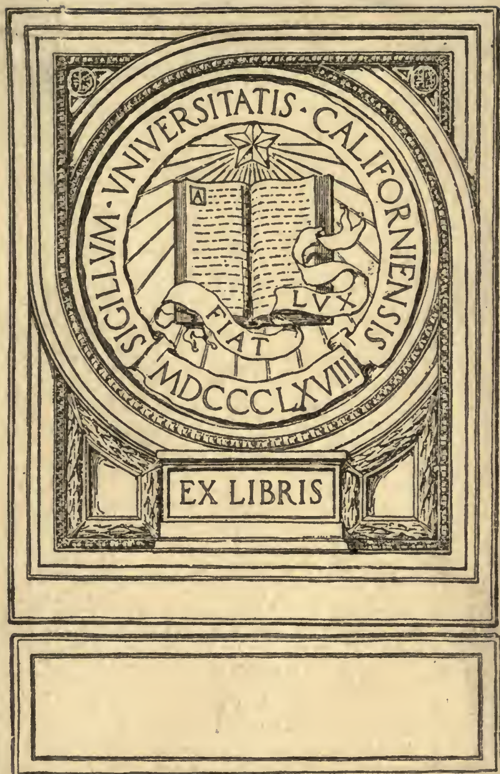
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$\frac{dH}{dv} = 0 \Leftrightarrow$ adiabatically. p 171

ELEMENTARY PRACTICAL MATHEMATICS

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ELEMENTARY PRACTICAL MATHEMATICS

WITH NUMEROUS EXERCISES FOR THE USE OF STUDENTS AND ESPECIALLY OF MECHANICAL AND ELECTRICAL ENGINEERING STUDENTS

BY

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CONTENTS

INTRODUCTION

Academic methods of teaching Mathematics succeed with about five per cent. of all students, the small minority who are fond of abstract reasoning: they fail altogether with the average student. Mathematical study may be made of great value to the average man if only it is made interesting to him. The name **Practical Mathematics** has been given to a new method of study, not because it describes the method but merely to differentiate it from the older method.

I began to use the new method forty years ago in an English public school, later in Japan. In 1881 I was bold enough to make it part of the curriculum at the City Guilds Technical College, Finsbury. It proved so successful that I induced the Board of Education to make it part of their scheme for Science classes. The number of Science class students of Practical Mathematics in Great Britain increases at a greater rate than the Compound Interest law, and at present there are more students of this subject than of any other. If a class is formed in September in elementary Pure Mathematics (Pure Mathematics is the name given to the older or academic method of study) and twenty students join it, at Christmas the number has dwindled to seven, and only one or two keep in attendance till May, whereas a class in Practical Mathematics keeps up its attendance almost intact to the end of the session.

There is always a difficulty in obtaining competent **teachers.** Any man who has learnt Pure Mathematics is thought by himself and others to be fit to teach, whereas his very fondness for and his fitness to study Pure Mathematics make it difficult for him to understand the simple principles underlying the new method.

We show a student how to work problems, exercising his common sense, and we give him experimental proof of the correctness of his results. Our methods of reasoning are those logical methods which are adopted in the teaching of Physics and in common affairs, and we claim that he understands and takes an interest in what he is doing, and feels confidence in his results.

When the Board established the new subject in 1899, I was asked to give a course of six lectures on Practical Mathematics to working men in London. A summary of these lectures, illustrated by exercises, was published by the Government. This has been republished by the Government with very large additions to the exercises, nearly all the important questions in the examinations from 1900 to 1909 being incorporated, as they are also in this book. It is found that, in spite of the cheapness of this publication, teachers do not consult it, nor do they seem to consult either the examination questions or the yearly reports of the examiners which are also published.

A great number of text-books of Practical Mathematics based upon the above summary have been published since 1900, and some of them are good, containing many excellent exercises like those I have mentioned; but, in spite of the guidance of these text-books, it is well known that there are many thousands of earnest hard-working students who are n the hands of teachers who do not understand our methods. They make their students work many text-book examples, but there is no real teaching. The best remedy is the doubling of salaries of the teachers.

The Board occasionally gives to a limited number of teachers the opportunity of coming to London to attend a three-weeks' course of instruction in July. Invariably I find that these teachers are the most earnest and hard working of students. I shall try in the following pages to reproduce part of the information which I give them, as well as the exercises which they are asked to give to their students, and to work themselves. I have usually one assistant for every ten students, his duty being that of helping in the exercise work.

I usually begin by directing the attention of students to some of my own published addresses, copies of which may be borrowed, and I make some introductory remarks which are merely amplifications of the following statements.

Essential things are neglected in our methods of primary and secondary education. Teachers have been brought up on bad old systems unsuited to modern needs. There are too many pupils per teacher, and teachers are badly paid. Useful reform is taking place in primary schools, but in secondary schools we still teach as if all average boys were proceeding to classical studies at the University, and preparing to be clergymen or schoolmasters.

The **average English boy** remains uneducated except through his sports; he learns nothing in the class rooms. There is hardly any business in which it is not essential to have a knowledge of English, a knowledge of computation, and a training in Natural Science, and success is hardly possible for any man who does not love books. It is easy to give these qualifications to any English boy, yet he almost never gets them. Again, the most prominent Englishmen understand nothing of those sciences which are transforming all the conditions of civilisation. We adhere to medieval ideals.

Five hundred years ago all books were in Latin. A man could not read unless he knew Latin; now, English Literature (including translations), is greater than any other ever known; but we still assume that a man is illiterate if he does not know Latin. Five hundred years ago only one man in a hundred could write his name; hardly anybody could compute. We have discarded some old methods of teaching, and now every boy in the country can read and write, and he can perform arithmetical work which was quite beyond the powers of the greatest Alexandrian philosophers. No one dreams of philosophising over Euclid's 7th, 8th, 9th, or 10th books, and in the same way all the books of Euclid ought to be superseded. For much of the 2nd and 5th books of Euclid we need only a page of easy algebra. After a man has become a Cambridge wrangler he will find the fifth book of Euclid, and indeed the whole of Euclid, a most fascinating study, but it is a cruel jest to call a boy stupid because he finds the study impossible.

The average boy cannot take to abstract reasoning, and he is called stupid; I think him much wiser than the boy who is usually called clever. He ought to be actually acquainted with concrete *things* before he is asked to reason about them. **Children** should in their play be accustomed to measurement; playing at keeping shop, selling things to each other by weight and measurement, paying for things in actual money.

Measurement of things with callipers and scales would accustom boys of eight to the use of decimals. Boys of ten will sketch and draw plans of their schoolhouse and the roads or streets about it; they soon learn to use maps, and they know that maps may be of different scales. They ought to be led up to the vector subject of Geometry only slowly, through maps useful to themselves, through problems on heights and distances and the like, and almost intuitively they would get an exact knowledge of the subjects of the sixth book of Euclid. Easy Mensuration is a fascinating subject to the average boy, and cultivates his reasoning powers if he measures and computes and tests his answers by experiment. I consider that no boy can get mental training through any subject of study unless he is interested in it and happy, and therefore I think that a boy's main business at school is to continue the study of observational and experimental science to which he has been accustomed since the day he was born. The methods of the ordinary schoolmaster are all opposed to this idea, and the worst of these methods is the teaching of many subjects in water-tight compartments, instead of teaching many subjects incidentally in the teaching of some one subject.

We make two very great mistakes: we painfully give a child the impression that an idea is difficult to understand, an idea perfectly familiar to it since it was three years of age, and again we assume that a child can understand quite readily some grown-up idea that looks to us simple. Thus we produce stupefaction, and the child is mentally shipwrecked; it seizes upon any raft for safety, and the raft ever ready is a formula, a rule. If schoolmasters studied their pupils as trainers of animals do, they would find the average boy capable of the highest kinds of intellectual work.

After a boy has made up by the use of objects, a multiplication table and can multiply and divide, I feel sure that he ought to leave mere number and get to quantity, so that he may use decimals quite early. To teach him the use of decimals and to educate his hand and eye and judgment, you allow him to measure things. But how do you do it? Often in the most uninteresting way! He is made to measure a certain length, to weigh an object, etc., and his answers are compared with the real length or weight, etc. Contrast this with the following exercise. He is given a block of iron, and he measures its length, 3·27 inches; breadth, 2·63 inches; thickness, 1·95 inches. He finds its volume to be 16·77 cubic inches. He readily uses contracted methods. Why will you try to stop him? He is given a cube 1 inch in edge of the same kind of iron. He takes it to the scales, and finds that it weighs 0·26 lb., so he computes the weight of his block to be 4·36 lb. He now goes and weighs it, and is delighted. Do you see where the difference comes in, and how interesting it is to find his computation agreeing with reality? And that same piece of iron is allowed to displace water in a vessel, and the displaced volume is found experimentally to be nearly 16·77 cubic inches. He now takes an irregularly shaped piece of iron, finds its volume by displacement, computes its weight, and weighs it by the scales. He sees at once how it is that perfect agreement cannot be expected.

Common-sense explanation accompanying experiment ought to be the rule. Do not teach abstract geometry at all; teach mensuration with the help of arithmetic and algebra and weighing and measuring. Do not be afraid to introduce a boy early to sines, cosines, and tangents, and the calculation and actual measurement by surveying, etc., of heights and distances. Boys are intensely interested in such work, and it is educational. But how awfully dull it may be made!

Besides this algebraic side of Practical Mathematics there is **Graphics**, which comprehends practical plane and solid geometry, and the summation of vectors and forces.

NOTE

If a sufficient number of the teachers attending a summer course desire to have some **advanced instruction**, I usually give them one lecture each day. They are expected to do most of the exercises given to the others, but special advanced exercises are also given. As such advanced work must take only one of several possible directions, I usually ask these students to choose the direction. This year, 1912, they chose the subject of harmonic functions, and the exercises which concern vibrating bodies, alternating currents of electricity, submarine telegraph and telephone circuits, heat conduction, etc., are given towards the end of this book. These 'advanced' exercises can all be readily worked by any student who faithfully works through the earlier or elementary ones. A student may think that they are of practical importance only to electricians; but it would be easy to set exactly the same mathematical exercises as mechanical engineering or general physics problems without introducing one technical term of the electrician.

The Board of Education examines now in two stages only. Candidates for the *lower* examination ought to be acquainted with almost all the work of Chapters I. to XVIII. of this book. Candidates for the *higher* examination ought to be acquainted with the whole of the book. I have given the answers to the papers set for the years 1910, 1911, and 1912.

The vile system of examinations created by such institutions as London University makes it to the interest of a student to attend the classes of a cheap crammer rather than those of a real teacher; it has also created that kind of text-book which exactly suits one examination and no other. The subject of Practical Mathematics is, I am happy to say, a subject which is not likely to commend itself to such institutions, nor are such text-books likely to be of much use to real students. This book is primarily for teachers, but it seems to me that a book so different from a cram text-book is the best of all text-books for a student.

Few people comprehend the difficulty of setting examples which are really practical. I have, myself, constructed all the exercises and questions given in this book. In many cases each of these questions has taken several hours, and in some cases several days, to construct.

ELEMENTARY PRACTICAL MATHEMATICS.

CHAPTER I.

ARITHMETIC.

1. When calculating from observed quantities it is **dishonest** to use more figures than we are sure of. A boy who experiments learns this very quickly.

Exercise. A boy measured a block of iron to be 6·56 inches long, 4·13 inches broad, and 2·67 inches thick. He took a cubic inch of what was presumably the same kind of iron, and he found it to weigh 0·26 lb. Therefore the weight of his block in pounds ought to be $6·56 \times 4·13 \times 2·67 \times 0·26$. Multiplying out, he obtained the answer 18·80763976.

Another boy measured the same block with the same measuring instruments, and found the length, breadth, and thickness to be 6·55, 4·14, and 2·68. He found the weight of the cubic inch of iron to be 0·26 lb. His answer is therefore

$$6·55 \times 4·14 \times 2·68 \times 0·26 = 18·895....$$

The two answers differ because, with the instruments employed, there was a possibility of error in the last figure of each of the measurements, and it is evident that the first boy ought to say that his answer is 18·8 and the second boy 18·9, because, even when he only gives three significant figures in his answer, there is a possibility that the last figure is in error. Of course all figures after the first three are worthless.

In a leading newspaper a few days ago I saw the indicated horse-power of a marine engine quoted as 3562·74 horse-power. Well, it is very probable that this measurement is in error at least 5 per cent. That is, the person who made the measurements and calculations is not sure whether the answer might not be 3700 or 3400, and yet he pretends that his last figure 4 has a meaning. I am sorry to say that many misleading figures of this kind are published in the best books written on the steam engine.

Sailors being examined for their Mate Certificates calculate their traverse tables to seconds; but their leeway is estimated in points; the error of several degrees is almost certain to occur. They use six or seven-figure logarithms in their work; four-figure logarithms are sufficiently accurate. In the examinations of the Institution of Civil Engineers, although only three figures are needed in the answers, it is imperatively required that seven-figure logarithms should be used.

When I was at school the mean distance from the earth to the sun was stated as 95,142,357 miles. I wonder why furlongs and inches were not mentioned. The best knowledge we now have of this distance is that it is not greater than 93 nor less than $92\frac{1}{2}$ millions of miles.

2. It will greatly prevent this sort of error if students will get into the way of writing $9{\cdot}3 \times 10^7$ or $9{\cdot}25 \times 10^7$, instead of 93,000,000 or 92,500,000. This is convenient in other ways.

In the following table note that we count how many places the decimal point must be moved to convert 6·548 into the number in question.

654,800,000,000	$6{\cdot}548 \times 10^{11}$
654,800	$6{\cdot}548 \times 10^{5}$
65·48	$6{\cdot}548 \times 10^{1}$
6·548	$6{\cdot}548 \times 10^{0}$
0·6548	$6{\cdot}548 \times 10^{-1}$
0·006548	$6{\cdot}548 \times 10^{-3}$.

When I calculate I seldom trouble my head about the position of the decimal point in my answer until everything else is finished. There are many cleverly contrived rules about the position of the decimal point, but we forget them in practical work. Better never learn them.

3. Multiplication. In multiplying $2{\cdot}714 \times 15{\cdot}68$, let us first neglect the positions of the decimal points and multiply 2714 by 1568. Let us suppose that we only want four significant figures in the answer. I here give the ordinary method; then I put 0 for the unnecessary figures; then I show how I actually work. Let a student study, understand, and practise for himself.

```
    2714           2714         2714
    1568           1568         8651
  -------        -------       ------
   21:712         22000         2714
  162:84          16300         1357
 1357:0          13570           163
 2714:           2714             22
 -----------     -------       ------
 4255:552        4256000        4256
```

I write the multiplier backwards. In multiplying by the 6, say, I am supposed to multiply only on the 27, the figure above the 6 and all to the left of it; but I add 1 because $6 \times 1 = 6$, and I cannot reject this as it is nearer to 10 than to 0, so I carry 1. Students must practise and get very familiar with this shortened method of multiplication. Some men prefer not to write the multiplier backwards, their work, however, being what I have given. Now, as to the position of the decimal point, we are multiplying a number which is between 2 and 3 by a number which is nearly 16, so that 42·56 is obviously our answer and not 4·256 or 425·6.

4. Division. Divide 2714 by 1568, giving only four significant figures in the answer. Here is the ordinary method: all the figures to the right of the dotted line and below *AB* are unnecessary; hence the merit of the other method, in which we cut off figures in the divisor.

```
1568 ) 2714   |173086          15̸6̸8̸ ) 2714  |1731*
       1568                           1568
       ----                           ----
       1146:0                         1146
       1097:6                         1098
       ------                         ----
         48:40                          48
         47:04                          47
         -----                          --
          1:3600                         1
        A 1:2544 B
          -------
          :10560
            9408
           -----
            1152
```

There is a further contraction of work, the above numbers 1568, 1098, and 47 not being written, which ought to be taught to boys. I give the above because I use it myself.

* Instead of cutting off figures in the divisor, some people write the answer backwards under the divisor like this $\begin{matrix}1568\\1371\end{matrix}$.

If you want to have no doubt about the correctness of your fourth figure in either of the above cases of multiplication or division, you had better use 27140 instead of 2714 ; in fact, get an answer with five significant figures and reject the last of them.

Ex. 1. Multiply the following numbers a and b, and also divide. I give the answers.

a	b	Answers.		
		ab	a/b	b/a
1323	24·32	32185	54·4	0·01838
17·56	143·5	2520	0·1224	8·172
0·5642	0·2471	0·1394	2·283	0·4380
4·326	0·003457	0·01495	1251·0	0·0007991
0·01584	2·104	0·03333	0·007530	132·8

Ex. 2. To find the Napierian logarithm of a number n (this is called $\log_e n$), we multiply the common logarithm (this is called $\log_{10} n$) by 2·3026. Convert the following:

$\log_{10} n$	Answers.
	$\log_e n$
2·1469	4·9435
0·3574	0·8229
−1·5178	−3·4949
−3·2005	−7·3695

When given $\log_e n$ we divide it by 2·3026, or, what is the same thing, we multiply it by 0·4343 to find the common logarithm.

Ex. 3. Convert the following:

$\log_e n$	Answers.	$\log_e n$	Answers.
	$\log_{10} n$		$\log_{10} n$
5·7152	2·4821	−2·1543	−0·9256
0·3513	0·1526	1·7216	0·7477
−2·1435	−0·9309	8·4175	3·6557
−4·7354	−2·0566	−0·1493	−0·0648

Let the student make himself quite sure that dividing by 2·3026 is really the same as multiplying by 0·4343.

5. Percentages. When we say 5 per cent., we mean 5 per hundred or $5 \div 100$ or 0·05. Thus 0·035 means $3\frac{1}{2}$ per cent., 0·02 means 2 per cent.

Some quite wrong expressions are in such common use that we do not condemn them. For example, a broker says he will charge you half-a-crown per cent. He means half-a-crown per hundred pounds: this is really $\frac{1}{8}$th of one per cent., whereas his language implies half-a-crown per hundred half-crowns, which is *one* per cent. One cow per cent. means one cow per hundred cows, or one per cent. But we forgive the broker.

Ex. 1. A piece of alloy weighing 3·28 lb. contains 2·65 lb. of copper, 0·46 lb. of tin, 0·17 lb. of lead; state these as percentages of the whole. Answer: $2{\cdot}65 \div 3{\cdot}28 = 0{\cdot}8079$ or 80·79 per cent. of copper; $0{\cdot}46 \div 3{\cdot}28 = {\cdot}1402$ or 14·02 per cent. of tin; $0{\cdot}17 \div 3{\cdot}28 = {\cdot}0518$ or 5·18 per cent. of lead.

Ex. 2. By measurement and computation two men find the weight of a body to be 15·72 and 15·59 tons. Assuming the mean of these two figures to be right, what is the percentage error in each?

The mean is $\frac{1}{2}(15{\cdot}72 + 15{\cdot}59)$ or 15·655; each differs from this by 0·065; the fractional error is $0{\cdot}065 \div 15{\cdot}655 = {\cdot}00415$ or 0·415 per cent.

Ex. 3. A man whose income is 230 pounds per annum has it increased by 5 per cent. What is the new income? Answer: add $\cdot 05 \times 230$ or 11·5; it is 241·5. Or we might have multiplied 230 by 1·05.

Ex. 4. A man's income is 241·5 pounds; it is reduced to 230; what is the percentage reduction? Answer: $11{\cdot}5 \div 241{\cdot}5 = {\cdot}0476$ or 4·76 per cent.

Ex. 5. I can buy £100 of 4 per cent. stock for £86. What income do I obtain? Answer: for each £86 I get £4 income; that is the fraction $4 \div 86$ or 0·0465 or 4·65 per cent. of my capital.

6. Commercial Arithmetic. As our sums of money, etc., are not expressed in a decimal system, before we can multiply or divide we find it necessary to express them in the decimal system. Bank clerks, grocers, and others who have many calculations to make of the same kind use labour-saving rules to effect such objects. When a grown-up student sees what the object is he easily understands such rules, but he need not hamper his memory with such rules unless they belong to his trade. The principle is easily understood from the following exercises.

1. Convert £182. 17*s.* 9*d.* to the decimal system.

Here 9*d.* is $\frac{9}{12}$ or 0·75*s.*; we have therefore 17·75*s.* or $\frac{17{\cdot}75}{20}$ or 0·8875 pounds. Our answer is therefore 182·8875 pounds.

2. Convert 3 tons 5 cwt. 2 qrs. 18 lb. to the decimal system.

$$\frac{18}{28} = \cdot 643\,; \ \frac{2\cdot 643}{4} = \cdot 6608\,; \ \frac{5\cdot 6608}{20} = \cdot 28304\,;$$

so that the answer is 3·28304 tons.

3. Convert 5 miles 3 furlongs 30 perches 4 yards to the decimal system.

$$\frac{4}{5\frac{1}{2}} \text{ or } \frac{8}{11} = \cdot 73\,; \ \frac{30\cdot 73}{40} = \cdot 768\,; \ \frac{3\cdot 768}{8} = \cdot 471\,;$$

so that the answer is 5·471 miles.

The converse process will be understood from the following examples.

4. Convert 7·828 pounds into pounds, shillings, and pence.

·828 pounds = ·828 × 20 shillings or 16·56, and ·56 shillings = ·56 × 12 pence or 6·72, so that the answer is £7. 16*s.* 6·72*d.*, or £7. 16*s.* 6¾*d.* nearly.

5. Convert 56·2154 tons into the usual absurd English form.

·2154 tons = ·2154 × 20 cwt. or 4·308, and ·308 cwt. = ·308 × 4 qrs. or 1·232; ·232 qrs. = ·232 × 28 lb. or 6½ lb., so that the answer is 56 tons 4 cwt. 1 qr. 6½ lb.

6. Convert 3·78085 miles. Here ·78085 miles = ·78085 × 8 furlongs or 6·2468; ·2468 furlongs = ·2468 × 40 perches or 9·872; ·872 perches = ·872 × 5½ yards = 4·796 yards; 0·796 yards = 2·388 feet; ·388 feet = 4·656 inches; so that our absurd-looking answer is 3 miles 6 furlongs 9 perches 4 yards 2 feet and 4·656 inches.

This kind of absurdity is flaunted in our faces in school books as if it were admirable, and there is no man to say that it is wicked. Truly it is only a special providence that can have prevented the ruin of the English people.

7. The rules called **Practice**, Interest, etc., are merely easy examples in multiplication and division in countries where decimal systems of money and weights and measures are employed.

Ex. 1. What is the cost of 33·24 yards of ribbon at 7·86 cents (or 0·0786 dollars) per yard? Answer: 33·24 × 0·0786 or 2·613 dollars. (This may be read 2 dollars 61 cents.)

Ex. 2. What is the cost of 11·275 kilogrammes at 1·234 francs per kilo? Answer: 11·275 × 1·234 or 13·913 francs. (This may be read 13 francs 91 centimes.)

Interest. Exercise. The sum of 5143·65 dollars is lent for 302 days at 4½ per cent. per annum. What is the interest? Here the interest for 365 days is 5143·65 × ·045, so that this multiplied by 302 and divided by 365 is the answer, or 191·51 dollars.

Proportion. If 3·275 tons cost 1·625 pounds, what will 1·164 tons cost? Here 1 ton costs 1·625 ÷ 3·275, and this multiplied by 1·164 is the answer, or 0·578 pounds.

Any person who examines an English Arithmetic will see that the troubles of boys are twenty times as great as they would be if we used decimal systems. And not only is this the case, but a boy who is expected to know the reasons for the rules he uses is introduced to complex logic which is far beyond his powers. I think that it is just here that a boy gives up all hope of being able to reason about such matters, and afterwards he refuses to make any serious effort to reason.

CHAPTER II.

LOGARITHMS.

8. The use of logarithms enables us to compute much more rapidly than by ordinary arithmetic.

Many kinds of computation seem almost hopelessly difficult except by the use of logarithms.

The symbol a^3 means $a \times a \times a$.

Hence $2^3 = 8$; $2^5 = 32$.

Many people say "a^3 means a multiplied on itself three times." Of course this is wrong. It is, however, right to say, "a^3 means 1 multiplied by a three times." Thus a^0 means 1 multiplied by a no times, or a^0 is really 1.

Definition of a Logarithm.

If $a^n = N$, then $n = \log_a N$, and we read this as "n is the logarithm of N to the base a." Thus $2^3 = 8$, and hence 3 is the logarithm of 8 to the base 2.

We almost always use only logarithms to the base 10 in arithmetical work because we use the decimal system of writing numbers; but in many important calculations we need to use Napierian logarithms whose base is 2·71828, a number so important that the letter e is generally used to denote it, just as the Greek letter π is used to denote 3·14159. It can be shown that if we multiply the common logarithm of any number by 2·30258 or divide by 0·43429 we get its Napierian logarithm.*

Whether n and m are integers or not, we *define* a^n to be such that $a^n \times a^m = a^{m+n}$.

* NOTE.—If $e^x = N = 10^n$, then x is the Napierian and n is the common logarithm of N; $\log_{10} N = n = x \log_{10} e$ and $\log_{10} e = 0·43429$. Hence we divide n by 0·43429 to get x.

It follows from this that

$$a^n \div a^m = a^{n-m},$$

and also that $$(a^n)^m = a^{nm}.$$

Hence $$a^n \div a^n = a^0 = 1,$$

and $$1 \div a^n = a^0 \div a^n = a^{0-n} = a^{-n}.$$

Let a student take any number which I shall call a; let him extract a very high root of it either by logarithms or by continued extraction of square roots; he will find that the higher the root, that is the nearer he approaches to a^0, the nearer does it get to 1. The student must not scorn an exercise like this; he need not work it if he thinks he knows this thing well enough already, but it is by working such exercises that a man gets a real intimate acquaintance with the subject.

We have then the rules:

1. Add the logarithms of two numbers and we have the logarithm of their product.

2. Subtract the logarithms of two numbers and we have the logarithm of their quotient.

Again, the logarithm of a^2 is *twice* the logarithm of a.
" " a^3 is *three* times " a.
" " $a^{\frac{1}{2}}$ or $\sqrt{a}$ is *half* " a.
" " $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$ is *one-third* " a.

In the same way, if b is any number whatsoever the logarithm of a^b is b times the logarithm of a.

(In the first 18 chapters of this book, instead of writing the common logarithm of a number as $\log_{10} n$, I shall write it $\log n$.)

9. I have given each of you a copy of a set of tables of four-figure logarithms and anti-logarithms published by the Board of Education; there is also a table of the sines, cosines, tangents, and radian measure of angles between 0° and 90°, and the first page of the little pamphlet gives various useful numbers and formulæ. See the tables, Chap. VIII.

Later, I shall tell you how to calculate a table of logarithms.

Now to use the table headed "logarithms," notice that there are no decimal points anywhere. Given the number 5204, we look at 52 on the left, and 0 above, this gives us 7160; our 4 causes us to look for the small number 3 on the right-hand columns, and we add it and so get 7163. The smallest amount of practice is surely enough to teach this.

Now I want you to understand that this means

$$\log 5{\cdot}204 = 0{\cdot}7163.$$

You will see, therefore, that to find the logarithm of any number, it is only necessary to study the following illustrative examples. Multiplying a number by 10 adds 1 to its logarithm.

Number.	Number as written in Art. 2.	The logarithm is	The value of the logarithm is more compactly written
520400	$5{\cdot}204 \times 10^5$	$\cdot 7163 + 5$	$5{\cdot}7163$
5204	$5{\cdot}204 \times 10^3$	$\cdot 7163 + 3$	$3{\cdot}7163$
5·204	$5{\cdot}204 \times 10^0$	$\cdot 7163 + 0$	$0{\cdot}7163$
·5204	$5{\cdot}204 \times 10^{-1}$	$\cdot 7163 - 1$	$\bar{1}{\cdot}7163$
·005204	$5{\cdot}204 \times 10^{-3}$	$\cdot 7163 - 3$	$\bar{3}{\cdot}7163$

The student notices that the whole number part of a logarithm depends on the position of the decimal point in the number.

If you like, in $5{\cdot}7163$ or $\bar{3}{\cdot}7163$ you may call the 5 or $\bar{3}$ (the whole number part) by such names as *index* or *characteristic*, and **the decimal part,** which **is always positive,** you may call the *mantissa*; but in truth you will do much better if you scorn the use of unnecessary technical terms like these.

10. Again, given the logarithm of a number, to find the number. Proceed backwards as from column 4 to column 1 of last table.

Thus, to find the number whose logarithm is 3·7163. In the antilogarithm table (we might do it with but little increase of trouble by means of the logarithm table itself), find what corresponds to ·7163; it is evidently 5200 + 4 or 5204. This means that

$$\log 5{\cdot}204 = \cdot 7163,$$
$$\text{and hence } \log 5204 = 3{\cdot}7163.$$

A little practice in multiplication and division will make you perfectly familiar with and give you a thorough understanding of this subject. Without such practice it is as useless to read minute instructions as it is to try to learn to swim or to ride a bicycle by reading instructions.

The student ought now, using logarithms, to work the exercises given in Art. 4.

Exercise. Find the common logarithms of the following numbers:

Number.	Logarithm.	Number.	Logarithm.
7·135	0·8534	1·065	0·0273
713·5	2·8534	0·01065	$\bar{2}$·0273
0·7135	$\bar{1}$·8534	10·65	1·0273
0·007135	$\bar{3}$·8534	106500	5·0273
71350	4·8534	1065	3·0273

If we are asked to multiply or divide a logarithm itself it is necessary to remember that the whole number part is sometimes negative.

Exercise. Find the values of

$$\sqrt{8574},\quad \sqrt{8{\cdot}574},\quad \sqrt{0{\cdot}8574},\quad \sqrt{0{\cdot}0008574}.$$

The logarithms of these numbers have to be divided by 2, as the logarithm of a square root is just half the logarithm of the number.

Number N.	log N.	$\frac{1}{2}$ log N.	Answers.
8574	3·9332	1·9666	92·60
8·574	0·9332	0·4666	2·928
0·8574	$\bar{1}$·9332	$\bar{1}$·9666	·9260
0·0008574	$\bar{4}$·9332	$\bar{2}$·4666	·02928

Thus to divide $\bar{1}$·9332 by 2, I remember that it is really $-1 + {\cdot}9332$; this is the same as $-2 + 1{\cdot}9332$, and the half of this is $-1 + {\cdot}9666$ or $\bar{1}$·9666.

To divide $\bar{1}$·9332 by 3, I call it $-3 + 2{\cdot}9332$, so that the answer is $-1 + 0{\cdot}9777$ or $\bar{1}$·9777.

To divide or multiply $\bar{3}$·9332 by such a number as 4·56, it is necessary to convert it all into the negative form. It is $-3 + 0{\cdot}9332$, and this is really $-2{\cdot}0668$. Dividing this by 4·56 we get $-0{\cdot}4532$, and this is $-1 + 0{\cdot}5468$ or $\bar{1}$·5468.

Thus $({\cdot}00857)^{\frac{1}{4\cdot56}}$ is 0·3522.

I think that perhaps, when a logarithm like $\bar{2}$·4666 has to be divided or multiplied, the student ought at first *always* to put it, not as $-2 + {\cdot}4666$ but as $-1{\cdot}5334$, even when he multiplies or divides by 2 or 3. This may not be so quick as the other method, but the student knows better what he is doing, and is less likely to make mistakes.

EXERCISES.

1. Calculate a^b. That is, the number a raised to the power indicated by b.

Find the logarithm of a, multiply it by b, and this is the logarithm of the answer :

Let $a=20·52$	and $b=2$.	*Ans.* 421·1.
„ $a=1·564$	„ $b=1\frac{1}{2}$.	„ 1·955.
„ $a=0·5728$	„ $b=3$.	„ 0·1879.
„ $a=60·71$	„ $b=\frac{1}{3}$.	„ 3·930.

Note here that to multiply by $\frac{1}{3}$ means that we are to divide by 3.

$a=0·2415$	and $b=\frac{1}{3}$.	*Ans.* 0·6227.
$a=1·671$	„ $b=2$.	„ 2·793.
$a=·5014$	„ $b=3\frac{1}{2}$.	„ 0·08919.

2. Without using logarithms, find $4·326\times 0·003457$ to four significant figures, leaving out all unnecessary figures in the work.

Find $0·01584\div 2·104$ to four significant figures.

Also do these, using logarithms. Find $\log_e 7$.

Calculate $5^{2·43}$, $3^{-0·246}$, $·042^{0·476}$, $\sqrt[5]{246·3}$, $30·01^{\frac{2}{3}}\times 0·02641^{\frac{1}{7}}$.

Ans. 0·01495, 0·007529, 1·94592, 49·95, 0·7632, 0·2211, 3·008, 5·745.

Teachers make a mistake when, in calculating a^b, they regard a negative value of b as creating an exercise for advanced students only. The most elementary students ought to be accustomed to such exercises; they will find no difficulty in them if their teachers do not introduce difficulties.

11. EXERCISES ON **CONTRACTED METHODS.**

1. Assuming that $\sqrt{3}=1·7321$ and $\sqrt{2}=1·4142$, find by contracted methods, as accurately as the given numbers will allow, $\sqrt{6}$ and $\sqrt{1·5}$.

Ans. 2·4495 ; 1·2248.

2. Having given that to five significant figures $\sqrt{5}=2·2361$ and $\sqrt{2}=1·4142$, find by contracted methods $\sqrt{10}$ and $\sqrt{2·5}$.

Ans. 3·1623 ; 1·5812.

3. Find by contracted methods correct to four significant figures, without using logarithms, $0·30103\times 0·026007$. *Ans.* 0·007829.

4. A rectangle measures $23·59\times 18·64$ cms. Using contracted methods, find the area and the ratio of its length to its breadth.

Ans. 439·7 ; 1·265.

5. Having given that $e^{\frac{1}{3}}=1·3956$, find by contracted methods $e^{\frac{2}{3}}$ and $e^{-\frac{1}{3}}$.

Ans. 1·9477 ; 0·7165.

6. Having given that $\sqrt[3]{3}=1·4422$ and that $\sqrt[3]{2}=1·2599$, find by contracted methods, without using logarithms, $\sqrt[3]{6}$ and $\sqrt[3]{1·5}$.

Ans. 1·8170 ; 1·1447.

7. To five significant figures $10^{\frac{1}{2}}=3·1623$ and $10^{\frac{1}{8}}$ is 1·3335, find by contracted methods $10^{\frac{5}{8}}$ and $10^{\frac{3}{8}}$. *Ans.* 4·217 ; 2·3715.

8. The diameter of a circle is 27·35 inches. Using contracted multiplication, show that the area of the circle lies between 587 and 588 square inches. The area of a circle is the square of its diameter multiplied by 0·7854.

9. If to five significant figures $e=2·7183$, find e^2 by a contracted method. *Ans.* 7·3891.

12. EXERCISES USING LOGARITHMS.

1. For the following values of x calculate x^n in the following cases :

Given Values of x.	ANSWERS. Values of x^n.					
	When n is 4.	When n is 1.	When n is $\frac{1}{4}$.	When n is $-\frac{1}{4}$.	When n is -1.	When n is -4.
0	0	0	0	∞	∞	∞
0·1	0·0001	0·1	0·5623	1·778	10	10,000
0·2	0·0016	0·2	0·6688	1·495	5	625
0·3	0·0081	0·3	0·7401	1·351	3·333	123·5
0·4	0·0256	0·4	0·7952	1·257	2·5	39·06
0·5	0·0625	0·5	0·8410	1·190	2	16
0·6	0·1296	0·6	0·8802	1·136	1·667	7·711
0·7	0·2401	0·7	0·9147	1·093	1·429	4·165
0·8	0·4096	0·8	0·9457	1·057	1·250	2·443
0·9	0·6561	0·9	0·9740	1·027	1·111	1·524
1·0	1·000	1·0	1·0	1·0	1·0	1·0
1·1	1·4641	1·1	1·024	0·976	0·9091	0·6830
1·2	2·0736	1·2	1·046	0·955	0·8333	0·482
1·3	2·8561	1·3	1·068	0·937	0·7692	0·3501
1·4	3·8416	1·4	1·088	0·919	0·7143	0·2603
1·5	5·0625	1·5	1·107	0·904	0·667	0·1975
1·6	6·5536	1·6	1·125	0·889	0·625	0·1526
1·7	8·352	1·7	1·142	0·876	0·588	0·1197
1·8	10·50	1·8	1·158	0·863	0·556	0·0953
1·9	13·03	1·9	1·174	0·852	0·526	0·0767
2·0	16·00	2·0	1·189	0·841	0·500	0·0625

To find $x^{-\frac{1}{4}}$ the student had better first find $x^{\frac{1}{4}}$ and then find its reciprocal. The teacher ought to give the student only a few of the above exercises on any one night.

2. Calculate a^b and a^{-b} in the following cases :

a	b	ANSWERS.	
		a^b	a^{-b}
5	2·43	49·95	·02002
3	0·246	1·310	0·7632
0·042	0·476	0·2211	4·522
246·3	0·2	3·008	·3324
30·01	2/3	9·657	·1036
0·02641	1/7	·595	1·681

3. Calculate the squares, cubes, square roots, cube roots, and reciprocals of the following numbers:

Given Numbers.	Answers.				
	Square.	Cube.	Square Root.	Cube Root.	Reciprocal.
3931	1545×10^4	6074×10^7	62·70	15·78	$2·544 \times 10^{-4}$
227	51529	$1·170 \times 10^7$	15·07	6·100	$4·405 \times 10^{-3}$
8·65	74·82	647·2	2·941	2·053	0·1156
0·7854	0·6169	0·4845	0·8862	0·9226	1·273
0·00326	$1·063 \times 10^{-5}$	$3·465 \times 10^{-8}$	$5·71 \times 10^{-2}$	$1·483 \times 10^{-1}$	306·7
$5·426 \times 10^{-2}$	$2·944 \times 10^{-3}$	$1·597 \times 10^{-4}$	0·2329	$3·786 \times 10^{-1}$	18·43

4. Find $\log_e n$ if $\log_{10} n = \bar{2}·1563$.

You are told to multiply the common logarithm by 2·3026, so it is evidently good to write $\bar{2}·1563$ in the form $-1·8437$. The answer is $-4·2453$.

5. Convert the following:

$\log_{10} n$	Answers.	$\log_{10} n$	Answers.
	$\log_e n$		$\log_e n$
$\bar{2}·4822$	$-3·4949$	$\bar{1}·0213$	$-2·2535$
$\bar{4}·7995$	$-7·3695$	$\bar{3}·9812$	$-4·6485$

CHAPTER III.

THE SLIDE RULE.

13. To multiply 5623×1547 we say

$$\log 5623 = 3{\cdot}7499$$
$$\log 1547 = 3{\cdot}1895$$

Adding, we have $6{\cdot}9394$ = logarithm of $8{\cdot}698 \times 10^6$.

The essential part of the work is the adding of ·7499 and ·1895, and we can add numbers mechanically in many ways.

It would be laborious to put 7499 beans in a bag and then put in 1895 and count the whole. We might put 749·9 ounces and 189·5 ounces in a scale pan and weigh the lot, thus getting at the sum. A good plan is to measure off the distance 7·499 inches or centimetres on a scale as in Fig. 1, measure off the distance 1·895 inches or centimetres on another scale; and place the two distances so that their sum may be measured.

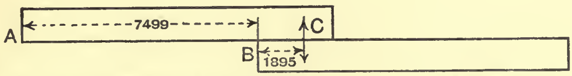

Fig. 1.—The distance AB added to the distance BC is the distance AC.

Now the slide rule does this very thing, only the slide rule has not the numbers of Fig. 1 written upon it. It has two sliding scales, and in the above position of things the marks upon the scale would be those shown in Fig. 2; that is, although the distance from

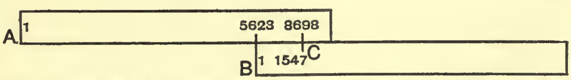

Fig. 2.—The distances AB, BC, and AC the same as in Fig. 1, but the numbers different.

A to B represents 7499 to some scale of measurement or other, it is not 7499 that is written on the upper scale but the number

5623, or 5·623 of which ·7499 is the logarithm, and it is not the logarithm ·9394 which we read off opposite C on the upper scale, although this really is the distance from A to C; it is the number 8698 or ·8698 which we see there.

14. Perhaps if I **make a slide rule** before you it will be easier to comprehend. Here are two black scales, A and B (Fig. 3), which

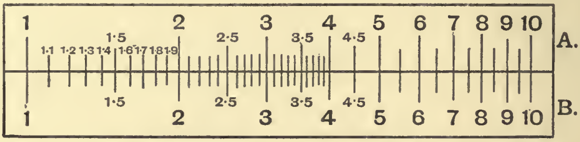

FIG. 3.—Slide rule manufactured in a few minutes in front of class.

slide past one another. Making their ends agree, I am going to put the same chalk marks on both. I have a sort of tape line here, on which I have taken a certain distance (you need not trouble yourselves about its actual length in inches) which I shall call a unit distance. I have divided it up into 10 and 100 equal parts so as to be able to set off any distance less than unity on my scales. To know where to put the number 4·5 on my scale I have made its distance from the point 1 to be ·6532. I have put the number 5 at a distance from 1 equal to ·6990.

Number on scale.	At a distance from the mark 1 of	Because
4·5	·6532	log 4·5 = ·6532
5·0	·6990	log 5·0 = ·6990
10·0	1·0000	log 10 = 1·0000

In fact my distances from the mark 1 are proportional to the logarithms of the numbers placed at those distances.

15. A slide rule has several scales. If you have a slide rule (Fig. 4), look at the two marked A and B. Slide them so that they

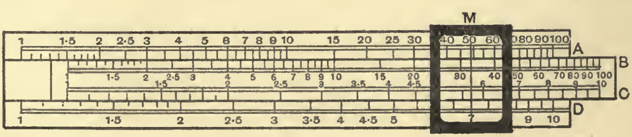

FIG. 4.

agree exactly. You will find that the distance from the mark 1 to the mark 2 or 3 or 4 or 12 is not 2 inches or 3 inches or 4 inches or

12 inches, but really ·3010 inch, ·4771 inch, ·6021 inch, or 1·0792 inch, because these four numbers are the *logarithms* of 2, 3, 4, 12. So if you place 1 of *B* against 3 of *A*, and look for the number on *A*, which is opposite 4 of *B*, you will see 12. You have **multiplied** 3×4 because you have merely added ·4771 and ·6021 to get 1·0792. Think it out for yourself; practise multiplying simple numbers; ask nobody to help you, and you will rapidly get familiar with and fond of the slide rule.

The lecturer manufactured a slide rule in two minutes, sufficiently good to illustrate multiplication and division. Every student ought to manufacture a slide rule, using two strips of paper.

To **divide** 12 by 4. Opposite 12 of *A*, place 4 of *B*, and the answer is on *A* opposite 1 of *B*. Again practise division.

Multiply $4 \times 3 \times 6 \times 7$, and so learn the value of the sliding marker (called a cursor). When you have multiplied 4×3, put the marker at the answer. You do not want to know this answer. Now put 1 of *B* at the marker, and remove the marker to 6 of *B* and so on. If you practise by yourself, you will need no telling. Again, if your answer is beyond the end of scale *A*, slide *B* back until the mark 10 or 100 occupies the place that 1 occupied; this is just as if you lengthened scale *A*. But instructions are of no use. Find all this out for yourself.

How do you find the **reciprocal** of a number? Divide 1 or 10 by the number; it is easy enough to do. Notice that when 1 of *B* is opposite 3 of *A*, then every number of *B* is opposite 3 times it of *A*. Thus one position of *B* enables us to multiply all the numbers in a table by the same number.

Test for yourself your accuracy in multiplying and dividing. Note how curiously the scales are divided, and accustom yourself to reading off numbers quickly.

In an actual slide rule (Fig. 4), we have not only the scales *A* and *B* which slide alongside one another, but another pair *C* and *D* which are alongside one another. These are prepared in the same way as *A* and *B*, only that

if the distance from 1 to 2 on scale *A* or *B* is ·3010 inch,
then „ „ 1 to 2 „ *C* or *D* is ·6020 „

You will see, therefore, that a number on *D* is just underneath its square on *A*, and you will see that the sliding marker *M* enables you to find **squares** or **square roots**.

Also it is easy to multiply or divide any number by the square or square root of any other number, and this again ought to be practised. So again, any number may be multiplied by its own square, and so we get its **cube**, and the student will see that we can reverse this process and so extract a **cube root.*** Read no book of elaborate instructions, you can find out everything for yourself by using the rule. Here on the lecture table are numerous contrivances called by all sorts of names, which are really all slide rules. Notice how the Fuller rule, instead of being straight, is spiral, so that we get what is equivalent to an exceedingly long rule in a small compass. Also these tables of Professor Everett are really a very long rule in the shape of two sheets of cardboard with slits in them.

16. When students are able to compute $a \times b$ or $a \div b$ or a^b, it is usual to give them about thirty exercises all of the same kind. They have their rule: to apply it needs no thought, and for the time being they are machines. Now I do not altogether object to a few examples of the same kind coming immediately after one another, but I submit that we ought as much as possible to vary the kind of question worked in any one lesson. For example, in one lesson I would ask a boy to compute $37{\cdot}56 \times 18{\cdot}23$, $37{\cdot}56 \div 18{\cdot}23$, $37{\cdot}56^{1{\cdot}823}$, $37{\cdot}56^{-1{\cdot}823}$, and along with these I would give quite different examples. If this is done, a boy has really to think of what he is doing all the time.

It is for this reason that I like to have boys compute from formulæ quite early, because each part of the work requires somewhat different thought from what the other parts require.

When a boy can multiply and divide accurately and readily he of course gets to do these things mechanically, and he ought not to

*Some years ago I designed a slide rule to calculate a^b. I found that Dr. Roget had used the same method in 1815. My rule is made by Messrs. Thornton of Manchester. Here is the principle:

If $x = a^b$, $\log x = b \log a$, and $\log(\log x) = \log b + \log(\log a)$. Keeping scale C as it is, I substitute for scale D one in which the distance to the mark a represents $\log(\log a)$. If 1 of scale C be placed opposite a of scale D, then opposite b on C will be found x on scale D. Even if b is less than 1 it is easy to find x.

The student ought to practise finding such things as 2^3, 2^4, etc., or 3^2, 3^3, etc., of which he knows the answers.

Evidently we can find the logarithm of any number to any base. We can find $x = a^{b/c}$ or $x = a^{1/c}$ at one operation, and there are many other uses of this new slide rule.

have to think about the details of such processes; his thought ought to be about the main question on which he is engaged.

EXERCISES.

1. Write down the common logarithms of 1·115; 1,115,000; 0·000,011,15. *Ans.* 0·0472, 6·0472, $\bar{5}$·0472.

2. Compute by logarithms and by slide rule (where convenient):

(i) $\dfrac{6837 \times 0{\cdot}002942}{52{\cdot}49}$; (ii) $\dfrac{54{\cdot}36 \times 763 \times 273}{760 \times 295} \times 0{\cdot}004136$;

(iii) $\dfrac{60{\cdot}8 \times 525 \times 10^7}{57750 \times 754 \times 0{\cdot}7}$; (iv) $\dfrac{5{\cdot}306 \times 0{\cdot}07632}{73{\cdot}15 \times \sqrt{0{\cdot}02164}}$;

(v) $\sqrt{251} \times \dfrac{0{\cdot}1639 \times 52100 \times 0{\cdot}0253}{0{\cdot}00035 \times 1{\cdot}0264}$;

(vi) $\dfrac{0{\cdot}0065 \times \sqrt{136}}{(0{\cdot}1324)^2 \times 0{\cdot}005621}$; (vii) $\dfrac{5{\cdot}473 \times 2{\cdot}517 \times \sqrt{3{\cdot}5}}{1{\cdot}324 \times 4{\cdot}768 \times (1{\cdot}69)^{1{\cdot}5}}$;

(viii) $\dfrac{5{\cdot}397 \times 307 \times 760 \times ({\cdot}589)^{2{\cdot}5}}{273 \times 673 \times \sqrt{1{\cdot}589}}$; (ix) $97{\cdot}43 \div (0{\cdot}3524 + 6{\cdot}321)^{2{\cdot}56}$;

(x) $(1{\cdot}342 \times 0{\cdot}01731 \div 0{\cdot}0274)^{0{\cdot}319}$.

Ans. (i) ·38320; (ii) ·2089; (iii) 10,472; (iv) ·03763; (v) $9{\cdot}528 \times 10^6$; (vi) 771; (vii) 1·858; (viii) 1·448; (ix) 0·7557; (x) 0·9487.

3. Evaluate, using logarithms:

(*a*) $\sqrt{\dfrac{31{\cdot}2 \times 0{\cdot}064}{25{\cdot}7 \times 18{\cdot}3}}$; (*b*) the cube root of $1782 \div 0{\cdot}3152$.

Ans. (*a*) 0·06516; (*b*) 17·81

4. Find the logarithm of 3 to the base e, where $e = 2{\cdot}718$. *Ans.* 1·0986.

5. Compute, using logarithms:

(*a*) $5{\cdot}36^{-1{\cdot}21}$, $0{\cdot}236^{2{\cdot}4}$; (*b*) $\sqrt[5]{237}$, $10{\cdot}5^{2{\cdot}3}$;

(*c*) $26{\cdot}12^3$, $26{\cdot}12^{-3}$; (*d*) $0{\cdot}3^{\frac{1}{3}}$; (*e*) $3{\cdot}59^{-\frac{1}{3}}$, ${\cdot}0359^{\frac{1}{3}}$.

(*f*) $71{\cdot}34^{1{\cdot}78}$, $0{\cdot}0632^{1{\cdot}78}$; (*g*) $3^{2{\cdot}59}$, $0{\cdot}3241^{0{\cdot}236}$;

(*h*) $25^{\frac{1}{3}}$, $0{\cdot}025^{0{\cdot}3}$, ${\cdot}25^{-1{\cdot}3}$; (*k*) $5^{2{\cdot}43}$, $3^{-0{\cdot}246}$, and $0{\cdot}042^{0{\cdot}476}$.

Ans. (*a*) 0·1311, ·03126; (*b*) 2·985, 223·2; (*c*) 17820, ·00005612; (*d*) ·6694; (*e*) ·6531, ·3299; (*f*) 10·99, 0·007333; (*g*) 17·21, ·7665; (*h*) 2·924, ·3307, 6·063; (*k*) 49·95, ·7632, 0·2211.

$(71.34)^{0.56169} = 10.99$. Aha! $0.56169 = \frac{1}{1.78}$

CHAPTER IV.

EVALUATION OF FORMULÆ.

17. Mathematical symbols are merely a very easy form of shorthand; they usually instruct us to perform certain arithmetical operations. Thus $+$, $-$, $\times$, and $\div$ are well known. Of course, $a \times b$ means the same as $a \, . \, b$ or ab when letters are used to represent quantities, but only the first of these ought to be used with numbers, for very obvious reasons. Again $a \div b$ or $a : b$ or $\frac{a}{b}$ or a/b mean the same. The use of brackets ought to be familiar to students. Again $\sqrt{a}$ is the same as $a^{\frac{1}{2}}$; $\sqrt[3]{a}$ the same as $a^{\frac{1}{3}}$; $a^{\frac{2}{3}}$ the same as $\sqrt[3]{a^2}$; or by $a^{p/q}$ we mean the q^{th} root of the p^{th} power of a or the p^{th} power of the q^{th} root of a. Again a^{-m} means $1 \div a^m$.

Sometimes in a formula we find a term like $\sin\theta$ or $\cos\theta$ or $\tan\theta$, and this means that, θ being some given angle, we are asked to look up the arithmetical value of the sine of this angle, or its cosine, or its tangent, in some suitable table, and use it in calculating with the formula.

Again, $\sin^{-1}a$ means "the angle whose sine is a," and if we know a it is easy from tables to find $\sin^{-1}a$, $\cos^{-1}a$, $\tan^{-1}a$, etc. It is a great pity that this symbol should be liable to be taken for another.

Again, if we use $\log_{10}x$ we know that we are expected to look up the value of the common logarithm of the number x in a table of common logarithms or logarithms to the base 10.

Again, if we see $\log_e x$ we know that we are required to use the Napierian, or, as some people call it, the Natural, or, as others very absurdly call it, the Hyperbolic Logarithm of the number x; a mathematician usually means $\log_e x$ when he writes $\log x$. To convert common into Napierian logarithms multiply by 2·3026.

Some formulæ tell us to look up other things, but there are always sufficient instructions to enable the necessary arithmetical work to be carried out if a teacher will only give exercises to his pupils and say nothing of philosophical difficulties which exist only in his imagination. The pupils have usually enough common sense to follow the plain instructions of even complicated formulæ.

The tables of Chap. VIII. will enable the following exercises to be worked. But some of the arithmetical work will be more easily done after Chapter V. is read.

We often use other symbols in our work: $a>b$ means "a is greater than b"; $a<b$ means "a is less than b"; $y \propto x$ means "y varies as x, or y is proportional to x; that is, y is equal to x multiplied by some constant number." The Greek letter s, which is Σ, is often used to mean "The sum of all such terms as . . ."

It ought to be made clear to students that the evaluation of a formula is merely arithmetic; there is nothing difficult about it. When, by working many exercises, a student has lost his fear of formulæ and knows that he can evaluate any formula, he will find Mathematics an easy study.

The most essential idea in the method of study called Practical Mathematics is that a student should become familiar with things before he is asked to reason about them. He must therefore become very familiar with algebraic formulæ by working such exercises as the following.

Once for all—these four chapters are merely about easy arithmetic. When a man uses a complex looking formula, it pleases his vanity to think that he is working in complex mathematics. A druggist when he makes up a prescription has this same vanity. The man who blows the bellows for an organ may have the same. In these modern days science has given us the most wonderful machinery, and every man who can turn a handle has the same vanity. I would impress on students that it is very easy to turn handles, and if one is not vain, if one is modest, if one desires knowledge, the turning of the handle may lead to a knowledge of the machine and to scientific discovery. Indeed, to use another metaphor, the turning of the handle is absolutely necessary if we wish the door of the scientific mansion to open for us.

EXERCISES.

1. If $m=(1+\log_e r)/r$, find m for the following values of r:

r	1·333	1·5	2	3	5	8	12	20
Answers m	0·965	0·937	0·846	0·70	0·522	0·385	0·290	0·20

2. If $m=(sr^{-1}-r^{-s})/(s-1)$, calculate m when s has the values 0·8, 0·9, 1·1, 1·2 for the following values of r:

r	Answers.			
	m when $s=0·8$.	m when $s=0·9$.	m when $s=1·1$.	m when $s=1·2$.
1·333	0·971	0·970	0·962	0·959
1·5	0·948	0·941	0·933	0·926
2	0·871	0·859	0·834	0·823
3	0·743	0·721	0·681	0·662
5	0·580	0·549	0·497	0·475
8	0·447	0·414	0·360	0·337
12	0·352	0·318	0·267	0·246
20	0·257	0·225	0·180	0·163

Notice that you can make no calculations if s is 1. There is a way of doing this, however, and the answers obtained are those given in Exercise 1.

3. If u cubic feet is the volume of 1 lb. of dry saturated steam, whose pressure is p lb. per sq. inch, experiment shows that, very nearly,

$$u=329\div p^{0·94}.$$

Calculate u for each of the following values of p.

For a certain purpose it is found necessary to use the incorrect formula

$$u_1=1\div(0·0171+0·0021p).$$

Calculate u_1 for the same values of p, and state the error in each case:

Given values of p.	Answers u.	Answers u_1.	Error u_1-u.
80	5·36	5·40	0·04
120	3·66	3·72	0·06
140	3·16	3·21	0·05
180	2·50	2·53	0·03
220	2·07	2·09	0·02
280	1·65	1·65	0·00

4. Find p for the following values of u, using the formula

$$p=479\div u^{1·0646}:$$

Given values of u	40	20	10	5	3	2
Answers p	9·436	19·73	41·28	86·34	148·7	229·1

The following two exercises are on very important approximations:

5. When a is very small compared with 1 we may write $(1+a)^n=1+na$, very nearly. Employ this to show that the following relations are nearly correct:

$$(1{\cdot}001)^3=1{\cdot}003;\quad (1{\cdot}01)^{\frac{1}{3}}=1{\cdot}0033;\quad (0{\cdot}99)^2=(1-{\cdot}01)^2=1-{\cdot}02={\cdot}98;$$

$$\frac{1}{{\cdot}99}=\frac{1}{1-{\cdot}01}=(1-{\cdot}01)^{-1}=1+{\cdot}01=1{\cdot}01.$$

$$\frac{1}{\sqrt[3]{1{\cdot}01}}=(1+{\cdot}01)^{-\frac{1}{3}}=1-{\cdot}0033={\cdot}9967,$$

$$\sqrt{99}=\sqrt{100(1-{\cdot}01)}=10(1-{\cdot}01)^{\frac{1}{2}}=10(1-{\cdot}005)=9{\cdot}95.$$

Find what the exact percentage errors are in these answers. For example, find the cube of 1·001 by multiplication, and observe how slightly it differs from 1·003.

6. How much error is there in the assumptions

$$\frac{1+\alpha}{1+\beta}=1+\alpha-\beta;\quad (1+\alpha)(1-\beta)=1+\alpha-\beta$$

when $\alpha={\cdot}01,\ \beta={\cdot}01;\quad \alpha=-{\cdot}003,\ \beta=-{\cdot}005$?

Ans. No error; ·01 per cent., ·001 per cent., ·0015 per cent.

7. Instead of calculating $a+\sqrt{a^2+b^2}$, we often substitute, as giving nearly the same answer, $1{\cdot}84a+0{\cdot}84b$. Take $a=1$, and, taking various values of b, show the two sets of answers in a table. Within what limiting values of b may we assume the error to be less than 3 per cent.?

8. P is the horse-power lost by friction when a disc D feet in diameter revolves n times per minute in an atmosphere of steam of the absolute pressure p lb. per sq. inch.

$$P=10^{-13}pD^5n^3.$$

Calculate P for the following values of p, D, and n:

D	n	p	Answers P.
5	1000	15	4·7
5	1000	1	0·3
1	1000	15	0·0015
1	1000	1	0·0001
5	500	15	0·6
5	500	1	0·04

9. A point A is h_0 feet above sea-level. A body is projected upwards from A with a velocity of v_0 feet per second, its height h above sea-level t seconds after starting and its velocity upwards are

$$h=h_0+v_0t-\tfrac{1}{2}gt^2,$$
$$v=v_0-gt.$$

If $h_0=100$ and $v_0=80$ and $g=32{\cdot}2$, calculate h and v for the following values of t. Here are the answers:

t	0	0·5	1	2	2·484	3	4·0	4·968
h	100	135·98	163·9	195·6	199·4	195·1	162·4	100
v	80	63·9	47·8	15·6	0	−16·6	−48·8	−80

Besides having the above vertical velocity, the body moves so that its horizontal distance x from the point A is ut; take $u=100$ and calculate x for the above values of t. Later, plot x and h on squared paper to get the real path of the body.

10. In a manufacturer's list of Thomson turbines, if P is the total horse-power of the waterfall, H the height of the fall in feet, n the revolutions per minute, R the outer radius of the wheel,

$$n=22{\cdot}75H^{\frac{5}{4}}P^{-\frac{1}{2}} \quad \text{and} \quad R=2{\cdot}373P^{\frac{1}{2}}H^{-\frac{3}{4}},$$

find n and R for the following values of P and H:

P	H	n	R
74	250	2629	0·3247
50	100	1017	0·5306
50	50	427·8	0·8924
70	25	152·0	1·7758

Other exercises of the above type will be found at the end of Chapter V.

CHAPTER V.

ALGEBRA.

18. Just as it is quite easy for a beginner to learn to calculate from a formula, so it is usually easy for him to do many other useful things **with formulæ.**

Elementary Algebra is made difficult by the mere statement of rules. Why should any fuss be made over addition, subtraction, and multiplication? Why, anybody who has used a formula with brackets knows these things already.

Tell a boy about ghosts, and the simplest things become complex and mysterious. Tell a boy that he is sure to find difficulty in simple Algebra, and of course he finds great difficulty with a problem that would be quite easy if you told him that it was easy. I would give a boy **simple equation** work at once, and especially **problems** which are solved by simple equations, for there is not much that makes a boy think for himself so quickly as such work. I know that the average boy will learn quite rapidly and understand well, simple equations in x, and also in x and y, and the problems leading to them.

He ought to be able to state in words the meaning of an algebraic formula if he has done such exercises as I have given in Chapter IV., and he ought to write out algebraically any rule that is stated to him in words. Let him practise this a little, and he will presently laugh at the difficulties which teachers find in making their students work problems. At the same time, simple problems will give him an interest in the work. I say, problems leading to easy equations. Complex-looking equations seldom arise in practical work, the exercises leading to them have been created by our examination systems. They are generally tricky, and are meant to test if a boy can work very rapidly without making mistakes, but really for the working of a practical question a man has plenty of time. Easy exercises like the following ought to be set:

Ex. 1. Divide 8·37 into two parts, one of which is 2·4 times the other. Here the two parts may be taken to be x and $8·37-x$. Let us say then that

$$8·37 - x = 2·4x \quad \text{or} \quad 3·4x = 8·37 \quad \text{or} \quad x = 2·4618$$
$$\text{and} \quad 8·37 - 2·4618 = 5·9082.$$

Ex. 2. Find two numbers whose sum is 22·5, and whose difference is 5·6. Let x be one of them and $22·5-x$ the other. Then $22·5-x-x=5·6$ or $x=8·45$.

Ex. 3. A father is 3·5 times as old as his son; in 20 years he will only be twice as old. What are their ages now?

Let x be the son's age, the father's is $3·5x$. In 20 years their ages will be $3·5x+20$ and $x+20$. The question states that

$$3·5x + 20 = 2(x + 20).$$

From which we find $x = 13·33$ years; this is the son's age. The father's age is $3·5 \times 13·33$ or 46·67 years.

Ex. 4. Divide a number 8·32 into two parts such that 5 times one part, subtracted from 6 times the other, leaves 1·56. Let x be one part, $8·32-x$ is the other. Then

$$6x - 5(8·32 - x) = 1·56,$$
$$6x - 41·6 + 5x \quad = 1·56,$$
$$11x = 43·16,$$
$$x = 3·924.$$

The parts are 3·924 and 4·396.

Sometimes a problem may be solved more easily if we use **simultaneous equations.** Take Ex. 2 above. Let the numbers be x and y, and the question tells us that

$$x + y = 22·5,$$
$$x - y = 5·6.$$

Adding these equations, we get $2x$, or, subtracting them, we get $2y$.

Many problems lead to **quadratic equations**; that is, equations of the form $\qquad x^2 + px + q = 0. \qquad$ ....(1)

The rule for working this is;—write the equation as

$$x^2 + px = -q.$$

To both sides add the square of half p or $\frac{p^2}{4}$,

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q.$$

This makes the left-hand side a perfect square.

Extract the square root of each side; and because when we extract the square root of, say 9, the answer is either $+3$ or -3, we write it ± 3. Hence

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q},$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}. \quad \text{.........................} (2)$$

There are, then, two answers, or roots as they are called, to a quadratic equation.

If we split the expression $x^2 + px + q$ in (1) into **factors**, let us call them $x - a$ and $x - b$, so that (1) is

$$(x-a)(x-b) = 0.$$

Now, how can this expression be 0? It is 0 if $x - a = 0$ or if $x = a$; and it is 0 if $x - b = 0$ or $x = b$.

If we see an expression like $x^2 + 11x + 30$, and if we are a little familiar with formulæ, we know that its factors are $x + 6$ and $x + 5$, because our eyes get accustomed to the idea that 30 is the product of 6 and 5, and 11 is the sum of 6 and 5. In fact

$$(x-a)(x-b) = x^2 - (a+b)x + ab.$$

Or if $(x-a)(x-b)$ is to be the same as $x^2 + px + q$, then q is ab and $-p$ is $a + b$.

In many cases mere inspection does not help us, but the above reasoning tells us what to do. We have to find the factors of

$$x^2 + px + q.$$

Put it equal to 0 and find the roots as in (2). If these roots are a and b, then $x - a$ and $x - b$ are the factors.

Example. Find the factors of $x^2 - 3{\cdot}8x - 11{\cdot}6$. Put it equal to 0 and find the roots, either by the method which led us to rule (2) or using (2) as a formula. In this case $p = -3{\cdot}8$, $q = -11{\cdot}6$. The roots are then $1{\cdot}9 \pm \sqrt{1{\cdot}9^2 + 11{\cdot}6} = 1{\cdot}9 \pm 3{\cdot}9$ or $5{\cdot}8$ and -2. The expression is then the same as

$$(x - 5{\cdot}8)(x + 2).$$

In solving quadratics we sometimes find in our answer such an expression as $\sqrt{-25}$, which is called an impossible or an **unreal quantity**. Although we cannot at present attach a meaning to such a thing, it is good algebra to work such questions, giving the unreal answers. To save trouble we write $\sqrt{-25}$ as $\sqrt{25} \times \sqrt{-1}$; we often use the letter i for $\sqrt{-1}$, so that $\sqrt{-25}$ is $5i$.

Solve the quadratic $x^2 - 4x + 5 \cdot 44 = 0$.

We find $x = 2 \pm 1 \cdot 2i$ as the two roots.

In Chapters XXVIII. to XXXVI. the student will find that we can make an important practical use of unreal quantities.

When the student gets an equation to solve, say $f(x) = 0$, he will now know that if, by guessing or by the use of squared paper or in any other way, he has found the answer, say $x = a$, then $x - a$ must be a factor of $f(x)$. If now he divides $f(x)$ by $x - a$, he may get a simpler equation to solve; if he finds $x = b$, then $x - b$ must be a factor of $f(x)$. If he again divides by $x - b$, he may get a still simpler equation, and so on.

Example. Solve the equation $x^3 + 5 \cdot 1x^2 + 2 \cdot 9x - 9 = 0$. Let us suppose that by guessing, or by the use of squared paper or in some other way, the student finds that $x = 1$ is a root of the equation. Now let him divide by $x - 1$, and he finds $x^2 + 6 \cdot 1x + 9 = 0$. The roots of this quadratic are $x = -3 \cdot 6$ and $x = -2 \cdot 5$. Therefore the three roots of the given equation are 1, $-3 \cdot 6$, and $-2 \cdot 5$.

A little, and only a very little, work on **simultaneous quadratics** ought to be given.

Example. $x^2 + y^2 = 37 \cdot 13$, $x + y = 8 \cdot 40$.

Squaring the second equation and subtracting the first, we get $2xy = 33 \cdot 43$. Subtracting this from the first equation and extracting the square root, we get $x - y = \pm 1 \cdot 92$. Adding this to the second equation and dividing by 2, we get $x = 5 \cdot 16$, $y = 3 \cdot 24$, and also $x = 3 \cdot 24$, $y = 5 \cdot 16$. Hence if the question had been:—Find two numbers such that their sum is 8·40 and the sum of their squares 37·13, our answer is 5·16 and 3·24.

Example. If $x^2 + y^2$ is given and also xy, we have only to add and subtract $2xy$, and in each case to extract the square to get $x + y$ and $x - y$.

Example. $x^3 - y^3 = 37$ and $x - y = 1$. Divide, and we have $x^2 + xy + y^2 = 37$, also $x^2 - 2xy + y^2 = 1$. Subtracting, $3xy = 36$ or $xy = 12$. Hence $x^2 + 2xy + y^2 = 49$ or $x + y = \pm 7$. Hence $x = 4$, $y = 3$ or $x = -3$, $y = -4$.

In the course of the above work, and not as a separate kind of work, the student will simplify expressions; he will intuitively recognise the factors of such expressions as

$$x^2 - 144, \quad x^2 - 5x - 66, \quad \text{or} \quad x^2 + 7x + 12.$$

He will multiply and divide without making any fuss about the matter, as if he were dealing with new and independent subjects.

In such multiplication and division he ought to try to remember the answers to $(a+x)^n$ when n is 3, 4, and 5;

also the series which are answers to $\frac{1}{1-x}$ and $\frac{1}{1+x}$.

Where the greatest error is made in teaching is in not introducing to a student, quite early in all this work, as a sort of relief work, the **plotting of functions** by means of squared paper. For if he takes any function of x and calls it y, and for any value of x calculates y and plots the corresponding values of x and y on the squared paper, he gets a curve. If he now notes the value of x which makes the function 0, he gets in the simplest fashion the very best knowledge of the solution of an equation, of the root, of the roots of an equation, and he can and will philosophise on the subject without any prompting from his teacher.

I must confess, however, that the compilers of modern school algebras must make the gods laugh over the uses to which they put this plotting of functions.

Just here it is well to point out that when we have a formula of several algebraic quantities, it is often easy to express any one of the quantities in terms of the others so as to make many useful calculations. In fact, then, I would ask teachers to mix together as one simple kind of algebraic work, easy to give, even to beginners, many parts of Algebra which are usually taken up very much later, so much later that the average student never reaches them in his study.

It will be seen, then, that I include in such work all sorts of work which goes under the name of "Rules in Arithmetic."

The following are only a few of the many hundreds of examples that may well be put before students:

EXERCISES.

1. If $Q=0{\cdot}00545d^2ln$, find d, having given $Q=4800$, $l=2$, $n=125$. *Ans.* 59·36.

2. If $y=wl^4/384EI$, where $w=\frac{W}{l}$ and $I=\frac{bd^3}{12}$, find w, I, and y if $W=3{\cdot}5\times 2240$, $l=147$, $b=3$, $d=9$, $E=1{\cdot}1\times 10^6$.
Ans. $w=53{\cdot}33$, $I=182{\cdot}25$, $y=0{\cdot}3234$.

3. The energy stored in similar flywheels is $E=ad^5n^2$, when d is the diameter in feet, n the speed in revolutions per minute, and a is a constant. If when $d=5$ and $n=100$, E is 18,500, find a for that kind of flywheel. Find d for a similar flywheel if E increases by 10,000 when n increases from 149 to 151. *Ans.* $a=0{\cdot}000592$, $d=7{\cdot}761$.

4. If $xy^n=a$; if x is 5 when y is 10 and also if x is 11 when y is 8, find n and a. What is the value of y when x is 7?

Ans. $n=3{\cdot}533$, $a=17060$, $9{\cdot}09$.

5. In a miner's handbook the following formula is given for the thickness of a cylindrical dam:

$$t=r(1-\sqrt{1-20p/T}).$$

If $r=144$, $T=1{\cdot}075\times10^5$ and $144p=20\times62{\cdot}5$, find t. *Ans.* 0·116.

6. If $x/y=e^{a\theta}$, where $e=2{\cdot}718$; if $a=0{\cdot}3$ and $\theta=2{\cdot}85$, and if $x-y=550$, find x. *Ans.* 957·0.

7. When x and y are small, we may take $(1+x)/(1+y)$ as being very nearly equal to $1+x-y$. What is the error in this when $x=0{\cdot}02$ and $y=0{\cdot}03$? *Ans.* 0·000291 or 0·0291 per cent.

8. $y=ax^2+bx^3$; when x is 1, y is 4·3, and when x is 2, y is 30. Find a and b. What is y when x is 1·5? *Ans.* $y=1{\cdot}1x^2+3{\cdot}2x^3$, 13·275.

9. If
$$\log_e\frac{853}{493}+0{\cdot}9\times\frac{836}{853}=\log_e\frac{677}{493}+\frac{961}{677}x,$$
find the value of x to three significant figures. *Ans.* 0·784.

10. Use the formula $\tan\frac{A+B}{2}\div\tan\frac{A-B}{2}=\frac{a+b}{a-b}$ to find the angles A and B of the triangle ABC, given that $C=29^\circ{\cdot}7$, $a=86{\cdot}92$, $b=54{\cdot}68$.

Ans. 115°·5, 34°·8.

11. If $q=\frac{t}{l}\left(\frac{q_1l_1}{t_1}+\log_e\frac{t_1}{t}\right)$; if $t=\theta+461$ and $t_1=\theta_1+461$; if $l=H-h$ and $l_1=H_1-h_1$; if $\theta=230$ and $\theta_1=338$; if $H=1152{\cdot}1$ and $h=198{\cdot}7$; if $H_1=1185{\cdot}0$ and $h_1=308{\cdot}7$, and if $q_1=1$, find the value of q. *Ans.* 0·9.

12. In a **thick cylinder** subjected to fluid pressure the radial compressive stress p and the hoop tensile stress f at a point which is at the distance r from the axis are such that

$$p=\frac{b}{r^2}-a, \quad \text{.....(1)}$$

$$f=\frac{b}{r^2}+a, \quad \text{.....(2)}$$

where a and b are constants.

(a) The internal pressure p_1 is 3 tons per sq. in., the inside radius r_1 being 5 inches; the outside pressure p_0 is 0, the outside radius r_0 being 10 inches. Find a and b in the formulæ (1) and (2), and calculate p and f for $r=5$, 6, 7, 8, 9, and 10.

Here
$$3=\frac{b}{25}-a,$$
$$0=\frac{b}{100}-a.$$

Subtracting, we find $3={\cdot}03b$ or $b=100$; therefore $a=1$.

(1) and (2) become $p=\frac{100}{r^2}-1$, $f=\frac{100}{r^2}+1$.

r	p	f
5	3	5
6	1·778	3·778
7	1·041	3·041
8	0·5625	2·5625
9	0·2346	2·2346
10	0	2

Later, the curves for p and f ought to be plotted on squared paper.
Instead of taking numbers for p_1 and r_1 and r_0, let the student now repeat the above work using these letters, giving p and f in formulæ applicable to all such cases. Take $p_0=0$.

(β) A gun tube $r_0=15$, $r_1=13$ is to have $p_0=0, f_1=20$; find p_1 and f_0.

$$0=\frac{b}{225}-a=0·004444b-a,$$

$$20=\frac{b}{169}+a=0·005917b+a.$$

Adding, we get $20=0·010361b$ or $b=1930·3$, $a=8·5783$,

$$p_1=\frac{1930·6}{169}-8·58=2·84,$$

$$f_0=\frac{1930·3}{225}+8·578=17·16.$$

The student ought to give general formulæ for this case also.

(γ) **A gun tube** $r_0=13$, $r_1=11$ is to have $p_0=2·84, f_1=20$; find p_1 and f_0.
Ans. $p_1=6·62$, $f_0=16·22$.

(δ) A cylinder, whose r_1 is 5 inches and r_0 is 10 inches, has an inside $f_1=10000$ lb. per sq. inch. If p_0 is 0, what is p_1? What is f everywhere? what is p everywhere?
Ans. $0=-a+\frac{b}{100}$ and $10000=a+\frac{b}{25}$.

We find from these equations $a=2000$, $b=200000$, and hence

$$f=2000+\frac{200000}{r^2} \text{ everywhere.}$$

The inside p is 6000 lb. per sq. inch,

$$p=-2000+\frac{200000}{r^2} \text{ everywhere.}$$

Show f and r and also p and r in two curves.

19. I want a student to practise using all sorts of formulæ, so that he shall cease to be afraid when he sees one. Of course, there may be some bit of shorthand, some symbol, which has not been explained yet to him, but he ought to know that there is nothing magical or uncanny about it. I have taken up some formulæ at random. I might call any one of them *a rule*, and so create difficulty, but indeed there is only one way with them all.

The average man who has worked through many rules in complex arithmetic, and algebra, and engineering, very quickly forgets them all, except the one or two that he constantly needs. It is only a teacher who remembers hundreds of rules. But if at the beginning a man knows that his rules are **all one rule**; all his separate rules are mere examples of one general principle; he never can forget it, for every common-sense calculation that he makes only fixes the general principle more firmly in his mind.

Have you not noticed that a great man has only a few simple principles on which to regulate all his actions? A great engineer keeps in his head just a few simple methods of calculation. But note that, through constant practice, the simple principles or methods are always ready for use in his mind. It may be that an expert may be quicker or neater in working some one kind of problem, but however clumsy or tedious may be the great man's method of working the problem, he gets the right answer, and he has no misgiving as he writes it out.

My one simple rule is to treat all numerical calculation as work with some formula, and all rules ought to be compactly stated as formulæ.

Of course, it is another thing to work out such formulæ, to prove them correct, and yet I say that, even here, my general principle introduces enormous ease, for, in most cases, to understand and feel unafraid of a formula is almost to see the proof of it.

20. Notice that if we have any equation, say $M=N$, we can say that $\log M=\log N$ or $M^n=N^n$, where n is any number. Again, if $x^n=y^m$ we may say that $n\log x=m\log y$.

Ex. 1. If $pu^{1 \cdot 0646}=479$, when $u=4$, find p.
Here $\log p+1 \cdot 0646 \log 4=\log 479$, and we find $p=109 \cdot 5$.

Ex. 2. Again, when $p=203$, find u. *Ans.* 2·24.

Ex. 3. Suppose $pv^{1 \cdot 13}=a$.

(α) If $p=100$ and $v=1$, find a. *Ans.* 100.
If v becomes 1·5, using the same value of a, find p. *Ans.* 63·24.

(β) If v becomes 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, in each case find the corresponding value of p.

Repeat the above work when $pv^{0 \cdot 9}=a$, taking $p=100$, and $v=1$ to start with.

(γ) Repeat again when $pv=a$.
Show the three sets of answers for p in a table.

Ex. 4. If $p_1 v_1^{1\cdot13} = p_2 v_2^{1\cdot13}$ and if v_2/v_1 be called r; if $p_2=6$, find r for the following values of p_1:

p_1	250	200	150	100	50
Answers, r	27·13	22·27	17·26	12·06	6·53

21. Proportion. The Ratio of a to b or $a:b$ is $a \div b$. Thus the ratio of 2 to 3 is 0·6667.

When we say that y varies as, or is proportional to x, we write the statement in the form $y \propto x$, and we know that this is the same statement as $y=ax$, where a is some constant number.

Simple Proportion. If $y \propto x$ and if $y=4$ when $x=3$, then as $y=ax$, $4=a\times 3$, or $a=\frac{4}{3}$, and hence $y=\frac{4}{3}x$ is the true law connecting y and x. If any value of x be now given, y may be found.

Compound Proportion. If 10 labourers dig 150 yards of trench in 5 days of 12 hours; how many labourers will dig 356 yards in 2 days of 9 hours? If we use l, y, d, and h, we see that the assumption made is

$$l \propto \frac{y}{dh} \quad \text{or} \quad l = a\frac{y}{dh}.$$

Hence $$10 = a\frac{150}{5\times 12}, \text{ so that } a = \frac{5\times 12\times 10}{150} = 4.$$

So that we have the formula $l=4\frac{y}{dh}$ for working any exercise. Thus the answer wanted is

$$l = 4\frac{356}{2\times 9} = 79\cdot 11 \text{ labourers.}$$

22. Now notice that all the following exercises are worked in the same way as those of Art. 21.

Ex. 1. In **steam vessels** of the same character, I being indicated horse-power, D displacement, v speed; $I \propto D^{\frac{2}{3}} v^3$:

(α) If $I=655$ when $D=1720$ tons and $v=10$ knots, find the exact rule. *Ans.* $I=D^{\frac{2}{3}}v^3 \div 219$.

(β) If D is 1500, $v=13$, find I. *Ans.* 1314.

(γ) If $I=800$, $D=1300$, find v. *Ans.* 11·37 knots.

At the highest speeds of modern ships, where the wave-making resistance is much more important than mere skin or eddy-making resistance, we may take Froude's law to be true for the whole resistance, and say that, at corresponding speeds (that is, speeds proportional to the square roots of lengths of ships), the indicated power being I and displacement D, $I = c\ D^{\frac{7}{6}}$, where c is a constant. Thus if a cross-Atlantic liner of 20 knots, $D = 10000$, $I = 20000$, crosses in 6 days;

If we want to cross in 5 days we must have $v = 24$ knots, and as $v \propto \sqrt{\text{length}} \propto D^{\frac{1}{6}}$, D must be increased to $10000 \times (\frac{6}{5})^6 = 29860$ tons and I to $20000\ (\frac{6}{5})^7 = 71660$.

The total coal used in passage $\propto D$, that is, it is always the same fraction of the displacement.

Ex. 2. The weights W of **similar objects** are as the cubes of their like dimensions and as the densities of their materials. Two similar objects are in the linear proportions of 1 to 4·37, and their densities are as 2 to 1·74. The weight of the first is 20 lbs. What is the weight of the second? *Ans.* $W = 1452$ lbs.

Ex. 3. If d is the calibre of a **gun**, it is usually assumed that the weight is proportional to d^3, and that the thickness of armour which its projectile will pierce is proportional to d. If an 8-inch gun weighs 14 tons, and can pierce 11 inches of armour, what thickness will be pierced by a 10-inch gun, and what is the weight of the gun? *Ans.* 13·75 inches; 27·34 tons.

Ex. 4. Two models of terrestrial globes are 1·22 feet and 3·14 feet diameter respectively. If the area of a country is 15 square inches on one, what is it on the other? *Ans.* 99·32 square inches.

Ex. 5. If $Q \propto h^{\frac{5}{2}}$ and if $Q = 0{\cdot}466$ when $h = 0{\cdot}5$, find the true law. *Ans.* $Q = 2{\cdot}636\ h^{\frac{5}{2}}$.

(α) Find Q when $h = 1{\cdot}3$. *Ans.* $Q = 5{\cdot}079$.

(β) Find h when $Q = 2{\cdot}46$. *Ans.* $h = 0{\cdot}9727$.

Ex. 6. If we know that $y = a + bx$, where a and b are constant numbers:

(α) If $y = 12$ when $x = 1$, and if $y = 15$ when $x = 5$, find a and b.

Here we have $\quad 12 = a + 1b, \quad 15 = a + 5b.$

Subtracting, we have

$$3 = 4b \text{ or } b = \tfrac{3}{4}, \quad 12 = a + \tfrac{3}{4} \text{ or } a = 11\tfrac{1}{4}.$$

Hence the law is really $\quad y = 11\frac{1}{4} + \frac{3}{4}x.$

(β) Find y if $x = 4$. *Ans.* $14\frac{1}{4}$.

(γ) Find x when $y = 20$. *Ans.* $11\frac{2}{3}$.

Ex. 7. If $y = ax + bz^{\frac{2}{3}}x^2$,

if $y = 49{\cdot}5$ when $x = 1$ and $z = 8$,

and $y = 356$ when $x = 1{\cdot}5$ and $z = 20$,

find a and b, and find the value of y when x is 2 and z is 17.

Ans. $y = -57{\cdot}1x + 26{\cdot}65z^{\frac{2}{3}}x^2$; 590·7.

23. Those parts of arithmetic called "Equation of Payments," "Barter," "Profit and Loss," "Fellowship," "Alligation" of many kinds, "Position," "Double Position," "Conjoined Proportion," and many others, are, when we strip them of their technical terms and artificial complexities, the simplest of algebraic exercises, and they ought to be treated as such.

24. Arithmetical Progression. It is easy to show that

$$l=f+(n-1)d \quad \text{and} \quad s=\tfrac{1}{2}n(f+l),*$$

if there are n terms with common difference d (that is, any term minus the preceding term), if their sum is s and if f is the first and l the last term.

It is, therefore, very easy to find any two of these when the other three are given.

(1) Find the 15^{th} term and the sum of 15 terms of 0·25, 0·50, 0·75, etc. *Ans.* 3·75, 30.

(2) What is the first term when the 59^{th} term is 70 and the 66^{th} term is 84? *Ans.* -46.

(3) Insert 4 arithmetic means between 3 and 18.
Ans. 6, 9, 12, 15.

25. Geometrical Progression. If f, l, s, n, are as in the last case; if r is the common ratio (that is, any term divided by the preceding term), it is easy to prove that

$$l=fr^{n-1}, \quad s=f\frac{r^n-1}{r-1}.\dagger$$

(1) Find the 10^{th} term and the sum of 10 terms of 2, 6, 18, etc.
Ans. 39366, 59048.

(2) Insert 3 geometric means between 3 and 243. *Ans.* 9, 27, 81.

* The student who writes the series down algebraically, f, $f+d$, $f+2d$, $f+3d$, etc., sees that the 2^{nd} term has d, the 3^{rd} term has $2d$, the 4^{th} term has $3d$, and so the 40^{th} term would have $39d$ and the n^{th} term would have $(n-1)d$.

Again the sum of the first and last terms is the same as the sum of the 2^{nd} and the 2^{nd} to last, of the 3^{rd} and the 3^{rd} to last, and so it is easy to write the sum of all the terms in the shape above given.

† The geometric series is evidently

$$f,\ fr,\ fr^2,\ fr^3 \ldots fr^{n-1} \text{ if there are } n \text{ terms.}$$

Now, by mere division, it is easy to show that

$$\frac{1-r^n}{1-r}=1+r+r^2+\text{etc.}+r^{n-1},$$

and the sum of the series is evidently this multiplied by f.

(3) The 6th term of a geometric series is 20·34, r is 0·26; find the first term and the sum of the 6 terms.

Ans. $1·712 \times 10^4$, $2·313 \times 10^4$.

(4) The sum of a geometric series of 5 terms is 534, r is 1·65; find the first and last terms. *Ans.* $f = 30·9$, $l = 229·1$.

26. Compound Interest. It is very easy to prove that if a sum of money P (the principal) is lent at compound interest at r per cent. per annum, it amounts in n years to

$$A = P\left(1 + \frac{r}{100}\right)^n,$$

and the increase $I = A - P$ may be called the *interest.**

Ex. 1. If $P = 255·75$, $r = 3\frac{1}{2}$, $n = 17$, find A. *Ans.* 459.

Ex. 2. If $A = 930$, $r = 4\frac{1}{2}$, $n = 10\frac{1}{2}$, find P. *Ans.* 585·82.

Ex. 3. If $P = 320$, $I = 456$, $r = 5$, find n. *Ans.* 18·156.

Ex. 4. If $P = 250$, $A = 420$, $n = 16$, find r. *Ans.* 3·3.

Ex. 5. $P =$ one farthing, $n = 1900$, $r = 5$; find A.

Ans. 2×10^{37} pounds nearly.

