Calculus Concentrate

Russell A. Gordon

Whitman College

Preface

There are many calculus books currently available on the market. Most of these are thorough, large, and expensive and, more or less, have the same table of contents. These books are useful to have around as references for those few people who actually use calculus on a regular basis. However, they are not necessarily beneficial to a student who is going through calculus for the first or second time. Students are easily overwhelmed by the amount of information, they suffer backaches from hauling around a heavy book, and sticker shock is inevitable at the beginning of the term. Calculus books have become so large because they are chosen by committee and mathematicians rarely agree on the proper way to teach calculus. There is no "right" way to teach calculus, but each person who teaches calculus has strong opinions concerning how it should be taught. In order for a calculus book to be marketable, it has to appeal to a wide variety of people by including their favorite topics and styles of presentation. Areas of disagreement include topics to cover, depth of coverage of each topic, variety of applications to other disciplines, amount of rigor, use of technology, and the number and type of examples and exercises. To satisfy many different people, the books of necessity become very large. The electronic text that follows is much shorter because it represents one person's attempt to put the essential ideas of calculus into a short and concise format. It may not appeal to a wide range of mathematicians, but it should provide most students with a good foundation in calculus.

A quick perusal of the text will reveal some of its key features. Almost every section is two pages long and the pages are arranged so that (when printed) an entire section appears when the book is held open. To keep the sections short, the number of examples and amount of discussion is kept to a minimum. The book is not intended to be a set of lecture notes, but rather a framework upon which a lecture can be constructed. Since the sections are short, the number of exercises is also limited. There are enough exercises to give most students good experience with the concepts in each section. Further exercises can be found at the end of each chapter. If more exercises are needed, there are many sources of problems available in other textbooks and at various sites on the Internet. The order and selection of topics is a bit different from standard books. Rather than justify each of these differences, I will just say that I have thought carefully about each topic and each sentence that appears in the book. I hope that a teacher who uses the book will be able to appreciate some of the guiding principles that have shaped it.

A Note to the Student

The title of this book, *Calculus Concentrate*, carries a double meaning depending on the definition of the word "concentrate". One meaning is similar to its use on frozen juice containers. Instead of adding water and stirring, you will need to add thought and contemplation. As you read the short sections, and it is highly recommended that you do so, think carefully about each sentence and each mathematical equation. Stop if necessary and do some calculations or spend some time absorbing the ideas. If you are still confused, get some assistance. Reading the book is where the more common meaning of concentrate comes into play; you will need to think hard while working through the book.

Many of the problems in elementary mathematics books can be solved by imitating the examples in the text rather than understanding the concepts. While learning skills in this way is useful, an important aspect of problem solving is knowing which skills to use and combining several skills in a multi-step problem. Learning to solve non-routine problems, those without an example to imitate and requiring more than one step, is one of the goals of this text. These sorts of problems require more time and effort and can be frustrating, but the satisfaction of solving such a problem is much greater than simply imitating the solution to a similar problem. Keep this in mind as you work the problems in this text. If you are not sure how to start a problem, review the ideas in the section and think about what you do know about the concepts mentioned in the problem.

Acknowledgments

Although the organization and wording of this text are my creations, there is very little original material to be found here. I have drawn on my 30 years experience teaching calculus and sources too numerous to mention. Conversations with colleagues have also impacted certain aspects of this work. David Guichard provided technical support for the preparation of the manuscript. A grant from the Stanley Rall summer research scholarship fund allowed Pat Cade to spend a summer reading the text and working through the exercises. However, it is inevitable that there will be some difficulties and errors in the text. Most of these will probably occur in the solutions section as sometimes the exercises were modified more quickly than the solutions. Feel free to contact me at the mathematics department of Whitman College, Walla Walla, WA 99362 or at gordon@whitman.edu with questions and comments.

Contents

A Very Brief History of Calculus

Calculus has its roots in two geometric problems: determining the areas of regions with curved boundaries and finding tangents to curves other than circles. Using geometric techniques, the ancient Greeks were able to solve many such problems for the conic sections and some other easily defined curves. An increased interest in these problems occurred during the Renaissance and it became evident that these problems had applications to physical problems. In the middle of the seventeenth century, René Descartes (1596–1650) and Pierre de Fermat (1601–1665) independently developed the x-y coordinate system. Although we take this idea for granted now, it was actually quite a leap at the time. The Cartesian coordinate system (named in honor of Descartes in spite of the fact that Fermat's approach was closer to the one we use today) provides a link between geometry and algebra. Geometric curves have algebraic equations and algebraic equations generate geometric curves. This connection between geometry and algebra makes it possible to use algebraic techniques to solve geometric problems; this area of mathematics is known as analytic geometry. Algebra provided new proofs of well-known geometry facts and it also led to the discovery of new results in geometry. More importantly, algebraic equations yielded a whole host of new curves to study. Previously, curves were defined by geometry or by the trajectory of a moving point, but now almost every algebraic equation provided another curve to study. The analysis of these curves using analytic geometry paved the way for the discovery of calculus.

During the time period 1640–1670, a number of mathematicians came tantalizingly close to discovering a connection between the area problem and the tangent problem. While working out various problems in physics, primarily the theory of gravitation, Isaac Newton (1642–1727) studied infinite series and through them saw how the area and tangent problems were connected. The date of this discovery is generally taken to be 1665–1666 when Newton was home from college due to an outbreak of the plague. Gottfried Leibniz (1646–1716) independently discovered this connection in the 1670's and formulated much of the notation for calculus that is in use today. He wrote the first published paper on calculus and it appeared in 1684. (Unfortunately, Newton's lack of formal publishing of his results led to a bitter dispute over priority of credit for discovering calculus.) This recognition of the connection between areas and tangents in the last part of the seventeenth century is taken to be the origin of calculus. The first calculus text, written by Guillaume de L'Hôpital (1661–1704) appeared in 1696. During these years and into the next century, there was an explosion of mathematical ideas and calculus led to the solutions of many different physical and geometric problems. The fact that many problems in the physical sciences can be reduced to finding areas or tangents has made calculus the cornerstone of the scientific revolution.

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The Derivative

Consider the linear function $y = 7x + 5$. The graph of this function is a line with slope 7 and y-intercept 5. If the variable x increases by 1, then the variable y increases by 7. This statement is valid whether x increases from 3 to 4 or from 150 to 151. It is easy to see why this is true; if x increases from a to $a + 1$, then the change in y is

$$\big(\text{value of } y \text{ when } x \text{ is } a+1\big) - \big(\text{value of } y \text{ when } x \text{ is } a\big) = \big(7(a+1)+5\big) - \big(7a+5\big) = 7.$$

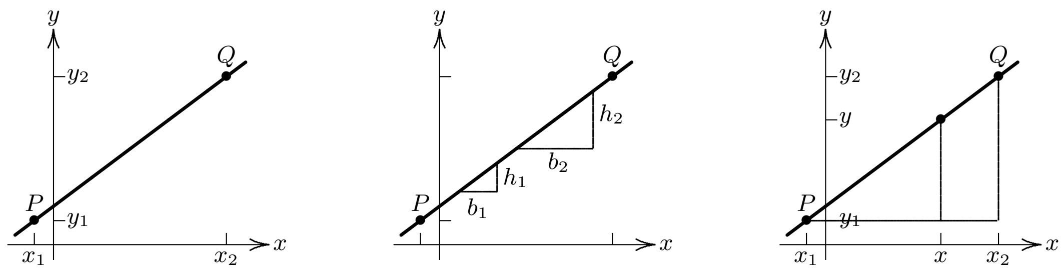
Thus the slope of the line gives a measure of the rate of change of y with respect to x. For the function $y = x^2$, if x increases from 3 to 4, then y increases by $4^2 - 3^2 = 7$, and if x increases from 150 to 151, then y increases by $151^2 - 150^2 = 301$. In this case, the rate of change of y depends on the original value of x; if x increases from a to $a + 1$, then the change in y is

$$(a+1)^2 - a^2 = 2a + 1.$$

In general, for functions that are not linear, the rate of change of y with respect to x depends on the original value of x, that is, the rate of change of y with respect to x is variable. Differential calculus provides a way to precisely measure the rate of change of y with respect to x for an arbitrary function. Since many physical quantities are rates of change (velocity is the rate of change of distance with respect to time, acceleration is the rate of change of velocity with respect to time, etc.), differential calculus has a variety of useful applications. In this lengthy chapter, we will define the derivative, develop formulas for computing derivatives, and explore some of the applications of the derivative.

1.1 Lines

Two distinct points in the plane determine a line. Consider the line through the points P and Q with coordinates (x_1, y_1) and (x_2, y_2), respectively. Assume that $x_1 \neq x_2$. Since the right triangles drawn on the middle graph in the figure below are similar, the ratios h_1/b_1 and h_2/b_2 are equal.



This shows that the change in y (often denoted by Δy) over the change in x (Δx) is the same for any two pairs of points chosen on the line. The constant value, usually denoted by m, is referred to as the **slope** of the line and can be determined by the formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

For a point (x, y) (other than (x_1, y_1)) to be on this line (see the graph on the right), we must have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1).$$

This form for the equation of a line is called the **point-slope form** of the line. The equation of a line is often written in the **slope-intercept** form $y = mx + b$, where m is the slope and b is the y-intercept of the line. Another useful formula for a line is $Ax + By + C = 0$, where A, B, and C are constants. This form allows for the possibility of vertical lines; a vertical line occurs when $B = 0$ and $A \neq 0$.

The slope of a line represents the rate of change of y with respect to x. If x changes by an amount Δx and the slope of the line is m, then y changes by an amount $\Delta y = m\Delta x$:

$$\Delta y = \big(\text{value of } y \text{ at } x + \Delta x\big) - \big(\text{value of } y \text{ at } x\big) = \big(m(x + \Delta x) + b\big) - \big(mx + b\big) = m\Delta x.$$

This is true no matter what the original value of x is; the change in y is always a constant multiple of the change in x. For general curves, the change in y with respect to x varies from point to point and we will later determine how to find this variable rate of change. For now, it is important to understand that a given line has the same slope at each point.

Two distinct lines are said to be **parallel** if they have no point in common. It is easy to verify that two distinct lines are parallel if and only if they have the same slope. Two distinct lines that are not parallel have exactly one point of intersection. Two intersecting lines form a unique acute or right angle. If the angle formed by the two intersecting lines is a right angle, the lines are said to be **perpendicular**. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1. You should be able to prove this fact using the Pythagorean Theorem.

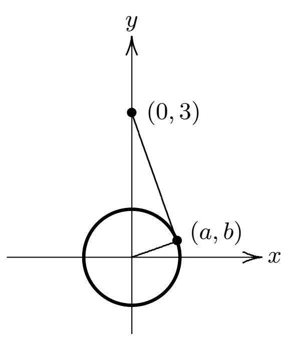
Problem: Find an equation for the line that goes through $(3,7)$ and is parallel to the line $2x+y-5=0$.

Solution: We first put the given line in slope-intercept form by solving the equation $2x+y-5=0$ for y to obtain $y=-2x+5$. The slope of the given line is thus -2. Since parallel lines have the same slopes, the requested line has a slope of -2 also. Using the point-slope form for the equation of a line, the equation of the line through $(3,7)$ and parallel to $2x+y-5=0$ is

$$y-7=-2(x-3) \quad \text{or} \quad y=-2x+13.$$

Problem: Find the points on the circle $x^2+y^2=1$ where the tangent line to the circle goes through $(0,3)$.

Solution: Let (a,b) be a point on the circle with the desired property. The solution to this problem hinges on the following fact concerning tangent lines to circles: the tangent line is perpendicular to the radial line. In the figure below, the line through (a,b) and the origin is perpendicular to the line through (a,b) and $(0,3)$. This means that the product of the slopes of these two lines is -1. The resulting equation, along with the equation $a^2+b^2=1$, makes it possible to find a and b. The computations are shown by the figure.



$$\begin{aligned} \frac{b}{a}\cdot\frac{b-3}{a-0}&=-1 \quad \Rightarrow \\ b^2-3b&=-a^2 \quad \Rightarrow \\ 3b&=a^2+b^2=1 \quad \Rightarrow \\ b&=\frac{1}{3} \quad \Rightarrow \\ a&=\pm\frac{\sqrt{8}}{3}. \end{aligned}$$

The two points on the circle for which the tangent line goes through $(0,3)$ are thus $\left(\frac{\sqrt{8}}{3},\frac{1}{3}\right)$ and $\left(-\frac{\sqrt{8}}{3},\frac{1}{3}\right)$.

Exercises

1. Find an equation for the line that goes through the points $(-1,2)$ and $(3,8)$.
2. Let ℓ be the line $2x+3y=6$ and let P be the point $(3,-2)$. Find an equation for the line through P that is parallel to ℓ and an equation for the line through P that is perpendicular to ℓ.
3. Find an equation for the perpendicular bisector of the segment joining the points $(-1,2)$ and $(3,8)$.
4. Find the slope, x-intercept, and y-intercept of the line $4x+3y=24$, then find its distance from the origin.
5. Find the point of intersection of the lines with equations $2x+y-5=0$ and $3x-4y-2=0$.
6. Let m be a negative number and let ℓ be the line through $(2,3)$ with slope m. The line ℓ cuts off a triangle in the first quadrant. Find the area of this triangle as a function of m.
7. Consider the trapezoid with vertices at $(0,0)$, $(3,2)$, $(6,2)$, and $(6,0)$. Find a line that goes through the origin and divides the trapezoid into two parts of equal area.
8. There are two points on the circle $x^2+y^2=1$ for which the tangent line passes through the point $(7,1)$. Find the (x,y) coordinates of both these points.

1.2 Numbers, Sets, and Functions

It is not possible to do mathematics without some reference to numbers. The **natural numbers** are the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$, which are used for counting objects. The **integers** are the numbers $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$, which extend indefinitely in either direction. In this context, the natural numbers are referred to as the **positive integers**. We will use the symbol $\mathbb{Z}$ to denote the set of integers and the symbol $\mathbb{Z}^+$ to denote the set of positive integers. When numbers are used for measurement, it is sometimes necessary to consider parts of a whole. This need for parts or fractions of an integer leads to the concept of a rational number. A **rational number** is a number of the form p/q, where p and q are integers with $q \neq 0$. In other words, a rational number is a ratio of two integers. The standard symbol for the set of rational numbers is $\mathbb{Q}$, where the use of the letter Q follows from the fact that rational numbers are quotients. The number $3/5$ can be interpreted as follows: cut a string into 5 equal pieces and take 3 of the pieces. The rational numbers include the integers since an integer n can also be represented as $n/1$. Since the rational numbers are the familiar numbers that arise in everyday life, a great deal of elementary school mathematics is devoted to a study of these numbers. However, not every number can be expressed as the ratio of two integers; the square root of 2 is one example. Such numbers are referred to as **irrational numbers**. One difference between the rational numbers and irrational numbers can be noted in their decimal expansions; the decimal expansion of a rational number has a repeating pattern whereas the decimal expansion of an irrational number does not. The rational and irrational numbers together form the set of **real numbers**, which is denoted by the symbol $\mathbb{R}$.

The reader should have some previous experience with sets. A **set** is loosely defined to be a collection of objects. The objects in a set usually have some features in common, such as the set of integers or the set of real numbers. The objects in a set are called **elements** or **points** in the set. If x belongs to a set S, then we write $x \in S$ and say "x is an element of S", "x belongs to the set S", or "x is a point in S". A set A is a **subset** of a set B, denoted $A \subseteq B$, if each element of A belongs to B. Two sets A and B are **equal** if and only if $A \subseteq B$ and $B \subseteq A$, that is, the sets A and B have the same elements. A set that has no elements is called the **empty set**, denoted by the symbol $\emptyset$. There are various ways to represent sets of numbers. For example, $\{1, 2, 3, \ldots, 98, 99\}$ and $\{n \in \mathbb{Z}^+ : n < 100\}$ both represent the set of all positive integers less than 100. The first representation is merely a listing of the elements of the set. The second representation is read as "the set of all positive integers n such that $n < 100$". In calculus, the most common type of set is an interval. An **interval** (of real numbers) has one of the following nine forms:

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} : a < x < b\}; & (-\infty,a) &= \{x \in \mathbb{R} : x < a\}\\
[a,b) &= \{x \in \mathbb{R} : a \le x < b\}; & (-\infty,a] &= \{x \in \mathbb{R} : x \le a\};\\
(a,b] &= \{x \in \mathbb{R} : a < x \le b\}; & (b,\infty) &= \{x \in \mathbb{R} : x > b\};\\
[a,b] &= \{x \in \mathbb{R} : a \le x \le b\}; & [b,\infty) &= \{x \in \mathbb{R} : x \ge b\};\\
& & (-\infty,\infty) &= \mathbb{R}\,.
\end{aligned}$$

The intervals in the first column are bounded intervals and those in the second are unbounded intervals; an unbounded interval contains numbers that are arbitrarily large. Note how the symbols (and [are used to denote whether or not the endpoint is included in the interval; $[a, b]$ is a closed interval since it contains both its endpoints and (a, b) is an open interval since it contains neither of its endpoints.

The objects of study in calculus are functions. A function provides a relationship between two or more quantities. The relationship can be exhibited with a graph, a table of values, a verbal description, or a formula. For the most part, calculus deals with the situation in which the function is given by a formula. Once again, it is assumed that the reader has some experience with functions. Let A and B be two nonempty sets. A **function** $f\colon A \to B$ is a rule of correspondence that assigns to each element of A exactly one element of B. If f assigns the element b of B to the element a of A, then we write $f(a) = b$. We say that b is the value of f at a or that b is the output that corresponds to the input a. In this class, we will be concerned with functions of the form $f\colon I \to \mathbb{R}$, where I is an interval of real numbers and $\mathbb{R}$ is the set of all real numbers. The rule of correspondence will almost always be given by a formula. For example, we can define a function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. This function assigns the real number x^2 to the real number x. It is important to understand the subtle distinction between f and $f(x)$; x represents the input, f is the function, and $f(x)$ is the output. It may help to think of f as a machine that accepts inputs x and turns them into outputs $f(x)$. However, the machine f is not the same as the output $f(x)$.

The **domain** of a function f is the set $\{x \in \mathbb{R} : f(x) \text{ is defined}\}$, that is, the set of inputs for which $f(x)$ makes sense. The **range** of a function f is the set $\{y \in \mathbb{R} : y = f(x) \text{ for some } x\}$, that is, the set of all possible outputs of the function. Finally, the **graph** of f is the set $\{(x, y) : y = f(x)\}$, the collection of all ordered pairs that satisfy the equation. A graph of the equation $y = f(x)$ provides an amazing link between algebra and geometry; an algebraic equation takes on a geometric shape. You should be able to sketch the graphs of simple functions such as $y = x^2$, $y = 1/x$, and $y = \sin x$ quickly and without a calculator as well as modify them slightly by translation and scaling.

Let n be a positive integer. A **polynomial** P of **degree** n is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are fixed real numbers and $a_n \neq 0$. The numbers a_i are known as the **coefficients** of P. A nonzero constant function, a function that has the same value at every point, is referred to as a polynomial of degree 0. For the record, a **root** of a polynomial is a real number r such that $P(r) = 0$. A **rational function** is a ratio of two polynomials. The collection of all rational functions includes all polynomials since the constant function 1 is a polynomial. The determination of the value of a rational function at a particular real number involves only the operations of addition, subtraction, multiplication, and division. An **algebraic function** is a function that involves these four operations along with the process of finding roots. For example, the function g defined by $g(x) = \sqrt[3]{4x^2 + 5x - 1}$ is an algebraic function. The five operations used to define algebraic functions are all familiar to students who have studied high school algebra, and, if necessary, simple examples of these operations can be performed by students without a calculator.

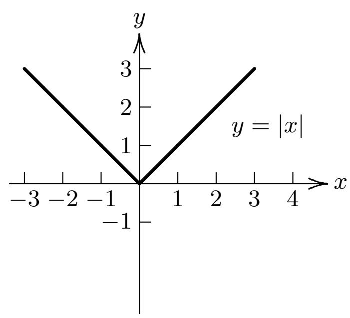
In addition to the algebraic functions, there are many other functions considered in algebra and calculus. The most familiar ones are **trigonometric functions**, **inverse trigonometric functions**, **exponential**

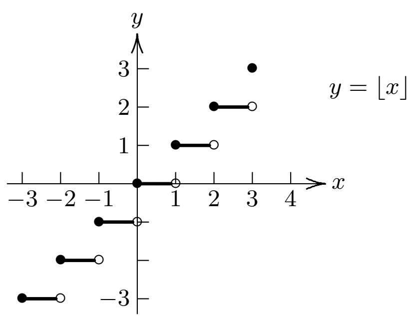
functions, and **logarithmic functions**. These functions are known as **transcendental functions** because the evaluation of these functions "transcends" the algebraic operations. There will be brief reviews of these functions later in the text, but it is assumed that the reader has worked with them in the past.

We will also make use of two less familiar functions; the absolute value function and the greatest integer function. Given a real number x, the **absolute value** of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. The number $|x|$ can be interpreted as the distance on the number line from the point x to the point 0. It follows that $|a - b|$ represents the distance between the numbers a and b. The **greatest integer function**, denoted by $\lfloor x \rfloor$, represents the greatest integer that is less than or equal to x. For example, $\lfloor 8.62 \rfloor = 8$, $\lfloor \pi \rfloor = 3$, and $\lfloor -4.1 \rfloor = -5$. The graphs of these two functions on the interval $[-3, 3]$ are given below.





Given two functions f and g, it is possible to combine them in several ways to obtain new functions:

addition :	$f + g$ is defined by $(f + g)(x) = f(x) + g(x)$
subtraction :	$f - g$ is defined by $(f - g)(x) = f(x) - g(x)$
multiplication :	fg is defined by $(fg)(x) = f(x)g(x)$
division :	f/g is defined by $(f/g)(x) = f(x)/g(x)$, assuming that $g(x) \neq 0$
composition :	$f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$

Pay particular attention to the notation for these combinations of functions.

Problem: Find the domain of each of the functions f, g, and h, where

$$f(x) = \sqrt{9 - x^2}, \quad g(x) = \frac{x}{\sqrt{9 - x^2}}, \quad \text{and} \quad h(x) = \sqrt[3]{9 - x^2}.$$

Solution: Since the square root of a negative number is not a real number, the domain of the function f consists of those values of x for which $9 - x^2 \geq 0$. Thus, the domain of f is the closed interval $[-3, 3]$. The domain of g is almost the same as the domain of f, but in this case we must also avoid division by 0. It follows that the domain of g is the open interval $(-3, 3)$. Finally, every real number has a cube root, so the domain of h is $\mathbb{R}$, the set of all real numbers.

Problem: Find the set of real numbers that satisfy the inequality $4 - |x - 1| \geq 1$.

Solution: The inequality can be written as $|x-1| \leq 3$. This inequality means that the distance from x to 1 is at most 3, or equivalently, $-3 \leq x-1 \leq 3$. The solution set of the inequality is thus $\{x \in \mathbb{R} : -2 \leq x \leq 4\}$.

Problem: For $f(x) = x^3 - 3x + 1$, find $f(x-1)$ and $f(x^2)$.

Solution: To find the value of a composite function, simply replace each occurrence of x with the given expression. Doing so yields

$$f(x-1) = (x-1)^3 - 3(x-1) + 1 = x^3 - 3x^2 + 3x - 1 - 3x + 3 + 1 = x^3 - 3x^2 + 3;$$
$$f(x^2) = (x^2)^3 - 3x^2 + 1 = x^6 - 3x^2 + 1.$$

Exercises

1. Express the rational number 4/7 as a repeating decimal.

2. Find the rational number represented by the repeating decimal $0.19191919\ldots$.

3. Let $A = \{0, 1\}$. For each set B, determine whether $A \subseteq B$, $B \subseteq A$, or $A = B$.
a) $B = \{x : x^3 = x\}$ **b)** $B = \{x : x = \sqrt{x}\}$ **c)** $B = \{x : x^5 = -x\}$

4. Find the domain of each of the following functions.
a) $f(x) = \dfrac{2x}{x-5}$ **b)** $g(x) = \sqrt{3 + 2x - x^2}$ **c)** $h(x) = \dfrac{x}{\sqrt{4 - x^2}}$

5. Sketch each of the following graphs. You should be able to sketch these graphs without using a calculator.
a) $y = x^2 + 4$ **b)** $y = (x+4)^2$ **c)** $y = \sqrt{x-2}$
d) $y = |x+3|$ **e)** $y = |2x-5|$ **f)** $y = \dfrac{1}{x-2}$

6. Sketch the graph on the given interval.
a) $y = \lfloor x/2 \rfloor + 3,\ [0, 6]$ **b)** $y = x - \lfloor x \rfloor,\ [-3, 3]$ **c)** $y = \lfloor x^2 - 1 \rfloor,\ [0, 2]$

7. Let $f(x) = x^2 - 3x$, $g(x) = 2x + 1$, and $h(x) = \lfloor x \rfloor$. Evaluate the following.
a) $(f-g)(-1)$ **b)** $(fg)(2)$ **c)** $(f/g)(1)$
d) $(f \circ g)(2)$ **e)** $(g \circ h)(\pi)$ **f)** $(h \circ f)(1/2)$

8. Given $f(x) = x^2 + 4x$ and $g(x) = 2x - 3$, find $(f \circ g)(x)$, $(g \circ f)(x)$, and $(g \circ g)(x)$.

9. Find functions f and g so that $h = f \circ g$.
a) $h(x) = \sqrt{x^4 + 4x^2 + 1}$ **b)** $h(x) = \sin 4x$ **c)** $h(x) = \cos^2 x$

10. Find all of the roots of the polynomial $x^3 + 3x^2 - 2x - 2$. (Find one root by inspection, then use long division.)

11. Find the points of intersection of the graphs $y = x^2 + 4x$ and $y = x + 10$.

12. For the function $f(x) = x^2$, find and simplify the expression $\dfrac{f(x+h) - f(x-2h)}{3h}$.

13. Express the area of a regular hexagon as a function of its perimeter.

14. Write each of the sets as an interval.
a) $\{x \in \mathbb{R} : |x-6| \leq 10\}$ **b)** $\{x \in \mathbb{R} : x^2 + 4x - 5 < 0\}$ **c)** $\{x \in \mathbb{R} : 21 - |2x-5| > 3\}$

15. Sketch a graph of the functions f and g defined by (one graph for each function)

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \leq 2; \\ 3 - 2x, & \text{if } x > 2; \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1/x, & \text{if } x < 0; \\ 3x + 1, & \text{if } x \geq 0. \end{cases}$$

1.3 The Limit Concept

The limit concept is the main idea which separates calculus from algebra. Limits are not easy—time and effort are required to understand them fully—but this concept lies behind the major ideas in calculus. In this section, we will work toward an intuitive understanding of limits.

The symbols $\lim_{x\to c} f(x) = L$ are read as "the limit of $f(x)$ as x approaches c is L". Intuitively, this means that the output values $f(x)$ are close to L when the input values x are close to c but not equal to c. As a simple example, the statement $\lim_{x\to 3} x^2 = 9$ means that if you take numbers near 3, such as 2.99 or 3.004, and square them, you get numbers close to 9. This statement says much more than $3^2 = 9$; it asserts that numbers near 3 have squares near 9 and that the closer x is to 3, the closer x^2 is to 9. This example may make the limit concept seem trivial, but consider the following limits:

$$\lim_{x\to 2}\frac{x-2}{x^2-4}, \qquad \lim_{x\to 1}\frac{\ln x}{x-1}, \qquad \lim_{x\to 0}\frac{10^x-1}{x}, \qquad \text{and} \qquad \lim_{x\to 0}\cos(\pi/x).$$

In each case, the function is undefined at the point in question. However, all that matters is what happens to the outputs when the inputs are close to c, not equal to c. In some cases, algebra can be used to find the exact value of a limit:

$$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}.$$

We will consider this method in more detail in Section 1.5. For other functions, a calculator can be used to provide an estimate of a limit (assuming that the limit exists). If $f(x) = (10^x - 1)/x$, then

$$f(0.01) = 2.3293, \quad f(0.001) = 2.3052, \quad f(0.0001) = 2.3029, \quad f(0.00001) = 2.3026,$$
$$f(-0.01) = 2.2763, \quad f(-0.001) = 2.2999, \quad f(-0.0001) = 2.3023, \quad f(-0.00001) = 2.3026,$$

so it appears that $\lim_{x\to 0}(10^x - 1)/x$ is approximately 2.3026. It is also possible to determine limits from a graph of a function; an example will be left for the exercises.

A function may not have a limit at a point. In this case, we say that $\lim_{x\to c} f(x)$ does not exist. For example, $\lim_{x\to 0}\cos(\pi/x)$ does not exist. To see this, let $f(x) = \cos(\pi/x)$ and note that $f(1/n) = (-1)^n$ for each positive integer n. As n increases, the numbers $1/n$ get closer to 0, but the output values are not getting close to a specific real number. Hence, the limit does not exist. (This fact is also apparent from a graph of the function.)

There are other variations on the limit concept. For example, the symbols $\lim_{x\to c^+} f(x) = L$ are read as "the limit of $f(x)$ as x approaches c from the positive side is L", and they mean that the output values $f(x)$ are close to L when the input values x are close to c but larger than c. This limit is called the right-hand limit of f at c since the input values of f are to the right of c on the number line. The symbols $\lim_{x\to c^-} f(x) = L$, which denote the left-hand limit of f at c, have an analogous interpretation. The left-hand and right-hand limits of f at c are known as one-sided limits since the values of x approach c from only one side. As simple examples, $\lim_{x\to 5^+}\lfloor x\rfloor = 5$ and $\lim_{x\to 5^-}\lfloor x\rfloor = 4$. It should be clear that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$. In particular, if the two one-sided limits are different, then the two-sided limit does not exist.

Some simple and useful properties of limits are listed below. It is possible to prove these rigorously (see Section 1.23 for the definition of a limit), but for now it is sufficient that the results should make intuitive sense. Suppose that f and g have limits at c and that k is a constant. Then

1. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$;

2. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;

3. $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;

4. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$;

5. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, assuming that $\lim_{x\to c} g(x) \neq 0$.

Analogous properties hold for one-sided limits. Note the use of these properties as you read the next few sections.

Exercises

1. Use algebra to evaluate each of the limits.

a) $\lim_{x\to 1} \dfrac{1-x^2}{x-1}$ **b)** $\lim_{x\to -2} \dfrac{x^2-4}{x^2+x-2}$ **c)** $\lim_{x\to 3} \dfrac{x^2-3x}{2x^2-3x-9}$

2. Use a calculator to estimate the limit to the nearest thousandth. Your solution should include a table of values; be sure to check numbers on either side of c. Radian measure is assumed whenever trigonometric functions appear.

a) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ **b)** $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$ **c)** $\lim_{x\to 0} \Big(1+x\Big)^{1/x}$

d) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ **e)** $\lim_{x\to 1} \dfrac{\log_{10} x}{x-1}$ **f)** $\lim_{x\to 0} \dfrac{x}{5^x - 3^x}$

3. Estimate $\lim_{x\to 0} \dfrac{a^x - 1}{x}$ to the nearest thousandth for $a = 2, 3, 4, 5$. Your solution should include a table of values.

4. Evaluate each of the following limits by thinking about the behavior of the functions and/or sketching a graph.

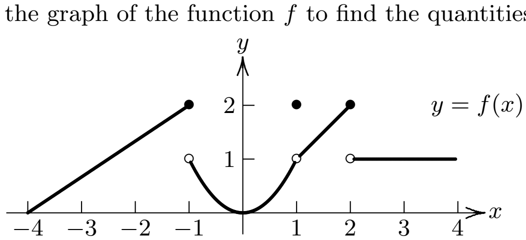
a) $\lim_{x\to 0^-} \dfrac{x}{|x|}$ **b)** $\lim_{x\to 2^-} \Big(x - \lfloor x \rfloor\Big)$ **c)** $\lim_{x\to 4^+} \Big(x - \lfloor x \rfloor\Big)$

d) $\lim_{x\to 5} \Big(\lfloor x \rfloor + \lfloor -x \rfloor\Big)$ **e)** $\lim_{x\to -1^-} \Big(\lfloor x/3 \rfloor - \lfloor x \rfloor\Big)$ **f)** $\lim_{x\to 0^+} \sqrt{x}\sin(\pi/x)$

5. Explain why each of the following limits does not exist.

a) $\lim_{x\to -3} \lfloor x \rfloor$ **b)** $\lim_{x\to 2} \dfrac{x-2}{|x-2|}$ **c)** $\lim_{x\to 0} \sin(2/x)$

6. Use the graph of the function f to find the quantities listed to the right of the graph.



a) $\lim_{x\to -1^-} f(x)$ **b)** $\lim_{x\to -1^+} f(x)$ **c)** $\lim_{x\to 1} f(x)$ **d)** $\lim_{x\to 2^-} f(x)$

e) $\lim_{x\to -4^+} f(x)$ **f)** $\lim_{x\to 3} f(x)$ **g)** $f(1)$ **h)** $f(2.1)$

7. Suppose that $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = 12$. Find each of the following limits.

a) $\lim_{x\to 2}(2f(x) - g(x) - 3x)$ **b)** $\lim_{x\to 2} 3f(x)g(x)$ **c)** $\lim_{x\to 2} \dfrac{4f(x)+1}{g(x)}$

1.4 Continuous Functions

When asked for a definition of a continuous function, most students say that it is a function whose graph has no breaks in it. Although it has intuitive merit, it should be clear that this visual representation of continuity is not suitable as a mathematical definition. To drive home this point, remember that a graph, even a computer-generated graph, is drawn by plotting some points and sketching a curve through those points. A sketch created in this fashion assumes in advance that the function is continuous. To use the visual idea to obtain a mathematical definition, note that a break in a graph occurs at some point on the graph. At such a point, something goes wrong with the limit of the function.

DEFINITION 1.1 A function f is **continuous** at a number c if $\lim_{x \to c} f(x) = f(c)$.

Continuous functions are predictable in the sense that inputs near c generate outputs near $f(c)$. The definition can be broken into three parts: (i) the function is defined at c, (ii) the function has a limit at c, and (iii) the value of the limit and the value of the function are the same. It follows that a function can fail to be continuous in one of three ways. These are illustrated by the following functions:

$$f(x) = \frac{2^x - 1}{x}, \quad g(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0; \end{cases} \quad \text{and} \quad h(x) = \lfloor x \rfloor + \lfloor -x \rfloor.$$

Each of these functions is not continuous at 0; f is not defined at 0 (but it does have a limit at 0), g does not have a limit at 0, and the limit of h at 0 (which is -1) does not equal $h(0)$.

The next two theorems can be proved using the definition of continuity and the properties of limits. The details will be omitted, but several of the exercises provide an indication of the nature of the proofs.

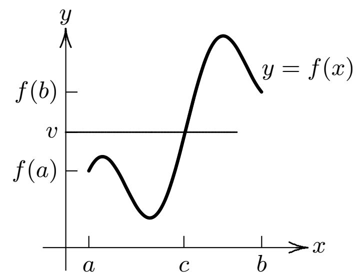
THEOREM 1.2 Let f and g be continuous functions and let k be a constant. Then the functions kf, $f + g$, $f - g$, fg, f/g (assuming $g \neq 0$), and $f \circ g$ are continuous functions. ■

THEOREM 1.3 Polynomials, rational functions, algebraic functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions and the absolute value function are continuous at all points for which they are defined. The greatest integer function has a discontinuity at each integer. ■

These two theorems indicate that most of the familiar functions are continuous at every point for which they are defined. We conclude this section with a very useful property of continuous functions.

THEOREM 1.4 **Intermediate Value Theorem** If $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and v is any number between $f(a)$ and $f(b)$, then there is a number c in (a, b) such that $f(c) = v$. ■

The Intermediate Value Theorem has a simple geometric interpretation. The graph of $y = v$ is a horizontal line that has a y-intercept that lies between $f(a)$ and $f(b)$. As indicated in the figure, the graph $y = f(x)$ of a continuous function that starts at the point $(a, f(a))$ and ends at the point $(b, f(b))$ must cross this horizontal line at some point c between a and b. Thinking of a continuous function as one whose graph can be sketched without raising the pencil makes this result plausible, but a proof using the definition of continuity involves ideas we have not discussed and will be omitted.



As a simple illustration of this theorem, we will prove that the equation $x^3 = 4x + 5$ has a solution. Let f be defined by $f(x) = x^3 - 4x - 5$. Since it is a polynomial, the function f is continuous on the interval $[2, 3]$. Note that $f(2) = -5$ and $f(3) = 10$. By the Intermediate Value Theorem, there exists a number c in $(2, 3)$ such that $f(c) = 0$. Since $0 = f(c) = c^3 - 4c - 5$, the number c is a solution to the equation $x^3 = 4x + 5$.

Exercises

1. List the discontinuities of the given function.

a) $f(x) = \dfrac{2x+3}{x^2-4x-5}$ **b)** $g(x) = \dfrac{2+\sin x}{x^3-6x}$ **c)** $h(x) = \dfrac{x^2+4}{2^x-1}$

2. Sketch the graph of a single function f with all the following properties: (i) f is defined on $[0, 5]$, (ii) f is continuous except at 1 and 4, (iii) f has different one-sided limits at 1, and (iv) f has a limit at 4.

3. Find a value for the constant a for which the given function is continuous for all real numbers. Explain your reasoning.

a) $f(x) = \begin{cases} \dfrac{4-x^2}{x-2}, & \text{if } x \neq 2; \\ a, & \text{if } x = 2. \end{cases}$ **b)** $g(x) = \begin{cases} ax^2, & \text{if } x \leq 3; \\ 4-ax, & \text{if } x > 3. \end{cases}$ **c)** $h(x) = \begin{cases} 2x^2, & \text{if } x \leq a; \\ x+3, & \text{if } x > a. \end{cases}$

4. Prove that the function $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$ is not continuous at any integer.

5. Consider the function f defined by $f(x) = \begin{cases} 4-x^2, & \text{if } x \leq 2; \\ x+2, & \text{if } x > 2. \end{cases}$ Show that f satisfies the conclusion of the Intermediate Value Theorem on the interval $[0, 5]$, but not on the interval $[1, 5]$. (Remember to check every value of v between $f(a)$ and $f(b)$.)

6. Prove that the polynomial $x^3 - 3x^2 - 100$ has a root. Find an interval of length one in which the root lies.

7. Prove that there is a number that is exactly 4 less than its cube and another number that is exactly 5 more than its cube.

8. Prove that the equation $x^4 = 3^x$ has two positive solutions.

9. Suppose that $f\colon [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and that $f(0) = f(2)$. Prove there is a point $c \in [0, 1]$ such that $f(c+1) = f(c)$. *Hint*: Consider the function g defined by $g(x) = f(x+1) - f(x)$ on the interval $[0, 1]$.

10. Suppose that f and g are continuous functions. Use the definition of continuity and the properties of limits listed at the end of Section 1.3 to prove that the functions $f + g$ and fg are continuous.

11. Let f be defined by $f(x) = C$, where C is a constant, and let g be defined by $g(x) = x$. Explain (using the definition) why f and g are continuous functions. Then use these functions and Theorem 1.2 to show that the polynomial P defined by $P(x) = 2x^2 - 3x + 4$ is continuous. The general result for polynomials is proved in a similar way.

12. Explain how the continuity of polynomials shows that rational functions are continuous at every point for which they are defined.

1.5 Evaluating Limits Algebraically

The limit of a continuous function is easy to evaluate. If f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$; the value of the limit is the value of the function. If the function is not continuous, then more work is involved in finding the limit. In such cases, the limit may or may not exist. If the limit does exist, it cannot be found by plugging a number into a function. Many of the limits that appear in calculus are of this latter type. Often direct substitution leads to $0/0$, a meaningless ratio. However, it is sometimes possible to use algebra to simplify the function so that direct substitution becomes possible. The most common algebraic techniques are factoring, expanding, finding a common denominator, and multiplying by the conjugate. Each of these is illustrated in the examples below.

The first example illustrates how factoring can simplify a limit (this technique has already been used in Section 1.3):

$$\lim_{x\to 2} \frac{x^2-4}{2x^2+3x-14} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(2x+7)} = \lim_{x\to 2} \frac{x+2}{2x+7} = \frac{4}{11}.$$

Direct substitution of $x = 2$ in the original limit results in $0/0$. After factoring, the function that remains is continuous and the limit can be found by substitution. The second example shows how expanding (multiplying out) a quantity can help find a limit.

$$\lim_{x\to 0} \frac{(2+x)^3-8}{6x} = \lim_{x\to 0} \frac{12x+6x^2+x^3}{6x} = \lim_{x\to 0} \frac{12+6x+x^2}{6} = 2.$$

A complex fraction is one in which there are fractions inside of fractions. When this occurs, it is usually a good idea to find a common denominator and simplify so that only one fraction appears. Remember that dividing by a is the same as multiplying by $1/a$.

$$\lim_{x\to 1} \frac{\dfrac{1}{2x-1}-1}{x-1} = \lim_{x\to 1} \left(\frac{1}{x-1}\cdot\frac{-2x+2}{2x-1}\right) = \lim_{x\to 1} \left(\frac{1}{x-1}\cdot\frac{-2(x-1)}{2x-1}\right) = \lim_{x\to 1} \frac{-2}{2x-1} = -2.$$

Finally, there are situations in which multiplying by a conjugate is useful. This is most often helpful when a square root is involved. Since the product $(a+b)(a-b) = a^2-b^2$ has no middle terms, multiplying a quantity such as $c-\sqrt{d}$ by $c+\sqrt{d}$ eliminates the square root: $\left(c-\sqrt{d}\right)\left(c+\sqrt{d}\right) = c^2-d$. This technique is illustrated in the next example.

$$\begin{aligned}\lim_{x\to 2} \frac{x-2}{\sqrt{6-x}-x} &= \lim_{x\to 2} \left(\frac{x-2}{\sqrt{6-x}-x}\cdot\frac{\sqrt{6-x}+x}{\sqrt{6-x}+x}\right) = \lim_{x\to 2} \frac{(x-2)\left(\sqrt{6-x}+x\right)}{6-x-x^2} \\ &= \lim_{x\to 2} \frac{(x-2)\left(\sqrt{6-x}+x\right)}{(2-x)(3+x)} = \lim_{x\to 2} \frac{-\left(\sqrt{6-x}+x\right)}{3+x} = -\frac{4}{5}.\end{aligned}$$

Another result that is sometimes useful for evaluating limits is the squeeze theorem. As with the other properties of limits, we will accept it on the basis of strong intuitive reasoning.

THEOREM 1.5 Squeeze Theorem for Functions Suppose that $g(x) \le f(x) \le h(x)$ for all x that are near c but not equal to c. If $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} f(x) = L$. ∎

To illustrate this theorem, consider $\lim_{x\to 0} x^4 \sin(\pi/x)$. Since the sine function has values between -1 and 1, it is easy to see that $-x^4 \le x^4 \sin(\pi/x) \le x^4$ for all nonzero values of x. Since both $\lim_{x\to 0} -x^4 = 0$ and $\lim_{x\to 0} x^4 = 0$, it follows from the squeeze theorem that $\lim_{x\to 0} x^4 \sin(\pi/x) = 0$.

As you might suspect, algebra alone is not sufficient to evaluate every limit. As an example, consider $\lim_{x\to 0} \frac{\sin x}{x}$. Substituting 0 for x gives $0/0$, which is not helpful, and there is no apparent algebra which will simplify the fraction. Using a calculator to make a table of values of the function $\sin x/x$ for x near 0, we find that a reasonable conjecture is that the limit is 1. (It is important to realize that radian measure is assumed here.) We will prove this fact in Section 1.17. Once this limit is known, other related limits can be found using some algebra. First, note that all of the following limits are 1:

$$\lim_{x\to 0} \frac{\sin 2x}{2x}, \quad \lim_{t\to 0} \frac{5t}{\sin 5t}, \quad \text{and} \quad \lim_{\theta\to 0} \frac{\sin \theta^2}{\theta^2}.$$

In each case, the argument of the sine function is the same as the other term in the fraction and this term is going to 0. Using this idea and the algebraic properties of limits, we find that

$$\lim_{x\to 0} \frac{\tan 4x}{x} = \lim_{x\to 0} \frac{\sin 4x}{x\cos 4x} = \lim_{x\to 0} \left(4 \cdot \frac{\sin 4x}{4x} \cdot \frac{1}{\cos 4x}\right) = 4 \cdot 1 \cdot 1 = 4$$

$$\text{and} \quad \lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{1-\cos x} = \lim_{x\to 0} (1+\cos x) = 2.$$

Exercises

1. Use algebra to evaluate each of the following limits.

a) $\lim_{x\to -3} \frac{x^2+2x-3}{x^2-x-12}$ **b)** $\lim_{x\to 2} \frac{x^3-4x}{x^4-16}$ **c)** $\lim_{x\to 1} \frac{x^2-1}{x^3-1}$

d) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$ **e)** $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$ **f)** $\lim_{h\to 0} \frac{(h+4)^2-16}{h}$

g) $\lim_{x\to 2} \frac{x^2-4}{\frac{1}{x}-\frac{1}{2}}$ **h)** $\lim_{h\to 0} \frac{(2h+5)^2-25}{3h}$ **i)** $\lim_{h\to 0} \frac{\frac{1}{(3+h)^2}-\frac{1}{9}}{h}$

2. Find a value of a so that $\lim_{x\to 2} \frac{x^2+ax-7}{x^2-2x}$ exists and find the value of the limit.

3. Use the squeeze theorem to evaluate each of the following limits.

a) $\lim_{x\to 0} x^2 \cos(1/x)$ **b)** $\lim_{x\to 0} x \arctan(4/x)$ **c)** $\lim_{x\to 0^+} \left(10+2^{1/x}\right)^x$

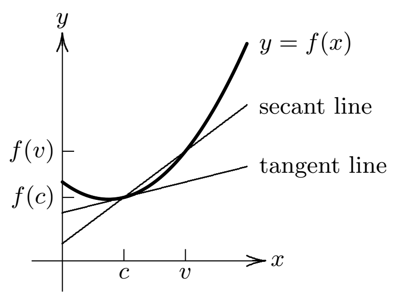
4. Evaluate each of the following trigonometric limits.

a) $\lim_{x\to 0} \frac{\tan x}{x}$ **b)** $\lim_{x\to 0} \frac{\sin 7x}{x}$ **c)** $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

d) $\lim_{x\to 0} \frac{\tan 3x}{\sin 5x}$ **e)** $\lim_{x\to 0} \frac{1-\cos x}{x}$ **f)** $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

1.6 Slope of a Curve

Consider the graph of a "smooth" continuous function (the graph has no sharp corners) and pick a point on the graph. If you look at a small portion of the graph that contains the point, the graph resembles a straight line. As the viewing area shrinks (and the graph is magnified), the effect becomes more dramatic. This sort of behavior is easy to see on a computer graph that has a zoom feature. The line resembling the graph is called the **tangent line** to the curve at the given point and its slope can be interpreted as the slope of the curve at the point. To compute the slope of the curve $y = f(x)$ at the point $(c, f(c))$, take any other point $(v, f(v))$ on the graph and determine the slope of the **secant line**—the line connecting the two points on the graph. The limit of the slopes of the secant lines as $v \to c$ is the slope of the tangent line at c (see the figure).



$$m_{\text{sec}} = \frac{f(v) - f(c)}{v - c}$$
$$m_{\text{tan}} = \lim_{v \to c} \frac{f(v) - f(c)}{v - c}$$

The slope of the secant line is given by the difference quotient $(f(v) - f(c))/(v - c)$; it is the quotient of two differences. Note that $v = c$ gives the meaningless ratio $0/0$. The slope of a curve is essentially the slope of the line joining two adjacent points on the curve, but since there are no adjacent points it is necessary to use a limit process as one point on the graph slides toward a fixed point on the graph. Thus, the geometric problem of determining the slope of the tangent line reduces to the algebraic problem of evaluating the limit $\lim_{v \to c} \frac{f(v) - f(c)}{v - c}$. This discussion is summarized in the following definition.

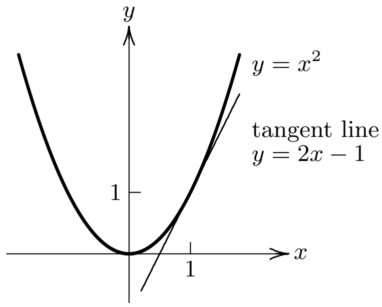
DEFINITION 1.6 The slope of the curve $y = f(x)$ when $x = c$ is given by

$$\lim_{v \to c} \frac{f(v) - f(c)}{v - c} \quad \text{or} \quad \lim_{h \to 0} \frac{f(c + h) - f(c)}{h},$$

provided that the limit exists. (The two limits give the same value.) This number, which is sometimes denoted by $\left.\frac{dy}{dx}\right|_{x=c}$, is the slope of the line tangent to the curve at the point $(c, f(c))$. It also represents the rate of change of $y = f(x)$ with respect to x at the point c.

The notation dy/dx for the slope of a curve is due to Leibniz and arises in a natural way from the equation $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, where Δx and Δy represent the change in x and change in y, respectively. Although the symbols dy and dx do not really have any meaning—it is only their ratio that does—it is sometimes quite useful to think of dy as a little bit of y and dx as a little bit of x. In fact, the "d" in the symbol dy stands for difference; dy is the difference between two extremely close y values.

As an example, let f be the function defined by $f(x) = x^2$. The point $(1,1)$ is on the graph of this function. The tangent line and the computations required to find its slope are shown below.



$$m_{\tan} = \lim_{v\to 1} \frac{f(v) - f(1)}{v - 1} = \lim_{v\to 1} \frac{v^2 - 1}{v - 1} = \lim_{v\to 1} (v + 1) = 2.$$

As a second example, let g be the function defined by $g(x) = 10/(x^2+6)$. The slope of the graph $y = g(x)$ when $x = 2$ is

$$\lim_{v\to 2} \frac{g(v) - g(2)}{v - 2} = \lim_{v\to 2} \frac{\dfrac{10}{v^2+6} - 1}{v - 2} = \lim_{v\to 2} \frac{4 - v^2}{(v-2)(v^2+6)} = \lim_{v\to 2} -\frac{2+v}{v^2+6} = -\frac{2}{5}.$$

An equation for the tangent line to the graph at the point $(2,1)$ is thus $y - 1 = -\frac{2}{5}(x-2)$. The number $-2/5$ represents the rate of change of y with respect to x for this function when $x = 2$. In particular, one would expect the y-values to be decreasing when the x-values are increasing near the point 2.

The fact that the slope of a graph can be interpreted as a rate of change opens the door to a number of physical applications. For instance, if x represents time and y represents distance, then the rate of change of y (distance) with respect to x (time) gives velocity. In other words, the slope of a distance graph is velocity. Similarly, the slope of a velocity graph is acceleration.

Exercises

1. Find the slope of the curve $y = f(x)$ at the given point.

a) $f(x) = x^2,\ x = 3$ **b)** $f(x) = x^3,\ x = 1$ **c)** $f(x) = \dfrac{1}{x},\ x = 5$

d) $f(x) = x^2 - x^3,\ x = 1$ **e)** $f(x) = \dfrac{x}{x+1},\ x = 2$ **f)** $f(x) = 4\sqrt{x},\ x = 4$

2. Find the x-intercept of the line tangent to $y = x^2$ when $x = c$, where c is any positive number.

3. Find equations for the tangent and normal lines to the curve $y = x^4$ when $x = 1$. (A **normal line** is perpendicular to the tangent line.)

4. Use both $\lim\limits_{v\to c}$ and $\lim\limits_{h\to 0}$ forms to find the slope of the graph $y = x^3$ at a generic point c.

5. Find the slope of the curve $y = f(x)$ at a generic point c in its domain.

a) $f(x) = x^3 + 2x^2$ **b)** $f(x) = 4/x$ **c)** $f(x) = \dfrac{10}{\sqrt{x}}$

6. Find the values of x for which the tangent line to the curve $y = x^3 + 2x^2$ is horizontal.

7. There are two points on the curve $y = 4/x$ for which the tangent line goes through $(3,1)$. Find these two points.

8. The velocity v in ft/sec of a particle at time t seconds is given by $v(t) = 10/\sqrt{t}$. What is the velocity of the particle when its acceleration is -5 ft/sec^2? What does the negative sign for acceleration indicate?

1.7 The Derivative

Let f be a continuous function and consider the graph of $y = f(x)$. Unless f is a linear function, it should be clear that the slope of the graph varies from point to point on the graph. For any given value of x, the graph has a certain slope. With a little thought, one can see that this defines a function; given an input x, an output—namely, the slope of the curve at the point $(x, f(x))$—is generated. This function is usually denoted by f' and called the derivative of f since it is derived from the function f. The derivative concept is given a formal definition below. As the reader should be certain to understand, the two forms of the limit given in the definition are equivalent.

DEFINITION 1.7 The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{v \to x} \frac{f(v) - f(x)}{v - x} \quad \text{or} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for each value of x in the domain of f for which the limit exists.

The number $f'(c)$ represents the slope of the graph $y = f(x)$ when $x = c$; it thus gives the rate of change of y with respect to x at the point c. If f has a derivative at the point c, then f is said to be differentiable at c. If f has a derivative at each point of an interval I, then f is said to be differentiable on I. As we will see, most of the familiar functions are differentiable at each point in their domains.

To illustrate the definition, the derivative of the function $f(x) = x^3 + 2x^2$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\big((x+h)^3 + 2(x+h)^2\big) - \big(x^3 + 2x^2\big)}{h} \\ &= \lim_{h \to 0} \frac{\big(x^3 + 3x^2h + 3xh^2 + h^3 + 2x^2 + 4xh + 2h^2\big) - \big(x^3 + 2x^2\big)}{h} \\ &= \lim_{h \to 0} \big(3x^2 + 3xh + h^2 + 4x + 2h\big) = 3x^2 + 4x. \end{aligned}$$

As is evident in this example, the computations involved in finding a derivative using the definition are not all that difficult but they can become a bit tedious.

It should be clear that a differentiable function is continuous; there can be no slope at a point on a graph where there is a break in the graph.

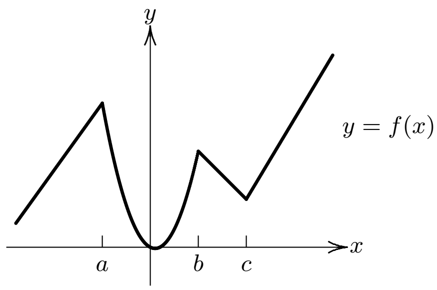
THEOREM 1.8 If f is differentiable at c, then f is continuous at c.

Proof. Suppose that f is differentiable at c. Performing some algebra and using the limit definition of the derivative, we find that

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}\,(x - c) + f(c) \right) = f'(c) \cdot 0 + f(c) = f(c).$$

This shows that f is continuous at c. ∎

The converse of this theorem is not true; there are continuous functions that are not differentiable. Any function whose graph is connected but has a sharp corner is an example of this type of function; the absolute value function at the origin is a simple example. The reason that corners create a problem lies in the fact that limits are two-sided. Consequently, in order for a derivative to exist, the slope of the curve must be the same on each side of the point. At a corner, the slopes are different on the left and the right. Such points are easy to identify on a graph:



The function f is not differentiable at the points a, b, and c, but it is differentiable at every other point.

For reasons which should now be clear, the graph of a differentiable function is said to be a smooth graph.

Exercises

1. Use the definition of the derivative to find $f'(x)$.

a) $f(x) = x^2$ **b)** $f(x) = x^3$ **c)** $f(x) = x^4$

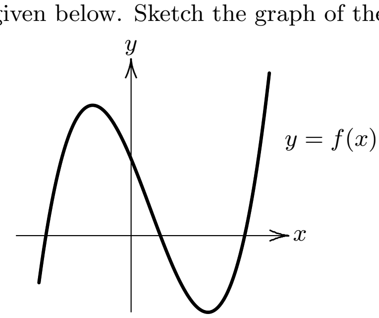
d) $f(x) = 3x^2 - 4x + 5$ **e)** $f(x) = \sqrt{x}$ **f)** $f(x) = 2x + \dfrac{5}{x}$

2. Find equations for the tangent and normal lines to the curve $y = x^2$ when $x = 3$.

3. Find the x-intercept of the line tangent to the graph of $y = x^3$ when $x = 2$.

4. Let $f(x) = x^3 - 4x^2 + x$. Find the values of x for which the slope of the graph of f is 4.

5. The graph of a function $y = f(x)$ is given below. Sketch the graph of the function f'.



6. Sketch the graph of a differentiable function f with the following properties: $f'(-2) = 0$, $f(1) = 3$, $f'(1) = -5$, $f(2) = 1$, and $f'(2) = 0$.

7. Sketch the graph of a continuous function that is not differentiable at the x values of 1, 3, and 6 but is differentiable at every other point.

8. Give a formula for a function that is continuous but not differentiable at $x = 3/2$.

9. Use the definition of the derivative to show algebraically that the given function is not differentiable at 0, that is, show that $\lim_{h\to 0} f(h)/h$ does not exist. Describe what is happening geometrically. Even though the derivative does not exist, can you find a tangent line for the graph at the origin?

a) $f(x) = x^{1/3}$ **b)** $f(x) = |x|$ **c)** $f(x) = x^{2/3}$

10. For each real number x, let $D(x)$ be the distance from x to the integer that is closest to x. Where does the function D fail to have a derivative? What is the derivative of D when it exists? Begin by sketching a graph of the function.

1.8 Differentiation of Polynomials

The definition of the derivative can be used to find the derivative of most functions, but as the formula for the function becomes more complicated so does the algebra required to simplify the quotient in the definition of the derivative. Once the derivative of a function has been found, there is no reason to use the definition again for that function; the result can be recorded for later use. In this way, a list of formulas for finding derivatives has been compiled. Each formula is discovered using the definition of the derivative, but then the formula rather than the definition is used to find the derivative. Formulas for derivatives come in two different types:

1) formulas for the derivatives of specific functions;
2) formulas for the derivatives of combinations of functions.

In this section, we will consider a few simple formulas for computing derivatives.

One of the simplest types of functions is the power function, $f(x) = x^r$, when r is a nonzero integer. The exercises in the previous section generate the following conjecture:

If r is a positive integer and $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

This formula is referred to as the **power rule** and is an example of a derivative formula for a specific function. A proof of this conjecture uses the definition of the derivative and a factoring formula:

$$f'(x) = \lim_{v\to x} \frac{v^r - x^r}{v - x} = \lim_{v\to x} \big(v^{r-1} + v^{r-2}x + v^{r-3}x^2 + \cdots + vx^{r-2} + x^{r-1}\big) = rx^{r-1}.$$

For example, if $f(x) = x^{2006}$, then $f'(x) = 2006x^{2005}$.

In Section 1.2, several ways of combining functions were discussed. It should be clear that the derivative of an algebraic combination of two functions f and g should somehow be related to the derivatives of f and g. The following theorem gives the relationship for some of these combinations.

THEOREM 1.9 Let F and G be functions and let k be a constant. If F and G are differentiable on an interval I, then the functions kF, $F + G$, and $F - G$ are differentiable on I and

a) constant multiple rule $(kF)' = kF'$,

b) sum rule $(F + G)' = F' + G'$,

c) difference rule $(F - G)' = F' - G'$.

Proof. We will give a proof of the result for sums. Using the definition of the derivative, we obtain

$$\begin{aligned} (F+G)'(x) &= \lim_{v\to x} \frac{(F+G)(v) - (F+G)(x)}{v-x} \\ &= \lim_{v\to x} \frac{\big(F(v)+G(v)\big) - \big(F(x)+G(x)\big)}{v-x} \\ &= \lim_{v\to x} \Big(\frac{F(v)-F(x)}{v-x} + \frac{G(v)-G(x)}{v-x}\Big) = F'(x) + G'(x). \end{aligned}$$

The proofs of the other parts are similar. ∎

In words, the sum rule says that the derivative of a sum is the sum of the derivatives. Combining these simple rules with the power rule (and the obvious fact that the derivative of a constant function is zero), it is possible to find the derivative of any polynomial without using the definition. For example,

$$\big(2x^4 - 3x^3 + 6x - 7\big)' = 2\big(x^4\big)' - 3\big(x^3\big)' + 6\big(x\big)' - \big(7\big)' = 2\cdot 4x^3 - 3\cdot 3x^2 + 6\cdot 1 - 0 = 8x^3 - 9x^2 + 6.$$

In other words, if $f(x) = 2x^4 - 3x^3 + 6x - 7$, then $f'(x) = 8x^3 - 9x^2 + 6$; all you need to do is find the derivative of each term and combine the results in the "obvious" way. It is immediately apparent that this is much easier than using the definition of the derivative.

If $y = f(x)$, then some common notations for the derivative are

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}\,f(x), \quad \frac{df}{dx}, \quad D_x f(x).$$

The first three notations are the only ones that will be used in this text. (Please note that, for the purposes of this course, y' is not an acceptable notation for the derivative; see the last exercise in this section.) For example, the power rule and sum rule can be written as

$$\frac{d}{dx}\,x^r = rx^{r-1} \qquad \text{and} \qquad \frac{d}{dx}\big(F(x) + G(x)\big) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x),$$

respectively. The derivative of $y = f(x)$ evaluated at the point c is denoted by $f'(c)$ or $\left.\dfrac{dy}{dx}\right|_{x=c}$. The latter notation is a bit awkward but some way of distinguishing between the derivative and its values is needed.

Exercises

1. Use formulas in this section to find the derivative of the given function. Pay particular attention to your notation.

a) $f(x) = 4x - 7$ **b)** $g(x) = 3x^2 + 13x - 8$ **c)** $h(x) = -2x^3 + 3x^2 - 5x + 6$

d) $F(t) = 3t^5 - 2t^4 + t - 2$ **e)** $G(z) = \frac{1}{2}z^4 + \frac{1}{3}z^3 - z^2$ **f)** $H(u) = \dfrac{2u^6 + 3u^4 - u^2}{12}$

g) $y = x^{100} + 100x$ **h)** $s = 0.01t^5 + 0.2t^3 - 4t$ **i)** $z = 2w^{30} - \frac{1}{33}\,w^{11}$

2. Find equations for the tangent and normal lines to the curve $y = x^6 - 4x^5 + 2x^4 + x + 3$ when $x = 1$.

3. Find the x-intercept of the line tangent to the graph of $y = x^4 - 3x^3 + 5x$ when $x = 2$.

4. Let $f(x) = 3x^5 - 10x^3 - 45x + 7$. Find the values of x for which the tangent line is horizontal.

5. The position s in meters of a particle at time t seconds is given by $s = 0.02t^5 - 0.4t^3 + 60t$. What is the velocity of the particle after $\sqrt{10}$ seconds?

6. Suppose that the height h in inches of a beanstalk after t hours is $h = \dfrac{t^5 + 2t^4 + 3t^3}{360}$. What is the rate of growth of the beanstalk at the end of two days? Give your answer in feet per minute.

7. Suppose that f and g are differentiable functions and that $f(2) = 9$, $f'(2) = 5$, $g(2) = 3$, and $g'(2) = -2$. Find $h(2)$ and $h'(2)$ for the given function h.

a) $h = 4f$ **b)** $h = f + g$ **c)** $h = f - g$

d) $h = 2f - 3g$ **e)** $h = (f + 2g)/5$ **f)** $h = f(2)g$

8. Prove the constant multiple rule and the difference rule.

9. For $y = 4x^2t^3$, find $\dfrac{dy}{dx}$ and $\dfrac{dy}{dt}$. Note that y' would be ambiguous in this situation.

1.9 Power Rule

In the last section, we proved that the derivative of x^r is rx^{r-1} for the case in which r is a positive integer. It turns out that this derivative formula is valid for any nonzero rational number r, but a little more effort is involved to prove this fact. We begin by proving the **reciprocal rule**: if f is a differentiable function, then

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2} \quad \text{or} \quad \frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{(f(x))^2},$$

for all values of x for which $f(x) \neq 0$. To prove this rule, we use the definition of the derivative, a little algebra, and the fact that f is a continuous function.

$$\begin{aligned}\frac{d}{dx}\frac{1}{f(x)} &= \lim_{v\to x}\frac{\dfrac{1}{f(v)}-\dfrac{1}{f(x)}}{v-x}\\ &= \lim_{v\to x}\left(\frac{1}{v-x}\cdot\frac{f(x)-f(v)}{f(x)f(v)}\right)\\ &= \lim_{v\to x}\left(-\frac{1}{f(x)f(v)}\cdot\frac{f(v)-f(x)}{v-x}\right)\\ &= -\frac{1}{(f(x))^2}\cdot f'(x) = -\frac{f'(x)}{(f(x))^2}\end{aligned}$$

Notice how algebra was used to manipulate the original fraction into one that contained the definition of the derivative of f. Where is the continuity of f used in this proof?

If r is a positive integer, then $\frac{d}{dx}x^r = rx^{r-1}$. Properties of exponents and the reciprocal rule then yield

$$\frac{d}{dx}x^{-r} = \frac{d}{dx}\frac{1}{x^r} = -\frac{rx^{r-1}}{(x^r)^2} = -\frac{rx^{r-1}}{x^{2r}} = -rx^{-r-1},$$

which shows that the power rule is valid for negative integers. The power rule for integers can then be used to prove the result for arbitrary rational numbers.

THEOREM 1.10 Power Rule If r is a nonzero rational number, then $\dfrac{d}{dx}x^r = rx^{r-1}$ for all values of x for which both sides of the equation are defined.

Proof. Let r be a nonzero rational number. Then $r = p/q$, where p is a nonzero integer and q is a positive integer. Define a function f by $f(x) = x^r = x^{p/q}$. We will use the definition of the derivative to find f'. Let $w = v^{1/q}$ and $z = x^{1/q}$. Since the function $x^{1/q}$ is continuous, it follows that $w \to z$ as $v \to x$. Hence,

$$\begin{aligned}f'(x) &= \lim_{v\to x}\frac{v^{p/q}-x^{p/q}}{v-x} = \lim_{w\to z}\frac{w^p-z^p}{w^q-z^q} = \lim_{w\to z}\left(\frac{w^p-z^p}{w-z}\cdot\frac{w-z}{w^q-z^q}\right)\\ &= pz^{p-1}\cdot\frac{1}{qz^{q-1}} = \frac{p}{q}z^{p-q} = \frac{p}{q}x^{(p/q)-1} = rx^{r-1}.\end{aligned}$$

(The values of the limits follow from the fact that they have already been established for integers.) We have thus shown that the derivative of x^r is rx^{r-1}. This completes the proof. ■

When using the power rule, it is important to be careful with exponents:

$$\frac{d}{dx}\left(4\sqrt{x}-\frac{6}{\sqrt{x}}\right) = \frac{d}{dx}\left(4x^{1/2}-6x^{-1/2}\right) = 2x^{-1/2}+3x^{-3/2} = \frac{2}{x^{1/2}}+\frac{3}{x^{3/2}} = \frac{2x+3}{x^{3/2}};$$

$$\frac{d}{dx}\left(4-\frac{6}{x^2}+\frac{5}{2x^3}\right) = \frac{d}{dx}\left(4-6x^{-2}+\frac{5}{2}x^{-3}\right) = 12x^{-3}-\frac{15}{2}x^{-4} = \frac{12}{x^3}-\frac{15}{2x^4} = \frac{3(8x-5)}{2x^4}.$$

Let's look at a different type of example. Let a and b represent nonzero real numbers and consider the function f defined by $f(x) = ax + \dfrac{b}{x}$. The numbers a and b are referred to as parameters; for each pair of numbers a and b, we obtain a different function f. The reader should sketch several graphs $y = f(x)$ for different choices of a and b. (A good place to start are the (a, b) pairs $(1, 2)$, $(-1, 2)$, $(1, -2)$, and $(-1, -2)$.) We will prove that none of the tangent lines to any of the curves represented in this way goes through the origin. Let w be any nonzero number (so w is in the domain of f) and consider the tangent line to the curve at $(w, f(w))$. The slope of the tangent line at this point is $f'(w)$, and the slope of the line through this point and the origin is $f(w)/w$. In order for the tangent line to go through the origin, these two slopes must be equal. This leads to the equation $wf'(w) = f(w)$. Since $f'(x) = a - \dfrac{b}{x^2}$, the number w must satisfy the equation

$$w\left(a - \frac{b}{w^2}\right) = aw + \frac{b}{w} \quad \Rightarrow \quad \frac{2b}{w} = 0.$$

Since no value of w can satisfy this last equation, there is no point on the graph for which the tangent line passes through the origin. By using the parameters a and b, we have shown in a single step that an infinite collection of functions has the same property. This is one of the advantages of using parameters.

Exercises

1. Find and simplify the derivative of the given function.

a) $f(x) = 2\sqrt[4]{x} - 6\sqrt[3]{x}$ **b)** $g(x) = \dfrac{7}{3x^4}$ **c)** $h(x) = \dfrac{5}{x} - \dfrac{12}{x^2}$

d) $F(t) = \dfrac{2t-5}{4\sqrt{t}}$ **e)** $G(z) = \dfrac{z^3 - 4z + 6}{z^2}$ **f)** $H(s) = s(3s - 2\sqrt{s})$

g) $y = 3x^2 - \dfrac{4}{x}$ **h)** $z = 2w^4 - 3w^3 + 6w^2 - 5w$ **i)** $s = \dfrac{7}{t} - \dfrac{2}{5\sqrt{t}}$

2. Find an equation for the line tangent to the graph of $f(x) = 3x + \dfrac{4}{x}$ when $x = 2$.

3. Let c and k be positive numbers. The tangent line to the curve $xy = k$ when $x = c$ cuts off a triangle in the first quadrant. Find the area of this triangle and note that its value is independent of c.

4. Find both points on the curve $y = \sqrt{x}$ for which the tangent line goes through the point $(5, 3)$.

5. Evaluate $\displaystyle\lim_{x\to 1} \frac{\sqrt[9]{x} - 1}{x - 1}$ by interpreting the limit as a derivative.

6. Given that f is a differentiable function, use the definition of the derivative to obtain the formula for the derivative of the function f^2. (The computations are similar to those in the proof of the reciprocal rule.)

