UNIVERSITY OF NEVADA, LAS VEGAS

Getting the Measure of Einstein's Space and Time

An Introduction to Special Relativity

Len Zane

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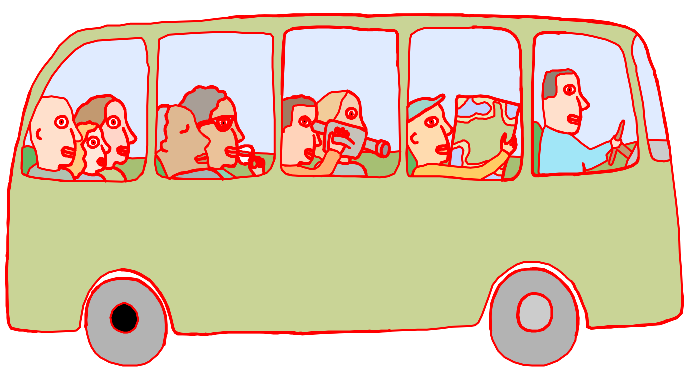


The space and time introduced by Albert Einstein in 1905 is explained by examining a series of simple thought or "gedanken" experiments. The development makes extensive use of spacetime diagrams to help readers appreciate the full extent of these changes.

Table of Contents

Introduction 4

Chapter One: Space and Time Before 1905 6

Speed and Velocity 6

The Experimenters are Introduced 6

The First Set of Experiments: Earth-Based 8

The Second Set of Experiments: Bus-Based 11

Bev and Anne Carefully Observe the Experiment Done on the Bus 12

The Addition of Velocities Formula 15

A Picture is Worth a Thousand Words 17

The Galilean Transformation Equations 22

Summary of Chapter One 26

The Principle of Relativity and the Isotropy of Space 26

Space and Time pre-1905 27

Chapter Two: The Speed of Light 28

Chapter Three: Space and Time After 1905 31

A New Set of Experiments are Proposed 31

New Stop Watches are distributed 31

Anne and Bev Measure the Speed of Light 31

The Super Bus Rolls into the Story 34

Anne and Bev Observe the Light Flash on the Bus: I 36

Anne and Bev Observe the Light Flash on the Bus: II 41

The Effect of Motion on Space 48

The Effect of Motion on Time 51

Light Moves through Space and Time in a Very Strange Way 54

Chapter Four: Generalizing the Observations 56

Defining the New Problem 57

Finding the Shrinkage Factor 58

The Ticking Rate of Moving Clocks 62

Quantifying the Disagreement over Synchronization 63
Testing the Theory 66
A Question about Synchronization 68
Global Positioning System (GPS) 70
Chapter Five: The Relationship between Bus and Earth Observers 72
Combining Spacetime Graphs 72
Bev Generalizes Chuck's Spacetime Diagram 79
A Simple Example for Dean 83
The Revised Equation for the Addition of Velocities 84
Chapter Six: Chuck Suggests an Experiment and Anne has a Dream 89
Experiment I 90
Experiment II 93
Answers for Experiment I 97
Answers for Experiment II 100
Anne has a Dream about a Superluminal Pigeon 104
Chapter Seven: The Bus and the Garage 108
Anne's Proposal 115
The Collision between Theory and Reality 116
R.I.P. Super Bus 118
Chapter Eight: The Solar System and Beyond 120
Synchronizing Watches Separated by Large Distances 121
Chapter Nine: Chuck and Dean Travel to Alpha Centauri 123
May 1, Thirteen Years Later 124
Using the Lorentz Equations 130
Looking at a Moving Watch – The Doppler Effect 132
Chapter Ten: Space Travel 136
The Equivalence Principle 136
Designing the Space Adventure 139
The Results of Chuck's Calculations 143

How does a Rocket Accelerate? 148
Bonus Topic: A Peek at General Relativity 152
Chapter Eleven: Does it Really Happen? 154
The Far-Away Observer (FAO) 154
A Dynamic Explanation for the Behavior of Watches and Rulers 158
A Model for a Hydrogen Atom 159
Chapter Twelve: $E = mc^2$ 162
Appendix A: Graphing 169
Appendix B: Scientific Notation 172
Appendix C: The Light Clock 173
Appendix D: The Gravitational Clock Effect 178

Introduction

This book was written after many years of teaching special and general relativity to students with varied backgrounds. The presentation emphasizes spacetime diagrams which in my experience helps students visualize space and time. The material is self-contained, allowing the book to be used as a tutorial for a person with little background in but some familiarity with algebra and a healthy curiosity about special relativity. There are questions scattered throughout the book to encourage the reader to take some time to review the material presented before moving on to new material. The book can also be used as a textbook for a course in space and time for non-science students, the audience I primarily had in mind when writing it, or as an introductory course for students planning to study science later in their undergraduate careers. Please feel free to contact me at len.zane@unlv.edu if you have any questions or suggestions.

Chapter One establishes a simple methodology for measuring speeds and velocities at non-relativistic values that are encountered in everyday life. In particular, the velocity of an object moving in a bus that is traveling down the road is measured by people on the bus and simultaneously by observers standing on the ground. These measurements are used to probe the rules that govern space and time.

Chapter Two presents a short summary of light's properties. For us, the most important property is light's astonishingly large velocity.

Chapter Three replaces the bus in Chapter One with an imaginary one that can move at relativistic speeds. This new "super" bus is used to probe space and time by doing experiments analogous to those done in Chapter One. These new experiments force our experimenters to drastically revise the rules governing space and time that emerged from Chapter One.

Chapter Four and Five generalize the results of the experiments done in Chapter Three culminating in a derivation of the Lorentz transformation equations and the relativistic version of the addition of velocity equation first encountered in Chapter One. Chapter Four ends with a section on how a GPS verifies that moving clocks run slow and that clocks further from the center of Earth run fast.

Chapter Six is primarily a tutorial designed to give readers a chance to review all the earlier material by using spacetime graphs and the Lorentz Transformation equations to analyze two imaginary experiments done with the super bus.

Chapter Seven describes and analyzes the well-known "Pole and Barn" problem and attempts to give a definitive answer to the question, does the pole fit in the barn or not? In this chapter, the super bus replaces the pole and a newly constructed garage takes the place of the barn.

Chapter Eight introduces the astronomical distance unit, the light year, and gives a brief description of our home galaxy, the Milky Way.

Chapter Nine analyzes in some detail the famous "Twin Paradox." This iconic paradox of special relativity arises when one of the twins travels away from Earth in a space ship and returns years later. Upon returning, it is discovered that the twin that remained on Earth aged more than the traveling twin.

Chapter Ten uses the laws governing space and time to study the motion of a rocket that moves with constant acceleration. The accelerating rocket is used to examine the possibilities of human travel to other galaxies. The last section of Chapter Ten explains why a clock further from the center of Earth runs faster than an identical closer to the center.

Chapter Eleven gives two separate arguments designed to show that rulers really do shrink and watches actually run slow and these effects are not just illusory or "theoretical."

Chapter Twelve gives a simple "derivation" of Einstein's famous equation, $E = mc^2$. Though not directly related to the primary theme of the book, it is difficult to write a book that purports to cover special relativity and not include that equation.

Chapter One: Space and Time Before 1905

This chapter examines some simple experiments that establish the space and time of Galileo and Newton. This view prevailed until 1905 when Albert Einstein introduced the world to Special Relativity. It is also the space and time of our everyday lives where rulers and watches are well behaved.

Speed and Velocity

The concepts of speed and velocity are colloquially understood to be a distance traveled divided by the elapsed time. In this chapter, velocities and speeds will be measured in feet/second or ft/s. To physicists, speed and velocity are related but not interchangeable ideas. Velocity includes a sense of direction in its definition. Throughout this book, anything or anybody moving from left to right will have a positive velocity, for example +10 ft/s. Note that a velocity of +10 ft/s is the same as 10 ft/s. While any object or person moving from right to left will have a negative velocity, for example -10 ft/sec,. The speed of an object is the size, or magnitude, of its velocity. In the above examples, both objects would have the same speed, 10 ft/sec, although those speeds would be in opposite directions.

Later we will look very carefully at how speed is measured. But for now, it is clear that to find the velocity or speed of an object, it is necessary to measure both a distance covered and the time it took to traverse that distance. Rulers and stop watches are the usual instruments used to measure distance and time. Imagine a warehouse full of excellently fabricated rulers and stop watches. All the rulers are identical to one another and the same is true for the stop watches. These instruments will be handed out to specially-trained observers who will collect the data used to determine the speed or velocity of test objects in a variety of circumstances. A key point to keep in mind is that the experiments described are all perfectly reasonable and doable, at least in principle, though some may be technologically too challenging to be done with currently available rulers, stop watches, and observers.

The Experimenters are Introduced

Anne, Bev, Chuck, and Dean are good friends, astute observers, and curious by nature. One evening, after watching an episode of **Star Trek**, they begin talking about space and time and Special Relativity, subjects none of them knows very much about. Finally Bev suggests that instead of speculating about the meaning of Special Relativity, they ought to do some careful experiments to get first-hand knowledge about space and time. Her three friends quickly agree, thinking it could be enlightening to get an experimentally based understanding of the nature of space and time.

Anne, Bev, Chuck, and Dean are the main characters in this book. Anne and Bev are stationed on Earth during all the experiments with Bev always being to the right of Anne. Conversely, Chuck and Dean are the traveling pair of experimenters. They ride in buses and rocket cruisers with Dean always situated to the right of Chuck. ***A & B*** *and* ***C & D are*** *shorthand for* ***Anne and Bev*** *and* **Chuck and Dean.**

The sections in italics represent the author interjecting himself into the narrative. The hope is that these asides will add to and not disrupt the main story, the experimental probing of space and time.

The next morning, bright and early, our friends meet to map out a set of experiments. They recall that Special Relativity has something to do with light and its velocity and the way it is perceived by different observers moving with respect to one another. Though they are not sure what it is about light that is so peculiar, they do know that the speed of light is very, very large. Anne suggests that they do some simple experiments with something that moves at a pedestrian speed; for example, one of the trained pigeons that she has seen in the park. These pigeons all fly at exactly 20 ft/s.

At the time scientists were grappling with the concepts of space and time, there was general agreement that light was a wave. One of the principle characteristics of a wave is that it travels through some medium. For example; sound travels through air and other material substances. The speed of a wave is the speed at which it moves through that medium. Scientists imagined space being permeated by an ethereal substance, the "lumeniferous ether," through which light moved. The speed of a pigeon is the speed that it moves through air.   This is analogous to the way a wave moves through a medium, or more specifically the way scientists pictured light moving through the ether. Thus Anne's choice of a pigeon as the object to study was reasoned and not fortuitous.

Dean volunteers to head to the park to sign up a pair of pigeons for their experiments. After he leaves, his friends decide to break themselves into two teams--Anne and Bev and Chuck and Dean. Anne and Bev will be the team that measures things from the perspective of Earth while C & D will ride in a laboratory that moves with respect to Earth. Chuck agrees with this plan and immediately heads down to the used bus lot to find a vehicle that could be appropriately modified to become the rolling laboratory for him and Dean.

After he leaves, A & B go to the warehouse to pick up four identical sets of rulers and stop watches. Later that day, the four friends reconvene to review what has been accomplished so far and what experiments ought to be done first. A & B distribute rulers and stop watches to C & D. They dutifully test the equipment and agree that they are all identical and establish that the watches are accurate to one tenth of a second, 0.1 second. Dean shows off the two trained pigeons that volunteered to help with the experiments. The pigeons demonstrate their skill by flying back and forth in tandem at identical speeds, each matching the other flap for flap. Finally Chuck gives a tour of the bus that will act as his and Dean's rolling lab. He points out that it has an excellent cruise control that will ensure that the bus maintains a constant speed during any experiment. Both teams measure the length of the bus and agree that it is 100 feet long.

In order to simplify the discussions that follow, the world of our four experimenters will need to have some peculiar, but not unreasonable, properties. The first is that the world has just one space dimension. That is physics talk for the fact that Anne, Bev, the pigeons, and the bus with Chuck and Dean aboard, can only move along a line. They can move to the right, the positive direction, or to the left, the negative direction. But motion perpendicular to that line is impossible and meaningless for the participants in the various experiments to be described in this and later chapters because there is no direction perpendicular to the line they move along! For a more concrete picture of a one dimensional world, imagine beads sliding on a wire. The beads can slide to or fro but it is impossible for them to move off the wire.

The First Set of Experiments: Earth-Based

In their first experiment, A & B use their rulers and watches to measure the speed of one of the pigeons. Meanwhile, the other pigeon is comfortably housed on the bus to be used by C & D in later experiments. A & B decide to have their pigeon fly 100 feet, the length of the bus, to make their experiments more directly comparable to those C & D will do. C & D watch as A & B carefully measure off 100 feet. Anne holds the pigeon while Bev stations herself 100 feet to Anne's right. The plan calls for Anne to release the pigeon while simultaneously starting her watch. Bev will start her watch as soon as Anne releases the pigeon and both will stop their watches the moment the pigeon reaches Bev.

Before we let A & B do the above experiment, let's make sure that the described procedure makes sense. There is a certain reaction time involved in starting and stopping a watch. Since Anne is starting her watch at the same instant that she releases the pigeon, we can assume that both of those actions take place at exactly the same instant. After all, that is the least we can expect from "well-trained observers!" On the other hand, Bev does not know exactly when the pigeon will be released. She will start her watch after a short delay that depends on how quickly she can react to the initial movements of the pigeon. Human reaction times for Olympic athletes are between 1 and 2 tenths of a second. When Bev catches the pigeon, she instantly stops her watch. Again we can assume that she is skilled enough to do both simultaneously. But now it is Anne watching Bev catch the pigeon who has to react, introducing some uncertainty on her end of the experiment.

Bev records 4.9 seconds for the pigeon's flight time while Anne gets 5.1 seconds. Being good scientists they decide to average the two times, concluding that the pigeon took 5.0 seconds to fly 100 feet for a speed of 20 feet/second or 20 ft/s. They repeat this experiment several times, and measure 20 ft/s each time for the speed of the pigeon. Just as advertised, their pigeon always flies at 20 ft/s. Note that the pigeon's velocity is +20 ft/s because it flies from Anne to Bev, left to right.

For the second experiment, the pigeon flies from Bev to Anne, from right to left. When they calculate the speed of the pigeon, they get, unsurprisingly since it is a well-trained pigeon, 20 ft/s. But the pigeon's velocity is now -20 ft/s. After this experiment, the friends decide to break for lunch.

The small difference in time measured by Bev and Anne was experimental error introduced because of the finite reaction times of our observers. In the future, we don't want to worry about whether or not well-trained observers can release/catch a pigeon while simultaneously starting/stopping a watch. Nor do we want to worry about human reaction times which needlessly complicate the results of our experiments. Consequently part of our definition of a well-trained observer will include the ability to perform all the duties required in a given experiment <u>without introducing any extraneous errors</u>.

Question 1.1: After lunch, A & B decide to redo the experiment with the pigeon. But now a breeze is blowing from Anne toward Bev at 10 ft/s. How does the wind affect the results of the experiment when the pigeon travels from Anne to Bev and then back to Anne? (Think about this for a bit before reading on.)

What did you conclude about the flying times for the pigeon with a breeze blowing from left to right at 10 ft/sec? The first thing to recognize is that the speed of the pigeon found during the first set of experiments was the speed of the pigeon with respect to the air it was flying through. Since the air was not moving with respect to the ground, the pigeon's speed with respect to the air was the same as the pigeon's speed with respect to the ground. That is the reason the pigeon's speed was the same flying from left to right as it was flying from right to left.

But now the air is moving with respect to the ground. In each second of flight from Anne to Bev, the pigeon advanced 20 feet through the air while the air moved 10 feet closer to Bev. Consequently the pigeon moved 30 feet closer to Bev for each second it flew. The pigeon's velocity with respect to the ground was +30 ft/s. Bev and Anne both recorded the flight time as 3.3 seconds.

The return trip was more difficult for the pigeon because it was now flying into a headwind. The measured flight time for the return trip was 10 seconds, making the pigeon's velocity -10 ft/s. Notice that the total time for the round trip in the 10 ft/s wind was 13.3 seconds compared to 10 seconds when there was no wind.

Question 1.2: What would be the total time for a round trip with a wind blowing +15 ft/s?

Question 1.3: What would happen to the roundtrip time if the wind were blowing at a velocity of +20 ft/s or larger?

Remember that the pigeon's motion through air is analogous to the motion of a wave through a medium. Scientists at the dawn of the Twentieth Century considered light a wave. Experiments analogous to those done by A & B when a wind was blowing were done by scientists on light, with the expectation that the speed of light would depend on the direction and speed of the "ether wind." These experiments on light will be discussed more in the next two chapters.

The idea that a pigeon flying in still air has exactly the same speed regardless of the direction flown is really a statement that space is "isotropic." That is, any experiment done with our instruments aligned left to right ought to give exactly the same result if instead we oriented our instruments right to left. The discussion is restricted to left/right because those are the only directions available to our experimenters. More generally, we expect the results of any experiment to be independent of the orientation of the apparatus because of the isotropy of space.

This expectation is taken by scientists as a fundamental property of space. Note that this expectation is false if the experiment is done on a windy day because the wind destroys the symmetry between left to right and right to left. Also, for those of us living on Earth, motion back and forth or from side to side is isotropic but gravity, like the wind, upsets the isotropy of space in the vertical direction.

The Second Set of Experiments: Bus-Based

After successfully measuring the speed of the pigeon with respect to Earth, Anne and Bev suggest that it is time for Chuck and Dean to do the same experiment in their rolling laboratory. This will give them a chance to test their experimental skill. C & D get in their bus, set the cruise control, and drive with a velocity of +30 ft/s toward A & B. Dean is in the front of the bus and Chuck in the back, exactly 100 feet separate them. When Dean peeks out of the window he sees the one-dimensional Earth, along with A & B, move past the bus from right to left with a velocity of -30 ft/s. For a minute, Dean forgets that he is in a moving bus and instead imagines that the Earth is moving.

It is important to note that this bus has an incredibly good suspension system, travels in a straight line, and its speed never deviates from 30 ft/s. Therefore, with the shades down, C & D have absolutely no sense of moving and could legitimately think of themselves as stationary.

Our Earthly bias makes it difficult in our heart-of-hearts to consider the bus stationary while the Earth ambles by at -30 ft/s. <u>It is essential that we let go of this bias</u>. The alternative is to have Bev and Anne in a bus identical to the one used by Chuck and Dean and then to have these two buses move relative to one another with a speed of 30 ft/s. For example, C & D can be traveling at +15 ft/s while A & B drove toward them at -15 ft/s. A & B would see the "other" bus moving with a velocity of +30 ft/s while C & D saw the "other" bus moving at -30 ft/s toward them. This scenario emphasizes the symmetry between A & B and C & D and the reason why they are both perfectly justified in claiming that they are on a <u>stationary</u> bus while the other bus is moving. Instead of creating a perfectly symmetric situation and wasting fuel by using two buses instead of one, you, the reader, will have to work to thwart your bias and become comfortable with the equivalence between experiments done on the bus and those done on Earth.

Chuck and Dean first measure the time it took the pigeon to fly from the back of the bus to the front. Then they carefully repeat the measurement with the pigeon flying from the front to the back. In the first case, the velocity of the pigeon is measured to be +20 ft/s while on the return trip the velocity is -20 ft/s. In both cases, the speed of their pigeon is identical to the speed of the pigeon used by A & B.

*Embedded in the discussion of these two sets of experiments is an important observation that was first explicitly stated by Galileo as the **Principle of Relativity**. The principle codifies our experience of flying through smooth air on a jet liner. If you fell asleep as soon as you boarded a plane, and woke some time later, you would be hard pressed to tell if you were still sitting on the runway waiting for clearance to takeoff or cruising at 500 mph at 35,000 feet. In fact, the Principle of Relativity states that there is no experiment that you could do in the airplane that would differentiate between the two states of motion. Before Einstein, the principle was meant to apply to any "mechanical" experiment and not to experiments involving light or more generally electro-magnetic phenomena. We will take it as a well established experimental truth that any experiment done by A & B can be repeated by C & D on their bus with identical results. The pigeon experiment is a particularly simple affirmation of the Principle of Relativity.*

More generally, the Principle of Relativity says that all "inertial" reference frames are the same. An inertial reference frame is a collection of observers moving through space and time with a constant velocity. A & B and C & D are observers in two distinct inertial reference frames. A & B claim that C & D are moving through their Earth-based reference frame at +30 ft/s while C & D see A & B moving through their bus-based reference frame at -30 ft/s. It is very important to keep in mind that Chuck and Dean can easily tell when the bus is accelerating. A simple pendulum hanging from the roof of the bus would tilt toward the rear of the bus as it sped up and tilt toward the front when the bus slowed down. During these times, experiments done in the bus will not give the same results as experiments done by Anne and Bev in the Earth frame. The Principle of Relativity only applies to reference frames moving at constant velocities. This distinction is central to understanding the famous Twin Paradox discussed in detail in Chapter Nine. All the conclusions reached in the first nine chapters are predicated on comparing observations made in equivalent reference frames, inertial reference frames.

Bev and Anne Carefully Observe the Experiment Done on the Bus

Before going on, let's fine tune the description of our one-dimensional world. During the experiments done with the bus, it was always moving from left to right. Therefore, before the start of each experiment the bus is first driven to some starting point to the left of Anne, turned around, and readied for its next trip past A & B. Every experiment will start when one of the Earth observers, A or B, is next to one of the bus observers, C or D. The start of each experiment *will also be marked by some particular event taking place as the two chosen observers pass one another, for example the release of a pigeon.*

Please don't ask how Anne and Chuck can be adjacent to one another in a one-dimensional world or how the bus can pass by Anne and Bev without obliterating them. If you need a concrete picture, imagine the bus with its inhabitants moving in a one-dimensional world parallel to the one occupied by A & B. Or going back to the bead and wire analogy, A & B and C & D are "beads" on parallel wires.

A & B decide to ask C & D to do their two experiments again but this time they want to determine how fast the pigeon on the bus flew with respect to them, that is, with respect to Earth. To do that, they need to measure the Earth distance covered by the pigeon as it flies from the rear to the front of the bus. The speed of the pigeon is that distance divided by the time it took the pigeon to fly from one end of the bus to the other.

For these experiments, Chuck agrees to release the pigeon when he is adjacent to Anne. At that same instant, Bev and Dean will start their watches. That instant also signals the start of the experiment. Before doing the experiment, Anne and Bev mull over the question of where along the road Bev ought to stand so that she will be adjacent to Dean just as he catches the pigeon.

Question 1.4: Before reading on, calculate how far to the right of Anne Bev has to be standing to witness the arrival of the pigeon at the front of the bus.

Bev observes that since the bus travels at 30 ft/s and the pigeon takes 5 seconds to fly from the back to the front of the bus, the bus will move 150 feet down the road while the pigeon is flying. Anne immediately agrees and adds, "Dean, in the front of the bus, will be 100 feet past me when Chuck releases the pigeon. Therefore he will be 100 plus 150 feet to my right when he catches the pigeon." Using this information, Bev stations herself down the road at the 250 foot marker. Now that A & B are ready, Anne gives Dean the signal to start the bus rolling down the road toward them.

Chuck releases the pigeon as he passes Anne. Bev stares intently at the bus as it comes towards her and sees Dean catch the pigeon just as the bus drives by. At that instant of passing, she and Dean simultaneously stop their watches. Bev excitedly waves to Anne that they had correctly calculated the place she needed to be standing to witness Dean catch the pigeon.

After the experiment, the four friends compare notes. Bev and Dean have both timed the pigeon's flight as lasting 5 seconds. Because Bev is standing at a spot 250 feet to the right of Anne, the pigeon's velocity is +50 ft/s with respect to Earth. C & D see their well-trained pigeon fly the length of the bus, 100 feet, with a speed of 20 ft/s, just like expected.

In the next experiment, the pigeon will fly from Dean to Chuck, from right to left. The plan is for Dean to release the pigeon the instant he is adjacent to Anne. Again A & B confer to figure out where Bev needs to stand in order to be adjacent to Chuck when he catches the pigeon in the bus.

Question 1.5: Where does Bev have to be standing in relation to Anne to be at the correct spot to observe Chuck catch the pigeon?

Bev points out that the bus will travel 150 feet during the 5 seconds it takes the pigeon to fly from Dean to Chuck just like in the previous experiment. Anne agrees but now the pigeon is flying in the other direction which makes it harder for her to think about. Bev is also having difficulty thinking about the pigeon but then she smiles and points out that the pigeon is not really relevant. "The pigeon flies from the front of the bus to the back of the bus, a distance of 100 feet. During that time, the back of the bus moves 150 feet, therefore the pigeon gets caught, from our perspective, 50 feet to the right of the place it is released."

Anne nods slowly. "When the pigeon is released, I will be standing next to Dean who is at the front of the bus. The rear of the bus where Chuck is sitting is 100 feet to my left. While the pigeon flies toward Chuck, the rear of the bus moves 150 feet. Therefore Chuck will be 50 feet to my right when he catches the pigeon." Now that A & B agree on the spot where Bev needs to be standing, they are ready to test their analysis against the actual experiment with the bus.

Bev stations herself 50 feet to the right of Anne. The bus comes rolling down the road. Dean releases the pigeon as he passes Anne. Chuck and Bev start their watches. Just as expected, Bev is adjacent to Chuck when he catches the pigeon and both their watches read 5 seconds. But this time Bev is located 50 feet to the right of Anne, making the velocity of the pigeon with respect to the Earth +10 ft/s. Of course, from the perspective of C & D, the pigeon's speed is still 20 ft/s.

The four friends review the results of these last two experiments. After some careful thought, they recognize that the results are consistent and make perfectly good sense. The pigeon flies in the still air of the bus which is moving at 30 ft/s. From the perspective of A & B, a bus moving at +30 ft/s with "stationary" air inside is analogous to a situation in which a pigeon flies when a wind blows from Anne towards Bev at a speed of 30 ft/s.

When the pigeon flies from the back to the front of the bus, the moving bus acts like a tail wind. During each second of flight, it flies +20 feet in the still air of the bus while the bus moves +30 feet along the road. Therefore, from the perspective of A & B, the pigeon flaps along at +50 feet every second.

On the other hand, for the return flight, the motion of the bus is like a head wind. Each second, the pigeon covers -20 feet of bus distance while the bus covers +30 feet of Earth distance. The net distance per second traveled by the pigeon flying from right to left in the bus was only +10 feet.

Before breaking up for the day, Chuck wonders aloud if there is some general principle or relationship that could explain the results of the two experiments done with the pigeon flying in the bus. Bev suggests that they sleep on it and meet the next morning to compare notes.

The Addition of Velocities Formula

The next day, Anne and Bev are anxious to share their thoughts with Chuck and Dean. C & D have a handful of papers they want to show A & B, but they let their friends have the floor first. Anne begins by writing down the following two equations:

$$+50 = +20 + 30 \tag{1.1}$$

$$+10 = -20 + 30. \tag{1.2}$$

C & D stare at her and the equations with blank expressions. Finally Dean, who is less adept at algebra than his three friends, says that even he recognizes those as "correct" equations, but he fails to see their relevance to yesterday's experiments! Bev impatiently jumps in and explains, "In the first equation +50 is the velocity of the pigeon with respect to Earth, or V_{PE} for shorthand, where the P stands for pigeon and the E for Earth. On the other side of the equation we have +20, the velocity of the pigeon with respect to the bus, or V_{PB}, and +30, the velocity of the bus with respect to

Earth, or V_{BE}. Therefore, that equation, in terms of the shorthand notation, can be written as,

$$V_{PE} = V_{PB} + V_{BE}.\text{''} \quad (1.3)$$

Chuck immediately recognizes equations 1.1 and 1.2 are special cases of the more general equation 1.3. The only difference is that in equation 1.2, the velocity of the pigeon with respect to the bus, V_{PB}, is - 20 ft/s instead of the +20 ft/s value it has in equation 1.1.

"Exactly!" said Bev.

Anne summarizes the conclusions that she and Bev have arrived at by writing down a more general version of equation 1.3,

$$V_{XY} = V_{XZ} + V_{ZY}. \quad (1.4)$$

This equation gives the relative velocity of X with respect to Y if the velocities of X with respect to Z and Z with respect to Y are known. Equation 1.4 succinctly explains the results of the experiments done with the pigeon and bus.

The connection between equations 1.4 and 1.3 requires the following associations: X → pigeon, Y → Earth, and Z → bus. Later in this chapter, it will be shown that equation 1.4, which relates velocities in one reference frame to those in another, encapsulates the pre-1905 concepts of space and time.

Use equation 1.3 or 1.4 to answer the following questions.

Question 1.6: Suppose a genetically modified pigeon that can fly 50 ft/s is used in the two experiments done on the bus. As the pigeon flies from Chuck to Dean, what is the pigeon's velocity with respect to Earth?

Question 1.7: What is this new pigeon's velocity with respect to Earth on the return flight from Dean to Chuck?

Question 1.8: What is the velocity of X with respect to X, V_{XX}?

Strange question. But the answer leads to a useful identity. The velocity of X with respect to X is a little vague. To be more specific, what is the velocity of the bus with respect to the bus? The *bus is not moving with respect to the bus so its velocity = 0.* V_{XX} *is by definition zero. But* V_{XX} = V_{XY} + V_{YX} = 0 *which leads naturally to,*

$V_{XY} = -V_{YX}$.

This result has already been used when it was pointed out that if the bus travels with a velocity of +30 ft/s with respect to Earth, V_{BE}*, then the bus riders see Earth traveling past them with a velocity of -30 ft/s,* V_{EB}*. This argument reaffirms that intuitively reasonable notion.*

A Picture is Worth a Thousand Words

Chuck and Dean are impressed with the conclusions reached by Anne and Bev. Now it is their turn to share their analysis of the experiments with A & B. C & D are not as analytically skilled as their friends, so they worked on a way to visualize the experiments done with the bus and pigeon.

Chuck shows A & B his effort at drawing a pigeon, figure 1.1. When A & B look at figure 1.1, they begin to giggle. Chuck tells A & B that the previous night Dean had burst out laughing when he looked at the pigeon. After he stopped laughing, Dean told Chuck that that his so-called pigeon could have been a hummingbird or a honey bee for that matter. At first Chuck's feelings were hurt by his friends comment, but then he explained to A & B, that Dean's remark led to a major breakthrough in his efforts at visualizing the experiments.

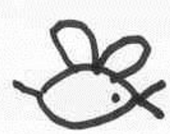


figure 1.1

Chuck tells A & B that the actual experiment would have been the same if it was done with a hummingbird or a honey bee instead of a pigeon. All that was needed was something that flew from the back to the front of the bus. In fact, the bus could have been replaced by an RV or a railroad car moving at 30 ft/s.

Bev and Anne both appear a bit perplexed by Chuck's statement. Chuck continues by pointing out that in an experiment with a honey bee in a bus, the honey bee would be so much smaller than the bus that it could not have been drawn with any detail. Instead, the honey bee would appear as a mere dot moving from one end of the bus to the other.

From the perspective of C & D as passengers on the bus, the experiment is even more simple. They see a pigeon fly across the bus while they remain stationary inside the bus. Chuck stays at *the rear of the bus and Dean at the front. From their bus-centric perspective, it is A & B that are moving past outside the bus. Therefore, from their perspective, A & B had nothing to do with the pigeon's flight inside the bus.*

With these simplifications in mind, Chuck shows A & B the following sketches, figures 1.2a, 1.2b, and 1.2c:

Figure 1.2a

"That is me on the left with the pigeon, the red dot, and Dean is 100 feet to my right. The blue line represents the bus. Notice that the pigeon has not been released yet. One second after I release the pigeon, the new situation can be represented by an analogous picture, figure 1.2b:

Figure 1.2b

The pigeon is now 20 feet to my right. Of course, we have not moved. Our positions are fixed with respect to the bus. Two seconds after releasing the pigeon, it flies 40 feet. After five seconds, Dean catches the pigeon at the front of the bus. Figure 1.2c represents that situation.

Figure 1.2c

These sketches helped Dean and I visualize the key five seconds of the experiment." A & B nod and agree that those sketches do a good job of representing the flight of the pigeon on the bus.

Dean then explains to Anne and Bev that he and Chuck wanted to consolidate the series of sketches, each of which represented a particular instant of time, into a single picture showing the motion of the pigeon through both space and time.

He describes to A & B how he hunched over his sketch pad for a few minutes while mumbling about capturing the essence, nothing but the essence of the experiment. He finally decided that the smiling faces in figures 1.2a through 1.2c are cute but not essential!

He shows A & B figure 1.3, the consolidated diagram showing the flight of the pigeon through space and time.

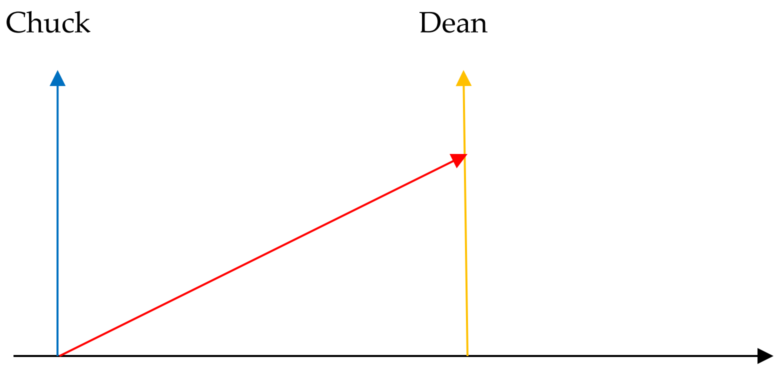


Figure 1.3

Anne comments, "When you said essence, you weren't kidding. I see four different color arrows; black, blue, red, and yellow. I assume they each stand for something 'essential.' Please, one of you, explain how this picture captures the flight of the pigeon through the bus."

Dean defers to Chuck who begins his description:

"The diagram shows the 'space' occupied by me and Dean, in this case the bus, along the horizontal direction. The horizontal black arrow is the space axis. I am represented by the vertical blue arrow and Dean by the yellow arrow. We are separated by 100 feet of space, the length of the bus. All of the important information about me and Dean are contained in those two lines. They show where we are with respect to the bus during the critical five seconds of the pigeon's flight. Consequently we decided to call the blue line \`Chuck's Worldline' and the yellow line \`Dean's Worldline.' The lines are vertical because neither of us is moving inside the bus. For example, if my position in the bus is labeled $X_{bus} = 0$ then Dean is located at $X_{bus} = 100$ feet.

“The slanted red line represents the pigeon and is the pigeon’s worldline. When the experiment began at time zero, T = 0 for shorthand, the pigeon was at the rear of the bus with me. That is the point where the pigeon’s worldline intersects my worldline. When I let the pigeon go, the pigeon flies toward Dean. The pigeon reaches Dean in 5 seconds at the spot where their worldlines meet; the place where the red line hits the yellow line. Just as space is represented on the diagram in the horizontal direction, time advances in the upward direction. The diagram simultaneously shows the space and time location of me, Dean, and the pigeon.

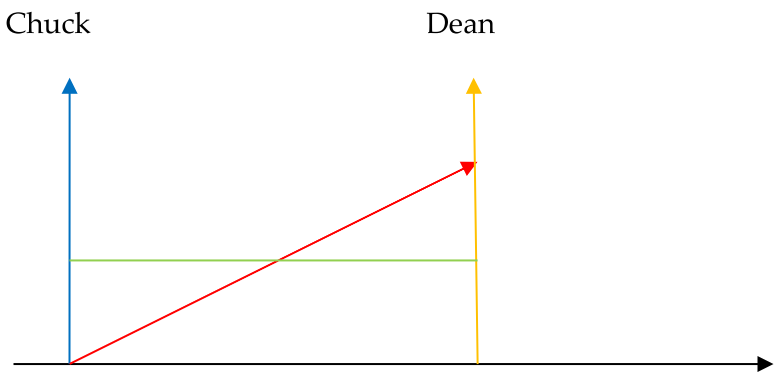


Figure 1.4

“Imagine the situation 2.5 seconds after the pigeon is released. I am at $X_{bus} = 0$ and Dean is at $X_{bus} = 100$ feet. The pigeon flying at 20 ft/s is halfway between us at $X_{bus} = 50$ feet. That instant in time is captured by the green horizontal line in the above figure. That line shows the experimental situation exactly 2.5 seconds after the release of the pigeon. The intersection of that 2.5 second line with my and Dean’s worldlines shows our location in the bus at that time. Because we are not moving, that information is self evident. On the other hand, the pigeon’s worldline is slanted because it is moving through the bus. The 2.5 second line hits the red line at a unique place in the bus, $X_{bus} =$ 50 feet. So the `distance’ between the horizontal black line, T = 0, and any other horizontal line drawn above it, represents the amount of time that has passed since the start of the experiment.”

Chuck stops talking and looks at Anne and Bev. Bev speaks first, “So if I understand what you are saying, the horizontal black arrow marks position with respect to the bus. You are at $X_{bus} = 0$ and Dean is at $X_{bus} = 100$ feet. But that line only represents your location at the start of the experiment.” At this point, Anne reminds Bev that the experiment starts at T = 0; therefore, the horizontal black line is the T = 0 line and the horizontal green line is the T = 2.5 second line.

Bev continues, "So in Figure 1.4, the forward march of time is represented by a horizontal line that continuously slides upward." She grimaces a little before proceeding. "So at any particular time, Dean is at the spot on the diagram that corresponds to the intersection of a `time line' with his worldline." Smiling she adds, "I can almost picture Dean's motion through time as a smiling face sliding up his worldline."

These "space and time" diagrams, or spacetime diagrams for shorthand, are going to play a central role in the analysis of space and time in later chapters. Take time to become comfortable with the idea that a single spacetime diagram captures the flow of events through time. In figure 1.3, the blue line, Chuck's worldline, represents his motion through space and time. He is not moving through the space of the bus since he stays at the rear but he cannot help but move through time. Dean's worldline is completely analogous to Chuck's. On the other hand, the pigeon is moving through both space, from the rear to the front of the bus, and time, the five seconds it took the pigeon to traverse the length of the bus. Since it is impossible to stop time, the various actors depicted on a spacetime diagram will never be completely stationary. At the very least, they will be moving up the spacetime diagram, going from earlier to later times.

Anne and Bev acknowledge that, while the space and time diagram drawn by Chuck captures the essence of the experiment from the perspective of Chuck and Dean, they saw a somewhat different experiment. From their perspective, the bus, with C & D and the pigeon riding along, rolls down the road at 30 ft/s. C & D are prepared for A & B's comment and after a second or two of searching, show them figure 1.5.

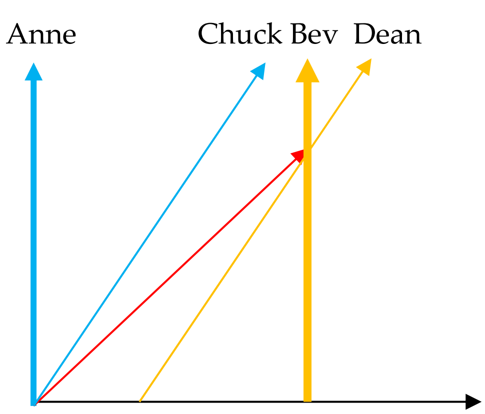


Figure 1.5

Chuck explains that, as before, the horizontal black arrow represents T = 0, the start of the experiment. But it no longer shows locations on the bus but instead shows where Anne and Bev are with respect to Earth. The heavy blue and yellow lines are Anne and Bev's worldlines. Those lines are vertical because they are not moving with respect to Earth. The worldlines of the pigeon, Dean, and him are the same as before, red, yellow, and blue. Those worldlines are all slanted because they are moving with respect to Earth. The intersection of each worldline with the T = 0 line gives the location of that participant in the experiment with respect to Earth at T = 0; Anne, Chuck, and the pigeon are at $X_{Earth} = 0$, Dean is at $X_{Earth} = 100$ feet, and Bev is waiting at $X_{Earth} = 250$ feet.

Question: 1.9: Imagine drawing the T = 1 second line on figure 1.5. At that instant, what are the X_{Earth} locations of Anne, Chuck, the pigeon, Dean, and Bev?

Question 1.10: Make a sketch analogous to figure 1.3 that shows the worldlines of Anne, Chuck, the pigeon, Dean, and Bev from the perspective of the bus observers. Remember C & D see the Earth "rolling" by at -30 ft/s and the horizontal black line on this spacetime diagram represents locations with respect to the bus, X_{bus}, instead of X_{Earth}.

Chuck asks Anne and Bev what they think of the spacetime diagrams. A & B immediately acknowledge that the diagrams really help to visualize the experiment with the bus traveling down the road while a pigeon flies from one end of the bus to the other. With relative velocities and spacetime diagrams to think about, they decide to call it a day; a very productive day at that.

On the walk home, Anne wonders aloud if there is a connection between the spacetime diagram in figure 1.5 and equation 1.4, $V_{XY} = V_{XZ} + V_{ZY}$, the relative velocity equation. Bev shrugs indifferently, and continues walking home at a brisk pace.

The Galilean Transformation Equations

This may be a good place for readers who have not thought much about graphs recently to review Appendix A, a primer on graphs. The appendix has a short discussion of the equation y = mx + b which represents a straight line with slope m and y-intercept b. Understanding the role m and b play will be useful in later chapters and becoming comfortable with graphs will pay immediate dividends in this chapter.

When they arrive home, Anne reproduces figure 1.5 and stares at it for a while. She points out to Bev that the diagram is really a graph though not the typical x versus y graph seen in algebra classes. Anne says, "If we think of the diagram as a graph, the black horizontal line that Chuck calls T = 0 is also the X_{Earth}-axis." Bev nods, and Anne continues, "My worldline on figure 1.5 is the X_{Earth} = 0 line and also the time axis of a graph. So Bev, Chuck's spacetime diagrams are actually x versus t graphs."

To emphasize this new perspective, Anne makes the following sketch, figure 1.6:

Figure 1.6

She explains to Bev that this is the start of a spacetime graph. The horizontal line is the x-axis and the vertical line is the t-axis. The axes intersect at the origin where x = 0 and t = 0. Bev, knowing that Anne is off and running, tries to slow her down by pointing out that the x-axis is also the t = 0 line and that the t-axis is the same as the x = 0 line. "Of course", says Anne, "but notice that the graph represents our <u>entire one-dimensional universe</u>!"

Anne continues her explanation, "When something happens, for example the pigeon gets caught by Dean riding at the front of the bus, that something is called an event. Events happen at a particular spot and a particular time. Dean caught the pigeon at the spacetime point X_{Earth} = 250 feet and T = 5 seconds. That event has a unique location on the spacetime graph given by the intersection of the X_{Earth} = 250 foot and T = 5 second lines. Any event in the past or future that happened or will happen is represented by a point on a graph like figure 1.6."

Bev stares with renewed interest at the cosmic scope of innocent looking figure 1.6. Anne gives Bev a little time to reflect before adding, "The x-axis in figure 1.6 could be either X_{Earth} or X_{bus} depending on who the stationary observers happen to be. Once that decision is made, it is possible to add the worldlines of the various participants in the experiment and to determine how large a slice of space and time is necessary to have those worldlines fit on the graph. For example, if figure 1.6 is drawn from our perspective, the space slice had to include the 250 feet separating us while the time slice required 5 seconds for the pigeon to fly across the bus. On the other hand, only 100 feet of bus space was needed for the spacetime diagram in figure 1.3."

Anne adds the worldlines of Chuck and Dean to her spacetime graph, figure 1.7, and reminds Bev that Chuck's worldline corresponds to the constant $X_{bus} = 0$ line while Dean's to the constant $X_{bus} = 100$ foot line. But those worldlines are slanted on our spacetime graph because the bus is moving them at +30 ft/s.

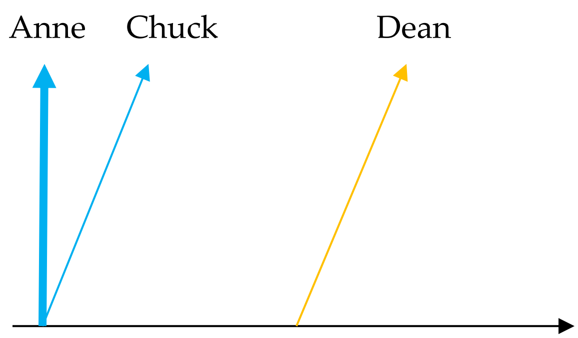


Figure 1.7

Question 1.11: Fill in the following table showing how X_{Earth} for Chuck and Dean change with time.

Time	0 s	1 s	2 s	3 s	4 s	5 s
X_{Earth} for Chuck						
X_{Earth} for Dean						

Chuck's worldline is described by the equation $X_{Earth} = 30\ T$ feet while Dean's is given by $X_{Earth} = 100 + 30T$ feet. Note that when $T = 0$, Chuck is at $X_{Earth} = 0$ while Dean is at $X_{Earth} = 100$ feet just as expected. But X_{Earth} for each of them increases by 30 feet each second, that increase is the 30 T term in the equations for their worldlines. The answers to question 1.11 ought to agree with those equations when T = 0, 1, 2, 3, 4, and 5 seconds.

Bev now takes over the conversation by making the following observation; "Every location in the bus reference frame is given by some value X_{bus} which can be drawn as a constant X_{bus} line of our spacetime graph, figure 1.7. On your sketch, you drew the particular constant X_{bus} lines for 0 and 100 feet and those slanted lines crossed the X_{Earth} axis, the T = 0 line, at 0 and 100 feet respectively. As you pointed out earlier, any point on our spacetime graph can be described by the unique pair of values X_{Earth} and T. For example Dean caught the pigeon at X_{Earth} = 250 feet and T = 5 seconds. But that point is also uniquely determined by the intersection of the X_{bus} = 100 feet and T = 5 second lines."

Anne agrees, and writes down the more general equation,

$$X_{Earth} = X_{bus} + 30\,T, \qquad (1.5)$$

where X_{Earth} shows how any constant bus location, X_{bus} moves through the Earth frame. The 30 in that equation is just the velocity of the bus with respect to Earth. Equation 1.5 connects the Earth-based coordinates, X_{Earth} and T, to the bus-based coordinates, X_{bus} and T, for any event.

Question 1.12: A passenger sitting in the middle of the bus, X_{bus} = 50 feet, sees the pigeon pass at T = 2.5 seconds. What is the Earth coordinate, X_{Earth} for that event?

Question 1.13: When Dean catches the pigeon, Anne claps her hands in glee. Where is the bus observer who is adjacent to Anne when she clapped her hands? (Anne is located at X_{Earth} = 0 and the pigeon gets caught at T = 5.)

The bus is located between X_{bus} = 0 and X_{bus} = 100 feet but we need to be able to imagine bus observers outside of that range because some experiments will require bus observers who are not actually on the bus! The bus observer who is adjacent to Anne when she clapped is riding 150 feet behind the bus at X_{bus} = -150 feet!

The bus is pushing observers along in front who have bus coordinates greater than 100 feet and is dragging others behind with bus coordinates less than zero. As Chuck brought up earlier, the important thing about the "bus" observers is the fact that they are moving at +30 ft/s with respect to Earth observers. Instead of riding on a 100 foot long bus, we could have had them riding on a mile long stretch of railroad cars. That would have given our moving observers a longer bit of space on which to arrange themselves. But since imagination is a prerequisite for making any sense of Special Relativity, we will stick to our 100 foot long bus with observers being pushed or pulled along as necessary.

Anne is now ready to think about the pigeon flying in the bus. The pigeon starts at X_{bus} = 0 and flies with a velocity of V_{PB}, where V_{PB} for the experiment we did was +20 ft/s. The location of the pigeon with respect to the bus is just $X_{bus} = V_{PB}\ T$. Analogously, the position of the pigeon with respect to Earth is just $X_{Earth} = V_{PE}\ T$. *Remember the pigeon started flying at $X_{Earth} = 0$.* Now she replaces X_{Earth} and X_{bus} in equation 1.5 with V_{PE} T and V_{PB} T to get,

$$V_{PE}\ T = V_{PB}\ T + 30\ T. \tag{1.6}$$

Dividing equation 1.6 by T gives $V_{PE} = V_{PB} + 30$. But the 30 in that equation is just the velocity of the bus with respect to Earth, V_{BE},

$$V_{PE} = V_{PB} + V_{BE}. \tag{1.7}$$

Bev is duly impressed with Anne's demonstration of the connection between the addition of velocity equation and spacetime graphs. She gives Anne a well-earned hug. Now they are both ready to call it a night though they look forward to sharing their new insights into space and time with Chuck and Dean.

In the general case of a bus moving with a velocity V_{BE} with respect to Earth, equation 1.6 becomes,

$$X_{Earth} = X_{bus} + V_{BE}\ T. \tag{1.7}$$

That equation connects the Earth-based coordinates, X_{Earth} and T, of any point on spacetime graph with the bus-based coordinates, X_{bus} and T, for the <u>same</u> point. Equation 1.7 is called the Galilean Transformation equation because it transforms Earth-based into bus-based coordinates.

Technically the extra equation, $T_{Earth} = T_{bus} = T$, is needed to complete the Galilean transformation of coordinates. But the uniformity of time was so embedded in human consciousness pre-1905, that including it seemed redundant.

