

*RICHARD C. DORF*

*ROBERT H. BISHOP*

**Instructor's Solutions Manual**  
**for**

*MODERN CONTROL SYSTEMS*

*TWELFTH EDITION*

# MODERN CONTROL SYSTEMS

## SOLUTION MANUAL

Richard C. Dorf  
University of California, Davis

Robert H. Bishop  
Marquette University

A companion to

**MODERN CONTROL SYSTEMS**

**TWELFTH EDITION**

Richard C. Dorf

Robert H. Bishop

**Prentice Hall**

Upper Saddle River Boston Columbus San Francisco New York  
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**Instructor's Solutions Manual**  
**for *Modern Control Systems, 12/E***  
**Richard C. Dorf, *University of California, Davis***  
**Robert H. Bishop, *University of Texas at Austin***

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# P R E F A C E

In each chapter, there are five problem types:

- Exercises
- Problems
- Advanced Problems
- Design Problems/Continuous Design Problem
- Computer Problems

In total, there are over 1000 problems. The abundance of problems of increasing complexity gives students confidence in their problem-solving ability as they work their way from the exercises to the design and computer-based problems.

It is assumed that instructors (and students) have access to MATLAB and the Control System Toolbox or to LabVIEW and the MathScript RT Module. All of the computer solutions in this *Solution Manual* were developed and tested on an Apple MacBook Pro platform using MATLAB 7.6 Release 2008a and the Control System Toolbox Version 8.1 and LabVIEW 2009. It is not possible to verify each solution on all the available computer platforms that are compatible with MATLAB and LabVIEW MathScript RT Module. Please forward any incompatibilities you encounter with the scripts to Prof. Bishop at the email address given below.

The authors and the staff at Prentice Hall would like to establish an open line of communication with the instructors using *Modern Control Systems*. We encourage you to contact Prentice Hall with comments and suggestions for this and future editions.

Robert H. Bishop    rhbishop@marquette.edu

# T A B L E - O F - C O N T E N T S

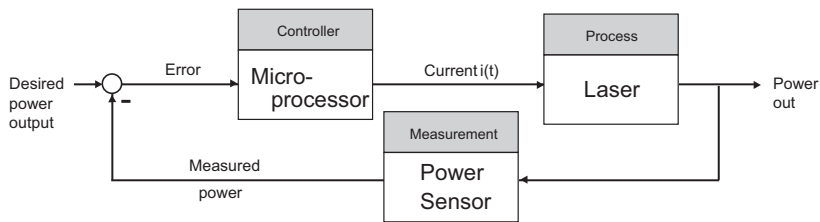
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# Introduction to Control Systems

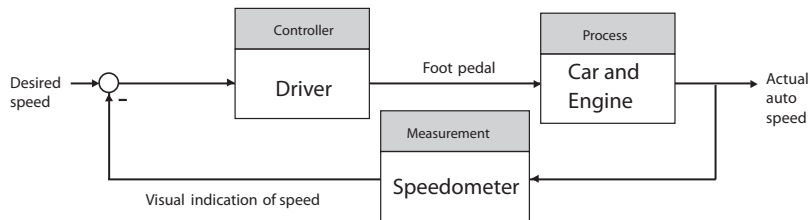
There are, in general, no unique solutions to the following exercises and problems. Other equally valid block diagrams may be submitted by the student.