1.10 Chain Rule

One of the most important formulas for differentiation is the formula for the derivative of a composite function. This formula is known as the chain rule. It indicates how the derivative of the composite function $F \circ G$ is related to the derivatives of the functions F and G. The formula itself is quite simple, but some practice is required before its use becomes second nature. As a start, note that the symbols used in the definition of the derivative simply represent quantities with certain properties. For a function F, both of the following equations are equivalent variations on the definition of the derivative:

$$F'(z) = \lim_{\theta \to z} \frac{F(\theta) - F(z)}{\theta - z} \quad \text{and} \quad F'\big(G(x)\big) = \lim_{w \to G(x)} \frac{F(w) - F\big(G(x)\big)}{w - G(x)}.$$

It is the last equation that is relevant for a proof of the chain rule. Suppose that G is a continuous function. Then $G(v) \to G(x)$ as $v \to x$. Using $G(v)$ for w, we find that

$$F'\big(G(x)\big) = \lim_{G(v) \to G(x)} \frac{F\big(G(v)\big) - F\big(G(x)\big)}{G(v) - G(x)} = \lim_{v \to x} \frac{F\big(G(v)\big) - F\big(G(x)\big)}{G(v) - G(x)}.$$

This fact will be used in the proof of the chain rule.

THEOREM 1.11 Chain Rule Let F and G be functions for which $F \circ G$ is defined. If F and G are differentiable, then $F \circ G$ is differentiable and $(F \circ G)' = (F' \circ G)\, G'$. That is ,

$$\frac{d}{dx} F\big(G(x)\big) = F'\big(G(x)\big)\, G'(x)$$

for each value of x in the domain of G.

Proof. Using the definition of the derivative, we obtain

$$\begin{aligned} \frac{d}{dx} F\big(G(x)\big) &= \lim_{v \to x} \frac{F\big(G(v)\big) - F\big(G(x)\big)}{v - x} \\ &= \lim_{v \to x} \left(\frac{F\big(G(v)\big) - F\big(G(x)\big)}{G(v) - G(x)} \cdot \frac{G(v) - G(x)}{v - x} \right) \\ &= F'\big(G(x)\big)\, G'(x). \end{aligned}$$

The last step uses the observation made before the theorem. ■

There is a potential problem with this proof; if $G(v) = G(x)$ infinitely often as $v \to x$, then there is an unavoidable division by zero in the middle step of the proof. This can only happen at a point for which $G'(x) = 0$ and the graph of G wiggles wildly. As such functions are extremely unusual and rarely occur in applications, we will not consider a general proof.

The chain rule states that the derivative of a composite function is the derivative of the "outside" function, with the "inside" function left alone, multiplied by the derivative of the "inside" function. The following examples illustrate the chain rule.

$$\frac{d}{dx}\big(x^4 - x + 1\big)^5 = 5\big(x^4 - x + 1\big)^4\big(4x^3 - 1\big);$$

$$\frac{d}{dx}\sqrt{2x^2 + 4x + 11} = \frac{1}{2}\big(2x^2 + 4x + 11\big)^{-1/2}\big(4x + 4\big) = \frac{2(x+1)}{\sqrt{2x^2 + 4x + 11}};$$

$$\frac{d}{dx}\frac{3}{x^2 + 5x} = \frac{d}{dx} 3\big(x^2 + 5x\big)^{-1} = -3\big(x^2 + 5x\big)^{-2}\big(2x + 5\big) = \frac{-3(2x+5)}{(x^2 + 5x)^2}.$$

The only specific derivative formula obtained thus far is for the function x^r. For composite functions of this type, the chain rule can be expressed as

$$\frac{d}{dx}\big(G(x)\big)^r = r\big(G(x)\big)^{r-1}G'(x).$$

This result is sometimes referred to as the **extended power rule**. However, the chain rule is quite general and will appear again when we find the derivatives of trigonometric and exponential functions.

Suppose that the variable y depends on a variable u and that the variable u depends on the variable x. Then y varies with x and has the form of a composite function; given x, first find u, then use it to find y. To find the rate of change of y with respect to x, we note that the equation

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x} \quad \text{becomes} \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

The latter equation provides a nice symbolic representation for the chain rule. It has the further appeal in that it looks like the du terms cancel and this makes the formula easy to remember. One of the drawbacks of this representation is that the points at which the derivatives should be evaluated are not indicated.

Exercises

1. Find and simplify the derivative of the given function.

a) $f(x) = (2x^2 - 3x + 1)^5$ **b)** $g(x) = (x^3 + 6x)^4$ **c)** $h(x) = \sqrt{x^4 + 3x^2 + 15}$

d) $F(t) = (t^3 - 3t^2 + t)^3$ **e)** $G(z) = \sqrt[3]{z^3 + 6z}$ **f)** $H(s) = (3s^2 + 15s - 8)^{4/3}$

g) $y = \dfrac{4}{(3x^2 - 4x + 6)^3}$ **h)** $z = \dfrac{6}{\sqrt{w^2 + 6}}$ **i)** $s = \dfrac{5}{t^3 + 3t^2}$

2. Find the derivative of the function s defined by $s(x) = \sqrt{x^2 + \sqrt{x^2 + 2x + 3}}$.

3. Find an equation for the line tangent to the graph of $f(x) = \dfrac{9}{\sqrt{4x + 1}}$ when $x = 2$.

4. Find a point on the curve $y = \sqrt{x^2 + 4x + 8}$ for which the tangent line goes through the origin.

5. Suppose that f and g are differentiable functions with $f(1) = 1$, $f'(1) = 3$, $g(1) = -2$, and $g'(1) = -1$. Determine $h'(1)$ for each function h.

a) $h(x) = (g(x))^3$ **b)** $h(x) = (f(x))^3 - 3g(x^2)$ **c)** $h(x) = g(f(7 - 6x))$

6. Let $f(x) = 2x^2 - 3x + 1$ and let $g = f \circ f \circ f$, that is, $g(x) = f(f(f(x)))$. Find $g'(2)$. (Do not find $g(x)$!)

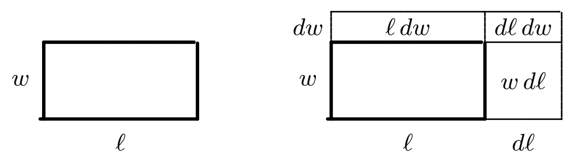
7. Explain why the reciprocal rule is a special case of the chain rule.

1.11 Product and Quotient Rules

Although the derivative of a sum is the sum of the derivatives, it is NOT true that the derivative of a product is the product of the derivatives. As an example, note that

$$\frac{d}{dx}\bigl(x^3 \cdot x^4\bigr) = \frac{d}{dx}\bigl(x^7\bigr) = 7x^6 \quad \text{and} \quad \frac{d}{dx}\bigl(x^3\bigr) \cdot \frac{d}{dx}\bigl(x^4\bigr) = 3x^2 \cdot 4x^3 = 12x^5$$

give different values. To motivate the correct formula, consider a rectangle with length ℓ and width w. Suppose that the length increases by a small amount $d\ell$ and that the width increases by a small amount dw. Subtracting the original area from the new area yields



$$\begin{aligned} dA &= (\ell + d\ell)\,(w + dw) - \ell w \\ &= \ell\,dw + w\,d\ell + d\ell\,dw, \end{aligned}$$

where dA is the increase in the area. If $d\ell$ and dw are quite small, say 10^{-8}, their product will be very small indeed and it follows that dA is essentially $\ell\,dw + w\,d\ell$. The change in the product is the first times the change of the second plus the second times the change in the first. The product rule results if the word 'change' is replaced by the word 'derivative'.

THEOREM 1.12 Product and Quotient Rules Let F and G be functions. If F and G are differentiable on an interval I, then FG and F/G are differentiable on I and

$$(FG)' = FG' + GF' \quad \text{and} \quad \left(\frac{F}{G}\right)' = \frac{GF' - FG'}{G^2}.$$

Of course, the quotient rule is only valid for those values of x for which $G(x) \neq 0$.

Proof. Using the definition of the derivative and the motivational idea discussed above, we obtain

$$\begin{aligned} (FG)'(x) &= \lim_{v \to x} \frac{F(v)G(v) - F(x)G(x)}{v - x} \\ &= \lim_{v \to x} \frac{\Bigl(F(x) + \bigl(F(v) - F(x)\bigr)\Bigr)\Bigl(G(x) + \bigl(G(v) - G(x)\bigr)\Bigr) - F(x)G(x)}{v - x} \\ &= \lim_{v \to x} \left(F(x) \cdot \frac{G(v) - G(x)}{v - x} + G(x) \cdot \frac{F(v) - F(x)}{v - x} + \bigl(F(v) - F(x)\bigr) \cdot \frac{G(v) - G(x)}{v - x}\right) \\ &= F(x)G'(x) + G(x)F'(x) + 0 \cdot G'(x) \\ &= F(x)G'(x) + G(x)F'(x). \end{aligned}$$

Note the use of the continuity of F in the evaluation of the third limit. The quotient rule follows from the product rule and the reciprocal rule by writing the quotient $\dfrac{F}{G}$ as $F \cdot \dfrac{1}{G}$. The details will be left as an exercise. ∎

In words, the product rule states that the derivative of a product is the first times the derivative of the second plus the second times the derivative of the first. The quotient rule states that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all over the denominator squared.

The following examples illustrate the product and quotient rules. As indicated, it is good practice to get into the habit of simplifying derivatives as much as possible.

$$\begin{aligned}\text{If } f(x) = (4x+3)^3(6-x)^4, \text{ then } f'(x) &= (4x+3)^3 \cdot 4(6-x)^3(-1) + (6-x)^4 \cdot 3(4x+3)^2 4 \\ &= 4(4x+3)^2(6-x)^3\big(-(4x+3)+3(6-x)\big) \\ &= 4(15-7x)(4x+3)^2(6-x)^3.\end{aligned}$$

$$\begin{aligned}\text{If } g(x) = x^3\sqrt{6x+1}, \text{ then } g'(x) &= x^3 \cdot \frac{1}{2}(6x+1)^{-1/2} \cdot 6 + (6x+1)^{1/2} \cdot 3x^2 \\ &= 3x^2(6x+1)^{-1/2}\big(x+(6x+1)\big) \\ &= \frac{3x^2(7x+1)}{\sqrt{6x+1}}.\end{aligned}$$

$$\text{If } y = \frac{2x+3}{x^2+4}, \text{ then } \frac{dy}{dx} = \frac{(x^2+4)\cdot 2 - (2x+3)\cdot 2x}{(x^2+4)^2} = \frac{8-6x-2x^2}{(x^2+4)^2} = \frac{-2(x-1)(x+4)}{(x^2+4)^2}.$$

Note that simplification means to combine terms and/or factor. You do not want to expand things that are already factored unless it is then possible to combine them with other terms. For instance, the denominator in the derivative of a quotient should NOT be multiplied out.

Exercises

1. Find and simplify the derivative of the given function. Be sure to factor as much as possible.

a) $f(x) = x\sqrt{x+2}$ **b)** $g(x) = \dfrac{x+3}{2x-1}$ **c)** $h(x) = \dfrac{x}{\sqrt{2x+5}}$

d) $u(x) = 4x\sqrt{10-x^2}$ **e)** $v(x) = (x+1)^3(2x-1)^4$ **f)** $w(x) = \dfrac{6x+1}{x^2+2x+4}$

g) $y = \dfrac{x^2}{\sqrt{4+x^2}}$ **h)** $s = \left(\dfrac{t-1}{t+1}\right)^4$ **i)** $z = \sqrt[3]{4+x^2(2-x)^5}$

2. Determine all the values of x for which the tangent line to the graph of $y = (x-2)^3(3x+1)^4$ is horizontal.

3. Find an equation for the line tangent to the graph $y = x^2/(3x+4)$ when $x = 4$.

4. The position s in meters of a particle at time t seconds is given by $s = 10t/(2t+1)$. When is the velocity of the particle 0.1 meters per second?

5. Consider the function f defined by $f(x) = x/(x+1)$.

a) Find equations for all of the tangent lines of f that are parallel to the line $x - 4y = 3$.

b) Find all of the points on the graph of f for which the tangent line goes through the point $(0,4)$.

6. Suppose that f and g are differentiable functions with $f(2) = 1$, $f'(2) = -3$, $g(2) = 4$, and $g'(2) = 5$. Determine $h'(2)$ for each function h.

a) $h = fg$ **b)** $h = f/g$ **c)** $h = f^2g^3$

7. Finish the proof of the quotient rule. Use the suggestion mentioned at the end of the proof of Theorem 1.12.

8. Suppose that F, G, and H are differentiable functions on an interval I. Find a derivative formula for $(FGH)'$. Can you extend this to the product of four or more functions?

1.12 Maximum and Minimum Outputs

Let f be a continuous function defined on a closed and bounded interval $[a, b]$. After sketching many possible graphs for $y = f(x)$, including some wiggly and jerky options, the following two observations seem to be true.

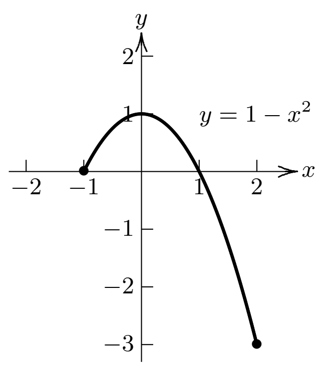
i. The graph always has a high point and a low point.

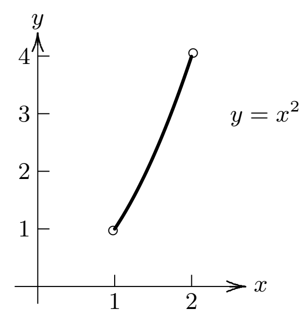
ii. The high point and low point occur either at the endpoints of the interval, at points for which $f' = 0$, or at points for which f' does not exist.

Both of these observations are indeed true. We begin with a mathematical definition of the maximum and minimum outputs of a function.

DEFINITION 1.13 Let f be a function defined on an interval I. The function f has a **maximum output** at $d \in I$ if $f(x) \leq f(d)$ for all $x \in I$, and f has a **minimum output** at $c \in I$ if $f(x) \geq f(c)$ for all $x \in I$.

For the most part, this is a simple concept. Consider the function f defined by $f(x) = 1 - x^2$ on the interval $[-1, 2]$. A quick sketch of the graph (see below) reveals that the maximum output of f is 1 and occurs when $x = 0$, while the minimum output of f is -3 and occurs when $x = 2$.





However, there are some subtleties going on here as well. For instance, consider the function g defined by $g(x) = x^2$ on the interval $(1, 2)$. This function has neither a maximum nor a minimum output on the given interval since the endpoints of the interval are not included in the domain. It is true that 4 is larger than all of the outputs of g, but 4 is not an output of g for x in the interval $(1, 2)$.

The maximum and minimum outputs of a function are known as the **extreme outputs** (or extreme values) of a function. As the second example illustrates, a function may not have extreme outputs on a given interval. The following theorem gives conditions that guarantee that extreme outputs exist.

THEOREM 1.14 **Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f has both a maximum output and a minimum output on $[a, b]$. In other words, there exist numbers $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. ■

The proof of this theorem will be omitted as it involves some properties of real numbers that we have not discussed. Our goal is to find the extreme outputs of a function once we know that they exist. This is where the second observation made earlier comes into play.

THEOREM 1.15 Let f be a function defined on an open interval I. If f has an extreme output at a point $c \in I$, then either f is not differentiable at c or $f'(c) = 0$.

Proof. Suppose that f has a minimum output when the input is c. This means that $f(v) - f(c) \geq 0$ for all $v \in I$. It follows that

$$\frac{f(v) - f(c)}{v - c} \quad \begin{cases} \text{is less than or equal to 0 if } v \in I \text{ and } v < c; \\ \text{is greater than or equal to 0 if } v \in I \text{ and } v > c. \end{cases}$$

If f happens to be differentiable at c, then $f'(c) = \lim\limits_{v \to c} \dfrac{f(v) - f(c)}{v - c} = 0$. ■

As a consequence of this theorem, it is a reasonably easy task to find the maximum and minimum outputs of a continuous function f defined on a closed interval $[a, b]$. The first step is to find the **critical inputs** of f; these are the inputs x for which either $f'(x) = 0$ or $f'(x)$ does not exist. The second step is to evaluate f at the critical inputs that belong to $[a, b]$ and at the endpoints of the interval. The largest of the outputs is the maximum output and the smallest is the minimum output.

Problem: Find the maximum and minimum outputs of $f(x) = 6x - x^3$ on the interval $[1, 3]$.

Solution: Since f is a polynomial, it is continuous and differentiable for all real numbers. To find the critical inputs of f, we first find its derivative: $f'(x) = 6 - 3x^2$. The function f' is zero when $x = \pm\sqrt{2}$. These are the critical inputs of f, but only $\sqrt{2}$ is in the given interval. Evaluating f at the endpoints and the appropriate critical input yields $f(1) = 5$, $f(\sqrt{2}) = 4\sqrt{2} \approx 5.657$, $f(3) = -9$. The maximum output of f on $[1, 3]$ is thus $4\sqrt{2}$, which occurs when $x = \sqrt{2}$, and the minimum output of f on $[1, 3]$ is -9, which occurs when $x = 3$.

Exercises

1. Find the maximum and minimum outputs of the function on the given interval.
 a) $f(x) = 16 + 4x - x^2,\ [-1, 4]$ **b)** $f(x) = 2x^3 - 3x^2 - 12x + 40,\ [-2, 3]$
 c) $f(x) = x^4 - 2x^2 + 8,\ [0, 4]$ **d)** $f(x) = 4x + \dfrac{3}{3x - 1},\ [0.4, 2]$ **e)** $f(x) = \dfrac{x}{2} + \dfrac{1}{x},\ [1, 4]$

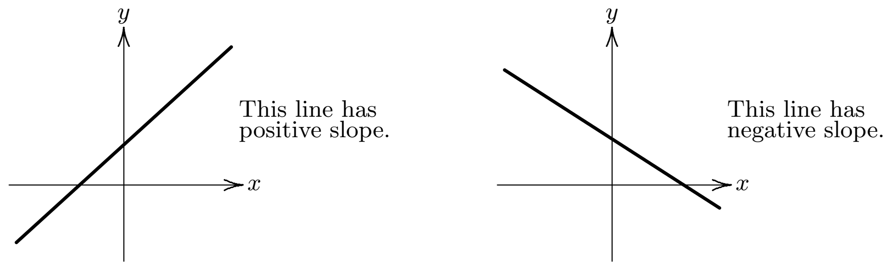
2. Sketch the graph of a function with the given properties.
 a) A function f that is continuous on $(2, 5)$ but has no maximum output.
 b) A function g that is not continuous on $[-1, 3]$ but has both a maximum and a minimum output.
 c) A function h that is continuous on $(1, \infty)$ and has both a maximum and a minimum output.

3. The sum of two nonnegative numbers is 10.
 a) Find such numbers so that the product of one with the cube of the other is as large as possible.
 b) Find such numbers so that the sum of one and the cube of the other is as small as possible.

4. Find the minimum distance from a point on the parabola $y = x^2$ to the point $(0, 2)$.

1.13 Increasing and Decreasing Functions

The slope of a line gives the rate of change of y with respect to x. If the line has positive slope, then y increases as x increases; the graph goes up as you move to the right. Similarly, if the line has negative slope, then y decreases as x increases; the graph goes down as you move to the right.



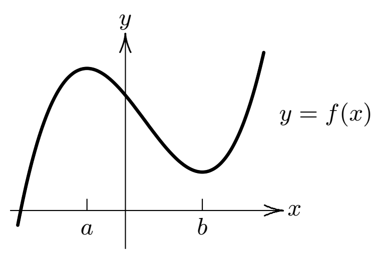
Putting the "going up" and "moving right" ideas into algebra yields the following definition.

DEFINITION 1.16 Let f be a function defined on an interval I.

a) The function f is **increasing** on I if $u < v$ implies $f(u) \le f(v)$ for all points u and v in I.

b) The function f is **decreasing** on I if $u < v$ implies $f(u) \ge f(v)$ for all points u and v in I.

The function f, whose graph is sketched below, is increasing on the interval $(-\infty, a]$, decreasing on the interval $[a, b]$, and increasing on the interval $[b, \infty)$.



Determining where a function is increasing or decreasing from a formula is more difficult than reading its graph. Of course, it is possible to first sketch the graph, then try to read off the values where the curve changes direction. This method works well in some cases, but it does have some disadvantages. First of all, you need to be certain you can see all of the interesting features of the graph on the same screen. If not, you will miss some important information. Secondly, there may be situations in which exact values for the turning points are needed, not just numerical approximations. Finally, if the formula for the function involves parameters rather than numbers ($x^3 + ax$ rather than $x^3 + 4x$), then the graphing method breaks down. Calculus provides a way to avoid all of these potential difficulties.

Recall that the derivative of a function f gives the slope of the graph $y = f(x)$. If f' is positive, then the tangent lines of f have positive slope. Since lines with positive slopes are increasing, it seems reasonable to expect that a function with a positive derivative is increasing. Similarly, we expect a function with a negative derivative to be decreasing. This result is included in the following theorem.

THEOREM 1.17 Suppose that f is differentiable on an interval I.

a) The function f' is nonnegative (≥ 0) on I if and only if f is increasing on I.

b) The function f' is nonpositive (≤ 0) on I if and only if f is decreasing on I. ■

Although this theorem should have a strong intuitive appeal, it is not a trivial matter to prove it; the details will be given in Section 1.26. The advantage of this property of derivatives is that it reduces the problem of determining where a differentiable function f is increasing or decreasing to the problem of finding the intervals on which its derivative f' is positive or negative. The function f' is negative when its graph is below the x-axis and positive when its graph is above the x-axis. There are only two ways for the graph of a function f' to switch sides of the x-axis at a point c: either the graph of f' crosses the x-axis ($f'(c) = 0$) or the graph of f' jumps across the x-axis (f' has a discontinuity at c). In between such points, the function f' must have the same sign. These ideas will be used in the following example.

Problem: Determine the intervals on which the function f defined by $f(x) = x^2/(x+2)$ is increasing and those on which it is decreasing.

Solution: As the reader should verify, $f'(x) = \dfrac{x(x+4)}{(x+2)^2}$. The graph of f' can only cross the x-axis at the points -4, -2, and 0. In between these points, the sign of f' stays the same. For example, between -2 and 0, the term x is negative, the term $x+4$ is positive, and the term $(x+2)^2$ is positive. It follows that f' is negative on the interval $(-2, 0)$. Using similar reasoning, we obtain the following information:

interval	$(-\infty, -4)$	$(-4, -2)$	$(-2, 0)$	$(0, \infty)$
sign of f'	positive	negative	negative	positive
property of f	increasing	decreasing	decreasing	increasing

Hence, the function f is increasing on the intervals $(-\infty, -4]$ and $[0, \infty)$, and decreasing on the intervals $[-4, -2)$ and $(-2, 0]$. Pay careful attention to whether or not the endpoints are included.

Exercises

1. Determine the intervals on which f is increasing and those on which it is decreasing given $f'(x) = x(x-1)^2(x-3)^3$.

2. Determine the intervals on which f is increasing and those on which it is decreasing. Treat a as a positive constant.

a) $f(x) = x^3 - 6x^2$ **b)** $f(x) = 3x^4 + 8x^3 + 24x^2$ **c)** $f(x) = x - 4\sqrt{x}$

d) $f(x) = 6x^5 - 110x^3 + 300x$ **e)** $f(x) = 2x + 1/x$ **f)** $f(x) = x - 3x^{2/3}$

g) $f(x) = \sqrt{12x - 2x^2}$ **h)** $f(x) = (x^2 - x + 4)/(x-1)$ **i)** $f(x) = 4x^3 + 21x^2 - 24x - 3$

j) $f(x) = x^3 - a^2 x$ **k)** $f(x) = x + a^2/x$ **l)** $f(x) = x^2 + a^3/x$

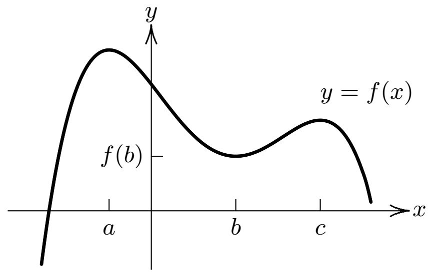
3. Suppose that f is increasing on I. Prove that $-f$ is decreasing on I.

4. Suppose that f and g are increasing on I. Prove that $f + g$ is increasing on I.

5. Give an example to show that the product of two increasing functions on $\mathbb{R}$ may not be an increasing function.

1.14 The First Derivative Test

Consider the graph of the differentiable function f shown below. It is clear that f is increasing on the intervals $(-\infty, a]$ and $[b, c]$ and decreasing on the intervals $[a, b]$ and $[c, \infty)$. The maximum output of the function occurs when $x = a$.



Although the graph changes direction when $x = b$ and $x = c$, the function f does not have a maximum or minimum output at these points. However, there is some sort of extreme behavior at the points b and c. For instance, on the interval $[0, c]$ the minimum output of the function f is $f(b)$. Extreme outputs such as these are known as relative extreme outputs. The function f has a relative minimum output at b and a relative maximum output at c. These terms are defined as follows.

DEFINITION 1.18 Let f be defined on an interval I and let c be a point in I. Then

a) f has a **relative maximum output** at c if $f(x) \le f(c)$ for all x in some open interval containing c.

b) f has a **relative minimum output** at c if $f(x) \ge f(c)$ for all x in some open interval containing c.

A brief study of Theorem 1.15 shows that if a continuous function f has a relative extreme output at a point c, then c is a critical input of f. That is, either $f'(c)$ does not exist or $f'(c) = 0$. However, a function may not have a relative extreme output at a critical input. For example, the function $f(x) = x^3$ has a critical input when $x = 0$, but the function does not have a relative extreme output at 0. The following theorem provides a way to test critical inputs to determine whether or not they correspond to relative extreme outputs.

THEOREM 1.19 **First Derivative Test** Suppose that f is continuous on an open interval containing the point c and that c is a critical input for f.

a) If f' is positive on an interval (a, c) for some $a < c$ and negative on an interval (c, b) for some $b > c$, then f has a relative maximum output at c.

b) If f' is negative on an interval (a, c) for some $a < c$ and positive on an interval (c, b) for some $b > c$, then f has a relative minimum output at c.

Proof. The proof of this theorem is quite easy. Given the conditions listed in part (a), the function f is increasing on (a, c) and decreasing on (c, b). It follows that $f(c) \ge f(x)$ for all $x \in (a, b)$. Hence, the function f has a relative maximum output at c. The proof of part (b) is similar. ■

Problem: Determine the nature of all the critical inputs for $f(x) = 3x^4 - 20x^3 - 18x^2 + 180x + 245$.

Solution: As usual, the first step is to solve the equation $f'(x) = 0$. (Since f is a polynomial, the derivative will exist at every point.) Since

$$f'(x) = 12x^3 - 60x^2 - 36x + 180 = 12(x^3 - 5x^2 - 3x + 15)$$
$$= 12\big(x^2(x-5) - 3(x-5)\big) = 12(x^2 - 3)(x - 5),$$

the critical inputs of f are $-\sqrt{3}$, $\sqrt{3}$, and 5. Proceeding as in the last section, we look at the intervals on which f' has constant sign.

interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, \sqrt{3})$	$(\sqrt{3}, 5)$	$(5, \infty)$
sign of f'	negative	positive	negative	positive
property of f	decreasing	increasing	decreasing	increasing

By the first derivative test, the function f has a relative maximum output when $x = \sqrt{3}$ and relative minimum outputs when $x = -\sqrt{3}$ and $x = 5$. Since $f(x)$ becomes arbitrarily large as x gets large in either the negative or positive direction, it is clear that f has no maximum output on its domain $(-\infty, \infty)$. Noting that $f(-\sqrt{3}) = 218 - 120\sqrt{3}$ and $f(5) = 70$, with a little more thought, we find that the minimum output of f on $(-\infty, \infty)$ is approximately 10.154 and occurs when $x = -\sqrt{3}$.

Problem: Find the minimum distance from a point on the curve $y = 2/x^2$ to the origin.

Solution: For each $x > 0$ (by symmetry, it is sufficient to consider this case only), the square $S(x)$ of the distance from the point $(x, 2/x^2)$ to the origin is given by $S(x) = x^2 + \dfrac{4}{x^4}$. As the reader may verify, $S'(x) = 0$ when $x = \sqrt{2}$ and the first derivative test indicates that S has a minimum at this value. Since $S(\sqrt{2}) = 3$, the minimum distance from a point on the curve $y = 2/x^2$ to the origin is $\sqrt{3}$. For the record, this minimum distance occurs at the points $(\pm\sqrt{2}, 1)$. (Note that minimizing the square of the distance is equivalent to minimizing the distance.)

Exercises

1. Determine the nature of all the critical inputs of the given function. Also check to see if the function has a maximum or minimum output on its domain. Treat a as a positive constant.

a) $f(x) = x^3 - 6x^2$ **b)** $f(x) = 3x^4 + 8x^3 + 24x^2$ **c)** $f(x) = x - 4\sqrt{x}$

d) $f(x) = 6x^5 - 110x^3 + 300x$ **e)** $f(x) = 2x + 6/x$ **f)** $f(x) = 3x^{2/3} - x$

g) $f(x) = x + a^2/x$ **h)** $f(x) = x^2 + a^4/x^2$ **i)** $f(x) = x/(x^2 + a^2)$

2. The product of two positive numbers is 10. Find the minimum value for their sum.

3. Suppose that x and y are two positive numbers for which $xy^2 = 10$. Find the minimum value of $x + y$.

4. Find the minimum distance from a point on the curve $y = 4/\sqrt{x}$ to the origin.

5. Find a cubic polynomial that has a relative minimum output when $x = -2$ and a relative maximum output when $x = 5$.

6. Prove part (b) of Theorem 1.19.

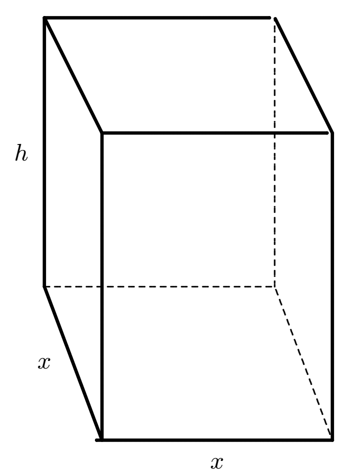
1.15 Optimization Problems

Many of the applications that involve mathematics begin with a verbal description. After developing an understanding of the problem, the next step is to translate the problem into the language of mathematics. It is then possible to use the tools of mathematics to solve the problem. Finally, the solution must be translated back into the original language of the problem.

One category of problems of this type are optimization problems; the goal is to maximize or minimize some quantity under certain restrictions. Since the derivative can be used to identify inputs that generate possible extreme outputs, calculus can be used to solve a number of these problems. The following problem and its solution illustrate the main ideas behind solving these types of problems.

Problem: Suppose that it is necessary to construct a rectangular box with a square base and a volume of twenty cubic feet. The material for the sides costs four dollars per square foot, the material for the base costs eight dollars per square foot, and the material for the top costs two dollars per square foot. Find the minimum cost of constructing the box.

Solution: The finished product will look something like this:



Since the base of the box is expensive, we do not want the base to be too large. However, a small base forces the sides to be rather large in order to maintain a volume of twenty cubic feet and the cost of construction increases again. There must be some dimensions in between these extremes that will minimize the cost. In general, a rectangular box has three dimensions, but in this case there are only two variables since the base is a square. Let

x be the length and width of the box, in feet;

h be the height of the box, in feet;

V be the volume of the box, in cubic feet;

C be the cost of construction for the box, in dollars.

The area of the base of the box is x^2, the area of the top of the box is also x^2, and each of the four sides of the box has an area of xh. It follows that $V = x^2h$ and $C = 2x^2 + 8x^2 + 4(4xh) = 10x^2 + 16xh$. We want to minimize the cost C, but it depends on both x and h. However, given a value for x, it is necessary to choose h so that $V = 20$. Hence, the height h is a function of the length x. It follows that C only depends on a choice of x. Given a value for x, we find that

$$h = \frac{20}{x^2} \quad \text{and} \quad C = 10x^2 + 16x \cdot \frac{20}{x^2} = 10x^2 + \frac{320}{x}.$$

Note that x may assume any positive value, that inputs of x near 0 generate large outputs for C, and that large inputs of x also generate large outputs for C. We want to find the value of x that will minimize C. Hence, we must solve the following mathematical problem:

$$\text{Find the minimum output of the function } C(x) = 10x^2 + \frac{320}{x} \text{ on the interval } (0, \infty).$$

To solve this problem, first find the values of x that satisfy the equation $C'(x) = 0$:

$$C'(x) = 20x - \frac{320}{x^2} \quad \text{and} \quad C'(x) = 0 \quad \text{implies} \quad x = \sqrt[3]{16} = 2\sqrt[3]{2}.$$

The analysis of the problem assures us that this value of x will generate the minimum cost, but the first derivative test can also be used. Since

$$C'(x) < 0 \text{ for } 0 < x < 2\sqrt[3]{2} \quad \text{and} \quad C'(x) > 0 \text{ for } 2\sqrt[3]{2} < x < \infty,$$

the function C has a minimum value when $x = 2\sqrt[3]{2}$. To find the minimum cost of the box and the height of the box, substitute this value of x into the equations for C and h to obtain $C = 120\sqrt[3]{4}$ and $h = 2.5\sqrt[3]{2}$. (Some effort is required to find these exact values.) Thus, the minimum cost of the box is about \$190.49 when the dimensions of the box are roughly 2.52 feet by 2.52 feet by 3.15 feet.

It should be pointed out that "real world" applications like this are always somewhat contrived. The construction of the box will require nails or screws (which cost a little), the sides must overlap some so that they can be fastened together, the box may need to fit in a certain corner of the basement, etc. In addition, plywood and other building materials are typically sold in fixed sizes, and small variations in the dimensions of the box will have little effect on the cost of the box. It is important to keep ideas such as these in mind and not turn off your common sense. However, the problems in this section are to be solved in the ideal mathematical world; a more careful analysis of the problem, which would be more complicated, will be deferred for now.

There are no definitive methods for solving optimization problems; practice and experience are the best teachers. The following list offers some suggestions that may be helpful.

1) Read the problem several times and make sure you understand what it says. (In other words, don't just glance at the problem and decide it is too hard or too confusing.) Draw a picture of the situation if that is appropriate; actually drawing several pictures representing different possibilities is a good idea. Be certain you understand which quantities vary and how they impact the problem.
2) Assign variables to unknown quantities and write down equations that link these variables. Finding mathematical connections between the variables may be the most difficult part of the problem as these

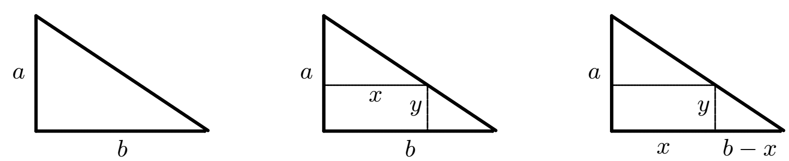
may not be immediately evident from the wording of the problem. Some common equations arise from known geometry formulas for area and volume, similar triangles, and the Pythagorean Theorem.

3) Express the "optimizing function" in terms of one variable and find the appropriate domain for this variable. The domain typically follows from the physical nature of the problem.
4) Solve the problem using calculus or perhaps some other mathematical technique such as completing the square. Be certain to verify that you obtain a maximum or a minimum as desired; you can use the closed interval method if appropriate or the first derivative test.
5) Be sure to answer the question as stated in the problem. This may involve translating your mathematics back into the language of the problem.

It is important to not give up on a problem too soon. Have confidence that you can find a solution and keep thinking about the problem until some useful insight occurs. Observing someone else solve one of these problems may make them look easy; once you see the key idea, the problem almost solves itself. The most crucial aspect of these problems is learning to find that key idea on your own. We conclude this section with one more example.

Problem: The legs of a right triangle have lengths a and b. Find the area of the largest rectangle that can be inscribed in this triangle assuming that one side of the rectangle is parallel to one of the legs.

Solution: Let x and y be the width and height, respectively, of a rectangle that satisfies the conditions of the problem (see the figure).



We want to maximize the area $A = xy$ of the rectangle. Using similar triangles, the relationship between x and y becomes apparent (see the far right triangle in the figure):

$$\frac{y}{a} = \frac{b-x}{b} \quad \text{or} \quad y = \frac{a}{b}(b-x).$$

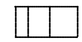
The problem can now be stated as follows:

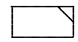
$$\text{Find the maximum value of } A(x) = \frac{a}{b}(bx - x^2) \text{ on the interval } [0, b].$$

It is easy to verify that the maximum value of A is $A(b/2) = ab/4$. Hence, the maximum area of the inscribed rectangle is $ab/4$. Note that the area of this largest rectangle is exactly half the area of the original triangle. The interested reader can show that the same result occurs if one side of the rectangle is parallel to the hypotenuse.

Exercises

1. A rectangular garden with area 400 m^2 is to be surrounded on three sides by a wall costing \$60/m and on the fourth side by a fence costing \$40/m. Find the dimensions for the garden that will minimize the cost.

2. Suppose you want to use 1000 feet of fencing to enclose a rectangular area that is further subdivided into three rectangles by two fences running parallel to one of the sides of the bordering rectangle (like this ). Find the largest total area that you can enclose.

3. A rectangular field with an area of 10,000 square feet is to be fenced off. In addition to going all around the rectangular field, the fence also cuts off an isosceles triangle inside one corner of the field. This part of the fence starts at the midpoint of one of the shorter sides of the rectangle (like this ). Find the dimensions of the field that minimize the amount of fencing required.

4. The area of the print on a book page is 45 square inches. The margins are one inch on the sides and bottom and one-half inch at the top. Find the dimensions of a page of this book if the only object is to use the minimal amount of paper. Also, find the minimum area of a page.

5. An open top rectangular box with a square base is to be constructed having a volume of 864 cm^3. Find the minimum surface area for the box.

6. You must construct a rectangular box with a square base. The material for the sides costs four dollars per square foot, the material for the base costs nine dollars per square foot, and the material for the top costs five dollars per square foot.

a) Find the volume of the largest box you can build for three hundred dollars.

b) Find the minimum cost of constructing a box with a volume of twenty-four cubic feet.

7. A can (with a top and a bottom) in the shape of a right circular cylinder is to be constructed with a volume of 300π cubic centimeters. Find the dimensions of the can that will minimize the surface area of the can. Also, determine the ratio of height to radius for this optimal can.

8. Find the maximum possible area of a rectangle with base that lies on the x-axis and with two upper vertices on the graph of the equation $y = 4 - x^2$.

9. Find the area of the largest rectangle that can be inscribed in an equilateral triangle of side length s. Assume that one side of the rectangle is parallel to a side of the triangle.

10. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius r. Find the lengths of all three sides of this optimal triangle. What fraction of the circle is occupied by the optimal triangle?

11. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r. Find the ratio of height to radius for this optimal cylinder. What fraction of the sphere is occupied by the optimal cylinder?

12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r. Find the ratio of height to radius for this optimal cone. What fraction of the sphere is occupied by the optimal cone?

13. Two vertical poles stand twenty feet apart. One is ten feet tall and the other is eight feet tall. Find the length of the shortest wire that can reach from the top of one pole to a point on the ground between the poles and then to the top of the other pole.

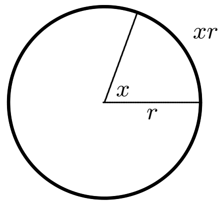
14. An island in a lake is located 600 yards opposite one end of a portion of the straight shoreline that is 3 miles long. Suppose you can swim at a rate of 3.2 miles per hour and can run at a rate of 9.5 miles per hour. Find the least amount of time required to get to the other end of the shoreline from the island. (Ignore transition time.)

15. Let a and b be positive constants. Find an equation for the line that passes through the point (a, b) and cuts off the least area in the first quadrant. (The line, which must have negative slope, becomes the hypotenuse of a right triangle with legs on the coordinate axes.) Find the area of this smallest triangle as well as the intercepts of the line.

16. Let a and d be positive numbers. Suppose that two light sources are separated by a distance d and that one source is a times as bright as the other. Find the point on the straight line between the light sources at which there is the least amount of light. Use the assumption that the intensity of the light at a point is proportional to the reciprocal of the square of the distance from the light source.

1.16 Trigonometric Functions

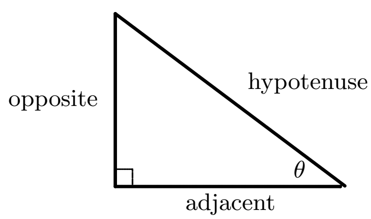
We assume that the reader is familiar with angles and angle measurement, both in degrees and radians. As a quick reminder, if x is a number between 0 and 2π, then the angle x radians is the angle cut off in a circle of radius r by an arc of length xr (see the figure).



$$360^\circ = 2\pi \text{ radians}$$
$$1^\circ = \frac{\pi}{180} \text{ radians}$$
$$\frac{180}{\pi}^\circ = 1 \text{ radian}$$

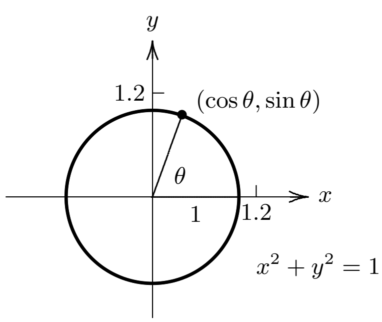
If $x > 2\pi$, then the angle is determined by "taking laps" in a counterclockwise direction. If $x < 0$, then the angle is determined by going in a clockwise direction.

The word "trigonometry" refers to the measurement of triangles. For acute angles, the trigonometric functions can be defined using the sides of a right triangle as in the figure below.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

However, in calculus, the trigonometric functions need to be defined for all real numbers. Given a real number θ, interpret θ as the radian measure of an angle with vertex at the origin and initial side the positive x-axis. The terminal side of this angle intersects the unit circle in a unique point. The x-coordinate of this point is defined to be $\cos\theta$ and the y-coordinate is defined to be $\sin\theta$. The other trigonometric functions are then defined in terms of $\sin\theta$ and $\cos\theta$.



$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

A number of relationships are clear from the definitions of the trigonometric functions. These include the fact that the trigonometric functions repeat every 2π units as well as the following identities:

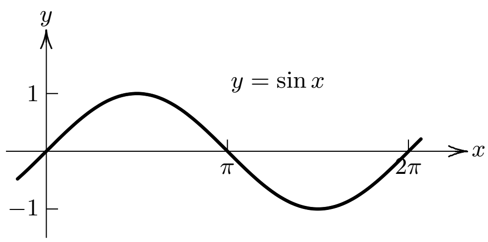
$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta \qquad \sin(\pi-\theta) = \sin\theta \qquad \sin(\pi+\theta) = -\sin\theta$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \cos(-\theta) = \cos\theta \qquad \cos(\pi-\theta) = -\cos\theta \qquad \cos(\pi+\theta) = -\cos\theta$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \tan(-\theta) = -\tan\theta \qquad \tan(\pi-\theta) = -\tan\theta \qquad \tan(\pi+\theta) = \tan\theta$$

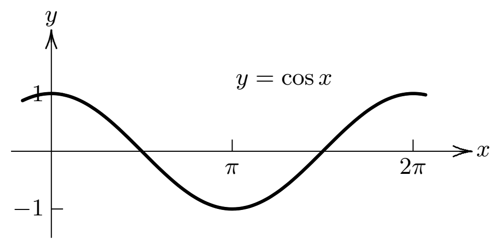
Another set of useful identities that follow from the symmetry of the circle are

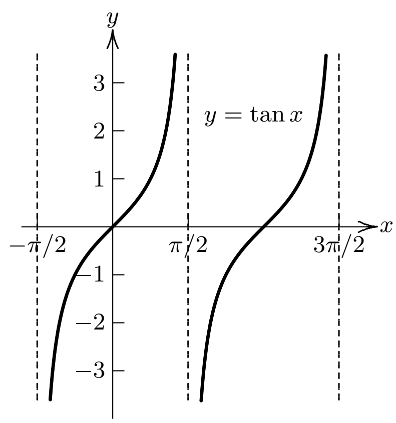
$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right); \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right); \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right).$$

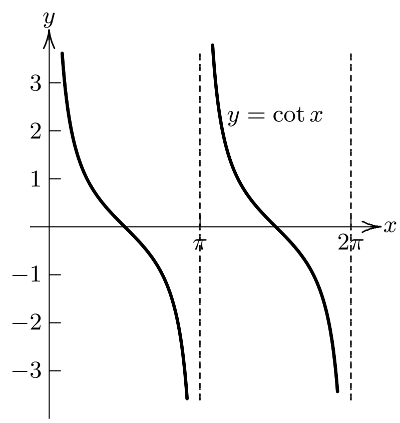
The prefix "co" in front of three of the trigonometric functions refers to the complement of an angle; for instance, the cosine of x is the sine of the complement of x.

In calculus, the angle will most often be denoted by x, where it is assumed that x is in radians. The graphs of the functions $\sin x$, $\cos x$, $\tan x$, and $\cot x$ are given below; graphs for $\sec x$ and $\csc x$ will be left for the reader. Since the graphs are periodic (that is, they repeat every 2π units) only a portion of each graph is given.









The exact values of the trigonometric functions can be determined easily for some angles. These values are recorded in the following table and should be used when they appear in problems.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	$*$

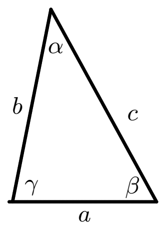
There are many other identities satisfied by the trigonometric functions. Some of these are listed below. There is no real need to memorize all of these formulas, but it is important to know that such formulas exist and to be able to use them when necessary.

$$\sin(x+y) = \sin x\cos y + \sin y\cos x \qquad \sin 2x = 2\sin x\cos x \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cos 2x = \cos^2 x - \sin^2 x \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y} \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

For triangles that do not have a right angle, the following relationships between the sides and angles of a triangle are sometimes useful.



law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

The proofs of these two properties of triangles are not difficult and will be requested in the exercises.

Finally, we give definitions for the **inverse trigonometric functions**. The number $\arcsin(1/2)$ represents the angle or arc (in radians) for which the value of the sine function is $1/2$. Since there are many angles for which this is true, we need to limit the range of potential answers in order to define a function. One way to proceed is the following.

1. For each real number $x \in [-1, 1]$, $\arcsin x$ is the unique real number taken from the interval $[-\pi/2, \pi/2]$ that satisfies $\sin(\arcsin x) = x$.

2. For each real number $x \in [-1, 1]$, $\arccos x = \dfrac{\pi}{2} - \arcsin x$.

3. For each real number x, $\arctan x = \arcsin\left(\dfrac{x}{\sqrt{x^2+1}}\right)$.

4. For each real number x, $\operatorname{arccot} x = \dfrac{\pi}{2} - \arctan x$.

5. For each real number x that satisfies $|x| \geq 1$, $\operatorname{arccsc} x = \arcsin(1/x)$.

6. For each real number x that satisfies $|x| \geq 1$, $\operatorname{arcsec} x = \dfrac{\pi}{2} - \operatorname{arccsc} x$.

Problem: Find the exact value of all the trigonometric functions when the angle is $2\pi/3$ radians.

Solution: The symmetry of the unit circle makes it possible to reduce problems such as this to a problem for which the angle lies in the first quadrant. Using identities and values listed in this section, we find that

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\left(\frac{2\pi}{3}\right) = -\cos\left(\pi - \frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}.$$

Once the sine and cosine are known, the other trigonometric functions are easily found:

$$\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}, \quad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}, \quad \sec\left(\frac{2\pi}{3}\right) = -2, \quad \text{and} \quad \csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}}.$$

Problem: Find all of the values of x in the interval $[-\pi, 2\pi]$ that satisfy $1 + 5\cos(3x) = 0$.

Solution: If we let $\theta = 3x$, then the given problem is equivalent to finding all of the solutions to $\cos\theta = -0.2$ that are in the interval $[-3\pi, 6\pi]$. (Be certain that you understand this first step.) There are several ways to proceed; here is one possibility. Let $z = \arccos 0.2$ and note that $z \in (0, \pi/2)$. The desired values of θ that lie in the interval $[0, 2\pi]$ are then $\pi - z$ and $\pi + z$. Since the trigonometric functions repeat every 2π units, we can just add or subtract multiples of 2π to each of these values to obtain further solutions to $\cos\theta = -0.2$. We limit ourselves to values of θ that belong to the interval $[-3\pi, 6\pi]$:

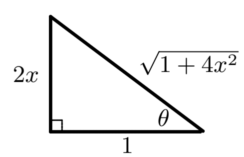
$$3x = \theta = -\pi + z,\ \pi - z,\ \pi + z,\ 3\pi - z,\ 3\pi + z,\ 5\pi - z,\ 5\pi + z.$$

Since $z \approx 1.3694$, the solutions to $1 + 5\cos(3x) = 0$ that lie in the interval $[-\pi, 2\pi]$ are

$$x \approx -0.591,\ 0.591,\ 1.504,\ 2.685,\ 3.598,\ 4.780,\ 5.692.$$

Problem: Simplify the expression $\sec(\arctan 2x)$.

Solution: Let $\theta = \arctan(2x)$. Then $\tan\theta = 2x$. This statement can be represented with a right triangle.



$$\tan\theta = 2x = \frac{2x}{1}$$
$$\sec\theta = \sqrt{1+4x^2}$$
$$\left(\theta \in (-\pi/2, \pi/2) \Rightarrow \sec\theta > 0\right)$$

Hence, the expression $\sec(\arctan 2x)$ simplifies to $\sqrt{1+4x^2}$.

Exercises

1. Find the exact value of all the trigonometric functions when the angle is $7\pi/6$ radians.

2. Find the exact value of all the trigonometric functions given that $\sin x = 2/3$ and $0 < x < \pi/2$.

3. Find the exact value of all the trigonometric functions given that $\tan x = -4$ and $\pi/2 < x < \pi$.

4. The hypotenuse of a right triangle is 12 and one of its angles is $38°$. Find the lengths of the other two sides.

5. Find all of the values of x in the interval $[-2\pi, 4\pi]$ that satisfy $\cos x = 1/2$.

6. Find all of the values of x in the interval $[0, 4\pi]$ that satisfy $\sin x = 0.7$.

7. Find all of the values of x in the interval $[0, 2\pi]$ that satisfy $5 - 7\sin(2x) = 0$.

8. Find all of the values of x in the interval $[0, 3\pi]$ that satisfy $2 + \cos(2x) = 5\sin x$.

9. Find three solutions to the equation $1 + \tan x = 0$.

10. Prove the law of sines. *Hint*: Compute the area of the triangle three different ways, using each side once as the base.

11. Prove the law of cosines. *Hint*: Referring to the figure in the text, drop a perpendicular from the top vertex to the side a, then use the Pythagorean Theorem on the right triangle having c as its hypotenuse.

12. Suppose a chord in a circle of radius 8 has length 12. Find the length of the arc cut off by the chord.

13. The sides of a triangle are 6, 15, and 16. Find the angles (to the nearest tenth of a degree) of the triangle.

14. Without a calculator, sketch a graph of each function.
a) $f(x) = 1 + 2\sin x$ **b)** $g(x) = 5\cos(2x)$ **c)** $h(x) = -2\tan(x/4)$

15. Sketch the graphs of $\sec x$ and $\csc x$.

16. List the ranges of the six inverse trigonometric functions.

17. Without a calculator, find the exact value of each of the following.
a) $\arcsin(1/\sqrt{2})$ **b)** $\arcsin(-1/2)$ **c)** $\arccos(-1/2)$
d) $\arctan(1/\sqrt{3})$ **e)** $\operatorname{arcsec}(-\sqrt{2})$ **f)** $\operatorname{arccsc}(-2/\sqrt{3})$

18. Simplify each of the following expressions. Indicate the values of x for which each is defined.
a) $\tan(\arcsin x)$ **b)** $\sin(\arctan x)$ **c)** $\cos(2\arcsin x)$

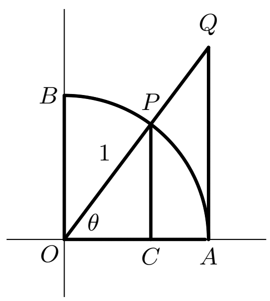
1.17 Derivatives of Trigonometric Functions

To find the derivative of the sine function, it is necessary to return to the definition of the derivative and determine some way of computing the limit of the difference quotient. In this case, some properties of the trigonometric functions and a few trigonometric identities provide the relevant information. The definition of the derivative yields

$$\begin{aligned} \frac{d}{dx}\sin x = \lim_{\theta\to 0}\frac{\sin(x+\theta)-\sin x}{\theta} &= \lim_{\theta\to 0}\frac{\sin x\cos\theta+\sin\theta\cos x-\sin x}{\theta} \\ &= \lim_{\theta\to 0}\Big(\cos x\,\frac{\sin\theta}{\theta}-\sin x\,\frac{1-\cos\theta}{\theta}\Big). \end{aligned}$$

To determine the limits of the quotients $\sin\theta/\theta$ and $(1-\cos\theta)/\theta$ as $\theta\to 0$, assume that θ is given in radians and consider the portion of the unit circle that lies in the first quadrant:

$A\ (1,0)$
$B\ (0,1)$
$C\ (\cos\theta, 0)$
$P\ (\cos\theta,\sin\theta)$
$Q\ (1,\tan\theta)$



area of triangle OCP is $\frac{1}{2}\sin\theta\cos\theta$
area of sector OAP is $\frac{1}{2}\,\theta$
area of triangle OAQ is $\frac{1}{2}\tan\theta$

From the figure, it is clear that the area of triangle OCP is less than the area of sector OAP which in turn is less than the area of triangle OAQ. Determining these areas in terms of θ and rearranging gives

$$\sin\theta\cos\theta<\theta<\tan\theta \;\Rightarrow\; \cos\theta<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}.$$

Although the figure indicates that θ is positive, this equation is valid for any small nonzero value of θ because $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)/(-\theta)=\sin\theta/\theta$. Since $\lim\limits_{\theta\to 0}\cos\theta=1$, the squeeze law gives

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1 \quad\text{or equivalently}\quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Using this limit and some algebra yields

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}=\lim_{\theta\to 0}\left(\frac{\sin\theta}{1+\cos\theta}\cdot\frac{\sin\theta}{\theta}\right)=0\cdot 1=0.$$

Given the values of these limits, the derivative of the function $\sin x$ is seen to be $\cos x$. Graphing $y=\sin x$, then using the graph to sketch its derivative makes this result seem very plausible.

The derivatives of the other five trigonometric functions can then be found using trigonometric identities and previous derivative formulas. For example, the chain rule and the quotient rule give

$$\begin{aligned} \frac{d}{dx}\cos x &= \frac{d}{dx}\sin\Big(\frac{\pi}{2}-x\Big)=\cos\Big(\frac{\pi}{2}-x\Big)(-1)=-\sin x; \\ \frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x}=\frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x. \end{aligned}$$

The derivatives of the six trigonometric functions are recorded in the following table.

$\dfrac{d}{dx}\sin x = \cos x$	$\dfrac{d}{dx}\tan x = \sec^2 x$	$\dfrac{d}{dx}\sec x = \sec x \tan x$
$\dfrac{d}{dx}\cos x = -\sin x$	$\dfrac{d}{dx}\cot x = -\csc^2 x$	$\dfrac{d}{dx}\csc x = -\csc x \cot x$

It is important to remember that all of the other derivative formulas still apply when differentiating functions that involve trigonometric functions. For example,

$$\begin{aligned}
\frac{d}{dx}\tan 5x &= \sec^2 5x \cdot 5 = 5\sec^2 5x;\\
\frac{d}{dx}x^2\sec x &= x^2\sec x\tan x + 2x\sec x = x\sec x(x\tan x + 2);\\
\frac{d}{dx}\cos^4(x^2) &= 4\cos^3(x^2)\cdot\big(-\sin(x^2)\cdot(2x)\big) = -8x\sin(x^2)\cos^3(x^2);\\
\frac{d}{dx}\frac{\cos x}{4+\sin x} &= \frac{(4+\sin x)(-\sin x) - \cos x(\cos x)}{(4+\sin x)^2} = \frac{-4\sin x - 1}{(4+\sin x)^2}.
\end{aligned}$$

For problems such as the third example, some students find it easier to rewrite trigonometric functions without the shorthand notation for powers:

$$\frac{d}{dx}\cos^4(x^2) = \frac{d}{dx}\big(\cos(x^2)\big)^4 = 4\big(\cos(x^2)\big)^3\cdot\big(-\sin(x^2)\cdot(2x)\big).$$

If you have trouble with this type of derivative, you might find this way of writing the powers helpful. Also, as in the fourth example, notice that trigonometric identities are sometimes used when simplifying derivatives of trigonometric functions. In this case, the basic identity $\sin^2 x + \cos^2 x = 1$ was used.

Exercises

1. Find and simplify the derivative of the given function.

a) $f(x) = \sin x - 2\cos 3x$	**b)** $g(x) = x^2\sin 2x$	**c)** $h(x) = \cos^3 2x$
d) $y = \dfrac{\cos x}{3+\sin x}$	**e)** $s = \sin^2 t\cos t$	**f)** $w = z - 2\sec 4z$
g) $u(x) = \sin x - \frac{1}{3}\sin^3 x$	**h)** $v(\theta) = \sec^2 5\theta$	**i)** $w(x) = \dfrac{\cos x}{2+\cot^2 x}$
j) $F(t) = t^2\sin(1/t)$	**k)** $G(t) = 5\tan 4t$	**l)** $H(x) = 3\sin^4(2x^2)$

2. Use the definition of the derivative to derive the derivative formula for $\cos x$.

3. Use the quotient rule or the chain rule to derive the derivative formula for the given function.

a) $\cot x$	**b)** $\sec x$	**c)** $\csc x$

4. Evaluate each of the following limits, where r is a nonzero real number.

a) $\displaystyle\lim_{\theta\to 0}\frac{\sin 2\theta}{\sin r\theta}$	**b)** $\displaystyle\lim_{\theta\to 0}\frac{\tan r\theta}{\theta}$	**c)** $\displaystyle\lim_{\theta\to 0}\frac{\tan r\theta}{\sin 7\theta}$

5. Find an equation for the line tangent to the curve $y = 2\sin x - 3\cos 2x$ when $x = \pi/6$.

6. Find all values of x in $[-\pi, 4\pi]$ for which the function $f(x) = x - 2\sin x$ has a horizontal tangent.

7. Show graphically that there is a point on the graph of $y = \sec x$ for which the tangent line goes through the origin. If the x-coordinate of such a point is a, what equation must a satisfy?

8. Find and simplify the derivative of $f(x) = \sqrt{1 + x^2 + x^4\cos^2 5x}$.

1.18 Derivative Applications of Trigonometric Functions

The derivatives of the inverse trigonometric functions will be useful in the chapter on integration. These formulas follow fairly easily from the derivatives of the trigonometric functions and some trigonometric identities. Assuming that $\arcsin x$ is differentiable (this fact does require proof, but we will not concern ourselves with it), its derivative can be found using an identity and the chain rule:

$$\sin(\arcsin x) = x \;\Rightarrow\; \frac{d}{dx}\sin(\arcsin x) = 1 \;\Rightarrow\; \cos(\arcsin x)\cdot\frac{d}{dx}\arcsin x = 1.$$

It follows that

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

The derivatives of the other inverse trigonometric functions can then be found using their definitions (see Section 1.16) and, if necessary, the chain rule. For instance,

$$\frac{d}{dx}\arccos x = \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) = -\frac{1}{\sqrt{1-x^2}}.$$

Proceeding in this way, we obtain the following formulas.

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

As one example, which incorporates the new formulas and the chain rule,

$$\frac{d}{dx}\arcsin(x^2/2) = \frac{1}{\sqrt{1-\left(x^2/2\right)^2}}\cdot x = \frac{x}{\sqrt{(4-x^4)/4}} = \frac{2x}{\sqrt{4-x^4}}.$$

The following problems illustrate these new derivative formulas (6 trigonometric functions and 6 inverse trigonometric functions). They also provide a review of the various applications of the derivative.

Problem: Find the maximum and minimum outputs of $f(x) = x - \sin 2x$ on the interval $[0, \pi]$.

Solution: We first find the critical inputs of f that lie in the interval $[0, \pi]$ by solving $f'(x) = 0$:

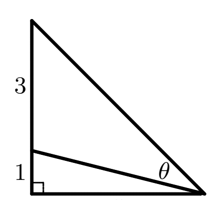
$$f'(x) = 1 - 2\cos 2x;\; f'(x) = 0 \;\Rightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Evaluating f at the critical inputs and the endpoints yields

$$f(0) = 0,\; f(\pi/6) = \frac{\pi}{6} - \frac{\sqrt{3}}{2},\; f(5\pi/6) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2},\; f(\pi) = \pi.$$

Using a calculator to approximate these values, the maximum output of f is $f(5\pi/6) \approx 3.4840$ and the minimum output of f is $f(\pi/6) \approx -0.3424$.

Problem: Find the value of x that will maximize θ.



Solution: Using right triangles, we can express θ as a function of x:

$$\theta(x) = \operatorname{arccot}(x/4) - \operatorname{arccot} x,\ 0 < x < \infty.$$

To find the value of x that maximizes θ, first find and simplify $\theta'(x)$. The result is

$$\theta'(x) = -\frac{4}{16+x^2} + \frac{1}{1+x^2} = \frac{12-3x^2}{(1+x^2)(16+x^2)}.$$

The only relevant critical input is 2. Since θ' is positive on the interval $(0, 2)$ and negative on the interval $(2, \infty)$, the function θ has a maximum output when $x = 2$. The corresponding maximum value for θ is approximately 36.87°. (Do you see the advantage of using $\operatorname{arccot} x$ rather than $\arctan x$ in this problem?)

Exercises

1. Find and simplify the derivative of the given function.

a) $f(x) = \arcsin(x/2)$ **b)** $g(t) = \arctan(t/4)$ **c)** $h(x) = \arccos(3/x)$

d) $y = \left(\arcsin x\right)^2$ **e)** $s = (1+t^2)\arctan t$ **f)** $r = \theta \arccos\theta - \sqrt{1-\theta^2}$

2. Use the chain rule to verify the derivative formulas for $\arctan x$ and $\operatorname{arccsc} x$.

3. Find the maximum and minimum outputs of the function $f(x) = \sin x + \sqrt{3}\cos x$ on the interval $[0, \pi]$.

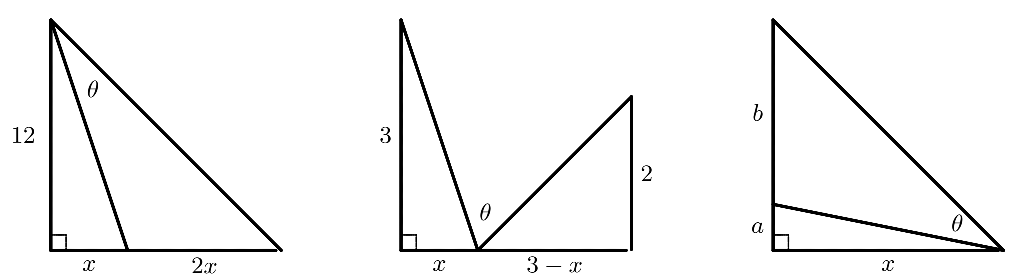
4. Find the maximum and minimum outputs of the function $g(x) = x - 2\cos x$ on the interval $[-\pi, \pi]$.

5. Find the maximum and minimum outputs of the function $h(x) = \sin^2 x + \cos x$ on the interval $[0, \pi]$.

6. Determine the intervals in $[0, 2\pi]$ on which the function $f(x) = \sin x + \sin^2 x$ is increasing and those on which it is decreasing.

7. A line of length 60 is split into equal thirds. The right and left thirds are then each bent upward through the same angle θ to form a (topless) trapezoid. Find the value of θ that will maximize the area of the trapezoid.

8. Consider each of the following figures with quantities as indicated.



a) For the left figure, find the value of $x \in (0, \infty)$ that will maximize θ.

b) For the middle figure, find the value of $x \in [0, 3]$ that will maximize θ.

c) For the right figure, find the value of $x \in (0, \infty)$ that will maximize θ. Treat a and b as constants.

1.19 Properties of Exponents and Logarithms

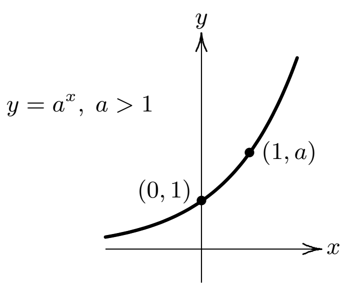
Let a be a real number and let n be a positive integer. The symbol a^n represents the product of a with itself n times; it is simply a shorthand notation. From this definition, it follows easily that

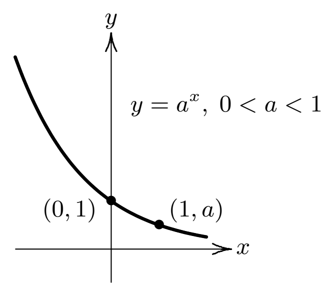
$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad \text{and} \quad \left(a^m\right)^n = a^{mn}$$

for positive integers m and n. Consistency with these rules yields the properties (give these some thought):

$$a^1 = a, \quad a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^{1/n} = \sqrt[n]{a}, \quad \text{and} \quad a^{m/n} = \sqrt[n]{a^m}.$$

In this way, the expression a^x can be defined for any rational number x. However, in order to apply calculus to an exponential function, the domain must be all real numbers. In other words, it is necessary to assign a meaning to a^x even when x is irrational. Since an irrational number can be approximated to any degree of accuracy by a rational number, the expression a^x can be approximated to any degree of accuracy for an irrational exponent x. For example, $a^{\sqrt{2}}$ is the limit of the sequence $a^{1.4}, a^{1.41}, a^{1.414}, \ldots$. It can be shown that the function f defined by $f(x) = a^x$, where a is any positive number, is continuous for all real numbers; the general shape of its graph is given below.





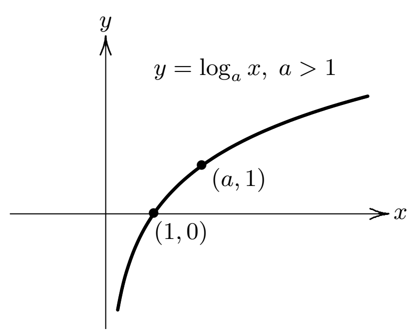
$$a^r a^s = a^{r+s} \qquad a^0 = 1$$

$$\frac{a^r}{a^s} = a^{r-s} \qquad a^1 = a$$

$$\left(a^r\right)^s = a^{rs} \qquad (ab)^r = a^r b^r$$

The properties of exponents for arbitrary real numbers are listed to the right of the graphs.

When working with exponential functions, it is sometimes necessary to solve equations such as $2^x = 21$ or $10^x = 280$. This leads to the concept of logarithm. The symbol $\log_a b$ represents the exponent to which a must be raised in order to obtain the number b. In other words, the equation $x = \log_a b$ is equivalent to $a^x = b$. The graph of $y = \log_a x$, where $a > 1$, is sketched below. Note that the domains of these functions are $(0, \infty)$.



$$\log_a(rs) = \log_a r + \log_a s$$

$$\log_a(r/s) = \log_a r - \log_a s$$

$$\log_a r^s = s \log_a r$$

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$\log_a a^r = r,\ r \in \mathbb{R}$$

$$a^{\log_a r} = r,\ r > 0$$

Since a logarithm is an exponent, logarithms have properties that are related to the properties of exponents. These are listed to the right of the graph.

Although logarithms can be defined using any positive number a as the base, the most common choices for bases are 10 and e. The choice of 10 should come as no surprise. The **number e**, which is a very important mathematical constant, can be defined by $e = \lim_{h\to 0}\left(1+h\right)^{1/h}$. To five decimal places, $e \approx 2.71828$. Logarithms to the base e are usually written as $\ln x$ rather than $\log_e x$, that is, $\ln x = \log_e x$.

Problem: Given that $\log_a 2 = r$ and $\log_a 3 = s$, express $\log_a 72$ in terms of r and s.

Solution: Using the properties of logarithms, we find that

$$\log_a 72 = \log_a(2^3 \cdot 3^2) = \log_a 2^3 + \log_a 3^2 = 3\log_a 2 + 2\log_a 3 = 3r + 2s.$$

Problem: Find the exact value of x that satisfies the equation $5 + 2e^{4x} = 89$.

Solution: To solve this problem, first isolate the exponential term, then rewrite the equation in logarithmic form:

$$5 + 2e^{4x} = 89 \ \Rightarrow\ e^{4x} = 42 \ \Rightarrow\ 4x = \ln 42 \ \Rightarrow\ x = \frac{1}{4}\ln 42.$$

This represents the exact value of x that satisfies the equation.

Exercises

1. Without a calculator, find $\log_{10} 0.01$, $\log_3(1/81)$, $\log_2 32$, $\log_4 8$, $\ln e^2$, $\ln(1/e)$, and $\log_{10}\sqrt{10}$.

2. Suppose that $\log_a 2 = r$, $\log_a 3 = s$, and $\log_a 10 = t$. Express $\log_a 6$, $\log_a 15$, $\log_a(1/9)$, $\log_a 4000$, $\log_a 24$, $\log_a 30$, $\log_a(9/20)$, and $\log_a 0.0003$ in terms of r, s, and t.

3. Find the exact value of x that satisfies $\ln(x-2) - \ln 5 = 1$.

4. Find the exact value of x that satisfies the equation, then use a calculator to approximate x to the nearest thousandth.

a) $4^x = 25$ **b)** $x = \log_6 1000$ **c)** $150e^{0.2x} = 1000$

d) $100 - e^{0.04x} = 20$ **e)** $\dfrac{10e^{-0.1x}}{1 + 0.3e^{-0.1x}} = 4$ **f)** $e^x + e^{-x} = 6$

5. The temperature of an object in degrees Celsius at time t minutes is given by $T(t) = 82 - 40e^{-0.3t}$. What is the initial temperature? What is the temperature when $t = 3$? When will the temperature be 80° Celsius?

6. Suppose that a function f is defined by $f(x) = Ce^{kx}$, where C and k are constants. Given that $f(2) = 100$ and $f(5) = 800$, find C and k.

7. Jack and Jill plant beans. Jack's beanstalk is t^2 inches tall after t days while Jill's beanstalk is 2^t inches tall after t days. How tall is each beanstalk after 3 weeks? Give your answers in suitable units.

8. Evaluate the expressions $\sqrt[10]{x}$ and $\log_{10} x$ for $x = 100$ and $x = 10^{1000}$. Which of the two expressions is greater for large values of x? Find an exact value of x for which the two expressions are equal.

