\varphi} \\ &= \sum_{i,j} \langle \xi_i, \varphi(A_i^* B_j) \eta_j \rangle_{\mathcal{H}} \\ &= \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix} \begin{bmatrix} \varphi(A_1^* B_1) & \varphi(A_1^* B_2) & \cdots & \varphi(A_1^* B_n) \\ \varphi(A_2^* B_1) & \varphi(A_2^* B_2) & \cdots & \varphi(A_2^* B_n) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi(A_n^* B_1) & \varphi(A_n^* B_2) & \cdots & \varphi(A_n^* B_n) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \end{aligned}$$

and it is not clear why the matrix  $(\varphi(A_i^* B_j))_{i,j=1}^n$  should be a positive operator acting in  $H \otimes \mathbb{C}^n$ . But we could very well put this extra requirement into an axiom, so the CP condition (5.1).

1. We only assume  $\mathfrak{A}$  is a  $*$ -algebra, not necessarily a  $C^*$ -algebra.  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is positive does not necessarily imply  $\varphi$  is completely positive. A counterexample for  $\mathfrak{A} = M_2(\mathbb{C})$ , and  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathbb{C}^2) \simeq M_2(\mathbb{C})$  given by taking transpose, i.e.,  $A \mapsto \varphi(A) = A^{tr}$ . Then  $\varphi$  is positive, but  $\varphi \otimes I_{M_2}$  is not.
2. The operator matrix  $(A_i^* A_j)$ , which is also written as  $\sum_{i,j} A_i^* A_j \otimes e_{ij}$  is a positive element in  $\mathfrak{A} \otimes M_n$ . All positive elements in  $\mathfrak{A} \otimes M_n$  are in such form. This notation goes back again to Dirac, for the rank-1 projections  $|v\rangle\langle v|$  are positive, and all positive operators are sums of these rank-1 operators.
3. Given a CP map  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ , we get a Hilbert space  $\mathcal{K}_{\varphi}$ , a representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{K}_{\varphi})$  and an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}_{\varphi}$ , such that

$$\varphi(A) = V^* \pi(A) V$$

for all  $A \in \mathfrak{A}$ .  $P = VV^*$  is a selfadjoint projection from  $\mathcal{K}_\varphi$  to the image of  $\mathcal{H}$  under the embedding. To see  $P$  is a projection, note that

$$P^2 = VV^*VV^* = V(V^*V)V^* = VV^*.$$

### Summary

Positive maps have been a recursive theme in functional analysis. A classical example is  $\mathfrak{A} = C_c(X)$  with a positive linear functional  $\Lambda : \mathfrak{A} \rightarrow \mathbb{C}$ , mapping  $\mathfrak{A}$  into a 1-d Hilbert space  $\mathbb{C}$ .

In Stinespring's formulation,  $\varphi : \mathfrak{A} \rightarrow \mathcal{H}$  is a CP map, then we may write  $\varphi(A) = V^*\pi(A)V$  where  $\pi : \mathfrak{A} \rightarrow \mathcal{K}$  is a representation on a bigger Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$ . The containment is in the sense that  $V : \mathcal{H} \hookrightarrow \mathcal{K}$  embeds  $\mathcal{H}$  into  $\mathcal{K}$ . Notice that

$$V\varphi(A) = \pi(A)V \implies \varphi(A) = V^*\pi(A)V$$

but not the other way around. In Nelson's notes [Nel69], we use the notation  $\varphi \subset \pi$  for one representation being the subrepresentation of another. To imitate the situation in linear algebra, we may want to split an operator  $T$  acting on  $\mathcal{H}$  into operators action on  $\mathcal{H}$  and its complement in  $\mathcal{K}$ . Let  $P : \mathcal{H} \rightarrow \mathcal{H}$  be the orthogonal projection. In matrix language,

$$\begin{bmatrix} PTP & PTP^\perp \\ P^\perp TP & P^\perp TP^\perp \end{bmatrix}.$$

A better looking would be

$$\begin{bmatrix} PTP & 0 \\ 0 & P^\perp TP^\perp \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix}$$

hence

$$\pi = \varphi_1 \oplus \varphi_2.$$

Stinespring's theorem is more general, where the off-diagonal entries may not be zero.

**Exercise 5.12** (Tensor with  $M_n$ ). Let  $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ ,  $\xi_1, \dots, \xi_n \in \mathcal{H}$ . The map

$$(\xi_1, \dots, \xi_n) \mapsto (A\xi_1, \dots, A\xi_n) \in \oplus^n \mathcal{H}$$

is a representation of  $\mathfrak{A}$  if and only if

$$\overbrace{id_{\mathcal{H}} \oplus \dots \oplus id_{\mathcal{H}}}^{n \text{ times}} \in \text{Rep}(\mathfrak{A}, \overbrace{\mathcal{H} \oplus \dots \oplus \mathcal{H}}^{n \text{ times}})$$

where in matrix notation, we have

$$\begin{aligned}
 & \begin{bmatrix} id_{\mathfrak{A}}(A) & 0 & \cdots & 0 \\ 0 & id_{\mathfrak{A}}(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & id_{\mathfrak{A}}(A) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \\
 = & \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} A\xi_1 \\ A\xi_2 \\ \vdots \\ A\xi_n \end{bmatrix}.
 \end{aligned}$$

In this case, the identity representation  $id_{\mathfrak{A}} : \mathfrak{A} \rightarrow \mathcal{H}$  has multiplicity  $n$ .

**Exercise 5.13** (Column operators). Let  $V_i : \mathcal{H} \rightarrow \mathcal{H}$ , and

$$V := \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} : \mathcal{H} \rightarrow \oplus_1^n \mathcal{H}. \tag{5.21}$$

Let  $V^* : \oplus_1^n \mathcal{H} \rightarrow \mathcal{H}$  be the adjoint of  $V$ . Prove that  $V^* = [ V_1^* \quad V_2^* \quad \cdots \quad V_n^* ]$ .

*Proof.* Let  $\xi \in \mathcal{H}$ , then  $V\xi = \begin{bmatrix} V_1\xi \\ V_2\xi \\ \vdots \\ V_n\xi \end{bmatrix}$ , and

$$\begin{aligned}
 \left\langle \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}, \begin{bmatrix} V_1\xi \\ V_2\xi \\ \vdots \\ V_n\xi \end{bmatrix} \right\rangle &= \sum_i \langle \eta_i, V_i\xi \rangle \\
 &= \sum_i \langle V_i^* \eta_i, \xi \rangle = \left\langle \begin{bmatrix} V_1^* & V_2^* & \cdots & V_n^* \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}, \xi \right\rangle.
 \end{aligned}$$

This shows that  $V^* = [ V_1^* \quad V_2^* \quad \cdots \quad V_n^* ]$ .  $\square$

**Exercise 5.14** (Row-isometry). Let  $V$  be as in (5.21). The following are equivalent:

1.  $V$  is an isometry, i.e.,  $\|V\xi\|^2 = \|\xi\|^2$ , for all  $\xi \in \mathcal{H}$ ;
2.  $\sum V_i^* V_i = I_{\mathcal{H}}$ ;

3.  $V^*V = I_{\mathcal{H}}$ .

*Proof.* Notice that

$$\|V\xi\|^2 = \sum_i \|V_i\xi\|^2 = \sum_i \langle \xi, V_i^*V_i\xi \rangle = \left\langle \xi, \sum_i V_i^*V_i\xi \right\rangle.$$

Hence  $\|V\xi\|^2 = \|\xi\|^2$  if and only if

$$\left\langle \xi, \sum_i V_i^*V_i\xi \right\rangle = \langle \xi, \xi \rangle$$

for all  $\xi \in \mathcal{H}$ . Equivalently,  $\sum_i V_i^*V_i = I_{\mathcal{H}} = V^*V$ .  $\square$

**Corollary 5.15** (Krauss). *Let  $\dim \mathcal{H} = n$ . Then all the CP maps are of the form*

$$\varphi(A) = \sum_i V_i^*AV_i.$$

(The essential part here is that for any CP mapping  $\varphi$ , we get a system  $\{V_i\}$ .)

This was discovered in the physics literature by Kraus. The original proof was very intricate, but it is a corollary of Stinespring's theorem. When  $\dim \mathcal{H} = n$ , let  $\{e_1, \dots, e_n\}$  be an ONB. Fix a CP map  $\varphi$ , and get  $(V, \mathcal{K}, \pi)$ . Set

$$V_i : e_i \mapsto Ve_i \in \mathcal{K}, \quad i = 1, \dots, n;$$

then  $V_i$  is an isometry. So we get a system of isometries, and

$$\varphi(A) = \begin{bmatrix} V_1^* & V_2^* & \cdots & V_n^* \end{bmatrix} \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}.$$

Notice that  $\varphi(1) = 1$  if and only if  $\sum_i V_i^*V_i = 1$ .

**Exercise 5.16** (Tensor products). Prove the following.

1.  $\oplus_1^n \mathcal{H} \simeq \mathcal{H} \otimes \mathbb{C}^n$
2.  $\sum_1^{\oplus \infty} \mathcal{H} \simeq \mathcal{H} \otimes l^2$
3. Given  $L^2(X, \mathfrak{M}, \mu)$ , then  $L^2(X, \mathcal{H}) \simeq \mathcal{H} \otimes L^2(\mu)$ ; where  $L^2(X, \mathcal{H})$  consists of all measurable functions  $f : X \rightarrow \mathcal{H}$  such that

$$\int_X \|f(x)\|_{\mathcal{H}}^2 d\mu(x) < \infty$$

and

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle_{\mathcal{H}} d\mu(x).$$

4. All the spaces above are Hilbert spaces.

**Exercise 5.17** (Using tensor product in representations). Let  $(X_i, \mathfrak{M}_i, \mu_i)$ ,  $i = 1, 2$ , be measure spaces. Let  $\pi_i : L^\infty(\mu_i) \rightarrow L^2(\mu_i)$  be the representation such that  $\pi_i(f)$  is the operator of multiplication by  $f$  on  $L^2(\mu_i)$ . Hence  $\pi_i \in \text{Rep}(L^\infty(X_i), L^2(\mu_i))$ , and

$$\begin{aligned} \pi_1 \otimes \pi_2 &\in \text{Rep}(L^\infty(X_1 \times X_2), L^2(\mu_1 \times \mu_2)), \\ \pi_1 \otimes \pi_2(\tilde{\varphi})\tilde{f} &= \tilde{\varphi}\tilde{f} \end{aligned}$$

for all  $\tilde{\varphi} \in L^\infty(X_1 \times X_2)$ , and all  $\tilde{f} \in L^2(\mu_1 \times \mu_2)$ .

Elementary tensors: Special form

$$\begin{aligned} \tilde{\varphi}(x_1, x_2) &= \varphi_1(x_1)\varphi_2(x_2), \\ \tilde{f}(x_1, x_2) &= f_1(x_1)f_2(x_2), \\ (\pi_1 \otimes \pi_2)(\tilde{\varphi})f &= \pi_1(\varphi_1)f_1 \otimes \pi_2(\varphi_2)f_2. \end{aligned}$$

**Exercise 5.18** (Transpose is not completely positive).

1. Let  $\mathfrak{A}$  be an abelian  $C^*$ -algebra; and let  $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  be a positive mapping; then show that  $\varphi$  is in fact automatically completely positive.
2. Show that there are positive mappings which are not completely positive. Hint: Let  $M_n$  be the  $n \times n$  complex matrices, and set

$$\varphi(A) = A^T, \quad A \in M_n$$

where  $A^T$  is the transpose matrix. If  $n > 1$ , show that  $M_n \xrightarrow{\varphi} M_n$  is positive but not completely positive.

## 5.5 Endomorphisms, Representations of $\mathcal{O}_N$ , and Numerical Range

Let  $\mathcal{H}$  be a Hilbert space, and consider endomorphisms in  $\mathcal{B}(\mathcal{H})$ , i.e.,  $\sigma : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ , linear, and and satisfy

$$\begin{aligned} \sigma(AB) &= \sigma(A)\sigma(B) \\ \sigma(A^*) &= \sigma(A)^*, \quad \forall A, B \in \mathcal{B}(\mathcal{H}), \text{ and} \\ \sigma(I) &= I. \end{aligned}$$

**Definition 5.19.** By a *representation*  $\pi$  of  $\mathcal{O}_N$  in  $\mathcal{H}$ ,  $\pi \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$ , we mean a system of isometries  $(S_i)_{i=1}^N$  in  $\mathcal{H}$  such that

$$\left\{ \begin{array}{l} S_i^* S_j = \delta_{ij} I \\ \sum_i S_i S_i^* = I \end{array} \right\} \quad (\text{Cuntz relations}) \quad (5.22)$$

See Figure 5.1.

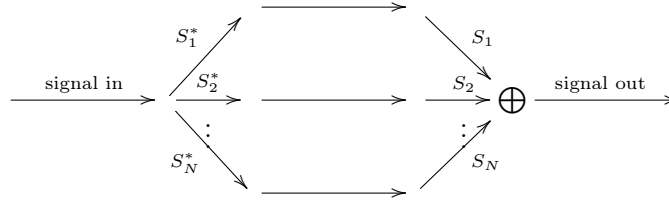


Figure 5.1: Orthogonal bands in filter bank, “in” = “out”. An application of representations of the Cuntz relations (5.22).

*Remark 5.20.* While the relations in (5.22), called the Cuntz relations, of Definition 5.19 are axioms, they have implications for a host of applications, and Figure 5.1 is a graphic representations of (5.22) stated in a form popular in applications to signal processing. Effective transmission of signals (speech, or images), is possible because the transmitted signals can be divided into frequency sub-bands; this is done with filters. A low-pass filter picks out the band corresponding to frequencies in a “band” around zero, and similarly with intermediate, and high bands. The horizontal lines in Figure 5.1 represent prescribed bands. The orthogonality part of (5.22) represents non-interference from one band to the next. Adding the projections on the LHS in (5.22) to recover the identity operator reflects perfect reconstruction, i.e, signal out equals signal in. The projections on the LHS in (5.22) are projections onto subspaces of a total Hilbert space (of signals to be transmitted), the subspaces thus representing frequency bands.

Thus Figure 5.1 represents such a filter design; there are many such, some good some not. Each one is called a “filter bank.” And each one corresponds to a representation of (5.22), or equivalently a representation of the Cuntz algebra  $\mathcal{O}_N$  where  $N$  is the number of band for the particular filter design.

**Exercise 5.21** ( $Rep(\mathcal{O}_N, \mathcal{H})$ ). Fix  $N \geq 2$ , and let  $\mathcal{O}_N$  denote the Cuntz- $C^*$ -algebra (see Example 4.4). By

$$\pi \in Rep(\mathcal{O}_N, \mathcal{H}) \tag{5.23}$$

we mean a homomorphism

$$\pi : \mathcal{O}_N \longrightarrow \mathcal{B}(\mathcal{H}),$$

(in particular, satisfying:  $\pi(AB) = \pi(A)\pi(B)$ ,  $\pi(A^*) = \pi(A)^*$ ,  $\forall A, B \in \mathcal{O}_N$ , and  $\pi(\mathbf{1}) = I_{\mathcal{H}}$ .)

Let  $\{S_i\}_{i=1}^N$  be a system of isometries in a Hilbert space  $\mathcal{H}$  satisfying (5.22), called Cuntz-isometries. For all multi-indices  $J = (j_1, j_2, \dots, j_m)$ ,  $j_i \in \{1, 2, \dots, N\}$ , set

$$\begin{aligned} s_J &:= s_{j_1} s_{j_2} \cdots s_{j_m}, \text{ and} \\ S_J &:= S_{j_1} S_{j_2} \cdots S_{j_m}. \end{aligned}$$

Show that, given a (5.22)-system  $\{S_i\}_{i=1}^N$  of isometries, there is then a unique  $\pi \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$  such that

$$\pi(s_J s_K^*) = S_J S_K^* \quad (5.24)$$

holds for all multi-indices  $J, K$ .

**Example 5.22** (Representation of the Cuntz algebra  $\mathcal{O}_2$ ). . Let  $\mathcal{H} = L^2(\mathbb{T})$ . In signal processing language  $\mathcal{H}$  is the  $L^2$ -space of frequency functions. Set

$$(S_0 f)(x) := \cos(x) f(2x) \quad (5.25)$$

$$(S_1 f)(x) := \sin(x) f(2x) \quad (5.26)$$

as the two Cuntz operators, where  $f \in \mathcal{H}$ , and

$$2x := 2x \pmod{2\pi\mathbb{Z}} (= \text{multiples of } 2\pi).$$

(The generators  $S_i$ ,  $i = 1, 2$  with up-sampling, and  $S_i^*$  with down-sampling.)

**Exercise 5.23** (Simplest Low/High filter bank). Show that (5.25)-(5.26) satisfy the  $\mathcal{O}_2$ -Cuntz relations, i.e.,

1.  $S_i$ ,  $i = 0, 1$  are isometries in  $\mathcal{H}$ ;
2.  $S_0^* S_1 = 0$  (orthogonality);
3.  $S_0 S_0^* + S_1 S_1^* = I_{\mathcal{H}}$ .

Hint: First show that

$$(S_0^* f)(x) = \frac{1}{2} \left( \cos\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + \cos\left(\frac{x+\pi}{2}\right) f\left(\frac{x+\pi}{2}\right) \right)$$

are similarly for  $S_1^* f$ . Then compute directly that

$$\|S_0^* f\|_{\mathcal{H}}^2 + \|S_1^* f\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2.$$

**Example 5.24.** Consider the Haar wavelet as in Example 1.78, with  $\phi_0$  (scaling function),  $\varphi_1$  and  $\psi_{j,k}$ ,  $j, k \in \mathbb{Z}$  be as in (1.51)-(1.52). Set

$$h(n) = \begin{cases} -\frac{1}{2} & n = -1 \\ \frac{1}{2} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \quad g(n) = \begin{cases} \frac{1}{2} & n = -1 \\ \frac{1}{2} & n = 0 \\ 0 & \text{otherwise} \end{cases};$$

so that  $h, g \in l^2$ ; where  $g$  is the low-pass filter (averaging data), and  $h$  is the high-pass filter (capturing high-frequency oscillations). Let  $m_0$  and  $m_1$  be Fourier transform of  $g$  and  $h$  respectively, i.e.,

$$m_0(x) = \sum_{n \in \mathbb{Z}} g(n) e^{-inx}$$

$$m_1(x) = \sum_{n \in \mathbb{Z}} h(n) e^{-inx}$$

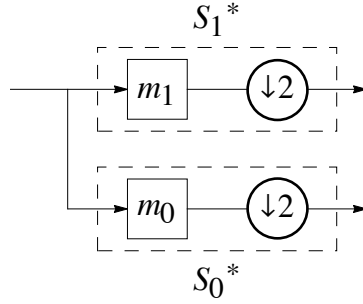


Figure 5.2: The Cuntz operators  $S_0$  and  $S_1$  in the Haar wavelet.

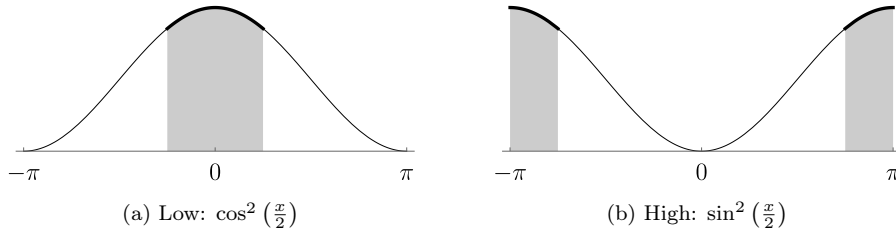


Figure 5.3: Low / high pass filters for the Haar wavelet (frequency mod  $2\pi$ ).

and  $m_0, m_1 \in L^2(\mathbb{T})$ . Finally set  $S_0^*$  and  $S_1^*$  as in Figure 5.2.

An input signal goes through the analysis filter bank (Figure 5.4) and splits into layers (frequency bands) of fine details. The original signal can be rebuilt through the synthesis filter bank (Figure 5.5), i.e., a perfect reconstruction.

Depending on the applications, the output of the analysis filter bank will go through other DSP device. For example, in data compression, insignificant coefficients are dropped; or if the task is to remove noise in the input signal, the coefficients corresponding to high frequency components (noise) are removed, and the remaining coefficients go through the synthesis filter bank. See Figures 5.7-5.8 for an illustration, and Figure 5.9 for an application in imaging processing.

We return to a much more detailed discussion of down-sampling and up-sampling in Chapter 6.

**Exercise 5.25** (Endomorphism vs representation). Let  $\mathcal{H}$  be a general separable Hilbert space. The purpose below is to point out that the study of  $Rep(\mathcal{O}_N, \mathcal{H})$  is essentially equivalent to that of the endomorphisms of  $\mathcal{B}(\mathcal{H})$ .

1. Let  $\sigma$  be an endomorphism in  $\mathcal{B}(\mathcal{H})$  of finite index. Show that there is a representation  $(S_i)$  of  $\mathcal{O}_N$  in  $\mathcal{H}$  such that

$$\sigma(A) = \sum_{i=1}^N S_i A S_i^*. \tag{5.27}$$



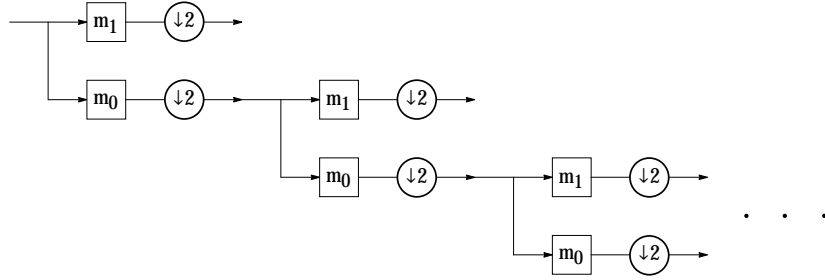


Figure 5.4: The two-channel analysis filter bank.

*Notation.* Given  $\sigma \in \text{End}(\mathcal{B}(\mathcal{H}))$ , the  $N$  in the corresponding representation (5.27) is called *Powers-index* of  $\sigma$ . It holds that for every  $\sigma \in \text{End}(\mathcal{B}(\mathcal{H}))$ , the relative commutant

$$\mathcal{B}(\mathcal{H}) \cap \sigma(\mathcal{B}(\mathcal{H}))'$$

is a type  $I_N$ , and this  $N$  coincides with the Powers-index.

2. Let  $\sigma, \{S_i\}$  be as in (1), and let  $A \in \mathcal{B}(\mathcal{H})$ ; then show that

$$NR_{\sigma(A)} \subseteq NR_A. \tag{5.28}$$

In other words, endomorphisms in  $\mathcal{B}(\mathcal{H})$  contract the numerical range.

Hint: Use the following three facts:

- (i) The numerical range  $NR_A$  is convex; and
- (ii) if  $x \in \mathcal{H}, \|x\| = 1$ , then (see Figure 5.6)

$$w_x(\sigma(A)) = \sum_{i=1}^N \|S_i^* x\|^2 w_{\frac{S_i^* x}{\|S_i^* x\|}}(A); \tag{5.29}$$

(iii) and lastly,

$$\sum_{i=1}^N \|S_i^* x\|^2 = 1. \tag{5.30}$$

**Exercise 5.26** (Convex sets that are not numerical ranges). Give an example of a bounded convex subset of the complex plane which is not  $NR_A$  for any  $A \in \mathcal{B}(\mathcal{H})$ , where  $\mathcal{H}$  is some Hilbert space.

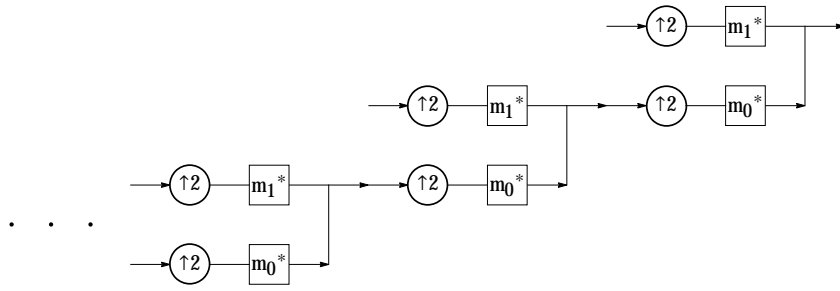


Figure 5.5: The two-channel synthesis filter bank.

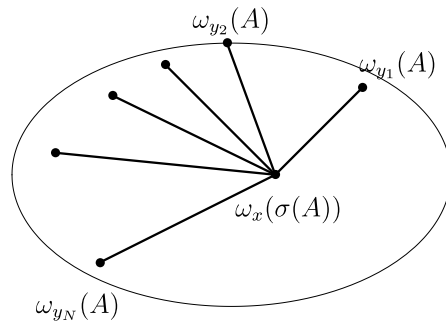


Figure 5.6: Illustration of eq. (5.29), with  $y_i := \frac{S_i^* x}{\|S_i^* x\|}$ ,  $i = 1, 2, \dots, N$ .

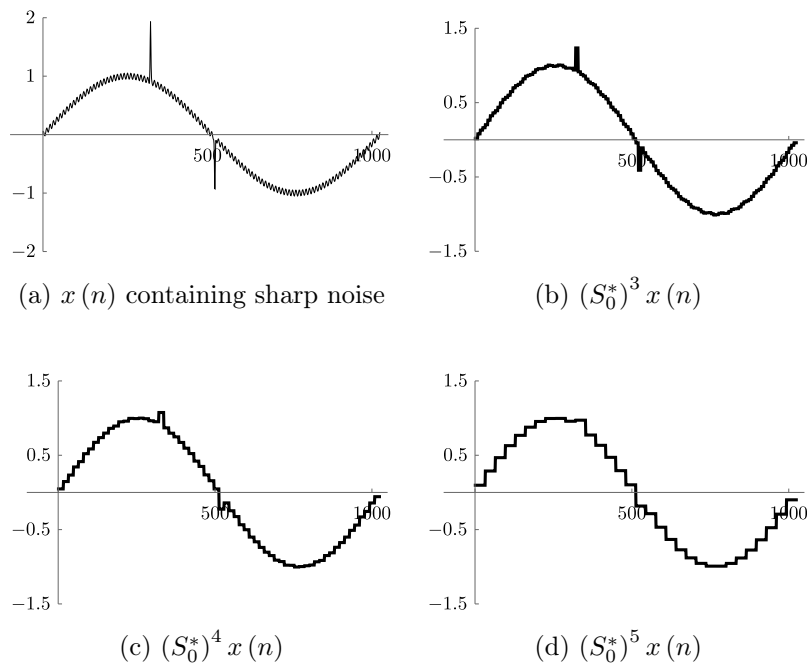


Figure 5.7: The outputs of  $(S_0^*)^n$ ,  $n = 3, 4, 5$ .

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest:

The paper [Sti55] is pioneering, starting the study of completely positive mappings in operator algebra theory. A more comprehensive list is: [Arv98, BR81b, Sti59, Tak79, BJ02, Jor06, Arv76, Fan10, BJKR84, Cun77, KR97b, Pow75, Sti55, MJD<sup>+</sup>15].

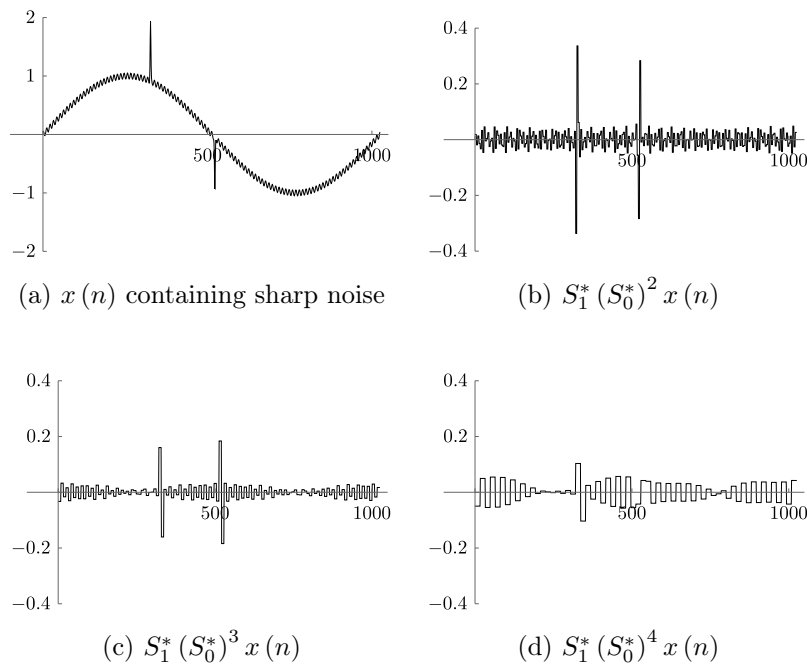


Figure 5.8: The outputs of high-pass filters.

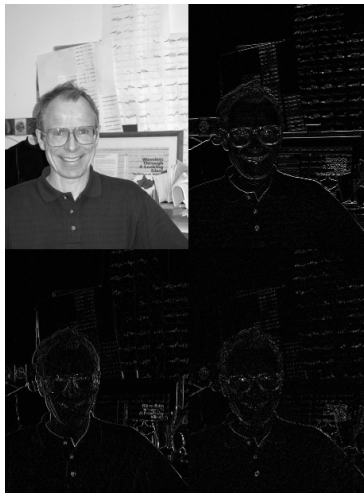


Figure 5.9: A coarser resolution in three directions in the plane, filtering in directions,  $x, y$ , and diagonal; – corresponding dyadic scaling in each coordinate direction. (Image cited from M.-S. Song, “*Wavelet Image Compression*” in [JLH06].)

## Chapter 6

# Brownian Motion

From its shady beginnings devising gambling strategies and counting corpses in medieval London, probability theory and statistical inference now emerge as better foundations for scientific models, especially those of the process of thinking and as essential ingredients of theoretical mathematics, even the foundations of mathematics itself.

— David Mumford

It is intriguing that the mathematics of Brownian motion was discovered almost simultaneously more than 100 years ago By Bachelier and by Einstein: In physics (Albert Einstein, 1905, "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen;" On the Motion of Small Particles Suspended in a Stationary Liquid, as Required by the Molecular Kinetic Theory of Heat). And in finance (Jean-Jacques Bachelier, 1900, "The Theory of Speculation"). Einstein

In Einstein's paper, Brownian motion offered one of the first experimental justification for the atomic theory. The Brownian motion model for financial markets is a continuous extension of the "one-period market model" of H. Markowitz, (in fact much later than 1900).

Bachelier: Continuous prices of financial asset-markets evolve in time according to a geometric Brownian motion.

"The glory of science is to imagine more than we can prove."

— Freeman Dyson

"Not only does God play dice, but... he sometimes throws them where they cannot be seen."

— Stephen Hawking

The concept of Brownian motion is not traditionally included in Functional Analysis. Below we offer a presentation which relies on almost all the big theorems from functional analysis, and especially on  $L^2$ -Hilbert spaces, built from probability measures on function spaces.

We have included a brief discussion of Brownian motion in order to illustrate infinite Cartesian products (sect 1.0.1) and unitary one-parameter group  $\{U(t)\}_{t \in \mathbb{R}}$  acting in Hilbert space. See [Jør14, Nel67, Nel59b, Hid80].

**Definition 6.1.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a *probability space*, i.e.,

- $\Omega$  = sample space
- $\mathcal{F}$  = sigma algebra of events
- $\mathbb{P}$  = probability measure defined on  $\mathcal{F}$ .

A function  $X : \Omega \rightarrow \mathbb{R}$  is called a *random variable* if it is a measurable function, i.e., if for all intervals  $(a, b) \subset \mathbb{R}$  the inverse image

$$X^{-1}((a, b)) = \{\omega \in \Omega \mid X(\omega) \in (a, b)\} \quad (6.1)$$

is in  $\mathcal{F}$ ; and we write  $\{a < X(\omega) < b\} \in \mathcal{F}$  in short-hand notation.

We say that  $X$  is *Gaussian* if  $\exists m, \sigma$  (written  $N(m, \sigma^2)$ ) such that

$$\mathbb{P}(\{a < X(\omega) < b\}) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} dx. \quad (6.2)$$

The function under the integral in (6.2) is called the Gaussian (or normal) distribution.

**Definition 6.2.** Events  $A, B \in \mathcal{F}$  are said to be *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Random variables  $X$  and  $Y$  are said to be independent iff (Def.)  $X^{-1}(I)$  and  $Y^{-1}(J)$  are independent for all intervals  $I$  and  $J$ .

**Definition 6.3.** A family  $\{X_t\}_{t \in \mathbb{R}}$  of random variables for  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to be a *Brownian motion* iff (Def.) for every  $n \in \mathbb{N}$ ,

1. the random variables  $X_{t_1}, X_{t_2}, \dots, X_{t_n}$  are jointly Gaussian with

$$\mathbb{E}(X_t) = \int_{\Omega} X_t(\omega) d\mathbb{P}(\omega) = 0, \quad \forall t \in \mathbb{R};$$

2. if  $t_1 < t_2 < \dots < t_n$  then  $X_{t_{i+1}} - X_{t_i}$  and  $X_{t_i} - X_{t_{i-1}}$  are independent;
3. for all  $s, t \in \mathbb{R}$ ,

$$\mathbb{E}(|X_t - X_s|^2) = |t - s|.$$

uniform	$a \leq x \leq b$	$\frac{1}{b-a}$
exponential ( $\lambda$ )	$x \geq 0$	$\lambda e^{-\lambda x}$
Gaussian normal $N(m, \sigma^2)$	$x \in \mathbb{R}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$
Cauchy	$x \in \mathbb{R}$	$\frac{1}{\pi(1+x^2)}$
$\chi^2$ (chi-square)	$x \geq 0$	$\frac{e^{-\frac{x}{2}} x^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})}$
Gamma	$x \geq 0$	$\frac{x^{\gamma-1} e^{-x}}{\Gamma(\gamma)}, \gamma > 0$

Table 6.1: Probability kernels (distributions)

*Remark 6.4.* There is a list of popular kernels in probability theory (Table 6.1). See any book on probability theory.

**Exercise 6.5** (Quadratic variation). Let  $\{X_t\}$  be the Brownian motion (Definition 6.3). Fix  $T \in \mathbb{R}_+$ , and consider partitions  $\pi : (t_i)_{i=0}^n$  of  $[0, T]$ , i.e.,

$$\pi : 0 = t_0 < t_1 < t_2 < \dots < t_n = T. \tag{6.3}$$

Set

$$\text{mesh}(\pi) (\text{:= } |\pi|) = \max_i \{t_i - t_{i-1}\}. \tag{6.4}$$

Then show that the limit,

$$\lim_{\text{mesh}(\pi) \rightarrow 0} \sum_i (X_{t_i} - X_{t_{i-1}})^2 = T \tag{6.5}$$

holds a.e. on  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega = C(\Omega)$ ,  $\mathcal{F} = \text{Cyl}$ , and  $\mathbb{P}$  = the Wiener measure.

Hint: Establish that

$$\mathbb{E}(|\Delta X_i|^2) = \Delta t_i, \tag{6.6}$$

$$\mathbb{E}(|\Delta X_i|^4) = 3(\Delta t_i)^2, \text{ and} \tag{6.7}$$

$$\mathbb{E}((\Delta X_i)^{2n-1}) = 0, n \in \mathbb{N}, \tag{6.8}$$

i.e., all the odd moments vanish; where

$$\begin{aligned} \Delta X_i &= X_{t_i} - X_{t_{i-1}}, \text{ and} \\ \Delta t_i &= t_i - t_{i-1}, \text{ for } i = 1, 2, \dots, n. \end{aligned}$$

Note that  $\sum_i (\ )^2$  on the LHS in (6.5) is a measurable function on  $(\Omega, \mathcal{F}, \mathbb{P})$ , while the RHS is deterministic, i.e., it is the constant function  $T$ .

*Remark 6.6.* Spectral Theorem and *functional calculus* is about the substitutions (see (6.9)).

$$\boxed{\begin{array}{c} A \\ \text{selfadjoint operator} \end{array}} \longrightarrow \boxed{\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ \text{scalar function} \\ \hookrightarrow f(A) \end{array}} \tag{6.9}$$

By contrast, Itô-calculus is about substitutions of Brownian motion (at least in a special case); as follows:

$$\boxed{\begin{array}{c} B_t \\ \text{Brownian motion} \end{array}} \longrightarrow \boxed{\begin{array}{c} f : \mathbb{R} \longrightarrow \mathbb{R} \\ \text{scalar function} \\ \hookrightarrow f(B_t) \end{array}} \tag{6.10}$$

See [Sto90, Yos95, Nel69, RS75, DS88c].



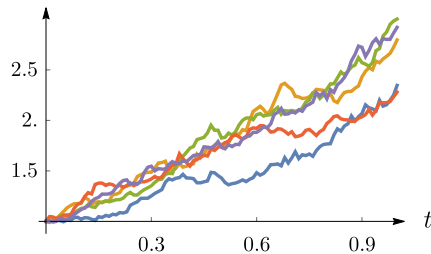


Figure 6.1: Geometric Brownian motion: 5 sample paths, with  $\mu = 1$ ,  $\sigma = 0.02$ , and  $X_0 = 1$ .

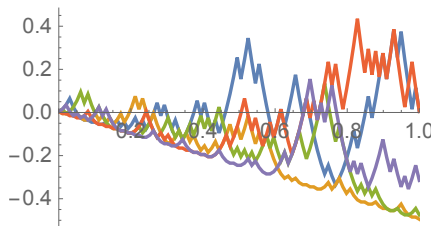


Figure 6.2: The process  $\frac{1}{2} (B_T^2 - T)$  in (6.13): 5 sample paths, with  $T = 1$ .

**Exercise 6.7** (Geometric Brownian motion). 1. Apply (6.10) to  $f(x) = \ln x$ ,  $x \in \mathbb{R}_+$ , together with (6.5) in Exercise 6.5, to show that, for  $T \in \mathbb{R}_+$ , the process,

$$X_T = X_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma B_T \right) \quad (6.11)$$

solves the SDE for geometric Brownian motion:

$$dX_t = X_t (\mu dt + \sigma dB_t). \quad (6.12)$$

See Figure 6.1.

2. Apply (6.10) to  $f(x) = x^2$ , together with (6.5) in Exercise 6.5 to establish the following:

$$\int_0^T B_t dB_t = \frac{1}{2} (B_T^2 - T). \quad (6.13)$$

See Figure 6.2.

In the previous exercise we explored stochastic processes derived from standard Brownian motion, but we now return to explore some additional properties for Brownian motion itself:

**Exercise 6.8** (A unitary one-parameter group). Using Definition 6.3, Corollary 1.35, Remark 1.36 and Example 1.37, show that there is a unique strongly

continuous unitary one-parameter group  $\{U(t)\}_{t \in \mathbb{R}}$  acting in  $L^2(C(\mathbb{R}), \text{Cyl}, \mathbb{P})$ , determined by

$$U(t)X_s = X_{s+t}, \quad \forall s, t \in \mathbb{R}. \quad (6.14)$$

Hint: By (3) in Definition 6.3, we have

$$\mathbb{E}(|X_{t_2} - X_{t_1}|^2) = \mathbb{E}(|X_{t_2+s} - X_{t_1+s}|^2), \quad \forall s, t_1, t_2 \in \mathbb{R}. \quad (6.15)$$

Hence, if  $U(t)$  is defined on the generator  $\{X_s : s \in \mathbb{R}\} \subset L^2(\mathbb{P})$  as in (6.14), it follows by (6.15) that it preserves the  $L^2(\mathbb{P})$ -norm. The remaining steps are left to the reader.

**Exercise 6.9** (Infinitesimal generator). Discuss the infinitesimal generator of  $\{U(t)\}_{t \in \mathbb{R}}$ .

**Exercise 6.10** (An ergodic action). Show that  $\{U(t)\}_{t \in \mathbb{R}}$  is induced by an ergodic action.

## 6.1 The Path Space

**The sample-space as a path-space.**

**Theorem 6.11** (see e.g., [Nel67]). Set  $\Omega = C(\mathbb{R}) =$  (all continuous real valued function on  $\mathbb{R}$ ),  $\mathcal{F} =$  the sigma algebra generated by cylinder-sets, i.e., determined by finite systems  $t_1, \dots, t_n$ , and intervals  $J_1, \dots, J_n$ ;

$$\text{Cyl}(t_1, \dots, t_n, J_1, \dots, J_n) = \{\omega \in C(\mathbb{R}) \mid \omega(t_i) \in J_i, i = 1, 2, \dots, n\}. \quad (6.16)$$

The measure  $\mathbb{P}$  is determined by its value on cylinder sets, and an integral over Gaussians; it is called the Wiener-measure. Set

$$X_t(\omega) = \omega(t), \quad \forall t \in \mathbb{R}, \omega \in \Omega (= C(\mathbb{R})).$$

If  $0 < t_1 < t_2 < \dots < t_n$ , and the cylinder set is as equation (6.16), then

$$\begin{aligned} & \mathbb{P}(\text{Cyl}(t_1, \dots, t_n, J_1, \dots, J_n)) \\ &= \int_{J_1} \dots \int_{J_n} g_{t_1}(x_1) g_{t_2-t_1}(x_2 - x_1) \dots g_{t_n-t_{n-1}}(x_n - x_{n-1}) dx_1 \dots dx_n \end{aligned}$$

where

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad \forall t > 0,$$

i.e., the  $N(0, t)$ -Gaussian. See Figure 6.3.

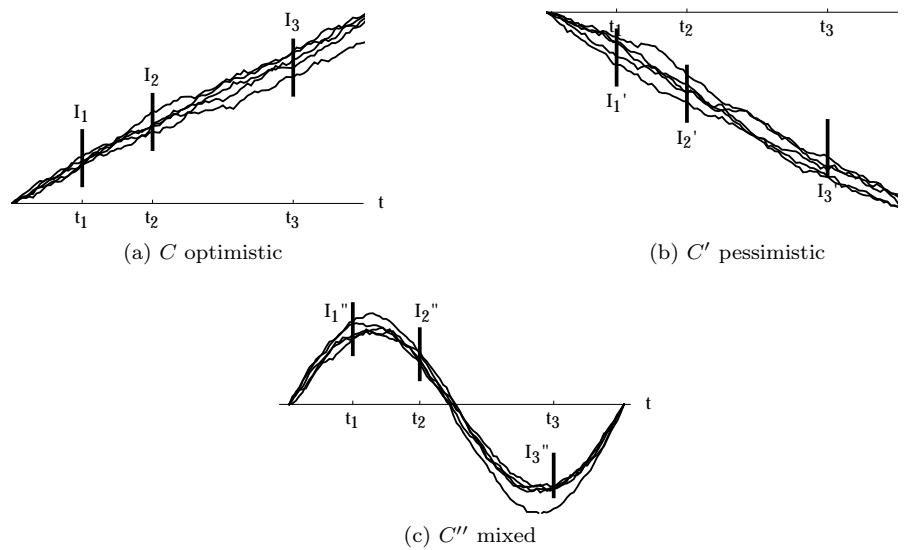


Figure 6.4: The cylinder sets  $C$ ,  $C'$ , and  $C''$ .

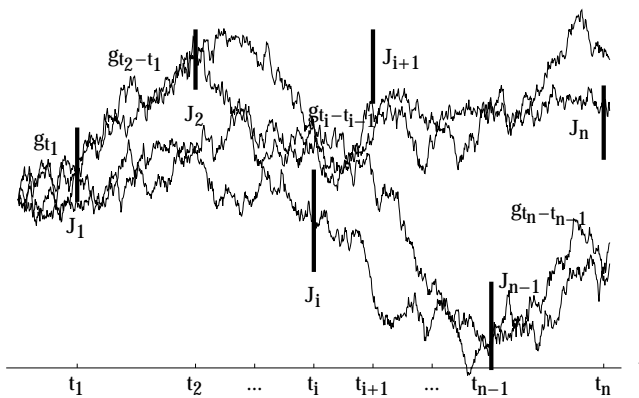


Figure 6.3: Stochastic processes indexed by time: A cylinder set  $C$  is a special subset of the space of all paths, i.e., functions of a time variable. A fixed cylinder set  $C$  is specified by a finite set of sample point on the time-axis (horizontal), and a corresponding set of “windows” (intervals on the vertical axis). When sample points and intervals are given, we define the corresponding cylinder set  $C$  to be the set of all paths that pass through the respective windows at the sampled times. In the figure we illustrate sample points (say future relative to  $t = 0$ ). Imagine the set  $C$  of all outcomes with specification at the points  $t_1, t_2, \dots$  etc.

**Infinite-product Measure** Let  $\Omega = \prod_{k=1}^{\infty} \{1, -1\}$  be the infinite Cartesian product of  $\{1, -1\}$  with the product topology.  $\Omega$  is compact and Hausdorff by Tychonoff's theorem.

For each  $k \in \mathbb{N}$ , let  $X_k : \Omega \rightarrow \{1, -1\}$  be the  $k^{\text{th}}$  coordinate projection, and assign probability measures  $\mu_k$  on  $\Omega$  so that  $\mu_k \circ X_k^{-1}\{1\} = a$  and  $\mu_k \circ X_k^{-1}\{-1\} = 1 - a$ , where  $a \in (0, 1)$ . The collection of measures  $\{\mu_k\}$  satisfies the consistency condition, i.e.,  $\mu_k$  is the restriction of  $\mu_{k+1}$  onto the  $k^{\text{th}}$  coordinate space. By Kolmogorov's extension theorem, there exists a unique probability measure  $P$  on  $\Omega$  so that the restriction of  $P$  to the  $k^{\text{th}}$  coordinate is equal to  $\mu_k$ .

It follows that  $\{X_k\}$  is a sequence of independent identically distributed (i.i.d.) random variables in  $L^2(\Omega, P)$  with  $\mathbb{E}(X_k) = 0$  and  $\text{Var}[X_k^2] = 1$ ; and  $L^2(\Omega, P) = \text{span}\{X_k\}$ .

*Remark 6.12.* Let  $\mathcal{H}$  be a separable Hilbert space with an orthonormal basis  $\{u_k\}$ . The map  $\varphi : u_k \mapsto X_k$  extends linearly to an isometric embedding of  $\mathcal{H}$  into  $L^2(\Omega, P)$ . Moreover, let  $\mathcal{F}_+(\mathcal{H})$  be the symmetric Fock space.  $\mathcal{F}_+(\mathcal{H})$  is the closed span of the algebraic tensors  $u_{k_1} \otimes \cdots \otimes u_{k_n}$ , thus  $\varphi$  extends to an isomorphism from  $\mathcal{F}_+(\mathcal{H})$  to  $L^2(\Omega, P)$ .

**Exercise 6.13** (The “fair-coin” measure). Let  $\Omega = \prod_1^{\infty} \{-1, 1\}$ , and let  $\mu$  be the “fair-coin” measure  $(\frac{1}{2}, \frac{1}{2})$  on  $\{\pm 1\}$  (e.g., “Head v.s. Tail”), let  $\mathcal{F}$  be the cylinder sigma-algebra of subsets of  $\Omega$ . Let  $\mathbb{P} = \prod_1^{\infty} \mu$  be the infinite-product measure on  $\Omega$ . Set  $Z_k(\omega) = \omega_k$ ,  $\omega = (\omega_i) \in \Omega$ . Finally, let  $\{\psi_j\}_{j \in \mathbb{N}}$  be an ONB in  $L^2(0, 1)$ , and set

$$X_t(\omega) := \sum_{j=1}^{\infty} \left( \int_0^t \psi_j(s) ds \right) Z_j(\omega), \quad t \in [0, 1], \omega \in \Omega.$$

Show that the  $\{X_t\}_{t \in [0, 1]}$  is Brownian motion, where time “ $t$ ” is restricted to  $[0, 1]$ .

Hint: Let  $t_1, t_2 \in [0, 1]$ , then

$$\begin{aligned} \mathbb{E}(X_{t_1} X_{t_2}) &= \mathbb{E} \left( \sum_j \left( \int_0^{t_1} \psi_j(s) ds \right) Z_j \cdot \sum_k \left( \int_0^{t_2} \psi_k(s) ds \right) Z_k \right) \\ &= \sum_j \sum_k \int_0^{t_1} \psi_j(s) ds \int_0^{t_2} \psi_k(s) ds \underbrace{\mathbb{E}(Z_j Z_k)}_{=\delta_{jk}} \\ &= \sum_j \langle \chi_{[0, t_1]}, \psi_j \rangle_{L^2} \langle \chi_{[0, t_2]}, \psi_j \rangle_{L^2} \\ &= \langle \chi_{[0, t_1]}, \chi_{[0, t_2]} \rangle_{L^2(0, 1)} = t_1 \wedge t_2 \quad (:= \min(t_1, t_2)). \end{aligned}$$

## 6.2 Decomposition of Brownian motion

The integral kernel  $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ ,  $K(s, t) = s \wedge t$ , is a compact operator on  $L^2[0, 1]$ , where

$$Kf(x) = \int_0^1 (x \wedge y) f(y) dy.$$

$Kf$  is a solution to the differential equation

$$-\frac{d^2}{dx^2} u = f$$

with zero boundary conditions.

$K$  is also seen as the covariance functions of Brownian motion process. A stochastic process is a family of measurable functions  $\{X_t\}$  defined on some sample probability space  $(\Omega, \mathfrak{B}, P)$ , where the parameter  $t$  usually represents time.  $\{X_t\}$  is a Brownian motion process if it is a mean zero Gaussian process such that

$$E[X_s X_t] = \int_{\Omega} X_s X_t dP = s \wedge t.$$

It follows that the corresponding increment process  $\{X_t - X_s\} \sim N(0, t - s)$ .  $P$  is called the Wiener measure.

Building  $(\Omega, \mathfrak{B}, P)$  is a fancy version of Riesz's representation theorem [Rud87, Theorem 2.14]. It turns out that

$$\Omega = \prod_t \bar{\mathbb{R}}$$

which is a compact Hausdorff space; (where  $\bar{\mathbb{R}} = (\mathbb{R} \cup \{\infty\})^\sim$  denotes the one-point compactification of  $\mathbb{R}$ .)

Now introduce random variable  $X_t : \Omega \rightarrow \mathbb{R}$ , defined as

$$X_t(\omega) = \omega(t), \quad t \in \mathbb{R};$$

i.e.,  $X_t$  is the continuous linear functional of evaluation at  $t$  on  $\Omega$ .

For Brownian motion, the increment of the process  $\Delta X_t$ , in some statistical sense, is proportional to  $\sqrt{\Delta t}$ , i.e.,

$$\Delta X_t \sim \sqrt{\Delta t}.$$

It is this property that makes the set of differentiable functions have measure zero. In this sense, the trajectory of Brownian motion is nowhere differentiable.

A very important application of the spectral theorem of compact operators is to decompose the Brownian motion process:

$$B_t(\omega) = \sum_{n=1}^{\infty} \frac{\sin(n\pi t)}{n\pi} Z_n(\omega) \quad (6.17)$$

where

$$s \wedge t = \sum_{n=1}^{\infty} \frac{\sin(n\pi s) \sin(n\pi t)}{(n\pi)^2}$$

and  $Z_n \sim N(0, 1)$ .

*Remark 6.14.* Consider the Hardy space  $\mathbb{H}_2$ , and the operator  $S$  from Exercise 4.123. Writing  $f(z) = \sum_{n=0}^{\infty} x_n z^n$ , we get

$$(Sf)(z) = f(z^N) = x_0 + x_1 z^N + x_2 z^{2N} + \dots; \quad (6.18)$$

and

$$(S^*f)(z) = x_0 + x_N z + x_{2N} z^2 + x_{3N} z^3 + \dots; \quad (6.19)$$

so in symbol-space,  $S^*$  acts as follows:

$$\begin{array}{c} (x_0, x_1, \dots, x_{N-1}, x_N, x_{N+1}, \dots, x_{2N}, x_{2N+1}, \dots) \\ S^* \downarrow \\ (x_0, x_N, x_{2N}, x_{3N}, \dots); \end{array} \quad (6.20)$$

so down-sampling  $\simeq$  “decimation”  $\simeq$  killing time-signals  $x_k$  when  $N + k$ , i.e.,  $k$  is not divisible by  $N$ .

The projection  $SS^*$  is:

$$\begin{array}{c} (x_0, x_1, \dots, x_{N-1}, x_N, x_{N+1}, \dots, x_{2N-1}, x_{2N}, x_{2N+1}, \dots, x_{3N-1}, x_{3N}, x_{3N+1}, \dots) \\ SS^* \downarrow \\ (x_0, 0, \dots, 0, x_N, 0, \dots, 0, x_{2N}, 0, \dots, 0, x_{3N}, 0, \dots) \end{array} \quad (6.21)$$

If the coordinates in  $\mathbb{H}_2$  label the i.i.d. random variables  $Z_k(\cdot)$  in the expansion (6.17) for Brownian motion, then downsampling corresponds to *conditional expectation*; conditioning on “less information”, i.e., leaving out the “decimated coordinates” in the expansion (6.17) for Brownian motion.

**Exercise 6.15** (The Central Limit Theorem). Look up the *Central Limit Theorem* (CLT), and prove the following approximation formula for Brownian motion:

Let  $\pi$  be the “fair-coin-measure” on the two outcomes  $\{\pm 1\}$ , i.e., winning or loosing one unit, and let  $\Omega = \times_{\mathbb{N}} \{\pm 1\}$ ,  $\mathbb{P} = \times_{\mathbb{N}} \pi$  be the corresponding infinite product measure. On  $\Omega$ , set

$$W_k(\omega) = \omega_k, \quad \omega = (\omega_1, \omega_2, \dots) \in \Omega, \quad k = 1, 2, \dots; \text{ and}$$

$$S_n(\cdot) = \frac{1}{\sqrt{n}} \sum_{k=1}^n W_k(\cdot). \quad (6.22)$$

Let  $X_t$  denote Brownian motion. Then show that

$$X_t(\cdot) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} W_k(\cdot) \quad (6.23)$$

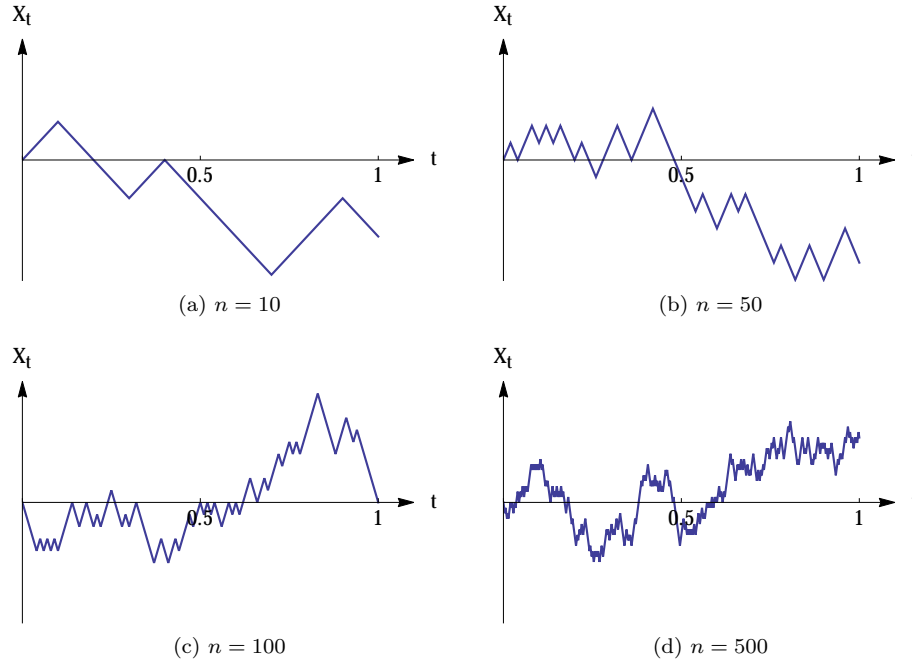


Figure 6.5: Monte-Carlo simulation of the standard Brownian motion process  $\{X_t : 0 \leq t \leq 1\}$ , where  $\mathbb{E}(X_t) = 0$ , and  $\mathbb{E}(X_s X_t) = s \wedge t = \min(s, t)$ .

For  $n = 10, 50, 100, 500$ , set  $X_0 = 0$  and  $X_{j/n}^{(n)} = n^{-1/2} \sum_{k=1}^j W_k$ . Applying linear interpolation between sample points  $\{j/n : j = 0, \dots, n\}$  yields the  $n$ -point approximation  $X_t^{(n)}$ , which converges in measure to  $X_t$  (standard BM restricted to the unit interval), as  $n \rightarrow \infty$ .

where  $\lfloor nt \rfloor$  denotes the largest integer  $\leq nt$ .

Hint: A good reference to the CLT is [CW14]. First approximate the CLT to the sequence  $S_n$  in (6.22). Figure 6.5 illustrates the approximation formula in (6.23). ( $X_1 = S_1$ .)

The Central Limit Theorem (CLT) states that the limit,  $n \rightarrow \infty$ , of the sequence  $S_n$  in (6.22) is a copy of  $N(0, 1)$ -random variable; i.e.,  $\lim_{n \rightarrow \infty} S_n(\cdot) = S(\cdot)$  exists; and

$$\mathbb{P}(\{\omega \mid a \leq S(\omega) \leq b\}) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx,$$

for all intervals  $(a, b) \subset \mathbb{R}$ . Applying this to (6.23), we get existence of  $X_t$  as a limit, and  $X_t \sim N(0, t)$ , i.e.,

$$\mathbb{P}(\{\omega \mid a \leq X_t(\omega) \leq b\}) = \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2} dx.$$

We claim that

$$\mathbb{E}_{\mathbb{P}}(X_s X_t) = s \wedge t. \quad (6.24)$$

Below we sketch the argument for the assertion in (6.24).

Fix  $s, t \in \mathbb{R}_+$ , say  $s < t$ , and  $n \in \mathbb{N}$ ; then

$$\begin{aligned} \mathbb{E} \left( \frac{1}{\sqrt{n}} \left( \sum_{j=1}^{\lfloor ns \rfloor} W_j \right) \frac{1}{\sqrt{n}} \left( \sum_{k=1}^{\lfloor nt \rfloor} W_k \right) \right) &= \frac{1}{n} \sum_{j=1}^{\lfloor ns \rfloor} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(W_j W_k) \\ &= \frac{1}{n} \sum_{j=1}^{\lfloor ns \rfloor} \sum_{k=1}^{\lfloor nt \rfloor} \delta_{j,k} \\ &= \frac{\lfloor ns \rfloor}{n} \rightarrow s, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence by the CLT, the desired conclusion in (6.24) follows.

This is the key step in proving that the limit  $X_t$  in (6.23) is Brownian motion. The remaining steps are routine left to the readers.

### 6.3 Large Matrices Revisited

Since large matrices and limit distributions have played a role in several topics in the present chapter, we mention here yet a different one; but now without proofs. Readers will find a complete treatment, for example in [Wig58, SS98].

Setting For all  $N \in \mathbb{N}$ , consider a symmetric (random) matrix

$$X = \begin{bmatrix} X_{1,1}^{(N)} & \cdots & X_{1,N}^{(N)} \\ \vdots & & \vdots \\ X_{N,1}^{(N)} & \cdots & X_{N,N}^{(N)} \end{bmatrix} \quad (6.25)$$

$$X_{i,j}^{(N)} = X_{j,i}^{(N)} \text{ (real valued)} \quad (6.26)$$

where the entries are i.i.d. random variables (“i.i.d” is short for independent identically distributed), mean 0, and variance  $m^2$ . And all the moments finite, and with at most exponential bounds.

Let  $a, b \in \mathbb{R}$ ,  $a < b$  be fixed, and set

$$V_N^{(a,b)} := \# \text{ of eigenvalues of } X^N \text{ that} \quad (6.27) \\ \text{fall in the interval } (a\sqrt{N}, b\sqrt{N}).$$

Then the following limit exists, i.e., the semicircle law holds for the limit distribution of the eigenvalues:

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}(V_N^{(a,b)})}{N} = \frac{1}{2\pi m^2} \int_a^b \sqrt{4m^2 - x^2} dx. \quad (6.28)$$



## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [[Hid80](#), [It04](#), [It06](#), [It07](#), [Nel67](#), [Par09](#), [Gro70](#), [Gro64](#), [Nel64](#), [AJ12](#), [AJS14](#), [BM13](#), [Sch58](#), [SS11a](#), [Jør14](#)].

## Chapter 7

# Lie Groups, and their Unitary Representations

Every axiomatic (abstract) theory admits, as is well known, an unlimited number of concrete interpretations besides those from which it was derived. Thus we find applications in fields of science which have no relation to the concepts of random event and of probability in the precise meaning of these words.