## Exercises

**E1.1** A microprocessor controlled laser system:

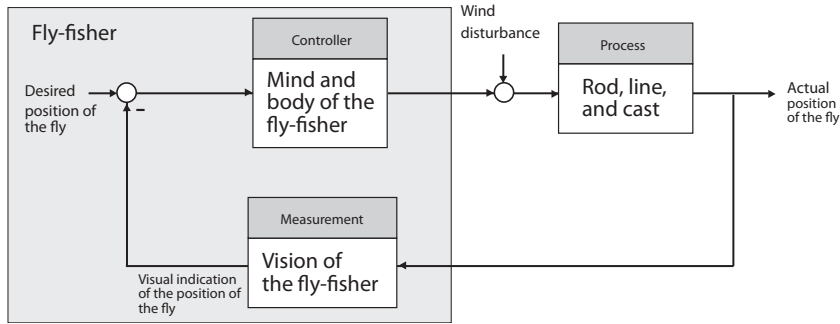


**E1.2** A driver controlled cruise control system:

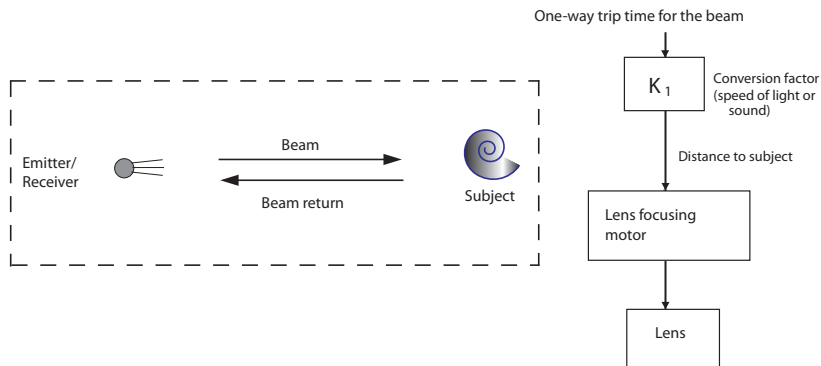


**E1.3** Although the principle of conservation of momentum explains much of the process of fly-casting, there does not exist a comprehensive scientific explanation of how a fly-fisher uses the small backward and forward motion of the fly rod to cast an almost weightless fly lure long distances (the

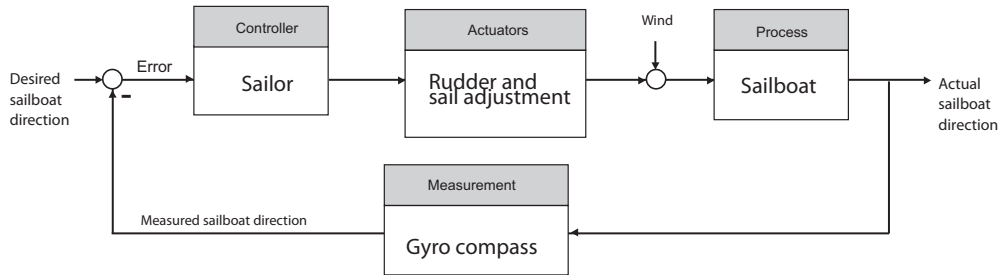
current world-record is 236 ft). The fly lure is attached to a short invisible leader about 15-ft long, which is in turn attached to a longer and thicker Dacron line. The objective is cast the fly lure to a distant spot with dead-eye accuracy so that the thicker part of the line touches the water first and then the fly gently settles on the water just as an insect might.



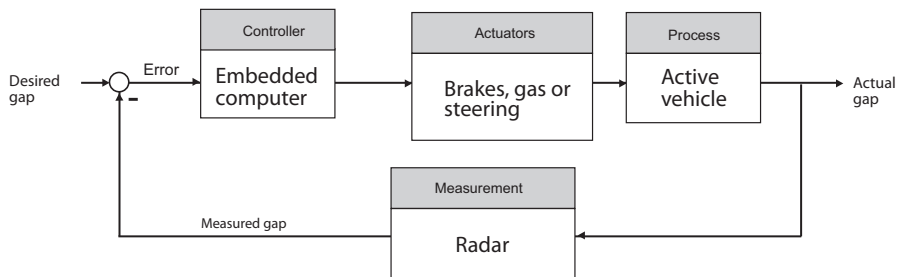
#### E1.4 An autofocus camera control system:



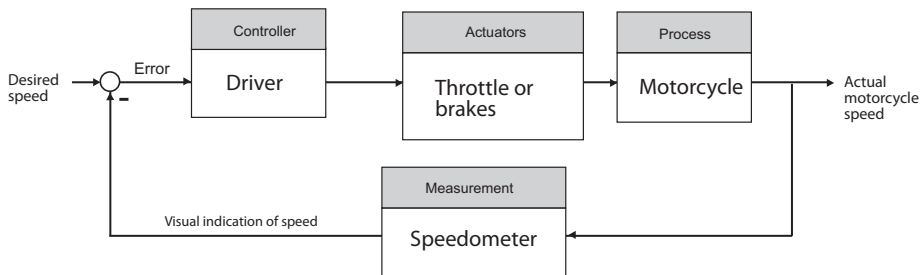
**E1.5** Tacking a sailboat as the wind shifts:



**E1.6** An automated highway control system merging two lanes of traffic:

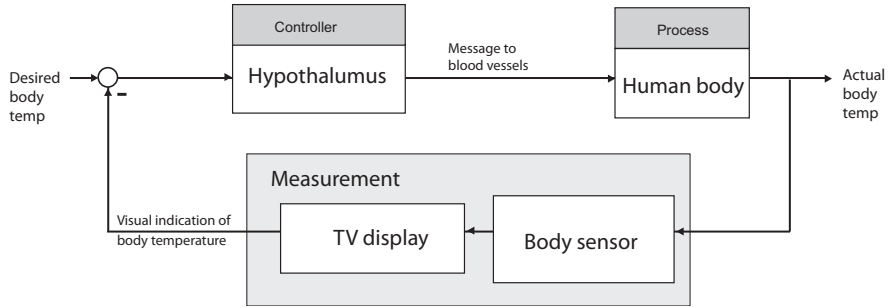


**E1.7** Using the speedometer, the driver calculates the difference between the measured speed and the desired speed. The driver throttle knob or the brakes as necessary to adjust the speed. If the current speed is not too much over the desired speed, the driver may let friction and gravity slow the motorcycle down.

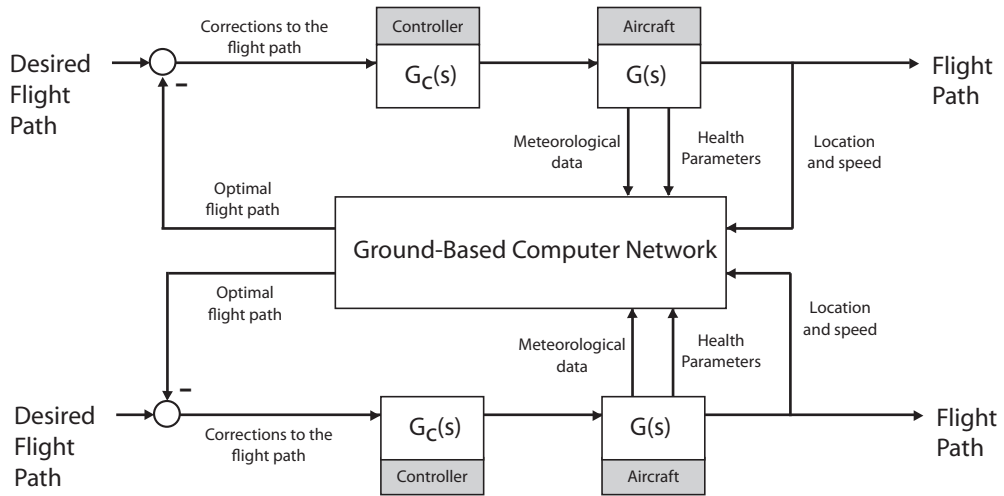




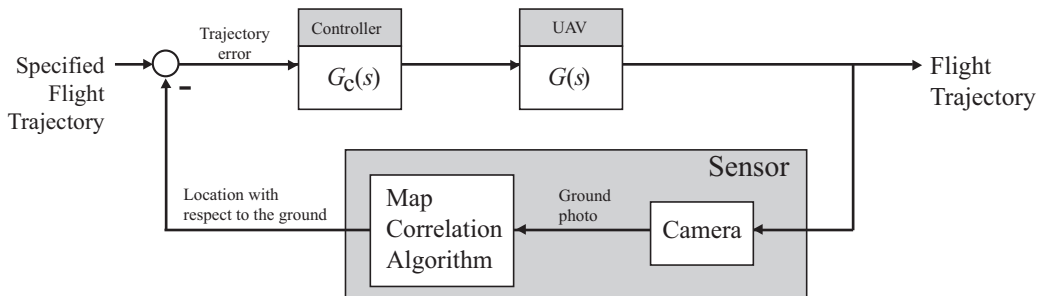
**E1.8** Human biofeedback control system:



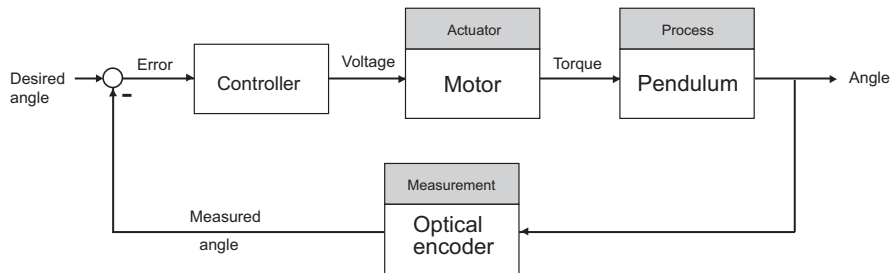
**E1.9** E-enabled aircraft with ground-based flight path control:



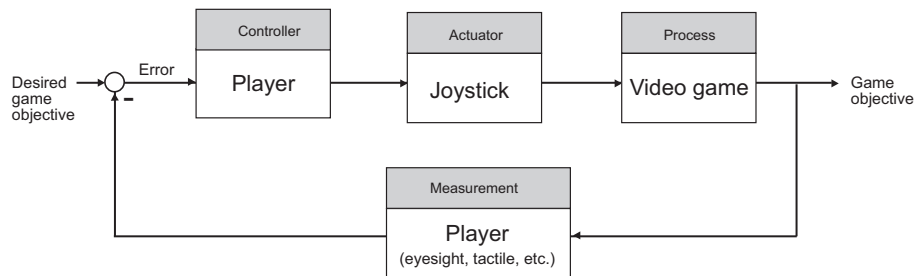
**E1.10** Unmanned aerial vehicle used for crop monitoring in an autonomous mode:



**E1.11** An inverted pendulum control system using an optical encoder to measure the angle of the pendulum and a motor producing a control torque:

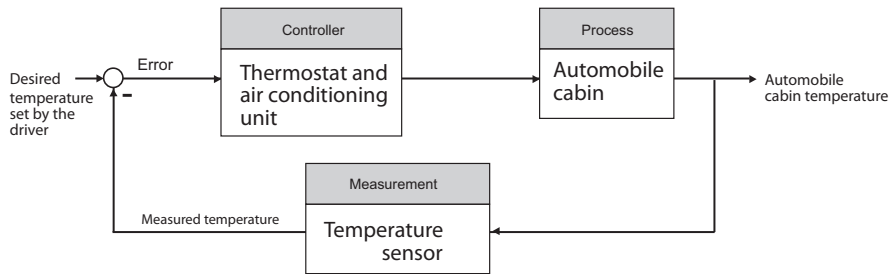


**E1.12** In the video game, the player can serve as both the controller and the sensor. The objective of the game might be to drive a car along a prescribed path. The player controls the car trajectory using the joystick using the visual queues from the game displayed on the computer monitor.

