Introduction to partial differential equations

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Lecture Notes

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0 Preliminaries

We consider Euclidean space $\mathbb{R}^n, n \geq 1$ with elements $x = (x_1, \dots, x_n)$. The Euclidean length of x is defined by

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and the standard inner product by

$$(x, y) = x_1 y_1 + \cdots + x_n y_n.$$

We use the Cauchy-Schwarz-Bunjakovskii inequality in $\mathbb{R}^n$

$$|(x, y)| \leq |x| \cdot |y|.$$

By $B_R(x)$ we denote the ball of radius $R > 0$ with center x

$$B_R(x) := \{y \in \mathbb{R}^n : |x - y| < R\}.$$

We say that $\Omega \subset \mathbb{R}^n, n \geq 2$ is an open set if for any $x \in \Omega$ there is $R > 0$ such that

$$B_R(x) \subset \Omega.$$

If $n = 1$ by open set we mean the open interval $(a, b), a < b$.

An n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers will be called a *multi-index*. We define

(i) $|\alpha| = \sum_{j=1}^{n} \alpha_j$

(ii) $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$ with $|\alpha + \beta| = |\alpha| + |\beta|$

(iii) $\alpha! = \alpha_1! \cdots \alpha_n!$ with $0! = 1$

(iv) $\alpha \geq \beta$ if and only if $\alpha_j \geq \beta_j$ for each $j = 1, 2, \dots, n$. Moreover, $\alpha > \beta$ if and only if $\alpha \geq \beta$ and there exists j_0 such that $\alpha_{j_0} > \beta_{j_0}$.

(v) if $\alpha \geq \beta$ then $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ and $|\alpha - \beta| = |\alpha| - |\beta|$.

(vi) for $x \in \mathbb{R}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

with $0^0 = 1$.

We will use the shorthand notation

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

This text assumes that the reader is familiar also with the following concepts:

1) Lebesgue integral in a bounded domain $\Omega \subset \mathbb{R}^n$ and in $\mathbb{R}^n$.

2) Banach spaces (L^p, $1 \le p \le \infty$, C^k) and Hilbert spaces (L^2): If $1 \le p < \infty$ then we set

$$L^p(\Omega) := \{f : \Omega \to \mathbb{C}\,\text{measurable} : \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < \infty\}$$

while

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{C}\,\text{measurable} : \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty\}.$$

Moreover

$$C^k(\overline{\Omega}) \quad := \quad \{f : \overline{\Omega} \to \mathbb{C} : \|f\|_{C^k(\overline{\Omega})} := \max_{x\in\overline{\Omega}} \sum_{|\alpha|\le k} |\partial^\alpha f(x)| < \infty\},$$

where $\overline{\Omega}$ is the closure of Ω. We say that $f \in C^\infty(\Omega)$ if $f \in C^k(\overline{\Omega_1})$ for all $k \in \mathbb{N}$ and for all bounded subsets $\Omega_1 \subset \Omega$. The space $C^\infty(\Omega)$ is not a normed space. The inner product in $L^2(\Omega)$ is denoted by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}dx.$$

Also in $L^2(\Omega)$, the duality pairing is given by

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

3) Hölder's inequality: Let $1 \le p \le \infty, u \in L^p$ and $v \in L^{p'}$ with

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then $uv \in L^1$ and

$$\int_\Omega |u(x)v(x)|dx \le \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^{p'} dx\right)^{\frac{1}{p'}},$$

where the Hölder conjugate exponent p' of p is obtained via

$$p' = \frac{p}{p-1}$$

with the understanding that $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$.

4) Lebesgue's theorem about dominated convergence:

Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_k\}_{k=1}^\infty$ be a sequence of measurable functions converging to $f(x)$ point-wise in A. If there exists function $g \in L^1(A)$ such that $|f_k(x)| \le g(x)$ in A, then $f \in L^1(A)$ and

$$\lim_{k\to\infty} \int_A f_k(x)dx = \int_A f(x)dx.$$

5) Fubini's theorem about the interchange of the order of integration:

$$\int_{X\times Y} |f(x,y)|dxdy = \int_X dx \left(\int_Y |f(x,y)|dy \right) = \int_Y dy \left(\int_X |f(x,y)|dx \right),$$

if one of the three integrals exists.

Exercise 1. Prove the generalized Leibnitz formula

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} C_\alpha^\beta \partial^\beta f \partial^{\alpha-\beta} g,$$

where the generalized binomial coefficients are defined as

$$C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!} = C_\alpha^{\alpha-\beta}.$$

Hypersurface

A set $S \subset \mathbb{R}^n$ is called *hypersurface* of class $C^k, k = 1, 2, \dots, \infty$, if for any $x_0 \in S$ there is an open set $V \subset \mathbb{R}^n$ containing x_0 and a real-valued function $\varphi \in C^k(V)$ such that

$$\nabla\varphi \equiv (\partial_1\varphi, \dots \partial_n\varphi) \neq 0 \quad \text{on} \quad \mathrm{S} \cap \mathrm{V}$$

$$S \cap V = \{x \in V : \varphi(x) = 0\}.$$

By implicit function theorem we can solve the equation $\varphi(x) = 0$ near x_0 to obtain

$$x_n = \psi(x_1, \dots, x_{n-1})$$

for some C^k function ψ. A neighborhood of x_0 in S can then be mapped to a piece of the *hyperplane* $\widetilde{x_n} = 0$ by

$$x \mapsto (x', x_n - \psi(x')),$$

where $x' = (x_1, \dots, x_{n-1})$. The vector $\nabla\varphi$ is perpendicular to S at $x \in S \cap V$. The vector $\nu(x)$ which is defined as

$$\nu(x) := \pm \frac{\nabla\varphi}{|\nabla\varphi|}$$

is called the *normal* to S at x. It can be proved that

$$\nu(x) = \pm \frac{(\nabla\psi, -1)}{\sqrt{|\nabla\psi|^2 + 1}}.$$

If S is the boundary of a domain $\Omega \subset \mathbb{R}^n, n \ge 2$ we always choose the orientation so that $\nu(x)$ *points out* of Ω and define the normal derivative of u on S by

$$\partial_\nu u := \nu \cdot \nabla u \equiv \nu_1 \frac{\partial u}{\partial x_1} + \dots + \nu_n \frac{\partial u}{\partial x_n}.$$

Thus ν and $\partial_\nu u$ are C^{k-1} functions.

Example 0.1. Let $S_r(y) = \{x \in \mathbb{R}^n : |x - y| = r\}$. Then

$$\nu(x) = \frac{x-y}{r} \quad \text{and} \quad \partial_\nu = \frac{1}{r}\sum_{j=1}^n (x_j - y_j)\frac{\partial}{\partial x_j}.$$

The divergence theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary $S = \partial\Omega$ and let F be a C^1 vector field on $\overline{\Omega}$. Then

$$\int_\Omega \nabla \cdot F dx = \int_S F \cdot \nu d\sigma(x).$$

Corollary (Integration by parts)**.** *Let f and g be C^1 functions on $\overline{\Omega}$. Then*

$$\int_\Omega \partial_j f \cdot g dx = -\int_\Omega f \cdot \partial_j g dx + \int_S f \cdot g\nu_j d\sigma(x).$$

Let f and g be locally integrable functions on $\mathbb{R}^n$, i.e. integrable on any bounded set from $\mathbb{R}^n$. The *convolution* $f * g$ of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = (g * f)(x),$$

provided that the integral in question exists. The basic theorem on the existence of convolutions is the following (*Young's inequality for convolution*):

Proposition 1 (Young's inequality)**.** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n), 1 \le p \le \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$

Proof. Let $p = \infty$. Then

$$|(f * g)(x)| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \le \|g\|_{L^\infty} \int_{\mathbb{R}^n} |f(x-y)|dy = \|g\|_{L^\infty} \|f\|_{L^1}.$$

Let $1 \le p < \infty$ now. Then it follows from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
\int_{\mathbb{R}^n} |(f * g)(x)|^p dx &\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \right)^p dx \\
&\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|dy \right)^{p/p'} \int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy dx \\
&\le \|f\|_{L^1}^{p/p'} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy dx \\
&\le \|f\|_{L^1}^{p/p'} \int_{\mathbb{R}^n} |g(y)|^p dy \int_{\mathbb{R}^n} |f(x-y)|dx \\
&= \|f\|_{L^1}^{p/p'} \|g\|_{L^p}^p \|f\|_{L^1} = \|f\|_{L^1}^{p/p'+1} \|g\|_{L^p}^p.
\end{aligned}$$

Thus, we have finally

$$\|f * g\|_{L^p} \le \|f\|_{L^1}^{1/p'+1/p} \|g\|_{L^p} = \|f\|_{L^1} \|g\|_{L^p}.$$

□

Exercise 2. Suppose $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Prove that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q .$$

In particular,

$$\|f * g\|_{L^\infty} \le \|f\|_{L^p} \|g\|_{L^{p'}} .$$

Definition. Let $u \in L^1(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} u(x)dx = 1.$$

Then $u_\varepsilon(x) := \varepsilon^{-n} u(x/\varepsilon), \varepsilon > 0$ is called an *approximation to the identity*.

Proposition 2. *Let $u_\varepsilon(x)$ be an approximation to the identity. Then for any function $\varphi \in L^\infty(\mathbb{R}^n)$ which is continuous at $\{0\}$ we have*

$$\lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)dx = \varphi(0).$$

Proof. Since $u_\varepsilon(x)$ is an approximation to the identity we have

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)dx - \varphi(0) = \int_{\mathbb{R}^n} u_\varepsilon(x)(\varphi(x) - \varphi(0))dx$$

and thus

$$\begin{aligned}\left| \int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)dx - \varphi(0) \right| &\le \int_{|x| \le \sqrt{\varepsilon}} |u_\varepsilon(x)||\varphi(x) - \varphi(0)|dx \\ &\quad + \int_{|x| > \sqrt{\varepsilon}} |u_\varepsilon(x)||\varphi(x) - \varphi(0)|dx \\ &\le \sup_{|x| \le \sqrt{\varepsilon}} |\varphi(x) - \varphi(0)| \int_{\mathbb{R}^n} |u_\varepsilon(x)|dx + 2\|\varphi\|_{L^\infty} \int_{|x| > \sqrt{\varepsilon}} |u_\varepsilon(x)|dx \\ &\le \sup_{|x| \le \sqrt{\varepsilon}} |\varphi(x) - \varphi(0)| \cdot \|u\|_{L^1} + 2\|\varphi\|_{L^\infty} \int_{|y| > 1/\sqrt{\varepsilon}} |u(y)|dy \to 0\end{aligned}$$

as $\varepsilon \to 0$. □

Example 0.2. Let $u(x)$ be defined as

$$u(x) = \begin{cases} \frac{\sin x_1}{2} \cdots \frac{\sin x_n}{2}, & x \in [0, \pi]^n \\ 0, & x \notin [0, \pi]^n. \end{cases}$$

Then $u_\varepsilon(x)$ is an approximation to the identity and

$$\lim_{\varepsilon \to 0} (2\varepsilon)^{-n} \int_0^{\varepsilon\pi} \cdots \int_0^{\varepsilon\pi} \prod_{j=1}^n \sin \frac{x_j}{\varepsilon} \varphi(x)dx = \varphi(0).$$

Fourier transform

If $f \in L^1(\mathbb{R}^n)$ its *Fourier transform* $\widehat{f}$ or $\mathcal{F}(f)$ is the bounded function on $\mathbb{R}^n$ defined by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Clearly $\widehat{f}(\xi)$ is well-defined for all ξ and $\left\|\widehat{f}\right\|_\infty \leq (2\pi)^{-n/2} \|f\|_1$.

The Riemann-Lebesgue lemma

If $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is continuous and tends to zero at infinity.

Proof. Let us first prove that $\mathcal{F}f(\xi)$ is continuous (even uniformly continuous) in $\mathbb{R}^n$. Indeed,

$$\begin{aligned} |\mathcal{F}f(\xi+h) - \mathcal{F}f(\xi)| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x)| \cdot |e^{-i(x,h)} - 1| dx \\ &\leq \int_{|x||h|\leq\sqrt{|h|}} |f(x)||x||h|dx + 2\int_{|x||h|>\sqrt{|h|}} |f(x)|dx \\ &\leq \sqrt{|h|}\,\|f\|_{L^1} + 2\int_{|x|>1/\sqrt{|h|}} |f(x)|dx \to 0 \end{aligned}$$

as $|h| \to 0$ since $f \in L^1(\mathbb{R}^n)$.

To prove that $\mathcal{F}f(\xi) \to 0$ as $|\xi| \to 0$ we proceed as follows. Since $e^{i\pi} = -1$ then

$$\begin{aligned} 2\mathcal{F}f(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i(x,\xi)}dx - (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-i(x-\pi\xi/|\xi|^2,\xi)}dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i(x,\xi)}dx - (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(y+\pi\xi/|\xi|^2)e^{-i(y,\xi)}dy \\ &= -(2\pi)^{-n/2}\int_{\mathbb{R}^n} (f(x+\pi\xi/|\xi|^2) - f(x))e^{-i(x,\xi)}dx. \end{aligned}$$

Hence

$$\begin{aligned} 2|\mathcal{F}f(\xi)| &\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n} |f(x+\pi\xi/|\xi|^2) - f(x)|dx \\ &= (2\pi)^{-n/2} \left\|f(\cdot+\pi\xi/|\xi|^2) - f(\cdot)\right\|_{L^1} \to 0 \end{aligned}$$

as $|\xi| \to \infty$ since $f \in L^1(\mathbb{R}^n)$. □

Exercise 3. Prove that if $f, g \in L^1(\mathbb{R}^n)$ then $\widehat{f * g} = (2\pi)^{n/2}\widehat{f}\widehat{g}$.

Exercise 4. Suppose $f \in L^1(\mathbb{R}^n)$. Prove that

1. If $f_h(x) = f(x+h)$ then $\widehat{f_h} = e^{ih\cdot\xi}\widehat{f}$.

2. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible then $\widehat{f \circ T} = |\det T|^{-1} \widehat{f}\,((T^{-1})'\xi)$, where T' is the adjoint matrix.

3. If T is rotation, that is $T' = T^{-1}$ (and $|\det T| = 1$) then $\widehat{f \circ T} = \widehat{f} \circ T$.

Exercise 5. Prove that

$$\partial^\alpha \widehat{f} = (\widehat{-ix)^\alpha} f, \quad \widehat{\partial^\alpha f} = (i\xi)^\alpha \widehat{f}.$$

Exercise 6. Prove that if $f, g \in L^1(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)d\xi.$$

For $f \in L^1(\mathbb{R}^n)$ define the inverse Fourier transform of f by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

It is clear that

$$\mathcal{F}^{-1} f(x) = \mathcal{F} f(-x), \quad \mathcal{F}^{-1} f = \mathcal{F}(\overline{f})$$

and for $f, g \in L^1(\mathbb{R}^n)$

$$(\mathcal{F} f, g)_{L^2} = (f, \mathcal{F}^{-1} g)_{L^2}.$$

The *Schwartz space* $S(\mathbb{R}^n)$ is defined as

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty, \text{ for any multi-indices } \alpha \text{ and } \beta \right\}.$$

The Fourier inversion formula

If $f \in S(\mathbb{R}^n)$ then $(\mathcal{F}^{-1}\mathcal{F})f = f$.

Exercise 7. Prove the Fourier inversion formula for $f \in S(\mathbb{R}^n)$.

The Plancherel theorem

The Fourier transform on S extends uniquely to a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto itself, i.e.

$$\left\| \widehat{f} \right\|_2 = \|f\|_2 .$$

This formula is called the Parseval equality.

The *support* of a function $f : \mathbb{R}^n \to \mathbb{C}$, denoted by $\operatorname{supp} f$, is the set

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

Exercise 8. Prove that if $f \in L^1(\mathbb{R}^n)$ has compact support then $\widehat{f}$ extends to an entire holomorphic function on $\mathbb{C}^n$.

Exercise 9. Prove that if $f \in C_0^\infty(\mathbb{R}^n)$ i.e. $f \in C^\infty(\mathbb{R}^n)$ with compact support, is supported in $\{x \in \mathbb{R}^n : |x| \le R\}$ then for any multi-index α we have

$$|(i\xi)^\alpha \hat{f}(\xi)| \le (2\pi)^{-n/2} e^{R|\operatorname{Im}\xi|} \left\|\partial^\alpha f\right\|_1,$$

that is, $\widehat{f}(\xi)$ is rapidly decaying as $|\operatorname{Re}\xi| \to \infty$ when $|\operatorname{Im}\xi|$ remains bounded.

Distributions

We say that $\varphi_j \to \varphi$ in $C_0^\infty(\Omega), \Omega \subset \mathbb{R}^n$ open, if φ_j are all supported in a common compact set $K \subset \Omega$ and

$$\sup_{x\in K} |\partial^\alpha \varphi_j(x) - \partial^\alpha \varphi(x)| \to 0, \quad j \to \infty$$

for all α. A *distribution* on Ω is a linear functional u on $C_0^\infty(\Omega)$ that is continuous, i.e.,

1. $u : C_0^\infty(\Omega) \to \mathbb{C}$. The action of u to $\varphi \in C_0^\infty(\Omega)$ is denoted by $\langle u, \varphi\rangle$. The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.
2. $\langle u, c_1\varphi_1 + c_2\varphi_2\rangle = c_1\langle u, \varphi_1\rangle + c_2\langle u, \varphi_2\rangle$
3. If $\varphi_j \to \varphi$ in $C_0^\infty(\Omega)$ then $\langle u, \varphi_j\rangle \to \langle u, \varphi\rangle$ in $\mathbb{C}$ as $j \to \infty$. It is equivalent to the following condition: for any $K \subset \Omega$ there is a constant C_K and an integer N_K such that for all $\varphi \in C_0^\infty(K)$,

$$|\langle u, \varphi\rangle| \le C_K \sum_{|\alpha|\le N_K} \left\|\partial^\alpha \varphi\right\|_\infty.$$

Remark. If $u \in L^1_{\text{loc}}(\Omega), \Omega \subset \mathbb{R}^n$ open, then u can be regarded as a distribution (in that case a *regular distribution*) as follows:

$$\langle u, \varphi\rangle := \int_\Omega u(x)\varphi(x)dx, \quad \varphi \in C_0^\infty(\Omega).$$

The Dirac δ-function

The δ-function is defined as

$$\langle \delta, \varphi\rangle = \varphi(0), \quad \varphi \in C_0^\infty(\Omega).$$

It is not a regular distribution.

Example 0.3. Let $u_\varepsilon(x)$ be an approximation to the identity. Then

$$\widehat{u_\varepsilon}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varepsilon^{-n} u(x/\varepsilon) e^{-i(x,\xi)} dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(y) e^{-i(y,\varepsilon\xi)} dy = \widehat{u}(\varepsilon\xi).$$

In particular,

$$\lim_{\varepsilon\to 0+} \widehat{u_\varepsilon}(\xi) = \lim_{\varepsilon\to 0+} \widehat{u}(\varepsilon\xi) = (2\pi)^{-n/2}.$$

Applying Proposition 2 we may conclude that

1) $\lim_{\varepsilon\to 0+}\langle u_\varepsilon, \varphi\rangle = \varphi(0)$ i.e. $\lim_{\varepsilon\to 0+} u_\varepsilon = \delta$ in the sense of distributions, and

2) $\widehat{\delta} = (2\pi)^{-n/2}\cdot 1$.

We can extend the operations from functions to distributions as follows:

$$\langle \partial^\alpha u, \varphi\rangle = \langle u, (-1)^{|\alpha|}\partial^\alpha \varphi\rangle,$$

$$\langle fu, \varphi\rangle = \langle u, f\varphi\rangle, \quad f\in C^\infty(\Omega),$$

$$\langle u*\psi, \varphi\rangle = \langle u, \varphi * \widetilde{\psi}\rangle, \quad \psi \in C_0^\infty(\Omega),$$

where $\widetilde{\psi}(x) = \psi(-x)$. It is possible to show that $u*\psi$ is actually a C^∞ function and

$$\partial^\alpha(u*\psi) = u*\partial^\alpha\psi.$$

A *tempered distribution* is a continuous linear functional on $S(\mathbb{R}^n)$. In addition to the preceding operations for the tempered distributions we can define the Fourier transform by

$$\langle \widehat{u}, \varphi\rangle = \langle u, \widehat{\varphi}\rangle, \quad \varphi\in S.$$

Exercise 10. Prove that if u is a tempered distribution and $\psi \in S$ then

$$\widehat{u*\psi} = (2\pi)^{n/2}\widehat{\psi}\widehat{u}.$$

Exercise 11. Prove that

1. $\widehat{\delta} = (2\pi)^{-n/2}\cdot 1, \quad \widehat{1} = (2\pi)^{n/2}\delta$
2. $\widehat{\partial^\alpha\delta} = (i\xi)^\alpha(2\pi)^{-n/2}$
3. $\widehat{x^\alpha} = i^{|\alpha|}\partial^\alpha(\widehat{1}) = i^{|\alpha|}(2\pi)^{n/2}\partial^\alpha\delta$.

1 Local Existence Theory

A *partial differential equation of order* $k \in \mathbb{N}$ is an equation of the form

$$F\left(x, (\partial^\alpha u)_{|\alpha| \le k}\right) = 0, \tag{1.1}$$

where F is a function of the variables $x \in \Omega \subset \mathbb{R}^n, n \ge 2, \Omega$ an open set, and $(u_\alpha)_{|\alpha| \le k}$.

A complex-valued function $u(x)$ on Ω is a *classical solution* of (1.1) if the derivatives $\partial^\alpha u$ occurring in F exist on Ω and

$$F\left(x, (\partial^\alpha u(x))_{|\alpha| \le k}\right) = 0$$

pointwise for all $x \in \Omega$. The equation (1.1) is called *linear* if it can be written as

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u(x) = f(x) \tag{1.2}$$

for some known functions a_α and f. In this case we speak about the (linear) *differential operator*

$$L(x, \partial) \equiv \sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha$$

and write (1.2) simply as $Lu = f$. If the coefficients $a_\alpha(x)$ belong to $C^\infty(\Omega)$ we can apply the operator L to any distribution $u \in \mathcal{D}'(\Omega)$ and u is called a *distributional solution* (or *weak solution*) of (1.2) if the equation (1.2) holds in the sense of distributions, i.e.

$$\sum_{|\alpha| \le k} (-1)^{|\alpha|} \langle u, \partial^\alpha (a_\alpha \varphi) \rangle = \langle f, \varphi \rangle,$$

where $\varphi \in C_0^\infty(\Omega)$. Let us list some examples. Here and throughout we denote $u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}$ and so forth.

1. The *eikonal equation*

   $$|\nabla u|^2 = c^2,$$

   where $\nabla u = (\partial_1 u, \dots, \partial_n u)$ is the *gradient* of u.

2. a) *Heat (or evolution) equation*

   $$u_t = k \Delta u$$

   b) *Wave equation*

   $$u_{tt} = c^2 \Delta u$$

   c) *Poisson equation*

   $$\Delta u = f,$$

   where $\Delta \equiv \nabla \cdot \nabla = \partial_1^2 + \cdots + \partial_n^2$ is the *Laplacian* (or the *Laplace operator*).

3. The *telegrapher's equation*

$$u_{tt} = c^2 \Delta u - \alpha u_t - m^2 u$$

4. *Sine-Gordon equation*

$$u_{tt} = c^2 \Delta u - \sin u$$

5. The *biharmonic equation*

$$\Delta^2 u \equiv \Delta(\Delta u) = 0$$

6. The *Korteweg-de Vries equation*

$$u_t + cu \cdot u_x + u_{xxx} = 0.$$

In the linear case, a simple measure of the "strength" of a differential operator is provided by the notion of *characteristics*. If $L(x, \partial) = \sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha$ then its *characteristic form* (or *principal symbol*) at $x \in \Omega$ is the homogeneous polynomial of degree k defined by

$$\chi_L(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

A nonzero ξ is called *characteristic* for L at x if $\chi_L(x, \xi) = 0$ and the set of all such ξ is called the *characteristic variety* of L at x, denoted by $\mathrm{char}_x(L)$. In other words,

$$\mathrm{char}_x(L) = \{\xi \neq 0 : \chi_L(x, \xi) = 0\}.$$

In particular, L is said to be *elliptic* at x if $\mathrm{char}_x(L) = \emptyset$ and elliptic in Ω if it is elliptic at every $x \in \Omega$.

Example 1.1. 1. $L = \partial_1 \partial_2, \mathrm{char}_x(L) = \{\xi \in \mathbb{R}^2 : \xi_1 = 0 \,\mathrm{or}\, \xi_2 = 0, \xi_1^2 + \xi_2^2 > 0\}.$

2. $L = \frac{1}{2}(\partial_1 + i\partial_2)$ is the *Cauchy-Riemann operator* on $\mathbb{R}^2$. It is elliptic in $\mathbb{R}^2$.

3. $L = \Delta$ is elliptic in $\mathbb{R}^n$.

4. $L = \partial_1 - \sum_{j=2}^n \partial_j^2, \quad \mathrm{char}_x(L) = \{\xi \in \mathbb{R}^n \setminus \{0\} : \xi_j = 0, j = 2, 3, \ldots, n\}.$

5. $L = \partial_1^2 - \sum_{j=2}^n \partial_j^2, \quad \mathrm{char}_x(L) = \left\{\xi \in \mathbb{R}^n \setminus \{0\} : \xi_1^2 = \sum_{j=2}^n \xi_j^2\right\}.$

Let $\nu(x)$ be the normal to S at x. A hypersurface S is called *characteristic* for L at $x \in S$ if $\nu(x) \in \mathrm{char}_x(L)$, i.e.

$$\chi_L(x, \nu(x)) = 0$$

and S is called *non-characteristic* if it is not characteristic at any point, that is, for any $x \in S$

$$\chi_L(x, \nu(x)) \neq 0.$$

Let us consider the linear equation of the first order

$$Lu \equiv \sum_{j=1}^{n} a_j(x)\partial_j u + b(x)u = f(x), \tag{1.3}$$

where a_j, b and f are assumed to be C^1 functions of x. We assume also that a_j, b and f are real-valued. Suppose we wish to find a solution u of (1.3) with given initial values $u = g$ on the hypersurface S (g is also real-valued). It is clear that

$$\mathrm{char}_x(L) = \left\{\xi \neq 0 : \vec{A} \cdot \xi = 0\right\},$$

where $\vec{A} = (a_1, \ldots, a_n)$. It implies that $\mathrm{char}_x(L) \cup \{0\}$ is the hyperplane orthogonal to $\vec{A}$ and therefore, S is characteristic at x if and only if $\vec{A}$ is tangent to S at x ($\vec{A} \cdot \nu = 0$). Then

$$\sum_{j=1}^{n} a_j(x)\partial_j u(x) = \sum_{j=1}^{n} a_j(x)\partial_j g(x), \quad x \in S,$$

is completely determined as certain directional derivatives of φ (see the definition of S) along S at x, and it may be impossible to make it equal to $f(x) - b(x)u(x)$ (in order to satisfy (1.3)). Indeed, let us assume that u_1 and u_2 have the same value g on S. This means that $u_1 - u_2 = 0$ on S or (more or less equivalently)

$$u_1 - u_2 = \varphi \cdot \gamma,$$

where $\varphi = 0$ on S (φ defines this surface) and $\gamma \neq 0$ on S. Next,

$$(\vec{A} \cdot \nabla)u_1 - (\vec{A} \cdot \nabla)u_2 = (\vec{A} \cdot \nabla)(\varphi\gamma) = \gamma(\vec{A} \cdot \nabla)\varphi + \varphi(\vec{A} \cdot \nabla)\gamma = 0,$$

since S is characteristic for L ($(\vec{A} \cdot \nabla)\varphi = 0 \Leftrightarrow (\vec{A} \cdot \frac{\nabla}{|\nabla|})\varphi = 0 \Leftrightarrow \vec{A} \cdot \nu = 0$). That's why to make the initial value problem well-defined we must assume that S is non-characteristic for this problem.

Let us assume that S is non-characteristic for L and $u = g$ on S. We define the *integral curves* for (1.3) as the parametrized curves $x(t)$ that satisfy the system

$$\dot{x} = \vec{A}(x), \quad x = x(t) = (x_1(t), \ldots, x_n(t)) \tag{1.4}$$

of ordinary differential equations, where

$$\dot{x} = (x_1'(t), \ldots, x_n'(t)).$$

Along one of those curves a solution u of (1.3) must satisfy

$$\frac{du}{dt} = \frac{d}{dt}(u(x(t))) = \sum_{j=1}^{n} \dot{x}_j \frac{\partial u}{\partial x_j} = (\vec{A} \cdot \nabla)u = f - bu \equiv f(x(t)) - bu(x(t))$$

or

$$\frac{du}{dt} = f - bu. \tag{1.5}$$

By the existence and uniqueness theorem for ordinary differential equations there is a unique solution (unique curve) of (1.4) with $x(0) = x_0$. Along this curve the solution $u(x)$ of (1.3) must be the solution of (1.5) with $u(0) = u(x(0)) = u(x_0) = g(x_0)$. Moreover, since S is non-characteristic, $x(t) \notin S$ for $t \neq 0$, at least for small t, and the curves $x(t)$ fill out a neighborhood of S. Thus we have proved the following theorem.

Theorem 1. *Assume that S is a surface of class C^1 which is non-characteristic for (1.3), and that a_j, b, f and g are C^1 and real-valued functions. Then for any sufficiently small neighborhood U of S in $\mathbb{R}^n$ there is a unique solution $u \in C^1$ of (1.3) on U that satisfies $u = g$ on S.*

Remark. The method which was presented above is called the *method of characteristics*.

Let us consider some examples where we apply the method of characteristics.

Example 1.2. In $\mathbb{R}^3$, solve $x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u$ with $u = g(x_1, x_2)$ on the plane $x_3 = 0$.

Since $S = \{x \in \mathbb{R}^3 : x_3 = 0\}$ then $\nu(x) = (0, 0, 1)$ and since $\chi_L(x, \xi) = x_1\xi_1 + 2x_2\xi_2 + \xi_3$ we have

$$\chi_L(x, \nu(x)) = x_1 \cdot 0 + 2x_2 \cdot 0 + 1 \cdot 1 = 1 \neq 0$$

so that S is non-characteristic. The system (1.4)-(1.5) to be solved is

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = 2x_2, \quad \dot{x}_3 = 1, \quad \dot{u} = 3u$$

with initial conditions

$$(x_1, x_2, x_3)|_{t=0} = (x_1^0, x_2^0, 0), \quad u(0) = g(x_1^0, x_2^0)$$

on S. We obtain

$$x_1 = x_1^0 e^t, \quad x_2 = x_2^0 e^{2t}, \quad x_3 = t, \quad u = g(x_1^0, x_2^0)e^{3t}.$$

These equations imply

$$x_1^0 = x_1 e^{-t} = x_1 e^{-x_3}, \quad x_2^0 = x_2 e^{-2t} = x_2 e^{-2x_3}.$$

Therefore

$$u(x) = u(x_1, x_2, x_3) = g(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{3x_3}.$$

Example 1.3. In $\mathbb{R}^3$, solve $\partial_1 u + x_1\partial_2 u - \partial_3 u = u$ with $u(x_1, x_2, 1) = x_1 + x_2$.

Since $S = \{x \in \mathbb{R}^3 : x_3 = 1\}$ then $\nu(x) = (0, 0, 1)$. That's why

$$\chi_L(x, \nu(x)) = 1 \cdot 0 + x_1 \cdot 0 - 1 \cdot 1 = -1 \neq 0$$

and S is non-characteristic. The system (1.4)-(1.5) for this problem becomes

$$\dot{x}_1 = 1, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = -1, \quad \dot{u} = u$$

with

$$(x_1, x_2, x_3)|_{t=0} = (x_1^0, x_2^0, 1), \quad u(0) = x_1^0 + x_2^0.$$

We obtain

$$x_1 = t + x_1^0, \quad x_2 = \frac{t^2}{2} + tx_1^0 + x_2^0, \quad x_3 = -t + 1, \quad u = (x_1^0 + x_2^0)e^t.$$

Then,

$$t = 1 - x_3, \quad x_1^0 = x_1 - t = x_1 + x_3 - 1,$$

$$x_2^0 = x_2 - \frac{(1-x_3)^2}{2} - (1-x_3)(x_1 + x_3 - 1) = \frac{1}{2} - x_1 + x_2 - x_3 + x_1x_3 + \frac{x_3^2}{2}$$

and, finally,

$$u = \left(\frac{x_3^2}{2} + x_1x_3 + x_2 - \frac{1}{2}\right) e^{1-x_3}.$$

Now let us generalize this technique to *quasi-linear equations* or to the equations of the form

$$\sum_{j=1}^{n} a_j(x, u)\partial_j u = b(x, u), \tag{1.6}$$

where a_j, b and u are real-valued. If u is a function of x, the normal to the graph of u in $\mathbb{R}^{n+1}$ is proportional to $(\nabla u, -1)$, so (1.6) just says that the vector field

$$\vec{A}(x, y) := (a_1, \ldots, a_n, b) \in \mathbb{R}^{n+1}$$

is tangent to the graph $y = u(x)$ at any point. This suggests that we look at the integral curves of $\vec{A}$ in $\mathbb{R}^{n+1}$ given by solving the ordinary differential equations

$$\dot{x}_j = a_j(x, y), \quad j = 1, 2, \ldots, n, \quad \dot{y} = b(x, y).$$

Suppose u is a solution of (1.6). If we solve

$$\dot{x}_j = a_j(x, u(x)), \quad j = 1, 2, \ldots, n,$$

with $x_j(0) = x_j^0$ then setting $y(t) = u(x(t))$ we obtain that

$$\dot{y} = \sum_{j=1}^{n} \partial_j u \cdot \dot{x}_j = \sum_{j=1}^{n} a_j(x, u)\partial_j u = b(x, u) = b(x, y).$$

Suppose we are given initial data $u = g$ on S. If we form the submanifold

$$S^* := \{(x, g(x)) : x \in S\}$$

in $\mathbb{R}^{n+1}$ then the graph of the solution should be the hypersurface generated by the integral curves of $\vec{A}$ passing through S^*. Again, we need to assume that S is non-characteristic in a sense that the vector

$$(a_1(x, g(x)), \ldots, a_n(x, g(x))), \quad x \in S,$$

should not be tangent to S at x. If S is represented parametrically by a mapping $\vec{\varphi} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ (for example $\vec{\varphi}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, \psi(x_1, \ldots, x_{n-1})))$ and we have the coordinates $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ this condition is just

$$\det \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_{n-1}} & a_1(\vec{\varphi}(x'), g(\vec{\varphi}(x'))) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_{n-1}} & a_n(\vec{\varphi}(x'), g(\vec{\varphi}(x'))) \end{pmatrix} \neq 0.$$

Remark. If S is parametrized as

$$x_n = \psi(x_1, \ldots, x_{n-1}), \quad x' = (x_1, \ldots, x_{n-1}) \in S' \subset \mathbb{R}^{n-1}$$

then S can be represented also by

$$\phi(x_1, \ldots, x_n) = 0,$$

where $\phi(x_1, \ldots, x_n) \equiv \psi(x') - x_n$ and $\nu(x)$ is proportional to

$$\nabla \phi = \left(\frac{\partial \psi}{\partial x_1}, \ldots, \frac{\partial \psi}{\partial x_{n-1}}, -1 \right).$$

Then S is non-characteristic if and only if

$$a_1 \frac{\partial \psi}{\partial x_1} + \cdots + a_{n-1} \frac{\partial \psi}{\partial x_{n-1}} - a_n \neq 0$$

or

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1(x, g(x)) \\ 0 & 1 & \cdots & 0 & a_2(x, g(x)) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1}(x, g(x)) \\ \frac{\partial \psi}{\partial x_1} & \cdots & \cdots & \frac{\partial \psi}{\partial x_{n-1}} & a_n(x, g(x)) \end{pmatrix} \neq 0,$$

where $x \in S$.