Ex. 6. Find n if $A = 2P$. *Ans.* $n = \log 2 \div \log\left(1 + \frac{r}{100}\right)$.

Later we shall find by squared paper that $n = 70 \div r$ is a good approximation, much used by practical men.

Ex. 7. If interest is added on m times a year, prove that

$$A = P\left(1 + \frac{r}{100m}\right)^{nm}.$$

Ex. 8. If interest is added every moment, $A = Pe^{nr/100}$.

Ex. 9. The population of England and Wales doubles itself every 50 years; what is the rate per cent. per annum of increase?

$$A = 2P, \; n = 50, \text{ and hence } r = 1·396.$$

If $A = 2P$ we see that $nr = 69·8$.

27. Present Value and Discount. Use the formula of Art. 26 if the sum A is due at the end of n years and P is its present value. Call $A - P$ the discount.

* Thus, at 5 per cent. per annum, the interest on a sum of money P for one year is $·05P$, so that if this interest is added to P the amount is $1·05P$. In fact a sum of money P at the beginning of a year becomes $1·05P$ at the end of the year. Therefore, at the end of the second year, the amount is $1·05^2P$, and at the end of the third year it is $1·05^3P$, and at the end of the n^{th} year it is $1·05^nP$.

It is easy to prove that an **annuity** of a per annum, payable for n years, has a present value P, or will amount to A if the use of money is worth r per cent. per annum, if

$$\frac{rA}{100a}=\left(1+\frac{r}{100}\right)^n-1, \text{ or if } \frac{rP}{100a}=1-\left(1+\frac{r}{100}\right)^{-n}.$$

One of these formulæ is enough if it is remembered that the connection between A and P is what was given in the rule for compound interest.

28. General Exercises on Formulæ. Students are asked to take up numerical exercises on all these. Such exercises are to be found in many books. But indeed almost any pocket book formula serves for the creation of good exercises.

Above all, I would impress upon teachers that they should introduce no artificial difficulties, no tricks, no conundrums into algebraic work. Let every exercise be straightforward and as easy as possible.

EXERCISES.

1. **Slipping of Belt on Pulley.** If N is the tension on the tight side, M on the slack side, θ radians the angle of lapping, μ the coefficient of friction between belt and pulley, then $N/M=e^{\mu\theta}$.

(1) If $\mu=0·3$, find N/M for the following values of θ. What is θ in degrees if 180 degrees are 3·14159 radians?

(2) Find also $N/(N-M)$.*

θ radians.	θ degrees.	N/M.	$N/(N-M)$.
1	57·30	1·35	3·86
2	114·59	1·823	2·21
3	171·88	2·46	1·69
4	229·18	3·320	1·43
5	286·48	4·482	1·29
6	343·78	6·050	1·20
7	401·07	8·167	1·14
8	458·37	11·03	1·10
9	515·66	14·88	1·07
10	572·96	20·09	1·05

* By Algebra we find that if $\frac{N}{M}=a$, $\frac{N}{N-M}=\frac{a}{a-1}$. Thus, if $\frac{N}{M}=1·35$, $\frac{N}{N-M}=\frac{1·35}{0·35}=3·86$.

2. **Compound Interest.** We have the formula

$$A = P\left(1 + \frac{r}{100}\right)^n.$$

The money P is lent at r per cent. per annum, and in n years A is its amount. The interest I is $A - P$.

(1) Calculate A/P, which is $\left(1+\frac{r}{100}\right)^n$ when $r=5$ or 4 or 3 or 2 or 1 for the following values of n:

n	Answers.				
	$r=5.$	$r=4.$	$r=3.$	$r=2.$	$r=1.$
10	1·629	1·48	1·344	1·219	1·105
50	11·47	7·107	4·384	2·692	1·645
100	$1·315 \times 10^2$	$5·050 \times 10$	19·22	7·245	2·705
200	$1·729 \times 10^4$	$2·551 \times 10^3$	$3·694 \times 10^2$	$5·249 \times 10$	7·316
400	$2·99 \times 10^8$	$6·506 \times 10^6$	$1·364 \times 10^5$	$2·755 \times 10^3$	$5·353 \times 10$
1000	$1·546 \times 10^{21}$	$1·08 \times 10^{17}$	$6·87 \times 10^{12}$	$3·983 \times 10^8$	$2·096 \times 10^4$
1500	$6·08 \times 10^{31}$	$3·543 \times 10^{25}$	$1·80 \times 10^{19}$	$7·949 \times 10^{12}$	$3·035 \times 10^6$
2000	$2·39 \times 10^{42}$	$1·166 \times 10^{34}$	$4·725 \times 10^{25}$	$1·585 \times 10^{17}$	$4·393 \times 10^8$

(2) If $A/P=2$, find n for the following values of r. This is the number of years in which a sum of money will double itself.

Values of r	1	2	2·5	3	3·5	4	4·5	5
Answers, n	69·66	35·0	28·07	23·45	20·15	17·67	15·75	14·21

The student will have seen that $n = 0·3010 \div \log\left(1+\frac{r}{100}\right)$.

(3) Let us take it that a sum of money will double itself in 14 years. Suppose that a glass of port was worth sixpence in the year 1800. At that time it was placed in a cellar and was not drunk till 1912; what is now its value? 112 years have lapsed, or eight times 14 years. So that the sixpence must be doubled eight times. *Ans.* 128 shillings.

3. Exercises, using the Tables, Chap. VIII.

(1)
$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B,$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Using $A = 55°$ and $B = 32°$, test these formulæ.
" $A = 50°$ " $B = 12°$ " "
Write out what the formulæ become if A is 90°.
" " " B is 90°.
" " " $A = B$.

(2)
$$\sin A \cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\},$$
$$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\},$$
$$\cos A \cos B = \tfrac{1}{2}\{\cos(A+B) + \cos(A-B)\}.$$

Using $A=52°$, $B=15°$, test these formulæ.
$A=28°$, $B=12°$ „ „

Write out what the formulæ become if $A=90°$.
„ „ „ $B=90°$.
„ „ „ $A=B$.

(3)
$$\sin\theta+\sin\phi=2\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi),$$
$$\sin\theta-\sin\phi=2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi),$$
$$\cos\theta+\cos\phi=2\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi),$$
$$\cos\theta-\cos\phi=2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\phi-\theta).$$

Illustrate these rules when $\theta=67°$ and $\phi=50°$; also when $\theta=60°$, $\phi=30°$. Also when $\theta=30°$, $\phi=60°$.

4. Exercises on $y=e^{bx}$; find its value for the following values of x. The student already knows that e is 2·71828, the base of the Napierian system of logarithms. In this case bx is $\log_e y$.

Given Values of x.	Answers. Values of e^{bx}.					
	When b is 4.	When b is 1.	When b is $\frac{1}{4}$.	When b is $-\frac{1}{4}$.	When b is -1.	When b is -4.
−1·1	0·01227	0·3329	0·7596	1·317	3·004	81·45
−1·0	0·01831	0·3679	0·7787	1·284	2·718	54·61
−0·9	0·02732	0·4065	0·7985	1·252	2·460	36·60
−0·8	0·04076	0·4493	0·8187	1·222	2·225	24·53
−0·7	0·06081	0·4965	0·8395	1·191	2·014	16·44
−0·6	0·09071	0·5488	0·8607	1·162	1·823	11·03
−0·5	0·1353	0·6065	0·8824	1·133	1·649	7·389
−0·4	0·2018	0·6703	0·9047	1·105	1·491	4·953
−0·3	0·3011	0·7408	0·9277	1·077	1·350	3·321
−0·2	0·4492	0·8188	0·9512	1·052	1·221	2·225
−0·1	0·6704	0·9047	0·9753	1·025	1·105	1·491
·0	1·000	1·000	1·000	1·000	1·000	1·000
0·1	1·491	1·105	1·025	0·9753	0·9047	0·6704
0·2	2·225	1·221	1·052	0·9512	0·8188	0·4492
0·3	3·321	1·350	1·077	0·9277	0·7408	0·3011
0·4	4·953	1·491	1·105	0·9047	0·6703	0·2018
0·5	7·389	1·649	1·133	0·8824	0·6065	0·1353
0·6	11·03	1·823	1·162	0·8607	0·5488	0·09071
0·7	16·44	2·014	1·191	0·8395	0·4965	0·06081
0·8	24·53	2·225	1·222	0·8187	0·4493	0·04076
0·9	36·60	2·460	1·252	0·7985	0·4065	0·02732
1·0	54·61	2·718	1·284	0·7787	0·3679	0·01831
1·1	81·45	3·004	1·317	0·7596	0·3329	0·01227

Only a few of these exercises ought to be given to any one student.

5. The **Binomial Theorem** is easy to prove. It is a pity that students are usually subjected to an unnecessary course in Permutations and Combinations before this proof is given to them. It is on account of this that many artificial and absurd exercises are given, such as "What is the r^{th} term?" or "Show that the r^{th} term from the beginning is equal to the r^{th} term from the end."

The theorem is :—For all values of x, a, and n,

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor\underline{2}}a^2x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor\underline{3}}a^3x^{n-3}$$

$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{4}}a^4x^{n-4} + \text{etc.}$$

Here $\lfloor\underline{4}$ means $1 \times 2 \times 3 \times 4$. Recently the symbol $4!$ has been used instead of $\lfloor\underline{4}$.

(1) Write out the theorem for the cases $n=2$, $n=3$, $n=4$.

Ans. $(x+a)^2 = x^2 + 2ax + a^2$.
„ $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$.
„ $(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$.

(2) Show that $(1+a)^{-1} = 1 - a + a^2 - a^3 + a^4 - \text{etc.}$,
and that $(1-a)^{-1} = 1 + a + a^2 + a^3 + a^4 + \text{etc.}$,
and test these by dividing 1 by $1+a$ or by $1-a$.

(3) Calculate $(1+a)^n$ for the following cases :—Where $n = 2, 3\,;\ -1,\ -2,\ -3\,;\ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\,;\ -\frac{1}{2}, -\frac{1}{3}$; letting $a = {\cdot}01$, ${\cdot}001$, $-{\cdot}01$, $-{\cdot}001$.

Only four significant figures are needed in the answers.

Let $1+a$ be called y.

y	Answers.									
	y^2	y^3	y^{-1}	y^{-2}	y^{-3}	$y^{\frac{1}{2}}$	$y^{\frac{1}{3}}$	$y^{\frac{1}{4}}$	$y^{-\frac{1}{2}}$	$y^{-\frac{1}{3}}$
1·01	1·020	1·030	0·9902	0·9804	0·9709	1·005	1·003	1·003	0·9955	0·9967
1·001	1·002	1·003	0·9991	0·9782	0·9983	1·001	1·000	1·0005	0·9991	0·9997
0·99	0·9801	0·9703	1·010	1·020	1·031	0·9950	0·9967	0·9975	1·005	1·00335
0·999	0·998	0·997	1·001	1·002	1·003	0·9995	0·9997	0·9998	1·001	1·0003

(4) Calculate $(1+a)^n$ in the following cases, giving more than four significant figures in the answer.

Values of y.	y^2	y^3	$\sqrt{y}$	$\sqrt[3]{y}$	$\frac{1}{y}$
1·001	1·002001	1·003003	1·0005	1·0003	0·9991
1·01	1·0201	1·0303	1·005	1·003	0·9902
1·1	1·21	1·331	1·049	1·032	0·909
1·5	2·25	3·375	1·225	1·145	0·6667
2·0	4·0	8·0	1·415	1·26	0·50

The values found in this laborious way ought to be tested by ordinary arithmetic or by using logarithms.

6. If $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc.}$, calculate e^x for the following values of x:

Values of x.	N or ϵ^x.
0·001	1·00100
0·01	1·01005
0·1	1·1052
0·2	1·2214
0·4	1·4918
0·7	2·014
1·0	2·71828
1·5	4·4817
2	7·389

x is here the Napierian logarithm of N, and when x is 1 we find the value of e itself. The symbol $\lfloor 5$ means $1 \times 2 \times 3 \times 4 \times 5$.

7. An angle α is usually supposed to be in radians. One radian is 57·30 degrees.

Using the following series, calculate $\sin \alpha$ and $\cos \alpha$ to four significant figures.

$$\sin \alpha = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \text{etc.},$$

$$\cos \alpha = 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \frac{\alpha^6}{\lfloor 6} + \text{etc.}$$

Calculate also $\tan \alpha$, which is $\sin \alpha \div \cos \alpha$.

α	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	Angle in degrees.
0·001	0·0010	1·0000	0·0010	0·0573
0·01	0·0100	0·9999	0·0100	0·573
0·1	0·0998	0·9950	0·1002	5·73
0·2	0·1987	0·9801	0·2028	11·46
0·4	0·3894	0·9207	0·4228	22·92
0·6	0·5646	0·8253	0·6843	34·38
1·0	0·8416	0·5403	1·5577	57·3
1·5	0·9975	0·0706	14·1589	85·95
2	0·9094	−0·4163	−2·1842	114·60
3	0·1411	−0·9900	−0·1423	171·9

Compare these values with those given in your tables.

To find the sine, cosine, and tangent of angles greater than 90° refer to Art. 145.

8. If $t = 273 + \theta$ and $\phi = \log_e \frac{t}{273}$, find ϕ for the following values of θ.

The more correct formula is

$$\phi^1 = 1·0565 \log_e \frac{t}{273} + 9 \times 10^{-7}\left(\frac{t^2}{2} - 503t\right) + 0·0902.$$

Calculate ϕ^1 and see what error there is in the usual way of calculating.

Given values of θ.	t	ANSWERS.	
		ϕ	ϕ^1
0	273	0	...
50	323	·168	·1683
100	373	·3118	·3134
150	423	·4375	·4415
200	473	·5494	·5570
250	523	·6497	·6628

I usually employ the symbol ϕ_w, as the above ϕ is the **entropy** of one pound of water at the temperature θ° C.

9. If
$$\phi_s = \log_e \frac{t}{273} + \frac{796 \cdot 2}{t} - 0 \cdot 695$$
and if
$$t = 273 + \theta,$$
find ϕ_s for the following values of θ.

Given values of θ	0	50	100	150	200	250
ϕ_s	2·223	1·938	1·752	1·6225	1·537	1·476

ϕ_s is the entropy of 1 lb. of steam at the temperature θ° C.

10. The work done by a perfect **Steam Engine**, working on the Rankine cycle, is, per pound of steam,
$$W = 1400\left\{(t - t_0) - t_0 \log_e \frac{t}{t_0} + L\left(1 - \frac{t_0}{t}\right)\right\},$$
where
$$L = 796 \cdot 2 - 0 \cdot 695 t.$$
Note that t is the absolute temperature of the supply steam, t_0 of the exhaust.

If $t_0 = 373$, find W for each of the following values of t. Find also $w = 1 \cdot 98 \times 10^6 \div W$ in each case: this w is the weight of steam used per hour per indicated horse-power.

Given t.	W	w
600	270,300	7·33
550	231,200	8·56
500	184,500	10·73
450	125,600	15·76
400	50,220	39·43

11. If p is the pressure in pounds per sq. inch of saturated **steam** at the temperature θ° C., there is a useful empirical formula,
$$\log_{10} p = 6 \cdot 1007 - \frac{1518}{t} - \frac{122500}{t^2} \text{ if } t = \theta + 273.$$

Find p in the following cases :

Given θ	60	100	120	140	160	170	180	190	200
Calculated p	2·88	14·70	28·83	52·52	89·86	115·1	145·8	182·4	225·9

12. If $e=1-r^{\gamma-1}$, where $\gamma=1·37$, find e for the following values of r :

r	0·4	0·3	0·25	0·2	0·17	0·14	0·12	0·10
Answers, e	0·2876	0·3594	0·4013	0·4487	0·481	0·5168	0·5435	0·5734

e is the efficiency of the hypothetical **gas or oil engine** using the Otto cycle. r is the ratio of clearance to the greatest volume.

13. If $e=1-P^{\frac{1}{\gamma}-1}$, where $\gamma=1·37$, find e for the following values of P:

P	2	6	10	14	18	22	26
Answers, e	0·1708	0·3835	0·4630	0·5095	0·5419	0·5660	0·585

e is the efficiency of the hypothetical gas or oil engine diagram using the Brayton cycle, where P is the pressure (in atmospheres) at which combustion takes place.

14. Uniform **beams** of length l, supported at the ends and loaded with W in the middle ; I the moment of inertia of cross-section about the neutral axis ; E Young's modulus of elasticity ; the deflection D at the middle is

$$D=Wl^3/48EI.$$

I for a circular section of diameter d is $\frac{\pi d^4}{64}$; for a tube whose outside diameter is d_0 and inside d_1 it is $\frac{\pi}{64}(d_0{}^4-d_1{}^4)$; for a rectangular section of breadth b and depth d it is $bd^3/12$. For a rolled girder section it is $\frac{1}{12}\{bd^3-(b-t)(d-2t)^3\}$ if b is breadth, d depth, and t is thickness of flanges and web.

If W' is the load which a beam of depth d will support at the centre, $W'=8If/ld$ if f is the safe stress in the material ; d is the depth of the section.

For wrought iron take $E=3\times10^7$ lb. per square inch ; $f=10^4$ lb. per square inch.

(1) If $l=60$ inches, rectangular section $b=1·2$ inch, $d=1·9$ inch, find W', and when W' is the load find the deflection.

Ans. $W'=481·4$ lb., $D=0·1053$ in.

(2) Circular section, diameter $d=3$ inches, $l=60$ inches, find W', and when W' is the load find the deflection.

Ans. $W'=1767$ lb., D$=0·0667$ in.

(3) Rolled girder section $l=350$ inches, $b=6$ inches, $d=16$ inches, $t=0{\cdot}64$ inch, find W', and when W' is the load find the deflection.
Ans. $W'=8900$ lb., $D=0{\cdot}4254$ in.

(4) Rectangular section, if $d=2b$ and $l=30d$. If the load $W=400$ produces a deflection $D=0{\cdot}1$ inch, find d, and therefore b and l.
Ans. $d=1{\cdot}8$ inch, $b=0{\cdot}9$ inch, $l=54$ inches.

(5) Uniform beam of hollow circular section, thickness one-tenth of outside diameter, length 25 times outside diameter; what is the diameter if 2000 lb. is the safe load? What is the deflection when this load is upon the beam? *Ans.* $d_0=4{\cdot}643$ inches, $D=0{\cdot}1613$ inch.

(6) A beam of English oak 1 foot long, of rectangular section, 1 inch broad, 1 inch deep, supported at the ends, has a safe load $W'=70$ lb. at its middle and the deflection is then 0·021 inch; find E and f for English oak. *Ans.* $E=1{\cdot}44\times 10^6$, $f=1260$.

15. Uniform beams of rectangular section, of breadth b and depth d and length l, loaded in the middle and supported at the ends; the deflection under the load W is

$$D=c\frac{l^3 W}{bd^3}$$

and the safe load $$W'=g\frac{bd^2}{l},$$

where c and g are constants for a particular material.

For beams loaded uniformly and supported at the ends, D has ·625 times the above value and W' has twice the above value.

For beams loaded uniformly and fixed at the ends, D has one-eighth of the above value, and W' is 3 times as great.

(1) The safe load W' at the middle of a beam of English oak 1 inch broad, 1 inch deep, 12 inches long, supported at the ends is 70 lb. and its deflection D is 0·021 inch. Find the safe load and deflection for a beam of English oak 20 feet long, 8 inches broad, and 11 inches deep.
Ans. $W'=3388$ lb., $D=0{\cdot}7635$ inch.

(2) If in the last case the beam is to be supported at the ends and loaded uniformly, find W' and D.
Ans. $W'=6776$, $D=0{\cdot}4772$ inch.

(3) Instead of the case stated in (1) the beam is to be fixed at the ends and loaded uniformly, find W' and D.
Ans. $W'=10164$ lb., $D=0{\cdot}0954$ inch.

16. $$x=6\tfrac{2}{3}\sin 3t, \quad\ldots\ldots(4)$$
$$x=6{\cdot}7\epsilon^{-0{\cdot}3t}\sin 2{\cdot}985t, \quad\ldots\ldots(3)$$
$$x=20t\epsilon^{-3t}, \quad\ldots\ldots(2)$$
$$x=2\tfrac{1}{2}(\epsilon^{-t}-\epsilon^{-9t}). \quad\ldots\ldots(1)$$

For the following values of t, calculate values of x from each of the four formulæ, and tabulate as here shown. Remember that an angle is supposed to be specified in radians.

Values of t.	x as calculated from (1).	x as calculated from (2).	x as calculated from (3).	x as calculated from (4).
0	0	0	0	0
0·2	1·634	2·1952	3·546	3·7647
0·4	1·6086	2·4088	5·525	6·21
0·6	1·3607	1·9836	5·461	6·492
0·8	1·1213	1·4515	3·607	4·5027
1·0	0·9195	0·9956	0·7729	0·9393
1·2	0·7525	0·6557	– 1·994	– 2·9520
1·4	0·6165	0·4197	– 3·792	– 5·8133
1·6	0·5045	0·2634	– 4·137	– 6·64
1·8	0·4133	0·1609	– 3·082	– 5·1493
2·0	0·3382	0·0971	– 1·135	– 1·86
2·2	0·2770	0·0598	0·9661	2·08
2·4	0·2267	0·0358	2·513	5·2927
2·6	0·1856	0·0213	3·057	6·0567
4	0·0459	0·00049	– 1·182	– 3·572

See note to exercise (21). When you plot these corresponding values of x and t on squared paper, you get four curves showing the motion of the same **vibrating body** with more and more damping. (See Art. 121.

17. If

$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi).$$

Take $a = 10$. For the following values of ϕ, calculate x and y and tabulate as here shown. ϕ is in radians; an angle of π radians is 180° or two right angles. Later plot x and y of the curve which is called a **cycloid.**

Values of ϕ.	Values of x.	Values of y.
0	0	0
$\frac{\pi}{6}$	0·236	1·338
$\frac{\pi}{4}$	0·783	2·929
$\frac{\pi}{3}$	1·81	5·0
$\frac{\pi}{2}$	5·708	10·0
$\frac{2\pi}{3}$	12·284	15·0
$\frac{3\pi}{4}$	16·491	17·07
$\frac{5\pi}{6}$	21·18	18·662
π	31·416	20·0

See note to exercise (21).

18. If
$$x = a\sin(pt+e).$$
Let $a=10$, $p=3\pi$, $e=\frac{\pi}{6}$ or 30°. Calculate x for the following values of t, ·06, ·061, ·062. See if your answers agree with the tabulated numbers of Art. 114.

Now calculate $ap\cos(pt+e)$ for $t=$·0605 and ·0615.

Also calculate $ap^2\sin(pt+e)$ for $t=$·061.

Compare your answers with the numbers tabulated, Art. 114.

19. If
$$c=\frac{v}{r}\left(1-e^{-\frac{r}{l}t}\right),$$
if $v=100$, $r=1$, $l=$·01, calculate c for the following values of t, 0, ·0001, ·0002; ·0010, ·0011, ·0012; ·0100, ·0101, ·0102. The true answers are tabulated in Ex. 10, Art. 99.

20. Calculate $c=\frac{a}{\sqrt{r^2+l^2p^2}}\sin(pt-e)$, where $\tan e=\frac{lp}{r}$, if $a=100$, $p=30\pi$, $r=1$, $l=$·01 for the following values of t, ·0150, ·0151, ·0152.

Also calculate $v=a\sin pt$ for $t=$·0151.

Compare your answers with the numbers tabulated in Ex. 11, Art. 99.

21.
$$y=a\sin(bx+c).$$
If $a=10$, $b=0$·8727, find the value of y for each of the following values of x: 1st, when c is 0; 2nd, when c is $\frac{\pi}{6}$; 3rd, when c is $\frac{\pi}{2}$; 4th, when c is π. Observe that the angle is in radians.

Given values of x.	Answers. Values of y.			
	When c is 0.	When c is $\pi/6$.	When c is $\pi/2$.	When c is π.
0	0·0	5·000	10·000	0·0
0·4	3·420	7·660	9·397	−3·420
0·8	6·428	9·397	7·660	−6·428
1·2	8·660	10·000	5·000	−8·660
:	:	:	:	:
8·0	6·428	9·397	7·660	−6·428

If you are not yet sure that you know how to find the sines of angles greater than 90°, work only those exercises in which the angle is less than 90°. After studying Art. 145, you can complete this exercise and draw the curve connecting y and x.

22. If $s=A\sin(nt+e)$, where $n^2=g/Wh$ and $g=32$·2.

If $h=$·01, $W=64$·4, $A=1$, $e=0$, calculate s for the following values of t:

·0700, ·0701, ·0702; ·1400, ·1401, ·1402,

use seven-figure tables.

Compare your answers with the numbers tabulated, Ex. 12, Art. 99.

23. When **gas flows** adiabatically from a vessel, where the pressure is p_0 lb. per sq. ft. and its weight per cubic foot is w_0, through an orifice

into an atmosphere; if we consider a stream tube which is of the small cross-sectional area A sq. feet at the place where the pressure is p; if v is the velocity at that place, g being 32·2 and γ being 1·4 for air, 1·13 for dry or wet steam,

$$v^2 = 2g\frac{p_0}{w_0}\frac{\gamma}{\gamma-1}\left(1-\alpha^{1-\frac{1}{\gamma}}\right), \quad \text{.............................(1)}$$

where α is p/p_0; and the weight of fluid per second in this stream being constant, we take

$$A\sqrt{\alpha^{\frac{2}{\gamma}}-\alpha^{1+\frac{1}{\gamma}}} \text{ as constant. } \text{.............................(2)}$$

It is not difficult to prove that dry or wet **steam** follows this law, as if it were a gas whose $\gamma = 1·13$.

I give the answers for steam only. Take $p_0 = 100 \times 144$, $w_0 = 0·23$, $\gamma = 1·13$. Put (2) equal to some small constant, say 0·00073, and calculate v and A for the following values of p.

Notice that A reaches a minimum value in the throat, and it can be shown that v there is the velocity which sound would have in fluid of the quality that exists there. A false analogy due to the observation of the flow of liquid caused physicists and engineers to imagine that the velocity of a gas could never be greater than the velocity of sound. Notice that the stream tubes outside the orifice get larger in section, and the velocity is also larger.

$p \div 144$	A	v
100	Very great	0
90	·00732	658
80	·00541	994
70	·00489	1245
60	·00483	1456
57·85	·00481	1512
55	·00484	1573
50	·00488	1708
40	·00519	1963
30	·00589	2252
20	·00743	2654
15	·00889	2910
10	·01170	3220
5	·01430	3506
2·5	·03306	4214

I have assumed no friction, and there is practically no friction till the throat is reached, but in the diverging streams outside there is considerable friction, which causes the calculated values of v to be too great.

24. The formula

$$\sqrt{\frac{kgr}{2}}\sqrt{\sqrt{\left(1+\frac{q^2l^2}{r^2}\right)\left(1+\frac{s^2}{k^2q^2}\right)} \mp \left(\frac{ql}{r}-\frac{s}{qk}\right)}$$

is the value of h if the minus sign is taken, and it is the value of g if the plus sign is taken. In the following five cases find h and g. Each of the five is a typical **telephone** or telegraph cable.

	q	r	l	k	s	Answers.	
						h	g
A	5000	88	0	$\cdot 05 \times 10^{-6}$	0	·105	·105
B	5000	18	·0039	$\cdot 008 \times 10^{-6}$	10^{-6}	·0122	·0302
C	5000	2·97	·0033	$\cdot 0096 \times 10^{-6}$	10^{-6}	·00281	·0282
D	5000	12	·0010	$\cdot 0714 \times 10^{-6}$	5×10^{-6}	·0382	·0564
E	60	2·88	0	$\cdot 4095 \times 10^{-6}$	0	·005948	·005948

These answers are tabulated also in Ex. 9, Art. 149.

25. In Exercise 24, a telephonic current 1 attenuates to ϵ^{-hx} in the distance x miles in each of the above cases, and it lags through the angle gx radians. Find the attenuation and the lag in degrees for each line in a distance of 50 miles. That is, calculate

$$\text{attenuation} = \epsilon^{-50h} \quad \text{and} \quad \text{lag} = 50g \times 57{\cdot}3.$$

Here are the answers :

	A	B	C	D	E
Attenuation	0·0052	0·54	0·868	0·14808	0·743
Lag in degrees	300°	87°·1	80°·5	161°·5	17°·04

26. Exercises on **Electrical Conductors.**

(1) A round copper wire of length l and diameter d (both in centimetres) has a resistance R ohms if

$$R = l\rho \div \frac{\pi}{4} d^2,$$

if $\rho = 1{\cdot}7 \times 10^{-6}$. Find R for one mile of wire of the following diameters. First show that $R = 0{\cdot}3482 \div d^2$.

d	1	0·3162	0·1	0·03162	0·01
R	0·3482	3·482	34·82	348·2	3482

These are the resistances when the currents are constant.

(2) The above wires when conveying alternating currents of frequency $q/2\pi$ undergo a fractional increase β of resistance if

$$\beta = \frac{1}{12} \frac{l^2 q^2}{R^2} \div 10^{18}.$$

Show that $\beta = 1{\cdot}779 \times 10^{-8} q^2 d^4$.

For the above diameters, and at frequencies such that $q = 500$, or 1000, or 2000, or 5000, or 10000, find the values of β.

ANSWERS. Values of β.

q	Values of d.		
	1	0·3162	0·1
500	·00445	·00004	·0000004
1000	·01779	·00018	·0000018
2000	·07116	·0007	·000007
5000	·4448	·0044	·000044
10000	1·7790	·0178	·000178

27. Loss in **Induction Coils.**

(1) The watts w_e lost per cubic c.m. in the iron by eddy currents are

$$w_e = 4{\cdot}8f^2 10^{-12}\beta^2 d^2 \quad \text{or} \quad 7{\cdot}6f^2 10^{-12}\beta^2 t^2,$$

if f=frequency, d=diameter of each iron wire if the iron is in wires, t=thickness of iron sheet if the iron is in sheets. All dimensions are in centimetres. β is the induction in C.G.S. units per sq. cm.

The watts w_h lost by hysteresis are

$$w_h = 2{\cdot}5 f \beta^{1{\cdot}6} 10^{-10}.$$

When $f=1000$, $\beta=25$, find the value of each of these.
Ans. $w_e=0{\cdot}003d^2$, $w_e=0{\cdot}00475t^2$, $w_h=0{\cdot}0000431$.

(2) When $d=0{\cdot}1$, or 0·03162, or 0·01, calculate w_e/w_h.
Ans. 0·696, 0·0696, and 0·00696.

(3) When $t=0{\cdot}1$, or 0·03162, or 0·01, calculate w_e/w_h.
Ans. 1·102, 0·1102, and 0·01102.

The iron is always thinner than $t=0{\cdot}3$ and the wire smaller than $d=0{\cdot}03$, so that hysteresis loss is much more important than eddy-current loss in all practical work.

28. In any class of water **turbine**, if P is the total horse-power of the waterfall, H the height of the fall in feet, and n the revolutions per minute; if R feet is the average radius of the place where water enters the wheel, then $n=aH^{5/4}P^{-1/2}$ and $R=bP^{1/2}H^{-3/4}$, where a and b are constants for a particular class of turbine.

(1) From the manufacturer's list of a certain kind of low fall turbine I take one example, in which $H=6$, $P=100$, $n=50$, $R=2{\cdot}51$. Find a and b. *Ans.* $a=53{\cdot}25$, $b=0{\cdot}963$.

We can now reproduce all the other figures of this manufacturer's list.

(2) If $H=11$, $P=80$, find n and R. *Ans.* $n=119{\cdot}6$, $R=1{\cdot}425$.

(3) If $n=93$ and $H=12{\cdot}5$, find P and R. *Ans.* $P=181$, $R=1{\cdot}947$.

(4) I find that for the Thomson turbine (called in America a Francis turbine), $a=22{\cdot}75$, $b=2{\cdot}373$.

Find n and R if $H=250$, $P=74$. *Ans.* $n=2628$, $R=0{\cdot}3247$.

Find H if $R=1{\cdot}84$ and $P=95$. *Ans.* $H=29{\cdot}22$.

RATHER MORE ADVANCED GENERAL EXERCISES.

1. If r is the radius of the earth, l the distance of the moon of mass M from the earth's centre, then $M/(l+r)^2$ and $M/(l-r)^2$ are the accelerations towards the moon of points of the earth most remote and nearest the moon. The **tide-producing** actions at these points are their differences from the acceleration of the centre of the earth M/l^2. Find these in a simple approximate form, as r/l is very small.

Ans. Let r/l be called a,

$$\frac{M}{l^2}-\frac{M}{(l+r)^2}=\frac{M}{l^2}\left\{1-\frac{1}{(1+a)^2}\right\}=\frac{M}{l^2}\{1-(1-2a)\}=\frac{2M}{l^2}a=2M\frac{r}{l^3}.$$

In the same way, $$\frac{M}{(l-r)^2}-\frac{M}{l^2}=2M\frac{r}{l^3}.$$

Thus the tide-producing effect is inversely proportional to the cube of the distance. The lunar tide of the earth is 2·1 times the solar tide, although the lunar attraction on the earth is very small compared with the solar attraction.

2. The correct formula to use in a certain practical investigation is

$$y=0·75+2·59x^2+\sqrt{0·5x^{-1}\sin 3x^2+x\log_e x+0·3x}. \quad \text{............(1)}$$

It was known that it would not be used for values of x less than ·5 nor greater than 1.

It was quite impossible to work with it mathematically, so the following, found by means of squared paper, was used to replace it:

$$y=0·11+3·833x. \quad \text{.................................(2)}$$

For various values of x compare the two formulæ.

3. In a certain armour-clad ship it is found that the easterly deviation D in degrees of the standard compass from the true magnetic north, when the ship's course makes the angle θ with the north of the standard compass (θ being measured clockwise looking downwards), is given by $D=15\sin(\theta+25^\circ)+2\sin 2\theta$; calculate D for sixteen values of θ between 0° and 360°, and plot on squared paper. Do this also for a certain wooden ship in which $D=5\sin\theta+0·8\sin 2\theta$.

If θ' is the angle between the ship's course and the *true* magnetic north, so that $\theta'=\theta+D$, plot θ' and D in each of the above cases.

4. On a Mercator's map, where a minute of longitude is always of the same length, a place in latitude l is at the distance m from the equator if $m=\frac{5400}{\pi}\log_e\frac{1+\sin l}{1-\sin l}$. (This is on the assumption that the earth is a sphere.) Calculate this for a few values of latitudes, and see that m is what sailors call "The Meridianal latitude" of a place.

Close to the equator m is equal to the minutes of latitude.

5. If $y=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ and if $z=\frac{ha^2/\sqrt{\pi}}{a^2+x^2}$, where $a=\frac{0·4769}{h}$, compare the values of y and z for many values of x from 0 to ∞.

1st, when $h=1$; 2nd, when $h=10$; 3rd, when $h=100$.

CHAPTER VI.

MENSURATION.

29. Geometry ought to be practised not earlier than algebra. As soon as the student knows how to evaluate formulæ he may go through some such course as the following, illustrating his study by practical geometry; and assuming the truth of certain things from his own observation and his teacher's statements, he ought to prove the truth of other things which may not be so obviously true. There is a danger that a teacher will give a long useless course on practical geometry or a long course of abstract reasoning. What is wanted is a common-sense treatment of the subject such as will interest the average student, using the kind of reasoning that every thinking man employs in matters in which he is interested. Here is a sort of syllabus:

The usual definitions. The measurement of angles in degrees, in radians, in right angles, and in revolutions. Draw any triangle, measure the angles, and add them together. Test the accuracy of your graphical work. Bisect any angle and any line; draw perpendiculars to lines. Construct triangles when given the three sides or two sides and the contained angle, or one side and the adjacent angles. Learn how to draw parallel lines. Get acquainted with the fact that a straight line drawn parallel to the base of a triangle divides the sides into proportionate segments. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the other sides. In equiangular triangles the sides are in the same proportions. Divide a straight line into parts proportional to those of a given divided line. The graphical methods of quickly dividing lines into many parts in certain proportions, by means of strips of squared tracing paper, and other labour-saving drawing office methods of working ought to be given to students, not as things to be well remembered, but merely as illustrative exercises, for the wise student will load his memory with only a few important principles. It is most important that early in this work the student should draw a right-angled triangle, measure the sides, calculate the sine, cosine, tangent, etc., of the angles to compare with what the tables give, and he will have no difficulty in working simple problems on heights and distances, if he is not frightened by the information that this is a new and very advanced science called Trigonometry.

There are only a few important principles which ought to be proved and illustrated and dwelt upon.

Define similar plane and other figures. Similar areas are in proportion to the squares of their like dimensions. Similar volumes are in proportion to the cubes of their like dimensions.

Prove that the area of a parallelogram is the length of a side multiplied by the perpendicular distance between this and the opposite side; that the area of a triangle is half the product of a side and the perpendicular on it from the opposite angle. Try if this gives the same answer as half the product of the lengths of two sides and the sine of the included angle.

Show that in a parallelogram the complements of the parallelograms about the diagonal are equal. Prove the relations between the squares of the lengths of the sides of a right-angled triangle, and show that

$$\sin^2 A + \cos^2 A = 1.$$

I am afraid that all this is frightfully unorthodox. The idea of replacing geometrical philosophy by arithmetical juggling is scorned by the modern mathematician. But who knows whether Euclid himself would not have used my method if he had not been so hopelessly ignorant of arithmetic.

Show that the first ten propositions of the 2nd book of Euclid are identical with certain elementary algebraic statements, and the 14th proposition is an extraction of a square root. Divide a number n into two parts x and $n-x$, so that $n(n-x)=x^2$. This is the 11th proposition. Given two sides of a triangle and the contained angle, show how we find (1) the third side, (2) the other angles.

The solution of triangles ought to be made as easy as possible, and there is no difficulty about this if teachers will only reflect on the great amount of unessential matter that is usually put before students. Hardly anybody nowadays needs any part of the unessential (except for examination purposes) excrescences which have grown round the subject of the solution of triangles. They were never used except by a few surveyors, and even the surveyor who gets paid for making surveys nowadays almost never makes a survey, because he has his ordnance map.

30. I have spoken of the **first and second books of Euclid**, so easy to make interesting to the average student, so stupefying as usually given. The **fifth** book is equivalent to two or three lines of the most elementary algebra, such as, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$, etc. I would impress upon a student the most general of all such propositions; that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{ma+nb}{pa+qb}=\frac{mc+nd}{pc+qd}$, whatever, m, n, p and q may be.

31. As for the **third** book of Euclid, I would simply illustrate propositions 1-19 by drawing, and assume the proofs to be unnecessary. On some of the following propositions there might be a little abstract reasoning, but not until the student has illustrated their truth by actual measurement.

Prove that the angle at the centre is double the angle at the circumference when they have the same part of the circumference for their base, and that angles in the same segment are equal. The angle in a semicircle is a right angle. Also that the sum of the opposite angles of a quadrilateral inscribed in a circle is 180°. Also, the angle between a tangent and a chord is equal to the angle in one of the segments of the circle cut off by the chord. Prove that the products of the parts of chords of a circle cutting one another internally or externally are equal, and consider the special case of one of these being a tangent.

In the **fourth** book of Euclid only one thing need be attended to—namely, how to

Inscribe a circle in a given triangle.

Of the 7th, 8th, 9th, and 10th books of Euclid I need not speak here, as they deal with arithmetic. Euclid did not, it is true, follow the mystical philosophy of numbers of Pythagoras and Plato, but these books are so utterly useless from every point of view that even the most orthodox of pedants allows them to be replaced by elementary arithmetic. There is some hope, therefore, of the rest of Euclid, and of all the recent pretences at reform of Euclid, disappearing from our school teaching in time.

32. Rules in **Mensuration** ought to be stated as formulæ, and proved, if the proofs are easy, as part of the geometrical work. Surely it is an abominable thing to maintain the present artificial distinction between geometry and mensuration when they are both really the same, and to scorn arithmetic, which we now know, because Euclid, being ignorant, did not use it. The centre of area of a parallelogram is at the intersection of the diagonals. The centre of area of a triangle is one-third of the way along the bisector of a side towards the opposite angle. The circumference of a circle is $3 \cdot 14159d$ or πd. The area of a circle is $\frac{\pi}{4} d^2$ or πr^2. The area of an irregular polygon is found by dividing it into triangles whose areas may be added up. The area of a trapezium is half the sum of the parallel sides multiplied by the perpendicular distance between them. The area of the segment of the circle:—Find the area of the sector having the same arc (or $\frac{1}{2}$ arc $\times$ radius, or $\frac{1}{2}r\theta^2$ if θ (in radians) is the central angle subtended), and find the area of the triangle formed by the chord and the two radii, and subtract.

There are some interesting rules which give the answer with a close approximation to accuracy, such as:

Area of segment smaller than a semicircle.

If h is the height and c is the chord,

$$\text{Area} = \frac{h^3}{2c} + \tfrac{2}{3}ch.$$

Approximate Rule for s, the Length of a Circular Arc. $s = \frac{1}{3}(8l - L)$, where l is the chord of half the arc and L is the chord of the whole arc.

We ought not to compel a student to keep in his memory more than a very few of the rules of mensuration. The most important thing is for a student, when he gets a problem to work, to be able to refer at once to some book in which the rules are clearly stated. In practical work a man may refer to any book for assistance; if I had my way, every candidate at an examination would be allowed to bring into the examination room his favourite text-books.

The 11th and 12th books of Euclid are now replaced by **Descriptive Geometry** and Mensuration.

Projections of Points and Lines. Traces of Lines and Planes. Given two or three planes, by their traces find their intersections and the angles they make with one another. Find a plane containing a given line and point, or containing three given points. Find where a given line and plane meet and the angle between them. Find the angle between two given lines, etc., etc. The average student soon finds it easy to work simple exercises on cones and cylinders and their intersections.

EXERCISES.

1. A circular disc of copper is 3·2 inches in diameter. What is its circumference? What is its area? *Ans.* 10·05 inches, 8·04 sq. inches.

2. The area of a circle is 20 sq. inches. What is its diameter? What is its circumference? *Ans.* 5·04 inches, 15·8 inches.

3. The parallel sides of a trapezium are 2·16 and 3·25 feet, their perpendicular distance apart is 1·57 feet. What is the area?
Ans. 4·25 sq. feet.

4. The segment of a circle has a chord of 8·56 inches and the height 3·14 inches. What is the length of the chord of half the arc? What is the area of the segment? *Ans.* 5·31 inches, 19·77 sq. inches.

5. The area of the curved surface of a right cylinder is the circumference multiplied by the length. What is the curved area of a cylinder 4·5 inches diameter, 7 inches long? *Ans.* 98·96 sq. inches.

6. The curved surface of a right cone is the circumference of the base multiplied by $\frac{1}{2}$ the slant height.
The radius of the base is 2·6 inches; the vertical height of the cone is 5·2 inches. What is the slant height? What is the curved area?
Ans. 5·815 inches, 47·5 sq. inches.

The student is supposed to know the shapes of **prisms and cylinders** bounded by parallel ends, and of cones and pyramids generally. The cubic content of a prism is always the area of the base multiplied by the perpendicular height, or what comes to the same thing, the area of cross-section by length along the axis. The axis of a prism goes through the centres of gravity of the ends. The centre of gravity of a prismatic body is half way along the axis.

The cubic content of a **pyramid or cone** is found by multiplying the area of the base by one-third of the *perpendicular* height from base to vertex. The centre of gravity of a right cone is one quarter of the way along the axis from the base. The axis of such a body joins the centre of area of the base and the vertex. The curved area is easily constructed by practical geometry. The curved area of a right circular cone is half the circumference of the base multiplied by the slant side.

If A is the area of the base of the frustum of a cone, the area of the other parallel end being a, the height being h, the volume is h multiplied by $\frac{1}{3}(A+a+\sqrt{Aa})$.

The volume of a **ring** is equal to the circumference of the circle which passes through the centres of area of the cross-sections, multiplied by the area of the cross-section. Thus, if R is the radius of the central circle and a is the radius of a circular section of a ring, the volume is

$$V=2\pi R\times\pi a^2 \text{ or } 2\pi^2 Ra^2. \quad\text{.......................(1)}$$

The proof of this rule is given in Art. 33.

The area of a ring is the perimeter of a cross-section multiplied by the circumference of the circle which passes through the centres of all the perimeters. Thus, in the above ring of circular section the area A is

$$A=4\pi^2 Ra. \quad\text{...............................(2)}$$

The proof of this rule is given in Art. 33.

Surface of a Sphere. Multiply the diameter by the circumference or take four times the area of a diametral circle, or $S=\pi d^2$ or $4\pi r^2$.

Curved Surface of a Right Circular Cylinder. Multiply the circumference by the length, or $S=2\pi rL$. The curved surface of a spherical segment is equal to the curved surface of a cylinder of the same height as the segment, its base being a great circle of the sphere.

Area of an Ellipse. Multiply the product of the major and minor axes by $\pi/4$. If the semi-axes are a and b, $A=\pi ab$.

Volume of a Sphere. $\dfrac{\pi d^3}{6}$ or $\dfrac{4\pi r^3}{3}$.

Volume of a Plate. Area of plate multiplied by its thickness.

Volume of the Segment of a Sphere. Subtract twice the height of the segment from three times the diameter of the sphere, multiply the remainder by the square of the height and by $\pi/6$ or ·5236.

The student ought to notice the importance in every case of writing out **his rule as a formula.**

For example. A hollow circular **cylinder**, of external radius R, internal radius r, and length l, has a volume V, a curved surface S, and a weight W (if w is the weight per unit volume):

$$V=\pi(R^2-r^2)l,$$

$$S=2\pi(R+r)l,$$

$$W=\pi w(R^2-r^2)l.$$

Now, not only may we be asked to calculate V or S or W, but we may be given V or S or W to calculate some of the dimensions.

7. If $R=2$, $r=1$, $l=5$, find V and S. *Ans.* $V=47{\cdot}12$, $S=94{\cdot}25$.

8. If $V=160$ cubic inches, $l=7$ inches, $r=2$ inches. *Ans.* $R=3{\cdot}358$.

9. If $W=20$ lb., $R=4$ inches, $r=2{\cdot}5$ inches, $w=0{\cdot}3$ lb. per cubic inch, find l. *Ans.* $l=2{\cdot}177$.

10. If $V=230$ cubic inches, $S=110$ square inches, find $R-r$.
Dividing the formula for V by the formula for S, we get

$$\tfrac{230}{110}=\tfrac{1}{2}(R-r), \text{ so that } R-r=4{\cdot}182.$$

Taking the formula for **a ring** given above,

$$V=2\pi^2Rr^2,\ A=4\pi^2Rr.$$

11. If $R=5$, $r=1$, find V and A. *Ans.* $V=98{\cdot}68$, $A=197{\cdot}4$.

12 If $A=230$ cubic inches, $V=190$ square inches, find r.

Ans. $\frac{V}{A}=\frac{1}{2}r$, so that $r=2\times\frac{190}{230}$ or $1{\cdot}652$ inches.

R is now found to be $3{\cdot}526$.

13. If $V=120$ cubic inches and $R=10$ inches, find r. *Ans.* $r={\cdot}7798$.

14. A spherical shell, outside diameter 12 inches, weighs 100 lb. The material is such that 1 cubic inch weighs ·26 lb. Find the internal diameter, which I here call x.

$$(12^3\times{\cdot}5236-x^3\times{\cdot}5236)\times{\cdot}26=100,$$

so that it is easy to find $x=9{\cdot}977$.

15. A hollow cylinder of brass (0·3 lb. per cubic inch) is 11 inches long and 4 inches internal diameter, its weight is 40 lb.; find its external diameter, which I here call x.

$$(x^2\times{\cdot}7854-4^2\times{\cdot}7854)11\times 0{\cdot}3=40,$$

so that it is easy to find $x=5{\cdot}607$.

16. The rim of a cast-iron wheel (0·26 lb. per cubic inch) weighs 20,000 lb.; the rim has a rectangular section, thickness radially x, size the other way $1{\cdot}5x$. The inside radius of the rim is $10x$. Find the actual sizes. Evidently

$$(x\times 1{\cdot}5x\times 2\pi\times 10{\cdot}5x)0{\cdot}26=20{,}000.$$

Hence $x=9{\cdot}194$ inches.

The following exercises are by intention unclassified:

17. Find the area and circumference of a circle of 3·2 inches diameter.
Ans. 8·041 square inches; 10·05 inches.

18. Find the diameter of a circle whose area is 15 square inches.
Ans. 4·37 inches.

19. Find the area of a triangle whose sides are 2 inches and 3·4 inches, the angle between being 74°. *Ans.* 3·268 square inches.

20. Segment of a circle, chord 30 inches, height $4\frac{1}{2}$ inches. Find the area. *Ans.* 91·35 square inches.
If this were a parabolic segment, its area would be two-thirds of the chord multiplied by the height, or 90 square inches.

21. Find the area of the sector of a circle and the length of its arc, radius 6 inches, angle 50°. *Ans.* 15·71 square inches ; 5236 inches.

22. Find the area and volume of a sphere of 10 inches radius.
Ans. 1256·6 square inches ; 4180 cubic inches.

23. Segment of sphere, height 6 inches, diameter of base 22·5 inches. Find volume. *Ans.* 1046 cubic inches.

24. A right circular cone, radius of base 3·42 inches, height 6·42 inches ; find the area of its curved surface ; also find its volume.
Ans. 78·14 square inches ; 78·63 cubic inches.

25. An anchor ring has a mean radius of 4·2 inches, and the radius of a transverse circular section is 1·05 inch ; find the area of its surface, and also its volume. *Ans.* 174·1 square inches ; 91·39 cubic inches.

26. A hollow cylinder is 8·5 inches long ; its external and internal diameters are 5 inches and 3·5 inches ; find its volume and the area of its curved surface. *Ans.* 96·38 cubic inches ; 220·4 square inches.

27. A piece of round copper wire 100 feet long weighs 4·3 lb. ; find its diameter. A cubic inch of copper weighs 0·32 lb. *Ans.* 0·119 inch.

28. Find the area of the convex surface of a conical frustum 7·2 feet high, the radii of whose ends are 4·24 feet and 5·16 feet ; also find the volume. *Ans.* 214·3 square inches ; 501·3 cubic inches.

29. Find the volume and weight of the rim of a cast-iron wheel of square section, the inside and outside diameters being 15 feet and 13 feet 6 inches. *Ans.* 25·19 cubic feet ; 5·05 tons.

30. The diameter of a circular cylinder is 4 feet ; find the area of the elliptic section made by a plane inclined to the axis of the cylinder at 50°. See Art. 36. *Ans.* 19·54 square feet.

31. Find the area of an ellipse whose major and minor axes are 3·6 feet and 2·14 feet. What is the volume of a cone 2·5 feet high having this ellipse for its base? *Ans.* 6·05 square feet ; 5·04 cubic feet.

32. A spherical shell has inside and outside diameters of 6·5 and 10·4 inches ; find its volume. *Ans.* 444 cubic inches.

33. How many gallons of water are there in a tank 24 feet long, 15 feet wide, and 3 ft. 6 in. deep, when full? *Ans.* 7875. See page 73.

34. A pipe 3 metres long has external and internal diameters of 15 and 11·5 centimetres ; find its weight in kilogrammes and pounds, if the specific gravity of the material is 7·2. See page 73.
Ans. 157·3 kilogrammes ; 346·8 pounds.

33. Ring Theorems. Volume of a Ring. AB (Fig. 5) is any area ; if it revolves about an axis OO in its own plane—the plane of the paper—it will generate a ring. The volume of this ring is equal to the area of AB multiplied by the circumference of the circle passed through by the centre of area of AB. Imagine an exceedingly small portion of the area a at the place P at the distance $PQ = r$ from the axis. The volume of the elementary ring generated by this is $a \times 2\pi r$, and the volume of the whole ring is the sum of all

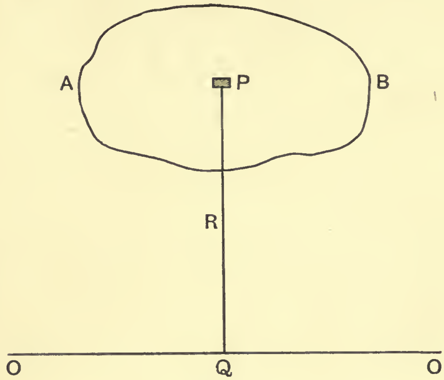

FIG. 5.

such terms, and may be written as $V = 2\pi\Sigma ar$. But $\Sigma ar = RA$ if A is the whole area of AB, if R is the r of the centre of area * [or centre of gravity of the area, as many people call it]. Hence $V = 2\pi R \times A$, the proposition to be proved.

Area of a Ring. The area of the ring surface is the length of the perimeter or boundary s of AB multiplied by the circumference of the circle passed through by the centre of gravity of the boundary. Imagine a very short length of the boundary δs at the distance r from the axis; this generates a strip of area $\delta s \times 2\pi r$. Hence the whole area is $2\pi\Sigma r . \delta s$, but $\Sigma r . \delta s = R . s$ if R is the distance of the centre of gravity of the boundary from the axis, and s is the whole perimeter. Hence the whole area of the ring is $s \times 2\pi R$.

Note that the centre of gravity of an area is not always the same as the centre of gravity of its boundary.

Ex. 1. Find the volume and area of the rim of a fly wheel, its mean radius being 10 feet, its section being a square whose side is 1·3 feet. *Ans.* $V = 1{\cdot}3^2 \times 2\pi \times 10 = 106{\cdot}2$ cub. feet.

$$\text{Area} = 4 \times 1{\cdot}3 \times 2\pi \times 10 = 326{\cdot}8 \text{ sq. ft.}$$

Ex. 2. An ellipse, whose principal diameters are 10 and 6, revolves about an axis in its own plane; the centre of the ellipse is at the distance 8 from the axis. What is the volume of the

* This is the definition of *centre of area*. See Art. 54.

ring? *Ans.* The area of the ellipse is $10 \times 6 \times \cdot 7854 = 47 \cdot 124$. This multiplied by $2\pi \times 8$ is 2369, the volume.

Ex. 3. At what distance x from the diameter is the centre of gravity of a semicircle? Regard a sphere as a ring generated by the revolution of a semicircle about its diameter. The volume of the sphere is $\frac{4}{3}\pi r^3$. The area of the semicircle is $\frac{1}{2}\pi r^2$.

$$\tfrac{4}{3}\pi r^3 = \tfrac{1}{2}\pi r^2 \times 2\pi x,$$

so that $x = 4r/3\pi$.

CHAPTER VII.

ANGLES.

34. An angle can be drawn: First, if we know its magnitude, in *degrees;* a right angle has 90 degrees. Second, if we know its magnitude, in *radians*; a right angle contains 1·5708 radians. Two right angles contain 3·1416 radians. One radian is equal to 57·2958 degrees. One radian has an arc BC equal in length to the radius AB or AC. It sometimes gets the clumsy name "a unit of circular measure." Third, we can draw an angle if we know either its *sine*, *cosine*, or *tangent*, etc. Draw any angle BAC (Fig. 6). Take any point, P in AB, and draw PR at right angles to AC. Then measure PR, AP, and AR in inches and decimals of an inch.

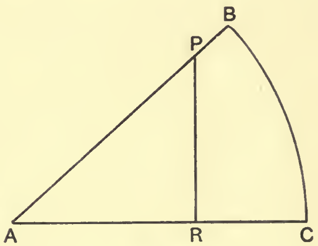

FIG. 6.

$PR \div AP$ is called the *sine* of the angle. $AR \div AP$ is called the *cosine* of the angle. $PR \div AR$ is called the *tangent* of the angle.

Calculate these for any angle you may draw, and measure with a protractor the number of degrees in the angle. You will find from the Mathematical Tables whether your three answers are exactly the sine, or cosine, or tangent of the angle. The exercise will impress on your memory the meaning of the three terms. It will also impress upon us the fact that if we know the angle in degrees, we can find, by means of our tables, its sine, or cosine, or tangent; and if we know any one of the sides AP or PR or AR of the right-angled triangle APR and the angle A, we can find the other sides. Divide the number of degrees in an angle by 57·2958, and we find the number of radians. Suppose we know the number of radians in the angle BAC, and we know the radius AB or AC, then

$$\text{the arc } BC \text{ is } AB \times \text{number of radians in the angle.}$$

Given, then, a radius to find the arc, or given an arc to find the radius, are very easy problems.

A student becomes accustomed, on seeing an angle drawn on paper, to judge from a mere glance how many degrees the angle contains. It would be an advantage to acquire the habit of judging how many radians there are in the angle.

What we mean is, that he ought to be as ready to *think* in *radians* as in *degrees*, and to do this he requires to be familiar with the size of a radian.

Ex. 1. Draw an angle of 35°. Find by measurement the sine, cosine, and tangent of the angle, and compare with the numbers in the tables.

Calculate the number of radians; that is, divide 35 by 57·296.

Try if $\sin^2 35° + \cos^2 35° = 1$; if $\sin 35° \div \cos 35° = \tan 35°$; if $\tan^2 35° + 1 = \sec.^2 35°$, where sec. A means $1 \div \cos A$.

Ex. 2. The sine of an angle is 0·25; find its cosine and tangent. Find the angle by actual drawing. How many radians?

Ans. ·9683, ·2582, 14°·5, ·2528.

Ex. 3. What are the sine, tangent, and radians of $1\frac{1}{2}$ degrees? Find them to four decimal places. *Ans.* Each ·0262.

Ex. 4. If in Fig. 6, A is 47° and AP is 5·23 feet, find AR and PR. *Ans.* $AR = 3·567$, $PR = 3·824$.

Ex. 5. The arc BC (Fig. 6) is 12 feet, AC is 16 feet; find the angle. Calculate AP and PR if AR is 10·5 feet.

Ans. ·75 radian or 42° 58′; $AP = 14·32$ ft., $PR = 9·779$ ft.

Ex. 6. The altitude of a tower observed from a point distant 200 yards horizontally from its foot is 19°·3; find its height.

Ans. 70·03 yards.

Ex. 7. A mast consists of two parts, AB and BC. From a point in the horizontal from A, AB subtends 35° and AC subtends 45°; find the ratio of AB to BC. *Ans.* 2·335.

Ex. 8. From the top of a hill the angles of depression from the horizontal of two consecutive milestones in a line with the hill on a straight level road were found to be 12° and 6°·2; find the vertical height of the hill above the road. *Ans.* 0·2220 mile.

Ex. 9. A point is in latitude 25°. If the earth were a sphere of 3960 miles radius, how far is the point from the axis? What is the length of the circumference of the circle called a parallel of latitude that passes through the point? What is the 360th part of this in length, and what is it called? What is the 360th part of a circle which is a meridian? What is it called?

Ans. Distance from axis, 3589 miles; circumference, 22,550 miles;

$\frac{\text{circum.}}{360} = 62 \cdot 63$ miles, the length of one degree of longitude;

$\frac{\text{meridian}}{360} = 69 \cdot 10$ miles, the length of one degree of latitude.

Ex. 10. What is the length of a degree of longitude in latitude 35°, taking the length on the equator as 60 nautical miles?

Ans. 49·12 miles.

Ex. 11. The gunners' rule is that 1 inch at 100 yards subtends an angle of 1 minute. What is the percentage error of this rule?

Ans. 4·5 per cent.

Ex. 12. A floating target is 20 feet in vertical height. A shell is descending at an angle with the horizontal of 4°. Within what limits of horizontal range will the shell hit the top or bottom of the target? *Ans.* 286 feet.

35. Angular Velocity. If a wheel makes 90 turns per minute, this means that it makes 1·5 turns per second. But in making one turn any radial line moves through the angle of 360 degrees, which is 6·2832 radians; so that 1·5 turns per second means $6 \cdot 2832 \times 1 \cdot 5$, or 9·4248 radians per second. This is the common *scientific way* in which the *angular velocity* of a wheel is measured—so many radians per second.

If a wheel makes 30 turns per minute, its angular velocity is π radians per second; n turns per minute mean $2\pi n$ radians per minute, or $2\pi n \div 60$ radians per second. One turn is the angular space traversed in one revolution.

Ex. Show that the linear speed in feet per second of a point in a wheel is equal to the angular velocity of the wheel multiplied by the distance in feet of the point from the axis.

Angular Acceleration. *The Increase of Angular Velocity per second.* If a wheel starts from rest, and has an angular velocity of 1 radian per second at the end of the first second, its average angular acceleration during this time is 1 radian per second per second.

Ex. 1. A shaft revolves at 800 revolutions per minute. What is its angular velocity in radians per second? *Ans.* 83·79.

Ex. 2. A point is 3000 miles from the earth's axis and revolves once in 23 hours 56 minutes 4 seconds. What is its velocity in miles per hour? *Ans.* 787·5.

Ex. 3. A point is in latitude 58°. Take the earth to be a sphere of 4000 miles radius; find the linear velocity of the point due to the earth's rotation. *Ans.* 556·4 miles per hour.

Ex. 4. The average radius of the rim of a fly-wheel is 10 feet. When the wheel makes 150 revolutions per minute, what is the average velocity of the rim? *Ans.* 157·1 per second.

Ex. 5. An acceleration of 1 turn per minute every second; how much is this in radians per second per second? *Ans.* 0·1047.

Ex. 6. A wheel is revolving at the rate of 90 turns a minute. What is its angular velocity in radians per second?

A point on the wheel is 6 feet from the axis; what is its linear speed? If its distance from the centre be increased by 50 per cent., what does its speed become? If, at the same time, the speed of the wheel increases 50 per cent., what is now the linear speed of the point?

Ans. 9·425 radians per second; 56·55 feet per second; 48·82 feet per second; 127·2 feet per second.

Ex. 7. There is a lever, OA, 30 inches long which works about an axis at O. The lever is made to turn by applying a force at a point B in OA, 15 inches from O, so that B receives a velocity of 2 feet per second. What is the angular velocity of the lever?

If the same velocity had been given to the point A instead of B, what would the angular velocity have been?

Ans. 1·6 radians per second; 0·8 radian per second.

36. If a line AB makes an angle θ with the horizontal, the **projection of its length** on the horizontal is $AB \cos\theta$.

Its projection on a vertical line is $AB \sin\theta$.

If a plane area of A square inches is inclined at an angle θ with the horizontal, its area as **projected on the horizontal** is $A \cos\theta$ square inches.

Try to prove that this must be so by dividing the area into strips by horizontal lines.

Ex. 1. A plane area of 35 square feet is inclined at 20° to the horizontal; find its horizontal projection. *Ans.* 32·82 square feet.

Ex. 2. The cross-section (a cross-section always means a section by a plane at right angles to the axis or line of centres of sections) of a cylinder is a circle of 0·7 inch radius. Find the areas of sections which make angles of 25° and 45° with the cross-section. Note that the cross-section is a projection of any other section.

Ans. 1·699, 2·177 square inches.

Ex. 3. The above cylinder is a tie bar of wrought iron. The total tensile load is 12,000 lb. How much is this per square inch of the cross-section? How much is it per square inch of either of the other sections? *Ans.* 7794 lb., 7063 lb., 5512 lb.

Ex. 4. The cross-section of a pipe is a circle of 15 inches diameter: what is the area in square feet? If 13 gallons flow per second, what is the velocity V_0? What is the area of a section at 28° to the cross-section? What is the velocity V normal to this section, if normal velocity × area = cubic feet per second? Show that V is the resolved part of V_0 in a direction normal to the section. *Ans.* 1·228, 1·7 feet per second; 1·39, 1·5 feet per second.

Ex. 5. Part of a roof, shown in plan as 4000 square feet, is inclined at 24° to the horizontal. What is its area?

Ans. 4378·7 square feet.

Ex. 6. A tie bar or short strut of 2 square inches cross-section; what is the area of a section making 45° with the cross-section? If the total tensile or compressive load is 20,000 lb., how much is this per square inch on each of the sections? Resolve the total load normal to and tangential to the oblique section, and find how much it is per square inch each way.

Ans. 2·828 square inches; 10,000 lb, 7070 lb., 5000 lb.

37. Vertical Line. A line showing the direction in which that force which we call the resultant force of gravity acts. It is a line at right angles to the surface of still water or mercury.

Level Surface. A surface like that of a still lake, everywhere at the same level, and everywhere at right angles to the force of gravity or other volumetric force which is acting upon matter. It is not a plane surface.

Curvature. For any curve we can find at any place what circle will best coincide with the curve just there. The radius of this circle is called the *radius of curvature* at the place. But since we say, for instance, that a railway line curves much, when we mean that the radius is small, the name *curvature* is always given to the reciprocal of the radius. Thus if the radius is 8 feet, we say that the curvature is $\frac{1}{8}$. If at another place the curvature is $\frac{1}{9}$, the change of curvature in going from the one place to the other is the difference between these two fractions.

Example. In making 100 steps round a curve, my compass showing the direction of motion changes for N. to N.E. What is the average curvature? *Ans.* From N. to N.E. is 45 degrees or 0·7854 radians, and this divided by 100 steps or ·007854 radians per step is the average curvature. The reciprocal of this, or 127·3 steps, is the radius of curvature, if the curvature is constant—that is, if the curve is an arc of a circle.

Exercise. Through what angle must a rail 10 feet long be bent to fit a curve of half a mile radius? *Ans.* 0·22 degree.

38. GENERAL EXERCISES.

1. Find the angle subtended at the centre of a circle of radius $6\frac{4}{11}$ inches by an arc which is 1 inch long. *Ans.* 0·1572 radians or 9°.

2. If an arc of 12 feet subtends an angle of 50°, what is the radius of the circle? *Ans.* 13·78 feet.

3. If one of the acute angles of a right-angled triangle is 1·2 radians, what is the other? *Ans.* 0·3708.

4. A certain arc subtends an angle of 1·5 radians if the radius of the circle is 2·5 feet. Find the radius of the circle of which an arc equal in length to the first subtends an angle of 3·75 radians. *Ans.* 1 foot.

5. Draw figures to show the following angles. Express them in radians: 152°, 205°, −270°, 300°, −840°, 1350°.

6. Draw a figure showing angles of 1, 2, 3, 4, 5 radians.

7. A fly-wheel of 6 feet diameter on a shaft of 6 inches diameter revolves at 260 revolutions per minute. What is the speed of a point on the rim? What is the speed of a point on the surface of the shaft? *Ans.* 81·7 feet per second, 6·81 feet per second.

8. The earth revolves about the sun, once a year, nearly in a circular path of $92·8 \times 10^6$ miles radius. Find its speed in miles per second. *Ans.* 18·5 miles per second.

9. Assume that the earth revolves about its axis once in 24 hours (this is slightly wrong). Through what angle (in radians) does it revolve in one second? Find the speed in feet per second of a point at the equator. Take the radius of the equator as 3963 miles. *Ans.* 0·0003, 1522.

10. Find the speed relatively to the sun, in feet per second, of a point on the earth's equator, (i) at midday, (ii) at midnight. *Ans.* 99,232 feet per second at midday, 96,128 at midnight.

11. A train is travelling in a curve of 0·5 mile radius, at the rate of 20 miles per hour; through what angle has it travelled in 10 seconds? *Ans.* 0·111 radian or 6°·36.

12. What angles do the large and small hands of a watch turn through between 11.15 a.m. and 2.30 p.m.? *Ans.* 1170°, 97°·5.

13. The gunner's rule is that a halfpenny (the diameter of a halfpenny is one inch) subtends an angle of 1 minute at a distance of 100 yards. What is the percentage error in this rule? *Ans.* 4·5 per cent.

14. Assuming the earth to be a sphere of 8000 miles diameter, what is the circumference of the parallel of latitude 51°·5? The earth makes one revolution in 24 hours (nearly); what is the speed in miles per hour of South Kensington, which is in latitude 51°·5? *Ans.* 15,640 miles; 652 miles per hour.

15. The inside of a hollow copper sphere is filled with water whose weight is 10 lb.; what is the inside radius? If the weight of the copper is 30 lb., what is its thickness? A cubic inch of copper weighs 0·32 lb.
Ans. 4·044 inches; 0·414 inches.

16. In a piece of coal there was found 0·1130 lb. of carbon, 0·0092 lb. of hydrogen, 0·0084 lb. of oxygen, 0·0056 lb. of nitrogen, 0·0071 lb. of ash. There being nothing else, find the percentage composition of the coal.
Ans. 78·9, 6·4, 5·9, 3·9, 4·9 per cent.

17. A pail is made in the shape of a frustum of a cone; the internal diameters are 10 inches at the top and 7 inches at the bottom, height 8 inches. Find its capacity and the weight of water which it will hold.
Ans. 459·4 cubic inches; 16·54 lb.

CHAPTER VIII.

SPEED.

39. I have given you exercises on velocity or speed. What do you mean by speed? When in a railway train we say that the speed is 30 miles per hour, what do we mean? Do we mean that we have gone 30 miles in the last hour, or that we are really going 30 miles in the next hour? Certainly not. We may have only left the terminus 10 minutes ago; there may be an accident in the next minute. It may be well to keep distances in feet and time in seconds.

Find the time in seconds taken by a body to traverse a certain distance measured in feet. This distance divided by the time is called the **average velocity**. Thus, if a railway train moves through 200 feet in four seconds, its average velocity during this time is $200 \div 4$, or 50 feet per second. If we find with careful measuring instruments that it moves through 20 feet in ·4 second, or through 2 feet in ·04 second, the velocity is $20 \div \cdot 4$, or $2 \div \cdot 04$, or 50 feet per second. It is important to remember that the velocity may be always changing during an interval of time, however short. To get **the velocity at any instant** we must make very exact measurements of the time taken to pass over a very short distance, and even this will only give us the average velocity during this short time. But if we make a number of measurements, using shorter and shorter periods of time, the average velocity becomes more and more nearly the velocity which we want. Thus, after 10 o'clock **a man in a railway train** making a careful measurement finds that the train passes over 200 feet in the next four seconds. He finds the average speed for four seconds after 10 o'clock to be $200 \div 4$, or 50 feet per second. Another man finds that it passes over 100·4 feet in the two seconds after 10 o'clock, and finds during these two seconds an average velocity of 100·4 ÷ 2, or 50·2 feet per second. Another man finds 50·25 feet passed over in one second after 10 o'clock, which shows an average velocity of 50·25 feet per second. Another man finds 25·132 feet passed over in half a second after 10 o'clock, and finds 25·132 ÷ 0·5, or 50·264 feet per second. Another man finds 12·567 feet in a quarter-second after 10 o'clock, and his observation gives 50·268 feet per second, and so on. It is evident that the values given by these various observations are approaching the **real value** of the velocity at 10 o'clock. Tabulating the results we have:

Intervals of time in seconds after 10 o'clock.	Average velocity in feet per second.
4	50·00
2	50·20
1	50·25
$\frac{1}{2}$	50·264
$\frac{1}{4}$	50·268

Plot the two sets of numbers on squared paper, and draw a curve through the points so found. Produce the curve, and we have the means of finding the average velocity for an indefinitely small interval of time after 10 o'clock. This is the required velocity.

The man who knows exactly what he means when he says "the speed of this train is 30 miles an hour," possesses the fundamental idea of the **differential calculus**, and can easily learn to use calculus methods. It is usually foolishly assumed that the idea cannot be possessed except by men who have devoted many years to mathematical study.

It is known that a bullet falls freely vertically through the following intervals of time after two seconds from rest, at London. That is, in the time between 2 and 2·1, or in the time between 2 and 2·01, or in the time between 2 and 2·001 seconds. I give here the distance fallen through:

Intervals of time in seconds	0·1	0·01	0·001
Distances fallen through -	6·601	0·6456	0·064416
Average velocity - -	66·01	64·56	64·416

We see that, as the interval of time after 2 seconds is taken less and less, the average velocity during the interval approaches more and more the true velocity at 2 seconds from rest, which is exactly 64·4 feet per second. We see that even an interval of 0·001 second is too long; we ought to take the average velocity during a very much smaller interval than this if we wish to call it the real velocity at $t=2$.

Acceleration. This is the time rate of change of the velocity of a body. Thus it is known that the velocity of a body falling freely in London :

At the end of	one second is	32·2	feet per	second,
,, ,,	two seconds is	64·4	,,	,,
,, ,,	three ,, ,,	96·6	,,	,,
,, ,,	four ,, ,,	128·8	,,	,,

and we see that there is an *increase* to the velocity of 32·2 every second. The acceleration in this case is always of the same amount, hence we call it *uniform acceleration*, and say it is 32·2 feet per second per second.

EXERCISES.

1. One mile per hour; also, one knot. Convert each of these into feet per minute and feet per second. *Ans.* 88, 1·467 ; 101·3, 1·689.

2. A destroyer travels at 32 knots. Convert this into English miles per hour. *Ans.* 36·85.

3. Prove that 60 miles per hour means 0·0268 kilometres per second.

4. Ten miles per hour. State this in feet per second, and in centimetres per second. *Ans.* $14\frac{2}{3}$, 447.

5. An acceleration of 32·2 feet per second per second. State this in miles per hour per second, state it in centimetres per second per second. *Ans.* 21·95, 981·4.

6. Two hundred gallons of water per minute. How many pounds per second? How many cubic feet per second? *Ans.* 33·3, 0·535.

7. A round pipe 6 inches diameter has 30 gallons per second flowing through it. What is the velocity? If the diameter becomes 10 inches, what is the velocity? Calculate in the two cases the kinetic energy of one pound of water, this being the square of the velocity divided by 64·4. *Ans.* 24·5 feet per second, 8·8 feet per second, 9·3 foot-pounds, 1·2 foot-pounds.

8. Two fine wires are 10 feet apart; a **bullet** breaks them both. The breaking of each wire causes an electric spark to make a mark underneath a fixed platinum pointer on a revolving drum. If the drum is 4 feet in

diameter, and revolves at 1000 revolutions per minute, and the spark-marks are found to be 1·32 feet asunder on the curved surface, assuming that the intervals of time between the breaking of the wires and making the marks were the same, find the time between the breaking of the wires and find the velocity of the bullet.

The surface velocity is $\frac{1000 \times 4\pi}{60}$, or 209·44 feet per second; 1·32 divided by this gives 0·006303 second; dividing this into 10 feet gives 1587 feet per second as the velocity of the bullet.

9. In some **gun experiments** screens 150 feet apart were cut by a bullet at the following times (in seconds), counting from the time of cutting the first screen: 0, 0·0666, 0·1343, 0·2031, 0·2729, 0·3439, 0·4159. Find the average velocity between every two successive screens.

Ans. 2252, 2216, 2180, 2149, 2113, and 2083 feet per second.

10. A body has passed through the space s feet, measured from some zero point in its path at the time t seconds, measured from some zero of time; the law of the motion is

$$s = 12{\cdot}2 - 3{\cdot}4t + 6{\cdot}7t^2.$$

Calculate s when t is 4; calculate s one-tenth of a second later, when t is 4·1. Now find the distance passed through in this tenth of a second, and so find the average velocity.

Thus s is 105·8 when t is 4, s is 110·887 when t is 4·1.

[The student will notice that when working with a law supposed to be exact we may use many significant figures.]

The space 5·087 feet being passed in 0·1 second, there is an average velocity during this tenth of a second

$$= \frac{\text{space}}{\text{time}} = \frac{5{\cdot}087}{0{\cdot}1} = 50{\cdot}87 \text{ feet per second.}$$

Now let the student calculate s for $t = 4{\cdot}01$, and for $t = 4{\cdot}001$, and find the average velocity for shorter and shorter intervals after $t = 4$, and so see what is the actual velocity at $t = 4$.

If s and t are plotted on squared paper (see Art. 87), the slope of the curve is a measure of the velocity.

40. It is inconvenient to refer to tables of numbers in bound books; therefore the Board of Education sells the following tables almost for nothing (about one halfpenny for each copy), with the idea that the very poorest student ought to have copies.

The exercise work of Chapters XXXIII. to XXXVI. will be made easier if students have tables of squares, square roots, and reciprocals.

There are no published tables which are just what we want. Unnecessary tables such as logarithmic sines, etc., are a great nuisance. We need to work only to four significant figures. In the following list each of the tables needs two pages; when will some enterprising publisher sell the collection for, say, a penny?

Logarithms and antilogarithms; Napierian logarithms and antilogarithms; sines, cosines, and tangents of angles from 0 to 90°; these angles ought to be in degrees and decimals of a degree, and make no reference to minutes. There ought to be a table such that, given $\tan\theta$, we find θ in degrees directly. A table to convert degrees into radians, and another to convert radians to degrees. Squares. Two tables of square roots. Reciprocals. These make 14 tables, 28 pages.

USEFUL CONSTANTS.

1 mile = 1·6093 kilometres.

1 inch = 25·40 millimetres.

1 foot = 30·48 cm.

1 gallon = ·1604 cubic foot = 10 lb. of water at 62° F.

1 knot = 6080 feet per hour = 1 nautical mile per hour = 1·15 miles per hour.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 kilogramme = 2·2046 lb.

1 cubic foot of fresh water weighs 62·3 lb.

1 cubic foot of air at 0° C. and 1 atmosphere weighs ·0807 lb.

1 cubic foot of hydrogen at 0° C. and 1 atmosphere weighs ·00559 lb.

1 foot-pound = 1·3562 × 10^7 ergs.

1 horse-power-hour = 33000 × 60 foot-pounds = 2·6853 × 10^{13} ergs.

1 electrical unit = 1000 watt-hours = 1·34 horse-power-hours.

1 Joule = 1 watt for 1 second = 10^7 ergs.

Joule's equivalent to suit Regnault's H is $\begin{cases} \text{774 ft.-lb.} = \text{1 Fah. unit.} \\ \text{1393 ft.-lb.} = \text{1 Cent. ,,} \end{cases}$

1 horse-power = 33000 foot-pounds per minute = 746 watts.

1 watt = 10^7 ergs per second.

Volts × amperes = watts.

1 atmosphere = 14·7 lb. per square inch = 2116 lb. per square foot = 760 mm. of mercury = 10^6 dynes per sq. cm. nearly.

A column of water 2·3 feet high corresponds to a pressure of 1 lb. per sq. inch.

Absolute temp., $t = \theta°$ C + 273°, or $\theta°$ F + 461°.

$\pi = 3·1416$.

One radian = 57°·30 ; π radians = 180°.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithms is $e = 2·7183$.

The value of g at London = 32·191 feet per sec. per sec. = 981 cm. per sec. per sec.

Velocity of light, 3 × 10^{10} cm. per second.

Mean density of the earth, 5·67 times that of water.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	22 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
		0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

CHAPTER IX.

USES OF SQUARED PAPER.

41. A sheet of squared paper is covered with equidistant horizontal and vertical lines. Every tenth line is very distinct, so that it is easy to measure off horizontal and vertical distances without using a scale. The paper has its scales on it everywhere, in fact.

Before 1876 sheets of squared paper were very expensive; they were only used by a few people in important work. In that year Prof. Ayrton and I began to use it extensively in Japan, and when we returned to London and introduced at the Finsbury Technical College our methods of teaching Mathematics and Mechanical and Electrical Engineering and laboratory work, which have now become so common, we saw that one essential thing was the manufacture of cheap squared paper. It can now be bought for 6*d*. a quire instead of 8*d*. per sheet. Our students treat it almost like scribbling paper. The candidates in three important subjects of the Board of Education write their answers upon books of squared paper. It is of importance that the student should use many sheets of squared paper, use them lavishly. Formerly many men knew how squared paper *might* be used, but they really never used it, or if they did use it, they used it not for solving problems, but for illustrating methods of solving problems.

I mean now to show you some of the uses of squared paper. It would be easy to divide this subject up into 150 propositions and lead you on from one to the next, and so build up a science; but here, at the very beginning, I want you to understand that, just as I said when describing the slide rule, all the following exercises are really one exercise. A student ought, after doing one or two of them, to see the general idea underlying them all, and if he will only practise by himself and exercise his common sense, he will be able to solve any such problem, and furthermore, he will need no elaborate proofs; things will be so self-evident as to require no proof.

Proofs indeed! Some people need proofs that they themselves exist. The mathematicians will tell you that this subject ought to be called Co-ordinate Geometry or Analytical Geometry. They will tell you that nobody ought to be allowed to begin it until he has mastered the most elaborate Algebraic and Trigonometric investigations. Now I want you to understand that I have known it to be used, and used very wisely and well, by a man who could neither read nor write.

42. I have here, cut from this morning's newspaper, this squared paper record of the rise and fall of the barometer and thermometer for the last week. Do you not all understand at once the meaning of this? Horizontal distances represent time since Sunday midnight. Vertical distances represent heights of the barometer in the one case, and heights of the thermometer in the other.

43. Trades' newspapers have many records on squared paper. Here are some squared paper records taken from the engineering papers showing fluctuations in the prices of some of the metals.

Here, again, are some curves showing the output of coal and of iron year by year since 1878 by Britain and America, Germany, France, Belgium, etc., which I find on exhibition in the museum. They show at a glance what you want to know. People interested in coal and iron will read an interesting story in every little fluctuation which you see. The general rate of growth of the industries is evident; what is most striking being their enormous development in America.

Think of a silk merchant in Yokohama putting on record in this way the price of silk per pound in Italy as it is telegraphed to him. **Why does he do it?** First, he has a record of the price for years back; a record read at a glance; a record showing at a glance the times when the price reached a maximum or a minimum value; times when the market was disturbed. Second, he sees by the *slope* of the curve the rate of increase or fall in price. Third, if he plots other things on the same sheet of paper at the same dates, he will note what effect their rise and fall have upon the price of his silk. Fourth, he gets so much information from his curve that he is able to prophesy with more certainty than a man who has no such records; indeed, he may actually be able to say with some certainty what price his silk will sell for in Italy a month hence, if he now sends a consignment.

Practical money-making men and philosophers may use squared paper for so many useful purposes that I shall not attempt to enumerate them.

I once read a clever article in the *Nineteenth Century*, by one of our greatest statesmen, concerning the rates of increase of the English population and wealth. The reasoning was most abstruse. On taking the author's figures, however, and plotting them on squared paper, every result which he had reasoned out so elaborately was plain upon the curves, so that a boy could understand them.

I was once sitting on a committee, when a manager was detailing reasons why attendance at certain classes was steadily dropping. In idleness, I manufactured some squared paper, quite roughly, and plotted the numbers, and it became at once evident that some curious event had happened at a particular date which had produced the mischief. This led at once to a rectification of the evil. Now I do not say that a man clever at figures would not have discovered this from the figures themselves, but the importance of the squared-paper method of working is that no worry over details of figures distracts one from the general story told by them.

44. A student ought at once to use squared paper for himself, and use it with numbers in which he is interested. He will find such a book as *Whitaker's Almanack* very useful.

You will see a table showing the price of Consols every year since 1790. Plot it as a curve, and get some notion in this way of the changing value of money. Or take this:

Ex. 1. A certain insurance office gives assurance of £100 at death for the following yearly premiums:

Age of insurer	21	25	30	35	40	45	50	55	60
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
Premium -	2 3 1	2 6 6	2 11 9	2 18 2	3 6 3	3 16 4	4 10 7	5 13 8	7 4 9

You had better convert the shillings and pence into decimals of a pound.

Thus take £5. 13*s.* 8*d.* Here 8*d.* is $\frac{8}{12}$ or 0·667 of a shilling, and 13·667 shillings = 0·6834 pound. So I would use £5·6834. [Perhaps you had better avoid exercises like this, which require tedious reduction to a decimal system.]

Now, having drawn a curve through your plotted points, note that you can **interpolate**—that is, you can make a very good guess at the premium which would be charged an insurer of any intermediate age. Besides, the curve itself will teach you a good deal by its mere appearance.

You might in the same way take an exercise from the table showing the price of an annuity for a person of a particular age.

You will find an excellent exercise in the table showing the average length of life which may be expected by persons of a particular age.

Plot the total amounts of gold or silver produced from mines every year since 1887. Draw the curve that lies most evenly among the points. See how nearly you can prophesy the amount of production during this coming year.

Plot the total number of letters and of postcards posted every year since 1885 in two curves. Note if one of them shows a peculiarity at any time whether the other was sympathetic.

Plot the increasing revenue or expenditure in India, or the increasing commerce of the United States; or the emigration from England; or the value of imported mutton or apples every year since 1885.

Ex. 2. The following are the numbers of half-time children in schools in twelve months following September 1, in each of the following years:

Year - - -	1892	1893	1894	1895	1896
Attendance -	164,018	140,831	126,896	119,747	110,654

Plot these and prophesy what the attendance will be in the following year. Also produce backwards and say what the probable attendance was from September 1st, 1891, to September 1st, 1892.

[This exercise was given in one of my lectures in 1899, because just then there was a Bill before the House of Commons dealing with the half-time system in schools.]

Plot the traffic receipts or the number of passengers of the Railways of the United Kingdom.

Show in a curve how the number of second-class passengers has diminished since 1883, and the number of third-class passengers increased.

I mention just a few exercises which one notices on picking up a book like *Whitaker's Almanack.*

45. The following exercise was given in one of my lectures in 1899. The tabulated numbers give some population statistics in millions; let us say in the middle of each specified year:

Year	1811	1821	1831	1841	1851	1861	1871	1881	1891
England and Wales	10·164	12·000	13·897	15·914	17·928	20·066	22·712	25·974	29·002
Scotland -	1·806	2·091	2·364	2·620	2·889	3·062	3·360	3·736	4·026
Ireland - -	—	6·802	7·767	8·175	6·552	5·779	5·412	5·175	4·705

Draw curves passing evenly through the plotted points.

Do such exercises as these: (*a*) What was the probable population, in millions, of England and Wales in 1845? *Ans.* 16·72. (*b*) What was the probable rate of increase per annum in the middle of 1845? *Ans.* 200,000 per annum. (*c*) What will be the probable population of England and Wales in 1901 and in 1911? *Ans.* 33·92 and 38·58 millions.

I give you this exercise (*c*) as it was given in 1899, because I now know the population in 1901 and 1911. In 1901 it was 32·5 and in 1911 it was 36·1. The above answers are therefore wrong, and this shows the danger of extrapolation.