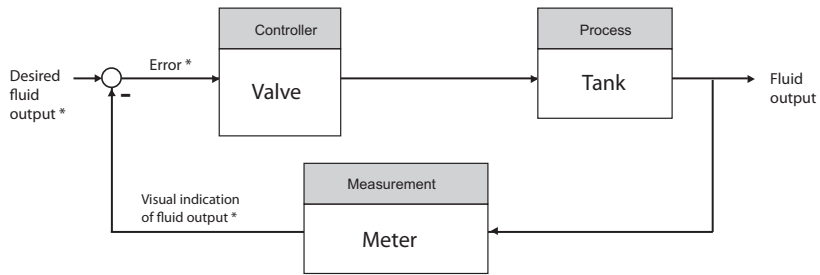


## Problems

**P1.1** An automobile interior cabin temperature control system block diagram:

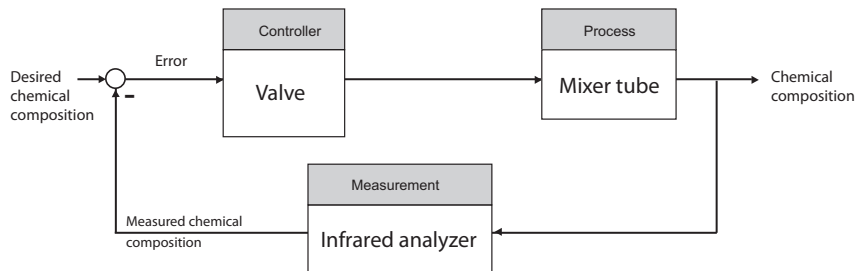


**P1.2** A human operator controlled valve system:

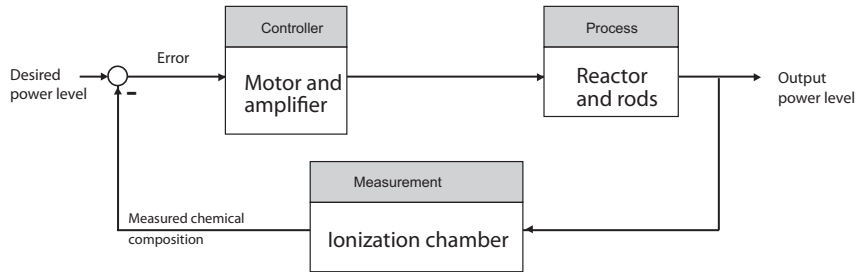


\* = operator functions

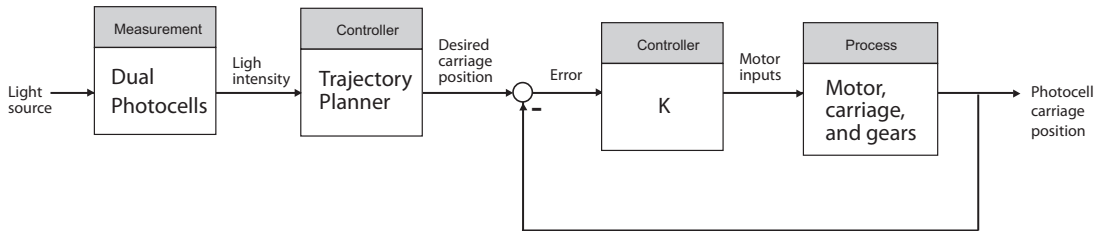
**P1.3** A chemical composition control block diagram:



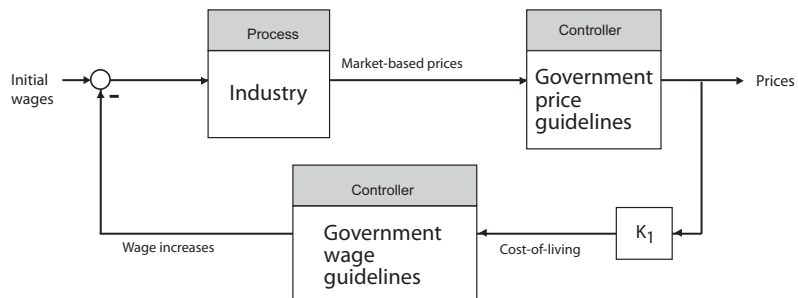
**P1.4** A nuclear reactor control block diagram:



**P1.5** A light seeking control system to track the sun:



**P1.6** If you assume that increasing worker’s wages results in increased prices, then by delaying or falsifying cost-of-living data you could reduce or eliminate the pressure to increase worker’s wages, thus stabilizing prices. This would work only if there were no other factors forcing the cost-of-living up. Government price and wage economic guidelines would take the place of additional “controllers” in the block diagram, as shown in the block diagram.