— A.N. Kolmogorov

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

— Eugene Paul Wigner

"Nowadays group theoretical methods—especially those involving characters and representations, pervade all branches of quantum mechanics."

— George Whitelaw Mackey

"The universe is an enormous direct product of representations of symmetry groups."

— Hermann Weyl

As part of our discussion of spectral theory and harmonic analysis, we had occasion to study unitary one-parameter groups  $\mathcal{U}(t)$ ,  $t \in \mathbb{R}$ . Stated differently (see Chapters 2-4), a unitary one-parameter group acting on a Hilbert space  $\mathcal{H}$ , is a strongly continuous unitary representation of the group  $\mathbb{R}$  with addition; – so it is an element in  $Rep(\mathbb{R}, \mathcal{H})$ . Because of applications to physics, to non-commutative harmonic analysis, to stochastic processes, and to geometry, it is of

interest to generalize to  $Rep(G, \mathcal{H})$  where  $G$  is some more general group, other than  $(\mathbb{R}, +)$ , for example,  $G$  may be a matrix group, a Lie group, both compact and non-compact, or more generally,  $G$  may be a locally compact group.

In Chapters 2-3 we studied the canonical commutation-relations for the quantum mechanical momentum and position operators  $P$ , respectively  $Q$ . Below we outline how this problem can be restated as a result about a unitary representation of the matrix group  $G_3$  of all upper triangular  $3 \times 3$  matrices over  $\mathbb{R}$ . This is a special unitary irreducible representation  $\mathcal{U}$  in  $Rep(G_3, L^2(\mathbb{R}))$ . It is called the Schrödinger representation, and the group  $G_3$  is called the Heisenberg group. We shall need the Stone-von Neumann uniqueness theorem, outlined in the appendix below. (Also see [vN32b, vN31].) Its proof will follow from a more general result which is included inside the present chapter. The Stone-von Neumann uniqueness theorem states that every unitary irreducible representation of  $G_3$  is unitarily equivalent to the Schrödinger representation.

We studied operators in Hilbert space of relevance to quantum physics. A source of examples is relativistic physics. The symmetry group of Einstein's theory is a particular Lie group, called the Poincaré group. The study of its unitary representations is central to relativistic physics. But it turns out that there is a host of diverse applications (including harmonic analysis) where other groups arise. Below we offer a glimpse of the theory of unitary representations, and its many connections to operators in Hilbert space.

Two pedantic points regarding unbounded operators. The first is the distinction between “selfadjoint” vs “essentially selfadjoint.” An operator is said to be essentially selfadjoint if its closure is selfadjoint. The second is the distinction between selfadjoint and skewadjoint. The difference there is just a multiple of  $i (= \sqrt{-1})$ .

These distinctions play a role in the study of unitary representations of Lie groups, but are often swept under the rug, especially in the physics literature. Every unitary representation of a Lie group has a derived representation of the corresponding Lie algebra. The individual operators in a derived representation are skewadjoint; – but to get a common dense domain for all these operators, we must resort to essentially skewadjointness. Nonetheless, indeed there are choices of common dense domains (e.g.,  $C^\infty$ -vectors), but the individual operators in the derived representation will then only be essentially skewadjoint there.

For more details on this, see e.g., [Pou72].

## 7.1 Motivation

The following non-commutative Lie groups will be of special interest to us because of their applications to physics, and to a host of areas within mathematics; they are: the Heisenberg group  $G = H_3$ , the  $ax + b$  group  $G = S_2$ ; and  $SL_2(\mathbb{R})$ . In outline:

- $G = H_3$ , in real form  $\simeq \mathbb{R}^3$ , with multiplication

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab'), \quad (7.1)$$

$\forall (a, b, c)$ , and  $(a', b', c') \in \mathbb{R}^3$ . This is also matrix-multiplication when  $(a, b, c)$  has the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

In complex form,  $z \in \mathbb{C}$ ,  $c \in \mathbb{R}$ , we have

$$(z, c)(z', c') = (z + z', c + c' + \Im(\bar{z}z')). \quad (7.2)$$

- $G = S_2$ , the group of transformation  $x \mapsto ax + b$  where  $a \in \mathbb{R}_+$ , and  $b \in \mathbb{R}$ , with multiplication

$$(a, b)(a', b') = (aa', b + ab'). \quad (7.3)$$

This is also the matrix-multiplication when  $(a, b)$  has the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

- $G = SL_2(\mathbb{R}) = 2 \times 2$  matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

over  $\mathbb{R}$ , with  $ad - bc = 1$ . Note that  $SL_2(\mathbb{R})$  is locally isomorphic to  $SU(1, 1) = 2 \times 2$  matrices over  $\mathbb{C}$ ,

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

such that  $|\alpha|^2 - |\beta|^2 = 1$ . In both cases, the multiplication in  $G$  is matrix-multiplication for  $2 \times 2$  matrices.

The four groups, and their harmonic analysis will be studied in detail inside this chapter.

An important question in the theory of unitary representations is the following: For a given Lie group  $G$ , what are its irreducible unitary representations, up to unitary equivalence. One aims for lists of these irreducibles. The question is an important part of non-commutative harmonic analysis. When answers are available, they have important implications for physics and for a host of other applications, but complete lists are hard to come by; and the literature on the subject is vast. We refer to the book [Tay86], and its references for an overview.

To begin with, the tools going into obtaining lists of the equivalence classes of irreducible representations, differ from one class of Lie groups to the other. Cases in point are the following classes, nilpotent, solvable, and semisimple. The Heisenberg group is in the first class, the  $ax + b$  group in the second, and  $SL_2(\mathbb{R})$  in the third. By a theorem of Stone and von Neumann, the classes of

irreducibles for the Heisenberg group are indexed by a real parameter  $h$ ; they are infinite-dimensional for non-zero values of  $h$ , and one dimensional for  $h = 0$ .

For the  $ax + b$  group, there are just two classes of unitary irreducibles. The verification of this can be made with the use of Mackey's theory of induced representations [Mac88]. But the story is much more subtle in the semisimple cases, even for  $SL_2(\mathbb{R})$ . The full list is divided up in series of representations (principal, continuous, discrete, and complementary series representations), and the paper [JØ00] outlines some of their properties. But the details of this are far beyond the scope of the present book.

We now turn to some:

**Exercise 7.1** (Semidirect product  $G \ltimes V$ ). Let  $V$  be a finite-dimensional vector space, and let  $G \subset GL(V)$  be a sub-group of the corresponding general linear group: Make the definition

$$(g, v)(g', v') = (gg', g(v') + v) \quad (7.4)$$

for all  $g, g' \in G$ , and  $v, v' \in V$ .

1. Show that with (7.4) we get a new group; called the semidirect product .
2. In the group  $G \ltimes V$ , show that

$$(g, v)^{-1} = (g^{-1}, -g^{-1}(v)), \quad g \in G, v \in V;$$

and conclude from this that  $V$  identifies as a normal subgroup in  $G \ltimes V$ .

3. Show that, if  $G$  is a Lie group, then so is  $G \ltimes V$ .
4. Show that, within the Lie algebra of  $G \ltimes V$ , the vector space  $V$  identifies as an ideal.

**General Considerations** Every group  $G$  is also a  $*$ -semigroup, with the  $*$  operation

$$g^* := g^{-1}.$$

$G$  is not a complex  $*$  algebra yet, in particular, multiplication by a complex number is not defined. As a general principal,  $G$  can be embedded into the  $*$ -algebra

$$\mathfrak{A}_G = G \otimes \mathbb{C} = \mathbb{C}\text{-valued functions on } G.$$

$\mathfrak{A}_G$  has a natural pointwise multiplication and scalar multiplication, given by

$$\begin{aligned} (g \otimes c_g)(h \otimes c_h) &= gh \otimes c_g c_h \\ t(g \otimes c_g) &= g \otimes t c_g \end{aligned}$$

for all  $g, h \in G$  and  $c_g, c_h, t \in \mathbb{C}$ . The  $*$  operation extends from  $G$  to  $\mathfrak{A}_G$  as

$$(g \otimes c_g)^* = g^{-1} \otimes \overline{c_g^{-1}}. \quad (7.5)$$

*Remark 7.2.* The  $*$  operation so defined (as in (7.5)) is the only way to make it a period-2, conjugate linear, anti-automorphism. That is,  $(tA)^* = \bar{t}A^*$ ,  $A^{**} = A$ , and  $(AB)^* = B^*A^*$ , for all  $A, B \in \mathfrak{A}_G$ , and all  $t \in \mathbb{C}$ .

There is a bijection between representations of  $G$  and representations of  $\mathfrak{A}_G$ . For if  $\pi \in \text{Rep}(G, \mathcal{H})$ , it then extends to  $\tilde{\pi} = \pi \otimes \text{Id}_{\mathbb{C}} \in \text{Rep}(\mathfrak{A}_G, \mathcal{H})$ , by

$$\tilde{\pi}(g \otimes c_g) = \pi_g \otimes c_g.$$

$\text{id}_{\mathbb{C}}$  denotes the identity representation  $\mathbb{C} \rightarrow \mathbb{C}$ . Conversely, if  $\rho \in \text{Rep}(\mathfrak{A}_G, \mathcal{H})$  then

$$\pi = \rho|_{G \otimes 1}$$

is a representation of the group  $G \simeq G \otimes 1$ .

*Remark 7.3.* The notation  $g \otimes c_g$  is usually written as  $c_g$ . Thus pointwise multiplication takes the form

$$c_g d_h = l_{gh}.$$

Equivalently, we write

$$l_g = \sum_h c_h d_{h^{-1}g}$$

i.e., the usual convolution. In more details:

$$\left( \sum_g c_g \pi(g) \right) \left( \sum_h d_h \pi(h) \right) = \sum_g \underbrace{\left( \sum_h c_h d_{h^{-1}g} \right)}_{=l_g} \pi(g).$$

The  $*$  operation now becomes

$$c_g^* = \overline{c_{g^{-1}}}.$$

More generally, every locally compact group has a left (and right) Haar measure. Thus the above construction has a “continuous” version. It turns out that  $\mathfrak{A}_G$  is the Banach  $*$ -algebra  $L^1(G)$ . Again, there is a bijection between  $\text{Rep}(G, \mathcal{H})$  and  $\text{Rep}(L^1(G), \mathcal{H})$ . In particular, the “discrete” version is recovered if the measure  $\mu$  is discrete, in which case  $\mathfrak{A}_G = l^1(G)$ .

**Definition 7.4.** Let  $G$  be a group, and  $\psi : G \rightarrow \mathbb{C}$  a function. We say that  $\psi$  is *positive definite* iff (Def.) for all  $n \in \mathbb{N}$ , all  $g_1, \dots, g_n \in G$ , and all  $c_1, \dots, c_n \in \mathbb{C}$ , we have

$$\sum_{j=1}^n \sum_{k=1}^n \bar{c}_j c_k \psi(g_j^{-1} g_k) \geq 0.$$

Often we also assume that  $\psi(e) = 1$ .

**Exercise 7.5** (Positive definite functions). Show that every positive definite function  $\psi$  on  $G$  extends by linearity to a positive definite function  $\tilde{\psi}$  on the group algebra  $\mathbb{C}[G]$ , i.e.,

$$\tilde{\psi} \left( \sum_g c_g g \right) := \sum_g c_g \psi(g)$$

on all (finite) linear expressions  $\sum_g c_g g$ .

**Exercise 7.6** (Contractions). Let  $\mathcal{H}$  be a Hilbert space, and let  $T$  be a *contraction*, i.e.,  $T \in \mathcal{B}(\mathcal{H})$ , satisfying one of the two equivalent conditions:

$$\|T\| \leq 1 \iff I - T^*T \geq 0 \text{ in the order on Hermitian operators.}$$

1. For  $G = \mathbb{Z}$ , define  $\psi : \mathbb{Z} \rightarrow \mathbb{C}$  as follows:

$$\psi(n) = \begin{cases} T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}} & \text{if } n \geq 0 \\ (T^*)^{|n|} & \text{if } n < 0. \end{cases} \quad (7.6)$$

Show that  $\psi$  is positive definite.

2. Conclude that  $\tilde{\psi}$  is completely positive (see Chapter 5).
3. Apply (1) & (2) to conclude the existence of a triple  $(V, \mathcal{K}, \mathcal{U})$ , where  $\mathcal{K}$  is a Hilbert space,  $V : \mathcal{H} \rightarrow \mathcal{K}$  is isometric,  $\mathcal{U} : \mathcal{K} \rightarrow \mathcal{K}$  is a unitary operator; and we have

$$T^n = V^* \mathcal{U}^n V, \quad \forall n \in \mathbb{N}. \quad (7.7)$$

*Historic Note:* The system  $(V, \mathcal{K}, \mathcal{U})$  above, satisfying (7.7), is called a *unitary dilation* [Sch55] (details in Chapter 5.)

**Exercise 7.7** (Central Extension). Let  $V$  be a vector space over  $\mathbb{R}$ , and let  $B : V \times V \rightarrow \mathbb{R}$  be a function.

1. Then show that  $V \times \mathbb{R}$  turns into a group  $G$  when the operation in  $G$  is defined as follows:

$$(u, \alpha)(v, \beta) = (u + v, \alpha + \beta + B(u, v)), \quad \forall \alpha, \beta \in \mathbb{R}, \forall u, v \in V, \quad (7.8)$$

if and only if  $B$  satisfies

$$\begin{aligned} B(u, 0) = B(0, u) = 0, \quad \forall u \in V; \text{ and} \\ B(u, v) + B(u + v, w) = B(u, v + w) + B(v, w), \quad \forall u, v, w \in V. \end{aligned} \quad (7.9)$$

2. Assuming that (7.9) is satisfied; show that the group inverse under the operation (7.8) is

$$(u, \alpha)^{-1} = (-u, -\alpha - B(u, -u)), \quad \forall \alpha \in \mathbb{R}, u \in V. \quad (7.10)$$

## 7.2 Unitary One-Parameter Groups

“The mathematical landscape is full if groups of unitary operators. The ..., strongly continuous one-parameter groups  $U(t)$ ,  $-\infty <$

$t < \infty$ , come mostly from three sources: processes where energy is conserved, such as those governed by wave equations of all sorts; process where probability is preserved, for instance, ones governed by Schrödinger equations; and Hamiltonian and other measure-preserving flows.”

— Peter Lax, from [Lax02]

Let  $\mathcal{H}$  be a Hilbert space, and let  $P(\cdot)$  be a projection valued measure (PVM),

$$P : \mathcal{B}(\mathbb{R}) \longrightarrow \text{Proj}(\mathcal{H}) \quad (7.11)$$

i.e., defined on the sigma-algebra of all Borel subsets in  $\mathbb{R}$ .

Then, as we saw, the integral (operator-valued)

$$U(t) = \int_{\mathbb{R}} e^{i\lambda t} P(d\lambda), \quad t \in \mathbb{R} \quad (7.12)$$

is well defined, and yield a strongly continuous one-parameter group acting on  $\mathcal{H}$ ; equivalently,

$$U \in \text{Rep}_{uni}(\mathbb{R}, \mathcal{H}) \quad (7.13)$$

where  $U$  is defined by (7.12).

The theorem of M.H. Stone states that the converse holds as well, i.e., every  $U$ , as in (7.13), corresponds to a unique  $P$ , a PVM, such that (7.12) holds.

**Exercise 7.8** (von Neumann’s ergodic theorem). Let  $\{U(t)\}_{t \in \mathbb{R}}$  be a strongly continuous one-parameter group with PVM,  $P(\cdot)$ . Denote by  $P(\{0\})$  the value of  $P$  on the singleton  $\{0\}$ .

1. Show that

$$P(\{0\})\mathcal{H} = \{h \in \mathcal{H} : U(t)h = h, \forall t \in \mathbb{R}\}. \quad (7.14)$$

(We set  $\mathcal{H}_0 := P(\{0\})\mathcal{H}$ .)

2. Establish the following limit conclusion:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T U(t) dt = P(\{0\}). \quad (7.15)$$

Hint:

$$\frac{1}{2T} \int_{-T}^T U(t) dt = \int_{\mathbb{R}} \frac{\sin(\lambda T)}{\lambda T} P(d\lambda)$$

holds for all  $T \in \mathbb{R}_+$ .



### 7.3 Group - Algebra - Representations

In physics, we are interested in representation of symmetry groups, which preserve inner product or energy, and naturally leads to unitary representations. Every unitary representation can be decomposed into irreducible representations; the latter amounts to elementary particles which can not be broken up further. In practice, quite a lot work goes into finding irreducible representations of symmetry groups. Everything we learned about algebras is also true for groups. The idea is to go from groups to algebras and then to representations.

We summarize the basic definitions:

- $\pi_G \in \text{Rep}(G, \mathcal{H})$

$$\begin{cases} \pi(g_1 g_2) &= \pi(g_1) \pi(g_2) \\ \pi(e_G) &= I_{\mathcal{H}} \\ \pi(g^{-1}) &= \pi(g)^* \end{cases}$$

- $\pi_{\mathfrak{A}} \in \text{Rep}(\mathfrak{A}, \mathcal{H})$

$$\begin{cases} \pi(A_1 A_2) &= \pi(A_1) \pi(A_2) \\ \pi(1_{\mathfrak{A}}) &= I_{\mathcal{H}} \\ \pi(A^*) &= \pi(A)^* \end{cases}$$

Case 1.  $G$  is discrete  $\longrightarrow \mathfrak{A} = G \otimes l^1$

$$\begin{aligned} \left( \sum_g a(g)g \right) \left( \sum_g b(h)h \right) &= \sum_{g,h} a(g)b(h)gh \\ &= \sum_{g'} \sum_h a(g'h^{-1})b(h)g' \\ \left( \sum_g c(g)g \right)^* &= \left( \sum_g \overline{c(g^{-1})}g \right) \end{aligned}$$

where  $c^*(g) = \overline{c(g^{-1})}$ . The multiplication of functions in  $\mathfrak{A}$  is a generalization of convolutions.

Case 1.  $G$  is locally compact  $\longrightarrow \mathfrak{A} = G \otimes L^1(\mu) \simeq L^1(G)$ .

**Definition 7.9.** Let  $\mathcal{B}(G)$  be the Borel sigma-algebra of  $G$ . A regular Borel measure  $\lambda$  is said to be *left (resp. right) invariant*, if  $\lambda(gE) = \lambda(E)$  (resp.  $\lambda(Eg) = \lambda(E)$ ), for all  $g \in G$ , and  $E \in \mathcal{B}(G)$ .  $\lambda$  is called a *left (resp. right) Haar measure* accordingly.

Note that  $\lambda$  is left invariant iff

$$\lambda'(E) := \lambda(E^{-1})$$

is right invariant, where  $E^{-1} = \{g \in G : g^{-1} \in E\}$ , for all  $E \in \mathcal{B}(G)$ . Hence one may choose to work with either a left or right invariant measure.

**Theorem 7.10.** *Every locally compact group  $G$  has a left Haar measure, unique up to a multiplicative constant.*

For the existence of Haar measures, one first proves the easy case when  $G$  is compact, and then extends to locally compact cases. For non compact groups, the left / right Haar measures could be different. Many non compact groups have no Haar measure. In applications, the Haar measures are usually constructed explicitly.

Given a left Haar measure  $\lambda_L$ , and  $g \in G$ , then

$$E \mapsto \lambda_L(Eg), \quad E \in \mathcal{B}(G)$$

is also left invariant. Hence, by Theorem 7.10,

$$\lambda_L(Eg) = \Delta_G(g) \lambda_L(E) \quad (7.16)$$

for some constant  $\Delta_G(g) \in \mathbb{R} \setminus \{0\}$ . Note that  $\Delta_G$  is well-defined, independent of the choice of  $\lambda_L$ . Moreover,

$$\begin{aligned} \lambda_L(Egh) &= \Delta_G(h) \lambda_L(Eg) = \Delta_G(h) \Delta_G(g) \lambda_L(E) \\ \lambda_L(Egh) &= \Delta_G(gh) \lambda_L(E) \end{aligned}$$

and it follows that  $\Delta_G : G \rightarrow \mathbb{R}_\times$  is a homomorphism, i.e.,

$$\Delta_G(gh) = \Delta_G(h) \Delta_G(g), \quad \forall g, h \in G. \quad (7.17)$$

**Definition 7.11.**  $\Delta_G$  is called the *modular function* of  $G$ .  $G$  is said to be *unimodular* if  $\Delta_G \equiv 1$ .

**Corollary 7.12.** *Every compact group  $G$  is unimodular.*

*Proof.* Since  $\Delta_G$  is a homomorphism,  $\Delta_G(G)$  is compact in  $\mathbb{R}_\times$ , so  $\Delta_G \equiv 1$ .  $\square$

**Corollary 7.13.** *For all  $f \in C_c(G)$ , and  $g \in G$ ,*

$$\int_G f(\cdot g) d\lambda_L = \Delta_G(g^{-1}) \int_G f d\lambda_L. \quad (7.18)$$

*Equivalently, we get the substitution formula:*

$$d\lambda_L(\cdot g) = \Delta_G(g) d\lambda_L(\cdot). \quad (7.19)$$

*Similarly,*

$$\int_G f(g^{-1}\cdot) d\lambda_R = \Delta_G(g^{-1}) \int_G f d\lambda_R; \quad (7.20)$$

*i.e.,*

$$d\lambda_R(g\cdot) = \Delta_G(g^{-1}) d\lambda_R(\cdot). \quad (7.21)$$

*Proof.* It suffices to check for characteristic functions. Fix  $E \in \mathcal{B}(G)$ , then

$$\begin{aligned}
 \int_G \chi_E(\cdot g) d\lambda_L &= \int_G \chi_{Eg^{-1}} d\lambda_L \\
 &= \lambda_L(Eg^{-1}) \\
 &\stackrel{(7.16)}{=} \Delta_G(g^{-1}) \lambda_L(E) \\
 &= \Delta_G(g^{-1}) \int_G \chi_E d\lambda_L
 \end{aligned}$$

hence (7.18)-(7.19) follow from this and a standard approximation.

For the right Haar measure, recall that  $E \mapsto \lambda_L(E^{-1})$  is right invariant, and so  $\lambda_R(E) = c\lambda_L(E^{-1})$ , for some constant  $c \in \mathbb{R} \setminus \{0\}$ . (In fact, more is true; see Theorem 7.14 below.) Therefore,

$$\begin{aligned}
 \lambda_R(gE) &= c\lambda_L(E^{-1}g^{-1}) \\
 &= c\Delta_G(g^{-1}) \lambda_L(E^{-1}) \\
 &= \Delta_G(g^{-1}) \lambda_R(E).
 \end{aligned}$$

This yields (7.20)-(7.21).  $\square$

**Theorem 7.14.** *Let  $G$  be a locally compact group, then the two Haar measures are mutually absolutely continuous, i.e.,  $\lambda_L \ll \lambda_R \ll \lambda_L$ .*

*Specifically, fix  $\lambda_L$ , and set*

$$\lambda_R(E) := \lambda_L(E^{-1}), \quad E \in \mathcal{B}(G);$$

*then*

$$\frac{d\lambda_R}{d\lambda_L}(g) = \Delta_G(g^{-1}) = \text{Radon-Nikodym derivative.}$$

*Proof.* Note that  $\Delta_G d\lambda_R$  is left invariant. Indeed,

$$\begin{aligned}
 \Delta_G(g \cdot) d\lambda_R(g \cdot) &= \underbrace{(\Delta_G(g) \Delta_G(\cdot))}_{(7.17)} \underbrace{(\Delta_G(g^{-1}) d\lambda_R(\cdot))}_{(7.21)} \\
 &= \Delta_G(\cdot) d\lambda_R(\cdot).
 \end{aligned}$$

Hence, by the uniqueness of the Haar measure, we have

$$\Delta_G d\lambda_R = c d\lambda_L$$

for some constant  $c \in \mathbb{R} \setminus \{0\}$ . One then checks that  $c \equiv 1$ .  $\square$

**Corollary 7.15.** *If  $\lambda_L$  is a left Haar measure on  $G$ , then*

$$d\lambda_L(g^{-1}) = \Delta_G(g^{-1}) d\lambda_L(g). \quad (7.22)$$

*Similarly, if  $\lambda_R$  is a right Haar measure, then*

$$d\lambda_R(g^{-1}) = \Delta_G(g) d\lambda_R(g). \quad (7.23)$$

*Remark 7.16.* In the case  $L^1(G)$ ,  $\lambda_L = \lambda_R =$  the counting measure, which is unimodular, hence  $\Delta_G$  does not appear.

In  $L^1(G)$ , we define

$$\begin{aligned} (\varphi \star \psi)(g) &:= \int_G \varphi(gh^{-1}) \psi(h) d\lambda_R(h) \\ &= \int_G \varphi(h^{-1}) \psi(hg) d\lambda_R(h) \\ &= \int_G \varphi(h) \psi(h^{-1}g) \underbrace{\Delta_G(h) d\lambda_R(h)}_{d\lambda_L(h)} \end{aligned} \quad (7.24)$$

and

$$\varphi^*(g) := \overline{\varphi(g^{-1})} \Delta_G(g). \quad (7.25)$$

The choice of (7.25) preserves the  $L^1$ -norm. Indeed,

$$\begin{aligned} \int_G |\varphi^*| d\lambda_R &= \int_G |\varphi(g^{-1})| \Delta_G(g) d\lambda_R(g) \\ &= \int_G |\varphi(g)| \Delta_G(g^{-1}) d\lambda_R(g^{-1}) \\ &= \int_G |\varphi| d\lambda_R \end{aligned}$$

where  $\Delta_G(g^{-1}) d\lambda_R(g^{-1}) = d\lambda_R(g)$  by (7.23).

$L^1(G)$  is a Banach  $*$ -algebra, and  $L^1(G) = L^1$ -completion of  $C_c(G)$ . (Fubini's theorem shows that  $f \star g \in L^1(G)$ , for all  $f, g \in L^1(G)$ .)

We may also use left Haar measure in (7.24). Then, we set

$$\begin{aligned} (\varphi * \psi)(g) &:= \int_G \varphi(h) \psi(h^{-1}g) d\lambda_L(h) \\ &= \int_G \varphi(gh) \psi(h^{-1}) d\lambda_L(h) \\ &= \int_G \varphi(gh^{-1}) \psi(h) \underbrace{\Delta_G(h^{-1}) d\lambda_L(h)}_{d\lambda_R(h)} \end{aligned} \quad (7.26)$$

and set

$$\varphi^*(g) := \overline{\varphi(g^{-1})} \Delta_G(g^{-1}). \quad (7.27)$$

There is a bijection between representations of groups and representations of algebras.

Given a unitary representation  $\pi \in \text{Rep}(G, \mathcal{H})$ , let  $dg$  denote the Haar measure in  $L^1(G)$ , then we get the group algebra representation  $\pi_{L^1(G)} \in \text{Rep}(L^1(G), \mathcal{H})$ , where

$$\begin{aligned} \pi_{L^1(G)}(\varphi) &= \int_G \varphi(g) \pi(g) dg \\ \pi_{L^1(G)}(\varphi^*) &= \pi_{L^1(G)}(\varphi)^* \end{aligned}$$

Indeed, one checks that

$$\pi_{L^1(G)}(\varphi_1 \star \varphi_2) = \pi_{L^1(G)}(\varphi_1) \pi_{L^1(G)}(\varphi_2).$$

Conversely, given a representation of  $L^1(G)$ , let  $(\varphi_i)$  be a sequence in  $L^1$  such that  $\varphi_i \rightarrow \delta_g$ . Then

$$\int \varphi_i(h) \pi(h) g dh \rightarrow \pi(g),$$

i.e., the limit is a representation of  $G$ .

*Remark 7.17.* Let  $G$  be a matrix group, then  $x^{-1}dx$ ,  $x \in G$ , is left translation invariant. For if  $y \in G$ , then

$$(yx)^{-1} d(yx) = x^{-1} (y^{-1}y) dx = x^{-1}dx.$$

Now assume  $\dim G = n$ , and so  $x^{-1}dx$  contains  $n$  linearly independent differential forms,  $\sigma_1, \dots, \sigma_n$ ; and each  $\sigma_j$  is left translation invariant. Thus  $\sigma_1 \wedge \dots \wedge \sigma_n$  is a left invariant volume form, i.e., the left Haar measure. Similarly, the right Haar measure can be constructed from  $dx \cdot x^{-1}$ ,  $x \in G$ .

### Example – $ax + b$ group

$$\text{Let } G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R}_+, b \in \mathbb{R} \right\}.$$

- Multiplication

$$\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a'a & a'b + b' \\ 0 & 1 \end{bmatrix}$$

- Inverse

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{bmatrix}.$$

$G$  is isomorphic to the transformation group  $x \mapsto ax + b$ ; where composition gives

$$x \mapsto ax + b \mapsto a'(ax + b) + b' = aa'x + (a'b + b').$$

*Remark 7.18.* Setting  $a = e^t$ ,  $a' = e^{t'}$ ,  $aa' = e^t e^{t'} = e^{t+t'}$ , i.e., multiplication  $aa'$  can be made into addition.

The left Haar measure is given as follows:

Let  $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in G$ , so that

$$\begin{aligned} g^{-1}dg &= \frac{1}{a} \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} \begin{bmatrix} da & db \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{a} \begin{bmatrix} da & db \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence we get two left invariant (linear independent) differential forms:

$$\frac{da}{a} \quad \text{and} \quad \frac{db}{a}.$$

Set

$$d\lambda_L(g) = d\lambda_L(x, y) := \frac{1}{x^2} dx \wedge dy; \quad \left( g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, x \in \mathbb{R}_+ \right).$$

Indeed,  $\lambda_L$  is left invariant. To check this, consider

$$g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$h^{-1}g = \begin{bmatrix} \frac{1}{a'} & -\frac{b'}{a'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{a}{a'} & \frac{b-b'}{a'} \\ 0 & 1 \end{bmatrix};$$

then

$$\begin{aligned} \int_G f(h^{-1}g) d\lambda_L(g) &= \int_0^\infty \int_{-\infty}^\infty f\left(\frac{a}{a'}, \frac{b-b'}{a'}\right) \frac{da \wedge db}{a^2} \\ &= \int_0^\infty \int_{-\infty}^\infty f(s, t) \frac{d(a's) \wedge d(a't + b')}{(a's)(a's)} \\ &= \int_0^\infty \int_{-\infty}^\infty f(s, t) \frac{ds \wedge dt}{s^2} \end{aligned}$$

where we set

$$\begin{aligned} s &= \frac{a}{a'}, \quad da = a' ds \\ t &= \frac{b-b'}{a'}, \quad db = a' dt \end{aligned}$$

so that

$$\frac{da \wedge db}{a^2} = \frac{a'^2 ds \wedge dt}{a'^2 s^2} = \frac{ds \wedge dt}{s^2}.$$

For the right Haar measure, note that

$$\begin{aligned} \int f(gh^{-1}) (dg) g^{-1} &= \int f(g') d(g'h) (g'h)^{-1} \\ &= \int f(g') (dg') (hh^{-1}) g'^{-1} \\ &= \int f(g') (dg') g'^{-1}. \end{aligned}$$

Since

$$(dg) g^{-1} = \begin{bmatrix} da & db \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{da}{a} & -\frac{b da}{a} + db \\ 0 & 0 \end{bmatrix}$$

we then set

$$d\lambda_R(g) := d\lambda_R(a, b) = \frac{da \wedge db}{a}.$$

Check:

$$g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, h = \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix},$$

$$gh^{-1} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a'} & -\frac{b'}{a'} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{a}{a'} & -\frac{ab'}{a'} + b \\ 0 & 1 \end{bmatrix}$$

and so

$$\begin{aligned} \int_G f(gh^{-1}) d\lambda_R(g) &= \int_0^\infty \int_{-\infty}^\infty f\left(\frac{a}{a'}, -\frac{ab'}{a'} + b\right) \frac{da \wedge db}{a} \\ &= \int_0^\infty \int_{-\infty}^\infty f(s, t) \frac{a' ds \wedge dt}{a's} \\ &= \int_0^\infty \int_{-\infty}^\infty f(s, t) \frac{ds \wedge dt}{s} \end{aligned}$$

with a change of variable:

$$s = \frac{a}{a'}, da = a' ds$$

$$t = -\frac{ab'}{a'} + b, db = dt$$

$$\frac{da \wedge db}{a} = \frac{(a' ds) \wedge dt}{a's} = \frac{ds \wedge dt}{s}.$$

## 7.4 Induced Representations

Most of the Lie groups considered here fall in the following class:

Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$  (or  $\mathbb{C}$ ). We will consider the real case here, but the modifications needed for the complex case are straightforward.

Let  $q : V \times V \rightarrow \mathbb{R}$  be a non-degenerate bilinear form, such that  $v \rightarrow q(v, \cdot) \in V^*$  is 1-1. Let  $GL(V)$  be the general linear group for  $V$ , i.e., all invertible linear maps  $V \rightarrow V$ . (If a basis in  $V$  is chosen, this will be a matrix-group.)

**Lemma 7.19.** *Set*

$$G(q) = \{g \in GL(V) \mid q(gu, gv) = q(u, v), \forall u, v \in V\}. \quad (7.28)$$

*Then  $G(q)$  is a Lie group, and its Lie algebra consists of all linear mappings  $X : V \rightarrow V$  such that*

$$q(Xu, v) + q(u, Xv) = 0, \forall u, v \in V. \quad (7.29)$$

*Proof.* Fix a linear mapping  $X : V \rightarrow V$ , and set

$$g_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n = \exp(tX)$$

i.e., the matrix-exponential. Note that  $g_X(t)$  satisfies (7.28) for all  $t \in \mathbb{R}$  iff  $X$  satisfies (7.29). To see this, differentiate: i.e., compute

$$\frac{d}{dt}q(\exp(tX), \exp(tX)v)$$

using that  $q$  is assumed bilinear. □

We will address two questions:

1. How to induce a representation of a group  $G$  from a representation of the a subgroup  $\Gamma \subset G$ ?
2. Given a representation of a group  $G$ , how to test whether it is induced from a representation of a subgroup  $\Gamma \subset G$ ? (See [Mac88].)

The main examples we will study are the Lie groups of

- $ax + b$
- Heisenberg
- $SL_2(\mathbb{R})$
- Lorentz
- Poincaré

Among these, the  $ax+b$ , Heisenberg and Poincaré groups are semi-direct product groups. Their representations are induced from normal subgroups.

In more detail, the five groups in the list above are as follows:

The  $ax + b$  group is the group of  $2 \times 2$  matrices  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$  where  $a \in \mathbb{R}_+$ , and  $b \in \mathbb{R}$ .

The Heisenberg group is the group of upper triangular  $3 \times 3$  real matrices  $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ ,  $x, y, z \in \mathbb{R}$ .

The group  $SL_2(\mathbb{R})$  is the group of all  $2 \times 2$  matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  satisfying  $a, b, c, d \in \mathbb{R}$ , and  $ad - bc = 1$ .

The Lorentz-group is the group  $L = G(q)$  defined by (7.28) where  $V = \mathbb{R}^4$ , (space-times in physics) and

$$q(x_0, x_1, x_2, x_3) = -x_0^2 + x_1^2 + x_2^2 + x_3^2;$$

so  $q$  is the non-degenerate quadratic form with one minus sign, and three plus signs.

The Poincaré group  $P$  is the semi-direct product  $P = L \ltimes \mathbb{R}^4$ , where the group-product in  $P$  is as follows:

$$(g, v)(g', v') = (gg', v + gv')$$

for all  $g, g' \in L$ , and all  $v, v' \in \mathbb{R}^4$ .



**Exercise 7.20** (The Heisenberg group as a semidirect product). Consider the following two subgroups  $A$  and  $B$  in the Heisenberg group  $H$ :

$$A = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Verify that  $A$  and  $B$  are both Abelian subgroups under the matrix-multiplication of  $H$ .
2. Show that  $H$  becomes a semidirect product

$$H = A \ltimes B$$

where the action  $\alpha_x$  of  $A$ , as a group of automorphisms in  $B$ , is as follows:

$$\alpha_x(y, z) = (y, z + xy), \quad \forall x, y, z \in \mathbb{R}.$$

**Exercise 7.21** (The invariant complex vector fields from the Heisenberg group). In its complex form, the Heisenberg group takes the form  $\mathbb{C} \times \mathbb{R}$ ,  $z \in \mathbb{C}$ ,  $c \in \mathbb{R}$ ; and with group multiplication:

$$(z, c)(z', c') = (z + z', c + c' + 2\Im(\bar{z}z')) \quad (7.30)$$

Set  $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ , and  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ , or in abbreviated form

$$\partial = \partial_z = \frac{1}{2} (\partial_x - i\partial_y), \quad \text{and} \quad \bar{\partial} = \bar{\partial}_z = \frac{1}{2} (\partial_x + i\partial_y), \quad (7.31)$$

so

$$4(\partial_x^2 + \partial_y^2) = \partial \bar{\partial} = \bar{\partial} \partial. \quad (7.32)$$

Show that a basis for the left-invariant vector fields on  $H$  is as follows:

$$\partial_z - i\bar{z}\partial_c, \quad \bar{\partial}_z + iz\partial_c, \quad \text{and} \quad i\partial_c, \quad (7.33)$$

with commutator

$$[\partial_z - i\bar{z}\partial_c, \bar{\partial}_z + iz\partial_c] = 2i\partial_c. \quad (7.34)$$

### Representation Theory.

It is extremely easy to find representations of abelian subgroups. Unitary irreducible representation of abelian subgroups are one-dimensional, but the induced representation [Mac88] on an enlarged Hilbert space is infinite dimensional.

**Exercise 7.22** (The Campbell-Baker-Hausdorff formula). The exponential function is arguably the most important function in analysis. The Campbell-Baker-Hausdorff formula (below) illustrates the role of non-commutativity in this.

Let  $G$  and  $\mathfrak{g}$  be as above, and let  $\mathfrak{g} \xrightarrow{\exp} G$  be the exponential mapping. See Figure 7.1.

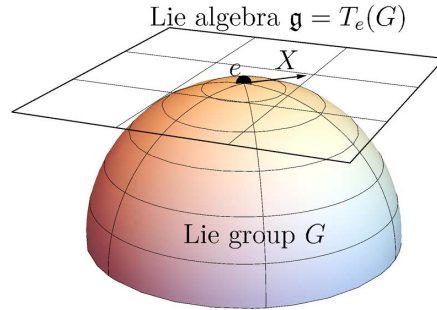


Figure 7.1:  $G$  and  $\mathfrak{g}$  (Lie algebra, Lie group, and exponential mapping).

1. Show that there is a convergent series with terms of degree  $> 1$  being iterated commutators  $Z(X, Y)$  with

$$\exp X \exp Y = \exp Z(X, Y), \text{ and} \quad (7.35)$$

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots \quad (7.36)$$

2. Use combinatorics and algebra in order to derive an algorithm for the terms “ $+\cdots$ ” in (7.36). This is the Baker–Campbell–Hausdorff formula; see, e.g., [HS68].
3. Show that

$$Z(X, Y) + Z(-X, -Y) = 0.$$

**Exercise 7.23** (The Lie algebra of a central extension). Let  $V$  be a vector space over  $\mathbb{R}$ , and  $B : V \times V \rightarrow \mathbb{R}$  a *cocycle*, i.e.,

$$B(u, v) + B(u + v, w) = B(u, v + w) + B(v, w), \text{ for } \forall u, v, w, \in V. \quad (7.37)$$

Let  $G_B = V \times \mathbb{R}$  be the corresponding Lie group:

$$(u, \alpha)(v, \beta) = (u + v, \alpha + \beta + B(u, v)), \text{ for } \forall \alpha, \beta \in \mathbb{R}, \forall u, v \in V. \quad (7.38)$$

1. Show that the Lie algebra  $La(G_B)$  of  $G_B$  is  $V \times \mathbb{R}$  itself with  $0 \times \mathbb{R} \subseteq$  the center; and with Lie bracket  $[\cdot, \cdot]$  given by

$$[u, v] = B(u, v) - B(v, u), \text{ for } \forall u, v \in V. \quad (7.39)$$

2. Show that the exponential mapping  $\exp_{G_B}$  is the trivial mapping

$$\exp_{G_B} \underbrace{(u, \alpha)}_{\in La(G_B)} = \underbrace{(u, \alpha)}_{\in G_B}, \text{ for } \forall u \in V, \alpha \in \mathbb{R}.$$

**Example 7.24** (The  $ax + b$  group ( $a > 0$ )).  $G = \{(a, b)\}$ , where  $(a, b) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ . The multiplication rule is given by

$$\begin{aligned} (a, b)(a', b') &= (aa', ab' + b) \\ (a, b)^{-1} &= \left(\frac{1}{a}, -\frac{b}{a}\right). \end{aligned}$$

$\Gamma = \{(1, b)\}$  is a one-dimensional abelian, normal subgroup of  $G$ . We check that

- abelian:  $(1, b)(1, c) = (1, c + b)$
- normal:  $(x, y)(1, b)(x, y)^{-1} = (1, xb)$ , note that this is also  $Ad_g$  acting on the normal subgroup  $\Gamma$
- The other subgroup  $\{(a, 0)\}$  is isomorphic to the multiplicative group  $(\mathbb{R}_+, \times)$ . Because we have

$$(a, 0)(a', 0) = (aa', 0)$$

by the group multiplication rule above.

- Notice that  $(\mathbb{R}_+, \times)$  is not a normal subgroup, since

$$(a, b)(x, 0)\left(\frac{1}{a}, -\frac{b}{a}\right) = (ax, b)\left(\frac{1}{a}, -\frac{b}{a}\right) = (x - bx + b).$$

$\Gamma$  is unimodular, hence it is just a copy of  $\mathbb{R}$ . Its invariant measure is the Lebesgue measure on  $\mathbb{R}$ .

The multiplicative group  $(\mathbb{R}_+, \times)$  acts on the additive group  $(\mathbb{R}, +)$  by

$$\begin{aligned} \varphi : (\mathbb{R}_+, \times) &\mapsto \text{Aut}((\mathbb{R}, +)) \\ \varphi_a(b) &= ab \end{aligned}$$

check:

$$\begin{aligned} (a, b)(a', b') &= (aa', b + \varphi_a(b')) = (aa', b + ab') \\ (a, b)^{-1} &= (a^{-1}, \varphi_{a^{-1}}(b^{-1})) = (a^{-1}, a^{-1}(-b)) = \left(\frac{1}{a}, -\frac{b}{a}\right) \\ (a, b)(1, x)(a, b^{-1}) &= (a, b + \varphi_a(x))(a, b^{-1}) \\ &= (a, b + ax)\left(\frac{1}{a}, -\frac{b}{a}\right) \\ &= (1, ax) = \varphi_a(x) \end{aligned}$$

**Example 7.25.** The Lie algebra of  $G$  is given by  $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

We check that

$$e^{tX} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$$

which is subgroup  $(\mathbb{R}_+, \times)$ ; and

$$e^{sY} = I + sY + 0 + \cdots + 0 = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

which is subgroup  $(\mathbb{R}, +)$ . We also have  $[X, Y] = Y$ .

**Example 7.26.** Form  $L^2(\mu_L)$  where  $\mu_L$  is the left Haar measure. Then  $\pi : g \rightarrow \pi(g)f(x) = f(g^{-1}x)$  is a unitary representation in  $L^2(\mu_L)$ . Specifically, if  $g = (a, b)$  then

$$f(g^{-1}x) = f\left(\frac{x}{a}, \frac{y-b}{a}\right).$$

Differentiate along the  $a$  direction we get

$$\begin{aligned} \tilde{X}f &= \frac{d}{da} \Big|_{a=1, b=0} f\left(\frac{x}{a}, \frac{y-b}{a}\right) = \left(-x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) f(x, y) \\ \tilde{Y}f &= \frac{d}{db} \Big|_{a=1, b=0} f\left(\frac{x}{a}, \frac{y-b}{a}\right) = -\frac{\partial}{\partial y} f(x, y) \end{aligned}$$

therefore we have the vector field

$$\begin{aligned} \tilde{X} &= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \\ \tilde{Y} &= -\frac{\partial}{\partial y} \end{aligned}$$

or equivalently we get the Lie algebra representation  $d\pi$  on  $L^2(\mu_L)$ . Notice that

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} \\ &= \left(-x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) \left(-\frac{\partial}{\partial y}\right) - \left(-\frac{\partial}{\partial y}\right) \left(-x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) \\ &= x \frac{\partial^2}{\partial x \partial y} + y \frac{\partial^2}{\partial y^2} - \left(x \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial y} + y \frac{\partial^2}{\partial y^2}\right) \\ &= -\frac{\partial}{\partial y} \\ &= \tilde{Y}. \end{aligned}$$

Notice that  $\tilde{X}$  and  $\tilde{Y}$  can be obtained by the exponential map as well.

$$\begin{aligned}
\tilde{X}f &= \left. \frac{d}{dt} \right|_{t=0} f(e^{-tX}x) \\
&= \left. \frac{d}{dt} \right|_{t=0} f((e^{-t}, 1)(x, y)) \\
&= \left. \frac{d}{dt} \right|_{t=0} f(e^{-t}x, e^{-t}y + 1) \\
&= \left( -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) f(x, y) \\
\tilde{Y}f &= \left. \frac{d}{dt} \right|_{t=0} f(e^{-tY}x) \\
&= \left. \frac{d}{dt} \right|_{t=0} f((1, -t)(x, y)) \\
&= \left. \frac{d}{dt} \right|_{t=0} f(x, y - t) \\
&= -\frac{\partial}{\partial y} f(x, y)
\end{aligned}$$

**Example 7.27.** We may parametrize the Lie algebra of the  $ax + b$  group using  $(x, y)$  variables. Build the Hilbert space  $L^2(\mu_L)$ . The unitary representation  $\pi(g)f(\sigma) = f(g^{-1}\sigma)$  induces the follows representations of the Lie algebra

$$\begin{aligned}
d\pi(\tilde{s})f(\sigma) &= \left. \frac{d}{ds} \right|_{s=0} f(e^{-sX}\sigma) = \tilde{X}f(\sigma) \\
d\pi(\tilde{t})f(\sigma) &= \left. \frac{d}{dy} \right|_{t=0} f(e^{-tY}\sigma) = \tilde{Y}f(\sigma).
\end{aligned}$$

Hence in the parameter space  $(s, t) \in \mathbb{R}^2$  we have two usual derivative operators  $\partial/\partial s$  and  $\partial/\partial t$ , where on the manifold we have

$$\begin{aligned}
\frac{\partial}{\partial s} &= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \\
\frac{\partial}{\partial t} &= -\frac{\partial}{\partial y}
\end{aligned}$$

The usual positive Laplacian on  $\mathbb{R}^2$  translates to

$$\begin{aligned}
-\Delta &= \left( \frac{\partial}{\partial s} \right)^2 + \left( \frac{\partial}{\partial t} \right)^2 \\
&= (\tilde{X})^2 + (\tilde{Y})^2 \\
&= \left( -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) \left( -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) + \left( -\frac{\partial}{\partial y} \right)^2 \\
&= x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + (y^2 + 1) \frac{\partial^2}{\partial y^2} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},
\end{aligned}$$

where we used  $(x \frac{\partial}{\partial x})^2 = x^2 (\frac{\partial}{\partial x})^2 + x \frac{\partial}{\partial x}$ . This is in fact an elliptic operator, since the matrix

$$\begin{bmatrix} x^2 & xy \\ xy & y^2 + 1 \end{bmatrix}$$

has trace  $trace = x^2 + y^2 + 1 \geq 1$ , and  $\det = x^2 \geq 0$ . If instead we have “ $y^2$ ” then the determinant is the constant zero.

The term “ $y^2 + 1$ ” is essential for  $\Delta$  being elliptic. Also note that all the coefficients are analytic functions in the  $(x, y)$  variables.

**Example 7.28.** Heisenberg group  $G = \{a, b, c\}$  where

$$(a, b, c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

The multiplication rule is given by

$$\begin{aligned} (a, b, c)(a', b', c') &= (a + a', b + b', c + ab' + c') \\ (a, b, c)^{-1} &= (-a, -b, -c + ab) \end{aligned}$$

The subgroup  $\Gamma = \{(0, b, c)\}$  where

$$(0, b, c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

is two dimensional, abelian and normal.

- abelian:  $(0, b, c)(0, b', c') = (0, b + b', c + c')$
- normal:

$$\begin{aligned} (a, b, c)(0, x, y)(a, b, c)^{-1} &= (a, b, c)(0, x, y)(-a, -b, -c + ab) \\ &= (a, b + x, c + y + ax)(-a, -b, -c + ab) \\ &= (0, x, y + ax + ab - ab) \\ &= (0, x, ax + y) \end{aligned}$$

Note that this is also  $Ad_g$  acting on the Lie algebra of  $\Gamma$ .

The additive group  $(\mathbb{R}, +)$  acts on  $\Gamma = \{(0, b, c)\} \simeq (\mathbb{R}^2, +)$  by

$$\begin{aligned} \varphi : (\mathbb{R}, +) &\rightarrow Aut(\Gamma) \\ \varphi(a) \begin{bmatrix} c \\ b \end{bmatrix} &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \end{bmatrix} \\ &= \begin{bmatrix} c + ab \\ b \end{bmatrix} \end{aligned}$$

check:

$$\begin{aligned}
 (a, (b, c))(a', (b', c')) &= (a + a', (b, c) + \varphi(a)(b', c')) \\
 &= (a + a', (b, c) + (b', c' + ab')) \\
 &= (a + a', b + b', c + c' + ab) \\
 (a, (b, c))^{-1} &= (-a, \varphi_{a^{-1}}(-b, -c)) \\
 &= (-a, (-b, -c + ab)) \\
 &= (-a, -b, -c + ab) \\
 (a, b, c)(0, b', c')(a, b, c)^{-1} &= (a, b + b', c + c' + ab')(-a, -b, -c + ab) \\
 &= (0, b', c' + ab') \\
 &= \varphi_a \begin{bmatrix} c' \\ b' \end{bmatrix}
 \end{aligned}$$

**Exercise 7.29** (Weyl's commutation relation). Let  $\mathcal{H}$  be a Hilbert space, and let  $P, Q$  be a pair of selfadjoint operators such that

$$e^{isP} e^{itQ} = e^{ist} e^{itQ} e^{isP} \quad (7.40)$$

holds for all  $s, t \in \mathbb{R}$ ; then prove that  $P$  and  $Q$  have a common dense invariant domain  $\mathcal{D}$  such that  $P = P|_{\mathcal{D}}$ ,  $Q = Q|_{\mathcal{D}}$ ; and

$$[P, Q] = -iI \quad (7.41)$$

holds on  $\mathcal{D}$ ; more precisely,

$$PQ\varphi - QP\varphi = -i\varphi, \quad (7.42)$$

holds for all  $\varphi \in \mathcal{D}$ .

Hint. One way of proving this is to show that when  $P$  and  $Q$  satisfy (7.40), then we automatically get a unitary representation of the Heisenberg group (see Sections 7.1, 7.4, and Example 7.28), and for  $\mathcal{D}$  we can take the corresponding Gårding Space (Section 7.7). However there is also a direct proof from first principles.

Caution. The converse implication does *not* hold: There are selfadjoint solutions to (7.42) which do not have a counterpart relation (7.40); see [JM84, KL14a, JM80]. (Eq (7.40) is called the Weyl relation.)

## Induced Representations

This also goes under the name of “Mackey machine” [Mac88]. Its modern formulation is in the context of completely positive map.

Let  $G$  be a locally compact group, and  $\Gamma \subset G$  be a closed subgroup. Let  $dx$  (resp.  $d\xi$ ) be the right Haar measure on  $G$  (resp.  $\Gamma$ ), and  $\Delta$  (resp.  $\delta$ ) be the corresponding modular function. Recall the modular function comes in when the translation is put on the wrong side, i.e.,

$$\int_G f(g^{-1}x) dx = \Delta(g^{-1}) \int_G f(x) dx$$

or equivalently,

$$\Delta(g) \int_G f(g^{-1}x) dx = \int_G f(x) dx.$$

Form the quotient  $M = \Gamma \backslash G$  space, and let  $\pi : G \rightarrow \Gamma \backslash G$  be the quotient map (the covering map).  $M$  carries a transitive  $G$  action.

group	right Haar measure	modular function
$G$	$dg$	$\Delta$
$\Gamma$	$d\xi$	$\delta$

*Note 7.30.*  $M$  is called a *fundamental domain* or a *homogeneous space*.  $M$  is a group if and only if  $\Gamma$  is a normal subgroup in  $G$ . In general,  $M$  may not be a group, but it is still a very important manifold.

*Note 7.31.*  $\mu$  is an *invariant measure* on  $M$ , if  $\mu(Eg) = \mu(E), \forall g \in G$ .  $\mu$  is *quasi-invariant*, if  $\mu(E) = 0 \Leftrightarrow \mu(Eg) = 0, \forall g$ . In general there is no invariant measures on  $M$ , but only quasi-invariant measures.

$G$  has an invariant measure if and only if  $G$  is unimodular (e.g. Heisenberg group.) Not all groups are unimodular. A typical example is the  $ax + b$  group.

Define  $\tau : C_c(G) \rightarrow C_c(M)$  by

$$(\tau\varphi)(\pi(x)) = \int_{\Gamma} \varphi(\xi x) d\xi. \tag{7.43}$$

*Note 7.32.* Since  $\varphi$  has compact support, the integral in (7.43) is well-defined.  $\tau$  is called *conditional expectation*. It is the summation of  $\varphi$  over the orbit  $\Gamma x$ . Indeed, for fixed  $x$ , if  $\xi$  runs over  $\Gamma$  then  $\xi x$  runs over  $\Gamma x$ . We may also say  $\tau\varphi$  is a  $\Gamma$ -periodic extension, by looking at it as a function defined on  $G$ . For if  $\xi_1 \in \Gamma$ , we have

$$(\tau\varphi)(\xi_1 x) = \int_{\Gamma} \varphi(\xi \xi_1 x) d\xi = \tau\varphi(x)$$

using the fact that  $d\xi$  right-invariant. Thus  $\tau\varphi$ , viewed as a function on  $G$ , is  $\Gamma$ -periodic, i.e.,  $(\tau\varphi)(\xi x) = (\tau\varphi)(x), \forall \xi \in \Gamma$ .

**Lemma 7.33.**  $\tau$  is surjective.

*Proof.* Suppose  $f$  is  $\Gamma$ -periodic, choose  $\psi \in C_c(G)$  such that  $\tau\psi \equiv 1$ . Then  $\psi f \in C_c(G)$ , and

$$\begin{aligned} (\tau(\psi f))(x) &= \int_{\Gamma} \psi(\xi x) f(\xi x) d\xi \\ &= \int_{\Gamma} \psi(\xi x) f(x) d\xi \\ &= f(x) \int_{\Gamma} \psi(\xi x) d\xi \\ &= f(x) (\tau\psi)(x) = f(x). \end{aligned}$$

□



**Example 7.34.** For  $G = \mathbb{R}$ ,  $\Gamma = \mathbb{Z}$ ,  $d\xi =$  counting measure on  $\mathbb{Z}$ , we have

$$(\tau\varphi)(\pi(x)) = \int_{\Gamma} \varphi(\xi x) d\xi = \sum_{n \in \mathbb{Z}} \varphi(n+x), \quad \forall \varphi \in C_c(\mathbb{R}).$$

Since  $\varphi$  has compact support,  $\varphi(n+x)$  vanishes for all but a finite number of  $n$ . Hence  $\tau\varphi$  contains a finite summation, and so it is well-defined. Moreover, for all  $n_0 \in \mathbb{Z}$ , it follows that

$$(\tau\varphi)(n_0+x) = \sum_{n \in \mathbb{Z}} \varphi(n_0+n+x) = \sum_{n \in \mathbb{Z}} \varphi(n+x) = (\tau\varphi)(x).$$

Hence  $\tau\varphi$  is translation invariant by integers, i.e.,  $\tau\varphi$  (as a function on  $\mathbb{R}$ ) is  $\mathbb{Z}$ -periodic.

Let  $L : \Gamma \rightarrow V$  be a unitary representation of  $\Gamma$  on a Hilbert space  $V$ . We now construct a unitary representation  $ind_{\Gamma}^G : G \rightarrow \mathcal{H}$  of  $G$  on an enlarged Hilbert space  $\mathcal{H}$ .

Let  $F_*$  be the set of function  $f : G \rightarrow V$  so that

$$f(\xi g) = \rho(\xi)^{1/2} L_{\xi} f(g), \quad \forall \xi \in \Gamma \quad (7.44)$$

where  $\rho = \frac{\delta}{\Delta}$ , i.e.,  $\rho(\xi) = \frac{\delta(\xi)}{\Delta(\xi)}$ ,  $\forall \xi \in \Gamma$ . For all  $f \in F_*$ , let

$$(R_g f)(\cdot) := f(\cdot g)$$

be the right-translation of  $f$  by  $g \in G$ .

**Lemma 7.35.**  $R_g f \in F_*$ . That is,  $F_*$  is invariant under right-translation by  $g \in G$ .

*Proof.* To see this, let  $f \in F_*$ ,  $\xi \in \Gamma$ , then

$$(R_g f)(\xi x) = f(\xi x g) = \rho(\xi)^{1/2} L_{\xi} f(xg) = \rho(\xi)^{1/2} L_{\xi}(R_g f)(x)$$

so that  $R_g f \in F_*$ . □

*Note 7.36.* We will defined an inner product on  $F_*$  so that  $\|f(\xi \cdot)\|_{new} = \|f(\cdot)\|_{new}$ ,  $\forall \xi \in \Gamma$ . Eventually, we will define the induced representation  $U^{ind} := ind_{\Gamma}^G(L)$  by

$$(U_g^{ind} f)(\cdot) := (R_g f)(\cdot)$$

not on  $F_*$ , but pass to a quotient space. The factor  $\rho(\xi)^{1/2}$  comes in as we are going to construct a quasi-invariant measure on  $\Gamma \backslash G$ .

To construct  $F_*$ , let  $\varphi \in C_c(G)$ , and set

$$f(g) := \int_{\Gamma} \rho^{1/2}(\xi^{-1}) L(\xi^{-1}) \varphi(\xi g) d\xi.$$

Now if  $\xi_1 \in \Gamma$  then

$$\begin{aligned}
 f(\xi_1 g) &= \int_{\Gamma} \rho^{1/2}(\xi^{-1}) L(\xi^{-1}) f(\xi \xi_1 g) d\xi \\
 &= \int_{\Gamma} \rho^{1/2}((\xi \xi_1^{-1})^{-1}) L((\xi \xi_1^{-1})^{-1}) f(\xi g) d\xi \\
 &= \rho^{1/2}(\xi_1) L(\xi_1) \int_{\Gamma} \rho^{1/2}(\xi^{-1}) L(\xi^{-1}) f(\xi g) d\xi \\
 &= \rho^{1/2}(\xi_1) L(\xi_1) f(g).
 \end{aligned}$$

The proof of Lemma 7.33 shows that all functions in  $F_*$  are obtained this way.

*Note 7.37.* Let's ignore the factor  $\rho(\xi)^{1/2}$  for a moment.  $L_\xi$  is unitary implies that for all  $f \in F_*$ ,

$$\|f(\xi \cdot)\|_V = \|L_\xi f(\cdot)\|_V = \|f(\cdot)\|_V, \quad \forall \xi \in \Gamma.$$

Since Hilbert spaces exist up to unitary equivalence,  $L_\xi f(g)$  and  $f(g)$  really are the same function. As  $\xi$  running through  $\Gamma$ ,  $\xi g$  running through  $\Gamma g$ . Thus  $\|f(\xi g)\|$  is a constant on the orbit  $\Gamma g$ . It follows that  $f(\xi g)$  is in fact a  $V$ -valued function defined on the quotient  $M = \Gamma \backslash G$  (i.e., quasi- $\Gamma$ -periodic). We will later use these functions as multiplication operators.

**Example 7.38.** The Heisenberg group is unimodular, so  $\rho \equiv 1$ .

**Example 7.39.** For the  $ax + b$  group,

$$\begin{aligned}
 d\lambda_R &= \frac{dad b}{a} \\
 d\lambda_L &= \frac{dad b}{a^2} \\
 \Delta &= \frac{d\lambda_L}{d\lambda_R} = \frac{1}{a}
 \end{aligned}$$

On the abelian normal subgroup  $\Gamma = \{(1, b)\}$ , we have  $a = 1$  and  $\Delta(\xi) = 1$ .  $\Gamma$  is unimodular,  $\delta(\xi) = 1$ . Therefore,  $\rho(\xi) = 1, \forall \xi \in \Gamma$ .