The student may here be troubled by my asking him for a rate of increase per annum at a particular time. He could give me the actual increase of population during the year 1845. The other idea is really familiar enough to him, but we must consider it in Chapter XVIII.

In exercises like this, students will notice that although the numbers may be perfectly exact, we may let our curve pass, not exactly through the points, but only evenly among them, if we are trying to see if there is some simple general law of growth of population.

46. Exercise. The following measured numbers are taken from a certain table very useful to engineers. I want to know p when θ is 152. Also, I want to know θ when p is 75.

θ	140	145	150	155	160	165
p	52·52	60·40	69·21	79·03	89·86	101·90

By plotting on squared paper [and a student ought to use a large sheet; some candidates in examinations waste a square inch of paper when they ought to waste a square foot of it], you can interpolate.

Ans. $\theta = 152, p = 73{\cdot}1$; $p = 75, \theta = 153$.

The student will notice that although the above numbers are derived from experiment, and are therefore probably slightly in error, yet, as they have already been corrected for errors, his curve ought to go exactly through the plotted points.*

In all the above work—when I ask a student to prophesy, what are the things that I have not warned him about, not told him about?—I have not told him to exercise his common sense. I have not told him that in crossing the street he is in danger of being knocked down by a cab. Of course I could spend hours in talking about self-evident things! But as for the philosophy of the idiosyncrasies of cab-drivers, it is too large a subject.

47. Ex. 1. A student has made experiments and tabulates his observations as follows. Thus he found that when x was 80 his y was 0·55. Never mind now what his actual x or y was. His x was perhaps the temperature of steam and y was its pressure. Or his x may have been ampères of electric current and y may have been volts of potential difference between some two places. In any laboratory experiment we are always finding how one thing depends upon another; x may be ampère turns on a magnet and y the magnetic field produced; x may be the gallons of water per second flowing through a water meter and y may be the angular motion of a pointer on a dial, and we are going to graduate that dial from our experimental results. Anyhow, the student knows that there are errors in his measurements. He will plot the numbers on squared paper; he will try to make some simple curve pass evenly among the points. Thus he will find the probable errors of his observations. If they seem to be too great he will probably reform his method of experimenting.†

x	80	100	150	190	250	300
y	·55	·78	·97	1·10	1·22	1·24

Now let a student take the above numbers and do as requested. He ought to know that he may use any scales whatever, and hence

* Another way. Suppose we know that certain tabulated numbers follow a certain simple law without great error from one end of the table to the other, the law may be used for very exact interpolation.

Example. The above values of θ (temperature of steam) and p (pressure of steam in pounds per square inch) are known to follow this law sufficiently well for interpolation:

$$p=a(\theta+b)^5,$$

where a and b are constants.

Given the following values, $\theta=150$, $p=69{\cdot}21$; $\theta=155$, $p=79{\cdot}03$, find a and b, and then calculate p when θ is 152; also θ when p is 75.

[Exact methods of interpolating and finding rates of increase by tabulating differences need not be taken up here. See Art. 100.]

† If he finds that they are greater than he thinks they ought to be, this is the beginning of a new kind of investigation, for it may be that his assumption that the curve is a simple one is wrong.

he had better use such scales as will give him points not merely in one corner of a sheet of paper. He will also notice that he may as well plot not the whole of y or of x, for there are no values of y less than ·55, nor of x less than 80. I make the probable errors of the above observed values of y to be respectively

·013, ·03, − ·03, ·01, ·02, − ·01.

The − sign means that I think the observed y too small.

Ex. 2. In the above case, y was not observed when x was 170; what is its probable value? I find 1·06.

48. Ex. 1. A man makes saucepans. He has only made them of three sizes, as yet, but he knows that other sizes will be wanted. He wishes to **publish a price list** of many sizes. I don't know much about the saucepan trade, but suppose he has fixed on the following as really correct prices from every point of view:

A 16 pint saucepan, price 87 pence.
A 10 pint saucepan, price 68 pence.
A 2 pint saucepan, price 28 pence.

Now let him plot these sizes and prices on squared paper and join his points by a curve, say by bending a straight-edge. Any point on the curve shows size, and probable best price, of a saucepan.

Thus I make the best price of a $1\frac{3}{4}$ gallon saucepan to be 81 pence, and of a 1 gallon saucepan to be 60 pence.

Many manufacturers merely find carefully the prices of two sizes of a thing, and plotting these properly, they use a straight line as giving the price of all other sizes.

Ex. 2. A man has made the following sizes of a certain type of small steam engine, and arranged their prices very carefully.

Horse-power 4, price £44.
Horse-power 12, price £108.

Plot these on squared paper. Join the points by a straight line, and make out the probable prices of other sizes.

Thus, I get, for 10 horse-power, the price £92.

The student ought to examine a few price lists and find the rule on which the prices are calculated.

49. When we are dealing with tables of numbers that are quite correct, our curve must pass exactly through the plotted points if we are going to interpolate. Thin battens of wood may be bent, weights resting on them here and there to enable a line to be drawn. Sometimes a bent straight-edge is found to do well enough. The following is an exercise exceedingly interesting, not merely as **illustrating interpolation**, but as a continuation of what I have already said about logarithms. I quote the words used by me in a lecture in 1899.

To calculate a Table of Logarithms, or of Antilogarithms.

Some friends of mine assert that no man or boy ought to be allowed to use logarithms until he knows how to calculate them. They say this, knowing that the calculation is a branch of Higher Mathematics, and that the average schoolboy after six years at mathematics finds it hopeless to even begin the study of the Exponential Theorem. It is a hard saying! It is exactly like saying that a boy must not wear a watch or a pair of trousers until he is able to make a watch or a pair of trousers. It is the sort of unfeeling statement which so well illustrates the attitude of the superior person.

Having recently discovered an easy way of calculating logarithms [I find that it is described in a book published about two days ago by Mr. Edser, an Associate of the Royal College of Science, and he has the priority over me as an inventor], and seeing that it is a very good illustration of our present work, I do not think that there can be any harm in giving it here.

I assume that a boy can extract square roots by arithmetic. Let him, then, extract the square root of 10, and the square root of this again, and so on. Thus he finds $10^1 = 10$, $10^{\frac{1}{2}} = 3{\cdot}1623$, $10^{\frac{1}{4}} = 1{\cdot}7783$, $10^{\frac{1}{8}} = 1{\cdot}3336$, $10^{\frac{1}{16}} = 1{\cdot}1548$, $10^{\frac{1}{32}} = 1{\cdot}0746$.

From these, by multiplication, he can find $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$, $10^{\frac{3}{16}}$, etc.

Thus $\frac{1}{16}$ is the logarithm of 1·1548; $\frac{1}{8}$ is the logarithm of 1·3336.

Let him use 0·125 instead of $\frac{1}{8}$; in fact, let him use only decimals, and he has a table of which I give here only the beginning, the middle, and the end.

Logarithm.	Number.
0	1·0000
·03125	1·0746
·06250	1·1548
·09375	1·2409
·12500	1·3336
...	...
...	...
...	...
·46875	2·9427
·50000	3·1623
·53125	3·3982
...	...
...	...
...	...
...	...
·90625	8·0584
·93750	8·6596
·96875	9·3057
1·00000	10·0000

If now he wants the logarithms of numbers between, say, 3 and 3·4, let him plot three points on squared paper, using the whole of his sheet.

Logarithm.	Number.
·46875	2·9427
·50000	3·1623
·53125	3·3982

Join these points by a curve; a slightly bent straight-edge enables this to be done very nicely. He will now be able to read off the logarithm of any number between 3 and 3·4, or the number for any logarithm between 0·47 and 0·53. In this way, even with cheap squared paper, he can calculate the tables correct to 4 figures. If he wants greater accuracy he must use $10^{\frac{1}{64}}$.

50. Exercise. The simplest form of the exponential theorem as given in algebra is

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \text{etc.},$$

where e is the base of the Napierian system of logarithms.

Calculate to five significant figures the values of $e^{0\cdot1}$, $e^{0\cdot2}$, $e^{1/3}$, $e^{0\cdot5}$, $e^{2/3}$, $e^{0\cdot8}$, e^1. Tabulate your answers.

x	e^x	e^{-x}
0·1	1·1052	0·90481
0·2	1·2214	0·81873
1/3	1·3956	0·71654
0·5	1·6487	0·60654
2/3	1·9477	0·51343
0·8	2·2255	0·44934
1·0	2·7183	0·36790

Check your answers by testing if $e^{0\cdot2} \times e^{0\cdot8} = e = e^{1/3} \times e^{2/3}$, and whether $e \div e^{0\cdot5} = e^{0\cdot5}$.

Put $y = e^x$. Then $x = \log_e y$. Plot x horizontally and y vertically on squared paper, and from your curve read off the values of the Napierian logarithms of 1·5, 2·0, 2·5, and 2·7 as accurately as possible. Check your answers by converting to common logarithms and comparing with your tables. [Multiply the Napierian logarithms by 0·4343 or divide by 2·3026 to get the common logarithms.]

y	$\log_e y$	$\log_{10} y$ from the tables.
1·5	0·407	0·1761
2·0	0·694	0·3010
2·5	0·9165	0·3979
2·7	0·994	0·4314

CHAPTER X.

SOME MENSURATION EXERCISES.

51. Before we take up the following exercises, it will be well to tell the student how we usually find the area of an irregular figure.

If three equidistant ordinates to three points P, Q, and R (Fig. 7), AP, BQ, and CR, are given, there are three approximate ways of finding the area $APQRCA$.

1. **Simpson's Rule** assumes that the curve PQR is a parabola.

$$(AP + 4BQ + CR) \div 6$$

is the average ordinate, and this multiplied by AC is the area.*

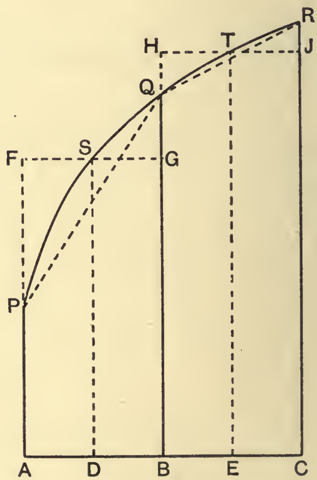

FIG. 7.

*The following proof of Simpson's rule will not be understood by students until they have read Chap. XVIII. If

$$z = a + bx + cx^2, \qquad \text{.............(1)}$$

and if z_1, z_2, and z_3 are three values of z for equidistant values of x, the average value of z between x_1 and x_3 is

$$\frac{1}{x_3 - x_1}\int_{x_1}^{x_3} z \, . \, dx = \frac{1}{6}(z_1 + z_3 + 4z_2). \qquad \text{......(2)}$$

To prove this:

Whatever z and x may be, we may represent their relationship by means of a curve, which, if (1) is true, is a parabola with its axis parallel to the axis of z. It is of no consequence what zero we take in measuring x, for if we take $x' = x + a$, we find that the law connecting z and x' is exactly like (1), only that the constants are different. We can imagine, therefore, in this general proof, that $x_2 = 0$, $x_1 = -h$, $x_3 = h$.

The average value of z is easily found to be

$$a + \frac{2}{3}ch^2. \qquad \text{.....................................(3)}$$

To find the values of a, b, and c in any case (but we here do not need b), we insert in (1) the values $x_1 = -h$, $z = z_1$; $x_2 = 0$, $z = z_2$; $x_3 = h$, $z = z_3$, and we find $a = z_2$ and $c = (z_1 + z_3 - 2z_2)/2h^2$.

Inserting these, we get the answer (2), which is Simpson's rule.

2. **The Mid-ordinate Rule.** Measure the mid-ordinates *DS* and *ET*. Take half their sum as the average ordinate of the whole curve. It is evident that this assumes the area of the curve to be the sum of the two rectangles *AFGB* and *BHJC*.

3. **The Trapezoidal Rule.** $(AP + CR + 2BQ) \div 4$ is the average ordinate. This assumes that the area is the sum of the areas of the two trapeziums *APQB* and *BQRC* as if the curve *PSQ* and the curve *QTR* were replaced by straight lines.

When we get any three points of this kind with any kind of curve connecting them, it must be remembered that its appearance may be very different from what is shown in the figure. The curve in the figure is convex upwards; the given curve may be concave upwards or may be partly concave and partly convex.

Just as we find an *area* having equidistant ordinates, so we can find a *volume* if we have areas of equidistant parallel sections.

For most of the simple solids, Simpson's rule gives the correct, and not merely an approximate answer. If A_1 and A_3 are the two parallel ends, and A_2 is the area of the section parallel to the ends, midway between them, then the average section is

$$\tfrac{1}{6}(A_1 + A_3 + 4A_2).$$

The volume of the frustum of a cone or pyramid; the volume of a sphere or of a paraboloid of revolution between two parallel cross-sections; or, indeed, of any ellipsoid and other surfaces of the second degree; the volume of the frustum of a wedge; the volume of a prismoid—these can all be computed exactly by Simpson's rule.

For example, a whole sphere may be said to have $A_1 = 0$, $A_3 = 0$, and $A_2 = \pi r^2$, so that the average section is $\tfrac{1}{6}(0 + 0 + 4\pi r^2)$ or $4\pi r^2/6$. This multiplied by $2r$ is the volume $= \tfrac{4}{3}\pi r^3$.

Definition of a Prismoid. Let there be two closed curves or irregular polygons on parallel planes, and let these be the ends of the prismoid. Imagine them joined by a surface which is made up of parts of cones or planes (a developable surface it is called). This is the most general definition of a prismoid that I can think of. Professor Harrison has proved that Simpson's rule is accurate for it.

Ex. 1. The area of each end of a barrel is 8 square feet, the middle area is 10 square feet; what is the average area of cross-section of the barrel? *Ans.* $(8 + 8 + 40) \div 6$ or 9·33 square feet. If the length of the barrel is 5 feet, its volume is $9{\cdot}33 \times 5$ or 46·67 cubic feet, by Simpson's rule.

Ex. 2. In a railway cutting, cross-sections 20 yards apart are 91, 110, and 112 square yards in area; what is the total volume of earthwork? *Ans.* By Simpson's rule, the average section is $(91+112+440) \div 6$ or 107·17 square yards; the volume is $107 \cdot 17 \times 40$ or 4287 cubic yards.

Ex. 3. The frustum of a right cone has a circular base 4 inches diameter, a circular top 2·5 inches diameter parallel to the base; the perpendicular distance between base and top is 5 inches; what is its volume? In this case Simpson's rule gives the correct answer. *Ans.* The mid area evidently has a diameter of $\frac{1}{2}(4+2 \cdot 5)$ or 3·25 inches. The three areas are the squares of the diameters multiplied by ·7854, so that the average area of section is

$$\cdot 7854(16+6 \cdot 25+42 \cdot 25) \div 6 \text{ or } 8 \cdot 443.$$

The volume is therefore $8 \cdot 443 \times 5$ or 42·22 cubic inches.

Ex. 4. A reservoir, whose sides are planes, is at the top a rectangle, 600 feet by 100 feet; at the bottom it is a rectangle, whose corresponding sides are 200 feet and 70 feet; the vertical depth is 50 feet. What is the volume? The mid section is evidently a rectangle whose sides are $\frac{1}{2}(600+200)$ and $\frac{1}{2}(100+70)$ or 400 and 85 feet. Therefore, $A_1 = 600 \times 100$ or 60,000 square feet, $A_2 = 400 \times 85$ or 34,000, $A_3 = 200 \times 70$ or 14,000. The average section is $\frac{1}{6}(60{,}000+14{,}000+136{,}000)$ or 35,000 sq. feet, and the volume is $35{,}000 \times 50 = 1{,}750{,}000$ cubic feet.

52. We may divide an irregular area up into many parts. The following rules may be at once derived from those given above. I do not give the trapezoidal rule, but it is easily found if wanted.

Simpson's Rule. Divide the area into any even number of parts by an odd number of equidistant parallel lines, or ordinates, the first and last being possibly of no length, for they must touch the boundary curve. Take the sum of the extreme ordinates (in many cases 0), four times the sum of the even ordinates, and twice the sum of the odd ones (omitting the first and last); multiply the total sum by one-third of the common distance asunder. This will give the area nearly.

The **mid-ordinate rule** is: divide the area into any number of parts by equidistant parallel lines, the first and last touching the bounding curve. Midway between every two, measure the breadth of the figure; add up these breadths and divide by the number of them; call this the average breadth. Multiply by the length or perpendicular distance between the extreme lines to get the area. In indicator diagram work we usually take ten parts.

The student ought to know the sort of error he may expect in using such methods. Let him draw a circle and divide it up; let him use Simpson's rule to find its area; he will find a considerable discrepance from the correct answer. This is a particularly bad case for the Simpson's rule. He ought to compare the two methods in other cases.

I prefer to use the mid-ordinate rule myself. I have often wondered why the Simpson's rule (extended to many ordinates) was so often mentioned (and always by people who never need to compute an irregular area), because it is not easy to remember. I have come to the conclusion that it is because it is arrived at through the use of the integral calculus, and it looks learned to speak about it.

A **planimeter** is an instrument which enables us to measure the area of a figure in square inches or square centimetres or other units. It is very valuable when one has many areas to find. We let the tracing point go round the boundary of our area exactly to the starting point; we can start anywhere; the increase in the dial reading gives the area. The principle of the Amsler planimeter is not difficult to understand, and ought to be given to students. When a curve has many loops, there is an interesting rule as to whether the area of a particular part ought to be called positive or negative. We need pay no attention to any such rule when working with the planimeter. Hence it is that the planimeter is so valuable when we require the average pressure on a gas or oil engine indicator diagram. We find the area by the planimeter, and divide by its extreme length parallel to the atmospheric line.

53. Just as we find an area having equidistant ordinates, so we can find a volume if we have areas of equidistant parallel sections, using either Simpson's rule or the other rule.

Ex. 1. A reservoir with irregular sides has the following dimensions. When filled with water to the vertical height h feet above a datum level, the following is the area A square feet of the water surface. The height of the lowest point above datum level $h=20$.

h	20	25	30	35	40	45	50	55	60	65	70
A	0	11,800	23,600	37,100	51,000	61,500	76,010	89,000	102,000	118,250	130,300

Find the volume in cubic feet when filled up to $h=70$.

Ans. $3{\cdot}177 \times 10^6$.

What is the volume between $h=44\frac{1}{2}$ and $h=45\frac{1}{2}$.

Ans. 61,500 cubic feet.

Ex. 2. A series of soundings taken across a river channel is given by the following table, x feet being the distance from one shore, and y feet the corresponding depth. Find (a) the area of the section, (b) the position of its centre of gravity. (See Art. 54.)

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	10	20	26	28	30	31	28	23	15	12	8	6	0

Ans. (a) 1583 sq. ft.; (b) $x=34{\cdot}1$ ft., $y=12{\cdot}1$ ft.

(c) If the average speed of the water normal to the cross-section is 4·5 feet per second, what is the quantity in cubic feet per second flowing? *Ans.* 7123 cub. ft. per sec.

(d) If there is an available fall of 10 feet, what is the horse-power? *Ans.* 8069.

Note.—A cubic foot of water weighs 62·3 lb.; if w lb. per minute can fall vertically h feet, the horse-power available is $wh \div 33{,}000$.

Ex. 3. Draw a figure something like a steam-engine indicator diagram, of length about four inches, and greatest height 3 inches. Find its area by means of a planimeter, after testing the accuracy of the planimeter by finding the area of a square.

Now compute the area by dividing it into a few strips, and then into many strips, using the Simpson's rule and also the mid-ordinate rule. Compare your answers, and thus get an idea of the relative accuracies of various methods.

Ex. 4. Describe a circle of, say, 6 inches diameter. Draw the diameter; divide it into six equal parts, and draw ordinates and measure them. Compute the area by Simpson's rule and then by the mid-ordinate rule. The true answer is of course 28·274 square inches. You will find that the Simpson's rule is not very satisfactory in this case. Now divide into 12 equal parts and repeat your work. Both answers will be more nearly right, but still the Simpson's method is the more inaccurate.

Ex. 5. The water plane of a vessel is 200 feet long; the central line is divided into 20 equal parts and the breadths at each place in feet are 0, 22, 27, 29, 30, 30·5, 30·5, 30·5, 30·5, 30, 29·5, 28, 26·2, 24·5, 21·5, 18, 14·5, 11, 7·1, 3·9, 0. Find its area by Simpson's rule. If the draught of the ship lessens by 1 inch in salt water, what is the loss of tonnage? *Ans.* 4477 sq. feet; 10·66 tons.

Ex. 6. The internal area of each of the two ends of a barrel is 12·35 sq. feet; the area of the middle section is 14·16 sq. ft.; the axial length of the barrel is 5 feet. What is its volume? What weight of water will it hold? *Ans.* 67·78 cub. ft.; 4223 lb.

CHAPTER XI.

MENSURATION EXERCISES.

54. Centres of Gravity. We often speak of the centre of gravity of a body, or of an area, or of a curve, when we mean centre of inertia, or centre of area, or linear centre.

1. Multiply each small portion of mass m of a body by its distance x from a plane; indicate the sum as Σmx. Divide this by the whole mass of the body M or Σm, and we have the distance of the centre of inertia or mass of the body from the same plane.

Make this calculation for three planes, and you get the exact position of the centre of mass.

If a body is symmetrical about a plane, one-third of the labour is saved. If the body is symmetrical about an axis, two-thirds of the labour is saved.

We write our rule as $\Sigma mx = M\bar{x}$.

2. Multiply each small portion a of an area by its distance x from a straight line in the same plane. Indicate the sum as Σax. Divide this by the whole area A or Σa, and we have the distance of the centre of area from the line. Make this calculation for two lines not parallel, and you get the exact position of the centre of area.

Ex. 1. Let $FBGC$ (Fig. 8) be an irregular figure. To find its area and the position of its centre of area, draw FD and OG, two parallel lines touching the extreme boundary, and let DAO be any line at right angles to these. Divide DO into many equal parts, and draw ordinates at the middle of each.

The sum of all such ordinates as BC divided by the number of them is the average breadth, and this multiplied by DO is the area. Or we may write it, if d is the distance AQ of the ordinates, asunder:

Area $A = d(HI + JK + LM + NP + BC + \text{etc.})$.

Also, if OX is the horizontal distance of the centre of gravity of the area from OG, by the definition

$$OX = \frac{d \,.\, HI \times \tfrac{1}{2}d + d \,.\, JK \times 1\tfrac{1}{2}d + d \,.\, LM \times 2\tfrac{1}{2}d + \text{etc.}}{\text{whole area}}.$$

This is evidently

$$OX = \tfrac{1}{2}d\frac{HI + 3 \,.\, JK + 5 \,.\, LM + 7 \,.\, NP + \text{etc.}}{HI + JK + LM + NP + \text{etc.}}.$$

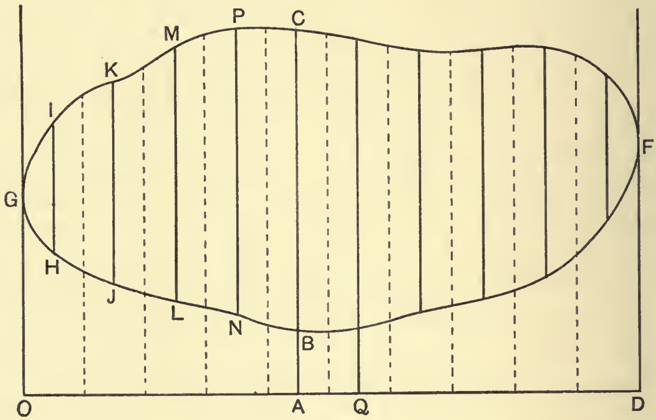

Fig. 8.

Another pair of boundary lines like OG and DF must now be chosen and the work repeated before we can find the actual position of the centre.

Ex. 2. The *half* ordinates of the load water-plane of a vessel are 12 feet apart, and their lengths are 0·5, 3·8, 7·7, 11·5, 14·6, 16·6, 17·8, 18·3, 18·5, 18·4, 18·2, 17·9, 17·2, 15·9, 13·4, 9·2, and 0·5 feet respectively.

Calculate (a) the total area of the plane. *Ans.* 5280 sq. ft.

(b) The longitudinal position of its centre of gravity.

Ans. 102 feet from 1st ordinate (0·5).

(c) The displacement in cubic feet per inch immersion at this water-plane (this is area in square feet ÷ 12). *Ans.* 440 cub. ft.

The moments of inertia of the area about its centre line and about a line through its centre of gravity at right angles to its centre line are of importance in calculating the stability of a vessel; to find them we have easy exercises of much the same kind, but perhaps I had better not give them here. See my *Applied Mechanics*, p. 138.

Ex. 3. There is a homogeneous body symmetrical about an axis; the following are its areas of cross-section A at the distances x inches from one end. Find its volume and its centre of gravity. As it is homogeneous, I shall take volume to mean mass in the calculation. The whole length is 200 inches.

x	10	30	50	70	90	110	130	150	170	190
A	320	304	311	297	292	279	287	274	263	251

Ans. The sum of the values of A is 2878, so that the average section is 287·8 square inches. This multiplied by the whole length is $287 \cdot 8 \times 200 = 57{,}560$ cubic inches.

To find the centre of gravity. We imagine slabs measuring 20 inches axially for each given section. Find, then, the sum

$$(20 \times 320 \times 10) + (20 \times 304 \times 30) + (20 \times 311 \times 50) + \text{etc.}$$

This is evidently

$$200\{(320 \times 1) + (304 \times 3) + (311 \times 5) + \text{etc.}\} \text{ or } 5{,}528{,}800.$$

This divided by the whole volume is 96 inches, the value of x of the centre of gravity.

55. Ex. 1. An irregular area has the following breadths measured at right angles to the direction OO, which we choose, at random, to call the direction of its length.

Two lines OA and $O'M$, 3·62 inches apart, are at right angles to OO' and touch the figure at its ends A and M. I measure the breadths of the figure at right angles to OO' at the distance x from OA.

Breadths in inches	0	·75	1·45	1·62	1·73	1·71	1·78	1·95	1·82	1·47	0·95	0
Corresponding values of x in inches	0	·123	0·426	0·823	1·22	1·72	2·34	2·57	2·97	3·25	3·47	3·62

Suppose these numbers to be given, but no other information. Of course, if the figure itself were given, we should not need to proceed in the following way:

Plot the breadths and values of x on squared paper and draw a curve.

This curve will enable us to get other breadths. For example, we can get breadths for equal increments of x and use Simpson's rule. Or we may find the area by the planimeter, and this is the area of the original figure. Or we may take the mid-ordinate rule.

Thus, on dividing the length into 10 equal parts, and measuring ordinates at the middles of each, I find the following:

x	0·181	0·543	0·905	1·267	1·629	1·991	2·353	2·715	3·077	3·439
Breadth	0·96	1·53	1·65	1·74	1·71	1·72	1·80	1·97	1·70	1·05

Now the sum of these is 15·83, and dividing by 10 we have the average breadth of 16·08. The length being 3 62 inches, the area is $1·583 \times 3·62 = 5·73$ square inches.

By the rule of Art. 54 we find the distance of the centre of area from OA to be 1·85 inches.

Ex. 2. The following are the areas of cross-section of a body at right angles to its straight axis:

A or area of cross-section in square inches	0	75	145	162	173	171	178	195	182	147	95	0
x inches the distance of the section from one end	0	12·3	42·6	82·3	122	172	234	257	297	325	347	362

Plot A and x on squared paper. The average value of A is easily found to be 158·3. This multiplied by the whole length 362 is $362 \times 158·3 = 5·73 \times 10^4$ cubic inches, the volume. The average value of A is, of course, found in the same way as in the last case.

The centre of gravity of the body is found to be 185 inches from the end from which x is measured.

Ex. 3. Find the volume of a reservoir when its greatest depth is 42 feet, given the following areas, A square yards of the surface of water when this surface is h feet vertically above the lowest point of the bottom:

A	0	2100	8200	13,100	15,500	19,500	25,400	32,400	47,100	52,000
h	0	5	10	17	21	25	29	33	38	42

Plotting A and h on squared paper and taking the average height of the curve, either by Simpson's rule or the mid-ordinate method, or the planimeter, we get the average A to be 20,000 square yards or 180,000 square feet. This multiplied by 42 gives 7,560,000 cubic feet, the answer.

Questions like the following are often set in "Naval Architecture," but the ordinates given are usually equidistant.

Ex. 4. The areas of successive water-planes of a vessel at the following draughts are as follows:

A = Area (sq. ft.) -	14,850	14,400	13,780	13,150	11,570	9,200	6,400
h = Draught in feet -	23·6	20·35	17·1	14·6	10·1	5·6	2·6

(1) Find the volume displaced and also the displacement of fresh water in tons, when the draught is 23·6 feet.

Ans. 265,000 cubic feet; 7370 tons.

(2) Draw a curve showing displacement V in cubic feet (or T tons if in fresh water) and draught.

(3) If the water-plane area is A square feet, (*a*) What is the extra tonnage displaced in fresh water when the draught is 1 inch more? (*b*) What is the extra displacement in cubic feet?

Ans. (*a*) $A \div 432$, (*b*) $A \div 12$.

(4) Draw curves on one sheet showing for all values of h the values of V, A, T in fresh, and V, A, T_1 in salt water.

Ex. 5. One man on the front of a tramcar looks at a well-damped spring balance inserted between the draw-bar and the car, thus measuring the pull on the car in pounds. I call this pull F. Another man has a means of measuring x, the number of feet passed through from some mark on the line. A third man has a watch and notes the time t seconds that have elapsed from some arbitrary time. They make simultaneous observations of F, x, and t, which are given in this table:

F	650	630	615	585	540	510	460	450	450	500	550
x	500	600	750	870	950	1100	1300	1400	1500	1650	1800
t	0	10	21·71	29·37	34·02	42·46	53·44	59·03	64·78	73·69	82·5

1. Plot F and x on one sheet of paper, and find the average value of F.

2. Plot F and t on another sheet of paper, and find the average value of F.

It is worth your while to think of the reason why these averages are not equal.

As an exercise on the work of Chapter XVIII.; if x and t are plotted as the co-ordinates of points on squared paper, the slope of the curve gives the speed v of the car in feet per second; let this be tabulated. $\frac{Fv}{550}$ is the horse-power; let this be tabulated.

56. Ex. 1. Give on one sheet curves which show roughly how the number of radians and the sine, cosine, and tangent of an angle alter as the angle alters from 0° to 90°. Fig. 9 shows the result.

Thus OC' represents 90°, OB represents 50°, BC shows to scale 0·6428 the cosine of 50°, BS is the sine of 50° or 0·7660, BR is the radians of 50° or 0·8727; BT is the tangent of 50° or 1·1918. OR' is straight. OSS' droops downwards and is horizontal at S'. OTT' goes up to infinity. CCC' is exactly like OSS' as seen in a looking glass.

When a student knows how to find the sine, etc., of angles, not merely between 0° and 90°, but of all sizes, he ought to plot the above curves, say from $-360°$ to $+360°$ as an exceedingly interesting exercise.

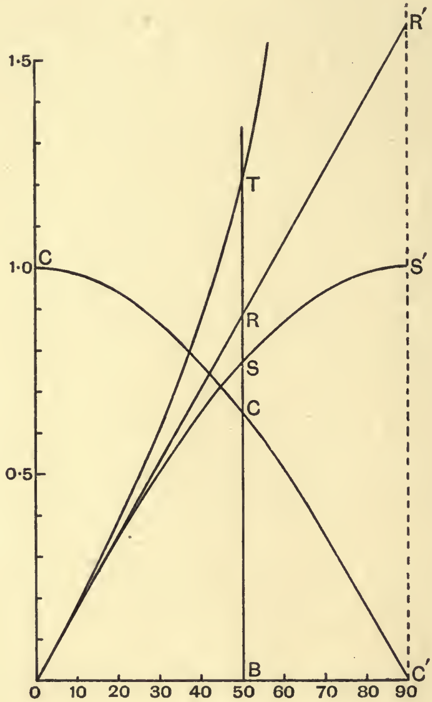

FIG. 9.

Ex. 2. The *average height* of the positive part of a sine or cosine curve is the fraction $\frac{2}{\pi}$ or 0·6366 of the greatest height or *amplitude*. Test this. I have only to add sin 5°, sin 15°, etc., sin 85°. I get 5·7369, and, dividing by 9, I find the average to be 0·6374. To get the answer more accurately, I may add sin 2°·5, sin 7°·5, etc., and find the average.

CHAPTER XII.

SQUARED PAPER.

57. When one quantity, say y, is expressed as an algebraic function of another, which I shall call x, take any value of x and calculate the corresponding y; plot on squared paper; draw the curve which passes through many such plotted points.

One most important function to plot is

$y = ax^n$, where a and n are any numbers whatsoever.

Thus $y = 9x$, or $y = 2x^2$, or $y = 3{\cdot}5x^3$, or $y = 10x^{\frac{1}{2}}$,

or $y = 5x^{\frac{1}{3}}$, or etc.,

or $y = 9x^{-1}$, which may be written $xy = 9$,

or $y = 9x^{-2}$, or $y = 4x^{-3}$, or etc.,

or $y = 2x^{-0{\cdot}246}$, or $y = 2x^{0{\cdot}246}$, etc.

Perhaps it might be well to use 1 as the value of a in every case, so that students in a class may draw the whole family of curves producible by using different values of n. It is good to prick them through on one sheet of paper.

As an example I will take, say,

$$y = x^{-0{\cdot}246}.$$

Choosing $x = {\cdot}1$, I find $y = ({\cdot}1)^{-0{\cdot}246} = 1{\cdot}762$,

,, $x = {\cdot}2$, ,, $y = ({\cdot}2)^{-0{\cdot}246} = 1{\cdot}486$,

and, in fact, by choosing the following values of x, I get the following values of y:

x	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0	1·1
y	1·762	1·486	1·345	1·253	1·186	1·133	1·091	1·056	1·027	1·000	0·977

Having plotted these points and drawn the curve, find the average value of y between $x = 0{\cdot}3$ and $x = 1{\cdot}1$.

Ans. Measuring the average breadth in the usual way, I find that the average y is 1·114.

The area then of the curve between $x=0·3$ and $x=1·1$ is

$$1·114 \times 0·8 = 0·891.$$

Ex. 1. In $$y = b + ax^n, \qquad \text{.......(1)}$$

we see that b is a mere constant addition; a merely determines the scale of measurement; hence, if we study

$$y = x^n, \qquad \text{.......(2)}$$

we may be said to have a complete study of (1).

The values of y for many values of x have already been computed in an exercise of Chap. II. for the following values of n, 4, 1, $\frac{1}{4}$, $-\frac{1}{4}$, -1, -4; so these six curves may be plotted at once.

Let a number of students get together; let each of them take one curve and plot it. When all are finished let them be pricked through upon one sheet of paper. Curves with various values taken for n are shown upon Fig. 10. They may be said to form one

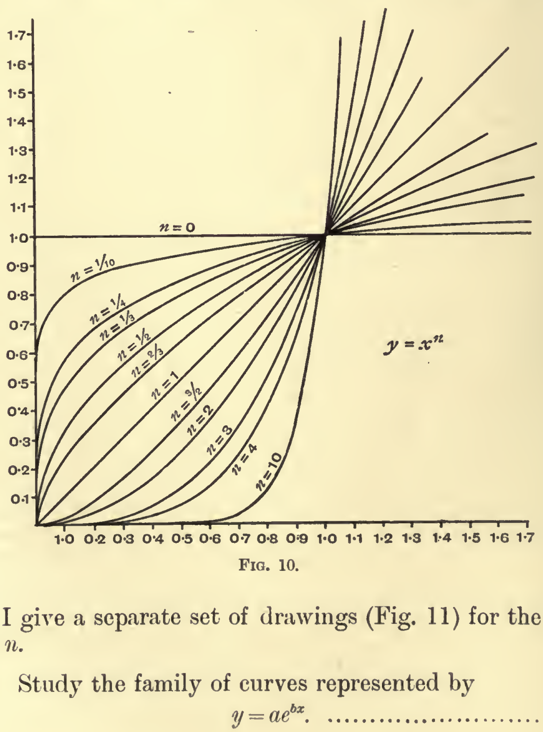

FIG. 10.

family. I give a separate set of drawings (Fig. 11) for the negative values of n.

Ex. 2. Study the family of curves represented by

$$y = ae^{bx}. \qquad \text{.......(1)}$$

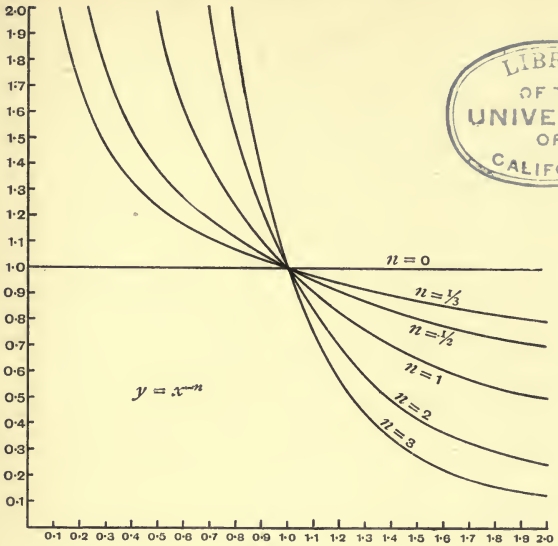
2·0
1·9
1·8
1·7
1·6
1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·1 0·2 0·3 0·4 0·5 0·6 0·7 0·8 0·9 1·0 1·1 1·2 1·3 1·4 1·5 1·6 1·7 1·8 1·9 2·0
n = 0
n = 1/3
n = 1/2
n = 1
n = 2
n = 3
y = x^{-n}

FIG. 11.

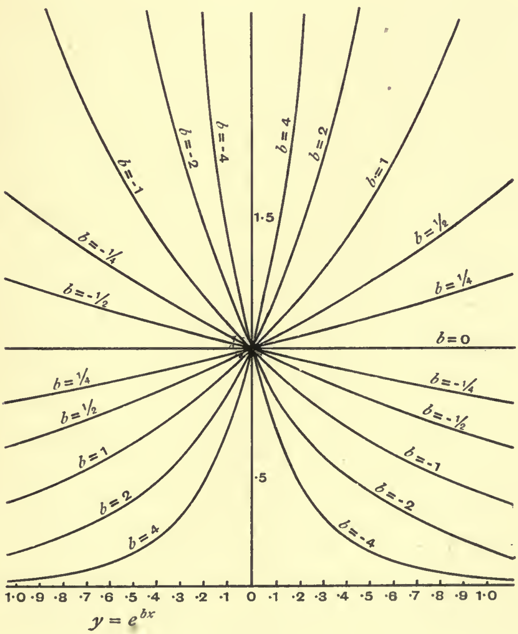
b = -1
b = -2
b = -4
b = 4
b = 2
b = 1
1·5
b = -1/4
b = -1/2
b = 1/2
b = 1/4
b = 0
b = -1/4
b = -1/2
b = 1/4
b = 1/2
b = 1
b = 2
b = 4
b = -1
b = -2
b = -4
·5
1·0 ·9 ·8 ·7 ·6 ·5 ·4 ·3 ·2 ·1 0 ·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8 ·9 1·0
y = e^{bx}

FIG. 12.

As before, a may be taken as 1, and the family of curves

$$y = e^{bx} \quad \text{..............................(2)}$$

may be drawn upon one sheet by students, each of whom takes a particular value of b. The work is easy if one has a table of Napierian logarithms, because

$$\log_e y = bx. \quad \text{..............................(3)}$$

Or if one has only a table of common logarithms,

$$\log_{10} y = 0{\cdot}4343bx. \quad \text{..............................(4)}$$

Curves with various values taken for b are shown (Fig. 12).

Values of y for various values of x have already been computed in Ex. 4, Art. 28, for the following values of b, 4, 1, $\frac{1}{4}$, $-\frac{1}{4}$, -1, -4; so these six curves may be plotted at once.

58. The Position of a Point.—The following story is not true. During the Seven Years' War, in 1760, in Saxony, a gentleman buried treasure in four different places on his estate. He was suddenly killed. His son did not know where the treasure was buried; he knew that it was buried. But there was a parchment document which his father had confided to him containing these symbols,

$$x = 2000,\ y = 977; \qquad x = -560,\ y = 700;$$
$$x = -750,\ y = -650; \quad x = 356,\ y = -274.$$

He sought for the treasure in vain.

In 1860, a descendant, a young American, found his way to the old estate, and was welcomed as a distant cousin. He was not very rich, yet he fell in love with a girl cousin as poor as himself. He had long known the old legend about the buried treasure—indeed, no member of the family ever forgot it. He happened to pick up a school-book one day and saw in it this figure (Fig. 13), which he had never seen before, albeit it is very common in mathematical books. It is so very common that no German who has gone to school from the age of seven to the age of 25, and had himself stuffed like a Strasburg goose with mind-training knowledge nine hours a day all that time, would ever dream of connecting it in any

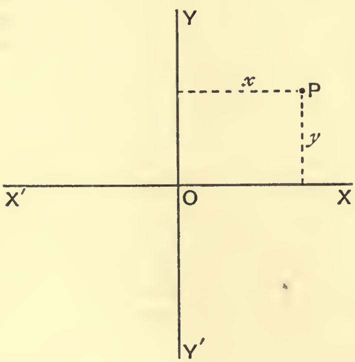

FIG. 13.

way with legend or romance, or, indeed, with anything but a school-book. But to this young American it told a story. Here were x's and y's evidently distances from two well-marked straight lines. So he and his cousin looked up an old 1760 map of the estate, and sure enough they saw two faint lines drawn in continuation of the sides of the very same old building in which they then were, something like Fig. 14. Then George W. went to Dorothea's father and said, "If I discover the old hidden treasure will you let me marry my cousin?" "Here are two discoveries," said old Heinrich, "one by you and one by me. I know well that an American, who has no such school-book knowledge as our Germans are hampered with, has imagination and ability to do things, and he is not dull; and therefore I will let you marry Dorothea, even if you have made a mistake in this thing." Then did George Washington Ollendorff make this sketch (Fig. 15) for his uncle.

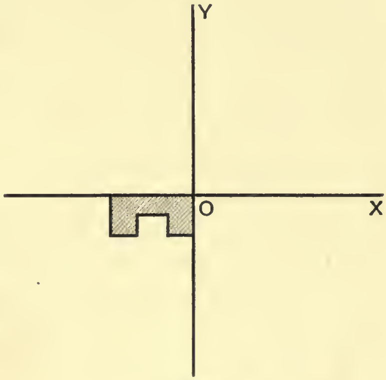

Fig. 14

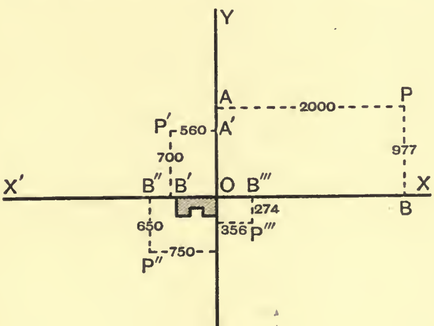

Fig. 15.

"What the distances are in," he said, "feet or yards, I do not know; but you know what measure they used here a hundred years ago, and the places I have marked are the spots to dig for buried treasure."

Then did old Count Heinrich von Ollendorff turn wrathfully upon the guileless George W. and say:

"Now do I see that you have been deceiving me all this time, for you know about Cartesian Co-ordinate Two-dimensional Analytical Geometry, and although nobody agrees with me on this matter, I affirm that you must be dull and stupid."

But Dorothea placated her father, and said, "You know, dear father, that I know nothing of learning, and yet even I see that if AP is 2000, being measured to the right of a line, $A'P'$ may well be called -560, being measured to the left of the same line. Also, if PB is 977, being measured above the line, $P'''B'''$ may well be called -274, being measured below the line."

Also, George W. spoke up and said, "I guess I know precious little of that terrible science you mentioned just now, but I say that this is a common-sense way of bringing together the legendary figures and the legendary map. Besides, I have your promise, and I here declare that for myself I will take none of this treasure; my treasure was buried here at the place of our **origin**."

But old Heinrich insisted on dividing the four buried treasures among the 207 Von Ollendorffs of Europe and the 310 Ollendorffs of America, and the joint shares of George W. and Dorothea amounted to 523·07 dollars.

59. When a point is not on a flat surface, we must still give two dimensions to find its position. Latitude and longitude are quite familiar to you all; thus, for example, if I say that a small island has been discovered in 56° N. latitude and 30° W. longitude, it is easy to find its place on a map or a globe. A shipmaster can tell his position with a possible error of only a mile or so, if his chronometer is to be depended upon, and if he can see the sun.

Ex. 1. Plot the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

It is easy to see that this is the same as

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \text{ or } y=\pm b\sqrt{1-\frac{x^2}{a^2}}.$$

There are then two values of y for each value of x, and again it is evident that $x=+4$ or $x=-4$ will produce values of y that are equal to one another. To make this clear one must work a few exercises.

(1) Suppose $b=5$ and $a=5$, the formula simplifies to

$$y=\pm\sqrt{25-x^2}$$

Taking $x=1$, then $y=+\sqrt{24}$ and also $y=-\sqrt{24}$; that is, there are two values of y for $x=1$, and so we get two points. Indeed we get four points by the one calculation, for $x=-1$ would give the same answers for y.

Now take $x=\pm 2$ and calculate y, and so get other four points, and so on. The curve is an ellipse whose diameters are equal to one another, that is, a circle.

(2) Suppose $a=5$, $b=3$.

Then $$y=\pm\tfrac{3}{5}\sqrt{25-x^2}.$$

If you have done (1) you see that this will be very easy.

Ex. 2. Plot the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

This is evidently the same as

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2}.$$

Take $b=3$ and $a=5$, and so plot the curve

$$y=\pm\tfrac{3}{5}\sqrt{x^2-25}.$$

Ex. 3. **The Cycloid.** Instead of giving y as a function of x or giving an equation connecting x and y, it is more convenient to state that $x=a(\phi-\sin\phi)$ and $y=a(1-\cos\phi)$.

Take a value for ϕ, calculate x and y. Remember that ϕ is in radians.

Take another value for ϕ, calculate a new pair of values for x and y.

In this way get the co-ordinates x and y and plot points of the curve; ϕ is an auxiliary angle.

These calculations were performed in Ex. 17, Art. 28, it only remains now to plot the curve.

CHAPTER XIII.

THE LINEAR LAW.

60. When a curve is simple looking, it may often be expressed by a simple algebraic formula.

When we plot corresponding values of y and x, and find that the points lie in a **straight line**, we always find that there is a simple law connecting them of the form

$$y = a + bx,$$

where a and b are constants.

It is worth while spending some time in plotting this function.

Thus, plot $y = 2 + 0{\cdot}75x$. By taking the following values of x, I have calculated the corresponding values of y. Let these be plotted on squared paper. You will find that the points lie exactly in a straight line.

x	0	1	2	3	4	5	6	7	8
y	2	2·75	3·5	4·25	5·0	5·75	6·5	7·25	8

I find that a stretched black thread gives the best test of straightness.

Now try $\quad y = 2 + 0{\cdot}5x, \quad y = 2 + 0{\cdot}9x,$

$$y = 2 - 0{\cdot}3x, \quad y = 2 - 0{\cdot}75x.$$

In every case you will find a straight line.

Now notice that in all these cases we have the same value of a, and consequently all your lines have some one thing in common. What is it? Find out for yourself.

61. Now plot another series of straight lines $y = a + bx$, in which b is common to all.

Take $\quad y = 1 + 0{\cdot}75x, \quad y = 3 + 0{\cdot}75x, \quad y = 4 + 0{\cdot}75x,$

$$y = 0 + 0{\cdot}75x, \quad y = -1 + 0{\cdot}75x, \quad y = -2 + 0{\cdot}75x.$$

You will find all such lines with the same b have the same slope, and indeed I usually call b **the slope** of the line. These lines are all parallel to one another.

62. Slope of a Line. If $y=a+bx$, take $x=x_1$, and find the corresponding y; it is

$$y_1=a+bx_1.$$

Now take a new x, call it x_2, and find the corresponding y; it is $y_2=a+bx_2$. Subtracting, we get

$$y_2-y_1=b(x_2-x_1),$$

or

$$\frac{y_2-y_1}{x_2-x_1}=b,$$

or in words,

$$\frac{\text{increase in } y}{\text{increase in } x}=b.$$

Hence whatever values we may take for x_1 and x_2, we find:

The **increase** of y divided by the increase of x is a constant, b. Now it is the rate of increase of one thing relatively to another which enters into most of our thinking about things, and we notice that this rate is constant in any case when on plotting the quantities on squared paper we get a straight line.

63. In the laboratory, when we have measured corresponding things, and on plotting we find the points lying in a straight line, we are usually glad, because we know that the things are connected by a very simple law. Besides, it is a law which it is very easy to test with a black thread.

Ex. 1. The following observed numbers are known to follow a law like

$$y=a+bx,$$

but there are errors of observation. Find by the use of squared paper the most probable values of a and b.

x	2	3	4·5	6	7	9	12	13
y	5·6	6·85	9·27	11·65	12·75	16·32	20·25	22·33

On plotting, as in Fig. 16, and stretching a black thread to get the straight line which lies most evenly among the points, I find that two points in my selected straight line are

$$x=2,\ y=5{\cdot}5;\ \text{and}\ x=10,\ y=17{\cdot}5.$$

Hence, to make $y=a+bx$ fit these numbers,

$$5{\cdot}5=a+2b, \quad 17{\cdot}5=a+10b.$$

Subtracting, we have

$$12 = 8b \text{ or } b = 1{\cdot}5;$$

therefore $$5{\cdot}5 = a + 2 \times 1{\cdot}5,$$

or $$a = 2{\cdot}5.$$

Hence the law required is

$$y = 2{\cdot}5 + 1{\cdot}5x.$$

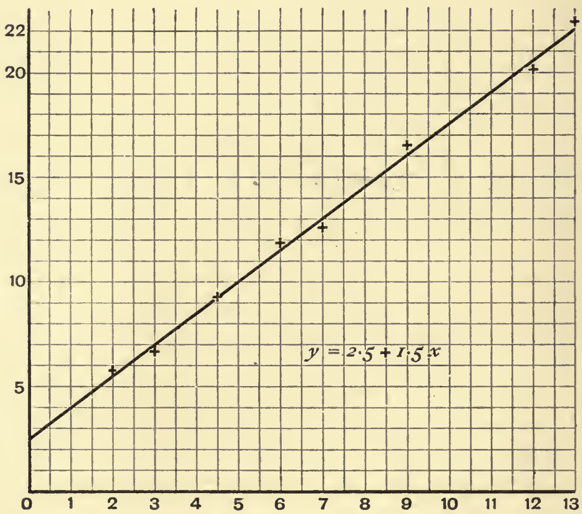

FIG. 16.

Note.—The wooden-headed, cock-sure academic person will tell you that he has an algebraic method of infinite exactness based on the laws of probability for finding the best values of a and b. Do not believe him; the black-thread method is easy to understand, and one therefore has one's wits about one when using it. To the average man using the other method it is occult; his belief in it is like our ancestors' belief in magic. Even the good mathematician forgets that it is based on the assumption that every observation is as likely to be in error as every other one. But with the black thread one cannot help adopting rules as to probability which suit the nature of the observations, especially if one has made them oneself. Very often the probable errors are not all equal, but rather the percentage error or the probable error may be in some curious relation to the observations.

Besides, there may be an absurdly large error in one of the observations. Our method shows a possible large error at once, and suggests a repetition of the experiment.

The student ought now to calculate y for each of the above values of x, and so see the probable errors of the observed values of y.

Ex. 2. In the price list of non-condensing steam turbines driving electric generators, I find the following:

Maximum output of plant or power in kilowatts $=K$.	Price in pounds sterling $=P$.	Speed in revolutions per minute $=n$.
2,000	15,785	1,200
1,000	8,085	1,800
500	4,235	2,500
100	1,155	3,500

Show that if we plot K and P we get a straight line. Now, in many motors it is $K \div n$, which, when plotted with P, gives a straight line; show that this is not so here. What is the list price of turbine plant of 700 kilowatts? *Ans.* $P = 385 + 7 \cdot 7K$; £5775.

Ex. 3. The following tests were made of a steam turbine (condensing) electric generator:

Output in kilowatts. K.	Steam consumption per hour in lb. W.
1,190	23,120
995	20,040
745	16,630
498	12,560
247	8,320
0	4,065

The steam was not all in the same state, being superheated from 10 to 20 degrees Centigrade. Find the best straight line law.

Ans. $W = 16 \cdot 2K + 4220$.

Ex. 4. I made the following tests of a large single cylinder gas engine. I is the indicated power and B the actual or brake horse-power given out by the engine. Plot I and B, and find the law connecting them. I did not measure B when I is 100; if I had made this measurement, what, in all probability, would my answer have been?

Brake horse-power B - -	16	57	95	99	117
Indicated horse-power I -	35	73	114	120	139

Ans. $B = 0 \cdot 96I - 16 \cdot 4$.

When $I = 100$, $B = 79 \cdot 6$.

Ex. 5. If P is the electric power in kilowatts sent out of an electric lighting station, and if C lb. is the amount of coal burnt in the boiler furnaces per hour, find the law connecting P and C if the following tests are correct:

Ans. $C=1{\cdot}67P+540$.

P	C
349	1121
291	1020
228	927
171	820
119	743
71	652

It is interesting for the student to calculate $C \div P$ in each of these cases, noting the diminution of efficiency of an electric light station when giving out small supplies of power, that is, during the greater part of the 24 hours. It is worth while to draw a curve showing the value of $C \div P$ for each value of P. Of course $C \div P$ means pounds of coal per kilowatt hour.

Ex. 6. When the weight A was being lifted by a laboratory crane, the handle effort B (the force applied at right angles to the handle) was measured and found to have the following values:

A	0	50	100	150	200	250	300	350	400
B	6·2	7·4	8·3	9·5	10·3	11·6	12·4	13·6	14·5

What law connects A and B? *Ans.* $B=0{\cdot}0207A+6{\cdot}3$.

Ex. 7. The speed ratio of the above crane being 80, a handle effort $A \div 80$ ought in every case to lift A if there were no friction or other source of unnecessary work done. Dividing $A \div 80$ by B, that is finding $A \div 80B$ and calling this the efficiency e, plot e and A, and so find a curve showing how the efficiency of the crane depends upon the load.

Ex. 8. At a certain electricity works, if W is the annual works cost in millions of pence and T is the annual total cost and U the number of millions of electrical units sold, it is found that approximately since 1894

$$W=0{\cdot}3+0{\cdot}6U, \quad T=0{\cdot}5+0{\cdot}97U.$$

If, in 1901, U was 3, and if U increases by 0·5 each year, what will be the value of U in 1904, and what will be the probable cost, and also the cost per unit? [This question was set in 1902.]

64. There are many operations of which the following is an example:

An examiner has given marks to papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law, converting the highest number of marks into 250 and the lowest into 100. How may he do this, and what is the converted number of marks of a paper, the original number being 140?

(1) *Laborious Method.* For any paper let y be the converted number of marks and x the original number.

The assumption is that $y = a + bx$, where a and b are constants

Then
$$250 = a + 185b$$
$$100 = a + \ \ 42b$$

Subtracting,
$$150 = \quad 143b$$
$$b = 1{\cdot}048,$$
$$100 = a + 42 \times 1{\cdot}048, \text{ so that } a = 56.$$

Hence he will convert all his marks by the rule
$$y = 56 + 1{\cdot}048x.$$

Thus, to convert $x = 140$,

he has $y = 56 + 1{\cdot}048 \times 140$

or $y = 203$.

(2) *Using Squared Paper.* Plot the two points $x = 185$, $y = 250$ and $x = 42$, $y = 100$; join these points by a straight line and read off the y corresponding to any x.

(3) Let him have two scales, one of them marked on a uniform strip of vulcanized india-rubber. Let him stretch the india-rubber scale alongside the other, making the marks 185 and 250 agree, and the marks 42 and 100 also. Fastening the scales together, he easily reads off the converted number corresponding to any original number. He is here depending upon the uniformity of stretch of the india-rubber.

(4) *Best Method.* Using squared paper and a handy scale. Let the scale lie sloping on the squared paper so that the divisions correspond.

There are other quite different ways of proceeding.

EXERCISES.

1. On testing the relation between the weight W lifted by the pulley blocks in the laboratory and the effort E required to lift it, the following numbers were obtained :

W	7	14	21	28	35	42	49	56
E	3·0	5·1	7·4	9·6	11·5	14·0	16·0	18·25

Try if they are connected by a law of the form $E = aW + b$, and if so, find the best values of a and b. *Ans.* $E = 0{\cdot}303\,W + 0{\cdot}94$.

2. Try whether the following experimental values are connected by the linear law, and if so, find the best constants.

x	14	28	42	56	70	84	98	112
y	3·5	5·0	6·75	8·25	9·75	11·5	13·25	14·78

Ans. $y = 0{\cdot}116x + 1{\cdot}75$.

3. The following numbers are from a student's laboratory note-book :

R	9·35	9·99	10·41	10·70	10·77	11·186	11·50
t	13	27	40	50	51	60	70

R and t are supposed to be connected by a law like $R=R_0(1+at)$. Try if this is so, approximately ; and if so, what are the best values of R_0 and a? *Ans.* $R_0=8·854$, $a=0·00431$.

4. An electrically driven pump, being tested, gave the following results. E is the electrical horse-power given to the pump, H is the horse-power actually spent in raising the water.

E	3·12	4·5	7·5	10·75
H	1·19	2·21	4·26	6·44

What is the law connecting E and H? The efficiency of the plant being $e=H\div E$, express it in terms of H.

Ans. $E=1·45H+1·38$, $e=H\div(1·45H+1·38)$.

5. K kilowatts being the average electrical power actually delivered to customers from an electric station during the 24 hours, W the average weight of coal consumed per hour, the following observations were made:

K	2560	2100	1800	1520	1300
W	7760	6740	6110	5480	5030

Find the law connecting K and W.

The maximum power which might be delivered to customers is 13,060 ; let $K/13,060$ be called f, the load factor ; let $\frac{W}{K}$ be called w, the coal per unit. [The Board of Trade 'unit' of energy is one kilowatt hour.] What seems to be the law connecting w and f? Calculate w when f has the values 0·25, 0·20, 0·15, 0·10, 0·05, and tabulate w and f.

f	0·25	0·20	0·15	0·10	0·05
w	2·843	3·012	3·293	3·856	5·545

Ans. $w=2·167+\frac{·1689}{f}$.

6. If $y=20+\sqrt{30+x^2}$. Take various values of x from 10 to 50, and calculate y. Plot on squared paper. What straight line agrees with the curve most nearly between these values? *Ans.* $y=0·97x+21·5$.

7. An engineer wanted to be able readily to state approximately the total cost K of steam plant, that is of the buildings, boilers, pipes, fittings, foundations, and engines for any given maximum indicated horse-power I. He obtained actual figures from a number of people who had recently

(1896) put up good steam plant under the average conditions existing in and about English manufacturing towns. Taking rough averages, he found:

If $I=200$, K may be as great as £4600 or as little as £3800; the average may be taken to be £4200.

If $I=30$, K may be as great as £900 or as little as £550; the average may be taken to be £725.

If $I=120$, K may be as great as £2600 or as little as £2300; the average may be taken to be £2450.

Plotting on squared paper, I find that a straight line will lie fairly well among the points, and this leads to the following rule:

$$K=100+20I.$$

What is the probable cost of steam plant whose indicated power is 160? *Ans.* £3300.

8. The following figures have just been published (May, 1912); tests of a Diesel oil engine using heavy oil. W is the weight of oil per hour, B is the brake horse-power. Full load means 250 brake horse-power. w is $W \div B$, or weight of oil per hour per brake horse-power.

	$\frac{1}{4}$ load.	$\frac{1}{2}$ load.	$\frac{3}{4}$ load.	Full load.	Ten per cent. overload.
w	0·64	0·50	0·45	0·448	0·458

From these figures I have computed the values of W and B:

B	275	250	187·5	125	62·5
W	126	112	84·4	62·5	40·0

Plotting on squared paper, I find a straight line, except for the overload case. Neglecting this case, which is evidently outside the simple law, I find

$$W=17{\cdot}6+0{\cdot}37B,$$

and therefore

$$w=\frac{W}{B}=\frac{17{\cdot}6}{B}+0{\cdot}37.$$

65. Exercise. The keeper of a restaurant can entertain at most 120 guests of the average kind in one hour; take it then that during the day he might have 1440 guests. After some time he had observed these average results:

Daily number of guests G.	Expenses, rent, rates, taxes, wages, and maintenance E pounds.	Cost of food and drink F pounds.	Total money collected M pounds.	Profit P pounds.
240	8	10	18	0
360	9	14·7	27	3·3

(*a*) Taking linear laws, show that

$P=0{\cdot}0275G-6{\cdot}6$, $M=0{\cdot}075G$, $F=0{\cdot}039G+0{\cdot}6$, $E=6+\frac{1}{120}G$.

(*b*) What number of guests will just produce a profit of £4 per day? *Ans.* 386.

(*c*) What is his profit per guest in the three cases when he has 300, 600, or 900 guests? *Ans.* 1·32, 3·96, 4·85 pence.

(*d*) It will be noticed that the average guest is supposed to pay 18 pence; suppose the above linear laws for E and F and G were to hold, whether the guests crowd in at particular hours or distribute themselves more uniformly during the day; suppose the guests at various hours are as follows, and in the first case that every guest pays 18 pence at whatever hour he comes, and in the second case he pays less according to the scale specified. Compare the daily profits:

	Time of day.	9–10.	10–11.	11–12.	12–1.	1–2.	2–3.	3–4.	4–5.	5–6.	6–7.	7–8.	8–9.	9–10.	10–11.
Old System.	Guests in one hour	2	3	10	90	120	30	3	2	5	50	80	10	2	2
	Pence paid by each guest	18	18	18	18	18	18	18	18	18	18	18	18	18	18
New System.	Guests in one hour	40	60	80	80	80	80	80	80	80	80	80	80	80	80
	Pence paid by each guest	12	12	14	17	18	17	16	14	16	17	18	17	16	16

The daily guests in the two cases are 409 and 1060 respectively, and we can calculate F and E from our formulæ.

Hence we have the results:

Daily number of guests G.	Expenses E pounds.	Cost of food, etc., F pounds.	Money collected. Pounds.	Daily profit. Pounds.
409	9·41	15·73	30·7	5·56
1,060	14·83	39·8	70·33	15·70

It is on this system of charging less when business is slack that lodging house and hotel keepers proceed; the Electric Light Companies soon discovered that the system was a good one, charging less per unit during the slack hours of the day. The above example shows that it might be important for a restaurant keeper or, indeed, almost any person in any kind of business to follow the same method.

If the student will use the name "load factor f" for $G \div 1440$, he will be using the language of the supply engineer.

CHAPTER XIV.

SQUARED PAPER.

66. Laws convertible into Linear Laws. Of course, if we have some theory to guide us, it is easy to find the algebraic law connecting x and y. The following are observed quantities:

x	1·1	1·8	2·5	2·9	3·6	4·3	4·8	5·4
y	1·91	2·13	2·42	2·65	3·09	3·66	4·09	4·73

Plotting on squared paper, we find a regular, simple curve. For many purposes this curve is enough; it allows us to correct observations, to interpolate, etc. But if we suspect that there is some simple algebraic law connecting y and x, we trust to our experience of curves and to lucky guessing and trial to determine the actual algebraic law.

After trying various things, let us suppose that we are lucky enough to think of plotting y and x^2, or let us suppose that theory tells us to plot y and x^2. Squaring all the values of x, we get:

x^2	1·21	3·24	6·25	8·41	12·96	18·49	23·04	29·16
y	1·91	2·13	2·42	2·65	3·09	3·66	4·09	4·73

I find on plotting now that I get a straight line. As there are evidently errors in the given values of x and y, I stretch my black thread and settle what is the best possible straight line: that is, I fix some two points as being probably quite correct. In this case I fix

$$y = 1{\cdot}9 \quad \text{when} \quad x^2 = 1,$$
$$y = 4{\cdot}3 \quad ,, \quad x^2 = 25.$$

Hence, if

$$y = a + bx^2,$$
$$1{\cdot}9 = a + b$$
$$4{\cdot}3 = a + 25b.$$

Subtract $2{\cdot}4 = 24b$, or $b = 0{\cdot}1$,

$1{\cdot}9 = a + 0{\cdot}1$, or $a = 1{\cdot}8$.

Hence $y = 1{\cdot}8 + 0{\cdot}1x^2$

is the law. Now taking the above values of x and calculating y, I can find the probable errors in the observed values.

67. Very often an experienced man who knows a great deal about curves fails to discover a simple law, although it may exist. If the points do not lie approximately in a straight line, I often try $y = ax^n$ or $y = ae^{bx}$; but it depends upon the appearance of the plotted curve **what I shall try.**

I remember the first time I discovered a law all by myself. It was in 1875.* I was a very inexperienced young man, and thought I had discovered a law of as much importance as Newton's law of gravitation. Of course I now know that many other empirical formulæ would have suited my numbers equally well; the merit of a formula lies not merely in its fitting the numbers, but also in its simplicity. Any single-valued function whatsoever may be represented by a formula like

$$y = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.},$$

if we only take enough terms.†

The merit of Newton's law lies in this, that, although it is so extremely simple, it seems to be wonderfully true throughout such space as we have had a chance of applying it to, the space occupied by our solar system. Beyond our system we know nothing about

* See *Proc. Royal Society*, 1874-5.

I had observed carefully and found the following results for the conductivity C of glass at varying temperatures θ:

Temp. Fah. θ	58°	86°	148°	166°	188°	202°	210°
C	0	0·004	0·018	0·029	0·051	0·073	0·090

I plotted θ and C on squared paper, and tried in all sorts of ways to find the algebraic law connecting them, for it was evident to me that some simple law did connect them. At length I was lucky enough to think of plotting $\log C$ with θ, and found that my points lay in a straight line, and it was then easy to show that $C = 0{\cdot}000124 \times 1{\cdot}032^{\theta}$.

† At the Royal Society, once, I heard a mathematical man talking about an empirical formula he had discovered. It was a good enough formula, within the range of values of his experiments; but he actually discussed the possibility of unreal values and curious roots of equations altogether out of the region of his experiments, as if he had found an infinitely exact law of nature. The exercise of a little common sense will prevent this kind of "extrapolation."

laws of gravitation, but in our usual optimistic way we always assume it to be true, just as we reasonably assume a great many other things to be true which may yet be proved to be false. Indeed, we do not know that the law of gravitation, as usually stated, is true inside our solar system; the weight of a body may really be dependent upon its temperature, for all we know, or some other property that we now assume it to be independent of.

It is astonishing how we go on believing things without proof, and so long as we are aware of the fact, there is no harm in it. Hundreds of thousands of lecturers on chemistry demonstrate the exact composition of the atmosphere to audiences, and then a man of original thought like Lord Rayleigh comes along and really tests these statements and finds them wrong.

Exercise. A pulley being fixed, a cord lapping round it by the amount l (a lap of amount $\frac{1}{4}$ means one quarter of a turn, or 90° or $\frac{\pi}{2}$ radians), the slack end being pulled by a force M; the force N on the tight side was found to be just sufficient to maintain a steady slipping; find the law connecting $\frac{N}{M}$ and l.

[First plot $\frac{N}{M}$ and l; now plot l and $\log\frac{N}{M}$.]

The force M was 2.

Lap l	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
N	3·17	5·06	7·90	12·68	19·90	32·10	50·12	80·3	125·8	201·5

Ans. $\frac{N}{M}=e^{1\cdot846l}$. (See Ex. 1, Art. 28.)

68. Now let me give you an example of a **lucky guess**. Here is the sort of result obtained in many thousands of cases of experimenting with steam engines. [I use one of the very best sets of experimental numbers ever obtained from a small condensing triple expansion steam engine, tested under seven steady loads, each lasting three hours.]

Indicated horse-power, I -	36·8	31·5	26·3	21	15·8	12·6	8·4
Pounds of steam used per hour per indicated horse-power, w - - - -	12·5	12·9	13·1	13·3	14·1	14·5	16·3

Plot I and w on squared paper, as in Fig. 17, and you will find the sort of result obtained by me and many other people year after year. Somehow we could make no use of our results; there was no simple law. But **Mr. Willans** was lucky enough to think of plotting with I, not w, but W, the whole weight of steam used per hour by the engine. Now try. Of course $I \times w$ is what I call W.

I	36·8	31·5	26·3	21	15·8	12·6	8·4
W	460	406·2	344·5	279·3	222·8	182·7	137·0

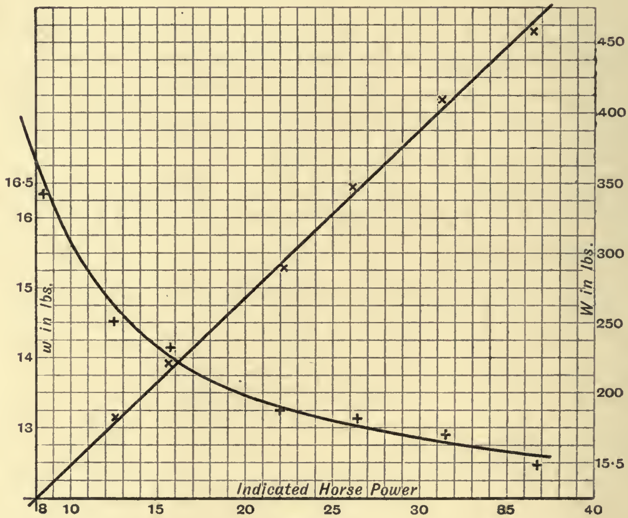

Fig. 17.

You see by Fig. 17 that the points lie sufficiently nearly in a straight line for us to be able to say that

$$W = 37{\cdot}5 + 11{\cdot}5I, \qquad (1)$$

a very simple law. If we use Iw for W and divide by I, we get

$$w = \frac{37{\cdot}5}{I} + 11{\cdot}5, \qquad (2)$$

and therefore if it had struck anybody to plot not w and I, but w and $\frac{1}{I}$, a simple law would have been discovered.*

* Note.—In every case tried by Mr. Willans, he found a linear law connecting W and I. It can be proved that in non-condensing engines it is reasonable to expect the linear law to hold, but that it is not true in condensing engines unless special precautions are adopted to keep the cylinders dry. Also, it is true in steam engines only when we regulate by letting the steam pressure vary; the cut-off is not supposed to be altered. See Ex. 41 and 42, Chap. XXVII.

69. Example. The following values of x and y were observed, and it was important to see if some simple algebraic law connected them.

x	0	·05	·1	·3	·5	1·0	2·0	1·4	2·5
y	0	·136	·26	·55	·78	·97	1·22	1·1	1·24

Plotting on squared paper, you will notice that when x and y are small, we may say roughly that

$$y \propto x,$$

but as x gets greater and greater, y seems not to get proportionately greater, but rather to be approaching a limiting value. In any case of this kind first try if

$$y = \frac{ax}{1 + bx}$$

is true, that is if $y + bxy = ax$;

or, dividing by x, if $\frac{y}{x} + by = a.$

This will be tested if we plot $\frac{y}{x}$ and y on squared paper.

In the present case, I find that the points so plotted do lie very nearly in a straight line, such that

$$\frac{y}{x} = -2y + 3,$$

and hence $$y = \frac{3x}{1 + 2x}.$$

70. The following values of x and y when plotted on squared paper give a curve differing in character from the last, principally for the smaller values.

x	0	0·5	1·0	1·5	2·0	2·5	3·0
y	0	0·48	1·37	2·12	2·58	2·93	3·07

The student ought to plot them, and then see the reasonableness of trying

$$y = \frac{ax^2}{1 + sx^2}.$$

This may be written $y + syx^2 = ax^2,$

or $$\frac{y}{x^2} + sy = a.$$

The law will therefore be very well tested if we plot y and $\frac{y}{x^2}$ on squared paper. If this is done, we find a straight line to lie evenly among the points, and I take

$$y = \frac{2 \cdot 2x^2}{1 + 0 \cdot 6x^2}$$

as representing the true law.

71. Exercise. Consider the following observed numbers:

x	1·70	2·24	2·89	4·08	5·63	6·80	8·42	12·4	16·3	19·0	24·3
y	320	411	491	671	903	1050	1270	1780	2250	2520	3180

If these are plotted on squared paper we get a curve. The shape of this curve would suggest to me,

1. To plot $\log x$ and $\log y$, or
2. To plot x and $\log y$, or
3. To plot $\log x$ and y, or
4. Try other tricks.

Now it will be found in the present case that the first of these plans succeeds. Thus I find:

$\log x$	0·2304	0·3502	0·4609	0·6107	0·7505	0·8325	0·9253	1·0934	1·2122	1·2788	1·3856
$\log y$	2·5051	2·6138	2·6911	2·8267	2·9557	3·0212	3·1038	3·2504	3·3522	3·4014	3·5024

and on plotting I find that the points lie very nearly in a straight line. In the way already described, I find the most probable line to be

$$\log y - 0 \cdot 876 \log x = 2 \cdot 299,$$

or

$$y = 199x^{-0 \cdot 876}.$$

72. Exercise. In some experiments in towing canal boats the following observations were made:

P Pull per ton in lbs.	V Speed in miles per hour.
0·70	1·68
1·70	2·43
2·35	3·18
3·20	3·60
3·50	4·03

Show that there is a law of the form

$$P = 0 \cdot 346 V^{1 \cdot 69}.$$

73. Exercise. The following measurements have been made upon a steamer:

I The indicated horse-power.	D The displacement in tons.	V The speed in knots.
140	1,748	7
410	1,748	10
820	1,748	13
1,500	1,748	16
442	2,030	10
351	1,400	10
294	1,000	10

Find if there is a law of the form

$$I \propto D^n v^m.$$

Ans. By plotting $\log I$ and $\log D$ for the speed of 10 knots, and by plotting $\log I$ and $\log v$ for the displacement 1748, we obtain

$$I = 0{\cdot}00513 D^{0{\cdot}595} v^{2{\cdot}97}.$$

74. I may say that plotting $\log x$ and $\log y$ on squared paper is so often successful, especially in gas and steam-engine work, that an old pupil of mine (Mr. Human) induced a stationer to manufacture sheets of **logarithmic paper**, so that one might lay out $\log x$ and $\log y$ without using a table. Here is a sheet of such paper. You will see that the figure 5 is not at the distance 5 inches from the axis, but at a distance proportional to log 5.

This paper is especially valuable when, instead of the law

$$yx^n = C,$$

the law that is most nearly true is

$$(y + \alpha)(x + \beta)^n = C.*$$

For, after we have plotted y and x upon this curious sheet and we find that no straight line will lie among our points, it is easy to study, not the plotted points, but points lying one or two or three divisions to the right or above, or to the left or below them.

Exercise. I have found that the expansion curve of an indicator diagram always seems to follow approximately a law:

$$pv^n = \text{constant}.$$

The following measurements (it is easy to prove that the scales of measurement are really of no consequence if we want to find n) are taken from a gas-engine diagram. But it is known that the clearance

* See Chap. XV. for a method of studying all such curves as this without using squared paper.

was not exactly measured, and hence all the values of v may need the addition or subtraction of a constant number. Find the law and the correction for clearance.

p	39·60	44·7	53·8	73·5	85·8	113·2	135·8	178·2
v	10·61	9·73	8·55	7·00	6·23	5·18	4·59	3·87

Ans. I find, using logarithmic paper, that 0·6 must be subtracted from every value of v as the correction for badly measured clearance. In fact, I find

$$p(v-0{\cdot}6)^{1{\cdot}346} = \text{constant}.$$

75. The following are tabulated values of p, the pressure in pounds per square inch of saturated steam; u being the number of cubic feet per pound. It was found that the law

$$pu^n = c, \text{ a constant, } \qquad \text{.........................(1)}$$

was very nearly true, and as a formula of this kind is very easy to use in calculations, it is important to find the best values of n and c within the following range of values:

p	6·86	14·70	28·83	60·40	101·9	163·3	250·3
u	53·92	26·36	14·00	6·992	4·28	2·748	1·853

If (1) is true, we have

$$\log p + n \log u = \log c.$$

Plotting $\log p$ and $\log u$, I find that a straight line lies very evenly among the points, and, in fact, we may say,

$$\log p + 1{\cdot}0646 \log u = 2{\cdot}68.$$

Hence $$pu^{1{\cdot}0646} = 479.$$

76. Ex. 1. Prove that the whole weight or displacement in tons of a vessel is the volume V cubic feet of displaced water divided by 36 if the water is fresh, or divided by 35 if the water is salt.

At the following draughts, h feet, a particular vessel has the following displacements in tons or in cubic feet found by calculation from the drawings:

Draught h feet	15	12	9	6·3
Length on water line l feet	300	280	270	265
Greatest breadth on water line b feet	36	35	32·5	27·3
Displacement V cubic feet	73440	52920	35640	20520
Displacement in sea water T tons	2098	1512	1018	586

Show that they satisfy the laws

$$T=44\cdot15h^{1\cdot42}, \quad V=1545h^{1\cdot42}.$$

Ex. 2. If for each draught h, the length l and greatest breadth of the vessel at the water line are as stated, show that

$$V=0\cdot45lbh.$$

Ex. 3. Taking the law $V=1545h^{1\cdot42}$ to be true, since A the horizontal sectional area in square feet of the vessel at the water line is dV/dh (after you have done Chap. XVIII.), show that $A=2194h^{0\cdot42}$, and calculate A in each of the above cases.

Ans. 6840, 6230, 5520, and 4750 sq. ft.

Ex. 4. Show that in each case $A=0\cdot45\times1\cdot42lb=0\cdot639lb$.

Ex. 5. Show that $A\div420$ or $lb\div657$ gives the diminution or increase of tonnage for one inch less or more draught in sea water.

Ex. 6. The water of a dock is such that the divisor of Ex. 1 is 35·5; that is, there is 35·5 cubic feet of it to the ton. What is the increased draught of the vessel if her weight is 2098 tons when she leaves the sea and enters the dock? *Ans.* 0·15 ft.

77. Exercise. For a certain investigation it would be very satisfactory if we could express $\frac{1}{u}$ as a linear function of p; try if this is possible from the numbers given in Art. 75.

Plotting $\frac{1}{u}$ and p on squared paper, I find that although there are discrepancies we may take

$$\frac{1}{u}=0\cdot0171+0\cdot0021p.$$

It depends upon the nature of the investigation what value we ought to place upon such a formula. When I use such an approximation I always test the importance of the discrepance.

78. Ex. 1. Plotting the following numbers, we find a regular curve. After various trials I found that on plotting x and $\log y$ on squared paper I obtained a straight line,* and so found that

$$\log y=1\cdot1955+0\cdot0843x,$$

or
$$y=15\cdot69\times10^{\cdot0843x},$$

or
$$y=15\cdot69e^{\cdot194x}.$$

* Of course if there is a theory to guide us our labour is greatly saved. For example, if we know that the rate of increase or diminution of y with regard

x	5	6·5	8	9·5	10·5	12·0	13·0	14·2
y	41·7	55·0	74·3	101·0	121	161	196	247

If we had many exercises like this there would evidently be a saving of labour if we had paper ruled logarithmically one way and with equal divisions the other way.

Ex. 2. Show from the table of Art. 45 [plotting t and $\log P$] that, leaving out the census result for 1811, if t is time in years since 1811, the population of England and Wales may be said to have closely followed the law:

$$P = 10^{7·03+0·00556t},$$

or

$$\log P = 7·03 + 0·00556t.$$

Calculate P from this formula, find the differences from it as given by the census returns, and plot these differences as a curve.* What will be the probable population in 1901 and 1911 ? †

Ans. $33·93 \times 10^6$ and $38·58 \times 10^6$.

Ex. 3. The following numbers were observed in the laboratory:

T	0·410	0·575	0·895	1·297	1·720	2·200	2·385
x	250	300	350	400	450	500	517

There were reasons for thinking that $T = cx^k$, where c and k are

---

to x is proportional to y itself, it is the compound interest law, and we know that

$$y = ae^{bx},$$

so that we plot x and $\log y$.

If we know that $\frac{dy}{dx}$ is proportional to $\frac{y}{x}$, we ought to try

$$y = ax^b.$$

If we know that $\frac{dy}{dx}$ is proportional to x, we ought to try

$$y = a + bx^2.$$

* It is to be remembered that we have had three kinds of problems:

(1) Plotting on squared paper exact numbers calculated from simple formulæ. Our curve goes exactly through the plotted points, and it is a simple curve.

(2) Plotting observed numbers: there being errors of observation, we assume that the simple curve going most evenly among the points gives us the correct law.

(3) Plotting correct numbers, as of population, the true curve goes exactly through the plotted points. But there is possibly a simple law complicated by perturbations; in studying the curve which goes evenly among the points, we look for the general law. Having it, we search for the perturbation law, if there is one.

† This extrapolation published in my 1899 lectures has turned out to be wrong. The population of England and Wales in 1901 was 32·5 millions, and in 1911 it was 36·1 millions.

constants. Test whether this is so within the range of the values given, and, if so, find the best values of k and c.

Ans. $T = 4{\cdot}624 \times 10^{-7} x^{2{\cdot}473}$.

Ex. 4. Q cubic feet of water were measured as flowing per second over a Thomson gauge notch when the difference of levels was H feet.

H	1·2	1·4	1·6	1·8	2·0	2·4
Q	4·2	6·1	8·5	11·5	14·9	23·5

Thomson's theory suggests that the numbers should follow the law $Q = aH^n$. Try this, and find the best values of a and n.

Ans. $Q = 2{\cdot}672 H^{2{\cdot}48}$.

Ex. 5. If t seconds is the *record* time of a trotting (in harness) race of m miles we have the following published records:

m	1	2	3	4	5	10	20	30	50	100
t	119	257	416	598	751	1575	3505	6479	14141	32153

Try if there is a law $t = am^b$. *Ans.* I find $t = 119 m^{1{\cdot}175}$.

It is interesting to learn that for all kinds of races of men and animals we have $t \propto m^{1{\cdot}175}$.

CHAPTER XV.

IMPORTANT CURVES.

79. When, instead of a sheet of squared paper, we may use a drawing board with T and set squares, the following properties of some of the curves mentioned in Chap. XIV. become useful. OX and OY (Fig. 18) are the axes of co-ordinates. Let any two angles

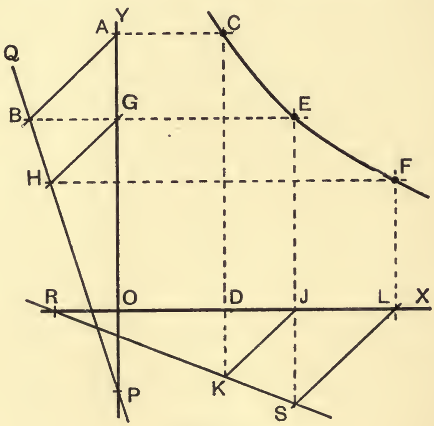

FIG. 18.

XRS and YPQ be set out from any points R and P in the axes. Choose any point C. Draw horizontal and vertical lines CA and CK. Make $BAO = 45°$. Make $KJD = 45°$, project from B and J to find the point E.* Find other points in the same way. The points CEF, etc., all lie upon a curve

$$y + a = c(x + b)^{-m}$$

* It is easy to show that the angles BAO and DJK need not be 45° nor be equal to one another. If all the angles PAB, PGH are equal to one another, each being, say, 60°, and if all the angles RLS, RJK are equal, each being, say, 45°, the curve still follows the above law.

where the distance OR is b, the distance OP is a, and the angles are such that $1+\tan QPY=(1+\tan XRS)^m$.

Again, in Fig. 19, starting with C, draw CD and CA. The

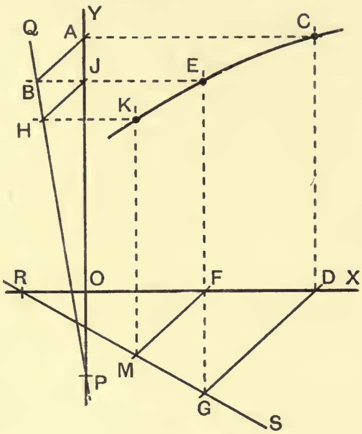

FIG. 19.

angles ODG and BAO are each made 45°. The point E is found by projecting from G and B. The points C, E, K, etc., all lie upon a curve

$$y+a=c(x+b)^n.$$

When n is positive, OR is b, OP is a, and

$$1+\tan QPY=(1+\tan XRS)^n.$$

If a curve is given, it is very easy to find whether it follows such a law as $y=c(x+b)^n$.

Thus, suppose the curve CEF (Fig. 18) and the axes OX, OY to be given; make OP zero, that is, let PQ start from O instead of P. Draw CA, AB, and BE as before. From E project to J. Make $OJK=45°$; let JK meet CD in K. Find other points, such as K; see if they meet in a straight line SKR.

In fact, if either a or b is zero the problem is easy, but if neither is zero the test requires some ingenuity.

Suppose, again, we have made observations of x and y, and we wish to test whether they follow the law $y=a+be^{cx}$.

Calculate $X=e^x$ and plot y and X.

We have to test for $y=a+bX^c$, which is the above case.

It will be noticed that $y=a+be^{x+c}$ is not more general than

$$y=a+be^x.$$

It would be easy to try for the law $y=b(e^x+c)^n$, as this becomes $y=b(X+c)^n$ when $X=e^x$.

Ex. 1. It is believed that the following observed quantities follow the law $y=a+bx^n$. Plot them on squared paper; draw the curve that lies most evenly among the points. Let CEK (Fig. 19) be the curve. In this case the point R is at O. Draw XRS (it will be XOS) any angle. From a point C on the curve draw CA horizontally and CD vertically. Make $OAB=45°$ and $ODG=45°$. Project vertically from G to F and E; make $OFM=45°$; project horizontally from E to J and B. Make $OJH=45°$. Proceed in this way, finding points B, H, etc. If the law is true, B, H, etc., ought to lie in a straight line. Drawing the straight line that lies most evenly among them, we find P. The distance OP is $-a$.

x	336·2	465	587·4	628	813
y	651	871	961	1011	1210

Ans. $y=210+2{\cdot}21x^{0{\cdot}914}$.