**P1.7** Assume that the cannon fires initially at exactly 5:00 p.m.. We have a positive feedback system. Denote by  $\Delta t$  the time lost per day, and the net time error by  $E_T$ . Then the following relationships hold:

$$\Delta t = 4/3 \text{ min.} + 3 \text{ min.} = 13/3 \text{ min.}$$

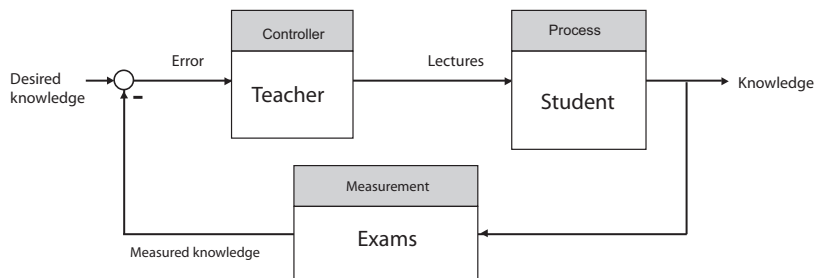
and

$$E_T = 12 \text{ days} \times 13/3 \text{ min./day} .$$

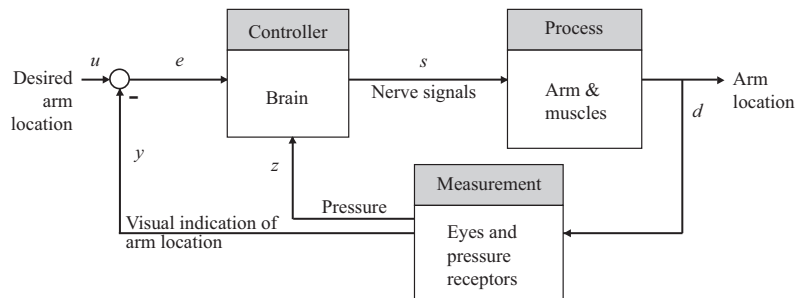
Therefore, the net time error after 15 days is

$$E_T = 52 \text{ min.}$$

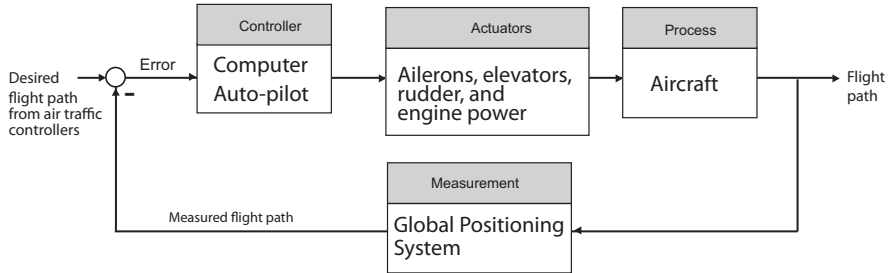
**P1.8** The student-teacher learning process:



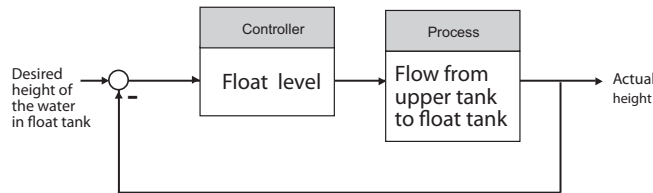
**P1.9** A human arm control system:



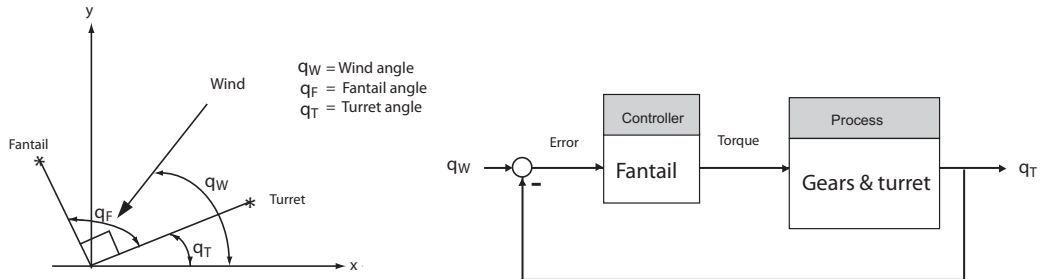
**P1.10** An aircraft flight path control system using GPS:



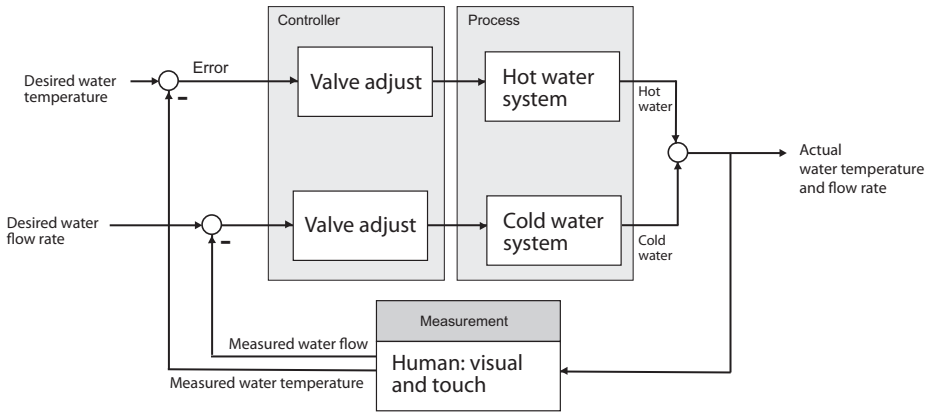
**P1.11** The accuracy of the clock is dependent upon a constant flow from the orifice; the flow is dependent upon the height of the water in the float tank. The height of the water is controlled by the float. The control system controls only the height of the water. Any errors due to enlargement of the orifice or evaporation of the water in the lower tank is not accounted for. The control system can be seen as:



**P1.12** Assume that the turret and fantail are at  $90^\circ$ , if  $\theta_w \neq \theta_F - 90^\circ$ . The fantail operates on the error signal  $\theta_w - \theta_T$ , and as the fantail turns, it drives the turret to turn.



**P1.13** This scheme assumes the person adjusts the hot water for temperature control, and then adjusts the cold water for flow rate control.

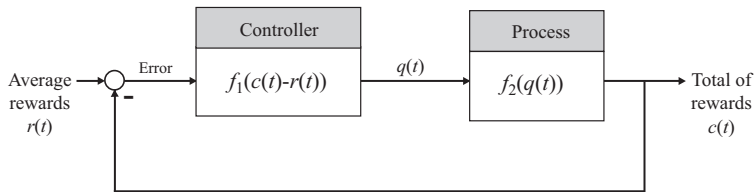


**P1.14** If the rewards in a specific trade is greater than the average reward, there is a positive influx of workers, since