Summary of Chapter One

The important things learned by our four friends are listed below:

The Principle of Relativity and the Isotropy of Space

No experiment can differentiate between the lab on the bus moving at constant velocity and the lab attached to Earth. Results of an experiment are independent of the orientation of the apparatus used to do the experiment. This pair of conclusions was tested by measuring the *velocity of the pigeon flying from left to right and back again in still air with respect to Earth, and comparing those results to the velocity of the pigeon flying in the bus from back to front and then from front to back. The two sets of experiments gave exactly the same results.*

Space and Time pre-1905

The space and time of Galileo, Newton, and all physicists prior to 1905 is accurately summarized by the experiments and conclusions reached by Anne, Bev, Chuck, and Dean in this chapter. The relative velocity equation derived by Anne, $V_{PE} = V_{PB} + V_{BE}$*, Chuck's spacetime graphs, and the transformation equation,* $X_{Earth} = X_{bus} + V_{BE}\,T$*, are three separate but equivalent ways of characterizing the space and time of this era.*

Chapter Two: The Speed of Light

The speed of light was measured in a series of experiments around 1850 by Hippolyte Fizeau, a French physicist. The history of efforts to measure the speed of light is worth pursuing, but doing that now would be a distraction. The primary thing to understand is that the speed of light is very, very large. Light traveling in a vacuum covers 300,000,000 (3×10^8) meters in a second or equivalently 186,000 (1.86×10^5) miles in a second. As David Mermin points out in his delightful book ***It's About Time****, this speed is very close to 1 foot in a nanosecond (ns), a billionth (10^{-9}) of a second. (For those of us living in a metric-challenged society, that coincidence is so fortuitous, that I have decided to use feet instead of meters as the standard of distance in this book. The actual value for the speed of light in feet is 0.98 ft/ns. A mere difference of 2% was not enough to deter me from using 1 ft/ns for the speed of light throughout this book!)*

Note that some of the numbers in the above paragraph were slyly written in scientific notation. Though it is not necessary to understand scientific notation, a short primer on scientific notation is given in Appendix B at the end of the book. This appendix ought to be helpful to people who are not so familiar with this useful way of dealing with large and small numbers.

Aside: It is very useful to be able to change the units used to describe the speed of light. For example, to find the speed of light in terms of feet/second, start with the speed in miles/second, 186,000 miles/second. Use the fact that one mile is equivalent to 5280 feet, 1 mile = 5280 feet. This means that the ratio,

$$\frac{5280\ feet}{1\ mile} = 1.$$

Any expression can be multiplied by one without changing it. To change the miles in 186,000 miles/second to feet, multiply that speed by $\frac{5280\ feet}{1\ mile}$. *The mile units cancel leaving feet in its stead. Next multiply 186,000 by 5280 to get 982 million. This shows that 186,000 miles/second is equal to 982 million ft/s which was rounded off to 10^9 ft/s or 1 ft/ns. In Chapter Four, the useful trick of multiplying an expression by 1 will be used to simplify algebraic expressions.*

The speed of light is so large that under normal circumstances the time it takes light to travel from here to there is "essentially" zero. For example, the rule for estimating how far you are from a lightning strike states that for every five seconds of time between seeing the flash and hearing the resulting thunder corresponds to a separation of one mile. If you hear the thunder 10 seconds after seeing the lightning, the bolt hit 2 miles away. This useful rule comes from the fact *that sound travels at about 1000 ft/s and a mile is 5280 feet. The assumption is that the flash arrives instantaneously so for each second you count off between seeing the flash and hearing the thunder, the sound travels 1000 feet. Of course if sound traveled faster than light, you would hear the thunder first and see the flash second!*

The time it takes the flash from a lightning strike to cover one mile is 1mile/(186,000 miles/s) or about 5 millionths (5×10^{-6}) of a second. Obviously, our senses are totally incapable of noticing times that small.

During the same period that Fizeau and others were doing careful experiments to accurately measure the speed of light, Clerk Maxwell, an English physicist was codifying all the assorted phenomena involving electricity and magnetism into a set of four equations, now known as Maxwell's equations. At that time, no one, including Maxwell, expected there to be any connection between light and electric and magnetic phenomena. But around 1860, he showed that electric and magnetic fields can travel as linked waves, electromagnetic waves, and the predicted velocity of these waves was given in terms of two well known physical constants, the permittivity, ε_0, and permeability, μ_0, which had no apparent connection to light or its speed. (The permittivity is a constant that shows up in the equation used to find the magnitude of the electric force between two point charges while the permeability is a constant that connects the current in a wire with the magnitude of the resulting magnetic field that surrounds the wire.) But when Maxwell calculated the speed of his "electromagnetic waves" by using the known values for ε_0 and μ_0 he found the speed eerily close to the known speed of light prompting him to observe:

"The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.[1]"

The historical evolution of our understanding of the nature of light is another curious story that would take us far afield from our main goal. But note that Maxwell speaks about light being a disturbance propagated through a substance which is just another way of saying light is a wave traveling through a medium, the lumeniferous ether.

The question of whether light is a wave or a particle also has a fascinating history. It turns out that the question is unanswerable because the behavior of light is particle-like under some circumstances and wave-like under others. Instead of trying to categorize light with labels like particle or wave, think of light as the stuff that behaves light-like. In this book, picture a flash of *light as something produced by a laser being turned on and off very quickly. For example, if the laser is on for 1/10 a nanosecond, the flash is 1/10 of a foot long, about an inch. So laser flashes in this book are short enough to be considered "objects" that travel in straight lines at 1 ft/ns. For experiments done over distances of 100 feet or so, a flash one inch long is short enough to qualify as a "particle of light."*

One last point, in 1983 scientists decided to define the speed of light to be exactly 299,792,458 meters/second. This changed the speed of light from an experimentally determined quantity to one that had a fixed value. Previously, the meter was defined as the distance between two scratch marks on a bar in a Paris vault. This change was made because scientists are able to measure time much more accurately than distance. With this definition for the speed of light, a meter became the distance light traveled in 1/299,792,458 of a second instead of the distance on that Parisian bar! This change has no impact on the conclusions reached by our intrepid explorers of space and time in Chapter Three, when they experimentally determine the speed of light.

1. James Clerk Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Philosophical Transactions of the Royal Society of London 155, 459-512 (1865).

Chapter Three: Space and Time After 1905

A New Set of Experiments are Proposed

The space and time explorers took off for a few days to enjoy their triumph over Galilean space and time in Chapter One. By their third day of chilling, Dean was getting restless and reminded them that the experiments with the pigeon and bus were just precursors to the more challenging experiments they needed to do with the pigeon being replaced by something moving at much larger speeds. After all, if they want to understand the space and time of Einstein they need an object moving at or near the speed of light.

Bev suggests that they cut to the chase and use a flash of light as the object to replace the pigeon. After all, they will be hard pressed to find anything else that can approach the speed of a light flash. Her friends agree.

New Stop Watches are distributed

The stop watches used in Chapter One will be of little use when trying to measure the speed of light. Remember that those watches read times to 1/10^th^ of a second. In 1/10^th^ of a second light travels 30 million meters or 18,600 miles or 100 million feet -- a distance larger than 2/3 the circumference of Earth.

New state-of-the-art stop watches are distributed to Anne, Bev, Chuck, and Dean. These new watches are accurate to 1/10^th^ of a <u>*nanosecond*</u> *(10^{-10} seconds). In order to take full advantage of these cutting edge watches, they will need reactions times of 1/10^th^ a nanosecond or better! Although real people or stop watches for that matter cannot be this accurate, it is easy enough to imagine well-trained genetically engineered observers with extremely accurate stop watches. Those people with the aforementioned stop watches are going to be doing experiments through much of the rest of the book. The experiments they perform are in the parlance of Einstein, "gedanken" experiments, or thought experiments. Though the experiments cannot be done as described because of human and technological limitations, the results of the experiments are completely consistent with the current scientific understanding of space and time.*

Anne and Bev Measure the Speed of Light

Anne and Bev requisition a laser that can be turned on and off in 0.1 ns. The light flash produced is 0.1 foot long, or about one inch. Their plan is to measure the speed of the light flash in a manner analogous to the method used to measure the speed of the pigeon. Consequently Bev walks to the 100 feet marker on the road while Anne remains at the zero foot mark. Bev stares intently at Anne and is ready to start her watch as soon as she sees Anne press the button on the laser that will start the one inch bit of light zooming toward her. She is standing next to a screen that will flash the instant the light arrives signaling her to stop her watch.

Anne presses the button and hears Bev exclaim, "RATS, I must not have been concentrating intently enough." Bev pulls herself together and tells Anne she is ready. Anne launches another light flash and Bev screams, "DOUBLE RATS!"

Anne walks over to Bev to find out what is causing her such consternation. Bev tells Anne, "Both times the light hit the screen at the same instant that I saw you press the button. It took NO time for the light to travel 100 feet."

Anne reminds Bev that light travels 1 foot in a nanosecond. Therefore it ought to take light 100 ns to cover the distance between us. She says to Bev, "We have been endowed with extraordinary reflexes and have the very best stop watches available that allow us to make measurements accurate to 0.1 ns. There is absolutely no reason why we cannot do this simple experiment!"

Chuck, who has been standing by watching quietly, suddenly yells out, "Experimenting with light is going to be trickier than we thought." Anne, Bev, and Dean simultaneously look at Chuck waiting for him to elaborate. Chuck is happy to oblige his friends; "Bev is watching Anne and waiting for her to press the button that sends the light flash on its way. The reason Bev can see Anne is because light is traveling from Anne to Bev. If it were pitch black outside, Bev would not even be able to see Anne let alone see her press the button. The image of Anne starting the laser pulse on its way travels to Bev at the speed of light, the same speed as the light flash. Therefore Bev `sees' Anne press the button at the same instant the light hits the screen."

The four friends sit down at a picnic table and mull over this difficulty in experimenting with objects moving at light speeds. Chuck points out that the same situation was taking place during the experiment with the pigeon. The image of Anne releasing the pigeon took 100 ns to reach Bev. So under the best of circumstances, Bev was starting her stop watch 100 ns late. But that late start had no discernible bearing on the time she measured for the pigeon to fly from Anne to her because 5 seconds – 100 nanoseconds = 4.9999999 seconds. This time appeared as 5.0 seconds on their old watches which were only accurate to tenths of a second. But this "error" will be a problem any time they try to apply this method to objects moving at or near the speed of light. Chuck suggests that in all future experiments each of them only pay attention to readings on their watch and to events in the space immediately adjacent to them to avoid the "light travel time problem."

Meanwhile Anne has been pacing around and mulling over the timing problem. She looks at her watch and starts and stops it a few times. Finally she calls her friends over to make the suggestion that saves the day. "Bev and I need to start our watches before the start of the experiment. We can do this by standing side-by-side when we start them. This eliminates the "light travel time problem." After starting them, we amble off to our stations 100 feet apart just like before except now we have running watches that have been synchronized. At the instant I press the button on the laser, I simultaneously read the time on my watch, T_{Anne}. Then Bev records the time the flash arrives as T_{Bev}. Afterwards we calculate the time difference, $T_{Bev} - T_{Anne}$, which is just the time it took for the flash to travel 100 feet." Chuck thinks Anne's idea is brilliant.

In the future, observers in the Earth or bus-frames will record the time that events happen at their fixed location in space. A & B and C & D will synchronize their watches using the method described by Anne. Before the start of any experiment, A & B will meet to synchronize their watches. C & D will meet on the bus as it travels towards A & B to synchronize their watches. All the observers will have watches that are running before the start of any experiment. During the experiment, each observer will be responsible for recording the time that events happen in his or her neighborhood. After the experiment is over, these local measurements will be shared to form a comprehensive view of the events that took place.

Now Anne and Bev redo the experiment with the light flash traveling from left to right. When they compare stop watch times, they discover that $T_{Anne} - T_{Bev} = 100$ ns just like expected. Anne gives the laser to Bev and they repeat the experiment with the light traveling from right to left. Bingo, again the light took 100 ns to travel 100 feet. So the speed of light was the same whether it traveled from Anne to Bev or Bev to Anne.

In the late 19th century, Michelson and Morley, American physicists, did very careful experiments to determine the speed of Earth with respect to the lumeniferous ether. The basic idea was that as Earth orbited the sun, its velocity with respect to the ether would change. Therefore at any given time, from the perspective of a stationary Earth, there would be an *"ether" wind blowing in some direction over Earth's surface. Consequently, Michelson and Morley expected to measure different speeds for light depending on how light was traveling with respect to the ether wind. Remember A & B did analogous experiments when they measured the speed of the pigeon flying with and against the wind that blew from Anne toward Bev.*

Michelson and Morley, through a series of careful experiments stretching over years, failed to see any effect on the speed of light due to the motion of Earth through the ether. From their result, it appeared that nature conspired to make the existence of the ether immune to experimental verification. As other physicists invented various schemes to explain this failure to detect the ether, Einstein decided on a more radical solution. He banished the lumeniferous ether to the dust bin of failed ideas. His explanation was that light propagated through empty space which made it very different and much stranger than all the other sorts of waves scientists had previously encountered, all of which traveled through a material medium.

The Super Bus Rolls into the Story

The bus used by Chuck and Dean in Chapter One is woefully inadequate to help during experiments with the laser. It takes light 100 ns to traverse the length of a bus 100 feet long. During the time it takes the flash to move 100 feet, the bus, moving at +30 ft/s with respect to Earth, would cover 30 ft/s times 100 ns (10^{-7} seconds) = 30 x 10^{-7} feet = 3 x 10^{-6} feet (3 millionths of a foot!). From the perspective of light, a bus moving 30 ft/s is stationary!

If our intrepid explorers of space and time want to study the speed of light in a reference frame moving with respect to A & B on Earth, the moving frame will have to have a speed comparable to the speed of light. Consequently, Chuck and Dean order a new "super bus" capable of zipping smoothly down the road at constant speeds up to $\frac{9}{10}$ the speed of light or 0.9 ft/ns. This new bus is exactly 100 feet long just like the original one. C & D borrow the laser from A & B and trade-in their old stop watches for the new more accurate ones.

The old rulers did not have to be upgraded because they were and are still good enough to measure the length of the bus accurately.

Question 3.1: A & B are surprised one day to see a rogue bus come rumbling down the road towards them. They see the bus soon enough to be able to synchronize their watches and station themselves along the road. The bus passes them and continues down the road out of sight.

Describe a method that A & B can use to measure the length and velocity of the rogue bus as it speeds through their reference frame. This question will be addressed later in this chapter in the section entitled *The Effect of Motion on Space.*

The plan for the bus-based experiment is straightforward. C & D will drive down the road at $\frac{3}{5}$ the speed of light, 0.6 ft/ns. While moving at a constant velocity with respect to Earth, Chuck and Dean will measure the speed of light inside the bus; first for a flash going from Chuck to Dean and then again for a flash going in the opposite direction, from the front to the rear of the super bus. They will synchronize their watches by the same method used by Anne and Bev. A & B sit down and watch as the bus heads down the road, turns around, and rumbles by with C & D inside measuring the velocity of light. In no time flat, C & D are back to share the results of the bus-based measurements of the speed of light.

Chuck summarizes their results for A & B, "When the laser flash traveled from the rear of the bus to the front, I noted the time, T_{Chuck}, when I pressed the button on the laser. Then Dean wrote down the time, T_{Dean}, that the flash arrived at the front of the bus. Using those times, we found the transit time for light, $T_{Dean} – T_{Chuck}$ = 100 ns. Then we repeated the experiment for a flash moving in the opposite direction and got an identical time difference, 100 ns. So for light moving in either direction, the speed of light was the same on the moving bus as it was on Earth."

No one is surprised by this result because it is consistent with the results they got using the pair of pigeons in Chapter One. The speed of the pigeon was the same on the bus as it was flying between A & B while standing on the ground. Also the speed did not depend on whether it was flying left to right or right to left. Our four friends decide to call it quits for the day, satisfied that they are making good progress in understanding space and time. Before heading home, they agree to meet early the next morning for another round of experiments.

The fact that the experiment on the bus with the light flash reproduced the results of the experiment done on Earth is another confirmation of the Principle of Relativity. Also because the speed of light in both cases did not depend on the direction of the light flash, these experiments add credence to the notion that space is isotropic.

The next day, Anne wants Chuck and Dean to redo the experiment with the laser on the bus. But this time, she and Bev will also collect data as the light flash travels from the rear to the front of the moving bus.

Chuck, thinking about his spacetime graphs, points out that this is a more complicated experiment because it involves observers in two different reference frames. The two experiments done the previous day only involved observers in a single reference frame: first A & B on Earth and then he and Dean in the bus.

Because of this added complication, Chuck suggests that they carefully go over the details of the experiment before actually having the bus zip by at 0.6 ft/ns. After all, it takes quite a bit of fuel to get the bus up to that speed so they ought to make sure they get it right the first time they try.

Anne and Bev Observe the Light Flash on the Bus: I

Chuck writes down what each of them will do during the experiment. When he is finished, he shares his list with his friends:

1. Anne and Bev will synchronize their watches before the start of the experiment. C & D will synchronize their watches on the bus while it zips toward A & B at 0.6 ft/ns. Chuck will be at the rear of the bus, $X_{bus} = 0$ and Dean will be at the front, $X_{bus} = 100$ feet.
2. The experiment will begin when Anne, standing at $X_{Earth} = 0$, and Chuck pass one another. At that precise instant, Chuck launches the light flash and he and Anne record the times on their watches. Those times are T_{Chuck} and T_{Anne}.
3. When the light flash reaches Dean, he will be adjacent to Bev. They will check their respective watches to note the time the flash arrived. Those times will be recorded as T_{Dean} and T_{Bev}.

Chuck's outline of the upcoming experiment meets with general agreement. Anne heads to the $X_{Earth} = 0$ spot while C & D board the bus and begin to go over their checklist. Bev begins to move off down the road but comes to a confused stop. Then she yells out in a voice loud enough for everyone to hear, "Where I am supposed to stand so that I will be next to Dean at the front of the bus when the light flash arrives?"

Question 3.2: Using what our friends learned about space and time in Chapter One, how far to the right of Anne should Bev stand so that she is adjacent to Dean when the light flash arrives?

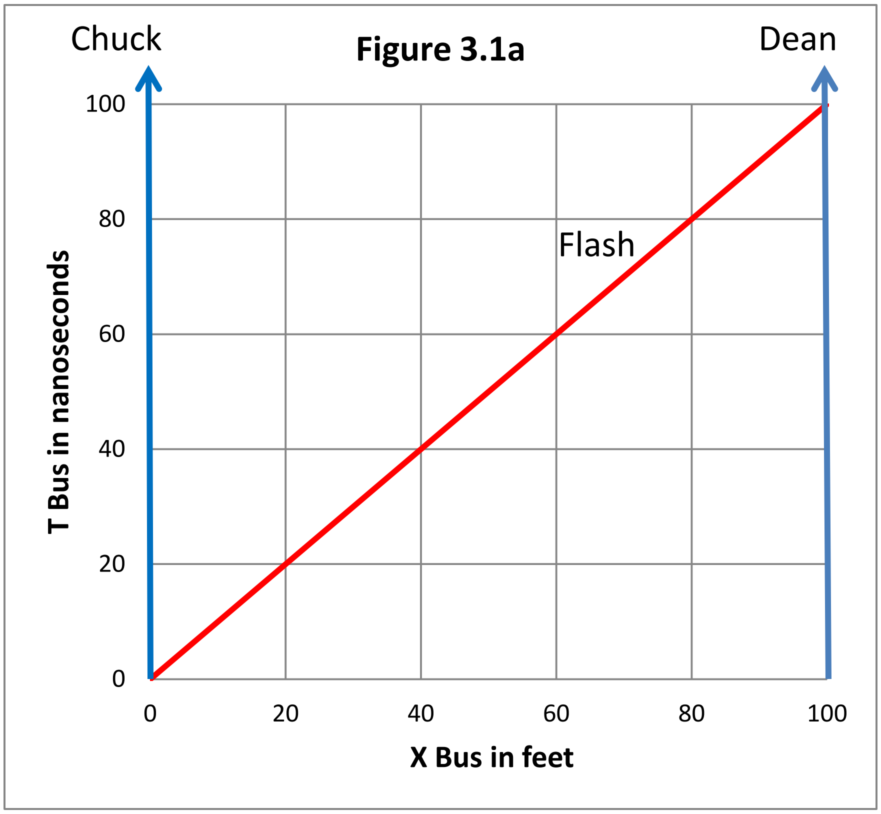
Chuck reviews his pre-experiment instructions and reluctantly admits that a crucial bit of information was missing. Namely, Bev's location so that she will be standing at just the right spot on the road to be adjacent to Dean just as the light flash reaches the front of the bus.

Anne points out that she and Bev have done this sort of thing before when they observed the pigeon flying from one end of the bus to the other. She mumbles to herself. "According the experiments done by C & D the previous day, the light flash will take 100 ns to cross the bus. During those 100 ns, the bus moves 60 feet down the road. Therefore the flash moves the length of the bus plus an extra 60 feet, for a total of 160 feet. Bev, you need to be standing 160 feet to my right." Bev nods in agreement.

After this careful preparation, Chuck and Dean board the bus, drive down the road, turn around, and head toward A & B ready to begin the experiment. At the instant Anne is adjacent to Chuck the laser flash begins its trip to the front of the bus. She and Chuck both record the times on their watches, T_{Anne} and T_{Chuck}. Bev, waiting anxiously 160 feet down the road, is horrified when the bus passes her before the light flash has gotten to Dean! Dean, intently waiting for the flash to arrive, does not see the bus zip by Bev. At the instant the flash arrives, he looks out the window expecting to see Bev but instead sees the 200 foot marker along the side of the road!

Anne and Bev wait for Chuck and Dean to return so the four of them can try to figure out what went wrong with the experiment. As soon as C & D step out of the bus, A & B ask them what happened. Dean looks bewildered by the turn of events but Chuck is ready to defend their experimental skill. Chuck says, "From the perspective of the bus, everything went perfectly. The light flash took exactly 100 ns to go from the back to the front of the bus."

A & B shake their heads and disagree with Chuck's rosy assessment of the experiment. Everything could not have gone perfectly because Bev, who was standing exactly at the 160 foot marker on the road, saw Dean pass <u>before</u> the light flash arrived! Chuck now admitted that there was one strange bit of evidence that he had not mentioned, namely that Dean thought he was next to the 200 foot road marker when the light flash arrived.



"Impossible," exclaim A & B simultaneously! Dean defensively asserts, "I know what I saw. And I also know what I did not see. Instead of Bev, I saw the 200 foot marker."

Anne points out that there is little to be gained by arguing. Clearly, although these experiments are expensive to do, this one has to be repeated. Chuck suggests they review the information gleaned from their first experiment before repeating it. This makes good sense to everyone.

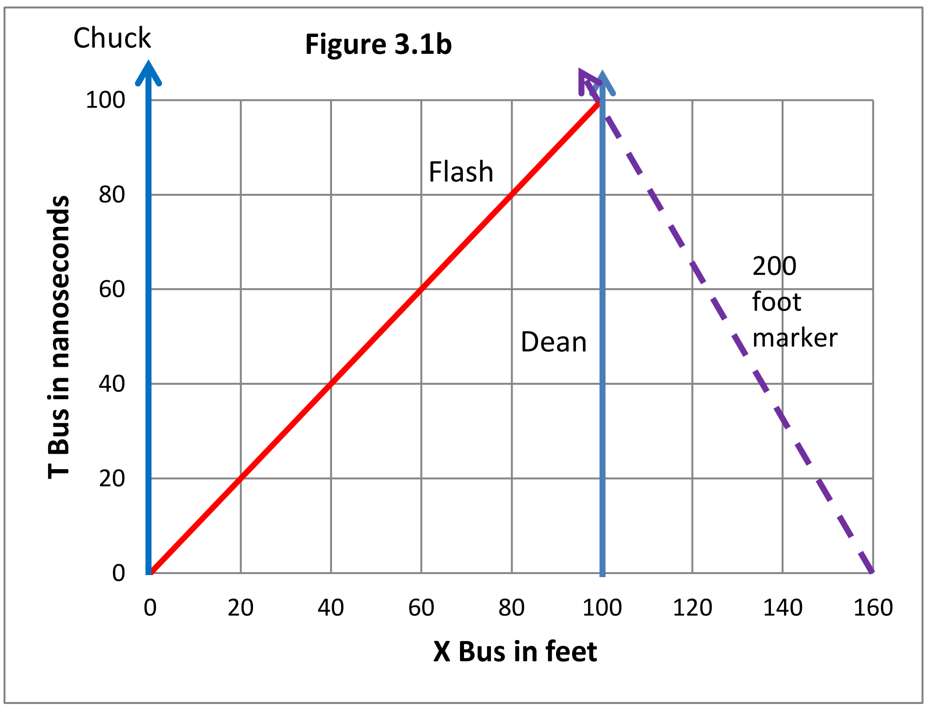
The failure of this simple experiment with the bus and light flash to confirm the Galilean nature of space and time marks the beginning of the development of the space and time introduced by Einstein in 1905. Our four experimenters are about to redo the experiment more carefully. In the process they will make some measurements of time. The data collected from this single simple experiment will be enough to develop the complete theory of Special Relativity.

While his three friends grab a bite to eat, Chuck draws the careful spacetime graph, figure 3.1a, which summarizes the experiment from the perspective of Dean and him, the bus riders. The experiment began at $T_{bus} = 0$ when he pressed the button that started the light flash zipping toward Dean, the blue worldline at $X_{bus} = 100$ feet on figure 3.1a. Exactly 100 ns later, Dean's screen recorded the arrival of the flash, the red worldline. Dean is exactly 100 feet to his right. Everything about figure 3.1a makes perfect sense.

Notice that on the graph drawn by Chuck, each box is 20 feet long and 20 nanoseconds high. This means that light traveling at 1 ft/ns moves 20 feet, one box to the right, in 20 ns, one box up. All the spacetime diagrams in the remainder of the book will have space and time scales that *make the slope of the worldline of a light flash either +1 for light moving toward the right or -1 for light moving toward the left.*

One other point: the experiment starts when Chuck passes Anne. At that instant, their watches read T_{Chuck} and T_{Anne} respectively. When the experiment is over, all the bus observers, including Chuck, subtract T_{Chuck} from whatever times they recorded for events at their location during the experiment. This adjustment shifts the bus starting time for the experiment from T_{Chuck} to zero. Analogously, all the Earth observers subtract T_{Anne} from their recorded times. These adjustments in time make the starting time for each experiment zero according to both sets of observers. This allows us to continue calling the starting time for every experiment T = 0.

Chuck, on the assumption that Dean did see the 200 foot road marker outside his window just as the flash arrived, decides to add the worldline for that road marker to the spacetime graph. From the perspective of the bus, that marker had a velocity of -0.6 ft/ns. So it traveled 60 feet, right to left, during the 100 ns it took the flash to cross the bus. Since the bus is 100 feet long, that put the 200 foot road marker 160 feet to the right of Chuck. The worldline for the marker is shown as the purple dashed line on figure 3.1b. Chuck rubs his eyes, frowns, and mumbles to himself, "This is very perplexing."



Question 3.3: From the perspective of the bus riders, figure 3.1b, how far in front of Chuck is the 200 foot road marker at the start of the experiment, $T_{bus} = 0$?

Question 3.4: Anne is adjacent to Chuck at the start of the experiment. At that instant, from her perspective, how far in front of Chuck is the 200 foot road marker?

The answers to these questions, though weird, are as simple as they seem!

Anne, Bev, and Dean come strolling over to Chuck. "What are you staring at?" asks Dean. He shows his three friends figure 3.1b and explains that it contains all the data collected from the experiment. Chuck points to the worldline of the road marker, "Notice that from my perspective on the bus, that marker is 160 feet away at the start of the experiment. But Anne, who was right next to me when the experiment began, would insist that the 200 foot marker is 200 feet in front of her! How can the same marker be 160 feet from me and 200 feet from her?" Because he was asking a rhetorical question, he did not expect an answer nor did he receive one!

Finally Dean, who actually saw the 200 foot road marker outside his window when the laser flash arrived at the front of the bus, suggests they redo the experiment with Bev 200 feet in front of Anne. A & B agree with that suggestion. Chuck points to figure 3.1b, and reminds his friends that from the perspective of bus observers, it sure looks like the 200 foot road marker was only 160 in front of him. Their next experiment ought to determine once and for all the distance between the 200 foot marker and he and Anne who were adjacent at the start of the experiment.

The four of them stare at the bus and wonder how to get a bus observer 160 feet to the right of Chuck which is actually 60 feet in front of the bus! Dean says, "No problem, I can attach a 60 foot boom with a seat to the front of the bus."

Chuck and Dean work on adding the boom to the front of the bus with a seat for the new observer. Meanwhile, Anne and Bev, scout around for a volunteer to ride in that seat. Ed happens to be passing by with nothing urgent on his agenda and thinks it will be exhilarating to be pushed along at 60% the speed of light in front of a bus.

C & D explain how the three of them will meet in the bus to synchronize their watches. Chuck tells Ed, "Once our three watches are running, Dean and I will stroll to our positions at the front and rear of the bus. Meanwhile, you'll crawl along the boom to reach your seat at X_{bus} = 160 feet."

Ed stares incredulously at the boom, looks at A & B for confirmation that Chuck was serious, and sighs, "After our watches are running properly, Chuck walks to the rear of the bus, Dean to the front, and I crawl along the 60 foot boom to reach my seat!" C & D nod in agreement. Being a trooper, he agrees but with less enthusiasm and more trepidation than he had had before.

Anne and Bev Observe the Light Flash on the Bus: II

The next morning, the five of them start getting ready for the re-run of the previous day's experiment. Chuck, Dean, and Ed practice synchronizing their watches a couple of times so that Ed can get used to crawling 60 feet along the boom to his observation post. C & D keep reminding him that from their perspective, the bus is stationary and it is Earth that is moving at 0.6 ft/ns. Their explanation does little to ease his anxiety.

Chuck gathers his friends together and carefully reviews the sequence of events that will transpire during the experiment:

1. Before the experiment begins, A & B will synchronize their watches. Using the same method, C & D & E will synchronize their watches.
2. When Anne and Chuck are adjacent to one another, Chuck will launch the laser flash and each of them will record the times on their respective watches, T_{Anne} and T_{Chuck}.
3. Ed and Bev will record the time on their watches as they pass, T_{Ed} and $T_{Bev,1}$.

Chuck reminds Bev that she has two times to record, hence the extra subscript.

4. Dean will note the time, T_{Dean}, that the laser flash reaches the front of the bus. And if things go according to plan, Bev will be adjacent to Dean when the flash arrives. She will note the time for that event as $T_{Bev,2}$.

When Chuck is finished, Anne says, "As we pass one another, each of us ought to note the time on the other person's watch. That will act as a check to insure that everyone recorded their times accurately." Chuck likes that idea and adds it to his list.

Chuck's step-by-step rehearsal of the experiment has gotten his friends excited and ready. The three bus observers pile in and start driving away as A & B yell, "*bon voyage*." Then A & B get down to business. They synchronize their watches, take up their stations at $X_{Earth} = 0$ and $X_{Earth} = 200$ feet, and concentrate on recording accurate times for the events that happen at their location.

In a little while, Ed comes into view, being pushed along by the bus. In the blink of an eye, Ed and the bus have passed A & B. Bev is very relieved because she was in exactly the right spot to observe the laser flash reach Dean at the front of the bus. Anne and Bev sit down under a tree and anxiously wait for the bus to return with their friends.

After the bus rolls to a stop and its passengers disembark, there is a round of congratulations for a job well done. But Anne impatiently cuts off the celebration and presents the data she and Bev collected. Seeing that Anne is in a no nonsense mood, Chuck writes down the times measured by the bus crew during the experiment.

Question 3.5: The experimental data collected by our space and time explorers are listed below:

t_{Chuck} = 200 ns, t_{Anne} = 105 ns, t_{Ed} = 200 ns, $t_{Bev,1}$ = 225 ns, t_{Dean} = 300 ns, and $t_{Bev,2}$ = 305 ns.

Those times have to be adjusted to make the start of the experiment T = 0 for both sets of observers. What are the values for the corrected times, T_{Chuck}, T_{Anne}, T_{Ed}, $T_{Bev,1}$, T_{Dean}, and $T_{Bev,2}$?

All of them study the times listed above and compare them to the times they saw as they passed one another. Everyone agrees that the times listed are correct. With that agreement in hand, Anne subtracts t_{Anne} from the data collected by her and Bev to get the adjusted times listed below:

T_{Anne} = 0

$T_{Bev,1}$ = 120 ns

$T_{Bev,2}$ = 200 ns

Without any fanfare, Chuck does the same for the times collected on the bus:

T_{Chuck} = 0

T_{Ed} = 0

T_{Dean} = 100 ns

They silently look at the data until Ed gets up to leave and says, "I have some errands to run." As he heads off, his four friends thank him for being a good sport and not complaining about his precarious perch in front of the bus.

Bev thinks the best way to try to make sense of the data is by using it to construct a spacetime graph of the experiment. Her friends agree since none of them have been able to make much headway by staring at the different times listed above. Chuck points out that there were three key events during the experiment.

A. He and Anne passed one another and the laser flash started across the bus.
B. Ed and Bev passed one another.
C. Bev and Dean witnessed the arrival of the laser flash at the front of the bus.

Chuck asks A & B, "Why don't you make a spacetime diagram representing the experiment from the perspective of Earth with those three events explicitly labeled on your graph. Dean and I will do the same from the perspective of the bus. Let's plan to meet tomorrow to see if the graphs can help us make sense of the experiment." A & B readily agree.

Tired, but feeling like much was accomplished, they head off for some well-earned rest.

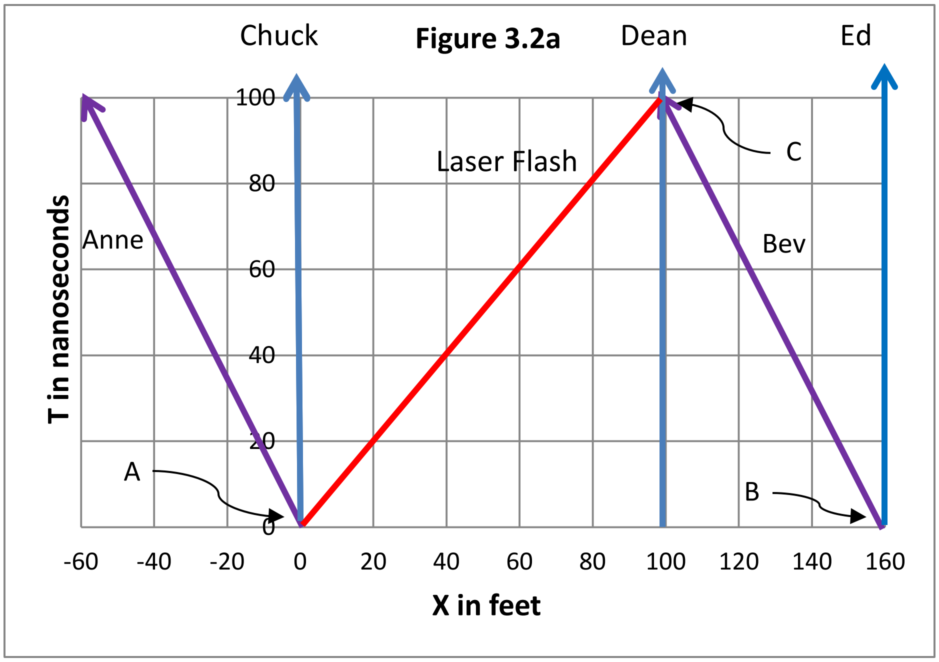


Figure 3.2a

By the next day, the four friends are eager to share their spacetime graphs. Chuck starts by showing everyone figure 3.2a; the spacetime graph from the perspective of observers on the bus. Of course he followed his own suggestion and labeled the three major events during the experiment as A, B, and C. Anne follows suit and presents the graph she and Bev drew from their Earth-based perspective, figure 3.2b. Four pairs of eyes scan the two spacetime diagrams. After a short time, there are nods all the way around acknowledging that the diagrams correctly represent the events that unfolded during the experiment.

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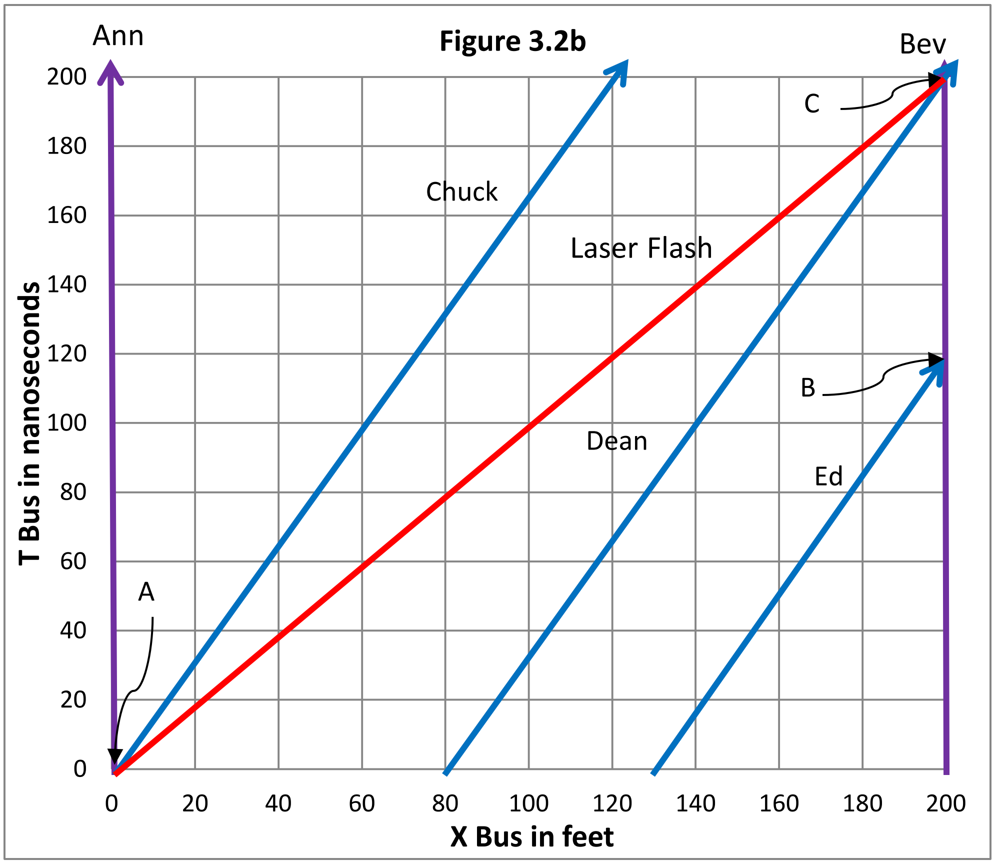


Before reading on, this is a good time to make sure you understand how these diagrams were constructed. The discussion will focus on figure 3.2a, but the same strategy was used to draw figure 3.2b. First, from the perspective of the bus, Chuck, Dean, and Ed are fixed in space at X_{bus} = 0, 100, and 160 feet, respectively. Events A, B, and C all happen in the space between Chuck and Ed. The experiment took 100 ns starting at T_{bus} = 0. Therefore the slice of spacetime that needed to be represented on this graph included X_{bus} from 0 to 160 feet and T_{bus} from 0 to 100 ns. This information allowed Chuck to figure out how many boxes, each 20 feet long and 20 ns high, were needed to capture the relevant chunk of spacetime. He included a little extra space to the left of the origin for Anne's worldline. The worldlines of the three stationary bus observers are vertical.

He also knew that Anne passed Chuck at event A. Therefore Anne's worldline had to pass through event A and slant to the left in a manner consistent with her velocity with respect to the bus, -0.6 ft/ns. That velocity corresponds to moving 3 horizontal boxes from right to left while moving up 5 boxes; -60 feet of space moved during 100 ns of time passed. More generally, the worldline of any observer moving with a known velocity can be drawn on a spacetime diagram by locating an event that observer passed through. The same strategy was used by Chuck to draw Bev's worldline through event B, the spacetime point where she and Ed passed.

Carefully look over the spacetime diagram constructed by A & B, figure 3.2b. Notice that event B, the place in spacetime where Ed and Bev passed, happened at X_{Earth} *= 200 feet and* T_{Earth} *= 120 ns. After finding that point, it was easy for A & B to draw Ed's worldline. Dean's worldline was constructed similarly since he passed Bev at* X_{Earth} *= 200 feet and* T_{Earth} *= 200 ns, event C.*

Dean looks at figure 3.2b, shakes his head, and groans out loud, "As Einstein might have said, the spacetime of Chapter One is *kaput!*" Continuing to shake his head, he adds, "the velocity of the light flash in the bus, V_{FB}, is the same as Chuck and I measured it in the earlier experiment, 1 ft/ns. The velocity of the bus with respect to Earth, V_{BE}, is 0.6 ft/ns. Using the handy-dandy relative velocity equation that A & B derived, equation 1.3, we expected the velocity of the flash with respect to Earth to be the sum of those two velocities, V_{FB} plus V_{BE}, or 1.6 ft/ns. But as figure 3.2b shows so very clearly, A & B, saw the flash go 200 ft in 200 ns for a velocity of 1 ft/ns – which makes no sense!" So as I said earlier, "The good old spacetime of Chapter One is dead, kaput, tossed in the dust bin of failed ideas along with the lumeniferous ether."

Chuck tells him to calm down, let's think about the implications of our accurately done experiment. First he points out that the light flash emitted by the laser in the bus traveled 100 feet in 100 nanoseconds, but when viewed from Earth, it traveled from Anne to Bev at X_{Earth} = 200 feet and T_{Earth} = 200 ns: 200 feet in 200 nanoseconds!

This last statement gets Anne's attention, who chimes in, "Suppose the above experiment was redone in exactly the same way <u>except</u> I had the laser instead of Chuck. That flash would take 200 nanoseconds to reach Bev standing 200 feet to my right. The spacetime diagram Bev and I would draw for that experiment would be identical to figure 3.2b. But that means the two spacetime diagrams, 3.2a and 3.2b, would be unchanged if I had the laser and sent the flash on its way!" Chuck thinks about that for a bit before agreeing with Anne. Anne continues, "The light flash travelled through spacetime in a manner that was totally independent of the state of motion of the laser, the source of the flash!" Dean pipes in again, "That makes no sense."

In Chapter One, the motion of the pigeon through spacetime depended on whether it flew in the bus from Chuck to Dean or from Anne to Bev who were stationary with respect to the ground. Light apparently behaves very differently.

Question 3.6: Convince yourself of the truth of Anne's assertion that the spacetime graphs, figures 3.2a and 3.2b, would be unchanged if the laser were with Anne instead of on the bus.

Bev reminds Dean that the arbiter of what makes sense or not, is experiment not expectation. But she agrees with Dean, that the results of the experiment are clearly contrary to their collective expectations. Their job now is to try to interpret what the experiment is saying about space and time.

Dean, undeterred says, "Imagine redoing the experiment in Chapter One with two pigeons. Pigeon A is in the bus riding with Chuck. Pigeon B stays with Anne. Bev is standing 100 feet to the right of Anne. Both pigeons are released simultaneously as Anne and Chuck pass one another. If the flight of the two pigeons mirrored the experiment done with the light flash, the two pigeons would have identical worldlines through space and time. Their paths would be exactly the same as they flapped from Anne and Chuck toward Bev and Dean. Do any of you think that would actually be the case if we did this experiment?" The unanimous answer was no! "Undoubtedly, pigeon A would reach me first," adds Bev.

Question 3.7: Draw the spacetime diagram, from the perspective of A & B, which corresponds to Dean's *gedanken* experiment. The velocity of the bus is + 30 ft/s and the pigeons fly with a speed of 20 ft/s. Label the worldlines for pigeons A and B and include enough time on the graph for both pigeons to reach Bev.

Question 3.8: Now draw the Earth frame spacetime graph that corresponds to Anne and Chuck simultaneously firing identical lasers towards the right as they pass one another. The 100 foot bus is moving at v = 0.6 ft/ns and Bev is 200 feet to the right of Anne.

Your answers to questions 3.7 and 3.8 highlight the difference between the way a pigeon moves through space and time compared to light. By the way, your answer to question 3.8 ought to be identical to figure 3.2b!

Chuck reminds Dean that all the experiments done by scientists to identify the lumeniferous ether, the substance through which light was supposed to travel, failed. So unlike the pigeon, whose speed is with respect to air, light apparently travels in a more mysterious manner. Their experiment showed that Earth and bus observers moving at $\frac{3}{5}$ the speed of light with respect to one another measure exactly the same speed for a light flash. And that speed does not depend on whether the laser producing the flash is on the bus or attached to Earth. Moreover, the measurements were made with identical rulers and watches and both sets of observers used the same protocol to synchronize their watches. Dean says nothing to this little speech by Chuck, but his body language makes it clear that from his perspective, light should not be allowed to behave this way!

Anne is pacing around with her eyes closed. Her three friends watch and wait. Finally Anne summarizes her picture of light as it moves through space and time. She says, "Apparently, light has to travel from here to there without violating the Principle of Relativity." That remark is met with puzzled stares from her three friends. She reminds them, "Measuring the speed of light is one particular experiment that can be done in the super bus or in an Earth-based lab, or for that matter in any two different reference frames moving with respect to one another. If light is going to travel through space and time in a manner consistent with the Principle of Relativity, every set of observers in an inertial reference frame measuring its speed are preordained to get the same value." Dean rolls his eyes, and whispers, "Preordained my foot."

Our space and time explorers have learned that the speed of light in a vacuum is independent of the state of motion of either the source of the light or the observers measuring it. This property is often referred to as the Principle of the Constancy of Light.

Bev says, "Our experiment certainly seems to force the conclusions voiced by Chuck and Anne. But the speed of light is the ratio of a measured distance (space) to a measured time that are determined using rulers and watches. Do the results of our experiment have implications for the way rulers and watches measure space and time?"

Bev's simple question changes the tone of the discussion. The ephemeral properties of light are left behind. The conversation shifts to the properties of real material things like rulers and watches and how they are used to measure velocity instead of the peculiar way light moves through space and time. But before they get too deeply into this new topic, they decide to take a short lunch break.

The Effect of Motion on Space

After lunch, Bev cuts to the chase by stating, "There is no doubt that I was standing 200 feet to the right of Anne. On the other hand, we all measured the boom that held Ed's rolling seat and agreed that it was 60 feet long putting him 160 feet in front of Chuck." Her three friends nod in agreement. She continues, "But it is abundantly clear from the spacetime graph in figure 3.2a, that when Chuck and Anne are adjacent, Ed and I are passing one another. At that moment of crossing, Ed's watch read zero, exactly the same time as Chuck's watch. Consequently, Chuck and Ed conclude that I am 160 feet in front of Anne. But neither Anne nor I moved during the experiment, so the distance between us never changed. It was 200 feet during the entire experiment. How could the distance between Anne and me be both 160 feet and 200 feet?" Bev's remarks are met with silence.

Dean, though still uneasy about the apparent way light moves through space and time, joins this discussion. In a questioning voice he says, "A & B measure their separation the same way Chuck and I measure the length of the bus. We take our certified rulers and mark off the distance between two locations that are fixed in our respective reference frames. But how are the lengths of moving objects measured? For example, how would A & B measure the length of our bus as it zipped past them at 0.6 ft/ns?"

Chuck points to the spacetime graph in figure 3.2b. My and Dean's slanted worldlines on that graph also represent the spacetime locations of the front and rear of the bus as it moves through A & B's reference frame. If A & B want to measure the length of the bus, they would simultaneously locate the two ends of the bus. The intersection of any horizontal constant time line with our two slanted worldlines determines the exact space locations of the front and rear of the bus at that time. Then A & B could use their rulers to measure the distance between those two points to determine the length of the bus.

Dean nods at Chuck, "So if A & B used the horizontal $T_{Earth} = 0$ line on figure 3.2b to determine the length of our 100 foot super bus, would I be correct to surmise that A & B measure our bus to be only 80 feet long?" Chuck looks at Dean and then at figure 3.2b, before nodding in agreement and adding, "It appears that A & B conclude that our bus is only 80 feet long."

Anne says, "Of course, it has to be that way and shows her friends the table 3.1. "

	Measurements made by Chuck & Dean in the bus frame	Measurements made by Anne & Bev in the Earth frame
Length of the Super Bus	100 feet	80 feet
Distance Between Anne and Bev	160 feet	200 feet

Table 3.1

She quickly continues, "When C & D measured the separation between me and Bev, they got 160 feet or 80% of 200 feet. On the other hand, when Bev and I measured the length of the bus we got 80 feet or 80% of 100 feet. We each saw the "moving" separation to be 80% as long as the stationary separation. Space separations moving through a reference frame with a speed of 0.6 ft/ns appear to be only 80% as long as the separation in their "home" frame. The Principle of Relativity survives another test since the amount of shrinkage measured is the same for bus and Earth observers."