Ex. 2. The following measurements of pressure p and volume v were made on a gas-engine indicator diagram. It is thought that the law is like pv^n constant.

Test for this by plotting $\log p$ and $\log n$. If you do not find

p	44·7	53·8	73·5	85·8	113·2	135·8
v	7·03	5·85	4·30	3·50	2·50	1·90

a straight line it may be that there is a constant error in v due to wrong measurement of the clearance. Try therefore by the above method for the law $$p=c(v+a)^{-n}.$$

Plot v horizontally as x in Fig. 18, and v vertically. Take the point P at O and proceed as directed.

Ans. I find points like SK, etc., lying in a straight line. Drawing this line I find R and $OR=2{\cdot}10$.

Ex. 3. In the last exercise, increase every v by the amount 2·10 and call it u. Plot $\log p$ and $\log u$ on squared paper and try for the law. I find $$p(v+2{\cdot}1)^{1{\cdot}356}=900.$$

Ex. 4. It is suspected that the following quantities observed in the laboratory follow the law

$$y=a+be^{cx}.$$

x	0·5	0·7	1·0	1·2	1·4	1·6	1·8	2·0
y	3·730	3·912	4·325	4·709	5·242	5·953	6·927	8·213

First compute the values of e^x and call them X. Now plot y and X and draw the best curve. Proceed as in Fig. 19, making $OR=0$. I find OP to be $-3{\cdot}2$. I would now write out a new table of y and X as corrected by the curve, and I would plot $\log(y+3{\cdot}2)$ and $\log X$. These ought to give a straight line. I find the answer

$$y=3{\cdot}2+0{\cdot}25e^{1{\cdot}5x}.$$

80. If three points only are given and I am asked to join them by means of a curve, and I do not know anything about the nature of the curve except that it ought to be simple, I usually make a choice between

$$y=a+bx+cx^2, \quad\ldots\ldots(1)$$
$$y=a+bx^n, \quad\ldots\ldots(2)$$
$$y=a+be^{cx}. \quad\ldots\ldots(3)$$

Exercise. Given the three points

x	1·5	3·5	5·0
y	6·24	16·45	27·07

Let (1), (2) or (3) be the equation to the curve passing through the three points and find the values of the constants.

Ans. $y=1{\cdot}54+2{\cdot}284x+0{\cdot}5643x^2, \quad\ldots\ldots(1)$

$y=2{\cdot}742+1{\cdot}823x^{1{\cdot}61}, \quad\ldots\ldots(2)$

$y=-15{\cdot}83+16{\cdot}57e^{0{\cdot}190x}. \quad\ldots\ldots(3)$

To make (2) or (3) pass through three points needs a squared paper method of solving an equation. Take for example the more tedious form (3) Let e^x be called X. Let X_1, X_2, and X_3 correspond to the three given values of x, and let y_1, y_2, and y_3 be the three given values of y.

$$y_1=a+bX_1^c,$$
$$y_2=a+bX_2^c,$$
$$y_3=a+bX_3^c.$$

Subtracting to get rid of a and dividing to get rid of b, we have

$$\frac{y_3-y_2}{y_2-y_1}=\frac{10{\cdot}62}{10{\cdot}21}=1{\cdot}0402=\frac{X_3^c-X_2^c}{X_2^c-X_1^c}.$$

Clearing of fractions and dividing by X_1^c, we have

$$e^{3{\cdot}5c}-2{\cdot}0402e^{2c}+1{\cdot}0402=0.$$

By trial and the use of squared paper, I find $c=0{\cdot}190$, and it is easy to find $b=16{\cdot}57$, $a=-15{\cdot}83$.

81. Use and Misuse of Empirical Formulæ. As a result of experiments, the following corresponding values of x and y were obtained:

x	4	5	6	7	8	9	10	11
y	6·29	5·72	5·22	4·78	4·39	4·06	3·75	3·48
y'	6·28	5·66	5·15	4·72	4·35	4·05	3·78	3·55
y''	6·23	5·72	5·26	4·83	4·44	4·08	3·75	3·45

These being plotted on squared paper and tested in various ways, it was found difficult to say which of the two empirical formulæ,

$$y = \frac{57{\cdot}12}{5{\cdot}1 + x} \quad \text{.......................(1)}$$

or

$$y = 8{\cdot}71\epsilon^{-0{\cdot}08445x}, \quad \text{.......................(2)}$$

more exactly represented the observed numbers. The values given by (1) are tabulated as y', and those given by (2) are tabulated as y'', and it will be seen that the discrepancies are much the same for both the formulæ, and in no case are they large.

It is not unusual for the man who discovers an empirical formula to use it for extrapolation. Now, if we use (1) or (2) for such a purpose, calculating y for $x=1$ or $x=20$, note the difference in the answers:

x	1	20
y'	9·36	2·28
y''	8·005	1·61

Either formula is, therefore, good enough for calculating y inside the experimental range; and good enough, no doubt, for many purposes, but it is evident that extrapolation is dangerous.

[The following remarks will be better understood after the student has reached Chapter XVIII.]

Furthermore, (1) and (2) represent quite different laws:

$-\frac{dy}{dx}$ from (1) is proportional to y^2, whereas $-\frac{dy}{dx}$ from (2) is proportional to y.

When, therefore, a man *smooths* his plotted points, drawing the curve which lies most evenly among them, it is most important to know the law of smoothing, for if he uses different kinds of curves he may obtain very different results. All of them may be accurate enough for some purposes and wholly misleading for others.

When there is no guiding from theory, an empirical formula may be adopted, deduction from which may be quite misleading. Again, when we have a hypothesis of which we desire to make a theory to fit the facts, it is often much too easy to do so.

It is of course mainly in finding $\frac{dy}{dx}$ or the higher differential coefficients that empirical formulæ are dangerous.

When men *smooth* their observations by means of a curve they are really doing the same thing as if they used an empirical formula.

I have not tried, but I should think that the three formulæ of the exercise in Art. 79, which probably all give nearly the same values of y for any value of x between 1·5 and 5, give quite different values of y for such values of x as 1 or 10.

CHAPTER XVI.

SQUARED PAPER.

82. When, as the result of an investigation, we have arrived at a formula connecting y and x, we sometimes try to replace it by an approximate **simpler formula**.

Ex. 1. Plot $y = 9{\cdot}9 + 0{\cdot}6167x + 0{\cdot}35x^2 - \frac{1}{15}x^3$. ....(1)

Find the nearest approximate linear law between the values $x = 0{\cdot}5$ and $x = 3{\cdot}0$.

Ans. $y = 9{\cdot}7 + 1{\cdot}14x$. ....(2)

Of course the student sees that he takes values of x, in each case calculates y, and plots (1) on squared paper. He then finds the straight line which lies most evenly among his points, and so finds (2). This simple formula must not be used for values of x outside the range 0·5 to 3·0.

Ex. 2. In my theory of Struts it simplified the reasoning enormously—in fact, without this, reasoning was practically impossible—to be able to use

$$\frac{a}{1 - bx} \text{ and } \frac{1}{\cos\sqrt{x}}$$

as if they were the same.

Let $\frac{1}{\cos\sqrt{x}}$ be called y. For values of x from 0 to 0·9 calculate y and tabulate. We must try if y is approximately represented by the other formula or

$$y = \frac{a}{1 - bx};$$

that is, $y - bxy = a$. If, now, we plot y and xy as the co-ordinates of points on squared paper, or if we plot x and $\frac{1}{y}$, we ought to get (approximately) a straight line. We choose the best straight line and calculate a and b. *Ans.* $a = 1{\cdot}003$, $b = 0{\cdot}471$.

Ex. 3. In Art. 26 we found that if n is the number of years in which a sum of money will double itself at compound interest at r per cent. per annum, then

$$n = \log 2 \div \log\left(1 + \frac{r}{100}\right).$$

Take various values of r, say 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, and in each case calculate n.

Now, to find if there is not a simple approximate law connecting n and r, try various systems of plotting, such as we have tried in other cases. If you plot n and $\frac{1}{r}$, you will find your points lying so nearly in a straight line that you know you may use the approximate formula

$$n = \frac{70}{r}$$

if r is not greater than 5. (See also Ex. 5, Art. 113.)

Ex. 4. Plot $$y = 1 + \sqrt{1 + x^2}.$$

Within what limits of values of x may we take

$$y = 1{\cdot}23 + 0{\cdot}98x$$

as equivalent to it. *Ans.* From $x = 2$ to $x = 10$ or more.

It may be deduced from this, that we may take $a + \sqrt{a^2 + x^2}$ as nearly equivalent to

$$1{\cdot}23a + 0{\cdot}98x,$$

between the limits $x = 2a$ and $x = 10a$.

83. Any equation in x, however complicated, **may be solved.**

Ex. 1. Find a value of x which will satisfy the equation

$$4{\cdot}3x^{2{\cdot}56} - 5{\cdot}4e^{-0{\cdot}3x} + 17x^{-3{\cdot}4} - 12 = 0.$$

To solve this equation, let us say that

$$y = 4{\cdot}3x^{2{\cdot}56} - 5{\cdot}4e^{-0{\cdot}3x} + 17x^{-3{\cdot}4} - 12.$$

Take various values of x, calculate the corresponding values of y, and plot on squared paper. It is easy to obtain from this an approximation to the value of x which will give $y = 0$. This leads me to take $x = 1{\cdot}2$, and I find $y = 0{\cdot}280$. Now I try $x = 1{\cdot}25$, and I find $y = -0{\cdot}140$. Hence the correct value of x is between these. Plotting these points to a large scale and joining by a straight line, I see that $x = 1{\cdot}23$ is probably right, and on trial I find it sufficiently correct for all practical purposes. By trying $1{\cdot}231$ and $1{\cdot}229$, and plotting the results to a still larger scale, I can obtain even a closer approximation; and, indeed, there is no limit to the accuracy with which one may solve such an equation but that due to want of accuracy of the tables used in calculation.

If there are several values of x which will cause y to be 0, of course they also are answers. Every value of x which satisfies an equation is called a *root* of the equation.

Ex. 2. The following equation may be very easily solved in another way, but the student ought to solve it as an illustration of our squared-paper method of working.

$$x^2 - 5{\cdot}11x + 5{\cdot}709 = 0.$$

Let $y = x^2 - 5{\cdot}11x + 5{\cdot}709$.

Taking the following values of x, I have calculated the values of y:

x	0	1	1·5	2·0	2·5	3	3·5	4	5
y	5·709	1·599	0·294	−0·511	−0·816	−0·621	0·074	1·269	5·159

Plotting these on squared paper, the axes being OX and OY, I get a curve.

Notice that it cuts OX in two points, P and Q, so that there are two roots to the equation, OP and OQ.

By closer approximation, as in the last case, near $x = 1{\cdot}64$ or $1{\cdot}66$, and $x = 3{\cdot}45$ or $3{\cdot}47$, I find that the two roots are 1·65 and 3·46. Solving the equation in the usual way (see Chap. V.), I find these same answers.

How many advanced students can solve even a cubic equation? In practical work we may have to solve any equation whatsoever, and we now have a method of doing this with any amount of accuracy that may be desired.

Ex. 3. Find a value of x to satisfy

$$5{\cdot}3e^{0{\cdot}104x}\sin^2 0{\cdot}8x + 0{\cdot}78x^{1{\cdot}52}\cos x - 2{\cdot}126 = 0.$$

The student must remember that $0{\cdot}8x$ is in radians, and must be multiplied by 57·296 to convert it to degrees. *Ans.* $x = 0{\cdot}74$.

Ex. 4. Find the value of x for which

$$\tan x = 2{\cdot}46x.$$

Let $$y = \tan x - 2{\cdot}46x.$$

Taking a few values of x at random, and appealing occasionally to squared paper, one is able to get an approximation quickly. Of course, an angle x radians is $57{\cdot}296\ x$ degrees.

x	y
1	− ·903
1·5	+ large
1·3	·4038
1·2	− ·381
1·25	− ·066
1·26	+ ·015

The horizontal lines show the places during the calculation at which I appealed to squared paper to assist me in my next guess. *Ans.* $x = 1{\cdot}258$.

Ex. 5. Find in each case a value of x which satisfies each of the following equations:

$$2x^{2\cdot 5} - 5x - 8 = 0, \qquad \textit{Ans. } x = 2\cdot 55.$$
$$x^2 - 10 \log_{10} x - 3 = 0, \qquad \textit{Ans. } x = 2\cdot 7.$$
$$x^3 - 2x^2 - 400 = 0. \qquad \textit{Ans. } x = 8\cdot 10.$$

Ex. 6. Find the three values of x which satisfy

$$x^3 - 15x^2 + 61x - 75 = 0.$$

Ans. 3, 2·822, 9·178.

Ex. 7. Find a value of x which satisfies

$$\frac{1}{5}\sqrt{x} + \log_{10} x = 0\cdot 8558. \qquad \textit{Ans. } x = 3\cdot 162.$$

Ex. 8. Find a value of x (in radians) which satisfies

$$\sin 2x + 5 \sin x + 3x = 1\cdot 38. \qquad \textit{Ans. } x = 0\cdot 1386.$$

Ex. 9. In Chapter XV., to work another problem we had to find a value of c which satisfies the equation

$$e^{3\cdot 5c} - 2\cdot 0402\, e^{2c} + 1\cdot 0402 = 0. \qquad \textit{Ans. } c = 0\cdot 190.$$

CHAPTER XVII.

MAXIMA AND MINIMA.

84. Ex. 1. Divide the number 10 into two parts such that the product is a maximum.

Let x be one part, then $10-x$ is the other.

Let the product be called y, or

$$y=x(10-x) \quad \text{or} \quad 10x-x^2.$$

Take various values of x, calculate y in each case, and plot on squared paper. The maximum value of y is evidently given by $x=5$.

Ex. 2. When is the sum of a number and its reciprocal a minimum? Let x be the number; when is

$$y=x+\frac{1}{x}$$

a minimum? Taking the following values of x, we find y in each case:

x	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{3}$	2
y	$2\frac{1}{2}$	$2\frac{1}{12}$	2	$2\frac{1}{12}$	$2\frac{1}{2}$

On plotting, we see that $x=1$ is the answer.

Ex. 3. What is the greatest cylindric parcel to be sent by parcel post? The Post Office regulation is that length plus girth must not be greater than 6 feet. Let x be the radius of the circular section, then $6-2\pi x$ is the length. The volume is

$$v=\pi x^2(6-2\pi x).$$

What value of x will make v a maximum? Taking $x=0{\cdot}4$, $0{\cdot}5$, $0{\cdot}6$, $0{\cdot}7$, $0{\cdot}8$, etc., and in each case calculating v, plotting on squared paper, we find that $x=0{\cdot}636$ feet.

Ex. 4. Divide 10 into two parts such that three times the square of one part plus four times the square of the other shall be a minimum.

If x is one part, $10 - x$ is the other. When is

$$y = 3x^2 + 4(10 - x)^2$$

a minimum? Calculating and plotting, we find $x = 5{\cdot}71$, and the other part 4·29.

Ex. 5. A man at A is at sea 4 miles distant from the nearest point C of a straight shore. He wishes to get to a place B, which is 10 miles distant from this nearest point. He can row and walk. Find at what point D he ought to land to get to B in the minimum time, if he rows at 3 miles per hour and walks at 4 miles per hour. Assume that he can equally well leave his boat at one place as at another. Here $AC = 4$, $CB = 10$; let $CD = x$, then $AD = \sqrt{16 + x^2}$ and $DB = 10 - x$, so that the whole time in hours is

$$y = \frac{1}{3}\sqrt{16 + x^2} + \frac{1}{4}(10 - x).$$

Calculate y for various values of x; plot and find that $x = 4{\cdot}535$ miles gives the minimum time.

Ex. 6. The strength of a rectangular beam of given length, loaded and supported in any particular way, is proportional to the breadth of the section multiplied by the square of the depth. If a cylindric tree is 15 inches in diameter, what is the strongest beam that may be cut from it? Let x be its breadth. Draw a rectangle inside the circle and you will see that the depth is $\sqrt{225 - x^2}$. Hence the strength is a maximum if y is a maximum, where

$$y = x(225 - x^2) \quad \text{or} \quad y = 225x - x^3 \ldots .$$

Calculating y for various values of x and plotting, we find that the strength is a maximum when the breadth x is 8·66 and the depth is 12·25 inches.

Ex. 7. The stiffness of a rectangular beam of given length, loaded and supported in any particular way, is proportional to the breadth and the cube of the depth of its section. If a cylindric tree is 15 inches in diameter, what is the stiffest beam that may be cut from it? If x is the depth, the breadth is $\sqrt{225 - x^2}$, and we desire that

$$y = x^3\sqrt{225 - x^2}$$

shall be a maximum. Calculating y for various values of x and plotting, we find that the stiffness is a maximum when the depth x is 13 inches and the breadth 7·5 inches. Observe the difference between this and the answer to Ex. 6.

Ex. 8. When a certain vessel moves through the water at x knots the total cost in wages, depreciation, and interest on capital, stores, coals, etc., in pounds per hour, is $4 + 0{\cdot}001x^3$. For a passage of

1000 miles, at what speed ought the vessel to go if the total cost on the passage is to be a minimum? Here the number of hours is $\frac{1000}{x}$, so that the total cost is

$$\frac{1000}{x}(4+0{\cdot}001x^3) \quad \text{or} \quad 1000\left(\frac{4}{x}+0{\cdot}001x^2\right).$$

Let $$y=\frac{4}{x}+0{\cdot}001x^2.$$

For various values of x calculate y and plot. The answer will be found to be 12·60 knots.

Ex. 9. If v, the speed of water in a river, is 4 miles per hour, and x is the speed of a steamer against stream *relatively to the water*; if the total cost of the steamer per hour, including coals, is $0{\cdot}5+0{\cdot}0003x^3$, find the speed x so as to make the cost of an upstream passage of 100 miles a minimum.

The speed of the steamer *relatively to the bank* is $x-v$; the time of the passage is $\frac{100}{x-v}$ hours, and hence the cost of the passage is

$$\frac{100(0{\cdot}5+0{\cdot}0003x^3)}{x-v} \quad \text{or} \quad \frac{100(0{\cdot}5+0{\cdot}0003x^3)}{x-4}.$$

Letting $y=\frac{0{\cdot}5+0{\cdot}0003x^3}{x-4}$, calculating y for various values of x and plotting, it will be found that the best x is about 8·5. That is, the speed of the steamer relatively to the bank is 4·5 miles per hour.

Exercises of this kind can be worked generally algebraically (see Chap. XXII.), but this plotting method of working ought always to be resorted to also. It tells us what is the extra loss if the best speed is not adhered to, and very often the extra loss is not great.

Ex. 10. In a submarine cable, if d is the diameter of the copper wire and D is the diameter of the gutta-percha covering; the distance to which readable signals may be sent is greater as

$$y=d^2\log\frac{D}{d}$$

is greater. Take D as 10; for various values of d calculate y and plot on squared paper; what value of d is best? It is quite evident that we can use common logarithms. *Ans.* $d=6{\cdot}065$.

CHAPTER XVIII.

THE INFINITESIMAL CALCULUS.

85. Slope of a Straight Line. In Art. 60 we saw that $y = a + bx$ is a straight line.

I have called b the slope of the line. We saw that if x increases by the amount 1, y increases by the amount b.

Let CD be the line; the distance OC represents a. P and R are two points in the line, and if PQ is 1, then QR is b.

b is the rise for 1 horizontal.

Note that when we say that a road rises $\frac{1}{20}$ or 1 in 20, we mean 1 foot rise for 20 feet *along* the sloping road. Thus $\frac{1}{20}$ is the sine of the angle of inclination of the road to the horizontal. Our *slope* is different, being the *tangent* of the angle RPQ. [For the meanings of such terms as sine and tangent the student is referred to Art. 34.] Looking upon y as a quantity whose value depends on that of x, observe that *the rate of increase of y relatively to the increase of x is constant,* being indeed b, the slope of the line. The symbol used

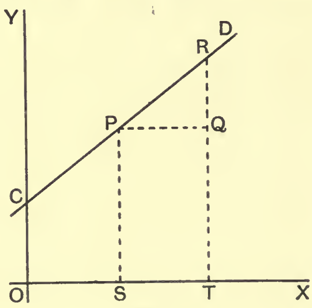

FIG. 20.

for this rate is $\frac{dy}{dx}$. Note that it is one symbol; it does not mean $\frac{d \times y}{d \times x}$. Try to recollect the statement that if $y = a + bx$, then $\frac{dy}{dx} = b$, and that if $\frac{dy}{dx} = b$, then it follows that $y = A + bx$, where A is some constant or other.

Exercise. If $4x + 3y = 7$, find $\frac{dy}{dx}$.

Here $y=\frac{7}{3}-\frac{4}{3}x$, so that $\frac{dy}{dx}=-\frac{4}{3}$.

Now, if a road is of perfectly constant slope, it is easy to measure this slope. It rises 1 foot, say, for 10 horizontal, or 2 feet for 20 horizontal, or 3 feet for 30 horizontal. $\frac{dy}{dx}$ is 0·1. But if the slope of a road is continually altering, and we want to know the slope at a particular place, we really only measure an average slope, however small our distances may be.

86. In the curve of Fig. 21 there is a positive slope (y increases as x increases) in the parts AB, CD, EF, GH, and negative slope

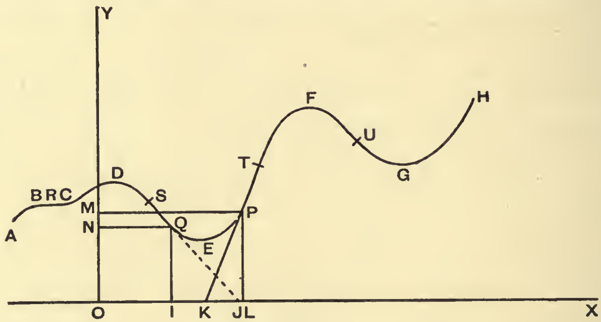

FIG. 21.

(y diminishes as x increases) in the parts DE and FG. The slope is zero at D and F, where y is said to reach *maximum* values, and it is also zero at E and G, where y is said to reach *minimum* values; it is also zero at R, which is neither a point of maximum nor minimum. It must be evident to a student that the slope of a curve at a point is the slope of the tangent to the curve at that point. Suppose we want to know the slope at the point P (Fig. 21). Imagine Fig. 22 to be a great magnification of the curve at point P.

If PD or OI is x and PI is y, let F be another point and let FC or OH be $x+\delta x$ and FH be $y+\delta y$. The student must note that δx is a symbol for a small increment to x; a new kind of symbol. It does not mean a quantity δ multiplied by a quantity x. Then PG is δx and FG is δy, so that $\frac{FG}{PG}$ or $\frac{\delta y}{\delta x}$ is the *average* slope from P to F, and this is really $\tan FPG$.

Now let F be nearer P, say at F', or nearer still, at F''. In every case the average slope is the tangent of the angle made by FP

or $F'P$ or $F''P$ with the horizontal. But it is not until δy and δx are imagined to get smaller and smaller without limit that $\frac{\delta y}{\delta x}$ can be called the actual slope at P. The line joining F and P or F' and

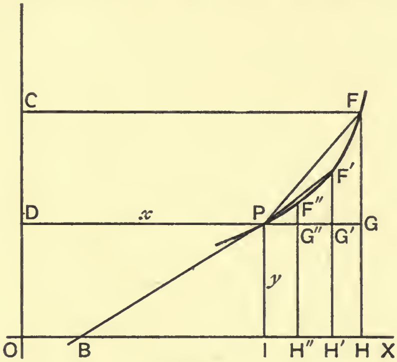

FIG. 22.

P or F'' and P gets to be more and more nearly what we call the tangent at P as F or F' or F'' gets nearer and nearer to P.

When we speak of δx getting smaller and smaller and $\frac{\delta y}{\delta x}$ reaching a limiting value, we distinguish this limiting value by the symbol $\frac{dy}{dx}$, and it is evidently $\tan PBX$ if PB is the tangent to the curve at P. I do not think that anybody can draw a tangent to any curve at a given point very accurately, but if he could he would have no difficulty in finding $\frac{dy}{dx}$ at the point. It is a pity that the word *tangent* is used for two different things. $\frac{dy}{dx}$ is the tangent of the angle that the tangent to the curve makes with the horizontal.

In Fig. 21 the slope at P is $\tan PKX$, and is positive. The slope at Q is $\tan QJX$, and the tangent of an obtuse angle is negative.

If, in Fig. 21, $PL=y$ is, say, **population**, and $OL=x$ means time after some date; the slope at P is $\frac{dy}{dx}$, the rate at which the population is increasing per annum. The scale to which this rate is represented by $\tan PKL$ or the slope, is easily found, because the rate is

$$\frac{\text{the population represented by } PL}{\text{time represented by } KL}.$$

87. If s is the distance in miles that a train has reached from Euston at the time t, how do we study the motion? If a Bradshaw's railway guide gave, not merely the time of reaching a few stations, but the time of reaching say fifty places between London and Northampton, the record would be something like this, only the student may imagine many more entries; entries perhaps for every half-mile

Hours and minutes.	t Time in hours since leaving Euston.	s Distance in miles from Euston.
10.20 o'clock - - - -	0	0
10.32 ,, - - - -	·2	6
10.35 ,, - - - -	·25	10
10.40 ,, - - - -	·33	14
10.44 ,, - - - -	·40	14
10.47 ,, - - - -	·45	18
10.53 ,, - - - -	·55	24
etc.	etc.	etc.

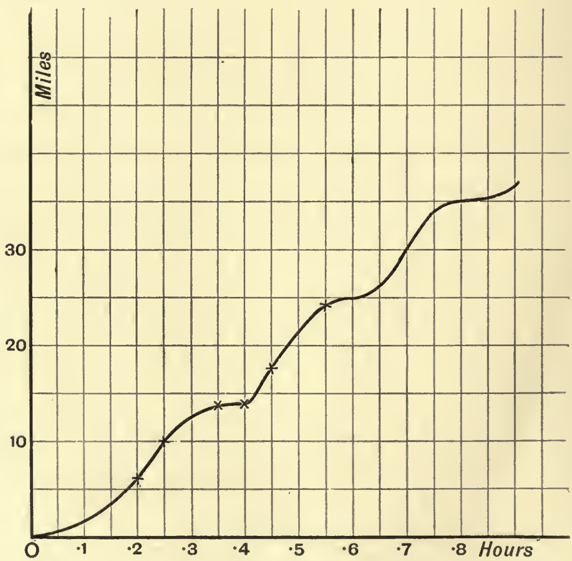

Fig. 23.

If we plot s and t on squared paper as in Fig. 23, it is evident that the slope of this curve everywhere represents the speed of the train. See how the speed quickens on leaving Euston, until from $s=5$ to $s=11$ miles it keeps nearly constant, but diminishes and becomes 0 at $s=14$ miles. In fact, the train has stopped; then we see the speed getting up again, then diminishing and increasing until at $s=24$ there is a crawl for a mile.

If we represent the train as coming back again, of course, there will be negative slope, as the velocity is negative.

Ex. 1. If s and t are carefully plotted, the slope of the tangent, which can be carefully drawn at every point, shows the speed; but tangents are not easy to draw with accuracy. The following method is better:—First plot carefully. Now draw a curve through the plotted points. Tabulate the values of s for *equidistant* values of t. Here is a particular case. In a new mechanism it was necessary for a certain purpose to know in every position of a point what its acceleration was. A skeleton drawing was made and the positions of the point marked at intervals of time from a time taken as 0. In the table I give at each instant the distance of the point from a fixed point of measurement, and I call it s feet.

t Seconds.	s Feet.	v Feet per second or $\delta s/\delta t$.	Acceleration in feet per second per second or $\delta v/\delta t$.
·06	·0880		
		14·74	
·07	·2354		−125
		13·49	
·08	·3703		−127
		12·22	
·09	·4925		−127
		10·95	
·10	·6020		−129
		9·66	
·11	·6986		−131
		8·35	
·12	·7821		−131
		7·04	
·13	·8525		

If a body of 200 lbs. has this motion, what force at every instant must be acting upon it to maintain the motion?

The mass in Engineers' units is $200 \div 32{\cdot}2$, or 6·2.

Multiply 6·2 by an acceleration 127, and we find the force to be 787 lbs.

When in the above table I find a velocity of, for example, 14·74 feet per second, remember that it is the average speed from $t=0{\cdot}06$ to $t=0{\cdot}07$; I have assumed that it is the *real* speed at $t=0{\cdot}065$. But this, generally, can only be approximately correct. I have found the assumption to be accurate enough for many practical purposes. There are obvious tedious ways of getting better approximations.

But I must consider what is the velocity at a particular instant more carefully, and I therefore take up the following exercise.

88. Exercise. The motion of a train leaving Euston is such that

$$s = 300t^2,$$

s being in miles from Euston, t in hours from leaving Euston.

At the end of 0·1 hour, find v, the velocity of the train in miles per hour.

Take $t = 0·01$, 0·02, etc., and in each case calculate s.

Plot on squared paper. I get the curve $OQPR$ (Fig. 24.) At P, $t = 0·1$ hour, $s = 3$ miles; drawing the tangent PK at P, I find that in the scale of time KL represents 0·05 hour and in the scale of distance PL is 3 miles. Hence the slope at P represents $\frac{3 \text{ miles}}{0·05 \text{ hour}}$, or 60 miles per hour. As a matter of fact, this is the correct answer, but I did not get it by drawing the tangent. Nobody could draw a tangent so as to get the perfectly correct answer. I got the answer in the following way. I calculated s for the following times:

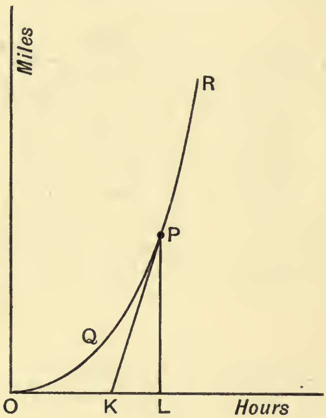

Fig. 24.

δt is the excess time beyond 0·1 hour.

δs is the distance in excess of 3 miles.

$\frac{\delta s}{\delta t}$ is the average speed during the interval δt, the space δs being passed over in this interval.

t	s	δt	δs	$\frac{\delta s}{\delta t}$
·1	3			
·11	3·63	·01	·63	63
·101	3·0603	·001	·0603	60·3
·1001	3·006003	·0001	·006003	60·03

It is evident that as we take δt less and less we are getting nearer and nearer to the actual speed at the time 0·1.

To be quite sure of this let the work be done algebraically.

At the time t calculate s; again, for the new time $t + \delta t$ calculate $s + \delta s$.

We have

$$s = 300t^2, \text{ and } s + \delta s = 300(t + \delta t)^2, \text{ or } s + \delta s = 300\{t^2 + 2t \cdot \delta t + (\delta t)^2\}.$$

Subtracting, we get

$$\delta s = 600t \cdot \delta t + 300(\delta t)^2.$$

Now divide by δt, and we get average speed

$$\frac{\delta s}{\delta t} = 600t + 300 \,.\, \delta t.$$

Please notice that this is exactly true for any value of δt. Now I come to the important idea; as δt gets smaller and smaller, $\frac{\delta s}{\delta t}$ approaches more and more nearly $600t$, the other term $300\delta t$ becoming smaller and smaller without limit.

Creatures of our limited senses and faculties cannot possibly comprehend *infinity*; but, for mathematical purposes, when we say that a thing is infinitely great, we mean that the thing is greater and greater without limit.

When we say that a thing becomes smaller and smaller without limit, we mean that it becomes 0; that is what we mean by *nothing*. Hence in the limit $\frac{\delta s}{\delta t}$ is truly $600t$. The limiting value of $\frac{\delta s}{\delta t}$ as δt gets smaller and smaller is called $\frac{ds}{dt}$ or the rate of change of s as t increases, or the differential coefficient of s with regard to t, or it is called the velocity at the time t. It is only when δt is made smaller and smaller without limit that we speak of the actual velocity $\frac{ds}{dt}$ or make the statement:

$$\text{Actual velocity at time } t \text{ is } \frac{ds}{dt} = 600t.$$

This formula enables us to find the velocity at any time. When t is 0·1, the velocity is $600 \times 0{\cdot}1$ or 60 miles per hour.

Now, surely there is no such great difficulty in catching the idea of a limiting value. Some people have the notion that we are stating something that is only approximately true; it is often because their teacher will say such things as "reject $300\delta t$ because it is small" or "let dt be an infinitely small amount of time," and they proceed to divide something by it. Men like this are incapable of acquiring common sense.

Another trouble is introduced by people saying, "Let $\delta t = 0$ and $\frac{\delta s}{\delta t}$ or $\frac{ds}{dt}$ is so and so." The true statement is, "As δt gets smaller and smaller without limit, $\frac{\delta s}{\delta t}$ approaches more and more nearly the finite value $600t$," and, as I have already said, everybody uses the important idea of a limit every day of his life.

The student ought to give a good deal of thought to this idea of a limiting value of $\frac{\delta s}{\delta t}$ as δt is made smaller and smaller. Of course δs gets smaller and smaller also.

He must not say, "Let δs be 0 and let δt be 0," because 0 divided by 0 may mean anything. Let him consider carefully what he means by velocity. If a body moves uniformly through 1 foot in 1 second it moves through 0·001 foot in 0·001 second, and it moves through the millionth of a foot in the millionth of a second. The ratio of the space to the time is the velocity, and if velocity is changing, the velocity at any instant can only be found when we measure an excessively short space and the excessively short time in which it is described.

The plain man of common sense finds no difficulty in catching the idea. Two thousand years ago neither he nor a small boy would have had a difficulty in understanding that a hare would beat a tortoise in a race; it is the mathematical philosopher who makes a difficulty about such matters, and in these days he says that this fundamental idea of the calculus can only be comprehended by a mathematician. This would not matter if these philosophers were not entrusted with the education of youth, a trust for which all their training has unfitted them. When they come to explain the essential idea of the limiting value of $\frac{\delta s}{\delta t}$ they talk foolishly.

89. Let me give the idea in another form. Suppose y and x to be any quantities whatsoever, connected by the law

$$y = ax^2. \quad \text{.................................}(1)$$

Take a particular value of x and calculate y. Now take a new value of x, say $x + \delta x$, and calculate the new y, calling it $y + \delta y$;

$$y + \delta y = a(x + \delta x)^2,$$

or

$$y + \delta y = a\{x^2 + 2x \,.\, \delta x + (\delta x)^2\}. \quad \text{.....................}(2)$$

Subtracting (1) from (2), we have

$$\delta y = 2ax \,.\, \delta x + a \,.\, (\delta x)^2;$$

divide by δx, and we have

$$\frac{\delta y}{\delta x} = 2ax + a \,.\, \delta x. \quad \text{..............................}(3)$$

Now (3) is true for any value of δx. It gives the *average* rate of increase of y in regard to x. But it is only when we let δx get smaller and smaller without limit that we say

$$\frac{dy}{dx} = 2ax, \quad \text{.................................}(4)$$

and the actual rate of increase of y with regard to x is known for any value of x.

90. Similarly, if

$$y = ax^3, \quad \text{.................................}(1)$$

$$y + \delta y = a(x + \delta x)^3,$$

$$y + \delta y = a\{x^3 + 3x^2 \,.\, \delta x + 3x \,.\, (\delta x)^2 + (\delta x)^3\}. \quad \text{...........}(2)$$

Subtracting (1) from (2), and dividing by δx, we get

$$\frac{\delta y}{\delta x} = 3ax^2 + 3ax \, . \, \delta x + a(\delta x)^2.$$

Now, letting δx get smaller and smaller without limit, we use the new symbol

$$\frac{dy}{dx} = 3ax^2.$$

91. With a little more knowledge of algebra than I presume in my hearers, it will be found easy to prove (see Art. 95) that if

$$y = ax^n,$$

where a and n are any numbers whatever, positive or negative,

$$\frac{dy}{dx} = nax^{n-1}.$$

This rule comprehends the others that I have given. In fact, if

$$y = a + bx + cx^2 + \text{etc.} + mx^n, \quad \text{......................(1)}$$

$$\frac{dy}{dx} = 0 + b + 2cx + \text{etc.} + nmx^{n-1}. \quad \text{..................(2)}$$

If y is given as a function of x, we are said to *differentiate* when we find $\frac{dy}{dx}$.

92. In some of the above exercises we had y given as a function of x, and we were asked to find $\frac{dy}{dx}$. We see that this rule is very much easier to apply than any rule that requires us to plot the curve on squared paper. $\frac{dy}{dx}$ is sometimes called the slope of the curve, or the tangent of the angle which the tangent to the curve makes with the axis of x, or it is sometimes called "the **Differential** Coefficient of y with regard to x," or "the rate of increase of y with regard to x."

93. Sometimes we are asked to **Integrate**; that is, when given $\frac{dy}{dx}$ we are asked to find y.

Thus given (2), above, we may be asked to find (1).

The integral of α is αx, the integral of βx is $\frac{1}{2}\beta x^2$, the integral of γx^2 is $\frac{1}{3}\gamma x^3$, and, generally, the integral of ax^m is $\frac{a}{m+1}x^{m+1}$ for any value of m.* When we integrate we add a constant, which may be *any* constant, because the differential coefficient of a constant is 0.

Ex. 1. Differentiate

$$y = 4 + 3x + 0{\cdot}7x^2 + 2{\cdot}15x^3 + 15x^{20} + 12x^{-1} + 24x^{-3{\cdot}04} + 2x^{0{\cdot}786}.$$

Ans. $$\frac{dy}{dx} = 0 + 3 + 1{\cdot}4x + 6{\cdot}45x^2 + 300x^{19} - 12x^{-2} - 72{\cdot}96x^{-4{\cdot}04} + 1{\cdot}572x^{-0{\cdot}214}.$$

Ex. 2. Integrate $3{\cdot}056 + 2x + 15x^2 + 12x^{15} + 1{\cdot}5x^{-2} + 17x^{-0{\cdot}46} + 2x^{4{\cdot}567}$.

Ans. $C + 3{\cdot}056x + x^2 + 5x^3 + \frac{3}{4}x^{16} - 1{\cdot}5x^{-1} + 17{\cdot}8x^{0{\cdot}954} + 0{\cdot}3592x^{5{\cdot}567}$.

The symbol $\frac{d^2y}{dx^2}$ means the differential coefficient of $\frac{dy}{dx}$. The differential coefficient of $\frac{d^2y}{dx^2}$ is indicated by $\frac{d^3y}{dx^3}$.

* This rule fails when $m = -1$. The integral of ax^{-1} is $a \log_e x$. See Art. 95.

CHAPTER XIX.

FORMULÆ AND PROOFS.

94. When the student knows how to differentiate and integrate x^n he can utilize the calculus in most practical engineering problems. The ordinary student spends months in differentiating and integrating all sorts of curious expressions for which he seldom has any practical use afterwards. Besides x^n, it may here be worth while to give a few others.

y	$\frac{dy}{dx}$	u	The integral of u, written $\int u\,.\,dx$.
a constant	0	0	a any constant
ax	a	b	bx
ax^2	$2ax$	bx	$\frac{1}{2}bx^2$
ax^n	nax^{n-1}	bx^m	$\frac{b}{m+1}x^{m+1}$
$\log x$	$\frac{1}{x}$	$\frac{1}{x}$	$\log x$
$\log(x+a)$	$\frac{1}{x+a}$	$\frac{1}{x+a}$	$\log(x+a)$
be^{ax}	abe^{ax}	be^{ax}	$\frac{b}{a}e^{ax}$
$A\sin(ax+b)$	$aA\cos(ax+b)$	$B\cos(ax+b)$	$\frac{B}{a}\sin(ax+b)$
$A\cos(ax+b)$	$-aA\sin(ax+b)$	$B\sin(ax+b)$	$-\frac{B}{a}\cos(ax+b)$

In future, $\log x$ will always mean the Napierian logarithm of x; the common logarithm will be written $\log_{10} x$.

Any other letters than x and y and u may be employed.

95. The only proofs which I can give for the present are these:

1. If $y=x^n$, $\qquad y+\delta y=(x+\delta x)^n$,

or $$y+\delta y=x^n+n\,.\,\delta x\,.\,x^{n-1}+\frac{n(n-1)}{1\,.\,2}(\delta x)^2x^{n-2}+\text{etc.},$$

by the Binomial Theorem. See Art. 28, Ex. 5.

Subtracting, $$\delta y=n\,.\,\delta x\,.\,x^{n-1}+\frac{n(n-1)}{1\,.\,2}(\delta x)^2\,.\,x^{n-2}+\text{etc.},$$

Therefore $$\frac{\delta y}{\delta x}=nx^{n-1}+\frac{n(n-1)}{1\,.\,2}\delta x\,.\,x^{n-2}+\text{etc.}$$

Now let δx get smaller and smaller without limit; we see that all the terms which have δx or $(\delta x)^2$ or higher powers of δx become zero, and so

$$\frac{dy}{dx}=nx^{n-1}.$$

2. If $y=e^x$. I have tried in many ways and failed to find a proof of our rule that does not involve the exponential theorem.

Here is the simplest form of this theorem (see Art. 28, Ex. 6):

$$y=e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\text{etc.}$$

Differentiating this term by term, we find

$$\frac{dy}{dx}=0+1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\text{etc.}=e^x=y.$$

In the same way we can show that if $y=e^{ax}$, then

$$\frac{dy}{dx}=ae^{ax}.$$

3. If $y=\log x$, then $\quad x=e^y,\ \dfrac{dx}{dy}=e^y=x,$

or $$\frac{dy}{dx}=\frac{1}{x}.$$

4. If $y=\sin x$, $$y+\delta y=\sin(x+\delta x),$$
$$\delta y=\sin(x+\delta x)-\sin x.$$

In elementary trigonometry it is shown (see Ex. 3, Art. 28) that this is

$$\delta y=2\cos(x+\tfrac{1}{2}\delta x)\sin\tfrac{1}{2}\delta x.$$

Hence $$\frac{\delta y}{\delta x}=\cos(x+\tfrac{1}{2}\delta x)\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

It is easy to see by drawing a small angle a and recollecting what $\sin a$ and a are, that $\sin a\div a$ becomes more and more nearly 1 as a gets smaller and smaller. Hence $\sin\frac{1}{2}\delta x\div\frac{1}{2}\delta x$ becomes 1 in the limit, and

$$\frac{dy}{dx}=\cos x.$$

We can in the same way show that if $y=\cos x$, then

$$\frac{dy}{dx}=-\sin x.$$

See Chapter XXXII. for an extension of these rules.

When preparing this course of lectures for the press I discovered new easy proofs of (1) and (2); they are given in Chapter XXXII.

CHAPTER XX.

THE CALCULUS.

96. I will now give an example of the usefulness of integration.

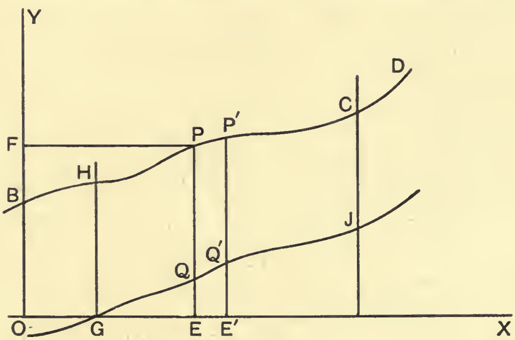

FIG. 25.

Let $BPCD$ be any curve; at the point P let FP or OE be called x, and PE be called y.

Let the ordinate QE of the curve GQJ represent A, the area of $HPEG$. I have taken HG anywhere as my starting ordinate from which the area is to be counted. Let OE' be called $x + \delta x$, so that $EE' = \delta x$.

Now $\qquad QE = A$ represents area $HPEG$,

$\qquad Q'E' = A + \delta A$ represents area $HP'E'G$.

Hence δA is the area $PP'E'E$.

But as δx is smaller and smaller, we find it to be more and more true that

$$\delta A = PE \times \delta x = y \,.\, \delta x.$$

Hence

$$\frac{dA}{dx} = y. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

That is, the ordinate of a curve is the differential coefficient of its area, or area A of a curve is the integral of y, the ordinate of the curve.

97. Exercise. Let OPM (Fig. 26) be the curve $y=ax^2$, a curve which is known as a parabola.

Then, integrating, $A=\frac{a}{3}x^3+C$,

where C is some unknown constant which depends upon where we count area from.

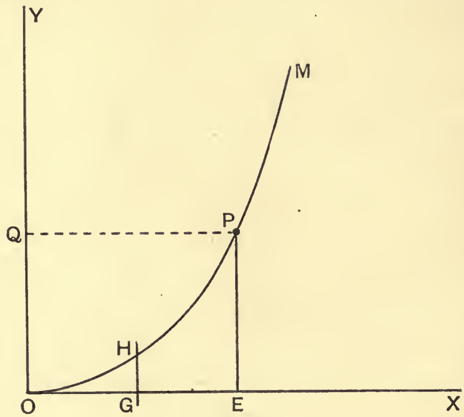

FIG. 26.

Thus if we count area from $x=0$, that is, let $A=0$ when $x=0$, then $C=0$, so that $A=\frac{a}{3}x^3$. Here we have a formula which gives us the area of the curve up to any value of x.

Again, find the area between the ordinates HG and PE.

$$A \text{ up to } PE \text{ from some ordinate} = \frac{a}{3}OE^3 + C,$$

$$A \text{ up to } HG \text{ from the same ordinate} = \frac{a}{3}OG^3 + C.$$

Hence area required $=\frac{a}{3}(OE^3-OG^3)$.

Suppose we want the above areas in terms of PE, etc., instead of a. As $y=ax^2$, and hence $PE=a\times OE^2$ or $a=PE/OE^2$, we can replace a in the above formulæ. Thus

$$\text{area of } POE = \frac{a}{3}OE^3 = \frac{1}{3}\frac{PE}{OE^2}\times OE^3 = \frac{1}{3}PE\,.\,OE$$

$$= \frac{1}{3} \text{ of the area of the rectangle } QPEO.$$

The proof of Simpson's rule given in a note to Art. 51 is a good exercise for students.

98. When using squared paper we saw that many other things are summed up exactly as areas are calculated. Thus, if y is the area of cross-section of a body with a straight axis at the distance x from one end; if there is a law connecting y and x; the integral of y with regard to x is the volume. Also the integral of xy divided by the volume gives the x of the centre of gravity.

Exercise. Suppose the rate of change of the velocity of a body per second $\frac{dv}{dt}$, which we call the acceleration, to be constant, being a feet per second per second. The integral of this with regard to t is v, the velocity, and it is

$$v = at + c, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where c is some constant. If v is v_0 when $t = 0$, then c is v_0. But v is $\frac{ds}{dt}$, so integrating again, we have

$$s = \frac{1}{2}at^2 + v_0 t + s_0, \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

for we add a constant on integrating, and we see that this constant must be s_0, the distance of the body from the zero of measurement when $t = 0$.

Newton used the symbol $\dot{s}$ for velocity; we may use this or $\frac{ds}{dt}$. He used $\ddot{s}$ or $\dot{v}$ for acceleration, which is our $\frac{d^2s}{dt^2}$ or $\frac{dv}{dt}$.

If we differentiate (2) we get (1); if we differentiate (1) we return to our old statement of acceleration $= a$.

99. The following values of x and y being given, find $\delta y/\delta x$.

Find also the values of $y\delta x$ and add to get A, the area of the y-curve.

The student must note that $\frac{\delta y}{\delta x}$ is the *average* rate in each interval, he does not know the exact value of x for which this is the rate. If we wish to know $\frac{dy}{dx}$ for any value of x from tabulated numbers very exactly, we must use some such rule as is given in Art. 100.

Note that in computing $y \,.\, \delta x$ I use the average value of y in the interval. $y \,.\, \delta x$ is the area of a strip like $EPP'E'$ of Fig. 25, and $\Sigma y \,.\, dx$ means the sum of the areas of strips, the Greek letter s or Σ being commonly used to express "the sum of all such terms as." When we find an area accurately and take an infinite number of infinitely narrow strips we replace Σ by the long English *s* or $\int$. It is most important to remember that when we express the integral of y we not only use the symbol $\int$ but also dx, that is, we use $\int y \,.\, dx$.

x	y	$\frac{\delta y}{\delta x}$	$y \,.\, \delta x$	A or $\Sigma y \delta x$
0	0			0·0000
		1·736	0·00868	
0·1	0·1736			0·00868
		1·684	0·02578	
0·2	0·3420			0·03446
		1·580	0·04210	
0·3	0·5000			0·07656
		1·428	0·05714	
0·4	0·6428			0·13370
		1·232	0·07044	
0·5	0·7660			0·20414
		1·000	0·08160	
0·6	0·8660			0·28574
		0·737	0·09028	
0·7	0·9397			0·37602
		0·451	0·09622	
0·8	0·9848			0·47224

The symbol $\int_a^b y \,.\, dx$ means "the area of the curve whose ordinate is y from $x=a$ to $x=b$." We already know how to find this. Or, what comes to the same thing, this symbol tells us, "Find the general integral of y, insert in it $x=b$, insert in it $x=a$, subtract." This is evident from Art. 97.

The student is again warned that he must not be frightened by algebraic symbols. A symbol is the handiest way of telling you exactly what to do. It is usually much easier to understand than the hieroglyphics scratched by tramps on a farmer's gate to give information about savage dogs or charitable women. It is the easiest thing imaginable to get familiar with innocent symbols like these that I have been using; yet the average student shuns them and never tries to understand them, because he has made up his mind that he cannot ever understand them.

Exercise. The parabola $y=0{\cdot}1x^{\frac{1}{2}}$ revolves about the axis of x and generates a paraboloid of revolution. Let the student plot the curve and study the shape of the surface Now the area of the cross-section at x is πy^2; the volume of a slice between two cross-sections at the distance dx asunder is

$$\pi y^2 \,.\, dx,$$

and the integral of this is the volume. Thus between sections at $x=a$ and $x=b$ the volume is

$$\pi\int_a^b y^2 \,.\, dx \quad \text{or} \quad \frac{\pi}{100}\int_a^b x \,.\, dx \quad \text{or} \quad \frac{\pi}{100}\left[\frac{1}{2}x^2\right]_a^b$$

$$\text{or} \quad \frac{\pi}{100}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right) \quad \text{or} \quad \frac{\pi}{200}(b^2 - a^2).$$

If $a=0$, so that we wish to know the volume from $x=0$ to $x=b$, the answer is $\frac{\pi}{200}b^2$. Call this V.

If instead of $y=0{\cdot}1x^{\frac{1}{2}}$ we had $y=mx^{\frac{1}{2}}$, then V would be $\frac{1}{2}\pi m^2 b^2$. If y_1 is the ordinate of the curve where x is b, it is easy to show that $V=\frac{1}{2}\pi y_1{}^2 b$. But $\pi y_1{}^2$ is the area of the end section, and we see that if a cylinder, a paraboloid, and a cone are on the same circular base and of the same vertical height their volumes are as 1 to $\frac{1}{2}$ to $\frac{1}{3}$.

Illustrations. Let the student manufacture a few illustrations like the following:

1. To illustrate that when $y=\sin x$, $\frac{dy}{dx}=\cos x$.

Angle in degrees.	x the angle in radians.	$y=\sin x$	δy	$\delta y/\delta x$	Average $\delta y/\delta x$.	$\cos x$
40	0·6981	0·6427876				
			0·0132714	0·7583		
41	0·7156	0·6560590			0·7547	0·7547
			0·0130716	0·7512		
42	0·7330	0·6691306				

2. To illustrate that when $y=\cos x$, $\frac{dy}{dx}= -\sin x$

Angle in degrees.	x the angle in radians.	$y=\cos x$	$-\delta y$	$\frac{\delta y}{\delta x}$	Average $\delta y/\delta x$.	$\sin x$
20	0·3491	0·9396926				
			0·0061122	−0·3513		
21	0·3665	0·9335804			−0·3584	0·3584
			0·0063965	−0·3656		
22	0·3840	0·9271839				

3. To illustrate that when $y = \log x$, $\frac{dy}{dx} = \frac{1}{x}$.

x	$y = \log x$	δy	$\frac{\delta y}{\delta x}$	$\frac{1}{x}$
2·13	0·7561			
		0·0047	0·47	
2·14	0·7608			0·47
		0·0047	0·47	
2·15	0·7655			

This illustration would be more interesting if I used seven-figure instead of only four-figure logarithms.

Remember that $\log x$ is the Napierian logarithm of x.

4. To illustrate that when $y = e^x$, $\frac{dy}{dx} = e^x$.

x	$y = e^x$	$\frac{\delta y}{\delta x}$	Average $\frac{\delta y}{\delta x}$
2·000	7·3889		
		7·3	
2·001	7·3962		7·4
		7·5	
2·002	7·4037		

5. There is a hyperbola $y = \frac{100}{x}$; find its area from the ordinate at $x = 10$ to any ordinate for greater values of x. Find also $\frac{dy}{dx}$ at any point. Tabulate as follows. Notice that to get $y.\delta x$ for the interval $\delta x = 1$ from 10 to 11, I take the average y, which is 9·546.

x	y	$\frac{\delta y}{\delta x}$	$y.\delta x$	Area or $\Sigma y.\delta x$.
10	10			0
		−0·909	9·546	
11	9·091			9·546
		−0·758	8·712	
12	8·333			18·258
		−0·641	8·013	
13	7·692			26·271
		−0·549	7.418	
14	7·143			33·689
		−0·476	6·905	
15	6·667			40·594
		−0·417	6·458	
16	6·250			47·052
		−0·368	6·066	
17	5·882			53·118
		−0·326	5·719	
18	5·556			58·837
		−0·293	5·409	
19	5·263			64·246
		−0·263	5·132	
20	5·000			69·378

The true answers are

$$\frac{dy}{dx} = -\frac{100}{x^2} \quad \text{or} \quad -\frac{y}{x},$$

and $\quad$ area $= 100 \log_e \frac{x}{10}$;

and it will be found that the tabulated answers are fairly correct.

EXERCISES.

1. The following numbers give x feet, the distance of a sliding piece measured along its path from a certain point to the place where it is at the time t seconds. Find the velocity and acceleration at various times, and draw three curves showing how x, v or $\frac{dx}{dt}$ and $\frac{dv}{dt}$ or the acceleration, depend upon t.

t	x	v	Acceleration.
0	1·000		
0·05		17·36	
0·1	2·736		−5·2
0·15		16·84	
0·2	4·420		−10·4
0·25		15·80	
0·3	6·000		−15·2
0·35		14·28	
0·4	7·428		−19·6
0·45		12·32	
0·5	8·660		−23·2
0·55		10·00	
0·6	9·660		−26·3
0·65		7·37	
0·7	10·397		−28·6
0·75		4·51	
0·8	10·848		−29·9
0·85		1·52	
0·9	11·000		−30·4
0·95		−1·52	
1·0	10·848		−29·9
1·05		−4·51	
1·1	10·397		

If a slider weighs 161 lb., what is the force acting upon it when $t=0·5$.

Ans. A retarding force of $\frac{161}{32·2}\times 23·2$ or 116 lb.

2. By tabulation, give approximately a table of values of $y.\delta x$ and $A=\int y.dx$, if the following values of x and y are given. Let A be 0 when x is 0. Plot y and x on squared paper and plot also A and x.

x	0	0·1	0·2	0·3	0·4	0·5	0·6
y	1·5663	1·6774	1·8002	1·9391	2·1000	2·2918	2·5281
$y.\delta x$	0·16219	0·17388	0·18697	0·20196	0·21959	0·24100	
A	0	0·16219	0·33607	0·52304	0·72500	0·94459	1·18559

3. The following numbers give v the speed of a train in miles per hour at the time t hours since leaving a terminus. In each interval of time, what is the distance $v.\delta t$ passed over by the train? At each of the times tabulated, what is x, the distance from the terminus?

t	v	$v.\delta t$	x
0	0		0
		0·048	
0·04	2·4		0·048
		0·142	
0·08	4·7		0·190
		0·238	
0·12	7·2		0·428
		0·336	
0·16	9·6		0·764
		0·432	
0·20	12·0		1·196
		0·526	
0·24	14·3		1·722
		0·624	
0·28	16·9		2·346
		0·716	
0·32	18·9		3·062
		0·792	
0·36	20·7		3·854
		0·858	
0·40	22·2		4·712
		0·912	
0·44	23·4		5·624
		0·954	
0·48	24·3		6·578
		0·984	
0·52	24·9		7·562

Thus, for example, at $t=0·12$, x is 0·428 mile. In the next interval of 0·04 hour the train goes 0·336 mile, so at $t=0·16$ it is at the distance $0·428+0·336$ or $x=0·764$.

4. A curve $y=bx^{2·5}$ passes through the point $(x=5, y=4)$; find b.

Ans. As $4=b\times 5^{2·5}$, $b=0·07155$.

Find the area of the curve between 0 and the ordinate at $x=5$.

$$\textit{Ans.}\ \int_0^5 bx^{2·5}.dx=\frac{b}{3·5}\left[x^{3·5}\right]_0^5=5·715.$$

5. A vessel is shaped like the frustum of a cone; the circular base is 10 inches diameter, the top is 5 inches diameter, the vertical height is 8 inches. If x is the height of the surface of a liquid from the bottom, express the diameter of its surface in terms of x; express A its area; express V the volume of the liquid in cubic inches.

$$\textit{Ans.}\ d=10-\frac{5}{8}x;\quad A=\frac{\pi}{4}\left(100-\frac{25}{2}x+\frac{25}{64}x^2\right);$$
$$V=78·54x-4·9085x^2+0·1023x^3.$$

6. There is a curve $y=ax^n$. If $y=2·34$ when $x=2$, and if $y=20·62$ when $x=5$, find a and n. *Ans.* $a=0·4511$, $n=2·3745$.

Let the curve rotate about the axis of x, forming a surface of revolution. Find the volume of the slice between sections at x and $x+\delta x$. What is the volume between the two sections at $x=2$ and $x=5$?

Ans. $\pi a^2x^{2n}.\delta x$ and 1156.

7. The curve $y=a+bx^{\frac{3}{2}}$ is such that if $y=1{\cdot}61$, $x=1$, and if $x=4$, $y=5{\cdot}32$; find a and b.

The curve rotates about the axis of x; find the volume enclosed by the surface of revolution between the two sections at $x=1$ and $x=4$.

Ans. $a=1{\cdot}08$, $b=0{\cdot}53$, volume $111{\cdot}85$.

8. If $y=0{\cdot}1x^2$, we know that $\frac{dy}{dx}=0{\cdot}2x$ and $A=\int y\,.\,dx=\frac{1}{30}x^3$, taking $A=0$ when $x=0$. Tabulate the following values of y; tabulate $\frac{dy}{dx}$ and $y\,.\,\delta x$ and A approximately. Show in three curves how y, $\frac{dy}{dx}$, and A depend upon x. Test your numbers against the correct formulæ.

x	y	$\frac{\delta y}{\delta x}$	Mean values of y.	$y\,.\,\delta x$	A	True A.
−0·2	0·004					
		−0·03				
−0·1	0·001					
		−0·01				
0	0·0				0	0
		0·01	0·0005	0·00005		
0·1	0·001				0·00005	0·000033
		0·03	0·0025	0·00025		
0·2	0·004				0·00030	0·000267
		0·05	0·0065	0·00065		
0·3	0·009				0·00095	0·000900
		0·07	0·0125	0·00125		
0·4	0·016				0·00220	0·00213
		0·09	0·0205	0·00205		
0·5	0·025				0·00425	0·00417
		0·11	0·0305	0·00305		
0·6	0·036				0·00730	0·00720
		0·13	0·0425	0·00425		
0·7	0·049				0·01155	0·01143
		0·15	0·0565	0·00565		
0·8	0·064				0·01720	0·017067
		0·17	0·0725	0·00725		
0·9	0·081				0·02445	0·02430
		0·19	0·0905	0·00905		
1·0	0·100				0·03350	0·03333

The student will find his $\frac{\delta y}{\delta x}$ to be the true $\frac{dy}{dx}$, but there are slight errors in his tabulative values of A.

9. Repeat the above tabulative process for $y=x^3$ from $x=0$ to $x=1$. We know that $\frac{dy}{dx}=3x^2$; and $A=\frac{1}{4}x^4$. There are slight errors in the tabulative values of $\frac{dy}{dx}$ and A.

10. C is the current in amperes flowing in an electric circuit of resistance r ohms and inductance l henries when v is the voltage at the time t seconds. If $r=1$, $l=0{\cdot}01$, c is known for the following values of t:

t	c	$\frac{\delta c}{\delta t}$	v
0	0		
		9950	100
0·0001	0·995		
		9850	100
0·0002	1·980		
0·0010	9·516		
		9010	100
0·0011	10·417		
		8900	100
0·0012	11·307		
0·0100	63·211		
		3660	100
0·0101	63·577		
		3630	100
0·0102	63·940		

From the tabulated numbers find $\frac{\delta c}{\delta t}$, and calculate

$$v=rc+l\frac{dc}{dt}. \qquad \text{.................................(1)}$$

We see that v is constant in this example.

11. It is known (see Art. 117) that if

$$v=a\sin pt, \qquad \text{.................................(2)}$$

then $$c=\frac{a}{\sqrt{r^2+l^2p^2}}\sin(pt-e), \text{ where } \tan e=\frac{lp}{r}.$$

Let $a=200$, $p=30\pi$, $r=1$, $l=0{\cdot}01$. Calculate c for the following values of t, and tabulate. From the tabulated numbers find $\frac{dc}{dt}$, and calculate v from (1) of last exercise. Now find the true values of v from (2), and compare your answers.

t	c	$\frac{\delta c}{\delta t}$	v	True v.
0·0150	88·96			
		10800	197·5	
0·0151	90·04			197·8
		10800	198·6	
0·0152	91·12			

12. A body of weight W lb. hangs from the end of a spiral spring whose stiffness is such that it extends h feet for a pulling force of 1 lb.; it vibrates, and at the time t seconds the body is s feet from its mid position; it is known that if $g=32\cdot2$,

$$\frac{W}{g}\frac{d^2s}{dt^2}+\frac{s}{h}=0 \quad\text{or}\quad \frac{d^2s}{dt^2}+\frac{g}{Wh}s=0. \quad\text{.....................(1)}$$

The solution (see Art. 121) of this equation is

$$s=A\sin(nt+e) \text{ if } n^2 \text{ is } \frac{g}{Wh}. \quad\text{.....................(2)}$$

A and e may have any values.

Let $h=0\cdot01$, $W=64\cdot4$, $A=1$, $e=0$; calculate s for the following values of t. From the tabulated numbers find $\frac{d^2s}{dt^2}$, and try if (1) is satisfied. Remember that the angle nt is in radians. If the sum of the terms in (1) is not zero, call it 'error.'

t	s	$v=\frac{\delta s}{\delta t}$	$\frac{\delta v}{\delta t}=\frac{d^2s}{dt^2}$	'Error.'
0·070	0·47500939			
		6·21044		
0·071	0·48121983		−23·96	+0·10
		6·18648		
0·072	0·48740631			
0·140	0·83599796			
		3·85945		
0·141	0·83985741		−42·33	−0·34
		3·81712		
0·142	0·84367453			

The student will find that even with seven-figure tables he does not get sufficient accuracy for the illustration of the true law.

Exercises like this are troublesome, but they teach many useful lessons.

100. To find $\frac{dy}{dx}$ with greater accuracy from tables of y and x.

Suppose that y has been tabulated for equidistant values of x. Let us take an example.

x	y	δy		
90	1463			
		302		
95	1765		49	
		351		**8**
100	2116		**57**	
		408		5
105	**2524**		62	
		470		8
110	2994		**70**	
		540		**8**
115	3534		78	
		618		
120	4152			

We tabulate the successive differences as shown. We want to know $\frac{dy}{dx}$ for $x=105$.

Now one-fifth of 408 is evidently too small and one-fifth of 470 is too great; the average of these is not usually correct either. There is a rule, deduced by an application of Taylor's theorem, which I shall not here prove (it is proved in my book on Steam, page 240), which may be employed in such cases. Let h be the x difference (in this case 5). Note the figures in clarendon type.

$$\frac{dy}{dx}=\frac{1}{h}\left\{\frac{1}{2}(408+470)-\frac{1}{10}(70-57)+\frac{1}{60}(8+8)\right\}.$$

In this case $\frac{dy}{dx}=87{\cdot}59$.

In particular cases we can find $\frac{dy}{dx}$ with great accuracy even from only two terms if we know a good empirical formula. For example, we know that with not very great but with some accuracy the pressure and temperature of steam are connected by the law

$$\theta=a+bp^{0{\cdot}2},$$

$$\frac{d\theta}{dp}=0{\cdot}2bp^{-0{\cdot}8}.$$

Therefore we need to know b only.

Suppose we know that when $p=2524$, θ is 105, and when $p=2994$, θ is 110, and we wish to find $\frac{dp}{d\theta}$ when θ is 105.

$$105=a+b2524^{0{\cdot}2},$$

$$110=a+b2994^{0{\cdot}2},$$

$$5=b(2994^{0{\cdot}2}-2524^{0{\cdot}2})=0{\cdot}1665b \quad \text{or} \quad b=30{\cdot}003,$$

$$\frac{d\theta}{dp}=0{\cdot}2\times 30{\cdot}003\times 2524^{-0{\cdot}8} \quad \text{or} \quad \frac{dp}{d\theta}=87{\cdot}7.$$

CHAPTER XXI.

ILLUSTRATIONS.

101. When we use the letters x and y, we are generally speaking of a curve.

Then $\frac{dy}{dx}$ is the slope of the curve at any place (that is, the tangent of the angle which the tangent to the curve makes with the axis of x) and the integral of y (denoted by the symbol $\int y \,.\, dx$) is the area of the curve. But it is to be remembered that x and y may be any physical quantities, and other letters may be used. Thus, if s is the space passed through by a body in the time t, $\frac{ds}{dt}$ is the velocity v of the body. Also $\frac{dv}{dt}$ is called the acceleration of the body. Various symbols are in use:

s and t for space and time.

Velocity v or $\frac{ds}{dt}$, but Newton used the symbol $\dot{s}$.

Acceleration $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$, or Newton's $\ddot{s}$ or Newton's $\dot{v}$.

Rate of change of acceleration would be $\frac{d^3s}{dt^3}$.

Note that $\frac{d^2s}{dt^2}$ is one symbol; it has nothing whatever to do with such an algebraic expression as $\frac{d^2 \times s}{d \times t^2}$. The symbol is supposed merely to indicate that we have differentiated s twice with regard to the time.

As an interesting example of using other letters than x and y, let us consider the **kinetic energy** stored up in a moving small body of mass m and velocity v which passes through the small space δs in the time δt, gaining velocity δv because the force F is acting upon it. The gain of energy by the body δE is $F.\delta s$, which is the work done by the force F acting through the space δs. But F is m multiplied by the acceleration, or $F = m\frac{\delta v}{\delta t}$ and $\delta s = v.\delta t$, so that

$$\delta E = F.\delta s = m\frac{\delta v}{\delta t}v.\delta t = mv.\delta v,$$

or

$$\frac{\delta E}{\delta v} = mv.$$

But our equations are only entirely true when δs, δt, etc., are made smaller and smaller without limit; hence

$$\frac{dE}{dv} = mv,$$

or, in words, "the differential coefficient of E with regard to v is mv." If, then, we integrate with regard to v,

$$E = \frac{1}{2}mv^2 + c,$$

where c is some constant. As there is no E when v is 0, c must be 0, and so we have

$$E = \frac{1}{2}mv^2.$$

Practise differentiation and integration, using other letters than x and y. In this case $\frac{dE}{dv}$ stands for our old $\frac{dy}{dx}$. If we had had $\frac{dy}{dx} = mx$ it might have been seen more easily that $y = \frac{1}{2}mx^2 + c$. But you must escape from the swaddling bands of x and y.

102. If the student knows anything about **electricity**, let him translate into ordinary language the improved Ohm's law

$$v = rc + l\frac{dc}{dt}.$$

Observe that r (ohms) and l (henries) are known constants, so that if c and $\frac{dc}{dt}$ are known variable functions of the time t, then v the voltage is known. The inductance l may be regarded as a back electromotive force in volts when the current increases at the rate of 1 ampere per second.

103. Ex. 1. Find the following integrals. The constants are not added in the answers:

$$\int x^2 . dx. \quad Ans. \ \frac{1}{3}x^3.$$

$$\int v^2 . dv. \quad Ans. \ \frac{1}{3}v^3.$$

$$\int v^{-s} dv. \quad Ans. \ \frac{1}{1-s}v^{1-s}.$$

$$\int \sqrt[3]{v^2} . dv \quad \text{or} \quad \int v^{\frac{2}{3}} . dv. \quad Ans. \ \frac{3}{5}v^{\frac{5}{3}}.$$

$$\int t^{-\frac{1}{2}} . dt. \quad Ans. \ 2t^{\frac{1}{2}}.$$

$$\int \frac{dv}{v+a}. \quad Ans. \ \log(v+a).$$

Ex. 2. It is proved in **thermodynamics** when ice and water or water and steam are together at the same temperature; if s_2 is the volume in cubic feet of 1 lb. of stuff in the higher state, and if s_1 is the volume of 1 lb. of stuff in the lower state, then

$$l = t(s_2 - s_1)\frac{dp}{dt},$$

where t is the absolute temperature (being $273 + \theta°$ C.), where l is the latent heat of one pound of stuff in foot pounds, p is the pressure in pounds per square foot.

(i) In ice water, $s_1 = 0{\cdot}01747$, $s_2 = 0{\cdot}01602$ at $t = 273$ (corresponding to 0° C.), p being 2116 lb. per square foot and $l = 79 \times 1400$. Hence

$$\frac{dp}{dt} = \frac{79 \times 1400}{273(0{\cdot}01602 - 0{\cdot}01747)} = -279400.$$

Hence the temperature of melting ice is less as the pressure increases; or pressure lowers the melting point of ice, that is, induces towards melting the ice. Observe the quantitative meaning of $\frac{dp}{dt}$; the melting point lowers at the rate of 1 degree for an increased pressure of 279400 lb. per sq. foot, or 132 atmospheres.

(ii) *Water-steam.* It seems almost impossible to measure accurately by experiment s_2, the volume in cubic feet of 1 lb. of steam at any temperature. s_1 for water is known. Calculate $s_2 - s_1$ from the above formula at a few temperatures, having from Regnault's experiments the following table. The figures explain themselves. My calculation is for 105° C.

θ° C.	t absolute.	Pressure in lb. per sq. foot.	$\frac{\delta p}{\delta t}$	Assumed $\frac{dp}{dt}$	l foot-pounds.	$s_2 - s_1$
100	373	2116·4				
			81·5			
105	378	2524		87·8	742500	22·38
			94			
110	383	2994				

It is here assumed that the value of $\frac{dp}{dt}$ for 105° C. is half the sum of 81·5 and 94, because it is not worth while for my present purpose to find it more accurately by the methods of Art. 100 or by drawing a curve.

$$s_2 - s_1 = 742500 \div (378 \times 87 \cdot 8) = 22 \cdot 26.$$

Now s_1 for cold water is 0·016, and it is not worth while making any correction for warmth. Hence we may take $s_2 = 22 \cdot 4$, which is sufficiently near the correct answer for my present purpose.

Ex. 3. The radius of **curvature** of any curve at the point (x, y) is

$$r = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \bigg/ \frac{d^2y}{dx^2}.$$

(1) Find the radius of curvature of the parabola $y = ax^2$ at the vertex, that is, when $x = 0$.

Here $\frac{dy}{dx} = 2ax$, $\frac{d^2y}{dx^2} = 2a$, and where $x = 0$, $\frac{dy}{dx} = 0$, $r = \frac{1}{2a}$.

(2) Find the radius of curvature of the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \quad \textit{Ans. } r = \frac{1}{c}y^2.$$

At the vertex when $x = 0$, $y = c$, $r = c$.

Ex. 4. The centre line of a uniform beam of length l fixed at one end loaded with the weight W at the other, x being distance of a section from the fixed end; it is known that y the deflection of the line from its unloaded straight form is

$$y = \frac{W}{EI}\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right).$$

(1) What is the curvature where $x = 0$?

Here $\frac{dy}{dx} = \frac{W}{EI}\left(lx - \frac{1}{2}x^2\right)$, being 0 where $x = 0$;

$\frac{d^2y}{dx^2} = \frac{W}{EI}(l - x)$, being $\frac{Wl}{EI}$ where $x = 0$.

The curvature is therefore $\frac{Wl}{EI}$.

(2) What is the curvature where $x = \frac{1}{2}l$? *Ans.* $\frac{Wl}{2EI}$.

(3) What is the curvature where $x = l$. *Ans.* 0.

In all practical cases in beams $\frac{dy}{dx}$ is negligible in comparison with 1, and I call it 0 in (2) and (3).

CHAPTER XXII.

MAXIMA AND MINIMA.

104. The student will see now that the problems of Art. 84 are easy to solve exactly.

In (1). Divide a number n into two parts such that the product is a maximum. Let x be one part, then $n - x$ is the other. Let the product be called y, then

$$y = x(n - x) \quad \text{or} \quad y = nx - x^2.$$

Now at a point of maximum, or minimum, the slope of the curve or $\frac{dy}{dx}$ is 0.

Here $\frac{dy}{dx} = n - 2x$. If we put this equal to 0,

$$n - 2x = 0, \quad 2x = n, \quad x = \frac{n}{2}.$$

We get the required answer.

It is true that we cannot in this way distinguish as to whether it is a maximum or a minimum that we have found, but there is no great difficulty in finding this out in other ways. I shall not give here the algebraic method of discriminating.

In (2). When is the sum of a number and its reciprocal a minimum? Let x be the number; when is $y = x + x^{-1}$ a minimum?

Ans. When $\frac{dy}{dx} = 0$ or $1 - x^{-2} = 0$ or $x^2 - 1 = 0$ or $x = 1$.

In (3). The volume of the cylindric parcel is a maximum when $\frac{dv}{dx}$ is 0. Now $v = 6\pi x^2 - 2\pi^2 x^3$, and $\frac{dv}{dx} = 12\pi x - 6\pi^2 x^2$. Putting this equal to 0 and dividing by $6\pi x$, we get $2 - \pi x = 0$ or $x = \frac{2}{\pi}$, the answer obtained by the use of squared paper.

In (4). Divide a number n into two parts, such that a times the square of one part plus b times the square of the other shall be a minimum. As before,

$$y = ax^2 + b(n-x)^2$$

$$\text{or} \quad y = ax^2 + bn^2 - 2bnx + bx^2$$

$$\text{or} \quad y = (a+b)x^2 + bn^2 - 2bnx.$$

Hence $$\frac{dy}{dx} = 2(a+b)x - 2bn.$$

Putting this equal to 0, we get

$$x = \frac{bn}{a+b}, \text{ the answer.}$$

In (5). The strength of a beam of rectangular section of given length, loaded and supported in any particular way, is proportional to the breadth of the section multiplied by the square of the depth. If the diameter of a cylindric tree is a, what is the strongest rectangular beam which may be cut from the tree?

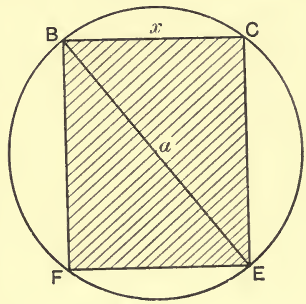

FIG. 27.

Let x be the breadth BC of the rectangle $BCEF$ (Fig. 27), BE being a. Then CE is $\sqrt{a^2 - x^2}$. Now the strength depends upon $y = BC \times CE^2$, and this is

$$y = x(a^2 - x^2) \quad \text{or} \quad y = a^2x - x^3.$$

When is y a maximum?

$\frac{dy}{dx} = a^2 - 3x^2$. Putting this equal to 0, we find

$$3x^2 = a^2 \quad \text{or} \quad x = \frac{a}{\sqrt{3}} \quad \text{or} \quad 0{\cdot}5774a.$$

In (6). y will be a maximum when its square is a maximum. The diameter of the tree being a, we desire that

$$z = x^6(a^2 - x^2) \quad \text{or} \quad z = a^2x^6 - x^8$$

shall be a maximum.

$$\frac{dz}{dx} = 6a^2x^5 - 8x^7.$$

Putting this equal to 0 and dividing by x^5, we find $x^2 = \frac{3}{4}a^2$ or the depth $x = \frac{1}{2}a\sqrt{3}$ and the breadth $= \frac{1}{2}a$.

These give the beam of greatest stiffness.

Of course in all these cases differentiation gives us the correct answer, but our old squared-paper method has merits of its own.

For **example**. Suppose there is a machine consisting of two parts whose weights are x and y. Now suppose the cost of the machine in pounds to be

$$c = 10x + 3y.$$

Also suppose the power of the machine to be proportional to xy. If the cost is fixed, find x and y, so that the power shall be a maximum.

Here $y = \frac{1}{3}(c - 10x)$,

and let $\qquad p = xy \quad \text{or} \quad p = x \times \frac{1}{3}(c - 10x) = \frac{c}{3}x - \frac{10}{3}x^2.$

$\frac{dp}{dx} = \frac{c}{3} - \frac{20}{3}x$, and if we put this equal to 0, we have $10x = \frac{1}{2}c$. This leads to $3y = \frac{1}{2}c$, that is, the costs of the two parts ought to be equal.

Now suppose it is an actual machine, say a dynamo, and that x is the weight of the armature part, y the weight of the rest, and that the numbers 10 and 3 were based on real workshop experience. As everything is evidently proportional to c, take $c = 3$. Then

$$p = x - \frac{10}{3}x^2.$$

Now plot x and p on squared paper. It is quite true that we find $x = 0{\cdot}15$ gives the maximum power. But it will be observed that if this exactly best value of x is not employed the power is not much less than the maximum. In fact the squared paper tells us that we do not suffer greatly even by departing considerably from the value of x which is here thought to be the best. Indeed I may say that I employ the differential calculus method to suggest the best answer; but I also use the squared paper method in practical problems.

x	p
0·1	0·0667
0·12	0·072
0·13	0·07367
0·14	0·07467
0·15	0·075
0·16	0·07467
0·17	0·07367
0·18	0·072

EXERCISES.

1. When is $ax-bx^2$ a maximum? *Ans.* When $x=a/2b$.

2. When is $ax-bx^3$ a maximum? *Ans.* When $x=\sqrt{a/3b}$.

3. The volume of a circular cylindric cistern being given (no cover). When is its surface a minimum?

Let x be its radius and y its length, the volume is

$$\pi x^2 y=a, \text{ say.} \quad\text{.......(1)}$$

The surface is
$$\pi x^2+2\pi xy. \quad\text{.......(2)}$$

From (1), y is $\frac{a}{\pi x^2}$; using this in (2) we see that we must make

$$\pi x^2+\frac{2a}{x} \text{ a minimum,}$$

$$2\pi x-\frac{2a}{x^2}=0 \quad\text{or}\quad x^3=a/\pi,$$

$$x^3=\frac{\pi x^2 y}{\pi} \quad\text{or}\quad x=y.$$

That is, the radius of the base is equal to the height of the cistern.

4. Let the cistern of the last exercise be closed top and bottom; find the condition that it shall have minimum surface with given volume. The answer is, that the diameter of the cistern is equal to its height.

5. When a vessel moves at v knots, the total cost in wages, depreciation, interest on capital, stores, coals, etc., is in pounds $a+bv^3$ per hour. For a passage of m miles, what is the speed which will cause the total cost to be a minimum? The number of hours is m/v, so that the total cost is $m(a+bv^3)/v$; therefore we try to make $av^{-1}+bv^2$ a minimum; that is,

$$-av^{-2}+2bv=0.$$

The best speed is therefore $v=\sqrt[3]{a/2b}$.

The student may take $a=4$ and $b=0{\cdot}001$ as being true of a certain cargo steamer whose speed I have studied.

6. The sum of the squares of two factors of n is a minimum; find them. If x is one of them, n/x is the other, and $y=x^2+\frac{n^2}{x^2}$ is to be a minimum. $\frac{dy}{dx}=2x-\frac{2n^2}{x^3}$, and this is 0 when $x^4=n^2$ or $x=\sqrt{n}$.

7. The weight of gas which will flow per second through an orifice from a vessel where it is at a pressure p_1 into an outer atmosphere of pressure p_0 is proportional to

$$x^{\frac{1}{\gamma}}\sqrt{1-x^{\frac{\gamma-1}{\gamma}}},$$

where x is p_0/p_1 and γ is a known constant; when is this a maximum?

That is, when is $x^{\frac{2}{\gamma}}-x^{1+\frac{1}{\gamma}}$ a maximum?

Differentiating with regard to x and equating to 0, we find

$$x=\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}.$$

In the case of air $\gamma=1{\cdot}41$, and we find $p_0=0{\cdot}527p_1$; that is, there is a maximum quantity leaving the vessel per second when the outside pressure is a little greater than half the inside pressure.

In the case of wet or dry steam we may assume that we are dealing with a perfect gas whose $\gamma=1{\cdot}13$. The answer in this case is $p_0=0{\cdot}578p_1$.

8. From a hypothetic steam-engine indicator diagram the work done per cubic foot of steam is

$$w=144p_1(1+\log r)-144p_3r,$$

where p_1 and p_3 are the initial and back pressures of the steam; r is the ratio of cut off (that is, cut off is at $\frac{1}{r}$th of the stroke).

The logarithm is Napierian. If p_1 and p_3 are given, find r so that w may be a maximum.

$\frac{dw}{dr}=144\frac{p_1}{r}-144p_3$. Putting this equal to 0, we find the best r to be p_1/p_3.

Taking $p_1=100$ lb. per sq. inch, find the best r in the following cases:

(*a*) If $p_3=2$, condensing engine. The best r is 50; cutting off at $\frac{1}{50}$ of the stroke gives most *indicated* energy per cubic foot of steam.

(*b*) If $p_3=10+2$, the 10 representing the friction of the engine. The best r is $100\div12$ or $8\frac{1}{3}$; that is, cutting off at $\frac{3}{25}$ of the stroke gives most shaft energy per cubic foot of steam.

(*c*) If $p_3=17$, non-condensing engine. The best r is $100\div17$; that is, cutting off at the 0·17 of stroke gives maximum *indicated* energy per cubic foot of steam.

(*d*) If $p_3=17+8$, where 8 represents the friction of the engine. The best r is $100\div25$ or 4; that is, cutting off at $\frac{1}{4}$ of the stroke gives most shaft energy per cubic foot of steam.

9. Taking the waste going on in an electric conductor as consisting of (1) the ohmic loss; the value of C^2r watts, where r is the resistance of a mile of going and coming conductor, and C is the current in amperes; (2) the loss due to interest and depreciation on the cost of the conductor. It is easy to show that the total loss of money per annum is proportional to

$$y=C^2r+\frac{a^2}{r}+b,$$

where a and b are constants depending upon the price of copper and cost of manufacture and laying of the cable, and also upon the value of electrical energy. We may take a as never less than 17 and never greater than 40. To make y a minimum,

$$\frac{dy}{dr}=C^2-a^2r^{-2}=0 \quad \text{or} \quad Cr=a.$$

Thus if a is 25, $r=\frac{25}{C}$.

When the cross-section of copper is α square inches, r is $0{\cdot}04/\alpha$ nearly, so that $0{\cdot}04\alpha C=25$ or $\alpha C=625$.

This means that in the most economical case we have 625 amperes per sq. inch of copper.

10. Find what positive value of x makes

$$4x^3+3x^2-168x+10$$

a minimum. *Ans.* $x=3{\cdot}5$.

11. For what value of x is $3\sin x+2\cos x$ a maximum, using (1) the calculus, (2) squared paper, (3) a geometrical method. *Ans.* 0·9826 or 56°·3.

CHAPTER XXIII.

CURVES.

105. When the equation to a new curve is given, the practical man ought greatly to rely upon his power of plotting it upon squared paper. Very often if we find $\frac{dy}{dx}$ or the slope, everywhere, it gives us a good deal of information.

If we are told that x_1, y_1 is a point on the curve, and we are asked to find the equation to the **tangent** there, we have simply to find the straight line which has the same slope as the curve there and which passes through x_1, y_1. The **normal** is the straight line which passes through the point x_1, y_1, and whose slope is minus the reciprocal of the slope of the curve there.

Ex. 1. The point $x=4$, $y=3$ is a point on the parabola $y=\frac{3}{2}x^{\frac{1}{2}}$. [Try if it really is a point on the curve.] Find the equation of the tangent there.

The slope is $\frac{dy}{dx}=\frac{1}{2}\times\frac{3}{2}x^{-\frac{1}{2}}$ or as $x=4$ there, the slope is $\frac{3}{4}\times\frac{1}{2}$ or $\frac{3}{8}$. The tangent is then $y=a+\frac{3}{8}x$. To find a we have $y=3$ when $x=4$, as this point is on the tangent, or $3=a+\frac{3}{8}\times 4$, so that a is $1\frac{1}{2}$ and the tangent is $y=1\frac{1}{2}+\frac{3}{8}x$.

Ex. 2. The point $x=32$, $y=3$ is a point on the curve $y=2+\frac{1}{2}x^{\frac{1}{5}}$. Find the equation to the normal there.

The slope of the curve there is $\frac{dy}{dx}=\frac{1}{10}x^{-\frac{4}{5}}=\frac{1}{160}$, and the slope of the normal is minus the reciprocal of this or -160.

Hence the normal is $y=a-160x$. But it passes through the point $x=32$, $y=3$, so that $3=a-160\times 32$.

Hence $a=5123$, and the normal is $y=5123-160x$.

Ex. 3. At what point in the curve $y=ax^{-n}$ is there the slope b? As $\frac{dy}{dx}=-nax^{-n-1}$, the point is such that its x satisfies $-nax^{-n-1}=b$. Knowing its x, we know its y from the equation to the curve.