For all  $f \in F_*$ , the map  $\mu_{f,f} : C_c(M) \rightarrow \mathbb{C}$  given by

$$\mu_{f,f} : \tau\varphi \mapsto \int_G \|f(g)\|_V^2 \varphi(g) dg, \quad \varphi \in C_c(G) \quad (7.45)$$

is a positive linear functional. By Riesz's theorem, there exists a unique Radon measure  $\mu_{f,f}$  on  $M$ , such that

$$\int_G \|f(g)\|_V^2 \varphi(g) dg = \int_M (\tau\varphi) d\mu_{f,f}.$$

**Lemma 7.40.** (7.45) is a well-defined positive linear functional.

*Proof.* Suppose  $\varphi \in C_c(G)$  such that  $\tau\varphi \equiv 0$  on  $M$ . It remains to verify that  $\mu_{f,f}(\tau\varphi) = 0$ , see (7.45). For this, we choose  $\psi \in C_c(G)$  such that  $\tau\psi \equiv 1$  on  $M$ , and so

$$\begin{aligned}
 \int_G \|f(g)\|_V^2 \varphi(g) dg &= \int_G \|f(g)\|_V^2 (\tau\psi)(\pi(g)) \varphi(g) dg \\
 &= \int_G \|f(g)\|_V^2 \varphi(g) \left( \int_\Gamma \psi(\xi g) d\xi \right) dg \\
 &\stackrel{\text{Fubini}}{=} \int_\Gamma \left( \int_G \|f(g)\|_V^2 \varphi(g) \psi(\xi g) dg \right) d\xi \\
 &= \int_\Gamma \left( \int_G \|f(\xi^{-1}g)\|_V^2 \varphi(\xi^{-1}g) \psi(g) \Delta(\xi) dg \right) d\xi \\
 &\stackrel{\text{Fubini}}{=} \int_G \psi(g) \left( \int_\Gamma \|f(\xi^{-1}g)\|_V^2 \varphi(\xi^{-1}g) \Delta(\xi) d\xi \right) dg \\
 &= \int_G \psi(g) \left( \int_\Gamma \|f(\xi g)\|_V^2 \varphi(\xi g) \Delta(\xi^{-1}) \delta(\xi) d\xi \right) dg \\
 &\stackrel{(7.44)}{=} \int_G \psi(g) \|f(g)\|_V^2 \left( \int_\Gamma \varphi(\xi g) d\xi \right) dg \\
 &= \int_G \psi(g) \|f(g)\|_V^2 \underbrace{(\tau\varphi)(\pi(g))}_{\equiv 0} dg = 0.
 \end{aligned}$$

□

*Note 7.41.* Recall that given a measure space  $(X, \mathfrak{M}, \mu)$ , let  $f : X \rightarrow Y$ . Define a linear functional  $\Lambda : C_c(Y) \rightarrow \mathbb{C}$  by

$$\Lambda\varphi := \int \varphi(f(x)) d\mu(x)$$

$\Lambda$  is positive, hence by Riesz's theorem, there exists a unique regular Borel measure  $\mu_f$  on  $Y$  so that

$$\Lambda\varphi = \int_Y \varphi d\mu_f = \int_X \varphi(f(x)) d\mu(x).$$

It follows that  $\mu_f = \mu \circ f^{-1}$ .

*Note 7.42.* Under current setting, we have a covering map  $\pi : G \rightarrow \Gamma \backslash G =: M$ , and the right Haar measure  $\mu$  on  $G$ . Thus we may define a measure  $\mu \circ \pi^{-1}$ . However, given  $\varphi \in C_c(M)$ ,  $\varphi(\pi(x))$  may not have compact support, or equivalently,  $\pi^{-1}(E)$  is  $\Gamma$  periodic. For example, take  $G = \mathbb{R}$ ,  $\Gamma = \mathbb{Z}$ ,  $M = \mathbb{Z} \backslash \mathbb{R}$ . Then  $\pi^{-1}([0, 1/2))$  is  $\mathbb{Z}$ -periodic, which has infinite Lebesgue measure. What we really need is some map so that the inverse of a subset of  $M$  is restricted to a single  $\Gamma$  period. This is essentially what  $\tau$  does: from  $\tau\varphi \in C_c(M)$ , get the inverse image  $\varphi \in C_c(G)$ . Even if  $\varphi$  is not restricted to a single  $\Gamma$  period,  $\varphi$  always has compact support.

Hence we get a family of measures indexed by elements in  $F_*$ . If choosing  $f, g \in F_*$  then we get complex measures  $\mu_{f,g}$  (using polarization identity.)

- Define  $\|f\|^2 := \mu_{f,f}(M)$ ,  $\langle f, g \rangle := \mu_{f,g}(M)$ .
- Complete  $F_*$  with respect to this norm to get an enlarged Hilbert space  $\mathcal{H}$ .
- Define the induced representation  $U^{ind} := \text{ind}_\Gamma^G(L)$  on  $\mathcal{H}$  as

$$(U_g^{ind} f)(x) = f(xg)$$

$U^{ind}$  is unitary, in particular,

$$\|U_g^{ind} f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}, \quad \forall g \in G.$$

*Note 7.43.*  $\mu_{f,g}(M) = \int_M (\tau\varphi)(\xi) d\xi$  with  $\tau\varphi \equiv 1$ . What is  $\varphi$  then? It turns out that  $\varphi$  could be constant 1 over a single  $\Gamma$ -period, or equivalently,  $\varphi$  could spread out to a finite number of  $\Gamma$ -periods. In the former case,

$$\begin{aligned} \|f\|^2 &= \int_G \|f(g)\|_V^2 \varphi(g) dg \\ &= \int_{1\text{-period}} \|f(g)\|_V^2 \varphi(g) dg \\ &= \int_{1\text{-period}} \|f(g)\|_V^2 dg \\ &= \int_M \|f(g)\|_V^2 dg. \end{aligned}$$

Define  $P(\psi)f(x) := \psi(\pi(x))f(x)$ , for  $\psi \in C_c(M)$ ,  $f \in \mathcal{H}$ ,  $x \in G$ . Note  $\{P(\psi) \mid \psi \in C_c(M)\}$  is the abelian algebra of multiplication operators.

**Lemma 7.44.** *We have*

$$U_g^{ind} P(\psi) U_{g^{-1}}^{ind} = P(\psi(\cdot g)).$$

*Proof.* One checks that

$$\begin{aligned} U_g^{ind} P(\psi) f(x) &= U_g^{ind} \psi(\pi(x)) f(x) \\ &= \psi(\pi(xg)) f(xg) \\ P(\psi(\cdot g)) U_g^{ind} f(x) &= P(\psi(\cdot g)) f(xg) \\ &= \psi(\pi(xg)) f(xg). \end{aligned}$$

□

Conversely, how to recognize induced representations? Answer:

**Theorem 7.45** (Imprimitivity [Ørs79]). *Let  $G$  be a locally compact group with a closed subgroup  $\Gamma$ . Let  $M = \Gamma \backslash G$ . Suppose the system  $(U, P)$  satisfies the covariance relation*

$$U_g P(\psi) U_{g^{-1}} = P(\psi(\cdot g)),$$

*and  $P(\cdot)$  is non-degenerate. Then, there exists a unitary representation  $L \in \text{Rep}(\Gamma, V)$  such that  $U \cong \text{ind}_{\Gamma}^G(L)$ .*

*Remark 7.46.*  $P(\cdot)$  is non-degenerate if  $P(C_c(M))\mathcal{H} = \{P(\psi)a : \psi \in C_c(M), a \in \mathcal{H}\}$  is dense in  $\mathcal{H}$ .

## 7.5 Example - Heisenberg group

Let  $G = \{(a, b, c)\}$  be the *Heisenberg group*, where

$$(a, b, c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

The multiplication rule is given by

$$\begin{aligned} (a, b, c)(a', b', c') &= (a + a', b + b', c + c' + ab') \\ (a, b, c)^{-1} &= (-a, -b, -c + ab) \end{aligned}$$

The subgroup  $\Gamma = \{(0, b, c)\}$  where

$$(1, b, c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

is two dimensional, abelian and normal.

- abelian:  $(0, b, c)(0, b', c') = (0, b + b', c + c')$
- normal:

$$\begin{aligned} (a, b, c)(0, x, y)(a, b, c)^{-1} &= (a, b, c)(0, x, y)(-a, -b, -c + ab) \\ &= (a, b + x, c + y + ax)(-a, -b, -c + ab) \\ &= (0, x, y + ax + ab - ab) \\ &= (0, x, ax + y) \end{aligned}$$

i.e.,  $Ad : G \rightarrow GL(\mathfrak{n})$ , as

$$\begin{aligned} Ad_g(n) &:= gng^{-1} \\ (x, y) &\mapsto (x, ax + y) \end{aligned}$$

the orbit is a 2-d transformation.

Fix  $h \in \mathbb{R} \setminus \{0\}$ . Recall the *Schrödinger representation* of  $G$  on  $L^2(\mathbb{R})$

$$U_g f(x) = e^{ih(c+bx)} f(x+a) \tag{7.46}$$

**Theorem 7.47.** *The Schrödinger representation is induced.*

*Proof.* We show that the Schrödinger representation is induced from a unitary representation  $L$  on the subgroup  $\Gamma$ .

Note the Heisenberg group is a non abelian unimodular Lie group ( $\Delta = 1$ ,  $\delta = 1$ , and so  $\rho \equiv 1$ .), The Haar measure on  $G$  is just the product measure  $dx dy dz$  on  $\mathbb{R}^3$ . Conditional expectation becomes integrating out the variables correspond to the subgroup.

1. Let  $L \in \text{Rep}(\Gamma, V)$  where  $\Gamma = \{(0, b, c)\}$ ,  $V = \mathbb{C}$ ,

$$L_{\xi(b,c)} = e^{ihc}.$$

The complex exponential comes in since we want a unitary representation. The subgroup  $\{(0, 0, c)\}$  is the center of  $G$ . (What is the induced representation? Is it unitarily equivalent to the Schrödinger representation?)

2. Look for the family  $F_*$  of functions  $f : G \rightarrow \mathbb{C}$  ( $V$  is the 1-d Hilbert space  $\mathbb{C}$ ), such that

$$f(\xi(b, c)g) = L_\xi f(g).$$

Since

$$f(\xi(b, c)g) = f((0, b, c)(x, y, z)) = f(x, b+y, c+z), \quad \text{and}$$

$$L_{\xi(b,c)} f(g) = e^{ihc} f(x, y, z)$$

so  $f$  satisfies

$$f(x, b+y, c+z) = e^{ihc} f(x, y, z).$$

That is, we may translate the  $y, z$  variables by arbitrary amount, and the only price to pay is the multiplicative factor  $e^{ihc}$ . Therefore  $f$  is really a function defined on the quotient

$$M = \Gamma \backslash G \simeq \mathbb{R}.$$

The homogeneous space  $M = \{(x, 0, 0)\}$  is identified with  $\mathbb{R}$ , and the invariant measure on  $M$  is simply the Lebesgue measure. It is almost clear at this point why the induced representation is unitarily equivalent to the Schrödinger representation on  $L^2(\mathbb{R})$ .

3. The positive linear functional  $\tau\varphi \mapsto \int_G \|f(g)\|_V^2 \varphi(g) dg$  induces a measure  $\mu_{f,f}$  on  $M$ . This can be seen as follows:

$$\begin{aligned} \int_G \|f(g)\|_V^2 \varphi(g) dg &= \int_{G \simeq \mathbb{R}^3} |f(x, y, z)|^2 \varphi(x, y, z) dx dy dz \\ &= \int_{M \simeq \mathbb{R}} \left( \int_{\Gamma \simeq \mathbb{R}^2} |f(x, y, z)|^2 \varphi(x, y, z) dy dz \right) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{R}} |f(x, y, z)|^2 \left( \int_{\mathbb{R}^2} \varphi(x, y, z) dydz \right) dx \\
 &= \int_{\mathbb{R}} |f(x, y, z)|^2 (\tau\varphi)(\pi(g)) dx \\
 &= \int_{\mathbb{R}} |f(x, 0, 0)|^2 (\tau\varphi)(x) dx.
 \end{aligned}$$

Note that

$$\begin{aligned}
 (\tau\varphi)(\pi(g)) &= \int_{\Gamma} \varphi(\xi g) d\xi \\
 &= \int_{\mathbb{R}^2} \varphi((0, b, c)(x, y, z)) dbdc \\
 &= \int_{\mathbb{R}^2} \varphi(x, b+y, c+z) dbdc \\
 &= \int_{\mathbb{R}^2} \varphi(x, b, c) dbdc \\
 &= (\tau\varphi)(x), \quad M = \Gamma \backslash G \simeq \mathbb{R}.
 \end{aligned}$$

Hence  $\Lambda : C_c(M) \rightarrow \mathbb{C}$  given by

$$\Lambda : \tau\varphi \mapsto \int_G \|f(g)\|_V^2 \varphi(g) dg$$

is a positive linear functional, therefore  $\Lambda = \mu_{f,f}$  and

$$\int_{\mathbb{R}^3} |f(x, y, z)|^2 \varphi(x, y, z) dx dy dz = \int_{\mathbb{R}} (\tau\varphi)(x) d\mu_{f,f}(x).$$

4. Define

$$\|f\|_{ind}^2 := \mu_{f,f}(M) = \int_M |f|^2 d\xi = \int_{\mathbb{R}} |f(x, y, z)|^2 dx = \int_{\mathbb{R}} |f(x, 0, 0)|^2 dx$$

$$U_g^{ind} f(g') := f(g'g)$$

By definition, if  $g = g(a, b, c)$  and  $g' = g'(x, y, z)$  then

$$\begin{aligned}
 U_g^{ind} f(g') &= f(g'g) \\
 &= f((x, y, z)(a, b, c)) \\
 &= f(x+a, y+b, z+c)
 \end{aligned}$$

and  $U^{ind}$  is a unitary representation.

5. To see that  $U^{ind}$  is unitarily equivalent to the Schrödinger representation on  $L^2(\mathbb{R})$ , we set

$$W : \mathcal{H}^{ind} \rightarrow L^2(\mathbb{R}), \quad (Wf)(x) = f(x, 0, 0)$$

(If put other numbers into  $f$ , as  $f(x, y, z)$ , the result is the same, since  $f \in \mathcal{H}^{ind}$  is really defined on the quotient  $M = \Gamma \backslash G \simeq \mathbb{R}$ .)

$W$  is unitary:

$$\|Wf\|_{L^2}^2 = \int_{\mathbb{R}} |Wf|^2 dx = \int_{\mathbb{R}} |f(x, 0, 0)|^2 dx = \int_{\Gamma \backslash G} |f|^2 d\xi = \|f\|_{ind}^2$$

The intertwining property: Let  $U_g$  be the Schrödinger representation, then

$$\begin{aligned} U_g(Wf) &= e^{ih(c+bx)} f(x+a, 0, 0) \\ WU_g^{ind} f &= W(f((x, y, z)(a, b, c))) \\ &= W(f(x+a, y+b, z+c+xb)) \\ &= W(e^{ih(c+bx)} f(x+a, y, z)) \\ &= e^{ih(c+bx)} f(x+a, 0, 0). \end{aligned}$$

6. Since  $\{U, L\}' \subset \{L\}'$ , the system  $\{U, L\}$  is reducible implies  $L$  is reducible. Equivalent,  $\{L\}$  is irreducible implies  $\{U, L\}$  is irreducible. Since  $L$  is 1-dimensional, it is irreducible. Consequently,  $U^{ind}$  is irreducible.  $\square$

**Exercise 7.48** (The Schrödinger representation). Prove that for  $h \neq 0$  fixed, the Schrödinger representation  $U^h$  (7.46) is irreducible.

Hint: Show that if  $A \in \mathcal{B}(L^2(\mathbb{R}))$  commutes with  $\{U_g^h : g \in G_{\text{Heis}}\}$ , then there exists  $\lambda \in \mathbb{C}$  such that  $A = \lambda I_{L^2(\mathbb{R})}$ , i.e., that the commutant of the representation  $U^h$  is one-dimensional.

### $ax + b$ group

$$a \in \mathbb{R}_+, b \in \mathbb{R}, g = (a, b) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

$$U_g f(x) = e^{iax} f(x+b)$$

could also write  $a = e^t$ , then

$$\begin{aligned} U_g f(x) &= e^{ie^t x} f(x+b) \\ U_{g(a,b)} f(x) &= e^{ia e^x} f(x+b) \end{aligned}$$

$$\begin{aligned} \left[ \frac{d}{dx}, ie^x \right] &= ie^x \\ [A, B] &= B \end{aligned}$$

or

$$\begin{aligned} U_{(e^t, b)} f &= e^{ite^x} f(x+b) \\ &\begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1-d representation.  $L_b = e^{ib}$ . Induce  $ind_L^G \simeq$  the Schrödinger representation.



## 7.6 Co-adjoint Orbits

It turns out that only a small family of representations are induced. The question is how to detect whether a representation is induced. The whole theory is also under the name of “Mackey machine” [Mac52, Mac88]. The notion of “machine” refers to something that one can actually compute in practice. Two main examples are the Heisenberg group and the  $ax + b$  group.

What is the mysterious parameter  $\hbar$  that comes into the Schrödinger representation? It is a physical constant, but how to explain it in mathematical theory?

### Review of some Lie theory

**Theorem 7.49** (Ado). *Every Lie group is diffeomorphic to a matrix group.*

The exponential function  $\exp$  maps a neighborhood of 0 into a connected component of  $G$  containing the identity element. For example, the Lie algebra of the Heisenberg group is

$$\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$$

All the Lie groups that we will ever encounter come from a quadratic form. Given a quadratic form

$$\varphi : V \times V \rightarrow \mathbb{C}$$

there is an associated group that fixes  $\varphi$ , i.e. we consider elements  $g$  such that

$$\varphi(gx, gy) = \varphi(x, y)$$

and define  $G(\varphi)$  as the collection of these elements.  $G(\varphi)$  is clearly a group. Apply the exponential map and the product rule,

$$\left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tX}x, e^{tX}y) = 0 \iff \varphi(Xx, y) + \varphi(x, Xy) = 0$$

hence

$$X + X^{tr} = 0$$

The determinant and trace are related so that

$$\det(e^{tX}) = e^{t \cdot \text{trace}(X)}$$

thus  $\det = 1$  if and only if  $\text{trace} = 0$ . It is often stated in differential geometry that the derivative of the determinant is equal to the trace.

**Example 7.50.**  $\mathbb{R}^n$ ,  $\varphi(x, y) = \sum x_i y_i$ . The associated group is the orthogonal group  $O_n$ .

There is a famous cute little trick to make  $O_{n-1}$  into a subgroup of  $O_n$ .  $O_{n-1}$  is not normal in  $O_n$ . We may split the quadratic form into

$$\sum_{i=1}^{n-1} x_i^2 + 1$$

where 1 corresponds to the last coordinate in  $O_n$ . Then we may identify  $O_{n-1}$  as a subgroup of  $O_n$

$$g \mapsto \begin{bmatrix} g & 0 \\ 0 & I \end{bmatrix}$$

where  $I$  is the identity operator.

Claim:  $O_n/O_{n-1} \simeq S^{n-1}$ . How to see this? Let  $u$  be the unit vector corresponding to the last dimension, look for  $g$  that fixes  $u$  i.e.  $gu = u$ . Such  $g$  forms a subgroup of  $O_n$ , and it is called isotropy group.

$$I_n = \{g : gu = u\} \simeq O_{n-1}$$

Notice that for all  $v \in S^{n-1}$ , there exists  $g \in O_n$  such that  $gu = v$ . Hence

$$g \mapsto gu$$

is onto  $S^{n-1}$ . The kernel of this map is  $I_n \simeq O_{n-1}$ , thus

$$O_n/O_{n-1} \simeq S^{n-1}$$

Such spaces are called homogeneous spaces.

**Example 7.51.** visualize this with  $O_3$  and  $O_2$ .

Other examples of homogeneous spaces show up in number theory all the time. For example, the Poincaré group  $G$ /discrete subgroup.

$G, N \subset G$  normal subgroup. The map  $g \cdot g^{-1} : G \rightarrow G$  is an automorphism sending identity to identity, hence if we differentiate it, we get a transformation in  $GL(\mathfrak{g})$ . i.e. we get a family of maps  $Ad_g \in GL(\mathfrak{g})$  indexed by elements in  $G$ .  $g \mapsto Ad_g \in GL(\mathfrak{g})$  is a representation of  $G$ , hence if it is differentiated, we get a representation of  $\mathfrak{g}$ ,  $ad_g : \mathfrak{g} \mapsto End(\mathfrak{g})$  acting on the vector space  $\mathfrak{g}$ .

$gng^{-1} \in N$ .  $\forall g, g \cdot g^{-1}$  is a transformation from  $N$  to  $N$ , define  $Ad_g(n) = gng^{-1}$ . Differentiate to get  $ad : \mathfrak{n} \rightarrow \mathfrak{n}$ .  $\mathfrak{n}$  is a vector space, has a dual. Linear transformation on vector space passes to the dual space.

$$\begin{aligned} \varphi^*(v^*)(u) &= v^*(\varphi(u)) \\ &\Downarrow \\ \langle \Lambda^* v^*, u \rangle &= \langle v^*, \Lambda u \rangle. \end{aligned}$$

In order to get the transformation rules work out, have to pass to the adjoint or the dual space.

$$Ad_g^* : \mathfrak{n}^* \rightarrow \mathfrak{n}^*$$

the coadjoint representation of  $\mathfrak{n}$ .

Orbits of co-adjoint representation amounts precisely to equivalence classes of irreducible representations.

**Example 7.52.** Heisenberg group  $G = \{(a, b, c)\}$  with

$$(a, b, c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

normal subgroup  $N = \{(0, b, c)\}$

$$(0, b, c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with Lie algebra  $\mathfrak{n} = \{(b, c)\}$

$$(0, \xi, \eta) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

$Ad_g : \mathfrak{n} \rightarrow \mathfrak{n}$  given by

$$\begin{aligned} gng^{-1} &= (a, b, c)(0, y, x)(-a, -b, -c + ab) \\ &= (a, b + y, c + x + ay)(-a, -b, -c + ab) \\ &= (0, y, x + ay) \end{aligned}$$

hence  $Ad_g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$Ad_g : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x + ay \\ y \end{bmatrix}.$$

The matrix of  $Ad_g$  is (before taking adjoint) is

$$Ad_g = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

The matrix for  $Ad_g^*$  is

$$Ad_g^* = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.$$

We use  $[\xi, \eta]^T$  for the dual  $\mathfrak{n}^*$ ; and use  $[x, y]^T$  for  $\mathfrak{n}$ . Then

$$Ad_g^* : \begin{bmatrix} \xi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ a\xi + \eta \end{bmatrix}$$

What about the orbit? In the example of  $O_n/O_{n-1}$ , the orbit is  $S^{n-1}$ .

For  $\xi \in \mathbb{R} \setminus \{0\}$ , the orbit of  $Ad_g^*$  is

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \mathbb{R} \end{bmatrix}$$

i.e. vertical lines with  $x$ -coordinate  $\xi$ .  $\xi = 0$  amounts to fixed point, i.e. the orbit is a fixed point.

The simplest orbit is when the orbit is a fixed point. i.e.

$$Ad_g^* : \begin{bmatrix} \xi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in V^*$$

where if we choose

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

it is a fixed point.

The other extreme is to take any  $\xi \neq 0$ , then

$$Ad_g^* : \begin{bmatrix} \xi \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \mathbb{R} \end{bmatrix}$$

i.e. get vertical lines indexed by the  $x$ -coordinate  $\xi$ . In this example, a cross section is a subset of  $\mathbb{R}^2$  that intersects each orbit at precisely one point. Every cross section in this example is a Borel set in  $\mathbb{R}^2$ .

We don't always get measurable cross sections. An example is the construction of non-measurable set as was given in Rudin's book. Cross section is a Borel set that intersects each coset at precisely one point.

Why does it give all the equivalent classes of irreducible representations? Since we have a unitary representation  $L_n \in Rep(N, V)$ ,  $L_n : V \rightarrow V$  and by construction of the induced representation  $U_g \in Rep(G, \mathcal{H})$ ,  $N \subset G$  normal such that

$$U_g L_n U_{g^{-1}} = L_{g n g^{-1}}$$

i.e.

$$L_g \simeq L_{g n g^{-1}}$$

now pass to the Lie algebra and its dual

$$L_n \rightarrow LA \rightarrow LA^*.$$

## 7.7 Gårding Space

**Definition 7.53.** Let  $\mathcal{U}$  be a strongly continuous representation of a Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ , and let  $\exp : \mathfrak{g} \rightarrow G$  denote the exponential mapping from Lie theory. For every  $\varphi \in C_c^\infty(G)$ , set

$$\mathcal{U}(\varphi) = \int_G \varphi(g) \mathcal{U}_g dg$$

where  $dg$  is a left-invariant Haar measure on  $G$ ; and set

$$\mathcal{H}_{\text{Gårding}} = \left\{ \mathcal{U}(\varphi) v \mid \varphi \in C_c^\infty(G), v \in \mathcal{H} \right\}.$$

**Lemma 7.54.** Fix  $X \in \mathfrak{g}$ , set

$$d\mathcal{U}(X)v = \lim_{t \rightarrow 0} \frac{\mathcal{U}(\exp(tX))v - v}{t}$$

then

$$\mathcal{H}_{\text{Gårding}} \subset \bigcap_{X \in \mathfrak{g}} \text{dom}(d\mathcal{U}(X))$$

and

$$d\mathcal{U}(X)\mathcal{U}(\varphi)v = \mathcal{U}(\tilde{X}\varphi)v,$$

for all  $\varphi \in C_c^\infty(G)$ ,  $v \in \mathcal{H}$ , where

$$(\tilde{X}\varphi)(g) = \left. \frac{d}{dt} \right|_{t=0} \varphi(\exp(-tX)g), \quad \forall g \in G.$$

*Proof.* (Hint)

$$\int_G \varphi(g)\mathcal{U}(\exp(tX))\mathcal{U}(g)dg = \int_G \varphi(\exp(-tX)g)\mathcal{U}(g)dg.$$

□

We talked about how to detect whether a representation is induced. Given a group  $G$  with a subgroup  $\Gamma$  let  $M := \Gamma \backslash G$ . The map  $\pi : G \rightarrow M$  is called a covering map, which sends  $g$  to its equivalent class or the coset  $\Gamma g$ .  $M$  is given its projective topology, so  $\pi$  is continuous. When  $G$  is compact, many things simplify. For example, if  $G$  is compact, any irreducible representation is finite dimensional. But many groups are not compact, only locally compact. For example, the groups  $ax + b$ ,  $H_3$ ,  $SL_n$ .

Specialize to Lie groups.  $G$  and subgroup  $H$  have Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  respectively.

$$\mathfrak{g} = \{X : e^{tX} \in G, \forall t \in \mathbb{R}\}$$

Almost all Lie algebras we will encounter come from specifying a quadratic form  $\varphi : G \times G \rightarrow \mathbb{C}$ .  $\varphi$  is then uniquely determined by a Hermitian matrix  $A$  so that

$$\varphi(x, y) = x^{tr} \cdot Ay$$

Let  $G = G(\varphi) = \{g : \varphi(gx, gy) = \varphi(x, y)\}$ , then

$$\left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tX}x, e^{tX}y) = 0$$

and with an application of the product rule,

$$\begin{aligned} \varphi(Xx, y) + \varphi(x, XY) &= 0 \\ (Xx)^{tr} \cdot Ay + x^{tr} \cdot AXy &= 0 \end{aligned}$$

$$X^{tr}A + AX = 0$$

hence

$$\mathfrak{g} = \{X : X^{tr}A + AX = 0\}.$$

Let  $U \in \text{Rep}(G, \mathcal{H})$ , for  $X \in \mathfrak{g}$ ,  $U(e^{tX})$  is a one parameter continuous group of unitary operator, hence by Stone's theorem (see [vN32b, Nel69]), it must have the form

$$U(e^{tX}) = e^{itH_X} \quad (7.47)$$

for some selfadjoint operator  $H_X$  (possibly unbounded). The RHS in (7.47) is given by the Spectral Theorem (see [Sto90, Yos95, Nel69, RS75, DS88c]). We often write

$$dU(X) := iH_X$$

to indicate that  $dU(X)$  is the directional derivative along the direction  $X$ . Notice that  $H_X^* = H_X$  but

$$(iH_X)^* = -(iH_X)$$

i.e.  $dU(X)$  is skew adjoint.

**Example 7.55.**  $G = \{(a, b, c)\}$  Heisenberg group.  $\mathfrak{g} = \{X_1 \sim a, X_2 \sim b, X_3 \sim c\}$ . Take the Schrödinger representation  $U_g f(x) = e^{ih(c+bx)} f(x+a)$ ,  $f \in L^2(\mathbb{R})$ .

- $U(e^{tX_1})f(x) = f(x+t)$

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} U(e^{tX_1})f(x) &= \frac{d}{dx}f(x) \\ dU(X_1) &= \frac{d}{dx} \end{aligned}$$

- $U(e^{tX_2})f(x) = e^{ih(tx)} f(x)$

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} U(e^{tX_2})f(x) &= ihx f(x) \\ dU(X_2) &= ihx \end{aligned}$$

- $U(e^{tX_3})f(x) = e^{iht} f(x)$

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} U(e^{tX_3})f(x) &= ihf(x) \\ dU(X_3) &= ihI \end{aligned}$$

Notice that  $dU(X_i)$  are all skew adjoint.

$$\begin{aligned} [dU(X_1), dU(X_2)] &= \left[\frac{d}{dx}, ihx\right] \\ &= ih\left[\frac{d}{dx}, x\right] \\ &= ih \end{aligned}$$

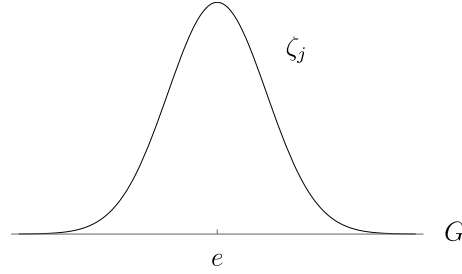


Figure 7.2: Approximation of identity.

In case we want selfadjoint operators, replace  $dU(X_i)$  by  $-idU(X_i)$  and get

$$\begin{aligned} -idU(X_1) &= \frac{1}{i} \frac{d}{dx} \\ -idU(X_2) &= hx \\ -idU(X_3) &= hI \\ \left[ \frac{1}{i} \frac{d}{dx}, hx \right] &= \frac{h}{i}. \end{aligned}$$

Below we answer the following question:

What is the space of functions that  $U_g$  acts on? L. Gårding /gor-ding/ (Swedish mathematician) looked for one space that always works. It's now called the Gårding space.

Start with  $C_c(G)$ , every  $\varphi \in C_c(G)$  can be approximated by the so called Gårding functions, using the convolution argument. Define convolution as

$$\begin{aligned} \varphi \star \psi(g) &= \int_G \varphi(gh)\psi(h)d_Rh \\ \varphi \star \psi(g) &= \int_G \varphi(h)\psi(g^{-1}h)d_Lh \end{aligned}$$

Take an approximation of identity  $\zeta_j$  (Figure 7.2), so that

$$\varphi \star \zeta_j \rightarrow \varphi, \quad j \rightarrow 0.$$

Define Gårding space as the span of the vectors in  $\mathcal{H}$ , given by

$$U(\varphi)v = \int \varphi(h)U(h)v d_Lh$$

where  $\varphi \in C_c(G)$ ,  $v \in \mathcal{H}$ , or we say

$$U(\varphi) := \int_G \varphi(h)U(h) d_Lh.$$

Since  $\varphi$  vanishes outside a compact set, and since  $U(h)v$  is continuous and bounded in  $\|\cdot\|$ , it follows that  $U(\varphi)$  is well-defined.

Every representation  $\mathcal{U}$  of a Lie group  $G$  induces a representation (also denote  $\mathcal{U}$ ) of the group algebra:

**Lemma 7.56.**  $U(\varphi_1 \star \varphi_2) = U(\varphi_1)U(\varphi_2)$  ( $U$  is a representation of the group algebra)

*Proof.* Use Fubini,

$$\begin{aligned} \int_G \varphi_1 \star \varphi_2(g)U(g)dg &= \iint_{G \times G} \varphi_1(h)\varphi(h^{-1}g)U(g)dhdg \\ &= \iint_{G \times G} \varphi_1(h)\varphi(g)U(hg)dhdg \quad (dg \text{ is r-Haarg } \mapsto hg) \\ &= \iint_{G \times G} \varphi_1(h)\varphi(g)U(h)U(g)dhdg \\ &= \int_G \varphi_1(h)U(h)dh \int_G \varphi_2(g)U(g)dg \end{aligned}$$

Choose  $\varphi$  to be an approximation of identity, then

$$\int_G \varphi(g)U(g)v dg \rightarrow U(e)v = v$$

i.e. any vector  $v \in H$  can be approximated by functions in the Gårding space. It follows that

$$\{U(\varphi)v\}$$

is dense in  $\mathcal{H}$ . □

**Lemma 7.57.**  $U(\varphi)$  can be differentiated, in the sense that

$$dU(X)U(\varphi)v = U(\tilde{X}\varphi)v$$

where we use  $\tilde{X}$  to denote the vector field.

*Proof.* need to prove

$$\lim_{t \rightarrow 0} \frac{1}{t} [(U(e^{tX}) - I)U(\varphi)v] = U(\tilde{X}\varphi)v.$$

Let  $v_\varphi := U(\varphi)v$ , need to look at in general  $U(g)v_\varphi$ .

$$\begin{aligned} U(g)v_\varphi &= U(g) \int_G \varphi(h)U(h)v dh \\ &= \int_G \varphi(h)U(gh)v dh \\ &= \int_G \Delta(g)\varphi(g^{-1}h)U(h)v dh \end{aligned}$$

set  $g = e^{tX}$ . □



*Note 7.58.* If assuming unimodular,  $\Delta$  does not show up. Otherwise,  $\Delta$  is some correction term which is also differentiable.  $\tilde{X}$  acts on  $\varphi$  as  $\tilde{X}\varphi$ .  $\tilde{X}$  is called the derivative of the translation operator  $e^{tX}$ .

**Exercise 7.59** (The Gårding space for the Schrödinger representation). Show that Schwartz space  $\mathcal{S}$  is the Gårding space for the Schrödinger representation.

**Exercise 7.60** (The Lie bracket). Let  $U$  be a representation of a Lie group  $G$ , and let  $dU(\cdot)$  be the derived representation, see Lemma 7.57. On the dense Gårding space, show that

$$\begin{aligned} dU([X, Y]) &= [dU(X), dU(Y)] \\ &= dU(X)dU(Y) - dU(Y)dU(X), \end{aligned}$$

where  $[X, Y]$  denotes the Lie bracket of the two elements  $X$  and  $Y$  in the Lie algebra.

## 7.8 Decomposition of Representations

We study some examples of duality.

- $G = \mathbb{T}$ ,  $\hat{G} = \mathbb{Z}$

$$\begin{aligned} \chi_n(z) &= z^n \\ \chi_n(zw) &= z^n w^n = \chi_n(z)\chi_n(w) \end{aligned}$$

- $G = \mathbb{R}$ ,  $\hat{G} = \mathbb{R}$

$$\chi_t(x) = e^{itx}$$

- $G = \mathbb{Z}/n\mathbb{Z} \simeq \{0, 1, \dots, n-1\}$ .  $\hat{G} = G$ .

This is another example where  $\hat{G} = G$ .

Let  $\zeta = e^{i2\pi/n}$  be the primitive  $n^{\text{th}}$ -root of unity.  $k \in \mathbb{Z}_n$ ,  $l \in \{0, 1, \dots, n-1\}$

$$\chi_l(k) = e^{i\frac{2\pi kl}{n}}$$

If  $G$  is a locally compact abelian group,  $\hat{G}$  is the set of 1-dimensional representations.

$$\hat{G} = \{\chi : g \mapsto \chi(g) \in \mathbb{T}, \chi(gh) = \chi(g)\chi(h), \text{ assumed continuous}\}.$$

$\hat{G}$  is also a group, with group operation defined by  $(\chi_1\chi_2)(g) := \chi_1(g)\chi_2(g)$ .  $\hat{G}$  is called the group characters.

**Theorem 7.61** (Pontryagin). *If  $G$  is a locally compact abelian group, then  $G \simeq \hat{\hat{G}}$  (isomorphism between  $G$  and the double dual  $\hat{\hat{G}}$ ), where “ $\simeq$ ” means “natural isomorphism.”*

*Note 7.62.* This result first appears in 1930s in the annals of math, when John von Neumann was the editor of the journal at the time. The original paper was hand written. von Neumann rewrote it, since then the theorem became very popular, see [Rud90].

There are many groups that are not abelian. We want to study the duality question in general. Examples:

- compact group
- finite group (abelian, or not)
- $H_3$  locally compact, nonabelian, unimodular
- $ax + b$  locally compact, nonabelian, non-unimodular

If  $G$  is not abelian,  $\hat{G}$  is not a group. We would like to decompose  $\hat{G}$  into irreducible representations. The big names in this development are Krein, Peter-Weyl, Weil, Segal. See [AD86, ARR13, BR79, Emc00, JÓ00, KL14b, KR97b, Rud73, Rud90, Seg50, Sto90].

Let  $G$  be a group (may not be abelian). The right regular representation is defined as

$$R_g f(\cdot) = f(\cdot g), \text{ (translation on the right) .}$$

Then  $R_g$  is a unitary operator acting on  $L^2(\mu_R)$ , where  $\mu_R$  is the right invariant Haar measure.

**Theorem 7.63** (Krein, Weil, Segal). *Let  $G$  be locally compact unimodular (abelian or not). Then the right regular representation decomposes into a direct integral of irreducible representations*

$$R_g = \int_{\hat{G}}^{\oplus} \text{''irrep''} d\mu$$

where  $\mu$  is called the Plancherel measure. See [Sti59, Seg50].

**Example 7.64.**  $G = T$ ,  $\hat{G} = \mathbb{Z}$ . Irreducible representations  $\{e^{in(\cdot)}\}_n \sim \mathbb{Z}$

$$\begin{aligned} (U_y f)(x) &= f(x + y) \\ &= \sum_n \hat{f}(n) \chi_n(x + y) \\ &= \sum_n \hat{f}(n) e^{i2\pi n(x+y)} \end{aligned}$$

$$(U_y f)(0) = f(y) = \sum_n \hat{f}(n) e^{i2\pi n y}$$

The Plancherel measure in this case is the counting measure.

**Example 7.65.**  $G = \mathbb{R}$ ,  $\hat{G} = \mathbb{R}$ . Irreducible representations  $\{e^{it(\cdot)}\}_{t \in \mathbb{R}} \sim \mathbb{R}$ .

$$\begin{aligned} (U_y f)(x) &= f(x + y) \\ &= \int_{\mathbb{R}} \hat{f}(t) \chi_t(x + y) dt \\ &= \int_{\mathbb{R}} \hat{f}(t) e^{it(x+y)} dt \\ (U_y f)(0) &= f(y) = \int_{\mathbb{R}} \hat{f}(t) e^{ity} dt \end{aligned}$$

where the Plancherel measure is the Lebesgue measure on  $\mathbb{R}$ .

As can be seen that Fourier series and Fourier integrals are special cases of the decomposition of the right regular representation  $R_g$  of a unimodular locally compact group.  $f^\oplus \implies \|f\| = \|\hat{f}\|$ . This is a result that was done 30 years earlier before the non abelian case. Classical function theory studies other types of convergence, pointwise, uniform, etc.

**Example 7.66.**  $G = H_3$ .  $G$  is unimodular, non abelian.  $\hat{G}$  is not a group.

Irreducible representations:  $\mathbb{R} \setminus \{0\}$  Schrödinger representation,  $\{0\}$  1-d trivial representation

Decomposition:

$$R_g = \int_{\mathbb{R} \setminus \{0\}}^\oplus U_{irrep}^h dh$$

For all  $f \in L^2(G)$ ,

$$(U_g f)(e) = \int^\oplus U^h f dh, \quad U^h \text{ irrep.}$$

Set

$$\begin{aligned} F(g) &= (R_g F)(e) \\ &= \int_{\mathbb{R} \setminus \{0\}}^\oplus e^{ih(c+bx)} f(x+a) dh; \text{ then} \\ \hat{F}(h) &= \int_G (U_g^h F) dg \end{aligned}$$

Plancherel measure:  $hdh$  and the point measure  $\delta_0$  at zero.

**Example 7.67.**  $G = ax + b$  group, non abelian.  $\hat{G}$  not a group. 3 irreducible representations:  $+$ ,  $-$ ,  $0$  but  $G$  is not unimodular.

The  $+$  representation is supported on  $\mathbb{R}_+$ , the  $-$  representation on  $\mathbb{R}_-$ , and the  $0$  representation is the trivial one-dimensional representation.

The duality question may also be asked for discrete subgroups. This leads to remarkable applications in automorphic functions, automorphic forms, p-adic numbers, compact Riemann surface, hyperbolic geometry, etc.

**Example 7.68.** Cyclic group of order  $n$ .  $G = \mathbb{Z}/n\mathbb{Z} \simeq \{0, 1, \dots, n-1\}$ .  $\hat{G} = G$ . This is another example where the dual group is identical to the group itself. Let  $\zeta = e^{i2\pi/n}$  be the primitive  $n^{\text{th}}$ -root of unity.  $k \in \mathbb{Z}_n, l = \{0, 1, \dots, n-1\}$

$$\chi_l(k) = e^{i\frac{2\pi kl}{n}}$$

In this case, Segal's theorem gives finite Fourier transform.  $U : l^2(\mathbb{Z}) \rightarrow l^2(\hat{\mathbb{Z}})$  where

$$Uf(l) = \frac{1}{\sqrt{N}} \sum_k \zeta^{kl} f(k)$$

## 7.9 Summary of Induced Representations, the Example of $d/dx$

We study decomposition of group representations. Two cases: abelian and non abelian. The non abelian case may be induced from the abelian ones.

non abelian

- semi product  $G = HN$  often  $N$  is normal.
- $G$  simple.  $G$  does not have normal subgroups, i.e., the Lie algebra does not have any ideals.

**Exercise 7.69** (Normal subgroups). (1) Find the normal subgroups in the Heisenberg group. (2) Find the normal subgroups in the  $ax + b$  group.

**Example 7.70.**  $SL_2(\mathbb{R})$  (non compact)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1$$

with Lie algebra

$$sl_2(\mathbb{R}) = \{X : tr(X) = 0\}.$$

Note that  $sl_2$  is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular,  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  generates the one-parameter group  $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \simeq \mathbb{T}$  whose dual group is  $\mathbb{Z}$ , where

$$\chi_n(g(t)) = g(t)^n = e^{itn}.$$

May use this to induce a representation of  $G$ . This is called principle series. Need to do something else to get all irreducible representations.

A theorem by Iwasawa states that simple matrix group (Lie group) can be decomposed into

$$G = KAN$$

where  $K$  is compact,  $A$  is abelian and  $N$  is nilpotent. For example, in the  $SL_2$  case,

$$SL_2(\mathbb{R}) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

The simple groups do not have normal subgroups. The representations are much more difficult.

### Induced Representations

Suppose from now on that  $G$  has a normal abelian subgroup  $N \triangleleft G$ , and  $G = H \ltimes N$ . The  $N \simeq \mathbb{R}^d$  and  $N^* \simeq (\mathbb{R}^d)^* = \mathbb{R}^d$ . In this case

$$\chi_t(\nu) = e^{it\nu}$$

for  $\nu \in N$  and  $t \in \hat{N} = N^*$ . Notice that  $\chi_t$  is a 1-d irreducible representation on  $\mathbb{C}$ .

Let  $\mathcal{H}_t$  be the space of functions  $f : G \rightarrow \mathbb{C}$  so that

$$f(\nu g) = \chi_t(\nu)f(g).$$

On  $\mathcal{H}_t$ , define inner product so that

$$\|f\|_{\mathcal{H}_t}^2 := \int_G |f(g)|^2 = \int_{G/N} \|f(g)\|^2 dm$$

where  $dm$  is the invariant measure on  $N \backslash G \simeq H$ .

Define  $U_t = \text{ind}_N^G(\chi_t) \in \text{Rep}(G, \mathcal{H}_t)$ . Define  $U_t(g)f(x) = f(xg)$ , for  $f \in \mathcal{H}_t$ . Notice that the representation space of  $\chi_t$  is  $\mathbb{C}$ , 1-d Hilbert space; however, the representation space of  $U_t$  is  $\mathcal{H}_t$  which is infinite dimensional.  $U_t$  is a family of irreducible representations indexed by  $t \in N \simeq \hat{N} \simeq \mathbb{R}^d$ .

*Note 7.71.* Another way to recognize induced representations is to see these functions are defined on  $H$ , not really on  $G$ .

Define the unitary transformation  $W : \mathcal{H}_t \rightarrow L^2(H)$ . Notice that  $H \simeq N \backslash G$  is a group, and it has an invariant Haar measure. By uniqueness on the Haar measure, this has to be  $dm$ . It would be nice to cook up the same space  $L^2(H)$  so that all induced representations indexed by  $t$  act on it. In other words, this Hilbert space  $L^2(H)$  does not depend on  $t$ .  $W_t$  is defined as

$$WF_t(h) = F_t(h).$$

So what does the induced representation look like in  $L^2(H)$  then? Recall by definition that

$$U_t(g) := W(\text{ind}_{\chi_t}^G(g))W^*$$

and the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H}_t & \xrightarrow{\text{ind}_{\chi_t}^G} & \mathcal{H}_t \\ w \downarrow & & \downarrow w \\ L^2(H) & \xrightarrow{U_t} & L^2(H) \end{array}$$

Let  $f \in L^2(H)$ .

$$\begin{aligned} U_t(g)f(h) &= W(\text{ind}_{\chi_t}^G(g))W^*f(h) \\ &= (\text{ind}_{\chi_t}^G(g)W^*f)(h) \\ &= (W^*f)(hg). \end{aligned}$$

Since  $G = H \ltimes N$ ,  $g$  is uniquely decomposed into  $g = g_N g_H$ . Hence  $hg = hg_N g_H = g_N g_N^{-1} h g_N g_H = g_N \tilde{h} g_H$  and

$$\begin{aligned} U_t(g)f(h) &= (W^*f)(hg) \\ &= (W^*f)(g_N \tilde{h} g_H) \\ &= \chi_t(g_N)(W^*f)(\tilde{h} g_H) \\ &= \chi_t(g_N)(W^*f)(g_N^{-1} h g_N g_H) \end{aligned}$$

This last formula is called the Mackey machine [Mac52, Mac88].

The Mackey machine does not cover many important symmetry groups in physics. Actually most of these are simple groups. However it can still be applied. For example, in special relativity theory, we have the Poincaré group  $\mathcal{L} \ltimes \mathbb{R}^4$  where  $\mathbb{R}^4$  is the normal subgroup. The baby version of this is when  $\mathcal{L} = SL_2(\mathbb{R})$ . V. Bargman formulated this baby version. Wigner pioneered the Mackey machine, long before Mackey was around.

Once we get unitary representations, differentiate it and get selfadjoint algebra of operators (possibly unbounded). These are the observables in quantum mechanics.

**Example 7.72.**  $\mathbb{Z} \subset \mathbb{R}$ ,  $\hat{\mathbb{Z}} = T$ .  $\chi_t \in T$ ,  $\chi_t(n) = e^{itn}$ . Let  $\mathcal{H}_t$  be the space of functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  so that

$$f(n+x) = \chi_t(n)f(x) = e^{int}f(x).$$

Define inner product on  $\mathcal{H}_t$  so that

$$\|f\|_{\mathcal{H}_t}^2 := \int_0^1 |f(x)|^2 dx.$$

Define  $\text{ind}_{\chi_t}^{\mathbb{R}}(y)f(x) = f(x+y)$ . Claim that  $\mathcal{H}_t \simeq L^2[0,1]$ . The unitary transformation is given by  $W : \mathcal{H}_t \rightarrow L^2[0,1]$

$$(WF_t)(x) = F_t(x).$$

Let's see what  $\text{ind}_{\chi_t}^{\mathbb{R}}(y)$  looks like on  $L^2[0, 1]$ . For any  $f \in L^2[0, 1]$ ,

$$\begin{aligned} (W(\text{ind}_{\chi_t}^G(y))W^*f)(x) &= (\text{ind}_{\chi_t}^G(y)W^*f)(x) \\ &= (W^*f)(x+y) \end{aligned}$$

Since  $y \in \mathbb{R}$  is uniquely decomposed as  $y = n + x'$  for some  $x' \in [0, 1)$ , therefore

$$\begin{aligned} (W(\text{ind}_{\chi_t}^G(y))W^*f)(x) &= (W^*f)(x+y) \\ &= (W^*f)(x+n+x') \\ &= (W^*f)(n+(-n+x+n)+x') \\ &= \chi_t(n)(W^*f)((-n+x+n)+x') \\ &= \chi_t(n)(W^*f)(x+x') \\ &= e^{itn}(W^*f)(x+x') \end{aligned}$$

*Note 7.73.* Are there any functions in  $\mathcal{H}_t$ ? Yes, for example,  $f(x) = e^{itx}$ . If  $f \in \mathcal{H}_t$ ,  $|f|$  is 1-periodic. Therefore  $f$  is really a function defined on  $\mathbb{Z} \setminus \mathbb{R} \simeq [0, 1]$ . Such a function has the form

$$f(x) = \left( \sum c_n e^{i2\pi n x} \right) e^{itx} = \sum c_n e^{i(2\pi n + t)x}.$$

Any 1-periodic function  $g$  satisfies the boundary condition  $g(0) = g(1)$ .  $f \in \mathcal{H}_t$  has a modified boundary condition where  $f(1) = e^{it}f(0)$ .

## 7.10 Connections to Nelson's Spectral Theory

In Nelson's notes [Nel69], a normal representation has the form (counting multiplicity)

$$\rho = \sum^{\oplus} n\pi|_{\mathcal{H}_n}, \mathcal{H}_n \perp \mathcal{H}_m$$

where

$$n\pi = \pi \oplus \cdots \oplus \pi \text{ (n times)}$$

is a representation acting on the Hilbert space

$$\sum^{\oplus} K = l_{\mathbb{Z}_n}^2 \otimes K.$$

In matrix form, this is a diagonal matrix with  $\pi$  repeated on the diagonal  $n$  times.  $n$  could be  $1, 2, \dots, \infty$ . We apply this to group representations.

Locally compact group can be divided into the following types.

- abelian
- non-abelian: unimodular, non-unimodular

- non-abelian: Mackey machine, semidirect product e.g.  $H_3$ ,  $ax + b$ ; simple group  $SL_2(\mathbb{R})$ . Even it's called simple, ironically its representation is much more difficult than the semidirect product case.

We want to apply these to group representations.

Spectral theorem says that given a normal operator  $A$ , we may define  $f(A)$  for quite a large class of functions, actually all measurable functions (see [Sto90, Yos95, Nel69, RS75, DS88c]). One way to define  $f(A)$  is to use the multiplication version of the spectral theorem, and let

$$f(A) = \mathcal{F}f(\hat{A})\mathcal{F}^{-1}.$$

The other way is to use the projection-valued measure version of the spectral theorem, write

$$\begin{aligned} A &= \int \lambda P(d\lambda) \\ f(A) &= \int f(\lambda)P(d\lambda). \end{aligned}$$

The effect is  $\rho$  is a representation of the abelian algebra of measurable functions onto operators action on some Hilbert space.

$$\begin{aligned} \rho : f \mapsto \rho(f) &= f(A) \\ \rho(fg) &= \rho(f)\rho(g) \end{aligned}$$

To imitate Fourier transform, let's call  $\hat{f} := \rho(f)$ . Notice that  $\hat{f}$  is the multiplication operator.

**Example 7.74.**  $G = (\mathbb{R}, +)$ , group algebra  $L^1(\mathbb{R})$ . Define Fourier transform

$$\hat{f}(t) = \int f(x)e^{-itx}dx.$$

$\{e^{itx}\}_t$  is a family of 1-dimensional irreducible representation of  $(\mathbb{R}, +)$ .

**Example 7.75.** Fix  $t$ ,  $\mathcal{H} = \mathbb{C}$ ,  $\rho(\cdot) = e^{it(\cdot)} \in \text{Rep}(G, \mathcal{H})$ . From the group representation  $\rho$ , we get a group algebra representation  $\tilde{\rho} \in \text{Rep}(L^1(\mathbb{R}), \mathcal{H})$  defined by

$$\tilde{\rho}(f) = \int f(x)\rho(x)dx = \int f(x)e^{itx}dx$$

It follows that

$$\begin{aligned} \hat{f}(\rho) &:= \tilde{\rho}(f) \\ \widehat{f \star g} &= \widehat{f \star g} = \hat{f}\hat{g} \end{aligned}$$

i.e. Fourier transform of  $f \in L^1(\mathbb{R})$  is a representation of the group algebra  $L^1(\mathbb{R})$  on to the 1-dimensional Hilbert space  $\mathbb{C}$ . The range of Fourier transform in this case is 1-d abelian algebra of multiplication operators, multiplication by complex numbers.



**Example 7.76.**  $\mathcal{H} = L^2(\mathbb{R})$ ,  $\rho \in \text{Rep}(G, \mathcal{H})$  so that

$$\rho(y)f(x) := f(x + y)$$

i.e.  $\rho$  is the right regular representation. The representation space  $\mathcal{H}$  in this case is infinite dimensional. From  $\rho$ , we get a group algebra representation  $\tilde{\rho} \in \text{Rep}(L^1(\mathbb{R}), \mathcal{H})$  where

$$\tilde{\rho}(f) = \int f(y)\rho(y)dy.$$

Define

$$\hat{f}(\rho) := \hat{\rho}(f)$$

then  $\hat{f}(\rho)$  is an operator acting on  $\mathcal{H}$ .

$$\begin{aligned} \hat{f}(\rho)g &= \tilde{\rho}(f)g = \int f(y)\rho(y)g(\cdot)dy \\ &= \int f(y)(R_y g)(\cdot)dy \\ &= \int f(y)g(\cdot + y)dy. \end{aligned}$$

If we have used the left regular representation, instead of the right, then

$$\begin{aligned} \hat{f}(\rho)g &= \tilde{\rho}(f)g = \int f(y)\rho(y)g(\cdot)dy \\ &= \int f(y)(L_y g)(\cdot)dy \\ &= \int f(y)g(\cdot - y)dy. \end{aligned}$$

Hence  $\hat{f}(\rho)$  is the left or right convolution operator.

Back to the general case. Given a locally compact group  $G$ , form the group algebra  $L^1(G)$ , and define the left and right convolutions as

$$\begin{aligned} (\varphi \star \psi)(x) &= \int \varphi(g)\psi(g^{-1}x)d_Lg = \int \varphi(g)(L_g\psi)d_Lg \\ (\varphi \star \psi)(x) &= \int \varphi(xg)\psi(g)d_Rg = \int (R_g\varphi)\psi(g)d_Rg \end{aligned}$$

Let  $\rho(g) \in \text{Rep}(G, \mathcal{H})$ , define  $\tilde{\rho} \in \text{Rep}(L^1(G), \mathcal{H})$  given by

$$\tilde{\rho}(\psi) := \int_G \psi(g)\rho(g)dg$$

and write

$$\hat{\psi}(\rho) := \tilde{\rho}(\psi).$$

$\hat{\psi}$  is an analog of Fourier transform. If  $\rho$  is irreducible, the operators  $\hat{\psi}$  forms an abelian algebra. In general, the range of this generalized Fourier transform gives rise to a non abelian algebra of operators.

For example, if  $\rho(g) = R_g$  and  $\mathcal{H} = L^2(G, d_R)$ , then

$$\tilde{\rho}(\psi) = \int_G \psi(g)\rho(g)dg = \int_G \psi(g)R_g dg$$

and

$$\begin{aligned} \tilde{\rho}(\psi)\varphi &= \int_G \psi(g)\rho(g)\varphi dg = \int_G \psi(g)(R_g\varphi)dg \\ &= \int_G \psi(g)\varphi(xg)dg \\ &= (\varphi \star \psi)(x) \end{aligned}$$

**Example 7.77.**  $G = H_3 \sim \mathbb{R}^3$ .  $\hat{G} = \{\mathbb{R} \setminus \{0\}\} \cup \{0\}$ .  $0 \in \hat{G}$  corresponds to the trivial representation, i.e.  $g \mapsto Id$  for all  $g \in G$ .

$$\rho_h : G \rightarrow L^2(\mathbb{R})$$

$$\rho_h(g)f(x) = e^{ih(c+bx)}f(x+a) \simeq ind_H^G(\chi_h)$$

where  $H$  is the normal subgroup  $\{b, c\}$ . It is not so nice to work with  $ind_H^G(\chi_h)$  directly, so instead, we work with the equivalent representations, i.e. Schrödinger representation. See Folland's book on abstract harmonic analysis.

$$\hat{\psi}(h) = \int_G \psi(g)\rho_h(g)dg$$

Notice that  $\hat{\psi}(h)$  is an operator acting on  $L^2(\mathbb{R})$ . Specifically,

$$\begin{aligned} \hat{\psi}(h) &= \int_G \psi(g)\rho_h(g)dg \\ &= \iiint \psi(a, b, c)e^{ih(c+bx)}f(x+a)dadbdc \\ &= \iint \left( \int \psi(a, b, c)e^{ihc}dc \right) f(x+a)e^{ihbx}dadb \\ &= \iint \hat{\psi}(a, b, h)f(x+a)e^{ihbx}dadb \\ &= \int \left( \int \hat{\psi}(a, b, h)e^{ihbx}db \right) f(x+a)da \\ &= \int \hat{\psi}(a, hx, h)f(x+a)da \\ &= \left( \hat{\psi}(\cdot, h, h) \star f \right)(x) \end{aligned}$$

Here the  $\hat{\psi}$  on the right hand side in the Fourier transform of  $\psi$  in the usual sense. Therefore the operator  $\hat{\psi}(h)$  is the one so that

$$L^2(\mathbb{R}) \ni f \mapsto (\hat{\psi}(\cdot, h, h) \star f)(x).$$

If  $\psi \in L^1(G)$ ,  $\hat{\psi}$  is not of trace class. But if  $\psi \in L^1 \cap L^2$ , then  $\hat{\psi}$  is of trace class.

$$\int_{\mathbb{R} \setminus \{0\}}^{\oplus} \text{tr} (\hat{\psi}^*(h)\hat{\psi}(h)) d\mu = \int |\psi|^2 dg = \int \bar{\psi}\psi dg$$

where  $\mu$  is the Plancherel measure.

If the group  $G$  is non unimodular, the direct integral is lost (not orthogonal). These are related to coherent states from physics, which is about decomposing Hilbert into non orthogonal pieces.

Important observables in QM come in pairs (dual pairs). For example, position - momentum; energy - time etc. The Schwartz space  $S(\mathbb{R})$  has the property that  $\widehat{S(\mathbb{R})} = S(\mathbb{R})$ . We look at the analog of the Schwartz space.  $h \mapsto \hat{\psi}(h)$  should decrease faster than any polynomials.

Take  $\psi \in L^1(G)$ ,  $X_i$  in the Lie algebra, form  $\Delta = \sum X_i^2$ . Require that

$$\Delta^n \psi \in L^1(G), \psi \in C^\infty(G).$$

For  $\Delta^n$ , see what happens in the transformed domain. Notice that

$$\frac{d}{dt} \Big|_{t=0} (R_{e^{tx}} \psi) = \tilde{X} \psi$$

where  $X \mapsto \tilde{X}$  represents the direction vector  $X$  as a vector field.

Let  $G$  be any Lie group.  $\varphi \in C_c^\infty(G)$ ,  $\rho \in \text{Rep}(G, \mathcal{H})$ .

$$d\rho(X)v = \int (\tilde{X}\varphi)(g)\rho(g)vdg$$

where

$$v = \int \varphi(g)\rho(g)wdg = \rho(\varphi)w. \text{ generalized convolution}$$

If  $\rho = R$ , the n

$$v = \int \varphi(g)R(g)$$

$$\tilde{X}(\varphi \star w) = (X\varphi) \star w.$$

**Example 7.78.**  $H_3$

$$\begin{aligned} a &\rightarrow \frac{\partial}{\partial a} \\ b &\mapsto \frac{\partial}{\partial b} \end{aligned}$$

$$c \mapsto \frac{\partial}{\partial c}$$

get standard Laplace operator.  $\{\rho_h(\varphi)w\} \subset L^2(\mathbb{R})$ . " = " due to Dixmier.  $\{\rho_h(\varphi)w\}$  is the Schwartz space.

$$\left(\frac{d}{dx}\right)^2 + (ihx)^2 + (ih)^2 = \left(\frac{d}{dx}\right)^2 - (hx)^2 - h^2$$

Notice that

$$-\left(\frac{d}{dx}\right)^2 + (hx)^2 + h^2$$

is the Harmonic oscillator. Spectrum =  $h\mathbb{Z}_+$ .