9. Let $a > 0$. Prove that $a^x = e^{x\ln a}$ for all real numbers x.

10. Let $a > 0$. Prove that $\log_a x = \dfrac{\ln x}{\ln a}$ for all positive real numbers x.

11. Use the fact that $e = \lim_{h\to 0}\left(1+h\right)^{1/h}$ to evaluate each of the following limits. Treat a as a nonzero constant.

a) $\lim_{h\to 0}\left(1+h\right)^{a/h}$ **b)** $\lim_{h\to 0}\left(1+ah\right)^{1/h}$ **c)** $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

1.20 Derivatives of Exponential and Logarithmic Functions

The derivative formula for an exponential function is quite simple. To derive this formula, recall that the number e is defined by $e = \lim_{h\to 0}(1+h)^{1/h}$. This means that $(1+h)^{1/h} \approx e$ or $(e^h - 1)/h \approx 1$ when h is near 0. Since the approximations improve as $h \to 0$, it follows that $\lim_{h\to 0}(e^h - 1)/h = 1$. This fact, along with the definition of the derivative, can then be used to find the derivative of the exponential function e^x:

$$\frac{d}{dx}\,e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x,$$

a very simple derivative formula. By a property of exponents and the chain rule,

$$\frac{d}{dx}\,a^x = \frac{d}{dx}\,e^{x\ln a} = e^{x\ln a}\ln a = (\ln a)\,a^x,$$

for any positive number a. Hence, the derivative of any exponential function is a constant multiple of the original function. In other words, the derivative of a^x is proportional to a^x and $\ln a$ is the constant of proportionality. The reason e is a special number is because the constant of proportionality for the derivative of e^x is 1.

The following examples illustrate this new derivative formula.

$$\frac{d}{dx}\,(3x^2 + 5e^x) = 6x + 5e^x;$$

$$\frac{d}{dx}\,(e^{4x} - 3e^{-2x}) = e^{4x}\cdot 4 - 3e^{-2x}\cdot(-2) = 4e^{4x} + 6e^{-2x};$$

$$\frac{d}{dx}\,xe^{x^2/4} = xe^{x^2/4}\cdot(x/2) + e^{x^2/4} = \frac{1}{2}\,(x^2+2)e^{x^2/4};$$

$$\frac{d}{dx}\,(2^x + 3^{\sqrt{x}}) = (\ln 2)\,2^x + (\ln 3)\,3^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}.$$

To make it explicit, the chain rule yields $\dfrac{d}{dx}\,e^{f(x)} = f'(x)e^{f(x)}$.

To find the derivative of the function $\ln x$, we begin with the identity $e^{\ln x} = x$, which is valid for all $x > 0$. Assuming that $\ln x$ is differentiable, differentiating both sides of this identity yields

$$e^{\ln x}\cdot\frac{d}{dx}\,\ln x = 1 \quad\Rightarrow\quad \frac{d}{dx}\,\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x},$$

another simple derivative formula. For example,

$$\frac{d}{dx}\,(x^3\ln x) = x^3\cdot\frac{1}{x} + 3x^2\ln x = x^2(1 + 3\ln x);$$

$$\frac{d}{dx}\,\ln(x^2+6x+10) = \frac{1}{x^2+6x+10}\cdot(2x+6) = \frac{2x+6}{x^2+6x+10}.$$

Note that $\dfrac{d}{dx}\,\ln(-x) = \dfrac{-1}{-x} = \dfrac{1}{x}$ for all $x < 0$. This fact, along with the chain rule, yields the formulas

$$\frac{d}{dx}\,\ln|x| = \frac{1}{x} \quad\text{and}\quad \frac{d}{dx}\,\ln\bigl|f(x)\bigr| = \frac{f'(x)}{f(x)}.$$

Problem: Determine the intervals on which the function f defined by $f(x) = x^2e^{-x/4}$ is increasing and those on which it is decreasing.

Solution: The function f is differentiable for all real numbers, so the only critical inputs will be those for which $f'(x) = 0$. Since

$$f'(x) = x^2 \cdot e^{-x/4}\Big(-\frac{1}{4}\Big) + 2x \cdot e^{-x/4} = \frac{x}{4}\,(8 - x)\,e^{-x/4},$$

we see that $f'(x) = 0$ when x is 0 or 8. Checking the sign of f' on the intervals $(-\infty, 0)$, $(0, 8)$, and $(8, \infty)$, it follows that f is decreasing on the intervals $(-\infty, 0]$ and $[8, \infty)$, and increasing on the interval $[0, 8]$.

Problem: Find the maximum output of the function g defined by $g(x) = x^{-1/4} \ln x$.

Solution: The domain of the function g is $(0, \infty)$. Note that $g(x) < 0$ for $0 < x < 1$ and $g(x) > 0$ for $x > 1$. Using the product rule to find g', we obtain

$$g'(x) = x^{-1/4} \cdot \frac{1}{x} - \frac{1}{4}\,x^{-5/4} \ln x = x^{-5/4}\Big(1 - \frac{1}{4}\ln x\Big) = \frac{4 - \ln x}{4x^{5/4}}$$

and conclude that g' is positive on $(0, e^4)$ and negative on (e^4, ∞). It follows that g has a maximum value when $x = e^4$. Substituting this value into the formula for g reveals that the maximum output of g is $4/e$.

Exercises

1. Find and simplify the derivative of the given function.

a) $f(x) = e^x + 2e^{-2x}$ **b)** $g(x) = 5x - 4e^{-x^2}$ **c)** $h(x) = (x + 1)e^{-x}$

d) $F(x) = \ln|2x + 1|$ **e)** $G(x) = e^{\sin 2x}$ **f)** $H(x) = 6\tan(e^{x/2})$

g) $y = \cos^2(e^{-x})$ **h)** $s = e^{-2t}\sin 3t$ **i)** $w = (\ln z)^4$

j) $u(x) = \ln|x^2 + 4x + 2|$ **k)** $v(x) = x\ln x$ **l)** $w(x) = \dfrac{\ln x}{x^3}$

m) $y = x^3 + 3^x$ **n)** $s = \dfrac{e^t}{1 + e^{2t}}$ **o)** $w = \sqrt{z^2 + (\ln z)^2}$

2. The velocity v in meters per second of a particle at time t seconds is given by $v = (t^2 + 100t)e^{-t}$. What is the acceleration of the particle after ten seconds?

3. Find the exact minimum output of the function $f(x) = e^x + e^{-2x}$.

4. Find the maximum output of the function $g(x) = a^2xe^{-ax}$, where a is a positive constant.

5. Find the maximum and minimum outputs of $(\ln x)/\sqrt{x}$ on the interval $[1, 25]$.

6. Find the intervals on which the function is increasing and those on which it is decreasing and determine the nature of the critical inputs.

a) $f(x) = x^2e^{-x^2}$ **b)** $g(x) = \dfrac{\ln x}{x^2}$ **c)** $h(x) = \dfrac{(\ln x)^2}{x}$

7. Find a point on the graph of $y = 2e^{x/3}$ at which the tangent line passes through the origin.

8. Prove that $\dfrac{d}{dx}\,x^r = rx^{r-1}$ for any nonzero real number r. *Hint*: Note that $x^r = e^{r\ln x}$.

9. Consider the rather unusual function $f(x) = x^x$, defined for $x > 0$. Find the derivative of this function, then find its minimum output. *Hint*: Write $x^x = e^{x\ln x}$.

10. Use a property of logarithms to find a derivative formula for $\log_a x$, where $a > 0$.

1.21 Differential Equations

In some cases, it is necessary to find a function given information about its derivative, usually in the form of an equation. An equation that involves the derivative of a function is known as a **differential equation**. A solution to a differential equation is a function that satisfies the equation. As an extremely simple example, suppose that we need to find a function f with the property that $f'(x) = 2x$. It is clear that the function $f(x) = x^2$ satisfies this differential equation. Note that $f(x) = x^2 + 1$ and $f(x) = x^2 - 8$ also satisfy the differential equation. In fact, any function of the form $f(x) = x^2 + C$, where C is a constant, satisfies the equation. This follows from the fact that the derivative of a constant is zero. In general, there are an infinite number of solutions to a given differential equation. However, if further information about the function is known, a unique solution can be determined. Suppose that $f'(x) = 2x$ and $f(1) = 4$. The only function f that satisfies both of these conditions is $f(x) = x^2 + 3$.

One way to find a function given its derivative is to just think about differentiation in reverse. For instance, for power functions differentiation reduces the exponent by one; the reverse process will add one to the exponent. If $f'(x) = x^5$, then $f(x) = \frac{1}{6}x^6 + C$, where C is any constant. Two further examples of simple differential equations follow.

If $f'(x) = 6x^2 - 4x + 1$ and $f(1) = 0$, then $f(x) = 2x^3 - 2x^2 + x - 1$.

If $g'(x) = 4\cos x$ and $g(\pi) = 2$, then $g(x) = 4\sin x + 2$.

Although there are many techniques for solving differential equations of this type (some will be considered in the next chapter), for now just think about differentiation in reverse. One of the advantages of differential equations is the fact that you can always check your answer by taking the derivative of your proposed solution and seeing if it satisfies the given differential equation.

Now consider the differential equation $f'(x) = f(x)$. This differential equation is of a different type than the ones just considered; the unknown function appears on both sides of the equation. A solution to this equation is a function whose derivative is the same as the original function. Thinking over the list of derivative formulas indicates that a solution is $f(x) = e^x$. Where should the generic constant C go? A quick check shows that $f(x) = e^x + C$ does not satisfy the equation $f'(x) = f(x)$, but $f(x) = Ce^x$ does. In general, if both C and k are constants, the chain rule gives

$$\frac{d}{dx}Ce^{kx} = Ce^{kx} \cdot k = k\big(Ce^{kx}\big).$$

This shows that the derivative of Ce^{kx} is k times itself. This discussion essentially proves the following theorem.

THEOREM 1.20 Let G be a function and let k be a constant. If $G'(x) = kG(x)$ for all x, then $G(x) = Ce^{kx}$, where C is a constant that can be determined from further information. In fact, it is easy to see that $C = G(0)$. ■

Here are two examples that illustrate the type of differential equation considered in the theorem.

If $f'(x) = 4f(x)$ and $f(0) = 20$, then $f(x) = 20e^{4x}$.

If $s'(t) = -s(t)$ and $s(0) = 5$, then $s(t) = 5e^{-t}$.

It is important to remember that this type of differential equation is different than the earlier type. In the first case, the derivative of the function is explicitly given, but in these last examples, it is given that the derivative of the function is a multiple of the function.

Using a little bit of algebra, we can solve slightly more complicated differential equations. The following two examples illustrate the main ideas.

Problem: Find a function f such that $f'(x) = f(x) + 3$ and $f(0) = 10$.

Solution: Since the functions $f(x)$ and $f(x) + 3$ have the same derivative, we find that

$$f'(x) = f(x) + 3 \;\Leftrightarrow\; \frac{d}{dx}\big(f(x) + 3\big) = f(x) + 3$$

The second equation is of the form given by the theorem with $G(x) = f(x) + 3$ and $k = 1$. It follows that the function f satisfies $f(x) + 3 = Ce^x$. Using the fact that $f(0) = 10$ gives $C = 13$. Thus the solution to the differential equation is $f(x) = 13e^x - 3$.

Problem: Find a function A such that $A'(t) = 10 - 2A(t)$ and $A(0) = 40$.

Solution: The first step is to factor out the coefficient of $A(t)$, then proceed as in the previous example.

$$A'(t) = -2\big(A(t) - 5\big) \;\Rightarrow\; \frac{d}{dt}\big(A(t) - 5\big) = -2\big(A(t) - 5\big) \;\Rightarrow\; A(t) - 5 = Ce^{-2t}.$$

(In this case, G is the function $A(t) - 5$ and $k = -2$.) Using the fact that $A(0) = 40$ gives $C = 35$. Thus the solution to the differential equation is $A(t) = 35e^{-2t} + 5$.

Exercises

1. Solve each of the following simple differential equations.

a) $f'(x) = 3x^2,\ f(1) = 2$ **b)** $g'(x) = \dfrac{2}{x},\ g(1) = 5$ **c)** $h'(x) = \sqrt{x},\ h(4) = 6$

d) $F'(x) = 3x^2 - 8x,\ F(0) = 1$ **e)** $G'(x) = x^4 + 2x^3 - x + 1,\ G(1) = 3$

f) $u'(t) = 4e^{2t},\ u(0) = 12$ **g)** $v'(t) = 8\sin(2t),\ v(\pi) = 5$ **h)** $w'(t) = \dfrac{t}{\sqrt{t^2+1}},\ w(0) = 2$

2. Solve each of the following differential equations.

a) $f'(t) = -2f(t),\ f(0) = 8$ **b)** $g'(t) = 4g(t),\ g(0) = 5$ **c)** $h'(t) = 5h(t),\ h(1) = 2$

d) $F'(x) = 2 + F(x),\ F(0) = 10$ **e)** $G'(x) = 4 - G(x),\ G(0) = 30$ **f)** $H'(x) = 3\big(1 + H(x)\big),\ H(0) = 46$

3. Solve each of the following differential equations.

a) $A'(t) = 8 + 2A(t),\ A(0) = 20$ **b)** $B'(t) = 15 - 5B(t),\ B(0) = 2$ **c)** $S'(t) = 80 - \dfrac{1}{3}S(t),\ S(0) = 40$

1.22 Applications of Differential Equations

Since a derivative can be interpreted as a rate of change, it can be used in many quantitative situations that involve change. A common situation that occurs in applications is one in which a quantity changes at a rate proportional to its size. Radioactive decay, population growth, chemical reactions, and debts or annuities involving interest can, under certain conditions, experience changes of this type. If $A(t)$ represents a function of time t, then the statement "the rate of change of A is proportional to A" can be translated mathematically as $A'(t) = kA(t)$, where k is a constant. The solution to this differential equation (see the last section) is $A(t) = Ce^{kt}$, where $C = A(0)$ is a constant. Two applications of this idea are given below.

Problem: Suppose a certain radioactive sample decays from 800 grams to 500 grams in 2 years. When will only 100 grams of the sample remain?

Solution: Let $A(t)$ be the number of grams in the sample after t years. It is given that $A(0) = 800$ and $A(2) = 500$. Since radioactive samples decay at a rate proportional to the amount present, there exists a constant k such that $A'(t) = kA(t)$. (Note that k will be a negative number.) It follows that $A(t) = 800e^{kt}$. To find k, we use the fact that $A(2) = 500$:

$$500 = 800e^{k\cdot 2} \;\Rightarrow\; k = \frac{1}{2}\,\ln\bigl(5/8\bigr).$$

Rather than obtain a decimal approximation for k (which can lead to roundoff errors), it is best to leave k in exact form. Thus $A(t) = 800e^{kt}$, where k has the value found above. To answer the question, solve the equation $A(t) = 100$ for t. This yields

$$100 = 800e^{kt} \;\Rightarrow\; t = \frac{\ln\bigl(1/8\bigr)}{k} = \frac{2\ln\bigl(1/8\bigr)}{\ln\bigl(5/8\bigr)} \approx 8.85.$$

Hence, there will be 100 grams of the sample about 8.85 years after the initial amount of 800 grams.

Problem: A 300 gallon tank contains 200 gallons of brine (salt dissolved in water) with a concentration of 1/4 pound of salt per gallon of water. A brine containing 1 pound of salt per gallon of water runs into the tank at the rate of 4 gallons per minute, and the well-stirred mixture runs out of the tank at the same rate. When will there be 100 pounds of salt in the tank?

Solution: Let $A(t)$ be the number of pounds of salt in the tank after t minutes. Since the initial concentration is 1/4 lb/gal, it is easy to see that $A(0) = 200 \cdot (1/4) = 50$. However, more effort is required to find a differential equation for $A(t)$. The units of $A'(t)$ are pounds per minute, so we need to determine the number of pounds of salt per minute entering the tank and subtract the number of pounds of salt per minute leaving the tank. The inflow concentration is 1 pound of salt per gallon and the outflow concentration is $A(t)/200$ pounds of salt per gallon. (This is where the well-stirred assumption is used; it is assumed that the salt is spread evenly throughout the tank.) Thus the rates at which salt enters and leaves the tank are

$$\bigl(4\,\text{gal/min}\bigr)\cdot\bigl(1\,\text{lb/gal}\bigr) = 4\,\text{lb/min} \quad\text{and}\quad \bigl(4\,\text{gal/min}\bigr)\cdot\Bigl(\frac{A(t)}{200}\,\text{lb/gal}\Bigr) = \frac{A(t)}{50}\,\text{lb/min}$$

respectively. It follows that $A'(t) = 4 - A(t)/50$. To find $A(t)$, we need to solve the differential equation

$$A'(t) = 4 - \frac{1}{50}A(t),\ A(0) = 50.$$

The techniques in the previous section yield $A(t) = 200 - 150e^{-t/50}$. To answer the question, solve $A(t) = 100$ for t to obtain $t = 50 \ln 1.5$. Therefore, there are 100 pounds of salt in the tank about 20.27 minutes after the process is started.

Exercises

1. The radioactive isotope thorium-234 has a half-life (the time required for half of a sample to decay) of about 590 hours. If 50 mg are present initially, after how many hours will there be only 40 mg left?

2. A population of bacteria increases at a rate proportional to the current population. Initially there were 100 bacteria and one hour later there are 130 bacteria. How many bacteria will there be after 24 hours?

3. The rate at which sugar dissolves in water kept at a constant temperature is proportional to the amount that remains to be dissolved. Suppose that 10 pounds of sugar is added to a vat of water and that 5 pounds dissolves in the first five minutes. How long does it take for the next 4 pounds to dissolve?

4. A 300 gallon tank contains 200 gallons of brine with a concentration of 0.4 pounds of salt per gallon of water. A brine containing 1.2 pounds of salt per gallon of water runs into the tank at the rate of 5 gallons per minute, and the well-stirred mixture runs out of the tank at the same rate. When will there be 180 pounds of salt in the tank?

5. A tank contains 300 gallons of fresh water. A brine containing 0.5 pounds of salt per gallon of water runs into the tank at the rate of two gallons per minute, and the well-stirred mixture runs out at the same rate. What is the concentration of salt in the tank at the end of ten minutes?

6. A 300 gallon tank contains 100 gallons of brine with a concentration of one pound of salt per gallon of water. A brine containing 0.5 pounds of salt per gallon of water runs into the tank at the rate of four gallons per minute, and the well-stirred mixture runs out of the tank at the same rate. When is the concentration of salt in the tank 0.7 pounds per gallon?

7. Five hundred gallons of pesticide is accidentally spilled into a lake with a volume of 8×10^7 gallons and uniformly mixes with the lake water. A river flows into the lake bringing 10,000 gallons of fresh water per minute and water leaves the lake at the same rate. Estimate how long will it take to reduce the pesticide in the lake to one part per billion.

8. A cup of tea is made by pouring boiling water (212° F) into a cup. Three minutes later, the tea is brewed but at 180° F is too hot to drink. Assuming that the air is a constant 68° F, how long will it take for the tea to become a drinkable temperature of 160° F? (Use Newton's Law of Cooling, which states that the rate of change of temperature of an object is proportional to the difference in temperature of the object and its surroundings.)

9. Suppose that you have acquired a college debt of \$12,000. The annual interest rate is 6% and you pay \$200 each month. How long does it take you to pay off your loan and how much do you pay altogether? Answer these same questions assuming you only pay \$100 each month. *Hint*: In order to obtain a differential equation, we need to assume that both interest and payments are made continuously. Let $A(t)$ be the amount you owe after t years. Since you are paying at the rate of \$2400 per year, the function A satisfies the differential equation $A'(t) = 0.06A(t) - 2400$.

10. Repeat Exercise 9 assuming that the interest rate is 9%.

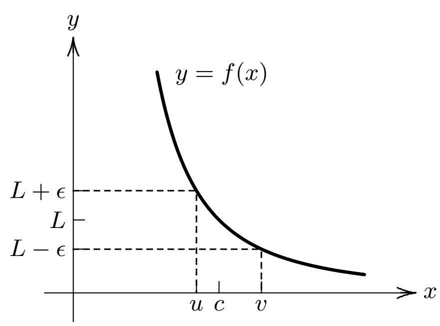
11. Repeat Exercise 9 assuming that the original debt is \$20,000.

12. Suppose that you have a credit card debt of \$1100. The annual interest rate is 12% and you pay \$15 each month. How long does it take you to pay off your loan and how much do you pay altogether? See the hint for Exercise 9.

1.23 Definition of Limit

An intuitive understanding of the symbols $\lim_{x\to c} f(x) = L$ is sufficient for many situations, but there are applications (both applied and theoretical) in which it is necessary to be more specific than the phrase "$f(x)$ is close to L when x is close to c". During the eighteenth and early nineteenth centuries, mathematicians struggled to come up with a suitable definition for the limit concept. The definition obtained at the end of this period is still in use today. This abstract but important concept will be the topic for this section.

A function f has a limit of L at a point c if the output values of f are near L when the input values are near c. In order to turn this sentence into a mathematical definition, it is necessary to explicitly quantify the phrases "near L" and "near c". The absolute value function can be used to determine when one number is near another since $|a-b|$ represents the distance between the points a and b. If $|x-c|$ and $|f(x)-L|$ are small, then x is near c and $f(x)$ is near L. In order for the limit of f at c to be L, the functional values $f(x)$ must be within a prescribed (small and arbitrary) distance of L for all input values of x close enough to c, but not including c. The value of the function at c does not come into play when finding the limit at c; the function may not even be defined at c. A function f has a limit L at c if given any positive number ϵ, it is possible to find another positive number δ so that $|f(x)-L|<\epsilon$ for all x that satisfy $0<|x-c|<\delta$. That is, if you want the functional values of f to be within ϵ of L, you must choose the input values of f to be within δ of c (see the figure).



To guarantee that $|f(x)-L|<\epsilon$ for all x that satisfy $0<|x-c|<\delta$, choose $\delta = \min\{v-c, c-u\}$.

DEFINITION 1.21 Let f be defined on some open interval containing the point c, except possibly at c. Then $\lim_{x\to c} f(x) = L$ if for each $\epsilon>0$ there exists $\delta>0$ such that $|f(x)-L|<\epsilon$ for all x that satisfy $0<|x-c|<\delta$.

This is not an easy definition and it takes time and effort to understand it well. For the time being, it is sufficient to understand the quantities that appear in the definition and how they correspond to the intuitive notion of a limit. For starters, the first problem below illustrates the relationship between ϵ and δ using actual numbers. The second problem provides a brief glance into the nature of limit proofs.

Problem: Consider $\lim_{x\to 4} \frac{1}{\sqrt{x}} = \frac{1}{2}$. Given $\epsilon = 0.01$, find (to 4 decimal places) a suitable value for δ.

Solution: According to the definition of limit, we must find $\delta>0$ so that

$$0<|x-4|<\delta \;\Rightarrow\; \left|\frac{1}{\sqrt{x}}-\frac{1}{2}\right|<0.01.$$

We begin with the second inequality and perform algebra until the quantity $x-4$ appears:

$$\left|\frac{1}{\sqrt{x}}-\frac{1}{2}\right|<0.01 \Leftrightarrow -0.01<\frac{1}{\sqrt{x}}-0.5<0.01 \Leftrightarrow 0.49<\frac{1}{\sqrt{x}}<0.51 \Leftrightarrow$$
$$\frac{1}{0.51}<\sqrt{x}<\frac{1}{0.49} \Leftrightarrow \frac{1}{0.51^2}<x<\frac{1}{0.49^2} \Leftrightarrow \frac{1}{0.51^2}-4<x-4<\frac{1}{0.49^2}-4.$$

Using a calculator, we find that $-0.1553248<x-4<0.1649312$. We can thus choose δ to be any positive number that is smaller than 0.1553248. Hence, a suitable choice for δ is 0.1553.

Problem: Prove that $\lim_{x\to 2}(4x-5)=3$.

Solution: Given $\epsilon>0$, we must choose $\delta>0$ so that

$$0<|x-2|<\delta \Rightarrow \big|(4x-5)-3\big|<\epsilon.$$

Noting that $\big|(4x-5)-3\big|=\big|4x-8\big|=4\big|x-2\big|$, it is clear that $\delta=\epsilon/4$ has the desired property. A formal proof reads as follows. Let $\epsilon>0$ and choose $\delta=\epsilon/4$. For all x that satisfy $0<|x-2|<\delta$, we find that

$$|(4x-5)-3|=|4x-8|=4|x-2|<4\delta=\epsilon.$$

This shows that $\lim_{x\to 2}(4x-5)=3$.

Exercises

1. For the given limit and value of ϵ, find (to 4 decimal places) a suitable choice for δ.

a) $\lim_{x\to 2}x^2=4,\ \epsilon=0.05$ **b)** $\lim_{x\to 1}\sqrt{x}=1,\ \epsilon=0.01$ **c)** $\lim_{x\to 4}x^3=64,\ \epsilon=0.2$

d) $\lim_{x\to 3}\frac{1}{x}=\frac{1}{3},\ \epsilon=\frac{1}{25}$ **e)** $\lim_{x\to\pi/6}\sin x=\frac{1}{2},\ \epsilon=0.1$ **f)** $\lim_{x\to 2}e^x=e^2,\ \epsilon=0.04$

2. For the limit $\lim_{x\to 2}x^2=4$, find (to 6 decimal places) a suitable δ for the given value of ϵ.

a) $\epsilon=0.01$ **b)** $\epsilon=0.004$ **c)** $\epsilon=0.0003$

3. For each limit, find (to 4 decimal places) a suitable δ for $\epsilon=1$.

a) $\lim_{x\to 2}x^2=4$ **b)** $\lim_{x\to 20}x^2=400$ **c)** $\lim_{x\to 100}x^2=10,000$

4. A square must have an area of 400 cm^2 give or take 5 cm^2.

a) What range of values for the side of the square will meet these specifications?

b) Formulate this problem in the form $\lim_{x\to c}f(x)=L$ and identify f, c, L, ϵ, and δ.

5. A spherical ball bearing must have a volume of 40 cm^3 give or take 1 cm^3.

a) What range of radius values will meet these specifications?

b) Formulate this problem in the form $\lim_{x\to c}f(x)=L$ and identify f, c, L, ϵ, and δ.

6. Use the definition of limit to prove each of the following.

a) $\lim_{x\to 1}(10x+3)=13$ **b)** $\lim_{x\to -1}(3-4x)=7$ **c)** $\lim_{x\to 3}x^2=9$

7. Suppose that $\lim_{x\to c}f(x)=L$ and that k is a nonzero constant. Prove that $\lim_{x\to c}kf(x)=kL$.

1.24 LIMITS INVOLVING ∞

The symbol ∞ represents a quantity that is larger than every real number. It is worth emphasizing that ∞ is a concept, not a number. To say that x goes to ∞ or to write $x \to \infty$ means that the variable x is eventually larger than any real number, that is, the values of x are growing without bound. The expression $\lim_{x\to\infty} f(x) = L$ means that the output values $f(x)$ are near L when the input values x are extremely large or, to say it another way, $f(x)$ gets closer and closer to L as x gets bigger and bigger. This statement can be made mathematically precise, but this intuitive version will be suitable for us. The expression $\lim_{x\to-\infty} f(x) = L$ is defined similarly. The simplest example of this concept is $\lim_{x\to\infty} 1/x = 0$.

To compute limits of the form $\lim_{x\to\infty} f(x)$ when f is a fraction made up of algebraic functions, a good procedure is the following: find the highest power of x that appears in the denominator and divide both numerator and denominator by that power of x. This is illustrated in the following examples.

$$\lim_{x\to\infty} \frac{2x^3+3x-1}{3x^3-4x^2+6x} = \lim_{x\to\infty} \frac{2x^3+3x-1}{3x^3-4x^2+6x} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{2+\dfrac{3}{x^2}-\dfrac{1}{x^3}}{3-\dfrac{4}{x}+\dfrac{6}{x^2}} = \frac{2}{3};$$

$$\lim_{x\to\infty} \left(\sqrt{x^2+7x+8}-x\right) = \lim_{x\to\infty} \frac{7x+8}{\sqrt{x^2+7x+8}+x} = \lim_{x\to\infty} \frac{7+\dfrac{8}{x}}{\sqrt{1+\dfrac{7}{x}+\dfrac{8}{x^2}}+1} = \frac{7}{2}.$$

Note the use of the conjugate in the second example.

The expression $\lim_{x\to c} f(x) = \infty$ means that the output values $f(x)$ become arbitrarily large when the input values x are near c, that is, $f(x)$ gets big as x approaches c. A limit of this type does not exist since there is no number which the output values get close to. However, the symbols give some indication as to why the limit does not exist and are a useful mathematical shorthand. Some examples include (one-sided limits of this type are defined in the expected way)

$$\lim_{x\to 0} \frac{1}{x^2} = \infty, \ \lim_{x\to 2} \frac{3}{|x-2|} = \infty, \ \lim_{x\to 0^-} \frac{1}{x} = -\infty, \ \lim_{x\to 3^+} \frac{2x-1}{x^2-9} = \infty, \text{ and } \lim_{x\to 0^+} e^{1/x} = \infty.$$

As indicated in the first four of these examples, if the numerator approaches a nonzero number while the denominator approaches 0, then an infinite limit occurs.

Limits involving ∞ can be used to define the concept of an asymptote for the graph of a function.

DEFINITION 1.22 Let f be a function.

- **a)** The line $x = c$ is a **vertical asymptote** for the graph $y = f(x)$ if either $\lim_{x\to c^-} |f(x)| = \infty$ or $\lim_{x\to c^+} |f(x)| = \infty$. (The graph goes off the scale vertically at c.)
- **b)** The line $y = d$ is a **horizontal asymptote** for the graph $y = f(x)$ if either $\lim_{x\to-\infty} f(x) = d$ or $\lim_{x\to\infty} f(x) = d$. (The graph resembles the horizontal line $y = d$ when x is large.)

Horizontal asymptotes are fairly easy to spot; just check the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. Vertical asymptotes require a little more effort. A rule of thumb is that vertical asymptotes occur when a division by zero appears in the formula for the function. However, it is still necessary to verify that the function has an infinite limit at such points. Furthermore, a function can have an infinite limit without a division by zero; the function $\ln|x|$ has a vertical asymptote at 0. It is thus necessary to think in terms of the definition and not only in terms of division by zero. For example, the function f defined by

$$f(x) = \frac{x^2 - 3x - 10}{x^2 - 4}$$

has a vertical asymptote at $x = 2$, but there is no vertical asymptote at $x = -2$. As the reader should verify, the function f has a limit as $x \to -2$.

Exercises

1. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{2x+5}{6x-1}$ **b)** $\displaystyle\lim_{x\to\infty} \frac{x-2x^3}{5x^3+6x-4}$ **c)** $\displaystyle\lim_{x\to\infty} \frac{x^2}{(4-x)(3x+11)}$

d) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2-3x+6}-x\right)$ **e)** $\displaystyle\lim_{x\to\infty} \left(2x-\sqrt{4x^2+9}\right)$ **f)** $\displaystyle\lim_{x\to\infty} e^{-x^2/3}$

g) $\displaystyle\lim_{x\to-\infty} \left(2+e^x\right)$ **h)** $\displaystyle\lim_{x\to-\infty} \frac{1-2x}{\sqrt{x^2+4x+5}}$ **i)** $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+8}}{3x+2}$

2. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to 2} \frac{1}{(x-2)^4}$ **b)** $\displaystyle\lim_{x\to 1} \frac{2x+3}{|x^2-x|}$ **c)** $\displaystyle\lim_{x\to 2^-} \frac{x^2}{x^2-4}$

d) $\displaystyle\lim_{x\to 3^+} \frac{2x+1}{\sqrt{x-3}}$ **e)** $\displaystyle\lim_{x\to -1^+} \frac{2x^2+3x-5}{x^2+3x+2}$ **f)** $\displaystyle\lim_{x\to 2^+} \frac{x+1}{x^2+x-6}$

g) $\displaystyle\lim_{x\to\pi^-} \cot x$ **h)** $\displaystyle\lim_{x\to 0} e^{2/x^2}$ **i)** $\displaystyle\lim_{x\to 2} \left(5-\ln|x-2|\right)$

3. Write a precise mathematical definition (see the previous section) for $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = L$.

4. Sketch the graph of a function with the given properties.

a) $\lim_{x\to 2} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1$, and $\lim_{x\to-\infty} f(x) = 1$

b) $\lim_{x\to 3^-} g(x) = \infty$, $\lim_{x\to 3^+} g(x) = -\infty$, and $\lim_{x\to\infty} g(x) = 0$

c) $\lim_{x\to -1} h(x) = -\infty$, $\lim_{x\to\infty} h(x) = 4$, and $\lim_{x\to-\infty} h(x) = 1$

5. Find all of the asymptotes, both vertical and horizontal, for the given function.

a) $f(x) = \dfrac{4x-1}{2x+3}$ **b)** $g(x) = \dfrac{2x^2+3}{x^2-5x+6}$ **c)** $h(x) = \dfrac{e^x}{e^x-2}$

6. Find all of the vertical asymptotes for the given function. Be careful.

a) $f(x) = \dfrac{x^2-4}{x^3-x^2-2x}$ **b)** $g(x) = \dfrac{\sin x}{x^2-2x}$ **c)** $h(x) = \ln|2x-1|$

7. Explain in words what the expression $\lim_{x\to\infty} f(x) = \infty$ means. Then evaluate the limits.

a) $\displaystyle\lim_{x\to\infty} \left(x^3-40x^2\right)$ **b)** $\displaystyle\lim_{x\to\infty} \left(10\sqrt{x}-x\right)$ **c)** $\displaystyle\lim_{x\to\infty} \frac{x^3}{x^2+6x+10}$

1.25 L'Hôpital's Rule

The derivative concept is defined in terms of limits; all derivative formulas have their roots in properties of limits. However, once derivative formulas have been established, it is possible to use derivatives to evaluate limits. For example,

$$\lim_{x\to 1}\frac{x^{1000}-1}{x^2-1}=\lim_{x\to 1}\left(\frac{x^{1000}-1}{x-1}\cdot\frac{1}{x+1}\right)=\frac{d}{dx}\,x^{1000}\Big|_{x=1}\cdot\frac{1}{2}=\frac{1000}{2}=500.$$

The use of derivatives to evaluate limits yields the following result known as L'Hôpital's Rule. The "suitable conditions" mentioned in the hypotheses involve continuity and differentiability properties that will always be satisfied by the functions we consider.

THEOREM 1.23 L'Hôpital's Rule Under suitable conditions on the functions f and g, if either the limits $\lim f(x)$ and $\lim g(x)$ are both 0 or both infinite, then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$, assuming that the latter limit exists. (The limits can be of any type: $x \to a$, $x \to a^+$, $x \to -\infty$, etc.)

Proof. Because it covers so many situations, this is not an easy result to prove. We offer here a proof that is valid under the assumption that f' and g' are continuous functions with $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$ and $g'(a) \neq 0$.

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && f, g \text{ are continuous at } a\\
&= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot\frac{x-a}{g(x)-g(a)}\right) && \text{multiply by 1}\\
&= \frac{f'(a)}{g'(a)} && \text{definition of derivative}\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} && f', g' \text{ are continuous}
\end{aligned}$$

Although not a general proof, this proof does give a good idea why L'Hôpital's Rule works. ■

We illustrate this rule with three examples. It is important to check that the limit is of the form $0/0$ or ∞/∞ before proceeding. If after one step, the limit is still of the form $0/0$ or ∞/∞, then the rule is just applied again.

$$\lim_{x\to 0}\frac{e^x-1}{\sin 2x}=\lim_{x\to 0}\frac{e^x}{2\cos 2x}=\frac{1}{2};$$

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}=\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}=\lim_{x\to 0}\frac{1}{3}\left(\frac{\tan x}{x}\right)^2=\frac{1}{3};$$

$$\lim_{x\to\infty}\frac{x^2}{e^{x/2}}=\lim_{x\to\infty}\frac{4x}{e^{x/2}}=\lim_{x\to\infty}\frac{8}{e^{x/2}}=0.$$

For the middle limit, we used the fact that $\lim\limits_{x\to 0}\sin x/x = 1$; see Section 1.17.

Limits that result in the forms $0/0$ or ∞/∞ are known as **indeterminate forms**. This term is used because the value of the limit is not determined by the form; the form gives no indication whatsoever as to the value of the limit. L'Hôpital's Rule can be used to determine the value of a limit of an indeterminate form of the type $0/0$ or ∞/∞. Other indeterminate forms include $\infty-\infty$, $0\cdot\infty$, 1^∞, ∞^0, and 0^0. L'Hôpital's

Rule cannot be used on these forms unless some algebra is first performed to convert them to a $0/0$ or ∞/∞ form. For example,

$$\lim_{x\to1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1}\frac{\ln x-x+1}{(x-1)\ln x}=\lim_{x\to1}\frac{\frac{1}{x}-1}{1-\frac{1}{x}+\ln x}=\lim_{x\to1}\frac{-\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}}=-\frac{1}{2}.$$

(The first limit has the form $\infty-\infty$, while the second and third limits have the form $0/0$.) For indeterminate forms that involve exponents, it is first necessary to find the limit of the natural logarithm of the function. To illustrate, suppose we are asked to find $\lim_{x\to0}\bigl(\cos x\bigr)^{1/x^2}$. This limit is of the form 1^∞. We first use the properties of logarithms to evaluate

$$\lim_{x\to0}\ln\Bigl(\bigl(\cos x\bigr)^{1/x^2}\Bigr)=\lim_{x\to0}\frac{\ln\bigl(\cos x\bigr)}{x^2}=\lim_{x\to0}\frac{-\tan x}{2x}=\lim_{x\to0}\frac{-\sec^2 x}{2}=-\frac{1}{2}.$$

The solution to the original limit is thus $\lim_{x\to0}\bigl(\cos x\bigr)^{1/x^2}=e^{-1/2}$. (We were asked to find a limit L and, after some effort, we found that $\ln L=-1/2$. It follows that $L=e^{-1/2}$.)

The symbols $0/0$, $\infty-\infty$, etc. are meaningless since they represent indeterminate forms. Consequently, they should not be used in equations. Do not write equations such as

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{0}{0},\quad \lim_{x\to c}\bigl(f(x)-g(x)\bigr)=\infty-\infty,\quad \lim_{x\to c}f(x)^{g(x)}=1^\infty.$$

You should simply make a note of the type of indeterminate form that appears, then continue with the problem.

Exercises

1. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to1}\frac{x^2+4x-5}{x^3+x^2+x-3}$ **b)** $\displaystyle\lim_{x\to2}\frac{x^5-32}{x^3+2x^2-6x-4}$ **c)** $\displaystyle\lim_{x\to-1}\frac{x^{47}+1}{x^5+x^4-2x-2}$

d) $\displaystyle\lim_{x\to0}\frac{\tan x}{e^x-1}$ **e)** $\displaystyle\lim_{x\to0}\frac{e^x-x-1}{x^2}$ **f)** $\displaystyle\lim_{x\to0}\frac{\cos x-\cos(3x)}{x^2}$

g) $\displaystyle\lim_{x\to0}\frac{x+\arctan(5x)}{2x+\arctan x}$ **h)** $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt[4]{x}}$ **i)** $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^x}$

2. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to0}(x^2+3x)\csc x$ **b)** $\displaystyle\lim_{x\to0^+}\sqrt{x}\ln x$ **c)** $\displaystyle\lim_{x\to\infty}\left(x-xe^{-2/x}\right)$

d) $\displaystyle\lim_{x\to0}\left(1+\sin 2x\right)^{1/x}$ **e)** $\displaystyle\lim_{x\to\infty}\left(e^{2x}+x^2\right)^{1/x}$ **f)** $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-2}\right)^x$

3. Let r be a positive constant. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to0}\frac{\tan(r^2x)}{\sin(rx)}$ **b)** $\displaystyle\lim_{x\to0^+}x^r\ln x$ **c)** $\displaystyle\lim_{x\to0}\left(1+rx\right)^{1/x}$

d) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^r}$ **e)** $\displaystyle\lim_{x\to\infty}\frac{x^r}{e^x}$ **f)** $\displaystyle\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$

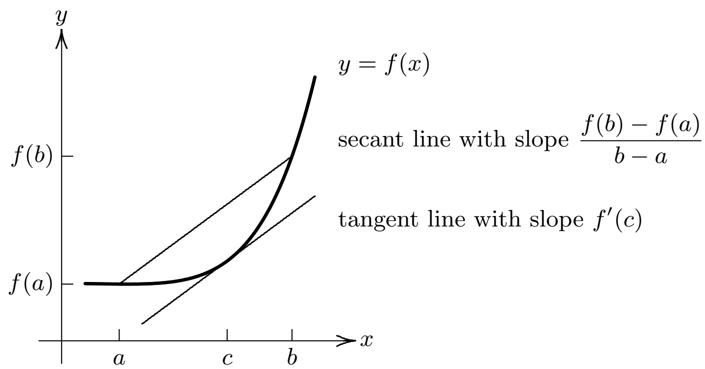
4. Let a be a positive constant. Evaluate $\displaystyle\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$, which is one of L'Hôpital's original examples.

1.26 The Mean Value Theorem

The Mean Value Theorem is an extremely important theorem in differential calculus. It is a simple result with far reaching consequences. In fact, it is used to prove most of the useful properties of the derivative.

THEOREM 1.24 Mean Value Theorem If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. ■

The graph below provides a geometric interpretation for the Mean Value Theorem. The difference quotient $(f(b)-f(a))/(b-a)$ represents the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ on the graph of f, and the number $f'(c)$ represents the slope of the tangent line to the graph of f at $(c, f(c))$. The conclusion of the Mean Value Theorem states that there exists a point in the interval where the tangent line is parallel to the secant line.



The slope of the secant line can be interpreted as the average rate of change of the function f on the interval $[a,b]$. It is from this perspective that the adjective "mean" appears in the Mean Value Theorem; there exists a point c in the interval at which the instantaneous rate of change of f at c equals the average rate of change of f on the interval $[a,b]$.

The Mean Value Theorem is a consequence of the following special case of the theorem in which the secant line is assumed to be horizontal. The details of the proof are requested in Exercise 10.

THEOREM 1.25 Rolle's Theorem If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

Proof. By the Extreme Value Theorem, the function f has a maximum output and a minimum output on the interval $[a,b]$. Unless $f(x) = f(a)$ for all $x \in [a,b]$ (in which case the conclusion is obvious), at least one of these extreme outputs occurs at a point $c \in (a,b)$. Since f is differentiable on (a,b), it follows from Theorem 1.15 that $f'(c) = 0$. ■

To illustrate the conclusion of the Mean Value Theorem, consider the function $f(x) = x^3 - 2x^2$ on the interval $[1,4]$. Since this function is a polynomial, it is continuous and differentiable on $[1,4]$. (Always make certain your function satisfies the continuity and differentiability conditions on the given interval.) Therefore, the Mean Value Theorem guarantees a point $c \in (1,4)$ for which

$$f'(c) = \frac{f(4)-f(1)}{4-1} \;\Leftrightarrow\; 3c^2 - 4c = \frac{32-(-1)}{4-1} = 11.$$

The quadratic formula yields $c = \big(2 \pm \sqrt{37}\big)/3$. Since $c \in (1, 4)$, we find that $c = \big(2 + \sqrt{37}\big)/3$.

As a second example, consider the function $g(x) = 1/x^2$ on the generic interval $[a, b]$ with $a > 0$. The point c guaranteed by the Mean Value Theorem satisfies

$$-\frac{2}{c^3} = \frac{\dfrac{1}{b^2} - \dfrac{1}{a^2}}{b - a} \;\Rightarrow\; \frac{2}{c^3} = \frac{a + b}{a^2 b^2} \;\Rightarrow\; c = \sqrt[3]{\frac{2a^2b^2}{a + b}}.$$

The formula for c represents a different way to obtain a mean value for two positive numbers a and b. (Two familiar ways are the **arithmetic mean** $(a + b)/2$ and the **geometric mean** $\sqrt{ab}$.)

Exercises

1. For the given function and interval, find the point c guaranteed by the Mean Value Theorem.

a) $f(x) = \sqrt{x}$, $[1, 9]$ **b)** $g(x) = x\sqrt{8 - x}$, $[0, 8]$ **c)** $h(x) = 3x^2 - 5x + 1$, $[1, 4]$

d) $F(x) = \dfrac{x}{x + 2}$, $[1, 4]$ **e)** $G(x) = \arctan x$, $[0, 1]$ **f)** $H(x) = x^3 - x^2 + 4x$, $[0, 2]$

2. Consider the function $f(x) = |x - 3|$ on the interval $[1, 5]$. Which of the hypotheses of Rolle's Theorem are satisfied? Is there a point c that satisfies the conclusion of Rolle's Theorem? Explain your answers.

3. Consider the function $f(x) = x^{2/3}$ on the interval $[-1, 27]$. Which of the hypotheses of the Mean Value Theorem are satisfied? Is there a point c that satisfies the conclusion of this theorem? Explain your answers.

4. Let a and b be constants with $a < b$. Find the point that is guaranteed by the Mean Value Theorem for the function f that is defined by $f(x) = (x - a)^2(b - x)^4$ on the interval $[a, b]$.

5. Let a and b be constants with $a < b$, and let r and s be positive constants. Find the point that is guaranteed by the Mean Value Theorem for the function f that is defined by $f(x) = (x - a)^r(b - x)^s$ on the interval $[a, b]$.

6. Find the point c guaranteed by the Mean Value Theorem for the function $f(x) = x^2$ on an arbitrary interval $[a, b]$. Simplify the value of c and note something interesting about it.

7. Find the point c guaranteed by the Mean Value Theorem for the function $f(x) = 1/x$ on an arbitrary interval $[a, b]$, where $a > 0$. Simplify the value of c and note something interesting about it.

8. Let f be a differentiable function defined on $[a, b]$, let A, B, and C be constants with $A \neq 0$, and consider the function g defined by $g(x) = Af(x) + Bx + C$. Show that a point c guaranteed by the Mean Value Theorem for the function g on $[a, b]$ is also a point guaranteed by this theorem for f.

9. Let f be a differentiable function. Suppose that the equation $f(x) = 0$ has n distinct solutions, where $n > 1$ is a positive integer. Prove that the equation $f'(x) = 0$ has at least $n - 1$ solutions.

10. Prove the Mean Value Theorem. Start by defining a function g by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a),$$

and showing that g satisfies the hypotheses of Rolle's Theorem. Then show that the point c guaranteed by Rolle's Theorem for g is the point c required for f. Also, explain what the function g represents in geometric terms.

1.27 Applications of the Mean Value Theorem

The Mean Value Theorem is used to prove many of the useful properties of the derivative. One such property and its simple corollary, both of which we have already been assuming to be true, is the following result.

THEOREM 1.26 Monotonicity Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b).

a) If $f' \geq 0$ on (a, b), then f is increasing on $[a, b]$.

b) If $f' \leq 0$ on (a, b), then f is decreasing on $[a, b]$.

c) If $f' = 0$ on (a, b), then f is constant on $[a, b]$.

Proof. We will prove part (a); the proofs of parts (b) and (c) are similar. Suppose that $a \leq u < v \leq b$. Applying the Mean Value Theorem to the function f on the interval $[u, v]$, there exists a point $c \in (u, v)$ such that $f(v) - f(u) = f'(c)(v - u)$. Since $f'(c) \geq 0$ and $v - u > 0$, it follows that $f(v) \geq f(u)$. Hence, the function f is increasing on $[a, b]$. ■

COROLLARY 1.27 Let f and g be two differentiable functions defined on $[a, b]$. If $f' = g'$ on $[a, b]$, then there exists a constant k such that $f(x) = g(x) + k$ for all $x \in [a, b]$. ■

As an example of the theorem, let f be the function defined by $f(x) = 2x + \sin x$. Since $f'(x) = 2 + \cos x > 0$ for all $x \in \mathbb{R}$, the function f is increasing on every interval $[a, b]$ and thus increasing on $\mathbb{R}$. As for the corollary, suppose we need to find a function g such that $g'(x) = 3x^2$ for all x. It is clear that both $g(x) = x^3$ and $g(x) = x^3 + 4$ will work. Is there another, possibly very complicated and/or unusual function, whose derivative is also $3x^2$? The answer is no; Corollary 1.27 guarantees that every function whose derivative is $3x^2$ for all values of x is of the form $x^3 + k$ for some constant k.

Theorem 1.26 appears to be an almost trivial result, especially if one is encouraged to visualize a graph as resembling its tangent lines. However, it is sometimes the case that seemingly obvious results are not all that easy to prove; this is just one example. By the way, it is possible for a function to have a positive derivative at a point without being an increasing function in an open interval containing that point. The function f defined by

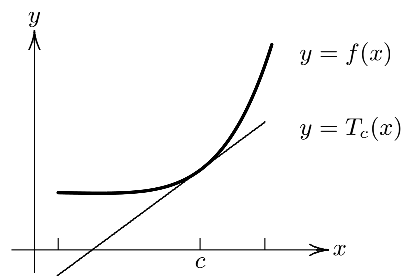
$$f(x) = \begin{cases} x/2 + x^2 \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0; \end{cases}$$

satisfies $f'(0) = 1/2$, but f is not increasing on any open interval that contains 0. The reader should try to obtain a graph of this function using a calculator.

The Monotonicity Theorem can be used to prove inequalities. Suppose a differentiable function h satisfies $h(a) = 0$ and $h'(x) > 0$ for all $x > a$. Since h is increasing for $x > a$, it follows that $h(x) > h(a) = 0$ for all $x > a$. As an example, suppose we would like to prove that $e^x > 1 + x$ for all $x > 0$. Define a function h by $h(x) = e^x - (x + 1)$. Then $h(0) = 0$ and $h'(x) = e^x - 1 > 0$ for all $x > 0$. It follows that $h(x) > 0$ or $e^x > x + 1$ for all $x > 0$. The following theorem provides a more theoretical example.

THEOREM 1.28 Let f be a differentiable function defined on an interval I. If f' is increasing on I, then the graph of f lies above its tangent lines on I.

Proof. Let $c \in I$. The function T_c that represents the tangent line to $y = f(x)$ when $x = c$ is defined by $T_c(x) = f'(c)(x - c) + f(c)$. We must show that $f(x) \geq T_c(x)$ for all $x \in I$ (see the figure).



Let $x \in I$ and assume that $x > c$; the case in which $x < c$ is similar. By the Mean Value Theorem, there exists $z_x \in (c, x)$ such that $f(x) - f(c) = f'(z_x)(x - c)$. Since f' is increasing on I and $z_x > c$, we find that $f'(z_x) \geq f'(c)$. Using the fact that $x - c > 0$,

$$f(x) = f'(z_x)(x - c) + f(c) \geq f'(c)(x - c) + f(c) = T_c(x),$$

as desired. This completes the proof. ■

Exercises

1. Prove that the given function is increasing on every interval $[a, b]$ and thus increasing on $\mathbb{R}$.

a) $f(x) = 2x^3 + 10x - 18$ **b)** $g(x) = x^5 - 5x^3 + 30x$ **c)** $h(x) = 8x - 3\cos 2x$

2. Find the most general function whose derivative is given.

a) $f'(x) = x^2 + x + 1$ **b)** $g'(x) = \dfrac{x}{x^2 + 4}$ **c)** $h'(x) = 8 \sin 2x$

3. Prove part (b) of Theorem 1.26.

4. Prove part (c) of Theorem 1.26, then use it to prove Corollary 1.27.

5. Prove that each of the following inequalities is valid for $x > 0$.

a) $\sin x < x$ **b)** $\sqrt{1 + x} < 1 + \dfrac{x}{2}$ **c)** $e^x > 1 + x + \dfrac{1}{2}x^2$

6. Write a version of Theorem 1.28 that is valid when f' is decreasing on I.

7. Use Theorem 1.28, or the corresponding version discussed in the previous exercise, to prove each of the following inequalities is valid for all $x > 0$.

a) $3x^{4/3} \geq 4x - 1$ **b)** $5x^{6/5} \geq 6x - 1$ **c)** $\ln x \leq x - 1$

8. During a 15 minute interval, a car started with a velocity of 35 mi/hr and ended with a velocity of 75 mi/hr. Show that at some time during this interval, the acceleration of the car was 160 mi/hr^2.

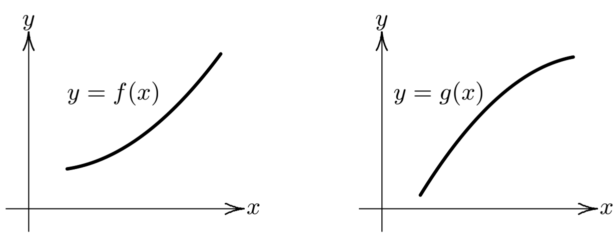
9. Suppose that $f(0) = 0$ and that f' is positive on $(-2, 5)$. Prove that f is negative on $(-2, 0)$ and positive on $(0, 5)$.

10. Prove that $x^4 + 6x + c$ has at most two real roots no matter what value the constant c has.

1.28 Concavity and Inflection Points

The derivative of a function f is another function f'. It is thus possible to find the derivative $(f')'$ of f'. This function is usually abbreviated f'' and referred to as the second derivative of f. Since f'' is also a function, its derivative $(f'')' = f'''$ can be computed. This process can be repeated indefinitely; the functions that are obtained in this way are known as the **higher derivatives** of f. Since the prime notation becomes awkward when there are many primes, the symbol $f^{(n)}$ is used to denote the n'th derivative of f. If $y = f(x)$, then the n'th derivative of y with respect to x is also denoted by $\dfrac{d^n y}{dx^n}$.

Since $f^{(n)}$ is the derivative of $f^{(n-1)}$, it gives the slope of the graph $y = f^{(n-1)}(x)$, but the more relevant question is whether or not $f^{(n)}$ gives any useful information about the function f. A physical interpretation of higher derivatives that applies to the motion of an object will be considered in the next section. For a geometric interpretation, consider the two curves sketched below.



f and g are increasing;
f' and g' are positive;
f' is increasing;
g' is decreasing.

The function f is increasing at an increasing rate, while the function g is increasing at a decreasing rate. To distinguish between functions such as these, we make the following definition.

DEFINITION 1.29 Let f be a differentiable function defined on an open interval I. The function f is **concave up** on I if f' is increasing on I and **concave down** on I if f' is decreasing on I.

Since a positive derivative indicates an increasing function and a negative derivative indicates a decreasing function, the following theorem holds.

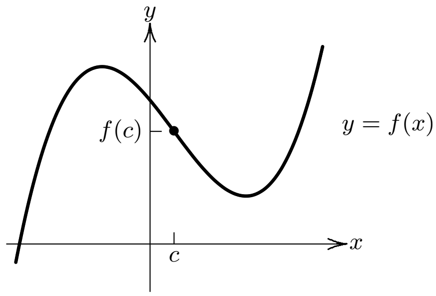
THEOREM 1.30 Suppose that f is twice differentiable on an open interval I.

a) If f'' is positive on I, then f is concave up on I.

b) If f'' is negative on I, then f is concave down on I. ■

Hence, the second derivative of a function f provides information about the concavity of f. There is a geometric interpretation of the third derivative, but it will not be discussed here as it is much more complicated. The other higher derivatives do not have a geometric interpretation for f, but they do reveal some useful information about a function. This will become clear when power series are discussed later in the book.

While sketching a graph with varying concavity, one can "feel" a turning point when the concavity changes. These points, known as inflection points, are not as dramatic as relative extreme points but they are important points on a graph. The point $(c, f(c))$ is an inflection point for the function f sketched below. By definition, an **inflection point** is a point on the graph of a continuous function at which the concavity changes.



The graph is concave down on $(-\infty, c]$ and concave up on $[c, \infty)$.

Possible inflection points for the graph of $y = f(x)$ occur at those values of x in the domain of f for which either f'' does not exist or $f'' = 0$. However, a function may not have an inflection point at a point with this property. For example, the function $f(x) = x^4$ satisfies $f''(0) = 0$, but the function does not have an inflection point at $(0, 0)$. It is necessary to check that the concavity actually changes at the point in question, that is, make certain that the second derivative changes sign at the point.

Problem: Determine the intervals on which the function f defined by $f(x) = xe^{-x}$ is concave up and those on which it is concave down. Find the (x, y) coordinates of any inflection points.

Solution: Using the product rule to find f' and f'' yields $f'(x) = (1 - x)e^{-x}$ and $f''(x) = (x - 2)e^{-x}$. It is clear that f'' is defined for all x and that $f''(x) = 0$ only when $x = 2$. On the interval $(-\infty, 2)$, we see that f'' is negative, while f'' is positive on the interval $(2, \infty)$. Hence, the function f is concave up on $[2, \infty)$ and concave down on $(-\infty, 2]$. The point $(2, 2/e^2)$ is an inflection point.

Exercises

1. Find and simplify the second derivative of the given function. Be careful computing the second derivative.

a) $f(x) = (x^2 + x + 1)^{10}$ **b)** $g(x) = \sqrt{x^2 + 4}$ **c)** $h(x) = \sin^3 x$

2. Find a general formula for $f^{(n)}(x)$. *Hint*: Find four or more derivatives, then look for a pattern.

a) $f(x) = e^{2x}$ **b)** $f(x) = xe^x$ **c)** $f(x) = 1/x$

3. Find $f^{(102)}$ for the function $f(x) = \cos x$. Do not take 102 derivatives!

4. For the given function, find the intervals on which the function is concave up and those on which it is concave down. Find the (x, y) coordinates of any inflection points.

a) $f(x) = x^2 - x^3$ **b)** $f(x) = x^2 - \dfrac{1}{x}$ **c)** $f(x) = x^4 - 6x^3 + 12x^2 + 4x$

d) $f(x) = \dfrac{8}{x^2 + 1}$ **e)** $f(x) = \dfrac{\ln x}{x}$ **f)** $f(x) = e^{-x^2/4}$

5. On what percentage (of the total length) of the interval $[0, \pi]$ is the function $f(x) = x^2 + \sin 2x$ concave down?

6. Let f be a twice differentiable function and suppose that the equation $f(x) = 0$ has 3 distinct solutions. Prove that the equation $f''(x) = 0$ has at least 1 solution. What can you say if $f(x) = 0$ has 7 distinct solutions?

7. Find a polynomial P of degree 3 so that P has an inflection point at $(2, 4)$ and a y-intercept of 12.

8. Discuss the relationship between concavity and Theorem 1.28.

1.29 Velocity and Acceleration

Suppose that a particle is moving along a straight line and that its distance s from a fixed point O on the line at time t is given by the function $s(t)$. Then its **velocity** v, **acceleration** a, and **jerk** j are given by

$$v = \frac{ds}{dt}, \qquad \text{rate of change of position with respect to time (velocity);}$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \qquad \text{rate of change of velocity with respect to time (acceleration);}$$

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}, \qquad \text{rate of change of acceleration with respect to time (jerk).}$$

Since there are two different directions to travel from the point O, these quantities may be either positive or negative; positive for one direction and negative for the other. On occasion, only the **speed** of a particle is important—this quantity is defined to be $|v|$.

Given a position function s, it is easy to find the functions v, a, and j simply by taking derivatives. In a typical application, however, information about the acceleration or velocity is given and the position function must be determined. This means that the derivative or second derivative of a function is known and it is necessary to find the function. The process of finding a function from its derivative is called **antidifferentiation**; an introduction to this process was presented in Section 1.21. Suppose that the acceleration function a of an object is known. Then an antiderivative gives the velocity function v and another antiderivative gives the position function s. The constants that arise in the antidifferentiation process can usually be found from further information about the motion of the object. To illustrate, suppose that the position function $s(t)$ must be determined from

$$a(t) = 12t,\ v(0) = 20,\ \text{and } s(1) = 8.$$

Since an antiderivative of $12t$ is $6t^2 + C$, we find that $v(t) = 6t^2 + 20$. (We choose the constant so that $v(0) = 20$.) To find the position function s, we take another antiderivative. An antiderivative of $6t^2 + 20$ is $2t^3 + 20t + C$. The fact that $s(1) = 8$ reveals that $C = -14$. Hence, the position function s is given by $s(t) = 2t^3 + 20t - 14$.

For an applied example, suppose that a ball is thrown straight upward from a height of 6 feet with an initial velocity of 80 feet per second. Near the surface of the earth, the acceleration due to gravity is essentially constant and this constant is denoted by g. (The value of g is approximately 9.8 m/sec^2 or 32 ft/sec^2.) Adopting the convention that up is positive, down is negative, and ground level is 0, the acceleration function of the ball is $a(t) = -32$. (This value of g is chosen because of the units given in the problem.) Hence (ignoring any effect of air resistance),

$$v(t) = -32t + c_1, \qquad v(0) = 80 \Rightarrow c_1 = 80;$$

$$s(t) = -16t^2 + 80t + c_2, \qquad s(0) = 6 \Rightarrow c_2 = 6.$$

Thus, the height of the ball (in feet) at any time t (seconds) is given by $s(t) = -16t^2 + 80t + 6$. This function can then be used to determine (a) the maximum height of the ball, (b) the length of time the ball is in the air, and (c) the speed with which the ball hits the ground.

To answer (a), note that the maximum height of the ball occurs at time t_1, where $v(t_1) = 0$. (The ball has an instantaneous velocity of 0 when it is at its peak.) Solving $v(t) = 0$ gives $t_1 = 2.5$. The maximum height of the ball is thus $s(2.5) = 106$ feet. The ball is on the ground when $s(t) = 0$. Using the quadratic formula,

$$s(t) = 0 \;\Rightarrow\; 8t^2 - 40t - 3 = 0 \;\Rightarrow\; t = \frac{40 \pm \sqrt{1696}}{16} = \frac{10 \pm \sqrt{106}}{4}.$$

Since $t > 0$, we find that the ball hits the ground after $\frac{1}{4}\big(10 + \sqrt{106}\big) \approx 5.074$ seconds. This is the length of time the ball is in the air. Finally, the solution to (c) is simply

$$\left| v\Big(\frac{1}{4}\big(10 + \sqrt{106}\big)\Big) \right| = 8\sqrt{106} \approx 82.365.$$

Therefore, the impact speed of the ball is approximately 82.365 ft/sec.

Exercises

1. Let $v(t)$ and $a(t)$ represent the velocity and acceleration functions of some particle and suppose that $a(t_1) < 0$ and $v(t_1) < 0$ for some time t_1. Is the speed of the particle increasing or decreasing at t_1?
2. Let $s(t) = \sqrt{t^2 + 11}$ denote the position function of a particle. Find the velocity and acceleration of the particle when $t = 5$.
3. Find the position function $s(t)$ from the given information.
   a) $a(t) = 50, v(0) = 10, s(0) = 20$
   b) $a(t) = 2t + 1, v(0) = -7, s(0) = 4$
   c) $a(t) = 3/\sqrt{t + 4}, v(0) = 2, s(5) = 14$
   d) $j(t) = 48t, a(0) = 0, v(0) = 10, s(0) = 0$
4. A toy rocket is shot straight upward from the ground with an initial velocity of 185 feet per second. How high does it go and how long is it in the air?
5. A ball thrown upward from the ground reaches a maximum height of 225 feet. What was its initial velocity in miles per hour?
6. Suppose that on earth you can throw a ball straight up to a height of 180 feet. If you went to a planet where the acceleration due to gravity is 10 ft/sec^2, how high could you throw the ball?
7. The Sears Tower in Chicago has a height of 1450 feet. How long does it take for an object dropped from the top of this building to hit the ground? How fast is it going when it hits the ground?
8. A certain car is able to brake with a deceleration of 4.92 m/sec^2. How long does such a car take to come to a stop if it is initially traveling at 24.6 m/sec? What is the distance traveled during the braking process?
9. The head of a rattlesnake can accelerate 50 m/sec^2 in striking a victim. If a car could do as well, how long would it take for it to reach a speed of 60 mi/hr from rest? (1 meter $\approx$ 3.28 feet)
10. Raindrops fall to earth from a cloud 5000 ft above the earth's surface. If they were not slowed by air resistance, how fast would the drops be moving when they struck the ground?
11. A ball thrown straight up from a height of 2 m takes 2.25 sec to reach a height of 36.8 m. What is its speed at this height? What was its initial speed? How much higher will the ball go?
12. A hot air balloon is ascending at the rate of 12 m/sec at a height 80 m above the ground when a brick is dropped. How long does it take for the brick to reach the ground? With what speed does it hit the ground?
13. From what height would a car have to be dropped in order for it to hit the ground at 60 mi/hr?
14. Suppose that the position of a particle is given by $s(t) = \dfrac{t^2}{2} - \dfrac{12}{t^2} + 15$, where t is measured in seconds and $s(t)$ is measured in meters. Find the minimum velocity of this particle for $2 \le t \le 10$.
15. Suppose that a particle is moving in a straight line with a velocity of 120 ft/sec. At time $t = 0$, the particle begins to decelerate at the rate of $6\,t$ ft/sec^2. How far does this particle travel before coming to a stop?

1.30 Polynomial Approximation

Consider the functions $f(x) = x^2 + 4x - 3$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$. Suppose that it is necessary to evaluate, without a calculator, each of these functions for $x = 5.2$. Finding $f(5.2)$ is relatively easy, but finding good approximations for $g(5.2)$ and $h(5.2)$ requires much more effort and a deeper understanding of these functions. Since calculators are readily available, this may seem like a silly exercise, but consider the following three observations:

1. It is important to have a historical perspective when it comes to knowledge. For many years, calculations such as $\sqrt{5.2}$ and $\sin 5.2$ had to be made without the assistance of calculators and the methods developed for doing so can provide useful insight.
2. Calculators make evaluations of functions very easy, but it is important to have some understanding of how calculators arrive at their answers. Since calculators rely on addition and multiplication, some formulas involving these operations are performed by a calculator when it is requested to evaluate trigonometric or logarithmic functions. Calculus has provided some of the tools required for determining how to best approximate these functions in this way.
3. There are instances in applications when it is necessary to replace a function with a simpler function. For example, in some physics problems, it is necessary to replace $\sin x$ with x (recall that $\lim\limits_{x \to 0} \sin x/x = 1$) in order to make it possible to solve the problem.

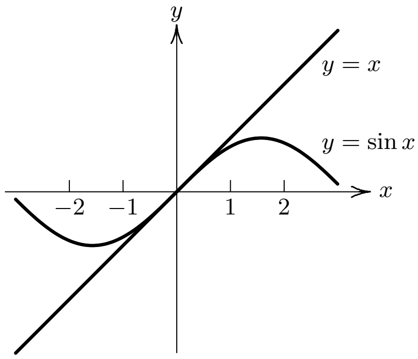
Since polynomials can be evaluated using only additions and multiplications, it is helpful to find polynomial approximations for functions that are not polynomials.

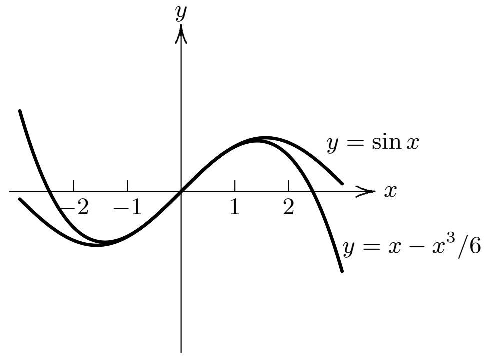
A moment's reflection makes it clear that different polynomials are needed to approximate a function at different points on its graph. Given a function f and a point a, the problem is to find a polynomial of a certain degree that approximates the function f for values of x near the point a. The linear function ℓ, the quadratic function q, and the cubic function c that approximate f for values of x near a are

$$\begin{aligned} \ell(x) &= f(a) + f'(a)(x-a); \\ q(x) &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2; \\ c(x) &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f'''(a)(x-a)^3. \end{aligned}$$

All three of these functions go through the point $(a, f(a))$. Note that the linear function ℓ is simply the tangent line to the graph at the point $(a, f(a))$; it has the same slope at a as the function f. The quadratic function q has both the same slope and concavity at a as the function f, that is, $q'(a) = f'(a)$ and $q''(a) = f''(a)$. Finally, the cubic function c satisfies $c'(a) = f'(a)$, $c''(a) = f''(a)$, and $c'''(a) = f'''(a)$.

One nice feature of these polynomial approximations is that each one builds on the previous one. In general, the higher the degree of the polynomial, the better the polynomial approximates the function f. A visual way to see this is to have a calculator plot the graphs of the function and its polynomial approximations. In the figure below, the linear approximation to $\sin x$ and the cubic approximation to $\sin x$ are shown (for x near the point 0). The reader should verify that the given polynomials for $\sin x$ are determined by the formulas presented above.





Note that as the degree of the polynomial increases, the polynomial graphs become closer to the graph of the function and the portions of the graph that are close expands.

As a numerical example, consider the function $f(x) = \sqrt{x}$ at the point $a = 9$. We then have

$$
\begin{aligned}
f(x) &= x^{1/2}; & f(9) &= 3; & \ell(x) &= 3 + \frac{1}{6}(x-9);\\
f'(x) &= \frac{1}{2}x^{-1/2}; & f'(9) &= \frac{1}{6}; & q(x) &= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2;\\
f''(x) &= -\frac{1}{4}x^{-3/2}; & f''(9) &= -\frac{1}{108}; & c(x) &= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3.\\
f'''(x) &= \frac{3}{8}x^{-5/2}; & f'''(9) &= \frac{1}{648}; & &
\end{aligned}
$$

To get a sense for how these functions approximate $\sqrt{x}$, we find that

$$\ell(8.5) \approx 2.916667, \quad q(8.5) \approx 2.915509, \quad c(8.5) \approx 2.915477, \quad \sqrt{8.5} \approx 2.915476.$$

Once again, it is clear that the approximations improve as the degree of the polynomial increases.

Exercises

1. Find the linear, quadratic, and cubic approximations to the function f at the point a.

a) $f(x) = 1/x, a = 1$ **b)** $f(x) = \sqrt{x}, a = 4$ **c)** $f(x) = \sqrt[3]{x}, a = 8$

d) $f(x) = e^x, a = 0$ **e)** $f(x) = \ln x, a = 1$ **f)** $f(x) = \arctan x, a = 0$

2. Use the linear, quadratic, and cubic approximations to e^x at 0 to approximate $\sqrt[10]{e}$ and $\sqrt{e}$. Then use a calculator to determine the accuracy of each approximation.

3. Use a quadratic approximation to estimate $\sqrt{220}$. (Choose a to be a perfect square near 220.)

4. Use a linear approximation to estimate $\sqrt[3]{220}$. (Choose a to be a perfect cube near 220.)

5. Determine the formula for the fourth degree polynomial approximation to a function f at a point a.

6. Suppose that f is a function with the property that $f'(x) = \sqrt{x^2+4}$ and $f(0) = 3$. Estimate $f(1/2)$ with both a linear and a cubic approximation. (Make no attempt to find $f(x)$.)