Example 1.4. In $\mathbb{R}^2$, solve $u \partial_1 u + \partial_2 u = 1$ with $u = s/2$ on the segment $x_1 = x_2 = s$, where $s > 0, s \neq 2$ is a parameter.

Since $\vec{\varphi}(s) = (s, s)$ then $(x' = x_1 = s)$

$$\det \begin{pmatrix} \frac{\partial x_1}{\partial s} & a_1(s, s, s/2) \\ \frac{\partial x_2}{\partial s} & a_2(s, s, s/2) \end{pmatrix} = \det \begin{pmatrix} 1 & s/2 \\ 1 & 1 \end{pmatrix} = 1 - s/2 \neq 0,$$

for $s > 0, s \neq 2$. The system (1.4)-(1.5) for this problem is

$$\dot{x}_1 = u, \quad \dot{x}_2 = 1, \quad \dot{u} = 1$$

with

$$(x_1, x_2, u)|_{t=0} = (x_1^0, x_2^0, \frac{x_1^0}{2}) = (s, s, s/2).$$

Then

$$u = t + s/2, \quad x_2 = t + s, \quad \dot{x}_1 = t + s/2$$

so that $x_1 = \frac{t^2}{2} + \frac{st}{2} + s$. This implies

$$x_1 - x_2 = t^2/2 + t(s/2 - 1).$$

For s and t in terms of x_1 and x_2 we obtain

$$\frac{s}{2} = 1 + \frac{1}{t}\left(x_1 - x_2 - \frac{t^2}{2}\right), \quad t = \frac{2(x_1 - x_2)}{x_2 - 2}.$$

Hence

$$\begin{aligned} u &= \frac{2(x_1 - x_2)}{x_2 - 2} + 1 + \frac{x_1 - x_2}{t} - \frac{t}{2} \\ &= \frac{2(x_1 - x_2)}{x_2 - 2} + 1 + \frac{x_2 - 2}{2} - \frac{x_1 - x_2}{x_2 - 2} \\ &= \frac{x_1 - x_2}{x_2 - 2} + 1 + \frac{x_2 - 2}{2} = \frac{x_1 - x_2}{x_2 - 2} + \frac{x_2}{2} \\ &= \frac{2x_1 - 4x_2 + x_2^2}{2(x_2 - 2)}. \end{aligned}$$

Exercise 12. In $\mathbb{R}^2$, solve $x_1^2 \partial_1 u + x_2^2 \partial_2 u = u^2$ with $u \equiv 1$ when $x_2 = 2x_1$.

Exercise 13. In $\mathbb{R}^2$, solve $u\partial_1 u + x_2 \partial_2 u = x_1$ with $u(x_1, 1) = 2x_1$.

Example 1.5. Consider the *Burgers equation*

$$u\partial_1 u + \partial_2 u = 0$$

in $\mathbb{R}^2$ with $u(x_1, 0) = h(x_1)$, where h is a known C^1 function. It is clear that $S := \{x \in \mathbb{R}^2 : x_2 = 0\}$ is non-characteristic for this quasi-linear equation, since

$$\det \begin{pmatrix} 1 & h(x_1) \\ 0 & 1 \end{pmatrix} = 1 \neq 0,$$

and $\nu(x) = (0, 1)$. Now we have to solve the ordinary differential equations

$$\dot{x}_1 = u, \quad \dot{x}_2 = 1, \quad \dot{u} = 0$$

with

$$(x_1, x_2, u)|_{t=0} = \left(x_1^0, 0, h(x_1^0)\right).$$

We obtain

$$x_2 = t, \quad u \equiv h(x_1^0), \quad x_1 = h(x_1^0)t + x_1^0$$

so that

$$x_1 - x_2 h(x_1^0) - x_1^0 = 0.$$

Let us assume that

$$-x_2 h_1'(x_1^0) - 1 \neq 0.$$

By this condition last equation defines an implicit function $x_1^0 = g(x_1, x_2)$. That's why the solution u of the Burgers equation has the form

$$u(x_1, x_2) = h(g(x_1, x_2)).$$

Let us consider two particular cases:

1. If $h(x_1^0) = ax_1^0 + b, a \neq 0$, then

$$u(x_1, x_2) = \frac{ax_1 + b}{ax_2 + 1}, \quad x_2 \neq -\frac{1}{a}.$$

2. If $h(x_1^0) = a(x_1^0)^2 + bx_1^0 + c, a \neq 0$, then

$$\begin{aligned} u(x_1, x_2) &= a\left(\frac{-x_2 b - 1 + \sqrt{(x_2 b + 1)^2 - 4ax_2(cx_2 - x_1)}}{2ax_2}\right) \\ &+ b\left(\frac{-x_2 b - 1 + \sqrt{(x_2 b + 1)^2 - 4ax_2(cx_2 - x_1)}}{2ax_2}\right) + c, \end{aligned}$$

with $D = (x_2 b + 1)^2 - 4ax_2(cx_2 - x_1) > 0$.

Let us consider again the linear equation (1.2) of order k i.e.

$$\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u(x) = f(x).$$

Let S be a hypersurface of class C^k. If u is a C^k function defined near S, the quantities

$$u, \partial_\nu u, \ldots, \partial_\nu^{k-1} u \tag{1.7}$$

on S are called the *Cauchy data* of u on S. And the *Cauchy problem* is to solve (1.2) with the Cauchy data (1.7). We shall consider $\mathbb{R}^n, n \geq 2$, as $\mathbb{R}^{n-1} \times \mathbb{R}$ and denote the coordinates by (x, t), where $x = (x_1, \ldots, x_{n-1})$. We can make a change of coordinates from $\mathbb{R}^n$ to $\mathbb{R}^{n-1} \times \mathbb{R}$ so that $x_0 \in S$ is mapped to $(0, 0)$ and a neighborhood of x_0 in

S is mapped into the hyperplane $t = 0$. In that case $\partial_\nu = \frac{\partial}{\partial t}$ on $S = \{(x,t) : t = 0\}$ and equation (1.2) can be written in the new coordinates as

$$\sum_{|\alpha|+j \le k} a_{\alpha,j}(x,t)\partial_x^\alpha \partial_t^j u = f(x,t) \tag{1.8}$$

with the Cauchy data

$$\partial_t^j u(x,0) = \varphi_j(x), \quad j = 0,1,\dots,k-1. \tag{1.9}$$

Since the normal $\nu = (0,0,\dots,0,1)$ then the assumption "S is non-characteristic" means that

$$\chi_L(x,0,\nu(x,0)) \equiv a_{0,k}(x,0) \neq 0.$$

Hence by continuity $a_{0,k}(x,t) \neq 0$ for small t, and we can solve (1.8) for $\partial_t^k u$:

$$\partial_t^k u(x,t) = (a_{0,k}(x,t))^{-1}\left(f - \sum_{|\alpha|+j\le k, j<k} a_{\alpha,j}\partial_x^\alpha \partial_t^j u\right) \tag{1.10}$$

with the Cauchy data (1.9).

Example 1.6. The line $t = 0$ is characteristic for $\partial_x \partial_t u = 0$ in $\mathbb{R}^2$. That's why we will have some problems with the solutions. Indeed, if u is a solution of this equation with Cauchy data $u(x,0) = g_0(x)$ and $\partial_t u(x,0) = g_1(x)$ then $\partial_x g_1 = 0$, that is, $g_1 \equiv$ constant. Thus the Cauchy problem is not solvable in general. On the other hand, if g_1 is constant, then there is no uniqueness, because we can take $u(x,t) = g_0(x) + f(t)$ with any $f(t)$ such that $f(0) = 0$ and $f'(0) = g_1$.

Example 1.7. The line $t = 0$ is characteristic for $\partial_x^2 u - \partial_t u = 0$ in $\mathbb{R}^2$. Here if we are given $u(x,0) = g_0(x)$ then $\partial_t u(x,0)$ is already completely determined by $\partial_t u(x,0) = g_0''(x)$. So, again the Cauchy problem has "bad" behaviour.

Let us now formulate and give "a sketch" of the proof of the famous *Cauchy-Kowalevski theorem* for linear case.

Theorem 2. *If $a_{\alpha,j}(x,t), \varphi_0(x), \dots, \varphi_{k-1}(x)$ are analytic near the origin in $\mathbb{R}^n$, then there is a neighborhood of the origin on which the Cauchy problem (1.10)-(1.9) has a unique analytic solution.*

Proof. The uniqueness of analytic solution follows from the fact that an analytic function is completely determined by the values of its derivatives at one point (see the Taylor formula or the Taylor series). Indeed, for all α and $j = 0,1,\dots,k-1$

$$\partial_x^\alpha \partial_t^j u(x,0) = \partial_x^\alpha \varphi_j(x).$$

That's why

$$\partial_t^k u|_{t=0} = (a_{0,k})^{-1}\left(f(x,0) - \sum_{|\alpha|+j\le k, j<k} a_{\alpha,j}(x,0)\partial_x^\alpha \varphi_j(x)\right)$$

and moreover

$$\partial_t^k u(x,t) = (a_{0,k})^{-1} \left(f(x,t) - \sum_{|\alpha|+j\le k, j<k} a_{\alpha,j}(x,t)\partial_x^\alpha \partial_t^j u \right).$$

Then all derivatives of u can be defined from this equation by

$$\partial_t^{k+1} u = \partial_t \left(\partial_t^k u\right).$$

Next, let us denote by $y_{\alpha,j} = \partial_x^\alpha \partial_t^j u$ and by $Y = (y_{\alpha,j})$ this vector. Then equation (1.10) can be rewritten as

$$y_{0,k} = (a_{0,k})^{-1} \left(f - \sum_{|\alpha|+j\le k, j<k} a_{\alpha,j} y_{\alpha,j} \right)$$

or

$$\partial_t \left(y_{0,k-1}\right) = (a_{0,k})^{-1} \left(f - \sum_{|\alpha|+j\le k, j<k} a_{\alpha,j} \partial_{x_j} y_{(\alpha-\vec{j}),j} \right)$$

and therefore the Cauchy problem (1.10)-(1.9) becomes

$$\begin{cases} \partial_t Y = \sum_{j=1}^{n-1} A_j \partial_{x_j} Y + B \\ Y(x,0) = \Phi(x), \quad x \in \mathbb{R}^{n-1}, \end{cases} \tag{1.11}$$

where Y, B and Φ are analytic vector-valued functions and A_j's are analytic matrix-valued functions. Without loss of generality we can assume that $\Phi \equiv 0$. Let $Y = (y_1, \ldots, y_N), B = (b_1, \ldots, b_N), A_j = (a_{ml}^{(j)})_{m,l=1}^N$. We seek a solution $Y = (y_1, \ldots, y_N)$ in the form

$$y_m = \sum C_{\alpha,j}^{(m)} x^\alpha t^j, \quad m = 1, 2, \ldots, N.$$

The Cauchy data tell us that $C_{\alpha,0}^{(m)} = 0$ for all α and m, since we assumed $\Phi \equiv 0$. To determine $C_{\alpha,j}^{(m)}$ for $j > 0$, we substitute y_m into (1.11) and get for $m = 1, 2, \ldots, N$

$$\partial_t y_m = \sum a_{ml}^{(j)} \partial_{x_j} y_l + b_m(x,y)$$

or

$$\sum C_{\alpha,j}^{(m)} j x^\alpha t^{j-1} = \sum_{j,l} \sum_{\beta,r} \left(a_{ml}^{(j)}\right)_{\beta r} x^\beta t^r \sum C_{\alpha,j}^{(m)} \alpha_j x^{\alpha-\vec{j}} t^j + \sum b_{\alpha j}^{(m)} x^\alpha t^j.$$

It can be proved that this equation determines uniquely the coefficients $C_{\alpha,j}^{(m)}$ and therefore the solution $Y = (y_1, \ldots, y_N)$. $\square$

Remark. Consider the following example in $\mathbb{R}^2$, due to Hadamard, which sheds light on the Cauchy problem:

$$\Delta u = 0, \quad u(x_1, 0) = 0, \quad \partial_2 u(x_1, 0) = ke^{-\sqrt{k}} \sin(x_1 k), \quad k \in \mathbb{N}.$$

This problem is non-characteristic on $\mathbb{R}^2$ since Δ is elliptic in $\mathbb{R}^2$. We look for $u(x_1, x_2) = u_1(x_1)u_2(x_2)$. Then

$$u_1'' u_2 + u_2'' u_1 = 0$$

which implies that

$$\frac{u_1''}{u_1} = -\frac{u_2''}{u_2} = -\lambda = \text{constant}.$$

Next, the general solutions of

$$u_1'' = -\lambda u_1$$

and

$$u_2'' = \lambda u_2$$

are

$$u_1 = A \sin(\sqrt{\lambda} x_1) + B \cos(\sqrt{\lambda} x_1)$$

and

$$u_2 = C \sinh(\sqrt{\lambda} x_2) + D \cosh(\sqrt{\lambda} x_2),$$

respectively. But $u_2(0) = 0, u_2'(0) = 1$ and $u_1(x_1) = ke^{-\sqrt{k}} \sin(kx_1)$. Thus $D = 0, B = 0, k = \sqrt{\lambda}, A = ke^{-\sqrt{k}}$ and $C = \frac{1}{k} = \frac{1}{\sqrt{\lambda}}$. So we finally have

$$u(x_1, x_2) = ke^{-\sqrt{k}} \sin(kx_1) \frac{1}{k} \sinh(kx_2) = e^{-\sqrt{k}} \sin(kx_1) \sinh(kx_2).$$

As $k \to +\infty$, the Cauchy data and their derivatives (for $x_2 = 0$) of all orders tend uniformly to zero since $e^{-\sqrt{k}}$ decays faster than polynomially. But if $x_2 \neq 0$ (more precisely, $x_2 > 0$) then

$$\lim_{k \to +\infty} e^{-\sqrt{k}} \sin(kx_1) \sinh(kx_2) = \infty,$$

at least for some x_1 and some subsequence of k. Hence $u(x_1, x_2)$ is not bounded. But the solution of the original problem which corresponds to the limiting case $k = \infty$ is of course $u \equiv 0$, since $u(x_1, 0) = 0$ and $\partial_2 u(x_1, 0) = 0$ in the limiting case. Hence the solution of the Cauchy problem may not depend continuously on the Cauchy data. It means by Hadamard that the Cauchy problem for elliptic operators is "ill-posed", even in the case when this problem is non-characteristic.

Remark. This example of Hadamard shows that the solution of the Cauchy problem may not depend continuously on the Cauchy data. By the terminology of Hadamard "the Cauchy problem for the Laplacian is not well-posed or it is ill-posed". Due to Hadamard and Tikhonov any problem is called *well-posed* if the following are satisfied:

1. existence
2. uniqueness
3. stability or continuous dependence on data

Otherwise it is called *ill-posed*.

Let us consider one more important example due to H. Lewy. Let L be the differential operator of the first order in $\mathbb{R}^3$ ($(x,y,t)\in\mathbb{R}^3$) given by

$$L \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial t}. \tag{1.12}$$

Theorem 3 (The *Hans Lewy example*)**.** *Let f be a continuous real-valued function depending only on t. If there is a C^1 function u satisfying $Lu = f$, with the operator L from (1.12), in some neighborhood of the origin, then $f(t)$ necessarily is analytic at $t = 0$.*

Remark. This example shows that the assumption of analyticity of f in Theorem 2 in the linear equation can not be omitted (it is very essential). It appears necessarily since $Lu = f$ with L from (1.12) has no C^1 solution unless f is analytic.

Proof. Suppose $x^2+y^2 < R^2, |t| < R$ and set $z = x+iy = re^{i\theta}$. Denote by $V(t)$ the function

$$V(t) := \int_{|z|=r} u(x,y,t)d\sigma(z) = ir\int_0^{2\pi} u(r,\theta,t)e^{i\theta}d\theta,$$

where $u(x,y,t)$ is the C^1 solution of the equation $Lu = f$ with L from (1.12). We keep denoting u in polar coordinates also by u. By the divergence theorem for $F := (u, iu)$ we get

$$\begin{aligned} i\int_{|z|<r}\nabla\cdot F dxdy &\equiv i\int_{|z|<r}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)dxdy = i\int_{|z|=r}(u,iu)\cdot\nu d\sigma(z) \\ &= i\int_{|z|=r}\left(u\frac{x}{r} + iu\frac{y}{r}\right)d\sigma(z) = i\int_{|z|=r} ue^{i\theta}d\sigma(z) \\ &= ir\int_0^{2\pi} ue^{i\theta}d\theta \equiv V(t). \end{aligned}$$

But on the other hand, in polar coordinates,

$$V(t) \equiv i\int_{|z|<r}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)dxdy = i\int_0^r\int_0^{2\pi}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(\rho,\theta,t)\rho d\rho d\theta.$$

This implies that

$$
\begin{aligned}
\frac{\partial V}{\partial r} &= ir\int_0^{2\pi}\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}\right)(r,\theta,t)d\theta = \int_{|z|=r}\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}\right)(x,y,t)2r\frac{d\sigma(z)}{2z} \\
&= 2r\int_{|z|=r}\left(i\frac{\partial u}{\partial t}+\frac{f(t)}{2z}\right)d\sigma(z) = 2r\left(i\frac{\partial V}{\partial t}+f(t)\int_{|z|=r}\frac{d\sigma(z)}{2z}\right) \\
&= 2r\left(i\frac{\partial V}{\partial t}+i\pi f(t)\right).
\end{aligned}
$$

That's why we have the following equation for V:

$$
\frac{1}{2r}\frac{\partial V}{\partial r} = i\left(\frac{\partial V}{\partial t}+\pi f(t)\right). \tag{1.13}
$$

Let us introduce now a new function $U(s,t) = V(s)+\pi F(t)$, where $s = r^2$ and $F' = f$. The function F exists because f is continuous. It follows from (1.13) that

$$
\frac{1}{2r}\frac{\partial V}{\partial r} \equiv \frac{\partial V}{\partial s}, \quad \frac{\partial U}{\partial s} = \frac{\partial V}{\partial s}, \quad \frac{\partial U}{\partial s} = i\frac{\partial U}{\partial t}.
$$

Hence

$$
\frac{\partial U}{\partial t}+i\frac{\partial U}{\partial s} = 0. \tag{1.14}
$$

Since (1.14) is the Cauchy-Riemann equation then U is a holomorphic (analytic) function of the variable $w = t+is$, in the region $0 < s < R^2, |t| < R$ and U is continuous up to $s = 0$. Next, since $U(0,t) = \pi F(t)$ ($V = 0$ when $s = 0 \Leftrightarrow r = 0$) and $f(t)$ is real-valued then $U(0,t)$ is also real-valued. Therefore, by the Schwarz reflection principle (see complex analysis), the formula

$$
U(-s,t) := \overline{U(s,t)}
$$

gives a holomorphic continuation of U to a full neighborhood of the origin. In particular, $U(0,t) = \pi F(t)$ is analytic in t, hence so is $f(t) \equiv F'(t)$. $\square$

2 Fourier Series

Definition. A function f is said to be *periodic* with period $T > 0$ if the domain $D(f)$ of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x), \quad x \in D(f). \tag{2.1}$$

It follows that if T is a period of f then mT is also a period for any integer $m > 0$. The smallest value of $T > 0$ for which (2.1) holds is called the *fundamental period* of f.

For example, the functions $\sin \frac{m\pi x}{L}$ and $\cos \frac{m\pi x}{L}, m = 1, 2, \ldots$ are periodic with fundamental period $T = \frac{2L}{m}$. Note also that they are periodic with the common period $2L$.

Definition. Let us assume that the domain of f is symmetric with respect to $\{0\}$, i.e. if $x \in D(f)$ then $-x \in D(f)$. A function f is called *even* if

$$f(-x) = f(x), \quad x \in D(f)$$

and *odd* if

$$f(-x) = -f(x), \quad x \in D(f).$$

Definition. The notations $f(c \pm 0)$ are used to denote the limits

$$f(c \pm 0) = \lim_{x \to c \pm 0} f(x).$$

Definition. A function f is said to be *piecewise continuous* on an interval $a \leq x \leq b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ such that

1. f is continuous on each subinterval $x_{j-1} < x < x_j$.
2. $f(x_j \pm 0)$ exists for each $j = 1, 2, \ldots, n-1$ and $f(x_0 + 0)$ and $f(x_n - 0)$ exist.

The following properties hold: if a piecewise continuous function f is even then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \tag{2.2}$$

and if it is odd then

$$\int_{-a}^{a} f(x)dx = 0. \tag{2.3}$$

Definition. Two real-valued functions u and v are said to be *orthogonal* on $a \leq x \leq b$ if

$$\int_{a}^{b} u(x)v(x)dx = 0.$$

A set of functions is said to be *mutually orthogonal* if each distinct pair in the set is orthogonal on $a \leq x \leq b$.

Proposition. *The functions* $1, \sin\frac{m\pi x}{L}$ *and* $\cos\frac{m\pi x}{L}, m = 1, 2, \ldots$ *form a mutually orthogonal set on the interval* $-L \le x \le L$. *In fact,*

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2.4}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \tag{2.5}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2.6}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}dx = \int_{-L}^{L} \cos\frac{m\pi x}{L}dx = 0. \tag{2.7}$$

Proof. Let us derive (for example) (2.5). Since

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$$

we have for $m \neq n$

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\sin\frac{(m+n)\pi x}{L}dx - \frac{1}{2}\int_{-L}^{L}\sin\frac{(m-n)\pi x}{L}dx \\
&= \frac{1}{2}\left\{\frac{-\cos\frac{(m+n)\pi x}{L}}{\frac{(m+n)\pi}{L}}\right\}\Bigg|_{-L}^{L} - \frac{1}{2}\left\{\frac{-\cos\frac{(m-n)\pi x}{L}}{\frac{(m-n)\pi}{L}}\right\}\Bigg|_{-L}^{L} \\
&= \frac{1}{2}\left\{\frac{-\cos(m+n)\pi}{\frac{(m+n)\pi}{L}} + \frac{\cos(m+n)\pi}{\frac{(m+n)\pi}{L}}\right\} \\
&- \frac{1}{2}\left\{\frac{-\cos(m-n)\pi}{\frac{(m-n)\pi}{L}} + \frac{\cos(m-n)\pi}{\frac{(m-n)\pi}{L}}\right\} = 0.
\end{aligned}$$

If $m = n$ we have

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\sin\frac{2m\pi x}{L}dx = 0$$

since sine is odd. Other identities can be proved in a similar manner and are left to the reader. □

Let us consider the infinite trigonometric series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\right). \tag{2.8}$$

This series consists of $2L-$periodic functions. Thus, if the series (2.8) converges for all x, then the function to which it converges will be periodic of period $2L$. Let us denote the limiting function by $f(x)$, i.e.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{2.9}$$

To determine a_m and b_m we proceed as follows: assuming that the integration can be legitimately carried out term by term, we obtain

$$\begin{aligned} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \end{aligned}$$

for each fixed n. It follows from the orthogonality relations (2.4),(2.5) and (2.7) that the only nonzero term on the right hand side is the one for which $m = n$ in the first summation. Hence,

$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx = La_n$$

or

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx. \tag{2.10}$$

A similar expression for b_n may be obtained by multiplying (2.9) by $\sin \frac{n\pi x}{L}$ and integrating termwise from $-L$ to L. Thus,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{2.11}$$

To determine a_0 we use (2.7) to obtain

$$\int_{-L}^{L} f(x) dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} dx = a_0 L.$$

Hence

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx. \tag{2.12}$$

Definition. Let f be a piecewise continuous function on the intervel $[-L, L]$. The *Fourier series* of f is the trigonometric series (2.9), where the coefficients a_0, a_m and b_m are given by (2.10), (2.11) and (2.12).

It follows from this definition and (2.2)-(2.3) that if f is even on $[-L, L]$ then the Fourier series of f has the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} \tag{2.13}$$

and if f is odd then

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L}. \tag{2.14}$$

The series (2.13) is called the *Fourier cosine series* and (2.14) is called the *Fourier sine series*.

Example 2.1. Find the Fourier series of

$$\operatorname{sgn}(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \le \pi \end{cases}$$

on the interval $[-\pi, \pi]$.

Since $L = \pi$ and $\operatorname{sgn}(x)$ is odd function we have a Fourier sine series with

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(x)\sin(mx)dx = \frac{2}{\pi}\int_0^{\pi}\sin(mx)dx = \frac{2}{\pi}\left\{-\frac{\cos(mx)}{m}\right\}\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left\{-\frac{\cos(m\pi)}{m} + \frac{1}{m}\right\} = \frac{2}{\pi}\left\{\frac{1-(-1)^m}{m}\right\} = \begin{cases} 0, & m = 2k, k = 1, 2, \ldots \\ \frac{4}{\pi m}, & m = 2k-1, k = 1, 2, \ldots. \end{cases} \end{aligned}$$

That's why

$$\operatorname{sgn}(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin((2k-1)x).$$

In particular,

$$\frac{\pi}{2} = \sum_{k=1}^{\infty} \frac{\sin((k-1/2)\pi)}{k-1/2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k-1/2}.$$

Example 2.2. Let us assume that $f(x) = |x|, -1 \le x \le 1$. In this case $L = 1$ and $f(x)$ is even. Hence we will have a Fourier cosine series (2.13), where

$$a_0 = \int_{-1}^{1} |x|dx = 2\int_0^1 xdx = 1$$

and

$$\begin{aligned} a_m &= 2\int_0^1 x\cos(m\pi x)dx = 2\left\{x\frac{\sin(m\pi x)}{m\pi}\right\}\Big|_0^1 - 2\int_0^1 \frac{\sin(m\pi x)}{m\pi}dx \\ &= 2\left\{\frac{\cos(m\pi x)}{(m\pi)^2}\right\}\Big|_0^1 = 2\left\{\frac{\cos(m\pi)}{(m\pi)^2} - \frac{1}{(m\pi)^2}\right\} \\ &= \frac{2((-1)^m - 1)}{(m\pi)^2} = \begin{cases} 0, & m = 2k, k = 1, 2, \ldots \\ -\frac{4}{(m\pi)^2}, & m = 2k-1, k = 1, 2, \ldots. \end{cases} \end{aligned}$$

So we have

$$|x| = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

In particular,

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Exercise 14. Find the Fourier series of $f(x) = x, -1 \leq x \leq 1$.

Let us consider the partial sums of the Fourier series defined by

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

We investigate the speed with which the series converges. It is equivalent to the question: how large value of N must be chosen if we want $S_N(x)$ to approximate $f(x)$ with some accuracy $\varepsilon > 0$? So we need to choose N such that the residual $R_N(x) := f(x) - S_N(x)$ satisfies

$$|R_N(x)| < \varepsilon$$

for all x, say, on the interval $[-L, L]$. Consider the function $f(x)$ from Example 2.2. Then

$$R_N(x) = \frac{4}{\pi^2} \sum_{k=N+1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}$$

and

$$\begin{aligned} |R_N(x)| &\leq \frac{4}{\pi^2} \sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^2} < \frac{4}{\pi^2} \left\{ \frac{1}{(2N)(2N+1)} + \frac{1}{(2N+1)(2N+2)} + \cdots \right\} \\ &= \frac{4}{\pi^2} \left\{ \frac{1}{2N} - \frac{1}{2N+1} + \frac{1}{2N+1} - \frac{1}{2N+2} + \cdots \right\} = \frac{4}{2N\pi^2} = \frac{2}{N\pi^2} < \varepsilon \end{aligned}$$

if and only if $N > \frac{2}{\varepsilon\pi^2}$. Since $\pi^2 \approx 10$ then if $\varepsilon = 0.04$ it is enough to take $N = 6$, for $\varepsilon = 0.01$ we have to take $N = 21$.

The function $f(x) = |x|$ is "good" enough with respect to "smoothness" and the smoothness of $|x|$ guarantees a good approximation by the partial sums. We would like to formulate a general result.

Theorem 1. *Suppose that f and f' are piecewise continuous on the interval $-L \leq x \leq L$. Suppose also that f is defined outside the interval $-L \leq x \leq L$ so that it is periodic with period $2L$. Then f has a Fourier series (2.8) whose coefficients are given by (2.10)-(2.12). Moreover, the Fourier series converges to $f(x)$ at all points where f is continuous, and to $\frac{1}{2}(f(x+0) + f(x-0))$ at all points x where f is discontinuous (at jump points).*

Corollary. *When f is a $2L-$periodic function that is continuous on $(-\infty,\infty)$ and has a piecewise continuous derivative, its Fourier series not only converges at each point but it converges uniformly on $(-\infty,\infty)$, i.e. for every $\varepsilon > 0$ there exists $N_0(\varepsilon)$ such that*

$$|f(x) - S_N(x)| < \varepsilon, \quad N \geq N_0(\varepsilon), \quad x \in (-\infty,\infty).$$

Example 2.3. For $\operatorname{sgn}(x)$ on $[-\pi,\pi)$ we had the Fourier series

$$\operatorname{sgn}(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1}.$$

Let us extend $\operatorname{sgn}(x)$ outside the interval $-\pi \leq x < \pi$ so that it is 2π-periodic. Hence, this function has jumps at $x_n = \pi n, n = 0, \pm1, \pm2, \ldots$ and

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi n)}{2k-1} = \frac{1}{2}(\operatorname{sgn}(\pi n+0) + \operatorname{sgn}(\pi n - 0)) = 0.$$

Example 2.4. Let

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}$$

and let f be defined outside this interval so that $f(x+2L) = f(x)$ for all x, except at the points $x = 0, \pm L, \pm 2L, \ldots$. We will temporarily leave open the definition of f at these points. The Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L L dx = L, \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \int_0^L \cos\frac{m\pi x}{L}dx = \left.\frac{\sin\frac{m\pi x}{L}}{\frac{m\pi}{L}}\right|_0^L = 0 \end{aligned}$$

and

$$\begin{aligned} b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \int_0^L \sin\frac{m\pi x}{L}dx = \left.\frac{-\cos\frac{m\pi x}{L}}{\frac{m\pi}{L}}\right|_0^L \\ &= \frac{L}{m\pi}(1-\cos(m\pi)) = \frac{L}{m\pi}(1-(-1)^m) = \begin{cases} 0, & m = 2k, k = 1, 2, \ldots \\ \frac{2L}{m\pi}, & m = 2k-1, k = 1, 2, \ldots. \end{cases} \end{aligned}$$

Hence

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{k=1}^{\infty}\frac{\sin\frac{(2k-1)\pi x}{L}}{2k-1}.$$

It follows that for any $x \neq nL, n = 0, \pm1, \pm2, \ldots$,

$$S_N(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{k=1}^{N}\frac{\sin\frac{(2k-1)\pi x}{L}}{2k-1} \to f(x), \quad N \to \infty,$$

where $f(x) = 0$ or L. At any $x = nL$,

$$S_N(x) \equiv \frac{L}{2} \to \frac{L}{2}, \quad N \to \infty.$$

But nevertheless, the difference

$$R_N(x) = f(x) - S_N(x)$$

cannot be made uniformly small for all x simultaneously. In the neighborhood of points of discontinuity ($x = nL$), the partial sums do not converge smoothly to the mean value $\frac{L}{2}$. This behavior is known as the *Gibbs phenomenon*. However, if we consider the pointwise convergence of the partial sums then Theorem 1 still applies.

Complex form of the Fourier series

Since

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \quad \text{and} \quad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

then the series (2.8) becomes

$$\sum_{m=-\infty}^{\infty} c_m e^{i\frac{m\pi x}{L}},$$

where

$$c_m = \begin{cases} \frac{a_m - ib_m}{2}, & m = 1, 2, \dots \\ \frac{a_0}{2}, & m = 0 \\ \frac{a_{-m} + ib_{-m}}{2}, & m = -1, -2, \dots. \end{cases}$$

If f is real-valued then $c_m = \overline{c_{-m}}$ and

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{m\pi x}{L}}\,dx, \quad m = 0, \pm 1, \pm 2, \dots.$$

In solving problems in differential equations it is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on the interval $[0, L]$ (instead of $[-L, L]$). Several alternatives are available.

1. Define a function g of period $2L$ so that

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0 \end{cases}$$

($f(-L) = f(L)$ by periodicity). Thus, $g(x)$ is even and its Fourier (cosine) series represents f on $[0, L]$.

2. Define a function h of period $2L$ so that

$$h(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, L \\ -f(-x), & -L < x < 0. \end{cases}$$

Thus, h is the odd periodic extension of f and its Fourier (sine) series represents f on $(0, L)$.

3. Define a function K of period $2L$ so that

$$K(x) = f(x), \quad 0 \le x \le L$$

and let $K(x)$ be defined on $(-L, 0)$ in any way consistent with Theorem 1. Then its Fourier series involves both sine and cosine terms, and represents f on $[0, L]$.

Example 2.5. Suppose that

$$f(x) = \begin{cases} 1 - x, & 0 < x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$$

As indicated above, we can represent f either by a cosine series or sine series. For cosine series we define an even extension of f as follows:

$$g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \\ 1 + x, & -1 \le x < 0 \\ 0, & -2 \le x < -1, \end{cases}$$

see Figure 1.

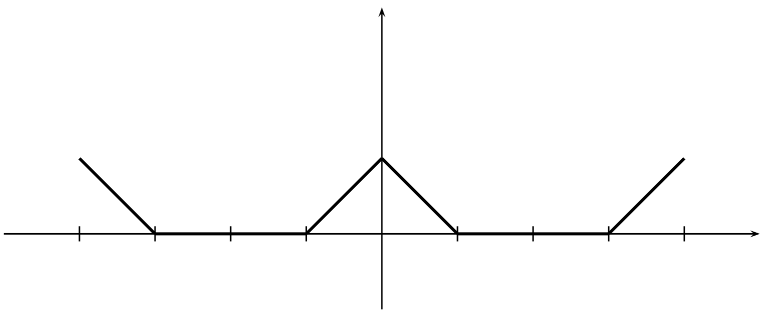


Figure 1: The extension of f.