Imagine that Bev and Anne made a 100 foot long life-size model of the super bus. C & D come zipping by passing A & B and their model super bus. A & B measure the passing super bus to be only 80 feet long. On the super bus, C & D, see the life-size model rushing toward them. They measure its length as it zooms past and they also get 80 feet. The Principle of Relativity requires that each set of observers measure equivalent amounts of shrinkage for the moving "bus." That is exactly what Anne concluded from the data in the table she showed her friends.

The separation between two locations in the reference frame in which those locations are fixed is called the "proper separation or distance." In the above example, the proper separation between A & B was 200 feet while the proper length of the bus was 100 feet. Any set of observers moving with respect to the proper frame measure the length or separation to be less than its value in the proper frame. In our example with observers moving at 0.6 ft/ns lengths appear to be 80% as long as their proper values.

Question 3.9: In figure 3.2b, Ed was adjacent to Bev when her watch read 120 ns. Use that information to find Ed's location, X_{Earth}, at $T_{Earth} = 0$.

Question 3.10: Is your answer to question 3.9, the separation between Chuck and Ed according to A & B, consistent with the 80% rule?

Dean reluctantly concedes that the experimental results, namely the constancy of the speed of light, forces moving rulers to appear shorter than identical stationary rulers. He points to Bev and says, "When Bev passed me, Bev's watch read 200 ns and mine read 100 ns. This difference arose despite the fact that Bev carefully synchronized her watch with Anne's and I, just as carefully, synchronized my watch with Chuck's. And as both spacetime graphs, figures 3.2a and 3.2b, make clear, Anne and Chuck passed one another when their watches read zero. My unnerving conclusion is that during the time interval between the start of the experiment and Bev and me passing one another, Bev's watch ran twice as fast as mine. Will someone please tell me what is going on here?"

Bev yawns, and says time can wait until tomorrow. No one argues and they pack it in for the day.

Earlier in this section, Chuck used the spacetime diagram to answer Dean's question about how to measure the length of an object moving through a reference frame. But there are other ways for A & B to measure the length of the bus. In particular, question 3.1 had an unknown bus passing A & B. Using their synchronized watches, they could take three straightforward time measurements to determine both the velocity and length of the bus. Anne notes the times the front and rear of the bus pass her, $T_{Anne,1}$ and $T_{Anne,2}$. Bev writes down the time the front of the bus reaches her, $T_{Bev,1}$.

After the rogue bus disappears, A & B compare notes and conclude that the front of the bus traveled the known distance between them, D, in time $T_1 = T_{Bev,1} - T_{Anne,1}$. Therefore the velocity of the bus was just $v = D/T_1$.

The bus took time $T_2 = T_{Anne,2} - T_{Anne,1}$ to pass her. Since she knows that the bus was traveling with a speed v, she can calculate the length of the bus, L, by multiplying v times T_2, $L = v\ T_2$.

This simple procedure can be used to find the length and velocity of any object moving through a reference frame.

Question 3.11: A bus comes zipping through A & B's reference frame. They carefully use their synchronized watches to record the information required to determine the velocity and length of bus. After the bus passes, Anne says that the front of the bus passed at $T_{Anne,1}$ = 13 seconds and the rear 150 ns later. Bev saw the front of the bus pass her at $T_{Bev,1}$ = 13 seconds plus 125 ns. A & B were standing 100 feet apart when they recorded the times listed above. What was the velocity of the bus and how long was it?

The Effect of Motion on Time

That night, all four dreamt of watches with wings flying about and ticking at different rates and reading strange times. The next morning, they are eager to explore whether or not the strange way light moves through space and time impacts the rate at which watches and clocks run. Of course, as Dean already pointed out, the fact that his watch read 100 ns when Bev passed him carrying an identical watch that read 200 ns, suggested that time was behaving in some strange and unnerving way.

Before they can begin, Anne points out that comparing the rates at which two watches run is going to be difficult if not impossible. She asks her friends to look at figures 3.2a which makes it very clear that the worldlines of Chuck and her cross only once when each of their watches read zero. Since they never pass one another again, it is impossible for her to directly compare the ticking rate of her watch with the ticking rate of Chuck's. The best they could do, which is exactly what they did do, was to compare the values on their respective watches as they zipped past one another.

More generally, in any given experiment with observers in two inertial reference frames moving with respect to each other, a given observer in one frame passes observers in the other frame only once. This makes the direct comparison of watch rates impossible.

Dean agrees but suggests a different method for comparing ticking rates. He refers to figure 3.2a, when Bev passed Ed, Bev's watch read 120 ns and Ed's read zero. When Bev reached him, her watch read 200 ns and his read 100 ns. Therefore, as she moved from Ed to him, Bev's watch ticked off 200 – 120 ns = 80 ns. But he and Ed, by comparing their watches, conclude that it took her 100 ns. Anne exclaims, "The 80% factor returns but now shows up in time measurements. Bev's watch ticked off only 80% as many nanoseconds as she moved from Ed to Dean."

Chuck joins the discussion and points out that in Dean's scenario, Bev in the Earth frame carried a watch from one bus observer, Ed, to another, Dean. They can also use figures 3.2a and 3.2b to compare the time ticked off his watch as he moved between two Earth observers, A & B. If the Principle of Relativity holds, A & B ought to see his moving watch run slow just like Ed and Dean's conclusion about Bev's watch.

Unfortunately, figure 3.2a does not quite have enough of a time slice to include the point where Bev passed Chuck, but the graph makes it obvious that she had to travel 160 feet to reach him. On the other hand, figure 3.2b makes it clear that Bev would claim that Chuck traveled 200 feet as he moved from Anne to her.

Question 3.12: Assuming the experiment went long enough, use figure 3.2a to calculate the time on Chuck's watch when Bev passed him.

Question 3.13: What did Bev's watch read when Chuck passed?

Dean does calculations out loud. "Bev is moving with a speed of 0.6 ft/ns and has to cover 160 feet to reach Chuck. The time it took Bev to cover the 160 feet is given by the equation, Time = Distance/Speed, which in this situation is 160/0.6 = 266.6 ns. But Bev, using figure 3.2b, concludes that Chuck moved 200 feet with a speed of 0.6 ft/ns to reach her. So her watch ticked off 333.3 ns during the time it took Chuck to travel from Anne to her. Therefore A & B say that Chuck's watch ticked off 266.6 ns during the time their watches recorded a difference of 333.3 ns." And guess what folks, "80% of 333.3 is exactly 266.6!"

So in each case, the moving watch ran slow. In the first case it was Bev's watch moving through the bus frame. In the second case, it was Chuck's watch moving through the Earth frame. And in both cases, the moving watch ticked of 0.8 ns for each 1 ns ticked off the "stationary" watches.

Remember the high tech watches used in these experiments are accurate to 0.1 ns. Consequently, the times calculated by Bev are listed to tenths of nanoseconds. The smaller decimal contributions, hundredths, thousandths, etc of nanoseconds have been dropped in the calculations since they represent fractions of time too small to be measured by the watches used by our spacetime probers.

Dean has followed these calculations very carefully and agrees with everything said so far about the ticking rates of clocks. But it seems impossible for him and Ed to observe Bev's watch run slow while at the same time A & B observe Chuck's watch to run slow. Dean asks his friends, "If observers in reference frame 1 conclude that observers in reference frame 2 have clocks that run slow, doesn't that imply that the observers in frame 2 would see the watches in frame 1 run fast? Since this apparently does not happen, I must be missing something important."

Anne volunteers to explain why observers in each frame can conclude that watches in the other frame run slow. "Look at figure 3.2a. Chuck sees Anne pass when both their watches read zero. At that same time according to Chuck, Bev is passing Ed whose watch also read zero. But as Bev passed, Ed saw that her watch read 120 ns. Later, when Bev passed Chuck his watch read 266.6 ns while hers read 333.3 ns. Since Ed's watch read zero when Bev passed, Chuck concludes that it took 266.6 ns for Bev to travel from Ed to him. But he and Ed claim that during that trip Bev's watch ticked off only 333.3 – 120 ns = 213.3 ns. And wonder of wonders, 213.3 is just 80% of 266.6 ns! Ed and Chuck conclude that Bev's watch ran slow as she moved from Ed to Chuck. From these three examples, it appears safe to conclude that moving watches run slow."

A watch carried by an observer is necessarily stationary with respect to that observer and ticks off what is called "proper time." Stationary watches tick off proper time just like any two stationary observers are separated by a proper distance.

Dean is still not satisfied but Anne ignores him and makes the following sketch, figure 3.3, based on figure 3.2b:

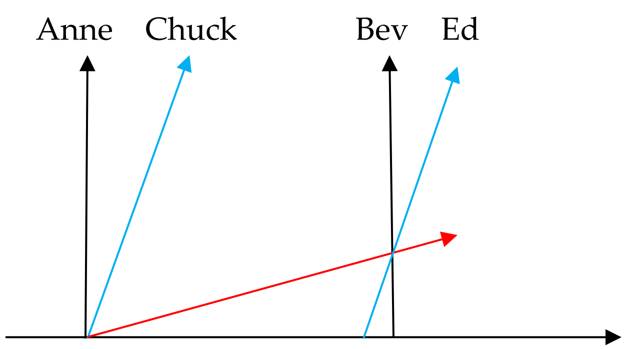


Figure 3.3

Then she quickly draws a table 3.2 summarizing the data collected during the experiment.

	Bus Observers		Earth Observers	
Event	X_{bus} (ft)	T_{bus} (ns)	X_{Earth} (ft)	T_{Earth} (ns)
A	0	0	0	0
B	160	0	200	120
C	100	100	200	200

Table 3.2

Anne reminds her colleagues that event B was the spacetime point where Ed and Bev passed one another. The bus observers claim that passing happened at $T_{bus} = 0$ while Earth observers saw that happen at $T_{Earth} = 120$ ns. For bus observers, that passing was simultaneous with the start of the experiment. On the other hand, she and Bev conclude that event B happened 120 ns after the start of the experiment.

Pointing back at figure 3.3, Anne says, "The red line connects events A and B which bus observers claim happened at $T_{bus} = 0$. Therefore the red line is actually the X_{bus}-axis where $T_{bus} = 0$. The horizontal black line is the X_{Earth}-axis where $T_{Earth} = 0$. So for Bev and me, any event that takes place simultaneously with the start of the experiment must have a spacetime point lying on the black line. C & D would dispute that because for them, events simultaneous with the start of the experiment have to have spacetime points lying on the red line. This disagreement about what constitutes simultaneous events is a consequence of the fact that the watches that were carefully synchronized by Bev and I are not synchronized according to C & D. And of course, Bev and I make the same claim about the watches that were carefully synchronized by C & D. This disagreement allows each of us to conclude that moving watches run slow."

Dean nods slowly, and thanks Anne and says, "If I have this right, our experiment with the light flash in the bus forces us to the following conclusions about space and time:

1. Moving clocks run slow.
2. Moving rulers shrink.
3. And watches that were properly synchronized in one inertial reference frame appear to be unsynchronized when viewed from another reference frame.

Have I missed anything?"

Her friends smile and give him pat on the back. That cogent summary is a good place to stop for the day, suggests Bev. Agreement is unanimous and they decide to celebrate their improved understanding of space and time with a dinner out.

Light Moves through Space and Time in a Very Strange Way

Before leaving this chapter, it might be useful to look at a more extreme situation where two sets of observers moving with respect to one another measure the speed of a light flash.

Imagine two super buses each moving at 0.9 ft/ns. The red bus is zipping away from Anne and the blue one towards her. Anne uses her laser to send a light flash, the red dot, off to the right as shown below.

The light flash passes through the windows of the red and blue buses before exiting stage right. Observers riding in each bus, measure the speed of the light flash as it passes through their bus.

Question 3.14: What is the speed of the light flash according to observers riding on the red bus?

Question 3.15: What is the speed of the light flash according to observers riding on the blue bus?

Remember light moves through space and time in a way that "forces" observers in any inertial reference frame to measure the same speed for light, 1 ft/ns. So as bizarre as it seems, riders on the red bus, who are moving away from the light flash at 0.9 ft/ns, and riders on the blue bus, who are moving toward the flash at 0.9 ft/ns, both measure the speed of the flash to be 1 ft/ns. Those are the properties of light and spacetime that were uncovered by Anne, Bev, Chuck, and Dean, with the help of Ed, in this chapter.

In order for light to have the same speed in all inertial reference frames, rulers and watches have to behave according to Dean's succinct summary at the end of the previous section.

If this still seems unbelievable or implausible, you may want to reread this chapter to review the results of the crucial experiment that led to these conclusions before moving on to Chapter Four.

Chapter Four: Generalizing the Observations

The essential details of the experiment done with the light flash on the super bus are captured by the spacetime graphs in figures 3.2a and 3.2b. That experiment was done with the bus moving at a specific speed, 0.6 ft/ns. In this chapter and the next, we will generalize those spacetime graphs by analyzing the results of a "gedanken" experiment done with a bus moving with a velocity v.

The experiment done in Chapter Three by our four friends demonstrated the important fact that light travels through space and time with a speed that is independent of:

1. *The velocity of the laser that emitted the flash -- nothing changed when the laser was moved from the bus to Earth.*
2. *The velocity of the observers measuring the speed – the bus and Earth observers measured the same speed for the flash*

As you work your way through this chapter, keep in mind that the constancy of the speed of light is central to the derivation of the rules governing space and time. One last admonition, there is a bit of messy algebra in this chapter. Although following all the algebraic manipulations is not essential to grasping the main thrust of the arguments presented, keeping up with the algebra will increase your confidence in the conclusions reached this chapter. Needless to say, it will also improve your algebraic skills.

After enjoying a vacation from thinking about space and time, our four friends meet for dinner to talk about the best way to continue their investigation of space and time. During that meeting, Bev points out that the strange behavior of rulers and watches discovered in Chapter Three were completely absent during the experiments they did in Chapter One with the pigeon and bus that cruised at a pedestrian 30 ft/s.

Chuck is not bothered by this because he is pretty confident that the slowing of clocks and the shrinking of buses must somehow depend on the speed of the bus. Anne and Dean agree with Chuck but are not satisfied with his conclusion that the observed effects must "somehow depend on speed." Anne asks Chuck, "What is the exact relationship between the speed of the bus and the behavior of watches and rulers?"

Before a serious discussion can get started, Dean suggests they get a good night's rest and wait until tomorrow before trying to answer Anne's question. His friends agree and break up for the evening with the promise to get an early start the next day.

Defining the New Problem

The next morning, the four friends meet to discuss how to generalize their conclusions from Chapter Three. Anne and Bev are the more analytic pair, so C & D wait for them to take the lead in directing the discussion. Bev tells her friends that one of the first things she learned in her algebra class in high school was to carefully define the quantities involved in the problem.

Before Bev can continue, Dean groans, "I was never any good at algebra."

Chuck encourages Dean by reminding him that he made some important contributions to their earlier discussions about space and time. Dean takes a deep breath, and nods toward Bev. Bev takes that as a signal to start up again.

She reminds her friends that the quantities involved in the experiment were the distance between Anne and her, the length of the bus, assorted times, and the velocity of the bus. These are the quantities that we need to define in terms of symbols to replace the particular numerical values they had in Chapter Three. Anne suggests that if they are going to symbolically define the quantities listed by Bev, they also ought to assign a symbol to represent the velocity of light. "Great idea," says Bev, "Especially since the numerical value for the velocity of light depends on the units used; 186,000 miles a second or 300 million meters a second as well as the 1 foot per nanosecond we used during our experiments." Dean and Chuck have nothing to add and let A & B continue to lead the discussion.

Before Bev can continue, Anne makes another suggestion. They ought to start by analyzing the effect motion has on space measurements before worrying about the effect on time. C & D enthusiastically endorse that idea since it may make the forthcoming analysis easier for them to follow.

Bev spends a couple of minutes writing in her pad. Then she rips a page out of the pad and holds it up for everyone to see. Bev uses her pencil as a pointer and says, "L will represent the length of the bus and D the distance between her and Anne. But since Earth and bus observers get different values for L and D, I added the subscripts "Earth" or "bus" to distinguish between the values measured by Earth and bus observers."

Anne, Chuck, and Dean stare at Bev's list shown below.

1. L_{bus} = length of the bus measured by bus observers (bus is stationary in this frame)
2. L_{Earth} = length of the bus measured by Earth observers (bus is moving in this frame)
3. D_{bus} = distance between Anne and Bev measured by bus observers (Anne and Bev are moving in this frame)
4. D_{Earth} = distance between Anne and Bev measured by Earth observers (Anne and Bev are stationary in this frame)
5. v = the relative speed between the two reference frames (Bus observers and Earth observers agree on this value – Principle of Relativity)
6. c = speed of light (Anne's suggestion)

Before going on it may be useful to explain the strategy that our friends are going to use to find an expression for the shrinkage factor that depends on the quantities defined above. The logical arguments are the same as used in Chapter Three except the values for the various quantities, 100 feet for the length of the bus according the C & D, the speed of light being 1 ft/ns, and the speed of the bus, 0.6 ft/ns, are going to be replaced by general values, L_{bus}, c, and v for the three examples listed.

In this chapter, the time it takes the light flash to cross the bus according to C & D is L_{bus}/c, distance/speed. In Chapter Three that same time was 100 ft/(1 ft/ns) = 100 ns. Both expressions use the same basic relation, namely speed = distance/time, which can be re-written as time = distance/speed. Sometimes it may be useful to think about a given calculation first in terms of the specific values from Chapter Three before converting to the more general values defined by Bev in the table she presented to Anne, Chuck, and Dean that defined L_{bus}, L_{Earth}, etc.

Finding the Shrinkage Factor

Dean reminds them that one of the key insights from last chapter was that the Principle of Relativity required the amount of shrinkage seen by the bus observers to equal the amount of shrinkage seen by Earth observers. Chuck nods in agreement. He and Dean measured the distance between A & B to be only 80% of the proper separation between them while A & B measured the bus to be 80 feet long, 80% of its proper length.

Bev points to her list and says, "L_{bus} and D_{Earth} are proper lengths while L_{Earth} and D_{bus} are the results of measurements made on distances moving through the Earth and bus frames respectively. Notice that I wrote the proper lengths in red and the moving ones in blue to remind us of the difference." Bev writes equation 4.1 on her pad and continues, "The Principle of Relativity requires that each of these ratios, $\frac{\textit{Bus Length Measured by A \& B}}{\textit{Proper Length of Bus}} = \frac{\textit{Distance Between A \& B Measrued by C \& D}}{\textit{Proper Distance Between A\& B}}$ have to equal the shrinkage factor." Then she points to her pad where those ratios are written in terms of the parameters she defined earlier.

$$L_{Earth}/L_{bus} = D_{bus}/D_{Earth} = \text{shrinkage factor} \qquad (4.1)$$

Chuck makes a couple of rough spacetime sketches, figures 4.1a and 4.1b, to help him picture the relationships between the quantities appearing in equation 4.1.

Chuck explains that figure 4.1a is from the perspective of observers on the bus and that figure 4.1b is from A & B's point of view. His worldline along with those of Dean and Ed are in blue while A & B's worldlines are in black. The laser flash is shown in red as usual.

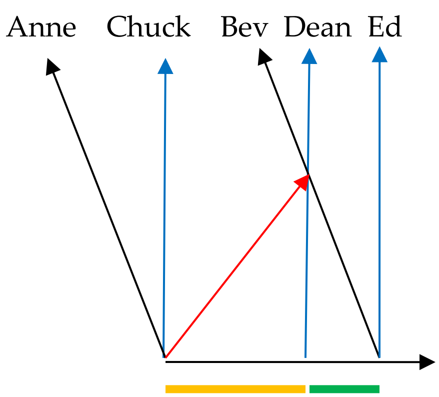


Figure 4.1a

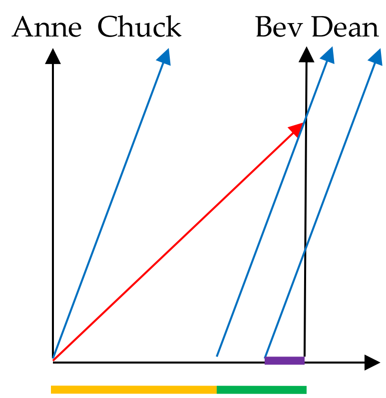


Figure 4.1b

Anne thinks they can use Chuck's sketch in figure 4.1a to write D_{bus}, the separation between her and Bev as measured by C & D, in terms of L_{bus}, the proper length of the bus. If they can do that, the same strategy applied to figure 4.1b will let them find an analogous relationship between D_{Earth}, the proper separation between her and Bev, and L_{Earth}, the length of the bus according to Bev and her. Those two expressions can then be used to eliminate D_{bus} and D_{Earth} from equation 4.1. If this works, the resulting equation would give them L_{Earth} in terms of L_{bus} and the shrinkage factor.

Dean, picking up on Anne's suggestion, immediately points to the figure 4.1a and adds two additional lines. Thinking out loud, he says, "Chuck is located at $X_{bus} = 0$ while I am at X_{bus} = 100 feet." Bev gently reminds Dean that in this *gedanken* experiment he is actually located at the more general location, $X_{bus} = L_{bus}$. Dean silently internalizes Bev's comment before continuing, "the distance Chuck and I measure between A & B, D_{bus}, is just L_{bus} plus the amount Bev moved during the time it took the laser flash to travel from Chuck to me. On figure 4.1a, that corresponds to adding the lengths of the heavy yellow and heavy green lines." Dean's remarks are greeted with nods of agreement.

Gaining confidence, he adds that since the laser flash traveled a distance L_{bus} while moving with a speed of c, it must have taken a time L_{bus}/c to traverse the length of the bus. During that time Bev was traveling with a speed of v. That makes the length of the heavy green line $v\,L_{bus}/c$. Now he combines these factors and writes the following equation in Bev's pad,

$$D_{bus} = L_{bus} + v\,L_{bus}/c = L_{bus}\,(1 + v/c) \qquad (4.2)$$

Chuck claps his hands and exclaims, Dean you just did algebra. Nice job."

This equation relates the distance measured by Chuck and Dean for the separation between Anne and Bev, D_{bus}, to the proper length of the bus, L_{bus}.

Anne looks at Chuck's right-hand sketch, figure 4.1b, and following Dean's strategy adds two new lines. Then while pointing to figure 4.1b, says, "D_{Earth} equals the sum of L_{Earth} (the heavy yellow line) and the distance Dean moved while the flash traveled from me at $X_{Earth} = 0$ to Bev at $X_{Earth} = D_{Earth}$ (the heavy green line). Bev and I saw the flash travel for a time D_{Earth}/c. During that time, Dean moved $v\,D_{Earth}/c$." Now she adds her equation to Bev's pad,

$$D_{Earth} = L_{Earth} + v\,D_{Earth}/c.$$

Bev quickly solves this equation for D_{Earth}, to get an equation analogous to 4.2 except this one relates D_{Earth} to L_{Earth},

$$D_{earth} = {L_{earth}}\Big/\left(1 - \frac{v}{c}\right) . \qquad (4.3)$$

Bev quickly substitutes the values found for D_{bus} and D_{Earth}, equations 4.2 and 4.3, back into equation 4.1 to get,

$$\frac{L_{earth}}{L_{bus}} = \frac{D_{bus}}{D_{earth}} = \frac{L_{bus}\left(1+\frac{v}{c}\right)}{\frac{L_{earth}}{\left(1-\frac{v}{c}\right)}} = \frac{L_{bus}}{L_{earth}}\left(1-\left(\frac{v}{c}\right)^2\right). \tag{4.4}$$

While her friends are straining to keep with her, she solves for the ratio $\frac{L_{Earth}}{L_{bus}}$,

$$\left(\frac{L_{earth}}{L_{bus}}\right)^2 = 1 - \left(\frac{v}{c}\right)^2,$$

And takes the square root of both sides to get,

$$L_{Earth}/L_{bus} = \sqrt{1-\frac{v^2}{c^2}} = \text{shrinkage factor.} \tag{4.5}$$

The expression, $\frac{L_{bus}(1+\frac{v}{c})}{\frac{L_{earth}}{(1-\frac{v}{c})}}$*, in equation 4.4 can be rewritten as* $\frac{L_{bus}}{L_{earth}}\frac{(1+\frac{v}{c})}{\frac{1}{(1-\frac{v}{c})}}$*. To simplify this further, multiply by* $\frac{(1-\frac{v}{c})}{(1-\frac{v}{c})}$ *which equals one so it does not change the value of the expression. Remember this same trick was used to change the units for the speed of light in Chapter Two. Here that trick simplifies the above expression to* $\frac{L_{bus}}{L_{earth}}\left(1+\frac{v}{c}\right)(1-\frac{v}{c})$*. Next, remember that when 1 – something is multiplied by 1 + something the result is just 1 – (something)*^{2}*. In this example the something is just* $\frac{v}{c}$*.*

While Anne, Chuck, and Dean stare at equation 4.5 with a growing sense of comprehension, Bev calculates the shrinkage factor for a super bus moving at $\frac{3}{5}$ the speed of light. She claps her hands and writes the results on the pad, $L_{Earth}/L_{bus} = 0.8$, in exact agreement with their experiment!

Then Anne points out that the since the shrinkage factor depends on v^2 it has the same value for bus observers who saw Anne and Bev move with velocity $v = -\frac{3}{5}$ ft/ns just as required by the Principle of Relativity.

Dean says, "Wow, aren't we the clever spaceketeers!" Three speechless people nod in agreement.

In Chapter Three v was $\frac{3}{5} = 0.6$ *ft/ns and c was 1 ft/ns so v/c = 0.6 =* $\frac{3}{5}$*. Also notice that the constancy of the speed of light snuck into this calculation when bus and Earth observers used the* <u>*same*</u> *value for the speed of light, c, for calculating how long it took a light flash to travel a given distance. Before they actually did the experiment with the super bus, A & B "predicted" that* *they would measure a different speed for light than their friends riding on the bus. Their misunderstanding of the behavior of light was the reason Bev was standing in the "wrong" place the first time they did the experiment. In fact, as Dean pointed out after the experiment was done correctly, their prior experience with the pigeon led them to believe that the speed of light with respect to Earth ought to have been 1.6 ft/ns instead of the measured 1 ft/ns!*

The Ticking Rate of Moving Clocks

The various lengths L and D will no longer be color coded but the subscripting will continue. Remember the subscript refers to the observers making the measurement. For example, D_{bus}, is the separation between Anne and Bev who are stationary in the Earth frame as measured by Chuck and Dean on the bus.

Full of confidence, the four friends are ready to tackle the impact of motion on time. Chuck points back to his two sketches, figures 4.1a and 4.1b. We can use the spacetime graph 4.1a to find the time ticked off my watch, T_{bus}, when I saw Bev pass me remembering that my watch read zero when Anne passed. Since Bev had to travel D_{bus}, that time is just,

$$T_{bus} = \frac{D_{bus}}{v} . \qquad (4.6)$$

But it is clear from diagram 4.1b that Anne and Bev conclude that I traveled a distance D_{Earth} to go from Anne to Bev. The time it took for me to travel from Anne to Bev according to their watches is just,

$$T_{earth} = \frac{D_{earth}}{v} . \qquad (4.7)$$

Bev jumps in to remind everyone that since it was Chuck's watch that was compared to two "stationary" watches, his watch ticked off less time that the time recorded by Anne and her. Consequently, she triumphantly writes the following equations on her pad,

$$\frac{T_{bus}}{T_{earth}} = shrinkage\ factor = \frac{D_{bus}/_{v}}{D_{earth}/_{v}} = \frac{D_{bus}}{D_{earth}} = \sqrt{1 - (\frac{v}{c})^2} . \qquad (4.8)$$

No one speaks for several seconds because Bev's demonstration that time is affected by the same algebraic expression that explained the shrinkage of distances was so straightforward and elegant. Finally Dean breaks the silence and says, "bravo!" Then he highlights Bev's conclusion by writing,

$T_{bus}/T_{Earth} = \sqrt{1 - (\frac{v}{c})^2}$, (4.9)

on Bev's pad. Chuck looks back at his notes from the experiment done in Chapter 3 to find the numerical values for T_{bus} and T_{Earth}. When he finds those values, T_{bus} = 266.6 ns and T_{Earth} = 333.3 ns, he quickly checks equation 4.9 for the bus speed of v/c = 0.6. He claps his hands and announces to his friends that the agreement is perfect.

Quantifying the Disagreement over Synchronization

Bev reminds her friends that even though the bus observers, Chuck, Dean and Ed, and the Earth observers, Anne and her, used exactly the same procedures to synchronize watches, the bus and Earth watches ended up out of synch. The lack of synchronization was vividly displayed by Anne in figure 3.3. Before they can claim to have generalized the results of the experiment with the super bus, they need to come to grips with the "synchronization problem."

Chuck directs his friends' attention back to figure 4.1b. Ed is to the left of Bev at the start of the experiment according to A & B. When he reaches Bev, his watch reads zero but Bev's watch reads a time equal to the distance Ed traveled to reach her, the heavy purple line, divided by his speed. He reminds his friends that in the experiment with the super bus going 0.6 ft/ns, Bev's watch read 120 ns when Ed passed. Now he looks at Anne, and asks if she can do some clever algebra to find an expression for the distance between Ed and Bev at the start of the experiment according to Bev and her.

Anne takes up Chuck's challenge and directs the attention of her friends to figure 4.1a, "The distance between Chuck and Ed in their reference frame, the sum of the heavy yellow and green lines, is just D_{bus}. Since Chuck and Ed are fixed in place in the bus frame, D_{bus} is the proper separation between them. On the other hand, Bev and I measure the separation between Chuck and Ed to be shorter because of the shrinkage factor, so we get for their separation D_{bus} times $\sqrt{1 - \frac{v^2}{c^2}}$."

She gives her friends a second to catch up before continuing, "But D_{bus} is also the bus observers' measurement of the proper distance between Bev and me, which for us is just D_{Earth}. The bus observers measure that distance shrunk by the same factor, so C & D conclude that $D_{bus} = \sqrt{1 - \frac{v^2}{c^2}}\, D_{Earth}$. When D_{bus} in the previous expression is replaced by

$\sqrt{1-\frac{v^2}{c^2}}$ D_{Earth}, Bev and I conclude that the distance between Chuck and Ed is just $(1-\frac{v^2}{c^2})D_{Earth}$. By the way, on figure 4.1b this distance is just the sum of the heavy yellow and green lines <u>minus the heavy purple line</u>."

Dean looks confused and gives Anne the timeout sign. She stops and looks at her friends. Finally, Dean nods and tells Anne to continue but to please speak more slowly. Anne takes a deep breath and begins, "From Bev's perspective, pointing to figure 4.1b, Ed was a distance to her left given by the distance between me and her minus the distance between me and Ed, the heavy purple line of figure 4.1b. Algebraically that distance is just D_{Earth} - $(1-\frac{v^2}{c^2})$ D_{Earth} which simplifies to $\frac{v^2}{c^2}$ D_{Earth}. That is how far Ed had to travel to reach Bev."

As his three friends peer over his shoulder, Chuck writes that distance down and divides it by v, Ed's speed according to A & B, to get the time it took Ed to travel the distance that separated him from Bev according to Anne and Bev. His answer is $\frac{v}{c^2}$ D_{Earth}, the time that Bev's watch should have read when Ed reached her. Chuck points to the pad and says, "Since Ed's watch read zero when he reached Bev, Bev concludes that Ed's watch is out of synch with hers by $\frac{v}{c^2}$ D_{Earth}."

Dean grabs a pencil and quickly calculates the time that Bev's watch ought to have read when they did the experiment with the 100 foot super bus moving at 0.6 ft/ns. He mumbles, v = 0.6 ft/ns and D_{Earth} was 200 feet, so Bev's watch ought to have read 0.6 ft/ns times 200 divided by $(1 \text{ ft/ns})^2$ which equals 120 ns. He blurts out, 'The equation works perfectly, $T = vD_{Earth}/c^2$ predicts the time on Bev's watch to a tee."

Bev adds that equation to her list of results.

$$T = \frac{vD_{earth}}{c^2} \qquad (4.10)$$

Since c = 1 ft/ns, the numerical value found for the time on Bev's watch does not depend on dividing by c^2 in the equation $T = vD_{Earth}/c^2$. On the other hand, the units of c, ft/ns, are needed to make the answer end up being in the correct units, namely nanoseconds. Although c was explicitly included in the algebraic calculation, it can be ignored in numerical calculations when using units that make c = 1.

Chuck asks his friends to look at figure 4.2, a reproduction of figure 3.3. "On that spacetime graph, my worldline is the both T_{bus}-axis and the $X_{bus} = 0$ line. A & B see me moving with a velocity v with respect to them and since I passed Anne when $T_{Earth} = 0$, my worldline on their spacetime diagram is represented by the equation, $X_{Earth} = v\ T_{Earth}$, or in the more standard form of $t = mx + b$, see Appendix A, as $T_{Earth} = \frac{1}{v} X_{Earth}$. So the T_{bus}-axis is represented by the line $T_{Earth} = \frac{1}{v} X_{Earth}$ on the Earth-based spacetime diagram."

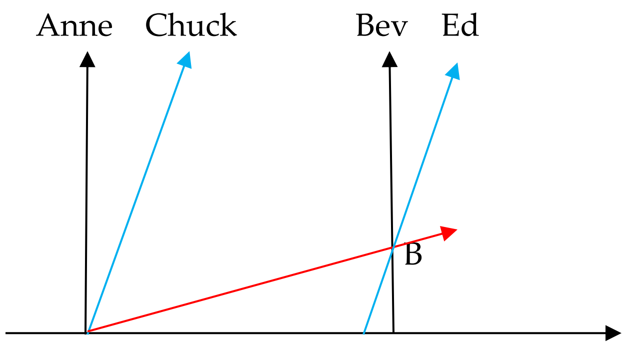


Figure 4.2

Now Chuck points to the red line on figure 4.2. "It is both the $T_{bus} = 0$ line and the X_{bus}-axis and passes through the origin and point B where Ed passed Bev. The coordinates of point B are $X_{Earth} = D_{Earth}$ and $T_{earth} = \frac{v}{c^2} D_{earth}$. I can use those two points to find the values of m and b in the general equation for a straight line, $T_{Earth} = m\ X_{Earth} + b$."

Question 4.1: Follow Chuck's directions and use the fact that the X_{bus}-axis passes through the origin, $X_{Earth} = T_{Earth} = 0$, and point B where $T_{Earth} = vD_{Earth}/c^2$ and $X_{Earth} = D_{Earth}$ to find m and b for the equation of the X_{bus}-axis, $T_{Earth} = mX_{Earth} + b$, on the X_{Earth} vs T_{Earth} graph.

Chuck takes a minute or two to finish his calculation. Then he adds the following two equations to Bev's list.

$$T_{earth} = \frac{1}{v} X_{earth} \quad (T_{bus}\text{-axis}) \tag{4.11a}$$

$$T_{earth} = \frac{v}{c^2} X_{earth} \quad (X_{bus}\text{-axis}) \tag{4.11b}$$

The four friends feel pretty pleased with themselves. Dean says that space and time are way stranger than he first thought. But he is gaining confidence that their analysis is correct though disconcerting.

Chuck, looking over the list of equations in Bev's pad, suggests they test their algebraic equations by redoing the experiment with the bus and laser, except this time he and Dean will drive the super bus at v = 0.8 ft/ns or $\frac{4}{5}$ the speed of light. "Great idea," says Anne, "Let's meet again tomorrow with a plan for the new experiment."

Questions 4.2a to 4.2d: The goal of these questions is find how far to the right of Anne Bev needs be standing to be adjacent to Dean when the light flash arrives when he is riding in a super bus traveling at 0.8 ft/ns?

a) What is the value of shrinkage factor, $\sqrt{1-(\frac{v}{c})^2}$, for a bus moving at $\frac{4}{5}$ the speed of light?
b) Using the value find in part a), how long is the 100 foot bus according to A & B?
c) The flash starts its trip when Anne and Chuck are next to one another. According to A & B, Dean is how many feet to the right of Anne, at that instant?

At the instant described in part c), a light flash leaves the back of the bus traveling at 1 ft/ns towards Dean at the front of the bus. Meanwhile Bev is waiting patiently down the road. Dean is moving toward Bev at 0.8 ft/ns, the velocity of the bus. The place where the worldlines of the flash and Dean intersect marks the spacetime point where Bev needs to be in order to be adjacent to Dean when the flash arrives.

d) The above information can be used to draw a careful Earth-based spacetime diagram that will show where Bev needs to be standing. Alternatively, the information can be used to write down the equations for the worldlines of the flash and Dean in terms of X_{Earth} and T_{Earth}. Solving those equations gives the coordinates, X_{Earth} and T_{Earth}, where the two worldlines meet. Use one of these methods, or any other method, to find the place along the road that Bev needs to be stationed at the start of the experiment.

Testing the Theory

Our four experimenters did an experiment, the laser flash in the bus moving at 0.6 ft/ns, which was carefully observed by C & D on the bus frame and A & B on Earth. The results conflicted with their view of space and time. Namely, A & B expected to measure the speed of light to be 1.6 ft/ns. Remember that was the underlying assumption used when it was decided that Bev needed to be 160 feet to the right of Anne.

When they enlisted Ed and redid the experiment with Bev stationed 200 feet from Anne, they discovered that the speed of light was the same for bus and Earth-frame observers. In this chapter they used the constancy of the speed of light to re-analyze the experiment. They found that lengths shrunk and watches ran slow by the same factor, $\sqrt{1-(\frac{v}{c})^2}$ *, and that clocks got unsynchronized by an amount* vD_{Earth}/c^2*. These results agreed perfectly with the numerical data collected during their experiment with the bus moving at 0.6 ft/ns.*

Having a theory that agrees with pre-existing data is very encouraging, but the real test of a theory is to make predictions about the outcome of an experiment before the experiment is done. That is the reason our four scientifically literate friends are so anxious to redo the experiment with the bus going at v = 0.8 ft/ns.

Bright and early the next morning, when Anne and Bev arrive at the super bus' garage, they find Chuck and Dean have already washed the super bus and have gotten everything ready for the experiment. A & B ask C & D if they have calculated the spot along the road where Bev needs to be standing. Chuck replies, "Of course, we drew a very nice spacetime graph to find the place Bev needs to stand." Anne replies, "Bev and I did a little algebra to find Bev's location at the start of the experiment."

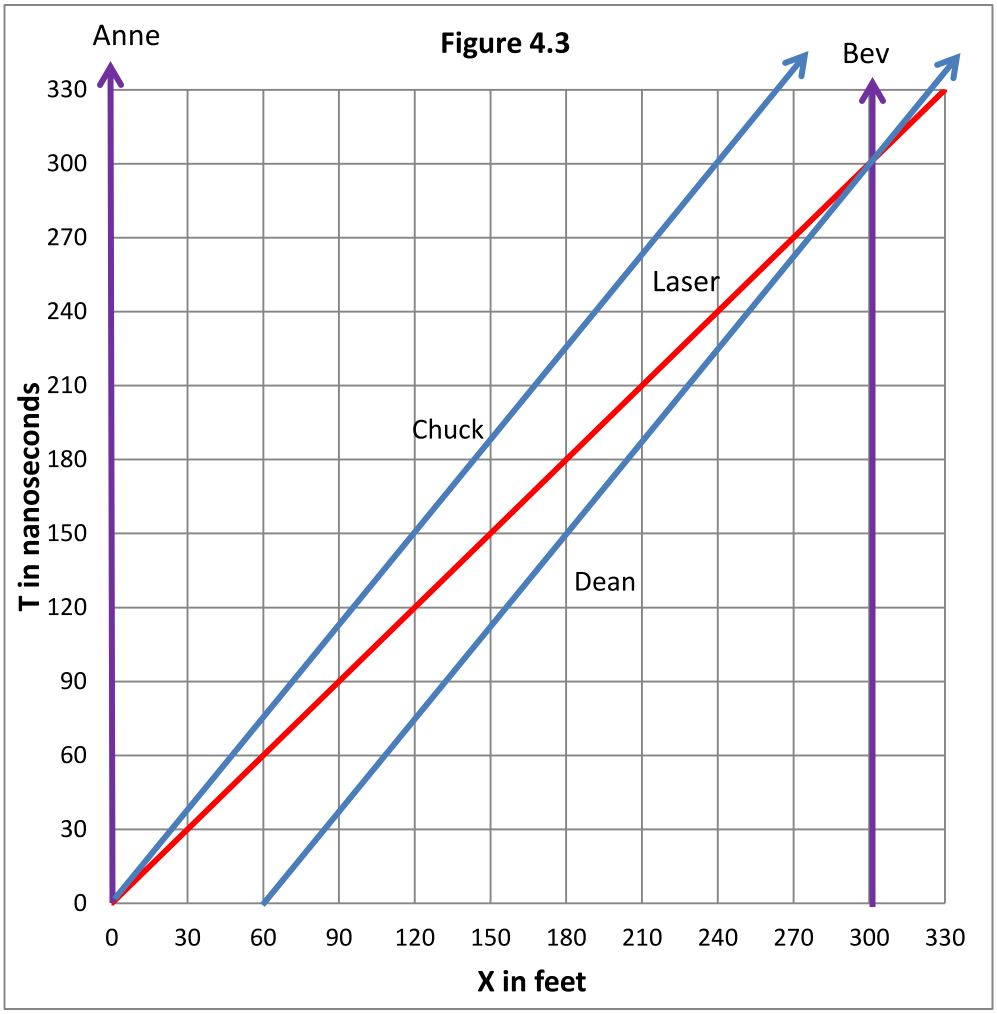
With that, Anne writes down the equations for the worldlines of the flash and Dean,

$$X_{flash,Earth} = T_{Earth} \tag{4.12a}$$

$$X_{Dean,Earth} = 60 + \frac{4}{5}T_{Earth}. \tag{4.12b}$$

She used the values c = 1 ft/ns and v = $\frac{4}{5}$ ft/ns in those equations. Also, Anne reminds C & D, that according to her and Bev, the bus is only 60 feet long. When the flash reached Dean, $X_{flash,Earth} = X_{Dean,Earth}$. Then they solved that equation for T_{Earth} to discover that Bev needed to be standing at X_{Earth} = 300 feet.

Dean smiles and points to figure 4.3, "We got the same answer by finding the place my worldline intersected the worldline of the light flash."



Since A & B and C & D agree on where Bev should stand, they are ready to do the experiment with the bus going 0.8 ft/ns. C & D head off in the bus. A & B synchronize their watches and Bev marches 300 feet down the road. On the bus, C & D also synchronize their watches.

Anne alerts Bev that the bus is coming. The bus zips by, Anne sees Chuck press the button on the laser to start the flash on its way, and an instant later she sees Bev jumping up and down with excitement. Bev yells to Anne, "the experiment confirmed our theory and calculations." Of course C & D were just as excited since Dean saw Bev out his window just as the flash arrived at the front of the bus.

When the bus got back to the garage, A & B have champagne ready for a little celebration. But to their surprise, Dean left the bus with a little bit of a frown.

A Question about Synchronization

Anne and Bev ask Dean, what's up? Dean replies, "On the way back to the garage, I began to think about how we synchronized our watches. On the bus, we meet at the back and simultaneously start our watches. Then I walk to the front of the bus. After collecting data, I walk back to Chuck to share the data collected." A & B nod in agreement and Bev says, "We do basically the same thing. What's the problem?"

Dean explains, "According to our theory, my watch runs slow compared to Chuck's while I am walking to the front of the bus and then back to Chuck after the experiment is over. That ought to cause our watches to become unsynchronized. But when I compared my watch with Chuck's after the experiment, our watches appeared to be perfectly synchronized. I don't understand how that is possible!"

Anne gives a sigh of relief, smiles, and says, "No problem. The answer has to do with the immense difference between your speed walking in the bus and the speed of light. Even if you walked at a quick 10 ft/s, compared to light, your speed would be miniscule."

Question 4.3: If Dean walked for 1 mile at 10 ft/s, how long would it take him? Find the answer in terms of minutes and seconds, x minutes and y seconds.

Anne works through the following calculation for Dean while her friends watch. Light travels a billion, 10^9, feet in a second, so for v = 10 ft/s, the ratio $\frac{v}{c}$ is ten over a billion, or in scientific notation, 10^{-8}. It takes Dean 10 seconds to walk from the back to the front of the 100 ft long bus. During those 10 seconds, his watch ran slow by the factor $\sqrt{1-(\frac{v}{c})^2}$ compared to Chuck's stationary watch. But $(\frac{v}{c})^2$ is $(10^{-8})^2 = (10^{-8})(10^{-8}) = 10^{-16}$, an unimaginably small number. So Dean's watch ticked off $10\sqrt{1-10^{-16}}$ seconds during his walk across the bus. Most hand held calculators will evaluate $1 - 10^{-16}$ to be exactly 1!

Question 4.4: Try to evaluate $\sqrt{1-10^{-16}}$ on a calculator. If you got an answer different than 1, make sure you did the evaluation correctly! If you are convinced your calculator gave you a correct answer, compare it with the value found below by using the approximation $\sqrt{1-x} \cong 1-\frac{x}{2}$ which works better and better the smaller the value of x. The agreement ought to be perfect.

A simple way of proving the useful equation $\sqrt{1-x} \cong 1-\frac{x}{2}$ *is to start with the expression* $\sqrt{(1-\frac{x}{2})^2}$ *which can be written in the following two ways:*

$$\sqrt{(1-\frac{x}{2})^2} = (1-\frac{x}{2})$$

$\sqrt{(1 - \frac{x}{2})^2} = \sqrt{1 - x + \frac{x^2}{4}}$

Now if x is small compared to 1 so that x^2 *is "much smaller" than x, the* x^2 *term in the radical can be ignored giving* $\sqrt{1 - x + \frac{x^2}{4}} \cong \sqrt{1-x}$ *. Combining the top equation with the approximation for the lower equation gives the desired relationship,*

$$\sqrt{1-x} \cong 1 - \frac{x}{2}. \quad (4.13)$$

In the numerical example worked out by Anne below x is 10^{-16} *making* $x^2 = 10^{-32}$ *which is unarguably much, much smaller than x! For that numerical example, this is an extremely accurate approximation.*

Anne points out that the difference, ($10 - 10\sqrt{1 - 10^{-16}}$) seconds, is the amount of time Dean's watch fell behind Chuck's while walking to the front of the bus. If we use equation 4.13 to find that difference, we get $\frac{10^{-15}}{2}$ seconds. After the experiment is over, Dean walked back to Chuck and his watch loses another $\frac{10^{-15}}{2}$ seconds. So at the end of the experiment, Dean's watch was 10^{-15} seconds out of synch with Chuck's.

Triumphantly, Anne reminds Dean that their watches, though wonderfully accurate, are only accurate to tenths of a nanosecond, 10^{-10} seconds. A time difference of 10^{-15} seconds is 100,000 times too small to be measurable by their watches. Consequently, to all intents and purposes, the watches on the bus stayed in perfect synchronization. And of course, the same can be said about the watches she and Bev used during the experiment.

Dean feels much, much better after this explanation and yells to no one in particular, "What's holding up the champagne?"

Global Positioning System (GPS)

GPS technology is now found in all sorts of useful devices. As it turns out, these devices are testing the correctness of both Special and General Relativity 24 hours a day 7 days a week. Satellites orbiting with very accurate clocks continually announce their position and time to GPS receivers on Earth. The devices on Earth figure out how far they are from three or four of these satellites. They do this by using their own internal clocks to calculate the time it took for the messages to arrive from each of the satellites. Knowing the time and the fact that radio waves *travel at 1 ft/ns, the GPS on the ground figures out how far it is from each satellite. Knowing the distance from three satellites fixes the location of the GPS on the ground. The system works because the time for the radio messages to travel from the satellite to the GPS can be determined very accurately.*

The satellites are approximately 16,000 miles from Earth's center and orbit at 8600 mph while the clocks on the ground, because of the Earth's rotation, move at about 1000 mph. Because moving clocks run slow, the clock in the satellite runs slower than an identical clock on the ground. At the end of Chapter Ten, we will learn that clocks further from the center of Earth run faster than identical clocks closer to the center of Earth. This is an effect of General Relativity, Einstein's theory of gravity. When calculated in detail, it turns out that the effects of General Relativity trump those of Special Relativity for orbiting clocks: the net result is that orbiting clocks gain about 40,000 nanoseconds a day compared to Earth-based clocks. At the speed of light, that corresponds to 40,000 feet or about 8 miles of GPS error if the orbiting clocks were not designed to correct for the effects of both Special and General Relativity. The fact that a good ground-based GPS can fix locations on Earth with an accuracy of a foot or less is daily confirmation that moving clocks run slow just as predicted by Special Relativity and clocks further from the center of Earth run fast in accordance with General Relativity.

Chapter Five: The Relationship between Bus and Earth Observers

Combining Spacetime Graphs

Over the next several days, Chuck keeps thinking about equations 4.11a and 4.11b, $T_{Earth} = \frac{1}{v} X_{Earth}$ (T_{bus}-axis) and $T_{Earth} = \frac{v}{c^2} X_{Earth}$ (X_{bus}-axis). Back in Chapter One, it was fairly straightforward to represent bus and Earth coordinates on the same spacetime graph. He recognizes that the primary reason that it was simple back then was because Earth and bus times were the same, $T_{Earth} = T_{bus} = T$. So not only was one second of bus time equal to one second of Earth time, but two events that happened at the same bus time also happened at the same Earth time. Consequently watches synchronized on the bus looked synchronized to Earth observers and *vice versa*. The constant time lines for bus and Earth observers were just horizontal lines crossing the vertical constant X_{Earth} lines and the slanted X_{bus} lines. Furthermore, there was no disagreement among bus and Earth observers about the length of the bus or anything else. All of that has changed. Their careful experiments with the super bus showed that his previous understanding of space and time were wrong. But Chuck is determined to find a new visualization of space and time that works for observers moving at relativistic speeds with respect to one another.