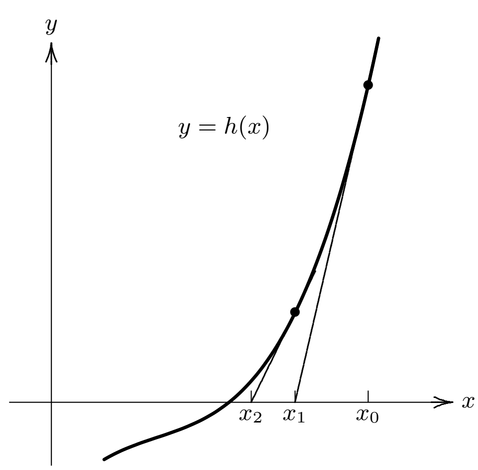
1.31 Newton's Method

Determining solutions to equations has been a practical problem for centuries. Ideally, an exact solution is sought, but when this is not possible, a method for generating an approximate solution to any degree of accuracy is desirable. It is only in recent years that the availability of calculators has made this problem quite a bit easier. Consider the following three equations:

$$(1)\ 2x^2 = 5x - 3; \qquad (2)\ x^5 - 4x^2 + 2x - 7 = 0; \qquad (3)\ 3\sin x = x.$$

It is easy to use the quadratic formula to find exact solutions to equation (1). An approximate solution to (2) can be obtained by trial and error; 1 is too small and 2 is too big, then 1.7 is too small and 1.8 is too big, etc. The same idea can be tried on (3), but it is much more difficult to evaluate $\sin x$ without some assistance. Of course, in this day and age these equations can be solved using a calculator. The purpose of this section is to discuss Newton's method for approximating solutions to equations so that you have some idea about what goes on behind the scenes when you use a calculator to solve an equation.

Newton's method provides a way of approximating solutions to an equation of the form $h(x) = 0$. The key idea behind this method is that a tangent line to a curve is a good local approximation of the curve. By looking at a graph of $y = h(x)$ or making some educated guesses, it is possible to find an approximate solution x_0 to the equation. The line tangent to the curve $y = h(x)$ at the point $(x_0, h(x_0))$ resembles the graph of $y = h(x)$ for x near x_0. Hence, the x-intercept of this tangent line should be a better approximation to a solution of $h(x) = 0$ (see the figure).



$$x_1 = x_0 - \frac{h(x_0)}{h'(x_0)}$$

$$x_2 = x_1 - \frac{h(x_1)}{h'(x_1)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$$

general iteration formula

The x-intercept of the tangent line to the curve at the point $(x_1, h(x_1))$ should be an even better approximation to a solution of $h(x) = 0$. This process can be continued to obtain approximations to any desired degree of accuracy. To find a formula for the x-intercepts of the tangent lines that appear in this process, let c be any real number and write down an equation for the line tangent to $y = h(x)$ at $(c, h(c))$, then find its x-intercept:

$$y - h(c) = h'(c)(x - c); \quad y = 0 \ \Rightarrow\ x = c - \frac{h(c)}{h'(c)}.$$

Using this formula, we can obtain x_1 from x_0, then x_2 from x_1, and so on (see the equations and the iteration formula to the right of the figure).

We will illustrate Newton's method by approximating the positive root of the polynomial $P(x) = x^4 - 3x - 8$. Since P is continuous and since $P(1) = -10 < 0 < 2 = P(2)$, the Intermediate Value Theorem guarantees a solution to $P(x) = 0$ in the interval $(1, 2)$. Since $P(2)$ is closer to 0 than $P(1)$, a good choice for x_0 is 2. The iteration formula and the first few terms it generates are given below.

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n - 8}{4x_n^3 - 3}$$

$$x_1 = 1.93103448276$$
$$x_2 = 1.92671132259$$
$$x_3 = 1.92669501894$$
$$x_4 = 1.92669501871$$

Hence, the number 1.926695 is a good approximation of the desired root. Note that x_2 is accurate to four decimal places and that not much computational effort is required to determine x_2. As in this example, Newton's method converges quite quickly in most cases. Typically, when x_n and x_{n+1} agree to d decimal places, the approximation x_{n+1} agrees with the actual root of the function to at least d decimal places.

Exercises

1. Use Newton's method to find a solution to the equation that is accurate to six decimal places.

a) $x^3 + 4x - 8 = 0$ **b)** $e^{-x} = x$ **c)** $2 \sin x = x$

d) $x^5 = 700 - 4x$ **e)** $\tan x = x,\ x > \pi$ **f)** $\ln x = x - 4,\ x > 4$

2. By applying Newton's method to the equation $x^2 - a = 0$, show that the square root of a can be calculated using the iteration formula $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, a formula used by the ancient Babylonians.

a) Give an explanation of this formula that does not involve calculus. *Hint*: It looks like an average.

b) Use this method to approximate the square roots of 2, 3, 5, and 6 in the following way. Let x_0 be $3/2$ for both 2 and 3 and let x_0 be 2 for both 5 and 6, then for each case, compute x_1 and x_2 as rational numbers (a ratio of integers) without using a calculator (to be in the proper spirit of the endeavor).

c) Use your calculator to compare your answers with the actual square roots. How close are the values?

3. Use Newton's method to find an iteration process for cube roots that is similar to the one for square roots. Use your formula to approximate the cube roots of 2 and 3 in the following way. Let $x_0 = 1$ in both cases and compute x_1 and x_2 as rational numbers. Compare your answers with the actual cube roots.

4. Show that Newton's method gives the formula $x_{n+1} = 2x_n - ax_n^2$ as an iteration formula to find the reciprocal of a. Does this formula work very well for approximating $1/7$? Use $x_0 = 0.1$.

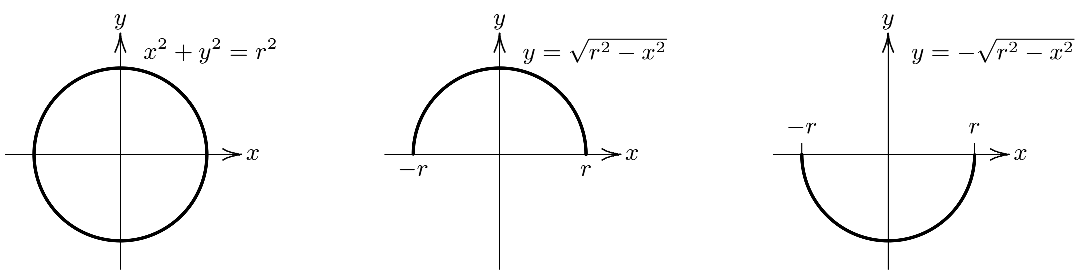
5. Here is an example of a practical problem that leads to an equation that cannot be solved exactly. A chord of length 4 cuts off an arc of length 5 in a circle. Find the radius of the circle and the central angle determined by the chord (the angle whose vertex is at the center of the circle and whose sides extend to the ends of the chord). You may use a calculator for this problem and not bother with Newton's method.

6. Provide a sketch for a situation in which Newton's method could fail to find a root. *Hint*: Look at the denominator of the iteration formula.

1.32 Implicit Differentiation

Some curves in the plane cannot be represented in the form $y = f(x)$; just imagine a fancy loop de loop scribble. Such a curve does not represent a function (think about the vertical line test), but the curve certainly appears to have a tangent line at most points. All the derivative formulas developed so far have been for curves expressed in the form $y = f(x)$, so a new technique is required for curves that are not or cannot be expressed in this form.

The most familiar example of a curve that is not the graph of a function is a circle of radius r generated by the equation $x^2 + y^2 = r^2$. This equation does not define a function since each value of x between $-r$ and r generates two values of y. This particular equation can be solved for y to obtain $y = \pm\sqrt{r^2 - x^2}$. The equation $y = \sqrt{r^2 - x^2}$ represents the top half of the circle while the equation $y = -\sqrt{r^2 - x^2}$ represents the bottom half of the circle (see the figure).



Hence, the equation $x^2 + y^2 = r^2$ gives rise to two functions and these functions are said to be defined implicitly by the equation $x^2 + y^2 = r^2$. A function is defined **implicitly** if some values of x determine one or more values of y but a formula for y in terms of x is not given explicitly in the form $y = f(x)$.

The technique of implicit differentiation makes it possible to find the derivative of a function that is defined implicitly. The basic idea is to assume that y is a differentiable function of x, even if it is not possible to find this function, and differentiate the entire equation with respect to x. For example, differentiating $x^2 + y^2 = r^2$ with respect to x, assuming y is a function of x and r is a constant, gives

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Note the use of the chain rule when differentiating the term y^2. The beauty of this method is that it gives the right answer no matter which function we use for y:

$$y = \sqrt{r^2 - x^2}; \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} = -\frac{x}{y};$$

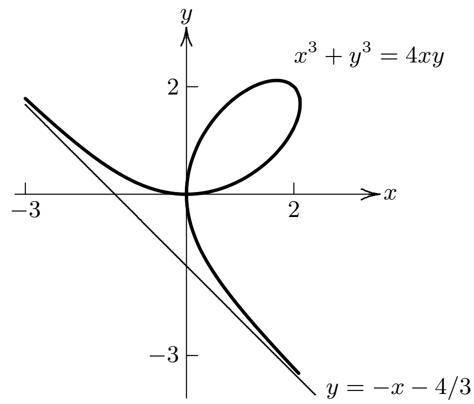
$$y = -\sqrt{r^2 - x^2}; \qquad \frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}} = -\frac{x}{y}.$$

In fact, the answer obtained by implicit differentiation is precisely the result given by geometry; if (x, y) is on the circle, then the radial line has slope y/x and the tangent line has slope $-x/y$. Implicit differentiation can also be used to find higher derivatives. In this case,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{y}\right) = -\frac{y \cdot 1 - x \cdot \dfrac{dy}{dx}}{y^2} = -\frac{y + x \cdot \dfrac{x}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{r^2}{y^3}.$$

Note the insertion of the formula for dy/dx and the use of the original equation $x^2 + y^2 = r^2$.

An interesting curve known as the folium of Descartes is defined by the equation $x^3 + y^3 = 3axy$, where a is a positive constant. A graph of this curve when $a = 4/3$ is given below; the line $y = -x - a$ is an asymptote for the folium of Descartes.



To find dy/dx for this curve, assume that y is a function of x and differentiate the equation $x^3 + y^3 = 4xy$ with respect to x to obtain

$$3x^2 + 3y^2 \frac{dy}{dx} = 4\Big(x \frac{dy}{dx} + y\Big) \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 4y}{4x - 3y^2}.$$

This formula can be used to find the slope of the curve at any point on the curve, but both the x and y coordinates of the point in question must be known. In particular, at the point $(2, 2)$, the slope is -1.

Exercises

1. The equation $4x^2 - y^2 = 1$ defines two functions of y implicitly. Solve for these two functions and find dy/dx for each of them. Then find dy/dx implicitly and compare the results.

2. Use implicit differentiation to find dy/dx.

a) $x^2 + y^2 = x + 4xy$ **b)** $x^3 + y^3 = 2x + y$ **c)** $x^4 + 3x^2y + y^3 = 10$

d) $(x^2 + y^2)^2 = 9(x^2 - y^2)$ **e)** $x^3 + xy + y^2 = e^y$ **f)** $\dfrac{y}{x} + \dfrac{x}{y+1} = 8$

3. Find an equation for the line tangent to the curve defined by $x^4 + xy + y^3 = 11$ at the point $(1, 2)$.

4. Consider the folium of Descartes $x^3 + y^3 = 9xy/2$.

a) Find the slope of this curve at the points $(1, 2)$, $(2, 1)$, and $(9/4, 9/4)$.

b) In addition to $(2, 1)$, find two more points on this curve for which the x coordinate is 2.

5. Find an equation for the line normal to the hyperbola $y^2 - x^2 = 7$ at the point $(3, 4)$.

6. For the curve defined by $x^3 + 4xy^2 + 6y = 8$, find an equation for each of the tangent lines to the curve at the points where the curve intersects the coordinate axes.

7. Find an equation for a tangent line to the curve $x^2 - y^2 = 5$ that passes through the point $(1, 1)$.

8. Find the points (other than the four points where the curve meets the x and y axes) on the curve $2x^4 + y^4 = 96$ for which the normal line goes through the origin.

9. For the curve $x^2 + xy + y^2 = 6$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{36}{(x+2y)^3}$.

10. Find the ordered pair (s, t) that satisfies the equation $x^2 - xy + y^2 = 1$ and has the largest possible value for t.

1.33 Related Rates

In a related rates problem, related quantities are changing over time. If the rates at which some quantities are changing are known, it is usually possible to determine the rates at which the related quantities are changing. Since rates can be determined using the derivative, calculus plays a role in solving such problems.

For example, suppose that at a given instant the sides of a square are 15 inches long and that all of the sides are increasing at the rate of 4 inches per minute. To find the rate at which the area of the square is increasing at this instant, we begin by noting that $A = s^2$, where s represents the length of the side of the square in inches and A represents the area of the square in square inches. Since both A and s are implicitly defined functions of time t (measured in minutes), we can differentiate this equation implicitly with respect to t to obtain

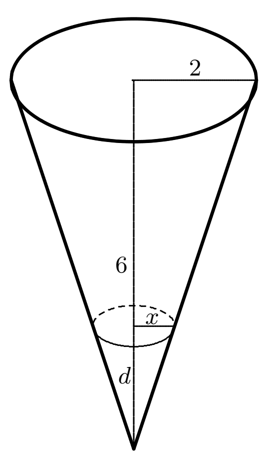
$$\frac{dA}{dt} = 2s\,\frac{ds}{dt} \quad \text{and thus} \quad \left.\frac{dA}{dt}\right|_{s=15} = 2 \cdot 15 \cdot 4 = 120.$$

(Note carefully the difference in notation between a general derivative and a derivative evaluated at a certain instant.) It follows that the area of the square is increasing at a rate of 120 in^2/min. (Pay attention to units while working these problems.) A more detailed and involved example is given in the following problem.

Problem: An inverted cone with a diameter of 4 meters and a height of 6 meters is being filled with water at a rate of 1 cubic meter per minute. At what rate is the depth of water in the cone changing when the water is 4 meters deep?

Solution: We are given the rate at which volume changes and are asked to determine the rate at which depth changes. It is thus necessary to find an equation that relates these two quantities. Let V be the volume (in cubic meters) of water in the cone, let d be the depth (in meters) of water in the cone, and let t be the elapsed time (in minutes). Then $dV/dt = 1$ and we must find dd/dt when $d = 4$. Using similar triangles and the formula for the volume of a cone (see the figure), we find that $V = \pi d^3/27$. Both V and d are functions of t, but they are not explicitly given. Differentiating the equation implicitly with respect to t, then substituting the specific values of the variables yields the computations to the right of the figure.

$$\frac{d}{6} = \frac{x}{2}$$
$$x = \frac{d}{3}$$



$$V = \frac{\pi}{3}\,x^2 d$$
$$V = \frac{\pi}{27}\,d^3$$
$$\frac{dV}{dt} = \frac{\pi}{9}\,d^2\,\frac{dd}{dt}$$
$$1 = \frac{\pi}{9}\,4^2\,\left.\frac{dd}{dt}\right|_{d=4}$$
$$\left.\frac{dd}{dt}\right|_{d=4} = \frac{9}{16\pi}$$

Thus, the depth of water is increasing at a rate of $9/(16\pi) \approx 0.179$ meters per minute when the depth is 4 meters. It is important to note that specific values of the variables are not substituted until derivatives are taken.

Some general guidelines that are useful for solving related rates problems are given below.

1. Read the problem carefully, draw a picture if it is helpful, and understand what is being asked.

2. Determine the quantities that vary with time and assign variables to them. Be careful not to mistake constants given in the problem with values that depend on time. Determine the rates that are known and those that are to be found.

3. Find a relationship between the quantities whose rates are known and those whose rates are to be determined. This step may require some extra effort.

4. Use implicit differentiation to differentiate the equation relating the quantities with respect to time. This will give a relationship involving the rates of change of the relevant quantities.

5. Substitute the given values of the variables and the known rates into the equation from (4) and solve for the unknown rate. Be certain to obtain the correct units for the rate.

Exercises

1. The radius of a circle is increasing at the rate of 3 cm/sec. At what rate is the area of the circle increasing when the radius is 20 cm?

2. The sides of an equilateral triangle are increasing at the rate of 1 cm/sec. At what rate is the area of the triangle increasing when the sides are 4 cm?

3. Gas is pumped into a spherical balloon at the rate of 1 ft^3/min. How fast is the diameter of the balloon increasing when the balloon contains 36π ft^3 of gas?

4. One airplane flew west over an airport at 300 km/hr. Ten minutes later another airplane flew south over the airport at 230 km/hr. Assuming that the airplanes were at the same altitude, determine the rate at which they were separating 20 minutes after the second airplane flew over.

5. A balloon leaves the ground 400 meters from a person standing at ground level. When the balloon is at a height of 200 meters, it is rising at the rate of 20 m/min.

a) How fast is the person's angle of observation increasing at this instant?

b) How fast is the distance between the person and the balloon increasing at this instant?

6. A upright cone (vertex up) with a radius of 4 m and height of 10 m is being filled with water at a rate of 2.5 m^3/min. How fast is the water level rising when the water is 6 m deep?

7. The minute hand on a clock is 4 inches long and the hour hand is 2 inches long. At what rate is the distance between the tips of the hands changing at 3:00? What is the rate at 3:40?

8. Suppose the surface area of a sphere is decreasing at a rate of 30 cm^2/min when its radius is 2 m. At what rate is the volume of the sphere decreasing at this instant?

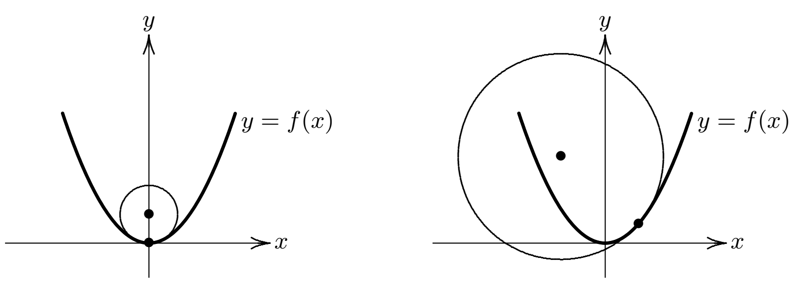
9. A lighthouse is 2 km away from a straight shoreline and its light makes 3 revolutions per minute. How fast is the beam of light moving along the shoreline when the light is 0.5 km from the nearest point on the shoreline?

10. Sand is leaking out of an elevated bin and falling into a conical pile at the rate of two cubic feet per minute. The radius of the base of the cone is always three times the height of the cone. At what rate is the circumference of the base increasing when the height of the pile is four feet?

1.34 Curvature

Consider the graph of a smooth function f. In addition to having a different slope at each point on the graph, the graph also has a different curvature (or amount of bend) at each point. It is both useful and interesting to find a numerical value for the curvature of a curve. For instance, the acceleration of a car moving along a road with constant speed is related to how much the road bends.

It should be clear that the curvature of a circle is the same at each point on the circle, and that the curvature of a circle decreases as the radius increases. Since a circle of radius r turns through an angle of 2π radians over a distance of $2\pi r$ units, the curvature is defined to be $1/r$. The curvature of a general curve will vary from point to point along the curve. Recall that the slope of a curve at a point is defined to be the slope of the line that best approximates the curve at that point. Similarly, the curvature of a curve at a point is defined to be the curvature of the circle that best approximates the curve at that point (see the figure).



The radius of the circle that best approximates the curve varies from point to point.

There are several ways to find the approximating circle; one method is presented in the next paragraph and another is outlined in the exercises.

Consider the graph $y = f(x)$ of a twice differentiable function f and suppose that $(a, f(a))$ is a point on this graph. The circle that best approximates f at $(a, f(a))$ goes through the point $(a, f(a))$ and has the same slope and concavity at a as the function f. Let the equation of the circle be $(x-h)^2+(y-k)^2 = r^2$. The goal is to find values for h, k, and r so that the circle has the desired properties. Using implicit differentiation on the equation $(x-h)^2 + (y-k)^2 = r^2$, we obtain

$$\frac{dy}{dx} = -\frac{x-h}{y-k} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{r^2}{(y-k)^3}.$$

In order for the circle to best approximate f at a, the three unknowns h, k, and r must satisfy the equations:

$$(a-h)^2 + \big(f(a)-k\big)^2 = r^2, \qquad f'(a) = -\frac{a-h}{f(a)-k}, \qquad f''(a) = -\frac{r^2}{\big(f(a)-k\big)^3}.$$

The first equation indicates that the point $(a, f(a))$ is on the circle, the second guarantees that the circle and the curve have the same slope at a, and the third gives the circle and the curve the same concavity at a. These three equations in the three unknowns h, k, and r look rather intimidating to solve. However, if we let $A = h - a$, $B = k - f(a)$, $c = f'(a)$, and $d = f''(a)$, the equations become

$$A^2 + B^2 = r^2, \qquad A = -cB, \qquad r^2 = dB^3,$$

which look much more manageable. We must solve for A, B, and r in terms of c and d. Substituting the second and third equations into the first equation yields $B = (c^2+1)/d$. It then follows fairly easily that

$$h = a - \frac{f'(a)}{f''(a)}\left(1+(f'(a))^2\right), \quad k = f(a) + \frac{1+(f'(a))^2}{f''(a)}, \quad \text{and} \quad r^2 = \frac{\left(1+(f'(a))^2\right)^3}{(f''(a))^2}.$$

If $f''(a) = 0$ (which corresponds to a possible inflection point), then the circle has infinite radius, which means that the curve is essentially a straight line at the point $(a, f(a))$.

The radius of the approximating circle is called the **radius of curvature** of the function and is usually denoted by ρ. The **curvature** of a function, which is the reciprocal of the radius of curvature, is usually denoted by κ. Thus, the radius of curvature ρ and curvature κ of a twice differentiable function f at $(x, f(x))$ are given by

$$\rho(x) = \frac{\left(1+(f'(x))^2\right)^{3/2}}{|f''(x)|} \qquad \text{and} \qquad \kappa(x) = \frac{f''(x)}{\left(1+(f'(x))^2\right)^{3/2}}.$$

Note that the numerator for $\kappa(x)$ is $f''(x)$ rather than $|f''(x)|$; this makes it possible to distinguish between curves that are concave up (positive curvature) and those that are concave down (negative curvature). The center of the circle that best approximates the curve $y = f(x)$ at a point, which is called the **center of curvature**, can be found using the formulas for h and k derived in the previous paragraph.

Exercises

1. For each curve, find the curvature at the indicated points.

a) $y = x^2,\ x = 0, 1$ **b)** $y = x^3,\ x = 0, 1$ **c)** $y = x^4,\ x = 0, 1$

d) $y = x^4 - 2x^2,\ x = 1, 2$ **e)** $y = e^{-x^2},\ x = 0, 1$ **f)** $y = \ln|\cos x|,\ x = 0, \pi/4$

2. For the curve $y = x^4 - 2x^2$, sketch the curve and its curvature function on the same graph.

3. For the curve $y = x^3 - 3x$, find the center of curvature when $x = 2$.

4. For the curve $y = x^2$, find the center of curvature when $x = a$.

5. For $y = \arccos(1-x) - \sqrt{2x - x^2}$, find a general formula for $\kappa(x)$. (After some algebra, it is quite simple.)

6. Find the maximum curvature of the curve $y = e^x$.

7. Let P be a polynomial of degree $n \geq 2$ and let κ be the curvature of P. Prove that $\lim_{x\to\infty} \kappa(x) = 0$ and $\lim_{x\to-\infty} \kappa(x) = 0$.

8. Consider the function $f(x) = -c\ln(1-x^2)$, where c is a positive constant. Sketch the graph of the curvature of this function for $c = 0.5$ and $c = 0.8$. You should find that the graphs look rather different on the interval $(-1, 1)$. Find the value of c for which the shape of the curvature graph changes. (The exact value is not complicated, but if you cannot find it, at least find an approximation.)

9. Here is a different approach to finding the center of curvature of a twice differentiable function f at a point a. For each $b \neq a$, let (u_b, v_b) be the point of intersection of the normal lines through $(a, f(a))$ and $(b, f(b))$. Use properties of circles to explain why it is reasonable to expect that $\lim_{b\to a}(u_b, v_b)$ is the center of curvature (h, k) of f at a. Show that

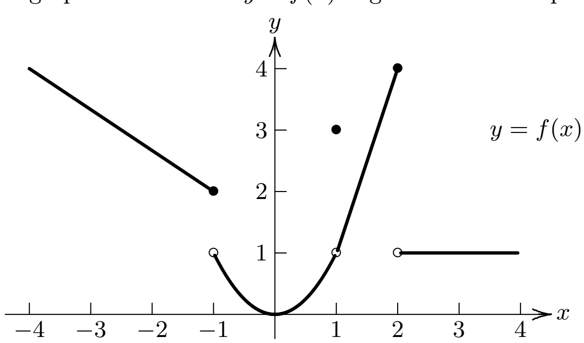
$$u_b = \frac{F(b) - F(a)}{G(b) - G(a)}, \quad \text{where} \quad F(x) = \frac{x}{f'(x)} + f(x) \quad \text{and} \quad G(x) = \frac{1}{f'(x)},$$

and thus $h = \lim_{b\to a} u_b = F'(a)/G'(a)$. Then find the value of k and show that these values are the same as those found by the method discussed in this section.

1.35 Supplementary Exercises

Remark. The exercises in this section follow the order of the text.

1. Find an equation for the line through the point $(2,4)$ that is parallel to the line through the points $(-1,10)$ and $(15,2)$.
2. Find an equation for the line tangent to the circle $x^2+y^2=34$ at the point $(3,5)$.
3. A certain concession stand sold 850 hot dogs on a day when the price was set at \$2 per hot dog and sold 1100 hot dogs on a day when the price was set at \$1.50 per hot dog. Assuming a linear relationship between number of hot dogs sold and sales price, how many hot dogs would be sold at \$1.65 per hot dog?
4. Consider the trapezoid with vertices at $(0,0)$, $(1,2)$, $(4,2)$, and $(6,0)$. Find a vertical line that divides the trapezoid into two parts of equal area. With a little more work, find a horizontal line with the same property.
5. Find the distance from the point $(1,2)$ to the line $y=3x+4$.
6. Find a point on the line $5x-7y=22$ that has integer coordinates.
7. Let A, B, and C be non-collinear points. Let ℓ_A be the perpendicular bisector of AB and let ℓ_C be the perpendicular bisector of BC. Explain why the point of intersection of ℓ_A and ℓ_C is the center of the circle that goes through A, B, and C. Use this fact to find an equation for the circle that goes through the points $(0,0)$, $(1,4)$, and $(3,2)$.
8. The graph of a function $y=f(x)$ is given. Find the quantities listed to the right of the graph.



a) $\lim_{x\to -1^-} f(x)$
b) $\lim_{x\to -1^+} f(x)$
c) $\lim_{x\to 1} f(x)$
d) $\lim_{x\to 2^-} f(x)$
e) $\lim_{x\to 2^+} f(x)$
f) $\lim_{x\to 3} f(x)$
g) $f(1)$
h) $f(2)$

9. Sketch a graph of the function $f(x)=\begin{cases} x^2, & \text{if } x<0; \\ x+1, & \text{if } 0\le x\le 4; \\ 1/x, & \text{if } x>4; \end{cases}$ then find $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 1} f(x)$, and $\lim_{x\to 4^+} f(x)$.
10. List the discontinuities of the given function.

a) $f(x)=\dfrac{x}{x^2-4}$ **b)** $g(x)=\dfrac{2}{1+\cos x}$ **c)** $h(x)=\dfrac{x^2+4}{x-1}$

11. Sketch the graph of a single function f with all the following properties: (i) f is defined on $\mathbb{R}$, (ii) f is continuous except at 3 and 8, (iii) f has different one-sided limits at 3, and (iv) $\lim_{x\to 8^+} f(x)$ does not exist.
12. Suppose that $f\colon [a,b]\to[a,b]$ is continuous on $[a,b]$. Prove there is a point $c\in[a,b]$ such that $f(c)=c$.
13. Use algebra to evaluate each of the following limits.

a) $\lim_{x\to 2}\dfrac{x-2}{2x^2-3x-2}$ **b)** $\lim_{x\to -1}\dfrac{x^2-1}{x^2-3x-4}$ **c)** $\lim_{x\to 1}\dfrac{x^4-1}{x^3-1}$

d) $\lim_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$ **e)** $\lim_{x\to 4}\dfrac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{2}}{x-4}$ **f)** $\lim_{h\to 0}\dfrac{\dfrac{1}{(1+h)^2}-1}{h}$

14. Find a positive constant a so that $\lim_{x\to 1}\dfrac{x^2-a^2}{x^2+2x-3}$ exists and find the value of the limit.
15. Suppose that f is a continuous function with the property that $\lim_{v\to 2}\dfrac{f(v)-7}{v-2}=3$. Find an equation for the line tangent to the graph of $y=f(x)$ when $x=2$.
16. Find equations for the tangent and normal lines to the curve $y=x^4$ when $x=1$.
17. Find two points on the curve $y=x^3$ for which the tangent line goes through $(4,0)$.
18. Suppose that the height h in inches of a beanstalk after t days is $h=t^3/3$. When is the beanstalk growing at a rate of 2 feet per hour?

19. Find and simplify the derivative of the given function.

a) $f(x) = 8\sqrt[4]{x} - 15\sqrt[5]{x}$ **b)** $g(x) = \dfrac{9}{5x^2}$ **c)** $h(x) = \sqrt{x} - \dfrac{6}{x}$

20. Find equations for the tangent and normal lines to the curve $y = 4/x$ when $x = 2$.

21. Find a point on the curve $y = \sqrt[4]{x}$ for which the tangent line goes through $(-15, 0)$.

22. Find a point on the curve $y = \sqrt[3]{x}$ for which the tangent line goes through $(0, 4)$.

23. Find and simplify the derivative of the given function.

a) $f(x) = (3x-1)^5$ **b)** $g(x) = \dfrac{1}{\sqrt{5x^2+4}}$ **c)** $h(x) = (x^2+x+1)^8$

24. Find both values of c for which the tangent line to $f(x) = (x^2+x+4)^4$ at $(c, f(c))$ goes through the origin.

25. Find and simplify the derivative of the given function. Be sure to factor as much as possible.

a) $f(x) = x\sqrt{x^2+2}$ **b)** $g(x) = \dfrac{2x-1}{3x+4}$ **c)** $h(x) = x^2(4x-15)^5$

26. Determine all the values of x for which the tangent line to the graph of $y = (2x+7)^3(3x-13)^5$ is horizontal.

27. Find an equation for the line tangent to the graph $y = 6x/(x^2+2)$ when $x = 1$.

28. Find the maximum and minimum outputs of the function on the given interval. Assume that $0 < a < b$.

a) $f(x) = \sqrt{x^2 - 4x + 10},\ [1,5]$ **b)** $g(x) = x^3 - 3x + 4,\ [0,5]$ **c)** $h(x) = x^2 + 4/x^2,\ [0.8, 4]$
d) $F(x) = x^3 - a^2x,\ [0, 2a]$ **e)** $G(x) = ax/(a^2+x^2),\ [a/2, 3a]$ **f)** $H(x) = x/a + b/x,\ [a, b]$

29. Find the volume of the largest cone with surface area 36π. Assume that the surface area of the cone includes the base as well as the sides.

30. Find the volume of the largest right circular cylinder that can be inscribed in a right circular cone of radius r and height h. Find the fraction of the volume of the cone that is taken up by the optimal cylinder. Also, find the ratio of height to radius for the optimal cylinder.

31. A wire with length L will be cut into two pieces and the pieces bent into different shapes.

a) Suppose that one piece is used to construct a square and the other piece is used to construct a circle. Where should the cut be made in order to minimize the sum of the two areas?

b) Suppose that one piece is used to construct a square and the other piece is used to construct an equilateral triangle. Where should the cut be made in order to minimize the sum of the two areas?

c) Suppose that one piece is used to construct a circle and the other piece is used to construct an equilateral triangle. Where should the cut be made in order to minimize the sum of the two areas?

d) Suppose that one piece is used to construct a square and the other piece is used to construct a hexagon. Where should the cut be made in order to minimize the sum of the two areas?

32. A company has plants that are located (in an appropriate coordinate system) at the points $(0,1)$, $(0,-1)$, and $(3,0)$. The company plans to build a distribution center on the x-axis. Where should the center be located in order to minimize the sum of the distances from the three plants?

33. Find a general formula for the distance from a point (x_0, y_0) to the line $Ax + By + C = 0$.

34. Let a, b, and d be positive numbers. Suppose that two vertical poles of heights a and b are separated by a distance d (all units are the same). Find the length of the shortest wire that can run from the top of one pole to a point on the ground between the poles and up to the top of the other pole. Do you notice anything interesting about the two triangles formed by the optimal wire, the two poles, and the ground?

35. Let a and b be positive constants and define a function f by $f(x) = b - ax^2$. For each value of x between 0 and the positive x-intercept of f, the tangent line to the graph cuts off a triangle in the first quadrant. Find the intercepts of the tangent line that cuts off the triangle of least area and the area of this triangle.

36. A pole leans away from the sun at an angle of 11° from the vertical. It casts a shadow of 25 feet when the angle of elevation of the sun with the horizontal is 68°. Find the length of the pole.

37. Find, to the nearest tenth of a degree, the acute angle formed by the lines $2x + y - 5 = 0$ and $3x - 4y - 2 = 0$.

38. Use the Intermediate Value Theorem to prove that the equation $4\sin x = x$ has a positive solution.

39. Find the maximum value for the sum of the sines of the three angles of an isosceles triangle.

40. Find the value of $\displaystyle\lim_{h\to 0} \frac{a^h - 1}{h}$. *Hint*: Interpret the limit as a derivative.

41. Evaluate each of the following limits.

a) $\lim_{x\to 1} \dfrac{x^{21}-1}{x^4-1}$ b) $\lim_{x\to 2} \dfrac{2x^2-3x-2}{x^3+x^2-6x}$ c) $\lim_{x\to -1} \dfrac{(x+1)^2}{1+\cos(\pi x)}$

d) $\lim_{x\to\infty} \dfrac{x^2}{3x^2-4x+1}$ e) $\lim_{x\to\infty} \dfrac{9x^4+3x}{(x^2-1)(2x^2+7)}$ f) $\lim_{x\to\infty} \dfrac{x^6}{e^x}$

42. Find the (x, y) coordinates of all inflection points. Treat a as a positive constant.

a) $f(x) = a^2/(a^2+x^2)$ b) $g(x) = e^{-x^2/a^2}$ c) $h(x) = (x^3 - a^3)/x$

43. Suppose a person has a vertical leap of 42 inches. What initial velocity (in miles per hour) is required for such a leap? How much hang time does the person have?

44. A person throws a ball straight upward along side of a tree. The ball is released from a height of 6 feet, just reaches the top of the tree and then is caught by the person at a height of 4 feet. If the ball remains in the air for 4.6 seconds, how tall is the tree?

45. Suppose that a particle is moving in a straight line with a velocity of 160 ft/sec. At time $t = 0$, the object begins to decelerate at the rate of $4\,t$ ft/sec^2. How far does this particle travel before coming to a stop?

46. For the curve $y = x^5/5 - x^3/3$, find the curvature when $x = 1$ and $x = 2$.

47. Find the maximum curvature of the curve $y = x^4/4$.

48. Show that the maximum curvature of $y = x^3 - 3x$ does not occur at its relative extreme points.

49. Give an example of a function that has zero curvature at a point on its graph that is not an inflection point.

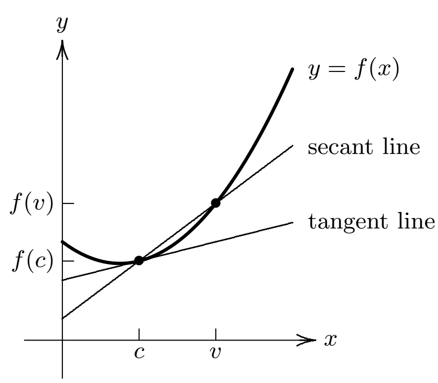
50. The curvature of a curve is related to the normal component of acceleration; think of getting pushed toward the door of a car as you round a corner quickly. Railroad tracks must be designed in such a way that the curvature does not change abruptly. Suppose that the x-axis for $x \le 0$ and the line $y = x$ for $x \ge 1$ represent two sections of railroad track that must be joined with a curve of the form $y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$. Find constants A, B, C, D, E, F so that the curve joins the two pieces of track and has the same tangent line and same curvature at the junction points.

51. Let b be a constant and let κ be the curvature of the function $f(x) = x^3 + bx$. Show that κ has exactly two relative extreme outputs. Find the values of x where these extreme outputs occur.

2

The Integral

The derivative concept is motivated by the geometric problem of finding the slope of the tangent line to a general curve. As a quick review, to compute the slope of the curve $y = f(x)$ at the point $(c, f(c))$, take any other point $(v, f(v))$ on the graph and determine the slope of the secant line. The limit of the slopes of the secant lines as $v \to c$ is the slope of the tangent line at c (see the figure).



Slope of secant line is $\dfrac{f(v) - f(c)}{v - c}$;

slope of tangent line is $\lim\limits_{v \to c} \dfrac{f(v) - f(c)}{v - c}$.

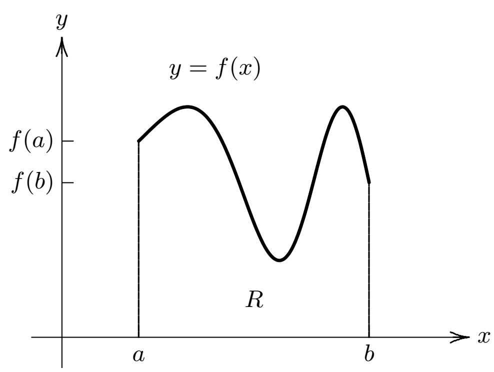
The slope of the secant line is given by the difference quotient $(f(v) - f(c))/(v - c)$. The slope of the tangent line is found by letting the point $(v, f(v))$ slide along the curve to the point $(c, f(c))$. In other words, the numbers v get closer to c. Hence, the geometric problem of determining the slope of a curve at $(c, f(c))$ reduces to the algebraic problem of evaluating $\lim\limits_{v \to c} \dfrac{f(v) - f(c)}{v - c}$. Since a general curve has a different slope at every point, this process gives rise to a function; the new function gives the slope of the original function at any particular point. Thus, the **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{v \to x} \frac{f(v) - f(x)}{v - x}$$

for each value of x in the domain of f for which the limit exists. The number $f'(c)$ represents the **slope** of the graph $y = f(x)$ at the point $(c, f(c))$. It also represents the **rate of change** of y with respect to x when

x is near c; this interpretation of f' opens the door to a number of physical applications. Differential calculus is concerned with finding simple ways to evaluate derivatives (rather than using the definition), using the derivative to study the graph and behavior of a function, and exploring various applications of the derivative to physical problems.

The integral concept is motivated by the geometric problem of finding the area of a region with curved boundaries. The most familiar region with curved boundaries is the circle and attempts to find its area can be traced back several thousand years. To phrase the problem in a modern setting, let f be a continuous nonnegative function defined on an interval $[a, b]$ and let R be the region under the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$ (see the figure).



Determine the area of the region R bounded by the curve $y = f(x)$ and three straight lines.

The problem is to find the exact value of the area of the region R. The solution to the area problem leads to the definition of the integral and, as with the derivative, the limit concept appears in the definition of the integral. In general, solving the area problem or, equivalently, evaluating integrals turns out to be more difficult than the problem of finding the slope of a curve. In the late seventeenth century, an amazing connection between the tangent problem and the area problem was discovered, thus making the area problem easier to solve. The discovery of this connection, which is known as the Fundamental Theorem of Calculus, is considered to be the origin of calculus. By taking advantage of this result, many great achievements in both mathematics and the physical sciences were accomplished. In spite of its geometric motivation as area, the integral has a wide variety of applications. These include computing geometric quantities such as volume and arc length, and physical quantities such as work and force. We will consider several of these applications in this chapter after first defining the integral and exploring its connection to the derivative.

2.1 Summation notation

Suppose you are asked to represent the sum of the squares of the first one hundred positive integers. Since this sum is rather long, you would probably abbreviate it as $1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2$. Since long sums of this type are common in mathematics, it is helpful to have a shorthand notation for them. The upper case Greek letter sigma Σ is used for this purpose. We write $\sum_{i=1}^{100} i^2$ for the sum of the squares of the first one hundred positive integers. Further (and more abstract) examples of this summation notation include

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n \quad \text{and} \quad \sum_{k=0}^{p+1} f(k) = f(0) + f(1) + f(2) + \cdots + f(p+1).$$

The letter i or k as used here is called the **index of summation** and the sum begins at the integer below the symbol Σ and ends at the integer above the symbol Σ, running through all of the integer values in between. The generic sum $1^2 + 2^2 + \cdots + n^2$ can be represented by $\sum_{i=1}^{n} i^2$ or $\sum_{k=0}^{n-1} (k+1)^2$ or even $\sum_{i=5}^{n+4} (i-4)^2$; there is more than one way to represent a sum in summation notation.

As a simple example of summation notation, consider the following sum:

$$\begin{aligned}
\sum_{i=1}^{4} (2i^3 - 3i) &= (2 \cdot 1^3 - 3 \cdot 1) + (2 \cdot 2^3 - 3 \cdot 2) + (2 \cdot 3^3 - 3 \cdot 3) + (2 \cdot 4^3 - 3 \cdot 4) \\
&= -1 + 10 + 45 + 116 = 170 \\
&= 2\big(1^3 + 2^3 + 3^3 + 4^3\big) - 3\big(1 + 2 + 3 + 4\big) \\
&= 2\sum_{i=1}^{4} i^3 - 3\sum_{i=1}^{4} i.
\end{aligned}$$

Of course, summation notation is not all that helpful for the sum of just a few terms, but this example illustrates the main idea. As indicated by the rewriting of this sum in a different way, the commutative, associative, and distributive properties of addition and multiplication can be applied to summation notation. These facts yield the following properties of sums; the letter C represents a quantity that does not depend on the index i.

1. $\sum_{i=1}^{n} C = nC$

2. $\sum_{i=1}^{n} Ca_i = C\sum_{i=1}^{n} a_i$

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

The proofs of these properties are not hard; simply write out both sides of the equations and note that they are equal. In addition to these properties of sums, we also need formulas for the sums of various powers of the first n positive integers. The formula for the sum of the first n positive integers can be derived rather easily by writing the sum forward and backward;

$$\begin{aligned}
S &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + (n-1) + n \\
S &= n + (n-1) + (n-2) + \cdots + \quad 2 \quad + 1
\end{aligned}$$

and then adding the two expressions. Thus $2S = n(n+1)$ and the formula for the sum is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \text{or} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

The technique used to find this sum is a useful one to remember. To find formulas for the sums of higher powers of positive integers, we begin by collecting some data.

n	$\sum_{i=1}^{n} i$	$\sum_{i=1}^{n} i^2$	$\sum_{i=1}^{n} i^3$	$\sum_{i=1}^{n} i^4$
1	1	$1 = 1 \cdot \frac{3}{3}$	$1 = 1^2$	$1 = 1 \cdot \frac{5}{5}$
2	3	$5 = 3 \cdot \frac{5}{3}$	$9 = 3^2$	$17 = 5 \cdot \frac{17}{5}$
3	6	$14 = 6 \cdot \frac{7}{3}$	$36 = 6^2$	$98 = 14 \cdot \frac{35}{5}$
4	10	$30 = 10 \cdot \frac{9}{3}$	$100 = 10^2$	$354 = 30 \cdot \frac{59}{5}$
5	15	$55 = 15 \cdot \frac{11}{3}$	$225 = 15^2$	$979 = 55 \cdot \frac{89}{5}$
6	21	$91 = 21 \cdot \frac{13}{3}$	$441 = 21^2$	$2275 = 91 \cdot \frac{125}{5}$
7	28	$140 = 28 \cdot \frac{15}{3}$	$784 = 28^2$	$4676 = 140 \cdot \frac{167}{5}$

We already have a formula for $\sum_{i=1}^{n} i$. Comparing these sums with those for $\sum_{i=1}^{n} i^3$ reveals a simple (and somewhat intriguing) relationship between them; the latter are the squares of the former. With a little more effort, a pattern appears in the $\sum_{i=1}^{n} i^2$ column; the sum $\sum_{i=1}^{n} i$ is multiplied by the fraction $(2n+1)/3$ to obtain $\sum_{i=1}^{n} i^2$. The last column is the most complicated one, but even there a pattern develops when the sums of the fourth powers are compared to the sums of the squares. Notice that the numerator of the multiplier goes up by the numbers $12, 18, 24, \ldots$. The numerators can then be written as

$$6\big(1\big) - 1,\ 6\big(1+2\big) - 1,\ 6\big(1+2+3\big) - 1,\ 6\big(1+2+3+4\big) - 1,$$

and so on. Since we have a formula for the sums in parentheses, these numbers are given by $6n(n+1)/2 - 1$ and the desired multiplier is thus $(3n^2 + 3n - 1)/5$. Putting all of this information together gives us the following table:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\
\sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \\
\sum_{i=1}^{n} i^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}
\end{aligned}$$

It is important to note that we have not proved these formulas; they are merely conjectures. After discussing mathematical induction in the next chapter, it will be possible to give rigorous proofs of these conjectures. For now, we will accept these as correct formulas. Studying the expanded form of each sum of the powers of the first n positive integers, a pattern begins to emerge. By focusing on the highest two powers of n, one is led to the conjecture that

$$\sum_{i=1}^{n} i^r = \frac{n^{r+1}}{r+1} + \frac{n^r}{2} + \text{ smaller positive integer powers of } n,$$

for every positive integer r. This conjecture is indeed true and it can be proved using mathematical induction.

As an example of these specific formulas and the general properties of sums, consider the following:

$$\sum_{i=1}^{75}(2i^3-5i) = 2\sum_{i=1}^{75} i^3 - 5\sum_{i=1}^{75} i = 2\left(\frac{75\cdot 76}{2}\right)^2 - 5\left(\frac{75\cdot 76}{2}\right) = 16,230,750.$$

For a second example, note how the "long" sum can be expressed in a simpler form:

$$\begin{aligned} 2+5+8+\cdots+(3n+2) &= \sum_{i=1}^{n+1}(3i-1) = 3\sum_{i=1}^{n+1} i - \sum_{i=1}^{n+1} 1 \\ &= \frac{3}{2}(n+1)(n+2) - (n+1) = \frac{1}{2}(n+1)(3n+4). \end{aligned}$$

Finally, it is important to note that these sum formulas are only valid when the starting value of the index is 1. When this is not the case, we can do some subtraction:

$$\begin{aligned} \sum_{k=9}^{34} k^2 &= \sum_{k=1}^{34} k^2 - \sum_{k=1}^{8} k^2 = \frac{34\cdot 35\cdot 69}{6} - \frac{8\cdot 9\cdot 17}{6} \\ &= 17(35\cdot 23 - 4\cdot 3) = 17(700+105-12) = 17(800-7) = 13600-119 = 13481. \end{aligned}$$

(The extended calculations are intended to reveal that the final answer can be obtained with minimal calculation and no electronic device.) In the following sections, we will see more interesting applications of these formulas.

Exercises

1. Write each of the following sums in summation notation.

a) $2+7+12+\cdots+997$ **b)** $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{29}{30}$ **c)** $1+3+9+\cdots+2187$

2. Write the sum $2+6+10+\cdots+(4n+6)$ in summation notation in two different ways: one starting with $i=1$ and another ending with $i=n$.

3. Use formulas in this section to find each of the following sums. Simplify your answers.

a) $\sum_{i=1}^{12} 2i^3$ **b)** $\sum_{k=1}^{15}(3k^2-4k+2)$ **c)** $\sum_{j=8}^{25} 3$

d) $\sum_{i=22}^{60} i^2$ **e)** $\sum_{i=1}^{n}(4i-2)$ **f)** $\sum_{k=1}^{m} k(k+1)$

4. Find simple formulas for each of the following sums. *Hint*: Write the sum in expanded form (without simplifying the terms) and notice lots of cancellation. Sums of this type are known as **telescoping sums**.

a) $\sum_{i=1}^{n}\left((i+1)^6-i^6\right)$ **b)** $\sum_{k=1}^{2n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$ **c)** $\sum_{j=1}^{n}\left(a_j-a_{j+2}\right)$

5. Find a formula, one that does not involve summation notation, for the indicated sum.

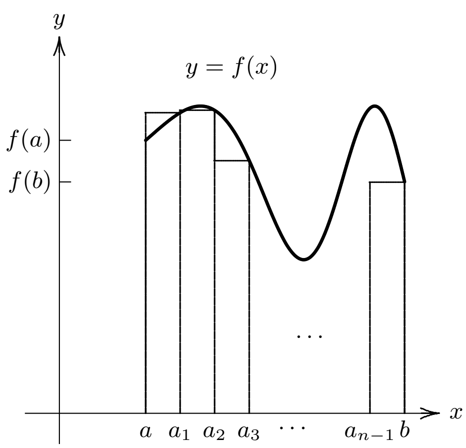
a) $4+6+8+\cdots+(2n+2)$ **b)** $1+4+7+\cdots+(3n+4)$ **c)** $1^2+4^2+7^2+\cdots+(3n-2)^2$

6. Evaluate the limit: $\lim_{n\to\infty}\frac{1^3+2^3+3^3+\cdots+n^3}{3n^4}$.

7. Prove that $r+r^2+r^3+\cdots+r^n = r\cdot\frac{r^n-1}{r-1}$, where r is any real number other than 1 and n is a positive integer. *Hint*: Let $z=r+r^2+r^3+\cdots+r^n$ and write out a similar expression for rz. Find and simplify $rz-z$, then solve for z.

2.2 Area under a curve

Let f be a continuous nonnegative function defined on an interval $[a,b]$. Consider the problem of finding the area A under the curve $y = f(x)$ and above the x-axis on the interval $[a,b]$. The basic idea for solving this problem is the following: use rectangles to approximate the area, then let the number of rectangles increase in such a way that the approximation to the area improves; the limiting value of the rectangular areas should give the area under the curve. The simplest way to carry out this process is to divide the interval $[a,b]$ into n equal subintervals, where n is an arbitrary positive integer, use the right endpoint of each subinterval to determine the height of a rectangle that approximates the curve (see the figure), find the sum of the areas of the n rectangles, then take the limit as n increases indefinitely.



$\frac{b-a}{n}$ is the width of each subinterval;

$a_1 = a + \frac{1}{n}(b-a)$ is the right endpoint of the first subinterval;

$a_2 = a + \frac{2}{n}(b-a)$ is the right endpoint of the second subinterval;

$a_3 = a + \frac{3}{n}(b-a)$ is the right endpoint of the third subinterval;

$\vdots$

$a_n = a + \frac{n}{n}(b-a) = b$ is the right endpoint of the nth subinterval.

Carrying out this process shows that the area A under the curve $y = f(x)$ and above the x-axis on the interval $[a,b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + \frac{i}{n}(b-a)\Big)\frac{b-a}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i)\frac{b-a}{n},$$

where $a_i = a + i(b-a)/n$ represents the ith hash mark on the interval $[a,b]$. The geometry problem of finding the area under a curve is reduced to the algebra problem of finding the limiting value of a sum.

As a simple example, let $f(x) = x^2$ and consider the interval $[0,b]$, where b is a positive number. The area A under this curve on $[0,b]$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Big(\frac{ib}{n}\Big)^2 \frac{b}{n} = \lim_{n\to\infty} \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \Big(\frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\Big) = \frac{b^3}{3};$$

the formula for the sum of the squares of the first n positive integers makes the limit easy to evaluate. More generally, if r is a positive integer and $f(x) = x^r$, then the area A_r under the graph of $y = f(x)$ and above the x-axis on the interval $[0,b]$ is given by (see the general formula presented in the previous section)

$$A_r = \lim_{n\to\infty} \sum_{i=1}^{n} \Big(\frac{ib}{n}\Big)^r \frac{b}{n} = \lim_{n\to\infty} \frac{b^{r+1}}{n^{r+1}} \sum_{i=1}^{n} i^r = \lim_{n\to\infty} \frac{b^{r+1}}{n^{r+1}} \Big(\frac{n^{r+1}}{r+1} + \text{smaller integer powers of } n\Big) = \frac{b^{r+1}}{r+1}.$$

Hence, a formula for the area under any curve of the form $y = x^r$, where r is a positive integer, follows pretty easily from the formula for the sum $\sum_{i=1}^{n} i^r$.

To illustrate how this general result can be used to find areas under specific curves, let $A_{[a,b]}$ represent the area under the curve $y = x^6$ and above the x-axis on the interval $[a, b]$. The formula derived above indicates that $A_{[0,b]} = b^7/7$ for any positive number b. Using symmetry and simple properties of areas, we find that

$$A_{[0,3]} = \frac{3^7}{7} = \frac{2187}{7}\,;$$

$$A_{[1,2]} = A_{[0,2]} - A_{[0,1]} = \frac{2^7}{7} - \frac{1^7}{7} = \frac{127}{7}\,;$$

$$A_{[-2,3]} = A_{[-2,0]} + A_{[0,3]} = A_{[0,2]} + A_{[0,3]} = \frac{2^7}{7} + \frac{3^7}{7} = \frac{2315}{7}\,.$$

Notice that once a formula for area has been obtained, it is no longer necessary to use the definition of area as a limit of a sum in order to find the area. This is particularly important to realize for curves that define elementary areas. For example, to find the area under the curve $y = \sqrt{1 - x^2}$ on the interval $[-1, 1]$, we simply need to recognize that the area under consideration is half the area of a circle of radius 1. It follows that the area of this region is $\pi/2$; there is no need to evaluate the limit of a sum.

Exercises

1. Find the area under the curve $y = x^2$ and above the x-axis on the given interval. You should use the $b^3/3$ result derived in this section along with some simple reasoning.

a) $[0, 12]$ **b)** $[3, 7]$ **c)** $[-3, 4]$

2. Find the area of the region bounded by the graph of $y = x^2$, the x-axis, and the tangent line to $y = x^2$ at the point (a, a^2), where a is a positive constant. *Hint*: Sketch a careful graph.

3. Use the definition of area under a curve as a limit of a sum to derive formulas for the area under the curves $y = x$ and $y = x^3$ on the interval $[0, b]$, where b is a positive constant. Imitate the example in the text.

4. Use the fact that the area under the curve $y = x^r$ and above the x-axis on the interval $[0, b]$ is $b^{r+1}/(r + 1)$ to find the area under the curve on the given interval.

a) $y = x^5$, $[0, 2]$ **b)** $y = x^3$, $[1, 5]$ **c)** $y = x^4$, $[-2, 4]$

5. Use facts from geometry to find the area under the curve on the given interval.

a) $y = 3 - |x - 1|$, $[-1, 2]$ **b)** $y = \sqrt{9 - x^2}$, $[0, 3]$ **c)** $y = \sqrt{9 - x^2}$, $[0, 2]$

6. Use the definition of area under a curve as a limit of a sum to find the area under the curve $y = 2^x$ on the interval $[0, 1]$. You will need to use the sum formula given in Exercise 7 of Section 2.1.

7. Find the area under the curve $y = \sqrt{x}$ and above the x-axis on the interval $[0, 4]$ and the area under the curve $y = \sqrt[3]{x}$ and above the x-axis on the interval $[0, 8]$. (Draw a graph and think creatively.)

8. Let r be an even positive integer. Show that the area under the curve $y = x^r$ and above the x-axis on the interval $[a, b]$ is given by the formula $\left(b^{r+1} - a^{r+1}\right)/(r + 1)$. *Hint*: The results in this section show that this formula is valid for intervals of the form $[0, b]$, where b is any positive number. Use this fact, along with properties of area and the symmetry of the curve $y = x^r$ (this is where the assumption that r is even is used), to derive the formula for a generic interval $[a, b]$. You must consider the cases $0 \le a < b$, $a < 0 < b$, and $a < b \le 0$ separately.

9. Let f be a continuous nonnegative function defined on $[a, b]$ and let k be a constant. Prove that the area under the curve $y = kf(x)$ and above the x-axis on the interval $[a, b]$ is k times the area under the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$. (This follows easily from the definition of area as a limit of a sum.)

2.3 The integral of a continuous function

The limit formula for the area under a curve that was derived in the last section forms the basis for the definition of the integral of a continuous function. Be certain you understand the significance of each term in the definition.

DEFINITION 2.1 The **integral** of a continuous function f on an interval $[a, b]$, denoted by $\int_a^b f(x)\,dx$, is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + \frac{i}{n}(b-a)\Big) \cdot \frac{b-a}{n}.$$

It can be shown that the limit that appears in the definition exists for every continuous function, but a proof of this fact is beyond the level of this text. We often write the definition of the integral more simply as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i) \cdot \frac{b-a}{n},$$

where it is assumed that $a_i = a + i(b-a)/n$ for $i = 0, 1, \ldots, n$. Note that the function f considered in the definition is an arbitrary continuous function; the condition that f be nonnegative (as it was in the last section) is not part of the definition. This impacts the area interpretation of the integral, but the limit and the sum still make sense from a mathematical point of view. Hence, although the integral can be used to represent certain areas, a general integral does not determine an area. Various interpretations of the integral will become apparent when we discuss applications of the integral.

To get a feeling for this definition, we will express a particular integral as a limit of a sum. If f is defined by $f(x) = \sqrt{10 + x^2}$ and the interval of interest is $[1, 3]$, then

$$\int_1^3 \sqrt{10 + x^2}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{10 + \Big(1 + \frac{2i}{n}\Big)^2} \cdot \frac{2}{n}.$$

As you can well imagine, directly evaluating a limit as complicated as this is almost a hopeless task. We will soon find a simple way to evaluate integrals that avoids the need for limits and sums.

The letters and symbols used in the notation for the integral are explained below.

The symbol $\int$ is a variation on the letter S, the first letter of the Latin word *summa* for sum.

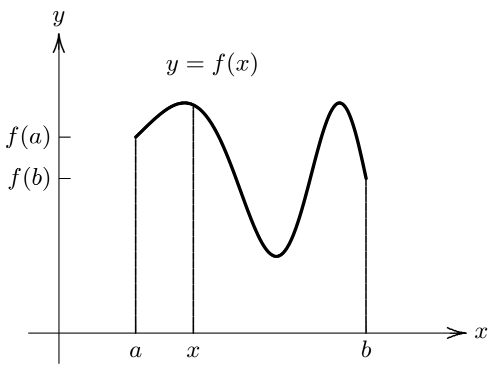
The numbers a and b are the lower and upper limits of integration, respectively.

The function f is sometimes referred to as the integrand.

The letter x is known as a dummy variable as it merely holds a place.

The quantity dx represents the small difference between two very close values of x (see Section 1.6).

Hence, the symbol $\int_a^b f(x)\,dx$ can be interpreted as the sum of products of the form $f(x)\,dx$, where the sum is taken over all the x values from $x = a$ to $x = b$. If f happens to be a nonnegative continuous function, then the product $f(x)\,dx$ can be interpreted as the area of a very skinny rectangle (see the figure).



The thin rectangle above position x has height $f(x)$ and width dx; its area is thus $f(x)\,dx$. Summing all these areas gives $\int_a^b f(x)\,dx$, which represents the area under the curve.

Therefore, when f is nonnegative, the integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$. In applications involving the integral, the product $f(x)\,dx$ often has other physical interpretations. For example, if x represents time in seconds and $f(x)$ represents the velocity of a particle in meters per second, then $f(x)\,dx$ is the displacement (measured in meters) of the particle. In this case, the integral $\int_a^b f(x)\,dx$ gives the total displacement of the particle during the time interval $[a, b]$.

Exercises

1. Use the definition of integral to express the given integral as a limit of a sum.

a) $\displaystyle\int_1^3 (4x^3 + x)\,dx$ **b)** $\displaystyle\int_0^\pi \sin x\,dx$ **c)** $\displaystyle\int_2^5 \frac{1}{x}\,dx$

2. Use the definition of integral to express the given limit as an integral.

a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n \left(4 + \frac{3i}{n}\right)^3 \frac{3}{n}$ **b)** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n \left(5\left(\frac{2i}{n}\right)^2 + \frac{2i}{n}\right)\frac{2}{n}$ **c)** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{n+i}$

3. Use the definition of the integral to evaluate $\int_1^4 (3x - x^2)\,dx$.

4. Use the definition of the integral to evaluate $\int_0^b e^x\,dx$, where b is a positive constant. (See Exercise 7 in Section 2.1.)

5. Suppose that f is continuous and negative on $[a, b]$. Give an area interpretation for $\int_a^b f(x)\,dx$.

6. Suppose that f is continuous on $[a, b]$ and assumes both positive and negative values.

a) Give area interpretations for both $\left|\int_a^b f(x)\,dx\right|$ and $\int_a^b |f(x)|\,dx$.

b) Give displacement interpretations for both $\left|\int_a^b f(x)\,dx\right|$ and $\int_a^b |f(x)|\,dx$, assuming x is time and $f(x)$ is velocity.

c) Which of the numbers $\left|\int_a^b f(x)\,dx\right|$ and $\int_a^b |f(x)|\,dx$ is larger?

7. The purpose of this exercise is to provide a proof of the fact that $\displaystyle\int_a^b x^r\,dx = \frac{b^{r+1} - a^{r+1}}{r+1}$, where r is a positive integer and $[a, b]$ is any interval.

a) Explain why Exercise 8 from the previous section establishes this result for even positive integers.

b) Adapt the proof for even powers to odd powers. The only real difference occurs when a or b are negative for then the integral (or a portion of the integral) must be interpreted as negative area. Remember to consider the cases $0 \le a < b$, $a < 0 < b$, and $a < b \le 0$ separately.

8. Let f be continuous on $[a, b]$. Prove that $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(a + \frac{i}{n}(b-a)\right)\frac{b-a}{n}$. *Hint*: Subtract the sum given here from the sum in the definition, then take the limit.

2.4 Evaluating the integral of a polynomial

As mentioned in the last section, finding the value of an integral from the definition can be an intimidating task. One way to simplify this task is to establish general properties that are satisfied by integrals. In this section, we will only consider a few basic properties of integrals, but they will be sufficient to allow us to find the integral of any polynomial with very little effort.

Let f and g be continuous functions defined on an interval $[a, b]$ and let k be an arbitrary constant. The following general properties of the integral are simple consequences of the definition of the integral and the corresponding properties of sums listed in Section 2.1.

1. $\displaystyle\int_a^b k\,dx = k(b-a)$

2. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4. $\displaystyle\int_a^b \big(f(x)-g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

The proofs of these properties involve more writing than thinking. For example, a proof of property (3) reads as follows.

$$\begin{aligned}
\int_a^b \big(f(x)+g(x)\big)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\big(f(a_i)+g(a_i)\big)\cdot\frac{b-a}{n} && \text{(definition of integral)}\\
&= \lim_{n\to\infty}\Big(\sum_{i=1}^{n} f(a_i)\cdot\frac{b-a}{n}+\sum_{i=1}^{n} g(a_i)\cdot\frac{b-a}{n}\Big) && \text{(property of sums)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(a_i)\cdot\frac{b-a}{n}+\lim_{n\to\infty}\sum_{i=1}^{n} g(a_i)\cdot\frac{b-a}{n} && \text{(property of limits)}\\
&= \int_a^b f(x)\,dx+\int_a^b g(x)\,dx && \text{(definition of integral)}
\end{aligned}$$

Proofs of the other properties will be left for the reader.

At this point in our development of the integral, we have three ways to evaluate integrals. The first way is to use the definition, but for most functions this is a rather complicated thing to do. The second way, which only works for a few simple situations, is to interpret the integral as an area and find the area some other way, most likely with a formula from geometry. The third way is to use formulas and properties that have been derived from the definition of the integral. The most relevant formula found thus far is

$$\int_a^b x^r\,dx = \frac{b^{r+1}-a^{r+1}}{r+1},$$

where r is a positive integer and a and b are real numbers with $a < b$. A proof of this fact was outlined in Exercise 7 of the previous section. Since this is an important result, we will include some of the details. Suppose that r is an odd positive integer and let $A_{[c,d]}$ denote the signed area of the region bounded by the curve $y = x^r$ and the x-axis on the interval $[a, b]$ (positive if above the x-axis and negative if below). If $a < 0 < b$, then

$$\begin{aligned}
\int_a^b x^r\,dx &= A_{[a,b]} = A_{[a,0]} + A_{[0,b]} && \text{(area interpretation of integral, "property" of area)}\\
&= -A_{[0,-a]} + A_{[0,b]} = -\frac{(-a)^{r+1}}{r+1}+\frac{b^{r+1}}{r+1} && \text{(symmetry of curve } y=x^r\text{, Section 2.2 result)}\\
&= \frac{b^{r+1}-a^{r+1}}{r+1} && (r+1 \text{ is an even integer})
\end{aligned}$$

Think carefully about each of the steps required in this proof. The most important step is the use of the formulas for the sums of the powers of the first n positive integers that were obtained earlier in this chapter.

This result, combined with properties (1) through (4), makes it possible to evaluate the integral of any polynomial. For example,

$$\begin{aligned}\int_1^3 (2x^4 + 3x^2 - 5x + 4)\,dx &= 2\int_1^3 x^4\,dx + 3\int_1^3 x^2\,dx - 5\int_1^3 x\,dx + \int_1^3 4\,dx \\ &= 2\cdot\frac{3^5 - 1^5}{5} + 3\cdot\frac{3^3 - 1^3}{3} - 5\cdot\frac{3^2 - 1^2}{2} + 4\cdot(3-1) \\ &= 96.8 + 26 - 20 + 8 = 110.8.\end{aligned}$$

Using the definition to write out this integral as the limit of a sum gives some appreciation for how much effort is avoided by these simple formulas.

Exercises

1. Write out a proof of property (4) of integrals using the definition of the integral.

2. Use properties and formulas from this section to evaluate each of the following integrals.

a) $\displaystyle\int_2^4 7x^3\,dx$ **b)** $\displaystyle\int_1^6 (8t+5)\,dt$ **c)** $\displaystyle\int_{-1}^2 (2x^3 - 6x + 3)\,dx$

d) $\displaystyle\int_1^3 (2x^2 + 4x + 7)\,dx$ **e)** $\displaystyle\int_{-2}^1 (4x^3 - 6x^2 - 2x + 1)\,dx$ **f)** $\displaystyle\int_0^1 (t^7 + 2t^3 - 6t - 8)\,dt$

3. Use properties of the integral and the area interpretation of an integral to evaluate each of the following integrals.

a) $\displaystyle\int_{-1}^2 (3 + 2|x-1|)\,dx$ **b)** $\displaystyle\int_0^r \left(t + 2\sqrt{r^2 - t^2}\right)dt$ **c)** $\displaystyle\int_0^1 \left(4\sqrt{1-x^2} - \sqrt{x - x^2}\right)dx$

4. Evaluate each of the following integrals. Use any method that has been considered thus far.

a) $\displaystyle\int_1^2 \left(\frac{x^2}{2} - \frac{x}{5} + 4\right)dx$ **b)** $\displaystyle\int_0^1 \left(x - 2\sqrt{1-x^2}\right)dx$ **c)** $\displaystyle\int_1^4 \left(2|x-3| - x^2\right)dx$

5. Suppose that $v(t) = 5 - 2t$ gives the velocity in meters per second of a particle at time t seconds. Find the distance traveled by the particle over each of the following time intervals.

a) $0 \le t \le 1$ **b)** $0 \le t \le 2$ **c)** $0 \le t \le 4$

6. Use formulas to evaluate $\displaystyle\int_1^2 tx^2\,dx$ and $\displaystyle\int_1^2 tx^2\,dt$. Note the importance of the differential term.

2.5 Further properties of the integral

In the definition of the integral, it is assumed that $a < b$. It is convenient to adopt the conventions that $\int_a^a f(x)\,dx = 0$ and $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. Furthermore, there is nothing magical about the value of the function at the right endpoint of each subinterval; any point in the subinterval will do. This observation leads to the following slight generalization of the definition of the integral. This form of the definition will be useful when we consider various applications of the integral.

DEFINITION 2.2 The **integral** of a continuous function f on an interval $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)(a_i - a_{i-1}),$$

where $a_{i-1} \le t_i \le a_i$ for $i = 1, 2, \ldots, n$. (Recall that $a_i = a + i(b-a)/n$.)

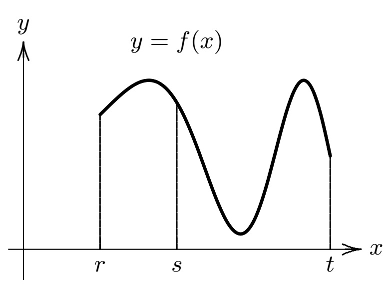
Let f and g be continuous functions defined on an interval $[a, b]$. The following further properties of integrals are valid. (The numbers are a continuation of those in the previous section.)

5. If $m \le f(x) \le M$ for all $x \in [a, b]$, then $m(b-a) \le \displaystyle\int_a^b f(x)\,dx \le M(b-a)$.

6. If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

7. If $r, s, t \in [a, b]$, then $\displaystyle\int_r^s f(x)\,dx = \int_r^t f(x)\,dx + \int_t^s f(x)\,dx$.

Assuming that $r < t < s$, property (7) is virtually obvious from the area interpretation of the integral. However, it is rather difficult to prove this fact using the definition of the integral, and the proof will therefore be omitted. It is important to realize that property (7) is valid for all choices of r, s, and t; the order does not matter. The following figure gives a geometric representation of this result; note the change of sign when the limits of integration are interchanged.



By interpreting the integrals as area,

$$\int_r^s f(x)\,dx = \int_r^t f(x)\,dx - \int_s^t f(x)\,dx$$
$$= \int_r^t f(x)\,dx + \int_t^s f(x)\,dx.$$

The inequality properties (5) and (6) follow easily from the definition of the integral. Values for m and M are typically chosen to be the minimum and maximum outputs, respectively, of the function f on the interval $[a, b]$. These properties can be used to find rough estimates for the value of an integral. For example, suppose we want to find lower and upper bounds for the value of the integral $\int_0^3 \sqrt{x^3 - 3x + 6}\,dx$. Noting that the minimum and maximum outputs of the function $\sqrt{x^3 - 3x + 6}$ on $[0, 3]$ are $\sqrt{4}$ and $\sqrt{24}$ (you need to use some simple ideas from differential calculus to see this), respectively, we find that

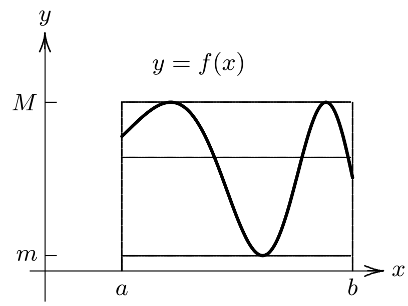
$$6 = 2 \cdot 3 \le \int_0^3 \sqrt{x^3 - 3x + 6}\,dx \le \sqrt{24} \cdot 3 = 6\sqrt{6}.$$

To illustrate property (6), since the inequality $\sqrt{x^6-2} < \sqrt{x^6}$ is valid on the interval $[2,5]$, we obtain

$$\int_2^5 \sqrt{x^6-2}\,dx \le \int_2^5 x^3\,dx = \frac{5^4-2^4}{4} = \frac{609}{4}.$$

Note that it is easy to find the value of the integral of the simpler function.