## 7.11 Multiplicity Revisited

Let  $\mathfrak{A}$  be a  $*$ -algebra, and let  $\pi$  and  $\rho$  be representations of  $\mathfrak{A}$ . To indicate the Hilbert space, we write  $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H}_\pi)$ , and  $\rho \in \text{Rep}(\mathfrak{A}, \mathcal{H}_\rho)$ .

**Definition 7.79.** Consider the following space of bounded linear operators  $s, t : \mathcal{H}_\pi \rightarrow \mathcal{H}_\rho$  which intertwine the respective representations, i.e., we have:

$$s\pi(a) = \rho(a)s, \quad \forall a \in \mathfrak{A}. \quad (7.48)$$

The set of solutions  $s$  to (7.48) forms a vector space, and it is denoted  $\text{Int}(\pi, \rho)$ , the intertwining operators. We check the following:

$$s \in \text{Int}(\pi, \rho) \iff s^* \in \text{Int}(\rho, \pi); \quad (7.49)$$

and therefore, if  $s, t \in \text{Int}(\pi, \rho)$ , we have:

$$s^*t \in \text{Int}(\pi, \pi). \quad (7.50)$$

Note

$$\begin{aligned} \text{Int}(\pi, \pi) &= \pi(\mathfrak{A})' \text{ (commutant)} \\ &= \{A \in \mathcal{B}(\mathcal{H}_\pi) : \pi(a)A = A\pi(a), \forall a \in \mathfrak{A}\}. \end{aligned} \quad (7.51)$$

If  $\pi$  is irreducible, therefore  $\text{Int}(\pi, \pi)$  is one-dimensional; hence, for  $\forall s, t \in \text{Int}(\pi, \rho)$ ,

$$s^*t = \langle s, t \rangle I_{\mathcal{H}_\pi}, \quad (7.52)$$

where  $\langle s, t \rangle \in \mathbb{C}$  is uniquely determined. This form  $\langle \cdot, \cdot \rangle$  is sesquilinear, and positive definite. We therefore get a Hilbert-completion of  $\text{Int}(\pi, \rho)$ . Let  $\mathcal{H}(\pi, \rho)$  be the corresponding Hilbert space.

**Definition 7.80.** Let  $\pi$  and  $\rho$  be as above, assume that  $\pi$  is irreducible, and let  $\mathcal{H}(\pi, \rho)$  be the corresponding Hilbert space; see (7.52). We say that  $\pi$  occurs in  $\rho$   $m$  times if

$$m = \dim \mathcal{H}(\pi, \rho). \quad (7.53)$$

**Exercise 7.81** (Multiplicity). Show that the definition of multiplicity (Definition 7.80) agrees with the one used inside Chapter 7.

**Exercise 7.82** (A Hilbert space of intertwiners). Let  $\pi$  and  $\rho$  be as above,  $\pi$  irreducible. Show that with the inner product defined in (7.52),  $\text{Int}(\pi, \rho)$  is a Hilbert space.

**Exercise 7.83** (An ONB in  $\text{Int}(\pi, \rho)$ ). Let  $(s_i)$  be an ONB in  $\text{Int}(\pi, \rho)$ .

1. Show that this is a system of isometries, satisfying:

$$s_i^* s_j = \delta_{i,j} I_{\mathcal{H}_\pi}. \quad (7.54)$$

2. For  $A \in \mathcal{B}(\mathcal{H}_\pi)$ , set

$$\alpha(A) := \sum_i s_i A s_i^*. \quad (7.55)$$

Show that

$$\begin{aligned} \alpha(AB) &= \alpha(A)\alpha(B), \text{ and} \\ \alpha(A^*) &= \alpha(A)^*, \forall A, B \in \mathcal{B}(\mathcal{H}_\pi). \end{aligned}$$

3. What can be said about

$$\alpha(I_{\mathcal{H}_\pi}) = \sum_i s_i s_i^* ?$$

**Exercise 7.84** (A Hilbert space of intertwining operators). Verify that the results above about  $\text{Int}(\pi, \rho)$  apply to *unitary representations*  $\pi$ , and  $\rho$  of some given *group*  $G$ ; i.e., with

$$\text{Int}(\pi, \rho) = \{s : \mathcal{H}_\pi \rightarrow \mathcal{H}_\rho : s\pi(g) = \rho(g)s, \forall g \in G\}.$$

Hint: Use the above on the group algebra  $\mathfrak{A}_G := \mathbb{C}[G]$ .

Now consider the Heisenberg group  $G$  of all  $3 \times 3$  matrices

$$g = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, (a, b, c) \in \mathbb{R}^3.$$

Recall its Haar measure is  $dg = da db dc = 3$ -dimensional Lebesgue measure.

Consider the following two representations  $\rho$  and  $\pi$  of  $G$  (the regular representations RR, and the Schrödinger representation SR):

- (RR)  $\mathcal{H}_\rho = L^2(G, dg)$ , Haar measure, and

$$(\rho(g)f)(h) = f(hg), \forall f \in \mathcal{H}_\rho, \forall g, h \in G.$$

And the Schrödinger representation ( $\hbar = 1$ ):

- (SR)  $\mathcal{H}_\pi = L^2(\mathbb{R})$ , Lebesgue measure, and

$$(\pi(g)F)(x) = e^{i(c+bx)}F(x+a), \forall F \in L^2(\mathbb{R}) = \mathcal{H}_\pi, \forall g \in G, x \in \mathbb{R}.$$

**Exercise 7.85** (Specify the operators in  $Int(\pi, \rho)$ ). Let  $G, \pi$ , and  $\rho$  be as above. What is the Hilbert space  $Int(\pi, \rho)$ ?

**Exercise 7.86** (A formula from Peter-Weyl [Mac92]). In case  $G$  is a compact group, look up and explain that the Peter-Weyl theorem states the following: If  $\rho$  is the regular representation, and if  $\pi$  is irreducible unitary, then

$$\dim(Int(\pi, \rho)) = \dim(\pi).$$

*Remark 7.87.* An important class of non-compact, non-commutative, locally compact groups  $G$ , and unitary representations  $\pi$ , for which the intertwining Hilbert spaces  $Int(\pi, \rho)$  are non-zero is the class of square-integrable representations: Suppose the representation  $\pi$  is irreducible and square-integrable, then  $Int(\pi, \rho)$  is non-zero. Here  $\rho$  denotes the regular representation of  $G$ . A representation  $\pi$  is square-integrable if its matrix coefficients are in  $L^2(G/Z)$ , where  $Z$  denotes the center of  $G$ .

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [JÓ00, Mac52, Mac85, Mac92, JM84, JPS01, JPS05, Jor11, Jor94, Jor88, Szaar, DJ08, Dix81, Jor02, Dud14, Nel59a, JM80, Seg50, Ørs79, Pou72, J LH06, DHL09, Tay86, Hal13, Hal15].

## 7.A The Stone-von Neumann Uniqueness Theorem

The “uniqueness” in the title above refers to “uniqueness up to *unitary equivalence*.”

**Definition 7.88.** Let  $\mathcal{H}_i, i = 1, 2$  be two Hilbert spaces, and let  $S_1 = \{A_\alpha\} \subset \mathcal{B}(\mathcal{H}_1)$ , and  $S_2 = \{B_\alpha\} \subset \mathcal{B}(\mathcal{H}_2)$  be systems of bounded operators, where the index set  $J = \{\alpha\}$  is the same for the two operator systems.

We say that  $S_1$  and  $S_2$  are *unitarily equivalent* iff (Def)  $\exists W : \mathcal{H}_1 \rightarrow \mathcal{H}_2, W$  a unitary isomorphism of  $\mathcal{H}_1$  onto  $\mathcal{H}_2$  such that

$$WA_\alpha = B_\alpha W, \forall \alpha \in J; \text{ see Fig. 7.3.} \tag{7.56}$$

We say that the system  $S_1 = \{A_\alpha\}$  is *irreducible* iff (Def) the following implication holds

$$\boxed{T \in \mathcal{B}(\mathcal{H}_1), TA_\alpha = A_\alpha T, \alpha \in J} \implies T = \lambda I_1, \text{ for some } \lambda \in \mathbb{C}; \tag{7.57}$$

i.e., the commutant is one-dimensional.

$$\begin{array}{ccc}
 \mathcal{H}_1 & \xrightarrow{W} & \mathcal{H}_2 \\
 A_\alpha \downarrow & & \downarrow B_\alpha \\
 \mathcal{H}_1 & \xrightarrow{W} & \mathcal{H}_2
 \end{array}
 \quad B_\alpha W = W A_\alpha$$

Figure 7.3:  $W$  intertwines  $S_1$  and  $S_2$ .

**Definition 7.89.** The Heisenberg group  $G_3$  is the matrix group

$$g = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad (a, b, c) \in \mathbb{R}^3,$$

of all upper triangular  $3 \times 3$  matrices.

Fix  $h \in \mathbb{R} \setminus \{0\}$ , and set

$$(\mathcal{U}_h(g)f)(x) = e^{ih(c+bx)}f(x+a) \tag{7.58}$$

$\forall g = (a, b, c) \in G_3, \forall f \in L^2(\mathbb{R}), \forall x \in \mathbb{R}$ .

It is easy to see that  $\mathcal{U}_h$  is a unitary irreducible representation of  $G_3$  acting on  $L^2(\mathbb{R})$ , i.e.,  $\mathcal{U}_h \in \text{Rep}_{uni}(G_3, L^2(\mathbb{R}))$  for all  $h \in \mathbb{R} \setminus \{0\}$ . It is called the *Schrödinger representation*.

**Theorem 7.90** (Stone-von Neumann). *Every unitary irreducible representation of  $G_3$  in a Hilbert space (other than the trivial one-dimensional representation) is unitarily equivalent to the Schrödinger representation  $\mathcal{U}_h$  for some  $h \in \mathbb{R} \setminus \{0\}$ .*

*Proof.* The proof follows from the more general result, Theorem 7.45 above; the Imprimitivity Theorem. Also see [vN32b, vN31].  $\square$

*Remark 7.91.* The center of  $G_3$  is the one-dimensional subgroup  $g = (0, 0, c)$ ,  $c \in \mathbb{R}$ , and so if  $\mathcal{U}_h \in \text{Rep}_{uni}(G_3, \mathcal{H})$ ,  $\dim \mathcal{H} > 1$ , then it follows that  $\exists! h \in \mathbb{R} \setminus \{0\}$  such that

$$\mathcal{U}(0, 0, c) = e^{ich}I_{\mathcal{H}}.$$

Hence  $\mathcal{U}$  is determined by two one-parameter groups

$$\begin{cases} \mathcal{U}_1(a) = \mathcal{U}(a, 0, 0), & a \in \mathbb{R}; \text{ and} \\ \mathcal{U}_2(b) = \mathcal{U}(0, b, 0), & b \in \mathbb{R} \end{cases} \tag{7.59}$$

such that

$$\mathcal{U}_1(a)\mathcal{U}_2(b)\mathcal{U}_1(-a) = e^{i h a b}\mathcal{U}_2(b), \quad \forall a, b \in \mathbb{R}. \quad (7.60)$$

The system (7.60) is called the *Weyl commutation relation*. It is the integrated form of the corresponding Heisenberg relation (for unbounded essentially selfadjoint operators). (We omit a systematic discussion of the interrelationships between the two commutation relations.)

Under the unitary equivalence  $W : \mathcal{H} \rightarrow L^2(\mathbb{R})$  from the Stone-von Neumann theorem, we get

$$\begin{cases} (W\mathcal{U}_1(a)W^*f)(x) = f(x+a), & \text{and} \\ (W\mathcal{U}_2(b)W^*f)(x) = e^{i h b}f(x), & \forall a, b, x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}). \end{cases} \quad (7.61)$$

We shall use the following

**Lemma 7.92.** *Let  $\mathcal{U}_1(\cdot)$  and  $\mathcal{U}_2(\cdot)$  be the two one-parameter groups from the Weyl relation (7.60), and let  $P_2$  be the projection valued measure corresponding to  $\{\mathcal{U}_2(b)\}_{b \in \mathbb{R}}$ , i.e.,*

$$\mathcal{U}_2(b) = \int_{\mathbb{R}} e^{i b \lambda} P_2(d\lambda), \quad \forall b \in \mathbb{R}. \quad (7.62)$$

Then the Weyl relation (7.60) is equivalent to

$$\mathcal{U}_1(a)P_2(\Delta)\mathcal{U}_1(-a) = P_2(\Delta - h a), \quad (7.63)$$

$\forall a \in \mathbb{R}, \forall \Delta \in \mathcal{B}(\mathbb{R})$ , where

$$\Delta - h a = \{s - h a \mid s \in \Delta\}. \quad (7.64)$$

*Proof.* The proof is an easy application of Stone's theorem ([vN32b, Nel69]); see Appendix 2.A.  $\square$



## Chapter 8

# The Kadison-Singer Problem

Born wanted a theory which would generalize these matrices or grids of numbers into something with a continuity comparable to that of the continuous part of the spectrum. The job was a highly technical one, and he counted on me for aid.... I had the generalization of matrices already at hand in the form of what is known as operators. Born had a good many qualms about the soundness of my method and kept wondering if Hilbert would approve of my mathematics. Hilbert did, in fact, approve of it, and operators have since remained an essential part of quantum theory.

— Norbert Wiener

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in the case of poetry, it's the exact opposite!

— Paul Adrien Maurice Dirac.

It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical equations of great beauty and power.

— Paul Adrien Maurice Dirac

**The Kadison-Singer Problem:** Does every pure state on the (abelian) von Neumann algebra  $\mathbb{D}$  of all bounded diagonal operators on  $l^2$  have a unique extension to a pure state on all  $\mathcal{B}(l^2)$ , the von Neumann algebra of all bounded operators on  $l^2$ ?

We shall begin by explaining the meaning, and the significance, of the terms used in the statement of the Kadison-Singer (abbreviated KS) problem. But on the whole, our discussion of the KS problem (or conjecture) in the present book will be modest in scope. The first to say is that it was just solved by Adam Marcus, Dan Spielman, and N. Srivastava, see [MSS15]. There are sev-

eral reasons for why we cannot go into proof-details for the solution: While the formulation of KS by Kadison and Singer in 1959 was in the language of operator algebras, as that subject back then was inspired by Dirac’s quantum theory, it turned out that the eventual solution to KS, five decades later [MSS15], by Adam Marcus, Dan Spielman, and N. Srivastava, surprisingly, involves themes that draw on new topics, quite outside the scope of the present book. And making the connection between tools from the 2015 solution, back to the original 1959 formulation of KS, is not at all trivial. The new tools employed by Marcus, Spielman, and Srivastava are from diverse mathematical areas, and with a heavy combinatorial component, and also involving mathematical notions which we have not even defined here; for example: interlacing families, random vectors, paving, probabilistic frames, discrepancy analysis, sparsification, . . . . We hope readers will find it interesting to see how apparently disparate areas can meet at the crossroads in the solution of a famous problem in mathematics<sup>1</sup>. In view of this, we stress that even a modest attempt on our part at going into a detailed discussion of the Marcus-Spielman-Srivastava solution to KS would take us far afield; and done properly it could easily become a separate book volume. Since the Marcus-Spielman-Srivastava paper has just now appeared in the Annals 2015 [MSS15], it may also be too early for a proper book presentation. Nonetheless, we feel that a discussion here, in the present chapter, of the original formulation of KS is in fact appropriate. Indeed, the initial motivation for KS derives from precisely the topics which central themes of our present book: Operator theory/algebra, positivity, states, spectral theory; and with how these mathematical themes intersect with quantum theory.

The authors of [MSS13, MSS15] have proved the Kadison-Singer conjecture in an indirect way, by proving instead Weaver’s conjecture [Wea04, Sri13].

**Conjecture 8.1** ( $KS_2$ ). *There exist universal constants  $\eta \geq 2$  and  $\theta > 0$  so that the following holds. Let  $w_1, \dots, w_m \in \mathbb{C}^d$  satisfy  $\|w_i\| \leq 1$  for all  $i$ , and suppose*

$$\sum_{i=1}^m |\langle w_i, u \rangle|^2 = \eta \tag{8.1}$$

for every unit vector  $u \in \mathbb{C}^d$ . Then there exists a partition  $S_1, S_2$  of  $\{1, \dots, m\}$  so that

$$\sum_{i \in S_j} |\langle w_i, u \rangle|^2 \leq \eta - \theta$$

for every unit vector  $u \in \mathbb{C}^d$  and each  $j \in \{1, 2\}$ .

Akemann and Anderson’s projection paving conjecture [AA91, Conj. 7.1.3] follows directly from  $KS_2$  (see [Wea04, p. 229]).

<sup>1</sup>The many interconnections between the disparate areas of mathematics coming together in the proof of KS are just emerging in the literature as of this point; in particular, there is a forthcoming paper by P. Casazza, M. Bownik, A. Marcus and D. Speegle; which promises to be an authoritative source. We are grateful to P. Casazza for updates on KS. On the same theme, see also the paper “Consequences of the Marcus/Spielman/Srivastava solution of the Kadison-Singer problem,” By Peter G. Casazza and Janet C. Tremain. [arXiv:1407.4768v2](https://arxiv.org/abs/1407.4768v2).

Anderson's original paving conjecture says:

**Conjecture 8.2** (Anderson Paving). *For every  $\varepsilon > 0$ , there is an  $r \in \mathbb{N}$  such that for every  $n \times n$  Hermitian matrix  $T$  with zero diagonal, there are diagonal projections  $P_1, \dots, P_r$  with  $\sum_{i=1}^r P_i = I$  such that*

$$\|P_i T P_i\| \leq \varepsilon \|T\|, \quad \text{for } i = 1, \dots, r.$$

The Kadison-Singer problem (KS) lies at the root of how questions from quantum physics take shape in the language of functional analysis, and algebras of operators.

A brief sketch is included below, summarizing some recent advances (in fact the KS-problem was recently solved.) It is known that the solution to KS at the same time answers a host of other questions; all with applications to engineering, especially to signal processing. The notion from functional analysis here is “frame.” A frame of vectors in Hilbert space generalizes the notion of orthonormal basis in Hilbert space.

The Kadison-Singer problem (KS) comes from functional analysis, but it was resolved (only recently) with tools from areas of mathematics quite disparate from functional analysis. More importantly, the solution to KS turned out to have important implications for a host of applied fields from engineering.<sup>2</sup>

This reversal of the usual roles seem intriguing for a number of reasons:

While the applications considered so far involve problems which in one way or the other, derive from outside functional analysis itself, e.g., from physics, from signal processing, or from anyone of a number of areas of analysis, PDE, probability, statistics, dynamics, ergodic theory, prediction theory etc.; the Kadison-Singer problem is different. It comes directly from the foundational framework of functional analysis; more specifically from the axiomatic formulation of  $C^*$ -algebras. Then  $C^*$ -algebras are a byproduct of a rigorous formulation of quantum theory, as proposed by P.A.M. Dirac.<sup>3</sup>

From quantum theory, we have such notions as *state*, *observable*, and *measurement*. See Figure 8.1. But within the framework of  $C^*$ -algebras, each of these same terms, “state”, “observable”, and “measurement” also has a purely mathematical definition, see Section 3.1 in Chapter 3. Indeed  $C^*$ -algebra theory was motivated in part by the desire to make precise fundamental and conceptual questions in quantum theory, e.g., the uncertainty principle, measurement, determinacy, hidden variables, to mention a few (see for example [Emc00]). The interplay between the two sides has been extraordinarily fruitful since the birth of quantum mechanics in the 1920ties.

Cited from [KS59]:

<sup>2</sup>Atiyah and Singer shared the Abel prize of 2004.

<sup>3</sup>P.A.M. Dirac gave a lecture at Columbia University in the late 1950's, in which he claimed without proof that pure states on the algebra of diagonal operators ( $\simeq l^\infty$ ) extends uniquely on  $\mathcal{B}(l^2)$ . Kadison and Singer sitting in the audience were skeptical about whether Dirac knew what it meant to be an extension. They later formulated the conjecture in a joint paper, made precise the difference between MASAs that are continuous vs discrete. They showed that non-uniqueness holds in the continuous case.

“The main concern of this paper is the problem of uniqueness of extensions of pure states from maximal abelian self-adjoint algebras of operators on a Hilbert space to the algebra of all bounded operators on that space. The answer, as many of us have suspected for several years, is in negative.” ... “We heard of it first from I.E. Segal and I. Kaplansky, though it is difficult to credit a problem which stems naturally from the physical interpretation and the inherent structure of a subject. This problem has arisen, in one form or another, in our work on several different occasions;...”

Now consider the following: (i) the Hilbert space  $\mathcal{H} = l^2(= l^2(\mathbb{N}))$ , all square summable sequences, (ii) the  $C^*$ -algebra  $\mathcal{B}(l^2)$  of all bounded operators on  $l^2$ , and finally (iii) the sub-algebra  $\mathfrak{A}$  of  $\mathcal{B}(l^2)$  consisting of all diagonal operators, so an isomorphic copy of  $l^\infty$ .

The Kadison-Singer problem (KS), in the discrete version, is simply this:

*Does every pure state of  $\mathfrak{A}$  have a unique pure-state extension to  $\mathcal{B}(l^2)$ ?*

We remark that existence (of a pure-state extension) follows from the main theorems from functional analysis of Krein and Krein-Milman, but the uniqueness is difficult. The difficulty lies in the fact that it’s hard to find all states on  $l^\infty$ , i.e., a dual of  $l^\infty$ . The pure states of  $\mathfrak{A}$  are points in the Stone-Ćech compactification  $\beta(\mathbb{N})$ . The problem was settled in the affirmatively (uniqueness in the discrete case) only a year ago, after being open for 50 years.

**Lemma 8.3.** *Pure normal states on  $\mathcal{B}(\mathcal{H})$  are unit vectors (in fact, the equivalent class of unit vectors.<sup>4</sup>) Specifically, let  $u \in \mathcal{H}$ ,  $\|u\| = 1$ , then*

$$\mathcal{B}(\mathcal{H}) \ni A \longmapsto \omega_u(A) = \langle u, Au \rangle$$

*is a pure state. All normal pure states on  $\mathcal{B}(\mathcal{H})$  are of this form.*

The pure states on  $\mathcal{B}(\mathcal{H})$  not of the form  $\omega_u$ , for  $u \in \mathcal{H}$ ,  $\|u\| = 1$ , are called *singular pure states*.

*Remark 8.4.* Since  $l^\infty$  is an abelian algebra Banach  $*$ -algebra, by Gelfand’s theorem,  $l^\infty \simeq C(X)$  where  $X$  is a compact Hausdorff space. Indeed,  $X = \beta\mathbb{N}$ , – the Stone-Ćech compactification of  $\mathbb{N}$ . Points in  $\beta\mathbb{N}$  are called *ultra-filters*. Pure states on  $l^\infty$  correspond to pure states on  $C(\beta\mathbb{N})$ , i.e., Dirac-point measures on  $\beta\mathbb{N}$ .

Let  $s$  be a pure state on  $l^\infty$ . Using Hahn-Banach theorem one may extend  $s$ , as a linear functional, from  $l^\infty$  to  $\tilde{s}$  on the Banach space  $\mathcal{B}(\mathcal{H})$ . However, Hahn-Banach theorem doesn’t guarantee the extension remains a *pure state*. Let  $E(s)$  be the set of all states on  $\mathcal{B}(\mathcal{H})$  which extend  $s$ .  $E(s)$  is non-empty, compact and convex in the weak  $*$ -topology. By Krein-Milman’s theorem,  $E(s) = \text{closure}(\text{Extreme Points})$ . Any extreme point will then be a pure state extension of  $s$ ; but which one to choose? It’s the uniqueness part that is the famous KS problem.

<sup>4</sup>Equivalently, pure states sit inside the projective vector space. If  $\mathcal{H} = \mathbb{C}^{n+1}$ , pure states is  $\mathbb{C}P^n$ .

Physics	Mathematics
	<p> <math>H^* = H, v \in \mathcal{H}, \ v\  = 1</math>                      (state). Spectral theorem:  <math>H \sim P_H(\cdot)</math>                      projection-valued measure.                 </p> <p>                     Measurement:  <math>Prob(H \in (a, b)) = \ P_H(a, b)v\ ^2</math> </p>

Figure 8.1: Observable, state, measurement. Left column: An idealized physics experiment. Right: the mathematical counterpart, a selfadjoint operator  $H$ , its associated projection-valued measure  $P_H$ , and a norm-one vector  $v$  in Hilbert space.

**Exercise 8.5** (Non-normal pure states on  $\mathcal{B}(l^2)$ ). Show that there are pure states on  $\mathcal{B}(l^2)$  which do not have the form given in Lemma 8.3.

Hint:

- Step 1. The states listed in Lemma 8.3 have cardinality  $c = 2^{\aleph_0}$ .
- Step 2. The pure states of  $C(\beta(\mathbb{N}))$  are given by points in  $\beta(\mathbb{N})$ , and the cardinality of  $\beta(\mathbb{N})$  is
 
$$2^{2^{\aleph_0}} > c. \tag{8.2}$$
- Step 3. By Krien-Milman, every pure state on  $\mathcal{D} (\simeq l^2(\mathbb{N}))$  has a pure state extension to  $\mathcal{B}(l^2)$ .
- Step 4. Use (8.2) in step 2 to conclude that some of these pure state extensions to  $\mathcal{B}(l^2)$  are *not* of the form given in Lemma 8.3.

**Exercise 8.6** (The Calkin algebra and Non-normal states on  $\mathcal{B}(l^2)$ ). Let  $\mathcal{K} \subset \mathcal{B}(l^2)$  be the ideal of all compact operators in  $l^2$ ; then the quotient

$$\mathcal{C} := \mathcal{B}(l^2) / \mathcal{K}$$

is called the *Calkin algebra*. Show that the quotient is a  $C^*$ -algebra.

Hint: Be careful in defining its  $C^*$ -norm.

**Exercise 8.7** (The pure state  $\varphi = s \circ \pi$  on  $\mathcal{B}(l^2)$ ). Let  $\pi : \mathcal{B}(l^2) \rightarrow \mathcal{C}$  be the natural quotient mapping, and let  $s$  be a pure state on  $\mathcal{C}$ . Show that the composition

$$\varphi := s \circ \pi \quad (\text{see Fig 8.2.})$$

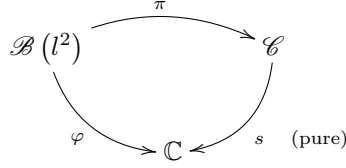


Figure 8.2: The pure state  $\varphi = s \circ \pi$  on  $\mathcal{B}(l^2)$

is a pure state on  $\mathcal{B}(l^2)$ , and that  $\varphi$  does not have the form in Lemma 8.3.

Hint: Suppose to the contrary, i.e., suppose  $\exists x \in l^2, \|x\| = 1$  such that

$$\varphi(A) = \langle x, Ax \rangle = \omega_x(A), \forall A \in \mathcal{B}(l^2). \tag{8.3}$$

We have  $\omega_x(|x\rangle\langle x|) = 1$ , but  $|x\rangle\langle x| \in \mathcal{K}$ , so  $\varphi(|x\rangle\langle x|) = s(0) = 0$ ; a contradiction. Hence (8.3) cannot hold for any state-vector  $x \in l^2$ .

**Exercise 8.8** (The Stone-Ćech compactification). Extend  $+$  on  $\mathbb{N}$  to a “ $+$ ” on  $\beta\mathbb{N}$  (the Stone-Ćech compactification).

Hint:

1. For subsets  $A \subset \mathbb{N}$ , and  $n \in \mathbb{N}$ , set  $A - n := \{k \in \mathbb{N} \mid k + n \in A\}$ .
2. Let  $F$  and  $G$  be ultra-filters on  $\mathbb{N}$ , and set

$$F + G := \{A \subset \mathbb{N} \mid \{n \in \mathbb{N}; A - n \in F\} \in G\}. \tag{8.4}$$

3. Show that  $F + G$  is an ultra-filter.
4. Show that the “addition” operation “ $+$ ” in (8.4) is an operation on  $\beta\mathbb{N}$ , i.e.,  $\beta\mathbb{N} \times \beta\mathbb{N} \rightarrow \beta\mathbb{N}$  which is associative, but not commutative, i.e.,  $F + G \neq G + F$  may happen.
5. Fix  $F \in \beta\mathbb{N}$ , and show that

$$\beta\mathbb{N} \ni G \mapsto F + G \in \beta\mathbb{N}$$

is continuous, where  $F + G$  is defined in (8.4).

Ultra-filters define pure states of  $l^\infty$  as follows: If  $(x_n)_{n \in \mathbb{N}} \in l^\infty$ , and if  $F \in \beta\mathbb{N}$ , i.e., is an ultra-filter, then there is a well-defined limit

$$\lim_F x_n = \varphi_F(x);$$

and this defines  $\varphi_F$  as a state on  $l^\infty$ .

## 8.1 The Dixmier Trace

A related use of ultra-filters yield the famous *Dixmier-trace*. For this we need ultra-filters  $\omega$  on  $\mathbb{N}$  with the following properties:

- (i)  $x_n \geq 0 \implies \lim_{\omega} x_n \geq 0$ .
- (ii) If  $x_n$  is convergent with limit  $x$ , then  $\lim_{\omega} x_n = x$ .
- (iii) For  $n \in \mathbb{N}$ , set

$$\sigma_N(x) = \left( \underbrace{x_1 \cdots x_1}_{N \text{ times}}, \underbrace{x_2 \cdots x_2}_{N \text{ times}}, \underbrace{x_3 \cdots x_3}_{N \text{ times}}, \cdots \right)$$

then  $\lim_{\omega}(x_n) = \lim_{\omega}(\sigma_N(x))$ .

Let  $A$  be a compact operator, and assume the eigenvalues  $\lambda_k$  of  $|A| = \sqrt{A^*A}$  as

$$\lambda_1 \geq \lambda_2 \geq \cdots, \lambda_k = \lambda_k(A)$$

and set

$$tr_{Dix,\omega}(A) = \lim_{\omega} \frac{1}{\log(n+1)} \sum_{k=1}^n \lambda_k(A). \quad (8.5)$$

**Definition 8.9.** We say that  $A$  has finite Dixmier trace if the limit in (8.5) is finite.

**Exercise 8.10** (The Dixmier trace). Show that (8.5) is well-defined and that:

1.  $A \mapsto tr_{Dix,\omega}(A)$  is linear, and positive.
2.  $tr_{Dix,\omega}(AB) = tr_{Dix,\omega}(BA)$  holds if  $B$  is bounded, and  $A$  has finite Dixmier trace.
3. If  $\sum_k \lambda_k(A) < \infty$ , then  $tr_{Dix,\omega}(A) = 0$ .

## 8.2 Frames in Hilbert Space

The proof of the KS-problem involves systems of vectors in Hilbert space called *frames*. For details we refer to [Cas13].

Below we include a sketch with some basic fact about frames; also called “generalized bases”, see Definition 8.11 below. The general idea is that a “frame expansion” inherits some (but not all) attractive properties and features of expansions in ONBs. In frame-analysis, this then offers the desirable feature of more flexibility in a host of applications; see e.g., [CFMT11] and [Chr96, HKLW07]. But we also give up something. For example, by contrast to what holds for an ONB, non-uniqueness is a fact of life for frame expansions.

Let  $\mathcal{H}$  be a separable Hilbert space, and let  $\{u_k\}_{k \in \mathbb{N}}$  be an ONB, then we have the following unique representation

$$w = \sum_{k \in \mathbb{N}} \langle u_k, w \rangle_{\mathcal{H}} u_k \quad (8.6)$$

valid for all  $w \in \mathcal{H}$ . Moreover,

$$\|w\|_{\mathcal{H}}^2 = \sum_{k \in \mathbb{N}} |\langle u_k, w \rangle_{\mathcal{H}}|^2, \quad (8.7)$$

the Parseval-formula.

**Definition 8.11.** A system  $\{v_k\}_{k \in \mathbb{N}}$  in  $\mathcal{H}$  is called a frame if there are constants  $A, B$  such that  $0 < A \leq B < \infty$ , and

$$A \|w\|_{\mathcal{H}}^2 \leq \sum_{k \in \mathbb{N}} |\langle v_k, w \rangle_{\mathcal{H}}|^2 \leq B \|w\|_{\mathcal{H}}^2 \quad (8.8)$$

holds for all  $w \in \mathcal{H}$ .

Note that (8.8) generalizes (8.7). Below we show that, for frames, there is also a natural extension of (8.6).

**Proposition 8.12.** Let  $\{v_k\}_{k \in \mathbb{N}}$  be a frame in  $\mathcal{H}$ ; then there is a dual system  $\{v_k^*\}_{k \in \mathbb{N}} \subset \mathcal{H}$  such that the following representation holds:

$$w = \sum_{k \in \mathbb{N}} \langle v_k^*, w \rangle_{\mathcal{H}} v_k \quad (8.9)$$

for all  $w \in \mathcal{H}$ ; absolute convergence.

*Proof.* Define the following operator  $T : \mathcal{H} \rightarrow l^2(\mathbb{N})$  by

$$Tw = (\langle v_k, w \rangle_{\mathcal{H}})_{k \in \mathbb{N}},$$

and show that the adjoint  $T^* : l^2(\mathbb{N}) \rightarrow \mathcal{H}$  satisfies

$$T^*((x_k)) = \sum_{k \in \mathbb{N}} x_k v_k.$$

Hence

$$T^*Tw = \sum_{k \in \mathbb{N}} \langle v_k, w \rangle_{\mathcal{H}} v_k. \quad (8.10)$$

It follows that  $T^*T$  has a bounded inverse, in fact,  $A I_{\mathcal{H}} \leq T^*T \leq B I_{\mathcal{H}}$  in the order of selfadjoint operators. As a result,  $(T^*T)^{-1}$  and  $(T^*T)^{-\frac{1}{2}}$  are well-defined bounded operators.

Substitute  $(T^*T)^{-1}$  into (8.10) yields:

$$\begin{aligned} w &= \sum_{k \in \mathbb{N}} \langle v_k, (T^*T)^{-1} w \rangle_{\mathcal{H}} v_k \\ &= \sum_{k \in \mathbb{N}} \langle (T^*T)^{-1} v_k, w \rangle_{\mathcal{H}} v_k \end{aligned}$$

which is the desired (8.9) with  $v_k^* := (T^*T)^{-1} v_k$ .  $\square$



**Exercise 8.13** (Frames from Lax-Milgram). Show that the conclusion in Proposition 8.12 may also be obtained from an application of the Lax-Milgram lemma (see Exercise 1.65.)

Specifically, starting with a frame and given frame constants (see (8.8)), write down the corresponding sesquilinear form  $B$  in Lax-Milgram, and verify that it satisfies the premise in Lax-Milgram. Relate the frame bounds to the constants  $b$ , and  $c$  in Lax-Milgram.

**Corollary 8.14.** Let  $\{v_k\}$  be as in Proposition 8.12, and set  $v_k^{**} := (T^*T)^{-\frac{1}{2}} v_k$ ,  $k \in \mathbb{N}$ ; then

$$w = \sum_{k \in \mathbb{N}} \langle v_k^{**}, w \rangle_{\mathcal{H}} v_k^{**}$$

holds for all  $w \in \mathcal{H}$ ; absolute convergence.

*Remark 8.15.* We saw in Chapter 3 (Section 3.3) that, if  $\{v_k\}_{k \in \mathbb{N}}$  is an ONB in some fixed Hilbert space  $\mathcal{H}$ , then

$$P(\Delta) = \sum_{k \in \Delta} |v_k\rangle\langle v_k|, \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (8.11)$$

is a projection valued measure (PVM) on  $\mathbb{R}$ .

Suppose now that some  $\{v_k\}_{k \in \mathbb{N}}$  in the expression (8.11) is only assumed to be a *frame*, see Definition 8.11.

**Exercise 8.16** (Positive operator valued measures from frames). Write down the modified list of properties for  $P(\cdot)$  in (8.11) which generalize the axioms of Definition 3.39 for PVMs.

*Remark 8.17.* It is possible to have *uniqueness* for *non-orthogonal* expansions in Hilbert space. The following theorem of Rota et al. is a case in point.

**Theorem 8.18** (Rota et al. [BR60, SN53]). Let  $\mathcal{H}$  be a separable Hilbert space; let  $\{e_k\}_{k \in \mathbb{N}}$  be an ONB in  $\mathcal{H}$ ; and let  $\{v_k\}_{k \in \mathbb{N}}$  be a linearly independent system of vectors in  $\mathcal{H}$  such that

$$\sum_{k=1}^{\infty} \|e_k - v_k\|^2 < \infty; \quad (8.12)$$

then every vector  $u \in \mathcal{H}$  has a unique representation

$$u = \sum_{k=1}^{\infty} x_k v_k, \quad x_k \in \mathbb{C}. \quad (8.13)$$

Moreover defining

$$B \left( \sum_k x_k e_k \right) := \sum_k x_k v_k, \quad (x_k) \in \ell^2; \quad (8.14)$$

we get the following conclusions:

- (i)  $B - I$  is compact; and
- (ii)  $\text{ran}(B) = \mathcal{H}$ .

**Exercise 8.19** (The operator  $B$ ). Fill in the missing details in the proof of Theorem 8.18.

The primary source on KS is the paper by R.V. Kadison and I.M. Singer [KS59]. An important early paper is [And79a] by Joel Anderson.

Since the 1970ties, the KS problem has been studied with the use of “pavings;” see e.g., [AAT14, SWZ11, CFMT11, Wea03, BT91]. While this [“pavings” and their equivalents] is an extremely interesting area, it is beyond the scope of the present book.

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest:

The pioneering paper [KS59] started the subject, and in the intervening decades there have been advances, and a discovery of the relevance of the KS-problem to a host of applied areas, especially harmonic analysis, frame theory, and signal processing. The problem was solved two years ago.

The most current paper concerning the solution to KS appears to be [MSS15] by Marcus, Spielman, and Strivastava. Paper [Cas14] by P. Casazza explains the problem and its implications. A more comprehensive citation list is: [Arv76, BR81b, Cas13, Cas14, AW14, MSS15, AAT14, And79a, BT91, Chr96, KS59, Wea03, CFMT11, Dix81, BP44, MJD<sup>+</sup>15].

## Part IV

# Extension of Operators

## Chapter 9

# Selfadjoint Extensions

Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe.

— Jacques Hadamard

It will interest mathematical circles that the mathematical instruments created by the higher algebra play an essential part in the rational formulation of the new quantum mechanics. Thus the general proofs of the conservation theorems in Heisenberg's theory carried out by Born and Jordan are based on the use of the theory of matrices, which go back to Cayley and were developed by Hermite. It is to be hoped that a new era of mutual stimulation of mechanics and mathematics has commenced. To the physicist it will seem first deplorable that in atomic problems we have apparently met with such a limitation of our usual means of visualisation. This regret will, however, have to give way to thankfulness that mathematics, in this field too, presents us with the tools to prepare the way for further progress.

— Niels Bohr

“Science is spectral analysis. Art is light synthesis.”

— Karl Kraus

Because of dictates from applications (especially quantum physics), below we stress questions directly related to key-issues for unbounded linear operators: Some operator from physics may only be “formally selfadjoint” also called Hermitian; and in such cases, one ask for selfadjoint extensions (if any).

The axioms of quantum physics (see e.g., [BM13, OH13, KS02, CRKS79, ARR13, Fan10, Maa10, Par09] for relevant recent papers), are based on Hilbert space, and selfadjoint operators.

A quantum mechanical observable is a Hermitian (selfadjoint) linear operator mapping a Hilbert space, the space of states, into itself. The values obtained in a physical measurement are in general described by a probability distribution; and the distribution represents a suitable “average” (or “expectation”) in a measurement of values of some quantum observable in a state of some prepared system. The states are (up to phase) unit vectors in the Hilbert space, and a measurement corresponds to a probability distribution (derived from a projection-valued spectral measure). The particular probability distribution used depends on both the state and the selfadjoint operator. The associated spectral type may be continuous (such as position and momentum; both unbounded) or discrete (such as spin); this depends on the physical quantity being measured.

Since the spectral theorem serves as the central tool in quantum measurements (see [Sto90, Yos95, Nel69, RS75, DS88c]), we must be precise about the distinction between linear operators with dense domain which are only Hermitian (formally selfadjoint) as opposed to selfadjoint. This distinction is accounted for by von Neumann’s theory of deficiency indices [AG93, DS88c, HdSS12]<sup>1</sup>.

## 9.1 Extensions of Hermitian Operators

In order to apply spectral theorem, one must work with self adjoint operators including the unbounded ones. Some examples first.

In quantum mechanics [Pol02, PK88, CP82], to understand energy levels of atoms and radiation, the energy level comes from discrete packages. The interactions are given by Coulomb’s Law where

$$H = -\Delta_{\vec{r}} + \frac{c_{jk}}{\|r_j - r_k\|}$$

and Laplacian has dimension  $3 \times \#(\text{electrons})$ .

In Schrödinger’s wave mechanics, one needs to solve for  $\psi(r, t)$  from the equation

$$H\psi = \frac{1}{i} \frac{\partial}{\partial t} \psi.$$

If we apply spectral theorem, then  $\psi(t) = e^{itH}\psi(r, t = 0)$ . This shows that motion in quantum mechanics is governed by unitary operators. The two parts in Schrödinger equation are separately selfadjoint, but justification of the sum being selfadjoint wasn’t made rigorous until 1957 when Kato wrote the book on “perturbation theory” [Kat95]. It is a summary of the sum of selfadjoint operators.

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<sup>1</sup>Starting with [vN32a, vN32c, vN32b], J. von Neumann and M. Stone did pioneering work in the 1930s on spectral theory for unbounded operators in Hilbert space; much of it in private correspondence. The first named author has from conversations with M. Stone, that the notions “deficiency-index,” and “deficiency space” are due to them; suggested by MS to vN as means of translating more classical notions of “boundary values” into rigorous tools in abstract Hilbert space: closed subspaces, projections, and dimension count.

In Heisenberg's matrix mechanics, he suggested that one should look at two states and the transition probability between them, such that

$$\langle \psi_1, A\psi_2 \rangle = \langle \psi_1(t), A\psi_2(t) \rangle, \quad \forall t.$$

If  $\psi(t) = e^{itH}\psi$ , then it works. In Heisenberg's picture, one looks at evolution of the observables  $e^{-itH}Ae^{itH}$ . In Schrödinger's picture, one looks at evolution of states. The two point of views are equivalent.

Everything so far is based on application of the spectral theorem, which requires the operators being selfadjoint in the first place.

von Neumann's index theory gives a complete classification of extensions of single Hermitian unbounded operators with dense domain in a given Hilbert space. The theory may be adapted to Hermitian representations of \*-algebras [Nel59a].

Let  $A$  be a densely defined Hermitian operator on a Hilbert space  $\mathcal{H}$ , i.e.  $A \subset A^*$ . If  $B$  is any Hermitian extension of  $A$ , then

$$A \subset B \subset B^* \subset A^*. \quad (9.1)$$

Since the adjoint operator  $A^*$  is closed, i.e.,  $\mathcal{G}(A^*)$  is closed in  $\mathcal{H} \oplus \mathcal{H}$ , it follows that  $\overline{\mathcal{G}(A)} \subset \mathcal{G}(A^*)$  is a well-defined operator graph, i.e.,  $A$  is closable and  $\overline{\mathcal{G}(A)} = \mathcal{G}(\overline{A})$ . Thus, there is no loss of generality to assume that  $A$  is closed and only consider its closed extensions.

The containment (9.1) suggests a detailed analysis in  $\mathcal{D}(A^*) \setminus \mathcal{D}(A)$ . Since  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ , the usual structural analysis in  $\mathcal{H}$  (orthogonal decomposition, etc.) is not applicable. However, this structure is brought out naturally when  $\mathcal{D}(A^*)$  is identified with the operator graph  $\mathcal{G}(A^*)$  in  $\mathcal{H} \oplus \mathcal{H}$ . That is,  $\mathcal{D}(A^*)$  is a Hilbert space under its graph norm. With this identification,  $\mathcal{D}(A)$  becomes a closed subspace in  $\mathcal{D}(A^*)$ , and

$$\mathcal{D}(A^*) = \mathcal{D}(A) \oplus (\mathcal{D}(A^*) \ominus \mathcal{D}(A)). \quad (9.2)$$

The question of extending  $A$  amounts to a further decomposition

$$\mathcal{D}(A^*) \ominus \mathcal{D}(A) = S \oplus K \quad (9.3)$$

in such a way that

$$\tilde{A} = A^*|_{\mathcal{D}(\tilde{A})}, \quad \text{where} \quad (9.4)$$

$$\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \oplus S \quad (9.5)$$

defines a (closed) Hermitian operator  $\tilde{A} \supset A$ .

The extension  $\tilde{A}$  in (9.4)-(9.5) is Hermitian iff the closed subspace  $S \subset \mathcal{D}(A^*)$  is symmetric, in the sense that

$$\langle A^*y, x \rangle - \langle y, A^*x \rangle = 0, \quad \forall x, y \in S. \quad (9.6)$$

**Lemma 9.1.** *Let  $S$  be a closed subspace in  $\mathcal{D}(A^*)$ , where  $\mathcal{D}(A^*)$  is a Hilbert space under the  $A^*$ -norm. The following are equivalent.*

1.  $\langle A^*y, x \rangle = \langle y, A^*x \rangle$ , for all  $x, y \in S$ .
2.  $\langle x, A^*x \rangle \in \mathbb{R}$ , for all  $x \in S$ .

*Proof.* If (1) holds, setting  $x = y$ , we get  $\langle x, A^*x \rangle = \langle A^*x, x \rangle = \overline{\langle x, A^*x \rangle}$ , which implies that  $\langle x, A^*x \rangle$  is real-valued.

Conversely, assume (2) is true. Since the mappings

$$\begin{aligned} (x, y) &\mapsto \langle y, A^*x \rangle \\ (x, y) &\mapsto \langle A^*y, x \rangle \end{aligned}$$

are both sesquilinear forms on  $S \times S$  (linear in the second variable, and conjugate linear in the first variable), we apply the polarization identity:

$$\begin{aligned} \langle y, A^*x \rangle &= \frac{1}{4} \sum_{k=0}^3 i^k \langle x + i^k y, A^*(x + i^k y) \rangle \\ \langle A^*y, x \rangle &= \frac{1}{4} \sum_{k=0}^3 i^k \langle A^*(x + i^k y), x + i^k y \rangle \end{aligned}$$

for all  $x, y \in \mathcal{D}(A^*)$ . Now, since  $A$  is Hermitian, the RHSs of the above equations are equal; therefore,  $\langle y, A^*x \rangle = \langle A^*y, x \rangle$ , which is part (2).  $\square$

Eqs (9.4)-(9.5) and Lemma 9.1 set up a bijection between (closed) Hermitian extensions of  $A$  and (closed) symmetric subspaces in  $\mathcal{D}(A^*) \ominus \mathcal{D}(A)$ . Moreover, by Lemma 9.1, condition (9.6) is equivalent to

$$\langle x, A^*x \rangle \in \mathbb{R}, \forall x \in \mathcal{D}(A). \quad (9.7)$$

Let  $\varphi \in \mathcal{D}(A^*)$ , such that  $A^*\varphi = \lambda\varphi$ ,  $\Im\{\lambda\} \neq 0$ ; then  $\langle \varphi, A^*\varphi \rangle = \lambda\|\varphi\|^2 \notin \mathbb{R}$ . By Lemma 9.1 and (9.7),  $\varphi \notin \mathcal{D}(\tilde{A})$ , where  $\tilde{A}$  is any possible Hermitian extension of  $A$ . This observation is in fact ruling out the “wrong” eigenvalues of  $\tilde{A}$ . Indeed, Theorem 9.4 below shows that  $A$  is selfadjoint if and only if ALL the “wrong” eigenvalues of  $A^*$  are excluded. But first we need the following lemma.

**Lemma 9.2.** *Let  $A$  be a Hermitian operator in  $\mathcal{H}$ , then*

$$\|(A - \lambda)x\|^2 = \|(A - a)x\|^2 + |b|^2\|x\|^2, \forall \lambda = a + ib \in \mathbb{C}. \quad (9.8)$$

*In particular,*

$$\|(A - \lambda)x\|^2 \geq |\Im\{\lambda\}|^2\|x\|^2, \forall \lambda \in \mathbb{C}. \quad (9.9)$$

*Proof.* Write  $\lambda = a + ib$ ,  $a, b \in \mathbb{R}$ ; then

$$\begin{aligned} &\|(A - \lambda)x\|^2 \\ &= \langle (A - a)x - ibx, (A - a)x - ibx \rangle \end{aligned}$$

$$\begin{aligned}
&= \|(A-a)x\|^2 + |b|^2 \|x\|^2 - i(\langle (A-a)x, x \rangle - \langle x, (A-a)x \rangle) \\
&= \|(A-a)x\|^2 + |b|^2 \|x\|^2 \\
&\geq |b|^2 \|x\|^2;
\end{aligned}$$

where  $\langle (A-a)x, x \rangle - \langle x, (A-a)x \rangle = 0$ , since  $A-a$  is Hermitian.  $\square$

**Corollary 9.3.** *Let  $A$  be a closed Hermitian operator acting in  $\mathcal{H}$ . Fix  $\lambda \in \mathbb{C}$  with  $\Im\{\lambda\} \neq 0$ , then  $\text{ran}(A-\lambda)$  is a closed subspace in  $\mathcal{H}$ . Consequently, we get the following decomposition*

$$\mathcal{H} = \text{ran}(A-\lambda) \oplus \ker(A^* - \bar{\lambda}). \quad (9.10)$$

*Proof.* Set  $B = A-\lambda$ ; then  $B$  is closed, and so is  $B^{-1}$ , i.e., the operator graphs  $\mathcal{G}(B)$  and  $\mathcal{G}(B^{-1})$  are closed in  $\mathcal{H} \oplus \mathcal{H}$ . Therefore,  $\text{ran}(B)$  ( $= \text{dom}(B^{-1})$ ) is closed in  $\|\cdot\|_{B^{-1}}$ -norm. But by (9.9),  $B^{-1}$  is bounded on  $\text{ran}(B)$ , thus the two norms  $\|\cdot\|$  and  $\|\cdot\|_{B^{-1}}$  are equivalent on  $\text{ran}(B)$ . It follows that  $\text{ran}(B)$  is also closed in  $\|\cdot\|$ -norm, i.e., it is a closed subspace in  $\mathcal{H}$ . The decomposition (9.10) follows from this.  $\square$

**Theorem 9.4.** *Let  $A$  be a densely defined, closed, Hermitian operator in a Hilbert space  $\mathcal{H}$ ; then the following are equivalent:*

$$\left\{ \exists \lambda, \Im\{\lambda\} \neq 0, \ker(A^* - \lambda) = \ker(A^* - \bar{\lambda}) = 0 \right\} \iff \left\{ A = A^* \right\}.$$

*Proof.*  $\implies$  By Corollary 9.3, the hypothesis in the theorem implies that

$$\text{ran}(A-\lambda) = \text{ran}(A-\bar{\lambda}) = \mathcal{H}.$$

Let  $y \in \mathcal{D}(A^*)$ , then

$$\langle y, (A-\lambda)x \rangle = \langle (A^* - \bar{\lambda})y, x \rangle, \quad \forall x \in \mathcal{D}(A). \quad (9.11)$$

Since  $\text{ran}(A-\bar{\lambda}) = \mathcal{H}$ ,  $\exists y_0 \in \mathcal{D}(A)$  such that

$$(A^* - \bar{\lambda})y = (A - \bar{\lambda})y_0.$$

Hence, RHS of (9.11) is

$$\langle (A - \bar{\lambda})y_0, x \rangle = \langle y_0, (A - \lambda)x \rangle. \quad (9.12)$$

Combining (9.11)-(9.12), we then get

$$\langle y - y_0, (A - \lambda)x \rangle = 0, \quad \forall x \in \mathcal{D}(A).$$

Again, since  $\text{ran}(A-\lambda) = \mathcal{H}$ , the last equation above shows that  $y - y_0 \perp \mathcal{H}$ . In particular,  $y - y_0 \perp y - y_0$ , i.e.,

$$\|y - y_0\|^2 = \langle y - y_0, y - y_0 \rangle = 0.$$

Therefore,  $y = y_0$ , and so  $y \in \mathcal{D}(A)$ . This shows that  $A^* \subset A$ .

The other containment  $A \subset A^*$  holds since  $A$  is assumed to be Hermitian. Thus, we conclude that  $A = A^*$ .  $\square$



To capture all the “wrong” eigenvalues, we consider a family of closed subspace in  $\mathcal{H}$ ,  $\ker(A^* - \lambda)$ , where  $\Im\{\lambda\} \neq 0$ .

**Theorem 9.5.** *If  $A$  is a closed Hermitian operator in  $\mathcal{H}$ , then*

$$\dim(\ker(A^* - \lambda))$$

*is a constant function on  $\Im\{\lambda\} > 0$ , and  $\Im\{\lambda\} < 0$ .*

*Proof.* Fix  $\lambda$  with  $\Im\{\lambda\} > 0$ . For  $\Im\{\lambda\} < 0$ , the argument is similar. We proceed to verify that if  $\eta \in \mathbb{C}$ , close enough to  $\lambda$ , then  $\dim(\ker(A^* - \eta)) = \dim(\ker(A^* - \lambda))$ . The desired result then follows immediately.

Since  $A$  is closed, we have the following decomposition (by Corollary 9.3),

$$\mathcal{H} = \text{ran}(A - \bar{\lambda}) \oplus \ker(A^* - \lambda). \quad (9.13)$$

Now, pick  $x \in \ker(A^* - \eta)$ , and suppose  $x \perp \ker(A^* - \lambda)$ ; assuming  $\|x\| = 1$ . By (9.13),  $\exists x_0 \in \mathcal{D}(A)$  such that

$$x = (A - \bar{\lambda})x_0. \quad (9.14)$$

Then,

$$\begin{aligned} 0 = \langle (A^* - \eta)x, x_0 \rangle &= \langle x, (A - \bar{\eta})x_0 \rangle \\ &= \langle x, (A - \bar{\lambda})x_0 - (\bar{\eta} - \bar{\lambda})x_0 \rangle \\ &= \|x\|^2 - (\bar{\eta} - \bar{\lambda})\langle x, x_0 \rangle \\ &\geq \|x\|^2 - |\bar{\eta} - \bar{\lambda}|\|x\|\|x_0\| \quad (\text{Cauchy-Schwarz}) \\ &= 1 - |\eta - \lambda|\|x_0\|^2 \end{aligned} \quad (9.15)$$

Applying Lemma 9.2 to (9.14), we also have

$$1 = \|x\|^2 = \|(A - \bar{\lambda})x_0\|^2 \geq |\Im\{\lambda\}|^2 \|x_0\|^2;$$

substitute this into (9.15), we see that

$$0 \geq 1 - |\eta - \lambda|\|x_0\|^2 \geq 1 - |\eta - \lambda| |\Im\{\lambda\}|^{-2}$$

which would be a contradiction if  $\eta$  was close to  $\lambda$ .

It follows that the projection from  $\ker(A^* - \eta)$  to  $\ker(A^* - \lambda)$  is injective. For otherwise,  $\exists x \in \ker(A^* - \eta)$ ,  $x \neq 0$ , and  $x \perp \ker(A^* - \lambda)$ . This is impossible as shown above. Thus,

$$\dim(\ker(A^* - \eta)) \leq \dim(\ker(A^* - \lambda)).$$

Similarly, we get the reversed inequality, and so

$$\dim(\ker(A^* - \eta)) = \dim(\ker(A^* - \lambda)).$$

□

$$\mathcal{H} \left\{ \begin{array}{ccc} \mathcal{D}_+ & \xrightarrow{\text{Partial Isometry}} & \mathcal{D}_- \\ \oplus & & \oplus \\ (A+i)\mathcal{D} & \xrightarrow{C_A=(A-i)(A+i)^{-1}} & (A-i)\mathcal{D} \end{array} \right\} \mathcal{H}$$

 Figure 9.1:  $\mathcal{D}_\pm = \text{Ker}(A^* \mp i)$ ,  $\mathcal{D} = \text{dom}(A)$ 

A complete characterization of Hermitian extensions of a given Hermitian operator is due to von Neumann. Theorem 9.5 suggests the following definition:

**Definition 9.6.** Let  $A$  be a densely defined, closed, Hermitian operator in  $\mathcal{H}$ . The closed subspaces

$$\begin{aligned} \mathcal{D}_\pm(A) &= \ker(A^* \mp i) \\ &= \{\xi \in \mathcal{D}(A^*) : A^*\xi = \pm i\xi\} \end{aligned} \quad (9.16)$$

are called the *deficiency spaces* of  $A$ , and  $\dim \mathcal{D}_\pm(A)$  are called the *deficiency indices*.

For illustration, see Figure 9.1.

The role of the *Cayley-transform*  $C_A := (A - i)(A + i)^{-1}$ , and its extensions by partial isometries  $\mathcal{D}_+ \rightarrow \mathcal{D}_-$ , is illustrated in Figure 9.1. The Figure further offers a geometric account of the conclusion in Theorem 9.7.

As a result we see that the two subspaces  $\mathcal{D}_\pm$ , also called *defect-spaces* (or *deficiency-spaces*), are non-zero precisely when the given symmetric operator  $A$  fails to be essentially selfadjoint. The respective dimensions

$$n_\pm := \dim \mathcal{D}_\pm \quad (9.17)$$

are called *deficiency indices*. The pair  $(n_+, n_-)$  in (9.17) is called the pair of *von Neumann indices*. We note that  $A$  has selfadjoint extensions if and only if  $n_+ = n_-$ .

**Theorem 9.7** (von Neumann). *Let  $A$  be a densely defined closed Hermitian operator acting in  $\mathcal{H}$ . Then*

$$\mathcal{D}(A^*) = \mathcal{D}(A) \oplus \mathcal{D}_+(A) \oplus \mathcal{D}_-(A); \quad (9.18)$$

where  $\mathcal{D}(A^*)$  is identified with its graph  $\mathcal{G}(A^*)$ , thus a Hilbert space under the graph inner product; and the decomposition in (9.18) refers to this Hilbert space.

*Proof.* By assumption,  $A$  is closed, i.e.,  $\mathcal{D}(A)$ , identified with  $\mathcal{G}(A)$ , is a closed subspace in  $\mathcal{D}(A^*)$ .

Note that  $\mathcal{D}_\pm(A) = \ker(A^* \mp i)$  are closed subspaces in  $\mathcal{H}$ . Moreover,

$$\|x\|_{A^*}^2 = \|x\|^2 + \|A^*x\|^2 = 2\|x\|^2, \quad \forall x \in \mathcal{D}_\pm(A);$$

and so  $\mathcal{D}_\pm(A)$ , when identified with the graph of  $A^* \Big|_{\mathcal{D}_\pm(A^*)}$ , are also closed subspaces in  $\mathcal{D}(A^*)$ .

Next, we verify the three subspaces on RHS of (9.18) are mutually orthogonal. For all  $x \in \mathcal{D}(A)$ , and all  $x_+ \in \mathcal{D}_+(A) = \ker(A^* - i)$ , we have

$$\begin{aligned} \langle x_+, x \rangle_{A^*} &= \langle x_+, x \rangle + \langle A^* x_+, A^* x \rangle \\ &= \langle x_+, x \rangle - i \langle x_+, Ax \rangle \\ &= -i (\langle x_+, ix \rangle + \langle x_+, Ax \rangle) \\ &= -i \langle x_+, (A + i)x \rangle = 0 \end{aligned}$$

where the last step follows from  $x_+ \perp \text{ran}(A + i)$  in  $\mathcal{H}$ , see (9.10). Thus,  $\mathcal{D}(A) \perp \mathcal{D}_+(A)$  in  $\mathcal{D}(A^*)$ . Similarly,  $\mathcal{D}(A) \perp \mathcal{D}_-(A)$  in  $\mathcal{D}(A^*)$ .

Moreover, if  $x_+ \in \mathcal{D}_+(A)$  and  $x_- \in \mathcal{D}_-(A)$ , then

$$\begin{aligned} \langle x_+, x_- \rangle_{A^*} &= \langle x_+, x_- \rangle + \langle A^* x_+, A^* x_- \rangle \\ &= \langle x_+, x_- \rangle + \langle ix_+, -ix_- \rangle \\ &= \langle x_+, x_- \rangle - \langle x_+, x_- \rangle = 0. \end{aligned}$$

Hence  $\mathcal{D}_+(A) \perp \mathcal{D}_-(A)$  in  $\mathcal{D}(A^*)$ .