106. It is easy to see and well to remember that if x_1, y_1 is a point in a straight line, and if the slope of the line is b, then the equation to the line most quickly written is

$$\frac{y-y_1}{x-x_1}=b.$$

Hence the equation to the tangent to a curve at the point x_1, y_1 on the curve is

$$\frac{y-y_1}{x-x_1}=\text{the } \frac{dy}{dx} \text{ at the point,}$$

and the equation to the normal is

$$\frac{x-x_1}{y-y_1}=-\text{the } \frac{dy}{dx} \text{ at the point.}$$

Ex. 1. Find the tangent and normal to the curve $x^m y^n=a$ at the point x_1, y_1 on the curve.

Ans. The tangent is $\frac{m}{x_1}x+\frac{n}{y_1}y=m+n$, and the normal is

$$\frac{n}{y_1}(x-x_1)-\frac{m}{x_1}(y-y_1)=0.$$

Ex. 2. Find the tangent and normal to the parabola $y^2=4ax$ at the point where $x=a$. *Ans.* $y=a+x$, $y=3a-x$.

Ex. 3. Find the tangent to the curve $y=a+bx+cx^2+ex^3$ at a point on the curve x_1, y_1. *Ans.* $\frac{y-y_1}{x-x_1}=b+2cx_1+3ex_1^2$.

107. P (Fig. 28) is a point on the curve BPC at which the tangent PA and the normal PD are drawn. OX and OY are the axes. $OR=x$, $RP=y$, $\tan PAX=\frac{dy}{dx}$; the distance AR is called the **subtangent**; prove that it is $y\div\frac{dy}{dx}$.

The distance RD is called the **subnormal**; it is evidently $y\frac{dy}{dx}$.

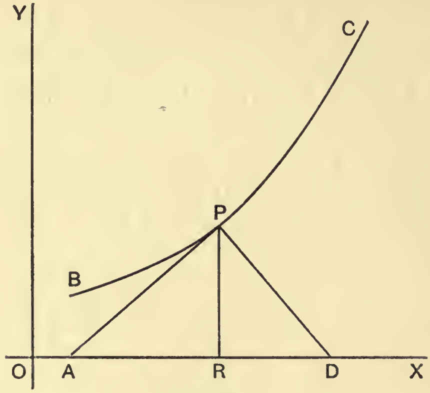

FIG. 28.

EXERCISES.

1. Find the length of the subtangent and subnormal of the parabola $y=mx^2$. Here $\frac{dy}{dx}=2mx$, so that

$$\text{subtangent}=mx^2\div 2mx \quad \text{or} \quad \frac{1}{2}x,$$

$$\text{subnormal}=y\times 2mx \quad \text{or} \quad 2m^2x^3.$$

2. Find the length of the subtangent of $y=mx^n$.

Here $\frac{dy}{dx}=mnx^{n-1}$, so that

$$\text{subtangent}=mx^n\div mnx^{n-1}=\frac{x}{n}.$$

3. For what curve is the subnormal constant in length? That is,

$$y\frac{dy}{dx}=a \quad \text{or} \quad \frac{dx}{dy}=\frac{1}{a}y.$$

The integral of y with regard to y is $\frac{1}{2}y^2$, so that $x=\frac{1}{2a}y^2+$ a constant b, say; and this is the equation to the curve, b having any value. It is evidently any one member of a family of parabolas.

CHAPTER XXIV.

ILLUSTRATIONS.

108. (1) The chain of a **suspension bridge** supports a load by means of detached rods; the loads are about equal and equally spaced. Suppose a chain to be continuously loaded, the load being w per unit of *horizontal* length. Any very flat uniform chain or telegraph wire is nearly in this condition. What is the shape of the chain? Let O be the lowest point. OX is tangential to the chain and horizontal at O. OY is vertical. Let P be any point in the

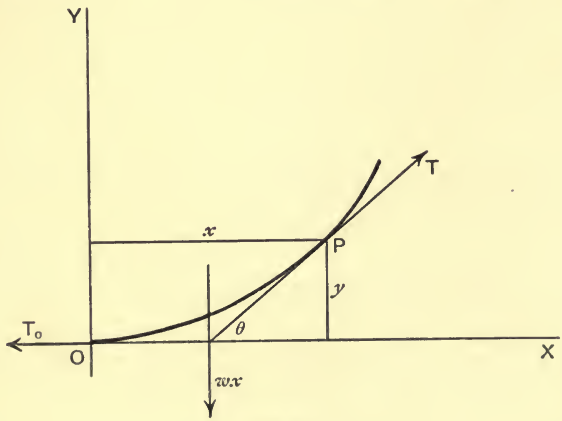

FIG. 29.

chain, its co-ordinates being x and y. Consider the equilibrium of the portion OP. OP is in equilibrium under the action of T_0 the horizontal tensile force at O, T the tangential force at P, and wx the resultant load upon OP acting vertically. We employ the laws of forces acting upon rigid bodies. A rigid body is a body which is acted on by forces and is no longer altering its shape.

If we draw a triangle whose sides are parallel to these forces, these sides represent the forces to the same scale. If θ is the inclination of T to the horizontal (see fig. 30).

$$\frac{T_0}{T} = \cos\theta, \quad \text{.....................................(1)}$$

and

$$\frac{wx}{T_0} = \tan\theta. \quad \text{.....................................(2)}$$

But $\tan\theta$ is $\frac{dy}{dx}$, so that $\frac{dy}{dx}$ is $\frac{w}{T_0}x$. ....(3)

Hence, integrating, $y = \frac{1}{2}\frac{w}{T_0}x^2 + \text{constant}.$

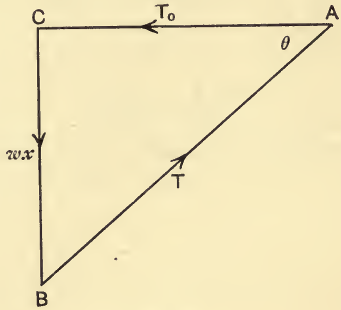

FIG. 30.

Now we see that y is 0 when x is 0, so that the constant is 0.
Hence the equation to the curve is

$$y = \frac{1}{2}\frac{w}{T_0}x^2, \quad \text{.....................................(4)}$$

and it is a parabola. From these statements all sorts of calculations are easily made.

(2) The work done when fluid of the volume v and pressure p expands from the volume v_1 to the volume v_2 is

$$w = \int_{v_1}^{v_2} p \,.\, dv. \quad \text{.....................................(1)}$$

If we know p as a function of v, it is easy to calculate w.

(*a*) Let $p = cv^{-n}$, the general integral of this is $\frac{c}{1-n}v^{1-n}$. Following the instructions of Art. 99, the answer is evidently

$$\frac{c}{1-n}(v_2^{1-n} - v_1^{1-n}).$$

(b) The method fails when $n=1$; that is, when $pv=c$.

The general integral of cv^{-1} is $c\log v$, so that in this case the answer is $c(\log v_2 - \log v_1)$ or $c\log\frac{v_2}{v_1}$.

(3) The curve $y=ax^n$ revolves about the axis of x; find the volume enclosed by the surface of revolution between the cross-sections at $x=0$ and $x=b$.

Evidently the volume of a slice of thickness dx is $\pi y^2 . dx$. Our answer is

$$\pi\int_0^b a^2x^{2n}. dx \quad \text{or} \quad \frac{\pi a^2}{2n+1}b^{2n+1}.$$

(4) It can be proved that when a perfect gas (whose law is $pv=Rt$ if p is pressure, v volume, t absolute temperature, and R a constant) changes its volume and pressure in any way, the rate of reception of heat by it per unit change of volume or $\frac{dH}{dv}$ is

$$\frac{dH}{dv}=\frac{1}{\gamma-1}\left\{v\frac{dp}{dv}+\gamma p\right\},$$

where γ is the ratio of the two important specific heats. I always express heat in work units.

(a) When gas expands according to the law $pv^n=c$ a constant, find $\frac{dH}{dv}$. *Ans.* $\frac{dH}{dv}=\frac{\gamma-n}{\gamma-1}p.$

(b) When the gas expands adiabatically $\left(\text{that is } \frac{dH}{dv}=0\right)$, what is n? *Ans.* $n=\gamma$; that is, the adiabatic law for an expanding gas is $pv^\gamma=$ constant.

γ is 1·41 for air and 1·37 for the stuff inside a gas or oil engine cylinder.

(c) What is $\frac{dH}{dv}$ when $n=1$? *Ans.* $\frac{dH}{dv}=p.$

(d) Notice that when n is greater than γ the stuff is having heat withdrawn from it.

(5) On the indicator diagram of a gas engine, here are some readings of p, pressure, and v, volume. The rate of reception of heat (if the gases are supposed to be receiving heat from an outside source and not from their own chemical action) is

$$h=\frac{dH}{dv}=p+\frac{1}{\gamma-1}\left(p+v\frac{dp}{dv}\right), \text{ where } \gamma=1{\cdot}4.$$

Find $\frac{dH}{dv}$, which I call h. Plot p and v, and also h and v. Perhaps you had better plot h to a much smaller scale than p.

v	p	$\frac{\delta p}{\delta v}$	$v\frac{dp}{dv}$	h or $\frac{dH}{dv}$
2·0	84·5			
		255	523	1644
2·1	110			
		660	1419	4048
2·2	176			
		390	878	2877
2·3	215			
		160	376	1721
2·4	231			
		40	98	1060
2·5	235			
		−90	−229	232
2·6	226			
		−130	−345	−94
2·7	213			
		−110	−303	−31
2·8	202			
		−100	−285	−23
2·9	192			
		−90	−266	−8
3·0	183			
		−80	−244	+16
3·1	175			
		−80	−252	−31
3·2	167			
		−80	−260	−80
3·3	159			
		−70	−235	−42
3·4	152			
		−60	−207	+4
3·5	146			
		−60	−213	−32
3·6	140			

CHAPTER XXV.

ILLUSTRATIONS. BEAMS AND STRUTS.

109. The curvature of a circle is the reciprocal of its radius, and of any curve at any point it is the curvature of the circle which best agrees with the curve at that point. The curvature of a curve is also "the angular change (in radians) of the direction of the curve per unit length." Now draw a very flat curve, with very little slope $\left(\text{or } \frac{dy}{dx}\right)$ everywhere. Observe that the change in $\frac{dy}{dx}$ in going from a point P to a point Q is almost exactly equal to the change of angle [change of $\frac{dy}{dx}$ is really a change in the tangent of an angle, but when the angle is very small, the angle, its sine, and its tangent are all equal]. Hence the increase in $\frac{dy}{dx}$ from P to Q divided by the length of the curve PQ is the average curvature from P to Q, and as PQ is less and less we get more and more nearly the curvature at P. But the curve being very flat, the length of the arc PQ is really δx, and the change in $\frac{dy}{dx}$ divided by δx as δx gets less and less is the rate of change of $\frac{dy}{dx}$ with regard to x, and the symbol for this is $\frac{d^2y}{dx^2}$. Hence we take the curvature of a very flat curve such as the centre line of a beam or strut to be $\pm\frac{d^2y}{dx^2}$.

It is proved in books on mechanics that the curvature of a beam or strut at any section is the bending moment M, there, divided by EI, if E is Young's Modulus for the material and I is the moment of inertia of the section about its neutral axis.

Ex. 1. Uniform beam of length l fixed at one end, loaded with weight W at the other. Let x be the distance of a section from the fixed end of the beam, then M is $W(l-x)$ if l is the whole length of the beam. Our $\frac{d^2y}{dx^2}=\frac{M}{EI}$ may be written $\frac{EI}{W}\frac{d^2y}{dx^2}=l-x$.

Integrating, we have, as E and I are constants,

$$\frac{EI}{W}\frac{dy}{dx}=lx-\frac{1}{2}x^2+c.$$

From this we can calculate the slope everywhere, but to know c it is necessary to know the slope at some one place. Now $\frac{dy}{dx}$ is 0 at the fixed end, that is where $x=0$, so that $c=0$. Integrating again,

$$\frac{EI}{W}y=\frac{1}{2}lx^2-\frac{1}{6}x^3+C.$$

To find C, we know that y is 0 when x is 0, so that C is 0.

We have then the shape of the beam

$$y=\frac{W}{EI}\left(\frac{1}{2}lx^2-\frac{1}{6}x^3\right).$$

We usually want to know y when x is l, and this is called D, the deflection of the beam,

$$D=\frac{Wl^3}{3EI}.$$

Ex. 2. A beam of length l loaded with W at the middle and supported at the ends. Observe that if half of this loaded beam has a casting of cement made round it so that it is rigidly held, the other half is simply a beam of length $\frac{1}{2}l$ fixed at one end and loaded at the other with $\frac{1}{2}W$, and, according to the above result, its deflection is

$$D=\frac{\frac{1}{2}W(\frac{1}{2}l)^3}{3EI} \quad \text{or} \quad \frac{Wl^3}{48 \cdot EI}.$$

Exercises on this formula were given in Ex. 14, Art. 28; they may be referred to again.

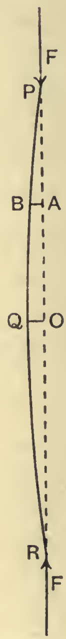

Fig. 31.

110. Euler's Theory of Struts. Consider a strut perfectly prismatic, of homogeneous material, its own weight neglected, the resultant force F at each end passing through the centre of the section there. Let PQR (Fig. 31) show the centre line of the bent strut. Let $AB=y$ be the deflection at A, where $OA=x$. Let $OP=OR=l$. y is supposed everywhere to be small in comparison with the length $2l$ of the strut. Fy is the bending moment at B and $\frac{Fy}{EI}$ is the curvature there if E is Young's

Modulus for the material and I is the least moment of inertia of the cross-section about a line through its centre. It is not difficult therefore to see that

$$\frac{Fy}{EI} = -\frac{d^2y}{dx^2} \quad \text{..............................(1)}$$

or

$$\frac{d^2y}{dx^2} + n^2y = 0 \quad \text{if} \quad n^2 = \frac{F}{EI}.$$

We find that $$y = a \cos nx \quad \text{..............................(2)}$$

satisfies (1) whatever value a may have. When $x = 0$ we see that $y = a$, so that the meaning of a is known to us; it is the deflection OQ of the strut in the middle. Again, when $x = l$, $y = 0$.

Hence $$a \cos nl = 0. \quad \text{..............................(3)}$$

Now, how can this be true? Either $a = 0$ or the cosine is 0. But *if bending occurs* so that a has some value, the *cosine must* be 0. That is, the angle nl must be

$$\frac{\pi}{2} \quad \text{or} \quad l\sqrt{\frac{F}{EI}} = \frac{\pi}{2} \quad \text{or} \quad F = \frac{EI\pi^2}{4l^2}$$

is the load which will produce bending. It will produce very little or very much bending equally well.

In my theory of struts I have shown why it is that this theory does not agree with experiment; experimental struts are not the perfect prisms perfectly loaded which the Euler theory assumes.

CHAPTER XXVI.

ILLUSTRATIONS. FLUID.

111. Suppose a mass of fluid to rotate like a rigid body about an axis with the angular velocity a radians per second. Let OO be the axis. Let P be a particle of fluid weighing w lb. Let $OP = x$.

Centrifugal force is mass $\frac{w}{g}$ multiplied by a^2x. Make PR represent this to scale and let PS represent w the weight, to the same scale; then the resultant force, represented by PT, is easily found and the angle RPT which PT makes with the horizontal. Thus

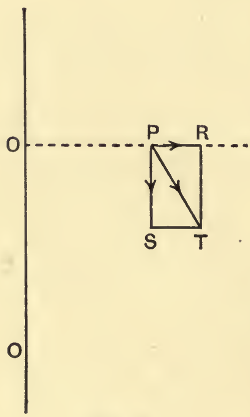

Fig. 32.

$$\tan RPT = w \div \frac{w}{g} a^2 x \quad \text{or} \quad g \div a^2 x,$$

being independent of w; we can therefore apply our results to a heterogeneous fluid. Now, if y is the distance of the point P above some datum level, and we imagine a curve drawn through P to which PT (at P) is tangential, and if at every point of the curve its direction (or the direction of its tangent) represents the direction of the resultant force; if such a curve were drawn, its slope $\frac{dy}{dx}$ is evidently $-g \div a^2x$.

Integrating this, we find

$$y = -\frac{g}{a^2} \log x + \text{constant}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The constant depends upon the datum level from which y is measured. This curve is called a line of force. Its direction at any place shows the direction of the total force there. It is a logarithmic curve.

Level Surfaces. If there is a curve to which PT is a normal at the point P, it is evident that its slope is positive, and in fact

$$\frac{dy}{dx}=\frac{a^2}{g}x,$$

so that the curve is $$y=\frac{a^2}{2g}x^2+\text{constant}, \quad\quad (2)$$

the constant depending upon the datum level from which y is measured. This is a parabola, and if it revolves about the axis we

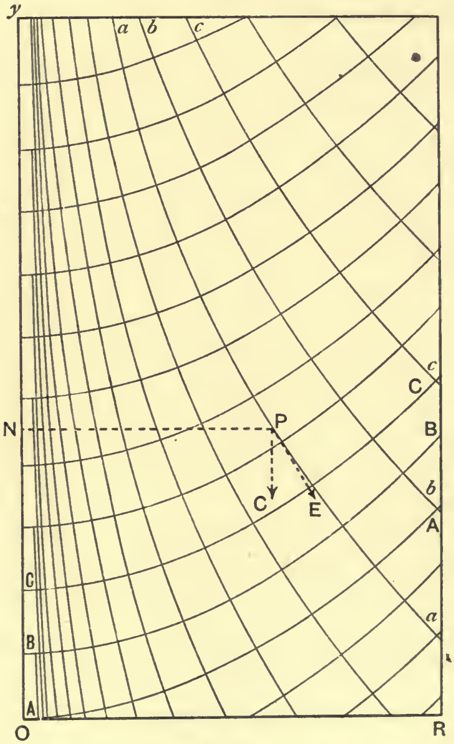

FIG. 33.

have a paraboloid of revolution. Any surface which is everywhere at right angles to the resultant force on the particle is called a level surface, and we see that the level surfaces in this case are paraboloids of revolution. They are sometimes called equi-potential surfaces. It is easy to prove that the pressure is constant everywhere in such a surface, and that it is a surface of equal density, so that if mercury, oil, water, and air are in a whirling vessel, their surfaces of separation are paraboloids of revolution.

The student ought to draw one of the lines of force (taking, say, $a = 8$, $g = 32{\cdot}2$) and cut out a template of it in thin zinc, *OO* being another edge. By sliding along *OO* he can draw many lines of force. Now cut out a template for one of the parabolas (its constant may be taken as *O*) and with it draw many level surfaces. The two sets of curves cut each other everywhere orthogonally. Fig. 33 shows the sort of result obtainable.

112. Motion of Fluid. If *AB* is a **stream** tube, in the vertical plane of the paper, consider the forces acting on the fluid which is between the sections at *P* and *Q*, of length δs feet along the stream and cross-section a square feet, where a and δs are in the limit supposed to be infinitely small. Let the pressure at *P* be p lb. per sq. foot, the velocity v feet per second, and let *P* be at the vertical height h above some datum level *R*. At *Q* let these quantities be $p + \delta p$, $v + \delta v$, $h + \delta h$. Let the fluid weigh w lb. per cubic foot. Find the forces urging *PQ* along the stream, that is, forces parallel to the stream direction at *PQ*.

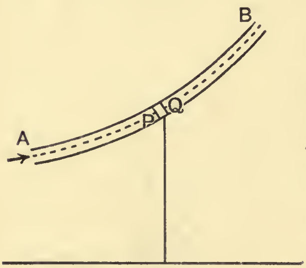

FIG. 34.

pa acts on one end *P* in the direction of motion, and $(p + \delta p)a$ acts at *Q*, retarding the motion. The weight of the portion between *P* and *Q* is $a \,.\, \delta s \,.\, w$, and as if on an inclined plane its retarding component is

$$\text{weight} \times \frac{\text{height of plane}}{\text{length of plane}} \quad \text{or} \quad a \,.\, \delta s \,.\, w \frac{\delta h}{\delta s}.$$

Hence we have altogether, accelerating the motion from *P* towards *Q*,

$$pa - (p + \delta p)a - a \,.\, \delta s \,.\, w \,.\, \frac{\delta h}{\delta s}.$$

But the mass is $a \,.\, \delta s \,.\, w/g$ and $\dfrac{dv}{dt}$ is its acceleration, so we have merely to put the force equal to $\frac{a \, . \, \delta s \, . \, w}{g} \cdot \frac{dv}{dt}$. We have then, dividing by a,

$$-\delta p - w \, . \, \delta h = \frac{\delta s \, . \, w}{g} \frac{dv}{dt}.$$

Now, if δt be the time taken by a particle in going from P to Q, we know that $v = \frac{\delta s}{\delta t}$ with greater and greater accuracy as δs is shorter and shorter. Also the acceleration is more and more nearly $\frac{\delta v}{\delta t}$. Hence, if δs is very small, $\delta s \, . \, \frac{dv}{dt} = \frac{\delta s}{\delta t} \delta v = v \, . \, \delta v$, so that we have

$$\frac{v}{g} dv + \frac{dp}{w} + dh = 0. \quad \text{..........................(1)}$$

This is the fundamental equation of fluid motion. Integrating,

$$\frac{v^2}{2g} + \int \frac{dp}{w} + h = \text{constant.} \quad \text{......................(2)}$$

I leave the sign of integration on the $\frac{dp}{w}$ because w may vary.

In a liquid where w is constant and in gases when pressures vary only a little (as in ventilation problems),

$$\frac{v^2}{2g} + \frac{p}{w} + h = \text{constant.} \quad \text{.......................(3)}$$

Ex. 1. Find what (2) becomes in the case of a gas in which the adiabatic law is followed, that is, $w = cp^{\frac{1}{\gamma}}$. Neglect h.

Ans. If $s = 1 - \frac{1}{\gamma}$, we have $\frac{v^2}{2g} + \frac{1}{cs} p^s = \text{constant.}$

Hence, if the gas flows from a vessel where $v = 0$, $p = p_1$ to a place where $p = p_0$ and $v = v_0$, we have

$$0 + \frac{1}{cs} p_1^s = \frac{v_0^2}{2g} + \frac{1}{cs} p_0^s \quad \text{or} \quad v_0^2 = \frac{2g}{cs}(p_1^s - p_0^s).$$

Ex. 2. Particles of water in a basin, flowing very slowly towards a hole in the centre, move in nearly circular paths, so that the velocity v is inversely proportional to the distance x from the central axis. Take $v = a/x$, where a is some constant. Then (3) becomes

$$h + \frac{a^2}{2gx^2} + \frac{p}{w} = \text{a constant.}$$

Now at the surface of the water p is constant, being the pressure of the atmosphere, so that there

$$h = c - \frac{a^2}{2gx^2}.$$

This gives us the shape of the curved surface. Assume c and a any values (for example take $c=1$, $a=0{\cdot}8$, $g=32$), and it is easy to calculate h for any value of x (say from $x=0{\cdot}05$ to $x=5$), and so plot the curve. This curve rotated about the axis gives the shape of the surface, which is a surface of revolution. $c-h$ is evidently the depth of any point below the level of the water at $x=\infty$.

Ex. 3. Water flowing spirally in a horizontal plane follows the law $v=a/x$ if x is distance from a central point. Show that

$$p=c-\frac{wa^2}{2gx^2}.$$

As an example, take $c=10000$, $w=62{\cdot}3$, $g=32{\cdot}2$, $a=70$, and draw the curve showing how p varies from $x=1$ to $x=2$.

CHAPTER XXVII.

THE COMPOUND INTEREST LAW.

113. If $y=be^{ax}$ ......(1), $\frac{dy}{dx}=abe^{ax}$ ......(2), or $\frac{dy}{dx}=ay$. ......(3)

Here is a function of x whose rate of increase is proportional to itself. If we ever see the statement (3) we can say that it is equivalent to the statement (1), where b is any constant whatsoever, an arbitrary constant, as we call it.

There are a great many phenomena in nature which have this property. Lord Kelvin's way of putting it was: "They follow the compound interest law." It is evident that b is the value of y when x is 0.

Ex. 1. An electric condenser of capacity k farads is discharging through a great resistance r ohms. If v is the potential difference in volts at any time t between the coatings of the condenser, the quantity of electricity in the condenser is $Q=kv$. The current from the condenser is v/r, and this is also $-\frac{dQ}{dt}$, being the rate of diminution of Q per second, or $-k\frac{dv}{dt}$, so that

$$-k\frac{dv}{dt}=\frac{v}{r}, \quad \frac{dv}{dt}=-\frac{1}{kr}v.$$

This is of the form (3) given above, v being our old y, t being our old x, and $-\frac{1}{kr}$ being our old a. Hence we can say that

$$v=be^{-\frac{t}{kr}},$$

so that

$$\log b-\log v=\frac{t}{kr}.$$

This knowledge gives us a means of measuring the leakage resistance of a cable or condenser. When a condenser is steadily losing its charge, if we measure v_1 at the time t_1 and we measure v_2 at the time t_2,

$$kr(\log b - \log v_1) = t_1,$$
$$kr(\log b - \log v_2) = t_2.$$

Subtracting, $$kr(\log v_1 - \log v_2) = t_2 - t_1,$$

so that $$r = \frac{t_2 - t_1}{k(\log v_1 - \log v_2)}.$$

These are Napierian logarithms. If we use common logarithms,

$$r = 0{\cdot}4343(t_2 - t_1)\Big/ k \log_{10}\frac{v_1}{v_2}.$$

Thus, suppose a condenser of 2 microfarads (or 2×10^{-6} farads) is discharging; we find v to be 10 volts and 20 seconds afterwards v is 8·2 volts; what is the leakage resistance?

Here $t_2 - t_1 = 20$, $k = 2 \times 10^{-6}$,

$$r = \frac{0{\cdot}4343 \times 20}{2 \times 10^{-6}(\log 10 - \log 8{\cdot}2)} = 50{\cdot}4 \times 10^6 \text{ ohms}.$$

Ex. 2. *Newton's Law of Cooling.* Imagine a body all at the temperature v (above the temperature of surrounding bodies) to lose heat at a rate which is proportional to v. Thus, let

$$\frac{dv}{dt} = -av,$$

where t is time. Then $$v = be^{-at}$$

or $$\log b - \log v = at.$$

Thus, let the temperature be v_1 at the time t_1 and v_2 at the time t_2; then $\log v_1 - \log v_2 = a(t_2 - t_1)$, so that a can be measured experimentally as being equal to

$$\log \frac{v_1}{v_2} \div (t_2 - t_1).$$

Ex. 3. A rod (like a tapering winding rope or like a pump rod of iron, but it may be like a tie rod made of stone to carry the weight of a lamp in a church) tapers gradually because of its own weight, so that it may have everywhere in it exactly the same tensile stress f lb. per sq. inch. If y is the area of the cross-section at the vertical distance x from its lower end, and if $y + \delta y$ is its cross-section at the distance $x + \delta x$ from its lower end, then $f . \delta y$ is evidently equal to the weight of the little portion between x and $x + \delta x$. This portion of volume is $y . \delta x$, and if w is the weight per unit volume, $f . \delta y = w . y . \delta x$, or rather

$$\frac{dy}{dx} = \frac{w}{f} \cdot y.$$

Hence, as before, $$y = be^{\frac{w}{f}x}.$$

If when $x=0$, $y=y_0$ the cross-section just sufficient to support a weight W hung on at the bottom (evidently $fy_0=W$), then $y_0=b$, because $e^0=1$.

It is, however, unnecessary to say more than that we have the law according to which the section of the rod alters.

Ex. 4. *Atmospheric Pressure.* At a place which is h feet above datum level, let the atmospheric pressure be p lb. per sq. foot; at $h+\delta h$, let the pressure be $p+\delta p$ (δp is negative, as will be seen). The pressure at h is really greater than the pressure at $h+\delta h$ by the weight of the air filling the volume δh cubic feet. If w is the weight in lb. of a cubic foot of air, $-\delta p = w \,.\, \delta h$. But $w=cp$, where c is some constant if the temperature is constant. Hence $-\delta p = c\,.\,p\,.\,\delta h$, or rather

$$\frac{dp}{dh} = -cp. \quad \text{......................... ....(1)}$$

Hence, as before, we have the compound interest law; the rate of fall of pressure as we go up being proportional to the pressure itself. Hence $p=be^{-ch}$, where b is some constant. If $p=p_0$ when $h=0$ (say at sea level), the law is

$$p=p_0e^{-ch}.$$

As for c it is w_0/p_0, w_0 being the weight of a cubic foot of air at the pressure p_0. If t is the constant (absolute) temperature, and w_0 is now the weight of a cubic foot of air at 0° C. or 273° absolute, then c is $\frac{w_0}{p_0}\frac{273}{t}$.

But it is absurd to imagine that the temperature is constant. The following assumption is much more likely to be true.

If w follows the adiabatic law, so that $pw^{-\gamma}$ is constant (γ is 1·4 for air), then (1) becomes

$$-\delta p = cp^{\frac{1}{\gamma}}\delta h \quad \text{or} \quad -p^{-\frac{1}{\gamma}}\,.\,dp = c\,.\,dh.$$

Integrating, we get $\quad -\frac{\gamma}{\gamma-1}p^{\frac{\gamma-1}{\gamma}} = ch + C.$

If $p=p_0$ when $h=0$, we can find C, and we have (if $\frac{\gamma-1}{\gamma}$ be called α),

$$p^{\alpha} = p_0{}^{\alpha} - \alpha ch$$

as the more usually correct law for pressure diminishing upwards in the atmosphere.

Observe that when we have the adiabatic law and we have also $p=Rtw$, it follows that the absolute temperature is proportional to p^{α}, so that (3) becomes

$$t=t_0-\frac{\alpha h}{R}.$$

That is, the rate of diminution of temperature is constant per foot upwards in such a mass of air.

Ex. 5. *Compound Interest.* £100 lent at 3 per cent. per annum becomes £103 at the end of the year. The interest during the second year being charged on the increased capital, the increase is greater the second year and is greater and greater every year. In fact, the increase of the principal every year is proportional to the amount of the principal.

Here the addition is made every 12 months; it might be made every 6 or 3 months or weekly or daily or every second. Nature's processes are, however, usually more continuous than even this.

Let us imagine compound interest to be added on to the principal continually and not by jerks every year, at the rate of r per cent. per annum. Let P be the principal at the end of t years. Then δP for the time δt is

$$\frac{r}{100}P\,.\,\delta t \quad \text{or} \quad \frac{dP}{dt}=\frac{r}{100}P, \quad \text{and hence} \quad P=P_0e^{\frac{rt}{100}}.$$

In what time will a sum of money double itself at compound interest at r per cent. per annum? Here

$$\frac{rt}{100}=\log_e 2 \quad \text{or} \quad rt=69{\cdot}31 \quad \text{or} \quad t=69{\cdot}31/r.$$

Students will find it interesting to refer to Arts. 26 and Ex. 3 of Art. 82.

To what will the principal £100 amount in 20 years at 5 per cent. per annum? Calculate first by the above formula. Now calculate by the formula of Art. 26, the interest being added on yearly. Compare the answers.

Answer by the above 271·8 pounds; the other answer is 265·3 pounds.

Ex. 6. Slipping of a belt on a pulley. When students make experiments on this slipping phenomenon, they ought to cause the

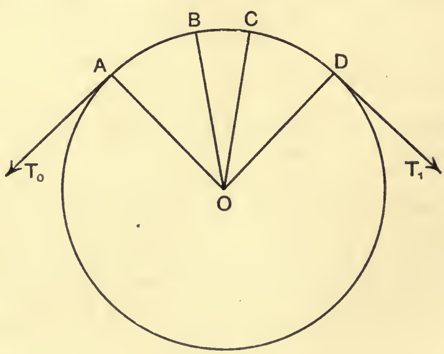

Fig. 35.

pulley to be fixed so that they may see the slipping when it occurs.

The pull on the belt at D is T_1, and this overcomes not only the pull T_0 at A, but also the friction between the belt and the pulley. Consider the tension T in the belt at B (Fig. 35), the angle AOB being θ; also the tension $T+\delta T$ at C, the angle AOC being $\theta+\delta\theta$.

Fig. 36 shows BC greatly magnified, $\delta\theta$ being very small. In calculating the force pressing the small portion of belt BC against

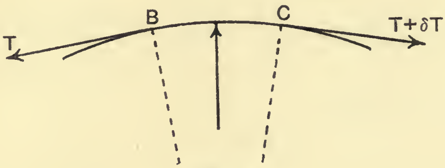

FIG. 36.

the pulley rim, as we think of BC as a shorter and shorter length, we see that the resultant pressing force is $T.\delta\theta$,* so that $\mu T.\delta\theta$ is the friction if μ is the coefficient of friction. It is this that δT is required to overcome. When $\mu.T.\delta\theta$ is exactly equalled by δT, sliding is about to begin. Then $\mu.T.\delta\theta=\delta T$ or $\frac{dT}{d\theta}=\mu T$, the compound interest law. Hence $T=be^{\mu\theta}$. Insert now $T=T_0$ when $\theta=0$, and $T=T_1$ when $\theta=AOD$ or θ_1, and we have $T_1=T_0e^{\mu\theta_1}$.

In calculating the horse-power H given by a belt to a pulley, we must remember that $H=(T_1-T_0)V\div 33{,}000$ if T_1 and T_0 are in pounds and V is the velocity of the belt in feet per minute. Again, whether a belt will or will not tear depends upon T_1; from these considerations we have the well-known rule for belting. (See Arts. 28 and 67.)

For other illustrations of the Compound Interest law, the student may be referred to my book on the Calculus.

Equation (3) is the simplest of a number of important statements which are referred to again in Arts. 134.

* When two equal forces T make a small angle $\delta\theta$ with one another, find their equilibrant or resultant. The three forces are parallel to the sides of an isosceles triangle like Fig. 37, where $FD=DE$ represents T, where $FDE=\delta\theta$

FIG. 37.

and EF represents the equilibrant. Now it is evident that as $\delta\theta$ is less and less, $EF\div DE$ is more and more nearly $\delta\theta$, so that the equilibrant is more and more nearly $T.\delta\theta$.

In the meantime, let the student consider the following two exercises :

Ex. 7. Show that the equation $\frac{d^2y}{dx^2} - n^2y = 0$ is satisfied by

$$y = Ae^{nx} + Be^{-nx},$$

where A and B are arbitrary constants.

Ex. 8. If i stands for $\sqrt{-1}$, and if i behaves like an algebraic quantity so that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc., the equation $\frac{d^2y}{dx^2} + n^2y = 0$ is satisfied by

$$y = Me^{inx} + Ne^{-inx}.$$

It will be seen later on, that as M and N may themselves involve the unreal i, this answer is equivalent to

$$y = A \sin nx + B \cos nx,$$

where A and B are arbitrary real constants. The student ought to try if this is the answer ; or what is equivalent,

$$y = a \sin(nx + b),$$

where a and b are arbitrary constants.

GENERAL EXERCISES.

The following exercises are on the subject matter of Chapters I.-XXVII. They are really questions which I have set in old examination papers.

1. The value of a ruby is proportional to the $1{\cdot}5^{\text{th}}$ power of its weight. If one ruby is exactly of the same shape as another but of 2·20 times its linear dimensions, of how many times the value is it ? *Ans.* 34·73 times.

2. An annuity of £a per annum, the first payment being due one year from now, the last at the end of the n^{th} year from now, has a present value

$$P = a\frac{100}{r}\left\{1 - \left(1 + \frac{r}{100}\right)^{-n}\right\}.$$

At 3 and at 5 per cent. per annum, find the present value of an annuity of £1 payable for 30 years. *Ans.* £19·57 at 3 %, £15·38 at 5 %.
The present value of an annuity is £2000, it is to last for 20 years ; 4 per cent. per annum. What is the annuity ? *Ans.* £148·8.
The present value of an annuity of £65 is £630 at $4\frac{1}{2}$ per cent. per annum. What is the number of years' duration of the annuity ?
Ans. 13 years.

3. £100 being lent at compound interest at 4 per cent. for 20 years, find its amount—(1) If interest is added on yearly. (2) Every half-year. (3) Every month. (4) Continuously.
Ans. (1) £219·11 ; (2) £220·8 ; (3) £222·25 ; (4) £222·56.

4. An insurance office asks a present payment P for a life annuity of £a per annum (1st payment 1 year from now). If a person's

expectation of further life is n years and if the value of money is r per cent. per annum,

$$P=\frac{100a}{r}\left\{1-\left(1+\frac{r}{100}\right)^{-n}\right\}.$$

If a is 1 and r is 3 %, find P in the following cases :

Age.	Expectation of life n. Men.	Answers : P Men.	Expectation of life n. Women.	Answers : P Women.
5	50·87	25·89	53·08	26·36
15	43·41	24·05	45·63	24·65
25	35·68	21·68	37·98	22·45
35	28·64	19·0	30·90	19·92
45	22·07	15·93	24·06	16·95
55	15·95	12·5	17·33	13·33

If a is 1 and P has the following values, find r.

Age - - - -	5	25	45
Men P - - -	25·95	21·6458	16·4958
Answers r - - -	2·992	3·025	2·682

A sum of money P being expended on a house, the lease of which has n years to run; what is the equivalent addition a to the rent, r per cent. per annum being the value of money? The above formula holds.

In the following cases find a:

(1) $P=1200$, $r=5$, $n=21$. *Ans.* $a=93{\cdot}57$.

(2) $P=1500$, $r=4\frac{1}{2}$, $n=20$. *Ans.* $a=115{\cdot}3$.

In the following cases find n:

$P=1200$, $r=5$, $a=90$. *Ans.* $n=22{\cdot}5$.

To insure for a sum A to be paid at death, the letters r and n being as before, the premium being a, the first instalment payable at once.

$$A=\frac{100a}{r}\left\{\left(1+\frac{r}{100}\right)^{n-1}-\left(1+\frac{r}{100}\right)^{-1}\right\}.$$

If $a=1$, $r=3$, find A in the following cases:

Present age.	Expectation of life. n for males.	Answer A.
5	50·87	112·57
25	35·68	60·27
45	22·07	29·90

In an insurance office if $A = £100$ and n has the following values, and also a, find r.

Age.	n	a	Answer r.
21	38·64	2·1542	1·013
30	32·10	2·5875	1·26
40	25·30	3·3125	1·54

5. A sum of money A has to be provided for; it is due n years hence. The annuity a is paid regularly to provide, the first payment being one year from now.

$$A = \frac{100a}{r}\left\{\left(1+\frac{r}{100}\right)^n - 1\right\}.$$

If $A = 100$, $r = 3·5$; find a in the following cases:

n	5	20	40
Answer a	18·72	3·55	1·189

I suppose that all insurance offices are really *borrowers* of money, and therefore r is low. If they paid P now, to be repaid by yearly payments a for n years they would charge a large r.

6. If
$$A = P\left(1+\frac{r}{100}\right)^n,$$
and if
$$A = 3P \text{ when } r = 3\tfrac{1}{2},$$
find n. *Ans.* 32·02.

7. Two sides of a triangle are measured and found to be 32·5 and 24·2 inches; the included angle being 57°, find the area of the triangle. Prove the rule used by you. If the true lengths of the sides are really 32·6 and 24·1, what is the percentage error in the answer?

Ans. 329·8, 0·12 per cent.

8. In a triangle ABC, C being a right angle, AB is 14·85 inches, AC is 8·32 inches. Compute the angle A in degrees, using your tables.

Ans. 55° 55′.

9. ABC is a triangle. The angle A is 37°, the angle C is 90°, and the side AC is 5·32 inches; find the other sides, the angle B, and the area of the triangle. *Ans.* 4·009, 6·662, 53°, 10·66 sq. in.

10. In a triangle ABC the angle C is 53°, the sides AC and AB are 0·523 and 0·942 miles respectively; find the side CB in miles and the area of the triangle in square miles, either by actual construction on your paper or by calculation. *Ans.* 1·159 miles, 0·2420 square mile.

11. In a triangle ABC, AD is the perpendicular on BC; AB is 3·25 feet; the angle B is 55°. Find the length of AD. If BC is 4·67 feet, what is the area of the triangle?

Find also BD and DC and AC. Your answers must be right to three significant figures.

Ans. 2·66 ft., 6·21 sq. ft., $DB = 1·86$, $DC = 2·81$, $AC = 3·87$.

12. A tube of copper (0·32 lb. per cubic inch) is 12 feet long and 3 inches inside diameter; it weighs 100 lb.; find its outer diameter, and the area of its curved outer surface.

Ans. 3·43 inches, 1552 square inches.

13. The inside diameter of a hollow sphere of cast iron is the fraction 0·57 of its outside diameter. Find these diameters if the weight is 60 lb. Take 1 cubic inch of cast iron as weighing 0·26 lb.

If the outside diameter is made 1 per cent. smaller, the inside not being altered, what is the percentage diminution of weight?

Ans. 8·148 in., 4·644 in., 3·64 per cent.

14. The cross-section of a ring is an ellipse whose principal diameters are 2 inches and $1\frac{1}{2}$ inches; the middle of this section is at 3 inches from the axis of the ring; what is the volume of the ring?

Prove the rule you use for finding the volume of any ring.

Ans. 44·43 cubic in.

15. Let x be multiplied by the square of y, and subtracted from the cube of z, the cube root of the whole is taken and is then squared. This is divided by the sum of x, y, and z. Write all this down algebraically.

Ans. $\dfrac{(z^3-xy^2)^{\frac{2}{3}}}{x+y+z}$.

16. Express

$$\frac{x-13}{x^2-2x-15}$$

as the sum of two simpler fractions. *Ans.* $\dfrac{2}{x+3}-\dfrac{1}{x-5}$.

What is the integral of the expression? *Ans.* $\log (x+3)^2/(x-5)$.

17. The sum of two numbers is 76, and their difference is equal to one-third of the greater; find them. *Ans.* 45·6, 30·4.

18. Suppose s the distance in feet passed through by a body in the time t seconds is $s=10t^2$. Find s when t is 2, find s when t is 2·01, and also when t is 2·001. What is the average speed in each of the two short intervals of time after $t=2$? When the interval of time is made shorter and shorter, what does the average speed approximate to?

Ans. 40 ft., 40·401 ft., 40·04001 ft., 40·1, and 40·01 ft. per sec., 40 ft. per sec.

19. It is known that the weight of coal in tons consumed per hour in a certain vessel is $0·3+0·001v^3$, where v is the speed in knots (or nautical miles per hour). For a voyage of 1000 nautical miles tabulate the time in hours and the total coal consumption for various values of v. If the wages, interest on cost of vessel, etc., are represented by the value of 1 ton of coal per hour, tabulate for each value of v the total cost, stating it in the value of tons of coal, and plot on squared paper. About what value of v gives greatest economy? *Ans.* $v=$ between $8\frac{1}{2}$ and $8\frac{3}{4}$.

20. Write down algebraically: Add twice the square root of the cube of x to the product of y squared and the cube root of z. Divide by the sum of x and the square root of y. Add 4 and extract the square root of the whole. *Ans.* $\left\{\dfrac{2x^{\frac{3}{2}}+y^2z^{\frac{1}{3}}}{x+}+4\right\}^{\frac{1}{2}}$.

21. Express
$$\frac{3x-2}{x^2-3x-4}$$
as the sum of two simpler fractions. *Ans.* $\frac{2}{x-4}+\frac{1}{x+1}$.

22. Find two numbers such that if four times the first be added to two and a half times the second, the sum is 17·3; and if three times the second be subtracted from twice the first, the difference is 1·2.

Ans. 3·229, 1·753.

23. There are two quantities, a and b. The square of a is to be multiplied by the sum of the squares of a and b; add 3; extract the cube root; divide by the product of a and the square root of b. Write down this algebraically. *Ans.* $\frac{\sqrt[3]{a^2(a^2+b^2)+3}}{a\sqrt{b}}$.

24. Express $\frac{1}{x^2-7x+12}$ as the sum of two simpler fractions and integrate. *Ans.* $\frac{1}{x-4}-\frac{1}{x-3}$, $\log\frac{x-4}{x-3}$.

25. A crew which can pull at the rate of six miles an hour finds that it takes twice as long to come up a river as to go down; at what rate does the river flow? *Ans.* 2 miles per hour.

26. In any class of turbine if P is power of the waterfall and H the height of the fall and n the rate of revolution, then it is known that for any particular class of turbines of all sizes
$$n \propto H^{1\cdot25}P^{-0\cdot5}.$$
In the list of a particular maker I take a turbine at random for a fall of 6 feet, 100 horse-power, 50 revolutions per minute. By means of this, I find I can calculate n for all the other turbines of the list. Find n for a fall of 20 feet and 75 horse-power. *Ans.* 260.

27. If H is proportional to $D^{\frac{2}{3}}v^3$, and if D is 1810 and v is 10 when H is 620, find H if D is 2100 and v is 13. *Ans.* 1503.

28. If $y=ax^{\frac{1}{2}}+bxz^2$,

if $\quad y=62\cdot3$ when $x=4$ and $z=2$,

if $\quad y=187\cdot2$ when $x=1$ and $z=1\cdot46$,

find a and b, and find the value of y when x is 9 and z is 0·5.

Ans. $a=243\cdot9$, $b=-26\cdot59$, $y=671\cdot9$.

29. If $z=ax-by^3x^{\frac{1}{2}}$.

If $z=1\cdot32$ when $x=1$ and $y=2$; and if $z=8\cdot58$ when $x=4$ and $y=1$; find a and b.

Then find z when $x=2$ and $y=0$. *Ans.* $a=2\cdot2$, $b=0\cdot11$, $z=4\cdot4$.

30. A cast-iron flywheel rim (0·26 lb. per cubic inch) weighs 13,700 lb. The rim is of rectangular section, thickness radially x, size the other way $1\cdot6x$. The inside radius of the rim is $14x$. Find the actual sizes.

Ans. Thickness radially 7·124 in., other way 11·4 in., and inside radius 99·7 in.

31. The electrical resistance of copper wire is proportional to its length divided by its cross-section. Show that the resistance of a pound

of wire of circular section all in one length is inversely proportional to the fourth power of the diameter of the wire.

32. A hollow cylinder is 4·32 inches long; its external and internal diameters are 3·150 and 1·724 inches; find its volume and the sum of the areas of its two curved surfaces.

Ans. 23·58 cubic inches, 66·16 square inches.

33. A circular anchor ring has a volume 930 cubic inches and an area 620 square inches; find its dimensions.

Ans. Radius of cross-section 3 in., mean radius of ring 5·2 in.

34. The mean radius of a ring is 2 feet. The cross-section of the ring is an ellipse whose major and minor diameters are 0·8 and 0·5 feet; what is its volume? *Ans.* 3·948 cubic feet.

35. The length of a plane closed curve is divided into 24 elements each of 1 inch long. The middles of successive elements are at the distances x from a line in the plane, as follows (in inches): 10, 10·5, 10·91, 11·24, 11·49, 11·67, 12·57, 11·67, 11·49, 11·24, 10·91, 10·5, 10, 10·5, 10·91, 11·24, 11·49, 11·67, 12·57, 11·67, 11·49, 11·24, 10·91, 10·5.

If the curve rotates about the line as an axis describing a ring, find approximately the area of the ring. *Ans.* 1687 square inches.

36. A prism has a cross-section of 50·32 square inches. There is a section making an angle of 20° with the cross-section; what is its area? Prove the rule that you use. *Ans.* 53·56 square inches.

37. Assuming the earth to be a sphere, if its circumference is 360 × 60 nautical miles, what is the circumference of the parallel of latitude 56°? What is the length there of a degree of latitude? If a small map is to be drawn in this latitude, with north and south and east and west distances to the same scale, and if a degree of latitude (which is of course 60 miles) is shown as 10 inches, what distance will represent a degree of longitude? *Ans.* 12,080 miles, 33·55, 5·592 inches.

38. The following table records the growth in stature of a girl A (born January, 1890) and a boy B (born May, 1894). Plot these records. Heights were measured at intervals of 4 months.

TABLE OF HEIGHTS IN INCHES.

Year.	1900.	1901.			1902.			1903.
Month.	Sept.	Jan.	May.	Sept.	Jan.	May.	Sept.	Jan.
A	54·75	55·55	56·6	57·95	59·2	60·2	60·9	61·3
B	48·25	49·0	49·75	50·6	51·5	52·3	53·1	53·9

Find in inches per annum the average rates of growth of A and B during the whole period of tabulation. What will be the probable heights of A and B at the end of another four months? Plot the *rate* of growth of A at all times throughout the period. At about what age was A growing most rapidly and what was her quickest rate of growth?

Ans. 2·8, 2·4, $11\frac{1}{2}$ years, 4·2 in. per annum.

39. At a certain place where all the months of the year are assumed to be of the same length (30·44 days each), at the same time in each month the length of the day (interval from sunrise to sunset in hours) was measured, as in this table.

Nov.	Dec.	Jan.	Feb.	Mar.	April.	May.	June.	July.
8·35	7·78	8·35	9·87	12	14·11	15·65	16·22	15·65

What is the average increase of the length of the day (state in decimals of an hour per day) from the shortest day, which is 7·78 hours, to the longest, which is 16·22 hours? When is the increase of the day most rapid, and what is it? *Ans.* 0·046 ; March, 0·072 hr. per day.

40. If $pv^n = a$.

If $p = 100$ when $v = 1$, find a. *Ans.* 100.

Taking $pv^n = 100$. Here are important exercises, in some of which n is 0·8, another 0·9, and so on. Find in each case the value of p when v has the following values:

Given values of v.	Answers.		
	Value of p if $n=0·9$.	Value of p if $n=1$.	Value of p if $n=1·13$.
1	100	100	100
1·5	69·4	66·7	63·2
2	53·6	50	45·7
2·5	43·8	40	35·5
3	37·2	33·3	28·9
3·5	32·4	28·6	24·3
4	28·7	25	20·9

The student ought to show these answers on one sheet of squared paper as three curves.

41. If $p_e = p_1 \dfrac{1+\log_e r}{r} - p_3$.

Let $r = 3$ and $p_3 = 17$. Find p_e in the following cases:

p_1	u	Answers.			
		p_e	P	W	W'
200	2·273	122·82	140·8	9241	11320
170	2·649	101·8	116·7	7929	9716
140	3·177	80·86	92·7	6610	8100
110	3·984	59·89	68·66	5272	6462
80	5·370	38·92	44·64	3911	4793
50	8·340	17·95	20·59	2519	3085

$$P = \frac{Ap_eRn}{33000}.$$

If $A=210$, $R=1{\cdot}2$, $n=150$, calculate P for each of the given values of p_1.

If $V=\frac{A}{144}\times\frac{4Rn}{r}\times 60$, show that V is 21000.

If W is V/u, where each value of u is tabulated above, find W in each case.

Plot the values of P and of W on squared paper, and try if there is a rule connecting them like

$$W=a+bP,$$

and find a and b. *Ans.* $a=1400$, $b=56$.

42. If $W'=(1+y)W$ and if $y=10\frac{1+r}{\sqrt{An}}$, find W' in each case of the last exercise. Now plot W' and P on the same sheet of squared paper as before, and see if there is a law connecting them like

$$P=0{\cdot}0144\,W'-24.$$

In Exercises 41 and 42, p_1 and p_3 are the initial and back pressures of steam in a cylinder, p_e the effective pressure; cut off at $1/r^{\text{th}}$ of the stroke; P the horse-power; W lb. the indicated steam per hour; W' the actual steam per hour.

43. $x=\tan\theta\div\tan(\theta+\phi)$, where ϕ is always 10°, find x when θ has the values 30°, 40°, 50°, 60°, and plot the values of x and of θ on squared paper. About what value of θ seems to give the largest value of x? *Ans.* 40°.

44. At speeds greater than the velocity of sound, the air resistance to the motion of a projectile of the usual shape, of weight w lb., diameter d inches, is such that when the speed diminishes from v_1 feet per second to v, if t is the time in seconds and x is the horizontal space passed over in feet,

$$t=7000\frac{w}{d^2}\left(\frac{1}{v}-\frac{1}{v_1}\right),$$

$$x=7000\frac{w}{d^2}\log_e\frac{v_1}{v}.$$

For a 12 lb. projectile of diameter 3 inches, if v_1 is 2000, find x and t for the values of v, 1900, 1800, 1700, 1600, etc., down to 1000, and tabulate. For each of these values of t find $y=2000at-16{\cdot}1t^2$. Using x and y plot the curved path of a projectile, (1st) when the initial inclination a is 0·12, (2nd) when a is 0·06 radians to the horizontal. It is well to magnify the vertical dimensions.* You have tabulated v, t, x and y. Now tabulate the values of x', which is $x'=2000t$. Plot x' also with y; this gives the parabolic path of the projectile if there is no air resistance. The effect of air resistance ought to be carefully noticed in the two cases.

45. One of my students in travelling to and from college sometimes noted times and distances on the railway. Here is a copy of one set of his observations. Find the average speed of his train in every interval.

* The muzzle velocities in the two cases are 2014·4 and 2003·6; the horizontal component of the muzzle velocity is 2000 in both cases.

Distances.		Time.			Miles $-\delta s$.	Sec. δt.	Miles per hour $-\frac{\delta s}{\delta t}$.
	Miles.	hrs.	min.	sec.			
Gravesend Central -	22·21	6	59	24			
					·21	60	12·6
	22	7	00	24			
					·50	81	22·2
	21½		1	45			
					1·00	116	31
	20½		3	41			
					·50	52	34·5
Northfleet -	20		4	33			
					·50	46	39·1
	19½		5	19			
					·50	48	37·5
	19		6	07			
					·50	40	45
	18½		6	47			
					·50	36	50
Greenhithe -	18		7	23			
					·75	52	51·9
	17¼		8	15			
					·25	19	47·3
	17		8	34			
					·50	41	43·9
	16½		9	15			
					·50	44	40·9
	16		9	59			
					·70	73	34·5
Dartford - -	15·3 arr.		11	12			
					·00	103	00·0
	15·3 dep.		12	55			
					·05	42	4·3
	15¼		13	37			
					·25	49	18·4
	15		14	26			
					·25	36	25
	14¾		15	02			
					·25	29	31
	14½		15	31			
					·50	49	36·8
	14		16	20			
					1·0	83	43·3
	13		17	43			
					1·0	86	41·8
Bexley - -	12		19	09			
					1·0	97	37·1
	11		20	46			
					1·0	94	38·3
Sidcup - -	10		22	20			
					1·0	84	42·9
	9		23	44			
					1·0	76	47·4
	8		25	00			
					1·0	70	51·5
	7		26	10			
					·50	38	47·4
	6½		26	48			
					·75	149	18·1
	5¾		29	17			
					·15	49	11·0
	5·6		30	06			

46. A summer student was given the dimensions, etc., of a mechanism. He made a skeleton drawing and found the following positions of a sliding piece (distance in feet from a point in its straight path) at the time t seconds. The periodic time is 2·4 seconds. Find the velocity and acceleration.

Time t seconds.	x feet, the distance of slider from a point in its straight path.	$\frac{\delta x}{\delta t}=v$	$\frac{\delta v}{\delta t}$ or a
0	5·96		
		−10·7	
0·1	4·89		−2
		−10·9	
0·2	3·80		14
		−9·5	
0·3	2·85		19
		−7·6	
0·4	2·09		25
		−5·1	
0·5	1·58		19
		−3·2	
0·6	1·26		24
		−0·8	
0·7	1·18		20
		1·2	
0·8	1·30		4
		1·6	
0·9	1·46		30
		4·6	
1·0	1·92		6
		5·2	
1·1	2·44		11
		6·3	
1·2	3·07		9
		7·2	
1·3	3·79		12
		8·4	
1·4	4·63		2
		8·6	
1·5	5·49		−1
		8·5	
1·6	6·34		−7
		7·8	
1·7	7·12		−15
		6·3	
1·8	7·75		−22
		4·1	
1·9	8·16		−24
		1·7	
2·0	8·33		−34
		−1·7	
2·1	8·16		−31
		−4·8	
2·2	7·68		−24
		−7·2	
2·3	6·96		−28
		−10·0	
2·4	5·96		

Plot v and also a with x as abscissa and draw curves. Do the same with t as abscissa.

47. The present value of a lease which would bring in a net yearly value or annuity of £100 is as follows, interest being calculated at 4 per cent. per annum :

Number of years to run	5	10	15	20	25	30
Present value - -	445	811	1112	1359	1562	1729

Plot on squared paper. Find the present value if the number of years is 13. *Ans.* £992.

48. The present price of an annuity of £100 to be paid to a person of a certain age is quoted from a certain insurance office advertisement.

Age of annuitant - -	20	30	40	50	60	70
Price to be paid now -	2279	2045	1789	1500	1148	797

Plot on squared paper, and find the price to be paid for a £100 annuity for a person who is now 55 years old. *Ans.* £1334.

49. By extracting square roots, we find successively $10^{0\cdot5}$, $10^{0\cdot25}$, $10^{0\cdot125}$, $10^{0\cdot0625}$, $10^{0\cdot03125}$, and $10^{0\cdot015625}$. By multiplying such results together we can find the following numbers :

Numbers.	Logarithms to the base 10.
5·8294	0·765625
6·0429	6·781250
6·2643	0·796875

Plot on squared paper, and find the logarithms of 5·83, 6·00, 6·25 correct to four significant figures.

50. An army of 5000 men costs a country £800,000 per annum to maintain it ; an army of 10,000 men costs £1,300,000 per annum to maintain it ; what is the annual cost of an army of 8000 men ? Take the simplest law which is consistent with the figures given. Use squared paper or not, as you please. *Ans.* $£1\cdot1 \times 10^6$.

51. An examiner has given marks to papers ; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law converting the highest number of marks into 250, and the lowest into 100 ; show how he may do this, and state the converted marks for papers already marked 60, 100, 150. Use squared paper or mere algebra, as you please. *Ans.* 118·9, 160·8, 213·3.

52. At the following draughts in sea water a particular vessel has the following displacements :

Draught h feet - -	15	12	9	6·3
Displacement T tons -	2098	1512	1018	586

What are the probable displacements when the draughts are 11 and 13 feet respectively ? *Ans.* 1350, 1700.

53. Work the following three exercises as if in each case one were alone given, taking in each case the simplest supposition which your information permits :

(*a*) The total yearly expense in keeping a school of 100 boys is £2100 ; what is the expense when the number of boys is 175 ?

(*b*) The expense is £2100 for 100 boys, £3050 for 200 boys ; what is it for 175 boys ?

(*c*) The expense for three cases is known as follows :

£2100 for 100 boys ; £2650 for 150 boys ; £3050 for 200 boys.

What is the probable expense for 175 boys ?

If you use a squared paper method, show all three solutions together.
Ans. (*a*) £3675 ; (*b*) £2812·5 ; (*c*) £2860.

54. The following are the areas of cross-section of a body at right angles to its straight axis :

A in square inches -	250	292	310	273	215	180	135	120
x inches from one end	0	22	41	70	84	102	130	145

What is the whole volume from $x=0$ to $x=145$?

At $x=70$, if a cross-sectional slice of small thickness δx has the volume δv, find $\frac{\delta v}{\delta x}$. *Ans.* 33,420 cubic inches ; 273 sq. inches.

55. h is the height in feet of the atmospheric surface of the water in a reservoir above the lowest point of the bottom ; A is the area of the surface in square feet.

When the reservoir was filled to various heights the areas were measured and found to be :

Values of h	0	13	23	33	47	62	78	91	104	120
Values of A	0	21000	27500	33600	39200	44700	50400	54700	60800	69300

How many cubic feet of water leave the reservoir when h alters from 113 to 65 ? *Ans.* $2{\cdot}635 \times 10^6$ cubic feet.

56. There is a curve whose shape may be drawn from the following values of x and y :

x in feet - -	3	3·5	4·2	4·8
y in inches -	10·1	12·2	13·1	11·9

Imagine this curve to rotate about the axis of x describing a surface of revolution. What is the volume enclosed by this surface, and the two end sections where $x=3$ and $x=4{\cdot}8$? *Ans.* 6 cubic feet.

57. The New Zealand Pension law for a person who has already lived from the age of 40 to 65 in the colony is :

If the private income I is not more than £34 a year, the pension P is £18 a year. If the private income is anything from £34 to £52, the pension is such that the total income is just made up to £52. If the private income is £52 or more there is no pension.

Show on squared paper, for any income I the value of P, and also the value of the total income. If a person's private income is say £50, how much of it has he an inducement to give away before he applies for a pension? Show on the same paper the total income, if the pension were regulated according to the rule.

$P=18-\frac{9}{26}I$. *Ans.* Anything up to £16.

58. If $x=\frac{a}{b-y}+cy-\frac{a}{b}$.

If y is positive and never greater than b. If $b=10$ and $c=1$. Plot the curve connecting x and y.

1st when $a=5$; 2nd when $a=1$; 3rd when $a=0·1$; 4th when $a=0·01$.

Plot these curves on the same sheet of paper after tabulating values of y and x.

59. If $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Calculate y for the following values of x, and tabulate as here shown:

Given values of x.	0	0·5 or −0·5	1·0 or −1·0	1·5 or −1·5	2·0 or −2·0	2·5 or −2·5	3 or −3	3·5 or −3·5	4 or −4	4·5 or −4·5	5 or −5	Any number greater than 5 or less than −5.
Answers, values of y.	±4	±3·98	±3·92	±3·816	±3·665	±3·464	±3·2	±2·856	±2·4	±1·743	0	Imaginary.

Plot x and y on squared paper.

60. The following corresponding values of two quantities, which we may call x and y, were measured:

x	0·5	1·7	3·0	4·7	5·7	7·1	8·7	9·9	10·6	11·8
y	148	186	265	326	388	436	529	562	611	652

It is known that there is a law like

$$y=a+bx$$

connecting these quantities, but the observed values are slightly wrong. Plot on squared paper; find the most probable values of a and b, and state the probable error in the measured value of y when $x=8·7$.

Ans. $a=119$, $b=45·7$, 2·4 per cent.

61. In a price list I find the following prices of a certain type of steam electric generator of different powers:

Kilowatts K.	Price P pounds.
200	2,800
600	7,160
1,000	11,520

According to what rule has this price list been made up? What is the list price of a generator of 400 Kilowatts?

Ans. $P=10{\cdot}9K+620$; when $K=400$, $P=£4980$.

62. The following quantities are thought to follow a law like $pv^n=$constant. Try if they do so; find the most probable value of n:

v	1	2	3	4	5
p	205	114	80	63	52

Ans. $n=0{\cdot}86$.

63. The following table gives corresponding values of two quantities x and y:

y	10·16	12·26	14·70	20·80	24·54	28·83
x	37·36	31·34	26·43	19·08	16·33	14·04

Try whether x any y are connected by a law of the form $yx^n=c$, and if so, determine as nearly as you can the values of n and c.
What is the value of x when $y=17{\cdot}53$? *Ans.* $yx^{1{\cdot}08}=501$, 22·4.

64. It is thought that the following observed quantities, in which there are probably errors of observation, follow a law like

$$y=ae^{bx}.$$

Test if this is so, and find the most probable values of a and b.

x	2·30	3·10	4·00	4·92	5·91	7·20
y	33·0	39·1	50·3	67·2	8·56	125·0

Ans. $y=18e^{0{\cdot}26x}$.

65. At an electricity works, where new plant has been judiciously added, if W is the annual works cost in millions of pence, and T is the annual total cost, and U the number of millions of electrical units sold, the following results have been found:

U	W	T
0·3	0·47	0·78
1·2	1·03	1·64
2·3	1·70	2·73
3·4	2·32	3·77

Find approximately the rules connecting T and W with U. Also find the probable values of W and T when U becomes 5, if there is the same judicious management.

Ans. $T=0{\cdot}95U+0{\cdot}525$, $W=0{\cdot}6U+0{\cdot}28$; when $U=5$, $T=5{\cdot}25$, $W=3{\cdot}28$.

66. There is a function

$$y = 5\log_{10} x + 6\sin \tfrac{1}{10}x + 0\cdot084(x - 3\cdot5)^2.$$

Find a much simpler function of x which does not differ from it in value more than 2 per cent. between $x=3$ and $x=6$. Remember that the angle $\frac{1}{10}x$ is in radians. *Ans.* $y = 1\cdot22x + 0\cdot49$.

67. Find accurately to three significant figures a value of x to satisfy the equation $$0\cdot5x^{1\cdot5} - 12\log_{10} x + 2\sin 2x = 0\cdot921.$$

Notice in $\sin 2x$ that the angle is in radians. *Ans.* 1·22.

68. The population of a country was $4\cdot35 \times 10^6$ in 1820, $7\cdot5 \times 10^6$ in 1860, $11\cdot26 \times 10^6$ in 1890. Test if the population follows the compound interest law of increase. What was the probable population in 1910?

Ans. $14\cdot78 \times 10^6$.

69. Find accurately to three significant figures a value of x which satisfies the equation

$$2x^2 - 10\log_{10} x - 3\cdot25 = 0.$$ *Ans.* 1·645.

70. If p_3/p_1 be called y.

Let $p_3 = 20$; find y for the following values of p_1. In each case find a value of x which satisfies the equation

$$y = \frac{1}{x} - 0\cdot23\log_{10} x.$$

p_1	200	150	100	70	50
Answers, y	0·10	0·1333	0·20	0·2837	0·40
Answers, x	4·135	3·763	3·172	2·60	2·110

71. If the values of p_1 are as here given. If $p_3 = 10$, find in each case a value of x which satisfies the equation

$$\frac{p_3}{p_1} = \frac{1}{x} - \frac{1\cdot25\log_{10} x}{\sqrt{p_1} - 1\cdot2}.$$

p_1	200	150	100	70	50
Answers, x	7·46	6·35	5·02	4·03	3·24

72. Find, correctly to three significant figures, a value of x which will satisfy this equation:

$$9x^3 - 41x^{0\cdot8} + 0\cdot5e^{2x} - 92 = 0.$$ *Ans.* 2·35.

73. If $y = \frac{1}{2}x^2 - 3x + 2$, show, by taking some values of x and calculating y and plotting on squared paper, the nature of the relationship between x and y. For what values of x is $y = 0$? *Ans.* $x = 5\cdot236$, $0\cdot764$.

74. The following values of p and u have been observed. It is thought that there is a law connecting p and u like

$$pu^a = b;$$

try if this is so, and if it is nearly true, find the most fitting values of a and b.

p	10·16	20·80	60·4	101·9	163·3	225·9	305·5
u	37·36	19·08	7·009	4·29	2·756	2·031	1·529

Ans. $pu^{1·0646}=479$.

75. Mr. Odell made measurements of the torque c (in pound inches) required to keep a disc of paper of d inches diameter revolving at the following speeds n (in revolutions per minute),

$d=22$ inches.

c	0·33	0·56	0·875	1·29	1·76	2·4
n	400	500	600	700	800	900

Plot $\log c$ and $\log n$ on squared paper, and see if there is a law,

$c \propto n^m$. *Ans.* m is 2·5.

76. Do the same in the following case :

Disc $d=27$ inches diameter.

c	0·41	0·575	0·895	1·297	1·72	2·2	2·385
n	250	300	350	400	450	500	517

Mean answer for both sets, $m=2·5$.

77. The following measurements were made of the power P (in watts) required to keep a disc of paper of diameter d inches revolving at 1000 revolutions per minute:

d	15·04	21·82	26·83	36	47·1
P	0·398	3·162	12·59	50	154·9

Show that $P \propto d^s$, and find s. *Ans.* $s=5·5$.

78. A firm is satisfied from its past experience and study that its expenditure per week in pounds is

$$120+3·2x+\frac{C}{x+5}+0·01C,$$

where x is the number of horses employed by the firm, and C is the usual turnover.

If C is 2150 pounds; find for various values of x what is the weekly expenditure, and plot on squared paper to find the number of horses which will cause the expenditure to be a minimum. *Ans.* 21 horses.

79. A number is added to 2·25 times its reciprocal; for what number is this a minimum? Use squared paper or the calculus as you please.
Ans. 1·5.

80. At the end of a time t seconds it is observed that a body has passed over a distance s feet reckoned from some starting point. If it is known that

$$s=25+150t-5t^2,$$

what is the velocity at the time t? *Ans.* $150-10t$.

Prove the rule that you adopt to be correct. If corresponding values of s and t are plotted on squared paper, what indicates the velocity and why?

81. $y=a+bx^n$ is the equation to a curve which passes through these three points,

$$x=0,\ y=1·24;\quad x=2·2,\ y=5·07;\quad x=3·5,\ y=12·64;$$

find a, b, and n.

Find $\frac{dy}{dx}$ when $x=2$. *Ans.* $a=1·24$, $b=0·602$, $n=2·348$, 3·593.

82. The quantities V and C both vary with the time. If we know that

$$V=RC+L\frac{dC}{dt},$$

and that $R=0·1$, $L=0·001$; and if the values of C are as tabulated, find V approximately. Plot on squared paper.

C	t	Calculated V.
342·0	0·00020	
		1675
358·4	0·00021	
		1657
374·6	0·00022	
		1648
390·7	0·00023	
		1640
406·7	0·00024	
		1632
422·6	0·00025	
		1623
438·4	0·00026	
		1605
454·0	0·00027	
		1596
469·5	0·00028	
		1578
484·8	0·00029	

83. There is a piece of a mechanism whose weight is 200 lbs. The following values of s in feet show the distance of its centre of gravity (as measured on a skeleton drawing) from some point in its straight path at

the time t seconds from some era of reckoning. Find its acceleration at the time $t=2{\cdot}05$, and the force in pounds which is giving this acceleration to it.

s	t
0·3090	2
0·4931	2·02
0·6799	2·04
0·8701	2·06
1·0643	2·08
1·2631	2·10

Ans. 9·25 feet per sec. per sec.; 57·5 lb.

84. The following observed numbers are thought to follow a law like $y=ax/(1+sx)$. Try by plotting the values of y/x and y on squared paper if this is so, and find the values of a and s.

x	0·5	1	2	0·3	1·4	2·5
y	0·78	0·97	1·22	6·55	1·1	1·24

Ans. $y=3x/(1+2x)$.

85. In the curve $y=cx^{\frac{1}{2}}$, find c if $y=m$ when $x=b$. Let this curve rotate about the axis of x; find the volume enclosed by the surface of revolution between the two sections at $x=a$ and $x=b$. Of course m, b, and a are given distances. *Ans.* $\dfrac{\pi m^2(b^2-a^2)}{2b}$.

86. Find $\int p\,.\,dv$, if $pv^s=c$, a constant,

(1) when $s=0{\cdot}8$; (2) when $s=1$.

Ans. (1) $5cv^{0\cdot2}$; (2) $c\log_e v$.

87. If $y=2{\cdot}4-1{\cdot}2x+0{\cdot}2x^2$, find $\dfrac{dy}{dx}$ and plot two curves from $x=0$ to $x=4$, showing how y and $\dfrac{dy}{dx}$ depend upon x.

88. Divide the number 20 into two parts, such that the square of one, together with three times the square of the other, shall be a minimum. Use any method you please. *Ans.* 15 and 5.

89. If the current C amperes in a circuit is changing as shown in this table:

C	279·3	296·7	314·2	331·6	349·1	366·5	384·0
t seconds	0·1002	0·1004	0·1006	0·1008	0·1010	0·1012	0·1014

and if $V=RC+L\dfrac{dC}{dt}$, where R is 0·3 and L is 8×10^{-4}, find V. Plot C and t, and also V and t on the same sheet of paper.

ANSWERS.

t	C	$L\frac{dC}{dt}$	V
0·1003	288·0	69·6	156·0
0·1005	305·45	70·0	161·63
0·1007	322·9	69·6	166·47
0·1009	340·35	70·0	172·10
0·1011	357·8	69·6	176·94
0·1013	375·25	70·0	182·60

90. In Ex. 5, Art. 108, the values of p and v on a gas-engine indicator diagram are given. The old assumption that γ is a constant, is there made. But recent experiments have shown that we ought to take $\frac{1}{\gamma-1}=2+\frac{pv}{300}$, the stuff not being a perfect gas.*

Repeat the work on this new assumption. I get the following answers :

v	2·05	2·15	2·25	2·35	2·45	2·55	2·65	2·75
h	1750	4868	39·5	2467	1426	353	–273	–103
v	2·85	2·95	3·05	3·15	3·25	3·35	3·45	3·55
h	–144	–112	–69	–137	–202	–140	–66	–115

These results ought to be compared with those of Ex. 5, Art. 108, by both being plotted on the same sheet of paper.

* $\frac{1}{\gamma-1}=2+\frac{t}{1000}$ if t is the absolute temperature. In the above case t was 563° when v was 2 and p was 84·5, so that $t=3\frac{1}{3}pv$.

CHAPTER XXVIII.

SIMPLE VIBRATION.

114. The simplest periodic or vibratory motion is called Simple Harmonic Motion or S.H.M. In this case, if x is the displacement of a body from its mean position at the time t,

$$x = a \sin (qt + e). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In studying this let the student refer to Arts. 34 and 145, where I speak of the sine of any angle. He ought to watch the to and fro motion of the bob of a long pendulum, or the up and down motion of a cork floating on the sea when the sea waves have their simplest form, or the up and down motion of a weight hung from a spring balance.

Ex. 1. Take $a = 10$, $q =$ say 3π; take $e = 0$ or $\frac{\pi}{6}$ or any other value, and plot the curve. I often write 30° instead of $\frac{\pi}{6}$; this is incorrect, but convenient. It is seen that x cannot be greater than a and cannot be less than $-a$, and so a is called the *amplitude* of the motion. e is called the *advance* or *lead* or *lag* by various kinds of engineer.

The angle $qt + e$ is in radians. Now the sine of an angle is the same in every respect as the sine of the same angle plus 2π; therefore, after the lapse of a time T (called the periodic time) the values of x repeat themselves. That is, $qT = 2\pi$ or $q = \frac{2\pi}{T}$ or $2\pi f$ if f is $\frac{1}{T}$ or the frequency or what musicians call the pitch.

Ex. 2. A pendulum bob makes a total swing of 1·5 feet and executes the swing in 1·6 seconds; if the motion is given by (1), what are a and q? *Ans.* Evidently $a = 0{\cdot}75$ foot. The periodic time T is that of two swings or $T = 3{\cdot}2$ seconds; $q = \frac{2\pi}{3{\cdot}2} = 1{\cdot}96$.

It is evident in (1) that x has the value $a \sin e$ when $t=0$. It is evident that $x=a\sin(qt+120)$ is the same as $x=a\cos(qt+30)$, so the curve plotted, although usually called a **sine curve**, may just as truthfully be called a cosine curve.

The student will notice that the total area of the curve for a whole period is zero. He ought to spend much time in studying the motion, not merely getting the mathematician's knowledge of it, but making the sympathetic acquaintance with it which is necessary in the student of nature.

Fig. 38 shows the shape of the curve. I advise the student to draw the curve by the following method. A little knowledge of

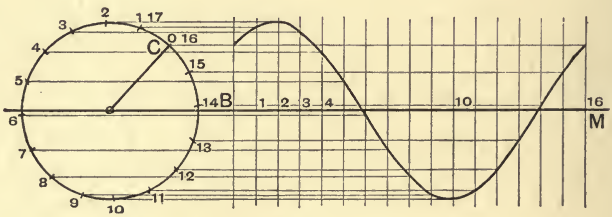

FIG. 38.

elementary trigonometry shows that it must give the same result as plotting. It is just what is done in drawing the elevation of a helical curve (as of a screw thread) in the drawing office. Draw a straight line OBM. Describe a circle with radius a. Set off the angle BOC equal to e. Divide the circumference of the circle from C into any convenient number of equal parts (16 in the figure), numbering the points of division 0, 1, 2, 3, etc. We may call the points 16, 17, 18, etc., or 32, 33, 34, etc., when we have gone once, twice or thrice or more times round. Take the distance BM to represent the periodic line of the motion and divide it into the same number of parts as you divided the circle. Number the points 0, 1, 2, 3, etc., 16, B being 0 and M being 16. Now project vertically and horizontally from corresponding points, and so get points on the curve.

If OC is imagined to be a **crank** rotating uniformly against the hands of a watch in the vertical plane of the paper, x in (1) means the distance of C above OM, qt means the angle that the crank makes at any time with its initial position OC, q being the angular velocity of the crank in radians per second, and of course $2\pi/q$ means the time of one revolution of the crank or the periodic time T of the motion; x is the displacement at any instant, from its mid-position, of a slider worked by an infinitely long connecting rod.

If a body has the motion (1), $\frac{dx}{dt}$ is its velocity v at any instant, and $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$ is its acceleration. On the table of Art. 94, we see that if

$$x = a\sin(qt+e), \quad \ldots\ldots(1)$$

$$v = \frac{dx}{dt} = aq\cos(qt+e) = aq\sin(qt+e+90°), \quad \ldots\ldots(2)$$

$$\text{accel. } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -aq^2\sin(qt+e) = aq^2\sin(qt+e+180°). \quad \ldots(3)$$

I give the proof of these formulae in Art. 95. But students ought to illustrate them in some such way as the following:

Illustration. Let $a=10$, $q=3\pi$, $e=\frac{\pi}{6}$ or 30°. Calculate x from (1) for the following values of t and tabulate. Now find $\frac{\delta x}{\delta t}$ in each interval, also find $\frac{\delta v}{\delta t}$. Calculate $aq\cos(qt+e)$, which is the true value of v, and calculate $-aq^2\sin(qt+e)$, which is the true value of the acceleration. Your tables are not accurate enough; you must work to six figures to get the acceleration approximately correct.

t	x	$v=\frac{\delta x}{\delta t}$	$\frac{\delta v}{\delta t}$	True v.	True $\frac{dv}{dt}$.
·060	8·86204				
		43·26			
·061	8·90530		780	42·87	791
		42·48			
·062	8·94778				

You see that the true v is not very different from the mean of the two tabulated values 42·875, and the tabulated acceleration is not very wrong.

Starting with (1), we might have written (2) as

$$\frac{dx}{dt} = aq\sin(qt+e+90°),$$

$$\frac{d^2x}{dt^2} = aq^2\sin(qt+e+180°),$$

and this fact cannot be too well remembered; when we are dealing with such a function as (1), differentiation means multiplication by q and increasing the lead by 90°. Integration means division by q and diminishing the lead by 90°.

115. We see from Art. 114 that a function $x = a \sin qt$ is analogous to the straight-line motion of a slider driven by a crank of length a (rotating with the angular velocity q radians per second), by an infinitely long connecting rod. x is the distance of the slider from the middle of its path at the time t. At the zero of time x is 0; $q = 2\pi f = \frac{2\pi}{T}$, if T is the periodic time or if f is the frequency or number of revolutions of the crank per second.

A function $x = a \sin(qt + e)$ is just the same except that the crank is the angle e radians in advance of the former position; that is, at time 0 the slider is at the distance $a \sin e$ past its mid position.

A function $x = a \sin(qt + e) + a' \sin(qt + e')$ is the same as

$$X = A \sin(qt + E);$$

that is, the **sum of two crank motions** can be given by a single crank of proper length and proper advance. To prove this:

Show on a drawing (Fig. 39) the positions of the first two when $t = 0$; that is, set off $YOP = e$, $OP = a$; $YOQ = e'$, $OQ = a'$. Complete

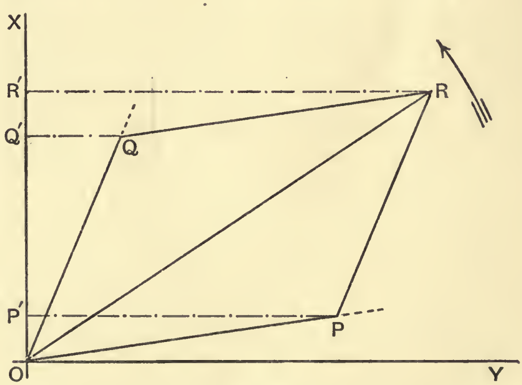

FIG. 39.

[In every case we draw a crank in the position it has when $t = 0$.]

the parallelogram $OPRQ$ and draw the diagonal OR, then the single crank of length $OR = A$ with the angle of advance $YOR = E$ would give to the slider the sum of the motions which OP and OQ would separately give. Imagine the slider to have a vertical motion. Draw OQ, OR, and OP in their relative positions *at any time*; these *relative* positions do not alter, as the angular velocities are the same; project P, R, and Q upon OX. The crank OP would cause the slider to be OP' above its mid position at this instant; the crank OQ would cause the slider to be OQ' above its mid position; the crank OR would cause the slider to be OR' above its mid position at this instant; observe that OR' is *always* equal to the algebraic sum of OP' and OQ'.

We may put it thus: "The S.H.M. which the crank OP would give, plus the S.H.M. which OQ would give, is equal to the S.H.M. which OR would give." Similarly, "The S.H.M. which OR would give, minus the S.H.M. which OP would give, is equal to the S.H.M. which OQ would give." We sometimes say: "The crank OR is the sum of the two cranks OP and OQ. In fact, cranks are added and subtracted just like vectors."

These propositions are of great value when dealing with valve motions and other mechanisms. They are of so much importance to electrical engineers that many practical men who are fond of graphical methods of calculation say, "Let the crank OP represent the current." They mean, "There is a current which alters with time according to the law $C = a \sin(qt + e)$; its magnitude is analogous to the displacement of a slider worked vertically by the crank OP, whose length is a and whose angular velocity is q, and OP is its position when $t = 0$ if the angle YOP is e."

116. A simple case of the above is that

$$a \sin qt + b \cos qt = a \sin qt + b \sin(qt + 90°) = A \sin(qt + e),$$

if $A^2 = a^2 + b^2$ and if $\tan e = b/a$.

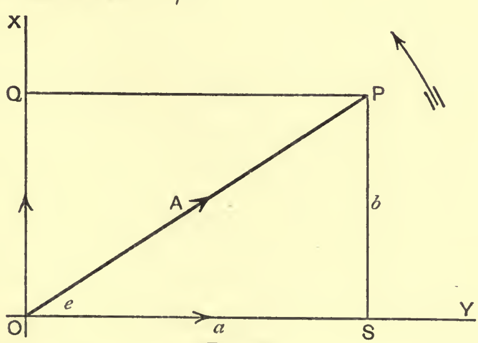

FIG. 40.

This is easily proved trigonometrically. Graphically (Fig. 40), let

$$OS = a, \quad OQ = b.$$

The crank $OP=A$ is the vector sum of OS and OQ and $\tan e$ or $\tan YOP=b/a$.

117. I have spoken of vibratory motion and the motion of a slider, but it is to be remembered that our algebra applies to many other phenomena.

The simplest alternating current of electricity follows the law $c=c_0 \sin(qt+e)$, having a periodic time T which is equal to $2\pi/q$; it passes from a positive value c_0 to the value $-c_0$, and the way in which it varies with the time is shown on the curve (Fig. 38). e is called its *lead*, and depends upon what time we count from. As an illustration of the formula of Art. 116, let us pursue this electrical example very slowly and carefully. Why not take a whole week to it?

If the voltage in an electric circuit is v volts, the current c amperes, the resistance r ohms, and the inductance l henries, then, t being time in seconds, there is a well-known electrical law,

$$v=rc+l\frac{dc}{dt}. \qquad \text{.............................(1)*}$$

The student ought to put this in non-algebraic language.

Now if $c=c_0 \sin qt$, $\frac{dc}{dt}=c_0 q \cos qt$ (see Art. 94), so that

$$v=rc_0 \sin qt + lc_0 q \cos qt.$$

By the above rule,

$$v=c_0\sqrt{r^2+l^2q^2}\sin(qt+e) \qquad \text{........................(2)}$$

where
$$\tan e=\frac{lq}{r}.$$

If the amplitude of v is called v_0, we see that $v_0=c_0\sqrt{r^2+l^2q^2}$, and that v leads c by the angle e. Or again, we can say that

$$c_0=\frac{v_0}{\sqrt{r^2+l^2q^2}},$$

and c lags behind v by the angle e.

In fact, if $v=v_0 \sin qt$, then $c=\frac{v_0}{\sqrt{r^2+l^2q^2}}\sin(qt-e)$. The student must think this out clearly; this statement is true if (2) is true.

* We may suppose a circuit closed on itself and v is the electromotive force of an alternator in the circuit. In that case r and l will include the resistance and inductance of the alternator; or we may suppose that we have only a branch connecting two points A and B, and v is the potential difference established somehow between A and B. I often speak of a branch as "a circuit," because in each case there is no chance of a mistake being made.

Remember that we can count from any zero of time, so long as we have the same zero of time when speaking of v and c.

$\sqrt{r^2 + l^2q^2}$ is called the **impedance** of the circuit.

It shortens our work often to write (1) in the *symbolic* form $v = \left(r + l\frac{d}{dt}\right)c$, and we call $r + l\frac{d}{dt}$ an *operator* on c; or we write θ for $\frac{d}{dt}$, and say $v = (r + l\theta)c$ or even that $c = v \div (r + l\theta)$. The student ought to get accustomed to this symbolic language.

118. Working **graphically**, let the crank OS (Fig. 40) represent $rc = rc_0 \sin qt$ at the time $t = 0$ and let all cranks be drawn in their position for $t = 0$. The length of OS is rc_0. Then, as

$$\frac{dc}{dt} = c_0 q \sin(qt + 90^\circ),\ l\frac{dc}{dt} \text{ is } lc_0 q \sin(qt + 90^\circ),$$

so let the crank OQ, set 90° in advance of OS, be of the length lqc_0. [In fact OS is r times the current and may be called the ohmic E.M.F.; OQ is lq times the current and may be called the **reactance** E.M.F.] The sum of these two cranks is OP, the total voltage. The length of OP is evidently

$$\sqrt{OS^2 + OQ^2} \text{ or } c_0\sqrt{r^2 + l^2q^2} \text{ and } \tan SOP = lq/r = \tan e.$$

Again, it is well known (see Arts. 27 and 126) that if a circuit has not only ohmic resistance r and inductance l, but also a condenser of capacity k in series (see Fig. 43), the condenser sets up a negative reactance. In Fig. 41 make $OS = rc_0$, make $OQ = lqc_0$, make $OK = \frac{c_0}{kq}$, the angles SOQ and SOK being right angles and OK just opposite to OQ; that is, OK is a crank lagging 90° behind the current. The vector sum of OS, OQ, and OK is the total E.M.F. In fact, add the three cranks to get a crank representing the total E.M.F.