It is also helpful to understand properties (5) and (6) from the perspective of area. For example, property (5) asserts that the area under a curve is trapped between the area of two rectangles (see the figure).



area of small rectangle ≤ area under curve ≤ area of large rectangle

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

From the figure, it makes sense that there is a rectangle with width $b-a$ whose area is exactly the area under the curve. This fact is made explicit in the following theorem; the proof (which will be left as an exercise) requires two properties of continuous functions, namely the Extreme Value Theorem and the Intermediate Value Theorem. This result will be used in the next section to prove the Fundamental Theorem of Calculus.

THEOREM 2.3 Mean Value Theorem for Integrals If f is continuous on $[a,b]$, then there exists a point $c \in [a,b]$ such that $f(c)(b-a) = \int_a^b f(x)\,dx$. ■

Exercises

1. Use property (5) to find upper and lower bounds for each integral.

a) $\displaystyle\int_2^5 \frac{1}{2x-1}\,dx$ **b)** $\displaystyle\int_1^4 \frac{1}{3+2\sqrt{x}}\,dx$ **c)** $\displaystyle\int_2^6 \sqrt[3]{x^2-6x+15}\,dx$

2. Use property (6) to determine the largest integral in each pair. Justify your answers.

a) $\displaystyle\int_2^6 \sqrt{x^4+5}\,dx,\ \int_2^6 x^2\,dx$ **b)** $\displaystyle\int_0^\pi \sin^3 x\,dx,\ \int_0^\pi \sin^5 x\,dx$ **c)** $\displaystyle\int_1^2 e^x\,dx,\ \int_1^2 e^{x^2}\,dx$

3. Use property (6) to prove that $\displaystyle\int_1^4 \sqrt{x^4-1}\,dx \le 21$. Do not evaluate the given integral.

4. Given that $\int_1^3 f(x)\,dx = 6$, $\int_2^3 f(x)\,dx = 2$, $\int_1^3 g(x)\,dx = 10$, and $\int_2^3 g(x)\,dx = -3$, evaluate

a) $\displaystyle\int_1^3 \big(2f(x)-3g(x)\big)\,dx$ **b)** $\displaystyle\int_3^2 \big(4f(x)+g(x)\big)\,dx$ **c)** $\displaystyle\int_1^2 \big(2g(x)-x^2\big)\,dx$

5. Use property (7) to evaluate $\displaystyle\int_0^3 |2x-x^2|\,dx$.

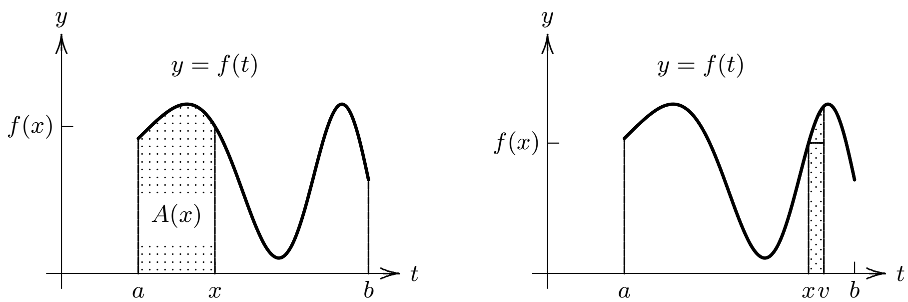
6. Give a proof of the Mean Value Theorem for integrals. Illustrate this theorem geometrically.

7. For $f(x)=x^2$ on $[1,4]$, find the point c guaranteed by the Mean Value Theorem for integrals.

8. Let $f(x) = \displaystyle\int_1^x (t^3-2t^2+5)\,dt$ for all $x>1$. Use formulas to find $f(x)$, then compute $f'(x)$.

2.6 The Fundamental Theorem of Calculus

As a reminder, the area problem is the following: find the area under the curve $y = f(t)$ and above the t-axis on the interval $[a, b]$, where f is any continuous and nonnegative function defined on $[a, b]$. The solution to this problem leads to the expression $\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i)\frac{b-a}{n}$. The quantity represented by this complicated expression is a number; the area of the region in question. When Newton considered the area problem, he made the insightful leap to treat area as a function. Let $A(x)$ be the area under the curve $y = f(t)$ and above the t-axis on the interval $[a, x]$; the number $A(x)$ clearly depends on x. Note that $A(a) = 0$ and the problem is to find $A(b)$.

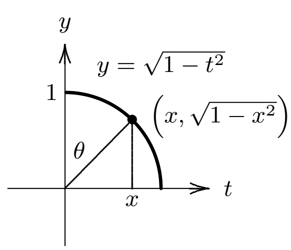


The shaded area is $A(v) - A(x)$; the rectangle has area $f(x)(v - x)$.

Since the area A is now a function, it is possible to consider its rate of change. To Newton, it was clear that the rate of change of the area function was the height of the curve, that is, $A'(x) = f(x)$. He simply noted that $A(v) - A(x) \approx f(x)(v - x)$ when $v > x$ is very close to x (see the figure: the area under the curve is close to the area of the rectangle). It follows that $A'(x) = \lim_{v\to x} \dfrac{A(v) - A(x)}{v - x} = f(x)$. A formal statement and proof of this result is given at the end of the section.

To get a sense of this result, for each $x \in [0, 1]$, let $A(x)$ denote the area under the curve $y = \sqrt{1 - t^2}$ and above the t-axis on the interval $[0, x]$. Using some simple facts from trigonometry and geometry, we find that

$\theta = \arcsin x$



$$\begin{aligned} A(x) &= \text{area under curve on } [0, x] \\ &= \text{area of sector } + \text{ area of triangle} \\ &= \tfrac{1}{2}\arcsin x + \tfrac{1}{2}x\sqrt{1 - x^2}. \end{aligned}$$

The reader should verify that $A'(x) = \sqrt{1 - x^2}$, as expected by the previous discussion.

In modern terms, Newton's observation is that the derivative of a function defined as an integral is simply the integrand. This fact indicates that the processes of integration and differentiation are very much related. Consequently, this result is known as the Fundamental Theorem of Calculus.

THEOREM 2.4 Fundamental Theorem of Calculus If f is continuous on an interval $[a, b]$ and a function F is defined by $F(x) = \int_a^x f(t)\,dt$ for all x in $[a, b]$, then $F'(x) = f(x)$ for all x in $[a, b]$.

Proof. Let $x \in [a, b]$. If $v \in [a, b]$ and $v \neq x$, then the Mean Value Theorem for integrals states that there is a point c_v between v and x such that $f(c_v)(v - x) = \int_x^v f(t)\,dt$. Note that $\lim_{v\to x} c_v = x$. We then have

$$\begin{aligned}
F'(x) &= \lim_{v\to x} \frac{F(v)-F(x)}{v-x} && \text{(definition of derivative)}\\
&= \lim_{v\to x} \frac{1}{v-x}\Big(\int_a^v f(t)\,dt - \int_a^x f(t)\,dt\Big) && \text{(definition of } F\text{)}\\
&= \lim_{v\to x} \frac{1}{v-x}\int_x^v f(t)\,dt && \text{(property (7) of integrals)}\\
&= \lim_{v\to x} f(c_v) && \text{(MVT for integrals)}\\
&= f(x). && (f \text{ is continuous})
\end{aligned}$$

This completes the proof. ■

For a couple of examples to illustrate this theorem, consider the following:

$$\frac{d}{dx}\int_1^x \sin(t^3)\,dt = \sin(x^3) \quad\text{and}\quad \frac{d}{dx}\int_0^{2x^2} e^{-u^2/2}\,du = e^{-(2x^2)^2/2}\cdot 4x = 4xe^{-2x^4}.$$

Note the use of the Chain Rule in the second example. We can also answer questions such as the following: find a function f so that $f(1)=5$ and $f'(x)=x\cot x$. By the Fundamental Theorem of Calculus, the function

$$f(x) = 5 + \int_1^x t\cot t\,dt$$

has the desired properties.

Exercises

1. Find the derivative of each of the following functions.

a) $f(x) = \int_0^x \sqrt{t^2+9}\,dt$ **b)** $g(x) = \int_3^x \frac{4}{2-t}\,dt$ **c)** $h(t) = \int_{-2}^t \frac{1}{\sqrt[3]{x^4+9}}\,dx$

d) $F(x) = \int_x^2 (1-e^{-t^3})\,dt$ **e)** $G(u) = \int_0^{u^2} s\sqrt{s^3+2}\,ds$ **f)** $H(x) = \int_{-1}^{e^x} \sin t^2\,dt$

g) $u(x) = \int_{-x^2}^4 e^{t^2}\,dt$ **h)** $v(x) = \int_x^{2x} \cos t^3\,dt$ **i)** $w(x) = \int_{x^2}^{e^x} \sqrt{t^3+4}\,dt$

2. Determine $F''(\pi/4)$ given that $F(x) = \int_x^1 f(t)\,dt$ and $f(x) = \int_1^{2x} \frac{\sin t}{t}\,dt$.

3. Evaluate $\lim_{x\to 0} \frac{1}{x^4}\int_0^x \sin(2t^3)\,dt$.

4. Determine the interval on which the function f defined by $f(x) = \int_0^x \frac{5}{2t^2-6t+19}\,dt$ is concave up.

5. Find an integral expression for a function f such that $f(2)=0$ and $f'(x)=\sin(x^4)$.

6. Suppose that $F(x) = \int_{-1}^x |t|\,dt$ for all $x \geq -1$. Find $F(0)$, $F'(0)$, $F(1)$, $F'(1)$, $F(3)$, and $F'(3)$.

7. Consider the function G defined by $G(x) = \int_{-2}^x |t|\,dt$ for all $x \geq -2$. Use the area interpretation of the integral to find a formula for the function G that does not involve an integral, then find $G'(x)$.

8. Explain how Newton's observation shows that $\int_0^b \cos t\,dt = \sin b$ for any $b > 0$.

2.7 The Fundamental Theorem of Calculus, continued

Leibniz discovered the connection between integration and differentiation by studying the properties of sums of differences. Consider any sequence of numbers such as $2, 5, 7, 11, 14, 20$. Summing all of the differences between the numbers gives the difference between the first and last numbers:

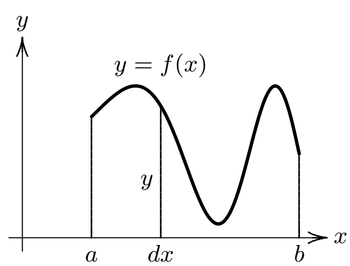
$$(5-2)+(7-5)+(11-7)+(14-11)+(20-14)=20-2.$$

Every sequence of numbers has this property. When a function is applied to a sequence and the corresponding differences are summed, the result is the last value of the function minus the first value of the function. For example, applying x^2 to the above sequence and summing the differences yields

$$(5^2-2^2)+(7^2-5^2)+(11^2-7^2)+(14^2-11^2)+(20^2-14^2)=20^2-2^2.$$

The term dx in the integral notation represents the difference between two successive x values. The equality $\int_a^b dx = b-a$, which follows easily from the definition of the integral, simply indicates that the sum of all the differences of the x values gives the length of the interval, the last value of x minus the first value of x. For Leibniz, a quantity such as dz represented a little difference of z values and it was clear to him that $\int dz = z_{\text{last}} - z_{\text{first}}$; the sum of all the little differences gave the total difference. This is a valid result even when z represents a function.

It was also evident to Leibniz that $\int y\,dx$ gave the area under a curve. The quantity $y\,dx$ represents the area of an extremely thin rectangle of height y and width dx and the sum of all these areas is the area under the curve (see the figure). If a function z can be found so that $dz = y\,dx$ (which is equivalent to $dz/dx = y$), then the area problem has been solved. The area problem thus simplifies to the following inverse tangent problem: given a function y of x, find a function z of x such that $dz/dx = y$.



The area is the sum of the $y\,dx$ areas;

$$\begin{aligned} A &= \int_a^b y\,dx \\ &= \int_a^b dz, \text{ assuming } dz = y\,dx \\ &= z(b)-z(a). \end{aligned}$$

It follows that integrals can be evaluated by finding an antiderivative of the integrand. A function F is an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all $x \in I$. This connection between integration and differentiation is another part of the Fundamental Theorem of Calculus.

THEOREM 2.5 Fundamental Theorem of Calculus If f is a continuous function defined on an interval $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

Proof. Let G be the function defined by $G(x) = \int_a^x f(t)\,dt$ for all $x \in [a, b]$. By Theorem 2.4, we know that $G'(x) = f(x)$ for all $x \in [a, b]$. Since F and G have the same derivative, there exists a constant C such that $G(x) = F(x) + C$ for all $x \in [a, b]$ (see Corollary 1.27 in Section 1.27). It follows that

$$\int_a^b f(x)\,dx = G(b)-G(a) = \big(F(b)+C\big) - \big(F(a)+C\big) = F(b)-F(a). \qquad \blacksquare$$

Writing down the definitions of the derivative and the integral side by side gives some indication of the surprising nature of this result. Furthermore, after spending some time using the definition of the integral to evaluate integrals, one can certainly appreciate how much this theorem simplifies the evaluation of integrals. Here are four examples illustrating this part of the Fundamental Theorem of Calculus; the symbol $F(x)\Big|_a^b$ is a common abbreviation for $F(b) - F(a)$.

$$\int_1^2 \big(2x^3 - 6x + 1\big)\,dx = \Big(\frac{x^4}{2} - 3x^2 + x\Big)\Big|_1^2 = \Big(8 - 12 + 2\Big) - \Big(\frac{1}{2} - 3 + 1\Big) = -\frac{1}{2};$$

$$\int_0^{\pi/3} (\cos x - 2\sin x)\,dx = \big(\sin x + 2\cos x\big)\Big|_0^{\pi/3} = \Big(\frac{\sqrt{3}}{2} + 1\Big) - \Big(0 + 2\Big) = \frac{\sqrt{3}}{2} - 1;$$

$$\int_1^3 \frac{12}{x^3}\,dx = -\frac{6}{x^2}\Big|_1^3 = -\frac{2}{3} - (-6) = \frac{16}{3};$$

$$\int_1^5 \frac{1}{1+x}\,dx = \ln\big|1+x\big|\Big|_1^5 = \ln 6 - \ln 2 = \ln 3.$$

Antiderivatives of many simple functions can be found by thinking about differentiation in reverse.

The Fundamental Theorem of Calculus provides the most common method for evaluating integrals. It states that an integral can be evaluated by finding an antiderivative of the integrand and plugging in the endpoints. However, it is important to remember that integration is not antidifferentiation. An integral represents the limit of a special type of sum. In many cases, integrals can be evaluated by finding an antiderivative. However, there are other ways to evaluate an integral; for instance, an integral sometimes represents a simple geometric area. In addition, the recognition of an integral as a limit of a sum will be necessary for applications of the integral.

Exercises

1. Evaluate the following integrals.

a) $\displaystyle\int_1^3 (3x^2 - 2x + 1)\,dx$ **b)** $\displaystyle\int_0^{\pi/2} \sin x\,dx$ **c)** $\displaystyle\int_0^{\pi/4} \sec x \tan x\,dx$

d) $\displaystyle\int_{-2}^3 (z^2 + z)\,dz$ **e)** $\displaystyle\int_1^2 \frac{1}{t^3}\,dt$ **f)** $\displaystyle\int_1^2 (x^3 - \sqrt{x})\,dx$

g) $\displaystyle\int_1^2 \frac{1}{x}\,dx$ **h)** $\displaystyle\int_{-3}^2 \Big(4 + 5|x|\Big)\,dx$ **i)** $\displaystyle\int_0^2 e^x\,dx$

j) $\displaystyle\int_2^5 \frac{5}{2x-1}\,dx$ **k)** $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$ **l)** $\displaystyle\int_0^2 \sqrt{4 - x^2}\,dx$

2. Find the area of the region under the curve $y = 2/(1+x^2)$ and above the x-axis on the interval $[-1, 1]$.

3. Find the area of the region under the curve $y = 1/x$ and above the x-axis on the interval $[1, b]$, where $b > 1$.

4. Find conditions on f and F so that the statements $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $\displaystyle\int_a^x F'(t)\,dt = F(x) - F(a)$ are valid. What do these statements say about the operations of integration and differentiation?

2.8 Finding antiderivatives: guess and check

The Fundamental Theorem of Calculus provides an excellent method for evaluating many integrals, but in order to use this theorem, it is first necessary to find an antiderivative for the integrand. Recall that an antiderivative of a function f is a function F with the property that $F'(x) = f(x)$ for all x in the domain of f. Antiderivatives are not unique since adding a constant to a function does not change its derivative. The standard notation to represent all of the antiderivatives of a function f is $\int f(x)\,dx$ and it is referred to as the **indefinite integral** of f; the symbol $\int_a^b f(x)\,dx$ is then called the **definite integral** of f. To illustrate this symbol, note that $\int 12x^3\,dx = 3x^4 + C$. The arbitrary constant C is called the **constant of integration**; every antiderivative of $12x^3$ is of the form $3x^4$ plus some constant.

Finding antiderivatives is much more challenging than finding derivatives; it involves techniques and skills rather than the simple use of formulas. As a start to finding antiderivatives, we simply note that every derivative formula becomes an antiderivative formula when read backward. (See the list of basic formulas in the table of integrals in Appendix B.) Even with these formulas, there is still a fair amount of work involved in finding antiderivatives. Sometimes basic algebra is enough:

$$\int (x^2+3)^2\,dx = \int (x^4+6x^2+9)\,dx = \tfrac{1}{5}x^5 + 2x^3 + 9x + C;$$

$$\int \frac{x-1}{x+1}\,dx = \int \Big(1 - \frac{2}{x+1}\Big)\,dx = x - 2\ln|x+1| + C.$$

But more often than not, the chain rule comes into play. As a reminder, the chain rule is a derivative formula for composite functions. It states that $\big(f(g(x))\big)' = f'(g(x))g'(x)$ and its reversal requires some practice. As a start, note that

$$\frac{d}{dx}\ln\big|x^4+x^2+1\big| = \frac{4x^3+2x}{x^4+x^2+1} \quad \text{implies} \quad \int \frac{2x^3+x}{x^4+x^2+1}\,dx = \frac{1}{2}\ln\big|x^4+x^2+1\big| + C;$$

$$\frac{d}{dx}\sin^3 2x = 3\sin^2 2x \cdot 2\cos 2x \quad \text{implies} \quad \int \sin^2 2x\cos 2x\,dx = \frac{1}{6}\sin^3 2x + C.$$

Look at these examples carefully and notice how the chain rule reels terms in when antidifferentiating. One method for reversing the chain rule is **guess and check**: make an educated guess as to what the antiderivative should look like, check the guess by taking its derivative, and modify the guess as necessary. This method emphasizes the reverse nature of antidifferentiation and is also a good review of differentiation. It also reinforces the idea that antiderivatives can always be checked.

As an example, consider $\displaystyle\int \frac{x^3}{\sqrt{4+x^4}}\,dx$. The major part of the function under consideration involves $(4+x^4)^{-1/2}$. Since differentiation reduces powers by one, antidifferentiation increases powers by one. Thus a reasonable guess for an antiderivative is $(4+x^4)^{1/2}$. Since

$$\frac{d}{dx}(4+x^4)^{1/2} = \frac{1}{2}(4+x^4)^{-1/2}\,4x^3 = \frac{2x^3}{\sqrt{4+x^4}}, \quad \text{we have} \quad \int \frac{x^3}{\sqrt{4+x^4}}\,dx = \frac{1}{2}\sqrt{4+x^4} + C.$$

It should be pointed out that this method only works when the guess is off by a constant; the reader should carefully consider why this is the case (see Exercise 5 below). Here is a second example.

$$\int xe^{-2x^2}\,dx, \qquad \begin{aligned}&\text{guess}: e^{-2x^2};\\ &\text{check}: \tfrac{d}{dx}e^{-2x^2} = -4xe^{-2x^2};\end{aligned} \qquad \text{hence} \int xe^{-2x^2}\,dx = -\frac{1}{4}e^{-2x^2} + C.$$

It is often a good idea to check (by differentiation) the final answer one more time. For a third example, consider the following.

$$\int \tan^4(x/3)\sec^2(x/3)\,dx, \qquad \begin{array}{l}\text{guess : } \tan^5(x/3);\\ \text{check : } \frac{d}{dx}\tan^5(x/3) = 5\tan^4(x/3)\sec^2(x/3)(1/3) = \dfrac{5}{3}\tan^4(x/3)\sec^2(x/3).\end{array}$$

It follows that

$$\int \tan^4(x/3)\sec^2(x/3)\,dx = \frac{3}{5}\tan^5(x/3) + C.$$

As a final comment, functions that look similar can have very different antiderivatives. For example,

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C; \quad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C; \quad \int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x + \sqrt{x^2-1}\right| + C.$$

Each of these results can be checked by differentiating the function to the right of the equal sign. It is thus very important to exercise caution when finding antiderivatives.

Exercises

1. Evaluate each of the following indefinite integrals.

a) $\int x^2\sqrt{x^3+1}\,dx$ **b)** $\int \frac{2}{\sqrt{3t-7}}\,dt$ **c)** $\int \sin(4x)\cos^3(4x)\,dx$

d) $\int \frac{\sec^2 t}{\tan^5 t}\,dt$ **e)** $\int (\sqrt{x}-1)^2\,dx$ **f)** $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\,dt$

g) $\int \frac{\sin t}{(2+3\cos t)^2}\,dt$ **h)** $\int \frac{1}{t^2+2t+1}\,dt$ **i)** $\int \frac{2}{5x+1}\,dx$

j) $\int \frac{5}{(4x-3)^2}\,dx$ **k)** $\int \frac{6x+6}{x^2+2x-3}\,dx$ **l)** $\int \frac{\ln x}{x}\,dx$

m) $\int (1-e^{2t})\,dt$ **n)** $\int \frac{1+e^t-e^{3t}}{e^{2t}}\,dt$ **o)** $\int \frac{6}{1+9x^2}\,dx$

2. Evaluate each of the following definite integrals.

a) $\int_1^4 \sqrt{x}(x+1)\,dx$ **b)** $\int_0^3 \frac{1}{4x+3}\,dx$ **c)** $\int_1^2 \frac{10}{(1-2t)^3}\,dt$

d) $\int_0^1 t\sin(\pi t^2)\,dt$ **e)** $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$ **f)** $\int_0^\pi \frac{\sin x}{1+3\cos^2 x}\,dx$

3. Evaluate $\int_0^1 (x+2)\sqrt{1-x^2}\,dx$ by first writing the integral as the sum of two integrals. Think carefully!

4. Evaluate $\int \frac{x^2}{\sqrt{4+x^3}}\,dx$, $\int \frac{x^2}{4+x^3}\,dx$, $\int \frac{x^2}{(4+x^3)^2}\,dx$, and $\int \frac{x^3}{4+x^2}\,dx$.

5. Consider $\int 6x^2\sin(x^2)\,dx$. After making the guess $\cos(x^2)$ for the antiderivative and checking it, Sally decided that the answer was $-3x\cos(x^2)+C$. Explain how Sally arrived at this conclusion and why it is incorrect.

2.9 Finding antiderivatives: integration by substitution

When finding antiderivatives, it may be helpful to make a change of variables. An appropriate substitution can make it clear what form the antiderivative should have. Consider the following example:

$$\int \frac{9x^2}{(4+x^3)^2}\,dx, \qquad \begin{pmatrix} \text{let } u = 4+x^3, \\ \text{then } du = 3x^2\,dx \end{pmatrix}; \qquad \int \frac{9x^2}{(4+x^3)^2}\,dx = \int \frac{3}{u^2}\,du = -\frac{3}{u} + C.$$

The substitution $u = 4+x^3$ transformed the integral from one in the variable x to a simpler one in the variable u. After finding the antiderivative in terms of the variable u, back substitution provides the antiderivative of the original function;

$$\int \frac{9x^2}{(4+x^3)^2}\,dx = -\frac{3}{4+x^3} + C.$$

Of course, this problem could have been solved using the guess and check method, but it illustrates the essential features of integration by substitution.

The key idea in finding antiderivatives using substitution is to completely transform the integral from one variable to another variable in such a way that the new integral is easier to evaluate than the old integral. There are no solid guidelines for a substitution—it is an acquired skill and thus requires a fair amount of practice. For some problems, there is more than one suitable choice for a substitution. If the original integrand (usually in the variable x) involves a composite function, then letting the new variable (usually u plays this role) be the inside function is often a good place to start. It is also important to keep track of the dx and du terms. Here is another example to illustrate this method.

$$\int \frac{x^2}{x^6+1}\,dx, \qquad \begin{pmatrix} \text{let } u = x^3, \\ \text{then } du = 3x^2\,dx \end{pmatrix}; \qquad \int \frac{x^2}{x^6+1}\,dx = \int \frac{1/3}{1+u^2}\,du = \frac{1}{3}\arctan u + C.$$

It follows that $\displaystyle\int \frac{x^2}{x^6+1}\,dx = \frac{1}{3}\arctan(x^3) + C$.

Integration by substitution can also be used to evaluate definite integrals. In this case, the limits of integration must also be transformed into appropriate values for the new variable. For example,

$$\int_0^1 54x^2\sqrt{4-3x}\,dx, \qquad \begin{pmatrix} \text{let } u = 4-3x, & u = 4 \text{ when } x = 0 \\ \text{then } du = -3\,dx; & u = 1 \text{ when } x = 1 \end{pmatrix};$$

$$\int_0^1 54x^2\sqrt{4-3x}\,dx = \int_4^1 54\Big(\frac{4-u}{3}\Big)^2\sqrt{u}\Big(-\frac{1}{3}\,du\Big) = 2\int_1^4 (4-u)^2\sqrt{u}\,du.$$

(Notice how the negative sign was used to change the order of the limits of integration.) To evaluate the new integral, expand the integrand:

$$2\int_1^4 (16-8u+u^2)u^{1/2}\,du = 2\int_1^4 (16u^{1/2} - 8u^{3/2} + u^{5/2})\,du = \frac{2468}{105}.$$

The last few steps (find an antiderivative and plug in the endpoints) are routine and have been omitted.

On occasion, a substitution of some sort is made with no real direction in mind other than a hope that the new integral will look more familiar. Consider the following example.

$$\int \frac{1}{t+\sqrt{t}}\,dt, \qquad \begin{pmatrix} \text{let } u = \sqrt{t}, \\ \text{then } t = u^2 \text{ and } dt = 2u\,du \end{pmatrix}.$$

Notice how we solved for t before taking differentials; this is sometimes an easier way to determine the relationship between the derivatives of the variables. We then have

$$\int \frac{1}{t+\sqrt{t}}\,dt = \int \frac{2u}{u^2+u}\,du = 2\int \frac{1}{u+1}\,du = 2\ln|1+u| + C = 2\ln|1+\sqrt{t}| + C.$$

The bottom line here is to try some sort of substitution and see what happens. If it appears the substitution did not make the integral any easier, try some other substitution.

Exercises

1. Evaluate each of the following indefinite integrals.

a) $\displaystyle\int x^3(x^4+6)^4\,dx$ **b)** $\displaystyle\int \frac{x}{\sqrt{4x^2+1}}\,dx$ **c)** $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

d) $\displaystyle\int \frac{x+2}{(x^2+4x-1)^3}\,dx$ **e)** $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx$ **f)** $\displaystyle\int (t-3)\sqrt{2t+1}\,dt$

g) $\displaystyle\int \frac{6(x+1)}{(x+3)(x-1)}\,dx$ **h)** $\displaystyle\int \frac{t}{(t^2+8)^2}\,dt$ **i)** $\displaystyle\int \frac{x^3}{\sqrt{a^2-x^2}}\,dx$

j) $\displaystyle\int \frac{e^{t/2}}{\sqrt{1-e^t}}\,dt$ **k)** $\displaystyle\int \frac{x}{1+x^4}\,dx$ **l)** $\displaystyle\int \frac{t}{(1-2t)^2}\,dt$

2. Evaluate each of the following definite integrals.

a) $\displaystyle\int_0^{\pi} \frac{\sin t}{(3+2\cos t)^2}\,dt$ **b)** $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$ **c)** $\displaystyle\int_0^1 \frac{x}{(1+2x^2)^3}\,dx$

d) $\displaystyle\int_0^{\sqrt{2}} x\sqrt{4-x^4}\,dx$ **e)** $\displaystyle\int_0^3 t\sqrt{t+3}\,dt$ **f)** $\displaystyle\int_{2/3}^3 \frac{9x^2}{\sqrt[3]{3x-1}}\,dx$

3. Evaluate $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$ two ways; first with $u = x+1$, then with $u = \sqrt{x+1}$. Are the solutions the same?

4. Evaluate $\displaystyle\int_1^4 \frac{x-1}{x+2}\,dx$ two ways; first using long division, then with a substitution.

5. Find the area of the ellipse defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are positive constants. *Hint*: Express the area as an integral, but do not use the Fundamental Theorem of Calculus to evaluate it.

6. Let f and g be functions defined on appropriate intervals and assume that both f and g' are continuous functions. Prove that

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

This fact offers a justification for the method of integration by substitution as applied to definite integrals. *Hint*: Let F be an antiderivative of f and use the Fundamental Theorem of Calculus to evaluate both integrals.

2.10 Finding antiderivatives: integration by parts

In an attempt to find an antiderivative of $x \cos x$, suppose that we try the function $F(x) = x \sin x$. It follows that $F'(x) = x \cos x + \sin x$; the product rule generates two terms, one desirable and the other not. To eliminate the unwanted term $\sin x$, it is necessary to add a term to $F(x)$ whose derivative is $-\sin x$. Since $\cos x$ is such a function, the modified guess becomes $F(x) = x \sin x + \cos x$ and

$$F'(x) = x \cos x + \sin x - \sin x = x \cos x,$$

as desired. The problem of finding the antiderivative of a function obtained as one part of the derivative of a product of two functions is a common one and the technique of integration by parts formalizes the guessing that was involved in this example.

Integration by parts is antidifferentiation's answer to the product rule. Suppose that u and v are differentiable functions of x. Then the product rule states that

$$\big(u(x)v(x)\big)' = u(x)v'(x) + v(x)u'(x) \quad \text{or} \quad u(x)v'(x) = \big(u(x)v(x)\big)' - v(x)u'(x).$$

Rewriting the last equation in terms of antiderivatives yields

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx, \quad \text{or simply} \quad \int u\,dv = uv - \int v\,du;$$

the differential form is the most common way to express integration by parts. This formula shows how to find the antiderivative of a function that can be identified as the product of one function with the derivative of another. It may appear to be a circular formula since both sides of the equation involve an indefinite integral, but if the integral on the right side of the equation is easier than the one on the left side, then progress has been made.

As an illustration of integration by parts, we will redo the opening example.

$$\int x \cos x\,dx, \qquad \begin{array}{lccc} \text{let} & u = x & \text{and} & dv = \cos x\,dx; \\ \text{then} & du = dx & \text{and} & v = \sin x; \end{array}$$

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C.$$

Note that when finding v from dv, the constant of integration is assumed to be 0. Two further examples should clarify the essential ideas.

$$\int xe^{2x}\,dx, \qquad \begin{array}{lccc} \text{let} & u = x & \text{and} & dv = e^{2x}\,dx; \\ \text{then} & du = dx & \text{and} & v = e^{2x}/2; \end{array}$$

$$\int xe^{2x}\,dx = \frac{1}{2}\,xe^{2x} - \int \frac{1}{2}\,e^{2x}\,dx = \frac{1}{2}\,xe^{2x} - \frac{1}{4}\,e^{2x} + C = \frac{1}{4}\,(2x-1)\,e^{2x} + C.$$

In this example, we are seeking an antiderivative for xe^{2x} and beginning with the guess $xe^{2x}/2$. Since $xe^{2x}/2$ is a product of two functions, its derivative will have two terms. One of these terms is exactly the function we desire (namely, xe^{2x}). Since the other term ($e^{2x}/2$) is unwanted, it needs to be removed by subtracting an appropriate function whose derivative is precisely this quantity, which in this case is $e^{2x}/4$. Checking the answer by differentiating indicates how this happens:

$$\frac{d}{dx}\Big(\frac{1}{2}\,xe^{2x} - \frac{1}{4}\,e^{2x} + C\Big) = xe^{2x} + \frac{1}{2}\,e^{2x} - \frac{1}{2}\,e^{2x} = xe^{2x}.$$

For the second example, consider the definite integral

$$\int_0^1 \arctan x\,dx, \qquad \begin{array}{llll} \text{let} & u = \arctan x & \text{and} & dv = dx; \\ \text{then} & du = dx/(1+x^2) & \text{and} & v = x; \end{array}$$

$$\int_0^1 \arctan x\,dx = x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{1}{2}\ln\big|1+x^2\big|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Note how the limits of integration are handled in the case of a definite integral.

Integration by parts is a viable option when the integrand consists of the product of two functions and one of the functions is not a constant multiple of the derivative of the other function (integration by substitution works in this case). It can even be used when the integrand consists of a single function as in the last example. There are no specific guidelines for splitting the integrand into u and dv, but dv should be easy to antidifferentiate and the new integral $\int v\,du$ should be simpler than the original one $\int u\,dv$. It is sometimes useful and/or necessary to first make a substitution before using integration by parts. Furthermore, it is sometimes necessary to perform integration by parts more than once to evaluate an integral.

Exercises

1. Evaluate each of the following indefinite integrals. Treat a as a nonzero constant.

a) $\int xe^{-x}\,dx$ **b)** $\int x\sin 4x\,dx$ **c)** $\int \arcsin x\,dx$

d) $\int \ln x\,dx$ **e)** $\int (\ln x)^2\,dx$ **f)** $\int x\sin(ax)\,dx$

g) $\int xe^{ax}\,dx$ **h)** $\int x\sqrt{2x+1}\,dx$ **i)** $\int x\sec^2 x\,dx$

j) $\int x^2e^{-x}\,dx$ **k)** $\int x^2\cos(ax)\,dx$ **l)** $\int x\arctan x\,dx$

2. Evaluate each of the following definite integrals.

a) $\int_0^{\pi/2} x\sin 2x\,dx$ **b)** $\int_0^1 x\cos(\pi x/2)\,dx$ **c)** $\int_1^e x\ln x\,dx$

d) $\int_0^{1/2} 4xe^{x/2}\,dx$ **e)** $\int_1^4 e^{\sqrt{x}}\,dx$ **f)** $\int_0^1 x^3\sin(\pi x^2)\,dx$

3. Find the area of the region under the graph of $y = \ln x$ and above the x-axis on the interval $[1, e^2]$.

4. Evaluate $\int \dfrac{8x}{\sqrt{4x+3}}\,dx$ in two ways; one using a substitution and one using integration by parts.

5. What happens if a constant is added when determining v from dv in the formula for integration by parts?

2.11 Improper integrals

The definition of the integral requires a continuous function f on a closed and bounded interval $[a, b]$. A definite integral is said to be **improper** if either the function f is unbounded on $[a, b]$ or the interval of integration is unbounded. For example, the integrals $\int_0^1 x^{-2}\,dx$ and $\int_2^\infty e^{-x}\,dx$ are both improper; the first because $1/x^2$ is unbounded on $(0, 1]$ and the second because the interval of integration is unbounded. Integrals of the latter type are easy to spot since ∞ and/or $-\infty$ appears as a limit of integration, but improper integrals involving an unbounded function are more difficult to recognize since it is necessary to consider the behavior of the function on the interval of integration. Although improper integrals do not fit the "proper" definition of an integral, the limit concept makes it easy to assign them a meaning.

Suppose that f is continuous for all $x \geq a$ for some number a. Then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

provided that the limit exists. When the limit exists, the integral is said to **converge** or be convergent; if not, the integral is said to **diverge** or be divergent. An integral of the form $\int_{-\infty}^b f(x)\,dx$, where f is continuous for all $x \leq b$, is defined analogously.

Two examples should be sufficient to illustrate this concept.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty;$$

$$\int_{-\infty}^2 e^{x/2}\,dx = \lim_{a\to-\infty} \int_a^2 e^{x/2}\,dx = \lim_{a\to-\infty} \left(2e^{x/2}\Big|_a^2\right) = \lim_{a\to-\infty} (2e - 2e^{a/2}) = 2e.$$

The first improper integral is divergent while the second is convergent. The first integral implies that the area under the curve $y = 1/x$ on the interval $[1, \infty)$ is infinite. However, the reader should verify that the area under the curve $y = 1/x^2$ on the interval $[1, \infty)$ is 1. This pair of results concerning the area of two regions that have an infinite length is rather surprising since the two graphs look very similar.

On occasion, it is useful to know whether or not an integral over an infinite interval is convergent even if its value cannot be determined exactly. The following intuitively clear result is helpful in this context. The proof involves some advanced ideas and will not be presented here.

THEOREM 2.6 Suppose that f and g are continuous functions and that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. If the integral $\int_a^\infty g(x)\,dx$ converges, then the integral $\int_a^\infty f(x)\,dx$ also converges. ■

To illustrate the use of this theorem, consider the improper integral $\int_0^\infty e^{-x}\cos^4 x\,dx$. It would be rather difficult to find an antiderivative for the function $e^{-x}\cos^4 x$ and thus find the exact value of this integral. However, since $e^{-x}\cos^4 x \leq e^{-x}$ for all $x \geq 0$ and since $\int_0^\infty e^{-x}\,dx$ converges (the reader should verify this), the integral $\int_0^\infty e^{-x}\cos^4 x\,dx$ converges by Theorem 2.6.

We now turn to the case in which f is unbounded on $[a, b]$. Suppose that a function f is continuous on an interval $[a, b)$ and that f becomes infinite at b. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx,$$

provided the limit exists. A similar definition is made if a function f is continuous on $(a, b]$ and becomes infinite at a. For example, the following improper integral is convergent since the appropriate limit exists:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\sqrt{x}\,\Big|_a^1 = \lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2.$$

This integral is improper since the graph of $y = 1/\sqrt{x}$ has a vertical asymptote at $x = 0$. As mentioned earlier, you need to look carefully at the integrand to spot this type of improper integral.

Exercises

1. Determine whether or not the integral converges. If it converges, find its value.

a) $\displaystyle\int_0^\infty e^{-x/4}\,dx$ **b)** $\displaystyle\int_0^\infty 12xe^{-3x}\,dx$ **c)** $\displaystyle\int_{-\infty}^{-3} \frac{1}{\sqrt[3]{5x+7}}\,dx$

d) $\displaystyle\int_2^\infty \frac{6}{x^3}\,dx$ **e)** $\displaystyle\int_1^\infty \frac{8}{2x+15}\,dx$ **f)** $\displaystyle\int_1^\infty \frac{x^2+6x+3}{x^4}\,dx$

g) $\displaystyle\int_0^1 \frac{4}{\sqrt[3]{x}}\,dx$ **h)** $\displaystyle\int_2^3 \frac{8}{(3-x)^2}\,dx$ **i)** $\displaystyle\int_1^{7.5} \frac{10}{\sqrt{15-2x}}\,dx$

2. Use Theorem 2.6 to show that the integral converges.

a) $\displaystyle\int_4^\infty \frac{100}{\sqrt{4+x^3}}\,dx$ **b)** $\displaystyle\int_0^\infty e^{-2x}\sin^2 x\,dx$ **c)** $\displaystyle\int_1^\infty \frac{5+2\sin x}{x^2}\,dx$

3. Find all values of r for which the integral converges.

a) $\displaystyle\int_1^\infty \frac{1}{x^r}\,dx$ **b)** $\displaystyle\int_0^\infty e^{rx}\,dx$ **c)** $\displaystyle\int_0^1 \frac{1}{x^r}\,dx$

4. Suppose that f is continuous for all real numbers. Give a definition for the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$.

5. Suppose that f is continuous on $[a, b]$ except at the point $c \in (a, b)$ and that f becomes infinite at c. Give a definition for the improper integral $\int_a^b f(x)\,dx$.

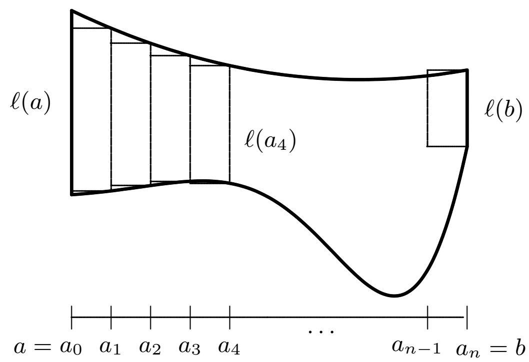
6. Apply the definitions from the previous two exercises to the following integrals. Assume that a is a positive constant.

a) $\displaystyle\int_{-\infty}^{\infty} \frac{a^3}{a^2+x^2}\,dx$ **b)** $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{4x-1}}\,dx$ **c)** $\displaystyle\int_{-2}^1 \frac{1}{x^2}\,dx$

7. Given the hypotheses of Theorem 2.6, what conclusion can be drawn if $\displaystyle\int_a^\infty f(x)\,dx$ diverges? Use this result to show that the improper integral $\displaystyle\int_1^\infty x^{-3}\sqrt{x^4+10}\,dx$ diverges.

2.12 Area between curves

Consider the region outlined in bold in the figure below and suppose that ℓ is a continuous function such that $\ell(x)$ gives the vertical distance across the region for each value of x in the interval $[a, b]$. Divide the interval $[a, b]$ into n subintervals of equal length and use the right endpoint of each subinterval to determine a rectangle that approximates the region on that subinterval.



For each x between a and b,

$\ell(x)$ is the distance across the figure.

$$a_i = a + i \cdot \frac{b-a}{n}$$

Let A be the area of this region. The sum of the areas of the n rectangles gives an approximation to A:

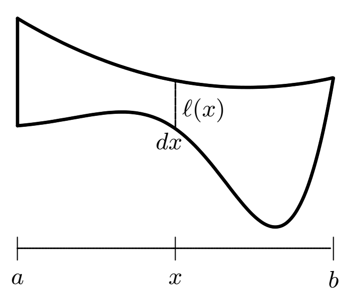
$$A \approx \sum_{i=1}^{n} \ell(a_i)(a_i - a_{i-1}) = \sum_{i=1}^{n} \ell(a_i)\,\frac{b-a}{n}.$$

It is intuitively clear that the approximation to A becomes more accurate as n increases. Since ℓ is a continuous function on $[a, b]$, the definition of the integral yields

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \ell(a_i)\,\frac{b-a}{n} = \int_a^b \ell(x)\,dx.$$

Thus, area is the integral of cross-sectional length, a generalization of the formula for the area of a rectangle (length $\times$ width), where the length is constant across the width of the figure.

A useful way to remember the integral version of the area formula is given in the following figure:



$\ell(x)$ is the distance across the figure at x;

dx is the small width of a rectangle;

$dA = \ell(x)\,dx$ is the area of a thin rectangle;

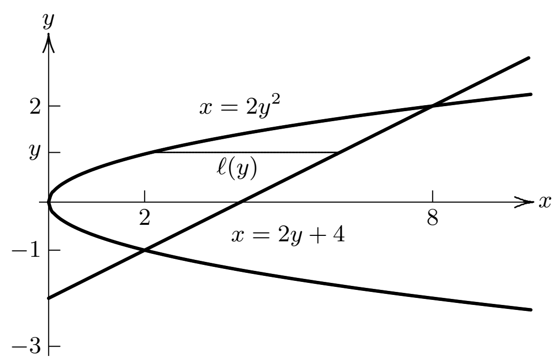
the total area is the sum of these little areas;

$$A = \int dA = \int_a^b \ell(x)\,dx.$$

The key step is the determination of a formula for the distance across the figure. In our illustrations, we have assumed that this distance is vertical, but it may be better to look at the figure horizontally (as in the example in the next paragraph) or in some other direction.

As an example, we will find the area of the region bounded by the curves $x = 2y^2$ and $x - 2y = 4$. The first curve is a parabola opening to the right and the second curve is a straight line. Substituting $x = 2y^2$ into the equation $x - 2y = 4$ yields $y^2 - y - 2 = 0$, and it follows that the curves intersect at the points

$(2,-1)$ and $(8,2)$. A glance at the graph (see the figure) indicates that the region between the curves is easier to describe horizontally than vertically. For each value of y between -1 and 2, the distance $\ell(y)$ across the figure is $\ell(y) = (2y+4) - 2y^2$, the x-value of the curve on the right minus the x-value of the curve on the left. The calculation of the area A of the region between the curves is given to the right of the graph.



$$\begin{aligned} A &= \int_{-1}^{2} (2y + 4 - 2y^2)\,dy \\ &= \left(y^2 + 4y - \frac{2}{3}y^3\right)\Big|_{-1}^{2} \\ &= \left(4 + 8 - \frac{16}{3}\right) - \left(1 - 4 + \frac{2}{3}\right) \\ &= 9. \end{aligned}$$

It is also possible to determine the area of this region using vertical cross-sections, but more effort is involved. The curves need to be in the form $y = f(x)$, and two integrals are needed because the lower boundary changes when $x = 2$. The reader should verify that

$$A = \int_0^2 2\sqrt{\frac{x}{2}}\,dx + \int_2^8 \left(\sqrt{\frac{x}{2}} - \frac{x-4}{2}\right) dx.$$

When finding the area between curves, it is a good idea to first decide whether vertical or horizontal cross-sections provide the best approach; one method is often easier than the other.

Exercises

1. Find the area of the region bounded by the given curves. Treat a as a positive constant.

a) $y = 2x - x^2,\ y = 2 - x$ **b)** $x = y^2,\ 2y = 3 - x$ **c)** $xy = 4,\ x + y = 5$

d) $y = x^2,\ y = ax$ **e)** $y = x^4,\ y = a^3x$ **f)** $xy = a^2,\ x + y = a^2 + 1,\ (a > 1)$

2. Find the area of the region bounded by the curves $y = x^2$ and $y = 6 - x$ in two different ways.

a) Find the vertical distance across the region and integrate with respect to x.

b) Find the horizontal distance across the region and integrate with respect to y.

3. Find the total area of the region bounded by the curves $y = 4x$ and $y = x^3$.

4. Find the area of the region bounded by the curves $y = e^{2x}$, $y = e^{x/2}$, and $y = 4$.

5. Let R be the region under the curve $y = 9/x^2$ and above the x-axis on the interval $[1, 3]$.

a) Find a vertical line that divides the region R into two pieces of equal area.

b) Find a horizontal line that divides the region R into two pieces of equal area.

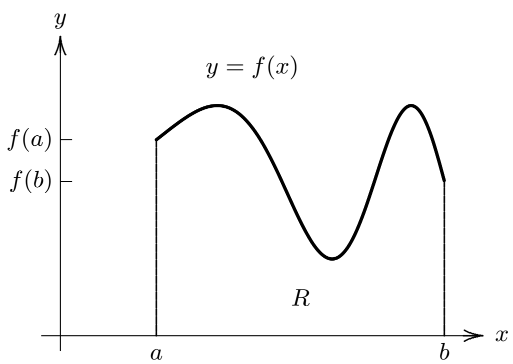
6. Let $a > 0$. Find the area of the region bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes.

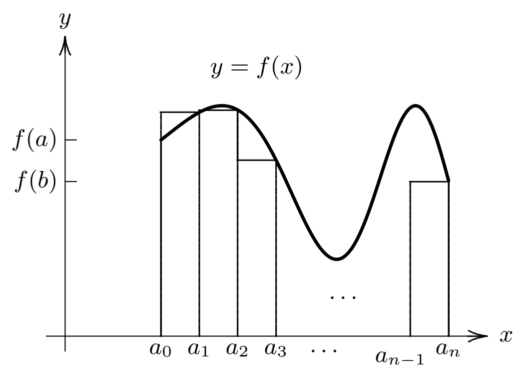
7. Let R be the region between the curves $y = 3/x^2$ and $y = 12/x^3$ on the interval $[1, \infty)$. Find the area of R. *Hint*: Sketch the graphs carefully to determine the region R.

8. Let R be the region in the first quadrant that is bounded by the curves $y = 1/x$, $y = x$, and $y = 3x$. Find a value for a so that the line $y = ax$ divides the region R into two pieces of equal area.

2.13 Volume: method of cross-sections

Let f be a continuous nonnegative function defined on an interval $[a, b]$ and let R be the region under the curve $y = f(x)$ and above the x-axis on the interval $[a, b]$ (see the left graph in the figure). Consider the three-dimensional solid S that is generated when the region R is revolved around the x-axis (a rather warped hourglass that is lying on its side in this case). A solid formed in this manner is called a **solid of revolution**. To determine the volume of S, we approximate the region R with rectangles, then approximate the volume of S by summing the volumes of the cylinders generated by the rectangles when they are revolved around the x-axis. As we have done before, divide the interval $[a, b]$ into n subintervals of equal length and use the right endpoint of each subinterval to determine the height of a rectangle that approximates the curve (see the right graph in the figure).

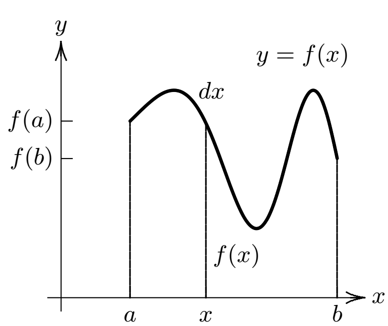




When the rectangles are revolved around the x-axis, flat cylinders or disks (think of a quarter or dime standing on its edge) are generated. The radius of the disk is the height of the curve and the small height of the cylinder is the length of the subinterval on the x-axis, namely, $(b - a)/n$. The volume of each cylinder is thus $\pi(f(a_i))^2(b - a)/n$. As n increases, the cylinders formed by the rotated rectangles better approximate the solid S. By the definition of the integral, the volume V of the solid S is given by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\big(f(a_i)\big)^2 \, \frac{b-a}{n} = \int_a^b \pi\big(f(x)\big)^2 \, dx.$$

As with area, there is an easy way to remember this volume formula. When a vertical line at position x is revolved around the x-axis, a thin disk is generated. Summing the volumes of all these disks gives the total volume of the solid.

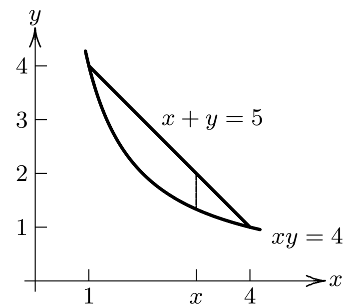


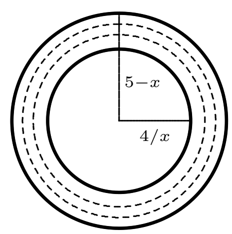
$f(x)$ is the radius of the disk;
dx is the height of the disk;
$\pi(f(x))^2$ is the area of the base of the disk;
$dV = \pi(f(x))^2 dx$ is the volume of the disk;
(rate of change of volume is cross-sectional area);
total volume is the sum of all the little volumes;

$$V = \int dV = \int_a^b \pi(f(x))^2 \, dx.$$

The cross-sections of a solid of revolution are circles. More generally, suppose a solid S extends from $x = a$ to $x = b$ and that for each value of x between a and b, the area of the cross-section perpendicular to the x-axis has area $A(x)$. For a solid of revolution, we would have $A(x) = \pi(f(x))^2$ since the cross-sections are circles. However, the cross-sections could be squares or triangles or some other shape. Then, assuming that the area function A is continuous on $[a, b]$, the volume V of S is given by $V = \int_a^b A(x)\,dx$. In other words, the volume of a solid is the integral of its cross-sectional area. The derivation of this formula is similar to the derivation for the special case in which the solid is a solid of revolution.

As an example, consider the region that is bounded by the curves $xy = 4$ and $x + y = 5$ (see the figure). When this region is revolved around the x-axis, the cross-sections of the resulting solid are washers. (A more technical term for this cross-section is **annulus**, the region between two concentric circles.) The area of the washer at position x is the area of the large circle (which has radius $5 - x$) minus the area of the small circle (which has radius $4/x$). Hence, for each value of x between 1 and 4, the area of the cross-section is $\pi(5 - x)^2 - \pi(4/x)^2$. The volume V of this solid is computed to the right of the figure.





washer cross-section

$$V = \int_1^4 \left(\pi(5-x)^2 - \pi(4/x)^2\right) dx$$
$$= \pi\left(-\frac{1}{3}(5-x)^3 + \frac{16}{x}\right)\Big|_1^4$$
$$= 9\pi.$$

Now suppose that the region bounded by the curves represents the base of a solid that is sitting on the page and projecting out toward you. Suppose further that each cross section of this solid taken perpendicular to the x-axis is a square. Since the side of each square is $5 - x - (4/x)$, the volume of the solid is

$$\int_1^4 \left(5 - x - \frac{4}{x}\right)^2 dx = \int_1^4 \left(x^2 - 10x + 33 - \frac{40}{x} + \frac{16}{x^2}\right) dx = \left(\frac{x^3}{3} - 5x^2 + 33x - 40\ln x - \frac{16}{x}\right)\Big|_1^4 = 57 - 80\ln 2.$$

Exercises

1. Let R be the region under the given graph and above the x-axis on the specified interval. Find the volume of the solid that is generated when R is revolved around the x-axis.

a) $y = x^2,\ [0, 3]$ **b)** $y = e^x,\ [0, 2]$ **c)** $y = 2/x,\ [1, 4]$

2. Find the volume of the solid that is generated when the region bounded by the curves $y = x^2$ and $y = 2x$ is revolved around the (a) x-axis, (b) y-axis.

3. Find the volume of the solid that is generated when the region bounded by the curves $y = x$ and $y = 3x - x^2$ is revolved around the x-axis.

4. Derive the formula for the volume of a cone of radius r and height h. *Hint*: Revolve an appropriate region.

5. Derive the formula for the volume of a sphere of radius r. *Hint*: Revolve an appropriate region.

6. A sphere of radius r is cut in two pieces by a plane perpendicular to a diameter of the sphere. The smaller piece has height h (when lying on its flat side). Find its volume.

7. Let R be the region bounded by the curves $y = x^2$ and $y = 4$. Find the volume of the solid that is generated when R is revolved around (a) the line $x = 3$, (b) the line $y = 4$.

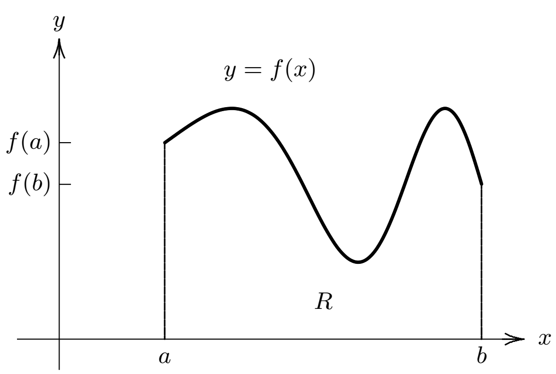
8. Suppose that the base of a solid is the part of the parabola $y = 80 - 0.1x^2$ that lies above the x-axis and that each cross section perpendicular to the y-axis is an equilateral triangle. Find the volume of this solid.

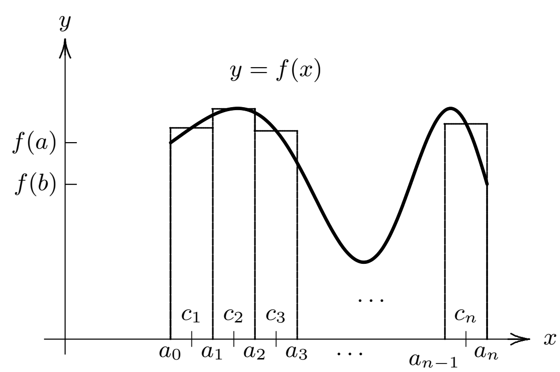
9. Suppose that the base of a solid is the region bounded by the curves $y = x^2$ and $y = 5x$ and that each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

10. Derive the formula for the volume of a square pyramid with base $a \times a$ and height h.

2.14 Volume of solids of revolution: method of cylindrical shells

Another way to find the volume of a solid of revolution is called the method of cylindrical shells or simply the **shell method**. Once again, let f be a continuous nonnegative function defined on an interval $[a,b]$, where $a > 0$, and let R be the region under the curve $y = f(x)$ and above the x-axis on the interval $[a,b]$. Consider the three-dimensional solid S that is generated when the region R is revolved around the y-axis. To determine the volume of S using the shell method, we approximate the region R with rectangles and sum the volumes of the cylindrical shells generated by the rectangles when they are revolved around the y-axis. Letting the number of rectangles increase indefinitely will give the volume V of S. To accomplish this, divide the interval $[a,b]$ into n subintervals of equal length and use the midpoint $c_i = (a_{i-1} + a_i)/2$ (where, as usual, $a_i = a + i\,(b-a)/n$) of each subinterval to determine the height of a rectangle that approximates the curve. (The use of the midpoint rather than the right endpoint will become clear in a moment.) When these rectangles are revolved around the y-axis, cylindrical shells (visualize a hollow tin can) are generated.





Let V_i be the volume of the cylindrical shell that is generated when the rectangle above the interval $[a_{i-1}, a_i]$ is revolved around the y-axis. The volume of a cylindrical shell is the area of the annulus that forms its base times the height of the cylinder. Since the inner radius of the annulus is a_{i-1} and the outer radius is a_i,

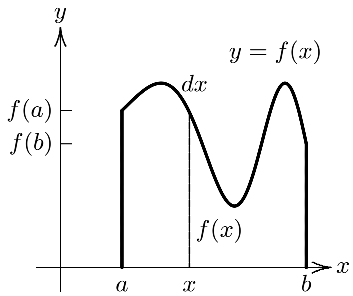
$$V_i = \left(\pi a_i^2 - \pi a_{i-1}^2\right) f(c_i) = \pi\,(a_i + a_{i-1})(a_i - a_{i-1}) f(c_i) = 2\pi c_i\,(a_i - a_{i-1}) f(c_i).$$

Using the more general definition of the integral (see Section 2.5), it follows that

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} V_i = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi c_i f(c_i)(a_i - a_{i-1}) = \int_a^b 2\pi x f(x)\,dx.$$

(You should now see the reason for the choice of the midpoint c_i.)

The shell method volume formula is easy to remember using the fact that an area sweeps out a volume as it moves. The $f(x)$ by dx "rectangle" (see the following figure) moves in a circle of radius x as it revolves around the y-axis and therefore sweeps out a volume of $2\pi x f(x)\,dx$, the product of distance traveled $(2\pi x)$ and area $(f(x)\,dx)$. The sum of all these tiny volumes gives the total volume. Rather than memorize a specific formula for the shell method, you should remember this essential idea, an idea that is especially helpful when the region is revolved around lines other than the coordinate axes (as in the following example).



$f(x)$ is the height of the rectangle;
dx is the width of the rectangle;
$2\pi x$ is the distance the rectangle travels around the y-axis;
$dV = 2\pi x f(x)\,dx$ is the volume swept out by the rectangle;
total volume V is the sum of all the little volumes;
$V = \int dV = \int_a^b 2\pi x f(x)\,dx.$

To illustrate the shell method, consider again (see the last section) the region bounded by the curves $xy = 4$ and $x + y = 5$, and suppose that this region is revolved around the line $x = -2$. A thin vertical rectangle at position x moves in a circle of radius $x + 2$ when it is revolved around the line $x = -2$. The height of the rectangle is the distance between the two curves; $(5 - x) - (4/x)$. The volume V of the resulting solid is thus

$$V = \int_1^4 2\pi(x+2)\Big(5 - x - \frac{4}{x}\Big)\,dx = (39 - 32\ln 2)\pi.$$

(Multiply out the integrand to evaluate the integral.) Note that this volume is different than the volume 9π obtained when the same region is revolved around the x-axis.

Exercises

1. Let R be the region under the given graph and above the x-axis on the specified interval. Find the volume of the solid that is generated when R is revolved around the y-axis.

a) $y = \cos x,\ [0, \pi/2]$ **b)** $y = 10/(1 + x^2),\ [0, 3]$ **c)** $y = e^x,\ [0, 2]$

2. Let R be the region in the first quadrant that is bounded by the curves $y = 7x$ and $y = 16x - x^3$. Find the volume of the solid that is generated when R is revolved around the y-axis.

3. Let R be the region bounded by the curves $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid that is generated when R is revolved around the line $x = 3$.

4. Use the shell method to derive the formula for the volume of a cone of radius r and height h.

5. Use the shell method to derive the formula for the volume of a sphere of radius r.

6. A cylindrical hole of radius r is bored through the center of a sphere with radius $R > r$. The remaining solid resembles a bead since it has a flat top and bottom with a hole through the middle.

a) Find the volume of the bead.

b) Find a value of r (in terms of R) that leaves a solid with exactly half of the original volume of the sphere.

c) Represent the volume of the bead as a function of the height h of the bead.

7. Suppose that $0 < r < a$ and consider the circle defined by the equation $(x - a)^2 + y^2 = r^2$. Let D be the region inside this circle and let R be the right half of D. Find the volumes of the solids that are generated when the regions D and R are revolved around the y-axis.

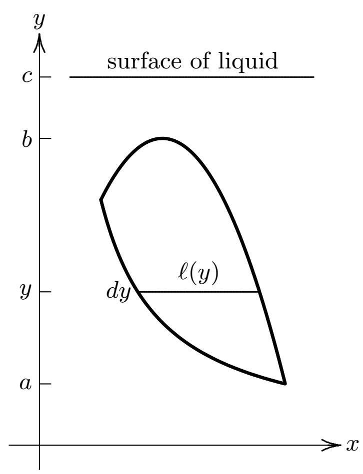
8. Let R be the region under the graph of $y = 2x^2$ and above the x-axis on the interval $[0, 3]$. Set up, but do not evaluate, an integral that represents the volume of the solid that is generated when R is revolved around (a) the line $x = 3$, (b) the line $y = 20$, (c) the line $x = -1$, and (d) the line $y = -4$.

9. Let R_a be the region under the curve $y = x^2$ and above the x-axis on the interval $[0, a]$, where a is a positive constant. Find a value for a so that the volume of the solid that is generated when R_a is revolved around the y-axis is one-half the volume of the solid that is generated when R_2 is revolved around the y-axis. Answer the analogous question for the case in which the regions are revolved around the line $x = 2$.

2.15 Force exerted by a liquid

Anyone who has had the pleasant experience of diving into a lake or swimming pool on a hot summer day has probably noticed some pain in their ears when they were underwater. This pain results from the pressure of the water against the eardrum and this pressure increases with depth. It has been experimentally verified that the pressure exerted by a liquid at a given point in the liquid is the same in all directions and has magnitude wd, where w is the weight density of the liquid and d is the depth of the given point. For our purposes, the units for w will be pounds per cubic foot, the units for d will be feet, and the units for pressure will be pounds per square foot. The problem considered in this section is to find the force (in pounds) exerted by a liquid on one side of a submerged plate.

This problem has a simple solution if the plate is horizontal—just multiply the pressure exerted by the liquid at the depth of the plate by the area of the plate. The problem is more difficult if the plate is vertical for then different parts of the plate occupy different depths and therefore experience different pressures. To make the discussion more precise, suppose that a plate is submerged vertically in a liquid with weight density w. Since points on the plate at the same depth experience the same pressure, the horizontal distance across the plate at a given depth is the most important quantity to know. Let $\ell(y)$ represent the distance across the plate at vertical position y and assume that ℓ is continuous on $[a, b]$ (see the figure).



At vertical position y,
$\ell(y)\,dy$ is the area,
$c - y$ is the depth,
$w(c-y)$ is the pressure,
$dF = w(c-y)\ell(y)\,dy$ is the force.
The total force F on the plate is the sum of all the forces;
$F = \int dF = \int_a^b w(c-y)\ell(y)\,dy.$

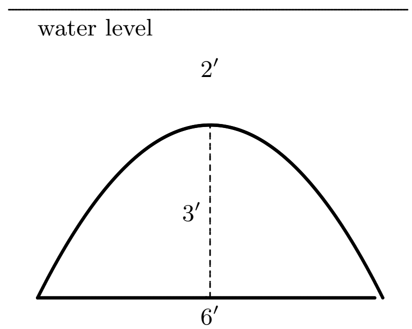
The data to the right of the figure provides a quick derivation of the formula

$$F = \int_a^b w(c-y)\ell(y)\,dy$$

for the force F exerted by the liquid on one side of the submerged plate. The details of this derivation using the definition of the integral will be left as an exercise.

As a specific example, suppose that a window on the side of a swimming pool is in the shape of an inverted parabola and has the dimensions and depth recorded on the figure below. To find the force on the window, we can place the origin in any convenient position. We will assume that the origin is the midpoint of the base of the window. The parabola then has the form $y = 3 - kx^2$, where k is some constant. Since $y = 0$ when $x = \pm 3$, we find that $k = 1/3$. Hence, an equation for the parabola is $y = (9 - x^2)/3$. We need to solve this equation for y in order to find the horizontal distance across the figure. Solving for y yields

$y = \pm\sqrt{9-3y}$ so the horizontal distance across the window is $2\sqrt{9-3y}$ for each value of y between 0 and 3. The force F of water on this window is computed to the right of the figure. (Take some time to understand each of the quantities that appears in the integrand and to determine the substitution that was used in the evaluation of the integral.)



$$\begin{aligned} F &= \int_0^3 w(5-y)\,2\sqrt{9-3y}\,dy \\ &= \frac{2w}{9}\int_0^9 (6+u)\,u^{1/2}\,du \\ &= 45.6w. \end{aligned}$$

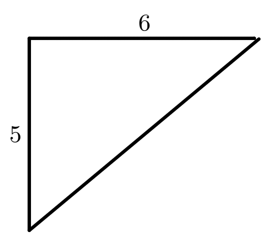
The weight density of fresh water is approximately 62.4 pounds per cubic foot so the water exerts a force of about 2845 pounds on the window.

Exercises

1. Suppose that the parabolic plate in the example is placed horizontally on the bottom of a swimming pool that is 12 feet deep. Find the force exerted by water on this plate. Use $w = 62.4$ lb/ft^3.

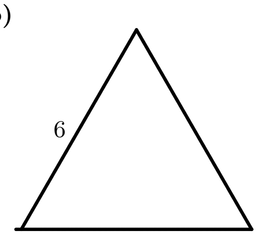
2. Find the force exerted by a liquid with weight density w on one side of each vertically submerged plate. The units on the figures are feet and the top of each plate is four feet beneath the surface of the liquid. For part (b), make certain that you correctly identify the height of the triangle.

a)



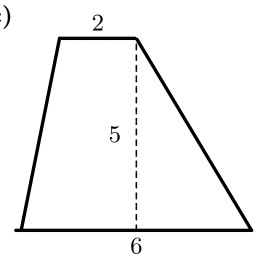
right triangle

b)



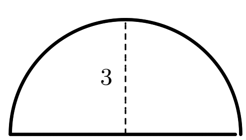
equilateral triangle

c)



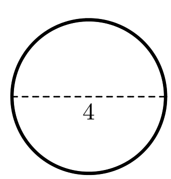
trapezoid

d)



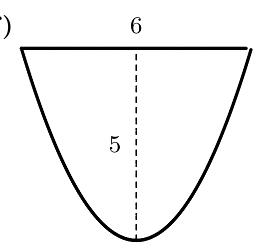
semicircle

e)



circle

f)



parabola

3. Let R be the region bounded by the curves $y = 4x$ and $y = x^2$. Suppose that R is a vertical plate with the water level at the line $y = 20$. Assuming all units are feet, find the force exerted by water on one side of R. Use $w = 62.4$ lb/ft^3.

4. Provide the details of the derivation of the formula for F that is given in the text. You will need to use words as well as equations; see the derivations in the last three sections for the correct style.

2.16 WORK

In physics, work is defined as the product of force and displacement. If a force of 8 lbs is applied over a displacement of 4 ft, then the work done by the force is 32 ft-lbs. Since force and displacement are actually vector quantities (they have both magnitude and direction), this sort of computation is only valid when the force and displacement have the same direction. We will make this assumption throughout this section. In this case, the work W done by a constant force F operating over a distance d is defined by $W = Fd$. If the force is variable, then this formula for work is no longer valid. As you should expect by now, the definite integral is needed to handle this situation.

Suppose that F is a continuous function defined on an interval $[a, b]$ and that $F(x)$ represents the force applied at the point x. We want to find the work done by this variable force F over the interval $[a, b]$. As usual, divide the interval $[a, b]$ into n subintervals of equal length and let $a_i = a+i(b-a)/n$ for $i = 0, 1, \ldots, n$. On the interval $[a_{i-1}, a_i]$, the force is almost constant and has a value very close to $F(a_i)$. Hence, the work W done by the force F over the interval $[a, b]$ is approximated by $W \approx \sum_{i=1}^{n} F(a_i)(a_i - a_{i-1})$. The approximation to W improves as n gets larger, so we conclude that

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(a_i)(a_i - a_{i-1}) = \int_a^b F(x)\,dx.$$

In other words, the work W done by the variable force F is $\int dW$, where $dW = F(x)\,dx$ is the little bit of work done by the force $F(x)$ over the short distance dx. The following problems illustrate the use of this formula.

Problem: A force of 8 pounds is required to hold a spring 6 inches beyond its natural length. How much work is required to stretch the spring 8 inches beyond its natural length?