*This may be a good time to review the discussion in section **The Galilean Transformation Equations** and figure 1.5, the spacetime graph Chuck is trying to "fix" to make it consistent with the real space and time described in Chapters Three and Four.*

After making several sketches with axes labeled X_{Earth} , X_{bus}, T_{Earth}, and T_{bus}, he throws up his hands and yells over to Dean, "The endless writing of subscripts is driving me crazy!" Dean wanders over, looks at the scattered pile of graphs, and suggests that he stop using subscripts. "Why not use lower and uppercase letters. For example let X_{Earth} = x, T_{Earth} = t, X_{bus} = X, and T_{bus} = T."

Chuck jumps at Dean's solution and enthusiastically gets back to work on finding a modification of the old spacetime graphs that will work regardless of the relative speed of the observers. After a while, he calls Dean over to act as a sounding board. As Dean watches, Chuck sketches and talks. "Notice it is easy to draw a spacetime diagram that represents the spacetime coordinates of any event with respect to EITHER the Earth or bus observers." He shows Dean the following two sketches:

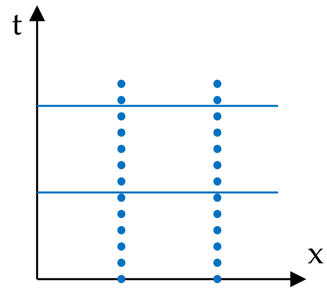


Figure 5.1a -- Earth-based x vs t

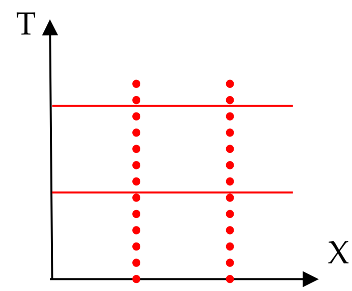


Figure 5.1b -- Bus-based X vs T

Dean looks at the two sketches and notices that Chuck has adopted his notational device to distinguish Earth and bus coordinates. But other than that he cannot see any difference between the two graphs except one has blue lines and the other red. But he decides to encourage Chuck and tells him the graphs are very, uh colorful. Chuck agrees and explains that the horizontal lines correspond to constant times, blue for Earth times and red for bus times. The vertical dotted lines correspond to constant x (blue) and X (red) values respectively.

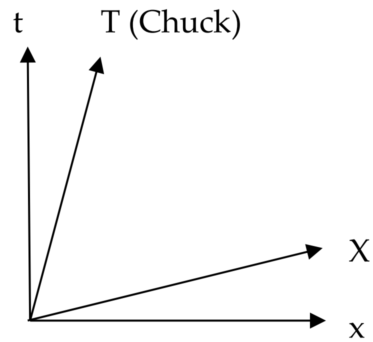
Chuck tells Dean, "But remember that if Earth observers conclude that two events are simultaneous for them, the bus observers will disagree. Also the Earth observers see bus clocks running slow and measure proper bus-frame lengths to be shorter and of course the bus observers come to exactly the same conclusions with respect to Earth times and distances. So I am having a hard time picturing lines of constant bus time and constant bus position on the Earth-based spacetime graph."

Dean points out that the spacetime graphs in figures 5.1a and 5.1b are indistinguishable as long as each set of observers keeps to their own coordinates. Earth observers plot only x and t values on their graph while bus people plot only X and T. This is another manifestation of the Principle of Relativity – the Earth and bus frames are indistinguishable so their spacetime diagrams also have to be indistinguishable.

Chuck nods in agreement, but is still fixated on drawing a single spacetime graph that can be used to find the coordinates of an event for either set of observers. He tells Dean, "I can use equations 4.11a and 4.11b to draw the X and T axes on the x vs t spacetime graph." Chuck shows Dean the sketch in figure 5.2 with both set of axes on the same graph.

This diagram is a little too abstract for Dean, so he asks Chuck to be more specific and to draw the diagram for the experiment they did with the super bus traveling at 0.6 ft/ns.

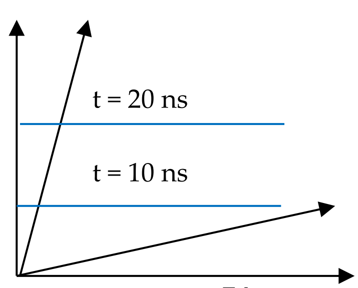
Figure 5.2



Chuck replies, "No problem," and inserts $v = \frac{3}{5}$ into equations 4.11a and 4.11b. "The T-axis is represented by the equation $t = \frac{5}{3}x$ and the X-axis by $t = \frac{3}{5}x$. But since my watches and rulers measure different elapsed times and distances than Earth observers, I am not sure how to fix the scales of the various axes."

Dean closes his eyes to help him picture Chuck moving through the spacetime of Earth observers. Eyes still closed, he says, "When you pass the origin, your watch agrees with the watch of the Earth observer at their origin. For concreteness, let's have Anne at the Earth origin. At that instant, when you are adjacent to Anne, both watches read zero. But as you travel through spacetime, you pass different Earth observers who notice that your watch is running slow compared to theirs. For example, since you are moving at 0.6 ft/ns, after 10 ns of Earth time, you have moved 6 feet of Earth distance putting you at the Earth spacetime point x = 6 feet and t = 10 ns. When the Earth observer at that spot looks at your passing watch, she sees it reading only 8 ns, 80% of the time on her watch. So the bus coordinates for that same spacetime point are X = 0 and T = 8 ns." Dean opens his eyes in time to see Chuck staring at him with something akin to awe. Chuck quickly adds the t = 10 and t = 20 ns lines to the spacetime graph (horizontal blue lines):

Figure 5.3



Chuck looks at Dean and pointing to figure 5.3 says, "So my watch reads 8 ns when I cross the t = 10 ns line and 16 ns when I cross the t = 20 ns line. For me, the "longer slanted distance" between those two lines corresponds to 8 ns but for Earth observers that "shorter distance" corresponds to a larger time interval, namely10 ns. Therefore we can use those known time intervals to calibrate the t and T axes." Chuck looks at Dean, and asks, "What about the X-axis?"

Dean closes his eyes again and imagines an Earth observer standing at the 10 foot road marker with bus observers moving by. Eyes still closed, and speaking in a trancelike voice, Dean says, "Two passing bus observers, making a simultaneous measurement of the distance between x = 0 and x = 10 feet, would measure that proper separation to be only 8 feet. Just as you pass the origin, another bus-frame observer is simultaneously in the bus frame passing the x = 10 foot marker. That bus observer has bus coordinates X = 8 feet and T = 0, a point that lies on the X-axis represented by the equation $t = \frac{3}{5}x$. But we know that x = 10 feet at that point so $t = \frac{3}{5} 10 = 6$ ns. Therefore, the spacetime point x = 10 feet and t = 6 ns corresponds to X = 8 feet and T = 0." Before Dean can open his eyes, he hears Chuck feverishly adding new lines to his spacetime graph.

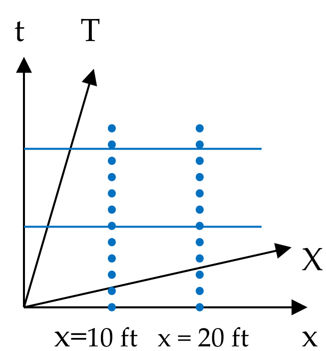


Figure 5.4

Dean opens his eyes just in time to see Chuck pointing to the blue dotted lines on figure 5.4. "As per your clever explanation," Chucks says to Dean, "we know that the constant x = 10 and 20 feet lines cross the X-axis at 8 and 16 feet respectively. So we can use those lines to calibrate the x and X-axes exactly like we calibrated the t and T-axes."

Chuck starts adding new lines to the sketch in figure 5.4 and talks simultaneously. "All the bus observers are moving at the same speed. So the bus observers at X = 8 and 16 feet have worldlines parallel to mine at X = 0 *(Remember Chuck's worldline is also the T-*

axis.). And of course those observers have watches that tick at the same rate as mine, so the distance between the T = 0 and 8 ns lines is the same for them as for me. That means the constant T = 8 and 16 ns lines are parallel to the X-axis." With that he jumps up and startles Dean by hugging him as he waves figure 5.5 in front of his face. Chuck says a little too loudly for Dean, "I have to call Anne and Bev and have them come over as soon as possible to see figure 5.5." Dean is happy to be released as Chuck scoots over to the phone.

By the time A & B arrive, Chuck has a larger version of figure 5.5 proudly sitting on an easel.

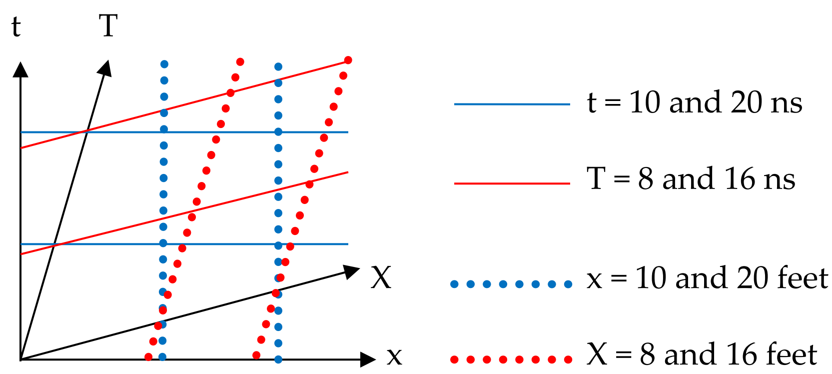


Figure 5.5

Chuck quickly brings A & B up to speed by describing how he and Dean figured out how to draw the constant X and T lines on the x vs t spacetime diagram. A & B ask some questions but after a short time come to fully appreciate the cleverness and utility of this combined spacetime graph.

The distance from the origin to the dotted or solid blue lines corresponds to 10 ft or 10 ns along the x and t-axes respectively. Meanwhile the place where those lines cross the X and T-axes corresponds to 8 ft and 8 ns in the bus frame. Notice that 8 ft or 8 ns along the X or T-axes are represented by a larger physical distance on the graph than the 10 ft or 10 ns along the x or t-axes. That difference in scale is required to make spacetime graphs like figure 5.5 consistent with the spacetime of Special Relativity, which is of course the spacetime we actually live in! In particular, since both Earth and bus observers measure the speed of light to be 1 ft/ns, a light flash passing through the origins of those coordinates has to pass through Earth points 10 ft and 10 ns, 20 ft and 20 ns, etc. Bus observers will also see that flash pass through equivalent points like 8 ft and 8 ns, 16 ft and 16 ns, etc.

Question 5.1: Using graph paper, draw a spacetime graph for a super bus traveling at v = 0.8 = $\frac{4}{5}$ ft/ns with respect to Earth that is analogous to figure 5.5. Include constant x = 10, 20, and 30 foot and t = 10, 20, and 30 ns lines along with the appropriate constant X and T lines. (What does Chuck's watch read when he crosses the t = 10 ns line? The t = 20 ns line? These two answers ought to help you decide what the "appropriate X and T lines are.)

Anne walks over to the easel and flips to a blank page. She picks up Chuck's colored markers and draws the spacetime graph in figure 5.6.

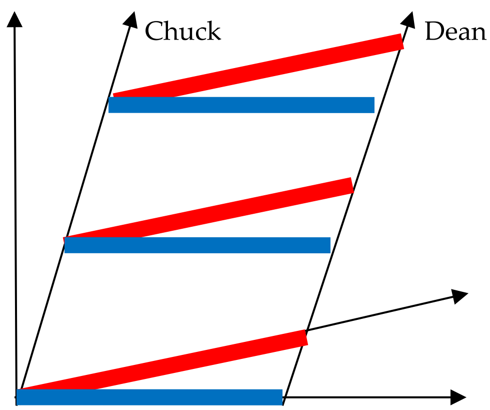


Figure 5.6

She looks at C & D, and says, "Your diagram highlights the fact that Earth and bus observers disagree about simultaneous events. This disagreement caused much of the confusion we had in understanding the results of our experiment with the super bus. When we compared data taken at the start of the experiment we were comparing apples with oranges."

Now Anne points to her sketch, figure 5.6, which shows the location of the bus at three different times. She lectures her friends, "The lowest pair of heavy blue and red lines shows the spacetime location of the bus at the beginning of the experiment when both t and T equaled zero. Of course that is also the instant when I saw Chuck pass me. At that moment, Earth-frame observers see the bus between the spacetime points x = 0 ft and t = 0 ns and x = 80 ft and t = 0 ns, the lowest heavy blue line on the graph. But Chuck and his bus riding friends would disagree and instead claim the bus was the heavy red line that goes from him at X = 0 ft and T = 0 ns to Dean at X = 100 ft and T = 0 ns."

Bev asks Anne, "What about the two other pairs of heavy blue and red lines?"

Anne answers, "A little later the bus has moved forward in time, up the spacetime graph, and towards the right. The middle pair of heavy lines represents the bus at that later time. The upmost pair of lines shows the bus at an even later time. And just like at the start of the experiment, Earth observers claim the bus is represented by the heavy blue line while bus observers insist it is the heavy red line."

Chuck, who has been very quiet, points to the middle pair of "buses" on Anne's graph and says, "Suppose I am at my usual spot in the rear of the bus. I glance down at my watch and see it reading 8 ns. That means the Earth observer who is adjacent to me has a watch reading 10 ns." He stops and sees Anne nodding in agreement, and takes that as a sign to continue, "So me and my bus riding colleagues would point to the heavy red line and say that is the spacetime location of the bus at the instant T = 8 ns. But the Earth observer who is adjacent to me looks at her watch and disagrees. For her and her colleagues, the bus at t = 10 ns is the heavy blue line." Anne agrees with Chuck that not only do bus and Earth observers disagree about how to represent the bus at a particular time on a spacetime graph, but they also disagree about the value of that particular time!

Bev squints at Anne's diagram and adds, "I can almost visualize the bus moving through spacetime as the heavy blue line sliding up the graph between the worldlines of C & D." C & D frown at Bev since they visualize the bus traveling through spacetime by the slanted heavy red line sliding up the T-axis.

Question 5.2: On the spacetime diagram drawn for question 5.1, there is not a large enough swatch of spacetime to draw a 100-foot bus. So instead, replace the bus passing Anne and Bev by a van with a proper length of 18 feet. On the figure you drew to answer exercise 5.1, draw a red line showing the spacetime location of the van according to A & B at t = 10 ns. Then add a blue line representing the van in spacetime according to C & D when their watches read 10 ns.

Notice that a question like "draw the location of the van in spacetime at 10 ns" is intrinsically ambiguous. To remove the ambiguity, the question needs to include 10 ns on whose watch. Analogously questions that ask for the distance or time separations between two events must specify a particular set of observers. For example, how long is the super bus, is a question with two valid answers, 100 feet or 80 feet. In fact sets of observers moving with different speeds with *respect to the super bus could, in principle, see the super bus have any length from 1 inch to 100 feet! But none of these inertial observers could ever measure the bus to be longer than its proper length, 100 feet.*

Question 5.3: How fast, relative to the bus, would observers have to be moving to measure the length of the bus to be only 1 inch?

Bev Generalizes Chuck's Spacetime Diagram

The algebra in this section is comparable to that used at the end of Chapter Four. The reward for persevering through this section is a derivation of the very useful and widely used Lorentz Transformation equations. Therefore you are encouraged to follow the arguments carefully. On the other hand, if you get too bogged down, read through material quickly and move on to Chapter Six. Most of the topics covered in the remainder of the book can be understood without using the Lorentz Transformation equations. And you can always return to this section later if you want to give it another go.

Bev spent some time admiring figure 5.5, the spacetime diagram drawn by Chuck. She noticed that any point on that spacetime graph is located at the intersection of a unique pair of constant x and t or X and T lines. This conveniently allows the Earth and bus coordinates for every event that happens during an experiment to be exhibited on a single spacetime graph instead of on two separate graphs. But figure 5.5 was drawn for the specific bus velocity of 0.6 ft/ns. A different spacetime diagram would be needed to describe the intertwining of space and time for a bus moving at any other velocity, see question 5.1.

After staring at the figure for a while, Bev begins to envision a way to develop a set of equations that would allow her to calculate the bus coordinates for any event given that she knew the Earth coordinates for the event and the velocity, v, of the bus.

Back in Chapter One, our intrepid quartet came up with the Galilean transformation equations that related Earth and bus coordinates for any point in spacetime under the assumption that the law for the addition of velocities, $V_{XY} = V_{XZ} + V_{ZY}$, was true. The relationship between coordinates was given by the simple equations, $X_{Earth} = X_{bus} + vT$ and $X_{bus} = X_{Earth} - vT$. What Bev is proposing to do, is to find a new set of equations connecting the bus coordinates X and T *to the Earth coordinates x and t that work for spacetime as it actually exists, the spacetime of shrinking buses and slow watches.*

By the way, the answer to question 5.3 is v = 0.99999965 ft/ns!

Bev flips the page on Chuck's pad sitting on the easel and sketches a spacetime diagram with the bus moving with a general velocity of v, figure 5.7. Bev explains that she included a general line of constant T (parallel to the X-axis) that passes through point A on the Earth t-axis, $(0, t_A)$ and a constant X line (parallel to the T-axis) that crosses the x-axis at point B, $(x_B, 0)$. Those two lines are examples of any of constant T and X lines that could be drawn on figure 5.7. Now she reminds her friends that any straight line on an x vs t graph can be described by an equation of the form t = mx + b, where m is the slope of the line and b is the t-intercept.

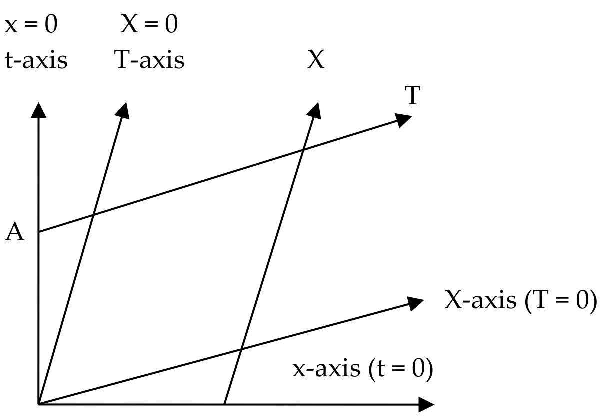


Figure 5.7

On the pad, Bev re-writes equations 4.11a and 4.11b in terms of the new notation: the T-axis is represented by the line $t = \frac{1}{v}x$ and the X-axis by the line $t = \frac{v}{c^2}x$. The slope of the T-axis is $\frac{1}{v}$ while the slope of the X-axis is $\frac{v}{c^2}$.

She asks her friends to focus their attention on the X-axis which is also the T = 0 line and tells them, "All constant T lines have the same slope, $m = \frac{v}{c^2}$, but intercept the t-axis at different points." Bev points out that the constant T line shown on the figure passes through the point $(0, t_A)$. Anne interrupts Bev to remind them that when she passed Chuck both watches read zero and then she saw a series of bus observers pass as she stood at the Earth origin. When she was at the spacetime point A, her watch read time

t_A while the bus observer adjacent to her had a watch that read T ns. Anne tells her friends, "In this scenario, it is my watch that appears to run slow compared to the two bus watches, so my watch reads time $T\sqrt{1-\frac{v^2}{c^2}}$ at point A. Therefore the x and t coordinates of point A are just (0, $\sqrt{1-\frac{v^2}{c^2}}$)."

Bev continues with Anne's chain of thought by substituting the values at point A into the equation, $t = \frac{v}{c^2}x + b$, to find the particular value of b for the constant T line shown in figure 5.7: $T\sqrt{1-\frac{v^2}{c^2}} = \frac{v}{c^2}0 + b$ and solving that equation for b. Bev writes the result on the pad,

$$b = T\sqrt{1-\frac{v^2}{c^2}} \qquad (5.1)$$

Bev uses equation 5.1 to re-write the equation for the constant T line on the x and t spacetime graph,

$$t = \frac{v}{c^2}x + T\sqrt{1-\frac{v^2}{c^2}}\,. \qquad (5.2)$$

Bev and Anne smugly point out that an analogous argument can be used to find the equation for any line of constant X. But before they can continue their display of dazzling algebra, Dean screams, "Stop, please stop. Would one of you kindly explain exactly what equation 5.2 means?"

Question 5.4: Earlier Chuck drew the spacetime graph for a bus velocity of 0.6 ft/ns, figure 5.1. Use equation 5.2 to find the equations for the T = 8 and 16 ns lines on that graph.

Question 5.5: On figure 5.1, the constant T = 8 and 16 ns lines pass through the points x = 6 and t = 10 and x = 12 feet and t = 20 ns respectively. Use those values of x and t to check your answers to question 5.4.

Before anyone can reply to Dean, Chuck solves equation 5.2 for T, and draws a box around it.

$$T = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{v}{c^2}x\right) \tag{5.3}$$

Pointing to equation 5.3, Chuck enthusiastically says, "This is exactly the equation I was searching for earlier. It gives me the bus time T for any spacetime point for which the Earth coordinates, x and t, are known. Once we have the equivalent equation for X in terms of x and t, we can translate the Earth coordinates of any event directly into bus coordinates without having to draw a spacetime graph like figure 5.1."

Questions 5.6a to 5.6e: Follow the outline below in conjunction with the steps that led to equation 5.3 to find the analogous equation that connects any spacetime point x and t to X. *(The primary purpose of this question is to help you better understand the derivation of equation 5.3. If the algebra is more than you bargained for, skip this question and read on.)*

a) Any constant X line can be written in the form t = mx + b. What is the slope m of this line?
b) The constant X line in figure 5.7 passes through point B, (x_B, 0), on the x-axis. What is the value of x_B in terms of v, c, and X.
c) Substitute m and the coordinates of point B into the equation t = mx + b and solve for b, the t-intercept of the constant X line in terms of X, v, and c.
d) Use the values found for m and b to find the specific equation t = mx + b that represents any line of constant X.
e) Solve your answer in d) for X. This equation for X in terms of x, t, v, and c is analogous to equation 5.3.

Chuck triumphantly writes down the equations for X and T in terms of x and t:

$$X = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) \tag{5.4a}$$

$$T = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{v}{c^2}\text{x}\right). \tag{5.4b}$$

Note that these equations can be solved to give x and t in terms of X and T. Those two equations are written below. Either pair, 5.4a and 5.4b or 5.5a and 5.5b, can be referred to as the Lorentz Transformation equations.

$$x = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(X + vT) \tag{5.5a}$$

$$t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(T + \frac{v}{c^2}X). \tag{5.5b}$$

From the perspective of bus observers, Earth is moving with a velocity of – v. Therefore equations 5.4a and 5.4b can be changed into equations 5.5a and 5.5b by merely switching the sign of v and interchanging the x and t's with X and T's.

Notice that when $\frac{v}{c}$ *is small, both* $\frac{v}{c^2}$ *and* $\frac{v^2}{c^2}$ *are very small. For the speeds encountered in our everyday lives both* $\frac{v}{c^2}$ *and* $\frac{v^2}{c^2}$ *can be replaced by zero in the Lorentz Transformation equations. When those substitutions are made, the Lorentz equations reduce to the Galilean equations from Chapter One, namely* $x = X + vT$ *and* $t = T$. *This had to be the case since the Galilean equations worked fine for velocities that were small compared to the speed of light.*

Dean, with glazed over eyes, says, "I am impressed with the algebraic virtuosity shown by the three of you, but would appreciate a bit of time to come to grips with the Lorentz Transformation equations before seeing any more algebraic manipulations."

A Simple Example for Dean

Anne reminds Dean that when they did the re-run of the first experiment of the super bus, Bev passed Dean when Bev's watch read 200 ns. Also Bev was located at the 200 foot marker along the road. Consequently, the spacetime point that represented Bev at that instant was x = 200 ft and t = 200 ns.

Table 3.1 on page 49 summarizes the data collected from that earlier experiment done in chapter three.

Those values could be used in equations 5.4a and 5.4b to find his coordinates at the instant Bev passed. Dean grabs a piece of paper and quickly calculates the values for X and T from those equations. Dean looks up quite amazed and says, "the equations worked perfectly, X = 100 ft and T = 100 ns."

Question 5.7: Check Dean's arithmetic and convince yourself that equations 5.4a and 5.4b transform the Earth coordinates of that event into the correct bus coordinates.

Anne also points out to Dean that when Bev saw, Ed at X = 160 feet and T = 0 ns, rush by, she looked at her watch and noticed that it read 120 ns.

Question 5.8: Use equations 5.5a and 5.5b to show that the spacetime point X = 160 feet and T = 0 converts to the correct Earth coordinates x and t, namely those occupied by Bev!

Dean says, "Thanks Anne, for those examples, I feel much better now."

The Revised Equation for the Addition of Velocities

Dean is now a tad more enthusiastic about the Lorentz Transformation equations. He asks, "I was wondering if these equations could help us find the relativistic version of the equation, $V_{XY} = V_{XZ} + V_{ZY}$."

While his three friends ponder that question, he remembers the experiment where he had to catch the pigeon flying at 20 ft/s, a speed that was too small to cause any conflict with the intuitively pleasing equation, $V_{XY} = V_{XZ} + V_{ZY}$ that worked just fine in Chapter One.

"Remember that pigeon we used earlier that was trained to fly at 20 ft/s," Dean asks. Now that he has recaptured the attention of his three friends, he points out that in the spirit of algebra, we can think of the pigeon flying at some unspecified speed U with respect to the bus. Later we can pick U to be large or small compared to the speed of light but for now, it is just U."

His friends smile as Dean tiptoes into using algebra to analyze a problem. He is too busy concentrating to notice their smiles, and continues developing his scenario with the pigeon, "If Chuck, at X = 0, releases the pigeon when T = 0, then for Chuck and me, the pigeon's position in the bus is given by the equation X = UT. The bus coordinates for the pigeon as it flies from Chuck towards me are just X = UT and T." Dean adds, "Anne is also at the spacetime point where the pigeon starts its flight. Her Earth coordinates at that instant are x = 0 and t = 0. According to her, the pigeon has a velocity u with respect to Earth. So Anne assigns coordinates x = ut and t for the pigeon as it flies from her to Bev."

Keep in mind that as the pigeon flies from the rear to the front of the bus, its position can be described in bus coordinates by noting that it passes through a series of spacetime points with X

= UT at time T. But that same sequence of spacetime points in Earth coordinates is just x = ut and t.

Bev runs to the easel, flips up a clean sheet, and takes over the conversation from Dean. Before beginning, she gives Dean a high five and a quick pat on the back. Anne and Chuck are still not quite up to speed, and Bev gives them no time to catch up. She starts right in, "Dean's analysis of the pigeon experiment was brilliant. The bus coordinates of the pigeon are X = UT and T. We can put those coordinates into equations 5.5a and 5.5b to find the Earth coordinates for the flying pigeon x and t." She writes those equations down:

$$x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(UT + vT)$$

$$t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(T + \frac{v}{c^2}\mathrm{UT}).$$

And continues, "But x = ut, so the velocity of the pigeon with respect to the bus is just the ratio of those equations, namely x/t." Bev, with a flourish, writes down the equation for u,

$$\mathrm{u} = \frac{U + v}{1 + \frac{vU}{c^2}} \cdot \qquad (5.6)$$

Dean asks, "What happened to the capital T's in the equations for x and t?" Chuck shows Dean how all the capital T's cancel leading to the equation written down by Bev. Dean nods with a satisfied look and mumbles, "Thanks."

Suppose the pigeon is replaced by a light flash. The equation then would give the velocity of the light flash with respect to Earth in terms of the velocity of the light flash with respect to the bus and the velocity of the bus with respect to Earth. That change is accomplished by replacing U, the velocity of the pigeon, with c, the speed of light, in equation 5.6. What answer do you expect for u, the speed of the light flash with respect to Earth? The question below helps you work through the calculation for u.

Question 5.9: Use equation 5.6 to find the speed of light with respect to Earth for a light flash on a bus moving with velocity v. (A nice way to do the algebra in this problem is to first multiply equation 5.6 by $\frac{c}{c}$, to get u = $\frac{c(U+v)}{c+\frac{vU}{c}}$. This is another example of the usefulness of the little trick of multiplying an expression by $\frac{something}{something} = 1$.

Question 5.10: Redo question 5.9 with the laser flash moving from the front of the bus to the rear, from right to left. In this case U is replaced by –c.

Of course the answers to questions 5.9 and 5.10 are obvious because of the constancy of the speed of light. Namely the Earth observers have to measure the same velocity for the flash as the bus observers, c in the first case and –c in the second. Equation 5.6 was derived from the Lorentz Transformation equations that came from analyzing the spacetime diagrams that arose from the super bus experiments that demonstrated that Earth and bus observers measure the same speed for light. Therefore it ought to come as no surprise that equation 5.6 "predicts" the constancy of the speed of light, since the constancy of the speed of light was the starting point for our explorations of space and time!

Before Bev has a chance to sit down and admire equation 5.6, Anne comes up to the easel and adds the following three definitions to the sheet containing the equation:

u = velocity of the pigeon with respect to Earth = V_{PE}

U = velocity of the pigeon with respect to the bus = V_{PB}

v = velocity of the bus with respect to Earth = V_{BE}

Anne points out that by using her definitions, equation 5.6 can be re-written as,

$$V_{PE} = \frac{V_{PB} + V_{BE}}{1 + \frac{V_{PB}V_{BE}}{c^2}}. \qquad (5.7)$$

This shows that for situations where V_{PB} and V_{BE} are small compared to c so that the term $\frac{V_{PB}V_{BE}}{c^2}$ in the denominator is very, very small compared to 1, equation 5.7 reduces to the familiar relative velocity equation,

$V_{PE} = V_{PB} + V_{BE}$.

The relativistic velocity addition equation 5.7 written in terms of X, Y, and Z is just,

$$V_{XZ} = \frac{V_{XY} + V_{YZ}}{1 + \frac{V_{XY}V_{YZ}}{c^2}}. \qquad (5.8)$$

Because the velocity of X with respect to X is still zero by definition, equation 5.8 with $V_{XX} = 0$ leads to the same conclusion found in Chapter One, namely that $V_{XY} + V_{YX} = 0$ and $V_{XY} = -V_{YX}$. If observers in reference frame X see reference frame Y moving with velocity v, then observers in Y see X moving with velocity –v. This conclusion remains true for reference frames moving at relativistic speeds with respect to one another.

Anne takes out a calculator to figure out what velocity equation 5.7 predicts for the pigeon flying in the bus in Chapter One. The velocity of the bus was +30 ft/s and the velocity of the pigeon with respect to the bus was 20 ft/s. The speed of light is 1 ft/ns or 10^9 ft/s.

When she punches these numbers into her calculator she gets 50 ft/s, the velocity measured in Chapter One!

The exact velocity for the pigeon with respect to Earth is given by,

$$u = \frac{20+30}{1+\frac{600}{10^{-18}}} = \frac{50}{1+6 \times 10^{-16}}\ .$$

The denominator in that expression is 1.0000000000000006 which Anne's calculator rounded off to 1.00000000. So Anne's calculator did the best it could when it came up with +50 ft/s!

In Chapter Four the handy approximation, $\sqrt{1-x} \cong 1 - \frac{x}{2}$, was used, equation 4.13. That equation is a specific example of the binomial approximation,

$$(1+x)^n = 1 + nx, \qquad (5.9)$$

where n can be any number and x is small compared to 1. In Chapter Four $n = \frac{1}{2}$.

In the above example we need an approximation for $\frac{1}{1+x} = (1+x)^{-1}$, so n equals -1. Using the binomial approximation,

$$\frac{1}{1+x} \cong 1 - x.$$

Question 5.11: Test the approximation $\frac{1}{1+x} \cong 1 - x$ for x = 1/100 = 10^{-2}, x = 1/10,000 = 10^{-4}, and x = 1/1,000,000 = 10^{-6}.

When that approximation is applied to $\frac{50}{1+6 \times 10^{-16}}$, the exact velocity of the pigeon with respect to Earth is found to be 49.99999999999997 ft/s instead of 50 ft/s. In order to have observed this *exact velocity, A & B would have needed to measure the time it took the pigeon to fly the exact distance of 250 ft with an accuracy of a millionth of a nanosecond!*

This was impossible in Chapter One with watches accurate to 0.1 s and remains impossible even with the super accurate watches introduced in Chapter Three which measured time to tenths of nanoseconds.

The "strange" conclusions of the experiments with light in Chapter Three seemed strange only because the earlier experiments were done with objects moving at pedestrian velocities compared to light.

Chapter Six: Chuck Suggests an Experiment and Anne has a Dream

Most of this chapter is a tutorial on using spacetime graphs, the velocity addition equation, and the Lorentz Transformation equations. Successfully working through this chapter will make you an expert in the spacetime of Special Relativity. The chapter ends with a discussion of the problems that arise if an object goes faster than the speed of light, superluminal travel.

While at a garage sale, Chuck finds a pair of extraordinary robotic pigeons that can fly at speeds up to 0.9 ft/ns. He immediately envisions a way to use the pigeons to test the new formula for adding velocities, $u = (U + v)/(1 + Uv/c^2)$. So he purchases the pair and heads off to tell Dean, Bev, and Anne his idea for a couple of new experiments with the super bus and his robotic pigeons.

First he reminds his spacetime comrades that they now know how to draw spacetime diagrams that contain both Earth and bus coordinates for any event in spacetime, figure 5.5, and also have equations, 5.4a & b and 5.5a & b, which connect the coordinates in one reference frame to those in another. He suggests that they perform the following two experiments to test their ever improving understanding of space and time.

Chuck outlines his idea, "In experiment I, I'll program both pigeons to fly at exactly 0.5 ft/ns. Then Dean and I will board the bus and drive it a ways down the road before turning it around and zipping by Anne and Bev at 0.6 ft/ns. The experiment will begin, when I'm adjacent to Anne. That will also be the origin, $x = X = 0$, of the coordinates for both reference frames. At that instant, according to our synchronized watches, Dean will release one of the pigeons from the front of the bus and I'll release the other from the back of the bus. Dean's pigeon flies to me and mine to him. The experiment ends when the speeding pigeons are caught. A & B will carefully note the Earth coordinates of those events."

Before anyone has a chance to respond, Chuck continues, "In experiment II, the pigeons will be reprogrammed to fly at exactly 0.75 ft/ns, ¾ the speed of light. This time, the pigeons will be flying in the Earth frame, one from Anne to Bev who will be standing 90 feet to the right of Anne and the other from Bev to Anne. The pigeons will be released simultaneously according to A & B who, of course, have carefully synchronized their watches. This experiment will also begin when I'm adjacent to Anne. But now it will be Dean and me watching where and when the pigeons are released and caught with respect to our bus coordinates."

Bev rolls her eyes a little and points out that they have become so adept at drawing spacetime diagrams that they do not really need to do the experiment. Instead they can analyze what would have happened if they did do the experiment. Anne and Dean like this because it's expensive to get the super bus zipping along at 0.6 ft/ns! Chuck is less sanguine because he is now the owner of two irrelevant but costly robotic pigeons. But after a little haggling they agree to analyze the situation first and only do the actual experiment if their analysis runs into a snag.

As a concession to the Earth-based observing team, Chuck and Dean agree that both of the experiments ought to be analyzed from the perspective of Earth. Chuck makes one final suggestion, "Since you won't let me actually do the experiment, let me devise a series of questions based on the two experiments for you guys to answer. Answering the questions will be equivalent to performing the experiment." Dean, Bev, and Anne think that is a very cool idea and look forward to seeing what Chuck comes up with.

This scenario is a tad contrived but seemed like a good way to give you the opportunity to check your understanding of the material covered up to now. Although no one will be watching to see if you actually take the time to answer Chuck's questions, Chuck would certainly appreciate it if you did! After all, he was up most of the night writing up a series of questions about the two experiments. Note that Chuck refers to himself in the two experiments as Chuck to avoid any confusion that might have arisen by using pronouns like I, me, mine, etc.

Experiment I

Chuck is adjacent to Anne at the start of the experiment when x = X = 0 and t = T = 0. Remember that lowercase letters are Earth coordinates and uppercase ones are bus coordinates. The spacetime diagram, figure 6.1, has labeled x and t axes. The bus is moving at v = 0.6 ft/ns which makes the shrinkage factor, $\sqrt{1 - v^2} = 0.8 = \frac{4}{5}$. (Use figure 5.5 as the model for this spacetime graph.)

a) Draw the X and T-axes on the graph, figure 6.1, or on a different piece of graph paper.
b) Use the scale on the x and t axes to calibrate the X and T-axes.
c) Use Dean's spacetime location at t = 0 to draw his worldline. His worldline ought to cross the X-axis at X = 100 feet.

When Chuck's clock reads zero, he releases the robotic pigeon which flies to Dean. According to Chuck and Dean the pigeon is zipping along at ½ the speed of light, 0.5 ft/ns.

d) Chuck and Dean assign what coordinates to the spacetime point B where Dean catches the pigeon? X_B =_________ and T_B =___________

e) On the spacetime diagram, find the starting point for the pigeon's trip from Chuck to Dean and label it point A. Also find and label point B on the graph. Use points A and B to draw a dashed line representing the worldline of the pigeon as it travels from Chuck to Dean.

At the same instant, <u>according to Chuck and Dean</u>, that Chuck releases his pigeon, Dean releases his which flies to Chuck.

f) Find the starting point for the pigeon's trip from Dean to Chuck and label it point C.

g) Chuck and Dean assign what coordinates to the spacetime point D where the Chuck catches the pigeon? X_D =_________ and T_D =_________.

h) Draw a dashed line representing the worldline of the pigeon as it travels from Dean to Chuck.

The velocity of an object in any reference frame is just the distance traveled divided by the time of flight in that frame, $\frac{x_f - x_i}{t_f - t_i}$ or $\frac{X_f - X_i}{T_f - T_i}$.

i) Use the graph to find the coordinates Anne and Bev assign to points A, B, C, and D.

   x_A = ________, x_B = ________, x_C = _________, x_D = _________,

   t_A = ________, t_B = ________, t_C = _________, t_D = _________.

j) Based on the coordinates in i), what is the velocity of the robotic pigeon according to Anne & Bev? (Leave the answers as fractions.)

   Velocity of pigeon traveling from Chuck to Dean = _____________

   Velocity of pigeon traveling from Dean to Chuck = _____________

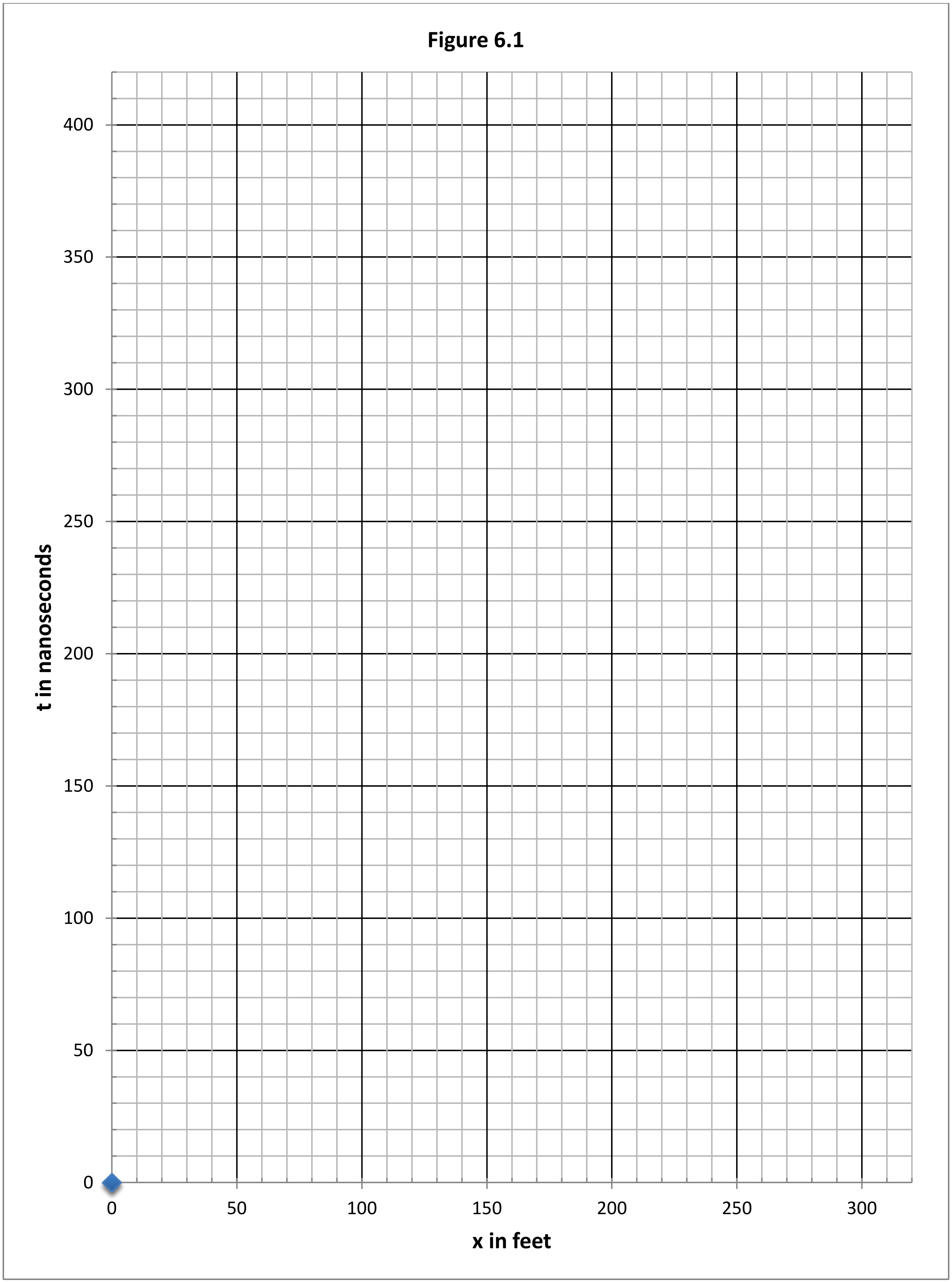


Figure 6.1
t in nanoseconds
400
350
300
250
200
150
100
50
0
0
50
100
150
200
250
300
x in feet

The addition of velocities equation, w = (v + U)/(1 + Uv), can also be used to find the velocity of the pigeon according to A & B. Since the velocities are all written as fractions of the speed of light, c is equal to 1 in the velocity addition equation.

w = velocity of the pigeon with respect to A & B

v = velocity of the bus with respect to A & B

U = velocity of the pigeon with respect to the bus

k) Use the velocity addition equation to find the velocity of the pigeon according to A & B. (Remember velocity is direction dependent: velocities toward the right are positive while velocities towards the left are negative.)
Velocity of the pigeon traveling from Chuck to Dean = __________
Velocity of the pigeon traveling from Dean to Chuck = __________

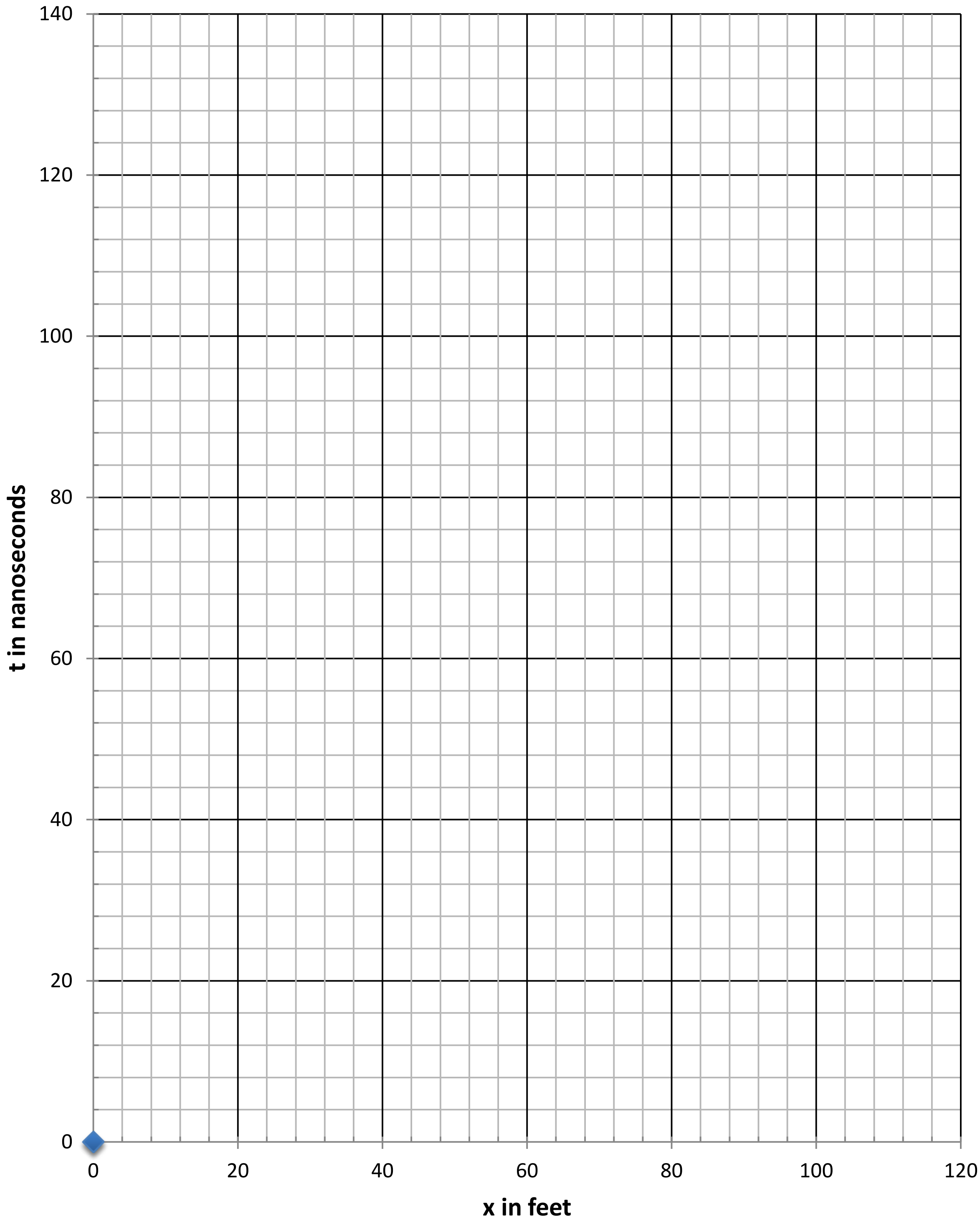
The velocities found in questions j) and k) ought to agree. If they do not, try to find the reason for the discrepancies. The answers to all these questions, including the spacetime diagram, can be found at the end of this chapter.

Experiment II

As usual, Chuck is adjacent to Anne at the start of the experiment but now it is A & B who release robotic pigeons. Bev is standing 90 feet to the right of Anne at x = 90 ft. Each pigeon flies with a speed of 0.75 or $\frac{3}{4}$ ft/ns with respect to Earth. The bus is going its usual speed, v = 0.6 ft/ns.

a) Draw the X and T axes on the figure 6.2 or on a separate piece of paper.
b) Use the scale on the x and t axes to calibrate the X and T axes.
c) Draw Bev's worldline on the graph.
d) Label the point where Anne releases the robotic pigeon as point A and the point where Bev catches the pigeon as point B. Draw a dashed line showing the path of the pigeon through spacetime.
e) Label the point where Bev launches her pigeon as point C and the point where Anne catches it as point D. Draw a dashed line representing the path of the robotic pigeon from Bev to Anne.

Figure 6.2



t in nanoseconds
140
120
100
80
60
40
20
0
0
20
40
60
80
100
120
x in feet

To use the spacetime diagram to find the velocity of the pigeon with respect to Chuck and Dean riding on the bus, the bus coordinates of the points A, B, C, and D need to be found. Remember that constant T lines are parallel to the X-axis and constant X-lines are parallel to the T-axis. For example, the line parallel to the X-axis through point B is the constant T_B line. The value of T_B can be found by noting where that line intersects the T-axis. X_B can be found analogously.

f) Use the procedure described above to find the coordinates Chuck & Dean assign to points A and B. (The more carefully the points of intersection are located, the more accurate the estimates will be for the values of X_B and T_B.)
X_A = ________, X_B = ________,
T_A = ________, T_B = ________.

g) Use the values found in f) to find the velocity that C & D assign to the pigeon that flew from Anne to Bev.
Velocity of pigeon traveling from Anne to Bev = ___________

Now use the addition of velocity formula, U = (u + V)/(1 + uV) to find the velocity Chuck & Dean assign to the pigeon going from Anne to Bev.