Finally, we show RHS of (9.18) yields the entire Hilbert space  $\mathcal{D}(A^*)$ . For this, let  $x \in \mathcal{D}(A^*)$ , and suppose (9.18) holds, say,  $x = x_0 + x_+ + x_-$ , where  $x \in \mathcal{D}(A)$ ,  $x_{\pm} \in \mathcal{D}_{\pm}(A)$ ; then

$$\begin{aligned} (A^* + i)x &= (A^* + i)(x_0 + x_+ + x_-) \\ &= (A + i)x_0 + 2ix_+. \end{aligned} \tag{9.19}$$

But, by the decomposition  $\mathcal{H} = \text{ran}(A + i) \oplus \ker(A^* - i)$ , eq. (9.10), there exist  $x_0$  and  $x_+$  satisfying (9.19). It remains to set  $x_- := x - x_0 - x_+$ , and to check  $x_- \in \mathcal{D}_-(A)$ . Indeed, by (9.19), we see that

$$A^*x - Ax_0 - ix_+ = -ix + ix_0 + ix_+; \text{ i.e.,}$$

$$A^*(x - x_0 - x_+) = -i(x - x_0 - x_+)$$

and so  $x_- \in \mathcal{D}_-(A)$ . Therefore, we get the desired orthogonal decomposition in (9.18).

Another argument: Let  $y \in \mathcal{D}(A^*)$  such that  $y \perp \mathcal{D}_{\pm}(A)$  in  $\mathcal{D}(A^*)$ . Then,  $y \perp \mathcal{D}_+(A)$  in  $\mathcal{D}(A^*) \implies$

$$\begin{aligned} 0 &= \langle y, x_+ \rangle + \langle A^*y, A^*x_+ \rangle \\ &= \langle y, x_+ \rangle + \langle A^*y, ix_+ \rangle \\ &= i(\langle iy, x_+ \rangle + \langle A^*y, x_+ \rangle) \\ &= i \langle (A^* + i)y, x_+ \rangle, \forall x_+ \in \mathcal{D}_+(A) = \ker(A^* - i) \end{aligned}$$

and so  $\exists x_1 \in \mathcal{D}(A)$ , and

$$(A^* + i)y = (A + i)x_1. \tag{9.20}$$

On the other hand,  $y \perp \mathcal{D}_-(A)$  in  $\mathcal{D}(A^*) \implies$

$$\begin{aligned} 0 &= \langle y, x_- \rangle + \langle A^* y, A^* x_- \rangle \\ &= \langle y, x_- \rangle + \langle A^* y, -i x_- \rangle \\ &= -i (\langle -i y, x_- \rangle + \langle A^* y, x_- \rangle) \\ &= i \langle (A^* - i) y, x_- \rangle, \quad \forall x_- \in \mathcal{D}_-(A) = \ker(A^* + i); \end{aligned}$$

hence  $\exists x_2 \in \mathcal{D}(A)$ , and

$$(A^* - i) y = (A - i) x_2. \quad (9.21)$$

Subtracting (9.20)-(9.21) then gives

$$y = \frac{x_1 + x_2}{2} \in \mathcal{D}(A).$$

□

*Remark 9.8.* More generally, there is a family of decompositions

$$\mathcal{D}(A^*) = \mathcal{D}(A) + \ker(A^* - z) + \ker(A^* - \bar{z}), \quad \forall z \in \mathbb{C}, \Im\{z\} \neq 0. \quad (9.22)$$

However, in the general case, we lose orthogonality.

*Proof.* Give  $z \in \mathbb{C}$ ,  $\Im\{z\} \neq 0$ , suppose  $x \in \mathcal{D}(A^*)$  can be written as

$$x = x_0 + x_+ + x_-;$$

where  $x_0 \in \mathcal{D}(A)$ ,  $x_+ \in \ker(A^* - z)$ , and  $x_- \in \ker(A^* - \bar{z})$ . Then

$$\begin{aligned} A^* x &= Ax_0 + zx_+ + \bar{z}x_- \\ \bar{z}x &= \bar{z}x_0 + \bar{z}x_+ + \bar{z}x_- \end{aligned}$$

and

$$(A^* - \bar{z})x = (A - \bar{z})x_0 + (z - \bar{z})x_+. \quad (9.23)$$

Now, we start with (9.23). By the decomposition

$$\mathcal{H} = \text{ran}(A - \bar{z}) \oplus \ker(A^* - z),$$

there exist unique  $x_0$  and  $x_+$  such that (9.23) holds. This defines  $x_0$  and  $x_+$ . Then, set

$$x_- := x - x_0 - x_+;$$

and it remains to check  $x_- \in \ker(A^* - \bar{z})$ . Indeed, by (9.23), we have

$$A^* x - Ax_0 - zx_+ = \bar{z}x - \bar{z}x_0 - \bar{z}x_+, \text{ i.e.,}$$

$$A^*(x - x_0 - x_+) = \bar{z}(x - x_0 - x_+)$$

thus,  $x_- \in \ker(A^* - \bar{z})$ . □

*Remark 9.9.* In the general decomposition (9.22), if  $f = x + x_+ + x_-$ ,  $g = y + y_+ + y_-$  where  $f, g \in \mathcal{D}(A)$ ,  $x_+, y_+ \in \ker(A^* - z)$ , and  $x_-, y_- \in \ker(A^* - \bar{z})$ ; then

$$\begin{aligned}
& \langle g, A^* f \rangle - \langle A^* g, f \rangle \\
&= \langle y + y_+ + y_-, A^*(x + x_+ + x_-) \rangle - \langle A^*(y + y_+ + y_-), x + x_+ + x_- \rangle \\
&= \langle y + y_+ + y_-, Ax + zx_+ + \bar{z}x_- \rangle - \langle Ay + zy_+ + \bar{z}y_-, x + x_+ + x_- \rangle \\
&= \underbrace{\langle y, Ax + zx_+ + \bar{z}x_- \rangle - \langle Ay, x + x_+ + x_- \rangle}_0 + \\
&\quad \underbrace{\langle y_+ + y_-, Ax \rangle - \langle zy_+ + \bar{z}y_-, x \rangle}_0 + \\
&\quad \langle y_+ + y_-, zx_+ + \bar{z}x_- \rangle - \langle zy_+ + \bar{z}y_-, x_+ + x_- \rangle \\
&= \langle y_+, zx_+ \rangle - \langle zy_+, x_+ \rangle + \langle y_-, \bar{z}x_- \rangle - \langle \bar{z}y_-, x_- \rangle \\
&\quad + \langle y_+, \bar{z}x_- \rangle + \langle y_-, zx_+ \rangle - \langle zy_+, x_- \rangle - \langle \bar{z}y_-, x_+ \rangle \\
&= (z - \bar{z}) \langle y_+, x_+ \rangle + (\bar{z} - z) \langle y_-, x_- \rangle + \\
&\quad \underbrace{\bar{z} \langle y_+, x_- \rangle + z \langle y_-, x_+ \rangle - \bar{z} \langle y_+, x_- \rangle - z \langle y_-, x_+ \rangle}_0 \\
&= (z - \bar{z}) (\langle y_+, x_+ \rangle - \langle y_-, x_- \rangle).
\end{aligned}$$

**Theorem 9.10** (von Neumann). *Let  $A$  be a densely defined closed Hermitian operator in  $\mathcal{H}$ .*

1. *The (closed) Hermitian extensions of  $A$  are indexed by partial isometries with initial space in  $\mathcal{D}_+(A)$  and final space in  $\mathcal{D}_-(A)$ .*
2. *Given a partial isometry  $U$  as above, the Hermitian extension  $\widetilde{A}_U \supset A$  is determined as follows:*

$$\begin{aligned}
& \widetilde{A}_U(x + (1 + U)x_+) = Ax + i(1 - U)x_+, \text{ where} \\
& \mathcal{D}(\widetilde{A}_U) = \{x + x_+ + Ux_+ : x \in \mathcal{D}(A), x_+ \in \mathcal{D}_+(A)\}
\end{aligned} \tag{9.24}$$

*Proof.* By the discussion in (9.6) and (9.7), and Lemma 9.1, it remains to characterize the closed symmetric subspaces  $S$  in  $\mathcal{D}_+(A) \oplus \mathcal{D}_-(A) (\subset \mathcal{D}(A^*))$ . For this, let  $x = x_+ + x_-$ ,  $x_\pm \in \mathcal{D}_\pm(A)$ , then

$$\begin{aligned}
\langle x, A^* x \rangle &= \langle x_+ + x_-, A(x_+ + x_-) \rangle \\
&= \langle x_+ + x_-, i(x_+ - x_-) \rangle \\
&= i \left( \|x_+\|^2 - \|x_-\|^2 - 2i\Im \{ \langle x_+, x_- \rangle \} \right) \\
&= i \left( \|x_+\|^2 - \|x_-\|^2 \right) + 2\Im \{ \langle x_+, x_- \rangle \}.
\end{aligned} \tag{9.25}$$

Thus,

$$\begin{aligned} \langle x, A^*x \rangle &\in \mathbb{R}, \quad \forall x \in S \\ &\Downarrow \\ S &= \{(x_+, x_-) : \|x_+\| = \|x_-\|, x_{\pm} \in \mathcal{D}_{\pm}(A)\} \end{aligned}$$

i.e.,  $S$  is identified with the graph of a partial isometry, say  $U$ , with initial space in  $\mathcal{D}_+(A)$  and final space in  $\mathcal{D}_-(A)$ .  $\square$

**Corollary 9.11.** *Let  $A$  be a densely defined, closed, Hermitian operator on  $\mathcal{H}$ , and set  $d_{\pm} = \dim(\mathcal{D}_{\pm}(A))$ ; then*

1.  $A$  is maximally Hermitian if and only if one of the deficiency indices is 0;
2.  $A$  has a selfadjoint extension if and only if  $d_+ = d_- \neq 0$ ;
3.  $\bar{A}$  is selfadjoint if and only if  $d_+ = d_- = 0$ .

*Proof.* Immediate from Theorem 9.7 and Theorem 9.10.  $\square$

**Example 9.12.**  $d_+ = d_- = 1$ . Let  $e_{\pm}$  be corresponding eigenvalues.  $e_+ \mapsto ze_-$  is the unitary operator sending one to the other eigenvalue. It is clear that  $|z| = 1$ . Hence the self adjoint extension is indexed by  $U_1(\mathbb{C})$ .

**Example 9.13.**  $d_+ = d_- = 2$ , get a family of extensions indexed by  $U_2(\mathbb{C})$ .

*Remark 9.14.* M. Stone and von Neumann are the two pioneers who worked at the same period. They were born at about the same time. Stone died at 1970's and von Neumann died in the 1950's.

There is a simple criterion to test whether a Hermitian operator has equal deficiency indices.

**Definition 9.15.** An operator  $J : \mathcal{H} \rightarrow \mathcal{H}$  is called a *conjugation* if

- $J$  is conjugate linear, i.e.,  $J(cx) = \bar{c}x$ , for all  $x \in \mathcal{H}$ , and all  $c \in \mathbb{C}$ ,
- $J^2 = 1$ , and
- $\langle Jx, Jy \rangle = \langle y, x \rangle$ , for all  $x, y \in \mathcal{H}$ .

**Theorem 9.16** (von Neumann). *Let  $A$  be a densely defined closed Hermitian operator in  $\mathcal{H}$ . Set  $d_{\pm} = \dim(\mathcal{D}_{\pm}(A))$ . Suppose  $AJ = JA$ , where  $J$  is a conjugation, then  $d_+ = d_-$ . In particular,  $A$  has selfadjoint extensions.*

*Proof.* Note that, by definition, we have  $\langle Jx, y \rangle = \langle Jx, J^2y \rangle = \langle Jy, x \rangle$ , for all  $x, y \in \mathcal{H}$ .

We proceed to show that  $J$  commutes with  $A^*$ . For this, let  $x \in \mathcal{D}(A)$ ,  $y \in \mathcal{D}(A^*)$ , then

$$\langle JA^*y, x \rangle = \langle Jx, A^*y \rangle = \langle AJx, y \rangle = \langle JAx, y \rangle = \langle Jy, Ax \rangle. \quad (9.26)$$

It follows that  $x \mapsto \langle Jy, Ax \rangle$  is bounded, and  $Jy \in \mathcal{D}(A^*)$ . Thus,  $J\mathcal{D}(A^*) \subset \mathcal{D}(A^*)$ . Since  $J^2 = 1$ ,  $\mathcal{D}(A^*) = J^2\mathcal{D}(A^*) \subset J\mathcal{D}(A^*)$ ; therefore,  $J\mathcal{D}(A^*) = \mathcal{D}(A^*)$ . Moreover, (9.26) shows that  $JA^* = A^*J$ .

Now if  $x \in \mathcal{D}_+(A)$ , then

$$A^*Jx = JA^*x = J(ix) = -iJx$$

i.e.,  $J\mathcal{D}_+(A) \subset \mathcal{D}_-(A)$ . Similarly,  $J\mathcal{D}_-(A) \subset \mathcal{D}_+(A)$ .

Using  $J^2 = 1$  again,  $\mathcal{D}_-(A) = J^2\mathcal{D}_-(A) \subset J\mathcal{D}_+(A)$ ; and we conclude that  $J\mathcal{D}_+(A) = \mathcal{D}_-(A)$ .

Since the restriction of  $J$  to  $\mathcal{D}_+(A)$  preserves orthonormal basis, we then get  $\dim(\mathcal{D}_+(A)) = \dim(\mathcal{D}_-(A))$ .  $\square$

## 9.2 Cayley Transform

There is an equivalent characterization of Hermitian extensions, taking place entirely in  $\mathcal{H}$  and without the identification of  $\mathcal{D}(A^*) \simeq \mathcal{G}(A^*)$ , where  $\mathcal{G}(A^*)$  is seen as a Hilbert space under its graph inner product. This is the result of the following observation.

**Lemma 9.17.** *Let  $A$  be a Hermitian operator acting in  $\mathcal{H}$ ; then*

$$\|(A \pm i)x\|^2 = \|x\|^2 + \|Ax\|^2, \quad \forall x \in \mathcal{D}(A). \quad (9.27)$$

*Proof.* See Lemma 9.2. Or, a direct computation shows that

$$\begin{aligned} \|(A + i)x\|^2 &= \langle (A + i)x, (A + i)x \rangle \\ &= \|x\|^2 + \|Ax\|^2 + i(\langle Ax, x \rangle - \langle x, Ax \rangle) \\ &= \|x\|^2 + \|Ax\|^2; \end{aligned}$$

where  $\langle Ax, x \rangle - \langle x, Ax \rangle = 0$  since  $A$  is Hermitian.  $\square$

**Theorem 9.18** (Cayley transform). *Let  $A$  be a densely defined, closed, Hermitian operator in  $\mathcal{H}$ .*

1. *The following subspaces in  $\mathcal{H}$  are isometrically isomorphic:*

$$\text{ran}(A \pm i) \simeq \mathcal{G}(A) \simeq \mathcal{D}(A).$$

*In particular,  $\text{ran}(A \pm i)$  are closed subspace in  $\mathcal{H}$ .*

2. *The map  $C_A : \text{ran}(A + i) \rightarrow \text{ran}(A - i)$  by*

$$(A + i)x \mapsto (A - i)x, \quad \forall x \in \mathcal{D}(A) \quad (9.28)$$

*is isometric. Equivalently,*

$$C_A x = (A - i)(A + i)^{-1}x \quad (9.29)$$

*for all  $x \in \text{ran}(A + i)$ .*

3. Moreover,

$$A = i(1 + C_A)(1 - C_A)^{-1}. \quad (9.30)$$

*Proof.* By (9.17),  $\text{ran}(A \pm i)$  are isometric to the graph of  $A$ , and the latter is closed (as a subset in  $\mathcal{H} \oplus \mathcal{H}$ ) since  $A$  is closed (i.e.,  $\mathcal{G}(A)$  is closed). Thus,  $\text{ran}(A \pm i)$  are closed in  $\mathcal{H}$ . Note this is also a result of Corollary 9.3.

The mapping (9.29) being isometric follows from (9.27).

By (9.28), we have

$$\begin{aligned} (1 - C_A)((A + i)x) &= (A + i)x - (A - i)x = 2ix \\ (1 + C_A)((A + i)x) &= (A + i)x + (A - i)x = 2Ax \end{aligned}$$

for all  $x \in \mathcal{D}(A)$ . It follows that

$$(1 + C_A)(1 - C_A)^{-1}(2ix) = (1 + C_A)((A + i)x) = 2Ax; \text{ i.e.,}$$

$$Ax = i(1 + C_A)(1 - C_A)^{-1}x, \quad \forall x \in \mathcal{D}(A)$$

which is (9.30).  $\square$

**Theorem 9.19.** *Suppose  $A$  is densely defined, closed, and Hermitian in  $\mathcal{H}$ . Then the family of (closed) Hermitian extensions of  $A$  is indexed by partial isometries  $U$  with initial space in  $\mathcal{D}_+(A)$  and final space in  $\mathcal{D}_-(A)$ . Given  $U$ , the corresponding extension  $\tilde{A}_U \supset A$  is determined by*

$$\begin{aligned} \tilde{A}_U(x + (1 - U)x_+) &= x + i(1 + U)x_+, \text{ where} \\ \text{dom}(\tilde{A}_U) &= \{x + (1 - U)x_+ : x \in \mathcal{D}(A), x_+ \in \mathcal{D}_+(A)\} \end{aligned}$$

Moreover,  $\tilde{A}_U$  is selfadjoint if and only if  $U$  is unitary from  $\mathcal{D}_+(A)$  onto  $\mathcal{D}_-(A)$ .

*Proof.* Since  $A$  is closed, we get the following decompositions (Corollary 9.3)

$$\begin{aligned} \mathcal{H} &= \text{ran}(A + i) \oplus \ker(A^* - i) \\ &= \text{ran}(A - i) \oplus \ker(A^* + i). \end{aligned}$$

By Theorem 9.18,  $C_A : \text{ran}(A + i) \rightarrow \text{ran}(A - i)$  is isometric. Consequently, getting a Hermitian extension of  $A$  amounts to choosing a partial isometry  $U$  with initial space in  $\ker(A^* - i) (= \mathcal{D}_+(A))$  and final space in  $\ker(A^* + i) (= \mathcal{D}_-(A))$ , such that

$$C_{\tilde{A}_U} := C_A \oplus U$$

is the Cayley transform of  $\tilde{A}_U \supset A$ .

Given  $U$  as above, for all  $x \in \mathcal{D}(A)$ ,  $x_+ \in \mathcal{D}_+(A)$ , we have

$$C_{\tilde{A}_U}((A + i)x \oplus x_+) = (A - i)x \oplus Ux_+.$$

Then,

$$(1 - C_{\tilde{A}_U})((A + i)x \oplus x_+) = ((A + i)x + x_+) - ((A - i)x + Ux_+)$$



$$\begin{aligned}
&= 2ix + (1 - U)x_+ \\
(1 + C_{\tilde{A}_U})(A + i)x \oplus x_+ &= ((A + i)x + x_+) + ((A - i)x + Ux_+) \\
&= 2Ax + (1 + U)x_+;
\end{aligned}$$

and so

$$i(1 + C_{\tilde{A}_U})(1 - C_{\tilde{A}_U})^{-1} \left( x + \frac{1}{2i}(1 - U)x_+ \right) = Ax + \frac{1}{2}(1 + U)x_+.$$

The theorem follows by setting  $x_+ := 2iy_+$ .  $\square$

### 9.3 Boundary Triple

In applications, especially differential equations, it is convenient to characterize selfadjoint extensions using boundary conditions. For recent applications, see [JPT12b, JPT12a, JPT14b]. A slightly modified version can be found in [dO09].

Let  $A$  be a densely defined, closed, Hermitian operator acting in a Hilbert space  $\mathcal{H}$ . Assume  $A$  has deficiency indices  $(d, d)$ ,  $d > 0$ , and so  $A$  has non-trivial selfadjoint extensions. By von Neumann's theorem (Theorem 9.7), for all  $x, y \in \mathcal{D}(A^*)$ , we have the following decomposition,

$$\begin{aligned}
x &= x_0 + x_+ + x_- \\
y &= y_0 + y_+ + y_-
\end{aligned}$$

where  $x_0, y_0 \in \mathcal{D}(A)$ ,  $x_+, y_+ \in \mathcal{D}_+(A)$ , and  $x_-, y_- \in \mathcal{D}_-(A)$ . Then,

$$\begin{aligned}
&\langle y, A^*x \rangle - \langle A^*y, x \rangle \\
&= \langle y_0 + y_+ + y_-, Ax_0 + i(x_+ - x_-) \rangle - \\
&\quad \langle Ay_0 + i(y_+ - y_-), x_0 + x_+ + x_- \rangle \\
&= \underbrace{\langle y_0, Ax_0 \rangle - \langle Ay_0, x_0 \rangle}_0 + \underbrace{\langle y_0, i(x_+ - x_-) \rangle - \langle Ay_0, x_+ + x_- \rangle}_0 + \\
&\quad \underbrace{\langle y_+ + y_-, Ax_0 \rangle - \langle i(y_+ - y_-), x_0 \rangle}_0 + \\
&\quad \langle y_+ + y_-, i(x_+ - x_-) \rangle - \langle i(y_+ - y_-), x_+ + x_- \rangle \\
&= 2i \{ \langle y_+, x_+ \rangle - \langle y_-, x_- \rangle \}. \tag{9.31}
\end{aligned}$$

Therefore, we see that

$$\left[ x, y \in \mathcal{D}(\tilde{A}), \tilde{A} \supset A, \text{ Hermitian extension} \right] \iff [\text{RHS of (9.31) vanishes}]$$

For selfadjoint extensions, this is equivalent to choosing a partial isometry  $U$  from  $\mathcal{D}_+(A)$  onto  $\mathcal{D}_-(A)$ , and setting

$$x_- = Ux_+, y_- = Uy_+; \text{ so that}$$

$$\begin{aligned}\langle y, A^*x \rangle - \langle A^*y, x \rangle &= 2i \{ \langle y_+, x_+ \rangle - \langle Uy_+, Ux_+ \rangle \} \\ &= 2i \{ \langle y_+, x_+ \rangle - \langle y_+, x_+ \rangle \} = 0.\end{aligned}$$

The discussion above leads to the following definition:

**Definition 9.20.** Let  $A$  be a densely defined, closed, Hermitian operator in  $\mathcal{H}$ . Suppose  $A$  has deficiency indices  $(d, d)$ ,  $d > 0$ . A boundary space for  $A$  is a triple  $(\mathcal{H}_b, \rho_1, \rho_2)$  consisting of a Hilbert space  $\mathcal{H}_b$  and two linear maps  $\rho_1, \rho_2 : \mathcal{D}(A^*) \rightarrow \mathcal{H}_b$ , such that

1.  $\rho_i(\mathcal{D}(A^*))$  is dense in  $\mathcal{H}_b$ ,  $i = 1, 2$ ; and
2. for all  $x, y \in \mathcal{D}(A^*)$ ,  $\exists c \neq 0$ , such that

$$\langle y, A^*x \rangle - \langle A^*y, x \rangle = c [\langle \rho_1(y), \rho_1(x) \rangle_b - \langle \rho_2(y), \rho_2(x) \rangle_b]. \quad (9.32)$$

*Remark 9.21.* In (9.31), we set

$$\begin{aligned}\mathcal{H}_b &= \mathcal{D}_+(A) \\ \rho_1(x_0 + x_+ + x_-) &= x_+ \\ \rho_2(x_0 + x_+ + x_-) &= Ux_+\end{aligned}$$

for any  $x = x_0 + x_+ + x_-$  in  $\mathcal{D}(A^*)$ . Then  $(\mathcal{H}_b, \rho_1, \rho_2)$  is a boundary space for  $A$ . In this special case,  $\rho_1, \rho_2$  are surjective. It is clear that the choice of a boundary triple is not unique. In applications,  $\mathcal{H}_b$  is usually chosen to have the same dimension as  $\mathcal{D}_\pm(A)$ .

Consequently, Theorem 9.10 can be restated as follows.

**Theorem 9.22.** Let  $A$  be a densely defined, closed, Hermitian operator in  $\mathcal{H}$ . Suppose  $A$  has deficiency indices  $(d, d)$ ,  $d > 0$ . Let  $(\mathcal{H}_b, \rho_1, \rho_2)$  be a boundary triple. Then the selfadjoint extensions of  $A$  are indexed by unitary operators  $U : \mathcal{H}_b \rightarrow \mathcal{H}_b$ , such that given  $U$ , the corresponding selfadjoint extension  $\widetilde{A}_U \supset A$  is determined by

$$\begin{aligned}\widetilde{A}_U &= A^* \Big|_{\mathcal{D}(\widetilde{A}_U)}, \text{ where} \\ \mathcal{D}(\widetilde{A}_U) &= \{x \in \mathcal{D}(A^*) : U\rho_1(x) = \rho_2(x)\}.\end{aligned}$$

Certain variations of Theorem 9.22 are convenient in the boundary value problems (BVP) of differential equations. In [DM91, GG91], a boundary triple  $(\mathcal{H}_b, \beta_1, \beta_2)$  is defined to satisfy

$$\langle x, A^*y \rangle - \langle A^*x, y \rangle = c' [\langle \beta_1(x), \beta_2(y) \rangle_b - \langle \beta_2(x), \beta_1(y) \rangle_b] \quad (9.33)$$

for all  $x, y \in \mathcal{D}(A^*)$ ; and  $c'$  is some nonzero constant. Also, see [JPT12b, JPT12a, JPT14b].

The connection between (9.32) and (9.33) is via the bijection

$$\begin{cases} \rho_1 &= \beta_1 + i\beta_2 \\ \rho_2 &= \beta_1 - i\beta_2 \end{cases} \iff \begin{cases} \beta_1 &= \frac{\rho_1 + \rho_2}{2} \\ \beta_2 &= \frac{\rho_1 - \rho_2}{2i} \end{cases}. \quad (9.34)$$

**Lemma 9.23.** *Under the bijection (9.34), we have*

$$\langle \rho_1(x), \rho_1(y) \rangle_b - \langle \rho_2(x), \rho_2(y) \rangle_b = 2i (\langle \beta_1(x), \beta_2(y) \rangle_b - \langle \beta_2(x), \beta_1(y) \rangle_b)$$

*Proof.* For convenience, we suppress the variables  $x, y$ . Then a direct computation shows that,

$$\begin{aligned} & \langle \rho_1, \rho_1 \rangle_b - \langle \rho_2, \rho_2 \rangle_b \\ &= \langle \beta_1 + i\beta_2, \beta_1 + i\beta_2 \rangle_b - \langle \beta_1 - i\beta_2, \beta_1 - i\beta_2 \rangle_b \\ &= i \langle \beta_1, \beta_2 \rangle_b - i \langle \beta_2, \beta_1 \rangle_b + i \langle \beta_1, \beta_2 \rangle_b - i \langle \beta_2, \beta_1 \rangle_b \\ &= 2i (\langle \beta_1, \beta_2 \rangle_b - \langle \beta_2, \beta_1 \rangle_b) \end{aligned}$$

which is the desired conclusion.  $\square$

**Theorem 9.24.** *Given a boundary triple  $(\mathcal{H}_b, \beta_1, \beta_2)$  satisfying (9.33), the family of selfadjoint extensions  $\widetilde{A}_U \supset A$  is indexed by unitary operators  $U : \mathcal{H}_b \rightarrow \mathcal{H}_b$ , such that*

$$\widetilde{A}_U = A^* \Big|_{\mathcal{D}(\widetilde{A}_U)}, \text{ where} \quad (9.35)$$

$$\mathcal{D}(\widetilde{A}_U) = \{x \in \mathcal{D}(A^*) : (1 - U)\beta_1(x) = i(1 + U)\beta_2(x)\}. \quad (9.36)$$

*Proof.* By Theorem 9.22, we need only pick a unitary operator  $U : \mathcal{H}_b \rightarrow \mathcal{H}_b$ , such that  $\rho_2 = U\rho_1$ . In view of the bijection (9.34), this yields

$$\beta_1 - i\beta_2 = U(\beta_1 + i\beta_2) \iff (1 - U)\beta_1 = i(1 + U)\beta_2$$

and the theorem follows.  $\square$

**Example 9.25.** Let  $A = -i \frac{d}{dx} \Big|_{\mathcal{D}(A)}$ , and

$$\mathcal{D}(A) = \{f \in L^2(0, 1) : f' \in L^2(0, 1), f(0) = f(1) = 0\}.$$

Then  $A^* = -i \frac{d}{dx} \Big|_{\mathcal{D}(A^*)}$ , where

$$\mathcal{D}(A^*) = \{f : f, f' \in L^2(0, 1)\}.$$

For all  $f, g \in \mathcal{D}(A^*)$ , using integration by parts, we get

$$\langle g, A^*f \rangle - \langle A^*g, f \rangle = -i \overline{g(x)} f(x) \Big|_0^1 = -i (\overline{g(1)}f(1) - \overline{g(0)}f(0)).$$

Let  $\mathcal{H}_b = \mathbb{C}$ , i.e., one-dimensional, and set

$$\rho_1(f) = f(1), \quad \rho_2(f) = f(0); \text{ then}$$

$$\langle g, A^*f \rangle - \langle A^*g, f \rangle = -i (\langle \rho_1(g), \rho_1(f) \rangle_b - \langle \rho_2(g), \rho_2(f) \rangle_b).$$

Therefore,  $(\mathcal{H}_b, \rho_1, \rho_2)$  is a boundary triple.

The family of selfadjoint extensions of  $A$  is given by the unitary operator

$$e^{i\theta} : \mathcal{H}_b \rightarrow \mathcal{H}_b, \text{ s.t. } \rho_2 = e^{i\theta} \rho_1;$$

i.e.,

$$\widetilde{A}_\theta = -i \frac{d}{dx} \Big|_{\{f \in \mathcal{D}(A^*) : f(0) = e^{i\theta} f(1)\}}.$$

**Example 9.26.** Let  $Af = -f''$ , with  $\mathcal{D}(A) = C_c^\infty(0, \infty)$ . Since  $A$  is Hermitian and  $A \geq 0$ , it follows that it has equal deficiency indices. Also,  $\mathcal{D}(A^*) = \{f, f'' \in L^2(0, \infty)\}$ , and  $A^*f = -f''$ ,  $\forall f \in \mathcal{D}(A^*)$ .

For  $f, g \in \mathcal{D}^*(A)$ , we have

$$\begin{aligned} \langle g, A^*f \rangle &= - \int_0^\infty \bar{g} f'' = - \left( [\bar{g}f' - \bar{g}'f]_0^\infty + \int_0^\infty \bar{g}'' f \right) \\ &= (\bar{g}f')(0) - (\bar{g}'f)(0) - \int_0^\infty \bar{g}'' f \\ &= (\bar{g}f')(0) - (\bar{g}'f)(0) + \langle A^*g, f \rangle \end{aligned}$$

and so

$$\langle g, A^*f \rangle - \langle A^*g, f \rangle = (\bar{g}f')(0) - (\bar{g}'f)(0).$$

Now, set  $\mathcal{H}_b = \mathbb{C}$ , i.e., one-dimensional, and

$$\beta_1(\varphi) = \varphi(0), \quad \beta_2(\varphi) = \varphi'(0); \text{ then}$$

$$\langle g, A^*f \rangle - \langle A^*g, f \rangle = \langle \beta_1(g), \beta_2(f) \rangle_b - \langle \beta_2(g), \beta_1(f) \rangle_b.$$

This defines the boundary triple.

The selfadjoint extensions are parameterized by  $e^{i\theta}$ , where

$$(1 - e^{i\theta}) \beta_1(f) = i(1 + e^{i\theta}) \beta_2(f); \text{ i.e.,}$$

$$f(0) = z f'(0), \quad f \in \mathcal{D}(A^*)$$

where

$$z = i \frac{1 + e^{i\theta}}{1 - e^{i\theta}}.$$

We take the convention that  $z = \infty \iff f'(0) = 0$ , i.e., the Neumann boundary condition.

**Example 9.27.**  $Af = -f''$ ,  $\mathcal{D}(A) = C_c^\infty(0, 1)$ ; then

$$\mathcal{D}(A^*) = \{f, f'' \in L^2(0, 1)\}.$$

Integration by parts gives

$$\langle g, A^*f \rangle = - \int_0^1 \bar{g} f''$$

$$\begin{aligned}
&= -[\bar{g}f' - \bar{g}'f]_0^1 - \int_0^1 \bar{g}''f \\
&= -[\bar{g}f' - \bar{g}'f]_0^1 + \langle A^*g, f \rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle g, A^*f \rangle - \langle A^*g, f \rangle &= [(\bar{g}f') (0) + (\bar{g}'f) (1)] - [(\bar{g}'f) (0) + (\bar{g}f') (1)] \\
&= \langle \beta_1 (g), \beta_2 (f) \rangle_b - \langle \beta_2 (g), \beta_1 (f) \rangle_b
\end{aligned}$$

where

$$\beta_1 (\varphi) = \begin{bmatrix} \varphi (0) \\ \varphi' (1) \end{bmatrix}, \quad \beta_2 (\varphi) = \begin{bmatrix} \varphi' (0) \\ \varphi (1) \end{bmatrix}.$$

The boundary space is  $\mathcal{H}_b = \mathbb{C}^2$ , i.e., 2-dimensional

The family of selfadjoint extensions is parameterized by  $U \in M(2, \mathbb{C})$ . Given  $U$ , the corresponding extension  $\widetilde{A}_U$  is determined by

$$\begin{aligned}
\widetilde{A}_U &= A^* \Big|_{\mathcal{D}(\widetilde{A}_U)}, \text{ where} \\
\mathcal{D}(\widetilde{A}_U) &= \{f \in \mathcal{D}(A^*) : (1 - U) \beta_1 (f) = i(1 + U) \beta_2 (f)\}.
\end{aligned}$$

*Remark 9.28.* Another choice of the boundary map:

$$\begin{aligned}
\langle g, A^*f \rangle - \langle A^*g, f \rangle &= [(\bar{g}f') (0) - (\bar{g}f') (1)] - [(\bar{g}'f) (0) - (\bar{g}'f) (1)] \\
&= \langle \beta_1 (g), \beta_2 (f) \rangle_b - \langle \beta_2 (g), \beta_1 (f) \rangle_b
\end{aligned}$$

where

$$\beta_1 (\varphi) = \begin{bmatrix} \varphi (0) \\ \varphi (1) \end{bmatrix}, \quad \beta_2 (\varphi) = \begin{bmatrix} \varphi' (0) \\ -\varphi' (1) \end{bmatrix}.$$

The selfadjoint boundary condition leads to

$$(1 - U) \begin{bmatrix} f (0) \\ f (1) \end{bmatrix} = i(1 + U) \begin{bmatrix} f' (0) \\ -f' (1) \end{bmatrix}.$$

For  $U = 1$ , we get the Neumann boundary condition:

$$f' (0) = f' (1) = 0.$$

For  $U = -1$ , we get the Dirichlet boundary condition:

$$f (0) = f (1) = 0.$$

**Exercise 9.29** (From selfadjoint extension to unitary one-parameter group).

1. For each of the selfadjoint extensions from Theorem 9.24, write down the corresponding unitary one-parameter group; and identify it as an induced representation; induction  $\mathbb{Z} \rightarrow \mathbb{R}$ ; see Section 7.4.
2. Same question for the selfadjoint extension operators computed in Examples 9.26 and 9.27.

### 9.4 The Friedrichs Extension

Let  $A : \mathcal{D} \rightarrow \mathcal{H}$  be an operator with dense domain  $\text{dom}(A) := \mathcal{D}$  in  $\mathcal{H}$ , such that

$$\langle \varphi, A\varphi \rangle \geq \|\varphi\|^2, \quad \forall \varphi \in \mathcal{D}. \tag{9.37}$$

Set  $\mathcal{H}_A :=$  Hilbert completion of  $\mathcal{D}$  with respect to the

$$\|\varphi\|_A := \langle \varphi, A\varphi \rangle^{\frac{1}{2}}. \tag{9.38}$$

Then  $\varphi \rightarrow \varphi$  defines a contraction  $J : \mathcal{H}_A \rightarrow \mathcal{H}$ , extending  $J\varphi = \varphi$ , for  $\varphi \in \mathcal{D}$ . Note that (9.37)  $\Leftrightarrow$

$$\|J\varphi\| \leq \|\varphi\|_A, \quad \forall \varphi \in \mathcal{H}_A,$$

(see (9.38).)

*Remark 9.30.* We will make use of two inner products:  $\langle \cdot, \cdot \rangle$  in  $\mathcal{H}$ , and  $\langle \cdot, \cdot \rangle_A$  (with subscript  $A$ ) in  $\mathcal{H}_A$ .

We have

$$\langle J\varphi, f \rangle = \langle \varphi, J^*f \rangle_A \tag{9.39}$$

see Figure 9.2:  $\varphi \in \mathcal{H}_A, f \in \mathcal{H}, J^*f \in \mathcal{H}_A$ .

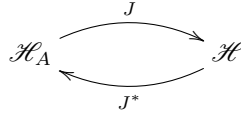


Figure 9.2: The operator  $J$  and its adjoint.

Note both  $J$  and  $J^*$  are contractions with respect to the respective norms, so

$$\|J^*f\|_A \leq \|f\|, \quad \forall f \in \mathcal{H}. \tag{9.40}$$

So  $JJ^* : \mathcal{H} \rightarrow \mathcal{H}$  is a contractive selfadjoint operator in  $\mathcal{H}$ , and  $(JJ^*)^{\frac{1}{2}}$  is well defined by the Spectral Theorem.

**Theorem 9.31.** *Let  $A, J, \mathcal{D}, \mathcal{H}, \mathcal{H}_A$  be as above. Then there is a selfadjoint extension  $\tilde{A} \supset A$  in  $\mathcal{H}$  such that*

$$\langle \tilde{A}x, y \rangle = \langle x, y \rangle_1, \quad \forall x \in \text{dom}(\tilde{A}), \forall y \in \text{dom}(A) (= \mathcal{D}).$$

*Proof.* The theorem is established in three steps:

Step 1.  $JJ^*A\varphi = \varphi, \forall \varphi \in \mathcal{D}$ .

Step 2.  $JJ^*$  is invertible (easy from (9.37).)

Step 3.  $A \subset (JJ^*)^{-1}$ , where  $(JJ^*)^{-1}$  is selfadjoint; it is the Friedrichs extension of  $A$ . Note Step 3 is immediate from step 1 by definition.

*Proof of Step 1.* Since  $\mathcal{D}$  is dense in  $\mathcal{H}$ , it is enough to prove that

$$\langle \psi, JJ^*A\varphi \rangle = \langle \psi, \varphi \rangle, \quad \forall \varphi, \psi \in \mathcal{D}. \quad (9.41)$$

Let  $\varphi, \psi \in \mathcal{D}$ , then

$$\begin{aligned} \text{LHS}_{(9.41)} &= \langle JJ^*\psi, A\varphi \rangle \\ &= \langle J^*\psi, A\varphi \rangle \\ &= \langle J^*\psi, \varphi \rangle_A \quad (\text{by (9.37) \& (9.38)}) \\ &= \langle \psi, J\varphi \rangle \quad (\text{by (9.39) and use } J^{**} = J) \\ &= \langle \psi, \varphi \rangle = \text{RHS}_{(9.41)}. \end{aligned}$$

Hence Step 1 follows.  $\square$

Let  $A$  be a densely defined Hermitian operator in a Hilbert space  $\mathcal{H}$ .  $A$  is *semi-bounded* if  $A \geq c > -\infty$ , in the sense that,  $\langle x, Ax \rangle \geq c \langle x, x \rangle$ ,  $\forall x \in \text{dom}(A)$ . Set

$$L_A := \inf \{ \langle x, Ax \rangle : x \in \text{dom}(A), \|x\| = 1 \} \quad (9.42)$$

and  $L_A$  is called the *lower bound* of  $A$ .

In the following discussion, we first assume  $A \geq 1$  and eventually drop the constraint.

Let  $\mathcal{H}_1 = \text{completion of } \text{dom}(A) \text{ with respect to the inner product}$

$$\langle x, y \rangle_1 := \langle x, Ay \rangle \quad (9.43)$$

**Theorem 9.32.** *Let  $A \geq I$ , i.e.,  $\langle x, Ax \rangle \geq \|x\|^2$ ,  $\forall x \in \text{dom}(A)$ ; and let  $\mathcal{H}$ , and  $\mathcal{H}_1$  be as above. Then*

1.  $\|x\| \leq \|x\|_1$ ,  $\forall x \in \text{dom}(A)$ .
2.  $\|\cdot\|$  and  $\|\cdot\|_1$  are topologically consistent, i.e., the identity map

$$\varphi : \text{dom}(A) \rightarrow \text{dom}(A)$$

extends by continuity to

$$\tilde{\varphi} : \mathcal{H}_1 \hookrightarrow \mathcal{H}$$

such that

$$\|x\| \leq \|x\|_1, \quad \forall x \in \mathcal{H}_1. \quad (9.44)$$

Therefore,  $\mathcal{H}_1$  is identified as a dense subspace in  $\mathcal{H}$ .

3. Moreover,

$$\langle y, x \rangle_1 = \langle y, Ax \rangle, \quad \forall x \in \text{dom}(A), \forall y \in \mathcal{H}_1. \quad (9.45)$$

4. Define

$$\begin{aligned} \tilde{A} &:= A^* \Big|_{\text{dom}(\tilde{A})}, \quad \text{where} \\ \text{dom}(\tilde{A}) &:= \text{dom}(\tilde{A}^*) \cap \mathcal{H}_1. \end{aligned}$$

Then  $\tilde{A} = \tilde{A}^*$ , and  $L_{\tilde{A}} = L_A$ .

*Proof.* (1)-(2) The assumption  $A \geq 1$  implies that

$$\|x\|^2 = \langle x, x \rangle \leq \langle x, Ax \rangle = \|x\|_1^2, \quad \forall x \in \text{dom}(A).$$

Hence  $\varphi$  is continuous and the norm ordering passes to the completions of  $\text{dom}(A)$  with respect to  $\|\cdot\|_1$  and  $\|\cdot\|$ . Therefore (9.44) holds.

Next, we verify that  $\tilde{\varphi}$  is injective (i.e.,  $\ker \tilde{\varphi} = 0$ .) Suppose  $(x_n) \subset \text{dom}(A)$  such that  $x_n \xrightarrow{\|\cdot\|_1} x \in \mathcal{H}_1$ , and  $x_n \xrightarrow{\|\cdot\|} 0$ . We must show that  $\|x\|_1 = 0$ . But

$$\begin{aligned} \|x\|_1^2 &= \lim_{m,n \rightarrow \infty} \langle x_m, x_n \rangle_1 \quad (\text{limit exists by assumption}) \\ &= \lim_{m,n \rightarrow \infty} \langle x_m, Ax_n \rangle \\ &= \lim_{n \rightarrow \infty} \langle 0, Ax_n \rangle = 0. \end{aligned}$$

In the computation, we used the fact that

$$\begin{aligned} |\langle x_m - x, Ax_n \rangle| &\leq \|x_m - x\| \|Ax_n\| \\ &\leq \|x_m - x\|_1 \|Ax_n\| \rightarrow 0, \quad \text{as } m \rightarrow \infty. \end{aligned}$$

(3) Let  $(y_n) \subset \text{dom}(A)$ , and  $\|y_n - y\|_1 \rightarrow 0$ . For all  $x \in \text{dom}(A)$ , we have

$$\langle y, x \rangle_1 = \lim_{n \rightarrow \infty} \langle y_n, x \rangle_1 = \lim_{n \rightarrow \infty} \langle y_n, Ax \rangle = \langle y, Ax \rangle.$$

Equivalently,

$$\begin{aligned} |\langle y_n, Ax \rangle - \langle y, Ax \rangle| &= |\langle y_n - y, Ax \rangle| \\ &\leq \|y_n - y\| \|Ax\| \\ &\leq \|y_n - y\|_1 \|Ax\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(4) For all  $x, y \in \text{dom}(\tilde{A})$ ,  $\exists (x_n), (y_n) \subset \text{dom}(A)$  such that  $\|x_n - x\|_1 \rightarrow 0$  and  $\|y_n - y\|_1 \rightarrow 0$ . Hence the following limit exists:

$$\lim_{m,n \rightarrow \infty} \langle x_m, Ay_n \rangle \left( = \lim_{m,n \rightarrow \infty} \langle x_m, y_n \rangle_1 \right).$$

Consequently,

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle x_m, Ay_n \rangle &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle Ax_m, y_n \rangle \\ &= \lim_{m \rightarrow \infty} \langle Ax_m, y \rangle \\ &= \lim_{m \rightarrow \infty} \langle x_m, A^* y \rangle \\ &= \lim_{m \rightarrow \infty} \langle x_m, \tilde{A} y \rangle = \langle x, \tilde{A} y \rangle \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \langle x_m, Ay_n \rangle = \lim_{n \rightarrow \infty} \langle x, Ay_n \rangle$$



$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \langle A^* x, y_n \rangle \\
&= \lim_{n \rightarrow \infty} \langle \tilde{A} x, y_n \rangle = \langle \tilde{A} x, y \rangle.
\end{aligned}$$

Thus,  $\tilde{A}$  is Hermitian.

Fix  $y \in \mathcal{H}$ . The map  $x \mapsto \langle y, x \rangle$ ,  $\forall x \in \text{dom}(A) \subset \mathcal{H}_1$ , is linear and satisfies

$$|\langle y, x \rangle| \leq \|y\| \|x\| \leq \|y\| \|x\|_1.$$

Hence it extends to a unique bounded linear functional on  $\mathcal{H}_1$ , as  $\text{dom}(A)$  is dense in  $\mathcal{H}_1$ .

By Riesz's theorem, there exists unique  $h_y \in \mathcal{H}_1$  such that

$$\langle y, x \rangle = \langle h_y, x \rangle_1, \quad \forall x \in \mathcal{H}_1. \quad (9.46)$$

In particular,

$$\langle y, x \rangle = \langle h_y, x \rangle_1 = \langle h_y, Ax \rangle, \quad \forall x \in \text{dom}(A).$$

Then,  $h_y \in \mathcal{H}_1 \cap \text{dom}(A^*) = \text{dom}(\tilde{A})$ , and  $\tilde{A}h_y = y$ . Therefore,  $\text{ran}(\tilde{A}) = \mathcal{H}$ . Note we have established the identity

$$\langle \tilde{A}y, x \rangle = \langle y, x \rangle_1, \quad \forall y \in \text{dom}(\tilde{A}), \forall x \in \text{dom}(A). \quad (9.47)$$

□

*Claim 9.33.*  $\text{ran}(\tilde{A}) = \mathcal{H}$  implies that  $\tilde{A}$  is selfadjoint. In fact, for all  $x \in \text{dom}(\tilde{A})$  and  $y \in \text{dom}(\tilde{A}^*)$ , we have

$$\langle y, \tilde{A}x \rangle = \langle \tilde{A}^* y, x \rangle = \langle \tilde{A}h, x \rangle = \langle h, \tilde{A}x \rangle$$

where  $\tilde{A}^* y = \tilde{A}h$ , for some  $h \in \text{dom}(\tilde{A})$ , using the assumption  $\text{ran}(\tilde{A}) = \mathcal{H}$ . Thus,

$$\langle y - h, \tilde{A}x \rangle = 0, \quad \forall x \in \text{dom}(\tilde{A});$$

i.e.,  $y - h \perp \text{ran}(\tilde{A}) = \mathcal{H}$ . Therefore,  $y = h$ , and so  $y \in \text{dom}(\tilde{A})$ .

*Proof.* Finally, we show  $L_{\tilde{A}} = L_A$ . By the definition of lower bound,  $\text{dom}(A) \subset \text{dom}(\tilde{A})$  implies  $L_{\tilde{A}} \leq L_A$ . On the other hand, let  $(x_n) \subset \text{dom}(A)$  such that  $x_n \xrightarrow{\|\cdot\|_1} x \in \text{dom}(\tilde{A})$ , then

$$\begin{aligned}
\langle x, \tilde{A}x \rangle &= \lim_{n \rightarrow \infty} \langle x, \tilde{A}x_n \rangle = \lim_{n \rightarrow \infty} \langle x, Ax_n \rangle \\
&= \lim_{n \rightarrow \infty} \langle x, x_n \rangle_1 = \langle x, x \rangle_1 \geq L_A \langle x, x \rangle
\end{aligned}$$

which shows that  $L_{\tilde{A}} \geq L_A$ . □

*Remark 9.34.* In the proof of Theorem 9.32, we established an embedding  $\psi : \mathcal{H} \hookrightarrow \mathcal{H}_1^*$  by  $\psi : y \mapsto h_y$  with the defining equation (9.46). And we define  $\tilde{A}h_y = y$ , i.e.,  $h_y = \tilde{A}^{-1}y$ . It follows that  $\text{dom}(\tilde{A}) = \text{ran}(\psi)$ , and  $\text{ran}(\tilde{A}) = \mathcal{H}$ .

**Theorem 9.35.** *The Friedrichs extension of  $A$  is the unique selfadjoint operator satisfying*

$$\langle \tilde{A}x, y \rangle = \langle x, y \rangle_1, \quad \forall x \in \text{dom}(\tilde{A}), \forall y \in \text{dom}(A).$$

See (9.47).

*Proof.* Suppose  $A \subset B, C \subset A^*$ , and  $B, C$  selfadjoint, satisfying

$$\begin{aligned} \langle Bx, y \rangle &= \langle x, y \rangle_1, \quad \forall x \in \text{dom}(B), \forall y \in \text{dom}(A) \\ \langle Cx, y \rangle &= \langle x, y \rangle_1, \quad \forall x \in \text{dom}(C), \forall y \in \text{dom}(A). \end{aligned}$$

Then, for all  $x, y \in \text{dom}(A)$ , we have

$$\langle Bx, y \rangle = \langle Cx, y \rangle = \langle x, Cy \rangle (= \langle x, Ay \rangle).$$

Fix  $x \in \text{dom}(A)$ , and the above identity passes to  $y \in \text{dom}(C)$ . Therefore,  $y \in B^* = B$ , and  $By = Cy$ . This shows  $C \subset B$ . Since

$$C = C^* \supset B^* = B$$

i.e.,  $C \supset B$ , it then follows that  $B = C$ .  $\square$

*Remark 9.36.* If  $A$  is only assumed to be semi-bounded, i.e.,  $A \geq c > -\infty$ , then  $B := A - c + 1 \geq 1$ , and we get the Friedrichs extension  $\tilde{B}$  of  $B$ ; and  $\tilde{B} - 1 + c$  is the Friedrichs extension of  $A$ .

## 9.5 Rigged Hilbert Space

In the construction of Friedrichs extensions of semi-bounded operators, we have implicitly used the idea of rigged Hilbert spaces. We study this method systematically and recover the Friedrichs extension as a special case.

Let  $\mathcal{H}_0$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle_0$ , and  $\mathcal{H}_1$  be a dense subspace in  $\mathcal{H}_0$ , which by itself, is a Hilbert space with respect to  $\langle \cdot, \cdot \rangle_1$ . Further, assume the ordering

$$\|x\|_0 \leq \|x\|_1, \quad \forall x \in \mathcal{H}_1. \quad (9.48)$$

Hence, the identity map

$$id : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \quad (9.49)$$

is continuous with a dense image.

Let  $\mathcal{H}_{-1}$  be the space of bounded conjugate linear functionals on  $\mathcal{H}_1$ . By Riesz's theorem,  $\mathcal{H}_{-1}$  is identified with  $\mathcal{H}_1$  via the map

$$\mathcal{H}_{-1} \rightarrow \mathcal{H}_1, \quad f \mapsto \xi_f, \text{ s.t.} \quad (9.50)$$

$$f(x) = \langle x, \xi_f \rangle_1, \quad \forall x \in \mathcal{H}_1. \quad (9.51)$$

Then  $\mathcal{H}_{-1}$  is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{-1} = \langle \xi_f, \xi_g \rangle_1, \quad \forall f, g \in \mathcal{H}_{-1}. \quad (9.52)$$

*Remark 9.37.* The map  $f \mapsto \xi_f$  in (9.50) is linear. For if  $c \in \mathbb{C}$ , then  $\langle x, \xi_{cf} \rangle_1 = cf(x) = c\langle x, \xi_f \rangle_1 = \langle x, c\xi_f \rangle_1$ , for all  $x \in \mathcal{H}_1$ ; i.e.,  $(\xi_{cf} - c\xi_f) \perp \mathcal{H}_1$ , and so  $\xi_{cf} = c\xi_f$ .

**Theorem 9.38.** *The mapping*

$$\mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}, \quad x \mapsto \langle \cdot, x \rangle_0, \quad \forall x \in \mathcal{H}_0 \quad (9.53)$$

*is linear, injective, continuous, and having a dense image in  $\mathcal{H}_{-1}$ .*

*Proof.* Since  $cx \mapsto \langle \cdot, cx \rangle = c\langle \cdot, x \rangle$ , the mapping in (9.53) is linear.

For all  $x \in \mathcal{H}_0$ , we have 8.6

$$|\langle y, x \rangle_0| \leq \|x\|_0 \|y\|_0 \stackrel{(9.48)}{\leq} \|x\|_0 \|y\|_1, \quad \forall y \in \mathcal{H}_1; \quad (9.54)$$

hence  $\langle \cdot, x \rangle_0$  is a bounded conjugate linear functional on  $\mathcal{H}_1$ , i.e.,  $\langle \cdot, x \rangle_0 \in \mathcal{H}_{-1}$ .

Moreover, by (9.54),

$$\|\langle \cdot, x \rangle_0\|_{-1} \leq \|x\|_0. \quad (9.55)$$

If  $\langle \cdot, x \rangle_0 \equiv 0$  in  $\mathcal{H}_{-1}$ , then  $\langle y, x \rangle_0 = 0$ , for all  $y \in \mathcal{H}_1$ . Since  $\mathcal{H}_1$  is dense in  $\mathcal{H}_0$ , it follows that  $x = 0$  in  $\mathcal{H}_0$ . Thus, (9.53) is injective.

Now, if  $f \perp \{\langle \cdot, x \rangle_0 : x \in \mathcal{H}_1\}$  in  $\mathcal{H}_{-1}$ , then

$$\langle f, \langle \cdot, x \rangle_0 \rangle_{-1} \stackrel{(9.52)}{=} \langle \xi_f, \xi_{\langle \cdot, x \rangle_0} \rangle_1 \stackrel{(9.50)}{=} \langle \xi_f, x \rangle_0 = 0, \quad \forall x \in \mathcal{H}_1. \quad (9.56)$$

Thus,  $\|\xi_f\|_0 = 0$ , since  $\mathcal{H}_1$  is dense in  $\mathcal{H}_0$ . Since  $id : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0$  is injective, and  $\xi_f \in \mathcal{H}_1$ , it follows that  $\|\xi_f\|_1 = 0$ . This, in turn, implies  $\|f\|_{-1} = 0$ , and so  $f = 0$  in  $\mathcal{H}_{-1}$ . Consequently, the image of  $\mathcal{H}_1$  (resp.  $\mathcal{H}_0$  as it contains  $\mathcal{H}_1$ ) under (9.53) is dense in  $\mathcal{H}_{-1}$ .  $\square$

Combining (9.48) and Theorem 9.38, we get the triple of Hilbert spaces

$$\mathcal{H}_1 \xrightarrow{(9.49)} \mathcal{H}_0 \xrightarrow{(9.53)} \mathcal{H}_{-1}. \quad (9.57)$$

The following are immediate:

1. All mappings in (9.57) are injective, continuous (in fact, contractive), having dense images.
2. The map  $x \mapsto \xi_{\langle \cdot, x \rangle_0}$  is a contraction from  $\mathcal{H}_1 \subset \mathcal{H}_0$  into  $\mathcal{H}_0$ . This follows from the estimate:

$$\begin{aligned} \|\xi_{\langle \cdot, x \rangle_0}\|_0 &\stackrel{(9.48)}{\leq} \|\xi_{\langle \cdot, x \rangle_0}\|_1 \\ &\stackrel{(9.52)}{=} \|\langle \cdot, x \rangle_0\|_{-1} \\ &\stackrel{(9.55)}{\leq} \|x\|_0 \stackrel{(9.48)}{\leq} \|x\|_1, \quad \forall x \in \mathcal{H}_1. \end{aligned} \quad (9.58)$$

In particular, for  $x \in \mathcal{H}_1$ ,  $x \neq \xi_{\langle \cdot, x \rangle_0}$  in general.

3. The canonical bilinear form  $\langle \cdot, \cdot \rangle : \mathcal{H}_1 \times \mathcal{H}_{-1} \rightarrow \mathbb{C}$  is given by

$$f(x) = \langle x, \xi_f \rangle_1, \quad \forall x \in \mathcal{H}_1, \forall f \in \mathcal{H}_{-1}. \quad (9.59)$$

In particular, if  $f = \langle \cdot, y \rangle_0$ ,  $y \in \mathcal{H}_0$ , then

$$\langle x, \xi_f \rangle_1 = \langle x, \xi_{\langle \cdot, y \rangle_0} \rangle_1 = \langle x, y \rangle_0, \quad \forall x \in \mathcal{H}_1.$$

4. By Theorem 9.38,  $\mathcal{H}_0$  is dense in  $\mathcal{H}_{-1}$ , and

$$\begin{aligned} \langle x, y \rangle_{-1} &:= \langle \langle \cdot, x \rangle_0, \langle \cdot, y \rangle_0 \rangle_{-1} \\ &= \langle \xi_{\langle \cdot, x \rangle_0}, \xi_{\langle \cdot, y \rangle_0} \rangle_1 \\ &= \langle x, \xi_{\langle \cdot, y \rangle_0} \rangle_0 = \langle \xi_{\langle \cdot, x \rangle_0}, y \rangle_0, \quad \forall x, y \in \mathcal{H}_0. \end{aligned} \quad (9.60)$$

Combined with the order relation  $\|x\|_{-1} \leq \|x\|_0$  for all  $x \in \mathcal{H}_0$ , we see that

$$\begin{aligned} |\langle x, y \rangle_{-1}| &= \left| \langle x, \xi_{\langle \cdot, y \rangle_0} \rangle_0 \right| \\ &\leq \|x\|_0 \|\xi_{\langle \cdot, y \rangle_0}\|_{-1} \\ &= \|x\|_0 \|y\|_{-1}. \\ &\leq \|x\|_0 \|y\|_0, \quad \forall x, y \in \mathcal{H}_0. \end{aligned}$$

Thus  $\langle \cdot, \cdot \rangle_{-1}$  is a continuous extension of  $\langle \cdot, \cdot \rangle_0$ .

**Theorem 9.39.** Let  $\mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  be the triple in (9.57). Define  $B : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  by

$$B : x \underset{\mathcal{H}_0}{\xrightarrow{(9.53)}} \langle \cdot, x \rangle_0 \underset{\mathcal{H}_{-1}}{\xrightarrow{(9.50)}} \xi_{\langle \cdot, x \rangle_0} \underset{\mathcal{H}_1}{\xrightarrow{(9.49)}} \xi_{\langle \cdot, x \rangle_0}. \quad (9.61)$$

Then,

1. For all  $x \in \mathcal{H}_1$ , and all  $y \in \mathcal{H}_0$ ,

$$\langle x, By \rangle_1 = \langle x, y \rangle_0. \quad (9.62)$$

In particular,

$$\begin{aligned} \langle x, y \rangle_{-1} &= \langle Bx, By \rangle_1 \\ &= \langle x, By \rangle_0 = \langle Bx, y \rangle_0, \quad \forall x, y \in \mathcal{H}_0; \end{aligned} \quad (9.63)$$

where  $\langle x, y \rangle_{-1} := \langle \xi_{\langle \cdot, x \rangle_0}, \xi_{\langle \cdot, y \rangle_0} \rangle_1$ , as defined in (9.60).

2.  $B$  is invertible.

3.  $\text{ran}(B)$  is dense in both  $\mathcal{H}_1$  and  $\mathcal{H}_0$ .

4.  $0 \leq B \leq 1$ . In particular,  $B$  is a bounded selfadjoint operator on  $\mathcal{H}_0$ .

*Proof.* □

1. For  $y \in \mathcal{H}_0$ ,  $By = \xi_{\langle \cdot, y \rangle_0} \in \mathcal{H}_1$ , where  $\xi_{\langle \cdot, y \rangle_0}$  is given in (9.50)-(9.51). Thus,

$$\langle x, By \rangle_1 = \langle x, \xi_{\langle \cdot, y \rangle_0} \rangle_1 = \langle x, y \rangle_0, \quad \forall x \in \mathcal{H}_1.$$

(9.63) follows from this.