Solution: The force $F(x)$ required to hold a spring (either compressed or stretched) x units from its equilibrium position (x is positive if stretched, negative if compressed) has been experimentally verified to be $F(x) = kx$, where k is a constant that depends on the size and composition of the spring. This fact is known as Hooke's Law. In order to compute work in ft-lbs, we will use ft and lbs as our units. Since a force of 8 lbs is required to hold the spring 0.5 ft beyond its natural length, Hooke's Law becomes $8 = 0.5k$. It follows that $k = 16$ lb/ft. The work W required to stretch this spring 8 inches beyond its natural length is thus

$$W = \int_0^{2/3} 16x\,dx = \frac{32}{9} \text{ ft-lb}.$$

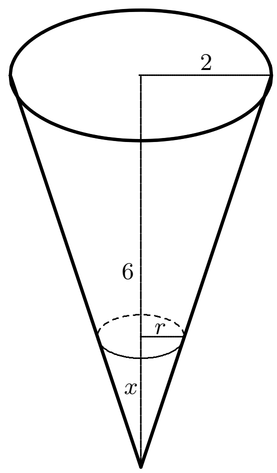
The reader should verify that it takes 8/9 ft-lb of work to stretch the spring the first four inches and 24/9 ft-lb of work to stretch the spring the next four inches.

Problem: An inverted cone with a diameter of four feet and a height of six feet is full of water. Find the minimum amount of work required to pump the water to a level three feet above the top of the cone.

Solution: The work here is done against gravity. Each layer of water is raised a different distance and thus requires a different amount of work. It is probably best to think in terms of the formula $W = \int dW$, where

dW represents the little bit of work required to raise a thin layer of water. Let x be the distance in feet from the bottom of the cone and let r be the radius in feet of the cone at position x (see the figure). The layer of water at position x has volume $\pi r^2\,dx$ ft^3 and this layer must be raised a distance of $9 - x$ ft. Using the fact that the weight density of water is 62.4 lbs/ft^3, it follows that the work dW required to lift this layer of water is $dW = 62.4\pi r^2\,dx(9 - x)$. The calculations to find the total work required are shown to the right of the figure.

$$\frac{x}{6} = \frac{r}{2}$$
$$r = \frac{x}{3}$$



$$\begin{aligned} W &= \int dW \\ &= \int 62.4\pi r^2\,dx(9 - x) \\ &= \int_0^6 62.4\pi \left(\frac{x}{3}\right)^2 (9 - x)\,dx \\ &= \frac{62.4\pi}{9} \int_0^6 (9x^2 - x^3)\,dx \\ &\approx 7057 \end{aligned}$$

Thus, the minimum amount of work required for this process is approximately 7057 ft-lbs.

Exercises

1. A force of 4 lbs is required to hold a spring 4 inches beyond its natural length. How much work is required to stretch the spring 10 inches beyond its natural length?

2. A force of 8 lbs is required to hold a spring 4 inches beyond its natural length. How much work is required to compress the spring 2 inches from its natural length?

3. Suppose that 40 ft-lbs of work are required to stretch a spring from its natural length to 6 inches beyond its natural length. How far from its natural length will 100 ft-lbs of work stretch it?

4. A hemispherical bowl with a radius of five feet is full of water. Find the minimum amount of work required to pump the water to a level two feet above the top of the bowl.

5. A tank in the shape of an inverted cone with a radius of 6 feet and a height of 10 feet contains water that is 9 feet deep. Find the minimum amount of work required to pump the water up and over the top of the tank.

6. The cross-section of a trough filled with water is an equilateral triangle (vertex down) with side length 4 feet and the trough is 8 feet long. Find the minimum amount of work required to pump the water up and over the top of the trough.

7. A chain weighing 1 lb/ft is used to raise a 100 lb object 40 ft out of a well. Find the work required to do this.

8. Recall that Newton's law of gravitation states that the force due to gravity is inversely proportional to the square of the distance from the center of the earth, that is, $F = k/d^2$. The constant k can be determined from the weight of the object on the surface of the earth. For the following problems, use 3960 miles for the radius of the earth and mi-lbs as the units of work.

a) Find the work required to launch a 500 lb satellite from the earth's surface to an orbit of 800 miles.

b) Find the work required to "lift" a 200 lb object from the earth's surface to a height of 100 miles.

c) Find the work required to move an 800 lb object from 100 miles to 600 miles above the earth.

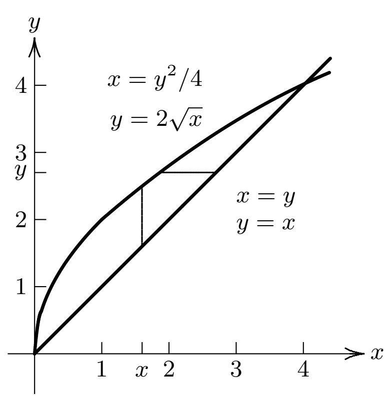
2.17 Center of mass

For the record, the discussion in this section will be quite informal. Also, we assume that the reader has some intuitive sense concerning center of mass. For example, the center of mass of a circle is its geometric center; a circle will be balanced on the tip of a pencil placed at its center. The center of mass of a collection of particles makes it possible to treat the collection as a single point. For instance, a point on the surface of a punted football travels in a very wobbly path, but (ignoring air resistance) the center of mass of the football travels in a parabola. The center of mass of an object makes a number of calculations involving Newton's second law ($F = ma$) much easier; it also shows up regularly in other applications. In many engineering and physics textbooks, the formulas for the coordinates of the center of mass of a collection of particles are given by

$$\overline{x} = \frac{\int x\,dm}{\int dm}, \qquad \overline{y} = \frac{\int y\,dm}{\int dm}, \qquad \overline{z} = \frac{\int z\,dm}{\int dm}.$$

The symbol dm in the integral $\int y\,dm$ represents the little bit of mass that occupies the space where the y-coordinate has the value y. (Some experience on a playground toy known as a teeter-totter should make it clear why the product $y\,dm$ of distance and mass, often called a moment, is relevant.) These formulas are quite general. They include the cases in which (i) the collection of particles occupies one, two, or three dimensions, (ii) the density varies from point to point, and (iii) the number of particles is finite or infinite. Because of all this variability, it takes some practice to learn how to apply them in various situations.

As an example, let R be the region bounded by the curves $y = x$ and $y = 2\sqrt{x}$ and assume that the density ρ of this region is constant. (The units of ρ in this case would be mass/area.) A sketch of the region is given below as well as the calculations required to find $(\overline{x}, \overline{y})$. The little bit of mass at position x is determined by the vertical distance between the curves at position x. Thus $dm = \rho\,dA = \rho\big(2\sqrt{x} - x\big)\,dx$. The little bit of mass at position y is determined similarly using the horizontal distance between the curves at position y.

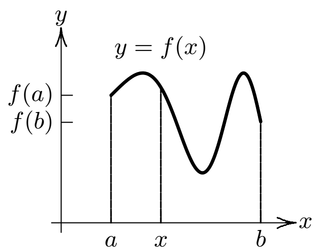


$$\overline{x} = \frac{\displaystyle\int_0^4 x\rho(2\sqrt{x} - x)\,dx}{\displaystyle\int_0^4 \rho(2\sqrt{x} - x)\,dx} = 1.6$$

$$\overline{y} = \frac{\displaystyle\int_0^4 y\rho\left(y - \frac{y^2}{4}\right)dy}{\displaystyle\int_0^4 \rho\left(y - \frac{y^2}{4}\right)dy} = 2$$

As in the example, for most of the problems we will consider, the density ρ of the object will be constant. Its units will be mass/length, mass/area, or mass/volume depending on the situation. Referring to the example, note how a constant ρ cancels in the formulas for the coordinates of the center of mass. When the density is constant and an object has symmetry, the center of mass always lies on any line of symmetry of the object. For example, the center of mass of a rectangle with constant density is at its geometric center.

We now consider a rather general three dimensional problem. Let f be a continuous nonnegative function defined on $[a, b]$, let R be the region under the curve $y = f(x)$, and let S be the solid that is generated when R is revolved around the x-axis. We want to find the center of mass of S. By symmetry, both $\overline{y}$ and $\overline{z}$ are zero. To find $\overline{x}$, note that the little bit of mass at position x depends on the size of the circular cross-section at x. It follows that $dm = \rho\pi(f(x))^2\,dx$ and thus



$$\overline{x} = \frac{\int_a^b x\rho\pi(f(x))^2\,dx}{\int_a^b \rho\pi(f(x))^2\,dx} = \frac{\int_a^b x(f(x))^2\,dx}{\int_a^b (f(x))^2\,dx}.$$

Exercises

Unless noted otherwise, assume that all "objects" have constant density.

1. Find the center of mass of the following collection of masses: a mass of 3 at $(3, 2, 1)$, a mass of 2 at $(2, -1, 2)$, a mass of 4 at $(5, -2, 4)$, and a mass of 1 at $(4, 6, 0)$.

2. A two kilogram mass is located at $(-1, 1)$ and a five kilogram mass is located at $(2, 4)$. Where should a one kilogram mass be located so that the collection has the origin as its center of mass?

3. Find the center of mass of the region under the curve and above the x-axis on the specified interval.

a) $y = x^2,\ [0, 3]$ **b)** $y = 4/x,\ [1, 4]$ **c)** $y = 1 - x^4,\ [0, 1]$

4. Find the center of mass of the region bounded by $y = x^2$ and $y = ax$, where a is a positive constant.

5. Find the center of mass of a quarter circle of radius r.

6. Find the center of mass of a solid cone with height h and radius r.

7. Find the center of mass of a solid hemisphere of radius r.

8. A stump with constant density and the shape of a chopped off cone has a top radius of one foot, a bottom radius of two feet, and a height of two feet. How far above the ground is its center of mass?

9. Suppose that a thin straight rod with a length of 1 meter has variable density. The density x meters from its left end is $\rho(x) = 10(x^2 + x + 1)$ grams per meter. Find the center of mass of this rod.

10. Let f be a continuous nonnegative function defined on an interval $[a, b]$ and let R be the region under the graph of $y = f(x)$ and above the x-axis on the interval $[a, b]$. Show that $\int y\,dm = \frac{\rho}{2}\int_a^b \left(f(x)\right)^2 dx$.

11. Use the previous exercise to find the center of mass of the region under the graph of $y = x^3$ on the interval $[0, 2]$.

12. Prove the **Theorem of Pappus**: the volume of a solid of revolution is the product of the area of the revolved region and the distance traveled by its center of mass during the revolution.

13. Consider the region bounded by the curves $y = 4x$ and $y = x^2$. Find the volume of the solid that is generated when this region is revolved around the line $y = 4x$. *Hint*: Use the Theorem of Pappus.

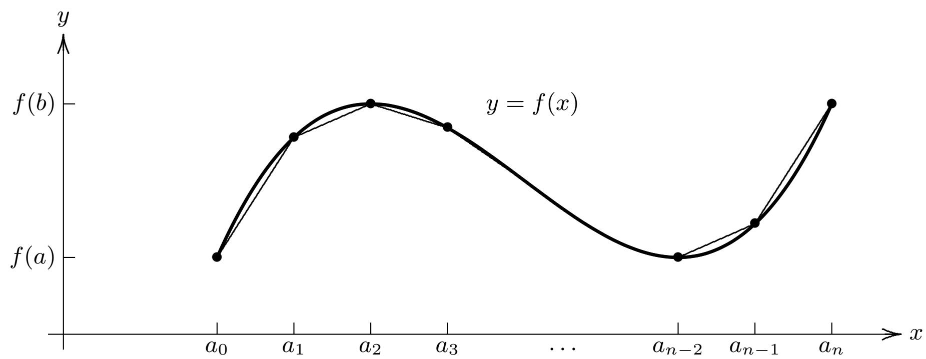
14. Prove that the force exerted by a liquid on one side of a submerged vertical plate is equal to the force exerted by the liquid on the same plate if the plate is placed horizontally at the depth of the center of mass of the vertical plate.

15. Find the x-coordinate of the center of mass of the solid that is generated when the region under the curve and above the x-axis on the specified interval is revolved around the x-axis.

a) $y = x^2,\ [0, 4]$ **b)** $y = e^x,\ [0, 2]$ **c)** $y = 9/x^2,\ [1, 3]$

2.18 Arc length

Let f be a continuous function defined on an interval $[a, b]$ and consider the following problem: find the length s of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$. As we have done in our previous work, let n be a positive integer and define $a_i = a + i(b-a)/n$ for $i = 0, 1, 2, \ldots, n$. An approximation to s can be obtained by finding the length of the polygonal path (a sequence of line segments) joining the points $(a_i, f(a_i))$ for $i = 0, 1, 2, \ldots, n$ (see the figure).



Since each portion of the polygonal path is a straight line, its length can be found using the distance formula. The sum of all these lengths gives an approximation to s:

$$s \approx \sum_{i=1}^{n} \sqrt{(a_i - a_{i-1})^2 + (f(a_i) - f(a_{i-1}))^2}.$$

The quantity in the sum does not fit the pattern in the definition of the integral because the term $a_i - a_{i-1}$ does not appear as a multiplier. However, a little algebra and a little theory yield

$$\sqrt{(a_i - a_{i-1})^2 + (f(a_i) - f(a_{i-1}))^2} = (a_i - a_{i-1})\sqrt{1 + \left(\frac{f(a_i) - f(a_{i-1})}{a_i - a_{i-1}}\right)^2} = \sqrt{1 + (f'(t_i))^2}\,(a_i - a_{i-1}),$$

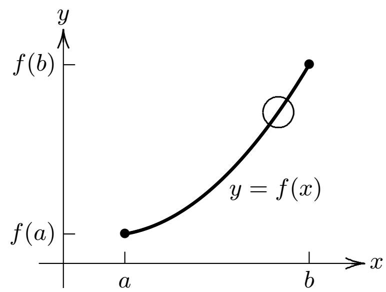
where t_i is some point between a_{i-1} and a_i. The existence of the point t_i is guaranteed by the Mean Value Theorem under the added assumption that f is differentiable on $[a, b]$. Since the approximation to s appears to improve as n increases, it follows that

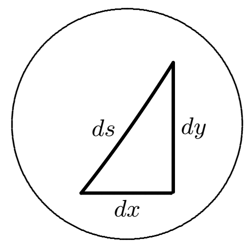
$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(t_i))^2}\,(a_i - a_{i-1}) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

The last step requires that f' be continuous on $[a, b]$. We have thus obtained the following result.

THEOREM 2.7 arc length If a function f has a continuous derivative on an interval $[a, b]$, then the length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is given by $\displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. ■

There is a quick and intuitive way to remember the arc length formula using differentials. By focusing on a tiny portion of the curve $y = f(x)$, we can consider the so-called **differential triangle** (see the figure); the base is dx, the height is dy, and the hypotenuse, which represents a very short, almost straight, portion of the curve, is ds. Thus, a little change dx in x generates a little change dy in y and these two determine ds, the little change in the arc length.





differential triangle

$$
\begin{aligned}
(ds)^2 &= (dx)^2 + (dy)^2 \\
s &= \int ds \\
&= \int \sqrt{(dx)^2 + (dy)^2} \\
&= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
\end{aligned}
$$

Although the integral representing the arc length of a curve is easy to write down, most such integrals are very difficult to evaluate. The reason for this lies in the fact that the square root makes it rather difficult to find an antiderivative. Except for a few contrived functions, most arc length integrals require advanced techniques of integration to evaluate in exact terms. To illustrate a contrived example, we will find the length of the curve $y = \dfrac{3x^4}{4} + \dfrac{1}{24x^2}$ on the interval $[1, 2]$. We first note that

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(3x^3 - \frac{1}{12x^3}\right)^2 = 1 + \left(3x^3\right)^2 - \frac{1}{2} + \left(\frac{1}{12x^3}\right)^2 = \left(3x^3 + \frac{1}{12x^3}\right)^2.$$

It should be clear how unique this example is! It follows that the length of the curve is

$$\int_1^2 \left(3x^3 + \frac{1}{12x^3}\right) dx = \left(\frac{3x^4}{4} - \frac{1}{24x^2}\right)\Bigg|_1^2 = \frac{361}{32}.$$

Exercises

1. Let $f(x) = \sqrt{r^2 - x^2}$, where $r > 0$. Use the arc length formula to find the length of the portion of the curve $y = f(x)$ on the interval $[0, r]$. (Note the appearance of an improper integral.) Why does the answer look familiar?
2. Find the length of the curve $y = x^{3/2}$ on the interval $[0, b]$, where $b > 0$. Use your formula to find a rational number b for which the length of this curve is also a rational number. Can you find an integer b for which the length of this curve is also an integer?
3. Find the length of the curve on the given interval.

   a) $y = \dfrac{x^{3/2}}{6} - 2x^{1/2}$, $[1, 4]$ **b)** $y = \dfrac{x^2}{8} - \ln x$, $[1, 4]$ **c)** $y = (e^x + e^{-x})/2$, $[0, \ln 10]$

   d) $y = \dfrac{2x^3}{3} + \dfrac{1}{8x}$, $[1, 2]$ **e)** $y = \dfrac{x^5}{10} + \dfrac{1}{6x^3}$, $[1, 2]$ **f)** $y = \ln|\cos x|$, $[0, \pi/3]$

4. Find the length of the entire curve given by the equation $x^{2/3} + y^{2/3} = 1$. (Note that this equation is similar to the equation of the unit circle; the only difference is the exponents on the variables x and y. The graph of this equation has a diamond shape.)
5. Find the exact length of the curve $y = x^{5/4}$ on the interval $[0, 16]$.
6. Find the center of mass of a wire bent in the shape of a quarter circle of radius r. Assume the wire has constant density.

2.19 Integrals involving quadratic polynomials

In order to find the length of a portion of the parabola $y = x^2$, we need to find an antiderivative of the function $\sqrt{1+4x^2}$. None of the techniques discussed thus far apply to this function. In the next few sections, we will consider more advanced techniques for finding antiderivatives and thus greatly expand our list of functions that can be antidifferentiated. Before beginning this section, the reader is encouraged to review the method of completing the square for quadratic polynomials and the technique for performing long division for the quotient of two polynomials. A brief summary of these can be found in Appendix A.

Integrals that contain a quadratic polynomial as part of the integrand occur often in applications. Using some algebraic techniques, an antiderivative can sometimes be found fairly easily. For example,

$$\int \frac{x+1}{x^2+9}\,dx = \int \Big(\frac{x}{x^2+9} + \frac{1}{x^2+9}\Big)\,dx = \frac{1}{2}\ln|x^2+9| + \frac{1}{3}\arctan(x/3) + C;$$

splitting the integral into two fractions leads to integrals that can be evaluated using basic formulas. Similarly, algebra in the form of long division yields

$$\int \frac{3x^2+8x+2}{x+4}\,dx = \int \Big(3x - 4 + \frac{18}{x+4}\Big)\,dx = \frac{3}{2}x^2 - 4x + 18\ln|x+4| + C.$$

Once again, some algebra reduces the problem to one that is much easier. Another useful algebraic tool that applies to quadratic polynomials is completing the square. For example,

$$\begin{aligned}
\int \frac{x+4}{x^2+12x+40}\,dx &= \int \frac{x+4}{(x+6)^2+4}\,dx && \text{complete the square}\\
&= \int \frac{u-2}{u^2+4}\,du && \text{let } u = x+6\\
&= \int \Big(\frac{u}{u^2+4} - \frac{2}{u^2+4}\Big)\,du && \text{split up}\\
&= \frac{1}{2}\ln|u^2+4| - \arctan(u/2) + C && \text{basic formulas}\\
&= \frac{1}{2}\ln|x^2+12x+40| - \arctan\Big(\frac{x+6}{2}\Big) + C && \text{return to } x
\end{aligned}$$

All three of these examples involve more algebra than calculus. As is evident in the last example, it is important to pay attention to details. Note that the final answer may be quite a bit more complicated than the original function. With some care, it is possible to avoid a substitution for these problems:

$$\begin{aligned}
\int \frac{4x-3}{\sqrt{24-2x-x^2}}\,dx &= \int \frac{4(x+1)-7}{\sqrt{25-(x+1)^2}}\,dx\\
&= \int \frac{4(x+1)}{\sqrt{25-(x+1)^2}}\,dx - \int \frac{7}{\sqrt{25-(x+1)^2}}\,dx\\
&= -4\sqrt{25-(x+1)^2} - 7\arcsin\Big(\frac{x+1}{5}\Big) + C.
\end{aligned}$$

If you find the algebra in this problem to be troublesome or confusing, you should use the substitution method (let $u = x+1$ in this case).

Since these integrals can sometimes cause difficulties, we give another, slightly more complicated example. Read each step carefully. When doing these problems, just take your time and write out the terms slowly and clearly.

$$
\begin{aligned}
\int \frac{x}{\sqrt{x-x^2}}\,dx &= \int \frac{x}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx && \text{complete the square}\\
&= \int \frac{2x}{\sqrt{1-(2x-1)^2}}\,dx && \text{multiply by } \tfrac{2}{2}\\
&= \frac{1}{2}\int \frac{u+1}{\sqrt{1-u^2}}\,du && \text{let } u = 2x-1\\
&= \frac{1}{2}\int \left(\frac{u}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-u^2}}\right) du && \text{split up}\\
&= \frac{1}{2}\left(-\sqrt{1-u^2} + \arcsin u\right) + C && \text{basic formulas}\\
&= \frac{1}{2}\left(-\sqrt{1-(2x-1)^2} + \arcsin(2x-1)\right) + C && \text{return to } x\\
&= -\sqrt{x-x^2} + \frac{1}{2}\arcsin(2x-1) + C && \text{simplify}
\end{aligned}
$$

Once again, note that there is quite a bit of algebra here and only a little bit of calculus.

Exercises

1. Complete the square as a first step to evaluate each of the following integrals.

a) $\displaystyle\int \frac{3x-2}{8+4x+x^2}\,dx$ **b)** $\displaystyle\int \frac{2x+1}{\sqrt{8+2x-x^2}}\,dx$ **c)** $\displaystyle\int \frac{4}{\sqrt{6x-x^2}}\,dx$

d) $\displaystyle\int \frac{4x+3}{x^2+3x+5}\,dx$ **e)** $\displaystyle\int \frac{5x+7}{x^2+4x+10}\,dx$ **f)** $\displaystyle\int \frac{6x-1}{\sqrt{4x-x^2}}\,dx$

2. Evaluate each of the following integrals.

a) $\displaystyle\int \frac{x}{3x+1}\,dx$ **b)** $\displaystyle\int \frac{x+3}{x^2+10}\,dx$ **c)** $\displaystyle\int \frac{x^2}{x^2+4}\,dx$

d) $\displaystyle\int \frac{2x^2+3x+1}{x^2+1}\,dx$ **e)** $\displaystyle\int \frac{x^3}{x^2+2x+2}\,dx$ **f)** $\displaystyle\int \frac{2x}{x-4}\,dx$

3. Use the fact that $\dfrac{12}{x^2-6x} = \dfrac{2}{x-6} - \dfrac{2}{x}$ to evaluate $\displaystyle\int \frac{12}{x^2-6x}\,dx$.

2.20 Using a table of integrals

Finding an antiderivative can be a lot of work and sometimes requires a certain amount of cleverness. In order to avoid repeating routine computations or remembering clever tricks, various mathematicians over the years have compiled lists of antiderivatives. Whenever an antiderivative is needed, one can consult the list rather than take the time to rediscover the result; the list is a labor-saving device. A list of antiderivatives is typically called a **table of integrals** (although table of indefinite integrals would be a more appropriate title) and extensive tables of integrals can be found in reference books. The formulas in a table of integrals are generally arranged according to the form of the integrand; formulas for exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions, etc. Tables also include many antiderivatives for functions involving the quadratic forms $a^2 + u^2$, $a^2 - u^2$, and $u^2 - a^2$ as these types occur frequently in applications. A short table of integrals suitable for this text can be found in Appendix B. Take a few minutes to look over this table and become familiar with the types of integrals it includes. In the next section, we will discuss one method that indicates how many of these integral formulas were obtained.

Even with a table of integrals, finding antiderivatives is not a simple process. It is often necessary to make a substitution before the table can be used and some care must be made to determine the appropriate formula to use. The following three examples illustrate some of these ideas.

$$\begin{aligned}
\int \frac{4}{\sqrt{3x^2-10}}\,dx &= \frac{4}{\sqrt{3}}\int \frac{1}{\sqrt{u^2-10}}\,du && \text{let } u=\sqrt{3}x \\
&= \frac{4}{\sqrt{3}}\ln\left|u+\sqrt{u^2-10}\right| + C && \text{integral formula 46} \\
&= \frac{4}{\sqrt{3}}\ln\left|\sqrt{3}x+\sqrt{3x^2-10}\right| + C && \text{return to } x
\end{aligned}$$

$$\begin{aligned}
\int x\sqrt{4-x^4}\,dx &= \frac{1}{2}\int \sqrt{4-u^2}\,du && \text{let } u=x^2 \\
&= \frac{1}{2}\left(\frac{u}{2}\sqrt{4-u^2}+\frac{4}{2}\arcsin(u/2)\right) + C && \text{integral formula 13} \\
&= \frac{1}{4}x^2\sqrt{4-x^4} + \arcsin(x^2/2) + C && \text{return to } x
\end{aligned}$$

$$\begin{aligned}
\int \sqrt{1+e^x}\,dx &= 2\int \frac{\sqrt{1+u^2}}{u}\,du && \text{let } u=e^{x/2} \\
&= 2\sqrt{1+u^2} - 2\ln\left|\frac{1+\sqrt{1+u^2}}{u}\right| + C && \text{integral formula 34} \\
&= 2\sqrt{1+e^x} - 2\ln\left|1+\sqrt{1+e^x}\right| + x + C && \text{return to } x
\end{aligned}$$

(You should pay careful attention to the last step in the third example; it involves properties of logarithms.) The general idea is to make a substitution that transforms the integral into one which appears in the table. As usual, it is important to pay attention to detail when evaluating more complicated integrals. In this regard, be very careful with the differential terms that appear in your substitution.

Integral formulas 62–65, 73, and 74 are known as **reduction formulas** because the formulas indicate how the integral of a function involving an exponent is related to a similar integral with the exponent reduced, that is, made smaller. By using a reduction formula a sufficient number of times, integrals of this type can be evaluated. As indicated in the next example, it is important to be aware of basic trigonometric identities when using the reduction formulas for trigonometric functions. The appropriate identity here is $\tan^2 x + 1 = \sec^2 x$ and the reduction formula for secant is integral formula 65.

$$\begin{aligned}
\int \sec^3 x \tan^2 x\, dx &= \int \sec^3 x\,(\sec^2 x - 1)\, dx \\
&= \int \sec^5 x\, dx - \int \sec^3 x\, dx \\
&= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx - \int \sec^3 x\, dx \\
&= \frac{1}{4}\sec^3 x \tan x - \frac{1}{4}\int \sec^3 x\, dx \\
&= \frac{1}{4}\sec^3 x \tan x - \frac{1}{4}\Big(\frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx\Big) \\
&= \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| + C.
\end{aligned}$$

Note that the reduction formula was not used on both integrals in the third step. Since the integrals in the following step can be combined, this ends up saving some work.

Exercises

1. Evaluate each of the following integrals using the table of integrals in Appendix B. Indicate by number the formula that was used.

a) $\displaystyle\int x^2\sqrt{4+x^2}\, dx$ **b)** $\displaystyle\int \frac{1}{(25+4x^2)^{3/2}}\, dx$ **c)** $\displaystyle\int \frac{6x}{x^4-9}\, dx$

d) $\displaystyle\int \frac{3}{\sqrt{x^2+4x+8}}\, dx$ **e)** $\displaystyle\int \frac{\sqrt{4x^2-9}}{x^2}\, dx$ **f)** $\displaystyle\int \frac{\sqrt{5-x^2}}{x}\, dx$

g) $\displaystyle\int \frac{\sqrt{e^{2x}-1}}{e^x}\, dx$ **h)** $\displaystyle\int \frac{6}{\sqrt{4-e^{2x}}}\, dx$ **i)** $\displaystyle\int \frac{\cos x \sin 2x}{\sqrt{9-\cos^2 x}}\, dx$

2. Find the area (give the exact value) of the region bounded by the hyperbola $x^2 - y^2 = 9$ and the line $x = 6$.

3. Find the perimeter (give the exact value) of the region bounded by the parabola $y = x^2$ and the line $y = 4x$.

4. Derive the reduction formula for sine. *Hint*: Use integration by parts.

5. Derive the reduction formula for secant. *Hint*: Use integration by parts.

6. Use reduction formulas to evaluate the integrals.

a) $\displaystyle\int \sec^5 x\, dx$ **b)** $\displaystyle\int \cos^4 x\, dx$ **c)** $\displaystyle\int \sec^5 x \tan^2 x\, dx$

d) $\displaystyle\int \sin^4 x \cos^2 x\, dx$ **e)** $\displaystyle\int x^7 e^{-x^2}\, dx$ **f)** $\displaystyle\int (\ln x)^3\, dx$

7. Evaluate $\displaystyle\int_0^{3b} x\sqrt{\frac{3b-x}{x+b}}\, dx$, where b is a positive constant. This integral appears when finding the area of the loop in the folium of Descartes. *Hint*: Try $u = \sqrt{x+b}$.

2.21 Trigonometric substitution

The integration technique known as trigonometric substitution can be used to evaluate integrals involving the quadratic forms a^2+u^2, a^2-u^2, and u^2-a^2. In fact, this technique of integration is how many of the entries under these headings in a table of integrals are obtained; one of the goals of this section is to help explain how some integration formulas were discovered. The idea is to take advantage of the basic trigonometric identities

$$1+\tan^2 x=\sec^2 x, \quad 1-\sin^2 x=\cos^2 x, \quad \text{and} \quad \sec^2 x-1=\tan^2 x.$$

Note that these identities show how a sum of two terms can be transformed into a perfect square. By making an appropriate trigonometric substitution and using one of these identities, the integral is converted into an integral involving trigonometric functions. Assuming this new integral can be evaluated, the original problem has been solved. To illustrate this technique, consider the following example.

$$\int \sqrt{4-x^2}\,dx; \qquad \begin{aligned} &\text{let} && x=2\sin\theta,\\ &\text{then} && dx=2\cos\theta\,d\theta \;\text{ and}\\ & && \sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}=2\cos\theta. \end{aligned}$$

We used a trigonometric identity to change the quantity in the square root into a perfect square. Using the reduction formula for cosine (integral formula 63), it follows that

$$\begin{aligned} \int \sqrt{4-x^2}\,dx &= \int 4\cos^2\theta\,d\theta\\ &= 2\sin\theta\cos\theta+2\theta+C\\ &= \frac{x}{2}\cdot\sqrt{4-x^2}+2\arcsin(x/2)+C. \end{aligned}$$

It is important to remember to convert your answer back to the original variable. In this case, the relationships between $\sin\theta$ and x, and $\cos\theta$ and x have already been noted in the above substitution:

$$x=2\sin\theta \;\Rightarrow\; \sin\theta=\frac{x}{2} \;\text{ and }\; \theta=\arcsin(x/2); \quad 2\cos\theta=\sqrt{4-x^2}.$$

In other problems (see the example at the end of the section), it may be necessary to use a right triangle and/or inverse trigonometric functions in order to find the appropriate relationships.

The basic idea behind **trigonometric substitution** is to use a trigonometric identity to reduce the quadratic form to a single trigonometric function. In the previous example, the quadratic form $\sqrt{4-x^2}$ was reduced to $2\cos\theta$ after the substitution $x=2\sin\theta$ was made. The appropriate substitution depends on the form of the quadratic:

$$\text{if the integral contains } \begin{cases} a^2-u^2\\ u^2+a^2\\ u^2-a^2 \end{cases}, \quad \text{then make the substitution } \begin{cases} u=a\sin\theta\\ u=a\tan\theta\\ u=a\sec\theta \end{cases}.$$

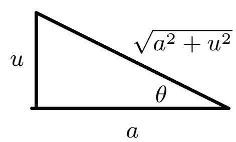
In each case, the substitution and a trigonometric identity turn the quadratic form into a single term. After the substitution, the new integral is a product of trigonometric functions that can often be evaluated using reduction formulas or some other technique.

As mentioned earlier, most of the entries in a table of integrals that involve quadratic forms can be discovered using the technique of trigonometric substitution. As an example, consider the following integral, where a is assumed to be a positive constant.

$$\int \frac{1}{(a^2+u^2)^{3/2}}\,du; \qquad \begin{aligned} \text{let} \quad & u = a\tan\theta, \\ \text{then} \quad & du = a\sec^2\theta\,d\theta \quad \text{and} \\ & (a^2+u^2)^{3/2} = (a^2+a^2\tan^2\theta)^{3/2} = a^3\sec^3\theta \end{aligned}$$

$$\int \frac{1}{(a^2+u^2)^{3/2}}\,du = \int \frac{a\sec^2\theta}{a^3\sec^3\theta}\,d\theta = \frac{1}{a^2}\int \cos\theta\,d\theta = \frac{1}{a^2}\,\sin\theta + C.$$

To express $\sin\theta$ as a function of u, use the fact that $\tan\theta = u/a$ and form a right triangle containing the angle θ:



$$\tan\theta = \frac{u}{a}; \qquad \sin\theta = \frac{u}{\sqrt{a^2+u^2}}.$$

It follows that

$$\int \frac{1}{(a^2+u^2)^{3/2}}\,du = \frac{1}{a^2}\,\sin\theta + C = \frac{u}{a^2\sqrt{a^2+u^2}} + C.$$

We have thus "discovered" integral formula 41 in the table of integrals.

One final comment is worth making. When performing trigonometric substitution, be very careful to include the differential terms dx or du. As seen in the above examples, the terms dx and du are non-trivial functions of θ and their omission introduces a large error into the early stages of an integration problem.

Exercises

1. Use trigonometric substitution to evaluate each of the following integrals.

a) $\displaystyle\int \frac{\sqrt{16-x^2}}{x}\,dx$ **b)** $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$ **c)** $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

d) $\displaystyle\int \frac{8}{(x^2+1)^3}\,dx$ **e)** $\displaystyle\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$ **f)** $\displaystyle\int \frac{x^2}{(1-x^2)^{5/2}}\,dx$

2. Use trigonometric substitution to evaluate each of the following integrals, where a is a positive constant. Use the table of integrals in Appendix B to check your answers.

a) $\displaystyle\int \sqrt{a^2+u^2}\,du$ **b)** $\displaystyle\int \sqrt{a^2-u^2}\,du$ **c)** $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du$

d) $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du$ **e)** $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du$ **f)** $\displaystyle\int \frac{u^3}{\sqrt{u^2-a^2}}\,du$

2.22 Integrating rational functions

In this section, we will indicate how to find an antiderivative of a rational function (a function that can be expressed as the ratio of two polynomials). To motivate the main idea, note that

$$\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x+2}=\frac{2x(x+2)}{x^2(x+2)}-\frac{x+2}{x^2(x+2)}+\frac{5x^2}{x^2(x+2)}=\frac{7x^2+3x-2}{x^3+2x^2}.$$

It follows that

$$\int \frac{7x^2+3x-2}{x^3+2x^2}\,dx=\int\Big(\frac{2}{x}-\frac{1}{x^2}+\frac{5}{x+2}\Big)\,dx=2\ln|x|+\frac{1}{x}+5\ln|x+2|+C.$$

Thus, a complicated looking rational function can be antidifferentiated by first writing it as a sum of simple fractions. In this example, we started with the simple fractions and added them together to obtain a rational function. It is a more difficult problem to begin with the rational function and break it into simpler fractions. Writing a rational function as a sum of simpler fractions is known as **partial fraction decomposition**.

As an example of this method for evaluating integrals, consider the following integration problem:

$$\int \frac{5x^3+x^2+x+2}{x^4+x^2}\,dx.$$

To find the partial fraction decomposition of the integrand, write

$$\frac{5x^3+x^2+x+2}{x^4+x^2}=\frac{5x^3+x^2+x+2}{x^2(x^2+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1},$$

for constants A, B, C, and D. Determining the general form of the partial fraction decomposition will be discussed in a moment; for now just accept this form as a reasonable sum to give the correct denominator. Clearing the denominators, we find that

$$\begin{aligned}5x^3+x^2+x+2&=Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2\\&=(A+C)x^3+(B+D)x^2+Ax+B.\end{aligned}$$

The only way that these two polynomials can be equal for all values of x is for the coefficients to be identical. Thus, $A+C=5$, $B+D=1$, $A=1$, and $B=2$. It follows that

$$\frac{5x^3+x^2+x+2}{x^4+x^2}=\frac{1}{x}+\frac{2}{x^2}+\frac{4x-1}{x^2+1},$$

(you can always check these answers by adding the fractions) and

$$\begin{aligned}\int \frac{5x^3+x^2+x+2}{x^4+x^2}\,dx&=\int\Big(\frac{1}{x}+\frac{2}{x^2}+\frac{4x}{x^2+1}-\frac{1}{x^2+1}\Big)\,dx\\&=\ln|x|-\frac{2}{x}+2\ln|x^2+1|-\arctan x+C.\end{aligned}$$

Note that this integral problem involves more algebra than it does calculus.

As is evident in this example, the difficult part of integrating a rational function is determining its partial fraction decomposition. The following steps will lead to the partial fraction decomposition of a rational function that is in proper form, that is, the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. (If this is not the case, perform long division and work with the remainder.)

1) Factor the denominator of the rational function into a product of linear terms and irreducible quadratic terms. (An **irreducible quadratic** is a second degree polynomial with no real roots.) The Fundamental Theorem of Algebra guarantees that such a factorization is always possible.

2) For each linear factor of the form $(ax+b)^n$, put

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

into the partial fraction decomposition.

3) For each irreducible quadratic factor of the form $(ax^2+bx+c)^n$, put

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

into the partial fraction decomposition.

For example, the partial fraction decomposition of the rational function

$$\frac{x^5+3x^2-6}{(x+2)(2x-3)^3(x^2+x+1)^2}$$

is given by

$$\frac{A_1}{x+2} + \frac{A_2}{2x-3} + \frac{A_3}{(2x-3)^2} + \frac{A_4}{(2x-3)^3} + \frac{A_5x+A_6}{x^2+x+1} + \frac{A_7x+A_8}{(x^2+x+1)^2}.$$

A proof of the existence and uniqueness of partial fraction decompositions (which really is an algebra problem and has nothing to do with calculus) is beyond the scope of this book, but the form of the decomposition will make more sense after you have worked several problems. The constants that appear in the partial fraction decomposition can be determined as in the example.

Exercises

1. Evaluate each of the following integrals.

a) $\displaystyle\int \frac{1}{x^2+5x}\,dx$ **b)** $\displaystyle\int \frac{2x+3}{x^2-3x-4}\,dx$ **c)** $\displaystyle\int \frac{x+10}{2x^2+5x-3}\,dx$

d) $\displaystyle\int \frac{4}{x^3-3x^2+2x}\,dx$ **e)** $\displaystyle\int \frac{x^2+2x+3}{x^3-2x^2+x}\,dx$ **f)** $\displaystyle\int \frac{6x^2+7x+20}{x^3+6x^2+10x}\,dx$

g) $\displaystyle\int \frac{x+4}{x^3+x}\,dx$ **h)** $\displaystyle\int \frac{1}{x^4-1}\,dx$ **i)** $\displaystyle\int \frac{2x^3+8x^2+2x+23}{x^4+5x^2+4}\,dx$

j) $\displaystyle\int \frac{12}{x^3-8}\,dx$ **k)** $\displaystyle\int \frac{x^3}{2x^2+3x-2}\,dx$ **l)** $\displaystyle\int \frac{2x+5}{(x^2+2x+2)^3}\,dx$

2.23 Numerical integration

The Fundamental Theorem of Calculus provides a way to evaluate definite integrals without resorting to the definition of the integral. However, if an antiderivative of the integrand cannot be found (which occurs regularly in applications), then it is necessary to consider the definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,(a_i - a_{i-1}) \approx \frac{b-a}{n} \sum_{i=1}^{n} f(t_i),$$

where t_i is any point in the interval $[a_{i-1}, a_i]$ (see Section 2.5). For large values of n, the sum should be a reasonable approximation for the value of the integral. Some more efficient ways of using this idea to approximate the value of an integral will be discussed in this section.

Suppose that f is a continuous function defined on an interval $[a, b]$. Let n be a positive integer, let $x_i = a + i\,(b-a)/n$ and $y_i = f(x_i)$ for $i = 0, 1, 2, \ldots, n$, and let $c_i = (x_{i-1} + x_i)/2$ for $i = 1, 2, \ldots, n$. Then each of the following represent an approximation to $\int_a^b f(x)\,dx$; the approximations generally become more accurate as n increases.

$$R_n = \frac{b-a}{n}\big(y_1 + y_2 + \cdots + y_{n-1} + y_n\big) \quad \textbf{right endpoint rule}$$

$$L_n = \frac{b-a}{n}\big(y_0 + y_1 + \cdots + y_{n-2} + y_{n-1}\big) \quad \textbf{left endpoint rule}$$

$$M_n = \frac{b-a}{n}\big(f(c_1) + f(c_2) + \cdots + f(c_n)\big) \quad \textbf{midpoint rule}$$

$$T_n = \frac{b-a}{2n}\big(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\big) \quad \textbf{trapezoid rule}$$

$$S_n = \frac{b-a}{3n}\big(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\big) \quad \textbf{Simpson's rule}\ (n \text{ must be even})$$

The first three rules follow immediately from the definition by choosing some routine values for t_i. It is easy to verify that $T_n = \frac{1}{2}(L_n + R_n)$, the average of the left and right endpoint estimates. A sketch of a graph of a continuous nonnegative function shows that this average is usually a much better approximation to the area under the curve; it also reveals that T_n corresponds to approximating the area under the curve using trapezoids. With a little care (see the exercises), it can be shown that $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$, a weighted average of T_n and M_n. Geometrically, Simpson's rule is obtained by using parabolas to approximate the function rather than lines. It follows that Simpson's rule is usually the most accurate of the five rules. The formula for Simpson's rule hinges on the following remarkable property of quadratic functions: if $Q(x) = px^2 + qx + r$, then

$$\int_a^b Q(x)\,dx = \frac{Q(a) + 4Q(c) + Q(b)}{6}\,(b-a),$$

where c is the midpoint of the interval $[a, b]$. In other words, the integral of a quadratic function is determined by its values at the endpoints and at the midpoint of the interval. A proof of this fact, as well as how it is used to generate Simpson's rule, will be left to the exercises.

To get a sense for these formulas, consider the integral $\int_0^2 \sqrt{1+x^3}\,dx$ and take n to be 6. Then

i	0	1	2	3	4	5	6
x_i	0	1/3	2/3	1	4/3	5/3	2
y_i	1	1.01835	1.13855	1.41421	1.83586	2.37268	3

The formulas for the trapezoid rule and Simpson's rule yield

$$T_6 = \frac{2}{2\cdot 6}\big(1\cdot 1 + 2\cdot 1.01835 + 2\cdot 1.13855 + 2\cdot 1.41421 + 2\cdot 1.83586 + 2\cdot 2.37268 + 1\cdot 3\big) \approx 3.2600;$$
$$S_6 = \frac{2}{3\cdot 6}\big(1\cdot 1 + 4\cdot 1.01835 + 2\cdot 1.13855 + 4\cdot 1.41421 + 2\cdot 1.83586 + 4\cdot 2.37268 + 1\cdot 3\big) \approx 3.2411.$$

The actual value of this integral is known to be 3.2413, accurate to 4 decimal places. This shows that Simpson's rule gives good accuracy without too much computation. If a much better approximation is required, a calculator or computer can be programmed to perform the routine calculations using a large value for n. Since Simpson's rule requires almost identical computations as the trapezoid rule and because for a given value of n it is usually much more accurate, it is the clear choice when an approximation to an integral is required. (Actually, there are other numerical procedures that are used to approximate the value of an integral. Questions concerning the accuracy of numerical estimations of many different types, including the theory behind the analysis of numerical algorithms, are addressed in a branch of mathematics called **numerical analysis**. Since most numerical algorithms require the use of a computer, numerical analysis must consider the methods of computation (such as evaluation techniques and round-off errors) as well as the mathematical theory of approximation.)

Exercises

1. For the given integral and value of n, find T_n and S_n accurate to four decimal places.

a) $\int_1^2 e^{-x^2}\,dx$, $n=8$ **b)** $\int_0^1 \sin x^2\,dx$, $n=4$ **c)** $\int_{0.4}^1 \frac{dx}{\sqrt{1+x^3}}$, $n=6$

2. Use Simpson's rule with $n=8$ to approximate $\int_0^1 \frac{\sin x}{x}\,dx$. What value should the integrand have at 0?

3. Use Simpson's rule with $n=6$ to approximate the length of the sine curve on the interval $[0,\pi]$.

4. Consider the following table of values for a function f:

x	1	2	3	4	5
$f(x)$	2	3	5	7	2

Assume that f is a nonnegative continuous function defined on $[1,5]$ and let R be the region under the curve $y=f(x)$ and above the x-axis on the interval $[1,5]$. Use Simpson's rule to approximate (a) the area of R, (b) the volume generated when R is revolved around the x-axis, and (c) the center of mass of R.

5. Suppose that the following table represents the velocity of a particle moving in a straight line.

t (sec)	0	1	2	3	4	5	6
v (m/sec)	0	5	12	15	14	6	0

Use Simpson's rule to approximate the distance traveled by the particle.

6. Prove that $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$ for each positive integer n.

7. Prove the property of quadratic functions stated in this section.

8. Use the property of quadratic functions stated in this section to derive Simpson's rule.

2.24 Supplementary Exercises

1. Find each of the following sums. Simplify your answers.

a) $\sum_{i=1}^{n}(2i-1)$ **b)** $\sum_{i=1}^{n}(2i-1)^2$ **c)** $\sum_{i=1}^{n}(2i-1)^3$

2. Evaluate the limits: $\lim_{n\to\infty}\dfrac{4n^3}{1^2+2^2+3^2+\cdots+n^2}$ and $\lim_{n\to\infty}\dfrac{(n+1)+(n+2)+(n+3)+\cdots+2n}{n^2}$.

3. Here is an unlikely application to bowling alleys.

a) Suppose that bowling pins are lined up in the customary triangular pattern except that there are 100 rows instead of the usual 4 rows. How many bowling pins are required?

b) Suppose that bowling balls are stacked up in a rather dangerous square pyramid formation that is 60 balls high. How many bowling balls are in the stack?

4. Use the definition of integral to evaluate $\int_1^2 (2x^2-x+3)\,dx$.

5. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\dfrac{2(n+2i)^2}{n^3}$.

6. Evaluate $\int_4^{10}|4t-24|\,dt$.

7. Use a property of integrals to determine the largest integral in each pair. Justify your answers.

a) $\int_2^3\sqrt{x^6-1}\,dx,\ \int_2^3 x^3\,dx$ **b)** $\int_\pi^{2\pi}\sin^3 x\,dx,\ \int_\pi^{2\pi}\sin^5 x\,dx$ **c)** $\int_0^1 e^x\,dx,\ \int_0^1 e^{x^2}\,dx$

8. Find the point c guaranteed by the Mean Value Theorem for integrals.

a) $f(x)=x^3,\ [1,2]$ **b)** $f(x)=x,\ [a,b]$ **c)** $f(x)=x^2,\ [a,b]$

9. Find the derivative of each of the following functions.

a) $f(x)=\int_0^x\dfrac{4t}{\sqrt{t^4+2}}\,dt$ **b)** $g(x)=\int_3^{4x}\cos(t^2)\,dt$ **c)** $h(t)=\int_{-2}^{2t^2}e^{-x^2/2}\,dx$

10. Evaluate $\lim_{x\to 0}\dfrac{1}{x^3}\int_0^x\left(e^{2t^2}-1\right)dt$.

11. Determine the interval on which the function f defined by $f(x)=\int_0^x\sqrt{4t^2-2t+7}\,dt$ is concave down.

12. Find an integral expression for a function f such that $f(1)=0$ and $f'(x)=e^{x^2/4}$.

13. Evaluate $\int e^x\sin x\,dx$ in the following way. Use integration by parts with $u=e^x$ and note that the new integral is not really any easier than the original one. Then use integration by parts on the new integral, again with $u=e^x$. It may seem that you are going in circles, but if you write out the equation you have obtained, you will see that it can be solved for the function $\int e^x\sin x\,dx$.

14. Repeat the steps in the previous exercise to evaluate $\int e^x\cos x\,dx$.

15. Find the area of the region bounded by the given curves. The x-coordinates of the points of intersection are integers.

a) $y=12/x,\ y=3+12x-3x^2$ **b)** $y=x^4,\ y=2/(1+x^2)$ **c)** $y=x^3,\ y=32\sqrt{x}$

16. Find the area of the region bounded by the curves $y=\sec x$ and $y=\sqrt{2}$ on the interval $(-\pi/2,\pi/2)$.

17. Let R be the region under the curve $y=4xe^{-x/2}$ and above the x-axis on $[0,\infty)$. Find the area of R.

18. Let R be the region under the curve $y=4/x$ and above the x-axis on the interval $[1,4]$. Find (a) a vertical line and (b) a horizontal line that divides the region R into two pieces of equal area.

19. Consider the region that lies under the graph of $y=2x-x^2$ and above the x-axis. Find a line of the form $y=mx$ that divides this region into two parts of equal area.

20. Find the center of mass of the following collection of masses: a mass of 3 is located at $(3,2)$, a mass of 2 is located at $(2,-1)$, a mass of 4 is located at $(5,-2)$, and a mass of 1 is located at $(-2,0)$.

21. Let R be the region under the curve $y=4/x^3$ and above the x-axis on the interval $[1,\infty)$. Find the center of mass of R. Is the center of mass of R in the region R?

22. Let T be the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,2)$. Suppose that the density of this region (in units of mass per area) is $\rho(x) = 1 + 5x$ at a distance x units from the y-axis. Find the center of mass of T.

23. Let R be the region under the graph of $y = e^x$ and above the x-axis on the interval $[0,2]$. Find the center of mass of the solid that is generated when R is revolved around the x-axis.

24. Let R be the region under the graph of $y = 4 - x^2$ and above the x-axis on the interval $[0,2]$ and let S be the solid that is generated when R is revolved around the y-axis. Find the center of mass of S.

25. Let f be a continuous nonnegative function defined on an interval $[a,b]$ and let R be the region under the graph of $y = f(x)$ and above the x-axis on the interval $[a,b]$. Show that

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \text{and} \qquad \overline{y} = \frac{1}{2A}\int_a^b \big(f(x)\big)^2\,dx,$$

where A is the area of the region R. Assume that the region has constant density ρ.

26. Use the previous exercise to find the center of mass of the region under the graph of $y = \sin x$ on the interval $[0,\pi]$.

27. Derive the reduction formula for tangent.

28. Use reduction formulas to evaluate the integrals.

a) $\displaystyle\int \sin^5 x\,dx$ **b)** $\displaystyle\int x^3 e^x\,dx$ **c)** $\displaystyle\int \tan^6 x\,dx$

d) $\displaystyle\int x^2 e^{-x}\,dx$ **e)** $\displaystyle\int x^2 e^{\sqrt{x}}\,dx$ **f)** $\displaystyle\int_1^e (\ln x)^4\,dx$

29. Use Simpson's rule with $n = 2$ to approximate $\displaystyle\int_2^6 4x^3\,dx$, then compare the approximation with the exact value of the integral. You should notice something interesting.

30. Show that $\displaystyle\int_a^b x^3\,dx = \frac{a^3 + 4c^3 + b^3}{6}(b-a)$, where c is the midpoint of the interval $[a,b]$. What does this result imply about Simpson's rule and cubic polynomials?

31. For the integral $\displaystyle\int_0^1 \sqrt{1+x^2}\,dx$, find M_2, M_4, T_2, T_4, S_2, and S_4 and compare all of these values to the actual value of the integral. Search the errors carefully and look for any patterns that you can find.

32. Suppose that the circle given by the equation $x^2 + y^2 = 6y - 2x$ is revolved around the line $x - y = 6$. Find the volume of the "donut" that is generated.

33. Suppose that the water is four feet deep in a hemispherical tank (flat side up) that has a radius of five feet. How many gallons of water are in the tank? One gallon is approximately 231 cubic inches.

34. Let R be the region under the curve $y = 6e^{-x/3}$ and above the x-axis on the interval $[0,\infty)$. Find the volume of the solid that is generated when R is revolved around the x-axis.

35. Evaluate each of the following integrals. You should not need a table of integrals.

a) $\displaystyle\int \frac{1}{9+x^2}\,dx$ **b)** $\displaystyle\int \frac{x}{9+x^2}\,dx$ **c)** $\displaystyle\int \frac{x^3}{9+x^2}\,dx$

d) $\displaystyle\int x^2\sqrt{x+2}\,dx$ **e)** $\displaystyle\int xe^{x^2}\,dx$ **f)** $\displaystyle\int x^2e^x\,dx$

g) $\displaystyle\int_0^1 \left(\sqrt{x}+3\right)^2 dx$ **h)** $\displaystyle\int_0^4 x\sqrt{16-x^2}\,dx$ **i)** $\displaystyle\int_0^4 (3x+5)\sqrt{16-x^2}\,dx$

j) $\displaystyle\int \frac{x+2}{(x^2+4x+9)^3}\,dx$ **k)** $\displaystyle\int 2x^3\sqrt[3]{x^4+6}\,dx$ **l)** $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$

m) $\displaystyle\int x\csc^2 x\,dx$ **n)** $\displaystyle\int \sin 2x\cos^3 2x\,dx$ **o)** $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

p) $\displaystyle\int_1^\infty \frac{x^2+2}{x^5}\,dx$ **q)** $\displaystyle\int_0^\infty xe^{-x/5}\,dx$ **r)** $\displaystyle\int_{-1}^2 \frac{1}{x^2}\,dx$

s) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$ **t)** $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$ **u)** $\displaystyle\int \frac{e^{2x}}{e^x+1}\,dx$

36. Let R be the region under the graph of $y = x^2$ and above the x-axis on the interval $[0,2]$. Find the exact value of each of the following.

a) the area of R

b) the volume of the solid generated when R is revolved around the x-axis

c) the volume of the solid generated when R is revolved around the y-axis

d) the volume of the solid generated when R is revolved around the line $x = 5$

e) the volume of the solid generated when R is revolved around the line $x = -3$

f) the volume of the solid generated when R is revolved around the line $y = 4$

g) the x-coordinate of the center of mass of R

h) the y-coordinate of the center of mass of R

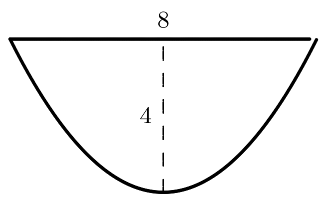
i) the x-coordinate of the center of mass of the solid generated when R is revolved around the x-axis

j) the perimeter of R

k) the force on one side of R if R is a submerged vertical plate with the fluid (weight density w) level at $y = 8$

37. Let R be the region under the graph of $y = \sin x$ and above the x-axis on the interval $[0, \pi]$. Repeat the previous exercise for this region R. You will need to approximate the value for part (j).

38. Suppose that the figure below represents a two-dimensional region with a parabola forming its curved boundary.



a) Find the area of this region.

b) Find the center of mass of this region. Specify its location carefully.

c) Find the force exerted by water on one side of this region, assuming it is a vertically submerged plate (assume the units are feet) with the top of the plate six feet below the surface of the water. Use $w = 62.4$ lb/ft^3.

39. Find the volume of the solid that is generated when the region on one side of a chord in a circle of radius r is revolved around the chord. Assume that the height of the chord is a, where $0 < a < r$.

3

Sequences and Series

For any real number $x \neq 1$ and any positive integer n, elementary algebra yields

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(To check this, you can either do long division on the fraction or multiply both sides by $1 - x$.) If $|x| < 1$, then $|x|^{n+1}$ gets closer to 0 as n increases, and it is tempting to write

(1) $$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

This equation asserts that it is possible to add an infinite number of numbers and find the exact value of the sum. For example, when $x = 0.1$, it asserts that $1 + 0.1 + (0.1)^2 + (0.1)^3 + \cdots = 10/9$, a believable result since the decimal expansion of $1/9$ is $0.1111\ldots$. Assuming that (1) is valid for $|x| < 1$, integration yields

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = -\ln|1 - x|,$$

which expresses the natural logarithm function as an infinite degree polynomial. If we substitute -1 for x (even though it is out of the specified range), we obtain

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

This is certainly a new way to look at logarithms. Replacing x with $-x^2$ in (1) and integrating gives an expression for $\arctan x$ as well as an interesting formula for π:

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2};$$
$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \arctan x;$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

(What value of x was used to get the formula for $\pi/4$?) Are these equations valid? We know that the integral of a finite sum of functions is the sum of the individual integrals (see property (3) of integrals in Section 2.4). Is the integral of a sum the sum of the integrals even when there are an infinite number of terms? Is it really possible to represent functions such as $\ln|1-x|$ and $\arctan x$ as infinite degree polynomials? Can all functions be represented in this way? Are there new functions that we have never seen before that can be represented in this way? These sorts of questions provide the motivation for the concepts considered in this chapter.

To obtain a better sense of these new kinds of equations, let's think about decimal expansions for a moment. The number $1/3$ is equal to the infinite decimal $0.33333\ldots$. In other words,

$$\frac{1}{3} = 3\cdot\frac{1}{10} + 3\cdot\frac{1}{10^2} + 3\cdot\frac{1}{10^3} + 3\cdot\frac{1}{10^4} + \cdots.$$

The irrational number $\sqrt{2}$ also has a decimal expansion, but there is no repeating pattern to its digits:

$$\sqrt{2} = 1.414213562\ldots = 1 + 4\cdot\frac{1}{10} + 1\cdot\frac{1}{10^2} + 4\cdot\frac{1}{10^3} + 2\cdot\frac{1}{10^4} + \cdots.$$

However, even if we do not know all of the digits, we do know that every real number x can be written as

$$x = n + d_1\cdot\frac{1}{10} + d_2\cdot\frac{1}{10^2} + d_3\cdot\frac{1}{10^3} + d_4\cdot\frac{1}{10^4} + \cdots,$$

where n is an integer and the digits d_i are integers between 0 and 9 inclusively. The calculations on the previous page seem to indicate that a function f can be written as

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots$$

for values of x in an appropriate interval, where the coefficients a_i are real numbers. Hence, in some sense, functions are determined by a sequence of numbers in much the same way that numbers are determined by a sequence of digits. Recognizing that many functions (however, as we will see, not all functions) can be represented in this way is extremely important in both pure and applied mathematics.

This chapter is more abstract than the first two chapters of this text. Many students find infinite series to be the most challenging topic in calculus. You will need to focus more on ideas and concepts rather than on calculations. Consequently, the way you study mathematics may need to adapt as you work through this chapter. These sentences are not intended to scare you, but rather to alert you to the fact that the nature of the course is about to change. The reward for this extra effort is an increased ability to reason abstractly and an opportunity to catch a glimpse of some fascinating aspects of mathematics. We begin our study of infinite series with a proof technique known as mathematical induction.

3.1 Mathematical Induction

Statements of the form, "for each positive integer n, something is true", occur in all branches of mathematics. Two simple examples are

1. For each positive integer n, the number $n^2 - n + 41$ is a prime number.

2. For each positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

To prove that statements such as these are false, it is only necessary to find one positive integer n for which the statement is false. For instance, statement (1) is false; you should be able to find a positive integer n for which $n^2 - n + 41$ is not a prime number. However, it is not possible to prove such statements are true by showing that they are true for several values of n; the formulas or statements must somehow be verified for every positive integer n. Since it is not possible to actually prove individually an infinite number of statements, some other method of proof is needed. The Principle of Mathematical Induction is a useful tool for proving some statements of this type. This important property is stated below. (Recall that the symbol $\mathbb{Z}^+$ represents the set of positive integers and that the symbols $a \in S$ mean that a is a member of the set S.)

Principle of Mathematical Induction: If S is a set of positive integers that contains 1 and satisfies the condition "if $k \in S$, then $k+1 \in S$", then $S = \mathbb{Z}^+$.

The Principle of Mathematical Induction can be compared to a chain reaction. If we know that each event will set off the next (the condition in quotes) and if the first event occurs ($1 \in S$), then the entire chain reaction will occur. Perhaps you have seen one of those amazing domino exhibits where thousands of dominoes fall over in interesting patterns. The dominoes must be set up in such a way that each one knocks over the next ("if $k \in S$, then $k+1 \in S$"), and someone must begin the process by pushing over the first domino (S contains 1).

Given a statement of the form "for each positive integer n, something is true", let S be the set of all positive integers n for which the statement is true. In order to prove that the statement is true for all positive integers, we must show that $S = \mathbb{Z}^+$. By the Principle of Mathematical Induction, it is sufficient to prove that S contains 1 and satisfies the condition "if $k \in S$, then $k+1 \in S$". In almost every situation of this type, the statement is easy to prove for $n = 1$. However, a proof that $k+1 \in S$ under the assumption that $k \in S$ requires more effort. For many of the induction arguments in calculus, the proof of this implication involves some form of algebraic manipulation. Two examples of such proofs are given below. The first proof is written rather formally, while the second is written in a more informal style. The second form is more common, but you may use whichever style you prefer.

THEOREM 3.1 For each positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. We will use the Principle of Mathematical Induction. Let S be the set of all positive integers n such that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since $1 = (1 \cdot 2 \cdot 3)/6$, it follows that $1 \in S$. Suppose that $k \in S$ for some positive integer k. This means that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We then have

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k+1}{6}\big(2k^2 + k + 6k + 6\big) \\ &= \frac{(k+1)(k+2)(2k+3)}{6}, \end{aligned}$$

which indicates that $k+1 \in S$. We have thus shown that "if $k \in S$, then $k+1 \in S$". By the Principle of Mathematical Induction, $S = \mathbb{Z}^+$. Hence,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n. ∎

THEOREM 3.2 For each positive integer n, the integer $9^n - 8n - 1$ is divisible by 64.

Proof. We will use the Principle of Mathematical Induction. Since 0 is divisible by 64, the statement is valid when $n = 1$. Suppose that $9^k - 8k - 1$ is divisible by 64 for some positive integer k. We want to show that $9^{k+1} - 8(k+1) - 1$ is divisible by 64. Since $9^k - 8k - 1$ is divisible by 64, there exists an integer j such that $64j = 9^k - 8k - 1$. We then have

$$9^{k+1} - 8(k+1) - 1 = 9 \cdot 9^k - 8k - 9 = 9(9^k - 1) - 8k = 9(64j + 8k) - 8k = 9(64j) + 64k = 64(9j + k).$$

Since $9j + k$ is an integer, it follows that $9^{k+1} - 8(k+1) - 1$ is divisible by 64. The result now follows by the Principle of Mathematical Induction. ∎

For the record, the Principle of Mathematical Induction can also be stated as follows.

Principle of Strong Mathematical Induction: If S is a set of positive integers that contains 1 and satisfies the condition "if $1, 2, \ldots, k \in S$, then $k+1 \in S$", then $S = \mathbb{Z}^+$.

This stronger form of induction (the statement is assumed to be true for all of the positive integers up to k, not just for k) is needed in some cases. An example of a proof that requires this stronger form of induction follows.

THEOREM 3.3 Suppose that $a_1 = 6$, $a_2 = 18$, and $a_{n+1} = a_n + 6a_{n-1}$ for each positive integer $n > 1$. Then $a_n = 2 \cdot 3^n$ for each positive integer n.

Proof. It is easy to see that the statement is true for $n = 1$ and $n = 2$. (We need to check both of these cases, since these numbers do not fit the general pattern for the generation of terms.) Suppose that $a_i = 2 \cdot 3^i$ for all the integers $i = 1, 2, \ldots, k$ for some positive integer $k \geq 2$. We must show that $a_{k+1} = 2 \cdot 3^{k+1}$. Using the assumption that all the terms up to k satisfy the pattern (all we really need to know is that the pattern is valid for the terms k and $k-1$), we find that

$$a_{k+1} = a_k + 6a_{k-1} = 2 \cdot 3^k + 6(2 \cdot 3^{k-1}) = 2 \cdot 3^k + 4 \cdot 3^k = 6 \cdot 3^k = 2 \cdot 3^{k+1},$$

as desired. By the Principle of Strong Mathematical Induction, the formula $a_n = 2 \cdot 3^n$ is valid for all positive integers n. ■

To prove a statement using the Principle of Mathematical Induction, it is necessary to first verify that the statement is true when $n = 1$ (or some other starting value), then to prove the induction hypothesis "if $k \in S$, then $k+1 \in S$" (or the modified form when using strong induction). In practice, it is usually a good idea to verify the statement for the first few positive integers, say $n = 1, 2, 3, 4$. This often gives you a better understanding of the problem and perhaps some insight into why the "formula" is true, and it increases your confidence that the statement is valid.

Exercises

1. Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for each positive integer n.

2. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for each positive integer n.

3. Prove that $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n \cdot (n+1)} = \dfrac{n}{n+1}$ for each positive integer n.

4. Prove that for each positive integer n, the integer $3^{2n+1} + 2^{n+2}$ is divisible by 7.

5. Let $a_1 = 1$ and $a_{n+1} = 3 - (1/a_n)$ for each integer $n \geq 1$. Prove that $1 \leq a_n \leq 3$ for each positive integer n.

6. Let $b_1 = 1$, $b_2 = 2$, and $b_n = 3b_{n-1} - 2b_{n-2}$ for each positive integer $n > 2$. Prove that $b_n = 2^{n-1}$ for each positive integer n. (You will need the stronger form of induction here.)

7. This exercise refers to the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. These numbers are defined by $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for each $n \geq 2$.

a) Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for each positive integer n.

b) Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for each positive integer n.

c) Prove that $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ for each positive integer n.

d) Prove that $f_1 f_2 + f_2 f_3 + f_3 f_4 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ for each positive integer n.

e) Prove that $f_{n+1} f_{n-1} = f_n^2 + (-1)^n$ for each positive integer $n > 1$.

f) Let $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$; the numbers α and β are the solutions to the equation $x^2 = x + 1$. Prove that $f_n = \left(\alpha^n - \beta^n\right)/\sqrt{5}$ for each positive integer n.

8. The following set of results provides a different way to think of factorials.

a) Use mathematical induction and L'Hôpital's Rule to prove that $\lim\limits_{x \to \infty} x^n e^{-x} = 0$ for all positive integers n.

b) Use part (a) and mathematical induction to prove that $\int_0^\infty x^n e^{-x}\, dx = n!$ for all positive integers n.

c) Use the result from part (b) to explain why $0!$ is defined to be 1.

3.2 Sequences

A **sequence** of real numbers is an ordered list of real numbers that continues indefinitely; one real number for each positive integer. Sequences will be denoted by symbols such as $\{a_n\}$, where it is assumed that n runs through the positive integers in increasing order: $\{a_n\} = a_1, a_2, a_3, a_4, a_5, a_6, \ldots$. The real numbers a_n are called the **terms** of the sequence and the letter n is called the **index** of the sequence. A sequence can often be represented as a function of n:

$$\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \qquad \left\{\frac{n}{2n+1}\right\} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \ldots$$

$$\{(-1)^n\} = -1, 1, -1, 1, -1, 1, \ldots \qquad \{n^2\} = 1, 4, 9, 16, 25, 36, \ldots$$

Sometimes a sequence is defined by writing out enough terms to generate a pattern; it is assumed that the pattern continues indefinitely. For example, $1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots$ represents a sequence.

It is useful to have some descriptive terminology to describe various types of sequences.

DEFINITION 3.4 Let $\{a_n\}$ be a sequence of real numbers.

a) $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \le M$ for all n.

b) $\{a_n\}$ is **bounded below** if there is a number m such that $a_n \ge m$ for all n.

c) $\{a_n\}$ is **bounded** if there is a number M such that $|a_n| \le M$ for all n.

d) $\{a_n\}$ is **increasing** if $a_{n+1} \ge a_n$ for all n.

e) $\{a_n\}$ is **decreasing** if $a_{n+1} \le a_n$ for all n.

f) $\{a_n\}$ is **monotone** if it is either increasing or decreasing.

g) $\{a_n\}$ is **convergent** if there is a number a such that $\lim_{n\to\infty} a_n = a$. This means that for each $\epsilon > 0$ there exists a positive integer N such that $|a_n - a| < \epsilon$ for all $n \ge N$.

For example, the sequence $\{1/n\}$ is bounded below by 0, the sequence $\{(-1)^n\}$ is bounded by 1, the sequence $\{n^2\}$ is increasing, and the sequence $\{n/(2n+1)\}$ converges to $\frac{1}{2}$. Informally, a sequence $\{a_n\}$ converges to a if the terms get closer to a as you go farther out in the sequence. Part (g) of the definition makes this somewhat vague idea precise by carefully quantifying the words 'closer' and 'farther'. For the most part, it is possible for a sequence to have one of the properties listed in the definition without having the others. (Some examples will be requested in the exercises.) However, one connection between these properties is given by the following theorem.

THEOREM 3.5 A convergent sequence is bounded.

Proof. Let $\{a_n\}$ be a sequence that converges to a. By the definition of convergent sequence, there is a positive integer N such that $|a_n - a| < 1$ for all $n \ge N$. This inequality is equivalent to the statement $|a_n| < 1 + |a|$ for all $n \ge N$. Letting $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a|\}$, it then follows that $|a_n| \le M$ for all n. Hence, the sequence $\{a_n\}$ is bounded. ∎

For simple sequences defined as a function of n, it is often easy to find the limit. For example, the sequence $\{n/(6n-5)\}$ converges to $1/6$ since the constant 5 becomes insignificant when n is large:

$$\lim_{n\to\infty}\frac{n}{6n-5} = \lim_{n\to\infty}\left(\frac{n}{6n-5}\cdot\frac{1/n}{1/n}\right) = \lim_{n\to\infty}\frac{1}{6-(5/n)} = \frac{1}{6}.$$

(If you can recognize these limits without including the extra steps, then you are encouraged to do so as long as you can explain your reasoning if necessary.) Sometimes it is helpful to convert a sequence into a function then use the properties of functions to analyze the sequence. To obtain a function from the sequence $\{a_n\}$, choose a function f defined for all $x \geq 1$ such that $f(n) = a_n$ for each positive integer n. A plot of the sequence $\{a_n\}$ can be represented as a collection of dots; one dot for each positive integer n. The graph of the corresponding function f is a curve that goes through all of the dots. If f is increasing on the interval $[1,\infty)$, then $\{a_n\}$ is an increasing sequence; if $\lim_{x\to\infty} f(x) = L$, then the sequence $\{a_n\}$ converges to L. It is then possible to use calculus to analyze the function f and thus obtain information about the sequence $\{a_n\}$. For the sequence considered above, the function would be $f(x) = x/(6x-5)$. Since

$$f'(x) = \frac{(6x-5)\cdot 1 - x\cdot 6}{(6x-5)^2} = -\frac{5}{(6x-5)^2},$$

the function f is decreasing on $[1,\infty)$. This means that the sequence $\{n/(6n-5)\}$ is decreasing.