This is an even 4-periodic function. The Fourier coefficients are

$$a_0 = \frac{1}{2}\int_{-2}^{2} g(x)dx = \int_0^2 g(x)dx = \int_0^1 (1-x)dx = \frac{1}{2}$$

and

$$
\begin{aligned}
a_m &= \frac{1}{2}\int_{-2}^{2} g(x)\cos\frac{m\pi x}{2}dx = \int_0^1 (1-x)\cos\frac{m\pi x}{2}dx \\
&= (1-x)\frac{\sin\frac{m\pi x}{2}}{\frac{m\pi}{2}}\Bigg|_0^1 + \frac{2}{m\pi}\int_0^1 \sin\frac{m\pi x}{2}dx \\
&= -\frac{2}{m\pi}\,\frac{\cos\frac{m\pi x}{2}}{\frac{m\pi}{2}}\Bigg|_0^1 = \frac{4}{m^2\pi^2}\left(1-\cos\frac{m\pi}{2}\right) \\
&= \begin{cases} \frac{4}{m^2\pi^2}, & m = 2k-1, k = 1, 2, \ldots \\ \frac{4}{m^2\pi^2}\left(1-(-1)^k\right), & m = 2k, k = 1, 2, \ldots. \end{cases}
\end{aligned}
$$

Hence the Fourier cosine series has the form

$$
\frac{1}{4} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi x}{2}}{(2k-1)^2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{1-(-1)^k}{(2k)^2}\right)\cos(k\pi x)
$$

or

$$
\frac{1}{4} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi x}{2}}{(2k-1)^2} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos((2k-1)\pi x)}{(2k-1)^2}.
$$

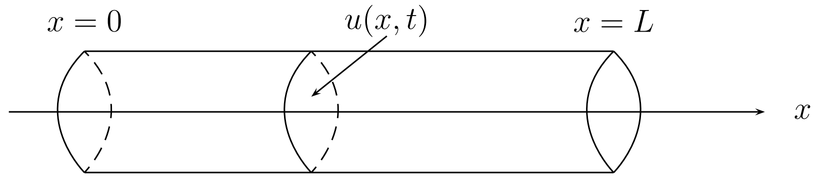
This representation holds for all $x \in \mathbb{R}$. In particular, for all $x \in [1, 3]$ we have

$$
\frac{1}{4} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi x}{2}}{(2k-1)^2} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos((2k-1)\pi x)}{(2k-1)^2} = 0.
$$

Exercise 15. Find the corresponding Fourier sine series of f.

3 One-dimensional Heat Equation

Let us consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let $x = 0$ and $x = L$ denote the ends of the bar (x-axis is chosen to lie along the axis of the bar). Suppose that no heat passes through the sides of the bar. We also assume that the cross-sectional dimensions are so small that temperature u can be considered the same on any given cross section.



Then u is a function only of the coordinate x and the time t. The variation of temperature in the bar is governed by a partial differential equation

$$\alpha^2 u_{xx}(x,t) = u_t(x,t), \quad 0 < x < L, t > 0, \tag{3.1}$$

where α^2 is a constant known as the thermal diffusivity. This equation is called the *heat conduction equation* or *heat equation*.

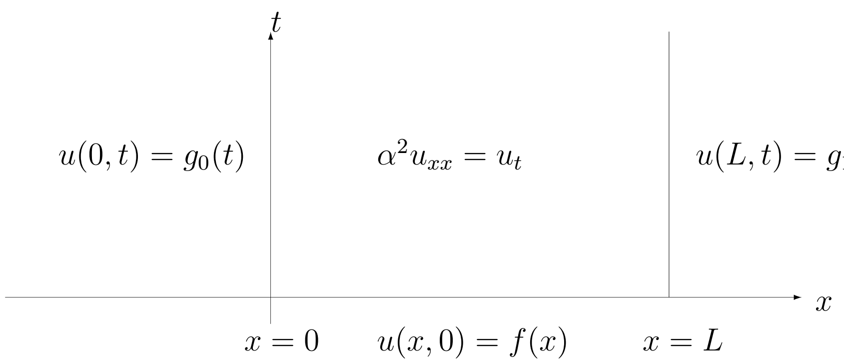
In addition, we assume that the initial temperature distribution in the bar is given by

$$u(x,0) = f(x), \quad 0 \leq x \leq L, \tag{3.2}$$

where f is a given function. Finally, we assume that the temperature at each end of the bar is given by

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad t > 0, \tag{3.3}$$

where g_0 and g_1 are given functions. The problem (3.1), (3.2), (3.3) is an initial value problem in time variable t. With respect to the space variable x it is a *boundary value problem* and (3.3) are called the *boundary conditions*. Alternatively, this problem can be considered as a boundary value problem in the xt-plane:



We start by considering the *homogeneous boundary conditions* when the functions $g_0(t)$

and $g_1(t)$ in (3.3) are identically zero:

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 \leq x \leq L. \end{cases} \tag{3.4}$$

We look for a solution to the problem (3.4) in the form

$$u(x,t) = X(x)T(t). \tag{3.5}$$

Such method is called a *separation of variables*. Substituting (3.5) into (3.1) yields

$$\alpha^2 X''(x)T(t) = X(x)T'(t)$$

or

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$$

in which the variables are separated, that is, the left hand side depends only on x and the right hand side only on t. This is possible only when both sides are equal to the same constant:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda.$$

Hence, we obtain two ordinary differential equations for $X(x)$ and $T(t)$

$$X'' + \lambda X = 0,$$

$$T' + \alpha^2 \lambda T = 0. \tag{3.6}$$

The boundary condition for $u(x,t)$ at $x = 0$ leads to

$$u(0,t) = X(0)T(t) = 0.$$

It follows that

$$X(0) = 0$$

(since otherwise $T \equiv 0$ and so $u \equiv 0$ which we do not want). Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0.$$

So, for the function $X(x)$ we obtain the homogeneous boundary value problem

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < L \\ X(0) = X(L) = 0. \end{cases} \tag{3.7}$$

The values of λ for which nontrivial solutions of (3.7) exist are called *eigenvalues* and the corresponding nontrivial solutions are called *eigenfunctions*. The problem (3.7) is called an *eigenvalue problem*.

Lemma 1. *The problem* (3.7) *has an infinite sequence of positive eigenvalues*

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots$$

with the corresponding eigenfunctions

$$X_n(x) = c \sin \frac{n\pi x}{L},$$

where c is an arbitrary nonzero constant.

Proof. Suppose first that $\lambda > 0$, i.e. $\lambda = \mu^2$. The characteristic equation for (3.7) is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$, so the general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

Note that μ is nonzero and there is no loss of generality if we assume that $\mu > 0$. The first boundary condition in (3.7) implies

$$X(0) = c_1 = 0,$$

and the second reduces to

$$c_2 \sin \mu L = 0$$

or

$$\sin \mu L = 0$$

as we do not allow $c_2 = 0$ too. It follows that

$$\mu L = n\pi, \quad n = 1, 2, \ldots$$

or

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots.$$

Hence the corresponding eigenfunctions are

$$X_n(x) = c \sin \frac{n\pi x}{L}.$$

If $\lambda = -\mu^2 < 0, \mu > 0$, then the characteristic equation for (3.7) is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$. Hence the general solution is

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x.$$

Since

$$\cosh \mu x = \frac{e^{\mu x} + e^{-\mu x}}{2} \quad \text{and} \quad \sinh \mu x = \frac{e^{\mu x} - e^{-\mu x}}{2}$$

this is equivalent to

$$X(x) = c_1' e^{\mu x} + c_2' e^{-\mu x}.$$

The first boundary condition requires again that $c_1 = 0$ while the second gives

$$c_2 \sinh \mu L = 0.$$

Since $\mu \neq 0$ ($\mu > 0$), it follows that $\sinh \mu L \neq 0$ and therefore we must have $c_2 = 0$. Consequently, $X \equiv 0$, i.e. there are no nontrivial solutions for $\lambda < 0$.

If $\lambda = 0$ the general solution is

$$X(x) = c_1 x + c_2.$$

The boundary conditions can be satisfied only if $c_1 = c_2 = 0$ so there is only the trivial solution in this case as well. □

Turning now to (3.6) for $T(t)$ and substituting $\frac{n^2\pi^2}{L^2}$ for λ we have

$$T(t) = ce^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}.$$

Hence the functions

$$u_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L} \tag{3.8}$$

satisfy (3.1) and the homogeneous boundary conditions from (3.4) for each $n = 1, 2, \ldots$. The *linear superposition principle* gives that any linear combination

$$u(x,t) = \sum_{n=1}^{N} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

is also a solution of the same problem. In order to take into account infinitely many functions (3.8) we assume that

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}, \tag{3.9}$$

where the coefficients c_n are yet undetermined, and the series converges in some sense. To satisfy the initial condition from (3.4) we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x), \quad 0 \leq x \leq L. \tag{3.10}$$

In other words, we need to choose the coefficients c_n so that the series (3.10) converges to the initial temperature distribution $f(x)$.

It is not difficult to prove that for $t > 0, 0 < x < L$, the series (3.9) converges (with any derivative with respect to x and t) and solves (3.1) with boundary conditions (3.4). Only one question remains: can any function $f(x)$ be represented by a Fourier sine series (3.10)? Some sufficient conditions for such representation are given in Theorem 1 of Chapter 2.

Remark. We can consider the boundary value problem for any linear differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{3.11}$$

of order two on the interval (a, b) with the boundary conditions

$$y(a) = y_0, \quad y(b) = y_1, \tag{3.12}$$

where y_0 and y_1 are given constants. Let us assume that we have found a fundamental set of solutions $y_1(x)$ and $y_2(x)$ to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then the general solution to (3.11) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

where $y_p(x)$ is a particular solution to (3.11) and c_1 and c_2 are arbitrary constants.

To satisfy the boundary conditions (3.12) we have the linear nonhomogeneous algebraic system

$$\begin{cases} c_1 y_1(a) + c_2 y_2(a) = y_0 - y_p(a) \\ c_1 y_1(b) + c_2 y_2(b) = y_1 - y_p(b). \end{cases} \tag{3.13}$$

If the determinant

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix}$$

is nonzero, then the constants c_1 and c_2 can be determined uniquely and therefore the boundary value problem (3.11)-(3.12) has a unique solution. If

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix} = 0$$

then (3.11)-(3.12) either has no solutions or has infinitely many solutions.

Example 3.1. Let us consider the boundary value problem

$$\begin{cases} y'' + \mu^2 y = 1, & 0 < x < 1 \\ y(0) = y_0, y(1) = y_1, \end{cases}$$

where $\mu > 0$ is fixed. This differential equation has a particular solution $y_p(x) = \frac{1}{\mu^2}$. Hence, the system (3.13) becomes

$$\begin{cases} c_1 \sin 0 + c_2 \cos 0 = y_0 - \frac{1}{\mu^2} \\ c_1 \sin \mu + c_2 \cos \mu = y_1 - \frac{1}{\mu^2} \end{cases}$$

or

$$\begin{cases} c_2 = y_0 - \frac{1}{\mu^2} \\ c_1 \sin \mu = y_1 - \frac{1}{\mu^2} - \left(y_0 - \frac{1}{\mu^2}\right) \cos \mu. \end{cases}$$

If

$$\begin{vmatrix} 0 & 1 \\ \sin\mu & \cos\mu \end{vmatrix} \neq 0$$

i.e. $\sin\mu \neq 0$ then c_1 is uniquely determined and the boundary value problem in question has a unique solution. If $\sin\mu = 0$ then the problem has solutions (actually, infinitely many) if and only if

$$y_1 - \frac{1}{\mu^2} = \left(y_0 - \frac{1}{\mu^2}\right)\cos\mu.$$

If $\mu = 2\pi k$ then $\sin\mu = 0$ and $\cos\mu = 1$ and the following equation must hold

$$y_1 - \frac{1}{\mu^2} = y_0 - \frac{1}{\mu^2}$$

i.e. $y_1 = y_0$. If $\mu = \pi + 2\pi k$ then $\sin\mu = 0$ and $\cos\mu = -1$ and we must have

$$y_1 + y_0 = \frac{2}{\mu^2}.$$

Suppose now that one end of the bar is held at a constant temperature T_1 and the other is maintained at a constant temperature T_2. The corresponding boundary value problem is then

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0 \\ u(0,t) = T_1, u(L,t) = T_2, & t > 0 \\ u(x,0) = f(x). \end{cases} \tag{3.14}$$

After a long time ($t \to \infty$) we anticipate that a steady temperature distribution $v(x)$ will be reached, which is independent of time and the initial condition. Since the solution of (3.14) with $T_1 = T_2 = 0$ tends to zero as $t \to \infty$, see (3.9), then we look for the solution to (3.14) in the form

$$u(x,t) = v(x) + w(x,t). \tag{3.15}$$

Substituting (3.15) into (3.14) leads to

$$\begin{cases} \alpha^2(v_{xx} + w_{xx}) = w_t \\ v(0) + w(0,t) = T_1, v(L) + w(L,t) = T_2 \\ v(x) + w(x,0) = f(x). \end{cases}$$

Let us assume that $v(x)$ satisfies the steady-state problem

$$\begin{cases} v''(x) = 0, & 0 < x < L \\ v(0) = T_1, & v(L) = T_2. \end{cases} \tag{3.16}$$

Then $w(x,t)$ satisfies the homogeneous boundary value problem for the heat equation:

$$\begin{cases} \alpha^2 w_{xx} = w_t, & 0 < x < L, t > 0 \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = \tilde{f}(x), \end{cases} \tag{3.17}$$

where $\tilde{f}(x) = f(x) - v(x)$. Since the solution of (3.16) is

$$v(x) = \frac{T_2 - T_1}{L} x + T_1 \tag{3.18}$$

the solution of (3.17) is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}, \tag{3.19}$$

where the coefficients c_n are given by

$$c_n = \frac{2}{L} \int_0^L \left[f(x) - \frac{T_2 - T_1}{L} x - T_1 \right] \sin \frac{n\pi x}{L} dx.$$

Combining (3.18) and (3.19) we obtain

$$u(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}.$$

Let us slightly complicate the problem (3.14), namely assume that

$$\begin{cases} \alpha^2 u_{xx} = u_t + p(x), & 0 < x < L, t > 0 \\ u(0,t) = T_1, u(L,t) = T_2, & t > 0 \\ u(x,0) = f(x). \end{cases} \tag{3.20}$$

We begin by assuming that the solution to (3.20) consists of a steady-state solution $v(x)$ and a transient solution $w(x,t)$ which tends to zero as $t \to \infty$:

$$u(x,t) = v(x) + w(x,t).$$

Then for $v(x)$ we will have the problem

$$\begin{cases} v''(x) = \frac{1}{\alpha^2} p(x), & 0 < x < L \\ v(0) = T_1, v(L) = T_2. \end{cases} \tag{3.21}$$

To solve this, integrate twice to get

$$v(x) = \frac{1}{\alpha^2} \int_0^x dy \int_0^y p(s) ds + c_1 x + c_2.$$

The boundary conditions yield $c_2 = T_1$ and

$$c_1 = \frac{1}{L}\left\{T_2 - T_1 - \frac{1}{\alpha^2}\int_0^L dy \int_0^y p(s)ds\right\}.$$

Therefore, the solution of (3.21) has the form

$$v(x) = \frac{T_2 - T_1}{L}x - \frac{x}{L\alpha^2}\int_0^L dy \int_0^y p(s)ds + \frac{1}{\alpha^2}\int_0^x dy \int_0^y p(s)ds + T_1.$$

For $w(x,t)$ we will have the homogeneous problem

$$\begin{cases} \alpha^2 w_{xx} = w_t, \quad 0 < x < L, t > 0 \\ w(0,t) = w(L,t) = 0, t > 0 \\ w(x,0) = \tilde{f}(x) := f(x) - v(x). \end{cases}$$

A different problem occurs if the ends of the bar are insulated so that there is no passage of heat through them. Thus, in the case of no heat flow, the boundary value problem is

$$\begin{cases} \alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0 \\ u_x(0,t) = u_x(L,t) = 0, t > 0 \\ u(x,0) = f(x). \end{cases} \tag{3.22}$$

This problem can also be solved by the method of separation of variables. If we let $u(x,t) = X(x)T(t)$ it follows that

$$X'' + \lambda X = 0, \quad T' + \alpha^2 \lambda T = 0. \tag{3.23}$$

The boundary conditions yield now

$$X'(0) = X'(L) = 0. \tag{3.24}$$

If $\lambda = -\mu^2 < 0, \mu > 0$, then (3.23) for $X(x)$ becomes $X'' - \mu^2 X = 0$ with general solution

$$X(x) = c_1 \sinh \mu x + c_2 \cosh \mu x.$$

Therefore, the conditions (3.24) give $c_1 = 0$ and $c_2 = 0$ which is unacceptable. Hence λ cannot be negative.

If $\lambda = 0$ then

$$X(x) = c_1 x + c_2.$$

Thus $X'(0) = c_1 = 0$ and $X'(L) = 0$ for any c_2 leaving c_2 undetermined. Therefore $\lambda = 0$ is an eigenvalue, corresponding to the eigenfunction $X_0(x) = 1$. It follows from (3.23) that $T(t)$ is also a constant. Hence, for $\lambda = 0$ we obtain the constant solution $u_0(x,t) = c_2$.

If $\lambda = \mu^2 > 0$ then $X'' + \mu^2 X = 0$ and consequently

$$X(x) = c_1 \sin \mu x + c_2 \cos \mu x.$$

The boundary conditions imply $c_1 = 0$ and $\mu = \frac{n\pi}{L}, n = 1, 2, \ldots$ leaving c_2 arbitrary. Thus we have an infinite sequence of positive eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ with the corresponding eigenfunctions

$$X_n(x) = \cos \frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

If we combine these eigenvalues and eigenfunctions with zero eigenvalue and $X_0(x) = 1$ we may conclude that we have the infinite sequences

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos \frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots.$$

and

$$u_n(x, t) = \cos \frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}, \quad n = 0, 1, 2, \ldots.$$

Each of these functions satisfies the equation and boundary conditions from (3.22). It remains to satisfy the initial condition. In order to do it we assume that $u(x, t)$ has the form

$$u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}, \tag{3.25}$$

where the coefficients c_n are determined by the requirement that

$$u(x, 0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} = f(x), \quad 0 \leq x \leq L.$$

Thus the unknown coefficients in (3.25) must be the Fourier coefficients in the Fourier cosine series of period $2L$ for even extension of f. Hence

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \ldots$$

and the series (3.25) provides the solution to the heat conduction problem (3.22) for a rod with insulated ends. The physical interpretation of the term

$$\frac{c_0}{2} = \frac{1}{L} \int_0^L f(x) dx$$

is that it is the mean value of the original temperature distribution.

Exercise 16. Let $v(x)$ be a solution of the problem

$$\begin{cases} v''(x) = 0, \quad 0 < x < L \\ v'(0) = T_1, v'(L) = T_2. \end{cases}$$

Show that the problem

$$\begin{cases} \alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0 \\ u_x(0,t) = T_1, u_x(L,t) = T_2, t > 0 \\ u(x,0) = f(x) \end{cases}$$

has a solution of the form $u(x,t) = v(x) + w(x,t)$ if and only if $T_1 = T_2$.

Example 3.2.

$$\begin{cases} u_{xx} = u_t, \quad 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) := f(x). \end{cases}$$

As we know the solution of this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-(n\pi)^2 t}.$$

Since

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x)$$

then we may conclude that $c_n = \frac{1}{n^2}$ necessarily (since the Fourier series is unique). Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) e^{-(n\pi)^2 t}.$$

Exercise 17. Find a solution of the problem

$$\begin{cases} u_{xx} = u_t, \quad 0 < x < \pi, t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, t > 0 \\ u(x,0) = 1 - \sin x \end{cases}$$

using the method of separation of variables.

Let us consider a bar with mixed boundary conditions at the ends. Assume that the temperature at $x = 0$ is zero, while the end $x = L$ is insulated so that no heat passes through it:

$$\begin{cases} \alpha^2 u_{xx} = u_t, \quad 0 < x < L, t > 0 \\ u(0,t) = u_x(L,t) = 0, t > 0 \\ u(x,0) = f(x). \end{cases}$$

Separation of variables leads to

$$\begin{cases} X'' + \lambda X = 0, \quad 0 < x < L \\ X(0) = X'(L) = 0 \end{cases} \tag{3.26}$$

and

$$T' + \lambda T = 0, \quad t > 0.$$

As above, one can show that (3.26) has nontrivial solutions only for $\lambda > 0$, namely

$$\lambda_m = \frac{(2m-1)^2\pi^2}{4L^2}, \quad X_m(x) = \sin\frac{(2m-1)\pi x}{2L}, \quad m = 1, 2, 3, \ldots.$$

The solution to the mixed boundary value problem is

$$u(x,t) = \sum_{m=1}^{\infty} c_m \sin\frac{(2m-1)\pi x}{2L} e^{-\left(\frac{(2m-1)\pi\alpha}{2L}\right)^2 t}$$

with arbitrary constants c_m. To satisfy the initial condition we have

$$f(x) = \sum_{m=1}^{\infty} c_m \sin\frac{(2m-1)\pi x}{2L}, \quad 0 \le x \le L.$$

This is a Fourier sine series but in some specific form. We show that the coefficients c_m can be calculated as

$$c_m = \frac{2}{L}\int_0^L f(x)\sin\frac{(2m-1)\pi x}{2L}dx$$

and such representation is possible.

In order to prove it, let us first extend $f(x)$ to the interval $0 \le x \le 2L$ so that it is symmetric about $x = L$, i.e. $f(2L-x) = f(x)$ for $0 \le x \le L$. Then extend the resulting function to the interval $(-2L, 0)$ as an odd function and elsewhere as a periodic function $\widetilde{f}$ of period $4L$. In this procedure we need to define

$$\widetilde{f}(0) = \widetilde{f}(2L) = \widetilde{f}(-2L) = 0.$$

Then the Fourier series contains only sines:

$$\widetilde{f}(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2L}$$

with the Fourier coefficients

$$c_n = \frac{2}{2L}\int_0^{2L} \widetilde{f}(x)\sin\frac{n\pi x}{2L}dx.$$

Let us show that $c_n = 0$ for even $n = 2m$. Indeed,

$$\begin{aligned} c_{2m} &= \frac{1}{L}\int_0^{2L}\widetilde{f}(x)\sin\frac{m\pi x}{L}dx \\ &= \frac{1}{L}\int_0^{L} f(x)\sin\frac{m\pi x}{L}dx + \frac{1}{L}\int_L^{2L} f(2L-x)\sin\frac{m\pi x}{L}dx \\ &= \frac{1}{L}\int_0^{L} f(x)\sin\frac{m\pi x}{L}dx - \frac{1}{L}\int_L^{0} f(y)\sin\frac{m\pi(2L-y)}{L}dy \\ &= \frac{1}{L}\int_0^{L} f(x)\sin\frac{m\pi x}{L}dx + \frac{1}{L}\int_L^{0} f(y)\sin\frac{m\pi y}{L}dy = 0. \end{aligned}$$

That's why

$$\widetilde{f}(x) = \sum_{m=1}^{\infty} c_{2m-1} \sin \frac{(2m-1)\pi x}{2L},$$

where

$$\begin{aligned} c_{2m-1} &= \frac{1}{L}\int_0^{2L} \widetilde{f}(x) \sin \frac{(2m-1)\pi x}{2L} dx \\ &= \frac{1}{L}\int_0^{L} f(x) \sin \frac{(2m-1)\pi x}{2L} dx + \frac{1}{L}\int_L^{2L} f(2L-x) \sin \frac{(2m-1)\pi x}{2L} dx \\ &= \frac{2}{L}\int_0^{L} f(x) \sin \frac{(2m-1)\pi x}{2L} dx \end{aligned}$$

as claimed. Let us remark that the series

$$\sum_{m=1}^{\infty} c_m \sin \frac{(2m-1)\pi x}{2L}$$

represents $f(x)$ on $(0, L]$.

Remark. For the boundary conditions

$$u_x(0,t) = u(L,t) = 0$$

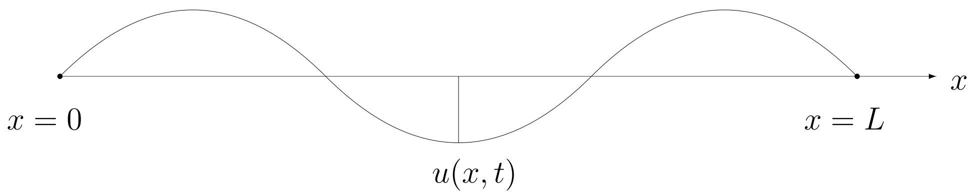
the function $f(x)$ must be extended to the interval $0 \le x \le 2L$ as $f(x) = -f(2L - x)$ with $f(L) = 0$. Furthermore, $\widetilde{f}$ is an even extension to the interval $(-2L, 0)$. Then the corresponding Fourier series represents $f(x)$ on the interval $[0, L)$.

4 One-dimensional Wave Equation

Another situation in which the separation of variables applies occurs in the study of a vibrating string. Suppose that an elastic string of length L is tightly stretched between two supports, so that the x-axis lies along the string. Let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. It turns out that if damping effects are neglected, and if the amplitude of the motion is not too large, then $u(x,t)$ satisfies the partial differential equation

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, t > 0. \tag{4.1}$$

Equation (4.1) is known as the one-dimensional *wave equation*. The constant $a^2 = T/\rho$, where T is the force in the string and ρ is the mass per unit length of the string material.



To describe the motion completely it is necessary also to specify suitable initial and boundary conditions for the displacement $u(x,t)$. The ends are assumed to remain fixed:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{4.2}$$

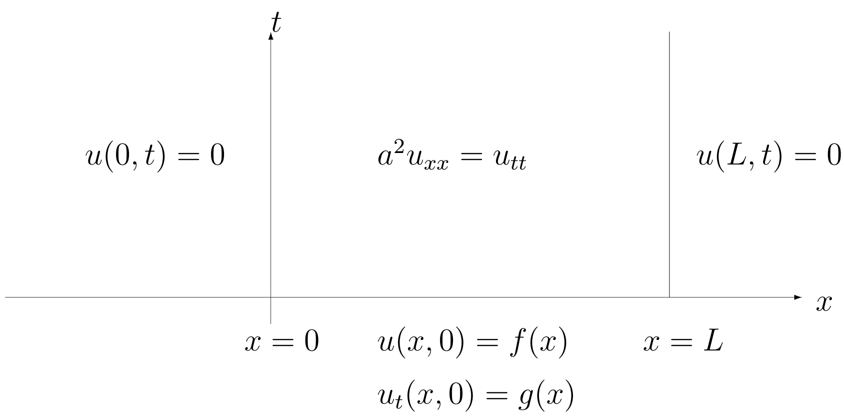
The initial conditions are (since (4.1) is of second order with respect to t):

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L, \tag{4.3}$$

where f and g are given functions. In order for (4.2) and (4.3) to be consistent it is also necessary to require that

$$f(0) = f(L) = g(0) = g(L) = 0. \tag{4.4}$$

Equations (4.1)-(4.4) can be interpreted as the following boundary value problem for the wave equation:



Let us apply the method of separation of variables to this homogeneous boundary value problem. Assuming that $u(x,t) = X(x)T(t)$ we obtain

$$X'' + \lambda X = 0, \quad T'' + a^2\lambda T = 0.$$

The boundary conditions (4.2) imply that

$$\begin{cases} X'' + \lambda X = 0, 0 < x < L \\ X(0) = X(L) = 0. \end{cases}$$

This is the same boundary value problem that we have considered before. Hence,

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

Taking $\lambda = \lambda_n$ in the equation for $T(t)$ we have

$$T''(t) + \left(\frac{n\pi a}{L}\right)^2 T(t) = 0.$$

The general solution to this equation is

$$T(t) = k_1 \cos\frac{n\pi at}{L} + k_2 \sin\frac{n\pi at}{L},$$

where k_1 and k_2 are arbitrary constants. Using the linear superposition principle we consider the infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi at}{L} + b_n \sin\frac{n\pi at}{L}\right), \tag{4.5}$$

where the coefficients a_n and b_n are to be determined. It is clear that $u(x,t)$ from (4.5) satisfies (4.1) and (4.2) (at least formally). The initial conditions (4.3) imply

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \quad 0 \le x \le L, \\ g(x) &= \sum_{n=1}^{\infty} \frac{n\pi a}{L} b_n \sin\frac{n\pi x}{L}, \quad 0 \le x \le L. \end{aligned} \tag{4.6}$$

Since (4.4) are fulfilled then (4.6) are the Fourier sine series for f and g, respectively. Therefore,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx, \\ b_n &= \frac{2}{n\pi a}\int_0^L g(x) \sin\frac{n\pi x}{L} dx. \end{aligned} \tag{4.7}$$

Finally, we may conclude that the series (4.5) with the coefficients (4.7) solves (at least formally) the boundary value problem (4.1)-(4.4).

Each displacement pattern

$$u_n(x,t) = \sin\frac{n\pi x}{L}\left(a_n\cos\frac{n\pi at}{L} + b_n\sin\frac{n\pi at}{L}\right)$$

is called *a natural mode* of vibration and is periodic in both space variable x and time variable t. The spatial period $\frac{2L}{n}$ in x is called *the wavelength*, while the numbers $\frac{n\pi a}{L}$ are called the *natural frequencies*.

Exercise 18. Find a solution of the problem

$$\begin{cases} u_{xx} = u_{tt}, 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0, t \geq 0 \\ u(x,0) = x(1-x), u_t(x,0) = \sin(7\pi x) \end{cases}$$

using the method of separation of variables.

If we compare the two series

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n\cos\frac{n\pi at}{L} + b_n\sin\frac{n\pi at}{L}\right)$$
$$u(x,t) = \sum_{n=1}^{\infty} c_n\sin\frac{n\pi x}{L}e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

for the wave and heat equations we can see that the second series has the exponential factor that decays fast with n for any $t > 0$. This guarantees convergence of the series as well as the smoothness of the sum. This is not true anymore for the first series because it contains only oscillatory terms that do not decay with increasing n.

The boundary value problem for the wave equation with free ends of the string can be formulated as follows:

$$\begin{cases} a^2u_{xx} = u_{tt}, 0 < x < L, t > 0 \\ u_x(0,t) = u_x(L,t) = 0, t \geq 0 \\ u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq L. \end{cases}$$

Let us first note that the boundary conditions imply that $f(x)$ and $g(x)$ must satisfy

$$f'(0) = f'(L) = g'(0) = g'(L) = 0.$$

The method of separation of variables gives that the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots$$

and the formal solution $u(x,t)$ is

$$u(x,t) = \frac{b_0 t + a_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi at}{L} + b_n \sin\frac{n\pi at}{L}\right).$$

The initial conditions are satisfied when

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

and

$$g(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \cos\frac{n\pi x}{L},$$

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,\dots \\ b_0 &= \frac{2}{L}\int_0^L g(x)dx \end{aligned}$$

and

$$b_n = \frac{2}{n\pi a}\int_0^L g(x)\cos\frac{n\pi x}{L}dx, \quad n = 1,2,\dots.$$

Let us consider the wave equation on the whole line. It corresponds, so to say, to the infinite string. In that case we no more have the boundary conditions but we have the initial conditions:

$$\begin{cases} a^2 u_{xx} = u_{tt}, -\infty < x < \infty, t > 0 \\ u(x,0) = f(x), u_t(x,0) = g(x). \end{cases} \tag{4.8}$$

Proposition. *The solution $u(x,t)$ of the wave equation is of the form*

$$u(x,t) = \varphi(x - at) + \psi(x + at),$$

where φ and ψ are two arbitrary C^2 functions of one variable.

Proof. By the chain rule

$$\partial_{tt}u - a^2\partial_{xx}u = 0$$

if and only if

$$\partial_\xi\partial_\eta u = 0,$$

where $\xi = x + at$ and $\eta = x - at$ (and so $\partial_x = \partial_\xi + \partial_\eta$, $\frac{1}{a}\partial_t = \partial_\xi - \partial_\eta$). It follows that

$$\partial_\xi u = \Psi(\xi)$$

or

$$u = \psi(\xi) + \varphi(\eta),$$

where $\psi' = \Psi$. $\square$

To satisfy the initial conditions we have

$$f(x) = \varphi(x) + \psi(x), \quad g(x) = -a\varphi'(x) + a\psi'(x).$$

It follows that

$$\varphi'(x) = \frac{1}{2}f'(x) - \frac{1}{2a}g(x), \quad \psi'(x) = \frac{1}{2}f'(x) + \frac{1}{2a}g(x).$$

Integrating we obtain

$$\varphi(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(s)ds + c_1, \quad \psi(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(s)ds + c_2,$$

where c_1 and c_2 are arbitrary constants. But $\varphi(x) + \psi(x) = f(x)$ implies $c_1 + c_2 = 0$. Therefore the solution of the initial value problem is

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds. \tag{4.9}$$

This formula is called the *d'Alembert formula*.

Exercise 19. Prove that if f is a C^2 function and g is a C^1 function, then u from (4.9) is a C^2 function and satisfies (4.8) in the classical sense.

Exercise 20. Prove that if f and g are merely locally integrable, then u from (4.9) is a distributional solution of (4.8) and the initial conditions are satisfied pointwise.

Example 4.1. The solution of

$$\begin{cases} u_{xx} = u_{tt}, -\infty < x < \infty, t > 0 \\ u(x,0) = f(x), u_t(x,0) = 0, \end{cases}$$

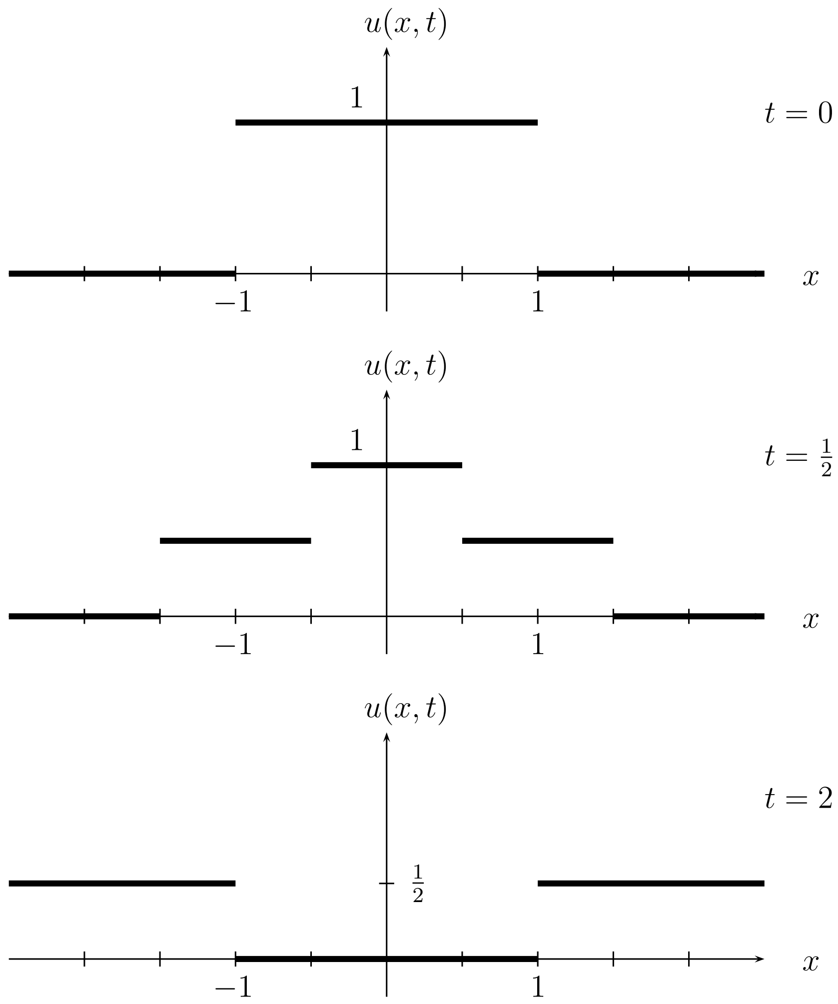
where

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right).$$

Some solutions are graphed below.