- (a) If  $\|Bx\|_0 = 0$ , then  $\|Bx\|_1 = 0$ , since  $Bx \in \mathcal{H}_1$  and  $\mathcal{H}_1 \hookrightarrow \mathcal{H}_0$  is injective. But then

$$\|Bx\|_1 \stackrel{(9.63)}{=} \|x\|_{-1} = 0 \implies \|x\|_0 = 0$$

since  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  is injective. This shows that  $B$  is injective.

- (b) Since  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  is dense, and  $\mathcal{H}_{-1} \simeq \mathcal{H}_1$ , it follows that  $\text{ran}(B)$  is dense in  $\mathcal{H}_1$ . Now if  $y \in \mathcal{H}_0$ , and  $\langle y, Bx \rangle_0 = 0$ , for all  $x \in \mathcal{H}_0$ , then

$$\langle By, Bx \rangle_1 = 0, \quad \forall x \in \mathcal{H}_0;$$

equivalently,

$$\langle y, x \rangle_{-1} = 0, \quad \forall x \in \mathcal{H}_0.$$

Since  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  is dense, we have  $\|y\|_{-1} = 0$ . But  $y \in \mathcal{H}_0$  and  $\mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  is injective, it follows that  $\|y\|_0 = 0$ , i.e.,  $y = 0$  in  $\mathcal{H}_0$ . Therefore,  $\text{ran}(B)$  is also dense in  $\mathcal{H}_0$ .

- (c) For all  $x \in \mathcal{H}_0$ , we have

$$\langle x, Bx \rangle_0 \stackrel{(9.62)}{=} \langle Bx, Bx \rangle_1 \geq 0 \implies B \geq 0.$$

On the other hand,

$$\langle x, Bx \rangle_0 = \langle Bx, Bx \rangle_1 \stackrel{(9.63)}{=} \langle x, x \rangle_{-1} \stackrel{(9.55)}{\leq} \langle x, x \rangle_0;$$

and so  $B \leq 1$ . Since  $B$  is positive and bounded, it is selfadjoint.

Another argument:

$$\|Bx\|_0 \leq \|Bx\|_1 = \|x\|_{-1} \leq \|x\|_0, \quad \forall x \in \mathcal{H}_0.$$

In view of applications, it is convenient to reformulate the previous theorem in terms of  $B^{-1}$ .

**Theorem 9.40.** *Let  $B$ ,  $\mathcal{H}_1$ ,  $\mathcal{H}_0$ ,  $\mathcal{H}_{-1}$ , be as in Theorem 9.39, set  $A := B^{-1}$ , then*

1.  $A = A^*$ ,  $A \geq 1$ .
2.  $\text{dom}(A)$  is dense in  $\mathcal{H}_1$  and  $\mathcal{H}_0$ , and  $\text{ran}(A) = \mathcal{H}_0$ .

3. For all  $y \in \text{dom}(A)$ ,  $x \in \mathcal{H}_1$ ,

$$\langle x, y \rangle_1 = \langle x, Ay \rangle_0. \quad (9.64)$$

In particular,

$$\begin{aligned} \langle x, y \rangle_1 &= \langle Ax, Ay \rangle_{-1} \\ &= \langle Ax, y \rangle_0 = \langle x, Ay \rangle_0, \quad \forall x, y \in \text{dom}(A). \end{aligned} \quad (9.65)$$

4. There is a unique selfadjoint operator in  $\mathcal{H}_0$  satisfying (9.64).

*Proof.* Part (1)-(3) are immediate by Theorem 9.39. For (4), suppose  $A, B$  are selfadjoint in  $\mathcal{H}_0$  such that

- (i)  $\text{dom}(A), \text{dom}(B)$  are contained in  $\mathcal{H}_1$ , dense in  $\mathcal{H}_0$ ;
- (ii)

$$\begin{aligned} \langle x, y \rangle_1 &= \langle x, Ay \rangle_0, \quad \forall x \in \mathcal{H}_1, y \in \text{dom}(A) \\ \langle x, y \rangle_1 &= \langle x, By \rangle_0, \quad \forall x \in \mathcal{H}_1, y \in \text{dom}(B). \end{aligned}$$

Then, for all  $x \in \text{dom}(A)$  and  $y \in \text{dom}(B)$ ,

$$\langle x, By \rangle_0 = \langle x, y \rangle_1 = \langle Ax, y \rangle_0.$$

Thus,  $x \mapsto \langle Ax, y \rangle_0$  is a bounded linear functional on  $\text{dom}(A)$ , and so  $y \in \text{dom}(A^*) = \text{dom}(A)$  and  $A^*y = Ay = By$ ; i.e.,  $A \supset B$ . Since  $A, B$  are selfadjoint, then

$$B = B^* \subset A^* = A.$$

Therefore  $A = B$ . □

**Theorem 9.41.** Let  $\mathcal{H}_1, \mathcal{H}_0, A$  as in Theorem 9.40. Then

1.  $\mathcal{H}_1 = \text{dom}(A^{1/2})$ , and

$$\langle x, y \rangle_1 = \left\langle A^{1/2}x, A^{1/2}y \right\rangle_0, \quad \forall x, y \in \mathcal{H}_1. \quad (9.66)$$

2. For all  $x, y \in \mathcal{H}_0$ ,

$$\langle x, y \rangle_{-1} = \left\langle A^{-1/2}x, A^{-1/2}y \right\rangle_0. \quad (9.67)$$

Since  $\mathcal{H}_0$  is dense in  $\mathcal{H}_{-1}$ , then  $\mathcal{H}_{-1} =$  completion of  $\mathcal{H}_0$  under the  $\|A^{-1/2} \cdot\|_0$ -norm.

3. For all  $x \in \text{dom}(A)$ ,

$$\|Ax\|_{-1} = \|x\|_1 \left( = \|A^{1/2}x\|_0 \right). \quad (9.68)$$

Consequently, the map  $\text{dom}(A) \ni x \mapsto Ax \in \mathcal{H}_0$  extends by continuity to a unitary operator from  $\mathcal{H}_1 (= \text{dom}(A^{1/2}))$  onto  $\mathcal{H}_{-1}$ , which is precisely the inverse of (9.50)-(9.51).

*Proof.* (1) This is the result of the following observations: □

- a.  $\text{dom}(A) \subset \text{dom}(A^{1/2})$ . With the assumption  $A \geq 1$ , the containment is clear. The assertion also holds in general: By spectral theorem,

$$x \in \text{dom}(A) \iff \int (1 + |\lambda|^2) \|P(d\lambda)x\|_0^2 < \infty;$$

where  $P(\cdot)$  is the projection-valued measure (PVM) of  $A$ , defined on the set of all Borel sets in  $\mathbb{R}$ , and  $d\mu_x := \|P(d\lambda)x\|_0^2$  is a finite positive Borel measure on  $\mathbb{R}$ . Thus,

$$\int (1 + |\lambda|) \|P(d\lambda)x\|_0^2 < \infty$$

since  $L^2 \subset L^1$  when the measure is finite. But this is equivalent to  $x \in \text{dom}(A^{1/2})$ .

- (a) For any Hermitian operator  $T$  satisfying  $T \geq c > 0$ , we have the estimate:

$$\|Tx\| \leq \|x\| + \|Tx\| \leq (1 + c) \|Tx\|$$

Thus, the graph norm of  $T$  is equivalent to  $\|T\cdot\|$ .

- (b)  $\text{dom}(A)$  is dense in  $\text{dom}(A^{1/2})$ . Note  $\text{dom}(A^{1/2})$  is a Hilbert space with respect to the  $A^{1/2}$ -graph norm. By the discussion above,  $\|\cdot\|_{A^{1/2}} \simeq \|A^{1/2}\cdot\|_0$ .  
Let  $y \in \text{dom}(A^{1/2})$ , then

$$\begin{aligned} y \perp \text{dom}(A) \text{ in } \text{dom}(A^{1/2}) \\ \iff \\ \langle A^{1/2}y, A^{1/2}x \rangle_0 = 0, \forall x \in \text{dom}(A^{1/2}) \\ \iff \\ \langle y, Ax \rangle_0 = 0, \forall x \in \text{dom}(A^{1/2}) \\ \iff \\ y = 0 \text{ in } \mathcal{H}_0 \end{aligned}$$

- (c)  $\text{dom}(A)$  is dense in  $\mathcal{H}_1$ . See theorems 9.39-9.40.

- (d)  $\|A^{1/2}x\|_0 = \|x\|_1, \forall x \in \text{dom}(A)$ . Indeed,

$$\langle x, x \rangle_1 \stackrel{(9.64)}{=} \langle x, Ax \rangle_0 = \langle A^{1/2}x, A^{1/2}x \rangle_0, \forall x \in \text{dom}(A).$$

**Conclusion:** (i)  $\text{dom}(A)$  is dense in  $\mathcal{H}_1$  and  $\text{dom}(A^{1/2})$ ; (ii)  $\|\cdot\|_1$  and  $\|A^{1/2}\cdot\|_0$  agree on  $\text{dom}(A)$ . Therefore the closures of  $\text{dom}(A)$  in  $\mathcal{H}_1$  and  $\text{dom}(A^{1/2})$  are identical. This shows  $\mathcal{H}_1 = \text{dom}(A^{1/2})$ . (9.66) is immediate.

*Proof.* (2) Given  $x, y \in \mathcal{H}_0$ ,

$$\langle x, y \rangle_{-1} \stackrel{(9.63)}{=} \langle A^{-1}x, A^{-1}y \rangle_1 \stackrel{(9.66)}{=} \langle A^{-1/2}x, A^{-1/2}y \rangle_0.$$

(3) Given  $x \in \text{dom}(A)$ , we have

$$\|Ax\|_{-1} \stackrel{(9.63)}{=} \|A^{-1}(Ax)\|_1 = \|x\|_1 \stackrel{(9.66)}{=} \|A^{1/2}x\|_0.$$

□

**Application: The Friedrichs extension revisited.**

**Theorem 9.42** (Friedrichs). *Let  $A$  be a densely defined Hermitian operator acting in  $\mathcal{H}_0$ , and assume  $A \geq 1$ . There exists a selfadjoint extension  $S \supset A$ , such that  $L_S = L_A$ , i.e.,  $A$  and  $S$  have the same lower bound.*

*Proof.* Given  $A$ , construct the triple  $\mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \hookrightarrow \mathcal{H}_{-1}$  as in Theorem 9.32, so that  $\mathcal{H}_1 = \text{cl}_{\langle \cdot, A \cdot \rangle_0}(\text{dom}(A))$ , and

$$\langle y, x \rangle_1 = \langle y, Ax \rangle_0, \quad \forall x \in \text{dom}(A), \forall y \in \mathcal{H}_1. \quad (9.69)$$

By Theorem 9.40, there is a densely defined selfadjoint operator  $S$  in  $\mathcal{H}_0$ , such that (i)  $\mathcal{H}_1 = \text{dom}(S^{1/2}) \supset \text{dom}(S)$ ; (ii)

$$\langle y, x \rangle_1 = \langle S^{1/2}y, S^{1/2}x \rangle_0, \quad \forall x, y \in \mathcal{H}_1. \quad (9.70)$$

Combing (9.69)-(9.70), we get

$$\langle y, Ax \rangle_0 = \langle S^{1/2}y, S^{1/2}x \rangle_0, \quad \forall x \in \text{dom}(A), \forall y \in \text{dom}(S^{1/2}).$$

Therefore,  $S^{1/2}x \in \text{dom}(S^{1/2})$ , i.e.,  $x \in \text{dom}(S)$ , and

$$\langle y, Ax \rangle_0 = \langle y, Sx \rangle_0, \quad \forall x \in \text{dom}(A), \forall y \in \text{dom}(S^{1/2}).$$

Since  $\mathcal{H}_1 = \text{dom}(S^{1/2})$  is dense in  $\mathcal{H}_0$ , we conclude that  $Sx = Ax$ , for all  $x \in \text{dom}(A)$ . Thus,  $S \supset A$ .

Clearly,  $S \supset A$  implies  $L_A \geq L_S$ . On the other hand,

$$\|x\|_1^2 = \langle x, Ax \rangle_0 \geq L_A \langle x, x \rangle_0 = L \|x\|_0^2, \quad \forall x \in \text{dom}(A)$$

and the inequality passes by continuity to all  $x \in \mathcal{H}_1$ ; i.e.,

$$\|x\|_1^2 \geq L_A \|x\|_0^2, \quad \forall x \in \mathcal{H}_1.$$



This is equivalent (by Theorem 9.40) to

$$\left\langle S^{1/2}x, S^{1/2}x \right\rangle_0 \geq L_A \langle x, x \rangle_0, \quad \forall x \in \text{dom}(S^{1/2}) (= \mathcal{H}_1).$$

In particular,

$$\langle x, Sx \rangle_0 \geq L_A \langle x, x \rangle_0, \quad \forall x \in \text{dom}(S)$$

and so  $L_S \geq L_A$ . Therefore,  $L_A = L_S$ . □

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [AG93, Nel69, Dev72, DS88c, Kre46, Jor08, Rud73, Sto51, Sto90, FL28, Fug82, GJ87, JM80, VN35, JPT12b, JP14, RS75, Hel13].

## Chapter 10

# Unbounded Graph-Laplacians

Mathematics is an experimental science, and definitions do not come first, but later on.

— Oliver Heaviside

It is nice to know that the computer understands the problem. But I would like to understand it too.

— Eugene Wigner

Knowing is not enough; we must apply.

— Göthe

Chance is a more fundamental conception than causality.

— Max Born

Below we study selfadjoint operators, and extensions in a particular case arising in the study of infinite graphs; the operators here are infinite discrete Laplacians.

As an application of the previous chapter, we consider the Friedrichs extension of discrete Laplacian in infinite networks [JP10, JP13a, JP13b, JT15a, JT15b].

By an electrical network we mean a graph  $G$  of vertices and edges satisfying suitable conditions which allow for computation of voltage distribution from a network of prescribed resistors assigned to the edges in  $G$ . The mathematical axioms are prescribed in a way that facilitates the use of the laws of Kirchhoff and Ohm in computing voltage distributions and resistance distances in  $G$ . It will be more convenient to work with prescribed conductance functions  $c$  on  $G$ . Indeed with a choice of conductance function  $c$  specified we define two crucial tools for our analysis, a graph Laplacian  $\Delta (= \Delta_c)$ , a discrete version of more classical notions of Laplacians, and an energy Hilbert space  $\mathcal{H}_E$ .

Because of statistical consideration, and our use of random walk models, we focus our study on infinite electrical networks, i.e., the case when the graph  $G$  is countable infinite. In this case, for realistic models the graph Laplacian  $\Delta_c$  will then be an unbounded operator with dense domain in  $\mathcal{H}_E$ , Hermitian and semibounded. Hence it has a unique Friedrichs extension.

Large networks arise in both pure and applied mathematics, e.g., in graph theory (the mathematical theory of networks), and more recently, they have become a current and fast developing research area; with applications including a host of problems coming from for example internet search, and social networks. Hence, of the recent applications, there is a change in outlook from finite to infinite.

More precisely, in traditional graph theoretical problems, the whole graph is given exactly, and we are then looking for relationships between its parameters, variables and functions; or for efficient algorithms for computing them. By contrast, for very large networks (like the Internet), variables are typically not known completely; – in most cases they may not even be well defined. In such applications, data about them can only be collected by indirect means; hence random variables and local sampling must be used as opposed to global processes.

Although such modern applications go far beyond the setting of large electrical networks (even the case of infinite sets of vertices and edges), it is nonetheless true that the framework of large electrical networks is helpful as a basis for the analysis we develop below; and so our results will be phrased in the setting of large electrical networks, even though the framework is much more general.

The applications of “large” or infinite graphs are extensive, counting just physics; see for example [BCD06, RAKK05, KMRS05, BC05, TD03, VZ92].

In discrete harmonic analysis, two operations play a key role, the Laplacian  $\Delta$ , and the Markov operator  $P$ . An infinite network is a pair of sets,  $V$  vertices, and  $E$ , edges. In addition to this, one specifies a conductance function  $c$ . This is a function  $c$  defined on the edge set  $E$ . There are then two associated operators  $\Delta$  and  $P$  are defined from, and they depend on the entire triple  $(V, E, c)$ . For many problems one of the two operators is even used in the derivation of properties of the other. Both represent actions (operations) on appropriate spaces of functions, i.e., functions defined on the infinite set of vertices  $V$ . For the networks of interest to us, the vertex set  $V$  will be infinite, and we are therefore faced with a variety of choices of infinite-dimensional function spaces. Because of spectral theory, the useful choices will be Hilbert spaces.

But even restricting to Hilbert spaces, there are at least three natural candidates: (i) the plain  $l^2$  sequence space, so an  $l^2$ -space of functions on  $V$ , (ii) a suitably weighted  $l^2$ -space, and finally (iii), an energy Hilbert space  $\mathcal{H}_E$ . (The latter is an abstraction of more classical notions of Dirichlet spaces.) Which one of the three to use depends on the particular operator considered, and also on the questions asked.

We note that in infinite network models, both the Laplacian  $\Delta$ , and the Markov operator  $P$  will have infinite by infinite matrix representations. Each of these infinite by infinite matrices is special, in that, as an infinite by infinite

matrix, it will have non-zero entries localized only in finite bands containing the infinite matrix-diagonal (i.e., they are infinite banded matrices). This makes the algebraic matrix operations well defined.

Functional analytic and spectral theoretic tools enter the analysis as follows: In passing to appropriate Hilbert spaces, we arrive at classes of Hilbert space-operators, and the operators in question will be Hermitian. But the Laplacian  $\Delta$  will be typically be an unbounded operator, albeit semibounded. By contrast we show that there is a weighted  $l^2$ -space such that the corresponding Markov operator  $P$  is a bounded, selfadjoint operator, and that it has its spectrum contained in the finite interval  $[-1, 1]$ . We caution, that in general this spectrum may be continuous, or have a mix of spectral types, continuous (singular or Lebesgue), and discrete.

## 10.1 Basic Setting

Let  $V$  be a countable discrete set, and let  $E \subset V \times V$  be a subset such that:

1.  $(x, y) \in E \iff (y, x) \in E$ ;  $x, y \in V$ ;
2.  $\#\{y \in V \mid (x, y) \in E\}$  is finite, and  $> 0$  for all  $x \in V$ ;
3.  $(x, x) \notin E$ ; and
4.  $\exists o \in V$  such that for all  $y \in V \exists x_0, x_1, \dots, x_n \in V$  with  $x_0 = o, x_n = y, (x_{i-1}, x_i) \in E, \forall i = 1, \dots, n$ . (This property is called connectedness.)
5. If a conductance function  $c$  is given we require  $c_{x_{i-1}x_i} > 0$ . See Definition 10.1 below.

**Definition 10.1.** A function  $c : E \rightarrow \mathbb{R}_+ \cup \{0\}$  is called *conductance function* if

1.  $c(e) \geq 0, \forall e \in E$ ; and
2. Given  $x \in V, c_{xy} > 0, c_{xy} = c_{yx}$ , for all  $(xy) \in E$ .

If  $x \in V$ , we set

$$c(x) := \sum_{(xy) \in E} c_{xy}. \quad (10.1)$$

The summation in (10.1) is denoted  $x \sim y$ ; i.e.,  $x \sim y$  if  $(xy) \in E$ .

**Definition 10.2.** When  $c$  is a conductance function (see also Definition 10.1) we set  $\Delta = \Delta_c$  (the corresponding graph Laplacian

$$(\Delta u)(x) = \sum_{y \sim x} c_{xy} (u(x) - u(y)) = c(x) u(x) - \sum_{y \sim x} c_{xy} u(y). \quad (10.2)$$

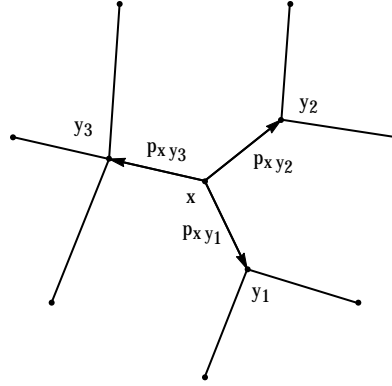


Figure 10.1: Transition probabilities  $p_{xy}$  at a vertex  $x$  (in  $V$ ).

Given  $G = (V, E, c)$  as above, and let  $\Delta = \Delta_c$  be the corresponding graph Laplacian. With a suitable ordering on  $V$ , we obtain the following banded  $\infty \times \infty$  matrix-representation for  $\Delta$  (eq. (10.3)). We refer to [GLS12] for a number of applications of infinite banded matrices.

$$\begin{bmatrix}
 c(x_1) & -c_{x_1x_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\
 -c_{x_2x_1} & c(x_2) & -c_{x_2x_3} & 0 & \cdots & \cdots & \cdots & \vdots & \cdots \\
 0 & -c_{x_3x_2} & c(x_3) & -c_{x_3x_4} & 0 & \cdots & \cdots & \mathbf{0} & \cdots \\
 \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \cdots \\
 \vdots & \mathbf{0} & \cdots & 0 & -c_{x_nx_{n-1}} & c(x_n) & -c_{x_nx_{n+1}} & 0 & \cdots \\
 \vdots & \vdots & \cdots & \cdots & 0 & \ddots & \ddots & \ddots & \ddots
 \end{bmatrix}
 \tag{10.3}$$

*Remark 10.3 (Random Walk).* If  $(V, E, c)$  is given as in Definition 10.2, then for  $(x, y) \in E$ , set

$$p_{xy} := \frac{c_{xy}}{c(x)}
 \tag{10.4}$$

and note then  $\{p_{xy}\}$  in (10.4) is a system of transition probabilities, i.e.,  $\sum_y p_{xy} = 1, \forall x \in V$ , see Figure 10.1.

A Markov-random walk on  $V$  with transition probabilities  $(p_{xy})$  is said to be *reversible* iff  $\exists$  a positive function  $\tilde{c}$  on  $V$  such that

$$\tilde{c}(x) p_{xy} = \tilde{c}(y) p_{yx}, \forall (xy) \in E.
 \tag{10.5}$$

**Theorem 10.4** ([Jor08, Woj09, Woj08]). *Let  $G = (V, E, c)$  be as above,  $V$ : vertices,  $E$ : edges, and  $c : E \rightarrow \mathbb{R}_+$  a given conductance function; we assume finite range so that the  $\infty \times \infty$  matrix in (10.3) is banded.*

Then the banded  $\infty \times \infty$  matrix in (10.3) defines an essentially selfadjoint operator in  $l^2(V)$ , with dense domain equal to all finitely supported functions on  $V$ .

## 10.2 The Energy Hilbert Spaces $\mathcal{H}_E$

Let  $G = (V, E, c)$  be an infinite connected network introduced above. Set  $\mathcal{H}_E :=$  completion of the space of all compactly supported functions  $u : V \rightarrow \mathbb{C}$  with respect to

$$\langle u, v \rangle_{\mathcal{H}_E} := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} (\overline{u(x)} - \overline{u(y)}) (v(x) - v(y)) \quad (10.6)$$

$$\|u\|_{\mathcal{H}_E}^2 := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} |u(x) - u(y)|^2 \quad (10.7)$$

then  $\mathcal{H}_E$  is a Hilbert space [JP10, JT15b].

**Lemma 10.5.** *For all  $x, y \in V$ , there is a unique real-valued dipole vector  $v_{xy} \in \mathcal{H}_E$  such that*

$$\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y), \quad \forall u \in \mathcal{H}_E.$$

*Proof.* Apply Riesz' theorem. □

**Exercise 10.6** (Gaussian free field (GFF) [SS11a]). Let  $G = (V, E, c)$  be as in the setting of Section 10.2, and let  $\mathcal{H}_E$  be the corresponding energy Hilbert space with inner product, and  $\mathcal{H}_E$ -norm as in (10.6)-(10.7).

1. Show that there is a probability space  $(\Omega, \mathcal{F}, \mathbb{P}^{(G)})$  and a Gaussian field  $X_\varphi$ , indexed by  $\varphi \in \mathcal{H}_E$  (real valued) such that  $E_{\mathbb{P}}(X_\varphi) = 0$ , and

$$\mathbb{E}_{\mathbb{P}^{(G)}}(e^{iX_\varphi}) = e^{-\frac{1}{2}\|\varphi\|_{\mathcal{H}_E}^2}; \quad (10.8)$$

in particular,

$$\mathbb{E}_{\mathbb{P}^{(G)}}(X_\varphi X_\psi) = \langle \varphi, \psi \rangle_{\mathcal{H}_E} \quad (10.9)$$

for all  $\varphi, \psi \in \mathcal{H}_E$ .

2. Show that  $X$  arises from a Gaussian point process  $\{X_x\}_{x \in V}$  such that

$$\mathbb{E}_{\mathbb{P}^{(G)}}(X_x X_\varphi) = \langle v_{xo}, \varphi \rangle_{\mathcal{H}_E} = \varphi(x) \quad (10.10)$$

for all  $\varphi \in \mathcal{H}_E$ , and all  $x \in V$ , where  $o$  is a fixed base-point in the vertex set  $V$ , and we normalize in (10.10) such that  $\varphi(o) = 0$ .

Hint: For (1), use Corollary 1.35; and for (2), use Lemma 10.5.

**Definition 10.7.** Let  $\mathcal{H}$  be a Hilbert space with inner product denoted  $\langle \cdot, \cdot \rangle$ , or  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  when there is more than one possibility to consider. Let  $J$  be a countable index set, and let  $\{w_j\}_{j \in J}$  be an indexed family of non-zero vectors in  $\mathcal{H}$ . We say that  $\{w_j\}_{j \in J}$  is a frame for  $\mathcal{H}$  iff (Def.) there are two finite positive constants  $b_1$  and  $b_2$  such that

$$b_1 \|u\|_{\mathcal{H}}^2 \leq \sum_{j \in J} |\langle w_j, u \rangle_{\mathcal{H}}|^2 \leq b_2 \|u\|_{\mathcal{H}}^2 \quad (10.11)$$

holds for all  $u \in \mathcal{H}$ . We say that it is a Parseval frame if  $b_1 = b_2 = 1$ .

For references to the theory and application of frames, see e.g., [HJL<sup>+</sup>13, KLZ09, CM13, SD13, KOPT13, EO13].

**Lemma 10.8.** *If  $\{w_j\}_{j \in J}$  is a Parseval frame in  $\mathcal{H}$ , then the (analysis) operator  $A = A_{\mathcal{H}} : \mathcal{H} \rightarrow l^2(J)$ ,*

$$Au = (\langle w_j, u \rangle_{\mathcal{H}})_{j \in J} \quad (10.12)$$

*is well-defined and isometric. Its adjoint  $A^* : l^2(J) \rightarrow \mathcal{H}$  is given by*

$$A^* \left( (\gamma_j)_{j \in J} \right) := \sum_{j \in J} \gamma_j w_j \quad (10.13)$$

*and the following hold:*

1. *The sum on the RHS in (10.13) is norm-convergent;*
2.  *$A^* : l^2(J) \rightarrow \mathcal{H}$  is co-isometric; and for all  $u \in \mathcal{H}$ , we have*

$$u = A^* Au = \sum_{j \in J} \langle w_j, u \rangle w_j \quad (10.14)$$

*where the RHS in (10.14) is norm-convergent.*

*Proof.* The details are standard in the theory of frames; see the cited papers above. Note that (10.11) for  $b_1 = b_2 = 1$  simply states that  $A$  in (10.12) is isometric, and so  $A^*A = I_{\mathcal{H}}$  = the identity operator in  $\mathcal{H}$ , and  $AA^*$  = the projection onto the range of  $A$ .  $\square$

**Theorem 10.9.** *Let  $G = (V, E, c)$  be an infinite network. Choose an orientation on the edges, denoted by  $E^{(ori)}$ . Then the system of vectors*

$$\left\{ w_{xy} := \sqrt{c_{xy}} v_{xy}, (xy) \in E^{(ori)} \right\} \quad (10.15)$$

*is a Parseval frame for the energy Hilbert space  $\mathcal{H}_E$ . For all  $u \in \mathcal{H}_E$ , we have the following representation*

$$u = \sum_{(xy) \in E^{(ori)}} c_{xy} \langle v_{xy}, u \rangle v_{xy}, \text{ and} \quad (10.16)$$

$$\|u\|_{\mathcal{H}_E}^2 = \sum_{(xy) \in E^{(ori)}} c_{xy} |\langle v_{xy}, u \rangle|^2 \quad (10.17)$$

*Proof.* See [JT15a, CH08]. □

Frames in  $\mathcal{H}_E$  consisting of our system (10.15) are not ONBs when resistors are configured in non-linear systems of vertices, for example, resistors in parallel. See Figure 10.2, and 10.10.

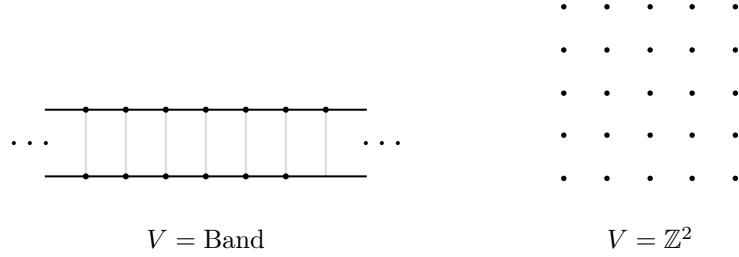


Figure 10.2: non-linear system of vertices

**Example 10.10.** Let  $c_{01}, c_{02}, c_{12}$  be positive constants, and assign conductances on the three edges (see Figure 10.3) in the triangle network.

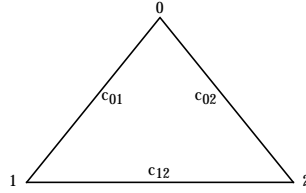


Figure 10.3: The set  $\{v_{xy} : (xy) \in E\}$  is not orthogonal.

In this case,  $w_{ij} = \sqrt{e_{ij}}v_{ij}$ ,  $i < j$ , in the cyclic order is a Parseval frame but not an ONB in  $\mathcal{H}_E$  [JT15a].

Note the corresponding Laplacian  $\Delta (= \Delta_c)$  has the following matrix representation

$$M := \begin{bmatrix} c(0) & -c_{01} & -c_{02} \\ -c_{01} & c(1) & -c_{12} \\ -c_{02} & -c_{12} & c(2) \end{bmatrix} \tag{10.18}$$

The dipoles  $\{v_{xy} : (xy) \in E^{(ori)}\}$  as 3-D vectors are the solutions to the equation

$$\Delta v_{xy} = \delta_x - \delta_y.$$

Hence,

$$\begin{aligned} Mv_{01} &= [1 \quad -1 \quad 0]^{tr} \\ Mv_{02} &= [1 \quad 0 \quad -1]^{tr} \\ Mv_{12} &= [0 \quad 1 \quad -1]^{tr} \end{aligned}$$



The Parseval frame from Lemma 10.8 is

$$\begin{aligned} w_{01} &= \sqrt{c_{01}} v_{01} = \left[ \frac{\sqrt{c_{01}} c_{12}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, -\frac{\sqrt{c_{01}} c_{02}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, 0 \right]^{tr} \\ w_{12} &= \sqrt{c_{12}} v_{12} = \left[ 0, \frac{\sqrt{c_{12}} c_{02}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, -\frac{\sqrt{c_{12}} c_{01}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}} \right]^{tr} \\ w_{20} &= \sqrt{c_{20}} v_{20} = \left[ \frac{-\sqrt{c_{20}} c_{12}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}}, 0, \frac{\sqrt{c_{20}} c_{01}}{c_{01}c_{02} + c_{01}c_{12} + c_{02}c_{12}} \right]^{tr}. \end{aligned}$$

*Remark 10.11.* The dipole  $v_{xy}$  is unique in  $\mathcal{H}_E$  as an equivalence class, not a function on  $V$ . Note  $\ker M = \text{harmonic functions} = \text{constant}$  (see (10.18)), and so  $v_{xy} + \text{const} = v_{xy}$  in  $\mathcal{H}_E$ . Thus, the above frame vectors have non-unique representations as functions on  $V$ .

### 10.3 The Graph-Laplacian

Here we include some technical lemmas for graph Laplacian in the energy Hilbert space  $\mathcal{H}_E$ .

Let  $G = (V, E, c)$  be as above; assume  $G$  is connected; i.e., there is a base point  $o$  in  $V$  such that every  $x \in V$  is connected to  $o$  via a finite path of edges.

If  $x \in V$ , we set

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad (10.19)$$

**Definition 10.12.** Let  $(V, E, c, o, \Delta)$  be as above. Let  $V' := V \setminus \{o\}$ , and set

$$v_x := v_{x,o}, \quad \forall x \in V'.$$

Further, let

$$\mathcal{D}_2 := \text{span} \{ \delta_x \mid x \in V \}, \quad \text{and} \quad (10.20)$$

$$\mathcal{D}_E := \left\{ \sum_{x \in V'} \xi_x v_x \mid \text{finite support} \right\}; \quad (10.21)$$

where by “span” we mean of all *finite* linear combinations.

Lemma 10.13 below summarizes the key properties of  $\Delta$  as an operator, both in  $l^2(V)$  and in  $\mathcal{H}_E$ .

**Lemma 10.13.** *The following hold:*

1.  $\langle \Delta u, v \rangle_{l^2} = \langle u, \Delta v \rangle_{l^2}, \quad \forall u, v \in \mathcal{D}_2;$
2.  $\langle \Delta u, v \rangle_{\mathcal{H}_E} = \langle u, \Delta v \rangle_{\mathcal{H}_E}, \quad \forall u, v \in \mathcal{D}_E;$
3.  $\langle u, \Delta u \rangle_{l^2} \geq 0, \quad \forall u \in \mathcal{D}_2, \quad \text{and}$
4.  $\langle u, \Delta u \rangle_{\mathcal{H}_E} \geq 0, \quad \forall u \in \mathcal{D}_E.$

Moreover, we have

5.  $\langle \delta_x, u \rangle_{\mathcal{H}_E} = (\Delta u)(x), \forall x \in V, \forall u \in \mathcal{H}_E$ .
6.  $\Delta v_{xy} = \delta_x - \delta_y, \forall v_{xy} \in \mathcal{H}_E$ . In particular,  $\Delta v_x = \delta_x - \delta_o, x \in V' = V \setminus \{o\}$ .

7.

$$\delta_x(\cdot) = c(x) v_x(\cdot) - \sum_{y \sim x} c_{xy} v_y(\cdot), \forall x \in V'.$$

8.

$$\langle \delta_x, \delta_y \rangle_{\mathcal{H}_E} = \begin{cases} c(x) = \sum_{t \sim x} c_{xt} & \text{if } y = x \\ -c_{xy} & \text{if } (xy) \in E \\ 0 & \text{if } (xy) \notin E, x \neq y \end{cases}$$

*Proof.* See [JP10, JP11a, JT15a]. □

## 10.4 The Friedrichs Extension

Fix a conductance function  $c$ . In this section we turn to some technical lemmas we will need for the Friedrichs extension of  $\Delta (= \Delta_c)$ .

It is known the graph-Laplacian  $\Delta$  is automatically essentially selfadjoint as a densely defined operator in  $l^2(V)$ , but not as a  $\mathcal{H}_E$  operator [Jor08, JP11b]. Since  $\Delta$  defined on  $\mathcal{D}_E$  is semibounded, it has the Friedrichs extension  $\Delta_{Fri}$  (in  $\mathcal{H}_E$ ).

**Lemma 10.14.** *Consider  $\Delta$  with  $\text{dom}(\Delta) := \text{span}\{v_{xy} : x, y \in V\}$ , then*

$$\langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = \sum_{(xy) \in E} c_{xy}^2 \left| \langle v_{xy}, \varphi \rangle_{\mathcal{H}_E} \right|^2.$$

*Proof.* Suppose  $\varphi = \sum \varphi_{xy} v_{xy} \in \text{dom}(\Delta)$ . Note the edges are not oriented, and a direct computation shows that

$$\langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = 4 \sum_{x, y} |\varphi_{xy}|^2.$$

Using the Parseval frames in Theorem 10.9, we have the following representation

$$\varphi = \sum_{(xy) \in E} \underbrace{\frac{1}{2} c_{xy} \langle v_{xy}, \varphi \rangle_{\mathcal{H}_E}}_{=: \varphi_{xy}} v_{xy}$$

Note  $\varphi \in \text{span}\{v_{xy} : x, y \in V\}$ , so the above equation contains a finite sum.

It follows that

$$\langle \varphi, \Delta \varphi \rangle_{\mathcal{H}_E} = 4 \sum_{(xy) \in E} |\varphi_{xy}|^2 = \sum_{(xy) \in E} c_{xy}^2 \left| \langle v_{xy}, \varphi \rangle_{\mathcal{H}_E} \right|^2$$

which is the assertion. □

**Theorem 10.15.** *Let  $G = (V, E, c)$  be an infinite network. If the deficiency indices of  $\Delta (= \Delta_c)$  are  $(k, k)$ ,  $k > 0$ , where  $\text{dom}(\Delta) = \text{span}\{v_{xy}\}$ , then the Friedrichs extension  $\Delta_{\text{Fri}} \supset \Delta$  is the restriction of  $\Delta^*$  to*

$$\text{dom}(\Delta_{\text{Fri}}) := \left\{ u \in \mathcal{H}_E \mid \sum_{(xy) \in E} c_{xy}^2 |\langle v_{xy}, \varphi \rangle_E|^2 < \infty \right\}. \quad (10.22)$$

*Proof.* Follows from Lemma 10.14, and the characterization of Friedrichs extensions of semibounded Hermitian operators (Chapter 9); see, e.g., [DS88c, AG93, RS75].  $\square$

### 10.5 A 1D Example

Consider  $G = (V, E, c)$ , where  $V = \{0\} \cup \mathbb{Z}_+$ . Observation: Every sequence  $a_1, a_2, \dots$  in  $\mathbb{R}_+$  defines a conductance  $c_{n-1, n} := a_n$ ,  $n \in \mathbb{Z}_+$ , i.e.,

$$0 \xleftrightarrow{a_1} 1 \xleftrightarrow{a_2} 2 \xleftrightarrow{a_3} 3 \quad \dots \quad n \xleftrightarrow{a_{n+1}} n+1 \quad \dots$$

The dipole vectors  $v_{xy}$  (for  $x, y \in \mathbb{N}$ ) are given by

$$v_{xy}(z) = \begin{cases} 0 & \text{if } z \leq x \\ -\sum_{k=x+1}^z \frac{1}{a_k} & \text{if } x < z < y \\ -\sum_{k=x+1}^y \frac{1}{a_k} & \text{if } z \geq y \end{cases}$$

See Figure 10.4.

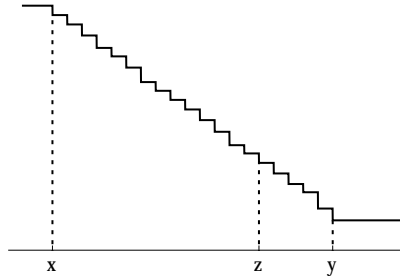


Figure 10.4: The dipole  $v_{xy}$ .

The corresponding graph Laplacian has the following matrix representation:

$$\left[ \begin{array}{cccccccc} a_1 & -a_1 & & & & & & \\ -a_1 & a_1 + a_2 & -a_2 & & & & & \\ & -a_2 & a_2 + a_3 & -a_3 & & & & \\ & & -a_3 & a_3 + a_4 & \ddots & & & \\ & & & \ddots & \ddots & & & \\ & & & & -a_n & a_n + a_{n+1} & -a_{n+1} & \\ 0 & & & & -a_{n+1} & & \ddots & \ddots \\ & & & & & -a_{n+1} & \ddots & \ddots \\ & & & & & & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{array} \right] \quad (10.23)$$

That is,

$$\begin{cases} (\Delta u)_0 &= a_1 (u_0 - u_1) \\ (\Delta u)_n &= a_n (u_n - u_{n-1}) + a_{n+1} (u_n - u_{n+1}) \\ &= (a_n + a_{n+1}) u_n - a_n u_{n-1} - a_{n+1} u_{n+1}, \quad \forall n \in \mathbb{Z}_+. \end{cases} \quad (10.24)$$

**Lemma 10.16.** *Let  $G = (V, c, E)$  be as above, where  $a_n := c_{n-1, n}$ ,  $n \in \mathbb{Z}_+$ . Then  $u \in \mathcal{H}_E$  is the solution to  $\Delta u = -u$  (i.e.,  $u$  is a defect vector of  $\Delta$ ) if and only if  $u$  satisfies the following equation:*

$$\sum_{n=1}^{\infty} a_n \langle v_{n-1, n}, u \rangle_{\mathcal{H}_E} (\delta_{n-1}(s) - \delta_n(s) + v_{n-1, n}(s)) = 0, \quad \forall s \in \mathbb{Z}_+; \quad (10.25)$$

where

$$\|u\|_{\mathcal{H}_E}^2 = \sum_{n=1}^{\infty} a_n \left| \langle v_{n-1, n}, u \rangle_{\mathcal{H}_E} \right|^2 < \infty. \quad (10.26)$$

*Proof.* By Theorem 10.9, the set  $\{\sqrt{a_n} v_{n-1, n}\}_{n=1}^{\infty}$  forms a Parseval frame in  $\mathcal{H}_E$ . In fact, the dipole vectors are

$$v_{n-1, n}(s) = \begin{cases} 0 & s \leq n-1 \\ -\frac{1}{a_n} & s \geq n \end{cases}; \quad n = 1, 2, \dots \quad (10.27)$$

and so  $\{\sqrt{a_n} v_{n-1, n}\}_{n=1}^{\infty}$  forms an ONB in  $\mathcal{H}_E$ ; and  $u \in \mathcal{H}_E$  has the representation

$$u = \sum_{n=1}^{\infty} a_n \langle v_{n-1, n}, u \rangle_{\mathcal{H}_E} v_{n-1, n}$$

see (10.14). Therefore,  $\Delta u = -u$  if and only if

$$\sum_{n=1}^{\infty} a_n \langle v_{n-1, n}, u \rangle_{\mathcal{H}_E} (\delta_{n-1}(s) - \delta_n(s)) = - \sum_{n=1}^{\infty} a_n \langle v_{n-1, n}, u \rangle_{\mathcal{H}_E} v_{n-1, n}(s)$$

for all  $s \in \mathbb{Z}_+$ , which is the assertion.  $\square$

Below we compute the deficiency space in an example with index values  $(1, 1)$ .

**Lemma 10.17.** *Let  $(V, E, c = \{a_n\})$  be as above. Let  $Q > 1$  and set  $a_n := Q^n$ ,  $n \in \mathbb{Z}_+$ ; then  $\Delta$  has deficiency indices  $(1, 1)$ .*

*Proof.* Suppose  $\Delta u = -u$ ,  $u \in \mathcal{H}_E$ . Then,

$$\begin{aligned} -u_1 &= Q(u_1 - u_0) + Q^2(u_1 - u_2) \iff u_2 = \left(\frac{1}{Q^2} + \frac{1+Q}{Q}\right)u_1 - \frac{1}{Q}u_0 \\ -u_2 &= Q^2(u_2 - u_1) + Q^3(u_2 - u_3) \iff u_3 = \left(\frac{1}{Q^3} + \frac{1+Q}{Q}\right)u_2 - \frac{1}{Q}u_1 \end{aligned}$$

and by induction,

$$u_{n+1} = \left(\frac{1}{Q^{n+1}} + \frac{1+Q}{Q}\right)u_n - \frac{1}{Q}u_{n-1}, \quad n \in \mathbb{Z}_+$$

i.e.,  $u$  is determined by the following matrix equation:

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} \frac{1}{Q^{n+1}} + \frac{1+Q}{Q} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$$

The eigenvalues of the coefficient matrix are

$$\begin{aligned} \lambda_{\pm} &= \frac{1}{2} \left( \frac{1}{Q^{n+1}} + \frac{1+Q}{Q} \pm \sqrt{\left(\frac{1}{Q^{n+1}} + \frac{1+Q}{Q}\right)^2 - \frac{4}{Q}} \right) \\ &\sim \frac{1}{2} \left( \frac{1+Q}{Q} \pm \left(\frac{Q-1}{Q}\right) \right) = \begin{cases} 1 \\ \frac{1}{Q} \end{cases} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Equivalently, as  $n \rightarrow \infty$ , we have

$$u_{n+1} \sim \left(\frac{1+Q}{Q}\right)u_n - \frac{1}{Q}u_{n-1} = \left(1 + \frac{1}{Q}\right)u_n - \frac{1}{Q}u_{n-1}$$

and so

$$u_{n+1} - u_n \sim \frac{1}{Q}(u_n - u_{n-1}).$$

Therefore, for the tail-summation, we have:

$$\sum_n Q^n (u_{n+1} - u_n)^2 = \text{const} \sum_n \frac{(Q-1)^2}{Q^{n+2}} < \infty$$

which implies  $\|u\|_{\mathcal{H}_E} < \infty$ .  $\square$

Next, we give a random walk interpretation of Lemma 10.17. See Remark 10.3, and Figure 10.1.

*Remark 10.18* (Harmonic functions in  $\mathcal{H}_E$ ). Note that in 10.5 (Lemma 10.17), the space of harmonic functions in  $\mathcal{H}_E$  is one-dimensional; in fact if  $Q > 1$  is fixed, then

$$\{u \in \mathcal{H}_E \mid \Delta u = 0\}$$

is spanned by  $u = (u_n)_{n=0}^\infty$ ,  $u_n = \frac{1}{Q^n}$ ,  $n \in \mathbb{N}$ ; and of course  $\|1/Q^n\|_{\mathcal{H}_E}^2 < \infty$ .

*Remark 10.19.* For the domain of the Friedrichs extension  $\Delta_{Fri}$ , we have:

$$\text{dom}(\Delta_{Fri}) = \{f \in \mathcal{H}_E \mid (f(x) - f(x+1))Q^x \in l^2(\mathbb{Z}_+)\} \quad (10.28)$$

i.e.,

$$\text{dom}(\Delta_{Fri}) = \left\{ f \in \mathcal{H}_E \mid \sum_{x=0}^\infty |f(x) - f(x+1)|^2 Q^{2x} < \infty \right\}.$$

*Proof.* By Theorem 10.9, we have the following representation, valid for all  $f \in \mathcal{H}_E$ :

$$\begin{aligned} f &= \sum_x \langle f, Q^{\frac{x}{2}} v_{(x,x+1)} \rangle_{\mathcal{H}_E} Q^{\frac{x}{2}} v_{(x,x+1)} \\ &= \sum_x (f(x) - f(x+1)) Q^x v_{(x,x+1)}; \end{aligned}$$

and

$$\langle f, \Delta f \rangle_{\mathcal{H}_E} = \sum_x |f(x) - f(x+1)|^2 Q^{2x}.$$

The desired conclusion (10.28) now follows from Theorem 10.15. Also see e.g. [DS88c, AG93].  $\square$

**Definition 10.20.** Let  $G = (V, E, c)$  be a connected graph. The set of transition probabilities  $(p_{xy})$  is said to be reversible if there exists  $c : V \rightarrow \mathbb{R}_+$  such that

$$c(x) p_{xy} = c(y) p_{yx}; \quad (10.29)$$

and then

$$c_{xy} := c(x) p_{xy} \quad (10.30)$$

is a system of conductance. Conversely, for a system of conductance  $(c_{xy})$  we set

$$c(x) := \sum_{y \sim x} c_{xy}, \text{ and} \quad (10.31)$$

$$p_{xy} := \frac{c_{xy}}{c(x)} \quad (10.32)$$

and so  $(p_{xy})$  is a set of transition probabilities. See Figure 10.5.

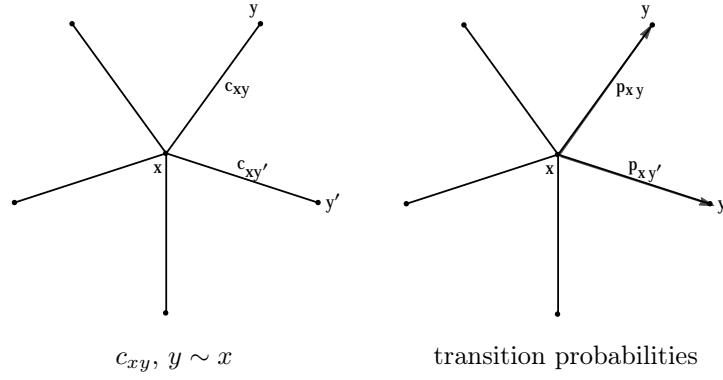


Figure 10.5: neighbors of  $x$

Recall the graph Laplacian in (10.24) can be written as

$$(\Delta u)_n = c(n) (u_n - p_-(n) u_{n-1} - p_+(n) u_{n+1}), \forall n \in \mathbb{Z}_+; \tag{10.33}$$

where

$$c(n) := a_n + a_{n+1} \tag{10.34}$$

and

$$p_-(n) := \frac{a_n}{c(n)}, p_+(n) := \frac{a_{n+1}}{c(n)} \tag{10.35}$$

are the left/right transition probabilities, as shown in Figure 10.6.

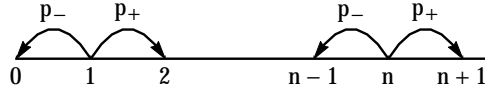


Figure 10.6: The transition probabilities  $p_+, p_-$ , in the case of constant transition probabilities, i.e.,  $p_+(n) = p_+$ , and  $p_-(n) = p_-$  for all  $n \in \mathbb{Z}_+$ .

In the case  $a_n = Q^n$ ,  $Q > 1$ , as in Lemma 10.17, we have

$$c(n) := Q^n + Q^{n+1}, \text{ and} \tag{10.36}$$

$$p_+ := p_+(n) = \frac{Q^{n+1}}{Q^n + Q^{n+1}} = \frac{Q}{1 + Q} \tag{10.37}$$

$$p_- := p_-(n) = \frac{Q^n}{Q^n + Q^{n+1}} = \frac{1}{1 + Q} \tag{10.38}$$

For all  $n \in \mathbb{Z}_+ \cup \{0\}$ , set

$$(Pu)_n := p_- u_{n-1} + p_+ u_{n+1}. \tag{10.39}$$

Note  $(Pu)_0 = u_1$ . By (10.33), we have

$$\Delta = c(1 - P). \quad (10.40)$$

In particular,  $p_+ > \frac{1}{2}$ , i.e., a random walker has probability  $> \frac{1}{2}$  of moving to the right. It follows that

$$\underbrace{\text{travel time}(n, \infty)}_{= \text{dist to } \infty} < \infty;$$

and so  $\Delta$  is not essentially selfadjoint, i.e., indices  $(1, 1)$ .

**Lemma 10.21.** *Let  $(V, E, \Delta(= \Delta_c))$  be as above, where the conductance  $c$  is given by  $c_{n-1, n} = Q^n$ ,  $n \in \mathbb{Z}_+$ ,  $Q > 1$  (see Lemma 10.17). For all  $\lambda > 0$ , there exists  $f_\lambda \in \mathcal{H}_E$  satisfying  $\Delta f_\lambda = \lambda f_\lambda$ .*

*Proof.* By (10.40), we have  $\Delta f_\lambda = \lambda f_\lambda \iff Pf_\lambda = (1 - \frac{\lambda}{c}) f_\lambda$ , i.e.,

$$\frac{1}{1+Q} f_\lambda(n-1) + \frac{Q}{1+Q} f_\lambda(n+1) = \left(1 - \frac{\lambda}{Q^{n-1}(1+Q)}\right) f_\lambda(n)$$

and so

$$f_\lambda(n+1) = \left(\frac{1+Q}{Q} - \frac{\lambda}{Q^n}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1). \quad (10.41)$$

This corresponds to the following matrix equation:

$$\begin{aligned} \begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} &= \begin{bmatrix} \frac{1+Q}{Q} - \frac{\lambda}{Q^n} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix} \\ &\sim \begin{bmatrix} \frac{1+Q}{Q} & -\frac{1}{Q} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}, \text{ as } n \rightarrow \infty. \end{aligned}$$

The eigenvalues of the coefficient matrix are given by

$$\lambda_\pm \sim \frac{1}{2} \left( \frac{1+Q}{Q} \pm \left( \frac{Q-1}{Q} \right) \right) = \begin{cases} 1 \\ \frac{1}{Q} \end{cases} \text{ as } n \rightarrow \infty.$$

That is, as  $n \rightarrow \infty$ ,

$$f_\lambda(n+1) \sim \left(\frac{1+Q}{Q}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1);$$

i.e.,

$$f_\lambda(n+1) \sim \frac{1}{Q} f_\lambda(n); \quad (10.42)$$

and so the tail summation of  $\|f_\lambda\|_{\mathcal{H}_E}^2$  is finite. (See the proof of Lemma 10.17.) We conclude that  $f_\lambda \in \mathcal{H}_E$ .  $\square$



**Corollary 10.22.** *Let  $(V, E, \Delta)$  be as in the lemma. The Friedrichs extension  $\Delta_{Fri}$  has continuous spectrum  $[0, \infty)$ .*

*Proof.* Fix  $\lambda \geq 0$ . We prove that if  $\Delta f_\lambda = \lambda f_\lambda$ ,  $f \in \mathcal{H}_E$ , then  $f_\lambda \notin \text{dom}(\Delta_{Fri})$ .

Note for  $\lambda = 0$ ,  $f_0$  is harmonic, and so  $f_0 = k \left(\frac{1}{Q^n}\right)_{n=0}^\infty$  for some constant  $k \neq 0$ . See Remark 10.18. It follows from (10.28) that  $f_0 \notin \text{dom}(\Delta_{Fri})$ .

The argument for  $\lambda > 0$  is similar. Since as  $n \rightarrow \infty$ ,  $f_\lambda(n) \sim \frac{1}{Q^n}$  (eq. (10.42)), so by (10.28) again,  $f_\lambda \notin \text{dom}(\Delta_{Fri})$ .

However, if  $\lambda_0 < \lambda_1$  in  $[0, \infty)$  then

$$\int_{\lambda_0}^{\lambda_1} f_\lambda(\cdot) d\lambda \in \text{dom}(\Delta_{Fri}) \quad (10.43)$$

and so every  $f_\lambda$ ,  $\lambda \in [0, \infty)$ , is a generalized eigenfunction, i.e., the spectrum of  $\Delta_{Fri}$  is purely continuous with Lebesgue measure, and multiplicity one.

The verification of (10.43) follows from (10.41), i.e.,

$$f_\lambda(n+1) = \left(\frac{1+Q}{Q} - \frac{\lambda}{Q^n}\right) f_\lambda(n) - \frac{1}{Q} f_\lambda(n-1). \quad (10.44)$$

Set

$$F_{[\lambda_0, \lambda_1]} := \int_{\lambda_0}^{\lambda_1} f_\lambda(\cdot) d\lambda. \quad (10.45)$$

Then by (10.44) and (10.45),

$$F_{[\lambda_0, \lambda_1]}(n+1) = \frac{1+Q}{Q} F_{[\lambda_0, \lambda_1]}(n) - \frac{1}{Q^n} \int_{\lambda_0}^{\lambda_1} \lambda f_\lambda(n) d\lambda - \frac{1}{Q} F_{[\lambda_0, \lambda_1]}(n-1)$$

and  $\int_{\lambda_0}^{\lambda_1} \lambda f_\lambda d\lambda$  is computed using integration by parts.  $\square$

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest: [Jor08, JP10, JP11a, JP11b, JP13a, JP13b, JT15a, RAKK05, Yos95, BKS13, Str12, JP14, LPW13, JP14, CZ07, AJSV13, SS11a, JP12, JT15b].

## Chapter 11

# Reproducing Kernel Hilbert Space

The simplicities of natural laws arise through the complexities of the language we use for their expression.

— Eugene Wigner

... an apt comment on how science, and indeed the whole of civilization, is a series of incremental advances, each building on what went before.

— Stephen Hawking

At a given moment there is only a fine layer between the ‘trivial’ and the impossible. Mathematical discoveries are made in this layer.

— Andrey Kolmogorov

... the distinction between two sorts of truths, profound truths recognized by the fact that the opposite is also a profound truth; by contrast to trivialities, where opposites are obviously absurd.

— Niels Bohr

A special family of Hilbert spaces  $\mathcal{H}$  are reproducing kernel Hilbert spaces (RKHSs). We say that  $\mathcal{H}$  is a RKHS if  $\mathcal{H}$  is a Hilbert space of functions on some set  $X$  such that for every  $x$  in  $X$ , the linear mapping  $f \mapsto f(x)$  is continuous in the norm of  $\mathcal{H}$ .

We begin with a general

**Definition 11.1.** Let  $S$  be a set. We say that  $\mathcal{H}$  is a  $S$ -reproducing kernel Hilbert space if:

1.  $\mathcal{H}$  is a Hilbert space of functions on  $S$ ; and

2. For all  $s \in S$ , the mapping  $\mathcal{H} \rightarrow \mathbb{C}$ , by  $E_s : h \mapsto h(s)$  is continuous on  $\mathcal{H}$ , i.e., by Riesz, there is a  $K_s \in \mathcal{H}$  such that  $h(s) = \langle K_s, h \rangle$ ,  $\forall h \in \mathcal{H}$ .

**Notation.** The system of functions  $\{K_s : s \in S\} \subset \mathcal{H}$  is called the associated reproducing kernel.

Let  $\mathcal{H}$  be a RKHS, see Definition 11.1, hence  $S$  is a set, and  $\mathcal{H}$  is a Hilbert space of functions on  $S$  such that (2) holds. Fix  $s \in S$ , and note that  $E_s : \mathcal{H} \rightarrow \mathbb{C}$  is then a bounded linear operator, where  $\mathbb{C}$  is a 1-dimensional Hilbert space. Hence its adjoint  $E_s^* : \mathbb{C} \rightarrow \mathcal{H}^* \simeq \mathcal{H}$  is well-defined.

*Claim 11.2.* The kernel  $K_s$  in Definition 11.1 is  $E_s^*(1) = K_s$ .

*Proof.* For  $\lambda \in \mathbb{C}$  and  $h \in \mathcal{H}$ , we have

$$\langle E_s^*(\lambda), h \rangle_{\mathcal{H}} = \bar{\lambda} E_s(h) = \bar{\lambda} h(s) \stackrel{\text{(by (2))}}{=} \bar{\lambda} \langle K_s, h \rangle_{\mathcal{H}} = \langle \lambda K_s, h \rangle_{\mathcal{H}};$$

and therefore  $E_s^*(\lambda) = \lambda K_s$  as desired.  $\square$

**Example 11.3.** For  $s, t \in [0, 1]$ , set

$$K(s, t) = s \wedge t = \min(s, t), \quad (11.1)$$

i.e., the covariance-kernel for Brownian motion on the interval  $[0, 1]$ , see Chapter 6.

**Exercise 11.4** (The Fundamental Theorem of Calculus and a RKHS). Show that the RKHS for the kernel  $K$  in (11.1) is

$$\mathcal{H} = \left\{ f : \begin{array}{l} \text{locally integrable with distribution-} \\ \text{derivative } f' \in L^2(0, 1) \text{ and } f(0) = 0 \end{array} \right\}$$

with the inner product

$$\langle f, g \rangle_{\mathcal{H}} := \int_0^1 \overline{f'(x)} g'(x) dx.$$

Hint:

Step 1 Show that  $K_t = K(t, \cdot)$  is in  $\mathcal{H}$

Step 2 Show that for all  $f \in \mathcal{H}$ , we have

$$\langle K_t, f \rangle_{\mathcal{H}} = \int_0^t f'(x) dx = f(t).$$

Suppose  $K : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$  is positive definite and continuous, i.e., for all finite sums:

$$\sum_j \sum_k \bar{c}_j c_k K(t_j, t_k) \geq 0, \quad (11.2)$$

$\{c_j\} \subset \mathbb{C}$ ,  $\{t_j\} \subset [0, 1]$ .

**Exercise 11.5** (Mercer [GBD94, Wit74]). Show that the operator  $T_K$ ,

$$(T_K \varphi)(t) = \int_0^1 K(t, s) \varphi(s) ds \quad (11.3)$$

in  $L^2(0, 1)$  is trace-class, and

$$\text{trace}(T_K) = \int_0^1 K(t, t) dt. \quad (11.4)$$

Hint: Apply weak-compactness, and the Spectral Theorem for compact self-adjoint operators, Theorem 3.58. Combine this with a choice of an ONB in the RKHS defined from (11.2).

**Exercise 11.6** (The Szegő-kernel). Let  $\mathbb{H}_2$  be the Hardy space of the disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , see Section 4.9. Show that  $\mathbb{H}_2$  is a RKHS with reproducing kernel (the Szegő-kernel):

$$K(z, w) = \frac{1}{1 - \bar{z}w} \quad (11.5)$$

i.e., that we have:

$$\langle K(z, \cdot), f \rangle_{\mathbb{H}_2} = f(z), \quad \forall f \in \mathbb{H}_2, \quad \forall z \in \mathbb{D}. \quad (11.6)$$

Hint: Substitute (11.5) into the formula from  $\langle \cdot, \cdot \rangle_{\mathbb{H}_2}$  inner product on the LHS in (11.6), and recall our convention: Inner products are linear in the second variable.