Exercises

1. Give examples of sequences that have the given properties: (a) monotone but not bounded; (b) bounded but not monotone; (c) monotone but not convergent; (d) convergent but not monotone; (e) bounded but not convergent; (f) both increasing and decreasing; (g) neither bounded above nor bounded below.

2. Find the first five terms of the given sequence.

   a) $\left\{\dfrac{2+(-1)^n}{3n}\right\}$ **b)** $\left\{\dfrac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)}{2^n}\right\}$ **c)** $\left\{\displaystyle\sum_{k=1}^{n}\frac{k+1}{k!}\right\}$

3. Represent the given sequence using a formula.

   a) $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \ldots$ **b)** $2, 0, 2, 0, 2, 0, \ldots$ **c)** $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \ldots$

4. Find the limit of the sequence.

   a) $\left\{\dfrac{2n+1}{7n-3}\right\}$ **b)** $\left\{\dfrac{n^2+2n+2}{2-3n^2}\right\}$ **c)** $\left\{\dfrac{k}{\sqrt{4k^2+3k+1}}\right\}$

   d) $\left\{\sqrt{n+5}-\sqrt{n}\right\}$ **e)** $\left\{\sqrt{k^2+k}-k\right\}$ **f)** $\left\{\dfrac{\ln n}{n}\right\}$

   g) $\left\{k\sin(\pi/k)\right\}$ **h)** $\left\{k\left(\sqrt[k]{2}-1\right)\right\}$ **i)** $\left\{\left(\dfrac{n}{n+1}\right)^n\right\}$

5. A sequence $\{x_n\}$ of positive numbers is increasing if $x_{n+1}/x_n \geq 1$ for all n and decreasing if $x_{n+1}/x_n \leq 1$ for all n. Use this fact to determine if the given sequence is increasing or decreasing.

   a) $\left\{\dfrac{n}{3n+1}\right\}$ **b)** $\left\{\dfrac{2^n}{n!}\right\}$ **c)** $\left\{\dfrac{(2n)!}{(n!)^2}\right\}$

6. Prove that the sequence $\left\{1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}\right\}$ is unbounded. (You may use the following fact: if $\{a_n\}$ is an unbounded sequence of positive numbers and $b_n \geq a_n$ for all n, then $\{b_n\}$ is unbounded.)

3.3 Properties of Sequences

Given two sequences $\{a_n\}$ and $\{b_n\}$, it is possible to perform algebraic operations on their terms and form new sequences. If the two sequences converge, then the new sequences converge as well and have the expected limits. Since the proofs of the results recorded in the following two theorems involve the abstract definition of a convergent sequence, they will be omitted. However, the results should be intuitively clear.

THEOREM 3.6 If $\{a_n\}$ converges to a, $\{b_n\}$ converges to b, and c is a constant, then

a) the sequence $\{c\,a_n\}$ converges to $c\,a$;

b) the sequence $\{a_n + b_n\}$ converges to $a + b$;

c) the sequence $\{a_n - b_n\}$ converges to $a - b$;

d) the sequence $\{a_n b_n\}$ converges to ab;

e) the sequence $\{a_n/b_n\}$ converges to a/b, assuming that $b \neq 0$ and $b_n \neq 0$ for all n. ■

THEOREM 3.7 **Squeeze Theorem** If $a_n \le x_n \le b_n$ for all n and the sequences $\{a_n\}$ and $\{b_n\}$ both converge to L, then the sequence $\{x_n\}$ also converges to L. ■

The next theorem lists the limits of some specific sequences that occur frequently. Because of their importance, these sequences and their limits should be intuitively understood and committed to memory.

THEOREM 3.8 **Limits of Specific Sequences**

1. If $p > 0$, then the sequence $\{1/n^p\}$ converges to 0.

2. If $|r| < 1$, then the sequence $\{r^n\}$ converges to 0.

3. If $|r| < 1$ and $p > 0$, then the sequence $\{n^p r^n\}$ converges to 0.

4. If $|r| > 1$, then the sequence $\{r^n\}$ is not bounded.

5. If $a > 0$, then the sequence $\{\sqrt[n]{a}\}$ converges to 1.

6. The sequence $\{\sqrt[n]{n}\}$ converges to 1.

7. If a is a real number, then the sequence $\{(1 + (a/n))^n\}$ converges to e^a.

Proof. Parts (1), (2), (4), and (5) are easy to visualize. Part (3) follows from the fact that exponential functions grow much more quickly than power functions. The details of these proofs will be omitted.

To prove part (6), we will use the Binomial Theorem in order to indicate how these sequence results can be proved without using calculus. Let $t_n = \sqrt[n]{n} - 1$ for each n. It is sufficient to prove that $\{t_n\}$ converges to 0. By the Binomial Theorem (see Appendix A),

$$n = (1 + t_n)^n = 1 + nt_n + \frac{n(n-1)}{2}\,t_n^2 + \cdots + t_n^n > \frac{n(n-1)}{2}\,t_n^2$$

for all $n \ge 3$. Rearranging this inequality yields $0 < t_n < \sqrt{2/(n-1)}$ for all $n \ge 3$. Hence, the sequence $\{t_n\}$ converges to 0 by the Squeeze Theorem.

To prove part (7), let $f(x) = \big(1 + (a/x)\big)^x$. Using L'Hôpital's Rule, we find that

$$\lim_{x\to\infty} \ln\big(f(x)\big) = \lim_{x\to\infty} \frac{\ln\big(1 + (a/x)\big)}{1/x} = \lim_{x\to\infty} \frac{\dfrac{-a/x^2}{1 + (a/x)}}{-1/x^2} = \lim_{x\to\infty} \frac{a}{1 + (a/x)} = a.$$

It follows that $\lim_{x\to\infty} f(x) = e^a$. Therefore, the sequence $\{(1 + (a/n))^n\}$ converges to e^a. ■

To illustrate the use of these theorems, consider the sequence $\left\{\sqrt[n]{3/n}\right\}$. Then

$\left\{\sqrt[n]{3}\right\}$ converges to 1 by part (5) of Theorem 3.8;

$\left\{\sqrt[n]{n}\right\}$ converges to 1 by part (6) of Theorem 3.8;

$\left\{\sqrt[n]{3/n}\right\} = \left\{\dfrac{\sqrt[n]{3}}{\sqrt[n]{n}}\right\}$ converges to $\dfrac{1}{1} = 1$ by part (e) of Theorem 3.6.

We have thus shown that $\left\{\sqrt[n]{3/n}\right\}$ converges to 1 by using previous results. We typically do not give this much detail when finding limits of sequences. For a second example, consider the sequence $\left\{\dfrac{8^n + n^3}{2^{3n+2} + n}\right\}$. Performing some algebra and using part (3) of Theorem 3.8, we find that

$$\lim_{n\to\infty} \frac{8^n + n^3}{2^{3n+2} + n} = \lim_{n\to\infty} \frac{8^n + n^3}{2^{3n} \cdot 2^2 + n} \cdot \frac{1/8^n}{1/8^n} = \lim_{n\to\infty} \frac{1 + n^3(1/8)^n}{4 + n(1/8)^n} = \frac{1+0}{4+0} = \frac{1}{4}.$$

It follows that the limit of the sequence $\left\{\dfrac{8^n + n^3}{2^{3n+2} + n}\right\}$ is $\dfrac{1}{4}$.

Exercises

1. Find, with proof (see the first of the above examples), the limit of the sequence.

a) $\left\{\sqrt[n]{4n}\right\}$ **b)** $\left\{3\sqrt[k]{8} + \dfrac{k}{2^k}\right\}$ **c)** $\left\{\dfrac{n^5}{3^n} - 7\sqrt[n]{2}\right\}$

2. Find the limit of the sequence. Justify your answers.

a) $\left\{\left(1 + \dfrac{4}{n}\right)^n\right\}$ **b)** $\left\{\left(1 - \dfrac{1}{2n}\right)^n\right\}$ **c)** $\left\{\left(\dfrac{k}{k+3}\right)^k\right\}$

d) $\left\{\dfrac{2^k}{3^k + 6^k}\right\}$ **e)** $\left\{\dfrac{4^n + n^2}{2^{2n+1} + n^3}\right\}$ **f))** $\left\{\sqrt[n]{2n^2 + 4n + 3}\right\}$

g) $\left\{(\sqrt[k]{k} - 1)^k\right\}$ **h)** $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}\right\}$ **i)** $\left\{\dfrac{2^n + n^4}{5^n - n}\right\}$

3. Use the squeeze theorem to prove that the sequences $\{2^n/n!\}$ and $\{n!/n^n\}$ converge to 0.

4. Suppose that $0 < a < b$. Find, with proof, the limit of the sequence $\left\{(a^n + b^n)^{1/n}\right\}$.

5. Prove that $\{|a_n|\}$ converges to 0 if and only if $\{a_n\}$ converges to 0. Give an example of a sequence $\{b_n\}$ such that $\{|b_n|\}$ converges but $\{b_n\}$ does not converge.

6. Suppose that $\{a_n\}$ converges to 0 and that $\{b_n\}$ is bounded. Prove that $\{a_n b_n\}$ converges to 0.

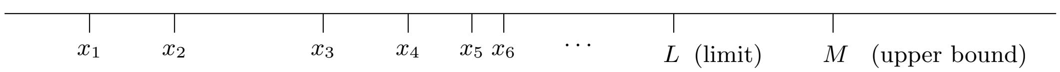
7. Prove that the sequence $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot \ \cdots \ \cdot (2n-1)}{2^n}\right\}$ is unbounded.

3.4 The Completeness Axiom

One of the disadvantages of the definition of a convergent sequence is that it is first necessary to find the limit of the sequence before proving the sequence converges. Fortunately, it is possible to show that a sequence converges without finding its limit. For the purposes of this book, we will treat the following statement as an axiom (a statement accepted without proof). It is called the Completeness Axiom since it indicates that there are no holes in the real number line.

Completeness Axiom: Every bounded monotone sequence of real numbers converges.

This statement is easy to accept from the perspective of the number line. If an infinite sequence of dots is moving to the right (an increasing sequence) and if the dots cannot go beyond a certain point (a bounded sequence), then the dots must be piling up at some point (the sequence converges).



The Completeness Axiom indicates that it is possible to prove that a sequence converges without finding its limit; simply show that the sequence is bounded and monotone.

Consider the sequence $\{a_n\}$ defined by $a_n = \sum\limits_{k=1}^{n} 1/k^2$ for each positive integer n. Since the expression $a_{n+1} - a_n$ simplifies to $1/(n+1)^2$, it follows that $\{a_n\}$ is an increasing sequence. To prove that $\{a_n\}$ is bounded, we need to find an upper bound for $\{a_n\}$. Using some estimation, for each $n \geq 2$, we find that

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \Big(\frac{1}{k-1} - \frac{1}{k}\Big) \\ &= 1 + \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \cdots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) = 2 - \frac{1}{n} < 2. \end{aligned}$$

It follows that $\{a_n\}$ is bounded by 2. Since $\{a_n\}$ is bounded and increasing, it must converge by the Completeness Axiom. Note that we have not found the limit of this sequence; we just know it is some number between 1 and 2.

A sequence $\{x_n\}$ may not be presented in the form $x_n = f(n)$ for some function f. As an example of another way to define a sequence, consider the sequence $\{b_n\}$ defined by $b_1 = 1$ and $b_{n+1} = 1/(1+b_n)$ for all $n \geq 1$. Using the formula for b_n, it is easy to write out the first few terms of this sequence;

$$\{b_n\} = 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots.$$

Sequences defined in this way, where successive terms depend on previous terms, are known as **recursively defined sequences**. Since an explicit formula is not given, it can be difficult to find terms further down the line; for instance, to find b_{1000}, we would need to successively list all of the terms b_1 through b_{1000} (or program a computer to do the work for us). However, in this specific case, you might notice the appearance of the Fibonacci numbers and thus be able to prove that $b_n = f_n/f_{n+1}$ for all n. Since there is a formula for f_n (see the exercises in Section 3.1), we could then find b_{1000} more quickly.

Consider the sequence $\{c_n\}$ defined recursively by $c_1 = 1$ and $c_{n+1} = \sqrt{1 + c_n}$ for $n \geq 1$. The first three terms of this sequence are 1, $\sqrt{2}$, and $\sqrt{1+\sqrt{2}}$ or, to the nearest thousandth, 1.000, 1.414, and 1.554. We will prove that $\{c_n\}$ is increasing and bounded. In particular, we will use the Principle of Mathematical Induction to prove that $c_n < c_{n+1}$ for all n and that $c_n < 2$ for all n. To prove that $c_n < c_{n+1}$ for all n, we begin by noting that $c_1 < c_2$. Suppose that $c_k < c_{k+1}$ for some positive integer k. We then have

$$c_k < c_{k+1} \;\Rightarrow\; 1 + c_k < 1 + c_{k+1} \;\Rightarrow\; \sqrt{1+c_k} < \sqrt{1+c_{k+1}} \;\Rightarrow\; c_{k+1} < c_{k+2},$$

which is the desired inequality for $n = k + 1$. By the PMI, we find that $c_n < c_{n+1}$ for all n, that is, the sequence $\{c_n\}$ is increasing. We now move on to a proof that $c_n < 2$ for all n. Once again, it is clear that $c_1 < 2$. Suppose that $c_k < 2$ for some positive integer k. Then

$$c_{k+1} = \sqrt{1+c_k} < \sqrt{1+2} = \sqrt{3} < 2,$$

and it follows from the PMI that $c_n < 2$ for all n. Since $\{c_n\}$ is bounded and increasing, the Completeness Axiom guarantees that it is a convergent sequence.

For recursively defined sequences, it is often possible to find the limit of the sequence once it is known to exist. We will illustrate the basic idea with the sequence $\{c_n\}$ defined in the previous paragraph. Let L be the limit of $\{c_n\}$. By properties of sequences, the sequence $\{\sqrt{1+c_n}\}$ converges to $\sqrt{1+L}$. Also, the sequence $\{c_{n+1}\}$ converges to L since it is just the sequence $\{c_n\}$ without the first term. Hence, the equality $c_{n+1} = \sqrt{1+c_n}$ yields $L = \sqrt{1+L}$ or $L^2 - L - 1 = 0$ when we take the limit as n goes to infinity. Since L must be positive, the sequence $\{c_n\}$ converges to $L = \big(1+\sqrt{5}\big)/2$.

Exercises

1. Prove that the sequence $\{A_n\}$ defined by $A_n = \sum_{k=1}^{n} 4k/(k^3+1)$ for each positive integer n converges. (Compare this sequence with the sequence $\{a_n\}$ considered in the text.)

2. For each positive integer n, let $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \cdots + \dfrac{1}{2n}$. Prove that the sequence $\{x_n\}$ converges.

3. Prove that the sequence $\{s_n\}$ does not converge, where $s_n = \sum_{k=1}^{n} k^{-3/4}$.

4. Consider the sequence $\{b_n\}$ defined in the third paragraph of this section. Suppose that it has been shown that $\{b_n\}$ converges. Find the limit of the sequence.

5. Let $r_1 = 8$ and $r_{n+1} = \dfrac{r_n}{2} + \dfrac{5}{r_n}$ for each $n \geq 1$. Assuming that $\{r_n\}$ converges, find its limit.

6. Let $d_1 = 2$ and $d_{n+1} = 3 + (d_n/8)$ for each $n \geq 1$. Prove that $\{d_n\}$ converges, then find its limit.

7. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = 3 - (1/a_n)$ for $n \geq 1$. Prove that $1 \leq a_n \leq 3$ for all n, then prove that $\{a_n\}$ is an increasing sequence. Conclude that $\{a_n\}$ converges and find its limit.

8. Let $z_1 = 3$ and let $z_{n+1} = \sqrt{3+z_n}$ for $n \geq 1$. Prove that the sequence $\{z_n\}$ converges, then find its limit.

3.5 Infinite Series

Given a sequence $\{a_k\}$, an **infinite series** (or simply, a series) is an expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots,$$

which represents the sum of all of the terms of the sequence $\{a_k\}$. As it is not possible to actually perform an infinite number of additions, some other approach is required to make sense of this expression. For each positive integer n, let $s_n = \sum_{k=1}^{n} a_k$. The sequence $\{s_n\}$ is known as the sequence of **partial sums** of the series. A series **converges** if and only if its corresponding sequence of partial sums converges. The sum of a convergent series is the limit of its sequence of partial sums: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$. If the sequence of partial sums does not converge, then the series is said to **diverge**. Consider the following four examples.

i. If $\{a_k\} = \{1\}$, then $\{s_n\} = \{n\}$. Note that $\{a_k\}$ converges to 1 and that $\{s_n\}$ does not converge.

ii. If $\{a_k\} = \{2^{-k}\}$, then $\{s_n\} = \{1 - 2^{-n}\}$. Note that $\{a_k\}$ converges to 0 and that $\{s_n\}$ converges to 1. (Refer to Exercise 2.1.7 to verify the formula for s_n.)

iii. If $\{a_k\} = \{1/\sqrt{k}\}$, then $\{s_n\} = \{\sum_{k=1}^{n} 1/\sqrt{k}\}$. Note that $\{a_k\}$ converges to 0 but that $\{s_n\}$ does not converge. (Refer to Exercise 3.2.6 to see that the sequence $\{s_n\}$ is unbounded.)

iv. If $\{a_k\} = \left\{\frac{1}{k+1} - \frac{1}{k+2}\right\}$, then $\{s_n\} = \left\{\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\right\} = \frac{1}{2} - \frac{1}{n+2}$. In this case, we see that $\{a_k\}$ converges to 0 and that $\{s_n\}$ converges to $1/2$.

Unlike examples (i), (ii), and (iv), the sequence of partial sums cannot usually be expressed in a simple form. Thus, an important but challenging problem is to determine when a particular infinite series converges. As a start, it should be intuitively clear that the terms of a convergent series must converge to 0.

THEOREM 3.9 Let $\sum_{k=1}^{\infty} a_k$ be an infinite series.

(a) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

(b) (**Divergence Test**) If $\lim_{k\to\infty} a_k$ does not exist or $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. Suppose that the series $\sum_{k=1}^{\infty} a_k$ converges. Then its corresponding sequence $\{s_n\}$ of partial sums converges to some number S, and we find that $\lim_{n\to\infty}(s_n - s_{n-1}) = S - S = 0$. For $n \geq 2$,

$$s_n - s_{n-1} = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = a_n.$$

It follows that $\lim_{n\to\infty} a_n = 0$, which proves (a). Part (b) is simply the contrapositive of part (a). ■

The usual algebraic properties of scalar multiplication and addition are valid for infinite series.

THEOREM 3.10 Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two convergent series. Then

a) the series $\sum_{k=1}^{\infty} ca_k$ converges and $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$, where c is any real number;

b) the series $\sum_{k=1}^{\infty}(a_k + b_k)$ converges and $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

Proof. A proof of part (b) is summarized in the following equations.

$$\sum_{k=1}^{\infty}(a_k+b_k) = \lim_{n\to\infty}\sum_{k=1}^{n}(a_k+b_k) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k\right) = \lim_{n\to\infty}\sum_{k=1}^{n}a_k + \lim_{n\to\infty}\sum_{k=1}^{n}b_k = \sum_{k=1}^{\infty}a_k + \sum_{k=1}^{\infty}b_k.$$

Be certain that you are able to justify each step in this equation. ∎

An important class of infinite series are the **geometric series**, which are series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots,$$

where a and r are constants. Each term of the series is r times the previous term. To determine whether or not this series converges, we need to look at its sequence $\{g_n\}$ of partial sums. Some algebra yields

$$g_n - rg_n = \left(a + ar + \cdots + ar^n\right) - \left(ar + ar^2 + \cdots + ar^{n+1}\right) = a - ar^{n+1} \quad \Rightarrow \quad g_n = \frac{a\left(1 - r^{n+1}\right)}{1-r}.$$

If $|r| < 1$, the sequence $\{r^{n+1}\}$ converges to 0 and the sequence $\{g_n\}$ converges to $a/(1-r)$. In other words (pay particular attention to this), the sum of a geometric series is simply the first term of the series divided by $1 - r$, where r is the common ratio.

THEOREM 3.11 Geometric Series Let a be a nonzero real number. The geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $a/(1-r)$ if $|r| < 1$ and diverges if $|r| \geq 1$. ∎

Exercises

1. Find an explicit formula or pattern for the sequence of partial sums for the given series.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{5}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{5^k}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+3} - \frac{1}{k+4}\right)$

2. Prove that each of the following series diverges.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{5k^2+60}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{6}{\sqrt[k]{3}+10}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{3/4}}$

3. Find the sum of each of the following series: for (c), assume that $|b| < 1$; for (i), use $\sum_{k=1}^{\infty} k^2 2^{-k} = 6$.

a) $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots$ **b)** $0.4 + 0.04 + 0.004 + \cdots$ **c)** $b^2 - b^4 + b^6 - b^8 + \cdots$

d) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{4^k}$ **e)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k 2^k}{5^{k-1}}$ **f)** $\displaystyle\sum_{k=2}^{\infty}\frac{6^k}{10^{2k-1}}$

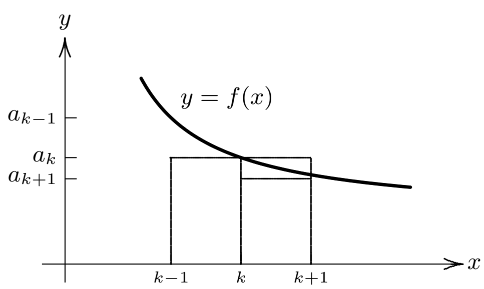
g) $\displaystyle\sum_{k=1}^{\infty}\frac{2^k+3^k}{4^k}$ **h)** $\displaystyle\sum_{k=1}^{\infty}\frac{2\cdot 3^k - (-2)^{k+1}}{6^{k-1}}$ **i)** $\displaystyle\sum_{k=2}^{\infty}\frac{3k^2+5}{2^k}$

4. Prove that the sequence $\{a_k\}$ converges if and only if the series $\sum_{k=1}^{\infty}(a_k - a_{k+1})$ converges.

5. Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and suppose that $s_n = \dfrac{2n+5}{3n-4}$ for all $n \geq 1$, where $\{s_n\}$ is its corresponding sequence of partial sums. Find a_1, a_2, a_{10}, and the sum of the series.

3.6 The Integral Test

If all of the terms of a series $\sum_{k=1}^{\infty} a_k$ are positive, then its corresponding sequence $\{s_n\}$ of partial sums is increasing. To prove that $\{s_n\}$ converges in this case, it is sufficient to prove that $\{s_n\}$ is bounded (see the Completeness Axiom). One way to do this involves relating the partial sums to the area under a curve. Suppose there is a function f such that $f(k) = a_k$ for all positive integers k and that f is continuous and decreasing on the interval $[1, \infty)$. Using the properties of the integral or the interpretation of the integral as area, we obtain the inequalities listed to the right of the following graph. Do not let the inequalities frighten you. The first inequality is just comparing the areas of rectangles to the area under the curve; some are overestimates and some are underestimates. The second inequality merely adds the inequalities listed in the first line with some care needed to match up the subscripts. The third inequality uses a property of integrals to combine the individual integrals and summarizes what has been found. This last inequality shows that the sequence $\{s_n\}$ is bounded if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges; a fact known as the Integral Test. It is stated a little more generally in the next theorem but do note that typically $K = 1$.



$$a_{k+1} \le \int_k^{k+1} f(x)\,dx \le a_k \le \int_{k-1}^{k} f(x)\,dx;$$
$$\sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \le a_1 + \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx;$$
$$\int_1^n f(x)\,dx \le \int_1^{n+1} f(x)\,dx \le s_n \le a_1 + \int_1^n f(x)\,dx.$$

THEOREM 3.12 Integral Test Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms and let f be a function that satisfies $f(k) = a_k$ for all k. Suppose that f is continuous and decreasing for all $x \ge K$ for some integer K. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the improper integral $\int_K^{\infty} f(x)\,dx$ converges. ■

The following two examples illustrate how easy the Integral Test is to apply.

1) The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges since the function $1/x^{3/2}$ is continuous and decreasing for $x \ge 1$ and $\displaystyle\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} -2x^{-1/2}\Big|_1^b = 2$. By looking back at the equations used to prove the Integral Test, the reader should convince themselves that the sum of this series is between 2 and 3.

2) The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ diverges since the function $1/x^{1/3}$ is continuous and decreasing for $x \ge 1$ and $\displaystyle\int_1^{\infty} \frac{1}{x^{1/3}}\,dx = \lim_{b\to\infty} \frac{3}{2}\,x^{2/3}\Big|_1^b = \infty.$

In these examples, the function related to the series is clearly continuous and decreasing; for more complicated series, it may be necessary to do some work to verify that the function is decreasing. Of course, the Integral Test is only useful when the terms of the series correspond to a function that can be easily integrated.

In addition to the geometric series, another important collection of infinite series are series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$; such series are known as ***p*-series**. As the two examples indicate ($p = 3/2$ and $p = 1/3$), the Integral Test provides an easy way of determining when a p-series converges.

THEOREM 3.13 A p-series $\sum_{k=1}^{\infty} 1/k^p$ converges if $p > 1$ and diverges if $p \le 1$.

Proof. The series clearly diverges if $p \le 0$ since the terms do not go to 0 in this case. When p is positive, the sequence $\{1/k^p\}$ is a decreasing sequence that converges to 0 and the function $1/x^p$ is continuous and decreasing on $[1, \infty)$. For $p \ne 1$, we find that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{1}{1-p} \cdot \frac{1}{x^{p-1}}\Big|_1^b = \lim_{b\to\infty} \frac{1}{1-p}\Big(\frac{1}{b^{p-1}} - 1\Big).$$

If $p > 1$, then the limit exists and has a value of $1/(p-1)$; the Integral Test then indicates that the series converges. If $0 < p < 1$, then the limit does not exist (it has a value of ∞) so the series diverges for these values of p. We leave the special case $p = 1$ as an exercise (see Exercise 1b). ∎

In contrast to the geometric series, there is no simple method for finding the sum of a p-series. A number of interesting techniques, often involving power series, Fourier series, or complex variables, have determined the sum of a p-series for some values of p. For example, it has been shown that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad\text{and}\quad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

It may come as a surprise that the number π appears in these sums. One method for verifying the first sum can be found in the supplementary exercises of this chapter.

Exercises

1. Use the Integral Test to determine whether or not the series converges.

a) $\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$ **b)** $\sum_{k=1}^{\infty} \frac{1}{k}$ **c)** $\sum_{k=1}^{\infty} \frac{k}{e^k}$ **d)** $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

2. Use the Integral Test (with $K = 2$) to show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

3. Use the ideas behind the Integral Test to estimate an integer n so that $\sum_{k=1}^{n} \frac{1}{k} > 20$.

4. Find all values of $p > 0$ for which the series converges.

a) $\sum_{k=1}^{\infty} \frac{1}{k^{p^2}}$ **b)** $\sum_{k=1}^{\infty} \frac{p}{k^p}$ **c)** $\sum_{k=1}^{\infty} \frac{1}{k^{4p-p^2}}$ **d)** $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$

5. For each positive integer n, let $z_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \int_1^n \frac{dx}{x}$. Use the ideas in this section to prove that $\{z_n\}$ is a decreasing sequence of positive terms and thus convergent.

3.7 Comparison Tests

As we saw in the last section, if all of the terms of a series $\sum_{k=1}^{\infty} a_k$ are positive, then the corresponding sequence $\{s_n\}$ of partial sums is increasing. It follows that the series converges if and only if $\{s_n\}$ is bounded. The two comparison tests stated in this section provide one way to determine when this occurs.

THEOREM 3.14 Comparison Test Suppose that $0 \le a_k \le b_k$ for all $k \ge K$, where $K \in \mathbb{Z}^+$.

a) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

b) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. To make the proof easier to follow, we assume that $K = 1$. Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums for the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, respectively. Both of these sequences are increasing. If $\sum_{k=1}^{\infty} b_k$ converges, then $\{t_n\}$ is bounded. Since $0 \le s_n \le t_n$ for all n, the sequence $\{s_n\}$ is also bounded. It follows that the series $\sum_{k=1}^{\infty} a_k$ converges. This proves (a); part (b) is the contrapositive of (a). ∎

In less concise but more user friendly language, the Comparison Test states that for positive term series, less than convergent is convergent and greater than divergent is divergent. In order to use the Comparison Test, we need a large collection of series that are known to converge or diverge: the geometric series and the p-series are the most common choices. Consider the following two examples.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^5+3k^4+2}$ converges since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges and $\displaystyle\frac{k^2}{k^5+3k^4+2} \le \frac{k^2}{k^5} = \frac{1}{k^3}$ for all $k \ge 1$.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{2k+5}$ diverges since $\displaystyle\sum_{k=1}^{\infty} \frac{1/3}{\sqrt{k}}$ diverges and $\displaystyle\frac{\sqrt{k}}{2k+5} \ge \frac{\sqrt{k}}{3k} = \frac{1/3}{\sqrt{k}}$ for all $k \ge 5$.

Note that care (even perhaps extreme care) must be taken with the inequalities that occur in the Comparison Test. If you think the series converges, then you must show that its terms are smaller than the terms of a known convergent series. Similarly, if you think the series diverges, then you must show that its terms are larger than the terms of a known divergent series.

The following variation on the Comparison Test avoids the use of inequalities; this is a real advantage in many cases. Read the statement of the theorem carefully since the inequality involving α is not the same in each part of the test.

THEOREM 3.15 Limit Comparison Test Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series for which a_k and b_k are positive for all $k \ge K$ and suppose that $\alpha = \lim_{k\to\infty} (a_k/b_k)$ "exists", either as a number or ∞.

a) If $\sum_{k=1}^{\infty} b_k$ converges and $0 \le \alpha < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges.

b) If $\sum_{k=1}^{\infty} b_k$ diverges and $0 < \alpha \le \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. Suppose that the series $\sum_{k=1}^{\infty} b_k$ converges and that $0 \le \alpha < \infty$. Since the sequence $\{a_k/b_k\}$ converges, it is bounded. Consequently, there exists a positive number M such that $|a_k/b_k| \le M$ for all k.

Since $0 \le a_k \le Mb_k$ for all $k \ge K$, the series $\sum_{k=1}^{\infty} a_k$ converges by the Comparison Test. The proof of part (b) is similar. ■

To illustrate the Limit Comparison Test, consider the series $\sum_{k=1}^{\infty} k^2/(k^3+5k+2)$. For large values of k, the numbers $5k$ and 2 are small in comparison to k^3, so the terms of the series are similar to $k^2/k^3 = 1/k$ when k is large. The series $\sum_{k=1}^{\infty} 1/k$ diverges and

$$\lim_{k\to\infty}\left(\frac{k^2}{k^3+5k+2} \div \frac{1}{k}\right) = \lim_{k\to\infty}\frac{k^3}{k^3+5k+2} = 1.$$

Hence, the series $\sum_{k=1}^{\infty} k^2/(k^3+5k+2)$ diverges by the Limit Comparison Test.

Exercises

1. Use the Comparison Test to determine whether or not the series converges.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3k^2+2k+6}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k+6\sqrt{k}}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{5}{\sqrt{k^3+8k}}$

2. Use the Limit Comparison Test to determine whether or not the series converges.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{5k^4+2k^3-3}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{2k+5}{k^2-5k+8}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{10}{\sqrt[3]{k^2+4k}}$

3. Determine whether or not the series converges.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{k-4}{k^2-3k+7}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{k^4+5k^2}{3k^6-7k^5+6k-1}$ **c)** $\displaystyle\sum_{k=2}^{\infty}\frac{k^2+3}{12k^2+8k+1}$

d) $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{2^k+5^k}$ **e)** $\displaystyle\sum_{k=1}^{\infty}\frac{3^k+k}{2^k+k^4}$ **f)** $\displaystyle\sum_{k=2}^{\infty}\frac{1}{\ln k}$

4. Let $\{d_k\}$ be the sequence of all positive integers (listed in increasing order) that do not have the digit 0 in their decimal representation. (For instance, 125 is in the sequence but 105 is not.) Prove that the series $\sum_{k=1}^{\infty} 1/d_k$ converges and that the sum of the series is less than 90.

3.8 Absolute Convergence

The tests for convergence considered thus far require series with positive terms, or at least with only a finite number of negative terms. For series with an infinite number of both positive and negative terms, the following theorem is sometimes useful.

THEOREM 3.16 If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ also converges.

Proof. The result follows from the inequality $0 \le a_k + |a_k| \le 2|a_k|$ and the Comparison Test. There are several missing details here; the reader should attempt to fill them in. ■

To illustrate this theorem, consider the series $\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2k}{k^3}$. Due to the oscillating nature of the cosine function, this series has an infinite number of positive and negative terms. However, we easily note that $\displaystyle\left|\frac{\cos 2k}{k^3}\right| \le \frac{1}{k^3}$ for all k. Since the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series, the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{\cos 2k}{k^3}\right|$ converges by the Comparison Test. By the previous theorem, we know that the series $\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2k}{k^3}$ converges.

As a result of this theorem, we are led to the following definition.

DEFINITION 3.17 Let $\sum_{k=1}^{\infty} a_k$ be a given series.

a) The series $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges.

b) The series $\sum_{k=1}^{\infty} a_k$ **converges conditionally** if $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} |a_k|$ diverges.

Given a series with positive and negative terms, it is a good idea to first determine whether or not the series converges absolutely. There are more tests available for series with positive terms, and if the series converges absolutely, it certainly converges. Note that two tests are necessary to conclude that a series converges conditionally; one to prove $\sum_{k=1}^{\infty} |a_k|$ diverges and another to prove $\sum_{k=1}^{\infty} a_k$ converges.

As we will discover when studying power series, many series that appear in applications have terms that alternate signs. The following test is often useful for such series.

THEOREM 3.18 Alternating Series Test Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where $a_k > 0$ for all $k \geq K$. If $a_{k+1} \leq a_k$ for all $k \geq K$ and $\lim_{k\to\infty} a_k = 0$, then the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Proof. We will assume that $K = 1$. Let $\{s_n\}$ be the sequence of partial sums of the alternating series. The hypotheses of the theorem indicate that the terms of this sequence are distributed on the number line as follows:

Thus, the sequence $\{s_{2n}\}$ is bounded and increasing, and $\{s_{2n-1}\}$ is bounded and decreasing. By the Completeness Axiom, each of these sequences converges. Since $s_{2n} = s_{2n-1} - a_{2n}$ and $\{a_{2n}\}$ converges to 0, the sequences $\{s_{2n}\}$ and $\{s_{2n-1}\}$ have the same limit. Therefore, the sequence $\{s_n\}$ converges. ∎

The Alternating Series Test is usually very easy to apply. For example, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges by the Alternating Series Test since $\{1/k\}$ decreases to 0. This series thus provides a simple example of a conditionally convergent series.

Let's consider a more involved example: determine if the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k}{4k^2 - 1}$ is absolutely convergent, conditionally convergent, or divergent. The first step is to consider the series $\sum_{k=1}^{\infty} \frac{k}{4k^2-1}$. Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent p-series and $\lim_{k\to\infty} \left(\frac{k}{4k^2-1} \div \frac{1}{k} \right) = \frac{1}{4}$, we see that $\sum_{k=1}^{\infty} \frac{k}{4k^2-1}$ diverges by the Limit Comparison Test. This shows that our given series is not absolutely convergent. However, the sequence $\{k/(4k^2-1)\}$ converges to 0 and is decreasing. (To verify the latter, note that

$$\frac{d}{dx}\left(\frac{x}{4x^2-1} \right) = \frac{4x^2 - 1 - 8x^2}{(4x^2-1)^2} = -\frac{4x^2+1}{(4x^2-1)^2} < 0$$

for all $x \geq 1$.) It follows that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,k}{4k^2-1}$ converges by the Alternating Series Test. Putting these two facts together shows that our original series is conditionally convergent. As mentioned previously, note that two tests are necessary to conclude that a series is conditionally convergent.

As a final comment, it is important to realize that a sequence of positive numbers can converge to 0 but not be a decreasing sequence. As one example, consider the sequence $\{a_k\}$ given by

$$1, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{9}, \frac{1}{4}, \frac{1}{16}, \frac{1}{5}, \frac{1}{25}, \frac{1}{6}, \frac{1}{36}, \ldots,$$

where the pattern for the terms should be clear. It is easy to see that $\{a_k\}$ converges to 0 and that the sequence $\{a_k\}$ is not monotone. For the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$, note that the partial sum corresponding to $2n$ satisfies

$$s_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}a_k = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k^2}.$$

Since the first sum goes to ∞ (a divergent p-series) and the second sum converges to a number (a convergent p-series), we see that the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ does not converge. Hence, when using the Alternating Series Test, it is important to verify that the appropriate sequence is decreasing.

Exercises

1. Show that the following series are absolutely convergent.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{\cos k}{k^2}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{2^k \sin k}{3^k}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{k^4+10}$ **d)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k \arctan k}{k^{3/2}}$

2. Determine whether or not the given series converges.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3k+7}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{4k-1}}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{4k-1}$ **d)** $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\ln k}{k}$

3. Prove that the series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{2k^2+3}$ is conditionally convergent.

4. Find all values of $p > 0$ for which the series $\sum_{k=1}^{\infty}(-1)^{k+1}/k^p$ is conditionally convergent.

5. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series. Prove that the series $\sum_{k=1}^{\infty} a_k^2$ converges.

6. Give an example of a series for which $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} a_k^2$ diverges.

7. Explain why $\sum_{k=1}^{\infty}(-1)^k a_k$ converges if and only if $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

8. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k+1)}{4k^3+7k^2-1}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k}{\sqrt{5k^2+4}}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k}{\sqrt{k^3-4k^2+20}}$ **d)** $\displaystyle\sum_{k=1}^{\infty}\frac{\sin^3 k}{k^2+3}$

9. Consider the two series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2+(-1)^k}}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{3+(-1)^k}}$. Does the Alternating Series Test apply to these series? Do these series converge or diverge?

3.9 Root and Ratio Tests

Two of the more useful tests for absolute convergence are the Ratio Test and the Root Test.

THEOREM 3.19 Ratio Test Let $\sum_{k=1}^{\infty} a_k$ be a series and suppose that $\ell = \lim_{k\to\infty} \left|a_{k+1}/a_k\right|$ exists. Then the series converges absolutely if $\ell < 1$ and diverges if $\ell > 1$.

Proof. Suppose $\ell < 1$. Let r be a number that satisfies $\ell < r < 1$. By the definition of a convergent sequence, there exists a positive integer p such that $\left|a_{k+1}/a_k\right| < r$ for all $k \geq p$. It follows that

$$|a_{p+1}| < r\,|a_p| = \left(\frac{|a_p|}{r^p}\right) r^{p+1}, \qquad |a_{p+2}| < r\,|a_{p+1}| < \left(\frac{|a_p|}{r^p}\right) r^{p+2},$$

$$|a_{p+3}| < r\,|a_{p+2}| < \left(\frac{|a_p|}{r^p}\right) r^{p+3}, \qquad |a_{p+4}| < r\,|a_{p+3}| < \left(\frac{|a_p|}{r^p}\right) r^{p+4},$$

and, in general, $|a_k| < \left(|a_p|/r^p\right) r^k$ for all $k > p$. Since the series $\sum_{k=1}^{\infty} \left(|a_p|/r^p\right) r^k$ is a convergent geometric series, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely by the Comparison Test.

Now suppose that $\ell > 1$. By the definition of a convergent sequence, there exists a positive integer q such that $\left|a_{k+1}/a_k\right| > 1$ for all $k \geq q$. It follows (as above) that $|a_k| > |a_q| > 0$ for all $k > q$, which indicates that the sequence $\{a_k\}$ does not converge to 0. By the Divergence Test, the series $\sum_{k=1}^{\infty} a_k$ diverges. ■

THEOREM 3.20 Root Test Let $\sum_{k=1}^{\infty} a_k$ be a series and suppose that $\ell = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists. Then the series converges absolutely if $\ell < 1$ and diverges if $\ell > 1$.

Proof. The proof of this theorem will be left as an exercise. ■

The Ratio Test is a convenient test to use when factorials are involved since a lot of cancellation occurs in the ratio of the terms. As an example, consider the series $\sum_{k=1}^{\infty} (-10)^k/k!$. Since

$$\ell = \lim_{k\to\infty} \left| \frac{(-10)^{k+1}}{(k+1)!} \div \frac{(-10)^k}{k!} \right| = \lim_{k\to\infty} \frac{10^{k+1}k!}{10^k(k+1)!} = \lim_{k\to\infty} \frac{10}{k+1} = 0,$$

the series converges absolutely by the Ratio Test. When using the Root Test, it is helpful to remember that $\lim_{k\to\infty} \sqrt[k]{k} = 1$. For example, the series $\sum_{k=1}^{\infty} 2^k/k^6$ diverges by the Root Test since

$$\ell = \lim_{k\to\infty} \sqrt[k]{\frac{2^k}{k^6}} = \lim_{k\to\infty} \frac{2}{(\sqrt[k]{k})^6} = 2.$$

For a slightly more complicated example involving these tests, consider the series

$$\sum_{k=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \;\cdots\; \cdot (4k-3)}{(-3)^k k!} = \frac{1}{(-3)^1 1!} + \frac{1 \cdot 5}{(-3)^2 2!} + \frac{1 \cdot 5 \cdot 9}{(-3)^3 3!} + \frac{1 \cdot 5 \cdot 9 \cdot 13}{(-3)^4 4!} + \cdots.$$

Due to the product, it is probably best to use the Ratio Test. Inverting and multiplying, we find that

$$\ell = \lim_{k\to\infty} \left| \frac{1 \cdot 5 \cdot 9 \cdot \;\cdots\; \cdot (4k+1)}{(-3)^{k+1}(k+1)!} \cdot \frac{(-3)^k k!}{1 \cdot 5 \cdot 9 \cdot \;\cdots\; \cdot (4k-3)} \right| = \lim_{k\to\infty} \frac{4k+1}{3(k+1)} = \frac{4}{3}.$$

(Pay careful attention to the cancellation that occurs in this example.) Since $\ell > 1$, the series diverges by the Ratio Test.

In essence, the Root and Ratio Tests compare the given series with a geometric series. (Recall that a geometric series converges when $|r| < 1$.) These tests give divergence only when the terms of the series get infinitely large. It follows that the Root Test and the Ratio Test will be inconclusive for a number of series, that is, series for which the value of the limit ℓ is 1. Inconclusive means that we have learned nothing at all; the series might converge absolutely or converge conditionally or not converge at all. It is not difficult to show that any series whose terms can be expressed as rational or algebraic functions of k will give $\ell = 1$. For example, both of the series

$$\sum_{k=1}^{\infty}(-1)^{k+1} \cdot \frac{4k^2+3k-2}{k^3-k^2+2k+4} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{4k+3}{\sqrt{k^5+k^3+8}}$$

give a value of $\ell = 1$ in either the Ratio Test or the Root Test. When this occurs, some other test must be used to determine whether or not the series converges. It is important to recognize this in advance so that you do not spend time reaching a result that is inconclusive. In such cases, it is probably best to try one of the comparison tests on the series $\sum_{k=1}^{\infty} |a_k|$. If this series diverges, the series $\sum_{k=1}^{\infty} a_k$ may still converge conditionally. The only test that we have considered that is useful at this stage of the process is the Alternating Series Test.

Exercises

1. Show that $\ell = 1$ when either the Ratio Test or the Root Test are applied to the series $\sum_{k=1}^{\infty} 1/k$ and $\sum_{k=1}^{\infty} 1/k^2$.

2. Determine whether or not the following series converge.

a) $\displaystyle\sum_{k=1}^{\infty} k^5 5^{-k}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{8^k}{k!}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{5^k (k!)^2}{(2k)!}$

d) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{2k+15}\right)^k$ **e)** $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)}$ **f)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k k!}{4\cdot 7\cdot 10\cdot \cdots \cdot (3k+1)}$

3. Determine if the series converges absolutely, converges conditionally, or diverges.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}+4}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-6)^{k+1}}{k!}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-4)^{k+1}}{k3^k}$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2-3k-7}$ **e)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k^2+5}$ **f)** $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1}\frac{k^3}{5^k}$

4. Prove the Root Test. (The proof is similar to that of the Ratio Test, but less complicated.)

5. Show that the Ratio Test is inconclusive for the series $\sum_{k=1}^{\infty} 2^{-k+1+(-1)^k}$, but that the Root Test shows that the series converges. *Hint*: Write out the first six terms of the series.

6. Find all values of x for which the series converges absolutely. *Hint*: Express ℓ as a function of x.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2 x^k}{2^k}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}2^k}{k!}\, x^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{5^k}\,(x-2)^k$

3.10 Power Series

Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let a be a real number. A **power series** centered at a is an expression of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots.$$

The constants c_k are known as the **coefficients** of the power series and the number a is called the **center** of the power series. As we will see, a power series looks and behaves like an infinite degree polynomial.

A power series represents a function of x; for each fixed value of x, a power series becomes an infinite series of real numbers and the sum of the series is the value of the function at x. The Root Test or the Ratio Test can be used to determine the domain of a function defined as a power series. As an example, consider the following power series centered at 3:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)\,4^k}(x-3)^k = 1 + \frac{1}{3\cdot 4}(x-3) + \frac{1}{5\cdot 4^2}(x-3)^2 + \frac{1}{7\cdot 4^3}(x-3)^3 + \cdots.$$

To determine the values of x for which this power series converges, we can apply the Root Test (in this case, we could just as easily use the Ratio Test);

$$\lim_{k\to\infty}\left|\frac{1}{(2k+1)\,4^k}(x-3)^k\right|^{1/k} = \lim_{k\to\infty}\frac{|x-3|}{4\sqrt[k]{2k+1}} = \frac{|x-3|}{4}.$$

The power series thus converges absolutely when $|x-3|<4$ and diverges when $|x-3|>4$. The convergence of the power series at the points $x=-1$ and $x=7$ has not been determined since the Root Test does not give any information at these points.

For $x=-1$, the series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$, which is convergent (use the Alternating Series Test);

for $x=7$, the series is $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2k+1}$, which is divergent (use the Comparison Test).

Therefore, the power series converges for all x in the interval $[-1,7)$. To phrase this conclusion in another way, the domain of the function defined by this power series is the interval $[-1,7)$.

In general, the set of values of x for which a power series converges is an interval centered at the center of the power series. The previous example illustrates this property quite well and provides a clear suggestion as to why it is true. A general proof of the following theorem is more complicated and will be omitted.

THEOREM 3.21 Given a power series $\sum_{k=0}^{\infty} c_k(x-a)^k$, one of the following occurs:

a) for some $\rho>0$, the series converges absolutely if $|x-a|<\rho$ and diverges if $|x-a|>\rho$;

b) the series converges absolutely for all real numbers;

c) the series converges only for $x=a$. ■

The number ρ is called the **radius of convergence** of the power series; this number is assumed to be ∞ in part (b) and to be 0 in part (c). The radius of convergence of a power series can usually be found using the Root Test or the Ratio Test. The set of all x values for which the power series converges is known as the **interval of convergence** of the power series. To determine the interval of convergence, it is necessary to check the endpoints separately (see the example prior to the theorem). Since these are the points for which the Root Test and the Ratio Test provide no information, some other test is required. It may seem like a lot of work to determine whether or not two additional points are in the domain of the function, but the series associated with these points are often the most interesting ones. As for the power series considered earlier in this section, its radius of convergence is 4 and its interval of convergence is $[-1, 7)$.

Exercises

1. Find the radius of convergence of the given power series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3k^2}(x-2)^k$ **b)** $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}(x+1)^k$ **c)** $\displaystyle\sum_{k=0}^{\infty} \frac{(k+2)3^k}{5^{k+1}} x^k$

2. Find the interval of convergence of the given power series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2 5^k} x^k$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{3^k}(x-4)^k$ **c)** $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k+3}(x+2)^k$

3. Suppose $\sum_{k=0}^{\infty} c_k (x-3)^k$ converges when $x = 8$ and diverges when $x = 9$.

a) Determine, with justification or an explanation that there is not sufficient information, whether or not the power series converges at the following values of x: $x = 4.4$, $x = \sqrt{110}$, $x = 8.5$, $x = 0$, and $x = -2$.

b) Determine the smallest possible interval of convergence for this power series (think about endpoints).

c) Determine the largest possible interval of convergence for this power series (think about endpoints).

4. The interval of convergence of a power series is $[-2, 9)$. Find its center and radius of convergence.

5. Find the interval of convergence of the given power series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} x^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k$ **d)** $\displaystyle\sum_{k=1}^{\infty} x^k$

6. Give an example of a power series with the specified interval of convergence.

a) $[-3, 3]$ **b)** $[2, 8)$ **c)** $(-4, 12)$ **d)** $(-3, 1]$

7. Find (in more familiar terms) the function represented by the given power series. In other words, find the sum of the series for all values of x in its interval of convergence. *Hint*: Use the geometric series.

a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} x^k$ **b)** $\displaystyle\sum_{k=0}^{\infty} (-2)^k (x+3)^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k} (x-1)^k$

3.11 Properties of Power Series

Since power series look very much like polynomials, it is tempting to treat them like polynomials when it comes to the calculus operations of differentiation and integration. This important property of power series is recorded in the next theorem; the technical proof will be omitted.

THEOREM 3.22 Power series can be differentiated and integrated term by term, just like polynomials. The resulting power series have the same radius of convergence as the original series. ∎

In symbols, this theorem states that for the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots = \sum_{k=1}^{\infty} k c_k (x-a)^{k-1}, \quad \text{and}$$

$$\int_a^x f(t)\,dt = c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \frac{c_3}{4}(x-a)^4 + \cdots = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-a)^{k+1}.$$

The theorem also asserts that all three of these power series have the same radius of convergence.

One application of this property of power series is finding formulas for the sums of certain series. The geometric series has one of the simplest formulas:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{valid for } |x| < 1.$$

Integrating both sides of this equation yields the formula

$$-\ln\left|1-x\right| = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty} \frac{1}{k} x^k, \quad \text{valid for } |x| < 1.$$

Differentiating both sides of the geometric series formula, then multiplying by x yields

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=1}^{\infty} k x^k, \quad \text{valid for } |x| < 1.$$

These examples indicate that many familiar functions can be represented as power series; see the next section for more on this topic. To illustrate how the power series just obtained can be used to find sums of series, note that

$$\sum_{k=1}^{\infty} \frac{3(-1)^{k+1}}{k2^k} = -3\sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{1}{2}\right)^k = 3\ln\left|1+\frac{1}{2}\right| = 3\ln(3/2) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} = \frac{1}{4}\cdot\frac{1/4}{\big(1-(1/4)\big)^2} = \frac{1}{9}.$$

As another example involving geometric series, consider carefully the following set of equalities:

$$\frac{x}{3-2x} = \frac{x}{3}\cdot\frac{1}{1-(2x/3)} = \frac{x}{3}\sum_{k=0}^{\infty}\left(\frac{2x}{3}\right)^k = \sum_{k=0}^{\infty} \frac{2^k x^{k+1}}{3^{k+1}} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} x^k.$$

These equations are valid for all values of x for which $|x| < 3/2$.

Suppose that $\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence $\rho > 0$. The function f defined by this power series can be differentiated any number of times to obtain

$$\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots; && f(0) = c_0;\\
f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \cdots; && f'(0) = c_1;\\
f''(x) &= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \cdots; && f''(0) = 2c_2;\\
f^{(3)}(x) &= 6c_3 + 24c_4 x + 60c_5 x^2 + 120c_6 x^3 + \cdots; && f^{(3)}(0) = 6c_3;\\
f^{(4)}(x) &= 24c_4 + 120c_5 x + 360c_6 x^2 + \cdots; && f^{(4)}(0) = 24c_4;
\end{aligned}$$

and so on indefinitely. A few moments of reflection reveals a pattern in the value of the derivatives evaluated at the center. These observations lead to the following result.

THEOREM 3.23 Let $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, where the power series has radius of convergence $\rho > 0$. Then f is infinitely differentiable on $(a-\rho, a+\rho)$ and $f^{(k)}(a) = k!\, c_k$ for each $k \geq 0$. ■

This theorem shows how the derivatives of a function f defined by a power series are related to the coefficients of the power series: the coefficient of the term $(x-a)^k$ is $f^{(k)}(a)/k!$. For example,

$$\text{if } f(x) = \sum_{k=0}^{\infty} \frac{k+1}{2^k}(x-4)^k, \text{ then } f^{(k)}(4) = k! \cdot \frac{k+1}{2^k} = \frac{(k+1)!}{2^k}.$$

We will see more interesting uses of this formula in the next section.

Exercises

1. Use the geometric series to find a power series expression (centered at 0) for the given function. Determine the interval of convergence for each power series.

a) $f(x) = \dfrac{1}{1-4x}$ **b)** $f(x) = \dfrac{x}{1+2x}$ **c)** $f(x) = \dfrac{1}{5-x}$

d) $f(x) = \dfrac{x}{1+x^2}$ **e)** $f(x) = \arctan x$ **f)** $f(x) = \dfrac{1}{(1-x)^3}$

2. By differentiating a power series found in this section, find a formula for the sum of the series $\sum_{k=1}^{\infty} k^2 x^k$.

3. By making a change of indices, rewrite each series in the form $\sum_{k=K}^{\infty} c_k x^k$. (The power on x is k).

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}\, x^{k-1}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{(2k+1)!}\, x^{k+1}$ **c)** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k (k-1)}{4^{k+1}}\, x^{k-2}$

4. Use power series found in this section to find the sum of the given series. Be careful on part (c).

a) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k2^k}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{k3^k}{4^k}$ **c)** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{4^k(k-1)}$

5. Find coefficients c_k so that the power series $\sum_{k=0}^{\infty} c_k x^k$ is unchanged when it is differentiated.

3.12 Taylor Series

Suppose that the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ has radius of convergence $\rho > 0$. We saw in the last section that the function f defined by this power series is infinitely differentiable on the interval $(a-\rho, a+\rho)$ and that $f^{(k)}(a) = k!\, c_k$ for each integer $k \geq 0$. Hence, the derivatives of f at the center of the power series are related to the coefficients of the power series. Turning things around, we can start with an infinitely differentiable function f, find its derivatives, then write the resulting power series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. This series is called the **Taylor series** of f centered at a or, in the frequently occurring case in which $a = 0$, the **Maclaurin series** of f. (For the record, it is possible that a function and its corresponding Taylor series are not equal. However, we will not consider such functions.)

The most direct way to express a function f as a Taylor series centered at a is to find a formula for $f^{(k)}(a)$. If the derivatives of a function have a particularly simple pattern, it is not too hard to do this. For example, since the functions e^x, $\sin x$, and $\cos x$ have derivatives that form simple patterns, it is relatively easy to determine their Maclaurin series; the details will be left to the reader. Note the similarities in the Maclaurin series for these functions.

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k = 1 + x + \frac{1}{2}\, x^2 + \frac{1}{6}\, x^3 + \cdots \\
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = x - \frac{1}{6}\, x^3 + \frac{1}{120}\, x^5 - \frac{1}{5040}\, x^7 + \cdots \\
\cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = 1 - \frac{1}{2}\, x^2 + \frac{1}{24}\, x^4 - \frac{1}{720}\, x^6 + \cdots
\end{aligned}
$$

The radius of convergence of each of these series is ∞, that is, these series converge for all values of x.

As a more complicated example, we will find the Taylor series centered at 8 for the function $f(x) = \sqrt[3]{x}$. The first step is to write out the first few derivatives and try to spot a pattern.

$$
\begin{aligned}
f(x) &= x^{1/3} \\
f'(x) &= \frac{1}{3}\, x^{-2/3} \\
f''(x) &= -\frac{1}{3} \cdot \frac{2}{3}\, x^{-5/3} \\
f^{(3)}(x) &= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3}\, x^{-8/3} \\
f^{(4)}(x) &= -\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3}\, x^{-11/3} \\
f^{(5)}(x) &= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3} \cdot \frac{11}{3}\, x^{-14/3}
\end{aligned}
$$

for $k \geq 2$,

$$
\begin{aligned}
f^{(k)}(x) &= (-1)^{k+1} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3k-4)}{3^k}\, x^{-(3k-1)/3} \\
f^{(k)}(8) &= (-1)^{k+1} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3k-4)}{3^k\, 2^{3k-1}}
\end{aligned}
$$

Notice that when searching for a pattern, it is best not to multiply out terms. Even so, it takes some practice to learn how to identify the patterns that appear. Since in this case, the first two terms of the Taylor series do not fit the pattern, they are pulled out separately. We thus obtain

$$\sqrt[3]{x} = 2 + \frac{1}{12}\,(x-8) + \sum_{k=2}^{\infty} (-1)^{k+1} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3k-4)}{3^k\, 2^{3k-1}\, k!}\,(x-8)^k.$$

The reader should verify that the radius of convergence of this power series is 8.

It is often possible to find the Maclaurin series for a function by using the known Maclaurin series of another function and thus avoid finding a pattern for the derivatives. For example, composition of functions and term by term operations yield all of the following results:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k; \qquad e^{3x} = \sum_{k=0}^{\infty} \frac{3^k}{k!}\, x^k;$$

$$x^2 e^{3x} = \sum_{k=0}^{\infty} \frac{3^k}{k!}\, x^{k+2} = \sum_{k=2}^{\infty} \frac{3^{k-2}}{(k-2)!}\, x^k; \qquad e^{3x} - 1 = \sum_{k=1}^{\infty} \frac{3^k}{k!}\, x^k;$$

$$\frac{e^{3x}-1}{x} = \sum_{k=1}^{\infty} \frac{3^k}{k!}\, x^{k-1} = \sum_{k=0}^{\infty} \frac{3^{k+1}}{(k+1)!}\, x^k; \qquad \int_0^x \frac{e^{3t}-1}{t}\, dt = \sum_{k=1}^{\infty} \frac{3^k}{k \cdot k!}\, x^k.$$

We have thus found power series representations centered at 0 for all these functions. However, a cautious person would ask whether or not these are actually the Maclaurin series of the functions. This is a valid question since the Maclaurin series formula was not used to obtain these series. But a function f can have only one power series representation centered at a (the coefficients must be of the form $f^{(k)}(a)/k!$), so any method that generates a power series centered at a for a function f actually finds the Taylor series for f centered at a.

Exercises

1. Use known series to find the Maclaurin series for the given function.

a) $f(x) = e^{2x}$ **b)** $f(x) = \sin 5x$ **c)** $f(x) = \cos x^2$

d) $f(x) = x^2 \cos(4x)$ **e)** $f(x) = xe^{-x}$ **f)** $f(x) = \left(\sin x - x\right)/x^3$

g) $f(x) = \int_0^x \sin t/t\, dt$ **h)** $f(x) = \left(e^x + e^{-x}\right)/2$ **i)** $f(x) = \cos^2 x$

2. Use known Maclaurin series to determine in more familiar terms the function represented by the power series.

a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, x^k$ **b)** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, x^{2k}$ **c)** $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^k}{(2k)!}\, x^{2k+1}$

3. Find the indicated Taylor series and determine its radius of convergence.

a) $f(x) = \sqrt{x}$, $a = 9$ **b)** $f(x) = \sqrt[3]{x}$, $a = 1$ **c)** $f(x) = 1/(3-x)$, $a = 5$

4. Find the Maclaurin series for the function $f(x) = 1/\sqrt{1-x}$.

5. For the given function, find $f^{(100)}(0)$. *Hint*: First find the Maclaurin series of the function.

a) $f(x) = xe^{-2x}$ **b)** $f(x) = x \sin x^3$ **c)** $f(x) = \left(1 - \cos x^2\right)/x^4$

6. Explain why the function $x^{17/3}$ does not have a Maclaurin series.

7. Let $i = \sqrt{-1}$ and assume that i behaves just like any other constant. Use known Maclaurin series to prove that $e^{ix} = \cos x + i \sin x$. Use this equation to prove that $e^{i\pi} + 1 = 0$ and also to find i^i.

3.13 Taylor's Theorem

Suppose that a function f can be expressed as a Taylor series centered at a. It follows that the partial sums of the Taylor series can be used to approximate the function. That is, $f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = P_n(x)$, with the approximation to f improving as n increases. The polynomial P_n is called the **nth Taylor polynomial** of f at a. Since polynomials can be evaluated using only the operations of addition and multiplication (operations that can be performed by hand or easily programmed into a computer), a Taylor polynomial provides a good way for approximating the values of a function. However, if the exact value of a function is not known, it is difficult to determine when an approximation is accurate. The following theorem, a generalization of the Mean Value Theorem, provides one measure of the accuracy of a Taylor polynomial approximation.

THEOREM 3.24 Taylor's Formula Suppose that f is defined on some open interval I that contains the point a and let n be a positive integer. If $f^{(n+1)}$ exists on I, then for each $x \in I$ there exists a point z between x and a such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

Proof. Fix $x \in I$ with $x \neq a$ and define a function F on I by

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + B(x-t)^{n+1}$$

for each $t \in I$, where the constant B is chosen so that $F(a) = f(x)$. Note that $F(t)$ is like the Taylor series of f centered at t and evaluated at x. By the hypotheses, the function F is continuous and differentiable on the interval with endpoints x and a, and $F(x) = f(x) = F(a)$. By the Mean Value Theorem, there exists a point z between x and a such that $F'(z) = 0$. Differentiating F using the product rule, then simplifying yields

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n - (n+1)B(x-t)^n.$$

(The details will be left as an exercise.) Since $F'(z) = 0$ and $z \neq x$, we find that $B = \frac{f^{(n+1)}(z)}{(n+1)!}$. Hence,

$$f(x) = F(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$ ■

Taylor's Formula provides a way to estimate the magnitude of the quantity $|f(x) - P_n(x)|$:

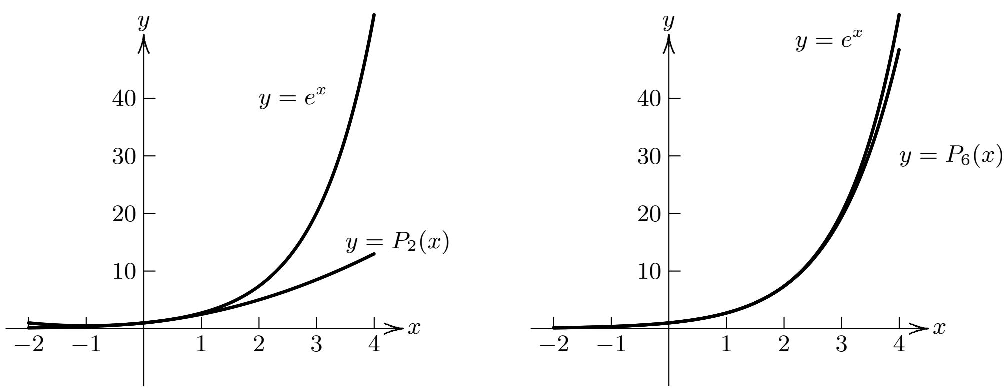
$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \right| \leq \frac{M_{n+1}|x-a|^{n+1}}{(n+1)!},$$

where M_{n+1} is the maximum value of $|f^{(n+1)}(x)|$ on the interval with endpoints x and a. This quantity represents the distance between the functional value $f(x)$ and its approximation $P_n(x)$. In practice, the actual value of $f(x)$ is not known. The error term lets us know roughly how close $P_n(x)$, which is easily evaluated, is to $f(x)$. As an example, suppose we need to find a Taylor polynomial centered at 0 that approximates e^x with an accuracy of 10^{-4} on the interval $[-2, 2]$. Since all of the derivatives of e^x are e^x, it is easy to verify that $M_{n+1} = e^2 < 7.5$ on the interval $[-2, 2]$ for all n. To obtain the desired degree of accuracy, we must choose a positive integer n such that $\dfrac{7.5 \cdot 2^{n+1}}{(n+1)!} < 10^{-4}$. Some trial and error reveals that $n = 11$ has the desired property. In other words,

$$\left| e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{11!}x^{11}\right) \right| < 10^{-4}$$

for all $x \in [-2, 2]$. To illustrate this numerically, we can compute $P_{11}(2) \approx 7.3890460$ and compare it to the known value of $e^2 \approx 7.3890561$. You can see that the approximation is actually much better than 10^{-4}.

Another way to get a sense for how the Taylor polynomials approximate a function is to graph the function and its Taylor polynomials on the same set of axes. The figure below indicates how the approximation to e^x improves as the degree of the Taylor polynomial (centered at 0) increases.



In order to evaluate quantities such as e^2 and $\sin 2$, calculators use some sort of approximation. Taylor polynomials are just one way to obtain approximations; other techniques are discussed in the field of numerical analysis.

Exercises

1. Supply the missing details in the proof of Taylor's Formula.
2. Use the theory in this section to answer the following questions.
   a) Determine the accuracy of the approximation $x - x^3/6$ for $\sin x$ on the interval $[-1, 1]$.
   b) Find a Taylor polynomial that approximates $\sin x$ to 10^{-5} on the interval $[-3, 3]$.
   c) Find a Taylor polynomial that approximates e^x to 10^{-3} on the interval $[-2, 2]$.
   d) Determine the accuracy of the approximation $1 - x^2/2 + x^4/24$ for $\cos x$ on the interval $[0, 1.6]$.
3. Find the fifth degree Taylor polynomial for $\sqrt{x}$ centered at 4. Use the polynomial to approximate $\sqrt{3}$, $\sqrt{4.2}$, and $\sqrt{5}$. How accurate are your estimates?
4. Find the fourth degree Taylor polynomial for $\sqrt[3]{x}$ centered at 8. Use the polynomial to approximate $\sqrt[3]{6}$, $\sqrt[3]{7}$, and $\sqrt[3]{7.6}$. How accurate are your estimates?
5. Find the seventh degree Taylor polynomial centered at 0 for the function $\ln\left|\dfrac{1+x}{1-x}\right|$, then use this polynomial to approximate $\ln 2$, $\ln 3$, and $\ln 5$.

3.14 Supplementary Exercises

Remark. The following exercises involve ideas concerning sequences.

1. Find the limit of the sequence.

a) $\left\{\dfrac{5n+1}{8n-3}\right\}$ **b)** $\left\{\dfrac{2n^2+7}{5n-8n^2}\right\}$ **c)** $\left\{\sqrt{4k^2+k}-2k\right\}$

d) $\left\{\sqrt[3]{k^3+k^2}-k\right\}$ **e)** $\left\{k\tan(\pi/4k)\right\}$ **f)** $\left\{k\left(\sqrt[k]{7}-1\right)\right\}$

g) $\left\{\dfrac{9^k\sqrt[k]{4}}{3^{2k-1}+5^k}\right\}$ **h)** $\left\{\left(1+\dfrac{1}{4n}\right)^n\right\}$ **i)** $\left\{\left(\dfrac{2k}{2k+5}\right)^k\right\}$

2. Find the limit of the sequence $\left\{\dfrac{n^3}{1+9+25+\cdots+(2n-1)^2}\right\}$.

3. Find a sequence $\{x_k\}$ such that $\{x_k\}$ converges to 0 and $\{\cos(\pi/x_k)\}$ converges to $-1/2$.

4. Let $b_1 = 4$ and $b_{n+1} = 1 + (b_n/7)$ for each $n \ge 1$. Prove that $\{b_n\}$ converges, then find its limit.

5. For each positive integer n, let $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{5n}$. Prove that the sequence $\{x_n\}$ converges.

6. Define a sequence $\{c_n\}$ by $c_1 = 2$ and $c_n = 1/(3 - c_{n-1})$ for $n \ge 2$. Prove that $\{c_n\}$ converges and find its limit.

7. Let t be a positive real number and let $s > 1$. Prove that the sequence $\{n^t/s^n\}$ converges to 0.

8. Let z be any real number and consider the sequence $\left\{\lfloor nz\rfloor/n\right\}$, where $\lfloor x\rfloor$ represents the greatest integer less than or equal to x. Prove that this sequence converges and find its limit.

9. For each positive integer n, let $a_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} - \displaystyle\int_1^n \frac{dx}{\sqrt{x}}$. Use the area interpretation of the integral to show that the sequence $\{a_n\}$ is decreasing and bounded.

Remark. The following exercises involve ideas concerning series.

10. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series and suppose that $\{b_k\}$ is a bounded sequence. Prove that $\sum_{k=1}^{\infty} a_k b_k$ is an absolutely convergent series.

11. Find a value of r for which $\sum_{k=0}^{\infty} r^k = 7/8$ and a value of r for which $\sum_{k=1}^{\infty} r^k = 7/8$.

12. Find the values of r for which the series $\sum_{k=0}^{\infty} r^k(1+r^k)$ converges and find the sum of the series for these values of r. What is the minimum possible sum?

13. Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive numbers. Prove that $\sum_{k=1}^{\infty} \sin a_k$ converges.

14. Determine if the series converges absolutely, converges conditionally, or diverges.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{4k+7}{k^2+2k+2}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{10k-3}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k^2)}{k^3}$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{4k^5+k^2+3}$ **e)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[k]{100}}$ **f)** $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

g) $\displaystyle\sum_{k=1}^{\infty} \frac{17^k (k!)^3}{(3k)!}$ **h)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-6)^{k+1}}{(2k)!}$ **i)** $\displaystyle\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{1\cdot 4\cdot 7\cdot\cdots\cdot(3k-2)}$

15. It is known that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Use this fact to find the sums of $\sum_{k=1}^{\infty} 1/(2k-1)^2$ and $\sum_{k=1}^{\infty} (-1)^{k+1}/k^2$.

16. Arnie, Beth, and Charles are playing a game with a fair coin. Each person takes a turn (in the order listed) flipping the coin. The first person to get a tails wins. Find the probability of winning for each person.

17. Danika, Kaden, and Yasmeen are playing a game with a fair six-sided die. Each person takes a turn rolling the die. Danika wins if she rolls a 1, Kaden wins if she rolls a 4 or 6, and Yasmeen wins if she rolls a 2, 3, or 5. Find the probability of winning for each person in each of the following cases.

a) The turns are taken in the order Danika, Kaden, and Yasmeen.

b) The turns are taken in the order Kaden, Danika, and Yasmeen.

c) The turns are taken in the order Yasmeen, Danika, and Kaden.

18. Let $\{a_k\}$ be a decreasing sequence that converges to 0 and consider the series $\sum_{k=1}^{\infty} a_k$.

a) Let S be the sum of the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$. Prove that $\left|\sum_{k=1}^{n}(-1)^{k+1}a_k - S\right| < a_{n+1}$.

b) Suppose that the function f for which $f(k) = a_k$ for all k is continuous and decreasing on $[1, \infty)$. Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and let T be its sum. Prove that $T - \sum_{k=1}^{n} a_k < \int_n^{\infty} f(x)\, dx$.