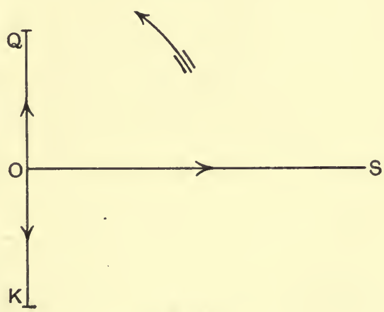

FIG. 41.

In work of this kind we may use c_0 and the other amplitudes, the length of crank in our answer being the amplitude of the voltage, or we can use the **effective** current instead of c_0, and the length of crank in the answer will be the effective voltage. Effective current is $0{\cdot}707c_0$ and effective voltage is $0{\cdot}707v_0$.*

119. Just as the rotating crank idea has produced an interesting graphical method of working exercises in alternating electric currents, so it has given great simplicity to algebraic calculation. In the above case, with no condenser, we have to add two vectors; write them algebraically as $OS = rc_0$ in the standard horizontal direction; lqc_0 in the direction OQ. Without Fig. 40 what are we to do? We may use clarendon type, and say,

$$\mathbf{OS} + \mathbf{OQ} = \mathbf{OP}.$$

This is my usual way of writing vector addition when I have no figure. And we may translate such a statement into

* Current c is measured by an electric dynamometer, the torque or couple in which is proportional at any instant to the square of c. But c varies so rapidly that it is the *average* or mean value of c^2 that is measured and the reading on the instrument is the square root of this, and this reading, what is called the *effective* current C, is "the square root of the mean square of c."

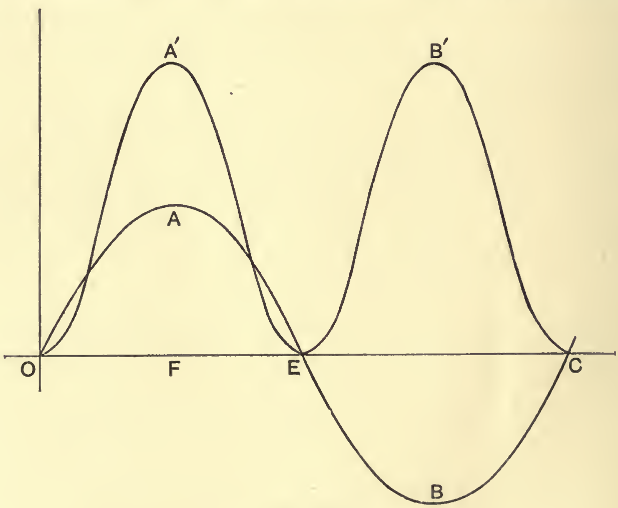

Fig. 42.

We graduate such an instrument using constant currents. If the current passes through a lamp of ohmic resistance r, the electric power converted into heat is C^2r watts. Now, if $c = c_0 \sin qt$, let the curve $OAEBC$ in Fig. 42 represent it for one period, AF being c_0. Let $OA'EB'C$ represent the square of c. $A'F$ is $c_0{}^2$. It is evident that the average value of $OA'EB'C$ (all whose ordinates are positive) is $\frac{1}{2}A'F$ or $\frac{1}{2}c_0{}^2$, and the square root of this is $c/\sqrt{2}$ or $0{\cdot}707c_0$, and this is what we call C, the *effective* current.

$$rc_0 \sin qt + lqc_0 \cos qt = v$$

or $$rc_0 \sin qt + lqc_0 \sin(qt + 90°) = v.$$

Now, if instead of this we write

$$(r + lqi)\, c_0 \sin qt = v,$$

and we understand that i is an operator which advances a crank through 90°, it is more easily written and has many advantages if we can be sure that to give this meaning to i is perfectly consistent with algebraic truth. "A vector **A** multiplied by i **turns it through** 90° in the positive direction."* Again multiplying by i, we turn it through another 90°, or $i^2\mathbf{A}$ means that **A** has been turned through 180°. Now i^2 is -1, so that $i^2\mathbf{A}$ is $-\mathbf{A}$, and this algebra agrees with our notions of vectors. Again, $i^3\mathbf{A}$ is $-i\mathbf{A}$ and $i^4\mathbf{A}$ is **A**; that is, we have brought it to its original position. Throughout we see that our algebra agrees with our notions of vectors.

When the student has worked the exercises which will be given later, on the properties of i, which is $\sqrt{-1}$, and exercises on such expressions as $a + bi$ and complex functions of such expressions, he will see the great importance of this way of thinking.

We shall take real quantities and get real answers to problems, but we shall get these answers through the medium of unreal quantities. If the student is very mathematical, he will probably demand a proof that our methods of working are legitimate. I do not think that he would be satisfied with such a proof as I might give. But in any particular case it is always easy to prove that our answer is correct, and that there can only be one correct answer. In the above case, our symbolic operator $r + l\frac{d}{dt}$ or $r + l\theta$ has become $r + lqi$ by our assumption that qi may be used instead of $\frac{d}{dt}$ or θ.† We see that the operator $a + bi$ when it multiplies or operates upon $M \sin qt$ multiplies M by $\sqrt{a^2 + b^2}$ and gives a lead e where $\tan e = \frac{b}{a}$.

* The positive direction of angular increase is anti-clockwise. A Congress of Electricians last year determined to change this and to make the clockwise direction positive, but I adhere to the usual rule of all the mathematicians of the world.

† Remember that it is only when we are dealing with simple periodic currents or functions of the time, of the type $\sin qt$ or $\cos qt$, that we replace θ or $\frac{d}{dt}$ by qi.

Similarly, when $M\sin qt$ is divided by $a+bi$, it divides by $\sqrt{a^2+b^2}$ and gives a lag e where $\tan e = \frac{b}{a}$.

The operator $\frac{a+bi}{\alpha+\beta i}$ evidently multiplies by $\sqrt{a^2+b^2}$, divides by $\sqrt{\alpha^2+\beta^2}$, gives a lead $\tan^{-1}\frac{b}{a}$ and a lag $\tan^{-1}\frac{\beta}{\alpha}$.

In fact, the result of multiplying $M\sin qt$ by $\frac{a+bi}{\alpha+\beta i}$ is

$$M\sqrt{\frac{a^2+b^2}{\alpha^2+\beta^2}}\sin\left(qt+\tan^{-1}\frac{b}{a}-\tan^{-1}\frac{\beta}{\alpha}\right). \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Until we deal with distributed capacity, it will be found that this formula will enable us to solve nearly every problem in alternating electric currents or the forced vibrations of mechanical systems.

Ex. 1. A coil of $r=1$ ohm and inductance $l=0{\cdot}004$ is subjected to a voltage $v=141{\cdot}4\sin 600t$. What is the current? Here $q=600$ (a frequency of $\frac{600}{2\pi}$ or about 95 per second), $c=\frac{141{\cdot}4\sin 600t}{1+0{\cdot}004\times 600i}$. The denominator is $1+2{\cdot}4i$. Therefore the amplitude of c or c_0 is $\frac{141{\cdot}4}{\sqrt{1+2{\cdot}4^2}}$ or $\frac{141{\cdot}4}{2{\cdot}6}$ or $54{\cdot}38$. And if e is the lag, $\tan e=2{\cdot}4$ or $e=67^\circ{\cdot}38$. It would be more correct to give e in radians, but people have got accustomed to the use of degrees.

The answer is then $c=54{\cdot}38\sin(600t-67^\circ{\cdot}38)$.

The *effective* voltage is $\frac{141{\cdot}4}{\sqrt{2}}$ or $0{\cdot}707\times 141{\cdot}4$ or 100 volts.

The *effective* current is $\frac{54{\cdot}38}{\sqrt{2}}$ or $0{\cdot}707\times 54{\cdot}38$ or $38{\cdot}45$ amperes.

Ex. 2. Part of a circuit has $r=100$ ohms, inductance $l=0{\cdot}09$ henry in series with a condenser whose capacity is $k=0{\cdot}5\times 10^{-6}$ farad [this is called a capacity of $\frac{1}{2}$ a microfarad], and it has a potential difference or voltage established between its ends. [Or we may say, a whole circuit has the above r, l, and k, and the electromotive force in the circuit is v]. $v=14{\cdot}14\sin 5000t$, so that $q=5000$ [a frequency of $\frac{5000}{2\pi}$ or nearly 800 per second]. What is the current? The resistance is here $r+lqi+\frac{1}{kqi}$ or $r+lqi-\frac{i}{kq}$ or $100+i(450-400)$ or $100+50i$.

$$c=\frac{14{\cdot}14\sin 5000t}{100+50i}=\frac{14{\cdot}14}{\sqrt{100^2+50^2}}\sin(5000t-e).$$

$$c=0{\cdot}1264\sin(5000t-26^\circ{\cdot}57) \quad \text{as} \quad \tan e=\frac{50}{100}=0{\cdot}5.$$

The effective voltage is 10 volts, the effective current is $0 \cdot 707 \times 0 \cdot 2164$ or $0 \cdot 0894$ ampere. The student ought to work this and the other exercises graphically also.

Ex. 3. A circuit of $r = 1000$ ohms, no inductance, in series with a condenser of $0 \cdot 5 \times 10^{-6}$ farad has a voltage $14 \cdot 14 \sin 5000t$. What is the current? Here the resistance is

$$r - \frac{i}{kq} = 1000 - 400i$$

and $$c = \frac{14 \cdot 14 \sin 5000t}{1000 - 400i} = 0 \cdot 01313 \sin (5000t + 21^\circ \cdot 8).$$

Notice how a condenser produces a lead just as an inductance produces a lag.

CHAPTER XXIX.

MAINLY ABOUT NATURAL VIBRATIONS.

120. This chapter will seem to be difficult to a beginner, and some persons may think that it ought to be omitted in an elementary book, because in most practical problems, as the natural vibrations die out rapidly, we have only to deal with forced vibrations. But the man who studies it a little and, after working at forced vibrations, returns and makes a study of this chapter, will find that he is getting a very thorough knowledge of a subject exceedingly important in mechanics, electricity, acoustics, and light.

I repeat that if a body has the motion (1), $\frac{dx}{dt}$ is its velocity at any instant and $\frac{d^2x}{dt^2}$ is its acceleration; we thus have

$$\text{displacement } x = a \sin(qt + e), \quad \text{.........(1)}$$

$$\text{velocity } v = \frac{dx}{dt} = aq \cos(qt + e), \quad \text{.........(2)}$$

$$\text{acceleration } \alpha = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -aq^2 \sin(qt + e). \quad \text{.........(3)}$$

The most important peculiarity of this motion is that at every instant

$$\text{acceleration } \alpha \propto \text{displacement } x$$

or

$$\alpha = -q^2 x \quad \text{.........(4)}$$

or

$$\frac{d^2x}{dt^2} + q^2 x = 0. \quad \text{.........(5)}$$

If the equation (5) is given, we know that (1) is true, a and e being of any values whatever.

121. Let us examine one case carefully. A weight of W lb. hangs from a spring whose stiffness is such that a force of 1 lb. elongates it h feet. If the body is vibrating, when at the time t it is x feet below (we imagine it moving downwards) its position of equilibrium, the force urging it *to* its position of equilibrium is x/h lb.; this force is retarding the motion, and is equal to the mass of the body W/g multiplied by $-\frac{d^2x}{dt^2}$. Now imagine that the motion of the body is also retarded by a force of friction which is proportional to the velocity, being b times the velocity let us say; we have the total retarding force

$$\frac{x}{h}+b\frac{dx}{dt}=-\frac{W}{g}\frac{d^2x}{dt^2}$$

or
$$\frac{W}{g}\frac{d^2x}{dt^2}+b\frac{dx}{dt}+\frac{x}{h}=0 \text{ ....(1)}$$

or
$$\frac{d^2x}{dt^2}+\frac{bg}{W}\frac{dx}{dt}+\frac{g}{Wh}x=0.$$

Let $\frac{bg}{W}$ be called $2f$, let $\frac{g}{Wh}$ be called n^2, and we have

$$\frac{d^2x}{dt^2}+2f\frac{dx}{dt}+n^2x=0. \text{ ....(2)}$$

(1) or (2) expresses the damped motion of a vibrating body. I am neglecting the mass of the spring itself.

It is shown in Art. 133 that the solution of (2) may have various forms, depending upon the amount of the friction.

I. Assume no friction or b (or f) zero (2) now becomes

$$\frac{d^2x}{dt^2}+n^2x=0,$$

and the complete solution of this is

$$x=A\sin nt+B\cos nt, \text{ ....(3)}$$

where A and B are arbitrary constants. Of course (3) may be written in the shape

$$x=a\sin(nt+e),$$

and A and B or a and e may be chosen arbitrarily or to suit any particular problem. For example, we may take the body to be at the extremity of its path or $x=a$ at time 0, and then e will be 90°. Or we may take the body to be at the middle of its path at time 0, so that $e=0$, and if we state its velocity v_0 at that instant, it is evident that a is $v_0\div n$. As an example, take $e=90°$; this is the same as taking $A=0$. In fact, let us take

$$x=B\cos nt. \text{ ....(4)}$$

II. When f has a value but is less than n, let $\sqrt{n^2-f^2}$ be called p. The solution of (2) is (see Art. 133)

$$x=e^{-ft}(A\sin pt+B\cos pt). \qquad \text{.....................(5)}$$

Here again A and B are arbitrary constants. We can take them of any values we please to suit any particular problem. To compare with (4), let us take

$$x=Be^{-ft}\cos pt. \qquad \text{.............................(6)}$$

I will not now consider the case of f equal to or greater than n, because cases of such excessive damping seldom come before us in practice, but all cases are easily solved.

122. Let the student take up one case and consider it carefully. Let $W=64{\cdot}4$, $h=0{\cdot}01$, so that $n^2=g/Wh=\dfrac{32{\cdot}2}{64{\cdot}4\times 0{\cdot}01}=50$ or $n=7{\cdot}071$.

Let $B=10$, so that without friction

$$x=10\cos 7{\cdot}071t. \qquad \text{..........................(7)}$$

In (5) assume that $f=0{\cdot}5$, $p=\sqrt{n^2-f^2}=\sqrt{49{\cdot}75}=7{\cdot}054$, so that in this frictional case

$$x=10e^{-0{\cdot}5t}\cos 7{\cdot}054t. \qquad \text{........................(8)}$$

These two relations (7) and (8) ought to be plotted as curves on the same sheet of squared paper. The frictionless period T is $\dfrac{2\pi}{n}$ or $0{\cdot}8886$ second, and the friction increases it to $\dfrac{2\pi}{p}$ or $0{\cdot}8907$ second.

The amplitude of x diminishes as time goes on because of friction, so that at the end of a period it is only $0{\cdot}64$ of its value at the beginning of the period. (See Ex. 16, Art. 28.)

Damped and undamped vibrations ought to be studied experimentally.

In the above formulæ x may be angular displacement, $\dfrac{W}{g}$ may be the moment of inertia of a body vibrating about an axis, and the stiffness of the restraining spring or torsional wire would be such that a unit torque or couple would produce a twist of h radians. A disc or wheel or rod surrounded by air, water, or oil, vibrating at the end of a twisting and untwisting wire is an excellent piece of apparatus to play with, and so create true instincts.

The essential fact about simple vibration is that (neglecting friction) acceleration is proportional to displacement. Neglecting the minus sign, since q is $2\pi/T$, we see that to find the periodic time in any case

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}. \quad\text{.......................(9)}$$

This may be linear displacement and linear acceleration or angular displacement and angular acceleration. Whether we study the simple pendulum or a compound pendulum or a vibrating column of liquid in a U-tube or the motion of the balance of a watch or the rolling of a ship, we readily calculate the periodic time by applying this rule. See my *Applied Mechanics*, Chap. XXV.

Ex. 1. A body of 644 lb. has a simple periodic motion, the periodic time being 0·4 second; the amplitude of the motion being 1·5 feet (or total swing 3 feet). What forces are giving this motion to the body? We can work with (9), and find if x is displacement and a is acceleration (without troubling about the minus sign);—

$a = \frac{4\pi^2}{0 \cdot 16}x$. The inertia or mass of the body is $644 \div 32 \cdot 2$ or 20, and force F in pounds is $20a$.

In fact $F = 4935x$.

At the end of a swing F is $4935 \times 1 \cdot 5 = 7403$ lb. At the middle of the swing the force is 0. The force is always in a direction towards the mid-point.

Ex. 2. A body of 3·22 lb. is supported at the end of a strip of steel (whose inertia is neglected); the stiffness of the strip is such that a force of 1 lb. deflects the body 0·1 foot; the body is set swinging, what is its periodic time? If x is displacement in feet, the force in pounds acting on the body is $F = 10x$; the inertia or mass of the body is $3 \cdot 22 \div 32 \cdot 2$ or 0·1; the acceleration is $F \div$ mass or $100x$. Formula (9) gives

$$T = 2\pi\sqrt{\frac{x}{100x}} = \frac{2\pi}{10} = 0 \cdot 62832 \text{ second.}$$

Ex. 3. The steel rope to the cage of a mine moving downwards at 5 feet per second is suddenly stopped at its upper end; neglecting the inertia of the rope itself, the cage which weighs 3220 lb. is now set in simple vibration, down and up. If the length and section of the rope are such that a stretch of 1 foot is produced by a force of 6000 lb., what is the greatest additional pull in the rope due to the stoppage? What is the amplitude of the vibrational motion? What is the periodic time of the oscillation? The moment of stoppage is regarded by me as the zero of time. After this the cage has moved downward through x feet at the time t. I call x the displacement in the vibratory motion and take $x = a \sin qt$; v the velocity is $aq \cos qt$ and the acceleration α is $-aq^2 \sin qt$.

The inertia or mass is $3220 \div 32{\cdot}2 = 100$; mass multiplied by acceleration is force, so that $100aq^2 =$ greatest force $= 6000a$. Hence $q^2 = 60$ or $q = 7{\cdot}746$ and $T = 0{\cdot}8112$ second. The greatest velocity $aq = 5$, so that $a = 0{\cdot}6455$ foot; greatest force $= 6000a = 3873$ lb.

123. Forced Vibrations. In the case considered (Art. 121), let the upper or supporting point of the spring be vibrated up and down. At the time t let it be lower than its mean position by the distance y. The spring is now elongated $x - y$ more than if everything were at rest; the force retarding the body's motion is $\frac{x-y}{h} + b\frac{dx}{dt}$, and we have

$$\frac{x-y}{h} + b\frac{dx}{dt} = -\frac{W}{g} \cdot \frac{d^2x}{dt^2}.$$

Therefore
$$\frac{d^2x}{dt^2} + \frac{bg}{W}\frac{dx}{dt} + \frac{g}{Wh}x = \frac{g}{Wh}y \quad \text{......................(1)}$$

or
$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = n^2y. \quad \text{..........................(2)*}$$

Suppose now that y is a simple harmonic motion of any period, say that $y = a \sin qt$; it is found that the body has a simple harmonic motion of this period, and along with this it probably has its own natural vibrating motion discussed in Arts. 120–122. But as there is always some friction, the natural vibratory motion is so rapidly damped out that it is of importance only for a short time—although it may be of considerable importance for that short time—and in much scientific work we neglect it. I shall therefore, after this, consider only the forced vibration.

In most acoustic phenomena, although we assume that there is sufficient damping to destroy quickly the natural vibrations, we assume that f is 0 because of ease in calculation and simplicity. This is so also in the study of light and some other sciences. But in the study of electro-magnetic radiation and alternating electric currents, the study of the tides and many other natural phenomena, we must take the f term into account.

Exercise. If a body is hung from a spring, there being no friction, and if the support P gets a motion

$$y = a \sin qt,$$

* The interested student will consider another kind of forced vibration. Take $y = 0$, but assume that a downward force $F = F_0 \sin qt$ acts upon the body. Instead of n^2y in (2) we have n^2hF. The results are much the same.

find the forced vibration. Here (2) becomes

$$\frac{d^2x}{dt^2} + n^2x = n^2a \sin qt. \quad \text{.........................(1)}$$

It will be found that $x = A \sin qt$ is the solution; try it.

$$\frac{dx}{dt} = Aq \cos qt, \quad \frac{d^2x}{dt^2} = -Aq^2 \sin qt;$$

and trying these values in (1), we find

$$-Aq^2 \sin qt + n^2A \sin qt = n^2a \sin qt.$$

Therefore our guess is right if $A = \frac{n^2a}{n^2 - q^2}$.

It is well worth while to study this answer carefully. x is merely y multiplied by $\frac{n^2}{n^2 - q^2}$.

As n and q are proportional to the natural and forced frequencies, if the ratio of the forced to the natural frequency is called p, and if $a = 1$, the amplitude of W's motion being called A, we have

$$A = \frac{1}{1 - p^2}.$$

p	A	p	A
·1	1·01	1·01	−50
·5	1·333	1·03	−16·4
·8	2·778	1·1	−4·762
·9	5·263	1·5	−0·8
·95	10·26	2·0	−0·333
·97	16·92	5·0	−0·042
·98	25·25	10·0	−0·010
·99	50·25		
1	∞		

Note that when the forced frequency is a small fraction of the natural frequency, the forced vibration of W is a *faithful copy* of the motion of the point of support P; the spring and W move like a rigid body. When the forced frequency is increased, the motion of W is a faithful magnification of P's motion. As the forced gets nearly equal to the natural frequency, the motion of W is an enormous magnification of P's motion. There is always some friction, and hence the amplitude of the vibration cannot become infinite. When the forced frequency is greater than the natural, W is always *a half period behind* P, being at the top of its path when P is at the bottom. This is the explanation of various interesting physical phenomena. For example, the dynamical theory of the tides of the ocean as differing from the older theory.

When the forced is many times the natural frequency, the motion of W gets to be very small; W is nearly at rest.

Men who design earthquake recorders try to find a steady point which does not move when everything else is moving during an earthquake. For up and down motion, observe that in the last case just mentioned, W is like a steady point.

When the forced and natural frequencies are nearly equal, we have the state of things which gives rise to **resonance** in acoustic instruments, which causes us to fear for suspension bridges or rolling ships. It would be easy to give twenty examples of important ways in which this principle enters into practical scientific problems.

124. Electrical Vibrations. Suppose a condenser of capacity k farads has the potential difference v volts between its coatings which are connected by a circuit of r ohms, of inductance l henries The quantity Q of electricity in the condenser is kv; the current *out* of the condenser is $c = -\frac{dQ}{dt} = -k\frac{dv}{dt}$, but this current is such that $v = rc + l\frac{dc}{dt}$. (See Art. 117.)

Hence $$v = -rk\frac{dv}{dt} - lk\frac{d^2v}{dt^2} \quad \text{or} \quad lk\frac{d^2v}{dt^2} + rk\frac{dv}{dt} + v = 0. \quad \text{.........(1)}$$

If we assume that in the circuit we have an alternator acting, or any other source of varying electromotive force E charging the condenser, we may write $v - E$ instead of v in the above, and we find

$$lk\frac{d^2v}{dt^2} + rk\frac{dv}{dt} + v = E$$

or $$\frac{d^2v}{dt^2} + \frac{r}{l}\frac{dv}{dt} + \frac{v}{lk} = \frac{E}{lk}.$$

If we use $2f$ for $\frac{r}{l}$ and n^2 for $\frac{1}{lk}$, we have

$$\frac{d^2v}{dt^2} + 2f\frac{dv}{dt} + n^2v = n^2E, \quad \text{........................(2)}$$

the very same formula that we had for mechanical vibrations.

It will be found that **the analogy** between mechanical and electrical vibrations is very close. Notice that

the mass or inertia $\frac{W}{g}$ corresponds with inductance l.

The friction per foot per second b corresponds with the resistance r. The displacement x corresponds with voltage v or, to be seemingly more accurate, v is Q the electric displacement or, as some call it, the charge, divided by k.

The yieldingness of the spring h corresponds with the capacity of the condenser k.

The forced displacement y corresponds with the forced E.M.F. of the alternator.*

As before, if we consider the natural vibration of the system left to itself with $E=0$, if f has a value and it is less than n, if $\sqrt{n^2-f^2}$ is called p, we find that

$$v=E^{-ft}A\sin(pt+e), \quad \text{.........................(3)}$$

where A and e may have any values.

Exercise. A condenser of $k=10^{-8}$ farads (called one one-hundredth of a microfarad) is short circuited through a resistance of $r=1$ ohm † with an inductance $l=10^{-4}$ henries. What is the nature of the surging?

Here $2f=\frac{r}{l}=10^4$, $n^2=\frac{1}{10^{-4}\times10^{-8}}=10^{12}$, so that $n=10^6$. If r is 0, the periodic time $T=2\pi\div10^6=6{\cdot}28\times10^{-6}$ second or the frequency is 159100 per second [the wave length of the radiation into space is the velocity of light 3×10^{10} cm. per second divided by 159100 or it is 190000 cm. or 1·18 miles].

As r is not 0, in fact $f=0{\cdot}5\times10^4$, $p=\sqrt{n^2-f^2}=\sqrt{10^{12}-0{\cdot}25\times10^8}$, we may take it that p is really the same as n, so that the frequency is still 159100 per second. The answer is then, that v follows the law (3) or, what is a more suitable form,

$$v=v_0e^{-5000t}\cos 10^6t, \quad \text{.........................(4)}$$

* When E is 0, that is, when there are no forced vibrations, it is easy to see that instead of (1) we may write

$$lk\frac{d^2Q}{dt^2}+rk\frac{dQ}{dt}+Q=0$$

or

$$lk\frac{d^2c}{dt^2}+rk\frac{dc}{dt}+c=0.$$

† In wireless telegraphy we have to recollect that there is loss of energy by radiation as well as ohmic loss, but in this elementary exercise we may assume that the ohmic loss includes the radiation loss.

where v_0 is the voltage between the terminals of the condenser at the time which we choose to call 0. Or again,

$$c = c_0 e^{-5000t} \sin 10^6 t, \qquad (5)^*$$

a suitable form if we consider the surging current.

If there are two tanks of water A and B connected by a large pipe; the water in A higher than the water in B by the height v feet; as soon as the communication is made, v follows a law like (4) and the current of water in the pipe follows a law like (5). Too much water comes into B and then it runs back again, the surges getting less and less as time goes on.

It is evident that the periodic time of vibration of an electric circuit depends so little upon r that we may neglect r and say that $n = \frac{2\pi}{T}$, and as $n^2 = \frac{1}{lk}$, then $T = 2\pi\sqrt{lk}$ or the frequency $= 1 \div 2\pi\sqrt{lk}$, and wave length in the ether is $3 \times 10^{10} \times 2\pi\sqrt{lk}$ or $18{\cdot}85 \times 10^{10}\sqrt{lk}$ centimetres or $1{\cdot}17 \times 10^6\sqrt{lk}$ miles.

125. A mechanical vibrating system P, if its frequency or pitch is anything from say 30 to 5000, acts on the air and sends out waves of sound; if these act upon another mechanical vibrating system Q they set up forced vibrations in Q. If Q's natural frequency is exactly the same as that of P, the forced vibration of Q may be very great. Sometimes when no musical sound is audible among the street noises a wire of a piano in one's room gives out an audible note because its frequency of oscillation happens to agree exactly with that of some inaudible musical note that has reached it. In such a case, it takes time for the forcing influence to produce the resonance; indeed in our mathematical work we assume that the forcing influence has lasted for an infinite time.

In the same way an electrical vibrating system P in Ireland, with a natural frequency of say 159100 per second (in the above example) sends out waves through the ether; if these act upon another electrical system Q in America, they set up forced vibrations in Q. If Q's natural frequency is 159100 per second, Q gets a forced surging of current which may be great enough to affect instruments, and thus it is that we have wireless telegraphy. The condenser of the P system is regularly charged and discharged by sparks through

* (4) and (5) differ very slightly as to the zero of time. As $= c - k\frac{dv}{dt}$, if we assume (5) to be true we can easily find (4), so that there shall be exact correspondence, but is this worth while?

the circuit already described, and each spark causes waves to be sent out whose amplitudes rapidly diminish; in fact each set dies away in about one ten-thousandth of a second in the above case; but at regular intervals we have fresh charging and discharging. It is extraordinary that in spite of these dyings away and renewals there is the sympathetic action of which I have spoken. Many inventors are now trying to find a more continuous transmission than what is now set up by sparks. It needs a sudden shock to start such vibrations, just as it needs the shock or blow of air against the lip of an organ pipe to set it going. The shock initiates vibrations of all periods, and with one of these the system synchronises, and thus it responds and magnifies. This is exactly analogous with the sounding of an organ pipe.

CHAPTER XXX.

FORCED VIBRATIONS.

126. In what follows I shall assume that the natural vibrations of a system have been damped out, and they will not be considered.

Let Arts. 121-123 be carefully read again about the body suspended by a spring, the upper support of which is displaced downwards through the distance y at the time t. Let $y = a \sin qt$. Equation (2) is

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = n^2a \sin qt. \quad \text{.......................(1)}$$

It is known that the forced motion is of the form $x = A \sin (qt + e)$, and therefore $\frac{dx}{dt}$ is qix, $\frac{d^2x}{dt^2}$ is q^2i^2x or $-q^2x$, and hence (1) is

$$(-q^2 + 2fqi + n^2)x = n^2a \sin qt,$$

$$x = \frac{n^2a \sin qt}{n^2 - q^2 + 2fqi}.$$

We know from Art. 119 that this means

$$x = A \sin (qt - e),$$

$$\text{where } A = \frac{n^2a}{\sqrt{(n^2 - q^2)^2 + 4f^2q^2}} \quad \text{and} \quad \tan e = \frac{2fq}{n^2 - q^2}.$$

If we take $f/n = 0{\cdot}0707$ as in Art. 122, and if we let q/n be called p, and let $a = 1$,

$$A = 1 \div \sqrt{(1 - p^2)^2 + p^2/50} \quad \text{and} \quad \tan e = 0{\cdot}1414p/(1 - p^2).$$

Let the forced amplitude A be plotted with p. Let e also be plotted with p. These results ought to be compared with those tabulated in Art. 123, where there was no frictional damping. In the frictionless case, e was either 0 or 180°. If the study of these

p	A	e
0·1	1·01	0°·8
0·5	1·333	5°·4
0·8	2·655	17°·44
0·9	4·372	33°·82
0·95	6·024	54°·02
0·97	6·697	66°·7
0·98	6·934	74°·06
0·99	7·070	81°·97
1·00	7·071	90°
1·01	6·934	98°
1·03	6·331	112°·73
1·10	3·827	143°·5
1·5	0·7879	170°·37
2·0	0·3319	174°·6
5·0	0·04167	178°·32
10·0	0·0101	179°·2

results takes some weeks of time, it is probable that the time is well spent. The student ought to work another case, where there is more friction.

127. In the three examples at the end of Chapter XXVIII., I studied the electric currents which were forced on the system by an electromotive force or impressed voltage. In Chapter XXIX. we studied the currents which flow in electric circuits when they are left to themselves, the natural surgings which are rapidly damped out. I have returned now to the forced vibrations of systems.

If v is the voltage between the coatings of an **electric condenser** of capacity k, the charge in the condenser is $Q=kv$, and the current c *into* the condenser is $c=k\frac{dv}{dt}$. If v is of the form $a\sin(qt+e)$, we know that $\frac{dv}{dt}$ is qiv and $c=kqiv$. Now if R is the resistance of a contrivance, the current C flowing because of the voltage V is $C=\frac{V}{R}$. We see in the case of the condenser that $c=v\div\frac{1}{kqi}$, so we may say that a condenser has a resistance $\frac{1}{kqi}$.

Multiplying numerator and denominator by i, and remembering that $i^2=1$, we find the resistance of a condenser to be $-\frac{i}{kq}$. I have already used this idea in Arts. 118 and 119.

We know that a circuit which is of resistance R ohms, inductance

L henries, and which has a condenser of K farads in series, may be said to have the total resistance

$$R+Lqi-\frac{i}{Kq} \quad \text{or} \quad R+i\left(Lq-\frac{1}{Kq}\right).$$

Notice that when the i term is 0, we may expect large currents.

Ex. 1. In fact, we say that such a circuit is in tune when $Lq=\frac{1}{Kq}$ or $LKq^2=1$. Suppose we wish it to be in tune for a frequency f, so that $q=2\pi f=6000$ say [this would be a frequency of about 955]. Then $LK\times 36\times 10^6=1$.

Let us take $K=10^{-6}$ or one microfarad and $L=\frac{1}{36}$.

Fig. 43 shows a part of a circuit with a potential difference v

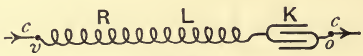

FIG. 43.

between its ends. Let us take $R=100$ ohms, which is rather high for such a circuit. Let the voltage be

$$v=1414\sin qt.$$

Then the current is $v\div$resistance or

$$c=\frac{1414\sin qt}{100+i\left(\frac{q}{36}-\frac{10^6}{q}\right)}.$$

For the following values of q, I give the current:

q	Resistance.	c
3000	$100-250i$	$5{\cdot}252\sin(qt+69°)$
4500	$100-97i$	$10{\cdot}152\sin(qt+44°)$
6000	100	$14{\cdot}14\ \sin qt$
7500	$100+75i$	$11{\cdot}312\sin(qt-37°)$
9000	$100+139i$	$8{\cdot}255\sin(qt-54°{\cdot}5)$
10500	$100+196i$	$6{\cdot}427\sin(qt-63°)$

If I had taken a smaller R, the greatness of the current when $q=6000$ would be more marked. The student ought to plot such answers on squared paper, showing how both the amplitude of c and also its lag or lead alters as q alters.

If we had no ohmic resistance the current would be infinite for $q=6000$.

Ex. 2. Two circuits in parallel. What is the total current? One has a current c_2, ohmic resistance R ohms, and inductance L. The other has a current c_3 and is merely a condenser of capacity K (see Fig. 44).

$$c_2 = \frac{v}{R + Lqi}, \quad c_3 = Kqiv, \quad C = c_2 + c_3 = \frac{1 - KLq^2 + RKqi}{R + Lqi} v.$$

It is evident that C will be small if $KLq^2 = 1$ or $q = \frac{1}{\sqrt{KL}}$, and if we depart from this value of q, we find C to be large.

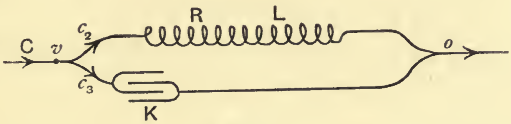

FIG. 44.

I will take a case of very great inductance, and I will tune the arrangement so that $KLq^2 = 1$. Let $R = 0{\cdot}5$, $L = 0{\cdot}01$, $K = 4 \times 10^{-6}$, so that the critical q is 5000 [a frequency of about 800]. I will only give the amplitude of C. Let $v = 1000 \sin qt$.

Notice the smallness of C for $q = 5000$.

q	Amplitude of C.
3000	21·30
4000	9·00
4500	4·22
5000	0·20
5500	3·82
6000	7·33
8000	19·50
10000	30·00

If I had calculated c_2 and c_3 both of them would have been large when $q = 5000$; their sum C is small because they differ in phase nearly 180°. In the same way the sum (or resultant, as it is called) of two great forces may be small when the forces are nearly opposed to one another.

Ex. 3. If a student will draw curves showing how c of Ex. 1 and C of Ex. 2 vary as q is altered, these curves will suggest to him Mr. Sidney Brown's old invention to cause a complex current which is the sum of many simple sine functions to divide itself, so that that part which is of nearly one frequency shall go along one circuit and all the rest of the complex current shall go in the other.

Suppose we give the total current (Fig. 45) three ways or circuits of passing from a point P to a point Q, there being the voltage v established between P and Q. The first circuit has r, l, and a capacity k, so that its total resistance is $r+i\left(lq-\frac{1}{kq}\right)$ and the current in it is

$$c=\frac{V}{r+i\left(lq-\frac{1}{kq}\right)}.$$

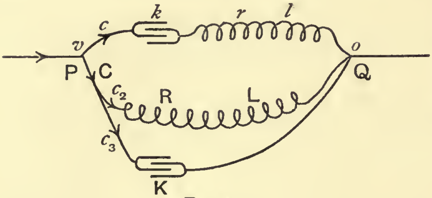

FIG. 45.

The second circuit has R and L, so that its resistance is $R+Lqi$ and its current is

$$c_2=\frac{V}{R+Lqi}.$$

The third circuit has a condenser K and its current is

$$c_3=KqiV.$$

Let c_2+c_3 be called C, then

$$C=\left(\frac{1}{R+Lqi}+Kqi\right)V.$$

Hence
$$\frac{C}{c}=\frac{(1-KLq^2+RKqi)\left\{r+i\left(lq-\frac{1}{kq}\right)\right\}}{R+Lqi}.$$

Now Brown desires that C shall be exceedingly small and c great for a particular value of q, say $q=5000$, but for all other values of q he desires that C shall be great and c small.

It is evident that for $q=5000$ we ought to make $KLq^2=1=klq^2$, as this makes the numerator small. Hence $KL=kl=4\times10^{-8}$. As Brown wished to have a telephone of considerable resistance in the c circuit, he was compelled to use a large l and a large r; take then $r=100$, $l=4$; therefore $k=10^{-8}$. But he could have the R and L of his other circuit small, so he took $R=0{\cdot}5$, $L=0{\cdot}01$; therefore $K=4\times10^{-6}$. Calculate the following results for various values of q:

q	$\frac{C}{c}$
3000	455·0
4000	81·02
4900	0·6778
5000	0·0200
5010	0·0278
6000	53·74
7000	188·1
10000	900·1

We need not trouble about anything but amplitudes. We see that if $q = 5000$, c is 50 times as great as C; if $q = 4000$ or 6000, c is from $\frac{1}{80}$ to $\frac{1}{50}$ of C; thus Brown's object is effected.

If we assume that R and r are so small as to be negligible, we get a much simpler expression for $\frac{C}{c}$ to study and far more striking figures.

Letting the circuit of Ex. 1, Fig. 43, be called I., let the compound circuit of Ex. 2, Fig. 44, be called II. Mr. Brown simply shunted a circuit I. by a circuit II.

128. The electrical exercises already given are all examples of the following general rule. Let θ stand for $\frac{d}{dt}$. A current having the ohmic resistance r, the inductance l, and the capacity k may be said to have the resistance $r + l\theta + \frac{1}{k\theta}$. In any network of conductors conveying electric currents, if there is a constant electromotive force E in any branch or if a potential difference of voltage V be established between any two points, then the constant current in any branch can be calculated, being E or V multiplied by an algebraic expression involving all the resistances of all the branches r_1, r_2, r_3, etc. If E or V is varying, we shall use the same algebraic expression; but now, instead of a mere ohmic resistance r_3, for example, we use $r_3 + l_3\theta + \frac{1}{k_3\theta}$ if r_3 is the ohmic resistance, l_3 the inductance, and k_3 a capacity in that branch. However complex the expression may be, when cleared of fractions, etc. (we treat θ as if it were a mere algebraic quantity *), it simplifies to this, that an operation like

$$\frac{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + \text{etc.}}{a' + b'\theta + c'\theta^2 + d'\theta^3 + e'\theta^4 + f'\theta^5 + \text{etc.}} \quad \text{.................(1)}$$

has to be performed upon some voltage which is a function of the time. If the function is of the type $\sin qt$, we substitute qi for θ in (1), and the complicated operation becomes

$$\frac{a + bqi + cq^2i^2 + dq^3i^3 + eq^4i^4 + fq^5i^5 + \text{etc.}}{a' + b'qi + c'q^2i^2 + d'q^3i^3 + e'q^4i^4 + f'q^5i^5 + \text{etc.}}$$

$$= \frac{a - cq^2 + eq^4 + \text{etc.} + iq(b - dq^2 + fq^4 - \text{etc.})}{a' - c'q^2 + e'q^4 + \text{etc.} + iq(b' - d'q^2 + f'q^4 - \text{etc.})}.$$

We may call this $\frac{A + Bi}{\alpha + \beta i}$, and we know from Art. 119 the effect of such an operator.

[* The proof of this is easy but tedious; it is given in my *Calculus*, pages 231-4.]

If two circuits r_1, r_2 are in parallel with constant currents c_1 and c_2, the whole current $C = c_1 + c_2$ divides itself so that

$$\frac{c_1}{C} = \frac{r_2}{r_1 + r_2} \quad \text{and} \quad \frac{c_2}{C} = \frac{r_1}{r_1 + r_2}.$$

Also the combined resistance of two circuits in parallel is

$$R = \frac{r_1 r_2}{r_1 + r_2},$$

so that if V is the voltage

$$C = \left(\frac{r_1 + r_2}{r_1 r_2}\right) V.$$

If the current is alternating, of the type $\sin qt$, we use these same expressions, only that r_1 and r_2 are now unreal quantities. Thus, suppose we use no condensers; take $r_1 + l_1 qi$ instead of r_1 and $r_2 + l_2 qi$ instead of r_2, and see what each formula becomes. Now take such an example as $r_1 = 1$, $l_1 = 0 \cdot 1$; $r_2 = 100$, $l_2 = 0 \cdot 01$; calculate the value of each of the above formulæ for many values of q, and you will become acquainted with very interesting results.

Example. Or we may proceed as in Ex. 2 above. I will, however, take different numbers. Let $v = 1414 \sin qt$. Let one circuit have a resistance $100 + qi$ (that is $R = 100$, $L = 1$). Then for $q = 1000$,

$$c_2 = \frac{1414 \sin qt}{100 + 1000i} \quad \text{or} \quad c_2 = 1 \cdot 407 \sin (1000t - 84^\circ \cdot 283).$$

Let the other circuit have a condenser, merely, one microfarad in capacity or $K = 10^{-6}$. Then $c_3 = Kqiv$; that is,

$$c_3 = 1 \cdot 414 \sin (1000t + 90^\circ),$$

and the total current $C = c_2 + c_3$ is

$$C = 0 \cdot 1407 \sin (1000t + 5^\circ \cdot 717).$$

It may disturb the mind of a beginner to find a great current in each branch but the total current small; but, as I said before, it is analogous with the sum of two great, nearly equal, but nearly opposite forces. In such a case as the above I usually say that the compound circuit is in tune with $q = 1000$ [a frequency of about 160]. All the exercises given in my *Calculus*, all the exercises that can be set on alternating currents through given coils and condensers, merely depend upon these simple formulæ which are so well known to us for constant currents.

129. I have only to add here the effect of mutual induction between two circuits. If there are two parts of circuits r_1 and r_2 with mutual induction m between them, let r_1 stand for $r_1 + l_1 \theta + \frac{1}{k_1 \theta}$

and r_2 for $r_2+l_2\theta+\frac{1}{k_2\theta}$; or if either circuit is without a condenser, let its $\frac{1}{k\theta}$ term be 0. Let v_1 be the voltage between the ends of the first circuit and v_2 between the ends of the second circuit. Then

$$\left.\begin{aligned}v_1=r_1c_1+m\theta c_2\\ v_2=r_2c_2+m\theta c_1\end{aligned}\right\}. \quad\text{.........................(1)}$$

So if we are given v_1 and v_2, we can calculate c_1 and c_2, and we can work a great many other more troublesome-looking and more curious problems.

m may be either negative or positive.

If the currents are of the type $\sin qt$, of course we substitute qi for θ in our work.

For examples of ordinary transformers I must refer the student to my *Calculus*. He will have no difficulty in working any of the exercises there given. He will note that I there use θ for $\frac{d}{dt}$. Perhaps he had better do as I have done in this book, use qi everywhere instead of θ or $\frac{d}{dt}$; but the actual numerical work is just the same.

130. It will be noticed in the above exercises that when I have a circuit containing r and l or r and l and k, I speak of the *resistances* as being $r+lqi$ and $r+i\left(lq-\frac{1}{kq}\right)$. Some people call these *impedances.* It is of no consequence which name is given, but strictly speaking the impedance is the amplitude part of $r+lqi$ or $\sqrt{r^2+l^2q^2}$.

For the transformers used in wireless telegraphy, and sometimes in telephones, we do not have the simple rules so familiar to the electrical engineer, because condensers may be in the circuits and also $l_1l_2-m^2$ is very far from being 0. It may be well to give the following example from wireless telegraphy.

131. Systems influencing each other. Let two pendulums, A and B, of very nearly the same length, be hung from a thin horizontal rod or string; or let them be hung from rigid supports, but sufficiently near one another to exercise force on one another, either by means of india-rubber threads, or even, without such threads, let them affect each other by the mere currents of air that they produce. If A is vibrating and B at rest, B begins to vibrate, and its swing increases to a maximum and then diminishes, whereas the swing of A decreases nearly to 0 and then increases again, and this exchange of energy continually goes on between them. In any of these pendulum cases it is easy to state the equations of motion and solve them. But I prefer to give the analogous case of two electrical circuits in presence of each other, each with its resistance, inductance, and capacity, left to themselves; that is, there is no applied electromotive force in either.

If r stands for $r + l\theta + \frac{1}{k\theta}$, where θ is $\frac{d}{dt}$, if m is the mutual induction, the two circuits and the currents in them being distinguished by the suffixes 1 and 2; then (1) of Art. 129 is

$$r_1 c_1 + m\theta c_2 = 0 = r_2 c_2 + m\theta c_1,$$

and we have
$$(r_2 r_1 - m^2\theta^2)c_1 = 0, \quad c_2 = -\frac{r_1 c_1}{m\theta}.$$

It is easy to develop these equations. The expressions are simplified by taking the ohmic resistances to be 0, as they usually are insignificant in comparison with the inductance terms. We shall therefore have no damping in our oscillations. The result is of the form
$$(\theta^4 + A\theta^2 + B)c_1 = 0,$$
and we have exactly the same equation for c_2. The auxiliary equation (2) of Art. 133 being $x^4 + Ax^2 + B = 0$, we find roots of the form $x = \pm \alpha i$ and $x = \pm \beta i$, so that the answer for c_1 is

$$c_1 = M_1 \sin(\alpha t + e_1) + N_1 \sin(\beta t + g_1),$$

where M_1 and N_1 are arbitrary constants and

$$c_2 = M_2 \sin(\alpha t + e_2) + N_2 \sin(\beta t + g_2).$$

In fact, c_1 has two frequencies and c_2 has the same two frequencies. It will be found that

$$\sqrt{\frac{k_1 l_1 + k_2 l_2 \pm \sqrt{(k_1 l_1 - k_2 l_2)^2 + 4k_1 k_2 m^2}}{2k_1 k_2 (l_1 l_2 - m^2)}}$$

gives the value α if the plus sign is taken and the value β if the minus sign is taken.

We may write this differently. Let $t_1 = 2\pi\sqrt{k_1 l_1}$ be the natural period of the first circuit and let $t_2 = 2\pi\sqrt{k_2 l_2}$ be the natural period of the second circuit when they are quite separate from one another. Let $\mu^2 = 4\pi^2 m\sqrt{k_1 k_2}$. Let $T' = \frac{2\pi}{\alpha}$ and $T'' = \frac{2\pi}{\beta}$ be the two periodic times which we find in c_1 and c_2. Then

$$\sqrt{\frac{2t_1^2t_2^2 - 2\mu^4}{t_1^2 + t_2^2 \pm \sqrt{(t_1^2 - t_2^2)^2 + 4\mu^4}}}$$

is the value of T' if the plus sign is taken, and it is the value of T'' if the minus sign be taken. All the above work is easy algebra, which any student can do for himself.

The usual case is to let $l_1k_1 = l_2k_2$, so that $t_1 = t_2 = T$ say. We find now that

$$T' = \sqrt{T^2 + \mu^2}, \quad T'' = \sqrt{T^2 - \mu^2}.$$

If, now, there is *loose coupling* as it is called, that is, m^2 much less than l_1l_2 (or we have much magnetic leakage), μ^2 will be small compared with T, so that T' and T'' are nearly equal.

Example. Let $l_1 = 10^{-6}$ henry, $l_2 = 20 \times 10^{-6}$, $m = 10^{-7}$, $k_1 = 10^{-9}$ farad, $k_2 = 0{\cdot}05 \times 10^{-9}$ farad.

$$T = 2\pi\sqrt{k_1l_1} = 2\pi\sqrt{k_2l_2} = 19{\cdot}90 \times 10^{-8}, \quad \mu^2 = 8{\cdot}83 \times 10^{-16};$$

$$T' = 20{\cdot}125 \times 10^{-8} \quad \text{and} \quad T'' = 19{\cdot}67 \times 10^{-8},$$

and

$$\alpha = 31{\cdot}23 \times 10^6, \quad \beta = 31{\cdot}94 \times 10^6.$$

We have then exactly the effect of beats in music when two notes of different pitches (we call them frequencies) are sounding together.

Each term in c_2 is connected with the corresponding term in c_1 by the relation $-c_2 = \frac{r_1c_1}{m\alpha i}$ or $\frac{r_1c_1}{m\beta i}$. In this case we find that

$$M_2 = 0{\cdot}1736M_1 \quad \text{and} \quad N_2 = 0{\cdot}6192N_1.$$

The algebra gets troublesome if we proceed further. But the general result is obvious enough if we look at the two swinging pendulums which I have described. If the pendulums are of exactly the same length, the illustration is very perfect. If one of the pendulums, when its swings are small, is quite stopped, the vibrations of the other no longer increase and diminish. This gives the reason why the quenched spark method is important in wireless telegraphy.

CHAPTER XXXI.

PERIODIC FUNCTIONS IN GENERAL.

132. I have considered the simplest kind of periodic function in Chap. XXVIII. A periodic function of the time is one which becomes the same in every particular (its actual value, its rate of increase, etc.) after a time T. Thus T is called the periodic time, and its reciprocal is the frequency. Algebraically the definition of a periodic function is

$$f(t)=f(t+nT),$$

where n is any positive or negative integer.

Fourier's Theorem can be proved to be true. It states that any periodic function whose complete period is T being called $f(t)$, and q being $2\pi/T$, may be expressed as

$$x=f(t)=A_0+A_1\sin(qt+e_1)+A_2\sin(2qt+e_2) + A_3\sin(3qt+e_3)+\text{etc.} \quad\text{.........(1)}$$

In the same way, the note of any organ pipe or fiddle string or other musical instrument consists of a fundamental tone and its overtones. We know from Art. 116 that (1) is really the same as

$$f(t)=A_0+a_1\sin qt+b_1\cos qt+a_2\sin 2qt+b_2\cos 2qt + a_3\sin 3qt+b_3\cos 3qt+\text{etc.}, \quad\text{......(2)}$$

if $A_1{}^2=a_1{}^2+b_1{}^2$ and $\tan e_1=\dfrac{b_1}{a_1}$, etc.

In all our forced vibration problems in mechanics or electricity, we have to operate with functions of $\dfrac{d}{dt}$ or θ upon a function of the time $f(t)$. In every case I assumed this to be of the shape $a\sin qt$. But if it is any periodic function whatsoever, and if we express it in the form (1), we operate upon each term of (1) as if it alone existed, and our complete answer is the sum of these partial answers.

For the theory of the Fourier development I must refer students to my *Calculus.*

In valve motion work and in electrical work, it is very important when given a curve showing $f(t)$ and t to be able to develop any periodic function in the forms (1) or (2). I described a graphical method in the *Electrician* newspaper of June 28th, 1895. In the same paper on Feb. 5th, 1892, I described a method which might be used when a table of numbers is given for the equidistant values of $f(t)$. Prof. Henrici's analysers are machines which give the correct results when we turn a handle. The following method is probably the quickest when we have tabulated values. We have greater and greater accuracy when we have more and more tabulated values.

Example. I here tabulate 24 equidistant values of x, and I assume that A_1, e_1, A_2, e_2, A_3, e_3, A_4, and e_4 have to be found. I had better use the letter ϕ for the angle qt; evidently ϕ passes from 0 to 360° or from 0 to 2π when t passes from 0 to T, that is through the whole period. First, add the 24 ordinates together, and divide by 24; we thus get $A_0 = 5$. We now subtract 5 from every ordinate and call the result x'. In this particular case I know that there are not more than four components. Then

$$x' = A_1 \sin(\phi + e_1) + A_2 \sin(2\phi + e_2) + A_3 \sin(3\phi + e_3) + A_4 \sin(4\phi + e_4).$$

Let the student imagine x' plotted vertically and ϕ horizontally on squared paper from $\phi = 0$ to $\phi = 360°$. Then if one half of the curve from 180° to 360° is superimposed on the other half from 0° to 180°, the 1st and 3rd components will be eliminated if corresponding ordinates are added [this is easy to see if anyone of these components be drawn separately with any value of e], and the resulting curve will be

$$x'' = 2[A_2 \sin(2\phi + e_2) + A_4 \sin(4\phi + e_4)].$$

Similarly if the original curve be divided into three equal portions by lines perpendicular to the axis of ϕ, and the three parts be superimposed and corresponding ordinates added, the 2nd and 4th components will be eliminated, and the resulting curve will be

$$x''' = 3a_3 \sin(3\phi + e_3).$$

It is an easy exercise for the student to prove this either graphically or analytically. If he has difficulty let him consult Mr. Wedmore's paper in the *Proceedings of the Institution of Electrical Engineers*, 1896. The table shows how the above method is used without drawing the curves. For instance, columns A, I, and J are the three equal parts superimposed, and when added give column K, which is 3 times component 3; in this case zero.

An examination of the table easily shows how it is all produced.

Component 1. Imagine column N to be continued to the top of the table; ordinate 0 will be 2·520; average of ordinates from 0 to

$\phi°$	No. of ordinate.	x	A	B	C	D	E	F	G	H
			x' or $x-5$.	This is A superimposed on itself.	$A+B$.	Half of C being sum of components 2 and 4.	Being D superimposed on itself.	Being $D+E$.	$\frac{1}{2}F$ or component 4.	$D-G$ being component 2.
0	0	8·02	3·02	−2·02	1·00	0·50	−0·50	0·00	0·00	0·50
15	1	8·37	3·37	−2·33	1·04	0·52	−0·345	0·175	0·0875	0·4325
30	2	8·33	3·33	−2·66	0·67	0·335	−0·165	0·170	0·085	0·250
45	3	7·93	2·93	−2·93	0·00	0·00	0·00	0·00	0	0·00
60	4	7·34	2·34	−3·01	−0·67	−0·335	0·165	−0·170	−0·085	−0·250
75	5	6·71	1·71	−2·75	−1·04	−0·52	0·345	−0·175	−0·0875	−0·4325
90	6	6·13	1·13	−2·13	−1·00	−0·50	G continued downwards ☞		0·00	−0·50
105	7	5·58	0·58	−1·27	−0·69	−0·345			0·0875	−0·4325
120	8	4·99	−0·01	−0·32	−0·33	−0·165			0·085	−0·250
135	9	4·38	−0·62	0·62	0·00	0·00			0·00	0·00
150	10	3·80	−1·20	1·53	0·33	0·165			−0·085	0·250
165	11	3·34	−1·66	2·35	0·69	0·345			−0·0875	0·4325
180	12	2·98	−2·02	M continued upwards ☞				0·50	−2·520	
195	13	2·67	−2·33					0·52	−2·850	
210	14	2·34	−2·66					0·335	−2·995	
225	15	2·07	−2·93					0·00	−2·930	
240	16	1·99	−3·01	−0·01	3·02	0		−0·335	−2·675	
255	17	2·25	−2·75	−0·62	3·37	0		−0·52	−2·230	
270	18	2·87	−2·13	−1·20	3·33	0		−0·50	−1·630	
285	19	3·73	−1·27	−1·66	2·93	0		−0·345	−0·925	
300	20	4·68	−0·32	−2·02	2·34	0		−0·165	−0·155	
315	21	5·62	0·62	−2·33	1·71	0		0·00	0·620	
330	22	6·53	1·53	−2·66	1·13	0		0·165	1·365	
345	23	7·35	2·35	−2·93	0·58	0		0·345	2·005	
Mean Ordinate } = 5				Being A superimposed.	Being A superimposed.	$A+I+J$ being 3 times component 3, which is 0.		D repeated.	$A-D$ being components 1 and 3 or comp. 1 only.	
			A	I	J	K		M	N	

No. 11 inclusive, treating all as positive, is $22·900 \div 12 = 1·908$. We use this method of finding A_1 because of the rule (see Ex. 2, Art. 56):

Maximum ordinate $A_1 = 1·908 \times \frac{\pi}{2} = 2·997$, say 3.

To get e_1, $\frac{\sin e_1}{\sin 90^\circ} = \frac{2 \cdot 520}{2 \cdot 997} = 0 \cdot 84$; therefore $e_1 = 57^\circ$.

Component 2. By inspection of column H, the maximum ordinate is 0·50, and $e_2 = 90^\circ$.

Component 3. Zero. See column K.

Component 4. Average of ordinates from 0 to No. 2 inclusive is $0 \cdot 1725 \div 3$ or 0·0575, and $A_4 = 0 \cdot 0575 \times \frac{\pi}{2} = 0 \cdot 091$.

By inspection $e_4 = 0$.

Hence $\quad x = 5 + 3 \sin(\phi + 57^\circ) + 0 \cdot 5 \cos 2\phi + 0 \cdot 09 \sin 4\phi$.

NOTE. When we have few ordinates of a sine curve it is not exact enough to take their mean value (taking them all as positive as I do above) as the mean ordinate of the real curve. I am not satisfied in such a case unless I plot the ordinates and draw the probable curve to give me A and e.

133. A differential equation like

$$\frac{d^4x}{dt^4} + P\frac{d^3x}{dt^3} + Q\frac{d^2x}{dt} + R\frac{dx}{dt} + Sx = T \quad \text{..................(1)}$$

if P, Q, R, S, and T are functions of t only, or constants, is said to be a linear differential equation.

Suppose we put $T = 0$; we shall find a solution $x =$ some function of t. Let us write it $x = f(t)$. It will involve four arbitrary constants, because the given equation is of the fourth order. Now, if when T is not 0 we *guess* at one solution and find it to be $x =$ some other function of t, which I shall call $F(t)$, then it is easily proved that the most complete solution of (1) is

$$x = f(t) + F(t).$$

I shall here consider only cases in which P, Q, R, and S are constants. First, let T be 0. Assume that $y = Me^{mt}$ is a solution, and we see that it is so, no matter what M may be, if

$$m^4 + Pm^3 + Qm^2 + Rm + S = 0. \quad \text{.....................(2)}$$

This is called the auxiliary equation. An equation of the fourth degree has four roots. Suppose these to be the values of m which satisfy the equation (2), a, b, $\alpha + \beta i$, $\alpha - \beta i$; the answer $x = f(t)$ is

$$x = Ae^{at} + Be^{bt} + Ce^{(\alpha+\beta i)t} + De^{(\alpha-\beta i)t},$$

where A, B, C, and D are arbitrary constants.

As C and D may be unreal, and as $e^{\beta ti}$ may be written

$$\cos \beta t + i \sin \beta t$$

(see Art. 137), it is easy to see that the solution most suitable for actual problems is

$$x = Ae^{at} + Be^{bt} + e^{\alpha t}(M\cos\beta t + N\sin\beta t),$$

where A, B, M and N are arbitrary constants, all real.

In Art. 121 I use this rule in the solution of an equation of the second order. A real root of (2) of the form $m = a$ gives rise to a term Ae^{at} in the answer; unreal roots of the form $\alpha \pm \beta i$ give rise to the terms

$$e^{\alpha t}(M\cos\beta t + N\sin\beta t).$$

There are rules about equal roots, but they are seldom needed. The theory will be found in my *Calculus*. If, for example, x is displacement of a body and t is time, the solution $x = f(t)$ is called the free or natural motion of the system, and the part $x = F(t)$ is called the forced motion. As I have said at Art. 120, we often neglect the free motion as it has damping terms, causing it to quickly cease to exist.

Excellent easy exercises on this subject will be found in (a) my "Theory of the *Hunting* of a Steam Engine," in the Appendix to my book on *Steam*, (b) the critical speeds of shafts with or without heavy wheels and the critical speeds of crank shafts, given in the Appendix to my book on *Applied Mechanics*.

If in Art. 121 we imagine a series of several springs and weights, we get a system of several natural frequencies; how will it behave when subjected to varying forces acting on the weights (*Applied Mechanics*, Appendix)? It is exceedingly interesting to watch the natural vibrations of such a system or its equivalent torsional system, and it is easy to study mathematically. Lord Kelvin was fond of this study, and made much use of his results in his famous Baltimore Lectures on "Light." The work is so easy that I am tempted to give it here, but this book is getting too large.

CHAPTER XXXII.

EXTENDED RULES AND PROOFS.

134. A knowledge of the differentiation of the few functions given in Art. 94 will suffice for nearly all the mathematical work that has to be done by the engineer. I mean by engineer, any man who applies the principles of Natural Science.

By means of a few rules it is easy to become able to differentiate any algebraic function of x and to integrate a great many.

Students often learn no more of this wonderful subject than to acquire these rules, and they rapidly lose their power after they have passed certain examinations, because they never have learnt really what dy/dx means and they have taken no interest in the subject.

I hope that our teaching will give students that kind of interest in the subject which is generally wanting, and that some of these students will pursue the subject in more orthodox treatises. I think, however, that they had better first study my book, *The Calculus.*

1. Let $y=u+v+w$ be the **sum** of any three functions of x.

Let x become $x+\delta x$, and in consequence let u become $u+\delta u$, v become $v+\delta v$, and w become $w+\delta w$. It results that if y becomes $y+\delta y$,

$$\delta y=\delta u+\delta v+\delta w$$

and

$$\frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}+\frac{\delta w}{\delta x},$$

and in the limit

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}.$$

In Art. 93 I assumed this without proof; that the differential coefficient of a sum of functions is equal to the sum of the differential coefficients of each.

A great extension of our table, Art. 94, may be made if the following rules are known.

2. Differential coefficient of a **product** of two functions.

Let $y = uv$, when u and v are functions of x. When x becomes $x + \delta x$, let
$$y + \delta y = (u + \delta u)(v + \delta v) = uv + u\,.\,\delta v + v\,.\,\delta u + \delta u\,.\,\delta v.$$

Subtracting, we find
$$\delta y = u\,.\,\delta v + v\,.\,\delta u + \delta u\,.\,\delta v$$
and
$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}.$$

We now imagine δx, and in consequence (for this is always assumed in our work) δu, δv, and δy to get smaller and smaller without limit. Consequently, whatever $\frac{dv}{dx}$ may be, $\delta u\frac{\delta v}{\delta x}$ must in the limit become 0, and hence
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The student ought to manufacture such exercises as this.
Let $u = 5x^3$ and $v = 2x^4$,
$$\frac{dv}{dx} = 8x^3, \quad \frac{du}{dx} = 15x^2, \text{ so that, by the rule,}$$
$$\frac{dy}{dx} = 5x^3 \times 8x^3 + 2x^4 \times 15x^2 = 70x^6.$$

But as $y = 10x^7$, we know that $\frac{dy}{dx} = 70x^6$.

3. Differential coefficient of a **quotient**.

Let $y = \frac{u}{v}$, where u and v are functions of x.

Then
$$y + \delta y = \frac{u + \delta u}{u + \delta v}.$$

Subtract, and we find
$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{v\,.\,\delta u - u\,.\,\delta v}{v^2 + v\,.\,\delta v},$$
$$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v\,.\,\delta v}.$$

Letting δx get smaller and smaller without limit, $v\,.\,\delta v$ becomes 0, and we have
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$

or in words, "Denominator into differential coefficient of numerator, minus numerator into differential coefficient of denominator, divided by denominator squared."

A few illustrations are easily manufactured. Thus, let
$$u = 24x^7 \text{ and } v = 3x^2 \text{ or let } u = 15x^2 \text{ and } v = 3x^4.$$

135. If y is given as a function of z and z is given as a function of x, of course y is a function of x. Under such circumstances, if instead of x we take $x+\delta x$, and so calculate $z+\delta z$, and with this *same* $z+\delta z$ we calculate $y+\delta y$, then we can say that our δy is in consequence of our δx, and

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta z}\cdot\frac{\delta z}{\delta x}. \quad\text{.................................(1)}$$

On the supposition that the two things written as δz remain the same however small they become, we see that the rule (1) is true even when δx gets smaller and smaller without limit, or

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}. \quad\text{.................................(2)}$$

This is an exceedingly important rule, and the student ought to illustrate it in ways of his own. I might suggest drawing three curves, a first showing y as a function of z, a second showing z as a function of x, and a third showing y as a function of x. The student ought to see clearly that at corresponding points the slope of the y, x curve is equal to the product of the slopes of the other two.

The following examples are applications of the rule (2)

Ex. 1. Let $y=e^{ax}$. Put it in the shape $y=e^u$, $u=ax$,

$$\frac{dy}{du}=e^u,\quad \frac{du}{dx}=a\,;$$

therefore $$\frac{dy}{dx}=e^u\times a=ae^{ax}\text{ (see Art. 113).}$$

Ex. 2. Let $y=\log(a+bx)$ or let $y=\log u$, where $u=a+bx$,

$$\frac{dy}{du}=\frac{1}{u},\quad \frac{du}{dx}=b,$$

so that $$\frac{dy}{dx}=\frac{1}{u}b=\frac{b}{a+bx}.$$

Ex. 3. Let $y=\sin(ax+b)$ or let $y=\sin u$, where $u=ax+b$,

$$\frac{dy}{du}=\cos u,\quad \frac{du}{dx}=a,$$

so that $$\frac{dy}{dx}=a\cos u=a\cos(ax+b).$$

Similarly, if $y=\cos(ax+b)$, $\frac{dy}{dx}=-a\sin(ax+b)$.

136. The following examples are not needed for any of the work of these classes, nevertheless I am weak enough to give them:

Ex. 1. Let $y = \tan x = \frac{\sin x}{\cos x}$. By the rule for a quotient, we have

$$\frac{dy}{dx} = \frac{\cos x . \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Also, if $y = \tan ax$, $\frac{dy}{dx} = \frac{a}{\cos^2 ax}$.

Ex. 2. If $x = a\phi - a\sin\phi$ and $y = a - a\cos\phi$, where x and y are the co-ordinates of points on a cycloid,

$$\frac{dy}{dx} = \frac{dy}{d\phi} \div \frac{dx}{d\phi} = \frac{a\sin\phi}{a - a\cos\phi} = \frac{\sin\phi}{1-\cos\phi}.$$

Ex. 3. Let $y = \sin^{-1} x$; in words, y is the angle whose sine is x.

Hence $x = \sin y$, $\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, if $y = \sin^{-1}\frac{x}{a}$, then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$.

Prove that if $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$;

and that if $y = \tan^{-1}\frac{x}{a}$, then $\frac{dy}{dx} = \frac{a}{a^2 + x^2}$.

Ex. 4. If $y = a^x$, then $\log y = x \log a$.
Differentiating with regard to y, we have

$$\frac{1}{y} = \frac{dx}{dy} \log a,$$

$$\frac{dy}{dx} = y \log a = a^x \log a.$$

Ex. 5. If $y = \sqrt{x^2 - a^2}$, or $y = u^{\frac{1}{2}}$, where $u = x^2 - a^2$,

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}, \quad \frac{du}{dx} = 2x,$$

so that $$\frac{dy}{dx} = \frac{1}{2} u^{-\frac{1}{2}} 2x = \frac{x}{\sqrt{x^2 - a^2}}.$$

Ex. 6. Example of a product. Let $y = e^{ax} \sin(bx + c)$.

Our rule is, if $y = uv$, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

Here $$u=e^{ax},\quad \frac{du}{dx}=ae^{ax},$$

$$v=\sin(bx+c),\quad \frac{dv}{dx}=b\cos(bx+c).$$

Hence $$\frac{dy}{dx}=be^{ax}\cos(bx+c)+ae^{ax}\sin(bx+c)$$

or $$\frac{dy}{dx}=e^{ax}\{b\cos(bx+c)+a\sin(bx+c)\}.$$

Ex. 7. Or let $y=e^{ax}(A\sin bx+B\cos bx)$.

$$\frac{dy}{dx}=e^{ax}(Ab\cos bx-Bb\sin bx)+ae^{ax}(A\sin bx+B\cos bx),$$

or $$\frac{dy}{dx}=e^{ax}\{(aA-Bb)\sin bx+(Ab+aB)\cos bx\}.$$

Ex. 8. Show that $y=e^{3x}\sin 5x$ or $e^{3x}\cos 5x$ satisfies the equation

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+34y=0.$$

Ex. 9. Show that $x=e^{-3t}(A\sin 5t+B\cos 5t)$ satisfies the equation

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+34x=0.$$

EXERCISES.

The average value of y from $x=a$ to $x=b$ is the area $\int_a^b y\,.\,dx$ divided by $b-a$.

A knowledge of some very elementary trigonometrical formulæ (those of Exercise 3, Art. 28) is necessary for the working of some of the examples in this page.

1. The average value of $a\cos(qt+e)$ or $a\sin(qt+e)$ for the whole periodic time T is zero.

2. The average value of $a\sin qt\times b\sin(qt\pm e)$ for the whole periodic time is $\frac{ab}{2}\cos e$.

3. The average value of $a\cos qt\times b\cos(qt\pm e)$ for the whole periodic time is $\frac{ab}{2}\cos e$.

4. The average value of the square of $a\sin qt$ for the whole periodic time is $a^2/2$. (See Art. 118, note.)

5. The effective value of a current $C=a\sin qt$ is the square root of the average value of the square of C; that is, the *effective* current $=\frac{a}{\sqrt{2}}$ or $0{\cdot}707a$.

Thus, if $C = 1414 \sin qt$, the electrical engineer says that the current is 1000, because he speaks of the effective current. So also with voltage.

It follows from (2) or (3) that the *effective* amperes × *effective* volts × cosine of the angle of lag or of lead between amperes and volts is electric power in watts, and $\cos e$ is what is called the *load factor*.

137. In the next two articles I propose to give **Proofs** which I discovered when this book was in the hands of the printer.

Some students may be dissatisfied because in Art. 95 I assumed the **Binomial** and **Exponential** theorems without proof, and they may complain of want of rigour, generally, in my methods. But from the beginning I have had to show that every student must first make acquaintance with formulæ by much exercise work; he must then learn to use calculus methods; afterwards even the man of non-mathematical mind will probably be interested in rigorous proofs. It must be understood that no elementary student, however mathematical, however clever, is introduced to proofs whose rigour will satisfy the great mathematicians. The following scheme is as rigorous as that which is usually given; it is much simpler because I use the calculus idea from the beginning. First, as in Arts. 134–5, prove the simple rule for the differentiation of a product, and that if u is a function of x,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

138. To show that when $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$. It can be shown at once (as in Art. 89) that this is true when n is 2 or 3. I shall first prove that it must be true when n is any positive integer.

(a) Suppose the rule true when $n = r$; I will show that it must also be true when $n = r + 1$.

Take $y = x^{r+1} = x \times x^r$. By the rule for a product, we have

$$\frac{dy}{dx} = x^r + rx^{r-1} \times x = (r+1)x^r.$$

As the rule is true for $r = 2$, we see that it must be true for $r = 3$ or 4 or any positive integer.

(b) If $y = x^{\frac{p}{q}}$, where p and q are any positive integers, then $y^q = x^p = z$ say.

Differentiating, we have

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

This simplifies to $\frac{dy}{dx}=\frac{p}{q}x^{\frac{p}{q}-1}$.

We have therefore proved the rule to be true when n is any positive number, integral or fractional.

(c) Let $y=x^{-m}$ or $yx^m=1$. Differentiating this product,

$$\frac{dy}{dx}x^m+ymx^{m-1}=0.$$

This simplifies to $\frac{dy}{dx}=-mx^{-m-1}$. Thus the rule is proved generally.

139. I will now prove the rule for **exponentials**, and incidentally I will prove the exponential theorem which is the expansion of a^x in a series of powers of x. I assume that there is such an expansion, namely,

$$y=a^x=A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}, \quad \text{.............}(1)$$

and we have to find the values of A, B, etc. As this is true for all values of x, it is true when $x=0$, so we have $a^0=A=1$.

$$y+\delta y=a^{x+\delta x}, \text{ and therefore } \delta y=a^{x+\delta x}-a^x=a^x(a^{\delta x}-1).$$

Expanding $a^{\delta x}$ according to (1), we have

$$\delta y=a^x\{B.\delta x+C(\delta x)^2+D(\delta x)^3+\text{etc.}\}. \quad \text{..............}(2)$$

So that $\frac{\delta y}{\delta x}=a^x\{B+C.\delta x+D.(\delta x)^2+\text{etc.}\}$.

Now let δx get smaller and smaller without limit, and we have

$$\frac{dy}{dx}=Ba^x. \quad \text{...............................}(3)$$

Apply this to (1), and we have

$$\frac{d}{dx}(1+Bx+Cx^2+Dx^3+\text{etc.})=B(1+Bx+Cx^2+Dx^3+\text{etc.}),$$

$$B+2Cx+3Dx^2+\text{etc.}=B+B^2x+BCx^2+BDx^3+\text{etc.}$$

Equating corresponding terms in this identity, we have

$$2C=B^2 \text{ or } C=\tfrac{1}{2}B^2; \quad 3D=BC \text{ or } D=\tfrac{1}{6}B^3, \text{ etc.}$$

Hence $$a^x=1+Bx+\frac{B^2x^2}{\underline{|2}}+\frac{B^3x^3}{\underline{|3}}+\frac{B^4x^4}{\underline{|4}}+\text{etc.} \quad \text{..............}(4)$$

Now let $Bx=z$, so that $x=\frac{z}{B}$, and

$$a^{\frac{z}{B}}=1+z+\frac{z^2}{\underline{|2}}+\frac{z^3}{\underline{|3}}+\text{etc.} \quad \text{....................}(5)$$

Let $a^{\frac{1}{B}}$ be called e, so that $\frac{1}{B}\log_e a=1$ or $B=\log_e a$.

With this value of B, (4) is the usual form of the exponential theorem, and (3) becomes, if $y=a^x$,

$$\frac{dy}{dx}=a^x \log_e a.$$

Or if $y=e^x$, $\frac{dy}{dx}=e^x$. Of course if we use x instead of z, (5) is

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\text{etc.} \quad\ldots\ldots\ldots\ldots(6)$$

This is the series given in Art. 95, without proof.

e can be calculated with any amount of accuracy from (6), by taking $x=1$, for then

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\text{etc.}$$

I have never known a thoughtful young man to be satisfied with the usual algebra proof of the exponential theorem [which is the modern basis of all calculation of logarithms]; and I feel sure that if he were only allowed to use the calculus idea as I have here done, he would have none of his usual feeling of dissatisfaction and insecurity. That feeling is mainly due to the complexity of the artifices employed.

140. Lastly, let $y=\log(a+x)$. This is the same statement as $a+x=e^y$. Differentiating with regard to x, we have

$$1=e^y\frac{dy}{dx} \text{ or } 1=(a+x)\frac{dy}{dx} \text{ or } \frac{dy}{dx}=\frac{1}{a+x}.$$

141. Exercises in Differentiation. (1) Expand $\sin x$ and $\cos x$ in series of powers of x.

Assume

$$\sin x=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+Hx^7+Ix^8+Jx^9+\text{etc.},$$

where A, B, C, etc., are unknown constants. Differentiating, we get

$$\cos x=B+2Cx+3Dx^2+4Ex^3+5Fx^4+6Gx^5+7Hx^6+8Ix^7+9Jx^8+\text{etc.}$$

Differentiating again, we get

$$-\sin x=2C+6Dx+12Ex^2+20Fx^3+30Gx^4+42Hx^5$$
$$+56Ix^6+72Jx^7+\text{etc.}$$

Equating like terms in the first and third of these, and recollecting that the third is all negative,

$$2C=-A,\ 6D=-B,\ 12E=-C,\ 20F=-D,\ 30G=-E,\ 42H=-F,$$
$$56I=-G,\ 72J=-H,\ \text{etc.}$$

Now, as $\sin x$ is 0 when x is 0, A is 0. As $\cos x = 1$ when x is 0, $B = 1$, so that

$$C=0,\ E=0,\ G=0,\ I=0,\ \text{etc.};\ D=-\frac{1}{6},\ F=\frac{1}{120},\ H=\frac{-1}{\underline{|7}},\ J=\frac{1}{\underline{|9}}\ \text{etc.}$$

Hence
$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} - \text{etc.},$$
$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \text{etc.}$$

These are the series given in Ex. 7, Art. 28, without proof.

(2) Expand $\log(1+x)$ in powers of x. Let
$$\log(1+x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.}$$

Differentiating both sides, we find
$$\frac{1}{1+x} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + 6Gx^5 + \text{etc.}$$

But by mere division, we get
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \text{etc.}$$

Equating the coefficients of corresponding terms in these identities, we find
$$B=1,\quad C=-\tfrac{1}{2},\quad D=\tfrac{1}{3},\quad E=-\tfrac{1}{4},\quad F=\tfrac{1}{5},\quad G=-\tfrac{1}{6},\quad \text{etc.}$$

Again, when $x=0$, $\log(1+x) = \log 1 = 0$, so that A is 0, and we have
$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \text{etc.}$$

By writing out $\log(1-x)$ and subtracting, and so finding $\log\frac{1+x}{1-x}$ and then writing $\frac{m-n}{m+n}$ for x and by other substitutions, we find series which are those used in the rapid calculation of tables of logarithms.

Many other interesting series may be obtained in the above way, but it is more usual to expand any function in series by what is called Taylor's theorem.

142. I will now prove **Taylor's theorem.** Lagrange was the first who introduced rigour into the proof of Taylor; but even to this day there is no proof which is perfectly rigorous. I give the usual non-rigorous proof. It has to be remembered that an expansion in an infinite series has no meaning to the mathematician if it is divergent. So in Algebra, $\sqrt{-1}$ has no meaning, but Chapters XXVIII. to XXXVI. show that when a certain meaning is given to $\sqrt{-1}$ which does not conflict with Algebra, it becomes a very useful thing. Mr. Heaviside has made great use of divergent series; he has thereby solved correctly most difficult problems, and it is acknowledged that his answers are correct, but the mathematician whose orthodox methods fail to solve the most elementary of these problems is very scornful of the Heaviside methods.

Proof of Taylor's Theorem. If a function of $x+h$ be differentiated with regard to x, h being supposed constant, we get the same answer as if we differentiate with regard to h, x being supposed constant.

For if the function is called $f(u)$, where $u=x+h$, then

$$\frac{d}{dx}f(u)=\frac{d}{du}f(u)\times\frac{du}{dx}=\frac{d}{du}f(u) \quad \text{as} \quad \frac{du}{dx} \quad \text{is} \quad 1,$$

and this is the same as $\frac{d}{dh}f(u)$ because

$$\frac{d}{dh}f(u)=\frac{d}{du}f(u)\times\frac{du}{dh}=\frac{d}{du}f(u) \quad \text{as} \quad \frac{du}{dh} \quad \text{is} \quad 1.$$

If $f(x+h)$ may be expanded in a series of ascending powers of h, let it be

$$f(x+h)=A+Bh+Ch^2+Dh^3+Eh^4+\text{etc.}, \quad \ldots\ldots\ldots\ldots(1)$$

where A, B, C, etc., are functions of x, but they do not contain h.

$$\frac{d}{dh}f(x+h)=B+2Ch+3Dh^2+4Eh^3+\text{etc.}, \quad \ldots\ldots\ldots\ldots(2)$$

$$\frac{d}{dx}f(x+h)=\frac{dA}{dx}+h\frac{dB}{dx}+h^2\frac{dC}{dx}+h^3\frac{dD}{dx}+\text{etc.} \quad \ldots\ldots\ldots(3)$$

As (2) and (3) are identical, we can equate corresponding terms, so that

$$B=\frac{dA}{dx}, \quad C=\frac{1}{2}\frac{dB}{dx}=\frac{1}{\underline{|2}}\frac{d^2A}{dx^2}, \quad D=\frac{1}{3}\frac{dC}{dx}=\frac{1}{\underline{|3}}\frac{d^3A}{dx^3}, \quad \text{etc.}$$

Also putting $h=0$ in (1), we find that $A=f(x)$. If the result of differentiating $f(x)$ once, twice, three times, etc., in regard to x be written $f'(x)$, $f''(x)$, $f'''(x)$, etc., we have Taylor's theorem

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\frac{h^3}{\underline{|3}}f'''(x)+\text{etc.} \quad \ldots\ldots\ldots(4)$$

Having differentiated $f(x)$ once, twice, etc., if we substitute 0 for x, let us call the results $f'(0)$, $f''(0)$, etc.; if we imagine 0 to be substituted for (x) in (4), we have

$$f(h)=f(0)+hf'(0)+\frac{h^2}{\underline{|2}}f''(0)+\frac{h^3}{\underline{|3}}f'''(0)+\text{etc.} \quad \ldots\ldots\ldots\ldots(5)$$

Observe that we have no longer anything to do with the quantity which we call x. We may if we please use any other letter than h

in (5); let us use the new letter x, and (5) becomes

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\underline{|2}}f''(0)+\frac{x^3}{\underline{|3}}f'''(0)+\text{etc.}, \quad \ldots\ldots\ldots(6)$$

which is called **Maclaurin's theorem.**

Ex. 1. Expand $(x+h)^n$ in powers of h by Taylor's theorem. The result is called the Binomial Theorem.

Ex. 2. Expand $\log(x+h)$ in powers of h by Taylor's theorem. In the result put $x=1$; this expansion of $\log(1+h)$ we obtained in another way in Art. 141.

Ex. 3. Expand $\sin(x+h)$ and $\cos(x+h)$ in powers of h by Taylor's theorem. In the results put $x=0$. We have already found these answers, which are also the answers of the two following exercises.

Ex. 4. Expand $\sin x$ in powers of x by Maclaurin's theorem.

Ex. 5. Expand $\cos x$ in powers of x by Maclaurin's theorem.

Ex. 6. Write out the expansions of $\sin\phi$ and $\cos\phi$ in powers of ϕ, and show that if i is $\sqrt{-1}$, expanding $e^{i\phi}$ and $e^{-i\phi}$ according to (6) of Art. 28,

$$e^{i\phi}=\cos\phi+i\sin\phi, \quad e^{-i\phi}=\cos\phi-i\sin\phi.$$

Ex. 7. Raising the expressions in Ex. 6 to the power n, show that

$$(\cos\phi+i\sin\phi)^n=\cos n\phi+i\sin n\phi,$$
$$(\cos\phi-i\sin\phi)^n=\cos n\phi-i\sin n\phi,$$

which is Demoivre's theorem.

The proof of this given in books on Trigonometry is quite easy; it begins with the proof that

$$(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi),$$

which is of course evident from the answer of Ex. 6.

Ex. 8. Using Demoivre's theorem, find the three cube roots of 27. If you like you can say that you are finding the three roots of the equation $x^3-27=0$. As one of the roots is 3, $x-3$ must be a factor of x^3-27. Dividing the equation by $x-3$, we find

$$x^2+3x+9=0.$$

And the roots of this quadratic are

$$x=-\tfrac{3}{2}\pm\sqrt{\tfrac{9}{4}-9}=-1{\cdot}5\pm2{\cdot}6i.$$

Using Demoivre, however, 27 may be written as

$$27(\cos 0+i\sin 0) \quad\text{or}\quad 27(\cos 2\pi+i\sin 2\pi) \quad\text{or}\quad 27(\cos 4\pi+i\sin 4\pi).$$

The cube roots of these are

$$3\cos 0 \text{ or } 3; \quad 3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \text{ or } 3(\cos 120^\circ + i\sin 120^\circ)$$

$$\text{or } 3(-0{\cdot}5+0{\cdot}866i) \text{ or } -1{\cdot}5+2{\cdot}6i;$$

$$3(\cos 240^\circ + i\sin 240^\circ) \text{ or } 3(-0{\cdot}5-0{\cdot}866i) \text{ or } -1{\cdot}5-2\,6i.$$

There are only three cube roots; if we write 27 as

$$27(\cos 6\pi + i\sin 6\pi)$$

or use higher values, we get these same answers.

143. Partial Differentiation. Hitherto we have been studying a function of one variable which we have generally called x. If u is a function of two independent variables, say $u=f(x, y)$, we may wish to find $\frac{du}{dx}$ on the assumption that y is a constant. I write this $\left(\frac{du}{dx}\right)$, but it may be written merely as $\frac{du}{dx}$ if there is a clear understanding that the differentiation is partial.

Thus in the Heat Conduction problem, Chap. XXXIV., v is the temperature at the time t at a point which is at the depth x. Now v is a function of x and t.

$\frac{d^2v}{dx^2}$ is calculated on the assumption that t is a constant, and $\frac{dv}{dt}$ is calculated on the assumption that x is constant. Thus, if

$$v=e^{ax+bt}, \quad \frac{dv}{dt}=be^{ax+bt}, \quad \frac{d^2v}{dx^2}=a^2e^{ax+bt}.$$

In our laboratories we try to make a thing u depend on one other thing x only. Thus, in observing the laws of gases: if p is the pressure and v the volume and t the temperature (where $t=\theta^\circ$ C. + 273) of say 1 lb. of gas, if t is kept constant we have the law $p \propto \frac{1}{v}$. If v is kept constant and the temperature alters, we find $p \propto t$. After much trial we find that the law $pv=Rt$ is very nearly true, R being a known constant, say 96 for air, if v is in cubic feet and p is in pounds per sq. foot. Now any one of the three, p, v, or t, is a function of the other two, and in fact if any values whatsoever be given to two of them, the other can be found. Thus

$$p=R\frac{t}{v}. \quad \text{.................................(1)}$$

We can say that p is a function of the two independent variables t and v. If any particular values whatsoever of t and v be taken in (1), we can calculate p. Now take new values, say $t+\delta t$ and $v+\delta v$, where δt and δv are perfectly independent of one another—they may be any values—then

$$p+\delta p=R\frac{t+\delta t}{v+\delta v} \quad \text{and} \quad \delta p=R\frac{t+\delta t}{v+\delta v}-R\frac{t}{v}.$$

We see therefore that the change δp can be calculated if the independent changes δt and δv are known.