**Definition 11.7.** Let  $X$  be a set and  $K : X \times X \rightarrow \mathbb{C}$  a fixed positive definite function; and let  $\mathcal{H}_K$  be the corresponding RKHS; (see Definition 11.1). Let  $\varphi : X \rightarrow \mathbb{C}$  be a function on  $X$ ; then we say  $\varphi$  is a *multiplier*, written “ $\varphi \in \text{Multp}(\mathcal{H}_K)$ ” if multiplication by  $\varphi$  defines a bounded linear operator in  $\mathcal{H}_K$ , so

$$\begin{aligned} M_\varphi : \mathcal{H}_K &\longrightarrow \mathcal{H}_K \\ (M_\varphi f)(x) &= \varphi(x) f(x), \quad \forall f \in \mathcal{H}_K, \quad \forall x \in X. \end{aligned} \quad (11.7)$$

**Exercise 11.8** ( $\text{Multp}(\mathcal{H}_K)$ ).

1. Show the following equivalence:

$$\begin{aligned} \varphi &\in \text{Multp}(\mathcal{H}_K) \\ &\Downarrow \end{aligned} \quad (11.8)$$

$\exists$  constant  $B < \infty$  such that we have the following estimate for all finite sums:

$$\sum_i \sum_j \bar{c}_i c_j \left( B - \overline{\varphi(x_i)} \varphi(x_j) \right) K(x_i, x_j); \quad (11.9)$$

i.e., computed for all systems  $\{c_i\} \subset \mathbb{C}$ , and  $\{x_i\} \subset X$ .

2. Let  $\varphi \in \text{Multp}(\mathcal{H}_K)$ , and let  $\{K_x\}_{x \in X}$  be the kernel functions. Prove that

$$M_\varphi^*(K_x) = \overline{\varphi(x)}K_x, \quad \forall x \in X, \quad (11.10)$$

where  $M_\varphi^*$  denotes the adjoint operator.

Hint: Verify the following identity:

$$\left\langle \overline{\varphi(x)}K_x, f \right\rangle_{\mathcal{H}_K} = \langle K_x, M_\varphi f \rangle_{\mathcal{H}_K}, \quad \forall x \in X, f \in \mathcal{H}_K. \quad (11.11)$$

3. Show directly from (1) and (2) that  $\text{Multp}(\mathcal{H}_K)$  is an algebra.  
 4. Apply (1) to the Hardy space  $\mathbb{H}_2$  of the disk to conclude that

$$\text{Multp}(\mathbb{H}_2) = \mathbb{H}_\infty. \quad (11.12)$$

#### Contents of the Chapter.

In this chapter, we study two extension problems, and their interconnections. The first class of extension problems concerns (i) positive definite (p.d.) continuous functions on Lie groups  $G$ , and the second deals with (ii) Lie algebras of unbounded skew-Hermitian operators in a certain family of reproducing kernel Hilbert spaces (RKHS). The analysis is non-trivial even if  $G = \mathbb{R}^n$ , and even if  $n = 1$ . If  $G = \mathbb{R}^n$ , we are concerned in (ii) with the study of systems of  $n$  skew-Hermitian operators  $\{S_i\}$  on a common dense domain in Hilbert space, and in deciding whether it is possible to find a corresponding system of strongly commuting selfadjoint operators  $\{T_i\}$  such that, for each value of  $i$ , the operator  $T_i$  extends  $S_i$ .

From the postulates of quantum physics, we know that measurements of observables are computed from associated selfadjoint operators—observables. From the corresponding spectral resolutions, we get probability measures, and of course uncertainty. There are many philosophical issues (which we bypass here), and we do not yet fully understand quantum reality. See for example, [Sla03, CJK<sup>+</sup>12].

The axioms are as follows: An *observable* is a Hermitian (selfadjoint) linear operator mapping a Hilbert space, the space of *states*, into itself. The values obtained in a *physical measurement* are, in general, described by a probability distribution; and the distribution represents a suitable “average” (or “*expectation*”) in a measurement of values of some quantum observable in a state of some prepared system. The states are (up to phase) unit vectors in the Hilbert space, and a measurement corresponds to a *probability distribution* (derived from a projection-valued spectral measure). The spectral type may be *continuous* (such as position and momentum) or *discrete* (such as spin).

Information about the measures  $\mu$  are computed with the use of *generating functions* (on  $\mathbb{R}$ ), i.e., spectral (Bochner/Fourier) transforms of the corresponding measure. Generating functions are positive definite continuous functions  $F (= F_\mu)$  on  $\mathbb{R}$ . One then tries to recover  $\mu$  from information about  $F$ . In this

chapter we explore the cases when information about  $F(x)$  is only available for  $x$  in a bounded interval.

In probability theory, normalized continuous positive definite functions  $F$ , i.e.,  $F(0) = 1$ , arise as *generating functions* for probability measures, and one passes from information about one to the other; – from generating function to probability measure is called “the inverse problem”, see e.g., [DM85]. Hence the study of partially defined p.d. functions addresses the inverse question: ambiguity of measures when only partial information for a possible generating function is available.

## 11.1 A Digression: Stochastic Processes

Below we continue the discussion of stochastic processes started in Section 1.4.

The interest in positive definite functions has at least three roots: (i) Fourier analysis, and harmonic analysis more generally, including the non-commutative variant where we study unitary representations of groups; (ii) optimization and approximation problems, involving for example spline approximations as envisioned by I. Schönberg; and (iii) the study of stochastic (random) processes.

A stochastic process is an indexed family of random variables based on a fixed probability space; in our present analysis, the processes will be indexed by some group  $G$ ; for example  $G = \mathbb{R}$ , or  $G = \mathbb{Z}$  correspond to processes indexed by real time, respectively discrete time. A main tool in the analysis of stochastic processes is an associated covariance function, see (11.13).

A process  $\{X_g \mid g \in G\}$  is called Gaussian if each random variable  $X_g$  is Gaussian, i.e., its distribution is Gaussian. For Gaussian processes we only need two moments. So if we normalize, setting the mean equal to 0, then the process is determined by the covariance function. In general the covariance function is a function on  $G \times G$ , or on a subset, but if the process is stationary, the covariance function will in fact be a positive definite function defined on  $G$ , or a subset of  $G$ . We will be using three stochastic processes in this book, Brownian motion, Brownian Bridge, and the Ornstein-Uhlenbeck process, all Gaussian, or Itô integrals.

We outline a brief sketch of these facts below.

Let  $G$  be a locally compact group, and let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $\mathcal{F}$  a sigma-algebra, and  $\mathbb{P}$  a probability measure defined on  $\mathcal{F}$ . A stochastic  $L^2$ -process is a system of random variables  $\{X_g\}_{g \in G}$ ,  $X_g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ . The covariance function  $c_X$  of the process is the function  $G \times G \rightarrow \mathbb{C}$  given by

$$c_X(g_1, g_2) = \mathbb{E}(\overline{X_{g_1}} X_{g_2}), \quad \forall (g_1, g_2) \in G \times G. \quad (11.13)$$

To simplify will assume that the mean  $\mathbb{E}(X_g) = \int_{\Omega} X_g d\mathbb{P}(\omega) = 0$  for all  $g \in G$ .

We say that  $(X_g)$  is stationary iff

$$c_X(hg_1, hg_2) = c_X(g_1, g_2), \quad \forall h \in G. \quad (11.14)$$

In this case  $c_X$  is a function of  $g_1^{-1}g_2$ , i.e.,

$$\mathbb{E}(X_{g_1}, X_{g_2}) = c_X(g_1^{-1}g_2), \quad \forall g_1, g_2 \in G. \quad (11.15)$$

(Just take  $h = g_1^{-1}$  in (11.14).)

We now recall the following theorem of Kolmogorov (see [PS75]). One direction is easy, and the other is the deep part:

**Definition 11.9.** A function  $c$  defined on a subset of  $G$  is said to be *positive definite* iff

$$\sum_i \sum_j \bar{\lambda}_i \lambda_j c(g_i^{-1}g_j) \geq 0$$

for all finite summation, where  $\lambda_i \in \mathbb{C}$  and  $g_i^{-1}g_j$  in the domain of  $c$ .

**Theorem 11.10** (Kolmogorov). *A function  $c : G \rightarrow \mathbb{C}$  is positive definite if and only if there is a stationary Gaussian process  $(\Omega, \mathcal{F}, \mathbb{P}, X)$  with mean zero, such that  $c = c_X$ .*

*Proof.* To stress the idea, we include the easy part of the theorem, and we refer to [PS75] for the non-trivial direction:

Let  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ , and  $\{g_i\}_{i=1}^N \subset G$ , then for all finite summations, we have:

$$\sum_i \sum_j \bar{\lambda}_i \lambda_j c_X(g_i^{-1}g_j) = \mathbb{E} \left( \left| \sum_{i=1}^N \lambda_i X_{g_i} \right|^2 \right) \geq 0.$$

□

## 11.2 Two Extension Problems

While each of the two extension problems has received a considerable amount of attention in the literature, our emphasis here will be the interplay between the two problems: Our aim is a duality theory; and, in the case  $G = \mathbb{R}^n$ , and  $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ , we will state our theorems in the language of Fourier duality of abelian groups: With the time frequency duality formulation of Fourier duality for  $G = \mathbb{R}^n$  we have that both the time domain and the frequency domain constitute a copy of  $\mathbb{R}^n$ . We then arrive at a setup such that our extension questions (i) are in time domain, and extensions from (ii) are in frequency domain. Moreover we show that each of the extensions from (i) has a variant in (ii). Specializing to  $n = 1$ , we arrive of a spectral theoretic characterization of all skew-Hermitian operators with dense domain in a separable Hilbert space, having deficiency-indices  $(1, 1)$ .

A systematic study of densely defined Hermitian operators with deficiency indices  $(1, 1)$ , and later  $(d, d)$ , was initiated by M. Krein [Kre46], and is also part of de Branges' model theory; see [dB68, dBR66]. The direct connection between this theme and the problem of extending continuous positive definite

(p.d.) functions  $F$  when they are only defined on a fixed open subset to  $\mathbb{R}^n$  was one of our motivations. One desires continuous p.d. extensions to  $\mathbb{R}^n$ .

If  $F$  is given, we denote the set of such extensions  $Ext(F)$ . If  $n = 1$ ,  $Ext(F)$  is always non-empty, but for  $n = 2$ , Rudin gave examples in [Rud70, Rud63] when  $Ext(F)$  may be empty. Here we extend these results, and we also cover a number of classes of positive definite functions on locally compact groups in general; so cases when  $\mathbb{R}^n$  is replaced with other groups, both Abelian and non-abelian.

The results in the framework of locally compact Abelian groups are more complete than their counterparts for non-Abelian Lie groups, one reason is the availability of Bochner's duality theorem for locally compact Abelian groups; – not available for non-Abelian Lie groups.

*Remark 11.11.* Even in one dimension the extension problem for locally defined positive definite functions is interesting. One reason is that among the Fourier transforms (generating functions) for finite positive Borel measures  $P$  on  $\mathbb{R}$ ,

$$g_P(u) = \int_{\mathbb{R}} e^{iux} dP(x), \quad u \in \mathbb{R}; \quad (11.16)$$

one wishes to identify the *infinitely divisible distributions*. We have:

**Theorem 11.12** (Lévy-Khinchin [Rit88]). *Infinite divisibility holds if and only if  $g_P$  has the following representation:  $g_P(u) = e^{\eta(u)}$ , such that for some  $a \in \mathbb{R}$ ,  $\sigma \in \mathbb{R}_+$ , and Borel measure  $L$  on  $\mathbb{R} \setminus \{0\}$ , we have:*

$$\eta(u) = i a u - \frac{\sigma^2}{2} u^2 + \int_{\mathbb{R} \setminus \{0\}} \left( e^{i u x} - 1 - \frac{i u x}{1 + x^2} \right) L(dx) \quad (11.17)$$

and the measure  $L$  satisfying

$$\int_{\mathbb{R} \setminus \{0\}} (1 \wedge x^2) L(dx) < \infty. \quad (11.18)$$

### 11.3 The Reproducing Kernel Hilbert Space $\mathcal{H}_F$

Reproducing kernel Hilbert spaces were pioneered by Aronszajn [Aro50], and subsequently they have been used in a host of applications; e.g., [Sza04, SZ09, SZ07]. The reproducing kernel property appeared for the first time in Zaremba's paper [Zar07].

As for positive definite functions, their use and applications are extensive and includes such areas as stochastic processes, see e.g., [JP13a, AJSV13, JP12, AJ12]; harmonic analysis (see [JÓ00]), and the references there); potential theory [Fug74, KL14b]; operators in Hilbert space [Alp92, AD86]; and spectral theory [AH13, Nus75, Dev72, Dev59]. We stress that the literature is vast, and the above list is only a small sample.



Associated to a pair  $(\Omega, F)$ , where  $F$  is a prescribed continuous positive definite function defined on  $\Omega$ , we outline a reproducing kernel Hilbert space  $\mathcal{H}_F$  which will serve as a key tool in our analysis. The particular RKHSs we need here will have additional properties (as compared to a general framework); which allow us to give explicit formulas for our solutions.

**Definition 11.13.** Let  $G$  be a Lie group. Fix  $\Omega \subset G$ , non-empty, open and connected. A continuous function

$$F : \Omega^{-1} \cdot \Omega \rightarrow \mathbb{C} \quad (11.19)$$

is *positive definite* (p.d.) if

$$\sum_i \sum_j \bar{c}_i c_j F(x_i^{-1} x_j) \geq 0, \quad (11.20)$$

for all finite systems  $\{c_i\} \subset \mathbb{C}$ , and points  $\{x_i\} \subset \Omega$ .

Equivalently,

$$\int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \varphi(y) F(x^{-1} y) dx dy \geq 0, \quad (11.21)$$

for all  $\varphi \in C_c(\Omega)$ ; where  $dx$  denotes a choice of left-invariant Haar measure on  $G$ .

For simplicity we focus on the case  $G = \mathbb{R}$ , indicating the changes needed for general Lie groups.

**Definition 11.14.** Fix  $0 < a < \infty$ , set  $\Omega := (0, a)$ . Let  $F : \Omega - \Omega \rightarrow \mathbb{C}$  be a continuous p.d. function. The *reproducing kernel Hilbert space (RKHS)*,  $\mathcal{H}_F$ , is the completion of the space of functions

$$\sum_{\text{finite}} c_j F(\cdot - x_j) : c_j \in \mathbb{C} \quad (11.22)$$

with respect to the inner product

$$\langle F(\cdot - x), F(\cdot - y) \rangle_{\mathcal{H}_F} = F(x - y), \quad \forall x, y \in \Omega, \text{ and}$$

$$\left\langle \sum_i c_i F(\cdot - x_i), \sum_j c_j F(\cdot - x_j) \right\rangle_{\mathcal{H}_F} = \sum_i \sum_j \bar{c}_i c_j F(x_i - x_j), \quad (11.23)$$

*Remark 11.15.* Throughout, we use the convention that the inner product is conjugate linear in the first variable, and linear in the second variable. When more than one inner product is used, subscripts will make reference to the Hilbert space.

**Notation.** Inner product and norms will be denoted  $\langle \cdot, \cdot \rangle$ , and  $\|\cdot\|$  respectively. Often more than one inner product is involved, and subscripts are used for identification.

**Lemma 11.16.** *The reproducing kernel Hilbert space (RKHS),  $\mathcal{H}_F$ , is the Hilbert completion of the space of functions*

$$F_\varphi(x) = \int_{\Omega} \varphi(y) F(x-y) dy, \quad \forall \varphi \in C_c^\infty(\Omega), x \in \Omega \quad (11.24)$$

with respect to the inner product

$$\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \psi(y) F(x-y) dx dy, \quad \forall \varphi, \psi \in C_c^\infty(\Omega). \quad (11.25)$$

In particular,

$$\|F_\varphi\|_{\mathcal{H}_F}^2 = \int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \varphi(y) F(x-y) dx dy, \quad \forall \varphi \in C_c^\infty(\Omega) \quad (11.26)$$

and

$$\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\varphi(x)} F_\psi(x) dx, \quad \forall \varphi, \psi \in C_c^\infty(\Omega). \quad (11.27)$$

*Proof.* Apply standard approximation, see Lemma 11.17 below.  $\square$

The remaining of this section is devoted to a number of technical lemmas which will be used throughout the chapter. Given a locally defined continuous positive definite function  $F$ , the issues addressed below are: approximation (Lemma 11.17), a reproducing kernel Hilbert space (RKHS)  $\mathcal{H}_F$  built from  $F$ , an integral transform, and a certain derivative operator  $D^{(F)}$ , generally unbounded in the RKHS  $\mathcal{H}_F$ . We will be concerned with boundary value problems for  $D^{(F)}$ , and in order to produce suitable orthonormal bases in  $\mathcal{H}_F$ , we be concerned with an explicit family of skew-adjoint extensions of  $D^{(F)}$ , as well as the associated spectra, see Corollaries 11.27 and 11.28.

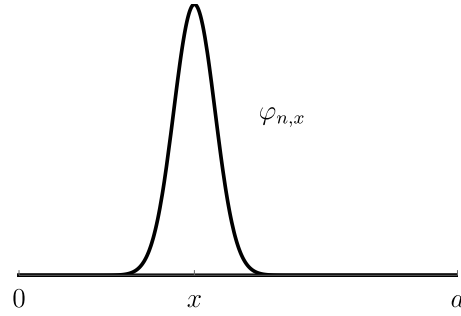
**Lemma 11.17.** *Let  $\varphi$  be a function such that*

1.  $\text{supp}(\varphi) \subset (0, a)$ ;
2.  $\varphi \in C_c^\infty(0, a)$ ,  $\varphi \geq 0$ ;
3.  $\int_0^a \varphi(t) dt = 1$ .

*Fix  $x \in (0, a)$ , and set  $\varphi_{n,x}(t) := n\varphi(n(t-x))$ . Then  $\lim_{n \rightarrow \infty} \varphi_{n,x} = \delta_x$ , i.e., the Dirac measure at  $x$ ; and*

$$\|F_{\varphi_{n,x}} - F(\cdot - x)\|_{\mathcal{H}_F} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (11.28)$$

*Hence  $\{F_\varphi : \varphi \in C_c^\infty(0, a)\}$  spans a dense subspace in  $\mathcal{H}_F$ . See Figure 11.1.*

Figure 11.1: The approximate identity  $\varphi_{n,x}$ 

Recall, the following facts about  $\mathcal{H}_F$ , which follow from the general theory [Aro50] of RKHS:

- $F(0) > 0$ , so we can always arrange  $F(0) = 1$ .
- $F(-x) = \overline{F(x)}$
- $\mathcal{H}_F$  consists of continuous functions  $\xi : \Omega - \Omega \rightarrow \mathbb{C}$ .
- The reproducing property:

$$\langle F(\cdot - x), \xi \rangle_{\mathcal{H}_F} = \xi(x), \quad \forall \xi \in \mathcal{H}_F, \forall x \in \Omega,$$

is a direct consequence of (11.23).

*Remark 11.18.* It follows from the reproducing property that if  $F_{\phi_n} \rightarrow \xi$  in  $\mathcal{H}_F$ , then  $F_{\phi_n}$  converges uniformly to  $\xi$  in  $\Omega$ . In fact

$$\begin{aligned} |F_{\phi_n}(x) - \xi(x)| &= \left| \langle F(\cdot - x), F_{\phi_n} - \xi \rangle_{\mathcal{H}_F} \right| \\ &\leq \|F(\cdot - x)\|_{\mathcal{H}_F} \|F_{\phi_n} - \xi\|_{\mathcal{H}_F} \\ &= F(0)^{1/2} \|F_{\phi_n} - \xi\|_{\mathcal{H}_F}. \end{aligned}$$

**Lemma 11.19.** *Let  $F : (-a, a) \rightarrow \mathbb{C}$  be a continuous and p.d. function, and let  $\mathcal{H}_F$  be the corresponding RKHS. Then:*

1. the integral  $F_\varphi := \int_0^a \varphi(y) F(\cdot - y) dy$  is convergent in  $\mathcal{H}_F$  for all  $\varphi \in C_c(0, a)$ ; and
2. for all  $\xi \in \mathcal{H}_F$ , we have:

$$\langle F_\varphi, \xi \rangle_{\mathcal{H}_F} = \int_0^a \overline{\varphi(x)} \xi(x) dx. \quad (11.29)$$

*Proof.* For simplicity, we assume the following normalization  $F(0) = 1$ ; then for all  $y_1, y_2 \in (0, 1)$ , we have

$$\|F(\cdot - y_1) - F(\cdot - y_2)\|_{\mathcal{H}_F}^2 = 2(1 - \Re\{F(y_1 - y_2)\}). \quad (11.30)$$

Now, view the integral in (1) as a  $\mathcal{H}_F$ -vector valued integral. If  $\varphi \in C_c(0, a)$ , this integral  $\int_0^a \varphi(y) F(\cdot - y) dy$  is the  $\mathcal{H}_F$ -norm convergent. Since  $\mathcal{H}_F$  is a RKHS,  $\langle \cdot, \xi \rangle_{\mathcal{H}_F}$  is continuous on  $\mathcal{H}_F$ , and it passes under the integral in (1). Using

$$\langle F(y - \cdot), \xi \rangle_{\mathcal{H}_F} = \xi(y) \quad (11.31)$$

the desired conclusion (11.29) follows.  $\square$

**Corollary 11.20.** *Let  $F : (-a, a) \rightarrow \mathbb{C}$  be as above, and let  $\mathcal{H}_F$  be the corresponding RKHS. For  $\varphi \in C_c^1(0, a)$ , set*

$$F_\varphi(x) = (T_F \varphi)(x) = \int_0^a \varphi(y) F(x - y) dy; \quad (11.32)$$

then  $F_\varphi \in C^1(0, a)$ , and

$$\frac{d}{dx} F_\varphi(x) = (T_F(\varphi'))(x), \quad \forall x \in (0, a). \quad (11.33)$$

*Proof.* Since  $F_\varphi(x) = \int_0^a \varphi(y) F(x - y) dy$ ,  $x \in (0, a)$ ; the desired assertion (11.33) follows directly from the arguments in the proof of Lemma 11.19.  $\square$

**Theorem 11.21.** *Fix  $0 < a < \infty$ . A continuous function  $\xi : (0, a) \rightarrow \mathbb{C}$  is in  $\mathcal{H}_F$  if and only if there exists a finite constant  $A > 0$ , such that*

$$\sum_i \sum_j \bar{c}_i c_j \xi(x_i) \overline{\xi(x_j)} \leq A \sum_i \sum_j \bar{c}_i c_j F(x_i - x_j) \quad (11.34)$$

for all finite system  $\{c_i\} \subset \mathbb{C}$  and  $\{x_i\} \subset (0, a)$ . Equivalently, for all  $\varphi \in C_c^\infty(\Omega)$ ,

$$\left| \int_0^a \varphi(y) \xi(y) dy \right|^2 \leq A \int_0^a \int_0^a \overline{\varphi(x)} \varphi(y) F(x - y) dx dy \quad (11.35)$$

We will use these two conditions (11.34) ( $\Leftrightarrow$  (11.35)) when considering for example the von Neumann deficiency-subspaces for skew Hermitian operators with dense domain in  $\mathcal{H}_F$ .

*Proof of Theorem 11.21.* Note, if  $\xi \in \mathcal{H}_F$ , then

$$\text{LHS}_{(11.35)} = \left| \langle F_\varphi, \xi \rangle_{\mathcal{H}_F} \right|^2,$$

and so (11.35) holds, since  $\langle \cdot, \xi \rangle_{\mathcal{H}_F}$  is continuous on  $\mathcal{H}_F$ .

If  $\xi$  is continuous on  $[0, a]$ , and if (11.35) holds, then

$$\mathcal{H}_F \ni F_\varphi \mapsto \int_0^a \varphi(y) \xi(y) dy$$

is well-defined, continuous, linear; and extends to  $\mathcal{H}_F$  by density (see Lemma 11.17). Hence, by Riesz' theorem,  $\exists! k_\xi \in \mathcal{H}_F$  such that

$$\int_0^a \varphi(y) \xi(y) dy = \langle F_\varphi, k_\xi \rangle_{\mathcal{H}_F}.$$

But using the reproducing property in  $\mathcal{H}_F$ , and  $F_\varphi(x) = \int_0^a \varphi(x) F(x-y) dy$ , we get

$$\int_0^a \overline{\varphi(x)} \xi(x) dx = \int_0^a \overline{\varphi(x)} k_\xi(x) dx, \quad \forall \varphi \in C_c(0, a)$$

so

$$\int_0^a \varphi(x) (\xi(x) - k_\xi(x)) dx = 0, \quad \forall \varphi \in C_c(0, a);$$

it follows that  $\xi - k_\xi = 0$  on  $(0, a) \implies \xi - k_\xi = 0$  on  $[0, a]$ .  $\square$

**Definition 11.22** (The operator  $D_F$ ). Let  $D_F(F_\varphi) = F_{\varphi'}$ , for all  $\varphi \in C_c^\infty(0, a)$ , where  $\varphi' = \frac{d\varphi}{dt}$  and  $F_\varphi$  is as in (11.24).

**Lemma 11.23.** *The operator  $D_F$  defines a skew-Hermitian operator with dense domain in  $\mathcal{H}_F$ .*

*Proof.* By Lemma 11.17,  $\text{dom}(D_F)$  is dense in  $\mathcal{H}_F$ . If  $\psi \in C_c^\infty(0, a)$  and

$$|t| < \text{dist}(\text{supp}(\psi), \text{endpoints}),$$

then

$$\|F_{\psi(\cdot+t)}\|_{\mathcal{H}_F}^2 = \|F_\psi\|_{\mathcal{H}_F}^2 = \int_0^a \int_0^a \overline{\psi(x)} \psi(y) F(x-y) dx dy \quad (11.36)$$

see (11.26), so

$$\frac{d}{dt} \|F_{\psi(\cdot+t)}\|_{\mathcal{H}_F}^2 = 0$$

which is equivalent to

$$\langle D_F F_\psi, F_\psi \rangle_{\mathcal{H}_F} + \langle F_\psi, D_F F_\psi \rangle_{\mathcal{H}_F} = 0. \quad (11.37)$$

It follows that  $D_F$  is well-defined and skew-Hermitian in  $\mathcal{H}_F$ .  $\square$

**Lemma 11.24.** *Let  $F$  be a positive definite function on  $(-a, a)$ ,  $0 < a < \infty$  fixed. Let  $D_F$  be as in Definition 11.22, so that  $D_F \subset D_F^*$  (Lemma 11.23), where  $D_F^*$  is the adjoint relative to the  $\mathcal{H}_F$  inner product.*

*Then  $\xi \in \mathcal{H}_F$  (as a continuous function on  $[0, a]$ ) is in  $\text{dom}(D_F^*)$  iff*

$$\xi' \in \mathcal{H}_F \text{ where } \xi' = \text{distribution derivative, and} \quad (11.38)$$

$$D_F^* \xi = -\xi' \quad (11.39)$$

*Proof.* By Theorem 11.21, a fixed  $\xi \in \mathcal{H}_F$ , i.e.,  $x \mapsto \xi(x)$  is a continuous function on  $[0, a]$  such that  $\exists C, \left| \int_0^a \varphi(x) \xi(x) dx \right|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2$ .

$\xi$  is in  $\text{dom}(D_F^*) \iff \exists C = C_\xi < \infty$  such that

$$\left| \langle D_F(F_\varphi), \xi \rangle_{\mathcal{H}_F} \right|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2 = C \int_0^a \int_0^a \overline{\varphi(x)} \varphi(y) F(x-y) dx dy \quad (11.40)$$

But LHS of (11.40) under  $|\langle \cdot, \cdot \rangle|^2$  is:

$$\left| \langle D_F(F_\varphi), \xi \rangle_{\mathcal{H}_F} \right|^2 = \langle F_{\varphi'}, \xi \rangle_{\mathcal{H}_F} \stackrel{(11.29)}{=} \int_0^a \overline{\varphi'(x)} \xi(x) dx, \quad \forall \varphi \in C_c^\infty(0, a) \quad (11.41)$$

So (11.40) holds  $\iff$

$$\left| \int_0^a \overline{\varphi'(x)} \xi(x) dx \right|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2, \quad \forall \varphi \in C_c^\infty(0, a)$$

i.e.,

$$\left| \int_0^a \overline{\varphi(x)} \xi'(x) dx \right|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2, \quad \forall \varphi \in C_c^\infty(0, a), \text{ and}$$

$\xi'$  as a distribution is in  $\mathcal{H}_F$ , and

$$\int_0^a \overline{\varphi(x)} \xi'(x) dx = \langle F_\varphi, \xi' \rangle_{\mathcal{H}_F}$$

where we use the characterization of  $\mathcal{H}_F$  in (11.35), i.e., a function  $\eta : [0, a] \rightarrow \mathbb{C}$  is in  $\mathcal{H}_F \iff \exists C < \infty, \left| \int_0^a \overline{\varphi(x)} \eta(x) dx \right| \leq C \|F_\varphi\|_{\mathcal{H}_F}, \forall \varphi \in C_c^\infty(0, a)$ , and then  $\int_0^a \overline{\varphi(x)} \eta(x) dx = \langle F_\varphi, \eta \rangle_{\mathcal{H}_F}, \forall \varphi \in C_c^\infty(0, a)$ . See Theorem 11.21.  $\square$

**Corollary 11.25.**  $h \in \mathcal{H}_F$  is in  $\text{dom}((D_F^2)^*)$  iff  $h'' \in \mathcal{H}_F$  ( $h''$  distribution derivative) and  $(D_F^*)^2 h = (D_F^2)^* h = h''$ .

*Proof.* Application of (11.41) to  $D_F(F_\varphi) = F_{\varphi'}$ , we have  $D_F^2(F_\varphi) = F_{\varphi''} = \left(\frac{d}{dx}\right)^2 F_\varphi, \forall \varphi \in C_c^\infty(0, a)$ , and

$$\begin{aligned} \langle D_F^2(F_\varphi), h \rangle_{\mathcal{H}_F} &= \langle F_{\varphi''}, h \rangle_{\mathcal{H}_F} = \int_0^a \overline{\varphi''(x)} h(x) dx \\ &= \int_0^a \overline{\varphi(x)} h''(x) dx = \langle F_\varphi, (D_F^2)^* h \rangle_{\mathcal{H}_F}. \end{aligned}$$

$\square$

**Definition 11.26.** [DS88c] Let  $D_F^*$  be the adjoint of  $D_F$  relative to  $\mathcal{H}_F$  inner product. The deficiency spaces  $DEF^\pm$  consists of  $\xi_\pm \in \text{dom}(D_F^*)$ , such that  $D_F^* \xi_\pm = \pm \xi_\pm$ , i.e.,

$$DEF^\pm = \left\{ \xi_\pm \in \mathcal{H}_F : \langle F_\psi, \xi_\pm \rangle_{\mathcal{H}_F} = \langle F_\psi, \pm \xi_\pm \rangle_{\mathcal{H}_F}, \forall \psi \in C_c^\infty(\Omega) \right\}.$$

**Corollary 11.27.** *If  $\xi \in DEF^\pm$  then  $\xi(x) = \text{constant } e^{\mp x}$ .*

*Proof.* Immediate from Lemma 11.24.  $\square$

The role of deficiency indices for the canonical skew-Hermitian operator  $D_F$  (Definition 11.22) in the RKHS  $\mathcal{H}_F$  is as follows: using von Neumann's conjugation trick [DS88c], we see that the deficiency indices can be only  $(0, 0)$  or  $(1, 1)$ .

We conclude that there exists proper skew-adjoint extensions  $A \supset D_F$  in  $\mathcal{H}_F$  (in case  $D_F$  has indices  $(1, 1)$ ). Then

$$D_F \subseteq A = -A^* \subseteq -D_F^* \quad (11.42)$$

(If the indices are  $(0, 0)$  then  $\overline{D_F} = -D_F^*$ ; see [DS88c].)

Hence, set  $U(t) = e^{tA} : \mathcal{H}_F \rightarrow \mathcal{H}_F$ , and get the strongly continuous unitary one-parameter group

$$\{U(t) : t \in \mathbb{R}\}, U(s+t) = U(s)U(t), \forall s, t \in \mathbb{R};$$

and if

$$\xi \in \text{dom}(A) = \left\{ \xi \in \mathcal{H}_F : \text{s.t. } \lim_{t \rightarrow 0} \frac{U(t)\xi - \xi}{t} \text{ exists} \right\}$$

then

$$A\xi = \text{s.t. } \lim_{t \rightarrow 0} \frac{U(t)\xi - \xi}{t}. \quad (11.43)$$

Now use  $F_x(\cdot) = F(x - \cdot)$  defined in  $(0, a)$ ; and set

$$F_A(t) := \langle F_0, U(t)F_0 \rangle_{\mathcal{H}_F}, \forall t \in \mathbb{R} \quad (11.44)$$

then using (11.28), we see that  $F_A$  is a continuous positive definite extension of  $F$  on  $(-a, a)$ . This extension is in  $Ext_1(F)$ .

**Corollary 11.28.** *Assume  $\lambda \in \mathbb{R}$  is in the point spectrum of  $A$ , i.e.,  $\exists \xi_\lambda \in \text{dom}(A)$ ,  $\xi_\lambda \neq 0$ , such that  $A\xi_\lambda = i\lambda\xi_\lambda$  holds in  $\mathcal{H}_F$ , then  $\xi_\lambda = \text{const} \cdot e_\lambda$ , i.e.,*

$$\xi_\lambda(x) = \text{const} \cdot e^{i\lambda x}, \forall x \in [0, a]. \quad (11.45)$$

*Proof.* Assume  $\lambda$  is in  $\text{spec}_{pt}(A)$ , and  $\xi_\lambda \in \text{dom}(A)$  satisfying

$$(A\xi_\lambda)(x) = i\lambda\xi_\lambda(x) \text{ in } \mathcal{H}_F, \quad (11.46)$$

then since  $A \subset -D_F^*$ , we get  $\xi \in \text{dom}(D_F^*)$  by Lemma 11.24 and (11.42), and  $D_F^*\xi_\lambda = -\xi'_\lambda$  where  $\xi'_\lambda$  is the distribution derivative (see (11.39)); and by (11.42)

$$(A\xi_\lambda)(x) = -(D_F^*\xi_\lambda)(x) = \xi'_\lambda(x) \stackrel{(11.46)}{=} i\lambda\xi_\lambda(x), \forall x \in (0, a) \quad (11.47)$$

so  $\xi_\lambda$  is the distribution derivative solution to

$$\xi'_\lambda(x) = i\lambda\xi_\lambda(x) \quad (11.48)$$

$$\begin{aligned}
 & \Downarrow \\
 - \int_0^a \overline{\varphi'(x)} \xi_\lambda(x) dx &= i\lambda \int_0^a \overline{\varphi(x)} \xi_\lambda(x) dx, \quad \forall \varphi \in C_c^\infty(0, a) \\
 & \Downarrow \\
 - \langle D_F(F_\varphi), \xi_\lambda \rangle_{\mathcal{H}_F} &= i\lambda \langle F_\varphi, \xi_\lambda \rangle_{\mathcal{H}_F}, \quad \forall \varphi \in C_c^\infty(0, a).
 \end{aligned}$$

But by Schwartz, the distribution solutions to (11.48) are  $\xi_\lambda(x) = \text{const} \cdot e^{i\lambda x} = \text{const} \cdot e^{i\lambda x}$ .  $\square$

In the considerations below, we shall be primarily concerned with the case when a fixed continuous p.d. function  $F$  is defined on a finite interval  $(-a, a) \subset \mathbb{R}$ . In this case, by a Mercer operator, we mean an operator  $T_F$  in  $L^2(0, a)$  where  $L^2(0, a)$  is defined from Lebesgue measure on  $(0, a)$ , given by

$$(T_F \varphi)(x) := \int_0^a \varphi(y) F(x-y) dy, \quad \forall \varphi \in L^2(0, a), \forall x \in (0, a). \quad (11.49)$$

**Lemma 11.29.** *Under the assumptions stated above, the Mercer operator  $T_F$  is trace class in  $L^2(0, a)$ ; and if  $F(0) = 1$ , then*

$$\text{trace}(T_F) = a. \quad (11.50)$$

*Proof.* This is an application of Mercer's theorem [LP89, FR42, FM13] to the integral operator  $T_F$  in (11.49). But we must check that  $F$ , on  $(-a, a)$ , extends uniquely by limit to a continuous p.d. function  $F_{ex}$  on  $[-a, a]$ , the closed interval. This is true, and easy to verify, see e.g. [JPT14a].  $\square$

**Corollary 11.30.** *Let  $F$  and  $(-a, a)$  be as in Lemma 11.29. Then there is a sequence  $(\lambda_n)_{n \in \mathbb{N}}$ ,  $\lambda_n > 0$ , such that  $\sum_{n \in \mathbb{N}} \lambda_n = a$ , and a system of orthogonal functions  $\{\xi_n\} \subset L^2(0, a) \cap \mathcal{H}_F$  such that*

$$F(x-y) = \sum_{n \in \mathbb{N}} \lambda_n \xi_n(x) \overline{\xi_n(y)}, \quad \text{and} \quad (11.51)$$

$$\int_0^a \overline{\xi_n(x)} \xi_m(x) dx = \delta_{n,m}, \quad n, m \in \mathbb{N}. \quad (11.52)$$

*Proof.* An application of Mercer's theorem [LP89, FR42, FM13]. See also Exercise 11.5.  $\square$

**Corollary 11.31.** *For all  $\psi, \varphi \in C_c^\infty(0, a)$ , we have*

$$\langle F_\psi, F_\varphi \rangle_{\mathcal{H}_F} = \langle F_\psi, T_F^{-1} F_\varphi \rangle_2. \quad (11.53)$$

Consequently,

$$\|h\|_{\mathcal{H}_F} = \|T_F^{-1/2} h\|_2, \quad \forall h \in \mathcal{H}_F. \quad (11.54)$$



*Proof.* Note

$$\begin{aligned} \langle F_\psi, T_F^{-1} F_\varphi \rangle_2 &= \langle F_\psi, T_F^{-1} T_F \varphi \rangle_2 = \langle F_\psi, \varphi \rangle_2 \\ &= \int_0^a \overline{\left( \int_0^a \psi(x) F(y-x) dx \right)} \varphi(y) dy \\ &= \int_0^a \int_0^a \overline{\psi(x)} \varphi(y) F(x-y) dx dy = \langle F_\psi, F_\varphi \rangle_{\mathcal{H}_F}. \end{aligned}$$

□

**Corollary 11.32.** *Let  $\{\xi_n\}$  be the ONB in  $L^2(0, a)$  as in Corollary 11.30; then  $\{\sqrt{\lambda_n} \xi_n\}$  is an ONB in  $\mathcal{H}_F$ .*

*Proof.* The functions  $\xi_n$  are in  $\mathcal{H}_F$  by Theorem 11.21. We check directly (Corollary 11.31) that

$$\begin{aligned} \langle \sqrt{\lambda_n} \xi_n, \sqrt{\lambda_m} \xi_m \rangle_{\mathcal{H}_F} &= \sqrt{\lambda_n \lambda_m} \langle \xi_n, T^{-1} \xi_m \rangle_2 \\ &= \sqrt{\lambda_n \lambda_m \lambda_m^{-1}} \langle \xi_n, \xi_m \rangle_2 = \delta_{n,m}. \end{aligned}$$

□

## 11.4 Type I v.s. Type II Extensions

When a pair  $(\Omega, F)$  is given, where  $F$  is a prescribed continuous positive definite function defined on  $\Omega$ , we consider the possible continuous positive definite extensions to all of  $\mathbb{R}^n$ . The reproducing kernel Hilbert space  $\mathcal{H}_F$  will play a key role in our analysis. In constructing various classes of continuous positive definite extensions to  $\mathbb{R}^n$ , we introduce operators in  $\mathcal{H}_F$ , and their *dilation* to operators, possibly acting in an enlargement Hilbert space [JPT14a, KL14b]. Following techniques from dilation theory we note that every dilation contains a minimal one. If a continuous positive definite extensions to  $\mathbb{R}^n$  has its minimal dilation Hilbert space equal to  $\mathcal{H}_F$ , we say it is type 1, otherwise we say it is type 2.

**Definition 11.33.** Let  $G$  be a locally compact group, and let  $\Omega$  be an open connected subset of  $G$ . Let  $F : \Omega^{-1} \cdot \Omega \rightarrow \mathbb{C}$  be a continuous positive definite function.

**Definition 11.34.** Consider a strongly continuous unitary representation  $U$  of  $G$  acting in some Hilbert space  $\mathcal{H}$ , containing the RKHS  $\mathcal{H}_F$ . We say that  $(U, \mathcal{H}) \in \text{Ext}(F)$  iff there is a vector  $k_0 \in \mathcal{H}$  such that

$$F(g) = \langle k_0, U(g) k_0 \rangle_{\mathcal{H}}, \quad \forall g \in \Omega^{-1} \cdot \Omega. \quad (11.55)$$

1. The subset of  $Ext(F)$  consisting of  $(U, \mathcal{H}_F, k_0 = F_e)$  with

$$F(g) = \langle F_e, U(g) F_e \rangle_{\mathcal{H}_F}, \quad \forall g \in \Omega^{-1} \cdot \Omega \quad (11.56)$$

is denoted  $Ext_1(F)$ ; and we set

$$Ext_2(F) := Ext(F) \setminus Ext_1(F);$$

i.e.,  $Ext_2(F)$ , consists of the solutions to problem (11.55) for which  $\mathcal{H} \not\supseteq \mathcal{H}_F$ , i.e., unitary representations realized in an enlargement Hilbert space.

(We write  $F_e \in \mathcal{H}_F$  for the vector satisfying  $\langle F_e, \xi \rangle_{\mathcal{H}_F} = \xi(e)$ ,  $\forall \xi \in \mathcal{H}_F$ , where  $e$  is the neutral (unit) element in  $G$ , i.e.,  $eg = g$ ,  $\forall g \in G$ .)

2. In the special case, where  $G = \mathbb{R}^n$ , and  $\Omega \subset \mathbb{R}^n$  is open and connected, we consider

$$F : \Omega - \Omega \rightarrow \mathbb{C}$$

continuous and positive definite. In this case,

$$Ext(F) = \left\{ \mu \in \mathcal{M}_+(\mathbb{R}^n) \mid \widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} d\mu(\lambda) \right. \quad (11.57)$$

$$\left. \text{is a p.d. extension of } F \right\}.$$

*Remark 11.35.* Note that (11.57) is consistent with (11.55): For if  $(U, \mathcal{H}, k_0)$  is a unitary representation of  $G = \mathbb{R}^n$ , such that (11.55) holds; then, by a theorem of Stone, there is a projection-valued measure (PVM)  $P_U(\cdot)$ , defined on the Borel subsets of  $\mathbb{R}^n$  such that

$$U(x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} P_U(d\lambda), \quad x \in \mathbb{R}^n. \quad (11.58)$$

Setting

$$d\mu(\lambda) := \|P_U(d\lambda) k_0\|_{\mathcal{H}}^2, \quad (11.59)$$

it is then immediate that we have:  $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ , and that the finite measure  $\mu$  satisfies

$$\widehat{\mu}(x) = F(x), \quad \forall x \in \Omega - \Omega. \quad (11.60)$$

Set  $n = 1$ : Start with a local p.d. continuous function  $F$ , and let  $\mathcal{H}_F$  be the corresponding RKHS. Let  $Ext(F)$  be the compact convex set of probability measures on  $\mathbb{R}$  defining extensions of  $F$ .

We now divide  $Ext(F)$  into two parts, say  $Ext_1(F)$  and  $Ext_2(F)$ .

All continuous p.d. extensions of  $F$  come from strongly continuous unitary representations. So in the case of 1D, from unitary one-parameter groups of course, say  $U(t)$ .

Let  $Ext_1(F)$  be the subset of  $Ext(F)$  corresponding to extensions when the unitary representation  $U(t)$  acts in  $\mathcal{H}_F$  (internal extensions), and  $Ext_2(F)$  denote the part of  $Ext(F)$  associated to unitary representations  $U(t)$  acting in a proper enlargement Hilbert space  $\mathcal{H}$  (if any), i.e., acting in a Hilbert space  $\mathcal{H}$  corresponding to a proper dilation of  $\mathcal{H}_F$ .

## 11.5 The Case of $e^{-|x|}$ , $|x| < 1$

Our emphasis is von Neumann indices, and explicit formulas for partially defined positive definite functions  $F$ , defined initially only on a symmetric interval  $(-a, a)$ . Among the cases of partially defined positive definite functions, the following example  $F(x) = e^{-|x|}$ , in the symmetric interval  $(-1, 1)$ , will play a special role. The present section is devoted to this example.

There are many reasons for this:

- (i) It is of independent interest, and its type 1 extensions (see Section 11.4) can be written down explicitly.
- (ii) Its applications include stochastic analysis [It606] as follows. Given a random variable  $X$  in a process; if  $\mu$  is its distribution, then there are two measures of concentration for  $\mu$ , one called “degree of concentration,” and the other “dispersion,” both computed directly from  $F(x) = e^{-|x|}$  applied to  $\mu$ .
- (iii) In addition, there are analogous relative notions for comparing different samples in a fixed stochastic process. These notions are defined with the use of example  $F(x) = e^{-|x|}$ , and it will frequently be useful to localize the  $x$ -variable in a compact interval.
- (iv) Additional reasons for special attention to example  $F(x) = e^{-|x|}$ , for  $x \in (-1, 1)$  is its use in sampling theory, and analysis of de Branges spaces [DM85], as well as its role as a Greens function for an important boundary value problem.
- (v) Related to this, the reproducing kernel Hilbert space  $\mathcal{H}_F$  associated to this p.d. function  $F$  has a number of properties that also hold for wider families of locally defined positive definite function of a single variable. In particular,  $\mathcal{H}_F$  has Fourier bases: The RKHS  $\mathcal{H}_F$  has orthogonal bases of complex exponentials  $e_\lambda$  with aperiodic frequency distributions, i.e., frequency points  $\{e_\lambda\}$  on the real line which do not lie on any arithmetic progression, see Figure 11.3. For details on this last point, see Corollaries 11.48, 11.49, 11.51, and 11.55.

### The selfadjoint Extensions $A_\theta \supset -iD_F$

The notation “ $\supset$ ” above refers to containment of operators, or rather of the respective graphs of the two operators; see [DS88c].

**Lemma 11.36.** *Let  $F(x) = e^{-|x|}$ ,  $|x| < 1$ . Set  $F_x(y) := F(x - y)$ ,  $\forall x, y \in (0, 1)$ ; and  $F_\varphi(x) = \int_0^1 \varphi(y) F(x - y) dy$ ,  $\forall \varphi \in C_c^\infty(0, 1)$ . Define  $D_F(F_\varphi) = F_\varphi$  on the dense subset*

$$\text{dom}(D_F) = \{F_\varphi : \varphi \in C_c^\infty(0, 1)\} \subset \mathcal{H}_F. \quad (11.61)$$

Then the skew-Hermitian operator  $D_F$  has deficiency indices  $(1, 1)$  in  $\mathcal{H}_F$ , where the defect vectors are

$$\xi_+(x) = F_0(x) = e^{-x} \quad (11.62)$$

$$\xi_-(x) = F_1(x) = e^{x-1}; \quad (11.63)$$

moreover,

$$\|\xi_+\|_{\mathcal{H}_F} = \|\xi_-\|_{\mathcal{H}_F} = 1. \quad (11.64)$$

*Proof.* (Note if  $\Omega$  is any bounded, open and connected domain in  $\mathbb{R}^n$ , then a locally defined continuous p.d. function,  $F : \Omega - \Omega \rightarrow \mathbb{C}$ , extends uniquely to the boundary  $\partial\Omega := \overline{\Omega} \setminus \Omega$  by continuity [JPT14a].)

In our current settings,  $\Omega = (0, 1)$ , and  $F_x(y) := F(x - y)$ ,  $\forall x, y \in (0, 1)$ . Thus,  $F_x(y)$  extends to all  $x, y \in [0, 1]$ . In particular,

$$F_0(x) = e^{-x}, \quad F_1(x) = e^{x-1}$$

are the two defect vectors, as shown in Corollary 11.27. Moreover, using the reproducing property, we have

$$\begin{aligned} \|F_0\|_{\mathcal{H}_F}^2 &= \langle F_0, F_0 \rangle_{\mathcal{H}_F} = F_0(0) = F(0) = 1 \\ \|F_1\|_{\mathcal{H}_F}^2 &= \langle F_1, F_1 \rangle_{\mathcal{H}_F} = F_1(1) = F(0) = 1 \end{aligned}$$

and (11.64) follows. For more details, see [JPT14a, lemma 2.10.14].  $\square$

**Lemma 11.37.** *Let  $F$  be any continuous p.d. function on  $(-1, 1)$ . Set*

$$h(x) = \int_0^1 \varphi(y) F(x - y) dy, \quad \forall \varphi \in C_c^\infty(0, 1);$$

then

$$h(0) = \int_0^1 \varphi(y) F(-y) dy, \quad h(1) = \int_0^1 \varphi(y) F(1 - y) dy \quad (11.65)$$

$$h'(0) = \int_0^1 \varphi(y) F'(-y) dy, \quad h'(1) = \int_0^1 \varphi(y) F'(1 - y) dy; \quad (11.66)$$

where the derivatives  $F'$  in (11.65)-(11.66) are in the sense of distribution.

*Proof.* Note that

$$\begin{aligned} h(x) &= \int_0^x \varphi(y) F(x - y) dy + \int_x^1 \varphi(y) F(x - y) dy; \\ h'(x) &= \int_0^x \varphi(y) F'(x - y) dy + \int_x^1 \varphi(y) F'(x - y) dy. \end{aligned}$$

and so (11.65)-(11.66) follow.  $\square$

We now specialize to the function  $F(x) = e^{-|x|}$  defined in  $(-1, 1)$ .

**Corollary 11.38.** For  $F(x) = e^{-|x|}$ ,  $|x| < 1$ , set  $h = T_F \varphi$ , i.e.,

$$h := F_\varphi = \int_0^1 \varphi(y) F(\cdot - y) dy, \quad \forall \varphi \in C_c^\infty(0, 1);$$

then

$$h(0) = \int_0^1 \varphi(y) e^{-y} dy, \quad h(1) = \int_0^1 \varphi(y) e^{y-1} dy \quad (11.67)$$

$$h'(0) = \int_0^1 \varphi(y) e^{-y} dy, \quad h'(1) = -\int_0^1 \varphi(y) e^{y-1} dy \quad (11.68)$$

In particular,

$$h(0) - h'(0) = 0 \quad (11.69)$$

$$h(1) + h'(1) = 0. \quad (11.70)$$

*Proof.* Immediately from Lemma 11.37. Specifically,

$$\begin{aligned} h(x) &= e^{-x} \int_0^x \varphi(y) e^y dy + e^x \int_x^1 \varphi(y) e^{-y} dy \\ h'(x) &= -e^{-x} \int_0^x \varphi(y) e^y dy + e^x \int_x^1 \varphi(y) e^{-y} dy. \end{aligned}$$

Setting  $x = 0$  and  $x = 1$  gives the desired conclusions.  $\square$

*Remark 11.39.* The space

$$\left\{ h \in \mathcal{H}_F \mid h(0) - h'(0) = 0, h(1) + h'(1) = 0 \right\}$$

is dense in  $\mathcal{H}_F$ . This is because it contains  $\{F_\varphi \mid \varphi \in C_c^\infty(0, 1)\}$ . Note

$$\begin{aligned} F_0 + F'_0 &= -\delta_0, \text{ and} \\ F_1 - F'_1 &= -\delta_1; \end{aligned}$$

however,  $\delta_0, \delta_1 \notin \mathcal{H}_F$ .

By von Neumann's theory [DS88c] and Lemma 11.24, the family of selfadjoint extensions of the Hermitian operator  $-iD_F$  is characterized by

$$\begin{aligned} A_\theta(h + c(e^{-x} + e^{i\theta} e^{x-1})) &= -ih' + ic(e^{-x} - e^{i\theta} e^{x-1}), \text{ where} \\ \text{dom}(A_\theta) &:= \{h + c(e^{-x} + e^{i\theta} e^{x-1}) \mid h \in \text{dom}(D_F), c \in \mathbb{C}\}. \end{aligned} \quad (11.71)$$

*Remark 11.40.* In (11.71),  $h \in \text{dom}(D_F)$  (see (11.61)), and by Corollary 11.38,  $h$  satisfies the boundary conditions (11.69)-(11.70). Also, by Lemma 11.36,  $\xi_+ = F_0 = e^{-x}$ ,  $\xi_- = F_1 = e^{x-1}$ , and  $\|\xi_+\|_{\mathcal{H}_F} = \|\xi_-\|_{\mathcal{H}_F} = 1$ .

**Proposition 11.41.** *Let  $A_\theta$  be a selfadjoint extension of  $-iD$  as in (11.71). Then,*

$$\psi(1) + \psi'(1) = e^{i\theta} (\psi(0) - \psi'(0)), \quad \forall \psi \in \text{dom}(A_\theta). \quad (11.72)$$

*Proof.* Any  $\psi \in \text{dom}(A_\theta)$  has the decomposition

$$\psi(x) = h(x) + c(e^{-x} + e^{i\theta}e^{x-1})$$

where  $h \in \text{dom}(D_F)$ , and  $c \in \mathbb{C}$ . An application of Corollary 11.28 gives

$$\begin{aligned} \psi(1) + \psi'(1) &= \underbrace{h(1) + h'(1)}_{=0 \text{ (by (11.70))}} + c(e^{-1} + e^{i\theta}) + c(-e^{-1} + e^{i\theta}) = 2ce^{i\theta} \\ \psi(0) - \psi'(0) &= \underbrace{h(0) - h'(0)}_{=0 \text{ (by (11.69))}} + c(1 + e^{-1}e^{i\theta}) - c(-1 + e^{-1}e^{i\theta}) = 2c \end{aligned}$$

which is the assertion in (11.72).  $\square$

**Corollary 11.42.** *Let  $A_\theta$  be a selfadjoint extension of  $-iD_F$  as in (11.71). Fix  $\lambda \in \mathbb{R}$ , then  $\lambda \in \text{spec}_{pt}(A_\theta) \iff e_\lambda(x) := e^{i\lambda x} \in \text{dom}(A_\theta)$ , and  $\lambda$  is a solution to the following equation:*

$$\lambda = \theta + \tan^{-1} \left( \frac{2\lambda}{\lambda^2 - 1} \right) + 2n\pi, \quad n \in \mathbb{Z}. \quad (11.73)$$

*Proof.* By assumption,  $e^{i\lambda x} \in \text{dom}(A_\theta)$ , so  $\exists h_\lambda \in \text{dom}(D_F)$ , and  $\exists c_\lambda \in \mathbb{C}$  such that

$$e^{i\lambda x} = h_\lambda(x) + c_\lambda(e^x + e^{i\theta}e^{x-1}). \quad (11.74)$$

Applying the boundary condition in Proposition 11.41, we have

$$\begin{aligned} e^{i\lambda} + i\lambda e^{i\lambda} &= e^{i\theta}(1 - i\lambda); \text{ i.e.,} \\ e^{i\lambda} &= e^{i\theta} \frac{1 - i\lambda}{1 + i\lambda} = e^{i\theta} e^{i \arg\left(\frac{1 - i\lambda}{1 + i\lambda}\right)} \end{aligned} \quad (11.75)$$

where

$$\arg\left(\frac{1 - i\lambda}{1 + i\lambda}\right) = \tan^{-1} \left( \frac{2\lambda}{\lambda^2 - 1} \right)$$

and (11.73) follows. For a discrete set of solutions, see Figure 11.2.  $\square$

**Corollary 11.43.** *If  $A_\theta \supset -iD_F$  is a selfadjoint extension in  $\mathcal{H}_F$ , then*

$$\begin{aligned} \text{spect}(A_\theta) &= \{ \lambda \in \mathbb{R} \mid e_\lambda \in \mathcal{H}_F \text{ satisfying (11.72)} \} \\ &= \{ \lambda \in \mathbb{R} \mid e_\lambda \in \mathcal{H}_F, e_\lambda = h_\lambda + c_\lambda(e^x + e^{i\theta}e^{x-1}), \\ &\quad h_\lambda \in \text{dom}(D_F), c_\lambda \in \mathbb{C} \}. \end{aligned}$$

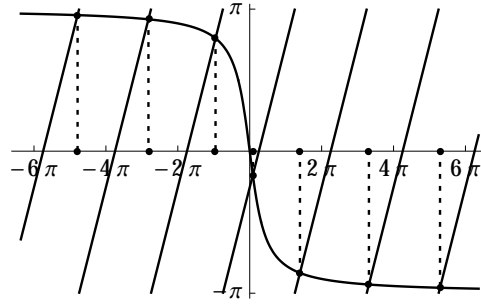


Figure 11.2: Fix  $\theta = 0.8$ ,  $\Lambda_\theta = \{\lambda_n(\theta)\}$  = intersections of two curves. (spectrum from curve intersections)

*Remark 11.44.* The corollary holds for all continuous p.d. functions  $F : (-a, a) \rightarrow \mathbb{C}$ .

**Corollary 11.45.** All selfadjoint extensions  $A_\theta \supset -iD_F$  have purely atomic spectrum; i.e.,

$$\Lambda_\theta := \text{spect}(A_\theta) = \text{discrete subset in } \mathbb{R}. \quad (11.76)$$

And for all  $\lambda \in \Lambda_\theta$ ,

$$\ker(A_\theta - \lambda I_{\mathcal{H}_F}) = \mathbb{C}e_\lambda, \text{ where } e_\lambda(x) = e^{i\lambda x} \quad (11.77)$$

i.e., all eigenvalues have multiplicity 1. (The set  $\Lambda_\theta$  will be denoted  $\{\lambda_n(\theta)\}_{n \in \mathbb{Z}}$  following Figure 11.2.)

*Proof.* This follows by solving eq. (11.73).  $\square$

**Corollary 11.46.** Let  $A$  be a selfadjoint extension of  $-iD_F$  as before. Suppose  $\lambda_1, \lambda_2 \in \text{spec}(A)$ ,  $\lambda_1 \neq \lambda_2$ , then  $e_{\lambda_i} \in \mathcal{H}_F$ ,  $i = 1, 2$ ; and  $\langle e_{\lambda_1}, e_{\lambda_2} \rangle_{\mathcal{H}_F} = 0$ .

*Proof.* Let  $\lambda_1, \lambda_2$  be as in the statement, then

$$(\lambda_1 - \lambda_2) \langle e_{\lambda_1}, e_{\lambda_2} \rangle_{\mathcal{H}_F} = \langle Ae_{\lambda_1}, e_{\lambda_2} \rangle_{\mathcal{H}_F} - \langle e_{\lambda_1}, Ae_{\lambda_2} \rangle_{\mathcal{H}_F} = 0;$$

so since  $\lambda_1 - \lambda_2 \neq 0$ , we get  $\langle e_{\lambda_1}, e_{\lambda_2} \rangle_{\mathcal{H}_F} = 0$ .  $\square$

For explicit computations regarding these points, see also Corollaries 11.52, 11.54, and 11.55 below.

### The Spectra of the s.a. Extensions $A_\theta \supset -iD_F$

Let  $F(x) = e^{-|x|}$ ,  $|x| < 1$ . Define  $D_F(F_\varphi) = F_{\varphi'}$  as before, where

$$F_\varphi(x) = \int_0^1 \varphi(y) F(x-y) dy$$

$$= \int_0^1 \varphi(y) e^{-|x-y|} dy, \quad \forall \varphi \in C_c^\infty(0,1).$$

And let  $\mathcal{H}_F$  be the RKHS of  $F$ .