19. Use the previous exercise to find an integer n so that the nth partial sum of the series is within 10^{-6} of its sum.

a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5}$ **c)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$ **d)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\ln(k+1)}$

e) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ **f)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^5}$ **g)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^8}$ **h)** $\displaystyle\sum_{k=1}^{\infty}\frac{k}{e^k}$

Remark. The following exercises involve ideas concerning power series.

20. (**Binomial Series**) The Binomial Theorem gives a formula for the expansion of $(1 + x)^n$ when n is a positive integer. This result can be extended to any real number r in the following way. In keeping with the notation for binomial coefficients, let

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

for all real numbers r and all positive integers k.

a) Show that the Maclaurin series for $(1 + x)^r$ is given by $1 + \sum_{k=1}^{\infty}\binom{r}{k}x^k$ with radius of convergence 1.

b) Let $f(x) = 1 + \sum_{k=1}^{\infty}\binom{r}{k}x^k$ for each $x \in (-1, 1)$ and show that $(1 + x)f'(x) = rf(x)$.

c) Solve the differential equation in part (b) to find the function f.

d) Write the Maclaurin series for $1/\sqrt{1-x}$ without the shorthand notation for binomial coefficients.

e) Use the result in (d) to find the Maclaurin series for $\arcsin x$.

21. Express the value of each of the following integrals as infinite series.

a) $\displaystyle\int_0^1 e^{-x^2}\, dx$ **b)** $\displaystyle\int_0^{0.5}\frac{\sin x}{x}\, dx$ **c)** $\displaystyle\int_0^1 \cos x^2\, dx$

22. Suppose that the power series $\sum_{k=0}^{\infty} c_k(x-5)^k$ converges when $x = 1$ and diverges when $x = 12$. Find the smallest and largest possible values for the radius of convergence of this power series.

23. Suppose that the radius of convergence of $\sum_{k=1}^{\infty} c_k x^k$ is a finite number ρ. Find the radius of convergence of $\sum_{k=1}^{\infty} c_k x^{3k}$.

24. Find Maclaurin series for each of the following functions. *Hint*: Begin with a geometric series.

a) $f(x) = \dfrac{x}{2-x}$ **b)** $g(x) = \dfrac{x}{(5-x)^2}$ **c)** $h(x) = \displaystyle\int_0^x \frac{t}{1+t^6}\, dt$

25. Find formulas for the sums of the power series $\sum_{k=1}^{\infty} k^3x^k$ and $\sum_{k=1}^{\infty} k^4x^k$.

26. Evaluate each limit by finding the first few terms of the Maclaurin series for the numerator and the denominator.

a) $\displaystyle\lim_{x\to 0}\frac{e^{x^3}-1}{x-\sin x}$ **b)** $\displaystyle\lim_{x\to 0}\frac{1-\cos x+\sin x^2}{e^{2x^2}-1}$ **c)** $\displaystyle\lim_{x\to 0}\frac{e^{x^2}-1-x^2}{x(\sin x-x)}$

27. Use results from this chapter to find the following sums.

a) $\sum_{k=1}^{\infty} k/5^k$ **b)** $\sum_{k=1}^{\infty} k^2/4^k$ **c)** $\sum_{k=1}^{\infty} k^3 2^k/3^k$

28. Use known Maclaurin series to determine in more familiar terms the function represented by the power series.

a) $\displaystyle\sum_{k=0}^{\infty}\frac{2^{2k}}{k!}\,x^k$ **b)** $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k 3^{2k}}{(2k+1)!}\,x^{2k+1}$ **c)** $\displaystyle\sum_{k=0}^{\infty}\frac{(-5)^{k+1}}{(2k)!}\,x^{2k+1}$

29. Use known Maclaurin series to determine the sum of the given series. Be careful!

a) $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}5^{2k}}{(2k+1)!}$ **b)** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3^k(2k)!}$ **c)** $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{2^k(k-1)!}$

30. For the given function, find $f^{(40)}(0)$. *Hint*: Use a Maclaurin series.

a) $f(x)=(1+2x)^{-1}$ **b)** $f(x)=x(3-x)^{-1}$ **c)** $f(x)=(1+3x^2)^{-1}$

31. Power series can be multiplied and divided just like polynomials, and the results give valid power series representations for the corresponding product and quotient. For example, using the Maclaurin series for e^x and $\sin x$, we can find the first few terms of the Maclaurin series for the product $e^x \sin x$:

$$\begin{aligned}
e^x\sin x &= \left(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\tfrac{1}{24}x^4+\cdots\right)\left(x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5-\tfrac{1}{5040}x^7+\cdots\right)\\
&= x+x^2+\tfrac{1}{2}x^3+\tfrac{1}{6}x^4+\tfrac{1}{24}x^5+\cdots\\
&\qquad\quad -\tfrac{1}{6}x^3-\tfrac{1}{6}x^4-\tfrac{1}{12}x^5+\cdots\\
&\qquad\qquad\qquad\qquad\quad \tfrac{1}{120}x^5+\cdots\\
&= x+x^2+\tfrac{1}{3}x^3-\tfrac{1}{30}x^5+\cdots.
\end{aligned}$$

Some care is required to guarantee that all of the terms of degree n have been obtained. Use this idea to find the first five nonzero terms of the Maclaurin series for the given function.

a) $e^x\cos x$ **b)** $e^{-x}\sin x$ **c)** $\tan x$

32. The purpose of this exercise is to find the sum of the series $\sum_{n=1}^{\infty} 1/n^2$.

a) Use a reduction formula and mathematical induction to prove that

$$\int_0^{\pi/2}\sin^{2n+1}x\,dx=\frac{4^n(n!)^2}{(2n+1)!}$$

for each positive integer n.

b) Use the result in part (a) and a trigonometric substitution to show that

$$\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx=\frac{4^n(n!)^2}{(2n+1)!}$$

for each positive integer n.

c) Use the Maclaurin series for $\arcsin x$ (see part (e) of Exercise 20) and term by term integration to show that

$$\frac{\pi^2}{8}=\int_0^1\frac{\arcsin x}{\sqrt{1-x^2}}\,dx=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

d) Prove that $\sum_{n=1}^{\infty}1/n^2=\pi^2/6$. *Hint*: Consider odd and even terms separately.

A

Algebra/Geometry Review

quadratic formula: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

factoring formulas:

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3)$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$

factorial notation: If n is a positive integer, then $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$. Also $0! = 1$.

Binomial Theorem: a formula for powers of $(a+b)^n$; for $(a-b)^n$, use $(a + (-b))^n$

	Pascal's Triangle
$(a+b)^0 = 1$	1
$(a+b)^1 = a + b$	1 1
$(a+b)^2 = a^2 + 2ab + b^2$	1 2 1
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	1 5 10 10 5 1

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2} a^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

completing the square: $x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}$

long division of polynomials: $\dfrac{x^3 - 4x^2 + 2x - 3}{x - 2} = x^2 - 2x - 2 - \dfrac{7}{x - 2}$ since

$$
\begin{array}{r|l}
 & x^2 - 2x - 2 \\
\hline
x - 2 & x^3 - 4x^2 + 2x - 3 \\
 & \underline{x^3 - 2x^2} \\
 & \quad -2x^2 + 2x - 3 \\
 & \quad \underline{-2x^2 + 4x} \\
 & \qquad\quad -2x - 3 \\
 & \qquad\quad \underline{-2x + 4} \\
 & \qquad\qquad\quad -7
\end{array}
$$

multiplying by the conjugate: use the fact that $(a+b)(a-b) = a^2 - b^2$ has no middle term

$$\frac{\sqrt{x^2+4x}-x}{x+1} = \frac{\sqrt{x^2+4x}-x}{x+1} \cdot \frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} = \frac{(x^2+4x)-x^2}{(x+1)\left(\sqrt{x^2+4x}+x\right)} = \frac{4x}{(x+1)\left(\sqrt{x^2+4x}+x\right)}$$

adding fractions: first find a common denominator

$$\frac{x}{2x+3} + \frac{5}{x-1} = \frac{x(x-1)}{(x-1)(2x+3)} + \frac{5(2x+3)}{(x-1)(2x+3)} = \frac{x(x-1)+5(2x+3)}{(x-1)(2x+3)} = \frac{x^2+9x+15}{(x-1)(2x+3)}$$

midpoint of line segment from (a_1, b_1) to (a_2, b_2) is $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$

distance between points (a_1, b_1) and (a_2, b_2) is $\sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}$

equation of circle with radius r and center (h, k) is $(x-h)^2 + (y-k)^2 = r^2$

Two Dimensional Figures

triangle: base b, height h, area $= \frac{1}{2}bh$

equilateral triangle: side s, area $= \frac{\sqrt{3}}{4}s^2$

similar triangles: Two triangles are similar if they have equal angles. Corresponding sides are in proportion.

trapezoid: bases B and b, height h, area $= \frac{1}{2}(B+b)h$

circle: radius r, circumference $= 2\pi r$, area $= \pi r^2$

circular sector: radius r, angle θ (radians): arc length $= \theta r$, area $= \frac{1}{2}\theta r^2$

Three Dimensional Figures

sphere: radius r, surface area$= 4\pi r^2$, volume $= \frac{4}{3}\pi r^3$

right circular cylinder: radius r, height h, lateral surface area $= 2\pi rh$, volume $= \pi r^2 h$

right circular cone: radius r, height h, lateral surface area $= \pi r\sqrt{r^2+h^2}$, volume $= \frac{1}{3}\pi r^2 h$

trigonometric identities: See Section 1.16.

properties of exponents and logarithms: See Section 1.19.

B

Table of Integrals

basic formulas

1. $\int u^r\,du = \frac{1}{r+1}\,u^{r+1},\ r \neq -1$
2. $\int \frac{du}{u} = \ln|u|$
3. $\int e^u\,du = e^u$
4. $\int \sin u\,du = -\cos u$
5. $\int \cos u\,du = \sin u$
6. $\int \sec^2 u\,du = \tan u$
7. $\int \csc^2 u\,du = -\cot u$
8. $\int \sec u \tan u\,du = \sec u$
9. $\int \csc u \cot u\,du = -\csc u$
10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin(u/a)$, where $a > 0$
11. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan(u/a)$, where $a > 0$
12. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,\mathrm{arcsec}(u/a)$, assuming that u is a positive function and $a > 0$

forms involving $a^2 - u^2$, where $a > 0$

13. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin(u/a)$

14. $\displaystyle\int u\sqrt{a^2-u^2}\,du = -\frac{1}{3}\,(a^2-u^2)^{3/2}$

15. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin(u/a)$

16. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin(u/a)$

17. $\displaystyle\int \frac{u}{\sqrt{a^2-u^2}}\,du = -\sqrt{a^2-u^2}$

18. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin(u/a)$

19. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|$

20. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \arcsin(u/a)$

21. $\displaystyle\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|$

22. $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du = -\frac{\sqrt{a^2-u^2}}{a^2u}$

23. $\displaystyle\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\arcsin(u/a)$

24. $\displaystyle\int u(a^2-u^2)^{3/2}\,du = -\frac{1}{5}\,(a^2-u^2)^{5/2}$

25. $\displaystyle\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|$

26. $\displaystyle\int \frac{1}{(a^2-u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2-u^2}}$

27. $\displaystyle\int \frac{u}{(a^2-u^2)^{3/2}}\,du = \frac{1}{\sqrt{a^2-u^2}}$

forms involving $a^2 + u^2$, where $a > 0$

28. $\displaystyle\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|$

29. $\displaystyle\int u\sqrt{a^2+u^2}\,du = \frac{1}{3}\,(a^2+u^2)^{3/2}$

30. $\displaystyle\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}\left(2u^2+a^2\right)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left|u+\sqrt{a^2+u^2}\right|$

31. $\displaystyle\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left|u+\sqrt{a^2+u^2}\right|$

32. $\displaystyle\int \frac{u}{\sqrt{a^2+u^2}}\,du = \sqrt{a^2+u^2}$

33. $\displaystyle\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|$

34. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|$

35. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left|u+\sqrt{a^2+u^2}\right|$

36. $\displaystyle\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|$

37. $\displaystyle\int \frac{1}{u^2\sqrt{a^2+u^2}}\,du = -\frac{\sqrt{a^2+u^2}}{a^2u}$

38. $\displaystyle\int (a^2+u^2)^{3/2}\,du = \frac{u}{8}(2u^2+5a^2)\sqrt{a^2+u^2} + \frac{3a^4}{8}\ln\left|u+\sqrt{a^2+u^2}\right|$

39. $\displaystyle\int u(a^2+u^2)^{3/2}\,du = \frac{1}{5}\,(a^2+u^2)^{5/2}$

40. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan(u/a)$

41. $\displaystyle\int \frac{1}{(a^2+u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2+u^2}}$

42. $\displaystyle\int \frac{u}{(a^2+u^2)^{3/2}}\,du = -\frac{1}{\sqrt{a^2+u^2}}$

forms involving $u^2 - a^2$, where $a > 0$

43. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right|$

44. $\displaystyle\int u\sqrt{u^2 - a^2}\, du = \frac{1}{3}(u^2 - a^2)^{3/2}$

45. $\displaystyle\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right|$

46. $\displaystyle\int \frac{1}{\sqrt{u^2 - a^2}}\, du = \ln\left|u + \sqrt{u^2 - a^2}\right|$

47. $\displaystyle\int \frac{u}{\sqrt{u^2 - a^2}}\, du = \sqrt{u^2 - a^2}$

48. $\displaystyle\int \frac{u^2}{\sqrt{u^2 - a^2}}\, du = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right|$

49. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\,\mathrm{arcsec}(u/a)$, assuming that u is a positive function

50. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right|$

51. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = a\,\mathrm{arcsec}(u/a)$, assuming that u is a positive function

52. $\displaystyle\int \frac{1}{u^2\sqrt{u^2 - a^2}}\, du = \frac{\sqrt{u^2 - a^2}}{a^2 u}$

53. $\displaystyle\int (u^2 - a^2)^{3/2}\, du = \frac{u}{8}(2u^2 - 5a^2)\sqrt{u^2 - a^2} + \frac{3a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right|$

54. $\displaystyle\int u(u^2 - a^2)^{3/2}\, du = \frac{1}{5}(u^2 - a^2)^{5/2}$

55. $\displaystyle\int \frac{1}{u^2 - a^2}\, du = \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right|$

56. $\displaystyle\int \frac{1}{(u^2 - a^2)^{3/2}}\, du = -\frac{u}{a^2\sqrt{u^2 - a^2}}$

57. $\displaystyle\int \frac{u}{(u^2 - a^2)^{3/2}}\, du = -\frac{1}{\sqrt{u^2 - a^2}}$

forms involving trigonometric, exponential, and logarithmic functions

58. $\int \tan u \, du = \ln|\sec u|$

59. $\int \cot u \, du = \ln|\sin u|$

60. $\int \sec u \, du = \ln|\sec u + \tan u|$

61. $\int \csc u \, du = \ln|\csc u - \cot u|$

62. $\int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u \, du$

63. $\int \cos^n u \, du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u \, du$

64. $\int \tan^n u \, du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \, du$

65. $\int \sec^n u \, du = \frac{1}{n-1} \sec^{n-2} u \tan u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$

66. $\int \sin au \sin bu \, du = \frac{\sin((a-b)u)}{2(a-b)} - \frac{\sin((a+b)u)}{2(a+b)}$

67. $\int \sin au \cos bu \, du = -\frac{\cos((a-b)u)}{2(a-b)} - \frac{\cos((a+b)u)}{2(a+b)}$

68. $\int \cos au \cos bu \, du = \frac{\sin((a-b)u)}{2(a-b)} + \frac{\sin((a+b)u)}{2(a+b)}$

69. $\int \arcsin u \, du = u \arcsin u + \sqrt{1-u^2}$

70. $\int \arctan u \, du = u \arctan u - \frac{1}{2} \ln\left|1 + u^2\right|$

71. $\int e^{au} \sin bu \, du = \frac{e^{au}}{a^2+b^2} \left(a \sin bu - b \cos bu\right)$

72. $\int e^{au} \cos bu \, du = \frac{e^{au}}{a^2+b^2} \left(b \sin bu + a \cos bu\right)$

73. $\int u^n \, e^{au} \, du = \frac{1}{a} u^n \, e^{au} - \frac{n}{a} \int u^{n-1} \, e^{au} \, du$

74. $\int (\ln u)^n \, du = u(\ln u)^n - n \int (\ln u)^{n-1} \, du$

C

Answers to Exercises

C.1 Chapter 1 Answers

Section 1.1: 1. $2y-3x=7$ 2. $2x+3y=0$, $3x-2y=13$ 3. $2x+3y=17$ 4. $m=-4/3$, $x=6$, $y=8$, $d=4.8$ 5. $(2,1)$ 6. $A(m)=-(3-2m)^2/(2m)$ 7. $y=x/4$ 8. $(0,1)$, $(7/25,-24/25)$

Section 1.2: 1. $0.571428\,571428\ldots$ 2. $19/99$ 3. $A\subseteq B$, $A=B$, $B\subseteq A$ 4. $\{x\in\mathbb{R}:x\neq 5\}$, $[-1,3]$, $(-2,2)$ 7. 5, -10, $-2/3$, 10, 7, -2 8. $4x^2-4x-3$, $2x^2+8x-3$, $4x-9$ 9. $f(x)=\sqrt{x}$, $g(x)=x^4+4x^2+1$; $f(x)=\sin x$, $g(x)=4x$; $f(x)=x^2$, $g(x)=\cos x$ 10. 1, $-2+\sqrt{2}$, $-2-\sqrt{2}$ 11. $(-5,5)$, $(2,12)$ 12. $2x-h$ 13. $\sqrt{3}\,P^2/24$ 14. $[-4,16]$, $(-5,1)$, $(-6.5,11.5)$

Section 1.3: 1. -2, $4/3$, $1/3$ 2. 0.500, -0.500, 2.718, 1.000, 0.434, 1.958 3. 0.693, 1.099, 1.386, 1.609 4. -1, 1, 0, -1, 1, 0 6. 2, 1, 1, 2, 0, 1, 2, 1 7. -8, 180, $7/4$

Section 1.4: 1. 5, -1; 0, $-\sqrt{6}$, $\sqrt{6}$; 0 3. -4, $1/3$, -1 or $3/2$ 6. $[5,6]$

Section 1.5: 1. $4/7$, $1/4$, $2/3$, 4, $1/4$, 8, -16, $20/3$, $-2/27$ 2. $3/2$, $11/4$ 3. 0, 0, 2 4. 1, 7, $2/3$, $3/5$, 0, 2

Section 1.6: 1. 6, 3, $-1/25$, -1, $1/9$, 1 2. $c/2$ 3. $y-1=4(x-1)$, $y-1=(1-x)/4$ 4. $3c^2$ 5. $3c^2+4c$, $-4/c^2$, $-5/c^{3/2}$ 6. 0, $-4/3$ 7. $(2,2)$, $(6,2/3)$ 8. 10 ft/sec

Section 1.7: 1. $2x$, $3x^2$, $4x^3$, $6x-4$, $1/2\sqrt{x}$, $2-5/x^2$ 2. $y-9=6(x-3)$, $y-9=(3-x)/6$ 3. $4/3$ 4. $-1/3$, 3 8. $f(x)=|2x-3|$ 10. $\{n/2:n\in\mathbb{Z}\}$, 1 or -1

Section 1.8: 1. $f'(x)=4$, $g'(x)=6x+13$, $h'(x)=-6x^2+6x-5$, $F'(t)=15t^4-8t^3+1$, $G'(z)=2z^3+z^2-2z$, $H'(u)=u^5+u^3-\frac{1}{6}u$, $\frac{dy}{dx}=100x^{99}+100$, $\frac{ds}{dt}=0.05t^4+0.6t^2-4$, $\frac{dz}{dw}=60w^{29}-\frac{1}{3}w^{10}$ 2. $y-3=5(1-x)$, $y-3=(x-1)/5$ 3. 0 4. $\sqrt{3}$,$-\sqrt{3}$ 5. 58 m/sec 6. 105.89 ft/min 7. 36,20; 12, 3; 6, 7; 9, 16; 3, 1/5; 27, -18

Section 1.9: 1. $f'(x)=\dfrac{1}{2\sqrt[4]{x^3}}-\dfrac{2}{\sqrt[3]{x^2}}$, $g'(x)=-\dfrac{28}{3x^5}$, $h'(x)=-\dfrac{5}{x^2}+\dfrac{24}{x^3}$, $F'(t)=\dfrac{2t+5}{8\sqrt{t^3}}$, $G'(z)=\dfrac{z^3+4z-12}{z^3}$, $H'(s)=3\sqrt{s}\left(2\sqrt{s}-1\right)$, $\dfrac{dy}{dx}=6x+\dfrac{4}{x^2}$, $\dfrac{dz}{dw}=8w^3-9w^2+12w-5$, $\dfrac{ds}{dt}=-\dfrac{7}{t^2}+\dfrac{1}{5t^{3/2}}$ 2. $y=2x+4$ 3. $2k$ 4. $(1,1)$, $(25,5)$ 5. $1/9$

Section 1.10: 1. $f'(x)=5(4x-3)(2x^2-3x+1)^4$, $g'(x)=12(x^2+2)(x^3+6x)^3$, $h'(x)=\dfrac{2x^3+3x}{\sqrt{x^4+3x^2+15}}$, $F'(t)=3(3t^2-6t+1)(t^3-3t^2+t)^2$, $G'(z)=\dfrac{z^2+2}{(z^3+6z)^{2/3}}$, $H'(s)=4(2s+5)(3s^2+15s-8)^{1/3}$, $\dfrac{dy}{dx}=-\dfrac{24(3x-2)}{(3x^2-4x+6)^4}$, $\dfrac{dz}{dw}=-\dfrac{6w}{(w^2+6)^{3/2}}$, $\dfrac{ds}{dt}=-\dfrac{15t(t+2)}{(t^3+3t^2)^2}$

2. $s'(x)=\dfrac{2x\sqrt{x^2+2x+3}+x+1}{2\sqrt{x^2+2x+3}\sqrt{x^2+\sqrt{x^2+2x+3}}}$ 3. $2x+3y=13$ 4. $(-4,2\sqrt{2})$ 5. -12, 15, 18 6. 1665

Section 1.11: 1. $f'(x) = \dfrac{3x+4}{2\sqrt{x+2}}$, $g'(x) = \dfrac{-7}{(2x-1)^2}$, $h'(x) = \dfrac{x+5}{(2x+5)^{3/2}}$, $u'(x) = \dfrac{8(5-x^2)}{\sqrt{10-x^2}}$, $v'(x) = (14x+5)(x+1)^2(2x-1)^3$, $w'(x) = \dfrac{-2(3x^2+x-11)}{(x^2+2x+4)^2}$, $\dfrac{dy}{dx} = \dfrac{x(x^2+8)}{(4+x^2)^{3/2}}$, $\dfrac{ds}{dt} = \dfrac{8(t-1)^3}{(t+1)^5}$, $\dfrac{dz}{dx} = \dfrac{x(4-7x)(x-2)^4}{3\left(4+x^2(2-x)^5\right)^{2/3}}$ 2. 2, $-1/3$, 1 3. $y = 1+5(x-4)/16$ 4. 4.5 s 5. $x-4y=-9$, $x-4y=-1$; $(-2,2)$, $(-2/3,-2)$ 6. -7, $-17/16$, -144

Section 1.12: 1. $M=20$, $m=11$; $M=47$, $m=20$; $M=232$, $m=7$; $M=83/5$, $m=16/3$; $M=9/4$, $m=\sqrt{2}$ 3. $2.5, 7.5$; $10-1/\sqrt{3}, 1/\sqrt{3}$ 4. $\sqrt{7}/2$

Section 1.13: 1. I on $(-\infty,0]$, $[3,\infty)$; D on $[0,3]$ 2a. I on $(-\infty,0]$, $[4,\infty)$; D on $[0,4]$ 2b. I on $[0,\infty)$; D on $(-\infty,0]$ 2c. I on $[4,\infty)$; D on $[0,4]$ 2d. I on $(-\infty,\sqrt{10}]$, $[-1,1]$, $[\sqrt{10},\infty)$; D on $[-\sqrt{10},-1]$, $[1,\sqrt{10}]$ 2e. I on $(-\infty,-1/\sqrt{2}]$, $[1/\sqrt{2},\infty)$; D on $[-1/\sqrt{2},0)$, $(0,1/\sqrt{2}]$ 2f. I on $(-\infty,0]$, $[8,\infty)$; D on $[0,8]$ 2g. I on $[0,3]$; D on $[3,6]$ 2h. I on $(-\infty,-1]$, $[3,\infty)$; D on $[-1,1)$, $(1,3]$ 2i. I on $(-\infty,-4]$, $[1/2,\infty)$; D on $[-4,1/2]$ 2j. I on $(-\infty,-a/\sqrt{3}]$, $[a/\sqrt{3},\infty)$; D on $[-a/\sqrt{3},a/\sqrt{3}]$ 2k. I on $(-\infty,-a]$, $[a,\infty)$; D on $[-a,0)$, $(0,a]$ 2l. I on $[a/\sqrt[3]{2},\infty)$; D on $(-\infty,0)$, $(0,a/\sqrt[3]{2}]$

Section 1.14: 1a. rel max at 0, rel min at 4 1b. rel min at 0, abs min is 0 1c. rel min at 4, abs min is -4 1d. rel max at $-\sqrt{10}$, 1, rel min at -1, $\sqrt{10}$ 1e. rel max at $-\sqrt{3}$, rel min at $\sqrt{3}$ 1f. rel max at 8, rel min at 0 1g. rel max at $-a$, rel min at a 1h. rel min at $-a$, a, abs min is $2a^2$ 1i. rel min at $-a$, rel max at a, abs min is $-1/2a$, abs max is $1/2a$ 2. $2\sqrt{10}$ 3. $1.5\sqrt[3]{20}$ 4. $2\sqrt{3}$ 5. $-2x^3+9x^2+60x$

Section 1.15: 1. $4\sqrt{30}$ m of fence 2. 31250 ft^2 3. 85.95 ft × 116.34 ft 4. 9.746 in × 7.309 in 5. 432 cm^2 6. 33.41 ft^3, \$240.64 7. $r=\sqrt[3]{150}$, $h=2r$ 8. $32\sqrt{3}/9$ 9. $\sqrt{3}s^2/8$ 10. $3\sqrt{3}r^2/4$, all $\sqrt{3}r$, 0.4135 11. $4\pi r^3/3\sqrt{3}$, $\sqrt{2}$, 0.57735 12. $32\pi r^3/81$, $\sqrt{2}$, 0.2963 13. $2\sqrt{181}$ 14. 24 min 58 s 15. $bx+ay=2ab$ 16. $d/(1+\sqrt[3]{a})$

Section 1.16: 1. $\sin(7\pi/6)=-1/2$, $\cos(7\pi/6)=-\sqrt{3}/2$ 2. $\sin x = 2/3$, $\cos x = \sqrt{5}/3$ 3. $\sin x = 4/\sqrt{17}$, $\cos x = -1/\sqrt{17}$ 4. 7.39, 9.46 5. $-5\pi/3$, $-\pi/3$, $\pi/3$, $5\pi/3$, $7\pi/3$, $11\pi/3$ 6. 0.7754, 2.3662, 7.0586, 8.6494 7. 0.3978, 1.1730, 3.5394, 4.3146 8. $\pi/6$, $5\pi/6$, $13\pi/6$, $17\pi/6$ 9. $-\pi/4$, $3\pi/4$, $7\pi/4$ 12. 13.57 13. 22.0°, 69.6°, 88.4° 17. $\pi/4$, $-\pi/6$, $2\pi/3$, $\pi/6$, $3\pi/4$, $-\pi/3$ 18. $x/\sqrt{1-x^2}$, $x/\sqrt{1+x^2}$, $1-2x^2$

Section 1.17: 1a. $f'(x) = \cos x + 6\sin 3x$ 1b. $g'(x) = 2x(\sin 2x + x\cos 2x)$ 1c. $h'(x) = -6\sin 2x\cos^2 2x$ 1d. $\dfrac{dy}{dx} = \dfrac{-(1+3\sin x)}{(3+\sin x)^2}$ 1e. $\dfrac{ds}{dt} = \sin t(2\cos^2 t - \sin^2 t)$ 1f. $\dfrac{dw}{dz} = 1-8\sec 4z\tan 4z$ 1g. $u'(x) = \cos^3 x$ 1h. $v'(\theta) = 10\tan 5\theta\sec^2 5\theta$ 1i. $w'(x) = \dfrac{-2\sin x + \cot^2 x(2\csc x - \sin x)}{(2+\cot^2 x)^2}$ 1j. $F'(t) = 2t\sin(1/t) - \cos(1/t)$ 1k. $G'(t) = 20\sec^2 4t$ 1l. $H'(x) = 48x\cos(2x^2)\sin^3(2x^2)$ 4. $2/r$, r, $r/7$ 5. $y+1/2 = 4\sqrt{3}(x-\pi/6)$ 6. $-\pi/3$, $\pi/3$, $5\pi/3$, $7\pi/3$, $11\pi/3$ 7. $a\tan a = 1$ 8. $f'(x) = \dfrac{x(1+2x^2\cos^2 5x - 5x^3\sin 5x\cos 5x)}{\sqrt{1+x^2+x^4\cos^2 5x}}$

Section 1.18: 1a. $f'(x) = 1/\sqrt{4-x^2}$ 1b. $g'(t) = 4/(16+t^2)$ 1c. $h'(x) = 3/\sqrt{x^4-9x^2}$ 1d. $dy/dx = 2\arcsin x/\sqrt{1-x^2}$ 1e. $ds/dt = 1+2t\arctan t$ 1f. $dr/d\theta = \arccos\theta$ 3. 2, $-\sqrt{3}$ 4. $\pi-2$, $-(\pi+6\sqrt{3})/6$ 5. $5/4$, -1 6. I on $[0,\pi/2]$, $[7\pi/6,3\pi/2]$, $[11\pi/6,2\pi]$; D on $[\pi/2,7\pi/6]$, $[3\pi/2,11\pi/6]$ 7. $\pi/3$ 8. $4\sqrt{3}$, $9-\sqrt{60}$, $\sqrt{a(a+b)}$

Section 1.19: 1. -2, -4, 5, $3/2$, 2, -1, $1/2$ 2. $r+s$, $s+t-r$, $-2s$, $2r+3t$, $3r+s$, $s+t$, $2s-r-t$, $s-4t$ 3. $5e+2$ 4. 2.322, 3.855, 9.486, 109.551, 7.885, ±1.317 5. 42°, 65.7°, 9.986 min 6. 25, $\ln 2$ 7. 36.75 ft, 33.099 miles 8. $\sqrt[10]{x}$, 10^{10} 11. e^a, e^a, e

Section 1.20: 1a. $f'(x) = e^x - 4e^{-2x}$ 1b. $g'(x) = 5+8xe^{-x^2}$ 1c. $h'(x) = -xe^{-x}$ 1d. $F'(x) = 2/(2x+1)$ 1e. $G'(x) = 2\cos 2x\, e^{\sin 2x}$ 1f. $H'(x) = 3e^{x/2}\sec^2(e^{x/2})$ 1g. $dy/dx = 2e^{-x}\sin(e^{-x})\cos(e^{-x})$ 1h. $ds/dt = e^{-2t}(3\cos 3t - 2\sin 3t)$ 1i. $dw/dz = 4(\ln z)^3/z$ 1j. $u'(x) = (2x+4)/(x^2+4x+2)$ 1k. $v'(x) = 1+\ln x$ 1l. $w'(x) = x^{-4}(1-3\ln x)$ 1m. $dy/dx = 3x^2 + (\ln 3)3^x$ 1n. $\dfrac{ds}{dt} = \dfrac{e^t(1-e^{2t})}{(1+e^{2t})^2}$ 1o. $\dfrac{dw}{dz} = \dfrac{z^2+\ln z}{z\sqrt{z^2+(\ln z)^2}}$ 2. -0.0445 m/sec^2 3. $3/\sqrt[3]{4}$ 4. a/e 5. 0, $2/e$ 6a. I on $(-\infty,-1]$, $[0,1]$; D on $[-1,0]$, $[1,\infty)$; abs max at ±1, abs min at 0 6b. I on $(0,\sqrt{e}]$; D on $[\sqrt{e},\infty)$; abs max at $\sqrt{e}$ 6c. I on $[1,e^2]$; D on $(0,1)$, $[e^2,\infty)$; abs min at 1, rel max at e^2 7. $(3,2e)$ 9. $1/\sqrt[e]{e}$ 10. $1/(\ln a)x$

Section 1.21: 1a. $f(x) = x^3+1$ 1b. $g(x) = 2\ln x + 5$ 1c. $h(x) = 2(x^{3/2}+1)/3$ 1d. $F(x) = x^3-4x^2+1$ 1e. $G(x) = \frac{1}{5}x^5 + \frac{1}{2}x^4 - \frac{1}{2}x^2 + x + \frac{9}{5}$ 1f. $u(t) = 2e^{2t}+10$ 1g. $v(t) = 9-4\cos(2t)$ 1h. $w(t) = \sqrt{t^2+1}+1$ 2a. $f(t) = 8e^{-2t}$ 2b. $g(t) = 5e^{4t}$ 2c. $h(t) = 2e^{5(t-1)}$ 2d. $F(x) = 12e^x - 2$ 2e. $G(x) = 4+26e^{-x}$ 2f. $H(x) = 47e^{3x}-1$ 3a. $A(t) = 24e^{2t}-4$ 3b. $B(t) = 3-e^{-5t}$ 3c. $S(t) = 240-200e^{-t/3}$ 4. $\sin 3x$, $\cos 3x$

Section 1.22: 1. 189.84 hr 2. 54280 3. 11.6 min 4. 39.23 min 5. 0.0322 lb/gal 6. 22.91 min 7. 48.56 days 8. 5.35 min 9. 5.94 yr, \$14,267; 15.27 yr, \$18,326 10. 6.64 yr, \$15,942; 25.58 yr, \$30,701 11. 11.55 yr, \$27,726; never

Section 1.23: 1. 0.0124, 0.0199, 0.0041, 0.3214, 0.1120, 0.0053 2. 0.002498, 0.000999, 0.000074 3. 0.2360, 0.0249, 0.0049 4. $19.875 < s < 20.124$ 5. $2.104 < r < 2.139$

Section 1.24: 1. 1/3, −2/5, −1/3, −3/2, 0, 0, 2, 2, 1/3 2. $\infty, \infty, -\infty, \infty, -\infty, \infty, -\infty, \infty, \infty$ 5. $x=-3/2$, $y=2$; $x=2$, $x=3$, $y=2$; $x=\ln 2$, $y=0$, $y=1$ 6. $x=0$, $x=-1$; $x=2$; $x=1/2$ 7. $\infty, -\infty, \infty$

Section 1.25: 1. 1, 40/7, −47, 1, 1/2, 4, 2, 0, 0 2. 3, 0, 2, e^2, e^2, e^2 3. r, 0, e^r, 0, 0, e^r 4. $16a/9$

Section 1.26: 1. 4, 16/3, 5/2, $3\sqrt{2}-2$, $\sqrt{(4-\pi)/\pi}$, $(1+\sqrt{7})/3$ 4. $(2a+b)/3$ 5. $(sa+rb)/(r+s)$ 6. $(a+b)/2$, arithmetic mean 7. $\sqrt{ab}$, geometric mean

Section 1.27: 2. $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2+x+C$, $g(x)=\frac{1}{2}\ln(x^2+4)+C$, $h(x)=-4\cos 2x+C$

Section 1.28: 1. $f''(x)=10(38x^2+38x+11)(x^2+x+1)^8$, $g''(x)=4(x^2+4)^{-3/2}$, $h''(x)=3\sin x(2\cos^2 x-\sin^2 x)$ 2. $2^n e^{2x}$, $(x+n)e^x$, $(-1)^n n! x^{-n-1}$ 3. $-\cos x$ 4a. CU on $(-\infty,1/3]$, CD on $[1/3,\infty)$, IP at $(1/3,2/27)$ 4b. CU on $(-\infty,0)$, $[1,\infty)$, CD on $(0,1]$, IP at $(1,0)$ 4c. CU on $(-\infty,1]$, $[2,\infty)$, CD on $[1,2]$, IP at $(1,11)$, $(2,24)$ 4d. CU on $(-\infty,-\sqrt{3}/3]$, $[\sqrt{3}/3,\infty)$, CD on $[-\sqrt{3}/3,\sqrt{3}/3]$, IP at $(-\sqrt{3}/3,6)$, $(\sqrt{3}/3,6)$ 4e. CU on $[e^{3/2},\infty)$, CD on $(0,e^{3/2}]$, IP at $(e^{3/2},3e^{-3/2}/2)$ 4f. CU on $(-\infty,-\sqrt{2}]$, $[\sqrt{2},\infty)$, CD on $[-\sqrt{2},\sqrt{2}]$, IP at $(-\sqrt{2},1/\sqrt{e})$, $(\sqrt{2},1/\sqrt{e})$ 5. $33\frac{1}{3}\%$ 7. $x^3-6x^2+4x+12$

Section 1.29: 1. increasing 2. 5/6, 11/216 3. $25t^2+10t+20$, $\frac{1}{3}t^3+\frac{1}{2}t^2-7t+4$, $4(t+4)^{3/2}-10t-44$, $2t^4+10t$ 4. 534.8 ft, 11.6 s 5. 81.8 mph 6. 576 ft 7. 9.5 s, 207.7 mph 8. 5 s, 61.5 m 9. 0.54 s 10. 385.7 mph 11. 4.4 m/s, 26.5 m/s, 1.0 m 12. 5.4 s, 41.4 m/s 13. 121 ft 14. 3.884 m/sec 15. $160\sqrt{10}$ ft

Section 1.30: 1a. $c(x)=1-(x-1)+(x-1)^2-(x-1)^3$ 1b. $c(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$ 1c. $c(x)=2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2+\frac{5}{20736}(x-8)^3$ 1d. $c(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ 1e. $c(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3$ 1f. $c(x)=x-\frac{1}{3}x^3$ 2. $\sqrt[10]{e}\approx 1.1, 1.105, 1.105167$, $\sqrt{e}\approx \frac{3}{2}, \frac{13}{8}, \frac{79}{48}$ 3. $14\frac{899}{1080}$ 4. $6\frac{1}{27}$ 6. 4, $4\frac{1}{96}$

Section 1.31: 1. 1.364656, 0.567143, 1.895494, 3.691203, 4.493409, 5.749031 2b. $x_2=577/408$, 97/56, 161/72, 49/20 3. $x_2=91/72$, 331/225 5. 2.21, 2.26

Section 1.32: 2a. $(2x-4y-1)/(4x-2y)$, 2b. $(3x^2-2)/(1-3y^2)$, 2c.$-(4x^3+6xy)/(3x^2+3y^2)$, 2d. $x(9-2x^2-2y^2)/y(9+2x^2+2y^2)$, 2e. $(3x^2+y)/(e^y-x-2y)$, 2f. $(y+1)(y^2+y-x^2)/x((y+1)^2-x^2)$ 3. $13(y-2)=6(1-x)$ 4a. 4/5, 5/4, −1 4b. $(2,(-1-\sqrt{33})/2)$, $(2,(-1+\sqrt{33})/2)$ 5. $4x+3y=24$ 6. $y=-2x+4$, $y=4(9-8x)/27$ 7. $3x+2y=5$ 8. $(\pm 2,\pm 2\sqrt{2})$ (four points) 10. $\left(1/\sqrt{3},2/\sqrt{3}\right)$

Section 1.33: 1. 120π cm^2/sec 2. $2\sqrt{3}$ cm^2/sec 3. $1/18\pi$ ft/min 4. 371.806 km/hr 5. 1/25 rad/min, $4\sqrt{5}$ m/min 6. 0.311 m/min 7. −0.1717 in/min, 0.1069 in/min 8. −3000 cm^3/min 9. 40.055 km/min 10. 1 in/min

Section 1.34: 1a. 2, $2\sqrt{5}/25$ 1b. 0, $3\sqrt{10}/50$ 1c. 0, $12\sqrt{17}/289$ 1d. 8, $44/577^{3/2}$ 1e. −2, $2e^2/(e^2+4)^{3/2}$ 1f. −1, $-\sqrt{2}/2$ 3. $(-119/2,53/6)$ 4. $(-4a^3,3a^2+1/2)$ 5. $1/\sqrt{8x}$ 6. $2\sqrt{3}/9$

Section 1.35: 1. $x+2y=10$ 2. $3x+5y=34$ 3. 1025 4. $x=11/4$, $y=4-\sqrt{10}$ 5. $\sqrt{10}/2$ 6. $(3,-1)$ 7. $(x-0.9)^2+(y-1.9)^2=4.42$ 8. 2, 1, 1, 4, 1, 1, 3, 4 9. 1, 2, 1/4 10. $\{-2,2\}$, $\{(2n-1)\pi : n\in\mathbb{Z}\}$, $\{1\}$ 13. $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{3}$, 6, $-\frac{1}{16}$, −2 14. $a=1$, $\frac{1}{2}$ 15. $y=3x+1$ 16. $4x-y=3$, $x+4y=5$ 17. $(0,0)$, $(6,216)$ 18. 24 19. $f'(x)=2/\sqrt[4]{x^3}-3/\sqrt[5]{x^4}$, $g'(x)=-18/5x^3$, $h'(x)=1/2\sqrt{x}+6/x^2$ 20. $x+y=4$, $y=x$ 21. $(5,\sqrt[4]{5})$ 22. $(216,6)$ 23. $f'(x)=15(3x-1)^4$, $g'(x)=-5x/\left(\sqrt{5x^2+4}\right)^3$, $h'(x)=8(2x+1)(x^2+x+1)^7$ 24. −1, 4/7 25. $f'(x)=2(x^2+1)/\sqrt{x^2+2}$, $g'(x)=11/(3x+4)^2$, $h'(x)=2x(14x-15)(4x-15)^4$ 26. −7/2, 13/3, −9/16 27. $3y-2x=4$ 28. $M=\sqrt{15}$, $m=\sqrt{6}$; $M=114$, $m=2$; $M=\frac{65}{4}$, $m=4$; $M=6a^3$, $m=-\frac{2\sqrt{3}}{9}a^3$; $M=\frac{1}{2}$, $m=\frac{3}{10}$; $M=(a+b)/a$, $m=2\sqrt{b/a}$ 29. $18\sqrt{2}\,\pi$ 30. $4\pi r^2h/27$, 4/9, $h/2r$ 31. $4L/(4+\pi)$ for square, $4L/(4+3\sqrt{3})$ for square, $\pi L/(3\sqrt{3}+\pi)$ for circle, $4L/(4+2\sqrt{3})$ for square 32. $x=1/\sqrt{3}$ 33. $|Ax_0+By_0+C|/\sqrt{A^2+B^2}$ 34. $\sqrt{(a+b)^2+d^2}$ 35. $x=2\sqrt{b/3a}$, $y=4b/3$ 36. 42.56 ft 37. 79.7° 39. $3\sqrt{3}/2$ 40. $\ln a$ 41. $\frac{21}{4}$, $\frac{1}{2}$, $2/\pi^2$, $\frac{1}{3}$, $\frac{9}{2}$, 0 42. $(\pm a/\sqrt{3},3/4)$; $(\pm a/\sqrt{2},1/\sqrt{e})$; $(a,0)$ 43. 10.2 mph, 0.94 s 44. 89.6 ft 45. $1280\sqrt{5}/3$ ft 46. 2, $28/145^{3/2}$ 47. $7\sqrt[6]{28}/9$ 50. $y=3x^5-8x^4+6x^3$ 51. $\pm\sqrt{\left(\sqrt{9b^2+5}-2b\right)/15}$

C.2 Chapter 2 Answers

Section 2.1: 1. $\sum_{i=1}^{200}(5i-3)$, $\sum_{i=1}^{29}\frac{i}{i+1}$, $\sum_{i=0}^{7}3^i$ 2. $\sum_{i=1}^{n+2}(4i-2)$, $\sum_{i=-1}^{n}(4i+6)$ 3. 12168, 3270, 54, 70499, $2n^2$, $\frac{1}{3}m(m+1)(m+2)$ 4. $(n+1)^6-1$, $2n/(2n+1)$, $a_1+a_2-a_{n+1}-a_{n+2}$ 5. $n(n+3)$, $\frac{1}{2}(n+2)(3n+5)$, $\frac{1}{2}n(6n^2-3n-1)$ 6. $\frac{1}{12}$

Section 2.2: 1. 576, $\frac{316}{3}$, $\frac{91}{3}$ 2. $\frac{1}{12}a^3$ 4. $\frac{32}{3}$, 156, $\frac{1056}{5}$ 5. $\frac{13}{2}$, $\frac{9}{4}\pi$, $\sqrt{5}+\frac{9}{2}\arcsin\frac{2}{3}$ 6. $1/\ln 2$ 7. $\frac{16}{3}$, 12

Section 2.3: 1. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(4\left(1+\frac{2i}{n}\right)^3+\left(1+\frac{2i}{n}\right)\right)\frac{2}{n}$, $\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n}$, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{2n+3i}$ 2. $\int_4^7 x^3\,dx$, $\int_0^2(5x^2+x)\,dx$, $\int_1^2 x^{-1}\,dx$ 3. $\frac{3}{2}$ 4. e^b-1

Section 2.4: 2. 420, 165, $\frac{15}{2}$, $\frac{142}{3}$, -27, $-\frac{83}{8}$ 3. 14, $\frac{1}{2}(1+\pi)r^2$, $\frac{7}{8}\pi$ 4. $\frac{73}{15}$, $\frac{1}{2}(1-\pi)$, -16 5. 4, 6, $\frac{17}{2}$ 6. $\frac{7}{3}t$, $\frac{3}{2}x^2$

Section 2.5: 1. $\frac{1}{3}$, 1; $\frac{3}{7}$, $\frac{3}{5}$; $4\sqrt[3]{6}$, $4\sqrt[3]{15}$ 2. 1^{st}, 1^{st}, 2^{nd} 4. -18, -5, $\frac{71}{3}$ 5. $\frac{8}{3}$ 7. $\sqrt{7}$ 8. $f'(x)=x^3-2x^2+5$

Section 2.6: 1. $f'(x)=\sqrt{x^2+9}$, $g'(x)=4/(2-x)$, $h'(t)=1/\sqrt[3]{t^4+9}$, $F'(x)=e^{-x^3}-1$, $G'(u)=2u^3\sqrt{u^6+2}$, $H'(x)=e^x\sin e^{2x}$, $u'(x)=2xe^{x^4}$, $v'(x)=2\cos 8x^3-\cos x^3$, $w'(x)=e^x\sqrt{e^{3x}+4}-2x\sqrt{x^6+4}$ 2. $-\frac{4}{\pi}$ 3. $\frac{1}{2}$ 4. $(-\infty,\frac{3}{2})$ 5. $\int_2^x \sin t^4\,dt$ 6. $\frac{1}{2}$, 0, 1, 1, 5, 3 7. $G(x)=2+\frac{1}{2}x|x|$

Section 2.7: 1. 20, 1, $\sqrt{2}-1$, $\frac{85}{6}$, $\frac{3}{8}$, $\frac{53}{12}-\frac{4}{3}\sqrt{2}$, $\ln 2$, $\frac{105}{2}$, e^2-1, $\frac{5}{2}\ln 3$, $\frac{1}{6}\pi$, π 2. π 3. $\ln b$

Section 2.8: 1. You should check these yourself. 2. $\frac{256}{15}$, $\frac{1}{4}\ln 5$, $-\frac{20}{9}$, $\frac{1}{\pi}$, $\left(\sqrt{2}-1\right)a$, $\frac{2\pi}{3\sqrt{3}}$ 3. $\frac{1}{3}+\frac{\pi}{2}$

Section 2.9: 1a. $\frac{1}{20}(x^4+6)^5$, 1b. $\frac{1}{4}\sqrt{4x^2+1}$, 1c. $2\sin\sqrt{t}$, 1d. $-\frac{1}{4}\left(x^2+4x-1\right)^{-2}$, 1e. $\frac{2}{3}(x-4)\sqrt{x+2}$, 1f. $\frac{1}{15}(3t-16)(2t+1)^{3/2}$, 1g. $3\ln\left|x^2+2x-3\right|$, 1h. $-\frac{1}{2(t^2+8)}$, 1i. $-\frac{1}{3}(x^2+2a^2)\sqrt{a^2-x^2}$, 1j. $2\arcsin\left(e^{t/2}\right)$, 1k. $\frac{1}{2}\arctan x^2$, 1l. $\frac{1}{4}\left(\frac{1}{1-2x}+\ln\left|2x-1\right|\right)$ 2. $\frac{2}{5}$, 2, $\frac{1}{9}$, $\frac{\pi}{2}$, $\frac{12}{5}\left(\sqrt{6}+\sqrt{3}\right)$, $\frac{1861}{40}$ 3. $\frac{2}{3}(x-2)\sqrt{x+1}$ 4. $3-\ln 8$ 5. $\pi\,ab$

Section 2.10: 1. Check by differentiation. 2. $\frac{\pi}{4}$, $(2\pi-4)/\pi^2$, $\frac{1}{4}(e^2+1)$, $16-12\sqrt[4]{e}$, $2e^2$, $(2\pi)^{-1}$ 3. e^2+1

Section 2.11: 1. 4, $\frac{4}{3}$, D, $\frac{3}{4}$, D, 5, 6, D, $10\sqrt{13}$ 3. $r>1$, $r<0$, $r<1$ 6. πa^2, $\frac{3}{8}\left(3^{2/3}-1\right)$, D

Section 2.12: 1. $\frac{1}{6}$, $\frac{32}{3}$, $\frac{15}{2}-8\ln 2$, $\frac{1}{6}a^3$, $\frac{3}{10}a^5$, $\frac{1}{2}(a^4-1)-2a^2\ln a$ 2. $\frac{125}{6}$ 3. 8 4. $12\ln 2-4.5$ 5. $x=\frac{3}{2}$, $y=12-6\sqrt{3}$ 6. $\frac{1}{6}a^2$ 7. $\frac{15}{4}$ 8. $\sqrt{3}$

Section 2.13: 1. $\frac{243}{5}\pi$, $\frac{\pi}{2}(e^4-1)$, 3π 2a. $\frac{64}{15}\pi$ 2b. $\frac{8}{3}\pi$ 3. $\frac{56}{15}\pi$ 6. $\frac{1}{3}\pi h^2(3r-h)$ 7. 64π, $\frac{512}{15}\pi$ 8. $32000\sqrt{3}$ 9. $\frac{625}{6}$ 10. $\frac{1}{3}a^2h$

Section 2.14: 1. $\pi(\pi-2)$, $10\pi\ln 10$, $2\pi(e^2+1)$ 2. $\frac{324}{5}\pi$ 3. 16π 6. $\frac{4}{3}\pi(R^2-r^2)^{3/2}$, $\sqrt{1-4^{-1/3}}\,R$, $\frac{1}{6}\pi h^3$ 7. $(2\pi a)(\pi r^2)$, $\frac{4}{3}\pi r^3+a\pi^2r^2$ 9. $\sqrt[4]{8}$, ≈ 1.2285

Section 2.15: 1. 8985.6 lbs 2. $85w$, $\left(54+36\sqrt{3}\right)w$, $\frac{415}{3}w$, $\left(\frac{63}{2}\pi-18\right)w$, $24\pi w$, $120w$ 3. 9052.16 lbs

Section 2.16: 1. 25/6 ft-lbs 2. 1/3 ft-lbs 3. $3\sqrt{10}$ in 4. 63303.1 ft-lbs 5. 55734.8 ft-lbs 6. 3993.6 ft-lbs 7. 4800 ft-lbs 8. 332773.1 mi-lbs; 19507.4 mi-lbs; 338812.5 mi-lbs

Section 2.17: 1. $(3.7, 0.2, 2.3)$ 2. $(-8,-22)$ 3. $(\frac{9}{4},\frac{27}{10})$, $(3/\ln 4, 3/\ln 16)$, $(\frac{5}{12},\frac{4}{9})$ 4. $(\frac{1}{2}a,\frac{2}{5}a^2)$ 5. $(\frac{4}{3\pi}r,\frac{4}{3\pi}r)$ 6. $\frac{1}{4}h$ 7. $\frac{3}{8}r$ 8. $\frac{11}{14}$ ft 9. $\frac{13}{22}$ 11. $(\frac{8}{5},\frac{16}{7})$ 13. ≈ 26 15. $\frac{10}{3}$, $(3e^4+1)/(2e^4-2)$, $\frac{18}{13}$

Section 2.18: 1. $\frac{1}{2}\pi r$ 2. $\frac{1}{27}((4+9b)^{3/2}-8)$ 3. $\frac{19}{6}$, $\frac{15}{8}+\ln 4$, $\frac{99}{20}$, $\frac{227}{48}$, $\frac{779}{240}$, $\ln|2+\sqrt{3}|$ 4. 6 5. $\frac{2048}{9375}+\frac{62176\sqrt{29}}{9375}$ 6. $(\frac{2}{\pi}r,\frac{2}{\pi}r)$

Section 2.19: 1a. $\frac{3}{2}\ln\left|8+4x+x^2\right|-4\arctan\left(\frac{x+2}{2}\right)$ 1b. $-2\sqrt{8+2x-x^2}+3\arcsin\left(\frac{x-1}{3}\right)$ 1c. $4\arcsin\left(\frac{x-3}{3}\right)$ 1d. $2\ln\left|x^2+3x+5\right|-\frac{6}{\sqrt{11}}\arctan\left(\frac{2x+3}{\sqrt{11}}\right)$ 1e. $\frac{5}{2}\ln\left|x^2+4x+10\right|-\frac{\sqrt{6}}{2}\arctan\left(\frac{x+2}{\sqrt{6}}\right)$
1f. $-6\sqrt{4x-x^2}+11\arcsin\left(\frac{x-2}{2}\right)$
2a. $\frac{1}{3}x-\frac{1}{9}\ln|3x+1|$ 2b. $\frac{1}{2}\ln|x^2+10|+\frac{3}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right)$ 2c. $x-2\arctan\left(\frac{x}{2}\right)$
2d. $2x+\frac{3}{2}\ln|x^2+1|-\arctan x$ 2e. $\frac{1}{2}x^2-2x+\ln|x^2+2x+2|+2\arctan(x+1)$ 2f. $2x+8\ln|x-4|$
3. $2\left(\ln|x-6|-\ln|x|\right)$

Section 2.20: 1a. $\frac{1}{4}x(x^2+2)\sqrt{4+x^2}-2\ln\left|x+\sqrt{4+x^2}\right|$ 1b. $\frac{x}{25\sqrt{25+4x^2}}$ 1c. $\frac{1}{2}\ln\left|\frac{x^2-3}{x^2+3}\right|$
1d. $3\ln\left|x+2+\sqrt{x^2+4x+8}\right|$ 1e. $-\frac{\sqrt{4x^2-9}}{x}+2\ln\left|2x+\sqrt{4x^2-9}\right|$ 1f. $\sqrt{5-x^2}-\sqrt{5}\ln\left|\frac{\sqrt{5}+\sqrt{5-x^2}}{x}\right|$
1g. $-e^{-x}\sqrt{e^{2x}-1}+\ln\left|e^x+\sqrt{e^{2x}-1}\right|$ 1h. $3\ln\left|\frac{e^x}{2+\sqrt{4-e^{2x}}}\right|$ 1i. $\cos x\sqrt{9-\cos^2 x}-9\arcsin\left(\frac{\cos x}{3}\right)$
2. $9\left(2\sqrt{3}-\ln\left(2+\sqrt{3}\right)\right)$ 3. $4\sqrt{17}+2\sqrt{65}+\frac{1}{4}\ln\left(\sqrt{65}+8\right)$ 6a. $\frac{1}{8}\left(2\sec^3 x\tan x+3\sec x\tan x+3\ln|\sec x+\tan x|\right)$
6b. $\frac{1}{8}\left(2\cos^3 x\sin x+3\cos x\sin x+3x\right)$ 6c. $\frac{1}{48}\left(8\sec^5 x\tan x-2\sec^3 x\tan x-3\sec x\tan x-3\ln|\sec x+\tan x|\right)$

6d. $\dfrac{1}{48}\Big(8\sin^5 x\cos x - 2\sin^3 x\cos x - 3\cos x\sin x + 3x\Big)$ 6e. $-\dfrac{1}{2}\,(x^6+3x^4+6x^2+6)e^{-x^2}$
6f. $x\Big((\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\Big)$ 7. $3\sqrt{3}\,b^2/2$

Section 2.21: 1a. $4\ln\left|\dfrac{4}{x} - \dfrac{\sqrt{16-x^2}}{x}\right| + \sqrt{16-x^2}$ 1b. $-\dfrac{(x^2+16)^{3/2}}{48x^3}$ 1c. $\dfrac{1}{2}\left(\dfrac{x}{x^2+1} + \arctan x\right)$
1d. $\dfrac{2x}{(x^2+1)^2} + \dfrac{3x}{x^2+1} + 3\arctan x$ 1e. $\dfrac{x}{\sqrt{1-x^2}} - \arcsin x$ 1f. $\dfrac{x^3}{3(1-x^2)^{3/2}}$

Section 2.22: 1a. $\dfrac{1}{5}\ln|x| - \dfrac{1}{5}\ln|x+5|$ 1b. $\dfrac{11}{5}\ln|x-4| - \dfrac{1}{5}\ln|x+1|$ 1c. $\dfrac{3}{2}\ln|2x-1| - \ln|x+3|$
1d. $2\ln|x| + 2\ln|x-2| - 4\ln|x-1|$ 1e. $3\ln|x| - 2\ln|x-1| - \dfrac{6}{x-1}$
1f. $2\ln|x| + 2\ln|x^2+6x+10| - 17\arctan(x+3)$ 1g. $4\ln|x| - 2\ln|x^2+1| + \arctan x$
1h. $\dfrac{1}{4}\ln\left|\dfrac{1-x}{1+x}\right| - \dfrac{1}{2}\arctan x$ 1i. $\ln|x^2+4| + 5\arctan x + \dfrac{3}{2}\arctan(x/2)$
1j. $\ln|x-2| - \dfrac{1}{2}\ln|x^2+2x+4| - \sqrt{3}\arctan\Big((x+1)/\sqrt{3}\Big)$ 1k. $\dfrac{1}{4}x^2 - \dfrac{3}{4}x + \dfrac{8}{5}\ln|x+2| + \dfrac{1}{40}\ln|2x-1|$
1l. $\dfrac{3x+1}{4(x^2+2x+2)^2} + \dfrac{9(x+1)}{8(x^2+2x+2)} + \dfrac{9}{8}\arctan(x+1)$

Section 2.23: 1. 0.1361, 0.1353; 0.3160, 0.3099; 0.5125, 0.5127 2. 0.9461 3. 3.8194 4. 18, $\frac{290}{3}\pi$, $(\frac{89}{27}, \frac{145}{54})$
5. 52 m

Section 2.24: 1. n^2, $\frac{1}{3}n(4n^2-1)$, $n^2(2n^2-1)$ 2. 12, $\frac{3}{2}$ 3. 5050, 73810 4. 37/6
5. 26/3 6. 40 7. 2^{nd}, 2^{nd}, 1^{st} 8. $\sqrt[3]{3.75}$, $(a+b)/2$, $\sqrt{(a^2+ab+b^2)/3}$
9. $4x/\sqrt{x^4+2}$, $4\cos(16x^2)$, $4te^{-2t^4}$ 10. $\frac{2}{3}$ 11. $(-\infty, \frac{1}{4})$ 12. $f(x) = \int_1^x e^{t^2/4}\,dt$
15. $36 - 24\ln 2$, $\pi - \frac{2}{5}$, $\frac{320}{3}$ 16. $\pi/\sqrt{2} - \ln\Big(3+2\sqrt{2}\Big)$ 17. 16 18. $x=2$, $y = \frac{4}{3}\ln 2$ 19. $2 - \sqrt[3]{4}$
20. $(3.1, -0.4)$ 21. $(2, 4/5)$ 22. $(12/13, 7/13)$ 23. $\overline{x} = (3e^4+1)/2(e^4-1)$ 24. $\overline{y} = 4/3$ 26. $(\pi/2, \pi/8)$
28a. $-\frac{1}{15}\cos x\Big(3\sin^4 x + 4\sin^2 x + 8\Big)$ 28b. $(x^3 - 3x^2 + 6x - 6)e^x$ 28c. $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$
28d. $-(x^2+2x+2)e^{-x}$ 28e. $2\Big(x^{5/2} - 5x^2 + 20x^{3/2} - 60x + 120x^{1/2} - 120\Big)e^{\sqrt{x}}$ 28f. $9e - 24$ 32. $100\sqrt{2}\,\pi^2$
33. 1378.7 gal 34. 54π 35g. $\frac{27}{2}$ 35h. $\frac{64}{3}$ 35i. $64 + 20\pi$ 35p. 1 35q. 25 35r. D, check the indefinite integrals by differentiation 36. $\frac{8}{3}$, $\frac{32}{5}\pi$, 8π, $\frac{56}{3}\pi$, 24π, $\frac{224}{15}\pi$, $\frac{3}{2}$, $\frac{6}{5}$, $\frac{5}{3}$, $6 + \sqrt{17} - \frac{1}{4}\ln\Big(\sqrt{17} - 4\Big)$, $\frac{272}{15}w$
37. 2, $\frac{1}{2}\pi^2$, $2\pi^2$, $2\pi(10-\pi)$, $2\pi(6+\pi)$, $\frac{1}{2}\pi(32-\pi)$, $\frac{1}{2}\pi$, $\frac{1}{8}\pi$, $\frac{1}{2}\pi$, 6.962, $\frac{1}{4}(64-\pi)w$ 38. $\frac{64}{3}$, $(0, 2.4)$ assuming vertex at $(0,0)$, 10117.12 lbs 39. $\frac{2}{3}\pi(2r^2+a^2)\sqrt{r^2-a^2} + a\pi r^2\Big(2\arcsin(a/r) - \pi\Big)$ for $0 < a < r$

C.3 Chapter 3 Answers

Section 3.2: 2. $\frac{1}{3}, \frac{1}{2}, \frac{1}{9}, \frac{1}{4}, \frac{1}{15}, \ldots$; $\frac{1}{2}, \frac{3}{4}, \frac{15}{8}, \frac{105}{16}, \frac{945}{32}, \ldots$; $2, \frac{7}{2}, \frac{25}{6}, \frac{35}{8}, \frac{177}{40}, \ldots$ 3. $\left\{n/2^n\right\}$, $\left\{1-(-1)^n\right\}$, $\left\{1/n(n+1)\right\}$ 4. $\frac{2}{7}$, $-\frac{1}{3}$, $\frac{1}{2}$, 0, $\frac{1}{2}$, 0, π, $\ln 2$, e^{-1} 5. I, D, I

Section 3.3: 1. 1, 3, -7 2. e^4, $e^{-1/2}$, e^{-3}, 0, $\frac{1}{2}$, 1, 0, 1, 0 4. b

Section 3.4: 4. $\left(\sqrt{5}-1\right)/2$ 5. $\sqrt{10}$ 6. 24/7 7. $\left(3+\sqrt{5}\right)/2$ 8. $\left(1+\sqrt{13}\right)/2$

Section 3.5: 1. $n/5$, $(5^n-1)/(4\cdot 5^n)$, $n/(4n+16)$ 3. $\frac{1}{5}$, $\frac{4}{9}$, $b^2/(1+b^2)$, $\frac{1}{3}$, $-\frac{10}{7}$, $\frac{9}{235}$, 4, 9, 19 5. -7, $\frac{23}{2}$, $-\frac{1}{26}$, $\frac{2}{3}$

Section 3.6: 1. C, D, C, C 3. $n = 485, 165, 195$ 4. $p>1$, $p>1$, $2-\sqrt{3}<p<2+\sqrt{3}$, $p>1$

Section 3.7: 1. C, D, C 2. C, D, D 3. D, C, D, C, D, D

Section 3.8: 2. C, C, D, C 4. $0<p\le 1$ 8. AC, D, CC, AC

Section 3.9: 2. C, C, D, C, C, C 3. CC, AC, D, AC, CC, AC 6. $-2<x<2$, $-\infty<x<\infty$, $-3<x<7$

Section 3.10: 1. 1, ∞, $\frac{5}{3}$ 2. $[-5,5]$, $(1,7)$, $[-2.5,-1.5)$ 3. C at $0, 4.4$, D at $\sqrt{110}$, not enough info for $8.5, -2$; $(-2,8]$, $[-3,9)$ 4. 3.5, 5.5 5. $[-1,1]$, $[-1,1)$, $(-1,1]$, $(-1,1)$ 6. $\sum_{k=1}^{\infty}\frac{x^k}{k^2 3^k}$, $\sum_{k=1}^{\infty}\frac{(x-5)^k}{k3^k}$, $\sum_{k=1}^{\infty}\frac{(x-4)^k}{8^k}$, $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k2^k}(x+1)^k$ 7. $2/(2-x)$, $1/(2x+7)$, $(x-1)/(4-x)$

Section 3.11: 1a. $\sum_{k=0}^{\infty}4^k x^k$, $(-\frac{1}{4},\frac{1}{4})$ 1b. $\sum_{k=1}^{\infty}(-2)^{k-1}x^k$, $(-\frac{1}{2},\frac{1}{2})$ 1c. $\sum_{k=0}^{\infty}5^{-k-1}x^k$, $(-5,5)$ 1d. $\sum_{k=0}^{\infty}(-1)^k x^{2k+1}$, $(-1,1)$ 1e. $\sum_{k=0}^{\infty}(-1)^k x^{2k+1}/(2k+1)$, $[-1,1]$ 1f. $\sum_{k=0}^{\infty}\frac{1}{2}(k+2)(k+1)x^k$, $(-1,1)$ 2. $(x+x^2)/(1-x)^3$ 3. $\sum_{k=0}^{\infty}\frac{k+1}{2^{k+1}}x^k$, $\sum_{k=2}^{\infty}\frac{5^{k-1}}{(2k-1)!}x^k$, $\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{4^{k+3}}x^k$ 4. $5\ln 2$, 12, $\frac{1}{4}\ln(0.8)$ 5. $c_k = 1/k!$

Section 3.12: 1a. $\sum_{k=0}^{\infty}\frac{2^k}{k!}x^k$ 1b. $\sum_{k=0}^{\infty}\frac{(-1)^k 5^{2k+1}}{(2k+1)!}x^{2k+1}$ 1c. $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{4k}$ 1d. $\sum_{k=1}^{\infty}\frac{(-16)^{k-1}}{(2k-2)!}x^{2k}$ 1e. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k-1)!}x^k$ 1f. $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+3)!}x^{2k}$ 1g. $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)(2k+1)!}x^{2k+1}$ 1h. $\sum_{k=1}^{\infty}\frac{1}{(2k)!}x^{2k}$ 1i. $1+\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}x^{2k}$ 2. e^{-x}, e^{-x^2}, $x\cos\left(\sqrt{2}\,x\right)$ 3a. $3+\frac{1}{6}(x-9)+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-3)}{2^k\,3^{2k-1}\,k!}(x-9)^k$, $\rho=9$ 3b. $1+\frac{1}{3}(x-1)+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3k-4)}{3^k\,k!}(x-1)^k$, $\rho=1$ 3c. $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2^{k+1}}(x-5)^k$, $\rho=2$ 4. $1+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2^k\,k!}x^k$ 5. $-100\cdot 2^{99}$, $100!/33!$, $-100!/52!$ 7. $i^i = e^{-\pi/2}$

Section 3.13: 2. $\frac{1}{120}$, $P_{14}=P_{13}$, P_{10}, 0.0233 3. $2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3-\frac{5}{16384}(x-4)^4+\frac{7}{131072}(x-4)^5$ 4. $2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2+\frac{5}{20736}(x-8)^3-\frac{5}{248832}(x-8)^4$ 5. $2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\frac{2}{7}x^7$

Section 3.14: 1. $\frac{5}{8}$, $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{\pi}{4}$, $\ln 7$, 3, $e^{1/4}$, $e^{-5/2}$ 2. $\frac{3}{4}$ 3. $x_k = 3/(6k+2)$ 4. $\frac{7}{6}$ 6. $\left(3-\sqrt{5}\right)/2$ 8. z 11. $-\frac{1}{7}$, $\frac{7}{15}$ 12. $(-1,1)$, $(2+r)/(1-r^2)$, $1+\frac{\sqrt{3}}{2}$ 14. D, CC, AC, AC, D, AC, AC, AC, AC 15. $\pi^2/8$, $\pi^2/12$ 16. $\frac{4}{7}$, $\frac{2}{7}$, $\frac{1}{7}$ 17. $\frac{3}{13}$, $\frac{5}{13}$, $\frac{5}{13}$; $\frac{6}{13}$, $\frac{2}{13}$, $\frac{5}{13}$; $\frac{9}{13}$, $\frac{3}{26}$, $\frac{5}{26}$ 19. 1000, 15, 10^{12}, $\exp(10^6)\approx 10^{434294}$, 10^6, 23, 6, 17 20d. $\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2}x^k$ 20e. $\sum_{k=0}^{\infty}\frac{(2k)!}{(2k+1)4^k(k!)^2}x^{2k+1}$ 21. $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\,k!}$, $\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)(2k+1)!}$, $\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)(2k)!}$ 22. 4, 7 23. $\sqrt[3]{\rho}$ 24. $\sum_{k=1}^{\infty}\frac{1}{2^k}x^k$, $\sum_{k=1}^{\infty}\frac{k}{5^{k+1}}x^k$, $\sum_{k=0}^{\infty}\frac{(-1)^k}{6k+2}x^{6k+2}$ 25. $(x^3+4x^2+x)/(1-x)^4$, $(x^4+11x^3+11x^2+x)/(1-x)^5$ 26. 6, $\frac{3}{4}$, -3 27. $\frac{5}{16}$, $\frac{20}{27}$, 222 28. e^{4x}, $\frac{1}{3}\sin 3x$, $-5x\cos\left(\sqrt{5}\,x\right)$ 29. $-\frac{1}{5}\sin 5$, $1-\cos\left(1/\sqrt{3}\right)$, $\frac{1}{2}\left(e^{-1/2}-1\right)$ 30. $40!\cdot 2^{40}$, $40!/3^{40}$, $40!\cdot 3^{20}$ 31a. $1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\frac{1}{630}x^7+\cdots$ 31b. $x-x^2+\frac{1}{3}x^3-\frac{1}{30}x^5+\frac{1}{90}x^6-\frac{1}{630}x^7+\cdots$ 31c. $x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\cdots$