When all the changes are considered to be smaller and smaller without limit, we have an easy way of expressing δp in terms of δt and δv. It is

$$\delta p=\left(\frac{dp}{dt}\right)\delta t+\left(\frac{dp}{dv}\right)\delta v. \quad \text{.........................(2)}$$

The proof of this is easy.

The student ought to put this in such words as these:—"The whole change in p is made up of two parts, (1st) the change that would occur in p if v did not alter, and (2nd) the change that would occur in p if t did not alter." The symbol $\left(\frac{dp}{dt}\right)$ means, the rate of increase of p with t when v is constant; the symbol $\left(\frac{dp}{dv}\right)$ means, the rate of increase of p with v when t is constant. Sometimes the symbols $\frac{\partial p}{\partial t}$ and $\frac{\partial p}{\partial v}$ are used instead of the brackets for this **partial** differentiation.

Now the student must understand that (2) is wrong, unless for exceedingly small values of δt and δv.

Illustration. Take 1 lb. of air, where R is 96.

As $p=96\frac{t}{v}$, $\left(\frac{dp}{dt}\right)=\frac{96}{v}$, $\left(\frac{dp}{dv}\right)=-\frac{96t}{v^2}=-\frac{p}{v}$.

Hence (2) is

$$\delta p=\frac{96}{v}\cdot\delta t-\frac{p}{v}\cdot\delta v. \quad \text{.........................(3)}$$

Example. Let $t=300$, $p=2000$, $v=14\cdot4$. These will be found to satisfy the law $p=96\frac{t}{v}$. Now let $t+\delta t=301$ (that is, let $\delta t=1$), let $v+\delta v=14\cdot5$ (that is, let $\delta v=0\cdot1$); it is easy to see that

$p+\delta p=1992{\cdot}82$ or $\delta p=-7{\cdot}18$. This is the true answer, but let us use (3), and we find

$$\delta p=\frac{96}{14{\cdot}4}\times 1-\frac{2000}{14{\cdot}4}\times{\cdot}1=-7{\cdot}22, \text{ a wrong answer.}$$

If we had used smaller values for δt and δv, we should have had an answer from (3) more nearly correct, but (3) gives a correct answer only when δt and δv are supposed to be smaller and smaller without limit.

CHAPTER XXXIII.

EXERCISES WITH UNREAL QUANTITIES.

144. The following exercises ought perhaps to be placed at the end of Chapter III., and there are few of them which may not be worked by the students who do the other numerical exercises in the early part of the book. But it is not necessary that all students should work them, and therefore I have placed them here. Until men get quite *familiar* with this kind of work it is quite useless for them to proceed further in the book.

To the end of the book, exercises will be found involving $\sqrt{-1}$, which I usually call i. Such exercises may be neglected by the ordinary student. But such students as proceed to the study of vibrations of bodies or alternating currents in electricity, to the study of telephonic circuits, for example, will find a knowledge of these unreal quantities necessary. For a few of the exercises in this article a student needs to be able to solve simple simultaneous quadratic equations, but such exercises will be worked, presently, in a much easier manner.

Ex. 1. If i means $\sqrt{-1}$, see that the following values are correct:

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = i^2 = -1,\ i^7 = i^3 = -i,\ i^8 = 1.$$

Ex. 2. Find $\sqrt{17+30i}$. We know that it is of the form $a+bi$, and we must find a and b,

$$17 + 30i = (a+bi)^2 = a^2 - b^2 + 2abi.$$

Equate the real and the unreal parts separately.
Hence $a^2 - b^2 = 17$, $2ab = 30$. We find $a = 5{\cdot}073$, $b = 2{\cdot}96$,

so that $$\sqrt{17+30i} = 5{\cdot}073 + 2{\cdot}96i.$$

Ex. 3. Show that $\sqrt{i} = (1+i)/\sqrt{2}$ or $\cdot 707 + \cdot 707i$.

Ex. 4. Show that $1 \div \sqrt{i} = (1-i)/\sqrt{2}$ or $\cdot 707 - \cdot 707i$.

Ex. 5. Find the value of $\sqrt{\frac{5+3i}{2-5i}}$. Let it be $a+bi$, and find a and b.

Thus, squaring, $\frac{5+3i}{2-5i}=(a+bi)^2=a^2-b^2+2abi$,

$$5+3i=2(a^2-b^2)+10ab+\{4ab-5(a^2-b^2)\}i.$$

Hence, equating real and unreal parts,

$$2(a^2-b^2)+10ab=5,$$
$$4ab-5(a^2-b^2)=3.$$

From these we find $a=0{\cdot}6747$, $b=0{\cdot}7927$, so that the answer is

$$0{\cdot}6747+0{\cdot}7927i.$$

Ex. 6. Extract the square root of $5+4i$. *Ans.* $2{\cdot}389+0{\cdot}8365i$.

Ex. 7. Extract the cube root of $-2{\cdot}35+1{\cdot}96i$.

Ans. $0{\cdot}9959+1{\cdot}0567i$.

Exercises like the above are much more easily worked in the following way.

145. The following way is easy, but only to a student who can readily find the sine, cosine, and tangent of any angle. Let him refer again to Art. 34 for the definition of these.

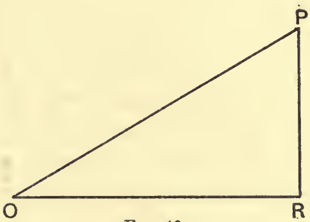

Fig. 46.

In Fig. 46, if OP is 1 inch, the length of PR in inches is the sine of the angle POR, and the length of OR is the cosine of the angle. Let PR be called y and OR be called x, then y is the sine and x is the cosine of the angle POR, and $y \div x$ is the tangent of the angle.

To find the sine, etc., of any angle.

Draw XOK and YOS (Fig. 47), both produced, at right angles to one another.

Let the angle be XOP' or XOP'' or XOP''' or XOP''''.

The angle is in every case measured from OX in the anti-clockwise direction. XOP' is said to be in the first quadrant; it is between 0 and 90°. XOP'' is said to be in the second quadrant; it is between 90° and 180°. XOP''' is said to be in the third quadrant; it is between 180° and 270°. XOP'''' is in the fourth quadrant; it is between 270° and 360°.

In every case make OP' or OP'' or OP''' or OP'''' one inch long. The vertical distance of P from XOK is called y or the sine of the angle; it is positive if P is *above* XOK, and it is negative if P is *below* XOK. The horizontal distance of P from YOS is called x or the cosine of the angle; it is positive if P is to the right of YOS, and it is negative if P is to the left of YOS, and $y \div x$ is the tangent of the angle.

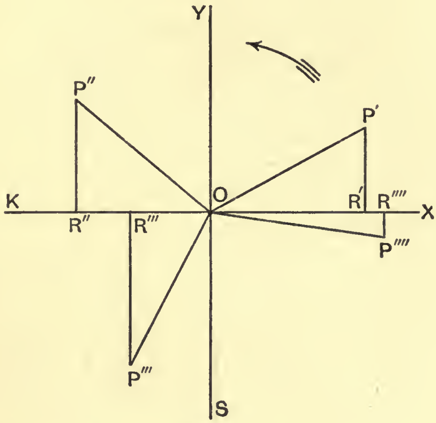

Fig. 47.

The student ought to practise finding the sine, cosine, and tangent of angles whose magnitudes are between 0° and 360°. First he finds by making a sketch in which of the four quadrants the angle is.

It helps greatly to remember that if A is an angle greater than 90° and less than 180°, so that it is in the second quadrant,

$$\sin A = \sin (180° - A),$$
$$\cos A = -\cos (180° - A),$$
$$\tan A = -\tan (180° - A).$$

Thus
$$\sin 160° = \sin 20° = 0{\cdot}3420,$$
$$\cos 160° = -\cos 20° = -0{\cdot}9397,$$
$$\tan 160° = -\tan 20° = -0{\cdot}3640.$$

Again, in the third quadrant,

$$\sin A = -\sin (A - 180°),$$
$$\cos A = -\cos (A - 180°),$$
$$\tan A = \tan (A - 180°).$$

Thus
$$\sin 210° = -\sin 30° = -0{\cdot}5000,$$
$$\cos 210° = -\cos 30° = -0{\cdot}8660,$$
$$\tan 210° = \tan 30° = 0{\cdot}5774.$$

Again, in the fourth quadrant,

$$\sin A = -\sin(360° - A),$$
$$\cos A = \cos(360° - A),$$
$$\tan A = -\tan(360° - A).$$

Thus $$\sin 320° = -\sin 40° = -0{\cdot}6428,$$
$$\cos 320° = \cos 40° = 0{\cdot}7660,$$
$$\tan 320° = -\tan 40° = -0{\cdot}8391.$$

Again, for negative angles we must remember that

$$\sin(-A) = -\sin A,$$
$$\cos(-A) = \cos A,$$
$$\tan(-A) = -\tan A.$$

Let us tabulate all the above answers:

Angle.	Sine.	Cosine.	Tangent.
20	0·3420	0·9397	0·3640
160	0·3420	−0·9397	−0·3640
210	−0·5000	−0·8660	0·5774
320	−0·6428	0·7660	−0·8391
−20	−0·3420	0·9397	−0·3640
−160	−0·3420	−0·9397	0·3640
−210	0·5000	−0·8660	−0·5774
−320	0·6428	0·7660	0·8391

I am afraid that there is no help for it; if a student is to do the following exercises with accuracy he must practise finding the sines, cosines, and tangents of angles so much that he is perfectly certain as to the + or − sign and the number in each case. Inaccuracy as to + or − is fatal in this work.

146. Demoivre's Theorem. It is easily proved in trigonometry (see also Exs. 6 and 7, Art. 142) that the n^{th} power of

$$r(\cos\phi + i\sin\phi) \quad \text{is} \quad r^n(\cos n\phi + i\sin n\phi),$$

where r is always positive, ϕ is any angle, and n any number, positive or negative. Of course ϕ is usually expressed in radians, but I often find it convenient to write it in degrees. To save time and for other good reasons, I often write $r(\phi)$ or $r[\phi]$, and indeed sometimes r_ϕ, instead of $r(\cos\phi + i\sin\phi)$.

Again,

$$r(\cos\phi + i\sin\phi) \times r_1(\cos\phi_1 + i\sin\phi_1) = rr_1\{\cos(\phi + \phi_1) + i\sin(\phi + \phi_1)\}$$

or $$r[\phi] \times r_1[\phi_1] = rr_1[\phi + \phi_1],$$

$$r(\cos\phi + i\sin\phi) \div r_1(\cos\phi_1 + i\sin\phi_1) = \frac{r}{r_1}\{\cos(\phi - \phi_1) + i\sin(\phi - \phi_1)\}$$

or $$r[\phi] \div r_1[\phi_1] = \frac{r}{r_1}[\phi - \phi_1].$$

Again, it is known that we get consistent answers if we use $e^{i\phi}$ instead of $\cos\phi + i\sin\phi$ in all calculations. Again, $e^{-i\phi}$ is the same as $\cos\phi - i\sin\phi$.

In fact $r(\cos\phi + i\sin\phi)$ is, algebraically, the same as

$$re^{i\phi} \quad \text{or} \quad a + bi \quad \text{if} \quad r\cos\phi = a \quad \text{and} \quad r\sin\phi = b.$$

EXERCISES.

1. Express $1{\cdot}452[46^\circ{\cdot}7]$ in the form $a + bi$;

that is, $$1{\cdot}452(\cos 46^\circ{\cdot}7 + i\sin 46^\circ{\cdot}7)$$

or $$1{\cdot}452(0{\cdot}6858 + 0{\cdot}7278i) \quad \text{or} \quad 0{\cdot}9958 + 1{\cdot}0567i.$$

2. Express $5 + 4i$ in the form $r[\phi]$ or $r(\cos\phi + i\sin\phi)$.
Here $r\cos\phi = 5$, $r\sin\phi = 4$. By division, therefore, $\tan\phi = 0{\cdot}8$, and from the tables $\phi = 38^\circ{\cdot}67$. Also $r^2 = 5^2 + 4^2$ or $r = 6{\cdot}403$.
The answer is then $6{\cdot}403[38^\circ{\cdot}67]$.

3. Extract the square root of $0{\cdot}9958 + 1{\cdot}0567i$. It is easy to show that the given expression is $1{\cdot}452[46^\circ{\cdot}7]$, and the square root of this is $1{\cdot}452^{\frac{1}{2}}[46^\circ{\cdot}7 \div 2]$ or $1{\cdot}205[23^\circ{\cdot}35]$, and it is easy to put this in the form $1{\cdot}107 + 0{\cdot}4776i$.

4. Raise $0{\cdot}9958 + 1{\cdot}0567i$ to the third power. As in the last case, convert the given expression to $1{\cdot}452[46^\circ{\cdot}7]$, and the answer is
$$1{\cdot}452^3[46^\circ{\cdot}7 \times 3] \quad \text{or} \quad 3{\cdot}062[140^\circ] \quad \text{or} \quad -2{\cdot}35 + 1{\cdot}96i.$$

5. Find $\sqrt{17 + 30i}$. Here $17 + 30i$ may be converted into $34{\cdot}48[60^\circ{\cdot}45]$, and its square root is $5{\cdot}872[30^\circ{\cdot}23]$, which may be converted into
$$5{\cdot}073 + 2{\cdot}96i.$$

6. Find $\sqrt{i}$. Now i is $1(\cos 90^\circ + i\sin 90^\circ)$ or $1[90^\circ]$, and its square root is $1[45^\circ]$ or $\cos 45^\circ + i\sin 45^\circ$ or $0{\cdot}707 + 0{\cdot}707i$.

7. Find $1 \div \sqrt{i}$. This is
$$i^{-\frac{1}{2}} \quad \text{or} \quad 1^{-\frac{1}{2}}[90^\circ \times (-\tfrac{1}{2})] \quad \text{or} \quad 1[-45^\circ] \quad \text{or} \quad 0{\cdot}707 - 0{\cdot}707i.$$

8. Find the value of $\sqrt{\dfrac{5+3i}{2-5i}}$. Converting separately the numerator and denominator of the given fraction,
$$\frac{5+3i}{2-5i} = \frac{5{\cdot}832[30^\circ{\cdot}96]}{5{\cdot}386[-68^\circ{\cdot}18]} = 1{\cdot}083[99^\circ{\cdot}14],$$
and the square root of this is $1{\cdot}041[49^\circ{\cdot}57]$ or $0{\cdot}6747 + 0{\cdot}7927i$.

9. Extract the square root of $5+4i$. This can be expressed as $6{\cdot}403[38^\circ{\cdot}67]$, and its square root is $2{\cdot}53[19^\circ{\cdot}34]$ or $2{\cdot}389+0{\cdot}8365i$.

10. Extract the cube root of $-2{\cdot}35+1{\cdot}96i$. This may be put in the form $3{\cdot}06[140^\circ]$; its cube root is $1{\cdot}452[46^\circ{\cdot}7]$, and this is converted into $0{\cdot}9958+1{\cdot}0567i$.

11. Expand $e^{i\phi}$ according to the rule of Ex. 5, Art. 28. Now expand $\cos\phi$ and $\sin\phi$ according to the rules of Ex. 6, Art. 28. Show that

$$e^{i\phi}=\cos\phi+i\sin\phi=[\phi].$$

In the same way show that $e^{-i\phi}=\cos\phi-i\sin\phi=[-\phi]$.

147. As in Art. 119, when $r[\phi]$ **operates** upon $a\sin qt$ (that is, when $a\sin qt$ is multiplied by $r[\phi]$), it converts it to

$$ar\sin(qt+\phi).$$

Ex. 1. Operate with $5+4i$ upon $10\sin qt$. Here we convert $5+4i$ into $6{\cdot}403[38^\circ{\cdot}67]$, so that the answer is

$$64{\cdot}03\sin(qt+38^\circ{\cdot}67).$$

The student ought to see that this agrees with the rule of Art. 119.

Ex. 2. The voltage applied at the sending end of a long telephone line being $v_0\sin qt$, the current entering the line is

$$\sqrt{\frac{s+kqi}{r+lqi}}\,v_0\sin qt,$$

where, per unit length of line, r is resistance, l is inductance, s is leakance, and k is permittance or capacity.

If, per mile, $r=6$ ohms, $l=0{\cdot}003$ henry, $k=5\times10^{-9}$ farad (or as it is usually stated $0{\cdot}005$ microfarad), $s=3\times10^{-6}$ mho; if $q=6000$, find the entering current.

Ans. The operator is $10^{-3}\sqrt{\dfrac{3+30i}{6+18i}}$, and is found to be

$$0{\cdot}00126[6^\circ{\cdot}35],$$

so that the entering current is $0{\cdot}00126v_0\sin(qt+6^\circ{\cdot}35)$.

148. In using θ or $\dfrac{d}{dt}$ or its equivalent qi, as if it were an ordinary algebraic quantity, it is to be observed that it is an operator upon a quantity which is a function of the time like $a\sin qt$. I have nowhere operated upon the product of two such functions.

Again, I have said that $a\sin(qt+e)$ may be represented by a vector when we add such functions, but I have been careful not to speak of the product of two such functions as the product of two vectors. The fact is, the name "product of two vectors" may get all sorts of meanings. Two of these are very useful in Vector Algebra, and neither of them has anything to do with the product of two algebraic expressions like $a \sin(qt+e)$. (See Chap. XXXVII.)

Operators like $a+bi$ and $\alpha+\beta i$ may be added like vectors, and the sum is $a+\alpha+(b+\beta)i$, or they may be multiplied algebraically, and their product is truly $a\alpha - b\beta + (a\beta + \alpha b)i$, but this product of operators need not be called a vector product. We have already two very important things that we call vector products; it is not wise to have a third and very different thing with the same name. But there is no harm in speaking of the product or quotient of *operators*.

149. $\cosh x$ means $\frac{1}{2}(e^x + e^{-x})$.
$\sinh x$ means $\frac{1}{2}(e^x - e^{-x})$.
$\tanh x$ means $\sinh x \div \cosh x$.*

These are called **hyperbolic** functions, because they originated in a study of the hyperbola just as $\sin x$, $\cos x$, and $\tan x$ are called circular functions, because they were first derived from the study of the circle.

There are some cases in which it is a pity that men know the history of a subject or the origin of a word; they get to think that this kind of knowledge is the only essential knowledge. I am sometimes thankful for the oblivion that hides from us the wonderful romance that must have been once attached to the invention of every common idea or every common word that we use.

* The student does not need to know the following formulæ for doing the exercises in this book; but he will need them if he pursues these subjects, and they are very easy to prove from the above definitions.

$$\sinh(a+b) = \sinh a \,.\, \cosh b + \cosh a \,.\, \sinh b.$$
$$\cosh(a+b) = \cosh a \,.\, \cosh b + \sinh a \,.\, \sinh b.$$
$$\sinh(a-b) = \sinh a \,.\, \cosh b - \cosh a \,.\, \sinh b.$$
$$\cosh(a-b) = \cosh a \,.\, \cosh b - \sinh a \,.\, \sinh b.$$
$$\sinh 2a = 2 \sinh a \,.\, \cosh a.$$
$$\cosh 2a = \cosh^2 a + \sinh^2 a = 2\cosh^2 a - 1 = 2\sinh^2 a + 1.$$
$$\cosh^2 a - \sinh^2 a = 1.$$

I wish we could drop such terms as *hyperbolic* functions; they have served their purpose and now do harm, for teachers will insist on deriving their properties from the hyperbola, and this is foolishness; the definitions are perfectly easy to deal with as I have given them.

EXERCISES.

1. Show that the differential coefficient of $\cosh ax = a \sinh ax$.

2. Show that the differential coefficient of $\sinh ax = a \cosh ax$.

3. Show that

$$\cosh 1{\cdot}013 = 1{\cdot}5562, \quad \sinh 1{\cdot}013 = 1{\cdot}1989, \quad \tanh 1{\cdot}013 = 0{\cdot}7704.$$

4. What are the values of $\cosh a$, $\sinh a$, and $\tanh a$ when a is small?

As $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \text{etc.}$ and $e^{-a} = 1 - a + \frac{a^2}{2} - \frac{a^3}{6} + \text{etc.}$,

$\cosh a = \frac{1}{2}(2 + a^2 + \text{etc.}) = 1 + \frac{1}{2}a^2 + \text{etc.}$, $\sinh a = \frac{1}{2}(2a + \frac{1}{3}a^2) = a + \frac{1}{6}a^3 + \text{etc.}$

Calculate the following values to four significant figures:

a	$\cosh a$	$\sinh a$	$\tanh a$
0	1	0	0
0·01	1	0·01	0·01
0·10	1·005	0·1002	0·09967

5. Show that we may take $\cosh a = \sinh a = \frac{1}{2}e^a$ and $\tanh a = 1$ if we are only using four significant figures, and if a is greater than 7.

6. If x is $a + bi$, find e^x. This is $e^{a+bi} = e^a e^{bi} = e^a[b]$. Express this in the following cases, stating b in degrees and also converting the answer into the $\alpha + \beta i$ form:

x	e^x
$0{\cdot}01 + 0{\cdot}01i$	$1{\cdot}01[0°{\cdot}573] = 1{\cdot}0099 + 0{\cdot}0101i$
$0{\cdot}1 + 0{\cdot}1i$	$1{\cdot}105[5°{\cdot}73] = 1{\cdot}0995 + 0{\cdot}11017i$
$0{\cdot}3 + 0{\cdot}3i$	$1{\cdot}350[17°{\cdot}19] = 1{\cdot}2898 + 0{\cdot}3989i$
$0{\cdot}6 + 0{\cdot}6i$	$1{\cdot}823[34°{\cdot}38] = 1{\cdot}5045 + 1{\cdot}0285i$
$1 + i$	$2{\cdot}7183[57°{\cdot}30] = 1{\cdot}4684 + 2{\cdot}2874i$
$1{\cdot}5 + 1{\cdot}5i$	$4{\cdot}482[85°{\cdot}95] = 0{\cdot}3169 + 4{\cdot}4712i$
$2 + 2i$	$7{\cdot}389[114°{\cdot}60] = -3{\cdot}0753 + 6{\cdot}7188i$
$3 + 3i$	$20{\cdot}086[171°{\cdot}90] = -19{\cdot}887 + 2{\cdot}8300i$
$5 + 5i$	$148{\cdot}41[286°{\cdot}50] = 42{\cdot}14 - 142{\cdot}28i$

7. Find e^{-x} in the following cases. If $x=a+bi$, $e^{-x}=e^{-a}e^{-bi}=e^{-a}[-b]$:

x	e^{-x}
$0{\cdot}01+0{\cdot}01i$	$0{\cdot}9901[-0^\circ{\cdot}573]=0{\cdot}9900-0{\cdot}0099i$
$0{\cdot}1+0{\cdot}1i$	$0{\cdot}9053[-5^\circ{\cdot}73]=0{\cdot}9008-0{\cdot}09026i$
$0{\cdot}3+0{\cdot}3i$	$0{\cdot}7407[-17^\circ{\cdot}19]=0{\cdot}7077-0{\cdot}2189i$
$0{\cdot}6+0{\cdot}6i$	$0{\cdot}5486[-34^\circ{\cdot}38]=0{\cdot}45275-0{\cdot}30979i$
$1+i$	$0{\cdot}3679[-57^\circ{\cdot}30]=0{\cdot}19874-0{\cdot}3096i$
$1{\cdot}5+1{\cdot}5i$	$0{\cdot}2231[-85^\circ{\cdot}94]=0{\cdot}01577-0{\cdot}22256i$
$2+2i$	$0{\cdot}1353[-114^\circ{\cdot}60]=-0{\cdot}05631-0{\cdot}12303i$
$3+3i$	$0{\cdot}0498[-171^\circ{\cdot}90]=-0{\cdot}04931-0{\cdot}07016i$
$5+5i$	$0{\cdot}00674[-286^\circ{\cdot}50]=0{\cdot}001914+0{\cdot}006462i$

8. Combining the answers to (6) and (7), find $\cosh x$ and $\sinh x$ for the following values of x:

x	$\cosh x$	$\sinh x$
$0{\cdot}01+0{\cdot}01i$	$1+0{\cdot}0001i=1{\cdot}0(0^\circ)$	$0{\cdot}01+0{\cdot}01i=0{\cdot}01414(45^\circ)$
$0{\cdot}1+0{\cdot}1i$	$1{\cdot}00015+0{\cdot}009955i=1{\cdot}0(0^\circ{\cdot}56)$	$0{\cdot}09935+0{\cdot}1002i=0{\cdot}1411(45^\circ{\cdot}2)$
$0{\cdot}3+0{\cdot}3i$	$0{\cdot}9988+0{\cdot}09i=1{\cdot}03(5^\circ{\cdot}2)$	$0{\cdot}2911+0{\cdot}3089i=0{\cdot}4244(46^\circ{\cdot}7)$
$0{\cdot}6+0{\cdot}6i$	$0{\cdot}9786+0{\cdot}35935i=1{\cdot}042(20^\circ{\cdot}2)$	$0{\cdot}5259+0{\cdot}6692i=0{\cdot}8511(51^\circ{\cdot}8)$
$1+i$	$0{\cdot}8336+0{\cdot}9889i=1{\cdot}293(49^\circ{\cdot}9)$	$0{\cdot}6349+1{\cdot}2985i=1{\cdot}445(63^\circ{\cdot}9)$
$1{\cdot}5+1{\cdot}5i$	$0{\cdot}16634+2{\cdot}1243i=2{\cdot}13(85^\circ{\cdot}5)$	$0{\cdot}15057+2{\cdot}3469i=2{\cdot}352(86^\circ{\cdot}3)$
$2+2i$	$-1{\cdot}5658+3{\cdot}2979i=3{\cdot}65(115^\circ{\cdot}4)$	$-1{\cdot}5095+3{\cdot}4209i=3{\cdot}74(113^\circ{\cdot}8)$
$3+3i$	$-9{\cdot}968+1{\cdot}3799i=10{\cdot}06(172^\circ{\cdot}1)$	$-9{\cdot}919+1{\cdot}4501i=10{\cdot}02(171^\circ{\cdot}1)$
$5+5i$	$21{\cdot}071-71{\cdot}137i=74{\cdot}19(286^\circ{\cdot}5)$	$21{\cdot}069-71{\cdot}143i=74{\cdot}19(286^\circ{\cdot}5)$

Tables of $\cosh a$ and $\sinh a$ are easily procurable. The student who wishes to calculate $\cosh x$ or $\sinh x$, where $x=a+bi$, may find the following formulæ convenient. Let him first prove them to be correct:

$$\cosh(a+bi)=\cosh a \, . \cos b+i\sinh a \, . \sin b,$$

$$\sinh(a+bi)=\sinh a \, . \cos b+i\cosh a \, . \sin b.$$

Prof. Kennelly has published a very complete table of $\cosh x$, $\sinh x$, and $\tanh x$, where $x=r[45^\circ]$ or $a+ai$.

9. The following values of r, l, s, and k are values per mile for certain known submarine and telephone lines. It is known that in telephone lines if a pure musical note of a frequency or pitch of about 800 per second is transmitted well, ordinary speech will also be transmitted well. I therefore take $q=5000$ in telephone lines, q being 2π times the frequency. Also in submarine telegraphy if a simple harmonic current of frequency about 9 is transmitted well, ordinary signalling at the rate of 160 letters per minute will be effectively written by the ordinary recorder. Therefore, for submarine cable work I use $q=60$.

Telephone lines differ greatly, and therefore the results of experiments are usually given in terms of what is called the "standard telephone cable," about thirty miles of which is about the limiting length along which speech may be transmitted so as to be heard articulately when the ordinary transmitters and receivers are used.

Calculate $n = h + qi = \sqrt{(r+qli)(s+qki)}$.

Calculate $z_0 = \sqrt{(r+qli) \div (s+qki)}$.

I often, instead of using z_0, call it r/n, really meaning $\frac{r+lqi}{n}$. The answers are given in the following table:

	q	r ohms.	l henries.	k farads.	s mhos.	$n=h+gi$	z_0 or $\frac{r}{n}$
Standard cable	5000	88	0	$0·05 \times 10^{-6}$	0	$0·105+0·105i$ $=0·1483[45°]$	$419·1-419·1i$ $=593[-45°]$
Open copper wire, telephone line	5000	18	0·0039	$0·008 \times 10^{-6}$	10^{-6}	$0·0122+0·0302i$ $=0·0326[67°·90]$	$761·9-287·9i$ $=815[-20°·67]$
Open copper wire, telephone line	5000	2·97	0·0033	$0·0096 \times 10^{-6}$	10^{-6}	$0·00281+0·0282i$ $=0·0284[84°·32]$	$589-46·38i$ $=591[-4°·50]$
Telephone cable	5000	12	0·0010	$0·0714 \times 10^{-6}$	5×10^{-6}	$0·0382+0·0564i$ $=0·0681[55°·9]$	$159·6-104·8i$ $=191[-33°·28]$
Submarine telegraph	60	2·88	0	$0·4095 \times 10^{-6}$	0	$0·005948+0·005948i$ $=0·00841[45°]$	$241·8-241·8i$ $=342[-45°]$

These lines in this order will later be called lines A, B, C, D and E.

Compare these answers for h and g with those of Ex. 24, Art. 28.

10. In this table I give distances L miles of each of the above lines. Calculate cosh Ln, sinh Ln, and tan Ln in each case:

Line.	L miles.	cosh Ln	sinh Ln	tanh Ln
A	30	11·685[180°·5]	11·645[180°·5]	0·9965[0°]
,,	5	1·025[15°·6]	0·7437[50°·3]	0·7256[34°·7]
B	150	3·041[259°]	3·19[259°·5]	1·049[0°·5]
,,	25	0·7911[15°·6]	0·7526[72°·5]	0·9513[56°·9]
C	200	0·995[339°·1]	0·8421[304°·2]	0·8457[325°·1]
,,	33	0·605[7°·3]	0·8077[85°·9]	1·335[78°·6]
D	60	4·993[193°·6]	4·904[194°·1]	0·9822[0°·5]
,,	10	0·9325[13°·1]	0·6642[59°·9]	0·7123[46°·8]
E	2432	952820[828°]	952820[828°]	1·0000[0]
,,	392	5·154[134°]	5·159[133°]	1·0008[1°·07]

11. For the standard cable A above, find the following answers ($q=5000$):

L	sinh Ln	cosh Ln	tanh Ln
1	0·1483[45°·4]	0·9998[0°·6]	0·1484[44°·8]
5	0·7437[50°·3]	1·025[15°·6]	0·726[34°·7]
10	1·525[65°·9]	1·348[53°·7]	1·131[12°·2]
20	4·097[119°·4]	4·034[120°·9]	1·016[−1°·5]
30	11·65[180°·5]	11·69[180°·5]	0·9965[0°·0]

12. For the standard cable A above, find the following answers:

q	$\frac{r}{n}$	n	cosh $40n$	cosh $60n$	cosh $80n$
3000	541·5$(1-i)$	0·08122$(1+i)$	12·875[186°·15]	65·5[280°·23]	332[372°]
5000	419·5$(1-i)$	0·10486$(1+i)$	33·156[240°·33]	270[360°·51]	2194[480°·45]
7000	354·5$(1-i)$	0·12408$(1+i)$	71·55[284°·40]	855[426°·77]	10246[568°·71]
10000	296·6$(1-i)$	0·1483$(1+i)$	188·5[339°·9]	3675[509°·9]	71130[679°·97]

These results are important when comparing the telephonic currents received at the distances of 40, 60, and 80 miles.

CHAPTER XXXIV.

FUNDAMENTAL EQUATIONS.

150. There are certain fundamental equations concerning the conduction of heat, the flow of electricity in telephone or submarine telegraph conductors, the flow of water or other fluid, the transmission of electric or electromagnetic conditions, the diffusion of fluids, etc., which all take the same mathematical shape. It results that when a man has worked a problem in any of these sciences, he has worked an analogous problem in all the other sciences. In my *Calculus* I have described some of the methods by which problems are solved; I shall here find the fundamental equation when heat is flowing in one direction.

151. Conduction of Heat. If material supposed to be uniform has a plane face AB (Fig. 48). If at the point P, which is at the distance x from AB, the temperature is v, and we imagine the temperature the same at all points in the same plane as P parallel to AB (that is, we are only considering flow of heat at right angles to the plane AB), and if $\frac{dv}{dx}$ is the gradient of *rise* of temperature per cm. at P, then $-k\,dv/dx$ is the amount of heat flowing per second per square cm. of area like PQ in the direction of increasing x. This is really a definition of what we mean by k, the conductivity of the material. I shall imagine k to be constant. k is the heat which flows per second per square cm. when the temperature gradient is 1.

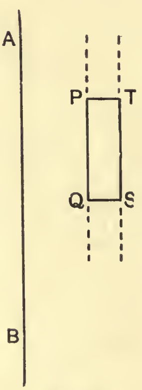

FIG. 48.

Let us imagine PQ exactly a square cm. in area, and PT or QS is δx. The heat current per second, $c = -k\frac{dv}{dx}$, flows into the block $PQTS$ through the face PQ; how much flows out at the face TS? Call it $c+\delta c$ or $c+\delta x\frac{dc}{dx}$. Then the amount of heat in the block diminishes by the amount $\delta x\frac{dc}{dx}$ per second, or the increase per second is

$$\delta x \,.\, k\frac{d^2v}{dx^2}.$$

Now the weight of the block is $1\times\delta x\times w$, if w is the weight per cubic cm., and if s is the specific heat of the material or the heat required to raise unit weight one degree in temperature, then if t is the time in seconds,

$$\delta x \,.\, w \,.\, s\frac{dv}{dt}$$

also measures the rate per second at which the heat in the block increases. Hence

$$\delta x \,.\, k\frac{d^2v}{dx^2}=\delta x \,.\, w \,.\, s\frac{dv}{dt}$$

or

$$\frac{d^2v}{dx^2}=\frac{ws}{k}\cdot\frac{dv}{dt}. \qquad \text{(1)}$$

We may call this $\frac{d^2v}{dx^2}=n^2v$, using n^2 for $\frac{ws}{k}\cdot\frac{d}{dt}$, the fundamental equation of which I spoke.

152. Telephone or Telegraph Cable. Imagine a metallic conductor insulated from an infinitely perfect return conductor (of no resistance) which we call earth, and which is everywhere at potential 0; at a place A (Fig. 49), which is at the distance x from the sending end, the potential is v and the current is c; at the place

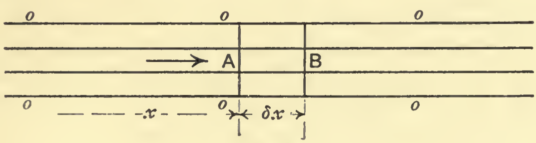

Fig. 49.

B, which is at the distance $x+\delta x$ from the sending end, the potential is $v+\delta v$ and the current is $c+\delta c$. I assume c to be in the direction of the arrow. The current at A is

$$c=-\frac{1}{r}\frac{dv}{dx},$$

where r is the resistance per unit length of conductor. The student must not proceed till he sees clearly this simple thing, that the current in any conductor is the voltage gradient divided by r. The current at B is $c+\delta c$ or $c+\delta x\dfrac{dc}{\delta x}$, so that $\delta c=-\delta x\dfrac{1}{r}\dfrac{d^2v}{dx^2}$.

But if there is leakance s mhos per unit length of conductor, there is a current leaking away sideways whose amount is $v.s.\delta x$. Hence the rate at which the quantity of electricity in the space between A and B is increasing per second is

$$\delta x\frac{1}{r}\frac{d^2v}{dx^2}-v.s.\delta x.$$

Now AB is one coating of a condenser whose capacity is $k.\delta x$, if k is the capacity per unit length; the charge or quantity of electricity in the space AB is $k.\delta x.v$, and it is increasing at the rate $k.\delta x\dfrac{dv}{dt}$, if t is time. We have two expressions for the same thing, and so

$$\delta x\frac{1}{r}\frac{d^2v}{dx^2}-v.s.\delta x=k.\delta x\frac{dv}{dt}$$

or
$$\frac{d^2v}{dx^2}=r(s+k\theta)v,$$

if we use θ as meaning $\dfrac{d}{dt}$.

But if the conductor has self-induction l per unit length (see Art. 128), we use $r+l\theta$ instead of r, and so we have the fundamental equation

$$\frac{d^2v}{dx^2}=(r+l\theta)(s+k\theta)v. \quad\text{.........................(1)}$$

We may call this
$$\frac{d^2v}{dx^2}=n^2v.$$

Notice that this is the same equation as we had for heat, only that in the case of heat we had no l, and as there was no side leak, we had no s. In studying heat conduction along a bar, we should give s a value, as heat would leave the rod sidewise.

In the following work v is voltage in the electrical problem and c is electrical current But v may mean temperature in the heat problem, and c will then mean the current of heat per sq. cm. per second. In any problem the student must not be worried about n being a function of θ.

When dealing with functions like $\sin qt$, he is already familiar with such a quantity as n, because θ is then merely qi.

CHAPTER XXXV.

TELEPHONE AND TELEGRAPH PROBLEMS.

153. If in (1) of Arts. 142 or 143, n were a real quantity, the solution would be

$$v = Pe^{nx} + Qe^{-nx}, \quad \text{..........................(1)}$$

where P and Q are arbitrary constants. But as n involves $\frac{d}{dt}$, instead of taking P and Q as mere constants, they ought to be regarded as functions of the time. In cases of x reaching to infinity P must be 0, and we have for an **infinitely long line**

$$v = e^{-nx}Q, \quad \text{................................(2)}$$

where $x = 0$, $v = Q$, so that Q is a function of the time expressing how v varies at $x = 0$, and it is evident that, to find the value of v for any value of x, we have to perform the operation indicated by e^{-nx} upon Q. In the cases which I shall consider, Q and P will be of the type $\sin qt$, and the exercises already given show how easily practical problems may be worked. Remember that in differentiating with regard to x, Q is a constant.

When x is **limited**, P in (1) has a value, and it is found more convenient to use what is equivalent to (1),

$$v = A \cosh nx + B \sinh nx. \quad \text{.....................(3)*}$$

As $$c = -\frac{1}{r}\frac{dv}{dx} \text{ in the electrical case,}$$

$$c = -\frac{n}{r}(A \sinh nx + B \cosh nx), \quad \text{...... ......(4)}$$

where r stands for $r + l\theta$ or $r + lqi$. Prof. Kennelly employs z_0 for $\frac{r + lqi}{n}$.

(3) and (4) enable us to work all sorts of problems.

* Cosh nx is an operator upon A which is a function of the time, so it might be more logical to write $\cosh nx \times A$, but there is less risk of algebraical mistakes if we write it $A \cosh nx$.

154. Suppose at the sending end of a telephone line $v=v_0 \sin qt$, and the line is **infinite** in length. I shall choose the formula (2) to work with. Here $c=\frac{n}{r}Qe^{-nx}$, where r stands for $r+lqi$. We can evaluate n when we know r, l, s, and k. Q is evidently $v_0 \sin qt$, but I shall consider $\sin qt$ as being everywhere understood, and I need not write it. I shall say that Q is v_0. In fact, everywhere

$$v=v_0 e^{-nx}$$

operating upon $\sin qt$, and $n=\sqrt{(r+lqi)(s+kqi)}$.

When r, l, s, k, and q are given numerically* it is easy to find n in the shape $h+gi$. Now $e^{-(h+gi)x}=e^{-hx}[-gx]$.

If at the sending end v (or the value of v when x is 0) is $v_0 \sin qt$, the value of v anywhere else is

$$v=v_0 e^{-hx}\sin(qt-gx).$$

That is, v attenuates because of the multiplier e^{-hx} and it lags by the amount gx. When the lag amounts to 2π we may, if we like, say that this gives a complete wave. If λ is the length of the wave,

$$g\lambda=2\pi \quad \text{or} \quad \lambda=2\pi/g.$$

It is to be observed that c attenuates and increases its lag in exactly the same way as v. In fact $c=\frac{nv}{r+lqi}$ or v/z_0.

Ex. 1. In what is called the Standard telephone cable, $r=88$ ohms per mile, $l=0$, $s=0$, $k=0{\cdot}05\times10^{-6}$ farad [this is usually called 0·05 microfarad, a microfarad being the millionth of a farad]. Find n either from the formula (1) or from (2). Also find $(r+lqi)/n$,

*I hope that the student does not mind doing algebraically what he has hitherto done, in the exercises of Chap. XXXIII., upon numbers. The lazy student will take my answer (2) here to be correct.

$$n=h+gi=\sqrt{(r+lqi)(s+kqi)}. \quad\ldots\ldots(1)$$

$$h^2-g^2+2hgi=rs-lkq^2+i(rkq+slq).$$

Then $$h^2-g^2=rs-lkq^2$$
and $$2hg=q(rk+sl).$$

Solving these equations for h and g, we find that

$$\sqrt{\frac{kqr}{2}}\sqrt{\sqrt{\left(1+\frac{q^2l^2}{r^2}\right)\left(1+\frac{s^2}{k^2q^2}\right)}\mp\left(\frac{ql}{r}-\frac{s}{qk}\right)} \quad\ldots\ldots(2)$$

is the value of h if the minus sign is taken, and it is the value of g if the plus sign is taken.

It is easy to show that when $\frac{q^2l^2}{r^2}$ is large compared with 1 and if s is 0, it is very nearly true that

$$h=\frac{r}{2}\sqrt{\frac{k}{l}} \text{ and } g=q\sqrt{kl}. \quad\ldots\ldots(3)$$

This is a good example of the fact that when a is small, $\sqrt{1+a}=1+\frac{1}{2}a$.

which I usually call r/n or z_0. Give three answers for the four values of q, 3000, 5000, 7000, 10000. Using formula (1), $n=\sqrt{rkqi}$.

q	$n=h+qi$	$\frac{r}{n}$ or z_0
3000	$0{\cdot}08124+0{\cdot}08122i=0{\cdot}1149[45°]$	$541{\cdot}5(1-i)=766[-45°]$
5000	$0{\cdot}10486+0{\cdot}10486i=0{\cdot}1483[45°]$	$419{\cdot}5(1-i)=593[-45°]$
7000	$0{\cdot}12408+0{\cdot}12408i=0{\cdot}1754[45°]$	$354{\cdot}5(1-i)=501[-45°]$
10000	$0{\cdot}1483\ +0{\cdot}1483i\ =0{\cdot}2097[45°]$	$296{\cdot}6(1-i)=419[-45°]$

Ex. 2. If $v_0=1$, find v and c at the end of 30 miles for each of the above values of q. Taking the case of $q=5000$, $h=g=0{\cdot}10486$, and as v anywhere is $v=e^{-0{\cdot}10486x}\sin(qt-0{\cdot}10486x)$, we have the answer given in the table. For the other values of q we have similar expressions. I give the lags in degrees instead of radians.

The current c anywhere is $v\times\frac{n}{r}$ or v/z_0.

q	v	c
3000	$0{\cdot}08744\sin(qt-139°{\cdot}6)$	$0{\cdot}0001142\sin(qt-94°{\cdot}6)$
5000	$0{\cdot}04303\sin(qt-180°{\cdot}2)$	$0{\cdot}00007253\sin(qt-135°{\cdot}5)$
7000	$0{\cdot}02418\sin(qt-213°{\cdot}27)$	$0{\cdot}0000482\sin(qt-168°{\cdot}3)$
10000	$0{\cdot}01169\sin(qt-255°{\cdot}0)$	$0{\cdot}000028\sin(qt-210°)$

Note that c leads v in every case by 45°. Note the greater attenuation and lag of the higher frequencies.

Ex. 3. If $v_0=1$, find v at the following distances x miles from the sending end when $q=5000$. The answers are:

$$e^{-0{\cdot}10486x}\sin(qt-0{\cdot}10486x).$$

x	v
1	$0{\cdot}9004\sin(qt-6°{\cdot}008)$
5	$0{\cdot}5920\sin(qt-30°{\cdot}04)$
10	$0{\cdot}3504\sin(qt-60°{\cdot}08)$
15	$0{\cdot}2074\sin(qt-90°{\cdot}12)$
20	$0{\cdot}1227\sin(qt-120°{\cdot}16)$
25	$0{\cdot}07269\sin(qt-150°{\cdot}2)$
30	$0{\cdot}04303\sin(qt-180°{\cdot}2)$
35	$0{\cdot}02548\sin(qt-210°{\cdot}24)$
40	$0{\cdot}01508\sin(qt-240{\cdot}3)$

Plot these values of v and x, and note how v attenuates.

λ the wave length is 60 miles.

155. It was discovered by Mr. Oliver Heaviside that the ordinary telephone or telegraph line whose l is usually so small as to be taken as 0, will attenuate current much less if l is large. Try this in the following example. As in the Standard cable take, per mile, $r=88$, $k=0{\cdot}05\times10^{-6}$, $s=0$, but $l=0{\cdot}1$ henry.

Take $q=5000$. Using either (1) or (2) or (3) of the note (Art. 154), we find $n=0{\cdot}0311+0{\cdot}3536i=0{\cdot}355[84^\circ{\cdot}97]$.

Hence, if $v_0=1$, $v=e^{-0{\cdot}0311x}\sin(qt-0{\cdot}3536x)$.

Note that z_0 or $\frac{r+lqi}{n}=\frac{88+500i}{n}=\frac{507{\cdot}7[80^\circ{\cdot}02]}{0{\cdot}355[84^\circ{\cdot}97]}=1430[-4^\circ{\cdot}95]$.

That is, the current everywhere is v divided by 1430, and it leads v only by about 5°; that is, v and c are nearly in synchronism.

x	v
1	$0{\cdot}9694\sin(qt-20^\circ{\cdot}26)$
5	$0{\cdot}8561\sin(qt-101^\circ{\cdot}3)$
10	$0{\cdot}7326\sin(qt-202^\circ{\cdot}6)$
15	$0{\cdot}6272\sin(qt-303^\circ{\cdot}9)$
20	$0{\cdot}5369\sin(qt-405^\circ{\cdot}2)$
25	$0{\cdot}4596\sin(qt-506^\circ{\cdot}5)$
30	$0{\cdot}3933\sin(qt-607^\circ{\cdot}8)$
35	$0{\cdot}3367\sin(qt-709^\circ{\cdot}1)$
40	$0{\cdot}2882\sin(qt-810^\circ{\cdot}4)$

The effect of the introduction of **Self-Induction** is obvious if we compare these numbers with those of the last example, and especially if we show the results in two curves.

This method is now carried out largely in practice.

In finding n in the above example a student can greatly shorten the work if he uses the formulæ given in the note of Art. 154.

When $\frac{ql}{r}$ is greater than 4, we have, approximately,

$$h=\frac{r}{2}\sqrt{\frac{k}{l}} \text{ and } g=q\sqrt{kl}.$$

In fact currents of all frequencies are attenuated to the same extent, and because of g being nearly a multiple of q, we have almost no distortion of complex currents.

Ex. 1. An infinite line, the standard cable A of Art. 149. Currents whose frequencies are $5000\div2\pi$ and $10000\div2\pi$ are sent into the line. Prove that the current of lower frequency is halved and lags 39° in every 6·6 miles; the current of higher frequency is halved and lags 39° in every 4·7 miles.

Ex. 2. The current sent into the above cable is

$$c_0 = \sin 5000t + \sin 10000t. \quad\text{.......................(1)}$$

Show that at the end of 30 miles the current is

$$c = 0{\cdot}04303 \sin(5000t - 3{\cdot}1458) + 0{\cdot}01169 \sin(10000t - 4{\cdot}449). \quad (2)$$

Show on squared paper (1) and (2) for a complete period; that is for the time $T = 2\pi/5000$.

Ex. 3. The case of Ex. (2), but the cable having also $l = 0{\cdot}1$ henry. Show that the current in 30 miles becomes

$$c' = 0{\cdot}3933 \sin(5000t - 10{\cdot}6) + 0{\cdot}3933 \sin(10000t - 21{\cdot}2). \quad\text{....(3)}$$

Show (3) on the same sheet of squared paper as (1) and (2).

Note that (2) is quite unlike (1) in shape, whereas (3) is exactly like (1) in shape. This is what we mean by *no distortion*. It is curious that distortion as great as what (2) shows should not prevent telephonic transmission.

156. Submarine Telegraphy. Assume the following cables to be infinitely long; the sending voltage is $30 \sin 60t$; what is v and what is the current when $x = 1000$ nautical miles in each case? r and k are given per nautical mile. In each case take $l = 0$, $s = 0$:

	r	k	n	$\frac{n}{r}$ or $1/z_0$
Suez-Aden cable -	10·42	$0{\cdot}3580 \times 10^{-6}$	$0{\cdot}01058 + 0{\cdot}01058i$	0·001435[45°]
Aden-Bombay - -	7·02	$0{\cdot}3610 \times 10^{-6}$	$0{\cdot}00872 + 0{\cdot}00872i$	0·001753[45°]
Persian Gulf, 1864 -	6·25	$0{\cdot}3486 \times 10^{-6}$	$0{\cdot}008085 + 0{\cdot}008085i$	0·001830[45°]
Atlantic, 1865 - -	4·27	$0{\cdot}3535 \times 10^{-6}$	$0{\cdot}00673 + 0{\cdot}00673i$	0·002229[45°]
French Atlantic, 1869	3·16	$0{\cdot}4295 \times 10^{-6}$	$0{\cdot}00638 + 0{\cdot}00638i$	0·002852[45°]
Direct U.S. cable -	2·88	$0{\cdot}4095 \times 10^{-6}$	$0{\cdot}005948 + 0{\cdot}005948i$	0·002920[45°]

In each of these cases

$$n = \sqrt{\frac{kqr}{2}}(1+i), \quad \frac{n}{r} = \sqrt{\frac{kq}{2r}}(1+i) = \sqrt{\frac{kq}{r}}\,[45°].$$

If $n = a + ai$, $\quad v = 30e^{-ax}\sin(qt - ax)$, $\quad c = \frac{n}{r}v$.

I find the following results for $x = 1000$ nauts (or 1000 nautical miles):

	v	c
Suez-Aden - -	$0{\cdot}000764 \sin(qt - 606°)$	$0{\cdot}0000011 \sin(qt - 561°)$
Aden-Bombay - -	$0{\cdot}00490 \sin(qt - 500°)$	$0{\cdot}0000086 \sin(qt - 455°)$
Persian Gulf - -	$0{\cdot}00924 \sin(qt - 463°)$	$0{\cdot}0000169 \sin(qt - 418°)$
Atlantic, 1865 - -	$0{\cdot}03583 \sin(qt - 386°)$	$0{\cdot}0000799 \sin(qt - 341°)$
French Atlantic -	$0{\cdot}05085 \sin(qt - 366°)$	$0{\cdot}0001450 \sin(qt - 321°)$
Direct U.S. - -	$0{\cdot}07832 \sin(qt - 341°)$	$0{\cdot}0002287 \sin(qt - 296°)$

It must be evident to students that in telegraph and telephone cables, if l and s are so small that we may neglect them, the received currents are the same if $L\sqrt{rk}$ or L^2rk is the same, L being the length of the cable. In fact if R is the whole resistance of a cable and K is its whole capacity, then the received currents are the same if RK is the same. But we have usually no such rule in telephone lines, because l and s are not negligible; if l is great enough any musical note lags for the same time as another of quite different pitch. The lag as an angle is gx, but the lag in time is gx/q.

I do not like here to interrupt the electrical problems. The student will find two heat problems analogous to the above electrical ones in Arts. 162 and 163.

157. Returning now to the more difficult case of a line of limited length L miles. As we have two arbitrary time functions A and B we must have two conditions given. See Art. 153.

Ex. 1. Let $v=v_0$ where $x=0$, and let $v=0$ where $x=L$. That is, let the line be put to earth at $x=L$. This is an easy exercise. It is to be remembered that when $x=0$, $\cosh nx=1$ and $\sinh nx=0$.

(3) gives $v_0=A+0$ or $A=v_0$, and also $0=A\cosh nL+B\sinh nL$.

Or $B=-v_0\dfrac{\cosh nL}{\sinh nL}$. Hence we have everywhere

$$v=v_0\left(\cosh nx-\frac{\cosh nL}{\sinh nL}\sinh nx\right), \quad\dots\dots\dots\dots\dots\dots(1)$$

$$c=-\left(\sinh nx-\frac{\cosh nL}{\sinh nL}\cosh nx\right)\frac{v_0 n}{r}. \quad\dots\dots\dots\dots(2)$$

I still use $\dfrac{r}{n}$ to mean $\dfrac{r+lqi}{n}$ or z_0.

Ex. 2. Let the given conditions be that both v_0 and c_0 are known when $x=0$. Here $A=v_0$. Also $c_0=-\dfrac{n}{r}B$ or $B=-\dfrac{r}{n}c_0$, so that

$$v=v_0\cosh nx-\frac{r}{n}c_0\sinh nx, \quad\dots\dots\dots\dots\dots\dots\dots(1)$$

$$c=-\frac{n}{r}v_0\sinh nx+c_0\cosh nx. \quad\dots\dots\dots\dots\dots\dots(2)$$

Ex. 3. Let it be known that $v=v_0$ where $x=0$, and also that at $x=L$, where v is v_1 and c is c_1, we have $\dfrac{v_1}{c_1}=R$, the resistance of a telephone or recorder or other receiving instrument. Note that R may have unreal terms.

Taking formulæ from the last example,

$$v_1 = v_0 \cosh nL - \frac{r}{n} c_0 \sinh nL,$$

$$c_1 = -\frac{n}{r} v_0 \sinh nL + c_0 \cosh nL.$$

Dividing and putting the quotient equal to R, we find c_0 to be

$$c_0 = v_0 \frac{\cosh nL + R\frac{n}{r}\sinh nL}{R\cosh nL + \frac{r}{n}\sinh nL} = v_0 \frac{1 + R\frac{n}{r}\tanh nL}{R + \frac{r}{n}\tanh nL}.$$

The expressions for v and c can now be written out. It will be found that the current through the receiving instrument or

$$c_1 = \frac{v_0}{R\cosh nL + \frac{r}{n}\sinh nL}, \quad \text{.....................(1)}$$

a very useful formula. In practical cases, where L is usually large, $\sinh nL = \cosh nL$, and then

$$c_1 = \frac{v_0}{\left(R + \frac{r}{n}\right)\cosh nL}. \quad \text{.........................(2)}$$

Ex. 4. It is to be remembered that

$$r_0 = \frac{v_0}{c_0} = \frac{R + \frac{r}{n}\tanh nL}{1 + R\frac{n}{r}\tanh nL}.$$

In a long line $\tan nL$ is 1, so that whatever R may be, $\frac{v_0}{c_0} = \frac{r}{n}$ or z_0.

It is, however, worth while to work a few exercises on this.

If $r = 88$, $k = 0{\cdot}05 \times 10^{-6}$, $l = 0$, $s = 0$, we have already found that when $q = 5000$, $n = 0{\cdot}1482[45°]$ and $\frac{r}{n} = 593[-45°]$.

Take lines of length $L = 1$, 5, 10, 20, 30 miles, and take $R = 0$, 50, 100, 200, 400 ohms. In each case calculate v_0/c_0 from (5).

Values of v_0/c_0 for

L	$R=0$	$R=50$	$R=100$	$R=200$	$R=400$
1	88·0[−0°·2]	138·0[−0°·7]	188·0[−1°·4]	287·7[−2°·9]	485·6[−5°·7]
5	429·4[−10°·3]	473·0[−12°·6]	513·3[−15°·1]	585·0[−19°·9]	696·0[−29°·1]
10	669·7[−32°·7]	675·8[−34°·9]	677·8[−36°·9]	677·0[−40°·3]	665·1[−44°·8]
20	602·4[−46°·5]	603·3[−43°·9]	604·0[−44°·1]	603·1[−44°·5]	602·0[−45°·1]
30	591·1[−45°]	591·2[−45°]	591·4[−45°·0]	591·6[−44°·9]	593·1[−44°·9]

Note that when a line is long, whatever R may be, $\frac{v_0}{c_0}$ becomes nearly $\frac{r}{n}$. This is evident from (5), for if L is large, $\tanh nL$ is nearly 1, and then $\frac{v_0}{c_0}=\frac{r}{n}$.

Ex. 5. Standard cable A of Ex. 12, Art. 149. From (2) of Ex. 3 calculate the current c_1 received by an instrument whose $R=420+0{\cdot}084qi$ [that is, its ohmic resistance is 420 with an inductance of 0·084 henry], for the following values of q and $L=40$, 60, and 80 miles. Take $v_0=1$. As

$$c_1=1\sin qt\div\left(420+0{\cdot}084qi+\frac{r}{n}\right)\cosh Ln,$$

the values of $\cosh Ln$ given in Ex. 10, Art. 149, will be found useful.

$$\frac{r}{n}=\frac{30028}{\sqrt{q}}(1-i).$$

q	c_1 for 40 miles.	c_1 for 60 miles.	c_1 for 80 miles.
3000	$70{\cdot}04\times10^{-6}\sin(qt-169°)$	$15{\cdot}09\times10^{-6}\sin(qt-263°)$	$2{\cdot}976\times10^{-6}\sin(qt-355°)$
5000	$35{\cdot}71\times10^{-6}\sin(qt-240°)$	$4{\cdot}385\times10^{-6}\sin(qt-360°)$	$0{\cdot}5396\times10^{-6}\sin(qt-480°)$
7000	$17{\cdot}21\times10^{-6}\sin(qt-300°)$	$1{\cdot}441\times10^{-6}\sin(qt-443°)$	$0{\cdot}1202\times10^{-6}\sin(qt-585°)$
10000	$5{\cdot}893\times10^{-6}\sin(qt-377°)$	$0{\cdot}3024\times10^{-6}\sin(qt-547°)$	$0{\cdot}01563\times10^{-6}\sin(qt-717°)$

Ex. 6. Submarine cable; per nautical mile, $r=3$, $k=0{\cdot}373\times10^{-6}$, $l=0$, $s=0$, $q=34$ (this corresponds to 95 letters per minute), $v_0=1\sin qt$. There is a recorder whose resistance is 344 and inductance 10·1 at the receiving station which is $L=2500$ miles away; what is the current c_1 through the recorder coil? Here $R=344+344i$, $n=0{\cdot}006168[45°]$, and $\frac{r}{n}=486{\cdot}3[-45°]=344-344i$, $Ln=15{\cdot}42[45°]=10{\cdot}90+10{\cdot}90i$, $\cosh Ln=27100[624°]$.

Hence, $$c_1=1\sin qt\div688\cosh Ln.$$

If the number of letters per minute is increased to 168, so that we take $q=60$, find the current c_1 through the recorder coil.

Letters per minute.	q	c_1
95	34	$5{\cdot}363\times10^{-8}\sin(qt-624°)$
168	60	$1{\cdot}732\times10^{-9}\sin(qt-860°)$

It has been found by practical experiment that the study of the sending of a sine current with $v_0=30$, where the arriving c_1 is 160×10^{-8}, tells us that when reversed *dot* elements are sent in actual word signals, the maximum sending voltage being 30, the maximum currents in the dots received are about 160×10^{-8} amperes.

Ex. 7. At the end of cable E of Ex. 9 of Art. 139, with $r = 2 \cdot 88$, $l = 0$, $k = 0 \cdot 4095 \times 10^{-6}$, $s = 0$, $L = 2432$ nauts, there is a recorder of resistance 263 ohms with inductance 4·4 henries. Between the cable and the recorder is a Varley condenser of 40×10^{-6} farad. If $v_0 = 30$, find c_1 the current through the recorder (1st) without the condenser, (2nd) with the condenser, when $q = 60$ and when $q = 4$. Notice that we wish to receive as little of the slowly varying current as possible, and this is the object of employing the Varley condenser.

I. (1) If $q = 60$, with the condenser as $n = 0 \cdot 00841[45°]$,

$$\cosh 2432\, n = 952820[828°], \quad \frac{r}{n} = 263 - 263i,$$

the resistance of the Varley condenser is

$$-\frac{i}{40 \times 10^{-6} q} = -417i, \quad R = 263 + 263i,$$

$$c_1 = 30 \sin qt \div (-417i + 263 + 263i + 263 - 263i) \cosh Ln.$$

This denominator is

$$(526 - 417i) \cosh Ln = 671[-38°] \times 952820[828°] = 6 \cdot 393 \times 10^8[790°],$$

therefore $\quad c_1 = 4 \cdot 692 \times 10^{-8} \sin(qt - 790°)$.

(2) Without the condenser

$$c_1 = 30 \sin qt \div 526 \cosh Ln = 5 \cdot 985 \times 10^{-8} \sin(qt - 828°).$$

II. (1) If $q = 4$, with condenser, the resistance of the condenser is

$$\frac{-i}{4 \times 40 \times 10^{-6}} = -6250i,$$

$$n = 0 \cdot 002172[45°], \quad \cosh Ln = 20 \cdot 93[214°],$$

$$\frac{r}{n} = 938 - 938i, \quad R = 263 + 18i.$$

The denominator is $(-6250i + 263 + 18i + 938 - 938i) \cosh Ln$

or $\quad (1201 - 7170i) \cosh Ln = 1 \cdot 521 \times 10^5[133°]$,

so that $\quad c_1 = 1 \cdot 972 \times 10^{-4} \sin(qt - 133°)$.

(2) If $q = 4$ and without condenser, the denominator is

$$(263 + 18i + 938 - 938i) \cosh Ln \quad \text{or} \quad 31660[176°],$$

so that $\quad c_1 = 9 \cdot 475 \times 10^{-4} \sin(qt - 176°)$.

Thus we see that the Varley condenser reduces the slowly varying current to the one-fifth part of itself, whereas it reduces the signals only *by* one-fifth.

Ex. 8. In the famous Submarine Telegraph Relay, the receiving *recorder* of resistance r is shunted by a great inductance L, the ohmic resistance of which is insignificant. A current C from the cable divides itself so that c goes through the recorder and the rest through the shunt. It is evident that $\frac{C}{c} = 1 - \frac{ri}{Lg}$. I shall only take amplitudes or effective currents.

Thus, let $L = 6$ henries, $r = 200$ ohms.

When $$q = 60, \quad \frac{c}{C} = 0{\cdot}7681.$$

When $$q = 4, \quad \frac{c}{C} = 0{\cdot}07974.$$

That is, of the good signalling currents the instrument receives 77 per cent., whereas, of the bad slowly varying currents, it only receives 8 per cent., the remainder passing through the shunt.

158. Exercise. To find the best winding of the receiving instrument at the end of a fairly long line. Ex. 3, Art. 157, gives the current received. But in all such instruments we desire *current turns* to be great. Now when any instrument is wound with different sizes of wire, if the insulation is always in the same proportion to the copper, $\rho = at^2$, $\lambda = bt^2$ if ρ is the ohmic resistance, λ the inductance, a and b constants, and t the number of turns.

In any case, let $$\frac{r}{n} = \alpha - \beta i.$$

The R of Art. 157 means $\rho + \lambda qi$, so that the denominator is

$$(\rho + \lambda qi + \alpha - \beta i)\cosh nL.$$

Neglect the $\cosh nL$, as our winding cannot affect this part; to make the current turns great, we must make the amplitude of

$$\{\rho + \alpha + i(\lambda q - \beta)\} \div t \quad \text{small}$$

or $$at + \frac{\alpha}{t} + i\left(btq - \frac{\beta}{t}\right) \quad \text{small.}$$

The square of this amplitude is

$$\left(at + \frac{\alpha}{t}\right)^2 + \left(btq - \frac{\beta}{t}\right)^2.$$

To make this as small as possible, we evidently ought to have

$$btq - \frac{\beta}{t} = 0;$$

that is, $\lambda q = \beta$. To make $at + \frac{\alpha}{t}$ a minimum, it will be found that $at^2 = \alpha$ or $\rho = \alpha$.

(1) What is the best ρ and λ for a telephone receiver at the end of a considerable length of standard cable, $r = 88$, $k = 0{\cdot}05 \times 10^{-6}$, $l = 0$, $s = 0$, $q = 5000$? *Ans.* We found $\frac{r}{n} = \alpha - \beta i = 419{\cdot}5 - 419{\cdot}5i$.

Hence $\rho = 419{\cdot}5$ ohms, $\lambda = \frac{419{\cdot}5}{5000} = 0{\cdot}084$ henries.

It is to be remembered, however, that eddy currents in the iron of the instrument (which is usually not nearly well enough divided) cause the static resistance of such a telephone receiver to be only 100 or 120 ohms instead of 419·5.

(2) What are the best ρ and λ for a recorder coil at the end of a considerable length of submarine cable; $r=3$, $k=0{\cdot}373 \times 10^{-6}$, $l=0$, $s=0$, $q=34$?

Here $$n = \sqrt{3 \times 0{\cdot}373 \times 10^{-6} \times 34i},$$

$$n = 0{\cdot}006168\sqrt{i} = 0{\cdot}006168[45°],$$

$$\frac{r}{n} = \frac{3}{0{\cdot}006168[45°]} = 486{\cdot}3[-45°] = 344 - 344i.$$

Hence the recorder coil ought to have a resistance of 344 ohms and an inductance of $\frac{344}{34}$ or 10·1 henries

159. When the receiving instrument has a resistance r_2 and it is in parallel with a resistance r_3, then

$$R = \frac{r_2 r_3}{r_2 + r_3} \quad \text{and} \quad c_2 = \frac{v_0}{\left(r_2 + \frac{r}{n}\frac{r_2 + r_3}{r_3}\right)\cosh Ln}.$$

A most important part of Brown's submarine relay is the great inductance r_3, which shunts his recorder r_2.

It is easy to state c_2 where there is a condenser in the r_2 part, and instead of this we may imagine a condenser through which c_1 passes before the shunt is reached [this is the famous Varley condenser], and compare such methods of working.

160. At the sending end of the line we may not be given v_0. Suppose there is a source of alternating current whose E.M.F. is E and resistance R_0, then $v_0 = E - R_0 c_0$ or $v_0 = \frac{E}{1 + \frac{R_0}{r_0}}$. So we replace v_0 in (2) of Ex. 3, Art. 157, by this expression. In long lines r_0 may be taken to be $\frac{r}{n}$ or z_0. Thus the current c_1 received by an instrument of resistance R at L miles will be $c_1 = \frac{E}{\left(1 + R_0\frac{n}{r}\right)\left(R + \frac{r}{n}\right)\cosh nL}$.

Whatever be the contrivance at the sending end of the line there is an alternating electromotive force E in it, and knowing the resistances it is easy to express v_0 in terms of E and r_0, and therefore to express the received current in terms of E.

The above formulæ suit at once a submarine cable, because the return conductor, the sea, may be taken to have no resistance. But in telephony, the return conductor is a separate wire about a foot away, of the same resistance as the going conductor. The formulæ are correct for this case if

$r =$ the resistance per loop mile.
$l =$ the inductance per loop mile.
$s =$ leakance per loop mile.
$k =$ capacity per loop mile.

These statements mean: Suppose we have two lengths of the conductor AB and CD placed side by side at the same distance apart as the real going and coming; and each is 1 mile in length. Then r is the resistance of AB and CD in series. That is, the resistance of two miles of conductor. The self-induction of the circuit $ABDC$, if the ends were joined, is l; or l is the mutual induction between AB and CD. s is the leakance between AB and CD, and k is the capacity between AB and CD regarded as the two coatings of a condenser.

L is the distance from transmitter to receiver; the length of the conductor is of course $2L$.

I need hardly say, that although I have given as the title to this chapter "Telephone Problems," the formulæ apply even more directly to problems on the transmission of power by alternating electric currents. But it is only on long lines that the effects of distributed capacity are great.

161. I recommend electrical students to read the papers published by Prof. Kennelly, of Harvard, as he has greatly developed this way of making calculations. In particular, his paper published in the *Proceedings of the American Institution of Electrical Engineers* in 1904 contains some interesting practical examples. He uses terms, however, which the student may have difficulty in understanding, nor indeed do I understand why he uses such terms.

My R he calls z_r, "the impedance of the receiving apparatus."

The denominator in equation (1) of Ex. 3, Art. 157, or

$$R \cosh Ln + \frac{r}{n} \sinh Ln$$

he calls z_1, "the receiving end impedance."

My $\frac{r}{n}$ or rather $\frac{r+lqi}{n}$ he calls z_0, "the sending end impedance."

My r_0 or $\frac{v_0}{c_0}$ he calls z_s, "the impedance of the circuit at the sending end."

The amplitude of my v_0 he calls E, "the maximum cyclic E.M.F. of the sending apparatus"; that is, he assumes that the sending apparatus has no resistance or impedance. I have found in the usual arrangement of transmitting apparatus on the common battery system, that the errors due to this wrong assumption may be considerable. See Art. 151.

CHAPTER XXXVI.

HEAT PROBLEMS.

162. The following heat problem is analogous with the electrical problem of Art. 146.

The temperature v at the hot surface of the metal of a steam-engine cylinder follows the law

$$v_0 \sin qt. \quad \text{.................................}(1)$$

Assume the periodic time to be $\frac{1}{2}$ a second (as if the engine made 2 revolutions per second), so that $q = 2\pi \times 2 = 6{\cdot}2832$. I shall use C.G.S. units. The conductivity of the iron is $k = 0{\cdot}20$, the capacity for heat of one cubic cm. of iron or $ws = 0{\cdot}87$; find v at any depth x and any time t. Assume that the surface of the metal is quite plane instead of being cylindric, and that the metal is infinitely thick. Here

$$n = \sqrt{\frac{ws}{k} qi} = 3{\cdot}7 + 3{\cdot}7i \quad \text{and} \quad v = e^{-nx} v_0 \sin qt,$$

or leaving the $\sin qt$ out, as being always understood,

$$v = e^{-nx} v_0,$$

$$c = -k\frac{dv}{dx} = nke^{-nx} v_0. \quad \text{.................................}(2)$$

Now $\quad e^{-nx} = e^{-3{\cdot}7x - 3{\cdot}7xi} = e^{-3{\cdot}7x}\left[-3{\cdot}7x\right].$

Hence $\quad v = v_0 e^{-3{\cdot}7x} \sin(qt - 3{\cdot}7x) \quad \text{........................}(3)$

and $\quad c = nkv.$

We may, if we please, introduce another term, and instead of (3) use $\quad v = v_0 e^{-3{\cdot}7x} \sin(qt - 3{\cdot}7x) + ax,$

because $v = ax$ is also a solution of the fundamental equation.

It is easy to see how this satisfies the case of a steady flow of heat inwards as from a steam jacket. I will, for the rest of this exercise, assume that $a = 0$.

If the heat enters the surface from an atmosphere at the temperature V, make some assumption as to the current of heat which would enter under the influence of the difference of temperature $V - v_0$. If e is the emissivity of the surface, the simplest assumption to take is $e(V - v_0) = c_0$. Neglecting the steam jacket term, that is, taking $a = 0$, $c_0 = nkv_0$, so that

$$e(V - v_0) = nkv_0 \quad \text{or} \quad V = v_0 \frac{nk + e}{e},$$

$$V = \left(1 + \frac{nk}{e}\right) v_0.$$

If there is a wet layer on the metal in a steam cylinder, e is very much larger than when the surface is dry, as it is when highly superheated steam is used. Take two cases: (1st) $\frac{k}{e} = 6$, a dry skin; (2nd) $\frac{k}{e} = 0{\cdot}1$, a wet skin.

1st case. $V = (1 + 6n)v_0 = (23{\cdot}2 + 22{\cdot}2i)v_0 = 32{\cdot}11[43^\circ{\cdot}74]v_0$.
or $V = 32{\cdot}11 v_0 \sin(qt + 43^\circ{\cdot}74)$.

That is, the range of temperature in the steam is 32·11 times the range of temperature in the skin of the metal.

2nd case. $V = (1{\cdot}37 + 0{\cdot}37i)v_0 = 1{\cdot}421[15^\circ{\cdot}1]v_0$.
or $V = 1{\cdot}421 v_0 \sin(qt + 15^\circ{\cdot}1)$.

That is, the range of temperature in the steam is only 1·421 times the range of temperature in the skin of the metal.

We may desire to know the amount of heat which enters the metal in one cycle. If at the surface $c = c_0 \sin qt - b$, the negative current b being due to the steam jacket, the amount of heat entering in the positive part of one cycle is $c_0/2q - \frac{1}{2}bT$, where T is the periodic time, in this case 0·5 second.

This enables an approximation to be made to the amount of water missing as indicated water, because of condensation. [See my book on *Steam*, pages 381-9.]

163. The following heat example is mathematically the same as the last, and is also the same as that of the telephone and telegraph cables of Arts. 154-6.

Lord Kelvin and Principal Forbes buried thermometers at various depths in the rock at Craigleith Quarry, Edinburgh. The changes of temperature were (1st) of 24 hours period, (2nd) of one year period. In such work as the above we ought therefore to study two terms with two values of q, the temperature v at any depth being the sum of what each term produces. But in this case it was possible in the observations to get results for the yearly period only, and these were the results:

Depth in feet below the surface.	Yearly range of temperature. Fahrenheit.	Time of highest temperature.
3 feet	16·138	August 14
6 feet	12·296	,, 26
12 feet	8·432	Sept. 17
24 feet	3·672	Nov. 7

[Observations at 24 feet below the surface at Calton Hill, Edinburgh, showed the highest temperature on January 6th, but I am not now studying the Calton Hill observations.]

I am sorry that as I write I cannot refer to the original paper by Kelvin and Forbes, and I do not recollect how they reduced their observations. A more tedious and more accurate method than the following might be adopted. First, multiplying the yearly range by 5 and dividing by 18, I get the amplitude in the Centigrade scale. Convert feet into centimetres. Write the lag in days, first assuming that the lag at the three feet depth is m days. Assume our theory to be true, and then

$$v = v_0 e^{-ax} \sin(qt - ax).$$

x feet.	x centimetres.	$y = v_0 e^{-ax}$ Centigrade.	lag in days $= d$.	ax the lag in radians calculated.	d as calculated if theory is correct.
0	0	v_0	0	0	0
3	91·44	4·483	m	0·207	12 + 0
6	182·88	3·4156	$m + 12$	0·414	12 + 12
12	365·76	2·3422	$m + 34$	0·828	12 + 36
24	731·52	1·020	$m + 85$	1·656	12 + 84

Plotting $\log_{10} y$ and x (feet) on squared paper, I find that the points lie very fairly well on a straight line [this verifies part of our theory], and I take $\log_{10} y = 0{\cdot}72 + 0{\cdot}03x$ or $y = 5{\cdot}25 e^{-0{\cdot}069x}$, x being in feet. Hence $a = 0{\cdot}069$. The lag in radians therefore ought to have the values calculated above. If the lags d given in days are correct,

$$\frac{d}{365{\cdot}25} = \frac{\text{lag in radians}}{2\pi},$$

and we can therefore calculate what d ought to be if our theory is correct. I find the answers given above. It is seen that the numbers agree with those observed, very nearly. The maximum temperature ought to have been found at the depth of 12 feet, not on Sept. 17 but on Sept. 19. And Nov. 7 ought to be Nov. 6. These are not great discrepancies.

If x is taken in centimetres, $ax = 0{\cdot}207$ when $x = 91{\cdot}44$, so that a in proper (or C.G.S.) units is 0·002264. Now we know that

$$a = \sqrt{\frac{wsq}{2k}} = 2{\cdot}264 \times 10^{-3}.$$

The periodic time is 1 year or $T = 31{\cdot}56 \times 10^6$ seconds; q is $2\pi/T$ or $q = 1{\cdot}991 \times 10^{-7}$; ws the capacity for heat, of the rock, per cubic cm. may be taken as 0·5. We have then

$$2{\cdot}264^2 \times 10^{-6} \times 2k = 0{\cdot}5 \times 1{\cdot}991 \times 10^{-7}.$$

This gives $k = 0{\cdot}00951$, the probable conductivity of the rock at Craigleith Quarry.

I find that the result published by Kelvin and Forbes is 0·01068. They may have taken a different value of ws from mine; or they may have carried out the above work more carefully than I have done.

Students of electric cables are fond of talking of wave-lengths. Here the heat wave-length is $2\pi \div a$ or $6{\cdot}2832 \div 0{\cdot}069$ or 91 feet. That is, at a depth of 91 feet the lag is just one year; would such a student say that heat is therefore found to travel in rock with the speed of 91 feet per annum? It seems rather absurd, and yet this is the very thing that he says about the speed of electricity in cables.

It is quite easy to consider heat problems which are analogous with that of the electric cable of limited length.

CHAPTER XXXVII.

VECTORS.

164. Hitherto I have only considered scalar quantities, quantities which are dealt with like mere numbers.

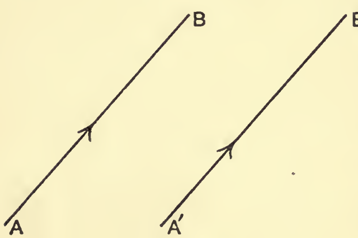

Fig. 50.

A vector quantity, like displacement, velocity, momentum, acceleration, force, impulse, strain, stress, flux or fluid, flux of electric current, magnetic force, magnetic induction, etc., has direction and magnitude. **A vector** can be represented by a straight line; its **magnitude** to some scale by the length of the line; its **clinure** or *ort* by the clinure of the line; its sense by an arrow-head. Thus the length of the line AB (Fig. 50) represents a vector to some scale of measurement; its sense is shown by the arrow-head.

If $A'B'$ is drawn parallel to AB and of the same length and sense, it represents exactly the same vector.

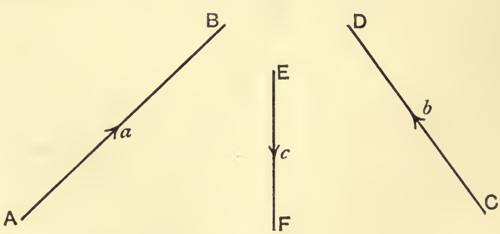

Fig. 51.

165. Addition of Vectors. If **a**, **b**, and **c** (Fig. 51) represent three vectors, to add them: make them the sides of a polygon

(Fig. 52); take care that their arrow-heads are circuital [follow my neighbour]. Thus AB is the same as **a**, CD is **b**, EF is **c**.

Then the last side of the polygon with a non-circuital arrow-head represents their sum. Or, as we write it,

$$AB + CD + EF = AF \quad \text{or} \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = AF. \qquad (1)$$

The student ought to take three or four vectors and add them according to the above rule, taking them in quite different order as sides of a polygon; in every case the same answer is obtained. In fact $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. We apply now the same symbols as we use in algebra, that is in dealing with scalar quantities, and say that (1) is the same as

$$\mathbf{a} + \mathbf{b} + \mathbf{c} - AF = 0 \qquad (2)$$

$$\text{or} \quad \mathbf{a} + \mathbf{b} = AF - \mathbf{c}. \qquad (3)$$

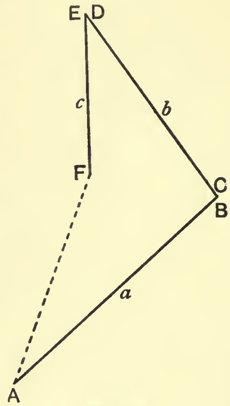

Fig. 52.

Notice that when a letter is used for a vector we employ Clarendon type. Notice that our vectors are not necessarily all in one plane, but when they have all sorts of clinures the polygon is a gauche polygon, and must be illustrated by bits of wire.

We may say that all the above statements make up our definition of a vector.

Vector quantities are such quantities as may be added and subtracted according to the above rule.

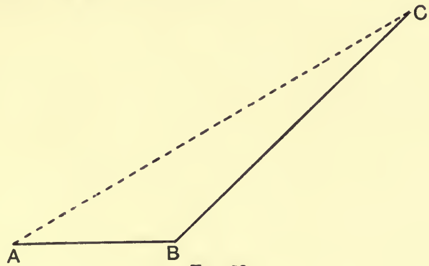

Fig. 53.

It is a self-evident truth that a displacement is a vector. If we say that a point is moved to the east 1 foot and to the north-east 2 feet, we draw AB and BC to represent these two displacements, and we see (Fig. 53) that the vector sum of the two, AC, gives the real sum of the two displacements.

Here $$AB + BC = AC$$
or $$AB = AC - BC$$
or $$BC = AC - AB,$$
so that subtraction of vectors is as easy to understand as addition of vectors.

It is evident that the sum of the displacements *AB*, *BC*, *CD*, *DE*, and *EF* (Fig. 54), is exactly the same as that of *AG*, *GH*, and *HF*; and *AF* expresses it.

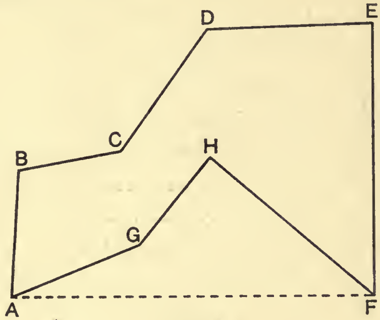

Fig. 54.

Similarly a velocity, being displacement in unit time, is a vector. We look upon the statement that all the above quantities are vectors as a self-evident truth.* There is nothing else to be done. We can no more prove the truth of such a statement than the truth of our own existence. *As soon as we comprehend* the statement we see that it requires no proof. In the same way, acceleration, being velocity added per second, is also a vector. Force is mass multiplied upon acceleration, and is therefore a vector also. In so far as a force has an actual position as well as mere clinure, it has a property in addition to that of a mere vector.

Notice that when we use a letter **a** to represent a vector, it is in Clarendon type, and when we see Clarendon type in an equation we know that it is a vector equation without special telling. Also $-\mathbf{a}$ is just the same as **a**, except that its sense, its arrow-headedness, is reversed.

When I was young I spent much time over Duchaylas' proof that forces are added in the above way. The old 15 page trouble of our youth has disappeared from the books, I am happy to say. It

*This statement is too sweeping. Advanced students will find that things not vectors may be mistaken for vectors. Thus a *finite* rotation is *not* a vector.

was not only illogical, but stupid, and everybody now recognises this fact. When will the remaining so-called "proofs by abstract reasoning" of the school-books disappear? Unfortunately they have a specious appearance of being useful in mind-training, and the stupid teachers can teach nothing else, and so Euclid and its trailing cloud of miseries can disappear only gradually.*

166. It is very difficult to set questions on this subject, because it is so difficult to state clinure or direction. If I use the points of the compass, only a few students will understand. I am afraid that I must ask students to remember the following way. Let OX (Fig. 55) be a given standard direction; say, towards the east from O. Now if I always assume the sense of a vector OP to be *out* from O, and if I state the angle XOP as traced out *anti*-clockwise, will the student be able to understand?

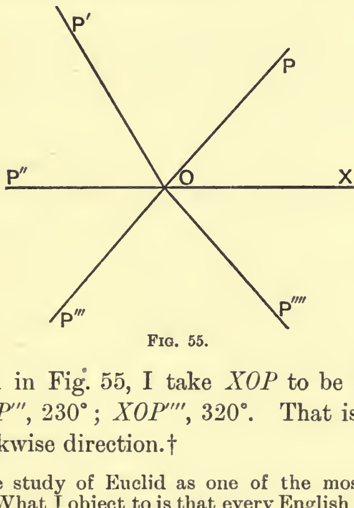

Fig. 55.

Thus, as drawn in Fig. 55, I take XOP to be 50°; XOP', 120°; XOP'', 180°; XOP''', 230°; XOP'''', 320°. That is, I never measure angles in the clockwise direction.†

*I look upon the study of Euclid as one of the most valuable of post-graduate studies. What I object to is that every English boy should be forced to pretend that he can follow Euclid's reasoning.

†In passing, I may say that in specifying a system of forces not passing through a point I use the following method: Let O be a known point and OX the standard direction. Let the direction of a force AB cut OX in the point A and let AB be the sense or arrowheadness; let the angle BAX be called θ and the distance OA be called a; if P be the magnitude of the force, then the force is specified by ${}_aP_\theta$. Thus, for a force of 50 lb., angle $\theta = 125°$; $OA = 2$, we use the symbol ${}_2 56_{125°}$.

Hitherto, to specify forces in setting questions, examiners have directed squares and triangles to be drawn, specifying their forces by means of these figures; the result has been that students believe problems in statics to have an occult connection with regular geometrical figures.

It will now be easy to set questions. I shall state the magnitude of each vector; assume that its sense is out from O, and I shall state the angle it makes with OX.

If $\mathbf{A}=20_{140^\circ}$; $\mathbf{B}=15_{30^\circ}$; $\mathbf{C}=30_{280^\circ}$; $\mathbf{D}=12_{330^\circ}$; $\mathbf{E}=25_{47^\circ}$.

The answers must be stated in the same way. Find by construction:

(1) $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}+\mathbf{E}$. *Ans.* $30{\cdot}5_{5^\circ{\cdot}5}$.

(2) $\mathbf{A}+\mathbf{B}+\mathbf{C}-\mathbf{D}-\mathbf{E}$. *Ans.* $32{\cdot}6_{220^\circ{\cdot}8}$.

(3) $-\mathbf{A}-\mathbf{B}-\mathbf{C}+\mathbf{D}+\mathbf{E}$. *Ans.* $32{\cdot}6_{40^\circ{\cdot}8}$.

(4) $-\mathbf{A}-\mathbf{B}-\mathbf{C}-\mathbf{D}-\mathbf{E}$. *Ans.* $30{\cdot}5_{185^\circ{\cdot}5}$.

As the letter **A** represents a vector, there is much meaning wrapped up in one letter; clinure and sense and the scalar part, which is mere magnitude. If we say $\mathbf{Q}=1\frac{1}{2}\mathbf{P}$, it can only be true if **Q** and **P** are of the same clinure (or *ort* some people call it) and sense.