U = velocity of pigeon with respect to bus

u = velocity of pigeon with respect to Anne and Bev

V = velocity of Earth with respect to bus

h) Velocity of pigeon traveling from Anne to Bev = ___________

The velocities found in g) and h) are not exactly the same because the values read off the graph were approximate values. The answer found in h), if done correctly, is the exact velocity of the flying from Anne to Bev according to Chuck and Dean.

A serious problem arises when attempting to find the bus coordinates for events C and D from the spacetime diagram. The intersection of the constant X and T lines that pass through spacetime points C and D lie outside the limits of the graph! Check to make sure this is true for X_C, T_C, X_D, and T_D.

The Lorentz equations 5.8a and 5.8b with c = 1 are written below,

$$X = \frac{1}{\sqrt{1-v^2}}(x - vt) \text{ and } T = \frac{1}{\sqrt{1-v^2}}(t - vx),$$

can be used to algebraically determine X_C, T_C, X_D, and T_D from the known values of x_C, t_C, x_D, and t_D.

i) Use the values of v and $\sqrt{1-v^2}$, $\frac{3}{5}$ and $\frac{4}{5}$, to rewrite the Lorentz equations with numerical coefficients in front of x and t.
X = (   ) x – (   ) t   and T = (   ) t – (   ) x.

j) Now use the equations for X and T in i) to find,
X_C = __________, T_C = ___________,
X_D = __________, T_D = ___________.

k) Use the values in j) to find the velocity of the pigeon, according to Chuck & Dean, which flew from Bev to Anne.
Velocity of pigeon traveling from Bev to Anne = __________

l) As a check of the answers to question j) and k), use the velocity addition formula to find the same velocity.
Velocity of pigeon traveling from Bev to Anne = __________

m) Lastly, use the Lorentz equations to check the accuracy of the values found graphically for X_B and T_B in f).
X_B from the Lorentz equation = ___________
T_B from the Lorentz equation = ___________
Velocity of pigeon traveling from Anne to Bev = __________

The velocity of the pigeon going from Anne to Bev found by using the coordinates in m) ought to agree exactly with the velocity found from the formula in h).

Answers for Experiment I

a) Draw the X and T axes on the graph, figure 6.1, or on a different piece of graph paper.

The X-axis is the line $t = \frac{3}{5}x$ *and the T axis is the line* $t = \frac{5}{3}x$. *The axes are drawn in red on figure 6.3.*

b) Use the scale on the x and t axes to calibrate the X and T-axes.

The t = 100 ns line intersects the T-axis at 80 ns. Analogously, the x = 100 foot line intersects the X-axis at 80 feet. Note that each 20 nanoseconds or feet on the t or x-axes correspond to 16 ns or feet on the T or X-axes. Those correspondences were used to scale the T and X-axes. The heavy red line is the T axis and Chuck's worldline.

c) Use Dean's spacetime location at t = 0 to draw his worldline. His worldline ought to cross the X-axis at X = 100 feet.

Earth observers see Dean at x = 80 feet when t = 0. Remember the bus appears shorter to Earth observers. Dean's worldline is in green. Note that Dean's worldline crosses the X axis, T = 0, right at X = 100 ft!

d) C & D assign what coordinates to the spacetime point B where Dean catches the pigeon?

Dean is located at $X_B = 100$ *ft. It takes 200 ns for the pigeon to cross the bus;* $T_B = 200$ *ns.*

e) On the spacetime diagram, find the starting point for the pigeon's trip from Chuck to Dean and label it point A. Also find point B on the graph. Use points A and B to draw a dashed line representing the worldline of the pigeon as it travels from Chuck to Dean.

Point A is just the origin. Point B can found by drawing the constant 200 ns line (dotted green line on the graph) until it intersects the X = 100 ft line. That is point B. The pigeon's worldline is the black dashed line going from A to B.

f) Find the starting point for the pigeon's trip from Dean to Chuck and label it point C.

$X_C = 100$ *feet and* $T_C = 0$ *ns.*

g) Chuck and Dean assign what coordinates to the spacetime point D where the Chuck catches the pigeon?

$X_D = 0$ *feet and* $T_D = 200$ *ns.*

h) Draw a dashed line representing the worldline of the pigeon as it travels from Dean to Chuck.

The black dashed line connecting C to D represents the pigeon's trip from Dean to Chuck.

i) Use the graph to find the coordinates A & B assign to points A, B, C, and D.

$x_A = 0$ $x_B = 275$ ft $x_C = 125$ ft $x_D = 150$ ft

$t_A = 0$ $t_B = 325$ ns $t_C = 75$ ns $t_D = 250$ ns

j) Based on the coordinates in i), what is the velocity of the robotic pigeon according to A & B? (Leave the answers as fractions.)

Velocity of robotic pigeon traveling from Chuck to Dean = 11/13

Velocity of robotic pigeon traveling from Dean to Chuck = 1/7

The addition of velocities equation, $w = (v + U)/(1 + Uv)$, can also be used to find the velocity of the pigeon according to Anne and Bev. Since the velocities are all given as fractions of the speed of light, c was set equal to 1 in the velocity addition equation.

w = velocity of the pigeon with respect to Anne and Bev

v = velocity of the bus with respect to Anne and Bev

U = velocity of the pigeon with respect to the bus

k) Use the velocity addition equation to find the velocity of the pigeon according to Anne and Bev. (Remember velocity is direction dependent: velocities toward the right are positive while velocities towards the left are negative.)

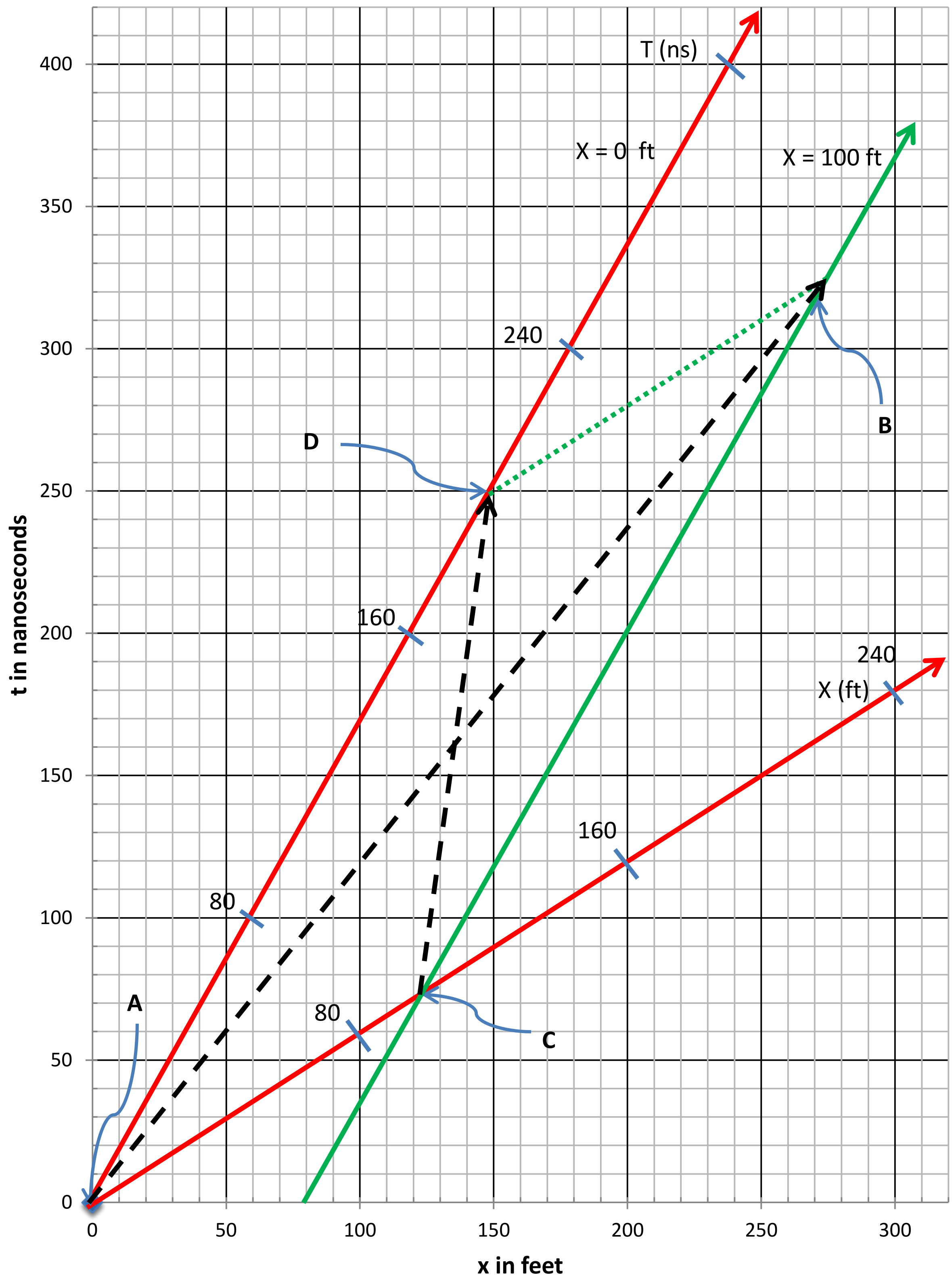
($v = 3/5$ ft/ns and $U = 1/$ ft/ns in the first case and -½ ft/ns in the second case)

Velocity of the pigeon traveling from Chuck to Dean, $w = 11/13$

Velocity of the pigeon traveling from Dean to Chuck, $w = 1/7$

Note that the answers for j) and k) agree perfectly.

Figure 6.3



T (ns)
X = 0 ft
X = 100 ft
240
D
B
160
240
X (ft)
160
80
A
80
C
t in nanoseconds
x in feet
0
50
100
150
200
250
300
350
400
0
50
100
150
200
250
300

Answers for Experiment II

a) Draw the X and T axes on the figure 6.2 or on a separate piece of paper.

The X and T axes are the red lines on figure 6.4

b) Use the scale on the x and t axes to calibrate the X and T axes.

Same as experiment I

c) Draw Bev's worldline on the graph.

Bev's worldline is in black at $x = 90$ ft.

d) Label the point where Anne releases the robotic pigeon as point A and the point where Bev catches the pigeon as point B. Draw a dashed line showing the path of the pigeon through spacetime.

Anne releases the pigeon at the origin. It takes the robotic pigeon 90 ft divided by 0.75 ft/ns to reach Bev, or 120 ns. So point b is $x_B = 90$ ft and $t_B = 120$ ns. The dashed line representing the path of the pigeon is in black.

e) Label the point where Bev launches her pigeon as point C and the point where Anne catches it as point D. Draw a dashed line representing the path of the robotic pigeon from Bev to Anne.

Point C is at $x_C = 90$ ft and $t_C = 0$. The pigeon released by Bev also takes 120 ns to reach Anne. Point D is at $x_D = 0$ and $t_D = 120$ ns. Again the path of the pigeon is in black.

f) Use the described procedure to find the coordinates C & D assign to points A and B. (The more carefully the points of intersection are located, the more accurate the estimates will be for the values of X_B and T_B.)

The green dashed lines going through point B on the graph denote the lines of constant X_B and T_B. The relevant parts of those lines are located at the place where the lines cross the X and T-axes to give the values of X_B and T_B respectively. Notice that there are 10 boxes between each tick mark, 32 ft or ns, on the X and T axes. Consequently each box on the graph represents 3.2 feet or nanoseconds.)

X_B falls about 7 boxes from the origin, 7 times 3.2 = 22.4 feet. So X_B is about 22.4 ft.

T_B is approximately 4 boxes before 96 ns. So TB is approximately 96 – 12.8 or 83.2 ns

$X_A = 0, \quad X_B \approx 22.4$ ft,

$T_A = 0, \quad T_B \approx 83.2$ ns.

g) Use the values found in question f) to find the velocity that Chuck and Dean assign to the pigeon that flew from Anne to Bev.

Velocity of pigeon traveling from Anne to Bev = $\frac{X_B - X_A}{T_B - T_A} \approx \frac{22.4 - 0}{83.2 - 0} \approx 0.269$

Now use the addition of velocity formula, U = (u + V)/(1 + uV) to find the velocity C & D assign to the pigeon going from Anne to Bev.

U = velocity of pigeon with respect to bus

u = velocity of pigeon with respect to Earth

V = velocity of Earth with respect to bus

h) Velocity of pigeon traveling from Anne to Bev

In this case, V is -3/5 ft/ns and u is ¾ ft/ns.

Using the formula, the velocity of pigeon traveling from E to H is 3/11 = 0.273 ft/ns.

We can compare the approximate velocity found from the graph in g) where we needed to estimate the values of X_B *and* T_B *with the exact answer found by using the velocity addition equation in h) by calculating the percent error.*

The percent error is given by $\frac{Velocity\ from\ h) - Velocity\ from\ g)}{Velocity\ from\ h)} \times 100$ = *1.5%*

The two answers are close enough to provide confidence that the questions were answered correctly.

The problem with using the graph in figure 6.4 to find X_C, T_C, X_D, *and* T_D *can be seen by following the dashed purple lines going through the spacetime points C and D as they travel towards the X and T-axes. None of those four lines reaches the appropriate axes within the confines of the graph! Consequently, the graph cannot be used to find the approximate values of* X_C, T_C, X_D, *and* T_D.

i) First use the values of v and $\sqrt{1 - v^2}$, $\frac{3}{5}$ and $\frac{4}{5}$, to rewrite those equations with numerical coefficients in front of x and t.

X = (5/4) x – (3/4) t and T = (5/4) t – (3/4) x.

j) Now use the equations for X and T in i) to find,

X_C = *225/2 ft,* X_D = *-90 ft,*

T_C = *-135/2 ns,* T_D = *150 ns.*

k) Use the values in j) to find the velocity of the pigeon, according to Chuck and Dean, which flew from Bev to Anne.

Velocity of pigeon traveling from Bev to Anne $= \frac{X_D - X_C}{T_D - T_{CA}} = \frac{-90 - 225}{150 - (-\frac{135}{2})} = -\frac{27}{29}$

l) As a check of the answers to question j) and k), use the velocity addition formula to find the same velocity.

In this situation V = - 3/5 ft/ns and u = - 3/4 ft/ns. Plugging those into the equation for U gives,

Velocity of pigeon traveling from Bev to Anne, C to D, U = -27/29 ft/ns.

m) Use the Lorentz equations to check the accuracy of the values found for X_B and T_B.

X_B from the Lorentz equation = 45/2 ft

T_B from the Lorentz equation = 165/2 ns.

Velocity of pigeon traveling from Anne to Bev, A to B = 45/165 = 3/11 ft/ns

As required, the velocity found by using the coordinates from the Lorentz equations agrees perfectly with the answer in h).

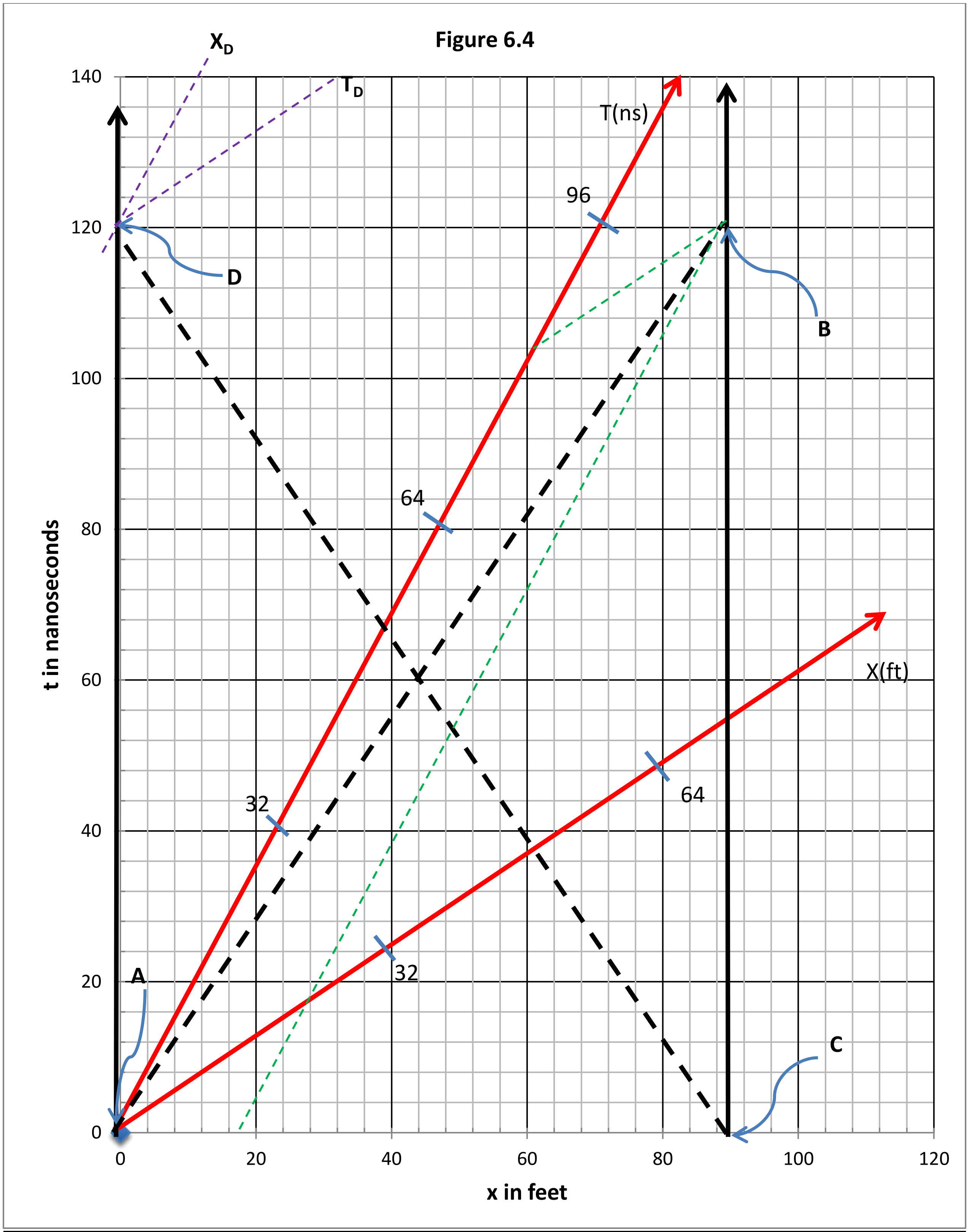
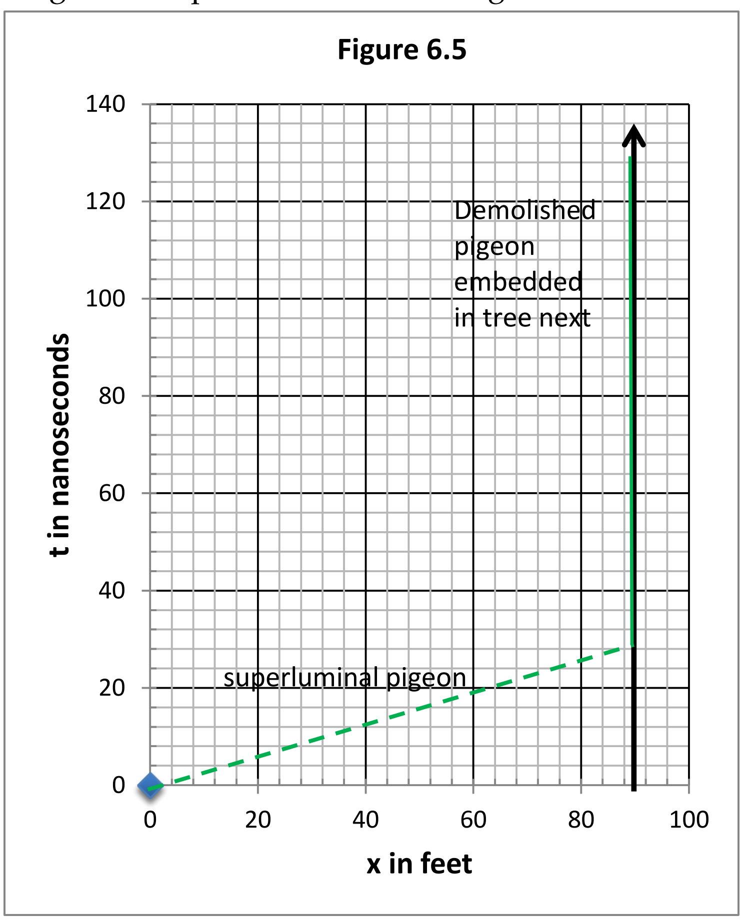


Figure 6.4
X_D
T_D
T(ns)
X(ft)
96
64
32
64
32
A
B
C
D
140
120
100
80
60
40
20
0
0
20
40
60
80
100
120
t in nanoseconds
x in feet

Anne has a Dream about a Superluminal Pigeon

That night, after a full day of "doing" Chuck's experiments and thinking about spacetime graphs and the velocity addition equation, Anne dreamt of robotic pigeons frenetically flying to and fro. At one point, she became fixated on a particular pigeon which appeared to fly faster than any of the other ones in her dream. Finally she realized that this pigeon was flying at <u>three</u> times the speed of light! In her dream, she tried to capture the superluminal pigeon but in kept eluding her grasp. Just as she was close to nabbing the elusive pigeon, it crashed into a tree next to Bev. The crash woke her up and the last thing she could remember from the dream were seeing pieces of the robotic pigeon scattered everywhere!

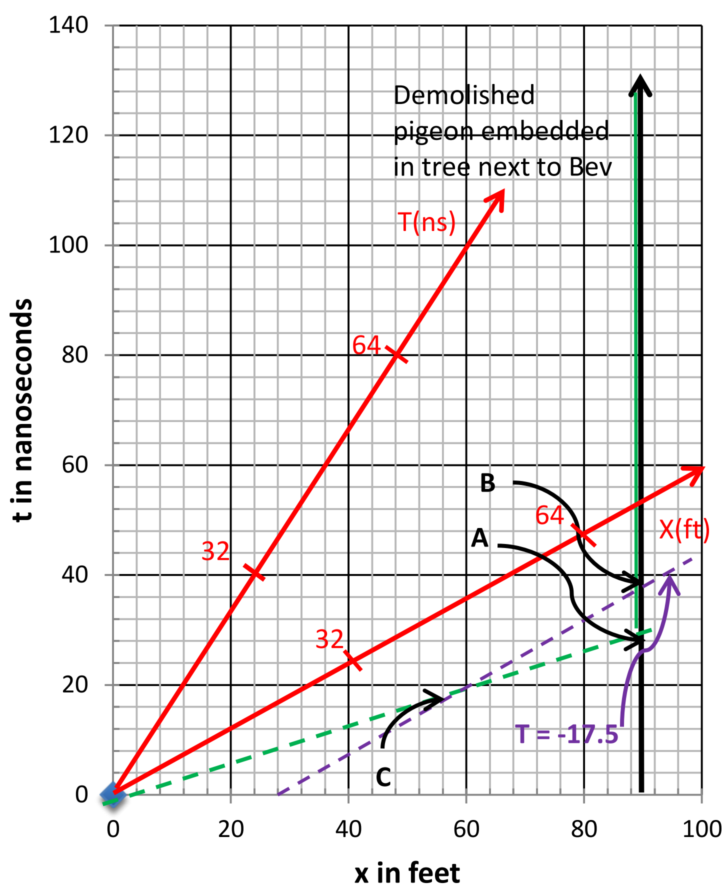
At breakfast, she told Bev about the dream. Bev suggested that it would be easy enough to draw the worldline of the superluminal pigeon on a spacetime diagram. So after eating, they sat down and drew a spacetime diagram depicting the flight of the pigeon from Anne to its untimely demise at the hands of the tree adjacent to Bev. That diagram is reproduced below as figure 6.5.



The flight of the pigeon is shown in green and appears perfectly reasonable to A & B. The pigeon flies at 3 ft/ns and crashes into the tree 30 ns after leaving Anne. They decide to show the diagram to C & D.

The first thing Chuck does is add the X and T-axes to the diagram for bus observers moving at 0.6 ft/ns. Those lines are shown in red on figure 6.6. Dean, looking at Chuck's spacetime graph, exclaims "The pigeon collided with the tree <u>before</u> Anne released it!" While Chuck, Anne, and Bev ponder what he means, Dean plugs the coordinates for the spacetime point where the pigeon hit the tree, point A on the graph, into the Lorentz equation for T and finds the pigeon hit the tree at T_A = -30 ns, 30 ns before Chuck saw Anne release the pigeon!

Figure 6.6



Dean shares his calculation of T_A with his friends. Anne and Bev point out that from their perspective the worldline of the superluminal pigeon traveling from Anne to Bev looks reasonable enough. It starts at x = 0 and t = 0 and arrives at x = 90 ft, at t = 30 ns.

Chuck and Dean, pointing to figure 6.6, totally disagree. Chuck says to Anne and Bev, "The bus observer at point A, X_A = 90 ft and T_A = -30 ns, sees the pigeon mysteriously appear out of thin air. Then the pigeon immediately starts to fly backwards towards the origin."

Question 6.1: Use the Lorentz equation to check that C & D have the correct bus coordinates for point A.

Dean adds the constant T = -17.5 ns line in purple to figure 6.6 and says, "Things get even more bizarre for Chuck and me. The two bus observers located at points B and C have watches reading identical times, T = -17.5 ns. The observer at C sees the pigeon flying backward toward the origin. Simultaneously, the observer at B sees pieces of the robotic pigeon scattered around the base of the tree! At T = 0, Chuck sees the pigeon arrive at the origin from two directions; flying toward him and being carried by Anne who, from Chuck's perspective, has not yet released the pigeon! At that the same time, the bus observer at X = 72 ft sees the smashed pigeon. But an instant after Chuck sees the two pigeons arrive at the origin, both pigeons are gone."

Question 6.2: Draw the constant T = 16 ns line on figure 6.6. According to bus observers, what is the X coordinate of the pigeon at that time?

Flying faster than light in the Earth frame violates cause and effect in the bus frame since there are a series of bus times where the pigeon <u>*is simultaneously scattered at the base of the tree and flying backward toward Chuck*</u>*. Any object moving faster than light in one reference frame will be seen to move backwards in time in other reference frames. Thus Special Relativity makes superluminal flight impossible unless we decide to not worry about cause and effect!*

Question 6.3: Sketch a spacetime graph from the perspective of Chuck and Dean that shows the worldline of the pigeon. Start with Anne carrying the pigeon toward the origin before she released it and end with the pigeon embedded in the tree. *(The answer to this question is at the very end of this chapter.)*

Question 6.4: Use the addition of velocity equation to find the velocity of the superluminal pigeon according to bus observers. *(If the algebra is done correctly, by sheer accident, the velocity measured by Chuck and Dean turns out to be -3 ft/ns, the opposite of the pigeon's velocity in the Earth frame!)*

Question 6.5: Imagine the flight of a slower superluminal pigeon flying at u = 2 ft/ns with respect to Earth. What do C & D measure for the velocity of this pigeon? *(This question shows that the answer to the previous question was mere coincidence.)*

Question 6.6: The denominator of the relativistic velocity addition equation is 1 + uV, where V is the velocity of Earth with respect to the bus, or $-\frac{3}{5}$ in our example. There is a particular pigeon speed with respect to Earth that makes the denominator zero. What is that velocity, u, for the pigeon?

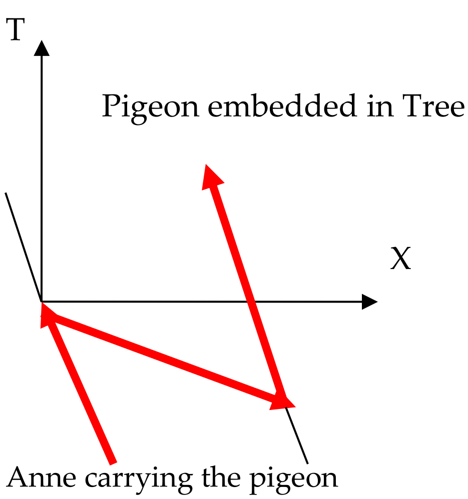
Note that if the pigeon flies a tiny bit slower than the speed found in question 6.5, the bus observers "see" the pigeon flying from left to right with an essentially infinite speed, the numerator, $\frac{5}{3} - \frac{3}{5}$*, divided by a positive number very, very close to zero is a very, very large positive number. On the other hand, if the pigeon is flying a tad faster than the speed found in question 6.5, the sign of the denominator switches from positive to negative. In this case, the bus people "see" the pigeon flying from right to left with an essentially infinite speed!*

The absurdity of these conclusions reinforces the idea that traveling faster than light violates the laws of physics as currently understood. The breakdown highlighted by question 6.6 occurs when the denominator of the velocity addition formula equals zero, $1 + uV = 0$, where u is the speed of the superluminal object with respect to Earth while V is the velocity of Earth with respect to the bus. Solving for V, $V = -1/u$.

This clearly demonstrates that for any object moving superluminally with respect to Earth, $u > 1$, there will be a physically realizable relative speed for the bus, $|V| < 1$, which makes the speed of the object with respect to the bus infinite.

The sketch below shows the path of the pigeon, heavy red line, through spacetime from the perspective of C & D. If we ignore the fact that time only goes in the forward direction, up on the spacetime graph, we could say that Anne carried the pigeon until she reached Chuck when she released it. At that point, it flew backwards in time from Chuck until its untimely demise at the hands of the tree traveling at $v = -0.6$ ft/ns through C & D's reference frame. After the collision, the pigeon continues to move with the tree.

The trouble with this reasonable sounding description of the pigeon's motion through spacetime time is that bus observers see events happen simultaneously along horizontal slices of time moving up the spacetime diagram. The sketch below clearly shows that the pigeon's earliest appearance in the bus frame is when Anne is seen carrying the pigeon. When $T = -30$ ns, there are now two pigeons; the pigeon being carried by Anne and the "same" pigeon suddenly appears by the tree. Each horizontal time slice from then until $T = 0$ ns, has the pigeon in three places at once; being carried by Anne, flying toward Chuck at the origin, and smashed into the tree! Time slices past $T = 0$ have a single pigeon stuck in the tree. So much for superluminal travel.



Chapter Seven: The Bus and the Garage

This chapter examines the famous problem of the "pole and the barn" with the super bus playing the role of the pole and the garage built by Anne and Bev the role of the barn. The primary goal of this chapter is to see how the tools and skills learned in the first six chapters can help to make sense of an apparently paradoxical situation.

While Chuck and Dean were vacationing in the super bus, Anne and Bev decided to surprise C & D by building a garage for the bus. Unfortunately, in their excitement to get the garage done before the bus returned, they built a 90 foot garage to house the 100 foot bus! Their error was apparent as soon as C & D returned from their vacation and pulled up alongside the garage. But it was still a very nice garage with sliding doors at both ends.

Dean pulled the first 90 feet of the bus into the garage and joined his friends outside. As the four of them walked around the garage examining the ten feet of bus sticking out, Bev made a curious observation. If the bus traveled toward the garage at 0.6 ft/ns, it would be only 80 feet long and could comfortably fit in the garage. On the other hand, because of the principle of relativity, C & D could just as legitimately assert that a runaway garage was approaching the stationary bus at v = -0.6 ft/ns! From their perspective, the onrushing 90 foot garage would be only 72 feet long, 28 feet shorter than their bus!

They agree that it seems strange that the bus could simultaneously fit and not fit in the garage. As experienced relativists, they decide to draw spacetime diagrams that faithfully mirrored the actual driving of the bus through the garage with open doors at both ends. As usual, the diagrams will be drawn with Chuck riding in the back of the bus while Dean drives. Anne will stand at the entrance of the garage while Bev is at the other end.

As they discuss the details of this *gedanken* experiment, it becomes clear there are four key events:

1. Event A is the spacetime point where the bus first enters the garage. This will also be the agreed upon origin on their spacetime diagrams. (Notice that this experiment begins when Dean is adjacent to Anne.)
2. Event B will be when the rear of the bus first enters the garage. (At this event, Chuck and Anne are adjacent.)

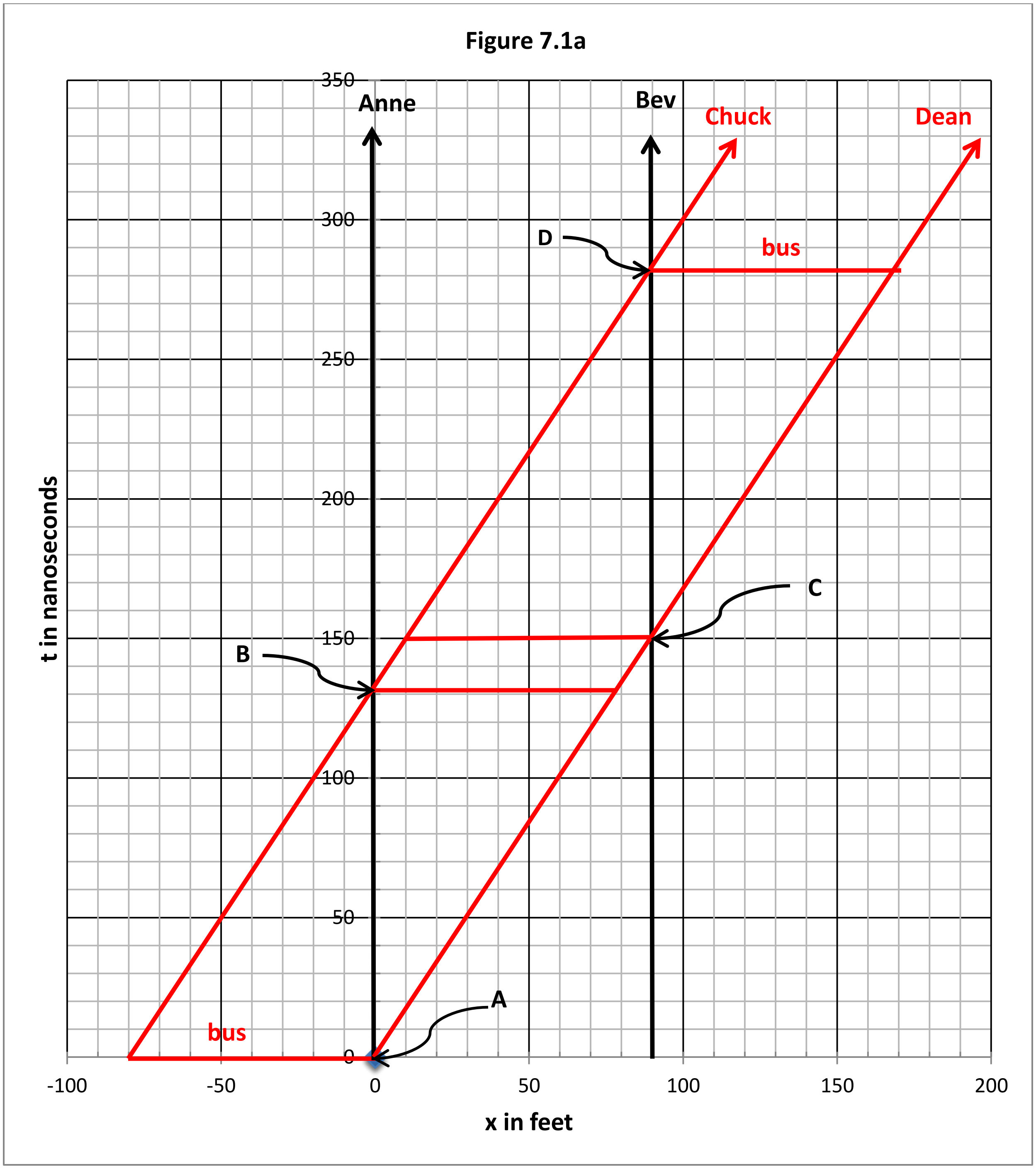
3. The bus begins to exit the garage at event C. (Dean and Bev are passing one another at event C.)
4. And for completeness, event D will be the place where the rear of the bus exits the garage. (Bev and Chuck are adjacent to one another.)

Each of the four experimenters is fixed in their respective reference frames. On the bus, Dean is at X = 0 feet and Chuck is at X = -100 feet. Chuck is to the left of the origin leading to a negative value for his location in the bus reference frame. On Earth, Anne is at x = 0 feet and Bev at x = 90 feet. And as usual, the watches are synchronized so that Anne and Dean pass one another at time zero in both frames.

Question 7.1: Use the information above and the equation, Time = $\frac{Distance}{Speed}$, to fill in the Earth and bus coordinates for events A, B, C, and D. *(For example, event B happens according to Anne & Bev at time t_B when the rear of the bus reaches the front of the garage. On the other hand, Chuck & Dean see the front of the garage reach the rear of the bus at T_B.)*

Event	Anne and Bev		Chuck and Dean	
	x	t	X	T
A				
B				
C				
D				

Anne and Bev grab a piece of graph paper and begin to plot the worldlines of the garage and bus. Chuck and Dean do the same. Bev yells over to Dean that they ought to include the location of the bus at the events labeled A, B, C and D. Dean agrees and suggests they represent the bus with heavy red lines. The finished spacetime diagrams are shown below, figures 7.1a and 7.1b.



The next two questions ask you to find the coordinates for events A, B, C, and D directly from the spacetime diagrams. If everything is done correctly, the answers in the three tables ought to agree reasonably well. But remember that oftentimes the coordinates read off a graph have to be estimated.

Question 7.2: Without using the table from question 7.1, fill in the table below by using the spacetime diagram drawn by Anne and Bev, figure, 7.1a. Your first step is to add the X-axis to the diagram. (Dean's worldline is the T-axis.) Be careful estimating the values of T_B and T_C. Note the box for T_D is excluded because that value lies outside the range of the graph. (Remember the scales on the x and t axes are different than the scales on the X and T axes. In figure 7.1a, each box on the t-axis corresponds to 10 ns while the intersection of the T-axis with each box corresponds to 8 ns.)

Event	Anne and Bev		Chuck and Dean	
	x	t	X	T
A				
B				
C				
D				

Question 7.3: Fill in the table below by using the spacetime diagram drawn by Chuck and Dean, figure, 7.1b, to find the coordinates of events A, B, C, and D. Start by adding the x-axis to the figure. (Anne's worldline is the t-axis.) Note that in this case it is t_D that lies off the graph. (In figure 7.1b, each box on the T-axis corresponds to 10 ns while the intersection of the t-axis with each box corresponds to 8 ns.)

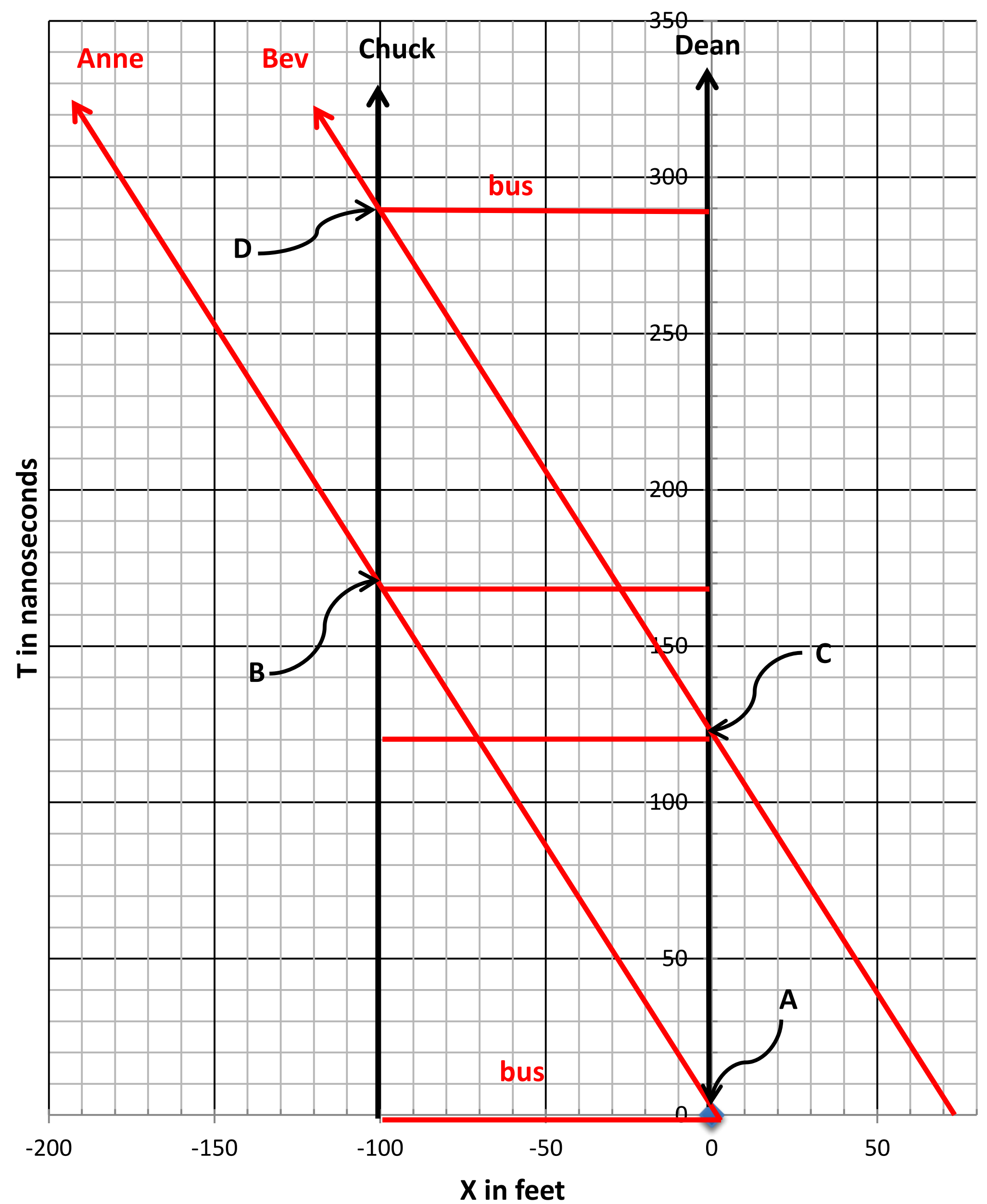
Event	Anne and Bev		Chuck and Dean	
	x	t	X	T
A				
B				
C				
D				

Chuck points out that it is clear that the bus, represented by the red horizontal lines at events A, B, C, and D, is never completely inside the garage which lies between the worldlines of Anne & Bev on figure 7.1b.

Anne on the other hand uses figure 7.1a as evidence that the bus was inside the garage between times t_B and t_C. Dean says, "That's crazy because the front of the bus left the garage, event C, before the back of the bus entered it, event B!"

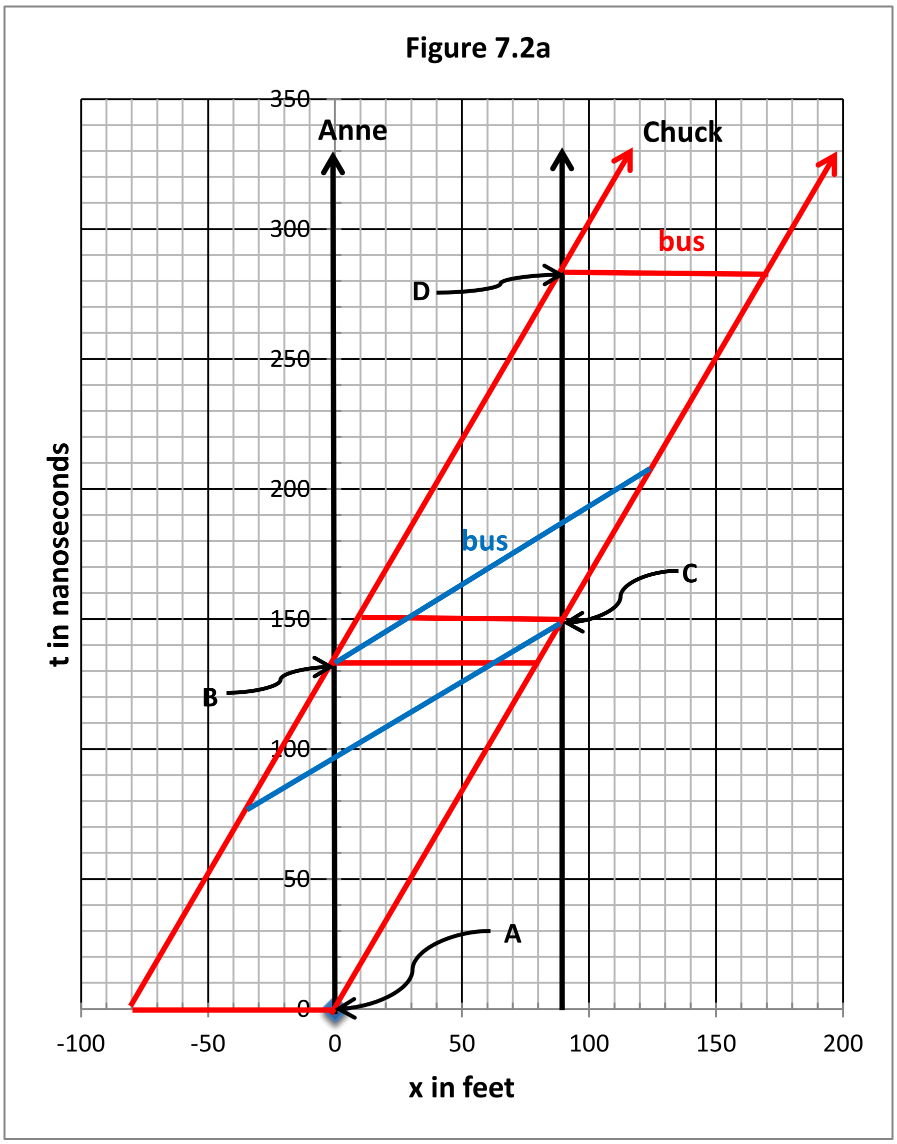
Anne shakes her head says to Dean, "Event B happened when the back of the bus, which is 80 feet long, reached the front of the garage. And since the garage is 90 feet long, the front of the bus was still 10 feet from the rear of the garage."

Figure 7.1b



The bus riders see this very differently. From their perspective, event B happened when the entrance of the garage traveled the length of the bus, 100 feet. They claim event C happen when the back of the 72 foot long garage reached Dean in the front of the bus. Clearly event C happened before the front of the garage had time to travel the 100 feet to reach Chuck at the back of the bus.

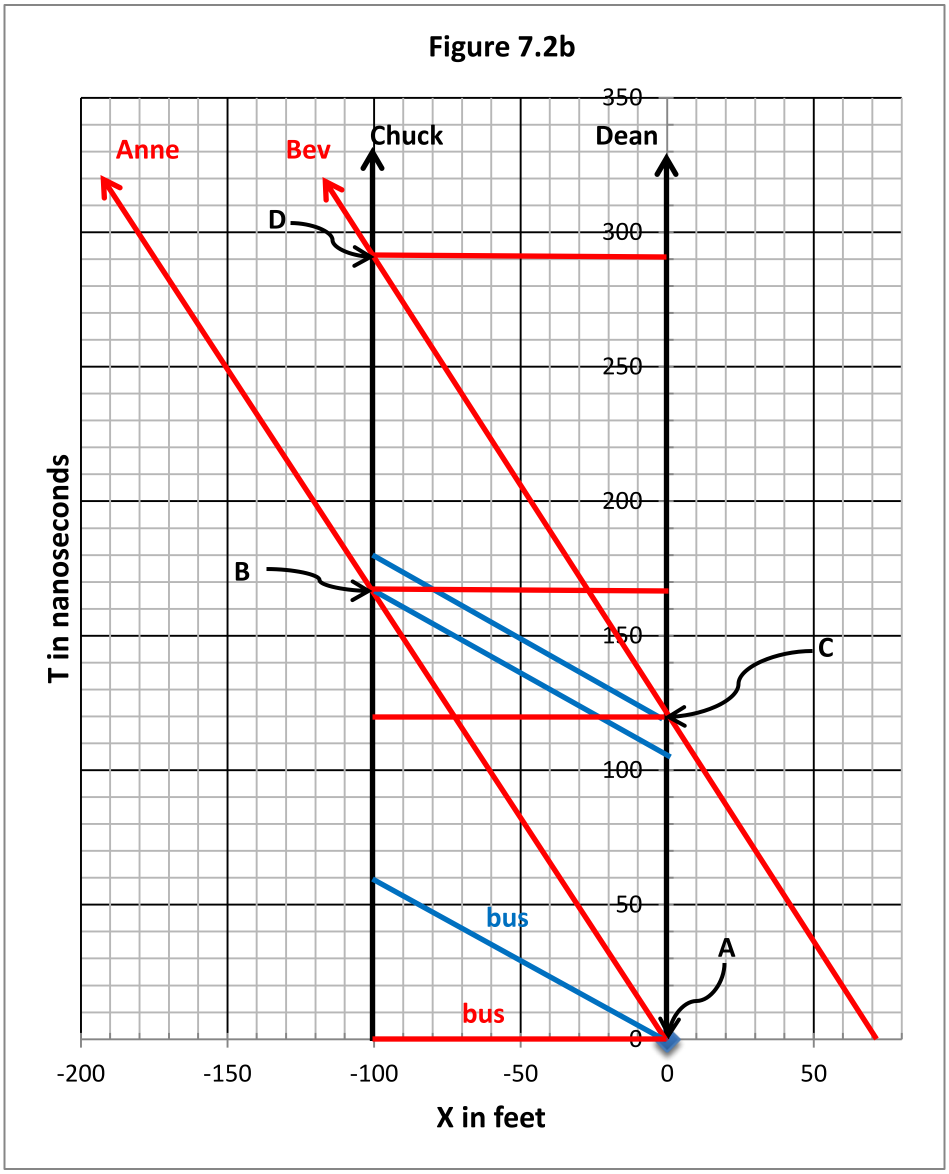
This conflict arises because Anne & Bev disagree with Chuck & Dean about the spacetime location of the bus at the events A, B, C, and D. Anne captured the essence of this problem in figure 5.6. Anne & Bev see event B happen, the back of the bus enter the garage, before C, the front of the bus leaving the garage, while Chuck and Dean see just the opposite, event C preceding event B. No amount of arguing can resolve this difference. Anne and Bev rightly argue that the bus is inside the garage for a whole 50/3 nanoseconds while Chuck and Dean correctly assert that the bus is never completely in garage.