We can apply the d'Alembert formula for the finite string also. Consider again the boundary value problem with homogeneous boundary conditions with fixed ends of the string.

$$\begin{cases} a^2u_{xx} = u_{tt}, 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, t \geq 0 \\ u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq L \\ f(0) = f(L) = g(0) = g(L) = 0. \end{cases}$$

Let $h(x)$ be the function defined for all $x \in \mathbb{R}$ such that

$$h(x) = \begin{cases} f(x), & 0 \leq x \leq L \\ -f(-x), & -L \leq x \leq 0 \end{cases}$$

and $2L$-periodic and let $k(x)$ be the function defined for all $x \in \mathbb{R}$ such that

$$k(x) = \begin{cases} g(x), & 0 \leq x \leq L \\ -g(-x), & -L \leq x \leq 0 \end{cases}$$

and $2L$-periodic. Let us also assume that f and g are C^2 functions on the interval $[0, L]$. Then the solution to the boundary value problem is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}\left(h(x-at) + h(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} k(s)ds.$$

Remark. It can be checked that this solution is equivalent to the solution which is given by the Fourier series.

5 Laplace Equation in Rectangle and in Disk

One of the most important of all partial differential equations in applied mathematics is the *Laplace equation*:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{2D-equation} \\ u_{xx} + u_{yy} + u_{zz} &= 0 \quad \text{3D-equation} \end{aligned} \tag{5.1}$$

The Laplace equation appears quite naturally in many applications. For example, a steady state solution of the heat equation in two space dimensions

$$\alpha^2(u_{xx} + u_{yy}) = u_t$$

satisfies the 2D-Laplace equation (5.1). When considering electrostatic fields, the electric potential function must satisfy either 2D or 3D equation (5.1).

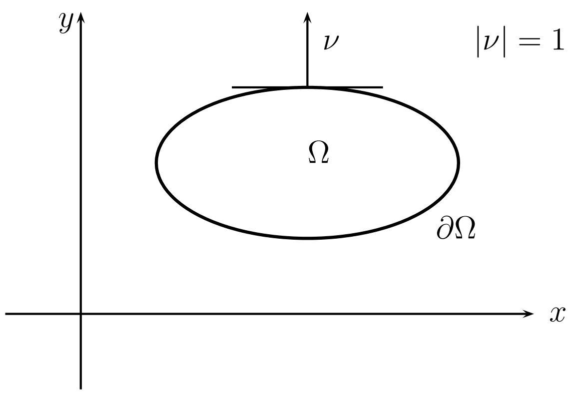
A typical boundary value problem for the Laplace equation is (in dimension two):

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \subset \mathbb{R}^2 \\ u(x, y) = f(x, y), & (x, y) \in \partial\Omega, \end{cases} \tag{5.2}$$

where f is a given function on the boundary $\partial\Omega$ of the domain Ω. The problem (5.2) is called the *Dirichlet problem* (Dirichlet boundary conditions). The problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \\ \frac{\partial u}{\partial \nu}(x, y) = g(x, y), & (x, y) \in \partial\Omega, \end{cases}$$

where g is given and $\frac{\partial u}{\partial \nu}$ is the outward normal derivative is called the *Neumann problem* (Neumann boundary conditions).



Dirichlet problem for a rectangle

Consider the boundary value problem in most general form:

$$\begin{cases} w_{xx} + w_{yy} = 0, & 0 < x < a, 0 < y < b \\ w(x, 0) = g_1(x), w(x, b) = f_1(x), & 0 < x < a \\ w(0, y) = g_2(y), w(a, y) = f_2(y), & 0 \le y \le b, \end{cases}$$

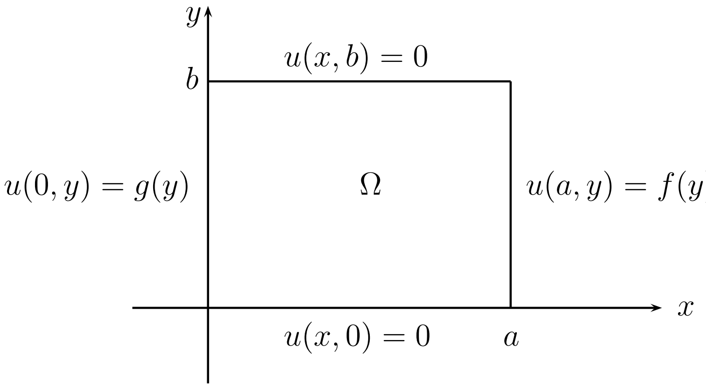
for fixed $a > 0$ and $b > 0$. The solution of this problem can be reduced to the solutions of

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b \\ u(x,0) = u(x,b) = 0, & 0 < x < a \\ u(0,y) = g(y), u(a,y) = f(y), & 0 \le y \le b, \end{cases} \tag{5.3}$$

and

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b \\ u(x,0) = g_1(x), u(x,b) = f_1(x), & 0 < x < a \\ u(0,y) = 0, u(a,y) = 0, & 0 \le y \le b. \end{cases}$$

Due to symmetry in x and y we consider (5.3) only.



The method of separation of variables gives for $u(x,y) = X(x)Y(y)$,

$$\begin{cases} Y'' + \lambda Y = 0, & 0 < y < b, \\ Y(0) = Y(b) = 0, \end{cases} \tag{5.4}$$

and

$$X'' - \lambda X = 0, \quad 0 < x < a. \tag{5.5}$$

From (5.4) one obtains the eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad Y_n(y) = \sin\frac{n\pi y}{b}, \quad n = 1, 2, \ldots.$$

Substitute λ_n into (5.5) to get the general solution

$$X(x) = c_1 \cosh\frac{n\pi x}{b} + c_2 \sinh\frac{n\pi x}{b}.$$

As above, represent the solution to (5.3) in the form

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b}\left(a_n \cosh\frac{n\pi x}{b} + b_n \sinh\frac{n\pi x}{b}\right). \tag{5.6}$$

The boundary condition at $x = 0$ gives

$$g(y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b},$$

with

$$a_n = \frac{2}{b} \int_0^b g(y) \sin \frac{n\pi y}{b} dy.$$

At $x = a$ we obtain

$$f(y) = \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} \left(a_n \cosh \frac{n\pi a}{b} + b_n \sinh \frac{n\pi a}{b} \right).$$

It is a Fourier sine series for $f(y)$. Hence,

$$a_n \cosh \frac{n\pi a}{b} + b_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} dy := \widetilde{b}_n.$$

It implies

$$b_n = \frac{\widetilde{b}_n - a_n \cosh \frac{n\pi a}{b}}{\sinh \frac{n\pi a}{b}}. \tag{5.7}$$

Substituting (5.7) into (5.6) gives

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} \left(a_n \cosh \frac{n\pi x}{b} + \frac{\widetilde{b}_n - a_n \cosh \frac{n\pi a}{b}}{\sinh \frac{n\pi a}{b}} \sinh \frac{n\pi x}{b} \right) \\ &= \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} \widetilde{b}_n \frac{\sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} \\ &+ \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} a_n \left(\frac{\cosh \frac{n\pi x}{b} \sinh \frac{n\pi a}{b} - \cosh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} \right) \\ &= \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} \widetilde{b}_n \frac{\sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} + \sum_{n=1}^{\infty} \sin \frac{n\pi y}{b} a_n \frac{\sinh \frac{n\pi (a-x)}{b}}{\sinh \frac{n\pi a}{b}}, \end{aligned}$$

because $\cosh \alpha \sinh \beta - \sinh \alpha \cosh \beta = \sinh(\beta - \alpha)$.

Exercise 21. Find a solution of the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 2, 0 < y < 1 \\ u(x,0) = u(x,1) = 0, & 0 < x < 2 \\ u(0,y) = 0, u(2,y) = y(1-y), & 0 \le y \le 1 \end{cases}$$

using the method of separation of variables.

Dirichlet problem for a disk

Consider the problem of solving the Laplace equation in a disk $\{x \in \mathbb{R}^2 : |x| < a\}$ subject to boundary condition

$$u(a, \theta) = f(\theta), \tag{5.8}$$

where f is a given function on $0 \leq \theta \leq 2\pi$. In polar coordinates $x = r\cos\theta, y = r\sin\theta$, the Laplace equation takes the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{5.9}$$

We apply again the method of separation of variables and assume that

$$u(r, \theta) = R(r)T(\theta). \tag{5.10}$$

Substitution for u in (5.9) yields

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0$$

or

$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ T'' + \lambda T = 0. \end{cases}$$

There are no homogeneous boundary conditions, however we need $T(\theta)$ to be 2π-periodic and also bounded. This fact, in particular, leads to

$$T(0) = T(2\pi), \quad T'(0) = T'(2\pi). \tag{5.11}$$

It is possible to show that (5.11) require λ to be real. In what follows we will consider the three possible cases.

If $\lambda = -\mu^2 < 0, \mu > 0$, then the equation for T becomes $T'' - \mu^2 T = 0$ and consequently

$$T(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

It follows from (5.11) that

$$\begin{cases} c_1 + c_2 = c_1 e^{2\pi\mu} + c_2 e^{-2\pi\mu} \\ c_1 - c_2 = c_1 e^{2\pi\mu} - c_2 e^{-2\pi\mu} \end{cases}$$

so that $c_1 = c_2 = 0$.

If $\lambda = 0$ then $T'' = 0$ and $T(\theta) = c_1 + c_2\theta$. The first condition in (5.11) implies then that $c_2 = 0$ and therefore $T(\theta) \equiv$ constant.

If $\lambda = \mu^2 > 0, \mu > 0$, then

$$T(\theta) = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta).$$

Now the conditions (5.11) imply that

$$\begin{cases} c_1 = c_1\cos(2\pi\mu) + c_2\sin(2\pi\mu) \\ c_2 = -c_1\sin(2\pi\mu) + c_2\cos(2\pi\mu) \end{cases}$$

or

$$\begin{cases} c_1\sin^2(\mu\pi) = c_2\sin(\mu\pi)\cos(\mu\pi) \\ c_2\sin^2(\mu\pi) = -c_1\sin(\mu\pi)\cos(\mu\pi). \end{cases}$$

If $\sin(\mu\pi) \neq 0$ then

$$\begin{cases} c_1 = c_2\cot(\mu\pi) \\ c_2 = -c_1\cot(\mu\pi). \end{cases}$$

Hence $c_1^2 + c_2^2 = 0$ i.e. $c_1 = c_2 = 0$. Thus we must have $\sin(\mu\pi) = 0$ and so

$$\lambda_n = n^2, \quad T_n(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta), \quad n = 0, 1, 2, \ldots. \tag{5.12}$$

Turning now to R, for $\lambda = 0$ we have $r^2R'' + rR' = 0$ i.e. $R(r) = k_1 + k_2\log r$. Since $\log r \to -\infty$ as $r \to 0$ we must choose $k_2 = 0$ in order for R (and u) to be bounded. That's why

$$R_0(r) \equiv \text{constant}. \tag{5.13}$$

For $\lambda = \mu^2 = n^2$ the equation for R becomes

$$r^2R'' + rR' - n^2R = 0.$$

Hence

$$R(r) = k_1r^n + k_2r^{-n}.$$

Again, we must choose $k_2 = 0$ and therefore

$$R_n(r) = k_1r^n, \quad n = 1, 2, \ldots. \tag{5.14}$$

Combining (5.10),(5.12), (5.13) and (5.14) we obtain

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta)). \tag{5.15}$$

The boundary condition (5.8) then requires

$$u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^n(a_n\cos(n\theta) + b_n\sin(n\theta)) = f(\theta).$$

Hence the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n &= \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \end{aligned}$$

and

$$b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta.$$

This procedure can be used also to study the Neumann problem, i.e. the problem in the disk with the boundary condition

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta). \tag{5.16}$$

Also in this case the solution $u(r,\theta)$ has the form (5.15). The boundary condition (5.16) implies that

$$\frac{\partial u}{\partial r}(r,\theta)\bigg|_{r=a} = \sum_{n=1}^{\infty} na^{n-1}(a_n\cos(n\theta) + b_n\sin(n\theta)) = f(\theta).$$

Hence

$$a_n = \frac{1}{\pi n a^{n-1}}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta$$

and

$$b_n = \frac{1}{\pi n a^{n-1}}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta.$$

Remark. For the Neumann problem a solution is defined up to an arbitrary constant $\frac{a_0}{2}$. Moreover, f must satisfy the consistency condition

$$\int_0^{2\pi} f(\theta)d\theta = 0$$

since integrating

$$f(\theta) = \sum_{n=1}^{\infty} na^{n-1}(a_n\cos(n\theta) + b_n\sin(n\theta))$$

termwise gives us zero.

6 The Laplace Operator

We consider what is perhaps the most important of all partial differential operators, the *Laplace operator* (*Laplacian*) on $\mathbb{R}^n$, defined by

$$\Delta = \sum_{j=1}^{n} \partial_j^2 \equiv \nabla \cdot \nabla.$$

We will start with a quite general fact about partial differential operators.

Definition. 1. A linear transformation T on $\mathbb{R}^n$ is called a *rotation* if $T' = T^{-1}$.

2. Let h be a fixed vector in $\mathbb{R}^n$. The transformation $T_h f(x) := f(x+h)$ is called *a translation*.

Theorem 1. *Suppose that L is a linear partial differential operator on $\mathbb{R}^n$. Then L commutes with translations and rotations if and only if L is a polynomial in Δ, that is, $L \equiv \sum_{j=0}^{m} a_j \Delta^j$.*

Proof. Let

$$L(x, \partial) \equiv \sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha$$

commute with a translation T_h. Then

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha f(x+h) = \sum_{|\alpha| \le k} a_\alpha(x+h) \partial^\alpha f(x+h).$$

This implies that $a_\alpha(x)$ must be constants (because $a_\alpha(x) \equiv a_\alpha(x+h)$ for all h), say a_α. Next, since L now has constant coefficients we have (see Exercise 5)

$$\widehat{Lu}(\xi) = P(\xi)\widehat{u}(\xi),$$

where the polynomial $P(\xi)$ is defined by

$$P(\xi) = \sum_{|\alpha| \le k} a_\alpha (i\xi)^\alpha.$$

Recall from Exercise 4 that if T is a rotation then

$$\widehat{u \circ T}(\xi) = (\widehat{u} \circ T)(\xi).$$

Therefore

$$\widehat{(Lu)(Tx)}(\xi) = \widehat{Lu}(T\xi)$$

or

$$P(\xi)\widehat{u(Tx)}(\xi) = P(T\xi)\widehat{u}(T\xi).$$

This forces

$$P(\xi) = P(T\xi).$$

Write $\xi = |\xi|\theta$, where $\theta \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the direction of ξ. Then $T\xi = |\xi|\theta'$ with some $\theta' \in \mathbb{S}^{n-1}$. But

$$0 = P(\xi) - P(T\xi) = P(|\xi|\theta) - P(|\xi|\theta')$$

shows that $P(\xi)$ does not depend on the angle θ of ξ. Therefore $P(\xi)$ is radial, that is,

$$P(\xi) = P_1(|\xi|) = \sum_{|\alpha|\le k} a'_\alpha |\xi|^{|\alpha|}.$$

But since we know that $P(\xi)$ is a polynomial then $|\alpha|$ must be even:

$$P(\xi) = \sum_j a_j |\xi|^{2j}.$$

By Exercise 5 we have that

$$\widehat{\Delta u}(\xi) = -|\xi|^2 \widehat{u}(\xi).$$

It follows by induction that

$$\widehat{\Delta^j u}(\xi) = (-1)^j |\xi|^{2j} \widehat{u}(\xi), \quad j = 0, 1, \ldots.$$

Taking the inverse Fourier transform we obtain

$$Lu = F^{-1}(P(\xi)\widehat{u}(\xi)) = F^{-1} \sum_j a_j |\xi|^{2j} \widehat{u}(\xi) = F^{-1} \sum_j a'_j \widehat{\Delta^j u}(\xi) = \sum_j a'_j \Delta^j u.$$

Conversely, let

$$Lu = \sum_j a_j \Delta^j u.$$

It is clear by the chain rule that Laplacian commutes with a translation T_h and a rotation T. By induction the same is true for any power of Δ and so for L as well. □

Lemma 1. *If $f(x) = \varphi(r), r = |x|$, that is, f is radial, then $\Delta f = \varphi''(r) + \frac{n-1}{r}\varphi'(r)$.*

Proof. Since $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$ then

$$\begin{aligned}
\Delta f &= \sum_{j=1}^n \partial_j(\partial_j \varphi(r)) = \sum_{j=1}^n \partial_j \left(\frac{x_j}{r} \varphi'(r) \right) \\
&= \sum_{j=1}^n \varphi'(r) \partial_j \left(\frac{x_j}{r} \right) + \sum_{j=1}^n \frac{x_j^2}{r^2} \varphi''(r) \\
&= \sum_{j=1}^n \left(\frac{1}{r} - \frac{x_j^2}{r^3} \right) \varphi'(r) + \sum_{j=1}^n \frac{x_j^2}{r^2} \varphi''(r) \\
&= \frac{n}{r}\varphi'(r) - \frac{1}{r^3} \sum_{j=1}^n x_j^2 \varphi'(r) + \varphi''(r) = \varphi''(r) + \frac{n-1}{r}\varphi'(r).
\end{aligned}$$

□

Corollary. *If* $f(x) = \varphi(r)$ *then* $\Delta f = 0$ *on* $\mathbb{R}^n \setminus \{0\}$ *if and only if*

$$\varphi(r) = \begin{cases} a + br^{2-n}, & n \neq 2 \\ a + b \log r, & n = 2, \end{cases}$$

where a *and* b *are arbitrary constants.*

Proof. If $\Delta f = 0$ then by Lemma 1 we have

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0.$$

Denote $\psi(r) := \varphi'(r)$. Since ψ solves the first order differential equation

$$\psi'(r) + \frac{n-1}{r}\psi(r) = 0$$

it can be found by the use of integrating factor. Indeed, multiply by $e^{(n-1)\log r} = r^{n-1}$ to get

$$r^{n-1}\psi'(r) + (n-1)r^{n-2}\psi(r) = 0$$

or

$$\left(r^{n-1}\psi(r)\right)' = 0.$$

It follows that

$$\varphi'(r) = \psi(r) = cr^{1-n}.$$

Integrate once more to arrive at

$$\varphi(r) = \begin{cases} \frac{cr^{2-n}}{2-n} + c_1, & n \neq 2 \\ c \log r + c_1, & n = 2 \end{cases} = \begin{cases} ar + b, & n = 1 \\ a \log r + b, & n = 2 \\ ar^{2-n} + b, & n \geq 3. \end{cases}$$

In the opposite direction the result follows from elementary differentiation. □

Definition. A C^2 function u on an open set $\Omega \subset \mathbb{R}^n$ is said to be *harmonic on* Ω if $\Delta u = 0$ on Ω.

Exercise 22. For $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and for $S = \partial\Omega$, which is a surface of class C^1, prove the following *Green's identities*:

a)

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_S (v\partial_\nu u - u\partial_\nu v)\, d\sigma$$

b)

$$\int_\Omega (v\Delta u + \nabla v \cdot \nabla u)\, dx = \int_S v\partial_\nu u d\sigma.$$

Exercise 23. Prove that if u is harmonic on Ω and $u \in C^1(\overline{\Omega})$ then

$$\int_S \partial_\nu u d\sigma = 0.$$

Corollary (from Green's identities). *If $u \in C^1(\overline{\Omega})$ is harmonic on Ω and*

1. *$u = 0$ on S, then $u \equiv 0$*
2. *$\partial_\nu u = 0$ on S, then $u \equiv$ constant.*

Proof. By resorting to real and imaginary parts it suffices to consider real-valued functions. If we let $u = v$ in part $b)$ of Exercise 22 we obtain

$$\int_\Omega |\nabla u|^2 dx = \int_S u \partial_\nu u d\sigma(x).$$

In the case 1) we get $\nabla u \equiv 0$ or $u \equiv$ constant. But $u \equiv 0$ on S implies that $u \equiv 0$. In the case 2) we can conclude only that $u \equiv$ constant. $\square$

Theorem 2 (The *mean value theorem*). *Suppose u is harmonic on an open set $\Omega \subset \mathbb{R}^n$. If $x \in \Omega$ and $r > 0$ is small enough so that $\overline{B_r(x)} \subset \Omega$, then*

$$u(x) = \frac{1}{r^{n-1}\omega_n} \int_{|x-y|=r} u(y) d\sigma(y) \equiv \frac{1}{\omega_n} \int_{|y|=1} u(x+ry) d\sigma(y),$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the unit sphere in $\mathbb{R}^n$.

Proof. Let us apply Green's identity $a)$ with u and $v = |y|^{2-n}$, if $n \neq 2$ and $v = \log |y|$ if $n = 2$ in the domain

$$B_r(x) \backslash \overline{B_\varepsilon(x)} = \{y \in \mathbb{R}^n : \varepsilon < |x-y| < r\}.$$

Then for $v(y-x)$ we obtain $(n \neq 2)$

$$\begin{aligned} 0 &= \int_{B_r(x)\backslash\overline{B_\varepsilon(x)}} (v\Delta u - u\Delta v) dy \\ &= \int_{|x-y|=r} (v\partial_\nu u - u\partial_\nu v) d\sigma(y) - \int_{|x-y|=\varepsilon} (v\partial_\nu u - u\partial_\nu v) d\sigma(y) \\ &= r^{2-n} \int_{|x-y|=r} \partial_\nu u d\sigma(y) - (2-n) r^{1-n} \int_{|x-y|=r} u d\sigma(y) \\ &- \varepsilon^{2-n} \int_{|x-y|=\varepsilon} \partial_\nu u d\sigma(y) + (2-n)\varepsilon^{1-n} \int_{|x-y|=\varepsilon} u d\sigma(y). \end{aligned} \tag{6.1}$$

In order to get (6.1) we took into account that

$$\partial_\nu = \nu \cdot \nabla = \frac{x-y}{r} \frac{x-y}{r} \frac{d}{dr} = \frac{d}{dr}$$

for the sphere. Since u is harmonic then due to Exercise 23 we can get from (6.1) that for any $\varepsilon > 0, \varepsilon < r$,

$$\varepsilon^{1-n} \int_{|x-y|=\varepsilon} u d\sigma(y) = r^{1-n} \int_{|x-y|=r} u d\sigma(y).$$

That's why

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{1-n} \int_{|x-y|=\varepsilon} u(y) d\sigma(y) &= \lim_{\varepsilon \to 0} \int_{|\theta|=1} u(x+\varepsilon\theta) d\theta \\ &= \omega_n u(x) = r^{1-n} \int_{|x-y|=r} u(y) d\sigma(y). \end{aligned}$$

This proves the theorem because the latter steps hold for $n = 2$ also. □

Corollary. *If u and r are as in Theorem 2 then*

$$u(x) = \frac{n}{r^n \omega_n} \int_{|x-y|\le r} u(y) dy \equiv \frac{n}{\omega_n} \int_{|y|\le 1} u(x+ry) dy, \quad x \in \Omega. \tag{6.2}$$

Proof. Perform integration in polar coordinates and apply Theorem 2. □

Remark. It follows from the latter formula that

$$\text{vol}\{y : |y| \le 1\} = \frac{\omega_n}{n}.$$

Exercise 24. Assume that u is harmonic in Ω. Let $\chi(x) \in C_0^\infty(B_1(0))$ be such that $\chi(x) = \chi_1(|x|)$ and $\int_{\mathbb{R}^n} \chi(x) dx = 1$. Define an approximation to the identity by $\chi_\varepsilon(\cdot) = \varepsilon^{-n} \chi(\varepsilon^{-1} \cdot)$. Prove that

$$u(x) = \int_{B_\varepsilon(x)} \chi_\varepsilon(x-y) u(y) dy$$

for $x \in \Omega_\varepsilon := \{x \in \Omega : \overline{B_\varepsilon(x)} \subset \Omega\}$.

Corollary 1. *If u is harmonic on Ω then $u \in C^\infty(\Omega)$.*

Proof. The statement follows from Excercise 24 since the function χ_ε is compactly supported and we may thus differentiate under the integral sign as often as we please. □

Corollary 2. *If $\{u_k\}_{k=1}^\infty$ is a sequence of harmonic functions on an open set $\Omega \subset \mathbb{R}^n$ which converges uniformly on compact subsets of Ω to a limit u, then u is harmonic on Ω.*

Theorem 3 (The *maximum principle*). *Suppose $\Omega \subset \mathbb{R}^n$ is open and connected. If u is real-valued and harmonic on Ω with $\sup_{x\in\Omega} u(x) = A < \infty$, then either $u < A$ for all $x \in \Omega$ or $u(x) \equiv A$ in Ω.*

Proof. Since u is continuous on Ω then the set $\{x \in \Omega : u(x) = A\}$ is closed in Ω. On the other hand due to Theorem 2 (see (6.2)) we may conclude that if $u(x) = A$ in some point $x \in \Omega$ then $u(y) = A$ for all y in a ball about x. Indeed, if $y_0 \in B'_\sigma(x)$ and $u(y_0) < A$ then $u(y) < A$ for all y from small neighborhood of y_0. Hence, by Corollary of Theorem 2,

$$\begin{aligned} u(x) &= \frac{n}{r^n \omega_n} \int_{|x-y|\le r} u(y)dy \\ &= \frac{n}{r^n \omega_n} \int_{|x-y|\le r, |y_0-y|>\varepsilon} u(y)dy + \frac{n}{r^n \omega_n} \int_{|y-y_0|\le \varepsilon} u(y)dy \\ &< A\left(\frac{n}{r^n \omega_n} \int_{|x-y|\le r, |y_0-y|>\varepsilon} dy + \frac{n}{r^n \omega_n} \int_{|y-y_0|\le \varepsilon} dy\right) \\ &= A\frac{n}{r^n \omega_n} \int_{|x-y|\le r} dy = A, \end{aligned}$$

that is, $A < A$. This contradiction proves our statement. This fact also means that the set $\{x \in \Omega : u(x) = A\}$ is also open. Hence it is either Ω (in this case $u \equiv A$ in Ω) or the empty set (in this case $u(x) < A$ in Ω). □

Corollary 1. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. If u is real-valued and harmonic on Ω and continuous on $\overline{\Omega}$, then the maximum and minimum of u on $\overline{\Omega}$ are achieved only on $\partial\Omega$.*

Corollary 2 (The uniqueness theorem)**.** *Suppose Ω is as in Corollary 1. If u_1 and u_2 are harmonic on Ω and continuous in $\overline{\Omega}$ (might be complex-valued) and $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ on $\overline{\Omega}$.*

Proof. The real and imaginary parts of $u_1 - u_2$ and $u_2 - u_1$ are harmonic on Ω. Hence they must achieve their maximum on $\partial\Omega$. These maximum are, therefore zero, so $u_1 \equiv u_2$. □

Theorem 4 (*Liouville's theorem*)**.** *If u is bounded and harmonic on $\mathbb{R}^n$ then $u \equiv$ constant.*

Proof. For any $x \in \mathbb{R}^n$ and $|x| \le R$ by Corollary of Theorem 2 we have

$$|u(x) - u(0)| = \frac{n}{R^n \omega_n} \left| \int_{B_R(x)} u(y)dy - \int_{B_R(0)} u(y)dy \right| \le \frac{n}{R^n \omega_n} \int_D |u(y)|dy,$$

where

$$D = (B_R(x) \backslash B_R(0)) \cup (B_R(0) \backslash B_R(x))$$

is the symmetric difference of the balls $B_R(x)$ and $B_R(0)$. That's why we obtain

$$\begin{aligned} |u(x) - u(0)| &\le \frac{n \|u\|_\infty}{R^n \omega_n} \int_{R-|x|\le |y| \le R+|x|} dy \le \frac{n \|u\|_\infty}{R^n \omega_n} \int_{R-|x|}^{R+|x|} r^{n-1} dr \int_{|\theta|=1} d\theta \\ &= \frac{(R+|x|)^n - (R-|x|)^n}{R^n} \|u\|_\infty = O\left(\frac{1}{R}\right) \|u\|_\infty . \end{aligned}$$

Hence the difference $|u(x) - u(0)|$ vanishes as $R \to \infty$, that is, $u(x) = u(0)$. □

Definition. A *fundamental solution* for a partial differential operator L is a distribution $K \in \mathcal{D}'$ such that

$$LK = \delta.$$

Remark. Note that a fundamental solution is not unique. Any two fundamental solutions differ by a solution of the homogeneous equation $Lu = 0$.

Exercise 25. Show that the characteristic function of the set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\right\}$$

is a fundamental solution for $L = \partial_1 \partial_2$.

Exercise 26. Prove that the Fourier transform of $\frac{1}{x_1+ix_2}$ in $\mathbb{R}^2$ is equal to $-\frac{i}{\xi_1+i\xi_2}$.

Exercise 27. Show that the fundamental solution for the Cauchy-Riemann operator $L = \frac{1}{2}(\partial_1 + i\partial_2)$ on $\mathbb{R}^2$ is equal to

$$\frac{1}{\pi}\frac{1}{x_1 + ix_2}.$$

Since the Laplacian commutes with rotations (Theorem 1) it should have a radial fundamental solution which must be a function of $|x|$ that is harmonic on $\mathbb{R}^n \setminus \{0\}$.

Theorem 5. *Let*

$$K(x) = \begin{cases} \frac{|x|^{2-n}}{(2-n)\omega_n}, & n \neq 2 \\ \frac{1}{2\pi}\log|x|, & n = 2. \end{cases} \tag{6.3}$$

Then K is a fundamental solution for Δ.

Proof. For $\varepsilon > 0$ we consider a smoothed out version K_ε of K as

$$K_\varepsilon(x) = \begin{cases} \frac{(|x|^2+\varepsilon^2)^{\frac{2-n}{2}}}{(2-n)\omega_n}, & n \neq 2 \\ \frac{1}{4\pi}\log(|x|^2 + \varepsilon^2), & n = 2. \end{cases} \tag{6.4}$$

Then $K_\varepsilon \to K$ pointwise ($x \neq 0$) as $\varepsilon \to +0$ and K_ε and K are dominated by a fixed locally integrable function for $\varepsilon \leq 1$ (namely, by $|K|$ for $n > 2$, $|\log|x|| + 1$ for $n = 2$ and $(|x|^2 + 1)^{1/2}$ for $n = 1$). So by the Lebesgue's dominated convergence theorem $K_\varepsilon \to K$ in L^1_{loc} (or in the topology of distributions) when $\varepsilon \to +0$. Hence we need to show only that $\Delta K_\varepsilon \to \delta$ as $\varepsilon \to 0$ in the sense of distributions, that is,

$$\langle \Delta K_\varepsilon, \varphi\rangle \to \varphi(0), \quad \varepsilon \to 0$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Exercise 28. Prove that

$$\Delta K_\varepsilon(x) = n\omega_n^{-1}\varepsilon^2(|x|^2 + \varepsilon^2)^{-\left(\frac{n}{2}+1\right)} \equiv \varepsilon^{-n}\psi(\varepsilon^{-1}x)$$

for $\psi(y) = n\omega_n^{-1}(|y|^2 + 1)^{-\left(\frac{n}{2}+1\right)}$.

Exercise 28 allows us to write

$$\langle \Delta K_\varepsilon, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x)\varepsilon^{-n}\psi(\varepsilon^{-1}x)dx = \int_{\mathbb{R}^n} \varphi(\varepsilon z)\psi(z)dz \to \varphi(0)\int_{\mathbb{R}^n} \psi(z)dz$$

as $\varepsilon \to +0$. So it remains to show that

$$\int_{\mathbb{R}^n} \psi(z)dz = 1.$$

Using Exercise 28 we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(x)dx &= \frac{n}{\omega_n}\int_{\mathbb{R}^n} (|x|^2+1)^{-\left(\frac{n}{2}+1\right)}dx = \frac{n}{\omega_n}\int_0^\infty r^{n-1}(r^2+1)^{-\left(\frac{n}{2}+1\right)}dr \int_{|\theta|=1} d\theta \\
&= n\int_0^\infty r^{n-1}(r^2+1)^{-\left(\frac{n}{2}+1\right)}dr = \frac{n}{2}\int_0^\infty t^{(n-1)/2}(1+t)^{-\frac{n}{2}-1}\frac{1}{\sqrt{t}}dt \\
&= \frac{n}{2}\int_0^\infty t^{n/2-1}(1+t)^{-\frac{n}{2}-1}dt = \frac{n}{2}\int_0^1 \left(\frac{1}{s}-1\right)^{n/2-1} s^{\frac{n}{2}+1}\frac{ds}{s^2} \\
&= \frac{n}{2}\int_0^1 (1-s)^{n/2-1}\,ds = \frac{n}{2}\int_0^1 \tau^{n/2-1}d\tau = 1.
\end{aligned}$$

It means that $\varepsilon^{-1}\psi(\varepsilon^{-1}x)$ is an approximation to the identity and

$$\Delta K_\varepsilon \to \delta.$$

But $K_\varepsilon \to K$ and so $\Delta K = \delta$ also. □

Theorem 6. *Suppose that*

1. $f \in L^1(\mathbb{R}^n)$ *if* $n \ge 3$
2. $\int_{\mathbb{R}^2} |f(y)|\left(|\log|y|| + 1\right)dy < \infty$ *if* $n = 2$
3. $\int_{\mathbb{R}} |f(y)|\left(1 + |y|\right)dy < \infty$ *if* $n = 1$.

Let K be given by (6.3). Then $f * K$ is well-defined as a locally integrable function and $\Delta(f * K) = f$ in the sense of distributions.

Proof. Let $n \ge 3$ and set

$$\chi_1(x) = \begin{cases} 1, & x \in B_1(0) \\ 0, & x \notin B_1(0). \end{cases}$$

Then $\chi_1 K \in L^1(\mathbb{R}^n)$ and $(1-\chi_1)K \in L^\infty(\mathbb{R}^n)$. So, for $f \in L^1(\mathbb{R}^n)$ we have that $f * (\chi_1 K) \in L^1(\mathbb{R}^n)$ and $f * (1-\chi_1)K \in L^\infty(\mathbb{R}^n)$ (see Proposition 1 of Chapter 0).

Hence $f * K \in L^1_{\text{loc}}(\mathbb{R}^n)$ by addition and we may calculate

$$\begin{aligned}
\langle \Delta(f * K), \varphi \rangle &= \langle f * K, \Delta\varphi \rangle, \quad \varphi \in C_0^\infty(\mathbb{R}^n) \\
&= \int_{\mathbb{R}^n} (f * K)(x)\Delta\varphi(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)K(x-y)dy\Delta\varphi(x)dx \\
&= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} K(x-y)\Delta\varphi(x)dxdy = \int_{\mathbb{R}^n} f(y)\langle K(x-y), \Delta\varphi(x)\rangle dy \\
&= \int_{\mathbb{R}^n} f(y)\langle \Delta K(x-y), \varphi(x)\rangle dy = \int_{\mathbb{R}^n} f(y)\langle \delta(x-y), \varphi(x)\rangle dy \\
&= \int_{\mathbb{R}^n} f(y)\varphi(y)dy = \langle f, \varphi \rangle.
\end{aligned}$$

Hence $\Delta(f * K) = f$.