It is very well worth while to express a vector **P** as $P_0\mathbf{p}$, where P_0 is the scaler magnitude, or *tensor* as it is sometimes called, and **p** is said to be the unit vector.

167. If a vector **a** changes to **b** in the time δt, we see that the vector AB (Fig. 56) has been added in this time, and we may say that there has been an average increase at the rate $AB \div \delta t$ per second during this time.

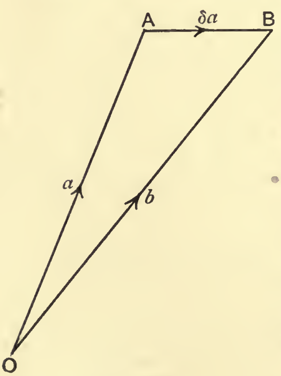

FIG. 56.

We here say $\quad \mathbf{a}+AB=\mathbf{b}$

or $\quad \mathbf{a}+\delta\mathbf{a}=\mathbf{b}$,

and $\dfrac{\delta\mathbf{a}}{\delta t}$ is the average rate of increase. This is a vector in the direction AB.

But if we want to know the actual rate of increase at any instant, we must take an interval of time δt which is smaller and smaller without limit.

The student might consider a very simple case; a body moves in a circular path of radius r at a constant speed v. Consider the body as it proceeds from a point a in the time δt to the position b. Let the distance ab be called δs. Draw lines (Fig. 57) OA parallel to the tangent at a, and OB parallel to the tangent at b and make them equal in length, each representing the velocity v to some scale.

Then, according to the above rule, the vector $AB \div \delta t$ is the acceleration.

$$\text{The angle } AOB = \frac{\delta s}{r} = \frac{v \,.\, \delta t}{r} \quad \text{and} \quad AB = v \times \frac{v \,.\, \delta t}{r},$$

so that acceleration $= \dfrac{v^2}{r}$ in a direction at right angles to v at any instant, that is towards the centre of the circular path.

It is quite easy to extend this to show that the total acceleration of a point moving with changing speed in a curved path, is the vector sum of the centripetal acceleration and of the acceleration of speed.

The study of tortuosity is made comparatively easy by vector algebra.

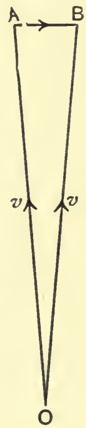

FIG. 57.

168. Multiplication of Vector Quantities. If I were asked to multiply 2 tables by 3 chairs, I would not refuse; I would say 6 chair-tables. But if I were asked to say what I mean by a chair-table, I would refuse to answer; because nobody has ever given a meaning to the term. But I do know that when this sort of thing comes into a Physical Problem we can always give a useful meaning. This is beyond ordinary Algebra, and yet our processes are carried on by the rules of Algebra. Observe that I do not say that

$$2 \text{ tables} \times 3 \text{ chairs} = 3 \text{ chairs} \times 2 \text{ tables}.$$

Whether this is or is not true depends upon the meaning we attach to the whole process. It is easy to multiply any quantity whatever by a mere number. For example, 2 tables $\times 3 = 6$ tables. Also a velocity of 6 feet per second due east multiplied by 3 is 18 feet per second due east. Again, if we multiply a vector by any scalar quantity we get a vector in the same direction, although the meaning, the name of its unit, may not yet have been fixed. A velocity of 6 centimetres per second due east multiplied by 3 grammes is a momentum of 18 gramme-centimetres per second due east. Thus, then, the multiplication of a vector by any scalar quantity is not difficult to understand; the result is a vector in the same direction. But what meaning are we to attach to the multiplication of vectors by one another?

Surely you are all acquainted with examples. If a vertical force of 4 lb. acts through a vertical distance of 3 feet, we say that it does the work

$$3 \text{ ft.} \times 4 \text{ lb.} = 12 \text{ foot-pounds.}$$

Now these are two vectors that I have multiplied together. In fact, the product of a force into a displacement is quite familiar to us. If there is any force **A**, and if the body that it acts upon has a displacement **B**, we say that the work done is **AB**.

Some people write this S.**AB**, because work done, energy, is a scalar quantity which has no direction and is not a vector, but I prefer to leave the S out. Now notice that we use **AB** to represent the work done, whatever may be the clinures and senses of **A** and **B**.

It is well worth while to study this example carefully, for this product of two vectors exactly illustrates what we mean by the scalar product of any two vectors.

Note that my unit of force is that used by the engineer, the weight of one pound at London.

Let us suppose that the force **A** is A lb., but it is evidently something more; I must state the direction. Let it be vertically upwards. If a force of ·1 lb. acting vertically upwards be completely indicated by the letter **a**, then

$$\mathbf{A} = A\mathbf{a}.$$

That is, **a** is the unit force vector and A is a mere numeric.

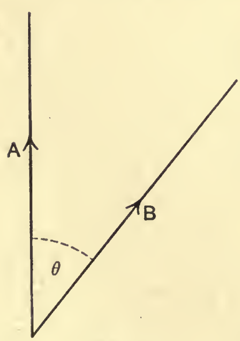

Fig. 58.

Again, suppose **B** (Fig. 58) to be a displacement of B feet in some direction which may not be vertically upwards; suppose that the letter **b** represents the vector displacement of 1 foot in that actual direction of **B**, that is, that the letter **b** signifies direction as well as 1 foot, in fact

$$\mathbf{B} = B\mathbf{b}.$$

Then we can say that

$$\mathbf{AB} = AB\,.\,\mathbf{ab}.$$

The product of two mere numbers A and B is easy enough, but what signification are we to give to the product **ab** of two unit vectors; **a** being one pound vertically upwards and **b** a displacement of 1 foot in some direction not necessarily vertically upwards?

It is exactly our chair-table question. We are asked to give a meaning to the product of our units **ab**. Now we know that meaning if in the above case we are going to say

$$\mathbf{AB} = \text{work done.}$$

Because the work done when the force **A** acts on a body when the body has the displacement **B** is obtained by finding the resolved part of **A** in the direction of **B** (we call this $A\cos\theta$ lb. if θ is the angle between the vectors), and multiplying by B feet, our answer being $AB\cos\theta$ foot pounds. Or we might have done it by finding the resolved part of **B** in the direction of **A** (this is $B\cos\theta$ feet), and multiplying by A pounds, our answer as before being $AB\cos\theta$ foot pounds.

Since then $$\mathbf{AB} = AB\,.\,\mathbf{ab} = AB\cos\theta \text{ foot pounds,}$$

we see that the product of

> **a**, which is a force of 1 lb. vertically upwards,
>
> **b**, which is a displacement of 1 foot making an angle θ with the upward direction,

is $\cos\theta$ foot pounds. The student should here note that **ab** is the same as **ba**.

Leaving out of account mere names of unit quantities, we may say then that our definition of the meaning of the scalar product of two unit vectors is

$$\cos\theta,$$

where θ is the angle between their directions, and the meaning of the product **AB** is $AB\cos\theta$, where A and B are the *tensors* and θ is the angle between the directions of the vectors. To measure θ, always draw the two vectors from a point 0 with their arrowheads going out from 0. As we deal only with a cosine of this angle, it will be found that it may be measured either clockwise or anti-clockwise.

Ex. 1. A force of 350 lb. acts on a tram-car in a direction towards 20° west of north; the velocity of the car is 20 feet per second due north (or $\theta = 20°$). Find the Power being given to the car. (Generally when we multiply any kind of force by any kind of velocity we are in the habit of calling the answer "the activity.")

Ans. 6578 foot pounds per second. In this case the activity is merely what the mechanical engineer calls *power*.

Ex. 2. In the above case the force acts towards due west (or $\theta = 90°$). What is the power or activity? *Ans.* 0.

Ex. 3. In the above case the force acts towards the south-west (or $\theta = 135°$). Find the power.

Ans. Minus 4949 foot pounds per second.

Ex. 4. In the above case the force acts towards the south (or $\theta = 180°$). Find the power.

Ans. Minus 7000 foot pounds per second.

169. We see that if OX (horizontal) and OY vertical are two standard lines at right angles in a plane, the position of a point P may be defined in two ways.

(1) If we know x its distance to the right of OY and y its distance above OX.

(2) If we know r its distance from the point O and the angle θ which r makes with OX.

We see that the connection between these co-ordinates is $x = r\cos\theta$, $y = r\sin\theta$, and of course $r^2 = x^2 + y^2$. Also $y/x = \tan\theta$. Given x and y we can find r and θ, or given r and θ we can find x and y.

170. When we deal with vectors in all sorts of directions in space it is necessary* to use three standard directions, OX, OY, and OZ mutually at right angles A vector OP is projected as OA in the direction OX. If α, β, γ, are the angles which OP makes with OX, OY, OZ, then $OA = x = OP \,.\, \cos\alpha$, $OB = y = OP \,.\, \cos\beta$, $OC = z = OP \,.\, \cos\gamma$. Two of these three angles ought to be given if we want to specify the direction of a vector.

Now it is easy to show that $OP^2 = OA^2 + OB^2 + OC^2$, so that $OP^2 \,.\, \cos^2\alpha + OP^2\cos^2\beta + OP^2\cos^2\gamma = OP^2$ or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

171. Hence, when a man, in telling me the direction of a vector, gives me its α, β, and γ, I have one check on the accuracy of his measurement.

Ex. 1. I have carefully measured and found in a certain case that $\alpha = 20°$, $\beta = 75°$, $\gamma = 76°{\cdot}5$.

Calculate what γ ought to be from the other two and state the percentage error in the measurement of γ, if the others are quite correct. *Ans.* $77°{\cdot}08$, percentage error $100 \times \frac{0{\cdot}58}{77{\cdot}08}$ or 0·75 per cent.

*Of course I do not mean that it is necessary. A single vector **a**, when given, enables us to specify all vectors in space parallel to **a**. Given another independent vector **b** (or parallel to **b**), we can specify any vector of any size in the plane of **a** and **b** (or in any parallel plane). Thirdly, any third independent vector **c** enables us to specify any vector whatever as $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$.

Ex. 2. The following values of OP and of α and β are given for three vectors. Find γ in every case. Find the projections in the three standard directions. Add each set up.

Values of OP.	α	β	γ as calculated.	OA	OB	OC
30	70°	37°	60°·3	10·26	23·96	14·86
25	150°	84°	60°·7	−21·65	2·612	12·23
15	85°	170°	81°·4	1·308	−14·77	2·242
Sums -				−10·08	11·80	29·33

Ex. 3. Find the vector whose three projections are the above sums.

Ans. $OP = \sqrt{(10{\cdot}08)^2 + (11{\cdot}8)^2 + (29{\cdot}33)^2} = 33{\cdot}18,$

and it makes angles α, β, and γ with OX, OY, and OZ such that

$$\cos\alpha = -0{\cdot}3038 \quad \text{or} \quad \alpha = 107^\circ{\cdot}7,$$
$$\cos\beta = 0{\cdot}3556 \quad \text{or} \quad \beta = 69^\circ{\cdot}2,$$
$$\cos\gamma = 0{\cdot}8840 \quad \text{or} \quad \gamma = 27^\circ{\cdot}9.$$

We have therefore found the sum of the above three vectors. The student who knows descriptive Geometry will find the sum of

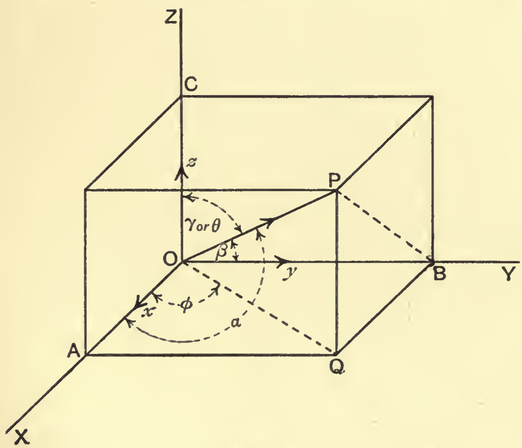

FIG. 59.

the above three vectors by construction, and see if he gets the same answer.

The projection of a vector OP upon the plane XOY (Fig. 59) is the line OQ or $OP \cos POQ$. The projection of OP upon the line OZ

is the line OC or $OP \cos POZ$. Let POZ be called θ (I called this γ just now); we have seen that POQ is the complement of θ,

$$OC = OP . \cos\theta, \quad OQ = OP . \sin\theta.$$

Let the angle QOX be called ϕ; then $OA = OQ . \cos\phi$ and $OB = OQ . \sin\phi$. If we let the length of OP be supposed to be always positive and we call it r; if OA is called x, OB, y; OC, z; these being the projections of r on OX, OY, and OZ, the three mutually rectangular standard directions, we have two ways of stating our results, both of which are in use.

$$x = r\cos\alpha \quad \text{or} \quad x = r\sin\theta . \cos\phi;$$
$$y = r\cos\beta \quad \text{or} \quad y = r\sin\theta . \sin\phi;$$
$$z = r\cos\gamma \quad \text{or} \quad z = r\cos\theta.$$

It will be noticed that $\quad x^2 + y^2 + z^2 = r^2$

and $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

172. If we think of the plane XOY as the equatorial plane of the earth, and OZ as the axis of the earth, then the position of a point P in or on or outside the earth, which moves with the earth, is given if we know its r, or distance to the centre of the earth, its θ or co-latitude, and its ϕ or east longitude. ϕ is evidently the angle which the plane containing OZ and OP and OQ (called a meridianal plane) makes with the standard meridianal plane ZOX (in our analogy this is the meridian through Greenwich). It is to be remembered, however, that the α, β, γ way is also very greatly used to specify direction. $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the "direction cosines" of a line, and the letters l for $\cos\alpha$, m for $\cos\beta$, n for $\cos\gamma$ are very greatly used. So that $x = lr$, $y = mr$, $z = nr$, and $l^2 + m^2 + n^2 = 1$.

We usually specify the inclination of a plane by stating the direction cosines of its normal, and of the surface of a body at any place by the direction cosines of the normal to the surface at that place.

I see that I have instinctively left the consideration of vectors in general, and applied my rules to mere displacement vectors. In fact, we have at length reached the subject of Geometry; that is, the relation of mere lines to one another. If we consider XOZ, ZOY, and YOX as three mutually perpendicular standard planes of reference, the position of a point P is fixed if we know its x, y, and z; or its r, θ, and ϕ; or its r, and any two direction cosines. And if we know the position of a point in any one of these three ways, we can calculate the other dimensions.

173. Ex. 1. The x and y co-ordinates of a point in a plane are 3 and 4; find r and θ. As $r\cos\theta=3$ and $r\sin\theta=4$,

$$\frac{4}{3}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta \quad\text{or}\quad \theta=53^\circ\cdot 13 \text{ nearly}; \quad r^2=x^2+y^2=25,$$

so that $r=5$.

Ex. 2. If a point has $r=10$ inches, $\theta=37^\circ$, find x and y; $x=10\cos 37^\circ=7\cdot986$ inches, $y=10\sin 37^\circ=6\cdot018$ inches.

Now going to three dimensions and Fig. 59:

Ex. 3. If $r=10$ inches, $\theta=35^\circ$, $\phi=62^\circ$, find x, y, z.

Here
$$x=10\sin 35^\circ\cos 62^\circ=2\cdot693 \text{ inches},$$
$$y=10\sin 35^\circ\sin 62^\circ=5\cdot064 \text{ inches},$$
$$z=10\cos 35^\circ=8\cdot192 \text{ inches}.$$

Ex. 4. If $x=3$, $y=5$, $z=6$, find r, θ, and ϕ.

Here
$$r^2=x^2+y^2+z^2, \text{ so that } r=8\cdot370.$$

$$z=r\cos\theta \quad\text{or}\quad \cos\theta=\frac{6}{8\cdot37}=0\cdot7171 \quad\text{or}\quad \theta=44^\circ\cdot18;$$

$$3=8\cdot370\sin 44^\circ\cdot18\times\cos\phi$$

or
$$\cos\phi=3\div(8\cdot370\times 0\cdot6969)=0\cdot5144$$

or
$$\phi=59^\circ\cdot03.$$

Ex. 5. If $x=3$, $y=5$, $z=6$, find r, l, m, n.

Answer as in last case, $r=8\cdot370$, $l=x/r$ or $0\cdot3585$, $m=y/r$ or $0\cdot5975$, $n=3/r$ or $0\cdot7171$.

174. Exercises on Scalar Products. (1) The flow of fluid per unit area through a surface is **nV**, where **n** is the unit vector normal to the surface, and **V** is the velocity of the fluid. If the fluid flows at 5 feet per second at an angle making 35° with the normal to the surface, find the flow per square foot.

Ans. $5\cos 35^\circ$ or $4\cdot096$ cubic feet per second.

(2) If **E** is electric force and **D** is electric displacement, it is important to calculate half of **ED**. If $\mathbf{E}=250$ vertically downwards, and if $\mathbf{D}=1\cdot56$ making an angle of 42° with the downward direction, find $\frac{1}{2}\mathbf{ED}$. *Ans.* $\frac{1}{2}\times 250\times 1\cdot56\times\cos 42^\circ=144\cdot9$.

(3) If **A** and **B** are the sides of a parallelogram, $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ are the diagonals.

$(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$. Show that this is really the ordinary formula for the square of the length of a diagonal.

$(\mathbf{A}-\mathbf{B})^2=\mathbf{A}^2-2\mathbf{AB}+\mathbf{B}^2$. Show that this is really the ordinary formula for the square of the length of the other diagonal.

Express the geometrical meaning of

$$(\mathbf{A}+\mathbf{B})^2+(\mathbf{A}-\mathbf{B})^2=2(\mathbf{A}^2+\mathbf{B}^2)$$

and

$$(\mathbf{A}+\mathbf{B})^2-(\mathbf{A}-\mathbf{B})^2=4\mathbf{AB}.$$

(4) If **A**, **B**, and **C** are the edges of a parallelopiped, show that the ordinary formula for the length of the diagonal is given by

$$(\mathbf{A}+\mathbf{B}+\mathbf{C})^2=\mathbf{A}^2+\mathbf{B}^2+\mathbf{C}^2+2\mathbf{AB}+2\mathbf{AC}+2\mathbf{BC}.$$

(5) If a plane figure moves, it traces out a volume (per unit of its area) equal to the scalar product **nD**, where **n** is the unit vector normal to the plane area and **D** is the displacement of the centre of area. If the area keeps parallel to itself (in the final position it may rotate about an axis through the centre of area, and this will make no change in the answer), **n** is the same in every position, and **D** will then be the resultant or total displacement.

175. When I was asked what I meant by

2 tables × 3 chairs = 6 chair-tables

I refused to answer. Algebra had done its duty; it could go no farther. I may now give a meaning to

1 chair-table

and invent a new science. But I may give another meaning and invent another science, and both sciences may be useful. So when we multiply vectors we can give a scalar meaning to our answers, as we have already seen. But there is no reason why we should not give another meaning, a vector meaning, and that in certain cases this also should be of value.

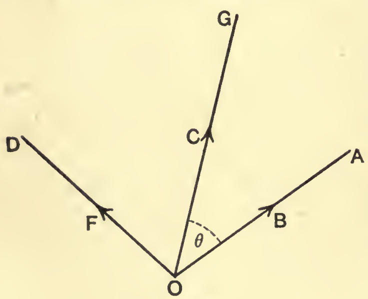

Fig. 60.

There is **another way*** in which the **product** of vectors enters into calculations. Suppose we have a magnetic field **B**. That is, it is of the amount B in the direction OA (Fig. 60), and amount and direction are both specified by the letter **B**. Let there be a conductor in the field with a current **C** flowing in it. Here again **C** signifies that there is a current of the amount C flowing in the direction OG. We know that there is a mechanical force acting on

*Any combination in products of the components of two vectors might perhaps be called some special kind of product of the vectors ; but our two are the only useful ones ; furthermore, as they are independent of the axes of co-ordinates, they are the ones that occur in Nature.

the conductor in the direction OD, which is at right angles to both OA and OB, and its amount is $BC \sin \theta$. If this force is denoted by **F**, we say that **F** is the vector product **CB**

or $$\mathbf{F} = V.\,\mathbf{CB}.$$

Definition. The **vector product** of two vectors **C** and **B** is a vector at right angles to both, its tensor being the product of the tensors of **C** and **B** multiplied by the sine of the angle between them; that is, the area of the parallelogram of which OA and OG are two sides.

Exercise. A body is spinning about an axis OX at A radians per second, and the letter **A** represents both amount of spin and direction OX. In fact A is the tensor of **A**, and **A** has the direction OX. A point P in the body is at the distance $r = OP$ from O, and the letter **R** signifies the amount r or OP and also the direction of OP. It is evident that the velocity **V** of P at any instant is specified both in magnitude and direction by

$$\mathbf{V} = V.\,\mathbf{AR},$$

for the amount of it (its tensor) is $Ar \,.\, \sin XOP$, and its direction is at right angles to the plane XOP.

It is outside the scope of this elementary book to dwell further upon the two kinds of products of vectors which enter so much into physical calculations. I may not even dwell upon the rule for the sign of $V.\,\mathbf{AB}$, or show that

$$V.\,\mathbf{AB} = -V.\,\mathbf{BA}.$$

As in other mathematical methods, the student has to get accustomed to the use of a few formulæ. For example, what is $\mathbf{a}V\mathbf{bc}$? We have the vector product of **b** and **c**, and we have the scalar product of this with **a**. If **a**, **b**, and **c** are the three edges of a parallelopiped meeting at a corner, $\mathbf{a}V\mathbf{bc}$ is evidently the volume of the parallelopiped, and hence $\mathbf{a}V\mathbf{bc} = \mathbf{b}V\mathbf{ca} = \mathbf{c}V\mathbf{ab}$, a set of very important identities. Again the student ought to show that

$$V\mathbf{c}V\mathbf{ab} = \mathbf{a}\,.\,\mathbf{bc} - \mathbf{b}\,.\,\mathbf{ca}.$$

I do not know of any book introducing the student to the use of Vector Algebra. An exceedingly interesting (but much too short) introduction will be found in Mr. Oliver Heaviside's *Electromagnetic Theory*, Vol. I., and electrical students in particular will find it valuable, for it is by means of Vector Algebra that electromagnetic theory can be most easily studied.

BOARD OF EDUCATION EXAMINATIONS.

PREVIOUS to 1912 the examinations were in Stages 1, 2, and 3. In future there will be no examination in Stage 1. There will be what is called the *Lower* examination ; it is of the same standard as the old Stage 2. There will be what is called the *Higher* examination ; it is of a standard higher than the old Stage 3. Examination papers for the years 1910, 1911, and 1912 are here given, with the answers, and where they are necessary, with suggestions as to methods of working. I do not give the papers of earlier years, as almost every question in them has been incorporated in this book already as an example or exercise.

1912. Lower Examination.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks :

(*a*) Without using logarithms compute by contracted methods

$$3{\cdot}207 \times 0{\cdot}01342 \div 9{\cdot}415.$$

(*b*) Using logarithms compute the square root of

$$62{\cdot}41 \times 0{\cdot}1352 \div 2{\cdot}416.$$

(*c*) State the values of the sine, cosine, and tangent of 230°.

(*d*) State the value of the Napierian logarithm of 13520.

Ans. (*a*) 0·004571 ; (*b*) 1·869 ; (*c*) −0·7660, −0·6428, 1·1918 ; (*d*) 9·5118.

2. The three parts (*a*), (*b*), and (*c*), must all be answered to get full marks :

(*a*) A hollow circular cylinder of cast iron is 10 inches long and 3 inches inside diameter ; what is the outside diameter if the cylinder weighs 30 lb.? [One cubic inch of cast iron weighs 0·26 lb.]

(*b*) ABC is a right-angled triangle, C being the right angle. If AC is 4 inches and the angle A is 40°, find BC and the area of the triangle.

(*c*) If x is 1·201, find $\frac{1}{2}(e^x + e^{-x})$.

Ans. (*a*) As $W = 30 = 0{\cdot}7854(D^2 - d^2) \times 10 \times 0{\cdot}26$, we have

$$D^2 - 9 = \frac{30}{2{\cdot}6 \times 0{\cdot}7854} = 14{\cdot}69, \quad D^2 = 23{\cdot}69, \quad D = 4{\cdot}867 \text{ inches.}$$

(*b*) As $\frac{BC}{4} = \tan 40 = 0{\cdot}8391$, $BC = 3{\cdot}3564$.

The area is $3{\cdot}3564 \times 2 = 6{\cdot}7128$ sq. inches.

(*c*) 1·813.

3. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks:

(*a*) The sum of x and y is 5·17 and the sum of their squares is 14·25; find x and y.

(*b*) What is the area of the curved surface of a right cone if its base is 3 inches in diameter and vertical height 5 inches?

(*c*) There are two perfectly similar statues of marble; the height of one is 2·13 times the height of the other: the smaller weighs 20 lbs. What is the weight of the other?

Ans. (*a*) As $x + y = 5{\cdot}17$ and $x^2 + y^2 = 14{\cdot}25$, we have

$$2xy = 5{\cdot}17^2 - 14{\cdot}25 = 12{\cdot}48. \quad \text{Hence } x^2 - 2xy + y^2 = 1{\cdot}77.$$

That is, $x - y = 1{\cdot}33$. Adding this to $x + y$, we get

$$2x = 6{\cdot}50 \quad \text{or} \quad x = 3{\cdot}25 \quad \text{and} \quad y = 1{\cdot}92.$$

(*b*) The slant height is $\sqrt{25 + 2{\cdot}25} = 5{\cdot}22$. The circumference of the base, which is the length of arc of a sector in the developed curved area, is 9·4248 inches; the area is $\frac{1}{2}(9{\cdot}4248 \times 5{\cdot}22)$ or 24·6 sq. inches.

(*c*) Volumes of similar things are as the cubes of their like dimensions, and heights are proportional to volumes if the materials are the same. The larger weighs $2{\cdot}13^3$ times or 9·664 times the other; that is, 193·28 lb.

4. If $y = x^2 - 3{\cdot}39x + 1{\cdot}95$ for values of x from 0 to 3, plot sufficient points of the curve on squared paper to show for what values of x, y is 0. What are these values of x? *Ans.* 2·655 and 0·735.

5. When the pointer of a planimeter is guided once round the boundary line of a plane figure, the reading of the instrument R is such that the area A is CR, where C is some constant.

If R is 22·48 for a circle of 3 inch radius, what is C, the area being required in square inches? On applying the instrument now to an indicator diagram, R is found to be 3·77; what is the area? The length of the diagram being 4·11 inches, what is its average breadth?

Ans. 1·258; 4·742 sq. inches; 1·1538 inches.

6. A disc whose outside radius is r_0 and inside radius r_1 is rotating; the radial stress P and the hoop stress Q, at any radius r, are

$$P = r_0^2 + r_1^2 - r^2 - \frac{r_0^2 r_1^2}{r^2},$$

$$Q = r_0^2 + r_1^2 + \frac{r_0^2 r_1^2}{r^2} - 0{\cdot}538r^2.$$

If $r_0 = 10$, $r_1 = 4$, write out the expressions for P and Q. Now calculate the values of P and Q for the following values of r: 4, 6, 8, 10, and show them in two curves.

Ans. $P = 116 - \frac{1600}{r^2} - r^2$, $Q = 116 + \frac{1600}{r^2} - 0{\cdot}538r^2$.

r	P	Q
4	0	207·4
6	35·56	141·1
8	27	106·6
10	0	78·2

When these are plotted I am afraid that more points are needed to show the curve P properly. It is evident that P is a maximum when $r=6{\cdot}325$.

7. The insulation resistance R of a piece of submarine cable is being measured; it has been charged and the voltage v is diminishing according to the law

$$v=be^{-t/KR},$$

where b is some constant, t is the time in seconds; K is known to be $0{\cdot}8\times10^{-6}$.

If v is noted to be 30 and in 15 seconds afterwards it is noted to be 26·43, find R.

Ans. When $t=0$ let $v=30$; when $t=15$ let $v=26{\cdot}43$,

$$v=30e^{-t/KR}, \text{ so that } 26{\cdot}43=30e^{-15/Kr}.$$

Therefore
$$\log_e\frac{30}{26{\cdot}43}=\frac{15}{KR},$$

$$R=15\div K\log_e\frac{30}{26{\cdot}43};$$

working this out, we have $R=148\times10^6$.

8. x and y are as tabulated. It is known that

$$u=5y+10\frac{dy}{dx};$$

find u approximately in the middle of each interval. Show y and u as two curves, x being abscissa:

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
y	5·000	5·736	6·428	7·071	7·660	8·192	8·660	9·063	9·397	9·659	9·848

u and y need not be plotted to the same scale.

Ans. Placing x and y in columns, I find in each interval:

x	y	$\frac{\delta y}{\delta x}$	u
0·05	5·368	7·36	100·44
0·15	6·082	6·92	99·61
0·25	6·750	6·43	98·05
0·35	7·366	5·89	95·73
0·45	7·926	5·32	92·83
0·55	8·426	4·68	88·93
0·65	8·862	4·03	84·61
0·75	9 230	3·34	79·55
0·85	9·528	2·62	73·84
0·95	9·754	1·89	67·67

The values of y here tabulated for the intermediate times are the means of those given in the question. Thus 5·368 is $\frac{1}{2}(5{\cdot}000+5{\cdot}736)$.

9. A pin at one end of a horizontal lever is 42 inches from the fulcrum. If the lever turns upwards 70°, find

(i) the length of the path traversed by the pin,
(ii) its distance from its original position,
(iii) its height above its original level.

What would these answers have been had the angle turned through been only 7°? *Ans.* 51·31, 48·17, 39·47 ; 5·132, 5·124, 5·120.

10. There is a probability that if a man stands at so short a distance as d from the muzzle of a gun which discharges a projectile of weight w, his sense of hearing will be hurt. If d is proportional to the sixth root of w and if d is 10 feet for the discharge of a 64 lb. shot, what is d for the discharge of a 9 lb. shot?

Ans. $d=cw^{\frac{1}{6}}$, c being a constant. Therefore

$$c=10\div 64^{\frac{1}{6}}=5 \quad \text{and} \quad d=5\times 9^{\frac{1}{6}},$$

so that the answer is 7·211 feet.

Perhaps this explains why so many artillery men are deaf. They think that because their ear is not hurt at, say, 11 feet from the muzzle of a large gun, they may come quite close to a small gun with impunity.

11. A square whose side is 4 inches has one diagonal parallel to an axis which lies in the plane of the square at the distance of 3″ from the diagonal. The square revolves about the axis and generates a ring. What are the volume and area of the ring?

Ans. 301·6 cubic inches ; 301·6 square inches.

12. The ends of a round barrel are 40 inches in diameter and the mid section is 48 inches in diameter: the barrel is 60 inches long. What is its volume? For what shape of barrel is your approximate rule quite exact?

Ans. The average section is $\frac{\pi}{4}(40^2+40^2+4\times 48^2)\div 6$, and 60 times this is the volume, or 97,513 cubic inches. If x is distance along the axis of the barrel from any point in the axis, for Simpson's rule to be correct we must have the cross-section A following a law

$$A=a+bx+cx^2,$$

where a, b and c may have any values. Of surfaces of revolution there are several for which the rule is exact, but the only one (I believe) that suits the barrel shape is the ellipsoid of revolution.

13. A ship going at 21 knots changes its direction steadily from due North to North-west in two minutes: what is the radius of its path?

The angle 45° is 0·7854 radians = arc ÷ radius and arc $=21\times 2\div 60$, so that $r=21\times\frac{2}{60}\div 0{\cdot}7854=0{\cdot}891$ nautical mile or 5420 feet.

14. The lengths of the intercepts of a plane on the three mutually perpendicular axes are $OA=3$, $OB=4$, $OC=5$. Find the length OP of the perpendicular from the origin on the plane. *Ans.* $OP=2{\cdot}164$.

1912. Higher Examination.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks:

(*a*) Without using logarithms, compute by contracted methods, so that four significant figures *shall be correct*—

$$10{\cdot}32 \times 0{\cdot}005231 \div 0{\cdot}02076.$$

(*b*) Using logarithms, compute cosh *a* and sinh *a*, where

$$a = 1{\cdot}013.$$

(*c*) Using logarithms, compute

$$(3{\cdot}062 \div 27{\cdot}15)^{-1{\cdot}23}.$$

(*d*) If i means $\sqrt{-1}$, express $-2{\cdot}35 + 1{\cdot}96i$ in the form $r(\cos\theta + i\sin\theta)$ and extract its cube root.

Ans. (*a*) 2·600.

(*b*) As $e^a = 2{\cdot}754$ and $e^{-a} = 0{\cdot}3632$, $\cosh a = 1{\cdot}5586$, $\sinh a = 1{\cdot}1954$.

(*c*) 14·65.

(*d*) See Art. 146. $3{\cdot}060(\cos 140^\circ{\cdot}167 + i\sin 140^\circ{\cdot}167)$.
One of its cube roots is

$$1{\cdot}452(\cos 46^\circ{\cdot}72 + i\sin 46^\circ{\cdot}72) \text{ or } 0{\cdot}9956 + 1{\cdot}056i.$$

2. When the pointer of a planimeter is guided once round the boundary of a plane figure, the reading R of the instrument gives the enclosed area in units which depend on the position S in which the roller frame is clamped on a graduated arm. For a circle of 3 inches radius and for three positions S, the readings are as follows:

S	24	27	30
R	25·29	22·48	20·23

There is a simple relation connecting S and R, find it. Find also the two positions S for which the readings will be the area in square inches and in square centimetres respectively.

Ans. It is well known that SR is constant; but a candidate ignorant of this would probably try plotting $\log S$ and $\log R$; and he will find that $SR = 607$. The area of the circle is 28·274 square metres or 182·42 sq. cm.; dividing these into 607, we find the two positions S to be 21·47 and 3·33.

3. The following values of x and y were observed in a laboratory, and theory suggested that there might be a law

$$y = ax + b\log x.$$

There are errors of observation. Using squared paper, try if there is such a law and, if so, find the most probable values of a and b. You may take $\log x$ as being $\log_{10} x$.

x	10·2	31·0	52	75	104	132	181
y	3·75	6·26	7·99	9·54	11·39	12·94	15·67

Ans. In any equation connecting x and y, if there are only two constants it is possible to plot things which will give a straight line. Thus, in this case we can

(1) Plot $\frac{x}{y}$ and $\frac{\log x}{y}$ and get a straight line.

(2) Plot $\frac{y}{x}$ and $\frac{\log x}{x}$ and get a straight line.

(3) Plot $\frac{y}{\log x}$ and $\frac{x}{\log x}$ and get a straight line.

I find $$y = 0{\cdot}045x + 3{\cdot}3 \log_{10} x.$$

4. If $y = Ae^{ax}$, what is $\frac{dy}{dx}$? An electric condenser, of capacity K farads and leakage resistance R ohms, has been charged, and the voltage v is diminishing according to the law

$$\frac{dv}{dt} = -\frac{v}{KR}.$$

Express v in terms of the time, t seconds. If $K = 0{\cdot}8$ micro-farad (that is, $0{\cdot}8 \times 10^{-6}$ farad); if v is noted to be 30, and 15 seconds afterwards it is noted to be 26·43, find R.

See Art. 113. If $y = Ae^{ax}$, $\frac{dy}{dx} = aAe^{ax} = ay$.

Hence $$v = v_0 e^{-t/KR} \quad \text{or} \quad \log_e \frac{v_0}{v} = \frac{t}{KR}.$$

If $t = 0$ when $v = 30$, evidently $v_0 = 30$; therefore

$$\log \frac{30}{26{\cdot}43} = \frac{15}{KR} \quad \text{or} \quad R = \frac{15}{K \log \frac{30}{26{\cdot}43}}.$$

The Napierian logarithm of $\frac{30}{26{\cdot}43}$ is 0·1266,

$$R = \frac{15 \times 10^6}{0{\cdot}8 \times 0{\cdot}1266} = 148 \times 10^6 \text{ ohms} = 148 \text{ megohms}.$$

5. A periodic function is given either as a curve or in a table of 12 or 18 or 24 or more equidistant values; describe how you would find the values of the various terms of its Fourier Series.

One answer to this is given in Art. 132. See also page 323.

6. There is a value of x between 2 and 2·5 which satisfies the equation

$$1{\cdot}5x - \frac{1}{3}x^2 + 3 \sin \frac{x}{2} - 5 \log_{10} x = 2{\cdot}6.$$

Find it, correct to three significant figures. *Ans.* 2·340.

7. Solve $$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2 x = 0.$$

Take $n^2 = 200$, $f = 7{\cdot}485$; let $x = 0$ and also $\frac{dx}{dt} = 10$ when $t = 0$.

See Arts. 121 and 133.

Ans. The auxiliary equation is $m^2 + 2fm + n^2 = 0$, and its roots are

$$m = -f \pm \sqrt{f^2 - n^2};$$

in this case $$m = -7{\cdot}485 \pm 12i \quad \text{if} \quad i = \sqrt{-1}.$$

Therefore $$x=e^{-7\cdot485t}(A\sin 12t+B\cos 12t).$$

If $x=0$ when $t=0$, it is evident that $B=0$, and if so,

$$\frac{dx}{dt}=Ae^{-7\cdot485t}(-7\cdot485\sin 12t+12\cos 12t).$$

If $\dfrac{dx}{dt}=10$ when t is 0, $A=\dfrac{10}{12}=0\cdot8333.$

Hence $$x=0\cdot8333e^{-7\cdot485t}\sin 12t.$$

8. The rate per unit increase in volume at which a pound of gas is receiving heat during its expansion is

$$\frac{dH}{dv}=\frac{1}{\gamma-1}\left(v\frac{dp}{dv}+\gamma p\right).$$

If $pv^n=c$, find $\dfrac{dH}{dv}$ in terms of p. γ, n, and c are constants.

For what value of n is $\dfrac{dH}{dv}=0$?

Ans. $p=cv^{-n}$; therefore $\dfrac{dp}{dv}=-ncv^{-n-1}=-n\dfrac{p}{v}$,

so that $$v\frac{dp}{dv}=-np \quad\text{and}\quad \frac{dH}{dv}=\frac{\gamma-n}{\gamma-1}p.$$

This is 0 when $n=\gamma$, and the expansion is then said to be adiabatic.

9. Describe a method of finding whether a given curve or the tabulated values of x and y follow, approximately, the law

$$y=a+bx^n$$

or

$$y=b(x+a)^n$$

or

$$y=a+be^{nx}.$$

This question is answered in Art. 79.

10. In a telephone line of length l, where q is $2\pi f$, if f is frequency or pitch of a musical note; let r, the resistance per mile, be 88; let k, the capacity per mile, be $0\cdot05\times10^{-6}$. Take $q=5000$. Let $n=\sqrt{rkqi}$, where i means $\sqrt{-1}$. Let $l=40$ miles. If $R=100+0\cdot04qi$ be the resistance of the receiving telephone, the current through it is

$$C=2V_0\div\left(R+\frac{r}{n}\right)e^{ln},$$

where V_0 is $10\sin 5000t$. C is of the form $a\sin(qt+b)$, where q is 5000. Find a and b.

[Suggestion: $\alpha+\beta i$ may be put in the form $\rho(\cos\theta+i\sin\theta)$; if this operates upon $m\sin qt$ the result is $m\rho\sin(qt+\theta)$. Note also that

$$e^{\beta i}=\cos\beta+i\sin\beta.]$$

See Art. 146 and Chap. XXXV.

$$n=\sqrt{88\times0\cdot05\times10^{-6}\times5000i}=0\cdot1483\sqrt{i}=0\cdot1049+0\cdot1049i=0\cdot1483[45°],$$

$$\frac{r}{n}=\frac{88}{0\cdot1483[45°]}=593[-45°]=419-419i.$$

Hence $R+\frac{r}{n}=100+200i+419-419i=519-219i=563[-22^\circ{\cdot}9]$,

$$ln=4{\cdot}196+4{\cdot}196i \quad \text{and} \quad e^{ln}=e^{4{\cdot}196}e^{4{\cdot}196i}.$$

Now 4·196 radians = 240°·43 and $e^{4{\cdot}196}=66{\cdot}42$, so that $e^{ln}=66{\cdot}42[240^\circ{\cdot}43]$.

$$C=\frac{20\sin 5000t}{563[-22^\circ{\cdot}9]\times 66{\cdot}42[240^\circ{\cdot}43]}=0{\cdot}0005348\sin(qt-217^\circ{\cdot}5).$$

11. We wish the current-turns in a certain telegraph coil to be as great as possible, and this will be the case when the following expression is a minimum:

$$\left(\frac{R}{t}+\frac{500}{t}\right)^2+\left(\frac{Lq}{t}-\frac{500}{t}\right)^2.$$

The resistance of the coil is $R=at^2$ ohms, where a is a constant and t is the number of turns of wire. The inductance of the coil is $L=bt^2$. The constant q is known.

Show that we get the best result when $R=500$ and $Lq=500$.

This is proved in Art. 158. It is, however, well for the practical man to remember that $R=500$ ohms may mean a resistance for steady constant currents of 120 ohms, there being a spurious resistance due to eddy currents and hysteresis in the iron or steel.

12. A telephone transmitter has a varying resistance

$$R=10+0{\cdot}1\sin 5000t,$$

the variation being due to a musical note; what is the frequency or pitch of this note? If there is a battery of electromotive force E which is 6 volts, the current is $E\div R$. Show that the current has a varying term of the above frequency, but that there are an octave and higher harmonics which are quite insignificant because 10, the constant part of R, is large. This explains why the common battery system gives better articulation than the local battery system.

Ans. The current is $6\div(10+0{\cdot}1\sin 5000t)=0{\cdot}6\div(1+0{\cdot}01\sin 5000t)$.

Now, as a sine can never exceed 1 or be less than -1, calling $0{\cdot}01\sin 5000t=\alpha$, we may say that

$$0{\cdot}6\div(1+\alpha)=0{\cdot}6(1-\alpha)=0{\cdot}6-0{\cdot}006\sin 5000t.$$

That is, the varying part of the current is what we desire it to be. The answer is really $0{\cdot}6(1-\alpha+\alpha^2-\alpha^3+\text{etc.})$, and the α^2, α^3 terms really mean the introduction of terms like $\sin 10000t$, $\sin 15000t$, etc., that is, the octave and higher harmonics; but it is seen that they are utterly insignificant in this case because α is so small, and this is due to 10, the constant part of R, being large.

13. The lengths of the intercepts of a plane on the axes are $OA=3$, $OB=4$, $OC=5$. Find the length of OP, the perpendicular from the origin on the plane. Find the angles which OP makes with the axes.

Ans. $OP=2{\cdot}17$, $\alpha=43^\circ{\cdot}7$, $\beta=57^\circ{\cdot}25$, $\gamma=64^\circ{\cdot}35$.

14. The curve $y=1+0{\cdot}2x^2$ rotates round the axis of x, generating a surface of revolution. What is its volume between the cross-section at $x=0$ and the cross-section at $x=10$?

Ans. The integral of πy^2 or $\pi(1+0{\cdot}4x^2+0{\cdot}04x^4)$ is

$$\pi\left(x+\frac{0{\cdot}4}{3}x^3+\frac{0{\cdot}04}{5}x^5\right);$$

taking $x=10$, the answer is $943{\cdot}3\pi$ or 2963.

1911. Stage 2.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks:

(*a*) Without using logarithms, compute by contracted methods, correct to four significant figures,

$$23{\cdot}56 \times 0{\cdot}1023 \div 2{\cdot}363.$$

Compute to *five* significant figures, and reject the last. *Ans.* 1·020.

(*b*) Using logarithms, compute the cube root of

$$1782 \div 0{\cdot}3152.$$ *Ans.* 17·81.

(*c*) State the values of $\sin 26°$, $\cos 110°$, $\sin 220°$.

Ans. 0·4384, −0·3420, −0·6428.

(*d*) A man is 68 inches in height; what angle does he subtend at the distance of 300 yards? Why are the sine and tangent of this angle and the angle itself, stated in radians, practically the same?

[If a figure of a very small angle is drawn, the reason can be put in a few words. The distance is 300×36 or 10,800 inches. The angle in radians is $68 \div 10800$, and we multiply by 57·3 to get it in degrees. *Ans.* 0·3608 degree or 21·65 minutes.]

2. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks:

(*a*) A hollow sphere of outside diameter D and inside diameter d. What is its volume? What is its weight W if the material weighs w lb. per unit volume?

The weight W is 10 lb., the inside diameter is 6 inches. What is the outside diameter if w is 0·3 lb. per cubic inch?

(*b*) What are the factors of $x^2 - 3{\cdot}25x + 1{\cdot}56$?

(*c*) The area of a rectangle is 22·4 square inches and its perimeter is 19·2 inches; what are the lengths of the sides?

Ans. (*a*) The volume of a sphere is $\frac{4}{3}\pi r^3$ or $0{\cdot}5236d^3$. Hence

$$W = 0{\cdot}5236w(D^3 - d^3).$$

Putting $W=10$, $d=6$, $w=0{\cdot}3$, we have $10 = 0{\cdot}1571(D^3 - 216)$ or $D^3 - 216 = 63{\cdot}65$ or $D^3 = 279{\cdot}65$ or $D = 6{\cdot}539$ inches.

(*b*) The factors cannot readily be seen by inspection. Let the expression be $(x-a)(x-b)$; then, if it is put equal to 0, the roots of the equation $x^2 - 3{\cdot}25x + 1{\cdot}56 = 0$ are a and b. Now the roots of this quadratic are 2·664 and 0·586. Therefore the factors are $x - 2{\cdot}664$ and $x - 0{\cdot}586$.

(*c*) Let x and y be the lengths of the sides. We have $xy = 22{\cdot}4$ and $x + y = 9{\cdot}6$. Squaring $x+y$,

$$x^2 + 2xy + y^2 = 92{\cdot}16,$$
$$4xy = 89{\cdot}6.$$

Therefore
$$x^2 - 2xy + y^2 = 2{\cdot}56,$$
$$x - y = 1{\cdot}6,$$
$$x + y = 9{\cdot}6,$$
$$2x = 11{\cdot}2 \text{ or } x = 5{\cdot}6 \text{ inches},$$
$$2y = 8 \quad \text{or } y = 4 \text{ inches}.$$

3. If sin $x \div x$ is 0·75, where x is in radians, what is x? Use squared paper.

Ans. The equation is $\sin x - 0{\cdot}75x = 0$. Let $y = \sin x - 0{\cdot}75x$. Take values of x and calculate y. Looking at your table, which gives x and $\sin x$, you see that x cannot be small; if you are quick in seeing whether $\frac{3}{4}x$ is nearly the same as $\sin x$, you will probably begin at about 65 degrees, and you are naturally led to work 70° and 75°.

Angle in degrees.	x	$\sin x$	y
65°	1·1345	0·9063	0·0556
70°	1·2217	0·9397	0·0234
75°	1·3090	0·9659	−0·0159
73°	1·2741	0·9563	0·0007

Plotting the three values of y and x roughly on squared paper, you will be directed to try $x = 73$, and you calculate $y = 0{\cdot}0007$. As the curve between $x = 73$ and $x = 70$ is nearly straight, you may take it to be really straight, as you are so nearly right; or by arithmetic;—for a difference of 3 degrees, the difference in y is 0·0227; for how many degrees is $\delta y = 0{\cdot}0007$? *Ans.* $3 \times \frac{7}{227}$ or 0·09 degree. The answer is then 73° − 0·09 or 73°·09. I might have worked in radians instead of degrees. The answer is 73°·09 ÷ 57·3 or 1·276 radians.

4. Part of a roof has an area of 150 sq. feet; its inclination to the horizontal is 37°: what is the area of its plan? Prove the rule which you use. *Ans.* $150 \times \cos 37°$ or 119·8 sq. feet. The proof is well known.

5. The following corresponding values of x and y were measured. There may be errors of observation. Test if there is a probable law

$$y = a + bx^2,$$

and, if this is the case, what are the probable values of a and b?

x	1·00	1·50	2·00	2·30	2·50	2·70	2·80
y	0·77	1·05	1·50	1·77	2·03	2·25	2·42

Ans. Tabulate the values of x^2, plot x^2 and y, and we see at once that the points lie nearly in a straight line. Taking the line that lies most evenly among the points, we find

$$y = 0{\cdot}54 + 0{\cdot}24x^2.$$

6. A railway train is at the distance s miles from the terminus at the time t hours from starting. Do not plot s and t. What is its average speed in miles per hour in each tabulated interval of time? Assume that this is really the speed in the middle of the interval, and now plot time and speed on squared paper. What is the speed and what is the acceleration when $t = 1{\cdot}07$?

t	1·03	1·05	1·07	1·09	1·11	1·13
s	20·15	20·76	21·42	22·13	22·88	23·67

Ans. Speed, 34·25 miles per hour ; acceleration, 125 miles per hour per hour.

7. If $s=10t^2$, where s is the space in feet which has been passed through by a body in t seconds, find s when $t=3$. Now find the space when $t=3+m$. What is the space passed in the interval m seconds after $t=3$? What is the average speed during this interval? What is this as m gets smaller and smaller?

$$s=10\times 9=90,$$
$$s+\delta s=10(9+6m+m^2),$$
$$\delta s=60m+10m^2,$$
$$\frac{\delta s}{\delta t}=60+10m.$$

This becomes $\frac{ds}{dt}=60$ feet per second, the true speed when $t=3$, more and more exactly as m gets smaller and smaller.

8. The speed of the rim of a centrifugal pump or fan being V feet per second, the head H feet produced when Q cubic feet of fluid per second are passing seems from theory to be such that

$$\frac{H}{V^2}=a+b\frac{Q}{V}-c\frac{Q^2}{V^2}.$$

The following observations were made on a Rateau Fan :

Q	V	H
1243	96·5	206·5
825	112	371
46	96	144

Find the values of a, b, and c. *Ans.* $a=0·0137$, $b=0·0048$, $c=0·000272$.

9. Three planes of reference mutually perpendicular meet at O. The distances of a point P from the three planes are $x=0·25$, $y=4·6$, $z=1·2$. The distances of a point Q are $x=0·57$, $y=6·82$, $z=2·5$. What is the distance from P to Q? *Ans.* 2·592.

10. A circle whose diameter is 2 inches rotates about a line in the same plane which is 3 inches from the centre of the circle ; thus a ring is generated. What is the area of the surface, and what is the volume of the ring? *Ans.* 118·5 sq. inches, 59·23 cubic inches.

11. The wheel on a vertical shaft is slightly out of balance ; it rotates at n revolutions per minute ; the bending moment M has the value

$$M=\frac{an^2}{1-bn^2},$$

where $a=10^{-3}$, $b=10^{-5}$.

What is the critical speed at which the shaft will fracture; that is, at which M is exceedingly great? *Ans.* $n=316$.

At a speed of twice this critical speed, what is the bending moment? *Ans.* $M=-133$.

At a speed of half this critical speed, what is the bending moment? *Ans.* $M=33$.

12. In any class of water turbines, if R is the mean radius of the wheel where the water enters, if H is the fall in feet and if P is the total power of the fall, it is found that
$$R \propto P^{\frac{1}{2}} H^{-\frac{3}{4}}.$$

In one case, where $P=100$ and $H=10$, R was 1·5 feet. Find R for a turbine of the same class when P is 250 and H is 50.

Ans. $1{\cdot}5 = c \times 100^{\frac{1}{2}} \times 10^{-\frac{3}{4}}$, so that $c=0{\cdot}845$. Therefore $R=0{\cdot}845 P^{\frac{1}{2}} H^{-\frac{3}{4}}$, so that in the special case $R=0{\cdot}7094$ feet.

1911. Stage 3.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks:

(*a*) Without using logarithms, compute by contracted methods, so that four significant figures *shall be correct*,
$$23{\cdot}56 \times 0{\cdot}01023 \div 0{\cdot}02563.$$

(*b*) Using logarithms, compute
$$(178{\cdot}2 \div 0{\cdot}3152)^{-0{\cdot}4}.$$

(*c*) State the values of $\cos 110°$, $\sin 220°$, $\tan 220°$, $\sin 320°$, $\tan 320°$, $\sin 580°$.

(*d*) If i means $\sqrt{-1}$, express $5+4i$ in the form
$$r(\cos\theta + i\sin\theta).$$

Now extract its square root.

Ans. (*a*) Working to five figures and rejecting the last, I get 9·404.

(*b*) The logarithm of $178{\cdot}2 \div 0{\cdot}3152$ is 2·7523. This multiplied by $-0{\cdot}4$ is $-1{\cdot}1009$ or $\bar{2}{\cdot}8991$, so that the answer is 0·07927.

(*c*) $-0{\cdot}3420$, $-0{\cdot}6428$, $0{\cdot}8391$, $-0{\cdot}6428$, $-0{\cdot}8391$, $-0{\cdot}6428$.

(*d*) As $r\cos\theta=5$, $r\sin\theta=4$, $\tan\theta=0{\cdot}8$, $\theta=38°{\cdot}66$, and $r^2=25+16$ or $r=6{\cdot}403$. The answer is $6{\cdot}403(\cos 38°{\cdot}66 + i\sin 38°{\cdot}66)$, and is often written $6{\cdot}403[38°{\cdot}66]$. The square root is
$$r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$
by Demoivre's theorem, or
$$2{\cdot}531[19°{\cdot}33] \text{ or } 2{\cdot}531(\cos 19°{\cdot}33 + i\sin 19°{\cdot}33) = 2{\cdot}388 + 0{\cdot}8377i.$$

2. A crank 1 foot in length rotates uniformly, making one revolution per second; the connecting rod is 5 feet long; state the distance of the cross-head from the end of the stroke as a function of the time. Show

that the motion is very nearly a simple harmonic motion combined with one of half the period.

Ans. In Fig. 61, let OP the crank be of length r, the connecting rod PQ of length l, let $AQ=s$ the distance of the cross-head from the end

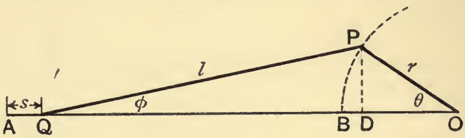

FIG. 61.

of its stroke. Let the angle QOP be θ. Let the angle OQP be ϕ. In any such problem, if we project a closed figure upon any two straight lines, we get two equations.

Projecting horizontally and then vertically, we have as $AO=l+r$,

$$\left.\begin{aligned} s+l\cos\phi+r\cos\theta=l+r,\\ l\sin\phi=r\sin\theta.\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(1)$$

If we eliminate ϕ, we get s in terms of r. As

$$\sin\phi=\frac{r}{l}\sin\theta,\quad \cos\phi=\sqrt{1-\frac{r^2}{l^2}\sin^2\theta},$$

so that the top equation becomes

$$s=l\left\{1-\sqrt{1-\frac{r^2}{l^2}\sin^2\theta}\right\}+r(1-\cos\theta).$$

Now l is five times r, and we may approximately consider the square root term as $\sqrt{1-a}=1-\frac{1}{2}a$. We find

$$s=\frac{r^2}{2l}\sin^2\theta+r-r\cos\theta.$$

This is the same as

$$s=r+\frac{r^2}{4l}-r\cos\theta-\frac{r^2}{4l}\cos 2\theta.$$

Let $\theta=qt$, where q is the angular velocity of the crank, so that we count time from the inner dead-point position, and we have

$$s=r+\frac{r^2}{4l}-r\cos qt-\frac{r^2}{4l}\cos 2qt$$

$$\text{or}\quad s=1-\cos qt+0{\cdot}05(1-\cos 2qt),\quad\text{where}\quad q=2\pi.$$

3. A fly-wheel is rotating at a radians per second at the time t seconds. If M is the moment acting, if fa is a fluid friction and is the only resistance, and I the moment of inertia of the wheel,

$$M=fa+I\frac{da}{dt}.$$

Take $f=200$ and $I=5000$. Find M if $a=20+0{\cdot}1\sin 12t$.

Ans. $\dfrac{da}{dt}=1{\cdot}2\cos 12t.$

$$M=4000+20\sin 12t+6000\cos 12t.$$

This may be put in the shape (I take $\sqrt{6000^2+20^2}$ to be 6000)

$$M=4000+6000\sin(12t+89^\circ{\cdot}81).$$

4. Find an expression for the area of the curve

$$y=a+bx+cx^n+ke^{mx}+h\sin qt.$$

Show clearly why you integrate y. What is the meaning of the constant which you add?

Ans. $A=\int y\,.\,dx=C+ax+\frac{1}{2}bx^2+\frac{c}{n+1}x^{n+1}+\frac{k}{m}e^{mx}-\frac{h}{q}\cos qt.$

5. Find two values of x to satisfy $\tan x\div x=3$.

Ans. $x=1{\cdot}324$ or $75^\circ{\cdot}875$. Also $x=4{\cdot}64$ or $265^\circ{\cdot}85$.

6. If R is the resistance, L the self-induction, K the capacity of a condenser in a part of an electric circuit, V the voltage between its ends, and C the current, show that

$$V=\left(R+L\theta+\frac{1}{K\theta}\right)C,$$

where θ means $\frac{d}{dt}$ if t is time. If $C=a\sin qt$, show that the effect of self-induction in a circuit may be destroyed by putting in a suitable condenser.

Ans. See Art. 127. In this case $\theta=qi$, where i is $\sqrt{-1}$. As $\frac{1}{K\theta}$ is $\frac{1}{Kqi}$, it is $\frac{-i}{Kq}$, so that the operator on C is $R+i\left(Lq-\frac{1}{Kq}\right)$. The condenser needed is such that $Lq-\frac{1}{Kq}=0$, and $K=\frac{1}{Lq^2}$ is its capacity.

7. The x, y, and z co-ordinates of a point P are 5, 4, and 3; find the r, θ, and ϕ co-ordinates. Find also the distance OP and the direction-cosines of OP, if O is the origin.

Ans. $OP=\sqrt{25+16+9}=7{\cdot}071$. The direction-cosines are

$$\cos\alpha \text{ or } l=\frac{5}{7{\cdot}071}=0{\cdot}7071,\quad \cos\beta \text{ or } m=\frac{4}{7{\cdot}071}=0{\cdot}5656,$$

$$\cos\gamma \text{ or } n=\frac{3}{7{\cdot}071}=0{\cdot}4242.$$

Also as $5=r\sin\theta\,.\cos\phi$, $4=r\sin\theta\,.\sin\phi$, $3=r\cos\theta$, and we have already found $r=OP=7{\cdot}071$, $\cos\theta=\frac{3}{7{\cdot}071}=0{\cdot}4242$ and $\theta=64^\circ{\cdot}9$ $\frac{4}{5}=0{\cdot}8=\tan\phi$, so that $\phi=38^\circ{\cdot}66$.

8. Suppose **A** to be a vector quantity and t the time, how do we measure $d\mathbf{A}/dt$? The value of the vector may be measured as a_θ, where a is the amount and θ is the angle measured anti-clockwise from a fixed direction. The vector keeps in a plane. A point has the following velocities in feet per second at the following times (seconds):

Velocity -	100_{30°	$103{\cdot}3_{35^\circ}$	$105{\cdot}7_{42^\circ}$	$107{\cdot}2_{51^\circ}$	$107{\cdot}8_{62^\circ}$
Time - -	10	10·01	10·02	10·03	10·04

Find approximately the value of the acceleration when t is 10·02. A graphical method is best.

Ans. Refer to Art. 167. I get 1480_{126} feet per second per second.

9. In a hollow cylinder of nickel steel subjected to internal pressure p, and no pressure outside, when the material is all yielding, if p is the radial compressive stress and f the hoop tensile stress at a point whose distance from the axis is r, and if $f+ap=b$ where a and b are constants for a particular kind of steel, and if we also have the usual relation

$$r\frac{dp}{dr}+p+f=0,$$

find p as a function of r. If the inside radius r_1 is 3 inches and the inside p_1 is 30 tons per sq. inch, what is r_0, the outer radius?

[Take for nickel steel $a=\frac{3}{4}$, $b=30$.]

Ans. Substituting for f from the first equation in the second,

$$r\frac{dp}{dr}+(1-a)p+b=0.$$

Multiplying by $\frac{dr}{r}$ and dividing by $(1-a)p+b$, we have

$$\frac{dp}{(1-a)p+b}+\frac{dr}{r}=0.$$

Integrating, $$\frac{1}{1-a}\log\{(1-a)p+b\}+\log r=\text{constant},$$

$$\log\{(1-a)p+b\}+(1-a)\log r=\text{constant},$$

$$\{(1-a)p+b\}r^{1-a}=\text{constant}.$$

Taking $\quad a=\frac{3}{4},\ b=30,\ (\frac{1}{4}p+30)r^{\frac{1}{4}}=C.$

Now $\quad p=30$ when $r=3$; $\quad\therefore\ 37\cdot5\times3^{\frac{1}{4}}=C.$

Where $\quad p=p_0=0$ at the outside, $30r_0^{\frac{1}{4}}=37\cdot5\times3^{\frac{1}{4}}$,

$$\frac{r_0}{3}=\left(\frac{37\cdot5}{30}\right)^4 \quad\text{or}\quad r_0=7\cdot323.$$

[The examiner has not asked this interesting question: Find p and f everywhere and plot on squared paper. We find that

$$p=150\left(\frac{3}{r}\right)^{\frac{1}{4}}-120,$$

$$f=120-112\cdot5\left(\frac{3}{r}\right)^{\frac{1}{4}}.$$

Such a question is of even greater interest when, as in the case of wrought iron and mild steel $a=1$. Then $p=p_1+b\log\frac{r_1}{r}$ and $f=b-p$.]

10. A hollow cylinder of cast iron of length 10 inches weighs 62 lb. The inside diameter is 0·78 of the outside diameter. If a cubic inch of cast iron weighs 0·26 lb., find the inside and outside diameters.

Ans. $W=0\cdot26\times0\cdot7854\times10(D^2-d^2)=2\cdot042(D^2-0\cdot6084D^2)$,

$$62=0\cdot7996D^2 \quad\text{or}\quad D=8\cdot805,\quad d=6\cdot868 \text{ inches.}$$

11. When air or steam is flowing from a vessel where the pressure is p_1 through a rounded orifice into an outside atmosphere, if p is the pressure in the orifice, the weight of fluid flowing per second is proportional to

$$x^{2n}-x^{1+n},$$

when x is p/p_1 and n is a known constant, being 0·8850 for nearly dry steam. For what value of x is the flow of nearly dry steam a maximum?

Ans. Calling the weight of fluid per second W, $\frac{dW}{dx}$ is

$$2nx^{2n-1}-(1+n)x^n.$$

Putting this equal to 0, we have $x^{n-1}=\frac{1+n}{2n}$. If we take the case of steam when $n=0·885$, we find

$$x^{-0·115}=1·885/1·770 \quad \text{or} \quad x=0·578.$$

That is, there is a maximum flow of steam when the throat pressure is 0·578 of the pressure inside the vessel.

12. The value of a periodic function of t is here given for twelve equidistant values of t covering the whole period. Express it in a Fourier Series. Terms of the fourth and higher orders are negligible: 2·340, 3·012, 3·685, 4·149, 3·685, 2·203, 0·825, 0·513, 0·875, 1·085, 1·189, 1·637.

Ans. I here give the full working out of this question in the manner described in Art. 132, but towards the end I modify the method, because there are few ordinates given.

			A	B	C	D	E	F	G	H
$\phi°$	No. of ordinate.	x	x' or $x-2·1$.	A superimposed on itself.	$A+B$	Half of C being sum of components 2 and 4.	Being D superimposed on itself.	Being $D+E$.	$\frac{1}{2}F$ or component 4 being 0.	$D-G$ being component 2.
0	0	2·340	0·240	−1·275	−1·035	−0·5175	0·5170	0	0	−0·5175
30	1	3·012	0·912	−1·587	−0·675	−0·3375	0·3370	0	0	−0·3375
60	2	3·685	1·585	−1·225	0·360	0·1800	−0·180	0	0	0·1800
90	3	4·149	2·049	−1·015	1·034	0·5170	G continued downwards ☞		0	0·5170
120	4	3·685	1·585	−0·911	0·674	0·3370			0	0·3370
150	5	2·203	0·103	−0·463	−0·360	−0·180			0	−0·180
180	6	0·825	−1·275	M continued upwards ☞				−0·5175	−0·7575	−0·5575
210	7	0·513	−1·587					−0·3375	−1·2495	−1·2495
240	8	0·875	−1·225	1·585	0·240	0·600	0·200	0·1800	−1·4050	−1·6050
270	9	1·085	−1·015	0·103	0·912	0·00	0·00	0·5170	−1·5320	−1·5320
300	10	1·189	−0·911	−1·275	1·585	−0·600	−0·200	−0·6630	−0·2480	−0·0480
330	11	1·637	−0·463	−1·587	2·049	0·00	0·00	−0·1800	−0·2830	−0·2830
Mean Ordinate		2·1		Being A superimposed.	Being A superimposed.	$A+I+J$ being 3 times component 3.	$\frac{1}{3}K$ being component 3.	D repeated.	$A-D$ being components 1 and 3.	$N-L$ being component 1.
			A	I	J	K	L	M	N	

To get the A and e of each component in this case, as we have so few ordinates, I thought it better to plot the ordinates. Thus for component 2 we have only three positive ordinates, 0·1800 at 60°, 0·5170 at 90°, and 0·3370 at 120°. I plotted these and found that A_2 is probably 0·527, $e=260$. And so for the others. My answer is

$$x=2{\cdot}10+1{\cdot}633\sin(\phi+20°)+0{\cdot}527\sin(2\phi+260)+0{\cdot}2\sin(3\phi+90°).$$

13. If r is the radius of the earth and l the distance from the centre of the earth to another heavenly body M whose mass is m, then $m/(l+r)^2$ and $m/(l-r)^2$ are the accelerations towards M at points of the earth farthest from and nearest to M. The tide-producing effects at these points are their differences from m/l^2, which is the acceleration at the earth's centre; prove that the tide-producing effect of M is inversely proportional to the cube of its distance from the centre of the earth: l is supposed to be very great in comparison with r.

This proof is given as an example at the end of Chapter V.

14. The total cost of a certain ship per hour (including interest, depreciation, wages, coal, etc.) is in pounds

$$c=8{\cdot}21+\frac{s^3}{1200},$$

where s is the speed in knots. Express the total cost of a passage of 3400 nautical miles in terms of s. What value of s will make this total cost a minimum? Show that at speeds 10 per cent. greater or less than this the total cost is not very much greater than what it is for the best speed.

This is like Ex. 5 of Art. 104. The passage lasts $3400/s$ hours. The total cost is c multiplied by $3400/s$ or $\frac{27914}{s}+2{\cdot}833s^2$. This is a minimum when $s=17$ knots. Calculating it for the values of 15·3, 17, and 18·7, I find

s	Cost.
15·3	2487
17	2462
18·7	2484

15. In the following table C denotes the radio-activity of a substance, t hours after the observations were commenced. There is reason for believing that

$$\frac{dC}{dt}=aC,$$

where a is a constant.

Try if this is so, and, if so, find the most probable value of a.

t	0	7·9	11·8	23·4	29·2	32·6	49·2	62·1	71·4
C	100	64	47·4	19·6	13·8	10·3	3·7	1·86	0·86

Ans. This is the compound interest law, and if $\frac{dC}{dt}=aC$, then by Art. 113 we see that

$$C=C_0e^{at} \quad \text{or} \quad \log_e C=\log_e C_0+at.$$

Using common logarithms, I plot t and $\log C$ on squared paper and find that the straight line which lies most evenly among the points gives $\log_{10} C = 2 - 0{\cdot}02933t.$

Multiplying by 2·3026 all across and altering, I find

$$C = 100\, e^{-0{\cdot}0675t}.$$

1910. Stage 2.

1. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks:

(*a*) Without using logarithms, compute by contracted methods to four significant figures

$$5{\cdot}306 \times 0{\cdot}07632 \div 73{\cdot}15.$$

(*b*) Using logarithms, compute

$$(22{\cdot}15 \div 4{\cdot}139)^{0{\cdot}86}.$$

(*c*) The value of g, the acceleration (in centimetres per second per second) due to gravity in latitude l is (approximately)

$$980{\cdot}62 - 2{\cdot}6 \cos 2l.$$

Calculate this for the latitude 52°.

(*d*) The gunners' rule is that one halfpenny (the diameter of a halfpenny is one inch) subtends an angle of one minute at the distance of 100 yards. What is the percentage error in this rule?

Ans. (*a*) 0·005536. (*b*) 4·232.

(*c*) $\cos 104° = -0{\cdot}2419$ and g is $980{\cdot}62 + 2{\cdot}6 \times 0{\cdot}2419$ or 981·25.

(*d*) 100 yards is 3600 inches, so that the angle in radians is $1 \div 3600$.

To get this in minutes multiply by $57{\cdot}3 \times 60$. The answer is 0·955 minute; the error is 0·045/0·955 or 4·72 per cent.

2. The four parts (*a*), (*b*), (*c*), and (*d*) must all be answered to get full marks:

(*a*) A hollow cylinder of outside diameter D and radial thickness t is of length l. What is its volume? If D is 4 inches and $t = 0{\cdot}5$ inch, if the volume is 20 cubic inches, find l.

(*b*) Two similar ships A and B are loaded similarly. B is twice the length of A. The wetted area of A is 12,000 square feet and its displacement 1500 tons. State the wetted area and displacement of B.

(*c*) The cross-section of a stream divided by the wetted perimeter of the channel in which it flows is called its Hydraulic Mean Depth. What are the Hydraulic Mean Depths when water flows in a pipe of diameter d (i) when the water fills the pipe, (ii) when it only half fills the pipe?

(*d*) What is the number of which 0·6314 is the Napierian logarithm?

Ans. (*a*) If d is the inner diameter, $V = \frac{\pi}{4} l (D^2 - d^2)$.

Write $d = D - 2t$, and we have $V = \pi l t (D - t)$,

$20 = \pi l \times 0{\cdot}5(4 - 0{\cdot}5) = 1{\cdot}75\pi l$, so that $l = 3{\cdot}638$ inches.

(*b*) The areas are as the squares, and the displacements are as the cubes of the lengths. The wetted area of $B = 12000 \times 4 = 48000$ sq. feet. The displacement of $B = 1500 \times 8 = 12000$.

(*c*) If r is the radius of the pipe. (1) The area is πr^2 and the perimeter $2\pi r$, so that $m = \pi r^2 \div 2\pi r = \frac{1}{2}r$. (2) The area is $\frac{1}{2}\pi r^2$, and the wetted perimeter is πr, so that m is still $\frac{1}{2}r$.

(*d*) As the common logarithm is $0{\cdot}6314 \times 0{\cdot}43429 = 0{\cdot}2742$, the number is 1·880.

3. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks:

(*a*) If $xy^n = a$; if x is 5 when y is 10, and if x is 11 when y is 8, find n and a. What is the value of y when x is 7?

(*b*) The velocity of sound in air is $66{\cdot}3\sqrt{t}$ feet per second, where t is the absolute temperature Centigrade, that is the ordinary temperature plus 273. What is the fractional change of velocity where the temperature alters from 10° C. to 15° C.?

(*c*) Assuming the earth to be a sphere of 8000 miles diameter, what is the circumference of the parallel of latitude 52°? The earth makes one revolution in 24 hours (approximately); what is the speed at latitude 52° in miles per hour?

Ans. (*a*) As $5 \times 10^n = 11 \times 8^n$, we have $\log 5 + n = \log 11 + n \log 8$ or $0{\cdot}6990 + n = 1{\cdot}0414 + 0{\cdot}9031n$, $0{\cdot}0969n = 0{\cdot}3424$, $n = 3{\cdot}5334$. Also

$$a = 5 \times 10^{3{\cdot}5334} = 17080,\quad 0{\cdot}8451 + n \log y = 4{\cdot}2324 \text{ or } y = 9{\cdot}093.$$

(*b*) The velocities are proportional to $\sqrt{283}$ and $\sqrt{288}$ or 1 and $\sqrt{288/283}$ or 1 and 1·009. So that the fractional change of velocity is 0·009 or 0·9 per cent.

(*c*) Circumference 15474 miles, 644·75 miles per hour.

4. There is a natural reservoir with irregular sides. When filled with water to the vertical height h feet above the lowest point, the following is the area A of the water surface in thousands of square feet:

h	0	5	10	20	30	42	50	65	75
A	0	220	322	435	505	560	586	617	624

Find the average value of A between $h = 10$ and $h = 65$.

What is A when h is 36? Find the volume of water which would raise the surface from $h = 35\frac{1}{2}$ to $h = 36\frac{1}{2}$. *Ans.* 520, 536, 536.

5. The energy stored in similar fly-wheels is $E = ad^5n^2$, where d is the diameter and n the revolutions per minute; a is a constant. A wheel whose diameter is 5 feet, revolving at 100 revolutions per minute, stores 18,500 ft.-lb.; find a. What is the diameter of a similar fly-wheel which will increase its store by 10,000 ft.-lb. when its speed increases from 149 to 151 revolutions per minute?

Ans. $a = 5{\cdot}92 \times 10^{-4}$. Also, as $10000 = ad^5(151^2 - 149^2)$ we have

$$d^5 = \frac{10000}{0{\cdot}35514} \quad \text{or} \quad d = 7{\cdot}761 \text{ feet.}$$

6. There is a root of $x^3+5x-11=0$ between 1 and 2; find it, using squared paper, accurately to four significant figures.

Writing $y=x^3+5x-11$, for many values of x calculate y, plot on squared paper, and find for what value of x, y is 0.

We first try $x=1$, and get $y=-5$; then $x=2$, and get $y=7$.

It is therefore evident that our answer lies between these two values of x.

Now try $x=1·5$, and get $y=-0·125$.

Plotting the three points now found on squared paper, I am induced to try $x=1·52$, and at once find that the answer is $x=1·511$.

x	y
1	−5
2	7
1·5	−0·125

7. A steamer is moving at 20 feet per second towards the east; the passengers notice that the smoke from the funnel streams off apparently towards the south-west with a speed of 10 feet per second; what is the real speed of the wind and what is its direction? If solved by actual drawing the work must be accurately done.

Ans. The whole velocity of the smoke is the velocity of the vessel plus the velocity relatively to the vessel; that is, drawing the vectors to scale, we find that the whole velocity of the smoke (that is, of the wind) is 14·73 feet per second in a direction from 28° 40′ north of west or in a direction *to* 28° 40′ south of east.

8. If $y=20+\sqrt{30+x^2}$, take various values of x from 10 to 50 and calculate y. Plot on squared paper. What straight line agrees with the curve most nearly between these values? Express it in the shape $y=a+bx$. *Ans.* $y=21·25+0·973x$.

9. If the force which retards the falling of an object in a fluid is proportional to vs, where v is the velocity of falling and s is the area of the surface of the object, and if the force which accelerates falling is the weight of the object, show that as objects are smaller they fall more and more slowly.

Recollect that of similar objects made of the same materials, the weights are as the cubes, the surfaces are as the squares of like dimensions.

Ans. It is evident that the weight is proportional to $s^{\frac{3}{2}}$ and the velocity of falling is such that $s^{\frac{3}{2}} \propto vs$ or $v \propto s^{\frac{1}{2}}$. Thus take spherical objects, if d is the diameter, $s \propto d^2$, so that $v \propto d$.

Therefore objects of diameters 1, 0·1, 0·01, or 0·001 fall with velocities as 1 to 0·1 or 0·01 or 0·001.

10. A sliding piece is at the distance s feet from a point in its path at the time t seconds. Do not plot s and t. What is the average speed in each interval of time? Assume that this is really the speed in the middle of the interval, and now plot time and speed on squared paper.

s	1·0000	1·1054	1·2146	1·3268	1·4432	1·5624	1·6857	1·8118
t	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7

What is the approximate increase in speed between $t=0·25$ and $=0·35$? What is approximately the acceleration when $t=0·3$?

Ans. The average speeds in the interval are

t	$\frac{\delta s}{\delta t}$
0	
	1·054
0·1	
	1·092
0·2	
	1·122
0·3	
	1·164
0·4	
	1·192
0·5	
	1·233
0·6	
	1·261
0·7	

The approximate increase in speed between $t=0·25$ and $t=0·35$ is from 1·122 to 1·164 feet per second in 0·1 second or 0·042 foot per second in 0·1 second, so that the acceleration at $t=0·3$ is approximately $\frac{0·042}{0·100}=0·42$ foot per second per second.

11. The sections of the two ends of a barrel are each 12·25 sq. feet; the middle section is 14·16 sq. feet; the axial length of the barrel is 5 feet. What is its volume?

Simpson's Rule gives $\frac{1}{6}(12·35+12·35+4\times 14·16)=13·558$ sq. feet as the average section, so that $13·558\times 5$ or 67·79 cubic feet is the volume.

12. There is a machine consisting of two parts, whose weights are x and y. The cost of the machine in pounds is $12x+5y$. The power of the machine is proportional to xy. Find x and y if the cost is £100, and if we desire to have the greatest power possible. Use squared paper if you please.

Ans. $12x+5y=100$, $y=20-2·4x$, $xy=20x-2·4x^2$, and we wish to find what value of x will make this a maximum. The calculus tells us that $x=4\frac{1}{6}$, $y=10$, gives the best result. But using squared paper, take the values of x, 3, 4, 4·5, 5, 6, etc., and calculate xy. Plot xy and x, and we find the crest of the curve where $x=4\frac{1}{6}$, and of course $y=20-2·4\times 4\frac{1}{6}=10$.

13. According to a certain hypothesis the tensile stress in a rectangular cross-section of an iron hook at a distance y from a certain line through the centre of the section is proportional to

$$p=\frac{y+c}{1-\frac{y+c}{R}}.$$

When $R=10$ and $c=1$, calculate p for various values of y from $y=5$ to $y=-5$, and plot on squared paper. What is the average value of p? For what value of y is the stress zero?

Ans. Evidently $p=0$ when $y=-1$. Average $p=2·55$.

1910. Stage 3.

1. The three parts (*a*), (*b*), and (*c*) must all be answered to get full marks:

(*a*) Without using logarithms, compute by contracted methods, so that four significant figures *shall be correct*,

$$5{\cdot}306 \times 0{\cdot}07632 \div 73{\cdot}15.$$

(*b*) Using logarithms, compute

$$(22{\cdot}15 \div 4{\cdot}139)^{-0{\cdot}86}.$$

(*c*) The lengths of a degree of latitude and longitude in centimetres in latitude l are

$$(1111{\cdot}317 - 5{\cdot}688 \cos l)10^4 \quad \text{and} \quad (1111{\cdot}164 \cos l - 0{\cdot}950 \cos 3l)10^4.$$

The length of a sea mile (or 6082 feet) is 185380 cm. What are the lengths of a *minute* of latitude and of a minute of longitude in sea miles in the latitude 52°?

Ans. (*a*) 0·005536; (*b*) 0·2363; (*c*) 0·99600, 0·61687.

2. A telephonic current of frequency $p/2\pi$ becomes of the value

$$C = C_0 e^{-hx} \sin(pt - gx)$$

in the distance of x miles, where

$$\sqrt{\frac{kpr}{2}}\sqrt{\sqrt{\left(1+\frac{p^2l^2}{r^2}\right)} \pm \frac{pl}{r}}$$

gives the value of h if the minus sign be taken, and the value of g if the plus sign be taken. When pl/r is very large, what are the values of h and g approximately? If $k = 0{\cdot}05 \times 10^{-6}$, $r = 88$, and $p = 5000$, take two cases, (i) when $l = 0$ and (ii) when $l = 0{\cdot}3$, and in each case find the distance x in which the amplitude of C is halved.

Ans. $h = \frac{r}{2}\sqrt{\frac{k}{l}}$, $g = p\sqrt{kl}$; $x = 6{\cdot}609$ and $38{\cdot}58$ miles.

3. Find the value of $\cosh 0{\cdot}1(1+i)$, where i means $\sqrt{-1}$.

Ans. $1 + 0{\cdot}01i$ or $1[0^\circ{\cdot}573]$. (See Art. 139.)

4. To find the volume of part of a wedge, the frustum of a pyramid or of a cone, of part of a railway cutting or embankment, etc., we use the "Prismoidal Formula," which is "the sum of the areas of the end sections and four times the mid section, all divided by 6, is the average section; this multiplied by the total length is the whole volume." Under what circumstances is this rule perfectly correct? Prove its correctness.

[The Prismoidal Formula is merely Simpson's Rule, and the question is fully answered in Art. 51.]

5. If $z = y + 2\frac{dy}{dx}$, and if y is tabulated, find z approximately.

Show both y and z as functions of x in curves:

x	4·0	4·1	4·2	4·3	4·4	4·5
y	3·162	3·548	3·981	4·467	5·012	5·623

Ans. The values of z in the mid intervals, that is for $x = 4{\cdot}05$, 4·15, etc., are 11·08, 12·42, 13·94, 15·64, 17·54.

6. A body capable of damped vibration is acted on by simply varying force which has a frequency f. If x is the displacement of the body at any instant t, and if the motion is defined by

$$\frac{d^2x}{dt^2}+b\frac{dx}{dt}+n^2x=a\sin 2\pi ft,$$

we wish to study the forced vibration.

Take $a=1$, $b=1{\cdot}5$, $n^2=4$; find x first when $f=0{\cdot}2547$, and second when $f=0{\cdot}3820$.

Ans. [Refer to Art. 126.] Let $q=2\pi f$, then $x=\sin qt/(n^2-q^2+bqi)$, $x=\sin qt/(4-q^2+1{\cdot}5qi)$. The values of q are 1·6 and 2·4.

$$x=\frac{\sin qt}{1{\cdot}44+2{\cdot}4i}\quad\text{and}\quad x=\frac{\sin qt}{-1{\cdot}76+3{\cdot}6i}$$

in the two cases,
or $x=\sin qt/2{\cdot}799[59^\circ{\cdot}03]$ and $x=\sin qt/4{\cdot}007[116^\circ{\cdot}05]$,
or $x=0{\cdot}3572\sin(qt-59^\circ{\cdot}03)$ and $x=0{\cdot}2496\sin(qt-116^\circ{\cdot}05)$.

7. The following values of x and y being given, tabulate $\frac{\delta y}{\delta x}$ in each interval, also $\delta A=y\,.\,\delta x$ and $A=\int y\,.\,dx$. Show in curves how the values of y, $\frac{dy}{dx}$, and A depend upon x.

x	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
y	6·428	7·071	7·660	8·192	8·66	9·063	9·397	9·659	9·848

Ans. For $\frac{\delta y}{\delta x}$ and $y\,.\,\delta x$ for each interval, and also the values of A for $x=0{\cdot}1$, $x=0{\cdot}2$, etc.

6·43	5·89	5·32	4·08	4·03	3·34	2·62	1·89
0·6750	0·7365	0·7926	0·8426	0·8862	0·9230	0·9528	0·9754
0·6750	1·4115	2·2041	3·0467	3·9329	4·8559	5·8087	6·7841

8. Here is a table giving values of y in terms of x, and another giving values of u in terms of y. What is u when $x=8{\cdot}3$?

x	y	y	u
7	14·914	15	0·8169
8	16·128	16	0·7118
9	17·076	17	0·5543

Ans. y is 16·44 and u is 0·6488.

9. If $pv=100t$, and $p=3000$ when $t=300$, find v. If $p=3010$ and $t=302$, find the new v. If the second set of values be called $3000+\delta p$, $300+\delta t$, and $v+\delta v$, what is δv? Now use the formula

$$\delta v=\left(\frac{dv}{dp}\right)\delta p+\left(\frac{dv}{dt}\right)\delta t,$$

and calculate δv in the new way. Why is there an error in the answer?

See Art. 143. $v=10$, true $\delta v=0{\cdot}033223$. The new formula gives 0·033333. The formula is true only when δp and δv are smaller and smaller without limit.

10. The value of y, a periodic function of t, is here given for 12 equidistant values of t covering the whole period. Express y in a Fourier Series.

13·602, 18·468, 20·671, 20·182, 17·820, 14·346,
10·130, 5·612, 1·877, 0·486, 2·500, 7·506.

It ought not to be necessary to say that 18·468 is the second value.
[See Art. 132 ; also Question 12, Stage 3, 1911.]

Ans. $y=11{\cdot}1+10\sin(\phi+10°)+\sin(2\phi+52°)$.

11. To solve $x^3-20x+9=0$ graphically, it is evident that we desire the value of x which will cause x^3 to be equal to $20x-9$; plot therefore the curve $y=x^3$, and plot the straight line $z=20x-9$. Where they intersect we have the value of x desired. When the trial is made it will be found that there are three answers ; what are they?

Ans. 4·23, 0·455, $-4{\cdot}68$.

12. On the indicator diagram of a gas engine the following are some readings of p pressure and v volume. The rate of reception of heat (if the gases are supposed to be receiving heat from an outside source and not from their own chemical action) is

$$\frac{dH}{dv}=p+\frac{k}{K-k}\left(p+v\frac{dp}{dv}\right),$$

where k and K the important specific heats are such that

$$\frac{k}{K-k}=2+\frac{pv}{300}.$$

v	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8
p	84·5	110	176	215	231	234	226	213	202

v	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6
p	192	183	175	167	159	152	146	140

Find $\frac{dH}{dv}$ at three places; where $v=2{\cdot}05$, 3·55, and at the place of highest pressure.

This question is answered in Ex. 89, Chap. XXVII.

Ans. 1750, -115, 1160.

13. When a shaft fails under the combined action of a bending moment M and a twisting moment T, according to what is called the internal friction hypothesis,

$$M+a\sqrt{M^2+T^2}$$

ought to be constant where a is constant. Test if this is so, using the following numbers which have been published. Considerable errors in the observations must be expected.

M	0	0	0	1200	1160	1240	2800	2840
T	4320	4360	4308	4338	4326	4368	3836	3846

M	2760	4400	4320	4600	5020	5180	5360
T	3804	2416	2438	2060	0	0	0

Ans. The student tabulates $\sqrt{M^2+T^2}$, and plots this with M. Using the straight line which lies most evenly among the plotted points, he will find

$$\frac{20}{3}\sqrt{M^2+T^2}-M=28830.$$

INDEX.

The references are to pages.

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