**Lemma 11.47.** *For all  $\varphi \in C_c^\infty(0,1)$ , and all  $h, h'' \in \mathcal{H}_F$ , we have*

$$\langle F_\varphi, h \rangle_{\mathcal{H}_F} = \langle F_\varphi, \frac{1}{2}(h - h'') \rangle_2 - \frac{1}{2} [W]_0^1 \quad (11.78)$$

where

$$W = \det \begin{bmatrix} h & F_\varphi \\ h' & F_{\varphi'} \end{bmatrix}. \quad (11.79)$$

Setting  $l := F_\varphi$ , we have

$$[W]_0^1 = -\bar{l}(1)(h(1) + h'(1)) - \bar{l}(0)(h(0) - h'(0)). \quad (11.80)$$

*Proof.* Note

$$\begin{aligned} \langle F_\varphi, h \rangle_{\mathcal{H}_F} &= \int_0^1 \varphi(x) h(x) dx \quad (\text{reproducing property}) \\ &= \left\langle \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) F_\varphi, h \right\rangle_2 \\ &= \langle F_\varphi, \frac{1}{2}(h - h'') \rangle_2 - \frac{1}{2} [W]_0^1. \end{aligned}$$

Set  $l := F_\varphi \in \mathcal{H}_F$ ,  $\varphi \in C_c^\infty(0,1)$ . Recall the boundary condition in Corollary 11.38:

$$l(0) - l'(0) = l(1) + l'(1) = 0.$$

Then

$$\begin{aligned} [W]_0^1 &= (\bar{l}h - \bar{l}h')(1) - (\bar{l}h - \bar{l}h')(0) \\ &= -\bar{l}(1)h(1) - \bar{l}(1)h'(1) - \bar{l}(0)h(0) + \bar{l}(0)h'(0) \\ &= -\bar{l}(1)(h(1) + h'(1)) - \bar{l}(0)(h(0) - h'(0)) \end{aligned}$$

which is (11.80). □

**Corollary 11.48.**  $e_\lambda \in \mathcal{H}_F$ ,  $\forall \lambda \in \mathbb{R}$ .

*Proof.* By Theorem 11.21, we need the following estimate:  $\exists C < \infty$  such that

$$\left| \int_0^1 \varphi(x) e_\lambda(x) dx \right|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2. \quad (11.81)$$

But

$$\begin{aligned} &\int_0^1 \varphi(x) e_\lambda(x) dx \\ &= \left\langle \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) F_\varphi, e_\lambda \right\rangle_2 \end{aligned}$$



$$\begin{aligned}
&= \langle F_\varphi, \frac{1}{2}(e_\lambda - e'_\lambda) \rangle_2 - \frac{1}{2} [W]_0^1 \\
&= \frac{1}{2} (1 + \lambda^2) \langle F_\varphi, e_\lambda \rangle_2 - \frac{1}{2} (-l(1)(1 + i\lambda)e^{i\lambda} - l(0)(1 - i\lambda));
\end{aligned}$$

see (11.78)-(11.80). Here,  $l := F_\varphi$ .

It suffices to show

(i)  $\exists C_1 < \infty$  such that

$$|l(0)|^2 \text{ and } |l(1)|^2 \leq C_1 \|F_\varphi\|_{\mathcal{H}_F}^2.$$

(ii)  $\exists C_2 < \infty$  such that

$$|\langle F_\varphi, e_\lambda \rangle_2|^2 \leq C_2 \|F_\varphi\|_{\mathcal{H}_F}^2.$$

For (i), note that

$$\begin{aligned}
|l(0)| &= |\langle F_0, l \rangle_{\mathcal{H}_F}| \leq \|F_0\|_{\mathcal{H}_F} \|l\|_{\mathcal{H}_F} = \|F_0\|_{\mathcal{H}_F} \|F_\varphi\|_{\mathcal{H}_F} \\
|l(1)| &= |\langle F_1, l \rangle_{\mathcal{H}_F}| \leq \|F_1\|_{\mathcal{H}_F} \|l\|_{\mathcal{H}_F} = \|F_1\|_{\mathcal{H}_F} \|F_\varphi\|_{\mathcal{H}_F}
\end{aligned}$$

and we have

$$\begin{aligned}
\|F_0\|_{\mathcal{H}_F} &= \|F_1\|_{\mathcal{H}_F} = 1 \\
\|l\|_{\mathcal{H}_F}^2 &= \|F_\varphi\|_{\mathcal{H}_F}^2 = \|T_F \varphi\|_2^2 \leq \lambda_1^2 \|\varphi\|_2^2 < \infty
\end{aligned}$$

where  $\lambda_1$  is the top eigenvalue of the Mercer operator  $T_F$  (Lemma 11.29).

For (ii),

$$\begin{aligned}
|\langle F_\varphi, e_\lambda \rangle_2|^2 &= |\langle T_F \varphi, e_\lambda \rangle_2|^2 \\
&= \left| \left\langle T_F^{1/2} \varphi, T_F^{1/2} e_\lambda \right\rangle_2 \right|^2 \\
&\leq \left\| T_F^{1/2} \varphi \right\|_2^2 \left\| T_F^{1/2} e_\lambda \right\|_2^2 \text{ (by Cauchy-Schwarz)} \\
&= \langle \varphi, T_F \varphi \rangle_2 \left\| T_F^{1/2} e_\lambda \right\|_2^2 \\
&\leq \|F_\varphi\|_{\mathcal{H}_F}^2 \|e_\lambda\|_2^2 = \|F_\varphi\|_{\mathcal{H}_F}^2;
\end{aligned}$$

where we used the fact that  $\left\| T_F^{1/2} e_\lambda \right\|_2^2 \leq \lambda_1 \|e_\lambda\|_2^2 \leq 1$ , since  $\lambda_1 < 1 =$  the right endpoint of the interval  $[0, 1]$  (see Lemma 11.29), and  $\|e_\lambda\|_2 = 1$ .

Therefore, the corollary follows.  $\square$

**Corollary 11.49.** *For all  $\lambda \in \mathbb{R}$ , and all  $F_\varphi, \varphi \in C_c^\infty(0, 1)$ , we have*

$$\begin{aligned}
\langle F_\varphi, e_\lambda \rangle_{\mathcal{H}_F} &= \frac{1}{2} (1 + \lambda^2) \langle F_\varphi, e_\lambda \rangle_2 \\
&\quad + \frac{1}{2} (\bar{l}(1)(1 + i\lambda)e^{i\lambda} + \bar{l}(0)(1 - i\lambda)).
\end{aligned} \tag{11.82}$$

*Proof.* By Lemma 11.47,

$$\langle F_\varphi, e_\lambda \rangle_{\mathcal{H}_F} = \langle F_\varphi, \frac{1}{2}(e_\lambda - e''_\lambda) \rangle_2 - \frac{1}{2}[W]_0^1.$$

where

$$\begin{aligned} \frac{1}{2}(e_\lambda - e''_\lambda) &= \frac{1}{2}(1 + \lambda^2)e_\lambda; \text{ and} \\ [W]_0^1 &\stackrel{(11.80)}{=} -\bar{l}(1)(1 + i\lambda)e^{i\lambda} - \bar{l}(0)(1 - i\lambda), \quad l := F_\varphi. \end{aligned}$$

□

**Lemma 11.50.** For all  $F_\varphi$ ,  $\varphi \in C_c^\infty(0, 1)$ , and all  $\lambda \in \mathbb{R}$ ,

$$\langle F_\varphi, e_\lambda \rangle_{\mathcal{H}_F} = \langle \varphi, e_\lambda \rangle_2. \quad (11.83)$$

In particular, set  $\lambda = 0$ , we get

$$\begin{aligned} \langle F_\varphi, \mathbf{1} \rangle_{\mathcal{H}_F} &= \int_0^1 \varphi(x) dx = \frac{1}{2} \int_0^1 (F_\varphi - F''_\varphi)(x) dx \\ &= \frac{1}{2} (\langle F_\varphi, \mathbf{1} \rangle_2 - \langle F''_\varphi, \mathbf{1} \rangle_2) \\ &\leq C \|F_\varphi\|_{\mathcal{H}} \end{aligned}$$

*Proof.* Eq. (11.83) follows from basic fact of the Mercer operator. See Lemma 11.29 and its corollaries. It suffices to note the following estimate:

$$\begin{aligned} \int_0^1 F''_\varphi(x) dx &= F'_\varphi(1) - F'_\varphi(0) \\ &= -e^{-1} \int_0^1 e^y \varphi(y) dy - \int_0^1 e^{-y} \varphi(y) dy \\ &= -F_\varphi(1) - F_\varphi(0) \leq 2 \|F_\varphi\|_{\mathcal{H}}. \end{aligned}$$

□

**Corollary 11.51.** For all  $\lambda \in \mathbb{R}$ ,

$$\langle e_\lambda, e_\lambda \rangle_{\mathcal{H}_F} = \frac{\lambda^2 + 3}{2}. \quad (11.84)$$

*Proof.* By Corollary 11.49, we see that

$$\begin{aligned} \langle F_\varphi, e_\lambda \rangle_{\mathcal{H}_F} &= \frac{1}{2}(1 + \lambda^2) \langle F_\varphi, e_\lambda \rangle_2 \\ &\quad + \frac{1}{2} (\bar{l}(1)(1 + i\lambda)e^{i\lambda} + \bar{l}(0)(1 - i\lambda)); \quad l := F_\varphi. \end{aligned} \quad (11.85)$$

Since  $\{F_\varphi : \varphi \in C_c^\infty(0, 1)\}$  is dense in  $\mathcal{H}_F$ ,  $\exists F_{\varphi_n} \rightarrow e_\lambda$  in  $\mathcal{H}_F$ , so that

$$\langle F_{\varphi_n}, e_\lambda \rangle_{\mathcal{H}_F} \rightarrow \langle e_\lambda, e_\lambda \rangle_{\mathcal{H}_F}$$

$$\begin{aligned}
&= \frac{1}{2}(1 + \lambda^2) + \frac{1}{2}(e^{-i\lambda}(1 + i\lambda)e^{i\lambda} + (1 - i\lambda)) \\
&= \frac{1}{2}(1 + \lambda^2) + 1 = \frac{\lambda^2 + 3}{2}.
\end{aligned}$$

The approximation is justified since all the terms in the RHS of (11.85) satisfy the estimate  $|\dots|^2 \leq C \|F_\varphi\|_{\mathcal{H}_F}^2$ . See the proof of Corollary 11.48 for details.  $\square$

Note Lemma 11.47 is equivalent to the following:

**Corollary 11.52.** For all  $h \in \mathcal{H}_F$ , and all  $k \in \text{dom}(T_F^{-1})$ , i.e.,  $k \in \{F_\varphi : \varphi \in C_c^\infty(0, 1)\}$ , we have

$$\langle h, k \rangle_{\mathcal{H}} = \frac{1}{2}(\langle h, k \rangle_0 + \langle h', k' \rangle_0) + \frac{1}{2}(\overline{h(0)}k(0) + \overline{h(1)}k(1)) \quad (11.86)$$

and eq. (11.86) extends to all  $k \in \mathcal{H}_F$ , since  $\text{dom}(T_F^{-1})$  is dense in  $\mathcal{H}_F$ .

**Example 11.53.** Take  $h = k = e_\lambda$ ,  $\lambda \in \mathbb{R}$ , then (11.86) gives

$$\langle e_\lambda, e_\lambda \rangle_{\mathcal{H}} = \frac{1}{2}(1 + \lambda^2) + \frac{1}{2}(1 + 1) = \frac{\lambda^2 + 3}{2}$$

as in (11.84).

**Corollary 11.54.** Let  $A_\theta \supset -iD$  be any selfadjoint extension in  $\mathcal{H}_F$ . If  $\lambda, \mu \in \text{spect}(A_\theta)$ , such that  $\lambda \neq \mu$ , then  $\langle e_\lambda, e_\mu \rangle_{\mathcal{H}_F} = 0$ .

*Proof.* It follows from (11.86) that

$$\begin{aligned}
2 \langle e_\lambda, e_\mu \rangle_{\mathcal{H}} &= \langle e_\lambda, e_\mu \rangle_0 + \lambda\mu \langle e_\lambda, e_\mu \rangle_0 + \left(1 + e^{i(\mu-\lambda)}\right) \\
&= (1 + \lambda\mu) \langle e_\lambda, e_\mu \rangle_0 + \left(1 + e^{i(\mu-\lambda)}\right) \\
&= (1 + \lambda\mu) \frac{e^{i(\mu-\lambda)} - 1}{i(\mu - \lambda)} + \left(1 + e^{i(\mu-\lambda)}\right) \quad (11.87)
\end{aligned}$$

By Corollary 11.42, eq. (11.75), we have

$$e^{i\lambda} = \frac{1 - i\lambda}{1 + i\lambda} e^{i\theta}, \quad e^{i\mu} = \frac{1 - i\mu}{1 + i\mu} e^{i\theta}$$

and so

$$e^{i(\mu-\lambda)} = \frac{(1 - i\mu)(1 + i\lambda)}{(1 + i\mu)(1 - i\lambda)}.$$

Substitute this into (11.87) yields

$$2 \langle e_\lambda, e_\mu \rangle_{\mathcal{H}} = \frac{-2(1 + \lambda\mu)}{(1 + i\mu)(1 - i\lambda)} + \frac{2(1 + \lambda\mu)}{(1 + i\mu)(1 - i\lambda)} = 0.$$

$\square$

**Corollary 11.55.** Let  $F(x) = e^{-|x|}$ ,  $|x| < 1$ . Let  $D_F(F_\varphi) = F_{\varphi'}$ ,  $\forall \varphi \in C_c^\infty(0,1)$ , and  $A_\theta \supset -iD_F$  be a selfadjoint extension in  $\mathcal{H}_F$ . Set  $e_\lambda(x) = e^{i\lambda x}$ , and

$$\Lambda_\theta := \text{spect}(A_\theta) \quad (= \text{discrete subset in } \mathbb{R} \text{ by Cor. 11.45}) \quad (11.88)$$

Then

$$\tilde{F}_\theta(x) = \sum_{\lambda \in \Lambda_\theta} \frac{2}{\lambda^2 + 3} e_\lambda(x), \quad \forall x \in \mathbb{R} \quad (11.89)$$

is a continuous p.d. extension of  $F$  to the real line. Note that both sides in eq. (11.89) depend on the choice of  $\theta$ .

The type 1 extensions are indexed by  $\theta \in [0, 2\pi)$  where  $\Lambda_\theta$  is given in (11.88), see also (11.73) in Corollary 11.42.

**Corollary 11.56** (Sampling property of the set  $\Lambda_\theta$ ). Let  $F(x) = e^{-|x|}$  in  $|x| < 1$ ,  $\mathcal{H}_F$ ,  $\theta$ , and  $\Lambda_\theta$  be as above. Let  $T_F$  be the corresponding Mercer operator. Then for all  $\varphi \in L^2(0,1)$ , we have

$$(T_F\varphi)(x) = 2 \sum_{\lambda \in \Lambda_\theta} \frac{\hat{\varphi}(\lambda)}{\lambda^2 + 3} e^{i\lambda x}, \quad \text{for all } x \in (0,1).$$

*Proof.* This is immediate from Corollary 11.55.  $\square$

*Remark 11.57.* Note that the system  $\{e_\lambda \mid \lambda \in \Lambda_\theta\}$  is orthogonal in  $\mathcal{H}_F$ , but *not* in  $L^2(0,1)$ .

*Proof.* We saw that  $A_\theta$  has pure atomic spectrum. By (11.51), the set

$$\left\{ \sqrt{\frac{2}{\lambda^2 + 3}} e_\lambda : \lambda \in \Lambda_\theta \right\}$$

is an ONB in  $\mathcal{H}_F$ . Hence, for  $F = F_0 = e^{-|x|}$ , we have the corresponding p.d. extension:

$$\begin{aligned} F_\theta(x) &= \sum_{\lambda \in \Lambda_\theta} \frac{1}{\|e_\lambda\|_{\mathcal{H}_F}^2} \langle e_\lambda, F \rangle_{\mathcal{H}_F} e_\lambda(x) \\ &= \sum_{\lambda \in \Lambda_\theta} \frac{2}{\lambda^2 + 3} e_\lambda(x), \quad \forall x \in [0,1]. \end{aligned} \quad (11.90)$$

where  $\langle e_\lambda, F \rangle_{\mathcal{H}_F} = \overline{e_\lambda(0)} = 1$  by the reproducing property. But the RHS of (11.90) extends to  $\mathbb{R}$ . See Figure 11.3.  $\square$

**Corollary 11.58.** Let  $F(x) = e^{-|x|}$  in  $(-1,1)$ , and let  $\mathcal{H}_F$  be the RKHS. Let  $\theta \in [0, 2\pi)$ , and let  $\Lambda_\theta$  be as above; then  $\left\{ \sqrt{\frac{2}{\lambda^2 + 3}} e_\lambda \mid \lambda \in \Lambda_\theta \right\}$  is an ONB in  $\mathcal{H}_F$ .

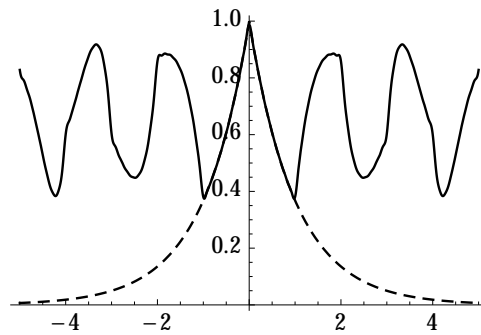


Figure 11.3:  $\theta = 0$ . A type 1 continuous p.d. extension of  $F(x) = e^{-|x|}|_{[-1,1]}$  in  $\mathcal{H}_F$ .

## A summary of relevant numbers from the Reference List

For readers wishing to follow up sources, or to go in more depth with topics above, we suggest:

The pioneering paper here is [Aro50] and the intervening decades have witnessed a host of applications. And by now there are books dealing with various aspects of reproducing kernel Hilbert spaces (RKHS). A more comprehensive citation list is: [AD86, JPT14a, Nus75, Rud63, Alp01, CZ07, AJSV13, Aro50, Nel59b, Sch64a, SZ07, SZ09].

Part V  
Appendix

## Appendix A

# An overview of Functional Analysis books (cast of characters)

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

— John von Neumann

Below we offer a list of related Functional Analysis books; they cover a host of diverse areas of functional analysis and applications, some different from what we select here: Our comments are telegraphic-review form (by P.J.):

Akhiezer and Glazman, “*Theory of linear operators in Hilbert space*” [AG93]

– a classic book set covering the detailed structure of unbounded operators and their applications; and now in a lovely Dover edition.

Arveson, “*An invitation to  $C^*$ -algebras*” [Arv76]

– an introduction to  $C^*$ -algebras and their representations on Hilbert spaces. – covers the most basic ideas, as simply and concretely as we could. – Hilbert spaces are separable and  $C^*$ -algebras are GCR. Representations are given a concrete parametric description, including the irreducible representations of any  $C^*$ -algebra, even if not GCR. For someone interested in Borel structures, see Chapter 3. Chapter 1 is a bare-bones introduction to  $C^*$ -algebras.

Bachman and Narici, “*Functional analysis*” [BN00]

The book by Bachman and Narici’s is a systematic introduction to the fundamentals of functional analysis. It is easier to follow than say Rudin’s Functional Analysis book, but it doesn’t go as far either. Rather it helps readers reinforcing

topics from real analysis and other masters level courses. It serves to bridge the gap between more difficult treatments of functional analysis. (Dover reprints classics in a cheap paper back format.)

Bratteli and Robinson, “*Operator algebras and quantum statistical mechanics*” [BR79, BR81a]

This is a widely cited two volume book-set, covering the theory of operator algebras, and its applications to quantum statistical mechanics. It is one of the more authoritative treatments of the many exciting applications of functional analysis to physics. Both books are self-contained; with complete proofs; – a useful text for students with a prior exposure to basic functional analysis. One of the main themes in v1 is decomposition theory, and the use of Choquet simplices. Example: the set of KMS-states for a  $C^*$ -algebraic dynamical system typically forms a Choquet simplex. An introductory chapter covers algebraic techniques and their use in statistical physics; this is followed up in v2. Indeed, a host of applications are covered in v2. The new edition has a more comprehensive discussion of dissipative operators and analytic elements; – and it includes a positive resolution of the question of whether maximal orthogonal probability measure on the state space of algebra is automatically maximal among all the probability measures on the space.

Conway, “*A course in functional analysis*” [Con90]

– a comprehensive introduction to functional analysis. The style is formal and rigorous. – is designed to be used in grad courses. Through its eleven chapters, J. Conway masterfully wrote a beautiful exposition of this core subject.

Dunford and Schwartz, “*Linear operators*” [DS88b, DS88c, DS88a]

This classic three-volume book set, the first Functional analysis, and the second the theory of linear operators. And for the theory of unbounded operators it is unsurpassed. – written by two notable mathematicians, it constitutes a comprehensive survey of the general theory of linear operations, and their diverse applications. Dunford and Schwartz are influenced by von Neumann, and they emphasize the significance of the relationships between the abstract theory and its applications. The two first volumes are for the students. – treatment is relatively self-contained. Now a paperback edition of the original work, unabridged, in three volumes.

Kolmogorov and Fomin, “*Introductory real analysis*” [KF75]

This book is two books bound as one; and in the lovely format from Dover. Part 1: metric spaces, and normed linear spaces. Part 2: Lebesgue integration and basic functional analysis. Numerous examples are sprinkled through the text. To get the most out of this book, it helps if you have already seen many of the results presented elsewhere. History: The book came from original notes from Andrei Kolmogorov’s lectures given at Moscow’s Lomonosov University in the 1940’s, and it still stands as timely introduction to real and functional analysis. Strengths: step by step presentation of all the key concepts needed in the



subject; proceeding all the way from set theory to Fredholm integral equations. Offers a wonderful and refreshing insight. Contents (sample): Elements of Set Theory; Metric and Topological Spaces; Normed and Topological Linear Spaces; Linear Functionals and Linear Operators; Elements of Differential Calculus in Linear Spaces; Measure, Measurable Functions, Integral; Indefinite Lebesgue Integral, Differentiation Theory; Spaces of Summable Functions; Trigonometric Series, Fourier Transformation; Linear Integral Equations.

Kadison and Ringrose, “*Fundamentals of the theory of operator algebras*” [KR97a, KR97b]

Here we cite the first two volumes in a 4-volume book-set. It begins with the fundamentals in functional analysis, and it aims at a systematic presentations of the main areas in the theory of operator algebras, both  $C^*$ -algebras, von Neumann algebras, and their applications, so including subfactors, Tomita-Takesaki theory, spectral theory, decomposition theory, and applications to ergodic theory, representations of groups, and to mathematical physics.

Lax, “*Functional analysis*” [Lax02]

The subject of functional analysis, while fundamental and central in the landscape of mathematics, really started with seminal theorems due to Banach, Hilbert, von Neumann, Herglotz, Hausdorff, Friedrichs, Steinhouse,...and many other of, the perhaps less well known, founding fathers, in Central Europe (at the time), in the period between the two World Wars. It gained from there because of its many success stories, – in proving new theorems, in unifying old ones, in offering a framework for quantum theory, for dynamical systems, and for partial differential equations. The Journal of Functional Analysis, starting in the 1960ties, broadened the subject, reaching almost all branches of science, and finding functional analytic flavor in theories surprisingly far from the original roots of the subject. Peter Lax has himself, – alone and with others, shaped some of greatest successes of the period, right up to the present. That is in the book!! And it offers an upbeat outlook for the future. It has been tested in the class room, – it is really user-friendly. At the end of each chapter P. Lax offers personal recollections; – little known stories of how several of the pioneers in the subject have been victims, – in the 30ties and the 40ties, of Nazi atrocities. The writing is crisp and engaged.

MacCluer, “*Elementary functional analysis*” [Mac09]

I received extremely positive student-feedback on MacCluer’s very nice book. It covers elementary functional analysis, is great for self-study, and easy to follow. It conveys the author’s enthusiasm for her subject. It includes apposite quotes, anecdotes, and historical asides, all making for a wonderful personal touch and drawing the reader into dialogue in a palpable way. Contents: six chapters, each introduced by a well-chosen quote, often hinting in a very useful manner at the material that is to follow. I particularly like MacCluer’s choice of Dunford and Schwartz to start off her third chapter: “In linear spaces with a suitable topology one encounters three far-reaching principles concerning contin-

uous linear transformations. . .” We find out quickly that these “Big Three” (as the chapter is titled) are uniform boundedness, the open mapping theorem, and Hahn-Banach. MacCluer quickly goes on to cover these three gems in a most effective and elegant manner, as well as a number of their corollaries or, in her words, “close cousins,” such as the closed graph theorem and Banach-Steinhaus. The book takes the reader from Hilbert space preliminaries to Banach- and  $C^*$ -algebras and, to the spectral theorem.

Nelson, “*Topics in Dynamics I: Flows*” [Nel69]

This is a book in the Princeton Math Lecture Notes series, appearing first in 1972, but since Prof Nelson kindly made it available on his website. In our opinion, it is the best account of general multiplicity for normal operators, bounded and unbounded, and for abelian  $*$ -algebras. In addition it contains a number of applications of functional analysis to geometry and to physics.

Riesz et al., “*Functional analysis*” [RSN90]

A pioneering book in F.A., first published in the early 50s, and now in a Dover edition, very readable. The book starts with an example of a continuous function which is not differentiable and then proves Lebesgue’s theorem which tells you when a function does have a derivative. The 2nd part of the book is about integral equations which again starts with some examples of problems from the 19th century mathematicians. The presentation of Fredholm’s method is a gem.

Rudin, “*Functional analysis*” [Rud73]

“Modern analysis” used to be a popular name for the subject of this lovely book. It is as important as ever, but perhaps less “modern”. The subject of functional analysis, while fundamental and central in the landscape of mathematics, really started with seminal theorems due to Banach, Hilbert, von Neumann, Herglotz, Hausdorff, Friedrichs, Steinhouse,...and many other of, the perhaps less well known, founding fathers, in Central Europe (at the time), in the period between the two World Wars. In the beginning it generated awe in its ability to provide elegant proofs of classical theorems that otherwise were thought to be both technical and difficult. The beautiful idea that makes it all clear as daylight: Wiener’s theorem on absolutely convergent (AC) Fourier series of  $1/f$  if you can divide, and if  $f$  has AC Fourier series, is a case in point. The new subject gained from there because of its many success stories, – in proving new theorems, in unifying old ones, in offering a framework for quantum theory, for dynamical systems, and for partial differential equations. And offering a language that facilitated interdisciplinary work in science! The topics in Rudin’s book are inspired by harmonic analysis. The later part offers one of the most elegant compact treatment of the theory of operators in Hilbert space, I can think of. Its approach to unbounded operators is lovely.

Sakai, “ *$C^*$ -algebras and  $W^*$ -algebras*” [Sak71]

The presentation is succinct, theorem, proof, ... qed; but this lovely book

had a profound influence on the subject. Its scope cover nearly all major results in the subject up until that time. In order to accomplish this goal (without expanding into multiple volumes), the author omits examples, motivation, . . . . It is for students who already have an interest in operator theory. As a student, myself (PJ), I learned a lot from this wonderful book.

Shilov, “*Elementary functional analysis*” [Shi96]

Elementary Functional Analysis by Georgi E. Shilov is suitable for a beginning course in functional analysis and some of its applications, e.g., to Fourier series, to harmonic analysis, to partial differential equations (PDEs), to Sobolev spaces, and it is a good supplement and complement to two other popular books in the subject, one by Rudin, and another by Edwards. Rudin’s book is entitled “Functional Analysis” includes new material on unbounded operators in Hilbert space. Edwards’ book “Functional Analysis: Theory and Applications;” is in the Dover series, and it is twice as thick as Shilov’s book. Topics covered in Shilov: Function spaces,  $L^p$ -spaces, Hilbert spaces, and linear operators; the standard Banach, and Hahn-Banach theorems. It includes many exercises and examples. Well motivated with applications. Book Comparison: Shilov book is gentler on students, and it is probably easier to get started with: It stresses motivation a bit more, the exercises are easier, and finally Shilov includes a few applications; fashionable these days.

Stein et al., “*Functional analysis*” [SS11b]

This book is the fourth book in a series: Elias Stein’s and Rami Shakarchi’s Princeton lectures in analysis. Elias Stein is a world authority on harmonic analysis. The book is of more recent vintage than the others from our present list. The book on functional analysis is actually quite different from other texts in functional analysis. For instance Rudin’s textbook on functional analysis has quite a different emphasis from Stein’s. Stein devotes a whole chapter to applications of the Baire category theory while Rudin devotes a page. Stein does this because it provides some insights into establishing the existence of a continuous but nowhere differentiable function as well as the existence of a continuous function with Fourier series diverging a point. A special touch in Stein: Inclusion of Brownian motion, and of process with independent increments, a la Doob’s. Stein’s approach to the construction of Brownian motion is different and closer to the approaches taken in books on financial math. Stein et al develop Brownian motion in the context of solving Dirichlet’s problem.

Stone, “*Linear Transformations in Hilbert Space and Their Applications to Analysis*” [Sto90]

Stone’s book is a classic, came out in 1932, and was the unique source on spectral multiplicity, and a host of applications of the theory of unbounded operators to analysis, to approximation theory, and to special functions. The last two chapters illustrate the theory with a systematic study of (infinite  $\times$  infinite) Jacobi matrices; i.e., tri-diagonal infinite matrices; assumed formally selfadjoint (i.e., Hermitian). Sample results: A dichotomy: Their von Neumann

indices must be  $(0, 0)$  or  $(1, 1)$ . Some of the first known criteria for when they are one or the other are given; plus a number of applications to classical analysis.

Takesaki, “*Theory of operator algebras*” [Tak79]

– written by one of the most prominent researchers of the area, provides an introduction to this rapidly developing theory. ... These books are recommended to every graduate student interested in this exciting branch of mathematics. Furthermore, they should be on the bookshelf of every researcher of the area.

Trèves, “*Topological vector spaces, distributions and kernels*” [Trè06b]

Covers topological vector spaces and their applications, and it is a pioneering book. It is antidote for those who mistakenly believe that functional analysis is about Banach and Hilbert spaces. It’s also about Fréchet spaces, LF spaces, Schwartz distributions (generalized functions), nuclear spaces, tensor products, and the Schwartz Kernel Theorem (proved by Grothendieck). Trèves’s book provides the perfect background for advanced work in linear differential, pseudodifferential, or Fourier integral operators.

Yosida “*Functional analysis*” [Yos95]

Yosida’s book is based on lectures given decades ago at the University of Tokyo. It is intended as a textbook to be studied by students on their own or to be used in a course on Functional Analysis, i.e., the general theory of linear operators in function spaces together with salient features of its application to diverse fields of modern and classical analysis. Necessary prerequisites for the reading of this book are summarized, with or without proof, in Chapter 0 under titles: Set Theory, Topological Spaces, Measure Spaces and Linear Spaces. Then, starting with the chapter on Semi-norms, a general theory of Banach and Hilbert spaces is presented in connection with the theory of generalized functions of S.L. Sobolev and L. Schwartz. The reader may pass, e.g., from Chapter IX (Analytical Theory of Semi-groups) directly to Chapter XIII (Ergodic Theory and Diffusion Theory) and to Chapter XIV (Integration of the Equation of Evolution). Such materials as “Weak Topologies and Duality in Locally Convex Spaces” and “Nuclear Spaces” are presented in the form of the appendices to Chapter V and Chapter X, respectively.

#### Some relevant books: Classics, and in the Dover series:

Banach, “*Theory of Linear Operations*” [Ban93]

Georgi, “*Weak Interactions and Modern Particle Theory*” [Geo09]

Prenter, “*Splines and Variational Methods*” [Pre89]

In chapters 4 and 8 above we have cited pioneers in quantum physics, the foundations of quantum mechanics. The most central here are Heisenberg (ma-

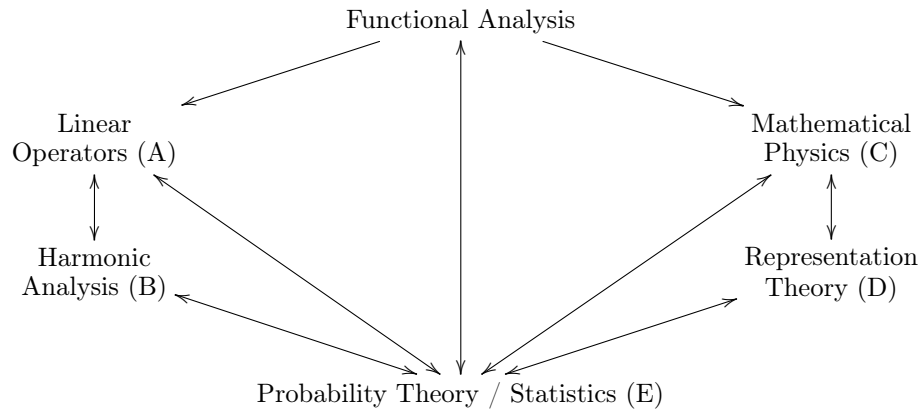


Table A.1

trix mechanics), Schrödinger (wave mechanics, the Schrödinger equation), and Dirac (Dirac’s equation is a relativistic wave equation, describes all spin- $\frac{1}{2}$  massive particles free form, as well as electromagnetic interactions). We further sketched von Neumann’s discovery of the equivalence of the answers given by Heisenberg and Schrödinger, and the Stone-von Neumann uniqueness theorem. The relevant papers and books are as follows: [Hei69, Sch32, vN31, HN28, Dir35, Dir47].

(A)		
↕	bounded	differential operators, ODE/PDE
↕	unbounded	generators of diffusion
↕	geometry	Schrödinger operators
↕	spectral theory	wave operators
	spectral representation	scattering operators
	single operators	
	system of operators	
	operator commutation relations	

Table A.2

(B)	
$\updownarrow$ commutative non-commutative	analysis / synthesis Fourier analysis, wavelet analysis
	Applications: signal processing physics statistics analysis on fractals

Table A.3

(C)	
$\updownarrow$ quantum physics classical mechanics	quantum information states and decomposition equilibrium: Gibbs, KMS, ...
$\updownarrow$ statistical physics quantum field theory	
$\updownarrow$ relativistic non-relativistic	

Table A.4

(D)	
$\updownarrow$ groups (abelian, non-abelian) algebras generators and relations	locally compact, non-locally compact  Lie groups $\longleftrightarrow$ Lie algebras induced representations decomposition of representations groups over $\mathbb{R}$ , $\mathbb{C}$ , or other local fields

Table A.5

(E)	
stochastic processes	discrete continuous Gaussian Brownian motion non-Gaussian Lévy
	solutions of diffusion equations with the use of functional integrals (i.e., probability measure on infinite-dimensional spaces such as $C(\mathbb{R})$ or Schwartz space $\mathcal{S}$ )

Table A.6

## Appendix B

# Terminology from neighboring areas

**Classical Wiener measure/space.** Classical Wiener space (named Norbert Wiener) is the sample-space part of a probability space, a triple (sample-space, sigma-algebra, and probability measure). The sample space may be taken to be the collection of all continuous functions on a given domain (usually an interval for time). So sample paths are continuous functions. The sigma-algebra is generated by cylinder-sets, and the probability measure is called the Wiener measure; (its construction is subtle, see Chapter 6 above.) It has the property that the stochastic processes which samples the paths in the model is a Gaussian process with independent increments, the so called Brownian motion. It is also called the Wiener process. And it should perhaps be named after L. Bachelier, whose work predates that of Einstein.

It, and the related process “white noise”, are important in pure and applied mathematics. It is a core ingredient in stochastic analysis: the study of stochastic calculus, diffusion processes, and potential theory. Applications include engineering and physics: models of noise in electronics engineering, in instrument errors in filtering theory, and in control theory. In atomic physics, it is used in the study of diffusion, the Fokker–Planck and Langevin equations; and in path-space integrals; the Feynman–Kac formula, in the solution of the Schrödinger equation. In finance, it is used in the solution to the Black–Scholes equation for option prices. References include [AJ12, AJL13, ARR13, CW14, GJ87, Gro70, Hid80, Itô06, Jor06, Jør14, Nel64, Nel67, Sch32, Sch58, SS11a]; and we refer to Sections 1.4, 6.1, 6.2, and 11.1.

**Hilbert’s sixth problem.** This is not a “yes/no problem”; rather the 6th asks for a mathematical axiomatization of physics. In a common English translation, it reads: 6. Give a Mathematical Treatment of the Axioms of Physics. A parallel is drawn to the foundations of geometry: *To treat in the same manner, by means of axioms, those physical sciences where mathematics plays an*



*important part; in the first rank are the theory of probabilities and mechanics.*

Hilbert: "As to the axioms of the theory of probabilities, it seems to me desirable that their logical investigation should be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics, and in particular in the kinetic theory of gases. ... Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua."

In the 1930s, probability theory was put on an axiomatic and sound foundation by Andrey Kolmogorov. In the 1960s, we have the work of A. Wightman, R. Haag, J. Glimm, and A. Jaffe on quantum field theory. This was followed by the Standard Model in particle physics and general relativity. Still unresolved is the theory of quantum gravity.

Ref. Sections [0.6](#) (pg. [11](#)), [1.5](#) (pg. [56](#)), [6.1](#) (pg. [213](#)); see also [[Wig76](#), [MSS13](#)].

**Quantum mechanics (QM); (quantum physics, or quantum theory)** is a branch of physics which describes physical phenomena at "small" scales, atomic and subatomic length scales. The action is on the order of the Planck constant. QM deals with observation of physical quantities that can only change and interact, by discrete amounts or "steps" (hence "quantum"), and behave probabilistically rather than deterministically. The "steps" are too tiny even for microscopes. Any description must be given in terms of a wave function, as opposed to particles. (For details, see [[Dir47](#), [BR81a](#)].)

Ref. Sections [1.5](#) (pg. [56](#)), [2.1](#) (pg. [75](#)), [3.1](#) (pg. [93](#)), and Chapter [8](#) (pg. [276](#)).

**Quantum field theory (QFT)** is a mathematical framework used in physics for constructing models of subatomic particles (quantum mechanical). It covers such areas as particle physics and condensed matter physics. A QFT treats particle-wave duality as excited states of an underlying physical field, called field quanta. Of interest are quantum mechanical interactions between particles and the corresponding underlying fields. (For details, see [[GJ87](#)].)

Ref. Section [0.6](#) (pg. [11](#)).

**Signal processing (SP)** is an engineering discipline dealing with transmission of signals (information, speech or images, over wires, or wireless. An important tool in the area involves subdivision of time signals into frequency bands, and it involves effective algorithms for implementations of processing or transferring information contained in a variety of different symbolic, or abstract formats broadly designated as signals. SP uses mathematical, statistical, computational tools. (For details, see [[Wol51](#), [BJ02](#)].)

Ref. Section [5.4](#) (pg. [194](#)), Chapters [5](#) (pg. [184](#)), [8](#) (pg. [276](#)).

**Stochastic processes (SP), or random process**, are part of probability theory. They are used when deterministic quantities are not feasible: random variables are measures in their respective probability distributions (also called "laws.") A SP is an indexed family of random variables (representing measure-

ments, or samples), for example, if a SP is indexed by time, it represents the evolution or dynamics of some system. A SP is the probabilistic counterpart to a deterministic process (or a deterministic system). An example is Brownian motion (BM), the random motion of particles (e.g., pollen) suspended in a fluid (a liquid or a gas). BM results from their collision of the pollen with atoms or molecules making up the gas or liquid. BM also refer to the mathematical model used to describe such random movements. (For details, see [Gro64, Itô04, Itô06, Sch58, SS11a].)

Ref. Sections 1.2 (pg. 22), 11.1 (pg. 337); and Chapters 6 (pg. 208), 7 (pg. 221).

**Unitary representations (UR)** of a groups  $G$  are homomorphisms from the group  $G$  in question into the group of all unitary operators in some Hilbert space; the Hilbert space depending on the UR. The theory is best understood in in the case of strongly continuous URs of locally compact (topological) groups. Applications include quantum mechanics. Books by Hermann Weyl, Pontryagin, and George Mackey have influenced our presentation. The theory of UR is closely connected with harmonic analysis; – for non-commutative groups, non-Abelian harmonic analysis. Important groups in physics are non-commutative, so this case is extremely important, although it is also rather technical. There is a vast literature, though. Important papers are cited in the books by George Mackey. In fact, versions of the Plancherel theorem exist for some non-commutative Lie groups (of direct relevance to physics), but they are subtle, and the non-commutative analysis is carried out on a case-by-case basis. The best known special case is that of compact groups where we have the Peter-Weyl theorem. But the important symmetry groups in relativistic physics are non-compact, see [GJ87].

Ref. Sections 1.4 (pg. 28), 2.1 (pg. 75), and Chapters 4 (pg. 123), 5 (pg. 184), 7 (pg. 221).

**Wavelets** are wave-like functions; they can typically be visualized as "brief oscillations" as one might see recorded in seismographs, or in heart monitors. What is special about wavelet functions is that they allow for effective algorithmic construction of bases in a variety of function spaces. The algorithms in turn are based on a notion of resolution and scale-similarity. The last two features make wavelet decompositions more powerful than comparable Fourier analyses. Wavelets can be localized, while Fourier bases cannot. Wavelets are designed to have specific properties that make them of practical use in signal processing. (For details, see [BJ02].)

Ref. Sections 1.4 (pg. 34), 4.8 (pg. 163), 5.4 (pg. 194), and Chapter 5 (pg. 184).

## Appendix C

### Often cited above

**mathematical ideas originate in empirics.** But, once they are conceived, the subject begins to live a peculiar life of its own and is ... governed by almost entirely aesthetical motivations. In other words, at a great distance from its empirical source, or after much “abstract” inbreeding, a mathematical subject is in danger of degeneration. Whenever this stage is reached the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas.

— von Neumann

Inside the book, the following authors are cited frequently, W. Arveson, M. Atiyah, L. Bachelier, S. Banach, H. Bohr, M. Born, N. Bohr, P. Dirac, W. Döbblin, F. Dyson, K. Friedrichs, I. Gelfand, L. Gårding, I. Gelfand, W. Heisenberg, D. Hilbert, K. Itō, R. Kadison, S. Kakutani, M. Krein, P. Lax, G. Mackey, E. Nelson, R. Phillips, F. Riesz, M. Riesz, E. Schrödinger, H.A. Schwarz, L. Schwartz, J. Schwartz, I. Segal, I. Singer, M. Stone, J. von Neumann, N. Wiener. Below a short bio.

**William Arveson** (1934 – 2011) [[Arv72](#), [Arv98](#)]. Cited in connection with  $C^*$ -algebras and their states and representations.

W. Arveson, known for his work on completely positive maps, and their extensions; powerful generalizations of the ideas of Banach, Krein, and Stinespring. An early results in this area is an extension theorem for completely positive maps with values in the algebra of all bounded operators. This theorem led to injectivity of von-Neumann algebras in general, and work by Alain Connes relating injectivity to hyperfiniteness. In a series of papers in the 60’s and 70’s, Arveson introduced non-commutative analogues of several concepts from classical harmonic analysis including the Shilov and Choquet boundaries.

**Sir Michael Francis Atiyah** (1929 – ). Of the Atiyah-Singer Index Theorem. The Atiyah-Singer index of a partial differential operator (PDO) is related to the Fredholm index; – it equates an index, i.e., the difference of the number of independent solutions of two geometric, homogeneous PDEs (one for the operator and the other for its adjoint) to an associated list of invariants in differential geometry. It applies to many problems in mathematics after they are translated into the problem of finding the number of independent solutions of some PDE. The Atiyah-Singer index theorem gives a formula for the index of certain differential operators, in terms of geometric and topological invariants.

The Hirzebruch-Riemann-Roch theorem is a special cases of the Atiyah-Singer index theorem. In fact the index theorem gave a more powerful result, because its proof applied to all compact complex manifolds, while Hirzebruch's proof only worked for projective manifolds.

Related: In 1959 by Gelfand noticed homotopy invariance via an index, and he asked for more general formulas for topological invariants. For spin manifolds, Atiyah suggested that integrality could be explained as an index of a Dirac operator (Atiyah and Singer, 1961).

**Louis Bachelier** (1870 – 1946), a French probabilist, is credited with being the inventor of the stochastic process, now called Brownian motion; it was part of his PhD thesis, *The Theory of Speculation*, (1900). It discusses use of random walks, and Brownian motion, to evaluate stock options, and it is considered the first paper in mathematical finance. Even though Bachelier's work was more mathematical, and predates Einstein's Brownian motion paper by five years, it didn't receive much attention at the time, and it was only "discovered" much later by the MIT economists Paul Samuelson, in the 1960ties.

**Stefan Banach** (1892 – 1945) [Ban93]. The Banach of "Banach space." Banach called them "B-spaces" in his book. They were also formalized by Norbert Wiener (who traveled in Europe in the 1920ties.) But the name "Banach space" stuck.

S. Banach, one of the founders of modern functional analysis and one of the original members of the Lwów School of Mathematics, in Poland between the two World Wars. His 1932 book, *Théorie des opérations linéaires* (Theory of Linear Operations), is the first monograph on the general theory of functional analysis.

**Harald August Bohr** (1887 – 1951) was a Danish mathematician and soccer player. Best known for his theory of almost periodic functions. – In modern language it became the Bohr-compactification. (Different from the alternative compactifications we discussed above.) He is the brother of the physicist Niels Bohr.

**Niels Henrik David Bohr** (1885 – 1962) was a Danish physicist who made foundational contributions to understanding atomic structure and quantum theory, the "Bohr-atom", justifying the Balmer series for the visible spectral lines of the hydrogen atom; received the Nobel Prize in Physics in 1922; – "for his services in the investigation of the structure of atoms, and of the radiation

emanating from them”. Based on his liquid drop model of the nucleus, Bohr concluded that it was the uranium-235 isotope, and not the more abundant uranium-238, that was primarily responsible for fission.

In September 1941, at the start of WWII, Heisenberg, who had become head of the German nuclear energy project, visited Bohr in Copenhagen. During this meeting the two had discussions about possible plans by the two sides in the War, for a fission bomb, the content of the discussions have caused much speculation. Michael Frayn’s 1998 play “Copenhagen” explores what might have happened at the 1941 meeting between Heisenberg and Bohr.

**Max Born** (1882 – 1970) [BP44], a German physicist and mathematician, a pioneer in the early development of quantum mechanics; also in solid-state physics, and optics. Won the 1954 Nobel Prize in Physics for his “fundamental research in Quantum Mechanics, especially in the statistical interpretation of the wave function.” His assistants at Göttingen, between the two World Wars, included Enrico Fermi, Werner Heisenberg, and Eugene Wigner, among others. His early education was at Breslau, where his fellow students included Otto Toeplitz and Ernst Hellinger. In 1926, he formulated the now-standard interpretation of the probability density function for states (represented as equivalence classes of solutions to the Schrödinger equation.) After the Nazi Party came to power in Germany in 1933, Born was suspended. Subsequently he held positions at Johns Hopkins University, at Princeton University, and he settled down at St John’s College, Cambridge (UK). A quote: “I believe that ideas such as absolute certitude, absolute exactness, final truth, etc. are figments of the imagination which should not be admissible in any field of science. On the other hand, any assertion of probability is either right or wrong from the standpoint of the theory on which it is based.” Max Born (1954.)

**Paul Adrien Maurice Dirac** (1902 – 1984) [Dir35, Dir47]. Cited in connection with the “Dirac equation” and especially our notation for vectors and operators in Hilbert space, as well as the axioms of observables, states and measurements.

P. Dirac, an English theoretical physicist; fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. He was the Lucasian Professor of Mathematics at the University of Cambridge. Notable discoveries, the Dirac equation, which describes the behavior of fermions and predicted the existence of antimatter. Dirac shared the Nobel Prize in Physics for 1933 with Erwin Schrödinger, “for the discovery of new productive forms of atomic theory.” A rare interview with Dirac; see the link: [http://www.math.rutgers.edu/~greenfie/mill\\_courses/math421/int.html](http://www.math.rutgers.edu/~greenfie/mill_courses/math421/int.html)

**Wolfgang Döblin** (1915 – 40), French-German mathematician, and probabilist. Studied probability theory in Paris, under Fréchet. Served in the French army in the Ardennes when World War II broke out in 1939. There, he wrote down his work on the Chapman-Kolmogorov equation. And he sent it in a sealed envelope to the French Academy of Sciences. In 1940, after burning his mathematical notes, he took his own life as the German troops came in sight.

In 2000, the sealed envelope was opened, revealing that, at the time, Döbblin had anticipated the theory of Markov processes, Itô's lemma (now the Itô–Döbblin lemma), and parts of stochastic calculus.

**Freeman John Dyson** (1923 – ) theoretical physicist and mathematician; – known for his contributions to quantum electrodynamics, solid-state physics, astronomy, and to nuclear engineering. Within mathematics, he is known for his work on random matrices; his discovery of a perturbation expansion (the Dyson expansion.) He is a regular contributor to *The New York Review of Books*. Awards: the Lorentz Medal, the Max Planck Medal, and the Enrico Fermi Award.

**Kurt Otto Friedrichs** (1901 – 1982) [Fri80, FL28]. Is the Friedrichs of the Friedrichs extension; referring to the following Theorem: Every semibounded operator  $S$  with dense domain in Hilbert space has a selfadjoint extension having the same lower bound as  $S$ . There are other semibounded and selfadjoint extensions of  $S$ ; – they were found later by M. Krein.

K. Friedrichs, a noted German-American mathematician; a co-founder of The Courant Institute at New York University and recipient of the National Medal of Science.

*A story:* Selfadjoint operators, and the gulf between the lingo and culture of mathematics and of physics:

Peter Lax relates the following conversation in German between K.O. Friedrichs and W. Heisenberg, to have been taken place in the late 1950ties, in New York, when Heisenberg visited The Courant Institute at NYU. (The two had been colleagues in Germany before the war.) As a gracious host, Friedrichs praised Heisenberg for having created quantum mechanics. – After an awkward silence, Friedrich went on: “..and we owe to von Neumann our understanding of the crucial difference between a selfadjoint operator and one that is merely symmetric.” Another silence, and then – Heisenberg: “What is the difference?”

**Lars Gårding** (1919 – 2014). The “G” in Gårding vectors (representations of Lie groups), and in Gårding-Wightman quantum fields.

**Israel Moiseevich Gelfand** (1913 – 2009) [GJ60, GS60, GG59]. Is the “G” in GNS (Gelfand-Naimark-Segal), the correspondence between states and cyclic representations.

I. Gelfand, also written Israïl Moyseyovich Gel'fand, or Izrail M. Gelfand, a Russian-American mathematician; major contributions to many branches of mathematics: representation theory and functional analysis. The recipient of numerous awards and honors, including the Order of Lenin and the Wolf Prize, – a lifelong academic, serving decades as a professor at Moscow State University and, after immigrating to the United States shortly before his 76th birthday, at Rutgers University.

**Werner Karl Heisenberg** (1901 – 1976) [Hei69]. Is the Heisenberg of the Heisenberg uncertainty principle for the operators  $P$  (momentum) and  $Q$  (position), and of matrix mechanics, as the first mathematical formulation of

quantum observables. In Heisenberg's picture, the dynamics, the observables are studied as function of time; by contrast to Schrödinger's model which have the states (wave-functions) functions of time, and satisfying a PDE wave equation, now called the Schrödinger equation. In the late 1920ties, the two pictures, that of Heisenberg and of Schrödinger were thought to be irreconcilable. Work of von Neumann in 1932 demonstrated that they in fact are equivalent.

W. Heisenberg; one of the key creators of quantum mechanics. A 1925 paper was a breakthrough. In the subsequent series of papers with Max Born and Pascual Jordan, this matrix formulation of quantum mechanics took a mathematical rigorous formulation. In 1927 he published his uncertainty principle. Heisenberg was awarded the Nobel Prize in Physics for 1932 "for the creation of quantum mechanics." He made important contributions to the theories of the hydrodynamics of turbulent flows, the atomic nucleus.

**David Hilbert** (1862 – 1943) [Hil24, Hil22, Hil02]. Cited in connection with the early formulations of the theory of operators in (what is now called) Hilbert space. The name Hilbert space was suggested by von Neumann who studied with Hilbert in the early 1930ties, before he moved to the USA. (The early papers by von Neumann are in German.)

D. Hilbert is recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. Discovered and developed invariant theory and axiomatization of geometry. In his 1900 presentation of a collection of research problems, he set the course for much of the mathematical research of the 20th century.

**Kiyoshi Itô** (1915 – 2008) [Itô07, Itô04]. Cited in connection with Brownian motion, Itô-calculus, and stochastic processes. Making connection to functional analysis via the theory of semigroups of operators (Hille and Phillips.)

**Richard V. Kadison** [KS59, KR97a] (1925 – ) . . . known for his contributions to the study of operator algebras. Is the "K" in the Kadison-Singer problem (see [MSS15]); and the "K" in the Fuglede-Kadison determinant. He is a Gustave C. Kuemmerle Professor in the Department of Mathematics of the University of Pennsylvania; was awarded the Leroy P. Steele Prize for Lifetime Achievement, in 1999.

**Shizuo Kakutani** (1911 – 2004). Of his theorems in functional analysis, there is the Kakutani fixed-point theorem (a generalization of Brouwer's fixed-point theorem); – with such applications as to the Nash equilibrium in game theory. Also notable is his solution of the Poisson equation using the methods of stochastic analysis; as well as his pioneering advances in our understanding of two-dimensional Brownian motion; and its applications to PDE, and to potential theory.

**Mark Grigorievich Krein** [Kre46, Kre55] (1907–1989). Is the Krein or Krein-Milman on convex weak \*-compact sets. Soviet mathematician; known for pioneering works in operator theory, mathematical physics, the problem of moments, functional and classical analysis, and representation theory. Winner

of the Wolf Prize, 1982. His list of former students includes David Milman, Mark Naimark, Izrail Glazman, Moshe Livshits.

**Peter David Lax** (1926 – ). The “L” in Lax-Phillips scattering theory, and the Lax-Milgram lemma. A pioneer in PDE, and in many areas of applied mathematics; – especially as they connect to functional analysis.

**George Whitelaw Mackey** (1916 – 2006). The first “M” in the “Mackey-machine,” a systematic tool for constructing unitary representations of Lie groups, as induced representations. A pioneer in non-commutative harmonic analysis, and its applications to physics, to number theory, and to ergodic theory.

**Edward (Ed) Nelson** (1932 – 2014) [Nel69, Nel59a]. Cited in connection with spectral representation, and Brownian motion.

**Ralph Saul Phillips** (1913 – 1998) [LP89]. Cited in connection with the foundations of functional analysis, especially the theory of semigroups of bounded operators acting on Banach space.

**Frigyes Riesz** (1880 – 1956) made fundamental contributions to functional analysis, and to the theory of operators in Hilbert space. We frequently use his Riesz representation theorem. He also did some of the fundamental work, developing functional analysis for applications, especially to spectral theory, and ergodic theory; both important in physics. And with his brother, Marcel Riesz, work in harmonic analysis.

**Marcel Riesz** (1886 – 1969), born in Hungary, was the younger brother of the mathematician Frigyes Riesz (the two are known for the F. and M. Riesz theorem). Both are pioneers in Functional Analysis. M. Riesz moved to Sweden in 1911 where he taught at Stockholm University and at Lund University. His former students include Harald Cramér, Einar Hille (of Hille-Phillips), Otto Frostman (potential theory), Lars Hörmander (PDE), and Olaf Thorin (harmonic analysis).

**Erwin Rudolf Josef Alexander Schrödinger** (1887 – 1961) [Sch99, Sch40, Sch32]. Is the Schrödinger of the Schrödinger equation; the PDE which governs the dynamics of quantum states (as wave-functions).

E. Schrödinger, a Nobel Prize in physics. – quantum theory forming the basis of wave mechanics: he formulated the wave equation (stationary and time-dependent Schrödinger equation) , and he Schrödinger proposed an original interpretation of the physical meaning of the wave function; formalized the notion of entanglement. He was critical the conventional Copenhagen interpretation of quantum mechanics (using e.g. the paradox of Schrödinger’s cat).

**Karl Hermann Amandus Schwarz** (1843 – 1921) [Sch70]. Is the Schwarz of the Cauchy-Schwarz inequality. H.A. Schwarz is German and is a contemporary of K. Weierstrass.

H.A. Schwarz, a German mathematician, known for his work in complex analysis. At Göttingen, he pioneered of function theory, differential geometry and the calculus of variations.



**Laurent-Moïse Schwartz** (1915 – 2002) [Sch95, Sch58, Sch57]. Is the Schwartz (French) of the theory of distributions (dating the 1950ties), also now named “generalized functions” in the books by Gelfand et al. Parts of this theory were developed independently on the two sides of the Iron-Curtain;– in the time of the Cold War.

**Jacob Theodore "Jack" Schwartz** (1930 – 2009) [DS88b, DS88c, DS88a]. Is the Schwartz of the book set “linear operators” by Dunford and Schwartz. Vol II [DS88c] is one of the best presentation of the theory of unbounded operators.

**Irving Ezra Segal** (1918 – 1998) [Seg50]. Cited in connection with the foundations of functional analysis, and pioneering research in mathematical physics. Is the “S” in GNS (Gelfand-Naimark-Segal). Segal proved the Plancherel theorem in a very general framework: locally compact unimodular groups. For any locally compact unimodular group, Segal established a Plancherel formula; see [Seg50]. Segal showed that there is a Plancherel formula, despite the fact that it may not be feasible, for all locally compact unimodular groups, to “write down” all the irreducible unitary representations.

**Isadore Manuel Singer** (1924 – ) is an Institute Professor at the Massachusetts Institute of Technology; He is the “S” in the Atiyah–Singer index theorem (1962), Michael Atiyah is the “A.” Also of note: The Atiyah–Hitchin–Singer theorem, and The Atiyah–Patodi–Singer eta-invariant.

**Marshall Harvey Stone** (1903 – 1989) [Sto51, Sto90]. Is the “S” in the Stone-Weierstrass theorem; and in the Stone-von Neumann uniqueness theorem (see [vN32b, vN31]); the latter to the effect that any two representations of Heisenberg’s commutation relations in the same (finite!) number of degrees of freedom are unitarily equivalent. Stone was the son of Harlan Fiske Stone, Chief Justice of the United States in 1941-1946. Marshall Stone completed a Harvard Ph.D. in 1926, with a thesis supervised by George David Birkhoff. He taught at Harvard, Yale, and Columbia University. And he was promoted to a full Professor at Harvard in 1937. In 1946, he became the chairman of the Mathematics Department at the University of Chicago. His 1932 monograph titled “Linear transformations in Hilbert space and their applications to analysis” develops the theory of selfadjoint operators, turning it into a form which is now a central part of functional analysis. Theorems that carry his name: The Banach-Stone theorem, The Glivenko-Stone theorem, Stone duality, The Stone-Weierstrass theorem, Stone’s representation theorem for Boolean algebras, Stone’s theorem for one-parameter unitary groups, Stone-Čech compactification, and The Stone-von Neumann uniqueness theorem.

**John von Neumann** (1903 – 1957) [vN31, vN32a]. Cited in connection with the Stone-von Neumann uniqueness theorem, the deficiency indices which determine parameters for possible selfadjoint extensions of given Hermitian (formally selfadjoint, symmetric) with dense domain in Hilbert space.

J. von Neumann, Hungarian-American; inventor and polymath. He made major contributions to: foundations of mathematics, functional analysis, er-

godic theory, numerical analysis, physics (quantum mechanics, hydrodynamics, and economics (game theory), computing (von Neumann architecture, linear programming, self-replicating machines, stochastic computing (Monte-Carlo<sup>1</sup>)), – was a pioneer of the application of operator theory to quantum mechanics, a principal member of the Manhattan Project and the Institute for Advanced Study in Princeton. – A key figure in the development of game theory, cellular automata, and the digital computer.

**Norbert Wiener** (1894 – 1964) [Wie53, WS53]. Cited in connection with Brownian motion, Wiener measure, and stochastic processes. And more directly, the “Wiener” of Paley-Wiener spaces; – at the crossroads of harmonic analysis and functional analysis. Also the Wiener of filters in signal processing; high-pass/low-pass etc.

Der skal et par dumheder  
 med i en bog ....  
 for at også de dumme  
 skal syns, den er klog.  
 — Piet Hein.

*Translation:*

*Your book should include a few stupidities  
 mixing them in, – this is art.  
 so that also the stupid will think it is smart.*

---

<sup>1</sup>“Monte-Carlo” means “simulation” with computer generated random number.

## Appendix D

# Prizes and Fame

### **Nobel Prize in Physics:**

- N. Bohr, M. Born, P. Dirac, A. Einstein, W. Heisenberg, E. Wigner.

### **Fields Medal (Math):**

- Sir Michael Atiyah, A. Connes, D. Mumford.

### **The National Medal of Science:**

- K. Friedrichs, P. Lax, I.M. Singer, N. Wiener, E. Wigner.

### **The Wolf Prize:**

- I. Gelfand, K. Itō, A.N. Kolmogorov, M.G. Krein, P. Lax.

### **The Abel Prize:**

- Sir Michael Atiyah, P. Lax, I.M. Singer.

### **Presidential Medal of Freedom:**

- J. von Neumann.

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