□

Exercise 29. Prove Theorem 6 for $n = 2$.

Exercise 30. Prove Theorem 6 for $n = 1$.

Theorem 7. *Let Ω be a bounded domain in $\mathbb{R}^n$ (for $n = 1$ assume that $\Omega = (a, b)$) with C^1 boundary $\partial\Omega = S$. If $u \in C^1(\overline{\Omega})$ is harmonic in Ω, then*

$$u(x) = \int_S \left(u(y)\partial_{\nu_y}K(x-y) - K(x-y)\partial_\nu u(y)\right) d\sigma(y), \quad x \in \Omega, \tag{6.5}$$

where $K(x)$ is the fundamental solution (6.3).

Proof. Let us consider K_ε from (6.4). Then since $\Delta u = 0$ in Ω, by Green's identity $a)$ (see Exercise 22) we have

$$\int_\Omega u(y)\Delta_y K_\varepsilon(x-y)dy = \int_S \left(u(y)\partial_{\nu_y}K_\varepsilon(x-y) - K_\varepsilon(x-y)\partial_\nu u(y)\right) d\sigma(y).$$

As $\varepsilon \to 0$ the right hand side of this equation tends to the right hand side of (6.5) for each $x \in \Omega$. This is because for $x \in \Omega$ and $y \in S$ there are no singularities in K. On the other hand, the left hand side is just $(u * \Delta K_\varepsilon)(x)$ if we set $u \equiv 0$ outside Ω. According to the proof of Theorem 5

$$(u * \Delta K_\varepsilon)(x) \to u(x), \quad \varepsilon \to 0,$$

completing the proof. □

Remark. If we know that $u = f$ and $\partial_\nu u = g$ on S then

$$u(x) = \int_S \left(f(y)\partial_{\nu_y}K(x-y) - K(x-y)g(y)\right) d\sigma(y)$$

is the solution of $\Delta u = 0$ with Cauchy data on S. But this problem is overdetermined because we know from Corollary 2 of Theorem 3 that the solution of $\Delta u = 0$ is uniquely determined by f alone.

The following theorem concerns spaces $C^\alpha(\Omega)$ and $C^{k,\alpha}(\Omega)$ which are defined by

$$\begin{aligned} C^\alpha(\Omega) &\equiv C^{0,\alpha}(\Omega) = \{u \in L^\infty(\Omega) : |u(x) - u(y)| \le C|x-y|^\alpha, x, y \in \Omega\} \\ C^{k,\alpha}(\Omega) &\equiv C^{k+\alpha}(\Omega) = \left\{u : \partial^\beta u \in C^\alpha(\Omega), |\beta| \le k\right\} \end{aligned}$$

for $0 < \alpha < 1$ and $k \in \mathbb{N}$.

Theorem 8. *Suppose $k \ge 0$ is an integer, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ open. If $f \in C^{k+\alpha}(\Omega)$ and u is a distributional solution of $\Delta u = f$ in Ω, then $u \in C^{k+2+\alpha}_{\mathrm{loc}}(\Omega)$.*

Proof. Since $\Delta(\partial^\beta u) = \partial^\beta \Delta u = \partial^\beta f$ we can assume without loss of generality that $k = 0$. Given $\Omega_1 \subset \Omega$ such that $\overline{\Omega_1} \subset \Omega$ pick $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on Ω_1 and let $g = \varphi f$.

Since $\Delta(g * K) = g$ (see Theorem 6) and therefore $\Delta(g * K) = f$ in Ω_1, then $u - (g * K)$ is harmonic in Ω_1 and hence C^∞ there. It is therefore enough to prove that if g is a C^α function with compact support, then $g * K \in C^{2+\alpha}$. To this end we consider $K_\varepsilon(x)$ and its derivatives. Straightforward calculations lead to following formulae ($n \ge 1$):

$$\begin{aligned} \frac{\partial}{\partial x_j} K_\varepsilon(x) &= \omega_n^{-1} x_j (|x|^2 + \varepsilon^2)^{-n/2} \\ \frac{\partial^2}{\partial x_i \partial x_j} K_\varepsilon(x) &= \omega_n^{-1} \begin{cases} -n x_i x_j (|x|^2 + \varepsilon^2)^{-n/2-1}, & i \ne j \\ (|x|^2 + \varepsilon^2 - n x_j^2)(|x|^2 + \varepsilon^2)^{-n/2-1}, & i = j. \end{cases} \end{aligned} \tag{6.6}$$

Exercise 31. Prove formulae (6.6).

Since $K_\varepsilon \in C^\infty$ then $g * K_\varepsilon \in C^\infty$ also. Moreover, $\partial_j(g * K_\varepsilon) = g * \partial_j K_\varepsilon$ and $\partial_i \partial_j (g * K_\varepsilon) = g * \partial_i \partial_j K_\varepsilon$. The pointwise limits in (6.6) as $\varepsilon \to 0$ imply

$$\begin{aligned} \frac{\partial}{\partial x_j} K(x) &= \omega_n^{-1} x_j |x|^{-n} \\ \frac{\partial^2}{\partial x_i \partial x_j} K(x) &= \begin{cases} -n \omega_n^{-1} x_i x_j |x|^{-n-2}, & i \ne j \\ \omega_n^{-1} (|x|^2 - n x_j^2) |x|^{-n-2}, & i = j, \end{cases} \end{aligned} \tag{6.7}$$

for $x \ne 0$. The formulae (6.7) show that $\partial_j K(x)$ is a locally integrable function and since g is bounded with compact support then $g * \partial_j K$ is continuous. Next, $g * \partial_j K_\varepsilon \to g * \partial_j K$ uniformly as $\varepsilon \to +0$. It is equivalent to $\partial_j K_\varepsilon \to \partial_j K$ in the topology of distributions (see the definition). Hence $\partial_j(g * K) = g * \partial_j K$.

This argument does not work for the second derivatives because $\partial_i \partial_j K(x)$ is not integrable. But there is a different procedure for these terms.

Let $i \ne j$. Then $\partial_i \partial_j K_\varepsilon(x)$ and $\partial_i \partial_j K(x)$ are odd functions of x_i (and x_j), see (6.6) and (6.7). Due to this fact their integrals over any annulus $0 < a < |x| < b$ vanish. For K_ε we can even take $a = 0$.

Exercise 32. Prove this fact.

That's why for any $b > 0$ we have

$$\begin{aligned} g * \partial_i \partial_j K_\varepsilon(x) &= \int_{\mathbb{R}^n} g(x-y) \partial_i \partial_j K_\varepsilon(y) dy - g(x) \int_{|y|<b} \partial_i \partial_j K_\varepsilon(y) dy \\ &= \int_{|y|<b} (g(x-y) - g(x)) \partial_i \partial_j K_\varepsilon(y) dy + \int_{|y| \geq b} g(x-y) \partial_i \partial_j K_\varepsilon(y) dy. \end{aligned}$$

If we let $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} g * \partial_i \partial_j K_\varepsilon(x) = \int_{|y|<b} (g(x-y) - g(x)) \partial_i \partial_j K(y) dy + \int_{|y| \geq b} g(x-y) \partial_i \partial_j K(y) dy.$$

This limit exists because

$$|g(x-y) - g(x)||\partial_i \partial_j K(y)| \leq c|y|^\alpha |y|^{-n}$$

(g is C^α) and because g is compactly supported. Then, since b is arbitrary, we can let $b \to +\infty$ to obtain

$$\begin{aligned} \partial_i \partial_j (g * K)(x) &= \lim_{b \to \infty} \int_{|y|<b} (g(x-y) - g(x)) \partial_i \partial_j K(y) dy \\ &+ \lim_{b \to \infty} \int_{|y| \geq b} g(x-y) \partial_i \partial_j K(y) dy \\ &= \lim_{b \to \infty} \int_{|y|<b} (g(x-y) - g(x)) \partial_i \partial_j K(y) dy. \qquad (6.8) \end{aligned}$$

A similar result holds for $i = j$. Indeed,

$$\partial_j^2 K_\varepsilon(x) = \frac{1}{n} \varepsilon^{-n} \psi(\varepsilon^{-1} x) + K_j^\varepsilon(x),$$

where $\psi(x) = n\omega_n^{-1}(|x|^2+1)^{-n/2-1}$ and $K_j^\varepsilon = \omega_n^{-1}(|x|^2 - nx_j^2)(|x|^2+\varepsilon^2)^{-n/2-1}$ (see (6.6)). The integral I_j of K_j^ε over an annulus $a < |y| < b$ vanishes. Why is it so? First of all, I_j is independent of j by symmetry in the coordinates, that is, $I_j = I_i$ for $i \neq j$. So nI_j is the integral of $\sum_{j=1}^n K_j^\varepsilon$. But $\sum_{j=1}^n K_j^\varepsilon = 0$. Hence $I_j = 0$ also. That's why we can apply the same procedure. Since

$$g * (\varepsilon^{-n} \psi(\varepsilon^{-1} x)) \to g, \quad \varepsilon \to 0,$$

(because $\varepsilon^{-n}\psi(\varepsilon^{-1}x)$ is an approximation to the identity) then

$$\partial_j^2 (g * K)(x) = \frac{g(x)}{n} + \lim_{b \to \infty} \int_{|y|<b} (g(x-y) - g(x)) \partial_j^2 K(y) dy. \qquad (6.9)$$

Since the convergence in (6.8) and (6.9) is uniform then at this point we have shown that $g * K \in C^2$. But we need to prove more.

Lemma 2 (Calderon-Zigmund)**.** *Let N be a C^1 function on $\mathbb{R}^n \setminus \{0\}$ that is homogeneous of degree $-n$ and satisfies*

$$\int_{a<|y|<b} N(y)dy = 0$$

for any $0 < a < b < \infty$. Then if g is a C^α function with compact support, $0 < \alpha < 1$, then

$$h(x) = \lim_{b\to\infty} \int_{|z|<b} (g(x-z) - g(x))N(z)dz$$

belongs to C^α.

Proof. Let us write $h = h_1 + h_2$, where

$$\begin{aligned} h_1(x) &= \int_{|z|\le 3|y|} (g(x-z) - g(x))N(z)dz, \\ h_2(x) &= \lim_{b\to\infty} \int_{3|y|<|z|<b} (g(x-z) - g(x))N(z)dz. \end{aligned}$$

We wish to estimate $h(x+y) - h(x)$. Since $\alpha > 0$ we have

$$|h_1(x)| \le c \int_{|z|\le 3|y|} |z|^\alpha |z|^{-n} dz = c'|y|^\alpha.$$

and hence

$$|h_1(x+y) - h_1(x)| \le |h_1(x+y)| + |h_1(x)| \le 2c'|y|^\alpha.$$

On the other hand

$$\begin{aligned} h_2(x+y) - h_2(x) &= \lim_{b\to\infty} \int_{3|y|<|z+y|<b} (g(x-z) - g(x))N(z+y)dz \\ &- \lim_{b\to\infty} \int_{3|y|<|z|<b} (g(x-z) - g(x))N(z)dz \\ &= \lim_{b\to\infty} \int_{3|y|<|z|<b} (g(x-z) - g(x))(N(z+y) - N(z))dz \\ &+ \lim_{b\to\infty} \int_{\{3|y|<|z+y|<b\}\setminus\{3|y|<|z|<b\}} (g(x-z) - g(x))N(z+y)dz \\ &= I_1 + I_2. \end{aligned}$$

It is clear that

$$\begin{aligned} \{3|y| < |z+y|\} \setminus \{3|y| < |z|\} &\subset \{2|y| < |z|\} \setminus \{3|y| < |z|\} \\ &= \{2|y| < |z| \le 3|y|\}. \end{aligned}$$

That's why

$$\begin{aligned}
|I_2| &\leq \int_{2|y|<|z|\leq 3|y|} |g(x-z)-g(x)||N(z+y)|dz \\
&\leq c\int_{2|y|<|z|\leq 3|y|} |z|^{\alpha}|z+y|^{-n}dz \\
&\leq c'\int_{2|y|<|z|\leq 3|y|} |z|^{\alpha-n}dz = c''|y|^{\alpha}.
\end{aligned}$$

Now we observe that for $|z| > 3|y|$

$$\begin{aligned}
|N(z+y)-N(z)| &\leq |y|\sup_{0\leq t\leq 1}|\nabla N(z+ty)| \\
&\leq c|y|\sup_{0\leq t\leq 1}|z+ty|^{-n-1} \leq c'|y||z|^{-n-1},
\end{aligned}$$

because ∇N is homogeneous of degree $-n-1$, since N is homogeneous of degree $-n$. Hence

$$|I_1| \leq c\int_{|z|>3|y|} |z|^{\alpha}|y||z|^{-n-1}dz = c'|y|\int_{3|y|}^{\infty} \rho^{\alpha-2}d\rho = c''|y|^{\alpha}.$$

Note that the condition $\alpha < 1$ is needed here. Collecting the estimates for I_1 and I_2 we can see that the lemma is proved. □

In order to end the proof of Theorem it remains to note that $\partial_i\partial_j K(x)$ satisfies all conditions of Lemma 2. Thus the Theorem is also proved. □

Exercise 33. Show that a function K_1 is a fundamental solution for $\Delta^2 \equiv \Delta(\Delta)$ on $\mathbb{R}^n$ if and only if K_1 satisfies the equation

$$\Delta K_1 = K,$$

where K is the fundamental solution for the Laplacian.

Exercise 34. Show that the following functions are the fundamental solutions for Δ^2 on $\mathbb{R}^n$:

1. $n = 4$:
$$-\frac{\log|x|}{4\omega_4}$$

2. $n = 2$:
$$\frac{|x|^2\log|x|}{8\pi}$$

3. $n \neq 2, 4$:
$$\frac{|x|^{4-n}}{2(4-n)(2-n)\omega_n}.$$

Exercise 35. Show that $(4\pi|x|)^{-1}e^{-c|x|}$ is the fundamental solution for $-\Delta + c^2$ on $\mathbb{R}^3$ for any constant $c \in \mathbb{C}$.

7 The Dirichlet and Neumann Problems

The Dirichlet problem

Given functions f in Ω and g on $S = \partial\Omega$, find a function u in $\overline{\Omega} = \Omega \cup \partial\Omega$ satisfying

$$\begin{cases} \Delta u = f, & \text{in}\, \Omega \\ u = g, & \text{on}\, S. \end{cases} \tag{D}$$

The Neumann problem

Given functions f in Ω and g on S, find a function u in $\overline{\Omega}$ satisfying

$$\begin{cases} \Delta u = f, & \text{in}\, \Omega \\ \partial_\nu u = g, & \text{on}\, S. \end{cases} \tag{N}$$

We assume that Ω is bounded with C^1 boundary. But we shall not, however, assume that Ω is connected. The uniqueness theorem (see Corollary of Theorem 3 of Chapter 6) shows that the solution of (D) will be unique (if it exists), at least if we require $u \in C(\overline{\Omega})$. For (N) uniqueness does not hold: we can add to $u(x)$ any function that is constant on each connected component of Ω. Moreover, there is an obvious necessary condition for solvability of (N). If Ω' is a connected component of Ω then

$$\int_{\Omega'} \Delta u dx = \int_{\partial\Omega'} \partial_\nu u d\sigma(x) = \int_{\partial\Omega'} g(x) d\sigma(x) = \int_{\Omega'} f dx,$$

that is,

$$\int_{\Omega'} f(x) dx = \int_{\partial\Omega'} g(x) d\sigma(x).$$

It is also clear (by linearity) that (D) can be reduced to the following *homogeneous* problems:

$$\begin{cases} \Delta v = f, & \text{in}\, \Omega \\ v = 0, & \text{on}\, S \end{cases} \tag{$\mathrm{D_A}$}$$

$$\begin{cases} \Delta w = 0, & \text{in}\, \Omega \\ w = g, & \text{on}\, S \end{cases} \tag{$\mathrm{D_B}$}$$

and $u := v + w$ solves (D). Similar remarks apply to (N), that is

$$\begin{cases} \Delta v = f, & \text{in}\, \Omega \\ \partial_\nu v = 0, & \text{on}\, S \end{cases}$$

$$\begin{cases} \Delta w = 0, & \text{in}\, \Omega \\ \partial_\nu w = g, & \text{on}\, S \end{cases}$$

and $u = v + w$.

Definition. The *Green's function* for (D) in Ω is the solution $G(x, y)$ of the boundary value problem

$$\begin{cases} \Delta_x G(x, y) = \delta(x - y), & x, y \in \Omega \\ G(x, y) = 0, & x \in S, y \in \Omega. \end{cases} \tag{7.1}$$

Analogously, the Green's function for (N) in Ω is the solution $G(x, y)$ of the boundary value problem

$$\begin{cases} \Delta_x G(x, y) = \delta(x - y), & x, y \in \Omega \\ \partial_{\nu_x} G(x, y) = 0, & x \in S, y \in \Omega. \end{cases} \tag{7.2}$$

This definition allows us to write

$$G(x, y) = K(x - y) + v_y(x), \tag{7.3}$$

where K is the fundamental solution of Δ in $\mathbb{R}^n$ and, for any $y \in \Omega$, the function $v_y(x)$ satisfies

$$\begin{cases} \Delta v_y(x) = 0, & \text{in } \Omega \\ v_y(x) = -K(x - y), & \text{on } S \end{cases} \tag{7.4}$$

in the case of (7.1) and

$$\begin{cases} \Delta v_y(x) = 0, & \text{in } \Omega \\ \partial_{\nu_x} v_y(x) = -\partial_{\nu_x} K(x - y), & \text{on } S \end{cases}$$

in the case of (7.2). Since (7.4) guarantees that v_y is real then so is G corresponding to (7.1).

Lemma 1. *The Green's function* (7.1) *exists and is unique.*

Proof. The uniqueness of G follows again from Corollary 2 of Theorem 3 of Chapter 6, since $K(x - y)$ in (7.4) is continuous for all $x \in S$ and $y \in \Omega$ $(x \neq y)$. The existence will be proved later. □

Lemma 2. *For both* (7.1) *and* (7.2) *it is true that* $G(x, y) = G(y, x)$ *for all* $x, y \in \Omega$.

Proof. Let $G(x, y)$ and $G(x, z)$ be the Green's functions for Ω corresponding to sources located at fixed y and z, $y \neq z$, respectively. Let us consider the domain

$$\Omega_\varepsilon = (\Omega \setminus \{x : |x - y| < \varepsilon\}) \setminus \{x : |x - z| < \varepsilon\},$$

see Figure 2.

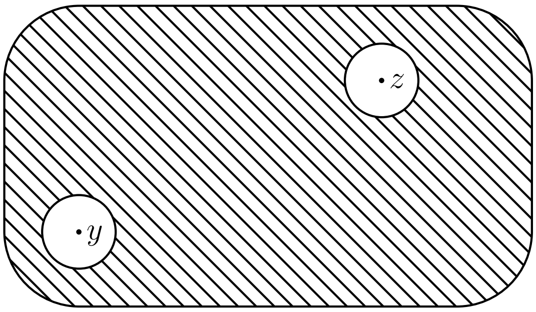


Figure 2: The domain Ω_ε.

If $x \in \Omega_\varepsilon$ then $x \neq z$ and $x \neq y$ and, therefore, $\Delta_x G(x,z) = 0$ and $\Delta_x G(x,y) = 0$. These facts imply

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} (G(x,y)\Delta_x G(x,z) - G(x,z)\Delta_x G(x,y))\, dx \\ &= \int_S (G(x,y)\partial_{\nu_x} G(x,z) - G(x,z)\partial_{\nu_x} G(x,y))\, d\sigma(x) \\ &- \int_{|x-y|=\varepsilon} (G(x,y)\partial_{\nu_x} G(x,z) - G(x,z)\partial_{\nu_x} G(x,y))\, d\sigma(x) \\ &- \int_{|x-z|=\varepsilon} (G(x,y)\partial_{\nu_x} G(x,z) - G(x,z)\partial_{\nu_x} G(x,y))\, d\sigma(x). \end{aligned}$$

Hence, by (7.1) or (7.2), for arbitrary $\varepsilon > 0$ (small enough)

$$\begin{aligned} &\int_{|x-y|=\varepsilon} (G(x,y)\partial_{\nu_x} G(x,z) - G(x,z)\partial_{\nu_x} G(x,y))\, d\sigma(x) \\ &= \int_{|x-z|=\varepsilon} (G(x,z)\partial_{\nu_x} G(x,y) - G(x,y)\partial_{\nu_x} G(x,z))\, d\sigma(x). \end{aligned}$$

Let $n \geq 3$. Due to (7.3) for $\varepsilon \to 0$ we have

$$\begin{aligned} &\int_{|x-y|=\varepsilon} (G(x,y)\partial_{\nu_x} G(x,z) - G(x,z)\partial_{\nu_x} G(x,y))\, d\sigma(x) \\ &\approx c_n \int_{|x-y|=\varepsilon} \varepsilon^{2-n} \left((2-n)\frac{(x-y, x-z)}{|x-y||x-z|^n} + \partial_{\nu_x} v_z(x) \right) d\sigma(x) \\ &- \int_{|x-y|=\varepsilon} G(x,z)\partial_{\nu_x} G(x,y) d\sigma(x) \\ &\approx c_n (2-n)\varepsilon^{2-n}\varepsilon^{n-1}\frac{1}{\varepsilon}\int_\theta \frac{(\varepsilon\theta, \varepsilon\theta + y - z)}{|\varepsilon\theta + y - z|^n} d\theta - I_1 \approx -I_1, \end{aligned}$$

where we have denoted

$$I_1 = \int_{|x-y|=\varepsilon} G(x,z)\partial_{\nu_x} G(x,y) d\sigma(x).$$

The same is true for the integral over $|x-z|=\varepsilon$, that is,

$$\int_{|x-z|=\varepsilon} (G(x,z)\partial_{\nu_x} G(x,y) - G(x,y)\partial_{\nu_x} G(x,z))\, d\sigma(x) \approx -I_2, \quad \varepsilon \to 0,$$

where

$$I_2 = \int_{|x-z|=\varepsilon} G(x,y)\partial_{\nu_x} G(x,z) d\sigma(x).$$

But using the previous techniques we can obtain that

$$I_1 \approx (2-n)c_n \varepsilon^{1-n}\varepsilon^{n-1} \int_{|\theta|=1} G(\varepsilon\theta + y, z) d\theta \to (2-n)c_n\omega_n G(y,z), \quad \varepsilon \to 0$$

and

$$I_2 \approx (2-n)c_n \varepsilon^{1-n}\varepsilon^{n-1} \int_{|\theta|=1} G(\varepsilon\theta + z, y) d\theta \to (2-n)c_n\omega_n G(z,y), \quad \varepsilon \to 0.$$

It means that $G(y,z) = G(z,y)$ for all $z \neq y$. This proof holds for $n=2$ (and even for $n=1$) with some simple changes. □

Lemma 3. *In three or more dimensions*

$$K(x-y) < G(x,y) < 0, \quad x, y \in \Omega, x \neq y$$

where $G(x,y)$ is the Green's function for (D).

Proof. For each fixed y, the function $v_y(x) := G(x,y) - K(x-y)$ is harmonic in Ω, see (7.4). Moreover, on $S = \partial\Omega$, $v_y(x)$ takes on the positive value

$$-K(x-y) \equiv -\frac{|x-y|^{2-n}}{\omega_n(2-n)}.$$

By the minimum principle, it follows that $v_y(x)$ is strictly positive in Ω. This proves the first inequality.

Exercise 36. Prove the second inequality in Lemma 3.

□

Exercise 37. Show that for $n=2$ Lemma 3 has the following form:

$$\frac{1}{2\pi} \log \frac{|x-y|}{h} < G(x,y) < 0, \quad x, y \in \Omega,$$

where $h \equiv \max_{x,y\in\overline{\Omega}} |x-y|$.

Exercise 38. Obtain the analogue of Lemma 3 for $n = 1$. Hint: show that the Green's function for the operator $\frac{d^2}{dx^2}$ on $\Omega = (0, 1)$ is

$$G(x, y) = \begin{cases} x(y-1), & x < y \\ y(x-1), & x > y. \end{cases}$$

Remark. $G(x, y)$ may be extended naturally (because of the symmetry) to $\overline{\Omega} \times \overline{\Omega}$ by setting $G(x, y) = 0$ for $y \in S$.

Now we can solve both problems (D_A) and (D_B). Indeed, let us set $f = 0$ in (D_A) outside Ω and define

$$v(x) := \int_\Omega G(x, y) f(y) dy \equiv (f * K)(x) + \int_\Omega (G(x, y) - K(x - y)) f(y) dy.$$

Then the Laplacian of the first term is f (see Theorem 6 of Chapter 6), and the second term is harmonic in x (since $v_y(x)$ is harmonic). Also $v(x) = 0$ on S because the same is true for G. Thus, this $v(x)$ solves (D_A).

Consider now (D_B). We assume that g is continuous on S and we wish to find w which is continuous on $\overline{\Omega}$. Applying Green's identity *a*) (together with the same limiting process as in the proof of Lemma 2) we obtain

$$\begin{aligned} w(x) &= \int_\Omega (w(y) \Delta_y G(x, y) - G(x, y) \Delta w(y))\, dy \\ &= \int_S w(y) \partial_{\nu_y} G(x, y) d\sigma(y) = \int_S g(y) \partial_{\nu_y} G(x, y) d\sigma(y). \end{aligned}$$

Let us denote the last integral by (P). Since $\partial_{\nu_y} G(x, y)$ is harmonic in x and continuous in y for $x \in \Omega$ and $y \in S$ then $w(x)$ is harmonic in Ω. In order to prove that this $w(x)$ solves (D_B) it remains to prove that $w(x)$ is continuous in $\overline{\Omega}$ and $w(x)$ on S is $g(x)$. We will prove this general fact later.

Definition. The function $\partial_{\nu_y} G(x, y)$ on $\Omega \times S$ is called the *Poisson kernel* for Ω and (P) is called the *Poisson integral*.

Now we are in the position to solve the *Dirichlet problem* in a half-space. Let

$$\Omega = \mathbb{R}^{n+1}_+ = \left\{ (x', x_{n+1}) \in \mathbb{R}^{n+1} : x' \in \mathbb{R}^n, x_{n+1} > 0 \right\},$$

where $n \geq 1$ now, and let $x_{n+1} = t$. Then

$$\Delta_{n+1} = \Delta_n + \partial_t^2, \quad n = 1, 2, \ldots$$

Denote by $K(x, t)$ a fundamental solution for Δ_{n+1} in $\mathbb{R}^{n+1}$, that is,

$$K(x, t) = \begin{cases} \frac{(|x|^2 + t^2)^{\frac{1-n}{2}}}{(1-n)\omega_{n+1}}, & n > 1 \\ \frac{1}{4\pi} \log(|x|^2 + t^2), & n = 1. \end{cases}$$

Let us prove then that the Green's function for $\mathbb{R}^{n+1}_+$ is

$$G(x,y;t,s) = K(x-y,t-s) - K(x-y,-t-s). \tag{7.5}$$

It is clear (see (7.5)) that $G(x,y;t,0) = G(x,y;0,s) = 0$ and

$$\Delta_{n+1}G = \delta(x-y,t-s) - \delta(x-y,-t-s) = \delta(x-y)\delta(t-s)$$

because for $t,s>0, -t-s<0$ and, therefore, $\delta(-t-s)=0$. Thus G is the Dirichlet Green's function for $\mathbb{R}^{n+1}_+$. From this we immediately have the solution of ($\mathrm{D_A}$) in $\mathbb{R}^{n+1}_+$ as

$$u(x,t) = \int_{\mathbb{R}^n}\int_0^\infty G(x,y;t,s)f(y,s)dsdy.$$

To solve ($\mathrm{D_B}$) we compute the Poisson kernel for this case. Since the outward normal derivative on $\partial\mathbb{R}^{n+1}_+$ is $-\frac{\partial}{\partial t}$ then the Poisson kernel becomes

$$\begin{aligned} -\frac{\partial}{\partial s}G(x,y;t,s)|_{s=0} &= -\frac{\partial}{\partial s}\left(K(x-y,t-s) - K(x-y,-t-s)\right)|_{s=0} \\ &= \frac{2t}{\omega_{n+1}(|x-y|^2+t^2)^{\frac{n+1}{2}}}. \end{aligned} \tag{7.6}$$

Exercise 39. Prove (7.6).

Note that (7.6) holds for any $n \ge 1$. According to the formula for (P), the candidate for a solution to ($\mathrm{D_B}$) is:

$$u(x,t) = \frac{2}{\omega_{n+1}}\int_{\mathbb{R}^n}\frac{tg(y)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}dy. \tag{7.7}$$

In other words, if we set

$$P_t(x) := \frac{2t}{\omega_{n+1}(|x|^2+t^2)^{\frac{n+1}{2}}}, \tag{7.8}$$

which is what is usually called the Poisson kernel for $\mathbb{R}^{n+1}_+$, the proposed solution (7.7) is simply equal to

$$u(x,t) = (g * P_t)(x). \tag{7.9}$$

Exercise 40. Prove that $P_t(x) = t^{-n}P_1(t^{-1}x)$ and

$$\int_{\mathbb{R}^n} P_t(y)dy = 1.$$

Theorem 1. *Suppose $g \in L^p(\mathbb{R}^n), 1 \le p \le \infty$. Then $u(x,t)$ from (7.9) is well-defined on $\mathbb{R}^{n+1}_+$ and is harmonic there. If g is bounded and uniformly continuous, then $u(x,t)$ is continuous on $\overline{\mathbb{R}^{n+1}_+}$ and $u(x,0) = g(x)$, and*

$$\|u(\cdot,t) - g(\cdot)\|_\infty \to 0$$

as $t \to +0$.

Proof. It is clear that for any $t > 0, P_t(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, see (7.8). Hence $P_t(x) \in L^q(\mathbb{R}^n)$ for all $q \in [1, \infty]$ with respect to x and for any fixed $t > 0$. Therefore, the integral in (7.9) is absolutely convergent and the same is true if P_t is replaced by its derivatives $\Delta_x P_t$ or $\partial_t^2 P_t$ (due to Young's inequality for convolution).

Since $G(x, y; t, s)$ is harmonic for $(x, t) \neq (y, s)$ then $P_t(x)$ is also harmonic and

$$\Delta_x u + \partial_t^2 u = g * (\Delta_x + \partial_t^2) P_t = 0.$$

It remains to prove that if g is bounded and continuous then

$$\|u(\cdot, t) - g(\cdot)\|_\infty \to 0$$

as $t \to +0$ and, therefore, $u(x, 0) = g(x)$ and u is continuous on $\overline{\mathbb{R}^{n+1}_+}$.

We have (see Exercise 40)

$$\begin{aligned}
\|g * P_t - g\|_\infty &= \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} g(x-y) P_t(y) dy - \int_{\mathbb{R}^n} g(x) P_t(y) dy \right| \\
&\leq \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |g(x-y) - g(x)| |P_t(y)| dy \\
&= \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |g(x-tz) - g(x)| |P_1(z)| dz \\
&= \sup_{x\in\mathbb{R}^n} \left(\int_{|z|<R} |g(x-tz) - g(x)| |P_1(z)| dz \right. \\
&+ \left. \int_{|z|\geq R} |g(x-tz) - g(x)| |P_1(z)| dz \right) \\
&\leq \sup_{x\in\mathbb{R}^n, |z|<R} |g(x-tz) - g(x)| + 2\|g\|_\infty \int_{|z|\geq R} |P_1(z)| dz < \varepsilon
\end{aligned}$$

for t small enough.

The first term in the latter sum can be made less than $\varepsilon/2$ since g is uniformly continuous on $\mathbb{R}^n$. The second term can be made less than $\varepsilon/2$ for R large enough since $P_1 \in L^1(\mathbb{R}^n)$. Thus, the theorem is proved. □

Remark. The solution of the considered problem is not unique: if $u(x, t)$ is a solution then so is $u(x, t) + ct$ for any $c \in \mathbb{C}$. However, we have the following theorem.

Theorem 2. *If $g \in C(\mathbb{R}^n)$ and $\lim_{x\to\infty} g(x) = 0$ then $u(x, t) := (g * P_t)(x) \to 0$ as $(x, t) \to \infty$ in $\mathbb{R}^{n+1}_+$ and it is the unique solution with this property.*

Proof. Assume for the moment that g has compact support, say $g = 0$ for $|x| > R$. Then $g \in L^1(\mathbb{R}^n)$ and

$$\|g * P_t\|_\infty \leq \|g\|_1 \|P_t\|_\infty \leq ct^{-n},$$

so $u(x, t) \to 0$ as $t \to \infty$ uniformly in x. On the other hand, if $0 < t \leq T$, then

$$|u(x, t)| \leq \|g\|_1 \sup_{|y|<R} |P_t(x-y)| = \|g\|_1 \sup_{|y|<R} \frac{2t}{\omega_{n+1}(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \leq cT|x|^{-n-1},$$

for $|x| > 2R$. Hence $u(x,t) \to 0$ as $x \to \infty$ uniformly for $t \in [0,T]$. This proves that $u(x,t)$ vanishes at infinity if $g(x)$ has compact support. For general g, choose a sequence $\{g_k\}$ of compactly supported functions that converges uniformly (in $L^\infty(\mathbb{R}^n)$) to g and let

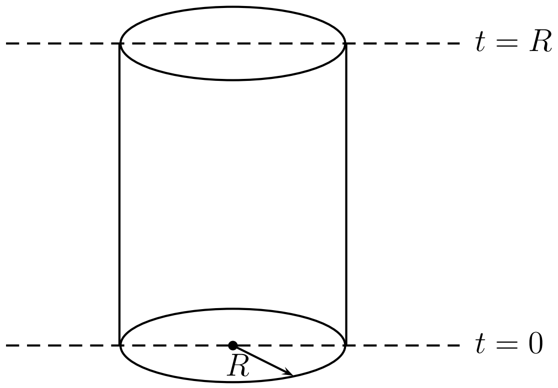
$$u_k(x,t) = (g_k * P_t)(x).$$

Then

$$\begin{aligned} \|u_k - u\|_{L^\infty(\mathbb{R}^{n+1})} &= \sup_{t,x} \left| \int_{\mathbb{R}^n} (g_k - g)(y) P_t(x-y) dy \right| \\ &\leq \sup_t \left(\|g_k - g\|_{L^\infty(\mathbb{R}^n)} \sup_x \int_{\mathbb{R}^n} |P_t(x-y)| dy \right) \\ &= \|g_k - g\|_{L^\infty(\mathbb{R}^n)} \sup_{t>0} \int_{\mathbb{R}^n} |P_t(y)| dy = \|g_k - g\|_{L^\infty(\mathbb{R}^n)} \to 0 \end{aligned}$$

as $k \to \infty$.