The differences in slope between lines of constant t and T on a given spacetime diagram is a pictorial representation that bus and Earth observers fundamentally disagree about the meaning of simultaneous events. Dean and Bev quickly amend the two figures to include the location of the bus at times T_B and T_C on figure 7.1a and times t_B and t_C on figure 7.1b. The heavy blue lines represent the bus at those times.

Now Anne and Bev clearly see that according to Chuck and Dean the bus is never completely inside the garage. It is also obvious from the picture that the constant time line representing T_B hits the T-axis, Dean's worldline, at a later time than T_C.

When Chuck and Dean examine figure 7.2b, they also get a deeper sense of how their view of the bus driving through the garage is fundamentally different than the view of Anne & Bev. According to Anne & Bev, the spacetime location of the bus at events B and C are represented by the slanted blue lines, one slanting downward from event B and other upward from event C. From figure 3.2b, it is clear that Anne & Bev rightly concluded that from their perspective event B happened before event C.



Spend some time studying all the information contained in these two spacetime diagrams, figures 7.2a and 7.2b. Like many of the supposed paradoxes in Special Relativity, this one is due to the fact that simultaneous events in one frame are not simultaneous in a different frame moving with respect to the first.

Anne is still puzzled because she can picture the bus inside the garage for $\frac{50}{3}$ ns.

She wonders why she and Bev can't close the doors of the garage during the $\frac{50}{3}$ ns the bus is inside. That surely would convince C & D that the bus was actually inside the garage for a time. The next day, Anne, who has secretly coveted the super bus, makes a proposal to Chuck & Dean. Bev, who has been enlisted to help Anne, is a little dubious about the proposal but agreed to help her.

Anne's Proposal

When Chuck & Dean arrive at the garage the next day, Anne makes her proposal. "If I can unequivocally demonstrate that the super bus fits in the garage, Bev and I get to keep it. On the other hand, if my demonstration fails to convince you, Bev and I will lengthen the garage to 120 feet and upgrade it by adding an automatic bus washing and waxing machine."

Dean and Chuck find the idea of having a super garage for their bus enticing but are nervous about the possibility of losing the bus to A & B. They go off for a while to study the spacetime diagrams depicting the trip of the bus through the garage. They finally convince themselves that it is impossible for Anne to unequivocally prove that the bus is completely inside the garage at any time during the experiment. So they agree to accept Anne's proposal and drive the bus through the garage.

While the bus is getting ready to drive back toward them, unbeknownst to C & D Bev closes the back door of the garage. Then Anne shuts off all the lights in the garage so that Dean won't notice the closed door at the back until the last possible moment. Anne's plan is simple as well as diabolical. As soon as the rear of the bus enters the garage, she will close the front door of the garage. Once that door is closed, the bus is completely inside the garage because the rear door was closed before the experiment began! The result of her clever scheme is that she and Bev will become the proud owners of the super bus.

Some important features of the super bus have not been mentioned previously because they were not relevant. The first is that Dean has a brake pedal that can instantly stop the front of the bus and Chuck has an equivalent pedal that can instantly stop the rear of the bus. If you ever traveled in a super bus, you know these safety features are necessary. The second feature is that Dean and Chuck communicate via radio while riding in the bus. The reason for this ought to be obvious. The speed of sound, 1000 ft/s, is way too slow.

For example, if Dean sees a nice restaurant 1000 feet in front of the bus and asks Chuck whether or not he wants to stop, traveling at the speed of sound the question takes about a tenth of a second to travel the 100 feet between them and another tenth for Chuck's answer to return to Dean. During 0.2 seconds, the bus has traveled 120 million feet and the restaurant is a distant memory! Communicating by radio waves, which travel at the speed of light, cuts the time down to a mere 200 ns. During that time, the bus travels only 120 feet, giving them plenty of time to stop for a bite at the restaurant.

Anne eagerly waits for the bus to come zipping down the road. Bev is less eager because she has more confidence in the logic of the spacetime diagrams than she does in Anne's scheme. From her perspective, the bus won't be so easy to catch inside the garage.

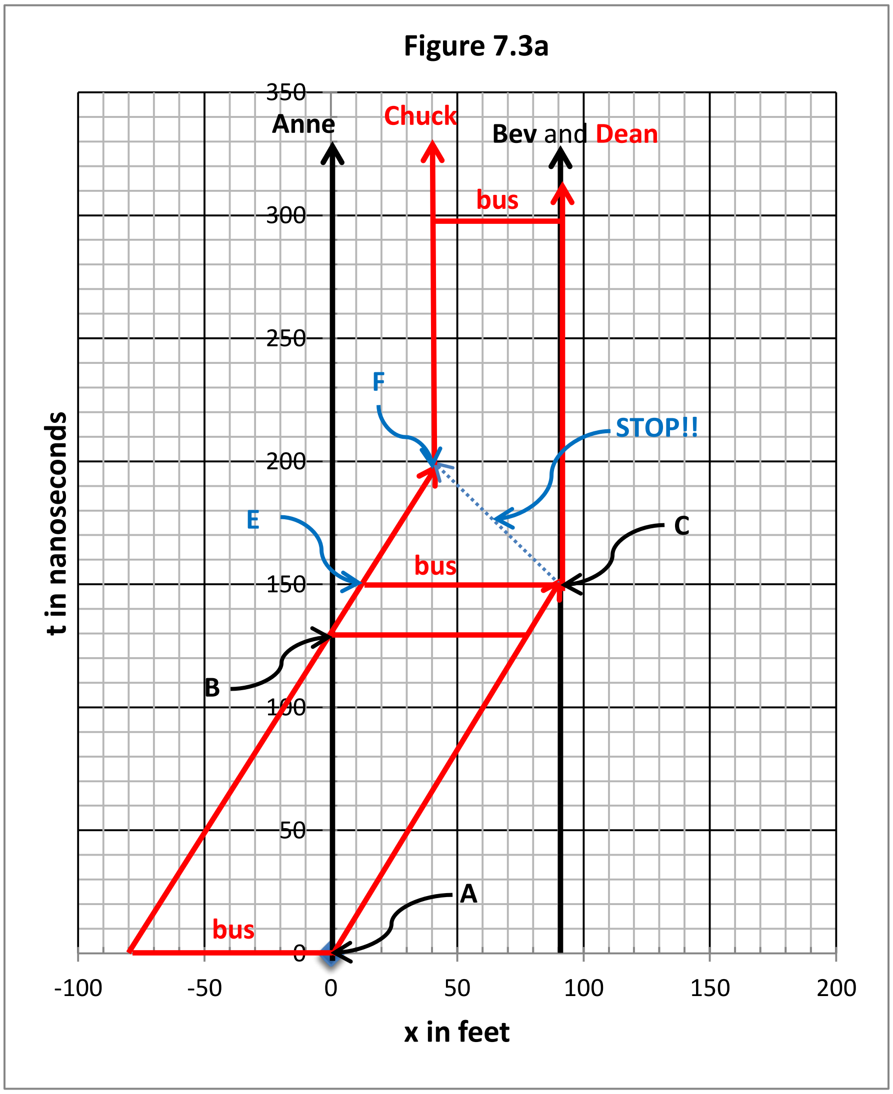
The Collision between Theory and Reality

Anne sees the bus approaching rapidly. As soon as Chuck passes her, she shuts the door of the garage. Meanwhile at that same instant; Bev sees the front of the bus 10 feet from the closed door at the rear of the garage. Dean does not notice the closed door rushing toward him until it is right at the front of the bus. He instantly applies the brake stopping the bus just as the front bumper touches the rear garage door.

The front of the bus has stopped. What about the rear of the bus? This leads to a strange question. How does the rear of the bus know that the front of the bus has stopped? The simple answer is that the front of the bus sends a message to the rear of the bus, "I've stopped and you ought to stop also!" Usually the message is sent from one atom to its neighboring atom along the length of the bus. This type of message sent from atom to atom through a material travels at the speed of sound in that particular material. Although the speed of sound varies from one material to the next, it is insignificant when compared to the speed of light. Consequently, at the same instant that Dean slammed the brake at the front of the bus, he sent a radio message to Chuck to stop the rear of the bus. This light speed message sent to Chuck from Dean is the <u>fastest possible way for the rear of the bus to learn that the front of the bus has stopped</u> and arrives long before the message being passed along atom by atom from the front to the rear of the bus!

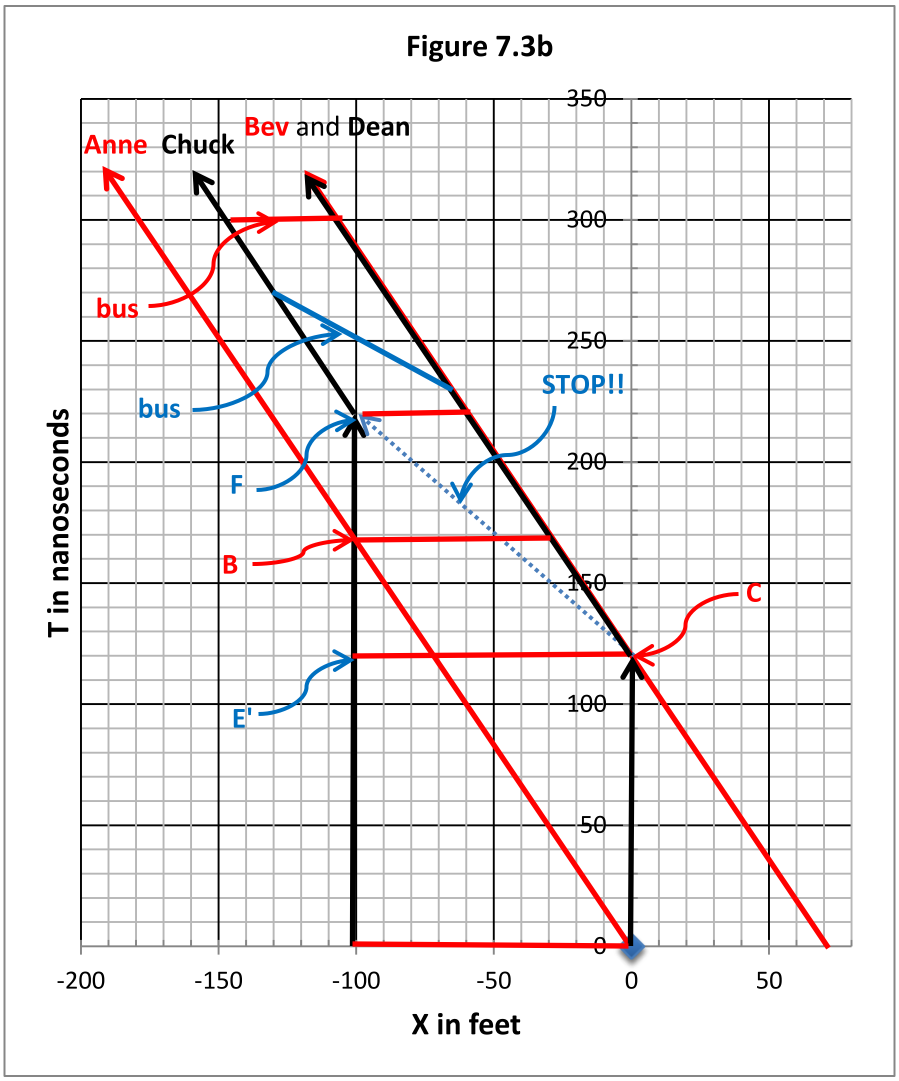
The events enfold in the following way for A & B. When the bus reaches the closed end of the garage, point C in figure 7.3a, Dean sends the frantic message to Chuck to stop. The front of the bus is now stationary and its worldline becomes vertical at point C. According to A & B, the back of the bus is 80 feet from Dean when the message is sent.

Chuck is moving toward the message at $\frac{3}{5}$ ft/ns while the message moves towards him at 1 ft/ns. It takes 50 ns for the message to reach him, point F on the spacetime diagram. During that time, the rear of the bus traveled 30 feet, from point E to point F. Chuck immediately stops the bus causing the worldline of the rear of the bus to become vertical at point F. The now stationary bus is only 50 feet long and uncomfortably fits in the garage! The spacetime diagram in figure 7.3a shows the worldlines representing the bus that ends up at rest completely in the garage!



Dean and Chuck describe the capture of the bus by the runaway garage somewhat differently but with the same outcome. At point C on figure 7.3b, Dean has stopped the front of the bus so that it now moves in tandem with Bev at the rear of the garage. From point C forward, the worldlines of the front of the bus and Bev are identical. At that instant in the bus frame, Chuck is still outside the garage at point E′. It is important to keep in mind that E′ is a different place in spacetime than point E. The first, E′, represents Chuck's position at the instant Dean sent the message to stop from the perspective of the bus frame. Point E represents Chuck's location at the instant Dean sends the message according to Earth-based observers. Chuck <u>receives</u> the message at point F, the same spacetime event in both frames, when the rear of the bus begins to move with the same velocity as the front of the bus. Figure 7.3b shows this sequence of events and includes the captured bus at two different times, the two short red horizontal lines. The bus trapped in the garage moves through the original bus frame with velocity $-\frac{3}{5}$ ft/ns. The tilted blue line represents the position of the captured bus at a single time in the Earth frame.

It takes more imagination to decipher figure 7.3b than figure 7.3a. Picture another group of observers traveling toward the garage on a flatbed truck adjacent to the bus. These observers will drive by the garage and continue to take data after the capture of the bus. From their perspective, the garage is rushing toward them and the bus. After the bus gets caught in the onrushing garage, these observers see the garage and captured bus as stationary in the Earth frame which is rushing by them at $-\frac{3}{5}$ *ft/ns. The observers on the flatbed truck are the ones that can follow the fate of the super bus during its fatal interaction with the marauding garage, figure 7.3b. These flatbed observers are the ones that see the captured bus as the short red horizontal lines on figure 7.3b.*

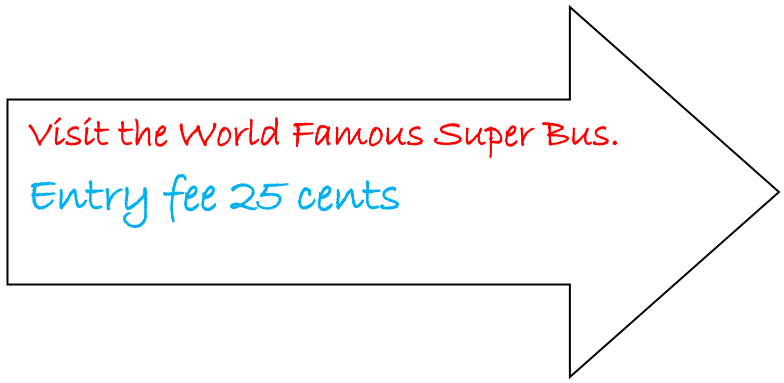


Question 7.4: The observers on the flatbed truck measure the length of the captured bus by examining figure 7.3b. Is the length they measure consistent with the 50 foot length reported by A & B for the stopped bus inside the garage?

R.I.P. Super Bus

Chuck and Dean crawl out of the remnants of the super bus, shaken but otherwise unhurt. Bev stares at the wreckage in awe and waits for someone to say something. Anne meekly breaks the ice by pointing out that she won the bet, the super bus is completely inside the garage. No one disagrees with that statement. Dean looks at Chuck, who nods, and he hands the keys to the bus over to Anne. The four friends dejectedly walk out of the garage. Bev says the obvious, "We won't be doing any more experiments with that super bus." Anne tries to cheer them up by pointing out that super bus served them well in their explorations of space and time.

The next day, Dean arrives at the garage with a big sign reproduced below:



Chuck, Anne, and Bev begin to smile at Dean's sign, and then all four start laughing.

Luckily, the super bus has served its purpose and won't be needed again. The remainder of the book will deal with distances and times substantially larger than the feet and nanoseconds used up to this point. In fact, Chuck and Dean end up using the insurance money from the wrecked bus to buy a space cruiser. Their new goal is to visit some nearby star systems.

Chapter Eight: The Solar System and Beyond

Our solar system resides in a rather ordinary spiral galaxy, the Milky Way. The sun is one of about 400 billion stars in the Milky Way. And the Milky Way, unsurprisingly, has to be very big to be home to so many stars. The usual unit used to describe the size of our galaxy is the light-year; the distance light travels in a year.

Question 8.1: What is the velocity of light in units of light-year/year? (The answer is as simple as it seems!)

Question 8.2: How many seconds are there in a year?

If you use the answer to question 8.2 and the velocity of light in meters per second, 3×10^8 m/s, it turns out that light travels about 9.5×10^{15} meters in one year. That is almost 10,000 trillion meters, a number that is essentially meaningless because it is ungraspable. The number of miles in a light-year is a little less daunting but still a mind numbing 5.9 trillion miles.

It seems a lot easier to picture a light-year (l-yr) as the distance light travels in one year without trying to describe it as this many meters or that many miles. For the next two chapters, distances will be measured in light years and time in years which conveniently make the speed of light 1! (How far does light travel in one year? One light-year. How long did it take light to travel that distance? One year. Therefore c = 1 l-yr/yr)

This makes the slope of the worldline of a light flash on a spacetime diagram with units of light-years replacing feet and years replacing nanoseconds plus one for light traveling to the right and minus one for light traveling to the left.

The Milky Way is roughly the shape of a disk with a diameter of 100,000 l-yrs and a thickness of about a 1,000 l-yrs. As immense as this is, the Milky Way is just one of something like 200 billion galaxies in the known universe. On average, each of those galaxies has about 200 billion stars. The upshot is that the universe is large and contains a dizzying amount of stars and galaxies.

The nearest galaxy that is similar to the Milky Way, a spiral galaxy with hundreds of billions of stars, is 2.5 million light years away. That neighboring galaxy is Andromeda. It is a little larger than the Milky Way and contains something like 1000 billion or one trillion stars.

The purpose for presenting this paralyzing array of numerical data is to give some notion of the immensity of space. It also gives some context for wondering about the possibility of human *travel to other stars in our galaxy or even venturing further from home to visit a neighboring galaxy. The rules governing space and time discovered earlier in the book give us the tools to make reasonable estimates of what is possible with respect to interstellar travel and what is impossible if we limit ourselves to the known laws of physics.*

Synchronizing Watches Separated by Large Distances

Up to now the method described in Chapter Three for synchronizing a pair of watches has been perfectly workable since Anne and Bev or Chuck and Dean were never very far apart. Therefore it was easy for them to meet, start their watches, and stroll off to their observing stations.

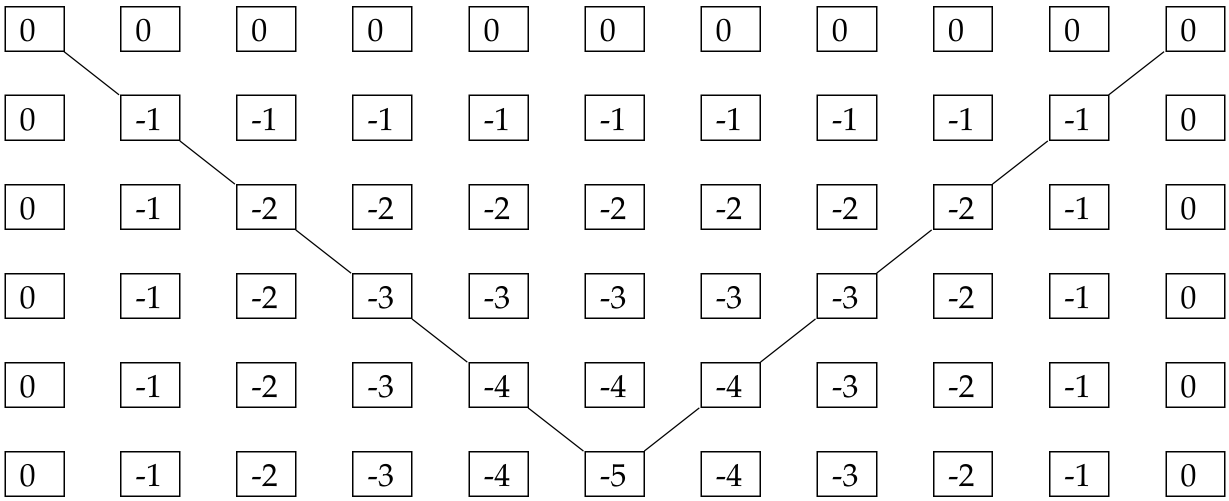
Now we will be dealing with observers who are light years apart but also need to have synchronized watches. Though we can imagine these observers meeting, starting their watches, and then traveling back to their outposts, a different more expedient plan for synchronizing watches is described below.

Suppose we have observers stationed one light-year apart located along the x-axis from x = -5 l-yrs to +5 l-yrs. The two watches that are 5 light-years from the origin are set to read 0, the pair 4 light-years from the origin are set to -1 year, and so on with the watch at the origin reading -5 years. When all the pre-set watches are in place, a light flash is sent out in both directions from the origin. That flash immediately starts the watch at the origin and then starts the watches at the different outposts as it passes them. When the watch at the origin reaches t = 0, each of the other ten watches in the remote locations will have been activated by the flash. At that instant, all eleven watches will read t = 0 and will be running synchronously. This procedure works perfectly for any number of observers in any given inertial reference frame. It is the gold standard method for synchronizing the watches of observers spread throughout a given reference frame.

Figure 8.1 graphically represents this procedure. Each box represents a watch and the watches are separated by one light-year. The two diagonal lines represent the light flashes traveling to the left and right that activate each watch. Note that the watches inside or between the light flashes have been activated and are running while those outside the flashes have not been activated. In the bottom row, none of the watches are ticking. As time moves forward, upward with one year between rows, more and more of the watches are activated. In the top row, all eleven watches are running and simultaneously read zero.

From this point forward in the book, all the observers in a given reference frame remain at fixed locations and with synchronized watches. These observers are the data collectors for events happening in spacetime.

Figure 8.1



Chapter Nine: Chuck and Dean Travel to Alpha Centauri

Our friends were sitting around discussing the famous "twin paradox" which involves a pair of twins, for example Sally and Sam. Sally ventures forth in a fast moving space cruiser and upon her return to Earth, discovers that she is younger than her stay at home sibling, Sam. Anne points out that this is reasonable because from Sam's perspective, Sally was moving at a large velocity causing her watch to run slow. Therefore when she returned, less time had ticked off her watch than his. Dean objects and counters that the Principle of Relativity ought to mean that from the perspective of Sally, it is Sam's clock that runs slow.

Chuck is ready to take a serious break from these discussions and could use a long vacation. So he is not paying much attention to the conversation. Dean suggests to Chuck that they use their new space cruiser, bought with the insurance money from the ill-fated experiment done to test whether or not the super bus fit into the garage, to both test the twin paradox and give Chuck an extended vacation. Chuck finds Dean's idea enticing and readily agrees. Anne and Bev are perfectly content to remain home and be the "Sam" of the twin paradox.

After checking a star chart, they decide to visit the Alpha Centauri system, a group of 3 stars about 4 light-years from Earth. Dean quickly calculates that if he and Chuck travel at 3/5 the speed of light, a roundtrip to Alpha Centauri will take more than 13 years! Chuck ignores him and begins to make a list of the provisions needed for the trip, including lots of books, DVDs, and video games. Dean reminds Chuck to make sure he plans on having plenty of food!

The four friends get together on New Year's Eve, the day before C & D will depart. Sitting around sipping champagne, they decide to keep in contact by sending annual greetings each January 1. A & B will keep the messages they receive and C & D will keep the messages they receive during the trip. Dean has prevailed upon Chuck to spend as little time as possible at the Alpha Centauri outpost. Consequently, their plan is to have a similar space cruiser fueled and ready for takeoff as soon as they arrive. Chuck figures it will take less than an hour to sign the necessary papers to trade their space cruiser in for the new one poised for the return flight.

Question 9.1: According to Anne and Bev, how long will it take Chuck and Dean to travel the 4 light-yrs to Alpha Centauri?

Question 9.2: When C & D arrive at Alpha Centauri, they see that according to Earth-frame observers, their trip took $6\frac{2}{3}$ years. But they disagree and claim that the trip actually took them how many years?

May 1, Thirteen Years Later

Anne and Bev nervously await the return of Chuck and Dean. Anne has sent 13 annual messages to Chuck while she and Bev have received only10 messages from them. Suddenly the ship appears overhead and makes a perfect landing at the spaceport. Peering through the windows of the terminal, A & B spot the returning travelers and notice that both are wearing serious looking neck braces! After a cursory stop at customs, C & D are reunited with their Earth-bound friends. On the drive home, A & B, ask about the neck braces, C & D tell their friends that every time the space cruiser started or stopped, they suffered a serious case of whiplash! Bev finds that curious but the topic is dropped. During the drive to take C & D home, the four friends decide to meet the next day with spacetime diagrams summarizing the trip to Alpha Centauri. The diagrams will be drawn from the perspective of a stationary Earth and will include the worldlines of the messages sent by A & B to the space cruiser and the messages C & D sent to Earth.

The following day, after exchanging some pleasantries, Anne quickly steers the conversation to the two spacetime graphs, figures 9.1a and 9.1b. She stares, shakes her head in wonder, and says, "We sent 13 annual messages to you and you sent 10 annual messages to us. Therefore, assuming everyone's watches ran correctly, the two of you aged three years less than we did back on Earth. Although the conclusion seems strange, it seems incontrovertible since Bev and I received only 10 messages during your roundtrip to Alpha Centauri. On the other hand, you received all 13 messages we sent. So there is no disagreement about the fact that a different amount of time passed on Earth for Bev and me than passed for you during the trip!"

The spacetime diagrams in figures 9.1a and 9.1b makes it appear perfectly reasonable that the space travelers age less than their stay at home friends.

Dean asks, "What happened to the Principle of Relativity?" Bev responds to Dean, "During the trip to Alpha Centauri, Anne and I saw the cruiser moving off at $\frac{3}{5}$ ft/ns and received one message for each two sent. The messages you sent reached Earth at half the frequency of the messages we sent to you. Meanwhile, you saw Earth receding at $\frac{3}{5}$ ft/ns and received messages from Earth at half the rate that you were sending them. Therefore during the trip to Alpha Centauri the Principle of Relativity appeared to apply perfectly."

Chuck picks up on that argument by pointing out to Bev that during the return trip, when he and Dean saw Earth rushing toward them at $\frac{3}{5}$ ft/ns, the frequency of the arriving messages was twice as high as the frequency of the sent messages. Bev says, "Anne and I saw exactly the same change in received to sent messages, two arrived for each one sent to the returning space cruiser.

The change in frequency caused by the relative motion of the source and observer which shows up here when C & D switch directions at Alpha Centauri is called the ***Doppler Effect****. When the source and observer are moving apart, the received frequency is lower than the sent frequency. On the other hand, when the source and receiver are moving toward one another, the received frequency is higher than the sent frequency. A derivation of the formula relating the sent to the received frequencies is presented at the end of this chapter.*

So Dean asks again, "If Chuck and I return about 3 years younger than you guys who stayed on Earth, doesn't that violate the Principle of Relativity?" Anne answers this time, "Dean, while the ship was cruising at constant velocity, the Principle of Relativity worked perfectly. There was no disagreement about the affect motion had on the rates of sent versus received messages. But, pointing to the neck braces worn by her friends, "That symmetry was broken when the space cruiser suddenly stopped at Alpha Centauri causing you to suffer neck injuries. Unfortunately for you, the same thing happened when you left Alpha Centauri to return to Earth. Meanwhile, Bev and I suffered no ill effects from your U-turn at Alpha Centauri!" "Ha, ha," says Bev.

Anne ignores Bev and continues. "A space traveler has to turn around to return home. During the turn around, the traveler feels the ship decelerate and then accelerate again. These are real measurable effects that act on the traveler but not on his stay at home friends on Earth. Consequently the Earth and space cruiser reference frames are not equivalent during the time the cruiser is changing velocity. This difference means that the Principle of Relativity does not apply during the time the ship is changing velocities. That break in the symmetry between you and Chuck and Bev and me accounts for the fact that you guys really did age less than Bev and I who remained on Earth."

Figure 9.1a

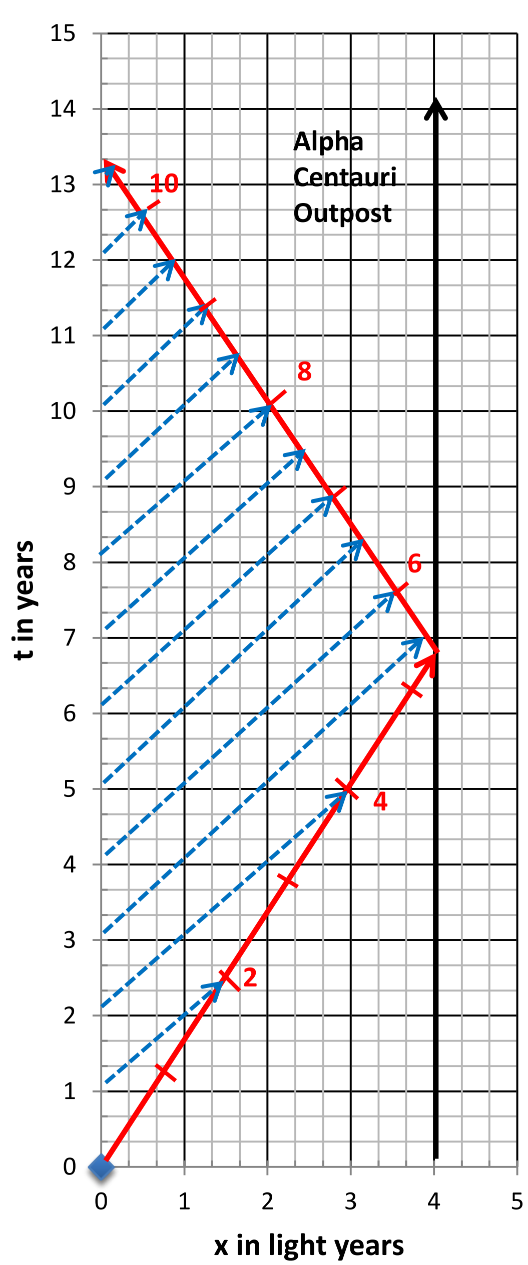
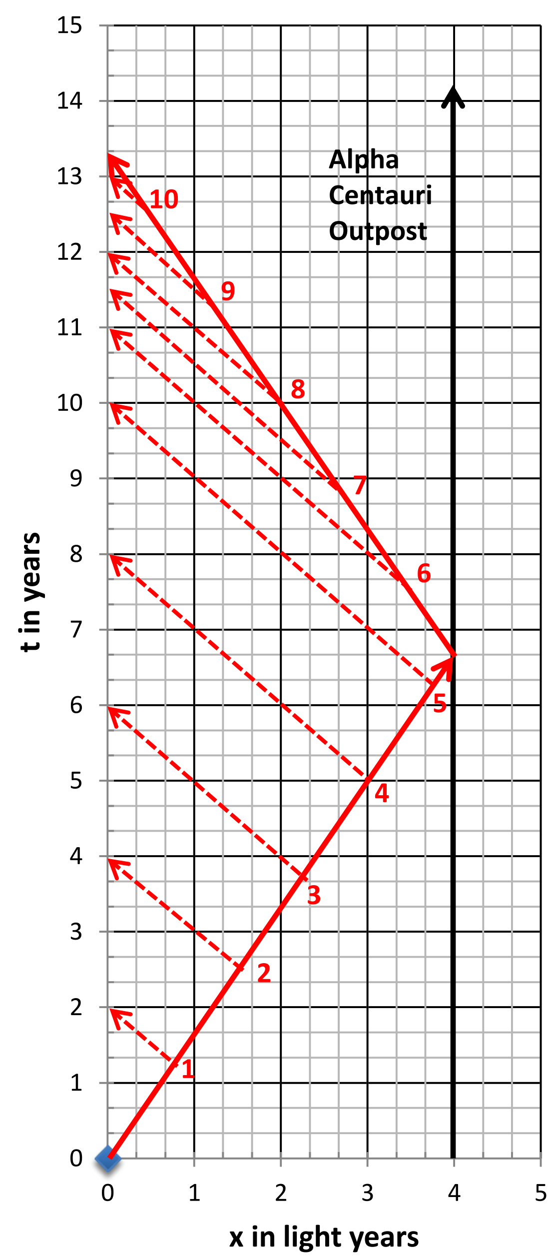


Figure 9.1b



Question 9.3: Make sure your answers to questions 9.1 and 9.2 are consistent with figures 9.1a and 9.1b.

Chuck and Dean try to nod in agreement but their heads can only move a limited amount because of the neck braces. Dean concedes, "Our injuries clearly confirm your clever analysis and shows why the so-called `Twin Paradox' is not really a paradox but instead a misunderstanding of the Principle of Relativity."

Now Anne points to the spacetime diagrams and speaking to Dean, says, "Everything that we have learned about space and time came about by comparing observers in two different reference frames that were moving with a constant velocity with respect to one another. When we drew the worldline for the space cruiser that you and Chuck traveled in, we forgot to consider the question of how the cruiser went from sitting on Earth to moving at $\frac{3}{5}$ ft/ns. And that drastic change in speed happened three more times before we met up again after your trip, twice at Alpha Centauri and a last time when you arrived back to Earth."

At Alpha Centauri, Chuck and Dean quickly changed space cruisers and reversed their velocity from v = +0.6 to -0.6 ft/ns. On the spacetime graph, that velocity change happened very abruptly when C & D jumped from the outgoing to incoming space cruiser. No wonder they suffered whiplash injuries! But their unfortunate injuries highlighted the distinction between the reference frame of the travelers and that of the stay at home observers.

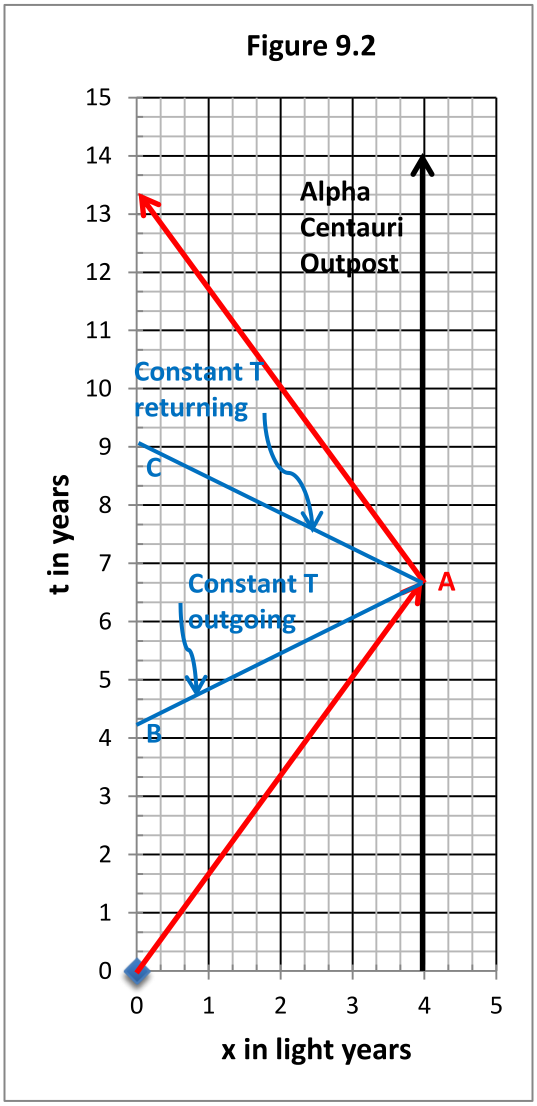
Chuck & Dean look at one another and then at their two friends with expressions that make it clear that they are not consoled by the fact that their injuries helped resolve the Twin Paradox. But they are happy to be back on Earth and finished with space travel and they will be even happier when the neck braces are removed for good!

The next chapter will show how to design a rocket trip that eliminates the disconcerting abrupt changes in velocity that played havoc with C & D. But for now, the discussion of the classic twin paradox continues.

Chuck is still curious about how the discrepancy between received and sent messages arose. He points out that the symmetry between Dean and him, the travelers, and A & B, their stay-at-home friends, was broken when he and Dean turned around at Alpha Centauri. Chuck asks no one in particular, "But how does the spacetime diagram know that the symmetry was broken when we turned around?"

Bev stares at the diagrams and observes that when C & D changed rockets they also changed reference frames. Bev says to her fellow spaceketeers, "During the trip out C & D were in a reference frame moving at $+\frac{3}{5}$ ft/ns with respect to Earth but on the return trip their velocity with respect to Earth was $-\frac{3}{5}$ ft/ns." Three blank pairs of eyes watched and waited for her to continue.

She quickly draws a new spacetime diagram, figure 9.2. Then she shows her friends the diagram and says, "Point A is the spacetime location of C & D when they made the switch from outgoing, v = $+\frac{3}{5}$ ft/ns to returning, v = $-\frac{3}{5}$ ft/ns, rocket cruiser. The two blue lines passing through that spacetime point represent the two constant time lines; one from the perspective of the outgoing rocket and the other from the returning rocket's perspective. From equation 4.11b, the slopes of those constant T lines are just $+\frac{3}{5}$ and $-\frac{3}{5}$, respectively. Knowing those slopes, I was able to add the two constant T lines to the spacetime graph. Those constant T lines intersect with Anne & Bev's worldline at B and C respectively"



Chuck some quick calculating and jumps in and replaces Bev as the speaker, "Point B in figure 9.2 marks the intersection of the constant T = 5⅓ year line with the x = 0 line, and represents the location of an observer in the outgoing rocket frame who is just passing Earth at the instant Dean and I arrived at Alpha Centauri, point A. That observer at B is in the same reference frame as Dean and me and has a watch that reads 5⅓ years."

Before anyone else can speak, Chuck continues, "On the other hand, the Earth observer at B has a watch reading $4\frac{4}{15}$ years. The observers in the outgoing rocket frame conclude that the Earth watch at x = 0 ticked off less time as <u>it moved through their rocket frame.</u>"

Bev points out, "But the Earth observer at A has a watch that reads 6⅔ years. She concludes that you and Dean had watches that ticked off less time than her Earth-frame watch during your trip to Alpha Centauri." At this point, all four friends exclaim together, "Observers in each frame see the watches moving through their frame as running slow, conclusions that are consistent with the Principle of Relativity."

The four friends feel confident that they understand why the travelers record less time on their watches than the stay-at-home people. But the neck braces on C & D also remind them that instantaneous changes in velocity are not suitable for real space travel, especially travel involving humans. Therefore they decide to meet in a week, after C & D have a chance to recuperate more fully, to think about how to plan a trip into space that is less stressful on vertebra!

The various watch readings in the last few paragraphs were calculated using the equation $d = vt$ and the effect motion has on watches and rulers. The next few questions give you the opportunity to check those. The answers can also be found by using the Lorentz equations. In either case, the algebraic answers ought to be consistent with the answers from figure 9.2.

Question 9.4: What is the reading on the watch of the outgoing rocket observer at point B? Call this time T_B.

Question 9.5: What is the reading on Anne's Earth watch at point B? This is t_B.

Question 9.6: Anne carried her watch through the outgoing rocket frame from the origin to point B. As the moving watch, time t_B ought to be only 80% as large as T_B. Do your answers for 9.4 and 9.5 agree with that conclusion?

Question 9.7: According to Anne & Bev, exactly how much time passed during Chuck & Dean's roundtrip to Alpha Centauri? Call this t_{trip}.

Question 9.8: According to C & D, exactly how long did their roundtrip take trip? This is T_{trip}.

Although the trip included some strange happenings during the turnaround at Alpha Centauri, there is a simple relationship between t_{trip} and T_{trip}, namely $T_{trip} = \sqrt{1-v^2}\, t_{trip}$. For the example in this chapter, the difference, $t_{trip} - T_{trip} = 2.67$ years. But the duration of the rocket trip according to the travelers can be made arbitrarily small by having the rocket travel at speeds closer and closer to the speed of light. For example, if the rocket traveled at $v = 0.99$, the shrinkage factor equals $\sqrt{1-v^2} = 0.14 \cong \frac{1}{7}$. At that speed, the travelers would age only one year for every seven that passed on Earth.

Question 9.9: Are your answers to 9.7 and 9.8 consistent with $T_{trip} = \sqrt{1-v^2}\ t_{trip}$?

In effect, Chuck & Dean not only made a trip to Alpha Centauri but they are also made a trip into the Earth's future since more Earth time ticked off than C & D time!

Question 9.10: Aliens, from some distant star, visit Earth periodically to check up on us Earthlings. Assume the alien rocket cruises at a constant speed of 0.99999999 ft/ns, eight 9's, and they conveniently keep track of time on their rocket in Earth years. If they return to Earth at intervals of one rocket year, how many years pass on Earth between their visits?

Using the Lorentz Equations

Finding T_A and t_B using the Lorentz equations is straightforward. For the trip to Alpha Centauri, the Lorentz equations are exactly the same as those used to answer part i) of Experiment II in Chapter Six.

$X = (5/4)\ x - (3/4)\ t$ and $T = (5/4)\ t - (3/4)\ x$. (9.1a and 9.1b)

From the perspective of A & B, it took $6\frac{2}{3}$ years for C & D to travel 4 lt-yrs at a speed of 0.6. So at point A, $x_A = 4$ lt-yrs and $t_A = 6\frac{2}{3}$ years. Substituting those values into equations 9.1a and 9.1b gives,

$X_A = 0$ and $T_A = 5\frac{1}{3}$ year, the expected results. To find t_B, note that at that spacetime point $x_B = 0$ and $T_B = 5\frac{1}{3}$ years. Equation 9.1b can be used to find t_B,

$5\frac{1}{3}$ years $= (5/4)t_B - (3/4)\ 0$.

Therefore t_B equals $\frac{64}{15}$ or $4\frac{4}{15}$ years. If the Lorentz equations are used to find times for C & D during the trip back to Earth, things are less straightforward. On this leg of the journey, $v = -0.6$ instead of 0.6. Switching the sign of v in equations 9.1a and 9.1b gives the Lorentz equations connecting coordinates in the reference frame of the returning rocket to those of A & B in Earth's frame,

$X^* = (5/4)\ x + (3/4)\ t$ and $T^* = (5/4)\ t + (3/4)\ x$. (9.2a and 9.2b)

The asterisks are to remind us that X^* and T^* are coordinates assigned to spacetime points by observers riding in the reference frame of the returning rocket. These coordinates are different from the X and T in equations 9.1a and 9.1b.

The Lorentz equations always require the origins of the two connected reference frames, x = X = 0, to be coincident at t = T* = 0. When x = t = 0 in equations 9.2a and 9.2b, X* = T* = 0 as required. Therefore the origin of the reference frame of the returning space cruiser is located at Earth when C & D start their trip. This means the returning rocket is someplace other than the origin when T* = 0!*

In order for the returning rocket to be at Alpha Centauri when Chuck & Dean arrive, the rocket must be at x = 8 lt-yrs when t = 0. These values can be substituted into equations 9.2a and 9.2b to find the coordinates of the returning cruiser in its reference frame. The results are X = 10 lt-yrs and T* = 6 years.*

If we use the coordinates Anne & Bev assign to the spacetime point A, we find that

$X_A^* = 10$ *lt-yrs and* $T_A^* = \frac{34}{3}$ *or* $11\frac{1}{3}$ *years.*

These answers though strange are perfectly reasonable. The location of the returning rocket does not change in the returning rocket's reference frame. So if it was at X = 10 lt-yrs when Chuck & Dean started their trip, it will still be at X* = 10 lt-yrs at Alpha Centauri and when it arrives back to Earth with its honored passengers. The clock on the returning cruiser read 6 years at the start of C & D's trip and* $11\frac{1}{3}$ *years when it arrived at Alpha Centauri. So the rocket time for the trip to Alpha Centauri was* $5\frac{1}{3}$ *years, exactly the same time it took the similar rocket to travel an equal distance, 4 lt-yrs, from Earth to Alpha Centauri.*

On the other hand, when Chuck & Dean exchanged their original cruiser for the one used to return to Earth, the local times on the cruisers found by using the Lorentz equations went from $5\frac{1}{3}$ *years to* $11\frac{1}{3}$ *years! Of course it did not take C & D 6 years to switch rockets and they sensibly reset the clocks on the returning cruiser to the* $5\frac{1}{3}$ *years it took them to reach Alpha Centauri.*

The primary motivation for the above discussion about the returning cruiser was to emphasize that care needs to be used when solving problems with the Lorentz equations. It is usually much safer to analyze a spacetime problem by using the fact that moving clocks run slow and moving rulers shrink in conjunction with an appropriate spacetime diagram. The Lorentz equations can help at the end of the analysis to make sure everything is consistent.

Looking at a Moving Watch – The Doppler Effect

We have learned that a watch moving through a reference frame runs slow compared to synchronized watches in that reference frame. What happens when an observer continuously monitors the time on a watch as it moves toward or away from him? This latter situation allows for the direct comparison of one watch with another. But in order for the observer to see the time on the moving watch, light has to travel from the moving watch back to the eyes of the observer.

Suppose Anne peered through a telescope to monitor the time on a clock in Chuck & Dean's rocket cruiser as it headed toward Alpha Centauri. The image of that clock traveled at light speed from the rocket back to Earth. The dashed red lines on figure 9.1b could have represented the images of that clock when it read 1, 2, 3, etc years instead of the yearly messages sent from C & D to A & B. As that spacetime graph makes clear, when Anne saw the rocket clock read one year, her Earth clock read two years. How can this be reconciled with the fact that Anne, Bev and the other Earth-frame observers note that Chuck and Dean carry clocks that run slow compared to their synchronized Earth clocks by a factor of $\sqrt{1-v^2}$ *which equals 4/5 when* v *= 3/5.*

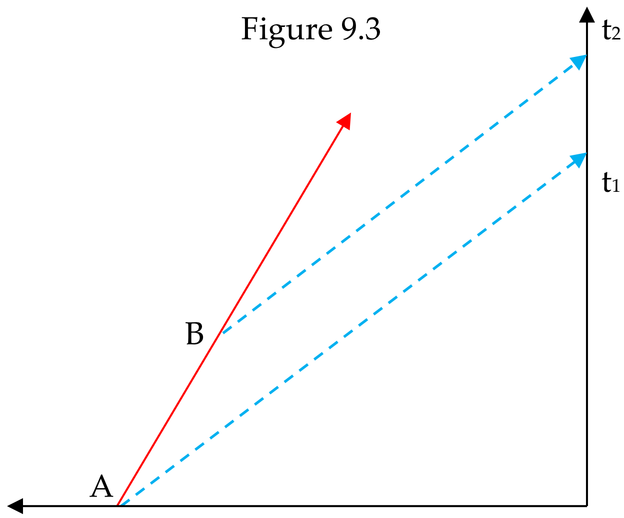
Chuck and Dean on the rocket see an identical situation; their watches read two and four years when they received the image of the Earth clock reading one and two years. Those image signals going from Earth to the rocket are shown in figure 9.1a. The clock receding at 3/5 the speed of light, when continuously observed is seen to run at half the rate of the stationary clock. As mentioned earlier, that difference in rates is a consequence of the famous Doppler Effect.

Question 9.11: Study figures 9.1a and 9.1b to convince yourself that during the return trip, the Earth observers see the rocket clock ticking twice as fast as the Earth clock and *vice versa*, the rocket people see the Earth clock ticking twice as fast as their clock.

The last thing to accomplish in this chapter is to generalize the Doppler Effect to clocks and watches moving at any velocity with respect to one another. Before beginning, the equation for the Doppler Effect is usually written in terms of frequency and not time differences. Frequency is just the <u>*rate*</u> *at which messages are received. In the earlier example, Anne on Earth received messages at ½ the rate that she was sending them to the receding rocket and received messages at twice the rate that she was sending them to the approaching rocket. Frequency is just the inverse of the time interval, T, between messages,* $f = 1/T$. *Our goal is find a relationship between* $f_{received}$ *and* f_{sent}. *Figure 9.3 below and some algebra are all that is needed to derive the general equation.*

The worldline of the approaching rocket is red and two Earth-bound messages are shown in blue. The messages were sent at rocket times T_A and T_B so that the interval between the sent messages is $\Delta T_{sent} = T_B - T_A$. Anne back on Earth received those messages at Earth times t_1 and t_2 and $\Delta t_{received} = t_2 - t_1$. The sent and received frequencies are just $f_{sent} = 1/\Delta T_{sent}$ and $f_{received} = 1/\Delta t_{received}$.

Figure 9.3



The first message was sent from the rocket from spacetime point A. At that instant, Earth observers noted that the rocket was a distance D from Earth and the Earth clock at A read zero. This first message traveled a distance D at the speed of light and reached Earth at,

$$t_1 = D/c. \tag{9.3}$$

The Earth observers at A and B note that the rocket clock is running slow. Therefore the time difference $t_B - t_A$ is larger than $T_B - T_A = \Delta T_{sent}$,

$$t_B - t_A = \Delta T_{sent} \Big/ \sqrt{1-(\tfrac{v}{c})^2}. \tag{9.4}$$

Since the first message was sent at Earth $t_A = 0$,

$$t_B = \Delta T_{sent} \Big/ \sqrt{1-(\tfrac{v}{c})^2}. \tag{9.5}$$

Earth observers saw the rocket move a distance $x = v\, t_B$ closer to Earth before sending the second message,

$$x = {}^{v\Delta T_{sent}}\Big/\sqrt{1-(\frac{v}{c})^2}\,. \qquad (9.6)$$

The second message reaches Earth a time (D – x)/c after it was sent or,

$$t_2 = t_B + (D - x)/c = \Delta T_{sent}/\sqrt{1-(\frac{v}{c})^2} + (D - x)/c. \qquad (9.7)$$

Use equations 9.3 and 9.7 to find $\Delta t_{received} = t_2 - t_1$,

$$\Delta t_{received} = \Delta T_{sent}/\sqrt{1-(\frac{v}{c})^2} - \frac{v}{c}\,\Delta T_{sent}/\sqrt{1-(\frac{v}{c})^2} = \Delta T_{sent}(1-\frac{v}{c})/\sqrt{1-(\frac{v}{c})^2} \qquad (9.8)$$

Note that D, the original distance between the rocket and Earth cancels out. The relationship between $\Delta t_{received}$ *and* ΔT_{sent} *depends only on the relative velocity between the sender and the receiver of messages and not on the distance separating them. The last step in the derivation is to recognize that in equation 9.8* $\sqrt{1-(\frac{v}{c})^2}$ *can be written as* $\sqrt{1-\frac{v}{c}}\,\sqrt{1+\frac{v}{c}}$ *and* $(1-\frac{v}{c})$ *as* $\sqrt{1-\frac{v}{c}}\,\sqrt{1-\frac{v}{c}}$. *Consequently the* $\sqrt{1-\frac{v}{c}}$ *terms in the numerator and denominator cancel to give,*

$$\Delta t_{received} = \Delta T_{sent}\,\frac{\sqrt{1-\frac{v}{c}}}{\sqrt{1+\frac{v}{c}}} \qquad (9.9)$$

In terms of frequency sent and received, equation 9.9 becomes,

$$f_{received} = f_{sent}\,\frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}. \qquad (9.10)$$

Notice that in equation 9.10 the velocity is positive when the rocket is approaching Earth.

Question 9.12: Show that equation 9.10 gives the correct answer when v = 3/5, namely that messages are received at twice the rate that they are sent.

Question 9.13: Redo the derivation of equation 9.10 for a rocket receding from Earth with a speed of v so the second message takes longer than the first to reach Earth.

Question 9.14: Does your answer to question 9.13 demonstrate that a rocket receding at v = 3/5 receives messages at half the rate that they are sent?

The answer to question 9.13 is identical to equation 9.10 with v replaced by $-v$. This is reasonable because a rocket approaching with a velocity of $-v$ is actually receding from Earth with a velocity of v!

Chapter Ten: Space Travel

Author's Note: This chapter has a little more mathematics than previous chapters. On the other hand, the rewards for sticking with it are commensurate with the extra effort; learning how to treat accelerating rockets within the framework of Special Relativity and developing a deeper understanding of the difficulty of space travel.

The material in the final two chapters does not depend on any of the results derived in this chapter. Therefore if you find yourself struggling in this chapter, move on to the final two chapters. You can always return to this chapter to give it another go at some later time.

The Equivalence Principle

After Chuck and Dean had their neck braces removed, they invite Anne and Bev over for dinner. After eating, the four discuss the potential limits placed on human space travel by the theory of Special Relativity. Not being engineers, they decide to ignore any possible technological limitations but agree instead that the imaginary travel has to be consistent with the known laws of physics.

Since Dean was still a little sore from the return trip to Earth, he immediately reminds them that their imaginary rocket has to move through space and time without the bone crushing changes of velocity that he and Chuck suffered through on their trip to Alpha Centauri. Chuck agrees that a discussion of changing velocities, especially changing them less abruptly, would be a good place to begin.

Acceleration is the technical term for the rate of change of velocity, $\frac{change\ of\ velocity}{change\ of\ time}$. *Up to now, we have only been considering reference frames and objects moving at constant velocity, situations where the "change in velocity" was zero, leading to zero acceleration. Accelerations snuck into the twin paradox because the travelers had to turn around at Alpha Centauri to return to Earth. Also, though less central to the understanding of the twin paradox, the trip to Alpha Centauri started with an acceleration at Earth and ended with a deceleration at Earth. Each time the velocity of the rocket changed abruptly, the occupants of the rocket were battered!*

Anne points out that Einstein had postulated an equivalence between gravity and acceleration. "If I remember correctly," she tells her friends, "a person standing inside a windowless rocket could not tell if the rocket was sitting on Earth or accelerating through empty space at 32 ft/s^2."

This equivalence means an experiment done by the person in the rocket would give exactly the same result regardless of whether the rocket was sitting on Earth or accelerating through empty space. This equivalence is analogous to the earlier one encountered with the Principle of Relativity. There we discovered that all reference frames moving at constant velocities were equivalent. This one connecting acceleration and gravity is known as the ***Equivalence Principle****. Einstein used the Equivalence Principle to develop his General Theory of Relativity, a theory that lies well outside the scope of this modest book!*

Dean wants to know what is so special about an acceleration of 32 ft/s^2. Anne answers: "Imagine an object dropped into the Grand Canyon. As the object falls, its speed increases by 32 ft/s every second due to the pull of gravity. In fact, any object dropped near the surface of Earth accelerates at that rate. So we call 32 ft/s^2 the acceleration due to gravity."

Anne tells Dean, "Imagine you're standing on a scale, gravity is still pulling on you but you aren't falling; something is holding you up against the pull of gravity that is trying make you fall at 32 ft/s^2. That push up is exerted by the scale and balances the pull of gravity. The size of that upward push is your weight. On a different planet, your weight would be more (less) than on Earth depending upon whether an object near the surface falls with an acceleration of more (less) than 32 ft/s^2."

Dean smiles and thanks Anne for the cogent explanation of why 32 ft/s^2 is special for Earthlings.

Question 10.1: A rock is dropped into the Grand Canyon. How fast is it going after 1, 2, 3, 4, and 5 seconds?

Chuck wants to know what this has to do with accelerating rockets. Bev decides to give Anne a rest and reminds Chuck of Newton's second law, $F = ma$, an equation that shows the connection between forces and accelerations.

Bev asks Chuck a series of questions to help him see how this early discussion is going to connect to the issue of accelerating rockets.

Question 10.2: You are standing on a scale inside a rocket sitting on the surface of Earth. What is your acceleration? *(Don't let Bev fool you with this question!)*

Question 10.3: Based on your answer to question 10.2, what is the total force acting on you while standing on the scale inside the rocket?

Question 10.4: What does the scale in the rocket read?

Question 10.5: Now you are standing on the same scale inside the same rocket except it is accelerating upward through empty space at 32 ft/s^2. What does the scale read in this case?

Chuck frowns and asks Bev to slow down to give him a chance to think. Chuck begins to answer Bev's questions, "Unless your first question had a hidden trick, my acceleration inside a stationary rocket is zero since I am standing still. If my acceleration is zero, the net force acting on me must also be zero. That means the scale is pushing up on me with a force equal to the downward pull of gravity. Therefore the scale reads my weight."

Bev smiles, gives Chuck a thumbs up, and says, "One more question to go, Chuck."

Chuck stops to think some more before answering Bev's last question. He answers talking to himself and Bev at the same time, "Now I am accelerating which means that some force is acting on me. Inside the rocket, I am standing on the scale, so the scale must be pushing up on me with just the right sized force to cause an acceleration of 32 ft/s^2. From Newton's Second Law, $F = ma$, the size of that force is just my mass times 32 ft/s^2. Okay, I get it now. My mass times 32 ft/s^2 is the same as my weight on Earth."

Now speaking directly to Bev, Chuck says, "So the scale reads the same value, my weight, whether the rocket is sitting on Earth or accelerating through empty space at 32 ft/s^2. For me inside the windowless rocket, there is no difference as long as the rocket accelerates at just the right rate, 32 ft/s^2!"

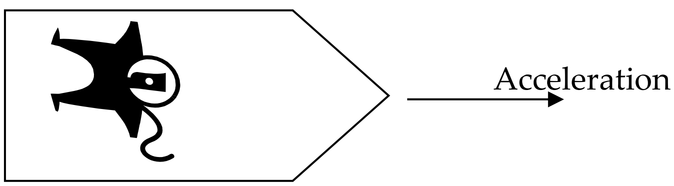
After listening to Bev and Chuck, Dean exclaims, "The equivalence between gravity and acceleration is very cool. I have to give Einstein credit for coming up with that idea."

Question 10.6: The engine in Chuck's accelerating rocket breaks down and the rocket begins to coast through empty space. Chuck looks at the scale, what does it read now?

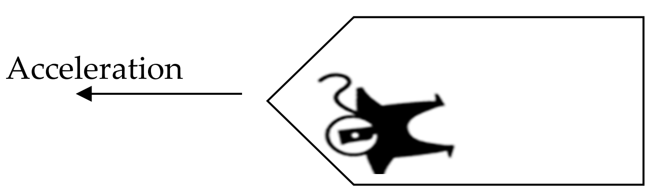
Now that Bev has brought Chuck and Dean up to speed, Anne shifts the conversation back to the imaginary rocket. As her three friends turn towards her, she says, "Humans are used to living on a planet where the acceleration of gravity is 32 ft/s^2. I suggest that our interstellar rocket move through space with a constant acceleration of 32 ft/s². That way, the spacefarers riding inside the rocket would be subject to a constant force equivalent to the pull of gravity at Earth's surface." Chuck & Dean enthusiastically endorse that idea recalling the jarring takeoff from Earth on their trip to Alpha Centauri.

When C & D switched space cruisers at Alpha Centauri, they changed their velocity from +0.6 ft/ns to -0.6 ft/ns, for a net change of 1.2 ft/ns or 1.2 billion ft/s in one hour. There are 3600 seconds in an hour, so during that switch in rockets, they were subject to an average acceleration of $\frac{1.2\ billion\ ft/s}{3600\ s}$ *= 333,333 ft/s² which is about 10,000 times the acceleration of gravity, 10,000 g's. No wonder they suffered from whiplash!*

Designing the Space Adventure



Our friends come up with the following scenario for the rocket's roundtrip. First it accelerates for a fixed time. This allows the rocket to slowly gain speed with respect to Earth. Halfway out, the rocket turns around and uses its engine to decelerate slowly back to zero. The second leg of the trip will take the same length of time as the first leg since the rate of change of speed is the same, 32 ft/s². After this leg of the journey, the rocket will be stationary and some large distance from Earth. The diagram above shows the first leg of the trip and the diagram below shows the second leg.



The return trip to Earth just repeats the first two legs except now the rocket is heading back home instead of away from Earth. The complete trip consists of four legs of equal duration. During the entire trip, the rocket maintains a constant acceleration designed to parrot the pull of gravity on the surface of Earth.

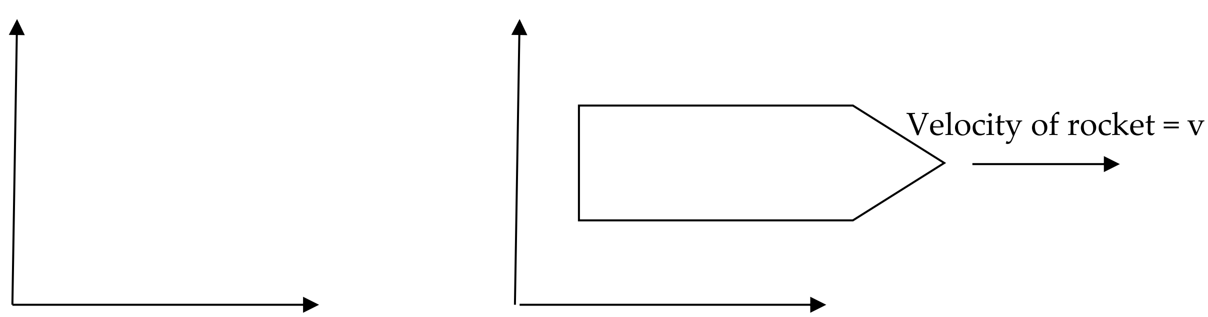
After agreeing to this basic outline, Chuck asks the group, “Whose time are we talking about?” Good question, they all agree. Bev says, “Based on our experience with the Twin Paradox, the watches on Earth will tick off more time than the watches carried by the occupants of the rocket.”

A consensus is soon reached that the relevant time is the rocket time. This led to the next obvious question, how long do our imaginary travelers spend on the trip? Bev reminds them that their goal is to test the extreme limits of space travel. After some haggling, they agree that the rocket ought to spend 10 rocket years during each of the four legs of the journey, for a total trip of 40 years. Assuming our travelers begin when they are 20 years old, they will return to Earth at age 60.

Anne feels good about the progress made creating a reasonable scenario for a trip into deep space. But she wonders how are they going to figure out how far from Earth the imaginary travelers can get in 20 years? She asks her fellow scientists, “Everything we have learned up to now about space and time assumed buses and rockets moving at constant speeds. How are we going to figure out how an accelerating rocket moves through space and time? Anne’s query is met with total silence. Bev, Chuck, and Dean stare at the floor hoping that Anne was asking a rhetorical question!

Anne wasn’t asking a rhetorical question but she finally says, “As long as the rocket’s speed is small compared to the speed of light, we don’t need to worry about the effects of Special Relativity because the factor $\sqrt{1-(\frac{v}{c})^2}$ will be essentially 1.” Chuck agrees but points out that as long as the rocket’s speed is small compared to the speed of light, the rocket is not going to get very far from Earth!

Bev mumbles to no one in particular, “The rocket is always going slow with respect to some inertial reference frame.” Then she draws the diagram below and holds it up for her friends to see.



Earth coordinates Local reference frame moving with velocity v

She suggests that they think of the first 10 year leg as being made up of many little trips, for example 3650 one day trips as recorded by the rocket occupants. Now Bev points to figure 10.1 and says, "Imagine that each new rocket day begins with the ship at rest in a local reference frame that is moving with respect to Earth." No one says anything, but the silence has an encouraging feel so Bev continues. "At the end of a given day of accelerating, the rocket will have a speed U in the local reference frame. That local frame with the moving rocket has a velocity v with respect to Earth." Bev gives her colleagues a chance to study figure 10.1. Before she can continue, Dean tells his friends, "We know how to find the rocket's new velocity with respect to Earth." And then he writes the relativistic addition of velocity equation derived in Chapter Five in Bev's pad under her diagram of the two reference frames.

$$u = \frac{U+v}{1+Uv} \tag{10.1}$$

Dean points to the equation and says, "v is just the velocity of the local reference frame with respect to Earth. U is the velocity of the rocket in the local frame after one day of accelerating. The velocity of the rocket with respect to Earth, u, is given by the relativistic sum of U and v."

At the start of each day, v, the velocity of the local frame with respect to Earth is a little larger than it was at the beginning of the previous day. On the other hand U is the same every day because it is just the increase in speed of a rocket <u>starting at rest and accelerating for one day at the rate of 32 ft/s²</u>. Remember that U, v, and u are all written in terms of units that make the speed of light one.

Question 10.7: How many seconds are in a day?

Question 10.8: Ignoring any relativistic effects, what is the speed of a rocket starting from rest and accelerating for a day at 32 ft/s^2? (*Convert the answer to mph. Your answer ought to be a very large compared to typical speeds encountered in normal life.)*

Question 10.9: Rewrite your answer to question 10.8 as a fraction of the speed of light, $\frac{answer\ to\ 10.8\ in\ ft/s}{speed\ of\ light\ in\ ft/s}$? *(Remember that the speed of light in this book has been decreed to be 1 ft/ns or 1 billion feet/second, 10^9 ft/s.)*

If the answer to question 10.9 is small compared to the speed of light, relativistic effects can be ignored during each day's acceleration. Then U, the speed of the rocket at the end of each day of accelerating in the local reference frame, is just your answer to question 10.8. On the other hand, if the answer to question 10.8 is some sizeable fraction of the speed of light, a day is too long of a time increment and the equation velocity = acceleration x *time cannot be used to find U, the rocket's speed in the local frame.*

The speed of the rocket after a day of accelerating at 32 ft/s^2 *is 0.00276 ft/ns or 0.276% the speed of light. The shrinkage factor,* $\sqrt{1 - v^2}$, *for v = 0.00276 is 0.999996 which is very, very close to 1! This means that we don't need to worry about the effects of Special Relativity when looking at the daily motion of the rocket in the local reference frame and U in equation 10.1 can be replaced by 0.00276 ft/ns, the answer to question 10.9. But Special Relativity will be extremely important in analyzing the cumulative effect of accelerating for 3650 days in a row!*

For the rest of this chapter, the unit of time will be one day and the unit of distance the light-day, the distance light travels in one day. These choices make the speed of light 1 light-day/day which is completely analogous to 1 ft/ns or 1 light-year/year. Also note that 1 light-year is equal to 365 light-days.

Anne summarizes the plan as she now understands it. The rocket starts from rest with respect to Earth on day one. Therefore v = 0 at the start of that day. At the end of that day, the rocket's speed with respect to Earth is u = $\frac{U+v}{1+Uv}$. But since v = 0, the speed of the rocket with respect to Earth after one day is just u = 0.00276. At the start of day two, the new local reference is moving with velocity v equal to the previous day's u, v = 0.00276. Therefore at the end of day two the velocity of the rocket with respect to Earth is,

$u = \frac{0.00276+0.00276}{1+0.00276*0.00276}$ = 0.00552.

Chuck nods sagely and says, "The u at the end of each day becomes the v used to start the next day. In this way, the speed of the rocket with respect to Earth increases a little each day for as long as the rocket keeps accelerating." Dean frowns and tells Chuck, "Doing that little calculation 3650 times could be a little tedious! " His three friends smile and point to the computer in the room. Dean feels better now that he has been reminded that his friends, unlike him, are very computer literate.

Bev asks Chuck, "Using equation 10.1, we can find the rocket's speed with respect to Earth after an arbitrary number of rocket days. But how do we determine the corresponding number of Earth days?"

Chuck answers immediately, "No problem. Remember, Earth observers see the clocks in the local reference frame moving with a velocity v and running slow by the factor $\sqrt{1-v^2}$. So for a day when the local reference frame has a velocity of v with respect to Earth, Earth observers record $\frac{1}{\sqrt{1-v^2}}$ days passing. Meanwhile the watches worn by the occupants of the rocket run at essentially the same rate as the watches in the local frame since $\sqrt{1-U^2}$ = 0.999996. So one day in the local reference frame is the same as one day traveling in the rocket."

Chuck finishes answering Bev's question, "Summing those daily contributions of $\frac{1}{\sqrt{1-v^2}}$ gives the Earth duration of the trip."

A rocket moving for a time t with an average velocity of v covers a distance v t during that time. For our rocket, the average velocity with respect to Earth during any given day is just the sum of the velocity at the start of that day, v, with the velocity at the end of that day, u, divided by 2. Dean uses this simple result to calculate how far the rocket has traveled after some given number of days.

Dean excitedly points out that each rocket day, Earth observers measure the rocket traveling for an Earth time of $\frac{1}{\sqrt{1-v^2}}$ days with an average velocity of $\frac{u+v}{2}$. That means that the rocket moved $\frac{u+v}{2}\frac{1}{\sqrt{1-v^2}}$ light-days further from Earth during each day of rocket travel. Adding together any number of these daily increments gives the rocket distance from Earth, in light-days.

Chuck is eager to work on this and suggests they call it quits for the night and meet again tomorrow. He promises to use his computer skills to create a spreadsheet that calculates the amount of Earth time passed, in days, and distance traveled from Earth, in light-days, for a rocket that continually accelerates at 32 ft/s^2 for 10 years or 3650 days.

The Results of Chuck's Calculations

The next day, Chuck starts by handing a copy of table 10.1, which has eight columns. Each row of the table corresponds to a rocket day. For example, the first row gives details of the rocket's trip during the first day; the second row summarizes what happened the second day, and so on row-by-row. The table has rows corresponding to the first three days of the rocket's trip. Below the table, Chuck has written a definition for each of the quantities in his table followed by a procedure to find those values in all the subsequent rows.

Table 10.1

1	2	3	4	5	6	7	8
T_{Rocket}	T_{Earth}	X_{Earth}	V_{Start}	V_{End}	$V_{Average}$	ΔT_{Earth}	ΔX_{Earth}
0	0	0	0	0.00276	0.00138	1.000001	00.00138
1	1	0.00138	0.00276				
2							

1. T_{Rocket} is the rocket time at the <u>start</u> of that day.
2. T_{Earth} is the corresponding Earth time. This is the time the travelers see on any Earth observer clock they pass.
3. X_{Earth} is rocket's distance from Earth at the start of that day.
4. V_{Start} is the rocket's speed with respect to Earth at the start of that day.

At the start of the trip the quantities in the first four columns are all zero.

5. V_{End} is the velocity of the rocket at the end of that rocket day. It is equal to $\frac{V_{Start}+U}{1+V_{Start}U}$ where U is the velocity gained each day. For this rocket U = 0.00276 and is the same for the first 3650 rows. When the rocket starts decelerating, U changes sign and becomes -0.00276 in the 3651th row.
6. $V_{Average} = \frac{V_{Start}+V_{End}}{2}$ is the average velocity of the rocket with respect to Earth during that rocket day.
7. $\Delta T_{Earth} = \frac{\Delta T_{Rocket}}{\sqrt{1-V_{Average}{}^2}} = \frac{1}{\sqrt{1-V_{Average}{}^2}}$ is the length of that rocket day according to Earth observers.
8. $\Delta X_{Earth} = V_{Average}\ \Delta T_{Earth}$ is the distance traveled during that rocket day as measured by Earth observers.

The quantities in columns 5 through 8 are calculated by using the values in the first four columns of that row. Those values are explicitly shown for the first row. The next series of steps describes the procedure used to find the values in the second row. That series of steps is repeated over and over for the duration of the acceleration phase of the rocket's trip.

9. T_{Rocket} becomes T_{Rocket} + 1. In the second row, T_{Rocket} = 2 days
10. T_{Earth} becomes T_{Earth} + ΔT_{Earth}. In the second row T_{Earth} = 2.000001 days
11. X_{Earth} becomes X_{Earth} + ΔX_{Earth}. In the second row, X_{Earth} = 0.00138 light-days
12. V_{Start} becomes the V_{End} from the previous day. In the second row, V_{Start} = 0.00276.

Those new values are then used to find V_{End}, $V_{Average}$, ΔT_{Earth}, and ΔT_{Earth} in that row.

Chuck gives his friends time to review the definitions of the quantities in each of the eight columns and his instructions for constructing the table. After a little while, Anne, Bev, and Dean shift their gazes from the table and definitions to Chuck. He takes this as a signal to continue. Chuck points at the table and says, "I thought it would be a useful exercise for each of you to calculate the missing values in the table." His colleagues nod in agreement, and each begins doing the necessary calculations to fill in the eleven blank spaces in the table.

Question 10.10: Follow Chuck's instructions and fill in the eleven missing values in Table 10.1. *(The answers are given later in this chapter.)*

Before Dean finishes doing the calculations, he says, " I can't wait any longer; how far from Earth is the rocket after it accelerates for 10 years?" Chuck smiles and notices that A & B have correctly filled in their tables. He hands each of them copies of Table 10.2.

Table 10.2

T_{Rocket}	T_{Earth}	X_{Earth}	V_{Start}	V_{End}	$V_{Average}$	ΔT_{Earth}	ΔX_{Earth}
3648	4273221	4272859	1	1	1	11810.42	11810.42
3649	4285032	4284670	1	1	1	11843.06	11843.06
3650	4296875	4296513	1	1	1	11875.80	11875.80

After giving them a few seconds to glance over this new table, Chuck says, "This table summarizes the last three days of the rocket's acceleration phase. On day 3651, the rocket starts using its engine to decelerate."

Use the values in Chuck's table to answer the following questions.

Question 10.11: After 10 years of travel, how much Earth time has passed in days? In years?

Question 10.12: After 10 years of travel, how far is the rocket from Earth in light-days? In light-years?

Question 10.13: The rocket reaches its maximum distance from Earth after traveling for 20 rocket years, speeding up for 10 years and then slowing down for 10 years. What is the maximum distance from Earth in light-years? How does this compare to the diameter of the Milky Way? *(Chapter Eight included the size of the Milky Way.)*

Question 10.14: After 40 years of rocket time, the ship returns to Earth. How much Earth time has passed in years?

The rocket travelers return more than 35,000 years into the Earth's future. This aspect of time travel makes for great science fiction stories as authors imagine Earth tens of thousands of years in the future. Traveling into the future is a natural consequence of Special Relativity, but making predictions about Earth's future is not!

This demonstration of the "twin paradox" avoids the abrupt changes of speed associated with the description in Chapter Nine. Also notice that with respect to intergalactic distances, the rocket does not get very far from Earth and is confined to stay in the neighborhood of the Milky Way. Remember, our nearby sister galaxy, Andromeda, is 2.5 million light years from Earth!

Dean, feeling smug, tells Chuck, "You must have done something wrong because the speed of the rocket, according to your table 10.2 is equal to 1, the speed of light!" Chuck was prepared for that question and shows Dean the table below with expanded values for the velocity. He points to the table and says, "Dean, the rocket's velocity gets closer and closer to one but always remains a tad less than one."

Table 10.3

T_{Rocket}	T_{Earth}	X_{Earth}	V_{Start}	V_{End}	$V_{Average}$	ΔT_{Earth}	ΔX_{Earth}
3648	4273222	4272859	0.999999996	0.999999996	0.999999996	11810.42	11810.42
3649	4285032	4284669	0.999999996	0.999999996	0.999999996	11843.06	11843.06
3650	4296875	4296512	0.999999996	0.999999996	0.999999996	11875.80	11875.80

Looking at table 10.3, Bev notices that during the last rocket day of acceleration, 11875.8 Earth days pass. "Yikes!" She says, "That is over 25 years of Earth time!" Anne points out that $\sqrt{1-v^2}$, the time stretching factor during the 3650th rocket day is very close to zero. Anne pulls out her calculator and says, "In fact, on that day $\sqrt{1-v^2}$ equals 0.000084 which stretches one rocket day to over 10,000 Earth days in perfect agreement with Chuck's table."

Dean, Bev, and Anne pat Chuck on the back for a job well done.

Question 10.15: Use a spreadsheet software program to reproduce the results in Chuck's table.

Chuck is a little embarrassed by the accolades of his friends. But he has one more thing to share with them. Since he and Dean were the ones who suffered the effects of the debilitating accelerations in Chapter Nine, he made a graph of the rocket's smooth shift in velocity at the turn around point. He shows his friends the graph in figure 10.1.

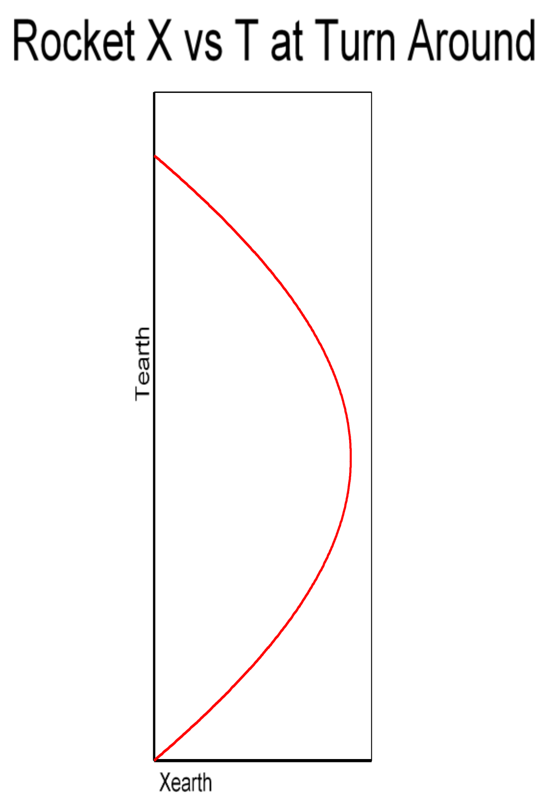


Figure 10.1

As Anne, Bev, and Dean look at the graph, Chuck says, "It shows the motion of the rocket 20 years into the trip, just as it reaches its maximum distance from Earth and turns around. The graph shows the motion of the rocket during the 500 days when its velocity changes from v = +0.6 to v = -0.6 ft/ns." Dean thinks back to the neck wrenching turn around at Alpha Centauri and says, "Those imaginary travelers have no idea how lucky they are!"

Question 10.16: How does a rocket accelerate while traveling through empty space? *(Our spaceketeers never asked this important question during their conversations about space travel. Before reading on, give this question some thought.)*

The table below includes the answers to Question 10.10 in the highlighted boxes.

T_{Rocket}	T_{Earth}	X_{Earth}	V_{Start}	V_{End}	$V_{Average}$	ΔT_{Earth}	ΔX_{Earth}
0	0	0	0	0.00276	0.00138	1.000001	0.00138
1	1	0.00138	0.00276	0.00552	**0.00414**	**1.000009**	**0.00414**
2	**2.00001**	**0.00552**	**0.00552**	**0.00828**	**0.0069**	**1.000024**	**0.0069**

How does a Rocket Accelerate?

About a week after Chuck's triumphant presentation, Anne asked Bev, "How does our rocket change its speed while moving through empty space?" Bev's response was, "Good question. We analyzed the kinematics of space travel but completely ignored the dynamics."

This is physics talk: kinematics is the study of motion while dynamics looks for the causes of motion, especially changes in motion like acceleration.

Earlier, we used Newton's Second Law to connect force to acceleration. An external agent exerts a force on an object causing it to accelerate. For example, the Earth pulls down on an object falling into the Grand Canyon causing to go faster and faster. But there is no external agent acting on a rocket moving through empty space.

Anne imagines the daily accelerations of the rocket. After thinking for a minute or two, she says, "Each day the rocket starts in a local reference frame with zero velocity. At the end of the day, the rocket's velocity is 0.00276 ft/ns. This happens day after day for ten years." Bev interrupts to remind Anne, "It happens every day for forty years, twenty years of accelerating and twenty more decelerating." Anne impatiently nods in agreement and points out to Bev, "All we need to do is figure out how the rocket can change its speed by either plus or minus 0.00276 ft/ns in one rocket day. Then we can use that speed changing trick over and over again for each day of the trip."

Notice that by starting each day in a new reference frame, the speed of the rocket is always small in that reference frame compared to the speed of light. This is very helpful because it means that the daily changes in speed can be understood without having to resort to Special Relativity.

Bev blurts out, "Momentum, conservation of momentum is what accelerates the rocket." Anne thinks about this for a couple of seconds before smiling in agreement. They decide to invite Chuck and Dean over for a picnic lunch. After lunch, they will share their new insight into the connection between conservation of momentum and acceleration with their friends. This will allow the four of them to wrap up their analysis of rockets and space travel as a team, just like all their earlier work on space and time.

After lunch, A & B begin to explain how the rocket accelerates by conserving momentum. But before they get very far, Dean asks, "Remind me please, what is momentum?"

The amount of momentum an object has is the product of the mass of the object times its velocity. When there are no external forces acting on an object, like our rocket moving through empty space, the object's momentum cannot change. Or said another way, it takes a force acting on an object to change its momentum.

Before Anne or Bev can reply, Chuck jumps in and says, "The momentum of the rocket is the product of the rocket's mass times its velocity. Our rocket starts each day with zero momentum in the local reference frame." Dean replies, "Thanks Chuck. I remember now. Momentum is that handy quantity that is conserved." But after remembering that, Dean looks confused, and adds, "But at the end of the rocket day, the rocket has momentum since it is moving with a speed of 0.00276 ft/ns. So is momentum conserved or not?"

While Chuck ponders Dean's question, Bev replies to Dean, "Imagine that the occupants of the rocket toss some mass out the back. That tossed mass has <u>negative</u> momentum since it was tossed out with a <u>negative velocity</u>. But the total momentum of the rocket plus tossed mass still has to add to zero, the initial momentum of the rocket. Tossing mass out the back causes the rocket to move in the positive direction." Bev looks at Dean and Chuck and emphatically states, "The rocket accelerates by ejecting mass out the back. Nothing could be simpler!"

A rocket engine is a machine designed to eject mass with the largest possible velocity or momentum causing the rocket to accelerate in the opposite direction. Building an efficient rocket engine is much more difficult that stating what such an engine ought to do! But remember, we agreed to not worry about technological details. Our rocket engine will operate with maximum possible efficiency.

Dean looks at his friends and says, "So momentum is like velocity, it has a direction. If the negative momentum of the ejected mass is going to be balanced by the positive momentum of the rocket, the magnitudes of those two momentums have to be equal." Since no one interrupts him, he continues by writing down the following equation,

$$M_{End}\,U = (M_{Start} - m)U = m\,v_{ejected}. \qquad (10.2)$$

When he looks up, Anne, Bev, and Chuck are staring at him. Dean says, "Equation 10.2 just summarizes in equation form what Bev told me. The mass of the rocket at the end of the day, M_{End}, times the speed of the rocket at the end of the day, U, has to equal the amount of mass ejected by the rocket, m, times the speed of the ejected mass, $v_{ejected}$. And of course the mass of the rocket at the end of the day is just equal to rocket's mass at the start of day minus the amount of ejected mass, $M_{Start} - m$." His three friends smile, and congratulate him for quickly and correctly turning the conversation about the conservation of momentum into an algebraic expression.

Chuck does a little mental algebra to solve for m, the amount of mass that needs to be ejected to give the rocket a forward speed of U. When he is finished, he writes the results below equation 10.2,

$$m = \frac{U}{U + v_{ejected}} M_{Start}. \qquad (10.3)$$

Bev stares at the equation and comments, "A `good' rocket is one that can change its speed by U every day by ejecting the smallest amount of mass, m, as possible."
"Right," says Chuck, "and that happens when the ejected mass has the largest possible

velocity." With that explanation he re-writes equation 10.3 with $v_{ejected}$ equal to 1, the speed of light, the maximum velocity that a rocket engine can give to the ejected mass,

$$m = \frac{U}{U+1} M_{Start}. \qquad (10.4)$$

Dean wonders how they are going to build such an engine, but he is cut short by Bev who points out that that is a problem for engineers. Their mission is to use the most optimistic assumptions, consistent with the known laws of physics, to maximize the capabilities of their rocket.

Anne has been silently mulling over equation 10.4. Finally she says, "The mass of the rocket at the end of the day is related to the mass at the beginning by the following equation,

$$M_{End} = M_{Start} - m = M_{Start} - \frac{U}{U+1} M_{Start} = \frac{1}{1+U} M_{Start}\ . \qquad (10.5)$$

Chuck sets U = 0.00276 in equation 10.5 and quickly calculates that $\frac{1}{1+U}$ is equal to 0.997 for their rocket. He looks at his friends and says, "So at the end of each day, the mass of the rocket is reduced to 99.7% of its starting mass."

If the mass at the start of the trip was M_0, the mass after day one would be 0.997 M_0. A day after that, the mass of the rocket would be 0.997(0.997 M_0) = $(0.997)^2$ M_0. After 10 days, the remaining mass of the rocket would be $(0.997)^{10}$ M_0.

Dean points out that the total trip is 14,600 days long and asks, "What is 0.997 raised to the 14,600th power!" Bev punches some buttons on her calculator and writes down the answer to Dean's question,

$$M_{Return} = 0.997^{14,600} M_0 = 0.000000000000000000089\ M_0. = 8.90 \times 10^{-20}\ M_0. \qquad (10.6)$$

Dean says the obvious, "That is an astonishingly tiny fraction of the starting mass!"

Remember that our friends used the theoretically most efficient possible rocket engine that ejected mass at the speed of light. Any rocket powered by a real engine would have to eject a considerably larger fraction of its mass to make this trip.

Question 10.17: The maximum landing weight of the Space Shuttle, NASA's work horse vehicle, is 100,000 kilograms. A rocket that travels for 40 years and returns to Earth would likely have a substantially larger returning mass, but for the sake of argument suppose the returning rocket had a mass equal to the Space Shuttle, 100,000 kg. Use equation 10.6 to calculate the initial mass, M_0, for this rocket?

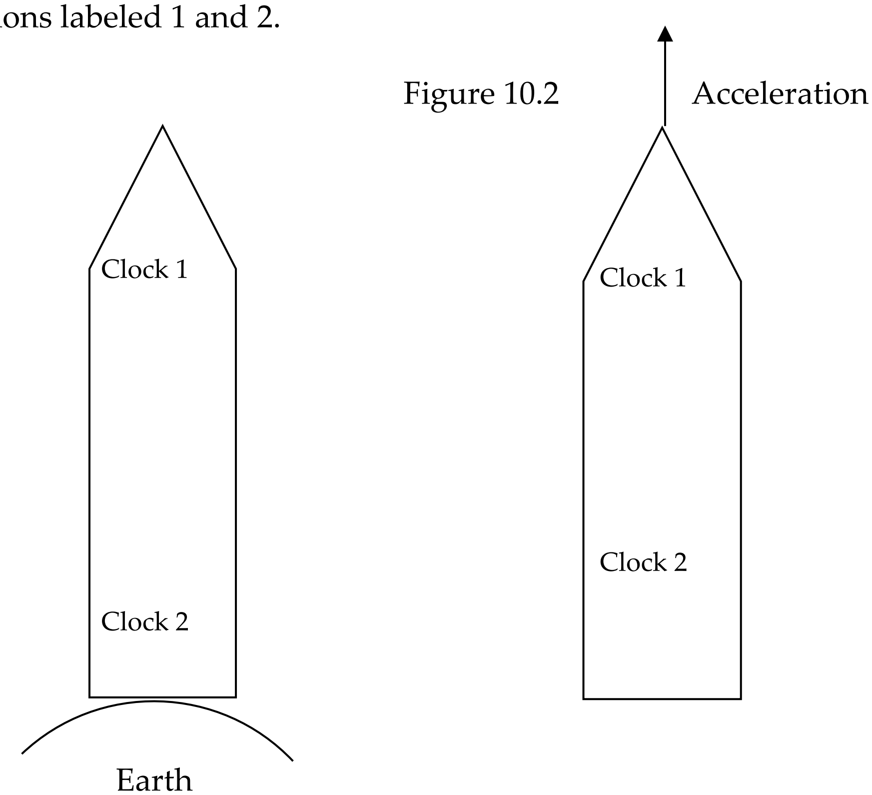
Your answer to question 10.13 ought to be very large, *but how large is very large. The mass of Earth is 5.98 x 10^{24} kilograms. Your answer to question 10.17 ought to be about 0.19 times the mass of Earth, or* 19 % *of the mass of our home planet!*

The primary purpose of this chapter was to show how relatively, pardon the pun, simple ideas can be used to establish practical limits on the ability of humans to explore space. For those of you dismayed by these limitations, remember the limitations arose because our spaceketeers insisted on building an imaginary rocket that operated within the known laws of physics.

Bonus Topic: A Peek at General Relativity

We can use the Equivalence Principle and the Doppler Effect from Chapter Nine to show that clocks further from the center of Earth run faster than those closer. This gravitational effect on clocks has to be corrected for in order for a GPS to accurately determine a location. Figure 10.2 shows two identical rockets with clocks located at the positions labeled 1 and 2.

Figure 10.2



The rocket on the left is sitting on Earth and the rocket on the right is accelerating through empty space. In the accelerating rocket, Clock 2 sends a signal at a fixed frequency to Clock 1. When the message arrives at Clock 1, the local inertial observers who were stationary within the rocket when the signal was emitted now see Clock 1 moving upward with a velocity v gained while the signal traveled the length of the accelerating rocket. Therefore the observers in the rocket located at Clock 1 see the signal from Clock 2 Doppler shifted to a lower frequency. They conclude that Clock 2 is ticking more slowly than their clock, Clock 1, or that their clock at the front of the rocket is ticking faster than clocks lower down in the rocket.

The equivalence of acceleration and gravity requires that observers in the left-hand rocket that is sitting on Earth see exactly the same effect. Namely, the clock at the top of the rocket, Clock 1, has to run faster than Clock 2 at the bottom of the rocket. This gravitational effect on clocks sitting on Earth has nothing to do with the Doppler Effect since both clocks in this rocket are stationary! In general, clocks further from the center of Earth will run faster than those that are closer. Therefore clocks in orbit around the Earth run faster than clocks on the ground: a consequence of General Relativity that has been confirmed by the GPS system. A quantitative analysis of this effect is presented in Appendix D.

This slowing of clocks near a source of gravity is extreme when the clock is close to a black hole. In fact General Relativity predicts that a clock will run at a slower and slower rate as it approaches a black hole. The place where the falling clock would appear to stop ticking is called the event horizon of the black hole. The clocks actually falls through the event horizon but observers outside a black hole can NEVER see the clock actually fall into the black hole. Instead the clock will appear to move slower and slower but never quite reach the event horizon. From the perspective of the clock, nothing strange happens at the horizon. It just zips by and enters the black hole to never be seen or heard from again.

Chapter Eleven: Does it Really Happen?

One question that always gets asked at the end of a semester goes something like this, "clocks don't really run slow, do they?" Or some equally unnerving question that convinces me that some students see the spacetime of special relativity as some sort of optical illusion or trick unconnected to reality. This chapter is my last and very best effort at convincing the die-hard skeptic that moving rulers and clocks really do shrink and run slow just like the rulers and clocks used by Anne, Bev, Chuck, and Dean. I will use two distinctly different arguments designed, for once and for all, to put to rest the notion that the effect motion has on space and time is just "theoretical." For the rest of this chapter, the rule that the author's comments are in italics will be suspended!

The Far-Away Observer (FAO)

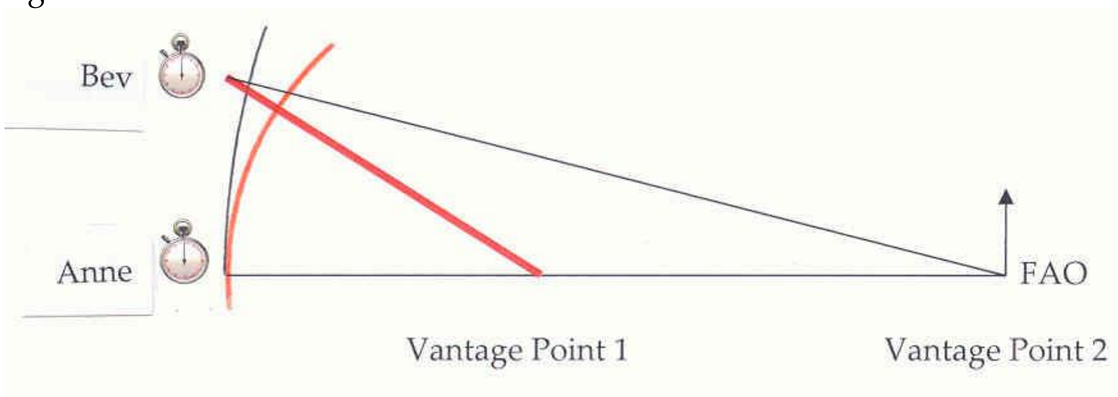
When Bev compared the time on her watch with the time she saw on Anne's watch, there was always a difference in readings because the image of Anne's watch took a finite time to reach Bev.

Of course under normal circumstances in our daily lives that difference is much too small to register on our inferior clocks and watches, but remember that our intrepid experimenters have watches that read to 0.1 ns with reflexes to match!

Therefore the time Bev saw was always a bit earlier than the time on her synchronized watch; the difference just being the distance between them divided by the speed of light. For example, if they were 100 feet apart the time delay would be 100 ns. If Bev saw exactly 10:00 a.m. on Anne's watch, her watch read exactly 100 ns after 10:00 a.m.

The FAO has a privileged vantage point that lets her see the same time on Anne and Bev's watches. This vantage point is gained by being far from both of them as shown in figure 11.1. Although Bev is further from the FAO than Anne, that difference in distance can be made arbitrarily small by having the FAO move further away. Notice that when the FAO shifts from point 1 to point 2 in the diagram below, the difference in distance between her and Anne and her and Bev gets smaller. Imagine a circle drawn by the FAO with her at the center and with a radius equal to her distance to Anne. The red line represents the arc of a circle centered at vantage point 1 and the black arc uses vantage point 2 as a center. As the diagram makes clear, the difference in distance, which is proportional to the difference in time it takes light to reach the FAO from Anne and Bev's watches, can be made arbitrarily small by having the FAO move further and further away.

Figure 11.1



The watches used in this book were accurate to 0.1 ns. We can easily calculate where the FAO has to stand to make the difference between the light travel times less than 0.01 ns, too small enough to be noticed by observers using watches and clocks ticking off tenths of nanoseconds. Of course, the FAO could move even further away to make the time difference even smaller than 0.01 ns if that was seen as prudent.

Algebraically, the difference in time observed by the FAO is just her distance to Bev minus her distance to Anne divided by the speed of light. The distance to Bev is the hypotenuse of the right triangle formed by Anne, Bev, and the FAO. If Anne and Bev are L feet apart and the FAO is D feet from Anne, the difference in time is just,

$$\Delta Time = [\sqrt{D^2 + L^2} - D]/c \quad (11.1)$$

Since light travels 1 foot in 1 nanosecond, a time difference of 0.01 ns corresponds to a difference in length of 0.01 ft, or about 1/8 of an inch. If Anne and Bev are 100 feet apart, the FAO has to be about 500,000 feet away, or about 100 miles.

Question 11.1: Do the calculation outlined above to show that 500,000 feet is the appropriate distance between the FAO and A & B.

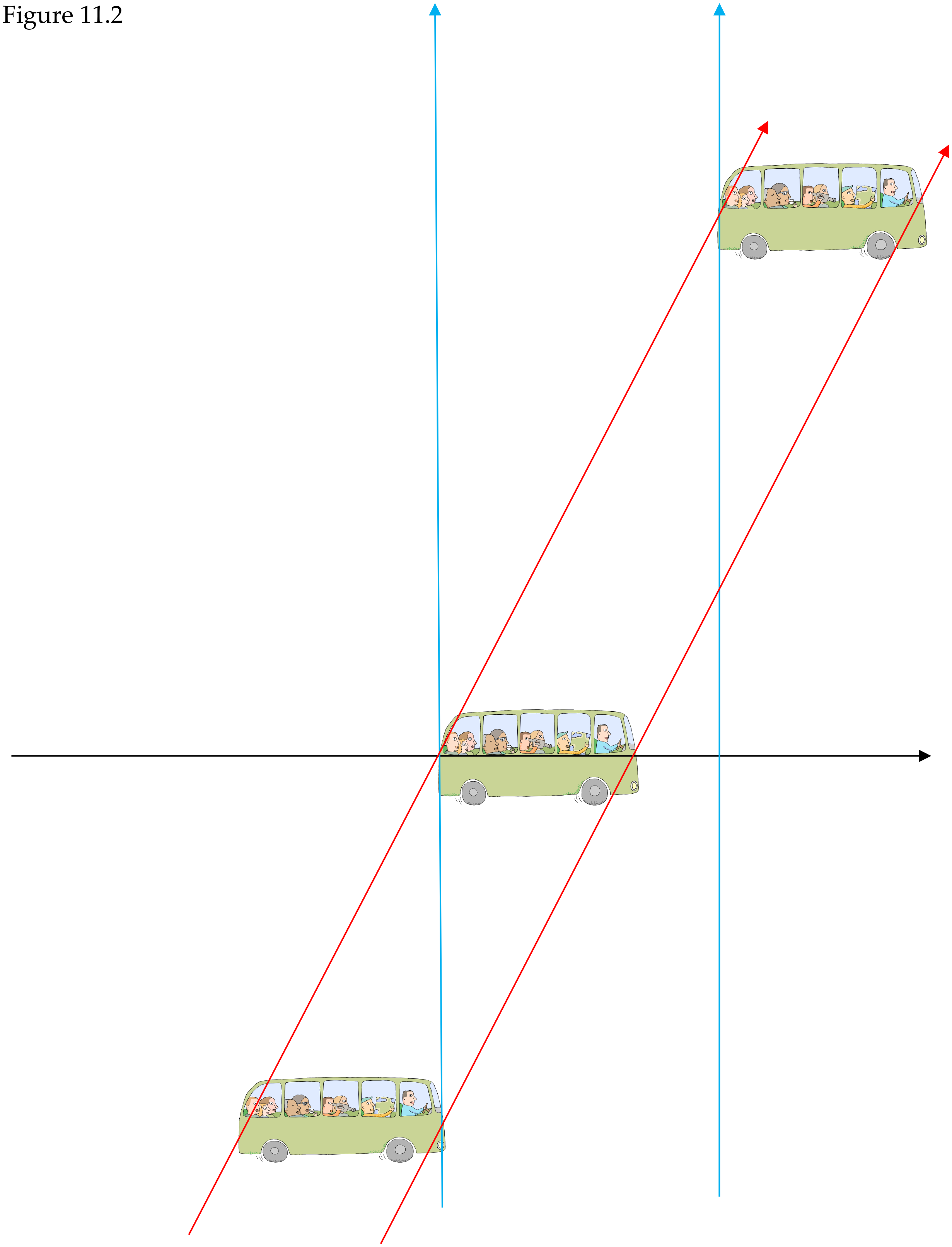
Imagine the FAO 100 miles from Bev and Anne peering through a powerful telescope. She is watching A & B and the super bus with Chuck and Dean. The FAO is stationary with respect to A & B and sees their watches reading "exactly" the same times though the images of their watches took 500,000 nanoseconds to reach her. Exactly is in quotation marks because it actually took 500,000.01 nanoseconds for the image of Bev's watch to reach the FAO! So in this case, "exactly" means accurate to within 0.01 ns. Therefore it is easy for the FAO to synchronize her watch with the readings she sees on the watches of A & B. This allows her to observe an experiment through her telescope that takes place in the neighborhood of A & B without worrying about the confusing effects of different light travel times. Her privileged vantage point makes the light travel time effectively the same from anyplace in the vicinity of Anne & Bev.