Hence $u(x,t)$ vanishes at infinity. Now suppose v is another solution and let $w := v - u$. Then w vanishes at infinity and also at $t = 0$ (see Theorem 1). Thus $|w| < \varepsilon$ on the boundary of the region $\{(x,t) : |x| < R, 0 < t < R\}$ for R large enough.



But since w is harmonic then by the maximum principle it follows that $|w| < \varepsilon$ on this region. Letting $\varepsilon \to 0$ and $R \to \infty$ we conclude that $w \equiv 0$. □

Let us consider now the *Dirichlet problem in a ball.* We use here the following notation:

$$B = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}, \quad \partial B = S.$$

Exercise 41. Prove that

$$|x - y| = \left| \frac{x}{|x|} - y|x| \right|$$

for $x, y \in \mathbb{R}^n, x \neq 0, |y| = 1$.

Now, assuming first that $n > 2$, define

$$\begin{aligned} G(x,y) &:= K(x-y) - |x|^{2-n} K\left(\frac{x}{|x|^2} - y\right) \\ &= \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left|\frac{x}{|x|} - y|x|\right|^{2-n}\right), \quad x \neq 0. \end{aligned} \tag{7.10}$$

Exercise 41 shows that $G(x,y)$ from (7.10) satisfies $G(x,y) = 0, x \in B, y \in S$. It is also clear that $G(x,y) = G(y,x)$. This is true because

$$\begin{aligned} \left|\frac{x}{|x|} - y|x|\right|^2 &= \left|\frac{x}{|x|}\right|^2 - 2(x,y) + |y|^2|x|^2 = 1 - 2(x,y) + |y|^2|x|^2 \\ &= \left|\frac{y}{|y|}\right|^2 - 2(y,x) + |x|^2|y|^2 = \left|\frac{y}{|y|} - x|y|\right|^2. \end{aligned}$$

Next, for $x, y \in B$ we have that

$$\left|\frac{x}{|x|^2}\right| = \frac{|x|}{|x|^2} = \frac{1}{|x|} > 1$$

and $y \neq \frac{x}{|x|^2}$. Hence,

$$G(x,y) - K(x-y) \equiv -|x|^{2-n} K\left(\frac{x}{|x|^2} - y\right)$$

is harmonic in y. But symmetry of G and K shows also that $G(x,y) - K(x-y)$ is harmonic in x. Thus, $G(x,y)$ is the Green's function for B. It also makes clear how to define G at $x = 0$ (and at $y = 0$):

$$G(0,y) = \frac{1}{(2-n)\omega_n}(|y|^{2-n} - 1)$$

since

$$\left|\frac{x}{|x|} - y|x|\right| \to 1$$

as $x \to 0$.

For $n = 2$ the analogous formulae are:

$$G(x,y) = \frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{x}{|x|} - y|x|\right|\right), \quad G(0,y) = \frac{1}{2\pi}\log|y|.$$

Now we can compute the Poisson kernel $P(x,y) := \partial_{\nu_y} G(x,y), x \in B, y \in S$. Since $\partial_{\nu_y} = y \cdot \nabla_y$ on S, then

$$P(x,y) = -\frac{1}{\omega_n}\left(\frac{(y, x-y)}{|x-y|^n} - \frac{\left(\frac{x}{|x|} - y|x|, y|x|\right)}{\left|\frac{x}{|x|} - y|x|\right|^n}\right) \equiv \frac{1 - |x|^2}{\omega_n |x-y|^n}, \quad n \geq 2. \tag{7.11}$$

Exercise 42. Prove (7.11).

Theorem 3. *If $f \in L^1(S)$ then*

$$u(x) = \int_S P(x,y)f(y)d\sigma(y), \quad x \in B,$$

is harmonic. If $f \in C(S)$ then u extends continuously to $\overline{B}$ and $u = f$ on S.

Proof. For each $x \in B$ (see (7.11)), $P(x,y)$ is a bounded function of $y \in S$, so $u(x)$ is well-defined for $f \in L^1(S)$. It is also harmonic in B, because $P(x,y)$ is harmonic for $x \neq y$. Next, we claim that

$$\int_S P(x,y)d\sigma(y) = 1. \tag{7.12}$$

Since P is harmonic in x then the mean value theorem implies ($y \in S$)

$$1 = \omega_n P(0,y) = \int_S P(ry',y)d\sigma(y')$$

for any $0 < r < 1$. But

$$P(ry',y) = P(y,ry') = P(ry,y')$$

if $y, y' \in S$. The last formula follows from

$$|ry' - y|^2 = r^2 - 2r(y',y) + 1 = |ry - y'|^2.$$

That's why we may conclude that

$$1 = \int_S P(ry',y)d\sigma(y') = \int_S P(x,y')d\sigma(y')$$

with $x = ry$. This proves (7.12). We claim also that for any $y_0 \in S$ and for the neighborhood $B_\sigma(y_0) \subset S$,

$$\lim_{r \to 1-0} \int_{S \setminus B_\sigma(y_0)} P(ry_0,y)d\sigma(y) = 0. \tag{7.13}$$

Indeed, for $y_0, y \in S$ and $0 < r < 1$,

$$|ry_0 - y| > r|y_0 - y|$$

and therefore

$$|ry_0 - y|^{-n} < (r|y_0 - y|)^{-n} \leq (r\sigma)^{-n}$$

if $y \in S \setminus B_\sigma(y_0)$, i.e., $|y - y_0| \geq \sigma$. Hence $|ry_0 - y|^{-n}$ is bounded uniformly for $r \to 1-0$ and $y \in S \setminus B_\sigma(y_0)$. In addition, $1 - |ry_0|^2 \equiv 1 - r^2 \to 0$ as $r \to 1-0$. This proves (7.13).

Now, suppose $f \in C(S)$. Hence f is uniformly continuous since S is compact. That's why for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad x, y \in S, |x - y| < \delta.$$

For any $x \in S$ and $0 < r < 1$, by (7.12),

$$\begin{aligned} |u(rx) - f(x)| &= \left| \int_S (f(y) - f(x)) P(rx, y) d\sigma(y) \right| \\ &\leq \int_{|x-y|<\delta} |f(y) - f(x)||P(rx, y)| d\sigma(y) \\ &+ \int_{S \setminus B_\delta(x)} |f(y) - f(x)||P(rx, y)| d\sigma(y) \\ &\leq \varepsilon \int_S |P(rx, y)| d\sigma(y) + 2 \|f\|_\infty \int_{S \setminus B_\delta(x)} |P(rx, y)| d\sigma(y) \\ &\leq \varepsilon + 2 \|f\|_\infty \int_{S \setminus B_\delta(x)} P(rx, y) d\sigma(y) \to 0, \end{aligned}$$

as $\varepsilon \to 0$ and $r \to 1 - 0$ by (7.13). Hence $u(rx) \to f$ uniformly as $r \to 1 - 0$. □

Corollary (without proof). *If $f \in L^p(S), 1 \leq p \leq \infty$, then*

$$\|u(r\cdot) - f(\cdot)\|_p \to 0$$

as $r \to 1 - 0$.

Exercise 43. Show that the Poisson kernel for the ball $B_R(x_0)$ is

$$P(x, y) = \frac{R^2 - |x - x_0|^2}{\omega_n R |x - y|^n}, \quad n \geq 2.$$

Exercise 44 (*Harnack's inequality*)**.** Suppose $u \in C(\overline{B})$ is harmonic on B and $u \geq 0$. Then show that for $|x| = r < 1$,

$$\frac{1 - r}{(1 + r)^{n-1}} u(0) \leq u(x) \leq \frac{1 + r}{(1 - r)^{n-1}} u(0).$$

Theorem 4 (The *Reflection Principle*)**.** *Let $\Omega \subset \mathbb{R}^{n+1}, n \geq 1$, be open and carry the property that $(x, -t) \in \Omega$ if $(x, t) \in \Omega$. Let $\Omega_+ = \{(x, t) \in \Omega : t > 0\}$ and $\Omega_0 = \{(x, t) \in \Omega : t = 0\}$. If $u(x, t)$ is continuous on $\Omega_+ \cup \Omega_0$, harmonic in Ω_+ and $u(x, 0) = 0$, then we can extend u to be harmonic on Ω by setting $u(x, -t) := -u(x, t)$.*

Definition. If u is harmonic on $\Omega \setminus \{x_0\}$, $\Omega \subset \mathbb{R}^n$ open, then u is said to have a *removable singularity* x_0 if u can be defined at x_0 so as to be harmonic in Ω.

Theorem 5. *Suppose u is harmonic on $\Omega \setminus \{x_0\}$ and $u(x) = o\left(|x - x_0|^{2-n}\right)$ for $n > 2$ and $u(x) = o\left(\log|x - x_0|\right)$ for $n = 2$ as $x \to x_0$. Then u has a removable singularity at x_0.*

Proof. Without loss of generality we assume that $\Omega = B := B_1(0)$ and $x_0 = 0$. Since u is continuous on ∂B then by Theorem 3 there exists $v \in C(\overline{B})$ satisfying

$$\begin{cases} \Delta v = 0, & \text{in } B \\ v = u, & \text{on } S. \end{cases}$$

We claim that $u = v$ in $B \setminus \{0\}$, so that we can remove the singularity at $\{0\}$ by setting $u(0) := v(0)$. Indeed, given $\varepsilon > 0$ and $0 < \delta < 1$ consider the function

$$g_\varepsilon(x) = \begin{cases} u(x) - v(x) - \varepsilon(|x|^{2-n} - 1), & n > 2 \\ u(x) - v(x) + \varepsilon \log|x|, & n = 2 \end{cases}$$

in $B \setminus \overline{B_\delta(0)}$. These functions are real (we can assume without loss of generality), harmonic and continuous for $\delta \leq |x| \leq 1$. Moreover $g_\varepsilon(x) = 0$ on ∂B and $g_\varepsilon(x) < 0$ on $\partial B_\delta(0)$ for all δ small enough. By the maximum principle, it is negative in $B \setminus \{0\}$. Letting $\varepsilon \to 0$ we see that $u - v \leq 0$ in $B \setminus \{0\}$. By the same arguments we may conclude that also $v - u \leq 0$ in $B \setminus \{0\}$. Hence $u = v$ in $B \setminus \{0\}$ and we can extend u to the whole ball by setting $u(0) = v(0)$. This proves the theorem. □

8 Layer Potentials

In this chapter we assume that $\Omega \subset \mathbb{R}^n$ is bounded and open, and that $S = \partial\Omega$ is a surface of class C^2. We assume also that both Ω and $\Omega' := \mathbb{R}^n \setminus \overline{\Omega}$ are connected.

Definition. Let $\nu(x)$ be a normal vector to S at x. Then

$$\partial_{\nu_-} u(x) := \lim_{t \to -0} \nu(x) \cdot \nabla u(x + t\nu(x))$$
$$\partial_{\nu_+} u(x) := \lim_{t \to +0} \nu(x) \cdot \nabla u(x + t\nu(x))$$

are called the interior and exterior normal derivatives, respectively, of u.

The interior Dirichlet problem (ID)

Given $f \in C(S)$, find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u = f$ on S.

The exterior Dirichlet problem (ED)

Given $f \in C(S)$, find $u \in C^2(\Omega') \cap C(\overline{\Omega'})$ such that $\Delta u = 0$ in Ω' and at infinity and $u = f$ on S.

Definition. We say that u is harmonic at infinity if

$$|x|^{2-n} u\left(\frac{x}{|x|^2}\right) = \begin{cases} o(|x|^{2-n}), & n \neq 2 \\ o(\log|x|), & n = 2 \end{cases}$$

as $x \to 0$.

The interior Neumann problem (IN)

Given $f \in C(S)$, find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $\partial_{\nu_-} u = f$ exists on S.

The exterior Neumann problem (EN)

Given $f \in C(S)$, find $u \in C^2(\Omega') \cap C(\overline{\Omega'})$ such that $\Delta u = 0$ in Ω' and at infinity and $\partial_{\nu_+} u = f$ exists on S.

Theorem 1 (Uniqueness). *1. The solutions of (ID) and (ED) are unique.*

2. The solutions of (IN) and (EN) are unique up to a constant on Ω and Ω', respectively. When $n > 2$ this constant is zero on the unbounded component of Ω'.

Proof. If u solves (ID) with $f = 0$ then $u \equiv 0$ because this is just the uniqueness theorem for harmonic functions (see Corollary 2 of Theorem 3 of Chapter 6). If u solves (ED) with $f = 0$ we may assume that $\{0\} \notin \overline{\Omega'}$. Then $\widetilde{u} = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$ solves (ID) with $f = 0$ for bounded domain $\widetilde{\Omega} = \left\{x : \frac{x}{|x|^2} \in \Omega'\right\}$. Hence $\widetilde{u} \equiv 0$ so that $u \equiv 0$ and part 1) is proved.

Exercise 45. Prove that if u is harmonic then $\widetilde{u} = |x|^{2-n} u\left(\frac{x}{|x|^2}\right), x \neq 0$, is also harmonic.

Concerning part 2), by Green's identity we have

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx + \int_S u \partial_{\nu_-} u d\sigma(x).$$

Thus $\nabla u = 0$ in Ω so that u is constant in Ω.

For (EN) let $r > 0$ be large enough so that $\overline{\Omega} \subset B_r(0)$. Again by Green's identity we have

$$\begin{aligned} \int_{B_r(0)\setminus\overline{\Omega}} |\nabla u|^2 dx &= -\int_{B_r(0)\setminus\overline{\Omega}} u\Delta u dx + \int_{\partial B_r(0)} u \partial_r u d\sigma(x) - \int_S u \partial_{\nu_+} u d\sigma(x) \\ &= \int_{\partial B_r(0)} u \partial_r u d\sigma(x), \end{aligned}$$

where $\partial_r u \equiv \frac{d}{dr} u$. Since for $n > 2$ and for large $|x|$ we have

$$u(x) = O\left(|x|^{2-n}\right), \quad \partial_r u(x) = O\left(|x|^{1-n}\right)$$

then

$$\left|\int_{\partial B_r(0)} u \partial_r u d\sigma(x)\right| \le c r^{2-n} r^{1-n} \int_{\partial B_r(0)} d\sigma(x) = c r^{3-2n} r^{n-1} = c r^{2-n} \to 0$$

as $r \to \infty$. Hence

$$\int_{\Omega'} |\nabla u|^2 dx = 0.$$

It implies that u is constant on Ω' and $u = 0$ on the unbounded component of Ω' because for large $|x|$,

$$u(x) = O\left(|x|^{2-n}\right), \quad n > 2.$$

If $n = 2$ then $\partial_r u(x) = O\left(r^{-2}\right)$ for function $u(x)$ which is harmonic at infinity.

Exercise 46. Prove that if u is harmonic at infinity then u is bounded and $\partial_r u(x) = O\left(r^{-2}\right)$ as $r \to \infty$ if $n = 2$ and $\partial_r u(x) = O(|x|^{1-n}), r \to \infty$, if $n > 2$.

Due to Exercise 46 we obtain

$$\left|\int_{\partial B_r(0)} u \partial_r u d\sigma(x)\right| \le c r^{-2} r = c r^{-1} \to 0, \quad r \to \infty.$$

Hence $\nabla u = 0$ in Ω' and u is constant in (each component of) Ω'. □

We now turn to the problem of finding the solutions (*existence problems*). Let us try to solve (ID) by setting

$$\widetilde{u}(x) := \int_S f(y)\partial_{\nu_y} K(x-y)d\sigma(y), \tag{8.1}$$

where K is the (known) fundamental solution for Δ.

Remark. Note that (8.1) involves only the known fundamental solution and not the Green's function (which is difficult to find in general) as in the Poisson integral

$$w(x) = \int_S f(y)\partial_{\nu_y} G(x,y)d\sigma(y). \tag{P}$$

We know that $\widetilde{u}(x)$ is harmonic in Ω, because $K(x-y)$ is harmonic for $x \in \Omega, y \in S$. It remains to verify the boundary conditions. Clearly $\widetilde{u}$ will not have the right boundary values but in a sense it is not far from right. We shall prove (very soon) that on S,

$$\widetilde{u} = \frac{f}{2} + Tf,$$

where T is a compact operator on $L^2(S)$. Thus, what we really want is to take

$$u(x) = \int_S \varphi(y)\partial_{\nu_y} K(x-y)d\sigma(y), \quad x \notin S, \tag{8.2}$$

where φ is the solution of

$$\frac{1}{2}\varphi + T\varphi = f.$$

Similarly, we shall try to solve (IN) (and (EN)) in the form

$$u(x) = \int_S \varphi(y)K(x-y)d\sigma(y), \quad x \notin S. \tag{8.3}$$

Definition. The functions $u(x)$ from (8.2) and (8.3) are called the *double and single layer potentials with moment (density)* φ, respectively.

Definition. Let $I(x,y)$ be continuous on $S \times S, x \neq y$. We call I a *continuous kernel* of order $\alpha, 0 \le \alpha < n-1, n \ge 2$, if

$$|I(x,y)| \le c|x-y|^{-\alpha}, \quad 0 < \alpha < n-1,$$

and

$$|I(x,y)| \le c_1 + c_2 \left|\log|x-y|\right|, \quad \alpha = 0,$$

where $c > 0$ and $c_1, c_2 \ge 0$.

Remark. Note that a continuous kernel of order 0 is also a continuous kernel of order α, $0 < \alpha < n-1$.

We denote by $\widehat{I}$ the integral operator

$$\widehat{I}f(x) = \int_S I(x,y)f(y)d\sigma(y), \quad x \in S$$

with kernel I.

Lemma 1. *If I is a continuous kernel of order $\alpha, 0 \le \alpha < n-1$, then*

1. *$\widehat{I}$ is bounded on $L^p(S), 1 \le p \le \infty$.*
2. *$\widehat{I}$ is compact on $L^2(S)$.*

Proof. It is enough to consider $0 < \alpha < n-1$. Let us assume that $f \in L^1(S)$. Then

$$\begin{aligned} \left\|\widehat{I}f\right\|_{L^1(S)} &\le \int_S \int_S |I(x,y)||f(y)|d\sigma(y)d\sigma(x) \\ &\le c\int_S |f(y)|d\sigma(y) \int_S |x-y|^{-\alpha} d\sigma(x) \\ &\le c\,\|f\|_{L^1(S)} \int_0^d r^{n-2-\alpha}dr = c'\,\|f\|_{L^1(S)}, \end{aligned}$$

where $d = \operatorname{diam} S = \sup_{x,y\in S}|x-y|$.

If $f \in L^\infty(S)$ then

$$\left\|\widehat{I}f\right\|_{L^\infty(S)} \le c\,\|f\|_{L^\infty(S)} \int_0^d r^{n-2-\alpha}dr = c'\,\|f\|_{L^\infty(S)}.$$

For $1 < p < \infty$ part 1) follows now by interpolation.

For part 2), let $\varepsilon > 0$ and set

$$I_\varepsilon(x,y) = \begin{cases} I(x,y), & |x-y| > \varepsilon \\ 0, & |x-y| \le \varepsilon. \end{cases}$$

Since I_ε is bounded on $S \times S$ then $\widehat{I_\varepsilon}$ is a Hilbert-Schmidt operator in $L^2(S)$ so that $\widehat{I_\varepsilon}$ is compact for each $\varepsilon > 0$.

Exercise 47. Prove that a Hilbert-Schmidt operator i.e. an integral operator whose kernel $I(x,y)$ satisfies

$$\int_S \int_S |I(x,y)|^2 dxdy < \infty$$

is compact in $L^2(S)$.

On the other hand, due to estimates for convolution,

$$\begin{aligned} \left\|\widehat{I}f - \widehat{I_\varepsilon}f\right\|_{L^2(S)} &\le c\left(\int_{|x-y|<\varepsilon} \left(\int |f(y)||x-y|^{-\alpha}d\sigma(y)\right)^2 d\sigma(x)\right)^{1/2} \\ &\le c\,\|f\|_{L^2(S)} \int_0^\varepsilon r^{n-2-\alpha}dr \to 0, \quad \varepsilon \to 0. \end{aligned}$$

Thus, $\widehat{I}$ as limit of $\widehat{I_\varepsilon}$, is also compact in $L^2(S)$. □

Lemma 2. *1. If I is a continuous kernel of order $\alpha, 0 \le \alpha < n-1$, then $\widehat{I}$ transforms bounded functions into continuous functions.*

2. If $\widehat{I}$ is as in part 1 then $u + \widehat{I}u \in C(S)$ for $u \in L^2(S)$ implies $u \in C(S)$.

Proof. Let $|x-y| < \delta$. Then

$$\begin{aligned}
|\widehat{I}f(x) - \widehat{I}f(y)| &\le \int_S |I(x,z) - I(y,z)||f(z)|d\sigma(z) \\
&\le \int_{|x-z|<2\delta} \left(|I(x,z)| + |I(y,z)|\right)|f(z)|d\sigma(z) \\
&+ \int_{S\setminus\{|x-z|<2\delta\}} |I(x,z) - I(y,z)||f(z)|d\sigma(z) \\
&\le c\,\|f\|_\infty \int_{|x-z|<2\delta} \left(|x-z|^{-\alpha} + |y-z|^{-\alpha}\right) d\sigma(z) \\
&+ \int_{S\setminus\{|x-z|<2\delta\}} |I(x,z) - I(y,z)||f(z)|d\sigma(z) := I_1 + I_2.
\end{aligned}$$

Since $|z-y| \le |x-z| + |x-y|$ then

$$I_1 \le c\,\|f\|_\infty \int_0^{3\delta} r^{n-2-\alpha} dr \to 0, \quad \delta \to 0.$$

On the other hand for $|x-y| < \delta$ and $|x-z| \ge 2\delta$ we have that

$$|y-z| \ge |x-z| - |x-y| > 2\delta - \delta = \delta.$$

So the continuity of I outside of the diagonal implies that

$$I(x,z) - I(y,z) \to 0, \quad x \to y,$$

uniformly in $z \in S\setminus\{|x-z| < 2\delta\}$. Hence, I_1 and I_2 will be small if y is sufficiently close to x. This proves part 1.

For part 2, let $\varepsilon > 0$ and let $\varphi \in C(S \times S)$ be such that $0 \le \varphi \le 1$ and

$$\varphi(x,y) = \begin{cases} 1, & |x-y| < \varepsilon/2 \\ 0, & |x-y| \ge \varepsilon. \end{cases}$$

Write $\widehat{I}u = \widehat{\varphi I}u + \widehat{(1-\varphi)I}u := \widehat{I}_0 u + \widehat{I}_1 u$. By the Cauchy-Schwarz-Buniakowsky inequality we have

$$|\widehat{I}_1 u(x) - \widehat{I}_1 u(y)| \le \|u\|_2 \left(\int_S |I_1(x,z) - I_1(y,z)|^2 d\sigma(z)\right)^{1/2} \to 0, \quad y \to x,$$

since I_1 is continuous (see the definition of φ). Now if we set

$$g := u + \widehat{I}u - \widehat{I}_1 u \equiv u + \widehat{I}_0 u$$

then g is continuous for $u \in L^2(S)$ by the conditions of this lemma. Since the operator norm of $\widehat{I_0}$ can be made on $L^2(S)$ and $L^\infty(S)$ less than 1 (we can do it due to the choice of $\varepsilon > 0$ small enough), then

$$u = \left(I + \widehat{I_0}\right)^{-1} g,$$

where I is the identity operator. Since g is continuous and the operator norm is less than 1, then

$$u = \sum_{j=0}^{\infty} \left(-\widehat{I_0}\right)^j g.$$

This series converges uniformly and therefore u is continuous. □

Let us consider now the double layer potential (8.2) with moment φ,

$$u(x) = \int_S \varphi(y) \partial_{\nu_y} K(x-y) d\sigma(y), \quad x \in \mathbb{R}^n \backslash S.$$

First of all

$$\partial_{\nu_y} K(x-y) = -\frac{(x-y, \nu(y))}{\omega_n |x-y|^n}. \tag{8.4}$$

Exercise 48. Prove that (8.4) holds for any $n \geq 1$.

It is clear also that (8.4) defines a harmonic function in $x \in \mathbb{R}^n \backslash S, y \in S$. Moreover, it is $O\left(|x|^{1-n}\right)$ as $x \to \infty$ $(y \in S)$ so that u is also harmonic at infinity.

Exercise 49. Prove that (8.4) defines a harmonic function at infinity.

Lemma 3. *There exists $c > 0$ such that*

$$|(x-y, \nu(y))| \leq c|x-y|^2, \quad x, y \in S.$$

Proof. It is quite trivial to obtain

$$|(x-y, \nu(y))| \leq |x-y||\nu(y)| = |x-y|.$$

But the latter inequality allows us to assume that $|x-y| \leq 1$. Given $y \in S$, by a translation and rotation of coordinates we may assume that $y = 0$ and $\nu(y) = (0, 0, \ldots, 0, 1)$. Hence $(x-y, \nu(y))$ transforms to x_n, and near y, S is the graph of the equation $x_n = \psi(x_1, \ldots, x_{n-1})$, where $\psi \in C^2(\mathbb{R}^{n-1}), \psi(0) = 0$ and $\nabla\psi(0) = 0$. Then by Taylor's expansion

$$|(x-y, \nu(y))| = |x_n| \leq c|(x_1, \ldots, x_{n-1})|^2 \leq c|x|^2 = c|x-y|^2.$$

□

We denote $\partial_{\nu_y} K(x-y)$ by $I(x, y)$.

Lemma 4. *I is a continuous kernel of order $n-2, n \geq 2$.*

Proof. If $x, y \in S$ then $I(x,y)$ is continuous, see (8.4), for $x \neq y$. It follows from Lemma 3 that

$$|I(x,y)| \leq \frac{c|x-y|^2}{\omega_n |x-y|^n} = c'|x-y|^{2-n}.$$

□

Lemma 5.

$$\int_S I(x,y)d\sigma(y) = \begin{cases} 1, & x \in \Omega \\ 0, & x \in \Omega' \\ \frac{1}{2}, & x \in S. \end{cases} \tag{8.5}$$

Proof. If $x \in \Omega'$ then $K(x-y)$ is harmonic in $x \notin S, y \in S$ and it is also harmonic in $y \in \Omega, x \in \Omega'$. Hence (see Exercise 23)

$$\int_S \partial_{\nu_y} K(x-y)d\sigma(y) = 0$$

or

$$\int_S I(x,y)d\sigma(y) = 0, \quad x \in \Omega'.$$

If $x \in \Omega$, let $\delta > 0$ be such that $\overline{B_\delta(x)} \subset \Omega$. Then $K(x-y)$ is harmonic in y in $\Omega \backslash \overline{B_\delta(x)}$ and therefore by Green's identity

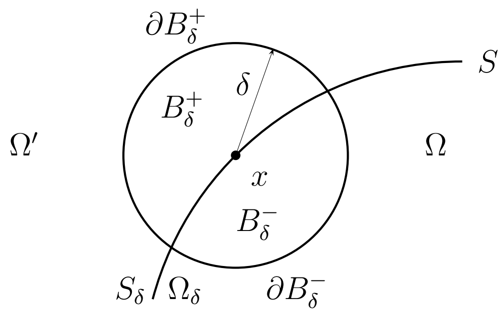
$$\begin{aligned} 0 &= \int_{\Omega \backslash \overline{B_\delta(x)}} \left(1 \cdot \Delta_y K(x-y) - K(x-y)\Delta 1\right) dy \\ &= \int_S \partial_{\nu_y} K(x-y)d\sigma(y) - \int_{|x-y|=\delta} \partial_{\nu_y} K(x-y)d\sigma(y) \\ &= \int_S I(x,y)d\sigma(y) - \frac{\delta^{1-n}}{\omega_n} \int_{|x-y|=\delta} d\sigma(y) \\ &= \int_S I(x,y)d\sigma(y) - 1 \end{aligned}$$

or

$$\int_S I(x,y)d\sigma(y) = 1.$$

Now suppose $x \in S$ and $S_\delta = S \backslash (S \cap B_\delta(x))$. In this case

$$\int_S I(x,y)d\sigma(y) = \lim_{\delta \to 0} \int_{S_\delta} I(x,y)d\sigma(y). \tag{8.6}$$



If $y \in \Omega_\delta := \Omega \backslash B_\delta(x)$ then for $x \in S$ we have that $x \neq y$. It implies that

$$\begin{aligned} 0 &= \int_{\Omega_\delta} \Delta_y K(x-y)dy \\ &= \int_{S_\delta} \partial_{\nu_y} K(x-y)d\sigma(y) - \int_{\partial B_\delta^-} \partial_{\nu_y} K(x-y)d\sigma(y). \end{aligned}$$

That's why, see (8.4),

$$\begin{aligned} \lim_{\delta\to 0} \int_{S_\delta} \partial_{\nu_y} K(x-y)d\sigma(y) &= \lim_{\delta\to 0} \int_{\partial B_\delta^-} \partial_{\nu_y} K(x-y)d\sigma(y) \\ &= \lim_{\delta\to 0} \frac{\delta^{1-n}}{\omega_n} \int_{\partial B_\delta^-} d\sigma(y) \\ &= \lim_{\delta\to 0} \frac{\delta^{1-n}}{\omega_n} \left(\delta^{n-1} \frac{\omega_n}{2} + o(\delta^{n-1}) \right) = \frac{1}{2}. \end{aligned}$$

It means that the limit in (8.6) exists and (8.5) is satisfied. □

Lemma 6. *There exists $c > 0$ such that*

$$\int_S |\partial_{\nu_y} K(x-y)| d\sigma(y) \le c, \quad x \in \mathbb{R}^n.$$

Proof. It follows from Lemma 4 that

$$\int_S |\partial_{\nu_y} K(x-y)| d\sigma(y) \le \frac{c}{\omega_n} \int_S |x-y|^{2-n} d\sigma(y) \le c_1, \quad x \in S.$$

Next, for $x \notin S$ define $\operatorname{dist}(x, S) = \inf_{y\in S} |x-y|$.

There are two possibilities now: if $\operatorname{dist}(x, S) \ge \delta/2$ then $|x-y| \ge \delta/2$ for all $y \in S$ and therefore

$$\int_S |\partial_{\nu_y} K(x-y)| d\sigma(y) \le c\delta^{1-n} \int_S d\sigma(y) = c', \tag{8.7}$$

where c' does not depend on $\delta > 0$ (because δ is fixed).

Suppose now that $\operatorname{dist}(x, S) < \delta/2$. If we choose $\delta > 0$ small enough then there is unique $x_0 \in S$ such that

$$x = x_0 + t\nu(x_0), \quad t \in (-\delta/2, \delta/2).$$

Denote $B_\delta = \{y \in S : |x_0 - y| < \delta\}$. We estimate the integral of $|I(x,y)|$ over $S\backslash B_\delta$ and B_δ separately. If $y \in S\backslash B_\delta$ then

$$|x-y| \geq |x_0 - y| - |x - x_0| > \delta - \delta/2 = \delta/2$$

and

$$|I(x,y)| \leq c\delta^{1-n}$$

so that the integral over $S\backslash B_\delta$ satisfies (8.7), where again c' does not depend on δ.

To estimate the integral over B_δ we note that (see (8.4)),

$$\begin{aligned} |I(x,y)| &= \frac{|(x-y,\nu(y))|}{\omega_n|x-y|^n} = \frac{|(x-x_0,\nu(y)) + (x_0-y,\nu(y))|}{\omega_n|x-y|^n} \\ &\leq \frac{|x-x_0| + c|x_0-y|^2}{\omega_n|x-y|^n}. \end{aligned} \tag{8.8}$$

The latter inequality follows from Lemma 3 since $x_0, y \in S$. Moreover, we have (due to Lemma 3)

$$\begin{aligned} |x-y|^2 &= |x-x_0|^2 + |x_0-y|^2 + 2(x-x_0, x_0-y) \\ &= |x-x_0|^2 + |x_0-y|^2 + 2|x-x_0|\left(x_0-y, \frac{x-x_0}{|x-x_0|}\right) \\ &\geq |x-x_0|^2 + |x_0-y|^2 - 2|x-x_0||(x_0-y,\nu(x_0))| \\ &\geq |x-x_0|^2 + |x_0-y|^2 - 2c|x-x_0||x_0-y|^2 \\ &\geq |x-x_0|^2 + |x_0-y|^2 - |x-x_0||x_0-y|, \end{aligned}$$

if we choose $\delta > 0$ such that $|x_0 - y| \leq \frac{1}{2c}$, where constant $c > 0$ is from Lemma 3.

Since $|x-x_0||x_0-y| \leq \frac{1}{2}\left(|x-x_0|^2 + |x_0-y|^2\right)$ then finally we obtain

$$|x-y|^2 \geq \frac{1}{2}\left(|x-x_0|^2 + |x_0-y|^2\right)$$

and (see (8.4) and (8.8))

$$|I(x,y)| \leq c\frac{|x-x_0| + |x_0-y|^2}{\left(|x-x_0|^2 + |x_0-y|^2\right)^{n/2}} \leq c\frac{|x-x_0|}{\left(|x-x_0|^2 + |x_0-y|^2\right)^{n/2}} + \frac{c}{|x_0-y|^{n-2}}.$$

This implies

$$\begin{aligned} \int_{B_\delta} |I(x,y)|d\sigma(y) &\leq c'\int_0^\delta \frac{|x-x_0|}{\left(|x-x_0|^2 + r^2\right)^{n/2}} r^{n-2}dr + c'\int_0^\delta \frac{r^{n-2}}{r^{n-2}}dr \\ &\leq c'\delta + c'\int_0^\infty \frac{ar^{n-2}}{(a^2+r^2)^{n/2}}dr, \end{aligned}$$

where $a := |x-x_0|$. For the latter integral we have ($t = r/a$)

$$\int_0^\infty \frac{ar^{n-2}}{(a^2+r^2)^{n/2}}dr = \int_0^\infty \frac{t^{n-2}}{(1+t^2)^{n/2}}dt < \infty.$$

If we combine all estimates then we may conclude that there is $c_0 > 0$ such that

$$\int_S |\partial_{\nu_y} K(x-y)| d\sigma(y) \le c_0, \quad x \in \mathbb{R}^n,$$

and this constant does not depend on x. □

Theorem 2. *Suppose $\varphi \in C(S)$ and u is defined by the double layer potential (8.2) with moment φ. Then for any $x \in S$,*

$$\lim_{t\to -0} u(x+t\nu(x)) = \frac{\varphi(x)}{2} + \int_S I(x,y)\varphi(y)d\sigma(y)$$
$$\lim_{t\to +0} u(x+t\nu(x)) = -\frac{\varphi(x)}{2} + \int_S I(x,y)\varphi(y)d\sigma(y)$$

uniformly on S with respect to x.