Consequently the FAO sees events unfold in real time, but delayed by 500,000 ns, in A & B's reference frame. For example if C & D drive the super bus through A & B's reference frame at 0.6 ft/ns, the FAO sees the entirety of the super bus at constant times according to A & B. In figure 11.2, the worldlines for the super bus with Chuck in the back and Dean in front with properly synchronized watches are shown in red. A & B are exactly 100 feet apart, the length of the stationary super bus, with Anne at the origin. They also have properly synchronized watches. This experiment, like most of the others discussed earlier, is arranged so that Chuck passes Anne when each of their running watches read exactly zero. A & B's worldlines are in blue. The super bus, as actually seen by the FAO, is shown at three different times.

The FAO sees the watches of A & B perfectly in synch with hers. On the other hand she sees C & D's watches running slow and not synchronized with one another. At any given moment, she sees Dean's watch reading 60 nanoseconds earlier than Chuck's watch. Also when she compares the length of the super bus to the 100 foot distance between A & B, it is clear to her that the bus is only 80 feet long. There is nothing illusory about any of these effects. If seeing is believing, the FAO is a believer!

The peculiar features of special relativity that were so bedeviling early on are just part of the FAO's normal observations. The only caveat is that she needs to be watching events in two different reference frames moving at relativistic speeds with respect to one another unfold from her privileged vantage point a large distance from the action. The only reason we have not observed the foreshortening of buses driving down the road is that there are no real buses that can travel at relativistic speeds!

Figure 11.2



For readers that need more convincing, Appendix C uses a device called a **Light Clock** to present a very clever alternative argument showing that the FAO sees moving clocks run slow and lengths shrink.

A Dynamic Explanation for the Behavior of Watches and Rulers

The FAO is an intelligent observer and asks herself the obvious question, "Why do moving clocks run slow and rulers shrink?" The FAO thinks about the internal structure of rulers and clocks. They are built of more fundamental entities like atoms and molecules. The atoms and molecules consist of electrons, protons, and neutrons. Even the protons and neutrons are made of more elementary units. She draws the outline of a ruler and scatters the inside of the ruler with some negative and positive charges representing electrons and nuclei. The nuclei contain both protons and neutrons.

+ - + -

She wonders, “How do the electrons and nuclei know how to arrange itself themselves to form a ruler?” Clocks and watches are much more complicated than rulers. Thinking about them makes her dizzy because of their internal complexity, so she concentrates on the structure of the ruler.

Appendix C describes a particularly simple sort of clock, the light clock, that is easy to analyze but that is not the sort of clock or watch that the FAO is thinking about in the above paragraph.

She begins by imagining a ruler sitting in Anne & Bev's laboratory which is not moving with respect to them or her. From her understanding of physics, she knows that the positive and negative charges try to arrange themselves into the lowest possible energy configuration. That energy is determined by the interactions between all the particles making up the ruler.

The FAO knows how to use Maxwell's equations to calculate the electric and magnetic fields set up by charged particles moving around inside the ruler. Then, at least in principle, she could use known laws of physics, to find the arrangement of charges that is optimum for the ruler. If the calculation is done correctly, that arrangement ought to result in the twelve inch ruler sitting in A & B’s lab.

Though she is a very talented and skillful physicist, she cannot actually do this calculation. It is way, way too complicated. But that does not stop her from thinking about what happens when she applies the same analysis to a ruler moving through Anne & Bev's lab. She immediately recognizes that the situation is very different because Maxwell's equations predict that the charges inside the ruler moving through A & B's lab produce very different electric and magnetic fields than those produced by the charges in the stationary ruler.

The FAO concludes that the fundamental interactions between electrons and nuclei that hold the ruler together, and when arranged optimally produce a ruler of a given length, are different for stationary and moving rulers. Might those differences cause the moving ruler to be shorter?

A Model for a Hydrogen Atom

Like all good physicists confronted by a problem too complicated to solve, she tries to come up with a simpler but related problem that is solvable. Instead of thinking about the actual atoms that make up the ruler, she focuses on hydrogen, the simplest atom, which consists of one electron and one proton. Starting with Newton's second law and Maxwell's equations, it is easy to show that the electron in this simplified version of a stationary hydrogen atom orbits the proton in circular orbits. In fact this problem is completely analogous to the way Earth orbits the sun. She can easily solve this problem but the FAO also knows the improved version of Newton's Second Law that includes effects due to Special Relativity. This version of the simplified hydrogen atom is more accurate than the first version because it turns out that the electron in Hydrogen orbits the nucleus at relativistic speeds.

The FAO sits down at her computer to solve the problem of an electron orbiting a moving proton. First she uses Maxwell's equations to find the electric and magnetic fields produced by the proton. Next she calculates the forces those fields exert on the electron as it moves through those fields. Finally she puts those forces into the relativistic version of F = ma and has the computer find the orbit of the electron around a proton moving with velocity v through A & B's lab.

To get a better picture of the problem she solved, imagine that the proton is being moved through the lab at a constant speed. The electron's path is calculated just as described above but the force the electron exerts on the proton is ignored because some entity is “holding” the proton and moving it with constant speed. This is a reasonable simplification because the proton is about 2000 times more massive than the electron so its motion is much less influenced by the fields of the electron than vice versa. Also remember that the protons we are interested in are “held” inside a ruler riding at relativistic speeds inside a super bus.

As a simple test of her solution, she first sets the proton’s velocity to zero. In this situation, she knows that the electron’s orbit must be a circle. She is gratified to see that the orbit generated by the computer is a circle. With a sense of anticipation, she plugs v = 0.8 and v = 0.95 into her equations to find the orbits around protons moving at relativistic speeds with respect to her and A & B. The orbits for these three situations, v = 0, 0.8, and 0.95, are shown in figure 11.3. Note that the orbits are drawn for protons moving from the bottom to the top of the page.

She stares at the result in disbelief and awe. The moving hydrogen atoms are foreshortened in the direction of motion by exactly the amounts predicted by the shrinkage factor, $\sqrt{1-v^2}$: 0.6 for a velocity of 0.8 and 0.31225 for a velocity of 0.95. The electron’s orbit becomes more and more elliptical as the velocity of the proton increases. *Note that the orbits are drawn so that the width of the atoms in all three cases goes from x = -1 to x = +1.* Although she made some simplifying assumptions, the results give her confidence that the reason moving rulers shrink is because the very atoms that constitute the ruler shrink along the direction of motion.

To picture the foreshortened atoms in three dimensions imagine the ellipses rotated about their vertical axes. The resulting shape is an ellipsoid.

Now she goes back and calculates how long it takes the electron in the three examples to make one orbit around the central proton. Again she finds results wonderfully consistent with special relativity. The electrons orbiting the protons moving at 0.8 and 0.95 take more time to complete one orbit than the electron orbiting the stationary proton. Moreover, the times agree with the predictions of Special Relativity. This suggests to the FAO that moving clocks run slow because the fundamental atomic time, the time for an electron to orbit its nucleus, is stretched out in atoms moving at relativistic speeds.

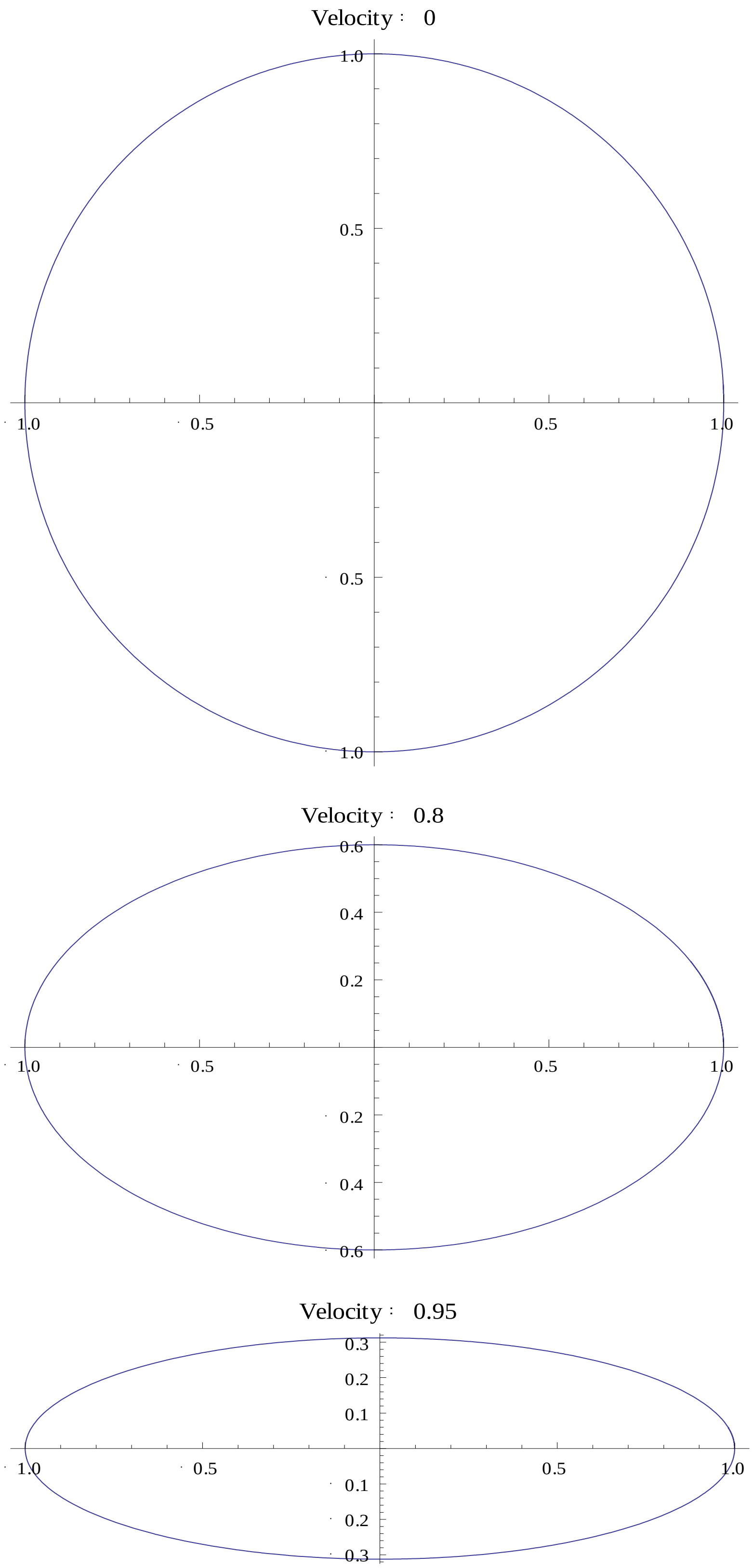
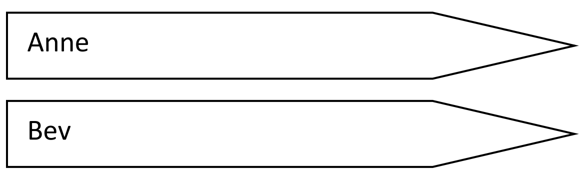

Velocity : 0
1.0
0.5
1.0
0.5
0.5
1.0
0.5
1.0
Velocity : 0.8
0.6
0.4
0.2
1.0
0.5
0.5
1.0
0.2
0.4
0.6
Velocity : 0.95
0.3
0.2
0.1
1.0
0.5
0.5
1.0
0.1
0.2
0.3

Figure 11.3

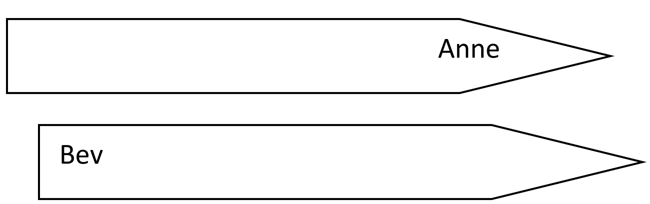
Chapter Twelve: $E = mc^2$

Author's Aside: Einstein's most famous equation, $E = mc^2$*, is not directly connected to the material covered in this book. On the other hand, to write a book about Special Relativity without saying anything about the relationship between mass and energy seemed negligent. The following argument is an attempt to provide a rationale for the equation,* $E = mc^2$*, that defines an equivalency between mass and energy. Anne and Bev agreed to come out of retirement to help by participating in another set of gedanken experiments. Chuck and Dean remained back on Earth but kept in radio contact with their friends.*

Anne and Bev are in separate but identical rockets floating adjacent to one another in empty space far from any source of gravity. From their perspective, the two rockets are perfectly still. At the start of the experiment, they are both standing at the rear of their respective rockets. Bev's job is to remain stationary while Anne is free to roam about her rocket.



The first thing they notice is that Anne's rocket moves back and forth with respect to Bev's rocket. At first, this seems very strange until they realize that the rocket only moves when Anne is strolling around. (Since Bev does not move in her rocket, the relative motion between the two rockets is assigned to Anne's rocket. It would be strange indeed if Bev's rocket moved back and forth while she remained stationary with the engines off!) Whenever Anne stopped moving, her rocket was also stationary. But Bev noticed that when Anne stopped, the two rockets were no longer perfectly aligned unless Anne stopped at her starting point at the rear of the rocket. The largest shift in the position of Anne's rocket occurred when Anne stopped at the front of her rocket.



Anne ran from the front to the back of the rocket, her starting position, and stopped by smashing into the wall. After picking herself up, she noticed that her rocket was perfectly aligned with Bev's. After making careful observations as Anne moved about at different speeds, it became clear that the shift in the relative positions of the two rockets was independent of her speed. The size of the shift only depended on where Anne decided to stop.

Bev finally recognized that their old friend conservation of momentum was at work here just like it was at work in the accelerating rocket problem in Chapter Ten. Except here, there was no ejected mass, just Anne moving about inside the rocket. Anne and her rocket had zero momentum to begin with so when Anne began to walk toward the front of the rocket, the rocket had to start moving backwards to keep the total momentum zero. For the same reason, anytime Anne stopped, the rocket had to stop.

Bev summarized the conservation of momentum statement in the following algebraic expression,

$$M_{rocket}V_{rocket} + m_{Anne}v_{Anne} = 0, \qquad (12.1)$$

where the velocities in the above equation are with respect to Bev's stationary rocket. *Note that in order to keep the total momentum zero, the two velocities have to be in opposite directions; if one is positive, the* *<u>other must be negative</u>*. *The validity of equation 12.1 requires that Vrocket and vanne are much smaller than the speed of light. This condition is obviously satisfied since Anne is meandering back and forth in the much more massive rocket.*

In terms of speed, equation 12.1 can be rewritten as,

$$M_{rocket}(Rocket's\ Speed) = m_{Anne}(Anne's\ Speed). \qquad (12.2)$$

Bev continues to think through the details of the experiment. Finally she says to Anne, "Suppose the distance your rocket shifted with respect to mine was x feet. During the time it took you to walk to the front of the rocket, L feet away, you only moved L – x with respect to me." Anne cogitates on this for a bit before agreeing with Bev. She remembers that when she reached the front of her rocket, the front of Bev's rocket was x feet in front of her rocket.

If Bev observed that it took T seconds for Anne to complete the trek from back to front, then she could easily find the ratio of the rocket's speed to Anne's speed. The rocket moved x feet and Anne moved L – x feet during T seconds.

$$\frac{Rocket's\ Speed}{Anne's\ Speed} = \frac{x/_T}{(L-x)/_T} = \frac{x}{L-x} \qquad (12.3)$$

Equation 12.2 can be re-arranged to give a different relationship for the ratio speeds, namely,

$$\frac{Rocket's\ Speed}{Anne's\ Speed} = \frac{m_{Anne}}{M_{rocket}}. \qquad (12.4)$$

Setting the two equations equal to one another, Bev ends up with an equation that gives Anne's mass in terms of the mass of the rocket,

$$m_{Anne} = \frac{x}{L-x} M_{rocket}. \qquad (12.5)$$

Anne laughs and says "Bev if you wanted to know my mass you ought to have just asked instead of coming up with this elaborate way of determining it!"

Normally the above equation is re-written so that x can be found when m_{Anne} *and* M_{rocket} *are known. A little algebra is needed to get,*

$$x = \frac{m_{Anne}}{M_{rocket}+m_{Anne}} L. \qquad (12.6)$$

This equation can be used to answer question 12.1 below. For our purposes, equation 12.5 is the preferred version as will become evident later in this chapter.

Since the rockets are in empty space far from any masses, Anne, Bev, and the rockets are just floating about and weightless. To walk from one end of the ship to the other, Anne would need the clunky magnetic shoes often seen in Sci-fi movies. Or she could launch herself toward the front by pushing on the back end of the rocket. When she arrived at the front, she could grab something to stop herself. The details of how Anne gets from one end of the rocket to the other are irrelevant since Bev observed the shift in Anne's rocket only depended on the place Anne stopped and not how she got there.

Question 12.1: Anne weighs 120 lbs and the 100 foot long rocket weighs 11,880 lbs. These weights were determined on Earth. Use equation 12.6 to find the distance Anne's rocket shifts, compared to Bev's stationary rocket, when Anne walks from one end to the other? *(Note that weight is just mass times 32 ft/s*^{2}*. So the ratio in equation 12.6 is the same whether masses or weights are used for Anne and the Rocket.)*

Your answer ought to give an x that is easily measured with ordinary rulers.

The weary reader may have begun to wonder what any of this has to do with the famous equation, $E = mc^2$. Stay patient, important progress has been made toward our goal but more patience is required.

Anne has a special spring powered laser in her rocket. To fire a pulse of light, Anne compresses a spring, like in a pinball machine. The compressed spring acts as an energy source analogous to a battery. When the spring is released, it is used to power a laser that produces a flash of light. All of the original energy in the spring ends up as light in the laser flash. For example if the spring stored an energy E, then after that spring energy was released to power the laser, the resulting light flash had an energy E.

At the front of the rocket, Anne has another clever bit of machinery that captures the energy in the laser flash with 100% efficiency and uses it to compress a spring that is identical to the spring at the rear of the rocket. Before Anne activates the laser at the rear of the rocket, there is a compressed spring at the rear with stored energy E and an identical <u>uncompressed</u> spring at the front of the rocket.

Bev had noticed these strange devices and gadgets in Anne's rocket but paid them no mind. But now she asks Anne for a demonstration. Anne was waiting for an excuse to show Bev how she could "magically" move the energy stored in the compressed spring from the rear to the front of the rocket.

Bev watched as Anne used the energy in the compressed spring to power the laser that then produced a light flash. When the flash reached the front of the rocket, its energy was captured by Anne's clever machine and used to compress the spring.

It is important to note that the laser flash was on stage for only about 100 ns, the time it took to go from the rear to the front of the rocket. All the energy carried across the rocket by the flash was converted into spring energy at the front of the rocket. The net result of this experiment was the transfer of the energy stored in a compressed spring at the rear of the rocket to an identical spring at the front. The laser and other required mechanical devices were introduced to facilitate this transfer of energy but after the dust settled, there was an uncompressed spring at the rear of the rocket and a compressed spring in the front. The net result is that energy E has been transferred from the rear to the front of the rocket.

Meanwhile, Bev who barely noticed the laser flash cross Anne's rocket, stared agape as she saw a spring at the rear of Anne's rocket uncompress and an instant later an uncompressed spring at the front of Anne's rocket became compressed.

The original stored energy could have been chemical instead of mechanical with a battery replacing the spring. Then the device at the front of the rocket would have had to convert the laser flash into chemical energy in the battery. The spring was chosen because it is easier to visualize the energy stored in a spring than it is to picture the energy stored in a battery. The key thing was that a <u>light flash</u> was used to move stored energy from one end of the rocket to the other.

After getting over the surprise produced by the transfer of energy from one spring to the other, Bev notices that the two ships are no longer perfectly aligned. Instead, just like when Anne walked from the rear to the front of her rocket, Anne's rocket has recoiled a very small distance, x.

Bev is amazed by this shift. She asks Anne, "How can the two ships end up displaced by a distance x when no mass was sent from the rear to the front of the ship? Equation 12.6 clearly shows that the displacement of the rocket was proportional to the mass that moved from the rear to the front of the rocket. The only thing that changed during your demonstration was the compressed spring at the rear of the rocket became uncompressed while the uncompressed spring at the front of the rocket ended up compressed." The two friends puzzle over this strange result.

The last bit of information needed for the derivation was well known in the nineteenth century. Light, even when envisioned as an electromagnetic wave, was known to carry both energy and momentum. The energy and momentum in light were simply connected by the equation, Momentum = Energy/Speed of Light, or

$$p = E/c, \qquad (12.7)$$

where p is the symbol for momentum.

Remember, the laser was attached to the rocket and stationary before the flash was emitted. If the flash carried momentum from the back to the front of the rocket, the laser and rocket had to recoil to conserve momentum. The rocket was recoiling only for the very short time that the flash was in flight, but that was long enough for Anne's rocket to shift with respect to Bev's. <u>This recoil distance is very tiny but is in principle measureable.</u>

Anne looking at Bev says, “Since the only thing that changed was the state of compression of the two springs, it seems like we are forced to conclude that the mass of a compressed spring must be larger than an uncompressed spring. So the laser flash effectively moved mass from the back to the front of the rocket by moving the energy in the rear spring to the front spring. Transferring energy must be equivalent to transferring mass!”

Bev jumps in and points to equation 12.5 and tells Anne, “I measured the shift x caused by moving the spring energy from the back to the front of the rocket. And we know the length and mass of the rocket. So we can easily calculate the amount of mass that had to move to cause the observed shift!”

Bev suggests they repeat the earlier calculation with the light flash replacing Anne. Just like before, from Bev’s perspective, in time T the rocket recoiled a distance x while the flash moved a distance L – x. So the ratio of speeds is given by an equation identical to equation 12.3 above,

$$\frac{\mathit{Speed\ of\ Rocket}}{\mathit{Speed\ of\ Light\ Flash}} = \frac{x/_T}{(L-x)/_T} = \frac{x}{L-x}\,. \qquad (12.8)$$

Conservation of momentum requires that the magnitude of the recoiling rocket’s momentum equal the magnitude of the light flash’s momentum,

$$M_{rocket}V_{rocket} = p_{flash} = E_{flash}/c. \qquad (12.9)$$

But E_{flash} is just equal to E_{sping}, the energy originally stored in the spring which is also the amount of energy transferred. Bev solves equation 12.9 for V_{rocket},

$$V_{rocket} = E_{spring}/M_{rocket}c,$$

and uses that to find another expression for the ratio of speeds,

$$\frac{\mathit{Speed\ of\ Rocket}}{\mathit{Speed\ of\ Light\ Flash}} = \frac{V_{rocket}}{c} = E_{spring}/M_{rocket}c^2. \qquad (12.10)$$

Getting excited, Bev equates the two different expressions for the ratio of the speeds, equations 12.8 and 12.10 to get,

$$\frac{x}{L-x} = \frac{E_{spring}}{M_{rocket}c^2}\,. \qquad (12.11)$$

Now she plugs equation 12.11 into 12.5 to get the mass equivalent of the transferred energy, E_{spring},

$$m_{equivalent} = \frac{E_{spring}}{c^2}.$$

Anne quickly solves for E_{spring} to get,

$$E_{spring} = m_{equivalent}c^2. \qquad (12.12)$$

This looks much like Einstein's famous equation relating mass to energy. But the above argument is much more limited in that it shows that the mass of a spring is increased by a tiny amount when it is compressed. In the end, the more general form of the equation that states that mass and energy are equivalent, $E = mc^2$, has been tested experimentally over and over and is now assumed to be true.

Note that the amount of mass associated with a given amount of energy is miniscule since c in the above equation is the speed of light in meters per second, 3 x 10^8 m/s. The mass equivalent of 1 Joule is 1.11 x 10^{-17} kg or written out with all the zeroes 0.0000000000000000111 kg! (Remember Appendix B is a primer on scientific notation.)

On the other hand, the energy equivalent of 1 kilogram is 9 x 10^{16} Joules, an astonishingly large amount of energy. To put that number into a better perspective, a typical house may use 1000 kilowatt hours in average month. In Joules, that is 3.6 x 10^9, or 3.6 billion Joules. The energy equivalent of 1 kilogram, if convertible to energy with 100% efficiency, could supply the typical house with electricity for about 2 million years!

Galileo established the equivalence between moving at constant speed and being stationary. Our intrepid spaceketeers saw this first hand in the experiments done with the bus and super bus. In Chapter Ten, the equivalence between acceleration and gravity was used to study the motion of an accelerating rocket. In this chapter, we established the equivalence between mass and energy. These last two equivalences were discovered by Einstein and represent great advances in our understanding of space and time.

This completes our journey into the strange world of Special Relativity which included a peek at General Relativity.

Appendix A: Graphing

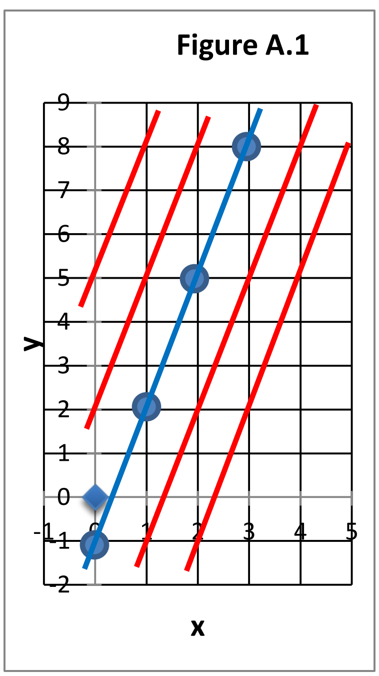
The typical graph, x vs y, in a beginning algebra class has x values plotted in the horizontal direction and y values in the vertical direction. Any straight line on an x vs y graph can be represented by the equation,

$y = mx + b.$

As a concrete example consider the line represented by the equation $y = 3x - 1$. In math parlance, x is the independent variable and y the dependent variable. Pick any value for x and use the equation to find the corresponding value of y. A good way to visualize the relationship between x and y is by making a table of associated values.

x	y
0	-1
1	2
2	5
3	8

A particular pair of associated values are written as (1, 2) or (3, 8) or more generally (x, y) with the understanding that the pair represent a point on the line. Although it only takes two points to specify a given straight line, it is a good idea to use more than two points to draw a line on a graph. Using more than two points acts as insurance against any numerical errors that might happen when using the equation $y = 3x - 1$. The four points in the table all have to lie on the same straight line. If an error was made, that will not happen. The graph, figure A.1, shows the four points as blue circles with the line representing $y = 3x - 1$ in blue.



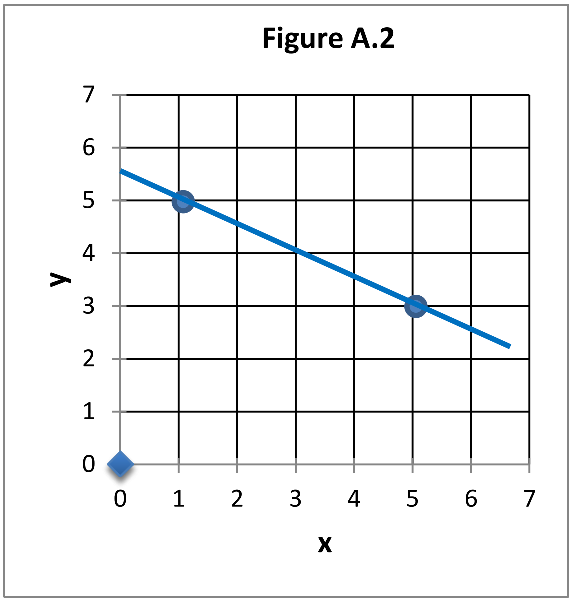
The table and the graph both show that each time x increases by 1, y increases by 3. That ratio, the change in y divided by the change in x, $\Delta y/\Delta x$, is just the coefficient of x in the equation $y = 3x - 1$. In the general equation, $y = mx + b$, that ratio is m and is called the **slope** of the straight line.

The red lines on figure A.1 represent other members of the family of lines with a slope of 3. Notice that each line intercepts the y-axis at a different place. The y-axis represents all the places on the graph where $x = 0$. Therefore the place where a given line intercepts the y-axis can easily be found by setting $x = 0$ in the equation $y = mx + b$. The parameter b is called the **y-intercept** of the straight line. For our example, $b = -1$.

Every straight line is characterized by those two parameters; the slope which is a measure of the steepness of the line and the y-intercept which locates the line on the y-axis. Whenever the equation of a straight line is given, it is easy to rearrange it to put in into the standard form y = mx + b.

Question A.1: What are the slope and y-intercept of the line represented by the equation, 3x – 2y = 6. Make a table of values and draw the line. (Hint: First manipulate the equation into the form y = something.)

Drawing a graph of the line represented by a given equation is straightforward. The inverse problem is to find the equation representing a given straight line. Consider the line shown in figure A.2.



It looks like the y-intercept is halfway between 5 and 6, 5 ½. We can use the two circled points to find the slope. Notice that y decreases by 2 as x increases by 4. That corresponds to a negative slope of -1/2. The equation y = mx + b for the line in figure A.2 is y = -(1/2)x + 5 ½. .

Another method for finding the equation of line in figure A.2 is to use the x and y values of two "convenient" points that lie on the line. Convenient values are ones that are easy to read off of the graph. In this example it is clear that the line goes through the pair of points (1, 5) and (5, 3). Therefore the equation y = mx + b has to be satisfied when those values are substituted in for x and y. Making the substitution leads to the two equations:

5 = 1 m + b

3 = 5 m + b.

These two equations can be solved for m and b in a variety of ways. For example, subtract the second equation from the first to cancel the b's. The result is,

2 = -4 m,

or m = - ½ . Now replace m by – ½ in either of the equations to find b,

$5 = 1 (- ½) + b$,

gives $b = 5 + ½ = 5 ½$.

These are the basic skills required to make sense of the spacetime graphs in this book. The only difference between an x vs y graph and a spacetime graph is that for a spacetime graph the vertical axis is time, t, instead of y. Therefore the standard equation for a straight line on a spacetime graph is, $t = mx + b$, instead of $y = mx + b$.

Appendix B: Scientific Notation

Scientific notation was developed to make it easier to deal with large numbers like 40,000,000,000 and small numbers like 0.000000000000053.

The basic form is,

$$N \times 10^n,$$

where N is a number between 1 and 10 that is multiplied by 10 raised to the n^{th} power. "and n is a positive or negative integer or zero.

When you see a number like 10^9, it really means 1×10^9.

Some examples: $10^3 = 1000$, $10^2 = 100$, $10^{-3} = 0.001$, $10^{-6} = 0.000001$

To change from scientific notation to regular notation for positive values of n all you need to do is move the decimal point to the right n times. Zeros are added to fill the places created by moving the decimal point. For example,

$3.4 \times 10^6 = 3{,}400{,}000.0$ (the decimal moved to the right 6 places)

To change from scientific notation to regular notation when n is negative all you need to do is move the decimal point to the left n times. Again zeros are added to fill the places created by moving the decimal. For example,

$5.87 \times 10^{-8} = 0.0000000587$ (the decimal moved to the left 8 places)

To multiply two numbers in scientific notation, you multiply the numbers in front of the powers of ten and add the numbers representing the powers of ten. After performing that calculation, you can than adjust the decimal and the powers of ten to make the number N fit into the range between 1 and 10.

Example: $(3.0 \times 10^7) \times (4.0 \times 10^{-10}) = 12.0 \times 10^{7-10} = 12.0 \times 10^{-3} = 1.2 \times 10^{-2}$

To divide two numbers in scientific notation, you divide the numbers in front of the powers of ten and subtract the denominator's power of ten from the numerator's. You can adjust the decimal and the powers of ten to make the number N greater than 1 and less than 10.

Example: $(3.0 \times 10^7) \div (4.0 \times 10^{-10}) = 0.75 \times 10^{7-(-10)} = 0.75 \times 10^{17} = 7.5 \times 10^{16}$

Note that $10^0 = 1$ (This is necessary if dividing a number by itself is going to give the correct answer, namely 1!)

Appendix C: The Light Clock

Clocks, watches, and bacteria are complicated things. Therefore trying to determine why a stationary Rolex watch ticks at a different rate than a moving one is daunting. Even more daunting is trying to come to grips with why a Petri dish with a colony of bacteria moving through the laboratory grows more slowly than a Petri dish with an identical strain of bacteria that is stationary in the lab.

The Principle of Relativity mandates that all the clocks, watches, and bacteria keep time at identical rates in any given laboratory. So if the FAO from Chapter Eleven, who is stationary with respect to Earth, sees one sort of clock running slow in the laboratory aboard a super bus moving at 0.8 ft/ns, then the whole collection of assorted time pieces on that bus have to run slow at the same rate.

Instead of the typical internally complicated time pieces that we are used to, imagine a device one-half foot high with mirrors on top and bottom. The bottom mirror goes "tick" every time light reflects off of it. Since light travels one foot in one nanosecond, this light clock ticks once every nanosecond. This wonderfully simple clock with no moving parts except for a light flash bouncing back and forth between the two mirrors has been given the obvious and descriptive name of "Light Clock."

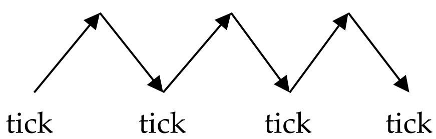
Four identical light clocks are constructed. Two are set up in an Earth-based lab and arranged perpendicular to one another, one parallel to the floor and the other stands vertically. The diagram below shows the spatial relationship of the clocks as seen by the FAO.

The other two clocks are arranged the same way in a super bus that will drive past the Earth-based lab with a relativistic velocity.

Of course the two clocks in the Earth-based lab tick at the same rate since the orientation of the clocks cannot affect their ticking rates. Likewise, the two clocks in the super bus tick at identical rates.

When the FAO watches the two vertical clocks, one stationary and the other zipping through the lab at v_{bus}, she sees the light flash in the stationary light clock going up and down, ticking once every nanosecond. On the other hand, she sees the light flash in the moving vertical clock taking a zigzag path as shown below in figure C.1. Obviously that path is longer than the simple up and down path followed by the light flash in the lab. Because of the Constancy of the Speed of Light, the FAO sees the vertical clock in the super bus ticking more slowly than the vertical light clock in the lab.

Figure C.1



Moreover, the FAO sees the horizontal clock in the super bus ticking at a slower rate that is identical to the rate she sees on the vertical clock. Some simple algebra will be used to show that the vertical clock in the super bus ticks off $\sqrt{1-(\frac{v}{c})^2}$ nanoseconds for each nanosecond ticked off the stationary light clock. It will also be shown that in order for the horizontal and vertical clocks in the super bus to tick at the same rate, the horizontal clock, as actually seen by the FAO, must be shorter than the identical horizontal clock in the lab.

As the above arguments are fleshed out below, keep in mind that these conclusions are based on the Principle of Relativity and the Constancy of the Speed of Light and are not dependent on any of the material presented earlier in this book!

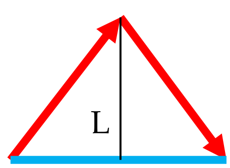


Figure C.2

The FAO sees the light in the vertical clock in the bus travel up and down the isosceles triangle in figure C.2. She uses her watch to measure how long it takes the flash to make one trip up and down, T_{FAO}, in the moving vertical light clock. The height of the clock is L. During that time, the FAO sees the light clock move a distance $v_{Bus}\, T_{FAO}$, the blue line, and the light flash move cT_{FAO}, the sum of the two red lines. As usual, c is the speed of light. Therefore the length of the base of either of the right triangles in the above diagram is just ½ $v_{Bus}\, T_{FAO}$ while either hypotenuse is ½ c T_{FAO}. The FAO uses the Pythagorean Theorem to relate the two legs of the right triangle to the hypotenuse,

$$(\frac{cT_{FAO}}{2})^2 = L^2 + (\frac{v_{Bus}T_{FAO}}{2})^2. \qquad \text{(C.1)}$$

When she solved equation C.1 for T_{FAO} she got,

$$T_{FAO} = \frac{2L/c}{\sqrt{1-(\frac{v_{Bus}}{c})^2}} \qquad \text{(C.2)}$$

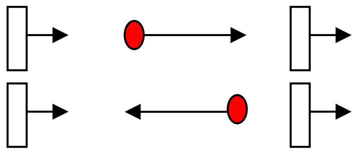
The factor 2L/c in the numerator is just the amount of time it takes a light flash to make one trip up and down in the stationary light clock. Therefore the FAO sees that it takes more time for the light flash in the moving light clock to make one round trip compared to the flash in the stationary one. And the difference in rates is proportional to $\sqrt{1-(\frac{v_{Bus}}{c})^2}$ just like expected.

Question C.1: Do the algebra necessary to go from equation C.1 to C.2.

In the discussion above it was implicitly assumed that the height of the moving and stationary clocks remain the same height. If the heights did change, then the top of the clock on the bus would pass above or below the top of the clock in the lab. Neither possibility is allowed by the Principle of Relativity. For example, if the top of the bus clock passed above the stationary clock, the FAO would conclude that moving objects perpendicular to the direction of motion appear taller than identical stationary objects. But a different FAO attached to the reference frame of the super bus would have seen the moving clock, the one in the lab, as shorter than the identical clock in the super bus. The only conclusion consistent with the Principle of Relativity is that the heights of the moving and stationary vertical clocks have to remain the same. Also notice that the isotropy of space, the equivalence of moving left and right, snuck into the argument because the two FAOs see the "moving" frame going in opposite directions, but the height of objects moving to the left have to behave just like objects moving to the right.

The FAO sees the moving horizontal light clock also tick every T_{FAO} seconds, synchronously with the moving vertical clock. The light flash in the horizontal clock follows an asymmetric path from the perspective of the FAO. It takes the light flash, the red oval, longer to reach the receding mirror on the right than it takes on the return trip where the left-hand mirror is moving toward the flash. See figure C.3 below. But the total time for the round trip has to be T_{FAO} seconds.

Figure C.3



The FAO views the light flash make a round trip in the horizontal clock and writes down the following equation,

$T_{FAO} = T_{Right} + T_{Left}$,

Where T_{Right} is the time it takes the light flash to reach the receding mirror, top picture, and T_{Left} is the time it takes to return to the starting point. She uses the above diagram as a guide to writing down the following two equations where L′ is the length of the horizontal light clock. She uses L′ for the length because, at this point in the book, she correctly suspects that the L′ will not be equal to L, the height of the moving vertical light clock.

$cT_{Right} = L' + v_{Bus}\, T_{Right}$ and $cT_{Left} = L' - v_{Bus}\, T_{Left}$

Question C.2: Solve the above equations separately for T_{Right} and T_{Left}.

Question C.3: The FAO knows that the total time for the light flash to make one roundtrip in the moving horizontal clock is just $T_{FAO} = \frac{2L/c}{\sqrt{1-(\frac{v_{Bus}}{c})^2}}$ and that time has to also equal $T_{Right} + T_{Left}$. Substitute your answers from question C.2 into the equation for T_{FAO} and solve for L′.

After doing a little algebra, the FAO arrived at the following equation,

$$\frac{2L/c}{\sqrt{1-(\frac{v_{Bus}}{c})^2}} = \frac{2L'/c}{(1-(\frac{v_{Bus}}{c})^2)},$$

which she immediately solved for L′ to get,

$$L' = L\sqrt{1-(\frac{v_{Bus}}{c})^2}\ . \qquad \text{(C.3)}$$

She concludes that in order for the moving horizontal light clock to tick at the same rate as the identical moving vertical light clock, the horizontal clock has to be shorter by the factor $\sqrt{1-(\frac{v_{Bus}}{c})^2}$. Again this is just what was expected.

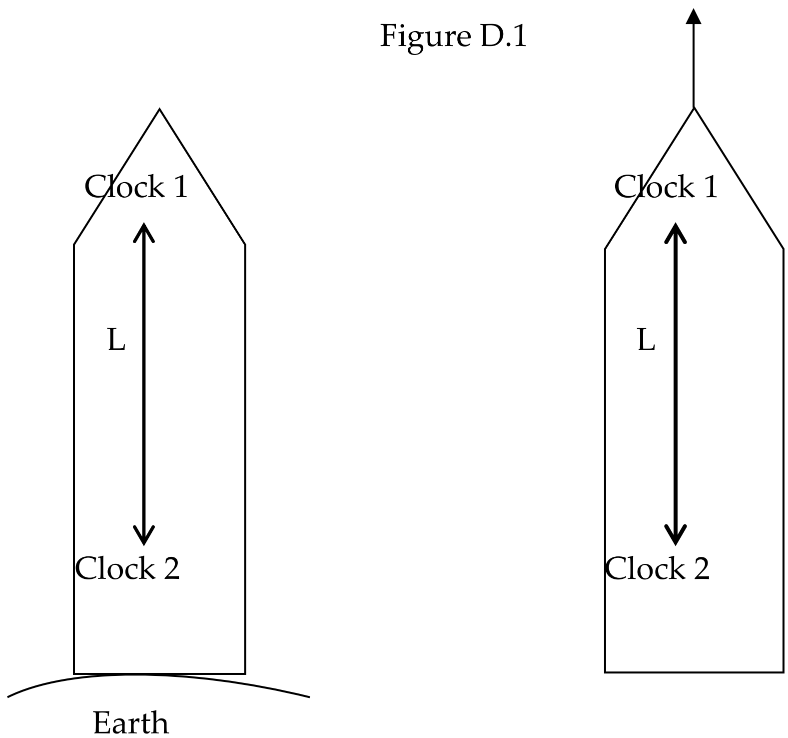
When the moving horizontal light clock passes the stationary horizontal clock in the lab, the FAO sees the moving clock in real time as being shorter by just this amount. This is not an optical illusion. The FAO, because of her distant vantage point sees the moving light clocks running slow compared to the identical clocks in the lab. Furthermore, she sees the moving horizontal clock foreshortened compared to the identical stationary light clock.

The Principle of Relativity forces her to conclude that all moving clocks have to run at the same rate as the light clocks in the super bus while all lengths parallel to the direction of motion have to shrink by the same amount as the horizontal light clock. These effects are exactly the ones observed by the FAO in Chapter Eleven.

Appendix D: The Gravitational Clock Effect

Figure D.1 shows two identical rockets, the one on the left is sitting on Earth and the one on the right is accelerating through empty space.

Figure D.1



The distance between the two clocks is L. The time it takes light to travel from Clock 2 to Clock 1 is L/c. During that time, the rocket's speed changes by gt = $\frac{gL}{c}$, where g is the acceleration of gravity, 32 ft/s^2 on Earth.

The local inertial observers who were stationary with respect to the rocket when Clock 2 sent the signal to Clock 1 conclude that the signal actually traveled L + $\frac{1}{2}$gt^2. But the extra distance, which is just the average velocity, $\frac{0+gt}{2}$, times t, is so small compared to L that it can be ignored.

Question D.1: Calculate the extra distance, $\frac{1}{2}$ gt^2, when L = 1000 ft. *This ought to convince you that ignoring that extra piece is more than reasonable!*

The observers at Clock 1 see the signals received from Clock 2 Doppler shifted to a lower frequency. That shift is easily calculated by replacing v in equation 9.10 with -gt,

$$f_{received} = f_{sent}\,\frac{\sqrt{1-\frac{gL}{c^2}}}{\sqrt{1+\frac{gL}{c^2}}}\,. \qquad \text{(D.1)}$$

The minus is used for the velocity because Clock 2 is receding from Clock 1. Now we can use the binomial approximation to write $\frac{\sqrt{1-\frac{gL}{c^2}}}{\sqrt{1+\frac{gL}{c^2}}}$ as $\left(1-\frac{1}{2}\frac{gL}{c^2}\right)(1-\frac{1}{2}\frac{gL}{c^2})$ which when multiplied out and put back into equation D.1 gives,

$$f_{received} = f_{sent}\left(1-\frac{gL}{c^2}\right). \qquad \text{(D.2)}$$

The squared term, $(\frac{1}{2}\frac{gL}{c^2})^2$, was dropped because it was much smaller than the $\frac{gt}{c}$ term in equation D.2. The received frequency is less than the sent frequency which means Clock 1 is running faster than Clock 2.

Because of the equivalence between acceleration and gravity, the two clocks in the rocket sitting on Earth have to run at the same rates as those in the rocket accelerating in empty space. Therefore equation D.2 not only shows that the clock at the front of the rocket runs faster than all the clocks below but also shows that clocks further from the center of a gravitating body run faster than those closer to the center.

It is also possible to use the equivalence between mass and energy to directly show that Clock 1 runs faster than Clock 2 in the rocket sitting on Earth. To do this we need to introduce two new ideas. The first is that the energy in a light flash is carried by photons, little bundles of energy. The energy of each photon is hf, where h is Planck's constant and f is the frequency of the light. But since energy and mass are equivalent, we can use $E_{photon} = hf_{photon} = m_{photon}c^2$ to define the mass of a photon sent from Clock 2 to Clock 1:

$$m_{sent} = \frac{hf_{sent}}{c^2}\,. \qquad \text{(D.3)}$$

The second idea is that as an object moves up against the pull of gravity it loses energy. The amount of energy lost is equal to the object's weight times the distance moved, mgy. Therefore the photon moving up from Clock 2 to Clock 1 loses energy which, for a photon, corresponds to a drop in frequency. This is the same effect we saw in the accelerating rocket which is not surprising since the Equivalence Principle mandates that clocks in the two rockets behave identically. The mass of the photon that arrives at Clock 1 is given by,

$$m_{received} = \frac{hf_{received}}{c^2}\,. \qquad \text{(D.4)}$$

The average mass of the photon that travels the length L up the stationary rocket is $\frac{m_{sent}+m_{received}}{2}$ so the energy lost by the photon as it traveled between the clocks is,

$\frac{m_{sent}+m_{received}}{2}gL$.

The energy of the photon arriving at Clock 1 is hf_{sent} minus the energy lost moving up against gravity,

$$hf_{received} = hf_{sent} - \frac{m_{sent}+m_{received}}{2}gL \ . \quad \text{(D.5)}$$

Now replace the two masses by equations D.3 and D.4 and cancel the h's,

$$f_{received} = f_{sent} - \frac{\frac{f_{sent}}{c^2}+\frac{f_{received}}{c^2}}{2}gL \ . \quad \text{(D.6)}$$

Now solve equation D.6 for $f_{received}$,

$$f_{received} = f_{sent}\frac{1-\frac{gL}{2c^2}}{1+\frac{gL}{2c^2}} \ . \quad \text{(D.7)}$$

The last step is to use the binomial approximation to simplify equation D.7. This step is analogous to the way equation D.1 was converted into equation D.2 and the results are the same. Namely, equation D.7 becomes,

$$f_{received} = f_{sent}\ \left(1-\frac{gL}{c^2}\right). \quad \text{(D.8)}$$

Equation D.8 is identical to D.2 showing that the difference in clock rates can be calculated by using the Doppler shift in the accelerating rocket or by using the connection between mass and energy in the rocket sitting on Earth. These two different ways of calculating the ticking rate difference between Clocks 1 and 2 reconfirms the consistency of our conclusions. First we used the Doppler Effect, which was derived by using Special Relativity, to calculate the rate difference between the two clocks in the accelerating rocket. The Equivalence Principle connected the clock rates in the accelerating rocket to the rates in the rocket sitting on Earth. Lastly, we used the equivalence between mass and energy to directly show the higher clock in the stationary rocket ran faster than the lower clock by the same factor found by using the Doppler shift in the accelerating rocket. This ends our peek at General Relativity, Einstein's theory of how mass and energy effect space and time.