Proof. If $x \in S$ and $t < 0$, with $|t|$ small enough, then $x_t := x + t\nu(x) \in \Omega$ and $u(x+t\nu(x))$ is well-defined by

$$\begin{aligned} u(x+t\nu(x)) &= \int_S \varphi(y) I(x_t,y)d\sigma(y) = \int_S (\varphi(y)-\varphi(x))I(x_t,y)d\sigma(y) + \varphi(x) \\ &\to \varphi(x) + \int_S \varphi(y)I(x,y)d\sigma(y) - \varphi(x)\int_S I(x,y)d\sigma(y) \\ &= \varphi(x) + \int_S \varphi(y)I(x,y)d\sigma(y) - \varphi(x)/2, \quad t \to -0. \end{aligned}$$

If $t > 0$, the arguments are the same except that

$$\int_S I(x_t,y)d\sigma(y) = 0.$$

The uniformity of convergence follows from the fact that S is compact and $\varphi \in C(S)$. □

Corollary. *For $x \in S$,*

$$\varphi(x) = u_-(x) - u_+(x),$$

where $u_\pm = \lim_{t\to\pm 0} u(x_t)$.

We state without a proof that the normal derivative of the double layer potential is continuous across the boundary in the sense of the following theorem.

Theorem 3. *Suppose $\varphi \in C(S)$ and u is defined by the double layer potential (8.2) with moment φ. Then for any $x \in S$,*

$$\lim_{t\to +0} (\nu(x)\cdot\nabla u(x+t\nu(x)) - \nu(x)\cdot\nabla u(x-t\nu(x))) = 0$$

uniformly on S with respect to x.

Let us now consider the single layer potential

$$u(x) = \int_S \varphi(y) K(x-y) d\sigma(y)$$

with moment $\varphi \in C(S)$.

Lemma 7. *The single layer potential u is continuous on $\mathbb{R}^n$.*

Proof. Since u is harmonic in $x \notin S$ we need only to show continuity for $x \in S$. Given $x_0 \in S$ and $\delta > 0$, let $B_\delta = \{y \in S : |x_0 - y| < \delta\}$. Then

$$\begin{aligned}
|u(x) - u(x_0)| &\leq \int_{B_\delta} (|K(x-y)| + |K(x_0-y)|) |\varphi(y)| d\sigma(y) \\
&+ \int_{S \setminus B_\delta} |K(x-y) - K(x_0-y)||\varphi(y)| d\sigma(y) \\
&\leq c\delta (\text{or}\, \delta \log \frac{1}{\delta} \,\text{for}\, n = 2) \\
&+ \|\varphi\|_\infty \int_{S \setminus B_\delta} |K(x-y) - K(x_0-y)| d\sigma(y) \to 0
\end{aligned}$$

as $x \to x_0$ and $\delta \to 0$. □

Exercise 50. Prove that

$$\int_{B_\delta} (|K(x-y)| + |K(x_0-y)|) |\varphi(y)| d\sigma(y) \leq c \|\varphi\|_\infty \begin{cases} \delta, & n > 2 \\ \delta \log \frac{1}{\delta}, & n = 2. \end{cases}$$

Definition. Let us set

$$I^*(x,y) := \partial_{\nu_x} K(x-y) \equiv \frac{(x-y, \nu(x))}{\omega_n |x-y|^n}.$$

Theorem 4. *Suppose $\varphi \in C(S)$ and u is defined on $\mathbb{R}^n$ by the single layer potential (8.3) with moment φ. Then for $x \in S$,*

$$\lim_{t \to -0} \partial_\nu u(x_t) = -\frac{\varphi(x)}{2} + \int_S I^*(x,y) \varphi(y) d\sigma(y),$$
$$\lim_{t \to +0} \partial_\nu u(x_t) = \frac{\varphi(x)}{2} + \int_S I^*(x,y) \varphi(y) d\sigma(y).$$

Proof. Consider the double layer potential on $\mathbb{R}^n \setminus S$ with moment φ

$$v(x) = \int_S \varphi(y) \partial_{\nu_y} K(x-y) d\sigma(y)$$

and define the function f on the tubular neighborhood V of S by

$$f(x) = \begin{cases} v(x) + \partial_\nu u(x), & x \in V \backslash S \\ \widehat{I}\varphi(x) + \widehat{I}^*\varphi(x), & x \in S, \end{cases} \tag{8.9}$$

where u is defined by (8.3).

Here the *tubular neighborhood* of S is defined as

$$V = \{x + t\nu(x) : x \in S, |t| < \delta\}.$$

We claim that f is continuous on V. It clearly is (see (8.9)) continuous on $V\backslash S$ and S, so it suffices to show that if $x_0 \in S$ and $x = x_0 + t\nu(x_0)$ then $f(x) - f(x_0) \to 0$ as $t \to \pm 0$. We have

$$\begin{aligned} f(x) - f(x_0) &= v(x) + \partial_\nu u(x) - \widehat{I}\varphi(x_0) - \widehat{I}^*\varphi(x_0) \\ &= \int_S I(x,y)\varphi(y)d\sigma(y) + \int_S \varphi(y)\partial_{\nu_x} K(x-y)d\sigma(y) \\ &- \int_S I(x_0,y)\varphi(y)d\sigma(y) - \int_S I^*(x_0,y)\varphi(y)d\sigma(y) \\ &= \int_S \left(I(x,y) + I^*(x,y) - I(x_0,y) - I^*(x_0,y)\right)\varphi(y)d\sigma(y). \end{aligned}$$

Write this expression as an integral over $B_\delta = \{y \in S : |x_0 - y| < \delta\}$ plus an integral over $S\backslash B_\delta$. The integral over $S\backslash B_\delta$ tends uniformly to 0 as $x \to x_0$, because $|y-x| \ge \delta$ and $|y - x_0| \ge \delta$ so that the functions I and I^* have no singularities in this case.

On the other hand, the integral over B_δ can be bounded by

$$\|\varphi\|_\infty \int_{B_\delta} \left(|I(x,y) + I^*(x,y)| + |I(x_0,y) + I^*(x_0,y)|\right) d\sigma(y).$$

Since

$$I(x,y) = -\frac{(x-y, \nu(y))}{\omega_n |x-y|^n}$$

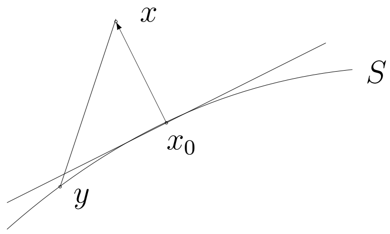
and $\nu(x) = \nu(x_0)$ for $x = x_0 + t\nu(x_0) \in V$ we have

$$I^*(x,y) = I(y,x) = \frac{(x-y,\nu(x))}{\omega_n|x-y|^n} \equiv \frac{(x-y,\nu(x_0))}{\omega_n|x-y|^n}. \tag{8.10}$$

Hence

$$\begin{aligned} |I(x,y) + I^*(x,y)| &= \left|\frac{(x-y,\nu(x_0)-\nu(y))}{\omega_n|x-y|^n}\right| \le \frac{|x-y||\nu(x_0)-\nu(y)|}{\omega_n|x-y|^n} \\ &\le c\frac{|x-y||x_0-y|}{\omega_n|x-y|^n} \le c'\frac{|x_0-y|}{|x_0-y|^{n-1}} = c'|x_0-y|^{2-n}, \end{aligned}$$

because $|x_0 - y| \le |x_0 - x| + |x - y| \le 2|x - y|$. Here we have also used the fact that $|\nu(x_0) - \nu(y)| \le c|x_0 - y|$ since ν is C^1.



This estimate allows us to obtain that the corresponding integral over B_δ can be dominated by

$$c\int_{|y-x_0|\le\delta} |x_0-y|^{2-n}d\sigma(y) = c'\int_0^\delta r^{2-n}r^{n-2}dr = c'\delta.$$

Thus $f := v + \partial_\nu u$ extends continuously across S. That's why for $x \in S$,

$$\widehat{I}\varphi(x) + \widehat{I}^*\varphi(x) = v_-(x) + \partial_{\nu_-}u(x) = \frac{1}{2}\varphi(x) + \widehat{I}\varphi(x) + \partial_{\nu_-}u(x).$$

It follows that

$$\partial_{\nu_-}u(x) = -\frac{\varphi(x)}{2} + \widehat{I}^*\varphi(x).$$

By similar arguments we obtain

$$\widehat{I}\varphi(x) + \widehat{I}^*\varphi(x) = v_+(x) + \partial_{\nu_+}u(x) = -\frac{1}{2}\varphi(x) + \widehat{I}\varphi(x) + \partial_{\nu_+}u(x)$$

and therefore

$$\partial_{\nu_+}u(x) = \frac{\varphi(x)}{2} + \widehat{I}^*\varphi(x).$$

This finishes the proof. □

Corollary.

$$\varphi(x) = \partial_{\nu_+}u(x) - \partial_{\nu_-}u(x),$$

where u is defined by (8.3).

Lemma 8. *If $f \in C(S)$ and*

$$\frac{\varphi}{2} + \widehat{I}^*\varphi = f$$

then

$$\int_S \varphi d\sigma = \int_S f d\sigma.$$

Proof. It follows from (8.10) and Lemma 5 that

$$\begin{aligned}\int_S f(x)d\sigma(x) &= \frac{1}{2}\int_S \varphi(x)d\sigma(x) + \int_S \varphi(y)d\sigma(y)\int_S I^*(x,y)d\sigma(x)\\ &= \frac{1}{2}\int_S \varphi(x)d\sigma(x) + \frac{1}{2}\int_S \varphi(y)d\sigma(y) = \int_S \varphi(y)d\sigma(y).\end{aligned}$$

□

Lemma 9. *Let $n = 2$.*

1. *If $\varphi \in C(S)$ then the single layer potential u with moment φ is harmonic at infinity if and only if*
$$\int_S \varphi(x)d\sigma(x) = 0.$$
2. *Let $\varphi \in C(S)$ with*
$$\int_S \varphi(x)d\sigma(x) = 0$$
and u as in part 1. If u is constant on $\overline{\Omega}$ then $\varphi \equiv 0$, and hence $u \equiv 0$.

Proof. Since $n = 2$ then

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_S \log|x-y|\varphi(y)d\sigma(y) \\ &= \frac{1}{2\pi}\int_S (\log|x-y| - \log|x|)\,\varphi(y)d\sigma(y) + \frac{1}{2\pi}\log|x|\int_S \varphi(y)d\sigma(y). \end{aligned}$$

But $\log|x-y| - \log|x| \to 0$ as $x \to \infty$ uniformly for $y \in S$ and therefore, this term is harmonic at infinity (we have a removable singularity). Hence u is harmonic at infinity if and only if $\int_S \varphi(x)d\sigma(x) = 0$ and in this case $u(x)$ vanishes at infinity. This proves part 1.

In part 2, u is harmonic at infinity. If u is constant on $\overline{\Omega}$ then it solves (ED) with $f \equiv$ constant on S. But a solution of such problem must be constant and vanish at infinity. Therefore this constant is zero. Thus $\varphi \equiv 0$ and, hence, $u \equiv 0$. □

We assume (for simplicity and without loss of generality) that Ω and Ω' are simply-connected, that is, $\partial\Omega$ has only one C^2 component. For $f \in C(S)$ consider the integral equations

$$\pm\frac{1}{2}\varphi + \widehat{I}\varphi = f, \qquad (1_\pm)$$

$$\pm\frac{1}{2}\varphi + \widehat{I}^*\varphi = f, \qquad (1^*_\pm)$$

where $I(x,y) = \partial_{\nu_y}K(x-y)$ and $I^*(x,y) = I(y,x)$. Theorems 2 and 4 show that the double layer potential u with moment φ solves (ID) (respectively (ED)) if φ satisfies (1_+) (respectively (1_-)) and the single layer potential u with moment φ solves (IN) (respectively (EN)) if φ satisfies (1^*_-) (respectively (1^*_+)). For $n = 2$ we need the extra necessary condition for (EN) given by Lemma 9.

We proceed to study the solvability of $(1_\pm)$ and $(1^*_\pm)$. Let us introduce

$$\begin{aligned} V_\pm &= \left\{\varphi : \widehat{I}\varphi = \pm\frac{1}{2}\varphi\right\} \\ W_\pm &= \left\{\varphi : \widehat{I}^*\varphi = \pm\frac{1}{2}\varphi\right\}, \end{aligned} \qquad (8.11)$$

where φ is allowed to range over either $L^2(S)$ or $C(S)$.

Fredholm's Theorem

Let A be a compact operator on a Hilbert space H. For each $\lambda \in \mathbb{C}$, let

$$V_\lambda = \{x \in H : Ax = \lambda x\}$$

and

$$W_\lambda = \{x \in H : A^* x = \lambda x\}.$$

Then

1. The set $\{\lambda \in \mathbb{C} : V_\lambda \neq \{0\}\}$ is finite or countable with only one possible accumulation point at $\{0\}$. Moreover, $\dim V_\lambda < \infty$ for $\lambda \neq 0$.
2. $\dim V_\lambda = \dim W_{\overline{\lambda}}$ if $\lambda \neq 0$.
3. $R(A - \lambda I)$ and $R(A^* - \overline{\lambda} I)$ are closed if $\lambda \neq 0$.

Here and throughout the use of symbol I for the identity operator is to be distinguished from a function $I = I(x, y)$ by the context in which it appears.

Corollary 1. *Suppose $\lambda \neq 0$. Then*

1. *$(A - \lambda I)x = y$ has a solution if and only if $y \perp W_{\overline{\lambda}}$.*
2. *$A - \lambda I$ is surjective (onto) if and only if it is injective (invertible).*

In other words, either $(A - \lambda I)\varphi = 0$ and $(A^* - \overline{\lambda} I)\psi = 0$ have only the trivial solutions $\varphi = \psi = 0$ for $\lambda \neq 0$ or they have the same number of linearly independent solutions $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m$, respectively. In the first case $(A - \lambda I)\varphi = g$ and $(A^* - \overline{\lambda} I)\psi = f$ have unique solutions ($A - \lambda I$ and $A^* - \overline{\lambda} I$ are invertible) for every $g, f \in H$. In the second case $(A - \lambda I)\varphi = g$ and $(A^* - \overline{\lambda} I)\psi = f$ have the solutions if and only if $\varphi_j \perp g$ and $\psi_j \perp f$ for every $j = 1, 2, \ldots, m$.

Proof. It is known and not so difficult to show that

$$\overline{R(A - \lambda I)} = \text{Ker}\,\left(A^* - \overline{\lambda} I\right)^\perp, \tag{8.12}$$

where $M^\perp$ denotes the *orthogonal complement* of $M \subset H$ defined by

$$M^\perp = \{y \in H : (y, x)_H = 0, x \in M\}.$$

Exercise 51. Prove (8.12)

But by part 3 of Fredholm's theorem ($\lambda \neq 0$) we know that $\overline{R(A - \lambda I)} = R(A - \lambda I)$ and, therefore $R(A - \lambda I) = \text{Ker}\,(A^* - \overline{\lambda} I)^\perp$. It is equivalent to the fact that

$$y \in R(A - \lambda I) \Leftrightarrow y \perp \text{Ker}\,(A^* - \overline{\lambda} I)$$

or

$$(A - \lambda I)x = y, x \in H \Leftrightarrow y \perp W_{\overline{\lambda}}.$$

For the second part, $A - \lambda I$ is surjective if and only if $R(A - \lambda I) = H$, that is, $\text{Ker}\,(A^* - \overline{\lambda} I) = 0$. But this is equivalent to $A^* - \overline{\lambda} I$ being invertible or $A - \lambda I$ being invertible (injective). □

Corollary 2.

$$\begin{aligned} L^2(S) &= V_+^{\perp} \oplus W_+ = V_-^{\perp} \oplus W_- \\ &= V_+ \oplus W_+^{\perp} = V_- \oplus W_-^{\perp} \end{aligned}$$

where $V_\pm$ and $W_\pm$ are defined by (8.11) and the direct sums are not necessarily orthogonal.

Proof. By Lemma 5 we know that

$$\int_S I(x,y)d\sigma(y) = \frac{1}{2}, \quad x \in S.$$

It can be interpreted as follows: $\varphi(x) \equiv 1$ belongs to V_+. Hence $\dim V_+ \geq 1$. But by part 2 of Fredholm's theorem $\dim V_+ = \dim W_+$. Since the single layer potential uniquely solves the (EN) and (IN) then $\dim W_+ \leq 1$. Hence $\dim V_+ = \dim W_+ = 1$.

Therefore, in order to prove the equality

$$L^2(S) = V_+^{\perp} \oplus W_+$$

it is enough to show that $V_+^{\perp} \cap W_+ = \{0\}$ (because $V_+^{\perp}$ is a closed subspace of codimension 1).

Suppose $\varphi \in V_+^{\perp} \cap W_+$. Then $\widehat{I}^*\varphi = \frac{1}{2}\varphi$ ($\varphi \in W_+$) and there is $\psi \in L^2(S)$ such that $\varphi = -\frac{\psi}{2} + \widehat{I}^*\psi$ ($\varphi \in V_+^{\perp}$), see Corollary 1 for $\lambda = 1/2$ and $A = \widehat{I}^*$. Next, since $\widehat{I}^*\varphi - \frac{1}{2}\varphi \equiv 0$ and $\varphi \in L^2(S)$ then part 2 of Lemma 2 gives that φ is continuous and hence ψ is continuous too.

Let u and v be the single layer potentials with moments φ and ψ, respectively. Then by Theorem 4

$$\begin{aligned} \partial_{\nu_-} u &= -\frac{\varphi}{2} + \widehat{I}^*\varphi = 0 \\ \partial_{\nu_-} v &= -\frac{\psi}{2} + \widehat{I}^*\psi = \varphi = \frac{\varphi}{2} + \widehat{I}^*\varphi = \partial_{\nu_+} u. \end{aligned}$$

It follows that

$$0 = \int_\Omega (u\Delta v - v\Delta u)dx = \int_S (u\partial_{\nu_-} v - v\partial_{\nu_-} u)d\sigma(x) = \int_S u\partial_{\nu_+} u d\sigma(x).$$

But on the other hand

$$\int_S u\partial_{\nu_+} u d\sigma(x) = -\int_{\Omega'} (u\Delta u + |\nabla u|^2)dx.$$

Hence

$$\int_{\Omega'} |\nabla u|^2 dx = 0$$

and so u is constant in Ω'. This gives finally $\varphi = \partial_{\nu_+} u = 0$. The other equalities can be proved in a similar manner. $\square$

Remark. Since we know that

$$W_{\mp}^{\perp} = \operatorname{Ker}\left(\widehat{I}^{*} \pm \frac{1}{2}I\right)^{\perp} = R\left(\widehat{I} \pm \frac{1}{2}I\right)$$

(see (8.12) and part 3 of Fredholm's theorem) we can rewrite Corollary 2 as

$$L^2(S) = V_{+} \oplus R\left(\widehat{I} - \frac{1}{2}I\right) = V_{-} \oplus R\left(\widehat{I} + \frac{1}{2}I\right).$$

Theorem 5. *[Main theorem]*

1. *(ID) has a unique solution for any* $f \in C(S)$
2. *(ED) has a unique solution for any* $f \in C(S)$
3. *(IN) has a solution for any* $f \in C(S)$ *if and only if* $\int_S f d\sigma = 0$. *The solution is unique up to a constant.*
4. *(EN) has a solution for any* $f \in C(S)$ *if and only if* $\int_S f d\sigma = 0$. *The solution is unique up to a constant.*

Proof. We have already proved uniqueness (see Theorem 1) and the necessity of the conditions on f (see Exercise 23). So all that remains is to establish existence. It turns out that in each case this question is reduced to the question of the solvability of an integral equation.

Note first that

$$\int_S f d\sigma = 0$$

if and only if

$$(f, 1)_{L^2(S)} = 0$$

or $f \in V_{+}^{\perp}$ since $1 \in V_{+}$ and $\dim V_{+} = 1$. But $f \in V_{+}^{\perp}$ is necessary and sufficient condition (see Corollary 1) to solve the integral equation

$$-\frac{\varphi}{2} + \widehat{I}^{*}\varphi = f.$$

If φ is a solution of this equation, φ is continuous (see part 2 of Lemma 2). Hence, by Theorem 4 the single layer potential with moment φ solves (IN)

Similarly, for (EN), we have that $\int_S f d\sigma = 0$ if and only if $f \in V_{-}^{\perp}$. In this case we can solve the equation

$$\frac{\varphi}{2} + \widehat{I}^{*}\varphi = f$$

and then solve (EN) by the single layer potential with moment φ, see again Theorem 4.

Consider now (ID). By Corollary 2 of Fredholm's Theorem and Remark after it we can write for $f \in C(S) \subset L^2(S)$,

$$f = \left(\frac{\varphi}{2} + \widehat{I}\varphi\right) + \psi, \tag{8.13}$$

where $\psi \in V_- \subset C(S)$ and φ is continuous since $f - \psi$ is continuous (part 2 of Lemma 2).

Since $\psi \in V_-$ then $\frac{1}{2}\psi + \widehat{I}\psi = 0$. Let us prove that this condition implies that $\psi = 0$. Consider the double layer potential

$$v(x) = \int_S \psi(y) I(x, y) d\sigma(y), \quad x \notin S.$$

It is harmonic outside of S and $v_- = \frac{1}{2}\psi + \widehat{I}\psi = 0$ (see Theorem 2). Hence $v \in C(\overline{\Omega})$ and the uniqueness result for the interior Dirichlet problem ensures that $v = 0$ in $\overline{\Omega}$. Therefore $\partial_{\nu_-} v = 0$ and hence $\partial_{\nu_+} v = 0$ follows from the jump relation $\partial_{\nu_+} v - \partial_{\nu_-} v = 0$ (Theorem 3). It means that v is constant in $\mathbb{R}^n \backslash \overline{\Omega}$. If $n > 2$ the uniqueness theorem for the exterior Neumann problem implies that $v = 0$ in $\mathbb{R}^n \backslash \overline{\Omega}$. If $n = 2$ the argument is slightly different. We know that $\Delta v = 0$ in $\mathbb{R}^n \backslash \overline{\Omega}$ and $\partial_{\nu_+} v = 0$. By part 4 the unique solution of this problem is

$$v(x) = \int_S \psi_1(y) K(x - y) d\sigma(y),$$

with $\int_S \psi_1(y) d\sigma(y) = 0$ since otherwise v is not harmonic at infinity and we do not even have uniqueness. Thus Lemma 9 implies that $v = 0$.

So $v \equiv 0$ which means that $\psi = v_- - v_+ = 0$. We have also proved above that the operator $\frac{1}{2}I + \widehat{I}$ is injective. Hence it is surjective too and the integral equation (8.13) is solvable for any $f \in C(S)$. By Theorem 2, the double layer potential v with moment φ now solves (ID).

Exercise 52. Prove part 2 of Theorem 5.

□

9 The Heat Operator

We turn our attention now to the *heat operator*

$$L = \partial_t - \Delta_x, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

The heat operator is a prototype of *parabolic operators*. These are operators of the form

$$\partial_t + \sum_{|\alpha| \le 2m} a_\alpha(x,t)\partial_x^\alpha,$$

where the sum satisfies the strong ellipticity condition

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x,t)\xi^\alpha \ge \nu|\xi|^{2m},$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ with $\nu > 0$ constant.

We begin by considering the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x). \end{cases}$$

This problem is a reasonable problem both physically and mathematically.

Assuming for the moment that $f \in S$, the Schwartz space, and taking the Fourier transform with respect to x only, we obtain

$$\begin{cases} \partial_t \widehat{u}(\xi,t) + |\xi|^2\widehat{u}(\xi,t) = 0 \\ \widehat{u}(\xi,0) = \widehat{f}(\xi). \end{cases} \tag{9.1}$$

If we solve the ordinary differential equation (9.1) we obtain

$$\widehat{u}(\xi,t) = e^{-|\xi|^2 t}\widehat{f}(\xi).$$

Thus (at least formally)

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-|\xi|^2 t}\widehat{f}(\xi)\right) = (2\pi)^{-n/2} f * \mathcal{F}^{-1}\left(e^{-|\xi|^2 t}\right)(x,t) = f * K_t(x),$$

where

$$K_t(x) = (2\pi)^{-n/2}\mathcal{F}^{-1}\left(e^{-|\xi|^2 t}\right) \equiv (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, t > 0 \tag{9.2}$$

is called the *Gaussian kernel*. We define $K_t(x) \equiv 0$ for $t \le 0$.

Exercise 53. Prove (9.2).

Let us first prove that

$$\int_{\mathbb{R}^n} K_t(x)dx = 1.$$

Indeed, using polar coordinates,

$$\begin{aligned}
\int_{\mathbb{R}^n} K_t(x)dx &= (4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}dx = (4\pi t)^{-n/2}\int_0^\infty r^{n-1}e^{-\frac{r^2}{4t}}dr\int_{|\theta|=1} d\theta \\
&= \omega_n(4\pi t)^{-n/2}\int_0^\infty r^{n-1}e^{-\frac{r^2}{4t}}dr \\
&= \omega_n(4\pi t)^{-n/2}\int_0^\infty (4st)^{\frac{n-1}{2}}e^{-s}\frac{1}{2}\sqrt{4t}\frac{ds}{\sqrt{s}} \\
&= \frac{\omega_n}{2}\pi^{-n/2}\int_0^\infty s^{n/2-1}e^{-s}ds \\
&= \frac{\omega_n}{2}\pi^{-n/2}\Gamma(n/2) = \frac{1}{2}\frac{2\pi^{n/2}}{\Gamma(n/2)}\pi^{-n/2}\Gamma(n/2) = 1.
\end{aligned}$$

Theorem 1. *Suppose that $f \in L^\infty(\mathbb{R}^n)$ is uniformly continuous. Then $u(x,t) := (f * K_t)(x)$ satisfies $\partial_t u - \Delta u = 0$ and*

$$\|u(\cdot,t) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)} \to 0$$

as $t \to +0$.

Proof. For fixed $t > 0$

$$\Delta_x K_t(x-y) = (4\pi t)^{-n/2}e^{-\frac{|x-y|^2}{4t}}\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right)$$

and for fixed $|x-y| \neq 0$

$$\partial_t K_t(x-y) = (4\pi t)^{-n/2}e^{-\frac{|x-y|^2}{4t}}\left(\frac{|x-y|^2}{4t^2} - \frac{n}{2t}\right).$$

Therefore $(\partial_t - \Delta_x)K_t(x-y) = 0$.

But we can differentiate (with respect to x and t) under the integral sign since this integral will be absolutely convergent for any $t > 0$. That's why we may conclude that

$$\partial_t u(x,t) - \Delta_x u(x,t) = 0.$$

It remains only to verify the initial condition. We have

$$\begin{aligned}
u(x,t) - f(x) &= (f * K_t)(x) - f(x) = \int_{\mathbb{R}^n} f(y)K_t(x-y)dy - f(x) \\
&= \int_{\mathbb{R}^n} f(x-z)K_t(z)dz - \int_{\mathbb{R}^n} f(x)K_t(z)dz \\
&= \int_{\mathbb{R}^n} (f(x-z) - f(x))K_t(z)dz \\
&= \int_{\mathbb{R}^n} (f(x-\eta\sqrt{t}) - f(x))K_1(\eta)d\eta.
\end{aligned}$$

The assumptions on f imply that

$$\begin{aligned} |u(x,t)-f(x)| &\le \sup_{x\in\mathbb{R}^n, |\eta|<R} |f(x-\eta\sqrt{t})-f(x)| \int_{\mathbb{R}^n} K_1(\eta)d\eta \\ &+ 2\|f\|_{L^\infty(\mathbb{R}^n)} \int_{|\eta|\ge R} K_1(\eta)d\eta < \varepsilon/2+\varepsilon/2 \end{aligned}$$

for small t and for R large enough. So we can see that $u(x,t)$ is continuous (even uniformly continuous and bounded) for $(x,t)\in\mathbb{R}^n\times[0,\infty)$ and $u(x,0)=f(x)$. □

Corollary 1. *$u(x,t)\in C^\infty(\mathbb{R}^n\times\mathbb{R}_+)$.*

Proof. We can differentiate under the integral defining u as often as we please, because the exponential function decreases at infinity faster than any polynomial. Thus, the heat equation takes arbitrary initial data (bounded and uniformly continuous) and smooths them out. □

Corollary 2. *Suppose $f\in L^p(\mathbb{R}^n), 1\le p<\infty$. Then $u(x,t):=(f*K_t)(x)$ satisfies $\partial_t u-\Delta u=0$ and*

$$\|u(\cdot,t)-f(\cdot)\|_{L^p(\mathbb{R}^n)}\to 0$$

as $t\to+0$. And again $u(x,t)\in C^\infty(\mathbb{R}^n\times\mathbb{R}_+)$.

Theorem 2 (Uniqueness)**.** *Suppose $u(x,t)\in C^2(\mathbb{R}^n\times\mathbb{R}_+)\cap C(\mathbb{R}^n\times\overline{\mathbb{R}_+})$ and satisfies $\partial_t u-\Delta u=0$ for $t>0$ and $u(x,0)=0$. If for every $\varepsilon>0$ there exists $c_\varepsilon>0$ such that*

$$|u(x,t)|\le c_\varepsilon e^{\varepsilon|x|^2}, \quad |\nabla_x u(x,t)|\le c_\varepsilon e^{\varepsilon|x|^2} \tag{9.3}$$

then $u\equiv 0$.

Proof. For two smooth functions φ and ψ it is true that

$$\varphi(\partial_t\psi-\Delta\psi)+\psi(\partial_t\varphi+\Delta\varphi)=\sum_{j=1}^n \partial_j(\psi\partial_j\varphi-\varphi\partial_j\psi)+\partial_t(\varphi\psi)=\nabla_{x,t}\cdot\vec{F},$$

where $\vec{F}=(\psi\partial_1\varphi-\varphi\partial_1\psi,\ldots,\psi\partial_n\varphi-\varphi\partial_n\psi,\varphi\psi)$. Given $x_0\in\mathbb{R}^n$ and $t_0>0$ let us take

$$\psi(x,t)=u(x,t), \quad \varphi(x,t)=K_{t_0-t}(x-x_0).$$

Then

$$\begin{aligned} \partial_t\psi-\Delta\psi=0, &\quad t>0 \\ \partial_t\varphi+\Delta\varphi=0, &\quad t<t_0. \end{aligned}$$

If we apply the divergence theorem in the region

$$\Omega=\{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+ : |x|<r, 0<a<t<b<t_0\}$$

we obtain

$$\begin{aligned} 0 &= \int_{\partial\Omega} \vec{F} \cdot \nu d\sigma = \int_{|x|\le r} u(x,b) K_{t_0-b}(x-x_0)dx - \int_{|x|\le r} u(x,a) K_{t_0-a}(x-x_0)dx \\ &+ \int_a^b dt \int_{|x|=r} \sum_{j=1}^n \left(u(x,t)\partial_j K_{t_0-t}(x-x_0) - K_{t_0-t}(x-x_0)\partial_j u(x,t)\right) \frac{x_j}{r} d\sigma(x). \end{aligned}$$

Letting $r \to \infty$ the last sum vanishes by assumptions (9.3). That's why we have

$$0 = \int_{\mathbb{R}^n} u(x,a) K_{t_0-a}(x-x_0)dx - \int_{\mathbb{R}^n} u(x,b) K_{t_0-b}(x-x_0)dx.$$

As we know from the proof of Theorem 1 for $b \to t_0 - 0$ the second term tends to $u(x_0, t_0)$ and for $a \to +0$ the first term tends to

$$\int_{\mathbb{R}^n} u(x,0) K_{t_0}(x-x_0)dx = 0$$

because $u(x,0) = 0$. Hence we have finally that $u(x_0,t_0) = 0$ for all $x_0 \in \mathbb{R}^n, t_0 > 0$. □

Theorem 3. *The kernel $K_t(x)$ is a fundamental solution for the heat operator.*

Proof. Given $\varepsilon > 0$, set

$$K_\varepsilon(x,t) = \begin{cases} K_t(x), & t \ge \varepsilon \\ 0, & t < \varepsilon. \end{cases}$$

Clearly $K_\varepsilon(x,t) \to K_t(x)$ as $\varepsilon \to 0$ in the sense of distributions. Even more is true, namely, $K_\varepsilon(x,t) \to K_t(x)$ pointwise as $\varepsilon \to 0$ and

$$\int_{\mathbb{R}^n} |K_\varepsilon(x,t)|\, dx = \int_{\mathbb{R}^n} K_\varepsilon(x,t)dx \le \int_{\mathbb{R}^n} K_t(x)dx = 1.$$

That's why we can apply the dominated convergence theorem and obtain

$$\lim_{e \to +0} \int_{\mathbb{R}^n} K_\varepsilon(x,t)dx = \int_{\mathbb{R}^n} K_t(x)dx.$$

So it remains to show that, as $\varepsilon \to 0$,

$$\partial_t K_\varepsilon(x,t) - \Delta_x K_\varepsilon(x,t) \to \delta(x,t),$$

or

$$\langle \partial_t K_\varepsilon - \Delta_x K_\varepsilon, \varphi \rangle \to \varphi(0), \quad \varphi \in C_0^\infty(\mathbb{R}^{n+1}).$$

Using integration by parts we obtain

$$\begin{aligned}
\langle \partial_t K_\varepsilon - \Delta_x K_\varepsilon, \varphi \rangle &= \langle K_\varepsilon, -\partial_t \varphi - \Delta \varphi \rangle = \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} K_t(x)(-\partial_t - \Delta)\varphi(x,t)dx \\
&= -\int_{\mathbb{R}^n} dx \int_\varepsilon^\infty K_t(x)\partial_t \varphi(x,t)dt \\
&- \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} K_t(x)\Delta_x \varphi(x,t)dx \\
&= \int_{\mathbb{R}^n} K_\varepsilon(x)\varphi(x,\varepsilon)dx + \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} \partial_t K_t(x)\varphi(x,t)dx \\
&- \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} \Delta_x K_t(x)\varphi(x,t)dx \\
&= \int_{\mathbb{R}^n} K_\varepsilon(x)\varphi(x,\varepsilon)dx + \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} (\partial_t - \Delta)K_t(x)\varphi(x,t)dx \\
&= \int_{\mathbb{R}^n} K_\varepsilon(x)\varphi(x,\varepsilon)dx \to \varphi(0,0), \quad \varepsilon \to 0
\end{aligned}$$

as we know from the proof of Theorem 1. □

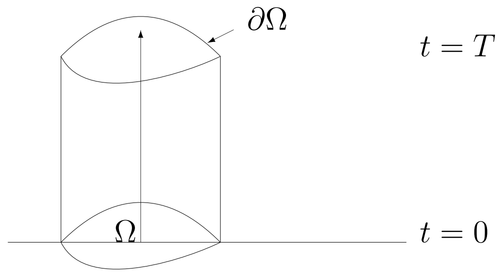
Theorem 4. *If $f \in L^1(\mathbb{R}^{n+1})$, then*

$$u(x,t) := (f * K_t)(x) \equiv \int_{-\infty}^t ds \int_{\mathbb{R}^n} K_{t-s}(x-y)f(y,s)dy$$

is well-defined almost everywhere and is a distributional solution of $\partial_t u - \Delta u = f$.

Exercise 54. Prove Theorem 4.

Let us now consider the heat operator in a bounded domain $\Omega \subset \mathbb{R}^n$ over a time interval $t \in [0,T], 0 < T \leq \infty$. In this case it is necessary to specify the initial temperature $u(x,0), x \in \Omega$, and also to prescribe a boundary condition on $\partial\Omega \times [0,T]$.



The first basic result concerning such problems is the *maximum principle*.

Theorem 5. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Suppose u is a real-valued continuous function on $\overline{\Omega} \times [0,T]$, that satisfies $\partial_t u - \Delta u = 0$ in $\Omega \times (0,T)$. Then u assumes its maximum and minimum either on $\Omega \times \{0\}$ or on $\partial\Omega \times [0,T]$.*

Proof. Given $\varepsilon > 0$, set $v(x,t) := u(x,t) + \varepsilon|x|^2$. Then $\partial_t v - \Delta v = -2n\varepsilon$. Suppose $0 < T' < T$. If maximum of v in $\overline{\Omega} \times [0,T']$ occurs at an interior point of $\Omega \times (0,T')$ then the first derivatives $\nabla_{x,t} v$ vanish there and the second derivative $\partial_j^2 v$ for any $j = 1,2,\dots,n$ is nonpositive (consider $v(x,t)$ as a function of one variable $x_j, j = 1,2,\dots,n$). In particular, $\partial_t v = 0$ and $\Delta v \le 0$, which contradicts with $\partial_t v - \Delta v = -2n\varepsilon < 0$ and $\Delta v = 2n\varepsilon > 0$.

Likewise, if the maximum occurs in $\Omega \times \{T'\}$, then $\partial_t v(x,T') \ge 0$ and $\Delta v(x,T') \le 0$ which contradicts with $\partial_t v - \Delta v < 0$. Therefore,

$$\max_{\overline{\Omega}\times[0,T']} u \le \max_{\overline{\Omega}\times[0,T']} v \le \max_{(\Omega\times\{0\})\cup(\partial\Omega\times[0,T'])} u + \varepsilon \max_{\overline{\Omega}} |x|^2.$$

It follows that for $\varepsilon \to 0$ and $T' \to T$,

$$\max_{\overline{\Omega}\times[0,T]} u \le \max_{(\Omega\times\{0\})\cup(\partial\Omega\times[0,T])} u.$$

Replacing u by $-u$ we can obtain the same result for the minimum. □

Corollary (Uniqueness)**.** *There is at most one continuous function $u(x,t)$ in $\overline{\Omega} \times [0,T], 0 < T < \infty$, which agrees with a given continuous function $f(x)$ in $\Omega \times \{0\}$, with $g(x,t)$ on $\partial\Omega \times [0,T]$ and satisfies $\partial_t u - \Delta u = 0$.*

Let us look now more closely at the following problem:

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in}\, \Omega \times (0,\infty) \\ u(x,0) = f(x), & \text{in}\, \Omega \\ u(x,t) = 0, & \text{on}\, \partial\Omega \times (0,\infty). \end{cases} \tag{9.4}$$

This problem can be solved by the method of separation of variables. We begin by looking for solution of the form

$$u(x,t) = F(x)G(t).$$

Then

$$\partial_t u - \Delta u = FG' - G\Delta_x F = 0$$

if and only if

$$\frac{G'}{G} = \frac{\Delta F}{F} := -\lambda^2$$

or

$$G' + \lambda^2 G = 0, \quad \Delta F + \lambda^2 F = 0,$$

for some constant λ. The first equation has the general solution

$$G(t) = ce^{-\lambda^2 t},$$

where c is an arbitrary constant. Without loss of generality we assume that $c = 1$. It follows from (9.4) that

$$\begin{cases} \Delta F = -\lambda^2 F, & \text{in } \Omega \\ F = 0, & \text{on } \partial\Omega, \end{cases} \tag{9.5}$$

because $u(x,t) = F(x)G(t)$ and $G(0) = 1$.

It remains to solve (9.5) which is an eigenvalue (spectral) problem for the Laplacian with Dirichlet boundary condition. It is known that the problem (9.5) has infinitely many solutions $\{F_j(x)\}_{j=1}^{\infty}$ with corresponding $\{\lambda_j^2\}_{j=1}^{\infty}$. The numbers $-\lambda_j^2$ are called *eigenvalues* and $F_j(x)$ are called *eigenfunctions* of the Laplacian. It is also known that $\lambda_j > 0, j = 1, 2, \ldots,$ $\lambda_j^2 \to \infty$ and $\{F_j(x)\}_{j=1}^{\infty}$ can be chosen as complete orthonormal set in $L^2(\Omega)$ (or $\{F_j(x)\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$). This fact allows us to represent $f(x)$ in terms of Fourier series

$$f(x) = \sum_{j=1}^{\infty} f_j F_j(x), \tag{9.6}$$

where $f_j = (f, F_j)_{L^2(\Omega)}$ are called the Fourier coefficients of f with respect to $\{F_j\}_{j=1}^{\infty}$.

If we take now

$$u(x,t) = \sum_{j=1}^{\infty} f_j F_j(x) e^{-\lambda_j^2 t}, \tag{9.7}$$

then we may conclude (at least formally) that

$$\partial_t u = -\sum_{j=1}^{\infty} f_j \lambda_j^2 F_j(x) e^{-\lambda_j^2 t} = \sum_{j=1}^{\infty} f_j \Delta F_j(x) e^{-\lambda_j^2 t} = \Delta u,$$

that is, $u(x,t)$ from (9.7) satisfies the heat equation and $u(x,t) = 0$ on $\partial\Omega \times (0,\infty)$. It remains to prove that $u(x,t)$ satisfies the initial condition and to determine for which functions $f(x)$ the series (9.6) converges and in what sense. This is the main question in the Fourier method.

It is clear that the series (9.6) and (9.7) (for $t \geq 0$) converge in the sense of $L^2(\Omega)$. It is also clear that if $f \in C^1(\Omega)$ vanishes at the boundary then u will vanish on $\partial\Omega \times (0,\infty)$ and one easily verifies that u is a distributional solution of the heat equation ($t > 0$). Hence it is a classical solution since $u(x,t) \in C^{\infty}(\Omega \times (0,\infty))$ (see Corollary 2 of Theorem 1).

Similar considerations apply to the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,0) = f(x), & \text{in } \Omega \\ \partial_\nu u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

This problem boils down to finding orthonormal basis of eigenfunctions for Laplacian with the Neumann boundary condition. Let us remark that for this problem, $\{0\}$ is always an eigenvalue and 1 is an eigenfunction.

Exercise 55. Prove that $u(x,t)$ of the form (9.7) is a distributional solution of the heat equation in $\Omega \times (0,\infty)$.

Exercise 56. Show that $\int_0^\pi |u(x,t)|^2 dx$ is a decreasing function of $t > 0$, where $u(x,t)$ is the solution of

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0. \end{cases}$$

10 The Wave Operator

The *wave equation* is defined as

$$\partial_t^2 u(x,t) - \Delta_x u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{10.1}$$

The wave equation is satisfied exactly by the components of the classical electromagnetic field in vacuum.

The characteristic variety of (10.1) is

$$\text{char}_x(L) = \left\{(\xi,\tau) \in \mathbb{R}^{n+1} : (\xi,\tau) \neq 0, \tau^2 = |\xi|^2\right\}$$

and it is called *the light cone.* Accordingly, we call

$$\{(\xi,\tau) \in \text{char}_x(L) : \tau > 0\}$$

and

$$\{(\xi,\tau) \in \text{char}_x(L) : \tau < 0\}$$

the forward and backward light cone, respectively.

The wave operator is a prototype of *hyperbolic operators.* It means that the main symbol

$$\sum_{|\alpha|+j=k} a_\alpha(x,t)\xi^\alpha \tau^j$$

has k distinct real roots with respect to τ.

Theorem 1. *Suppose $u(x,t)$ is C^2 function and that $\partial_t^2 u - \Delta u = 0$. Suppose also that $u = 0$ and $\partial_\nu u = 0$ on the ball $B = \{(x,0) : |x - x_0| \le t_0\}$ in the hyperplane $t = 0$. Then $u = 0$ in the region $\Omega = \{(x,t) : 0 \le t \le t_0, |x - x_0| \le t_0 - t\}$.*

Proof. By considering real and imaginary parts we may assume that u is real. Denote by $B_t = \{x : |x - x_0| \le t_0 - t\}$. Let us consider the following integral

$$E(t) = \frac{1}{2}\int_{B_t} \left((u_t)^2 + |\nabla_x u|^2\right) dx$$

which represents the energy of the wave in B_t at time t. Next,

$$\begin{aligned} E'(t) &= \int_{B_t} \left(u_t u_{tt} + \sum_{j=1}^n \partial_j u (\partial_j u)_t\right) dx \\ &- \frac{1}{2}\int_{\partial B_t} \left((u_t)^2 + |\nabla_x u|^2\right) d\sigma(x) := I_1 + I_2. \end{aligned}$$

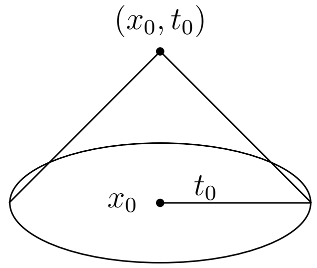
Straightforward calculations using the divergence theorem show us that

$$\begin{aligned} I_1 &= \int_{B_t} \left(\sum_{j=1}^{n} \partial_j[(\partial_j u)u_t] - \sum_{j=1}^{n} (\partial_j^2 u)u_t + u_t u_{tt} \right) dx \\ &= \int_{B_t} u_t(u_{tt} - \Delta_x u)dx + \int_{\partial B_t} \sum_{j=1}^{n} (\partial_j u)\nu_j u_t d\sigma(x) \\ &\le \int_{\partial B_t} |u_t| \, |\nabla_x u| d\sigma(x) \le \frac{1}{2} \int_{\partial B_t} \left(|u_t|^2 + |\nabla_x u|^2 \right) d\sigma(x) \equiv -I_2. \end{aligned}$$

Hence

$$\frac{dE}{dt} \le -I_2 + I_2 = 0.$$

But $E(t) \ge 0$ and $E(0) = 0$ due to Cauchy data. Therefore $E(t) \equiv 0$ if $0 \le t \le t_0$ and thus $\nabla_{x,t} u = 0$ in Ω. Since $u(x, 0) = 0$ then $u(x, t) = 0$ also in Ω. □



Remark. This theorem shows that the value of u at (x_0, t_0) depends only on the Cauchy data of u on the ball $\{(x, 0) : |x - x_0| \le t_0\}$.

Conversely, the Cauchy data on a region R in the initial ($t = 0$) hyperplane influence only those points inside the forward light cones issuing from points of R. Similar result holds when the hyperplane $t = 0$ is replaced by a space-like hypersurface $S = \{(x, t) : t = \varphi(x)\}$. A surface S is called *space-like* if its normal vector $\nu = (\nu', \nu_0)$ satisfies $|\nu_0| > |\nu'|$ at every point of S, i.e., if ν lies inside the light cone. It means that $|\nabla\varphi| < 1$.

Let us consider the Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = f(x), & \partial_t u(x, 0) = g(x). \end{cases} \tag{10.2}$$

Definition. If φ is a continuous function on $\mathbb{R}^n$ and $r > 0$, we define the *spherical mean* $M_\varphi(x, r)$ as follows:

$$M_\varphi(x, r) := \frac{1}{r^{n-1}\omega_n} \int_{|x-z|=r} \varphi(z) d\sigma(z) = \frac{1}{\omega_n} \int_{|y|=1} \varphi(x + ry) d\sigma(y).$$

Lemma 1. *If φ is a C^2 function on $\mathbb{R}^n$, then $M_\varphi(x,0)=\varphi(x)$ and*

$$\Delta_x M_\varphi(x,r)=\left(\partial_r^2+\frac{n-1}{r}\partial_r\right)M_\varphi(x,r).$$

Proof. It is clear that

$$M_\varphi(x,0)=\frac{1}{\omega_n}\int_{|y|=1}\varphi(x)d\sigma(y)=\varphi(x).$$

For the second part we have, by the divergence theorem, that

$$\begin{aligned}\partial_r M_\varphi(x,r) &= \frac{1}{\omega_n}\int_{|y|=1}\sum_{j=1}^n y_j\partial_j\varphi(x+ry)d\sigma(y)=\frac{1}{\omega_n}\int_{|y|\le 1} r\Delta\varphi(x+ry)dy\\ &= \frac{1}{r^{n-1}\omega_n}\int_{|z|\le r}\Delta\varphi(x+z)dz\\ &= \frac{1}{r^{n-1}\omega_n}\int_0^r\rho^{n-1}d\rho\int_{|y|=1}\Delta\varphi(x+\rho y)d\sigma(y).\end{aligned}$$

That's why we have

$$\partial_r\left(r^{n-1}\partial_r M_\varphi(x,r)\right)=\frac{r^{n-1}}{\omega_n}\int_{|y|=1}\Delta\varphi(x+ry)d\sigma(y)\equiv r^{n-1}\Delta_x M_\varphi(x,r).$$

It implies that

$$(n-1)r^{n-2}\partial_r M_\varphi(x,r)+r^{n-1}\partial_r^2 M_\varphi(x,r)=r^{n-1}\Delta_x M_\varphi(x,r)$$

and proves the claim. □

Corollary. *Suppose $u(x,t)$ is a C^2 function on $\mathbb{R}^{n+1}$ and let*

$$M_u(x,r,t)=\frac{1}{r^{n-1}\omega_n}\int_{|x-z|=r}u(z,t)d\sigma(z)=\frac{1}{\omega_n}\int_{|y|=1}u(x+ry,t)d\sigma(y).$$

Then $u(x,t)$ satisfies the wave equation if and only if

$$\left(\partial_r^2+\frac{n-1}{r}\partial_r\right)M_u(x,r,t)=\partial_t^2 M_u(x,r,t). \tag{10.3}$$

Lemma 2. *If $\varphi\in C^{k+1}(\mathbb{R}), k\ge 1$, then*

$$\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}\varphi(r)\right)=\left(\frac{\partial_r}{r}\right)^k(r^{2k}\varphi').$$

Proof. We employ induction with respect to k. If $k=1$ then

$$\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}\varphi(r)\right)=\partial_r^2(r\varphi)=\partial_r(\varphi+r\varphi')=2\varphi'+r\varphi''$$

and

$$\left(\frac{\partial_r}{r}\right)^k(r^{2k}\varphi')=\left(\frac{\partial_r}{r}\right)(r^2\varphi')=2\varphi'+r\varphi''.$$

Assume that

$$\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}\varphi(r)\right)=\left(\frac{\partial_r}{r}\right)^k(r^{2k}\varphi').$$

Then

$$\begin{aligned}
\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k}\left(r^{2k+1}\varphi(r)\right) &= \partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(\frac{\partial_r}{r}\left(r^{2k+1}\varphi\right)\right)\\
&= \partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left((2k+1)r^{2k-1}\varphi+r^{2k}\varphi'\right)\\
&= (2k+1)\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k-1}\varphi\right)+\partial_r^2\left(\frac{1}{r}\partial_r\right)^{k-1}\left(r^{2k}\varphi'\right)\\
&= (2k+1)\left(\frac{\partial_r}{r}\right)^k(r^{2k}\varphi')+\left(\frac{\partial_r}{r}\right)^k(r^{2k}(r\varphi')')\\
&= \left(\frac{\partial_r}{r}\right)^k\left((2k+1)r^{2k}\varphi'+r^{2k}(r\varphi')'\right)\\
&= \left(\frac{\partial_r}{r}\right)^k\left((2k+1)r^{2k}\varphi'+r^{2k}\varphi'+r^{2k+1}\varphi''\right)\\
&= \left(\frac{\partial_r}{r}\right)^k\left((2k+2)r^{2k}\varphi'+r^{2k+1}\varphi''\right)\\
&= \left(\frac{\partial_r}{r}\right)^{k+1}\left(r^{2k+2}\varphi'\right).
\end{aligned}$$

□

Corollary of Lemma 1 gives that if $u(x,t)$ is a solution of the wave equation (10.1) in $\mathbb{R}^n\times\mathbb{R}$ then $M_u(x,r,t)$ satisfies (10.3), i.e.,

$$\left(\partial_r^2+\frac{n-1}{r}\partial_r\right)M_u=\partial_t^2 M_u,$$

with initial conditions:

$$M_u(x,r,0)=M_f(x,r),\quad \partial_t M_u(x,r,0)=M_g(x,r), \tag{10.4}$$

since $u(x,0)=f(x)$ and $\partial_t u(x,0)=g(x)$.

Let us set

$$\begin{aligned} \widetilde{u}(x,r,t) &:= \left(\frac{\partial_r}{r}\right)^{\frac{n-3}{2}} \left(r^{n-2}M_u\right) \equiv TM_u, \\ \widetilde{f}(x,r) &:= TM_f, \quad \widetilde{g}(x,r) := TM_g \end{aligned} \tag{10.5}$$

for $n = 2k+1, k = 1, 2, \ldots$.

Lemma 3. *The following is true:*

$$\begin{cases} \partial_r^2 \widetilde{u} = \partial_t^2 \widetilde{u} \\ \widetilde{u}|_{t=0} = \widetilde{f}, \quad \partial_t \widetilde{u}|_{t=0} = \widetilde{g}, \end{cases} \tag{10.6}$$

where $\widetilde{u}, \widetilde{f}$ and $\widetilde{g}$ are defined in (10.5).

Proof. Since $n = 2k+1$ then $\frac{n-3}{2} = k-1$ and $n-2 = 2k-1$. Hence we obtain from Lemmata 1 and 2 that

$$\begin{aligned} \partial_r^2 \widetilde{u} &= \partial_r^2 TM_u = \partial_r^2 \left(\frac{\partial_r}{r}\right)^{k-1} \left(r^{2k-1}M_u\right) = \left(\frac{\partial_r}{r}\right)^{k} \left(r^{2k}\partial_r M_u\right) \\ &= \left(\frac{\partial_r}{r}\right)^{k-1} \left(2kr^{2k-2}\partial_r M_u + r^{2k-1}\partial_r^2 M_u\right) \\ &= \left(\frac{\partial_r}{r}\right)^{k-1} \left(r^{2k-1}\left(\partial_r^2 M_u + \frac{n-1}{r}\partial_r M_u\right)\right) = \left(\frac{\partial_r}{r}\right)^{k-1} \left(r^{2k-1}\partial_t^2 M_u\right) \\ &= \partial_t^2 \left(\frac{\partial_r}{r}\right)^{k-1} \left(r^{2k-1}M_u\right) = \partial_t^2 \widetilde{u}. \end{aligned}$$

Moreover, the initial conditions are satisfied due to (10.4) and (10.5). □

But now, since (10.6) is a one-dimensional problem, we may conclude that $\widetilde{u}(x,r,t)$ from Lemma 3 is equal to

$$\widetilde{u}(x,r,t) = \frac{1}{2}\left\{\widetilde{f}(x,r+t) + \widetilde{f}(x,r-t) + \int_{r-t}^{r+t} \widetilde{g}(x,s)ds\right\}. \tag{10.7}$$

Lemma 4. *If $n = 2k+1, k = 1, 2, \ldots,$ then*

$$M_u(x,0,t) = \lim_{r\to 0} \frac{\widetilde{u}(x,r,t)}{(n-2)!!r},$$

where $(n-2)!! = 1 \cdot 3 \cdot 5 \cdots (n-2)$, is the solution of (10.2). We have even more, namely,

$$u(x,t) = \frac{1}{(n-2)!!}\left(\partial_r \widetilde{f}|_{r=t} + \widetilde{g}(x,t)\right). \tag{10.8}$$

Proof. By (10.5) we have

$$\begin{aligned}\widetilde{u}(x,r,t) &= \left(\frac{\partial_r}{r}\right)^{k-1}\left(r^{2k-1}M_u\right) = \left(\frac{\partial_r}{r}\right)^{k-2}\left((2k-1)r^{2k-3}M_u + r^{2k-2}\partial_r M_u\right)\\ &= (2k-1)(2k-3)\cdots 1\cdot M_u r + O(r^2)\end{aligned}$$

or

$$\frac{\widetilde{u}(x,r,t)}{(n-2)!!r} = M_u + O(r).$$

Hence

$$M_u(x,0,t) = \lim_{r\to 0}\frac{\widetilde{u}(x,r,t)}{(n-2)!!r}.$$

But by definition of M_u we have that $M_u(x,0,t) = u(x,t)$, where $u(x,t)$ is the solution of (10.2). The initial conditions in (10.2) are satisfied due to (10.5). Next, since $\widetilde{u}(x,r,t)$ satisfies (10.7) then

$$\begin{aligned}\lim_{r\to 0}\frac{\widetilde{u}(x,r,t)}{(n-2)!!r} &= \frac{1}{2(n-2)!!}\left(\lim_{r\to 0}\frac{\widetilde{f}(x,r+t)+\widetilde{f}(x,r-t)}{r} + \lim_{r\to 0}\frac{1}{r}\int_{r-t}^{r+t}\widetilde{g}(x,s)ds\right)\\ &= \frac{1}{2(n-2)!!}\left(\partial_r\widetilde{f}|_{r=t} + \partial_r\widetilde{f}|_{r=-t} + \widetilde{g}(x,t) - \widetilde{g}(x,-t)\right),\end{aligned}$$

because $\widetilde{f}(x,t)$ and $\widetilde{g}(x,t)$ are odd functions of t. That's why we finally obtain

$$\lim_{r\to 0}\frac{\widetilde{u}(x,r,t)}{(n-2)!!r} = \frac{1}{(n-2)!!}\left(\partial_r\widetilde{f}|_{r=t} + \widetilde{g}(x,t)\right).$$

□

Now we are in the position to prove the main theorem for odd $n \geq 3$.

Theorem 2. *Suppose that* $n = 2k+1, k = 1,2,\ldots$. *If* $f \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ *and* $g \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$ *then*

$$\begin{aligned}u(x,t) = \frac{1}{(n-2)!!\omega_n}&\left\{\partial_t\left(\frac{\partial_t}{t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{|y|=1}f(x+ty)d\sigma(y)\right)\right.\\ &\left.+\left(\frac{\partial_t}{t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{|y|=1}g(x+ty)d\sigma(y)\right)\right\}\end{aligned} \qquad (10.9)$$

solves (10.2).

Proof. Due to Lemmata 3 and 4 $u(x,t)$ given by (10.8) is the solution of the wave equation. It remains only to check that this u satisfies the initial conditions. But (10.9) gives us for small t that

$$u(x,t) = M_f(x,t) + tM_g(x,t) + O(t^2).$$

It implies that

$$u(x,0)=M_f(x,0)=f(x), \quad \partial_t u(x,0)=\partial_t M_f(x,0)+M_g(x,0)=g(x).$$

The last equality follows from the fact that $M_f(x,t)$ is even in t and so its derivative vanishes at $t=0$. □

Remark. If $n=3$ then (10.9) becomes

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\left\{\partial_t\left(t\int_{|y|=1} f(x+ty)d\sigma(y)\right)+t\int_{|y|=1} g(x+ty)d\sigma(y)\right\} \\ &\equiv \frac{1}{4\pi}\left\{\int_{|y|=1} f(x+ty)d\sigma(y)+t\int_{|y|=1}\nabla f(x+ty)\cdot y d\sigma(y)\right. \\ &+ \left. t\int_{|y|=1} g(x+ty)d\sigma(y)\right\}. \end{aligned}$$

The solution of (10.2) for even n is readily derived from the solution for odd n by "the method of descent". This is just the trivial observation: if u is a solution of the wave equation in $\mathbb{R}^{n+1}\times\mathbb{R}$ that does not depend on x_{n+1} then u satisfies the wave equation in $\mathbb{R}^n\times\mathbb{R}$. Thus to solve (10.2) in $\mathbb{R}^n\times\mathbb{R}$ with even n, we think of f and g as functions on $\mathbb{R}^{n+1}$ which are independent of x_{n+1}.

Theorem 3. *Suppose that n is even. If $f\in C^{\frac{n+4}{2}}(\mathbb{R}^n)$ and $g\in C^{\frac{n+2}{2}}(\mathbb{R}^n)$ then the function*

$$\begin{aligned} u(x,t)=\frac{2}{(n-1)!!\omega_{n+1}}&\left\{\partial_t\left(\frac{\partial_t}{t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_{|y|\le 1}\frac{f(x+ty)}{\sqrt{1-y^2}}dy\right)\right. \\ &+\left.\left(\frac{\partial_t}{t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_{|y|\le 1}\frac{g(x+ty)}{\sqrt{1-y^2}}dy\right)\right\} \end{aligned} \tag{10.10}$$

solves the Cauchy problem (10.2).

Proof. If n is even then $n+1$ is odd and $n+1\ge 3$. That's why we can apply (10.9) in $\mathbb{R}^{n+1}\times\mathbb{R}$ to get that

$$\begin{aligned} u(x,t)=\frac{1}{(n-1)!!\omega_{n+1}}&\left\{\partial_t\left(\frac{\partial_t}{t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_{y_1^2+\cdots+y_n^2+y_{n+1}^2=1} f(x+ty+ty_{n+1})d\sigma(\widetilde{y})\right)\right. \\ &+\left.\left(\frac{\partial_t}{t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_{y_1^2+\cdots+y_n^2+y_{n+1}^2=1} g(x+ty+ty_{n+1})d\sigma(\widetilde{y})\right)\right\}, \end{aligned} \tag{10.11}$$

where $\widetilde{y}=(y,y_{n+1})$, solves (10.2) in $\mathbb{R}^{n+1}\times\mathbb{R}$ (formally). But if we assume now that f and g do not depend on x_{n+1} then $u(x,t)$ does not depend on x_{n+1} either and solves (10.2) in $\mathbb{R}^n \times \mathbb{R}$. It remains only to calculate the integrals in (10.11) under this assumption. We have

$$\begin{aligned} \int_{|y|^2+y_{n+1}^2=1} f(x+ty+ty_{n+1})d\sigma(\widetilde{y}) &= \int_{|y|^2+y_{n+1}^2=1} f(x+ty)d\sigma(\widetilde{y}) \\ &= 2\int_{|y|\le 1} f(x+ty)\frac{dy}{\sqrt{1-|y|^2}}, \end{aligned}$$

because we have the upper and lower hemispheres of the sphere $|y|^2 + y_{n+1}^2 = 1$. Similarly for the second integral in (10.11). This proves the theorem. □

Remark. If $n = 2$ then (10.10) becomes

$$u(x,t) = \frac{1}{2\pi}\left\{\partial_t\left(t\int_{|y|\le 1}\frac{f(x+ty)}{\sqrt{1-y^2}}dy\right) + t\int_{|y|\le 1}\frac{g(x+ty)}{\sqrt{1-y^2}}dy\right\}.$$

Now we consider the Cauchy problem for the inhomogeneous wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = w(x,t) \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x). \end{cases} \tag{10.12}$$

We look for the solution $u(x,t)$ of (10.12) as $u = u_1 + u_2$, where

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \\ u_1(x,0) = f(x), \quad \partial_t u_1(x,0) = g(x), \end{cases} \tag{A}$$

and

$$\begin{cases} \partial_t^2 u_2 - \Delta u_2 = w \\ u_2(x,0) = \partial_t u_2(x,0) = 0. \end{cases} \tag{B}$$

For the problem (B) we will use a method known as *Duhamel's principle*.

Theorem 4. *Suppose $w \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times \mathbb{R})$. For $s \in \mathbb{R}$ let $v(x,t;s)$ be the solution of*

$$\begin{cases} \partial_t^2 v(x,t;s) - \Delta_x v(x,t;s) = 0 \\ v(x,0;s) = 0, \quad \partial_t v(x,0;s) = w(x,s). \end{cases}$$

Then

$$u(x,t) := \int_0^t v(x,t-s;s)ds$$

solves (B).

Proof. By definition of $u(x,t)$ it is clear that $u(x,0)=0$. We also have

$$\partial_t u(x,t) = v(x,0;t) + \int_0^t \partial_t v(x,t-s;s)ds.$$

It implies that $\partial_t u(x,0) = v(x,0;0) = 0$. Differentiating once more in t we can see that

$$\begin{aligned} \partial_t^2 u(x,t) &= \partial_t(v(x,0;t)) + \partial_t v(x,0;t) + \int_0^t \partial_t^2 v(x,t-s;s)ds \\ &= w(x,t) + \int_0^t \Delta_x v(x,t-s;s)ds \\ &= w(x,t) + \Delta_x \int_0^t v(x,t-s;s)ds = w(x,t) + \Delta_x u. \end{aligned}$$

Thus u solves (B) and the theorem is proved. □

Let us consider again the homogeneous Cauchy problem (10.2). Applying the Fourier transform with respect to x gives

$$\begin{cases} \partial_t^2 \widehat{u}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t) = 0 \\ \widehat{u}(\xi,0) = \widehat{f}(\xi), \quad \partial_t \widehat{u}(\xi,0) = \widehat{g}(\xi). \end{cases}$$

But this ordinary differential equation with initial conditions can be easily solved to obtain

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(|\xi|t) + \widehat{g}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \equiv \widehat{f}(\xi)\partial_t\left(\frac{\sin(|\xi|t)}{|\xi|}\right) + \widehat{g}(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

It implies that

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left(\widehat{f}(\xi)\partial_t \frac{\sin(|\xi|t)}{|\xi|}\right) + \mathcal{F}^{-1}\left(\widehat{g}(\xi)\frac{\sin(|\xi|t)}{|\xi|}\right) \\ &= f * \partial_t\left((2\pi)^{-n/2}\mathcal{F}^{-1}\left(\frac{\sin(|\xi|t)}{|\xi|}\right)\right) + g * \left((2\pi)^{-n/2}\mathcal{F}^{-1}\left(\frac{\sin(|\xi|t)}{|\xi|}\right)\right) \\ &= f * \partial_t \Phi(x,t) + g * \Phi(x,t), \end{aligned} \tag{10.13}$$

where $\Phi(x,t) = (2\pi)^{-n/2}\mathcal{F}^{-1}\left(\frac{\sin(|\xi|t)}{|\xi|}\right)$.

The next step is to try to solve the equation

$$\partial_t^2 F(x,t) - \Delta_x F(x,t) = \delta(x)\delta(t).$$

By Fourier transform in x we obtain

$$\partial_t^2 \widehat{F}(\xi,t) + |\xi|^2 \widehat{F}(\xi,t) = (2\pi)^{-n/2}\delta(t).$$

That's why $\widehat{F}$ must be a solution of $\partial_t^2 u + |\xi|^2 u = 0$ for $t \neq 0$. Therefore

$$\widehat{F}(\xi, t) = \begin{cases} a(\xi)\cos(|\xi|t) + b(\xi)\sin(|\xi|t), & t < 0 \\ c(\xi)\cos(|\xi|t) + d(\xi)\sin(|\xi|t), & t > 0. \end{cases}$$

To obtain the delta function at $t = 0$ we require that $\widehat{F}$ is continuous at $t = 0$ but $\partial_t \widehat{F}$ has a jump of size $(2\pi)^{-n/2}$ at $t = 0$. So we have

$$a(\xi) = c(\xi), \quad |\xi|(d(\xi) - b(\xi)) = (2\pi)^{-n/2}.$$

This gives two equations for the four unknown coefficients a, b, c and d. But it is reasonable to require $F(x, t) \equiv 0$ for $t < 0$. Hence, $a = b = c = 0$ and $d = (2\pi)^{-n/2}\frac{1}{|\xi|}$. That's why

$$\widehat{F}(\xi, t) = \begin{cases} (2\pi)^{-n/2}\frac{\sin(|\xi|t)}{|\xi|}, & t > 0 \\ 0, & t < 0. \end{cases} \tag{10.14}$$

If we compare (10.13) and (10.14) we may conclude that

$$F(x, t) = (2\pi)^{-n/2}\mathcal{F}_\xi^{-1}\left(\frac{\sin(|\xi|t)}{|\xi|}\right), \quad t > 0$$

and

$$\Phi_+(x, t) = \begin{cases} \Phi(x, t), & t > 0 \\ 0, & t < 0 \end{cases}$$

is the fundamental solution of the wave equation, i.e., $F(x, t)$ with $t > 0$.

There is one more observation. If we compare (10.9) and (10.10) with (10.13) then we may conclude that these three formulae are the same. Hence, we may calculate the inverse Fourier transform of

$$(2\pi)^{-n/2}\frac{\sin(|\xi|t)}{|\xi|}$$

in odd and even dimensions respectively with (10.9) and (10.10). Actually, the result is presented in these two formulae.

When solving the wave equation in the region $\Omega \times (0, \infty)$, where Ω is a bounded domain in $\mathbb{R}^n$, it is necessary to specify not only Cauchy data on $\Omega \times \{0\}$ but also some conditions on $\partial\Omega \times (0, \infty)$ to tell the wave what to do when it hits the boundary. If the boundary conditions on $\partial\Omega \times (0, \infty)$ are independent of t, the method of separation of variables can be used.

Let us (for example) consider the following problem:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = f(x), \quad \partial_t u(x, 0) = g(x), & \text{in } \Omega \\ u(x, t) = 0, & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{10.15}$$

We can look for solution u in the form $u(x,t) = F(x)G(t)$ and get

$$\begin{cases} \Delta F(x) + \lambda^2 F(x) = 0, & \text{in } \Omega \\ F(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{10.16}$$

and

$$G''(t) + \lambda^2 G(t) = 0, \quad 0 < t < \infty. \tag{10.17}$$

The general solution of (10.17) is

$$G(t) = a\cos(\lambda t) + b\sin(\lambda t).$$

Since (10.16) has infinitely many solutions $\{F_j\}_{j=1}^{\infty}$ with corresponding $\{\lambda_j^2\}_{j=1}^{\infty}, \lambda_j^2 \to +\infty$ $(\lambda_j > 0)$ and $\{F_j\}_{j=1}^{\infty}$ can be chosen as an orthonormal basis in $L^2(\Omega)$, the solution $u(x,t)$ of (10.15) is of the form

$$u(x,t) = \sum_{j=1}^{\infty} F_j(x)\left(a_j\cos(\lambda_j t) + b_j\sin(\lambda_j t)\right). \tag{10.18}$$

At the same time $f(x)$ and $g(x)$ have the $L^2(\Omega)$ representations

$$f(x) = \sum_{j=1}^{\infty} f_j F_j(x), \quad g(x) = \sum_{j=1}^{\infty} g_j F_j(x), \tag{10.19}$$

where $f_j = (f, F_j)_{L^2}$ and $g_j = (g, F_j)_{L^2}$. It follows from (10.15) and (10.18) that

$$u(x,0) = \sum_{j=1}^{\infty} a_j F_j(x), \quad u_t(x,0) = \sum_{j=1}^{\infty} \lambda_j b_j F_j(x). \tag{10.20}$$

Since (10.19) must be satisfied also we obtain

$$a_j = f_j, \quad b_j = \frac{1}{\lambda_j} g_j.$$

Therefore, the solution $u(x,t)$ of (10.15) has the form

$$u(x,t) = \sum_{j=1}^{\infty} F_j(x)\left(f_j\cos(\lambda_j t) + \frac{1}{\lambda_j} g_j\sin(\lambda_j t)\right).$$

It is clear that all series (10.18),(10.19) and (10.20) converge in $L^2(\Omega)$, because $\{F_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$. It remains only to investigate the convergence of these series in stronger norms (which depends on f and g, or more precisely, on their smoothness).

The Neumann problem with $\partial_\nu u(x,t), x \in \partial\Omega$, can be considered in a similar manner.

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