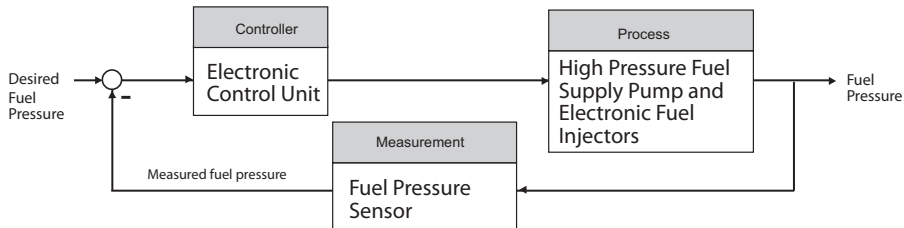
$$q(t) = f_1(c(t) - r(t)).$$

If an influx of workers occurs, then reward in specific trade decreases, since

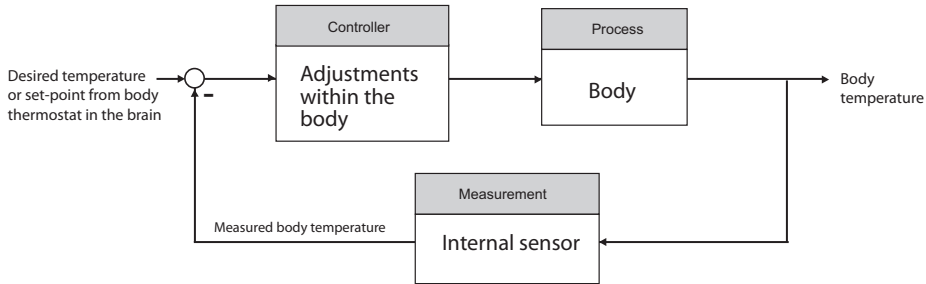
$$c(t) = -f_2(q(t)).$$



**P1.15** A computer controlled fuel injection system:

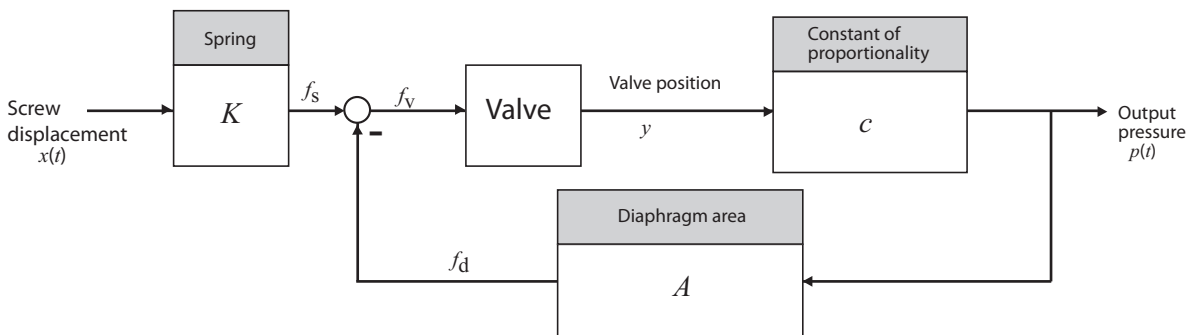


**P1.16** With the onset of a fever, the body thermostat is turned up. The body adjusts by shivering and less blood flows to the skin surface. Aspirin acts to lowers the thermal set-point in the brain.



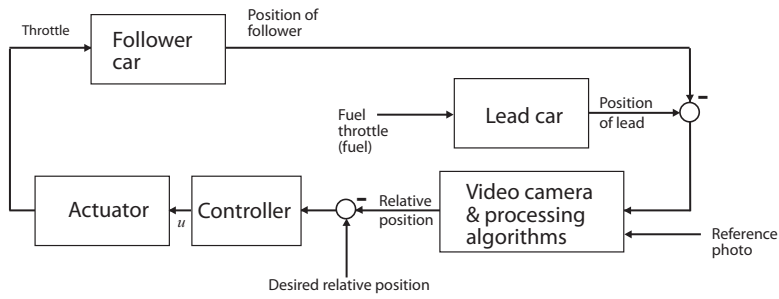
**P1.17** Hitting a baseball is arguably one of the most difficult feats in all of sports. Given that pitchers may throw the ball at speeds of 90 mph (or higher!), batters have only about 0.1 second to make the decision to swing—with bat speeds approaching 90 mph. The key to hitting a baseball a long distance is to make contact with the ball with a high bat velocity. This is more important than the bat’s weight, which is usually around 33 ounces (compared to Ty Cobb’s bat which was 41 ounces!). Since the pitcher can throw a variety of pitches (fast ball, curve ball, slider, etc.), a batter must decide if the ball is going to enter the strike zone and if possible, decide the type of pitch. The batter uses his/her vision as the sensor in the feedback loop. A high degree of eye-hand coordination is key to success—that is, an accurate feedback control system.

**P1.18** Define the following variables:  $p$  = output pressure,  $f_s$  = spring force =  $Kx$ ,  $f_d$  = diaphragm force =  $Ap$ , and  $f_v$  = valve force =  $f_s - f_d$ . The motion of the valve is described by  $\ddot{y} = f_v/m$  where  $m$  is the valve mass. The output pressure is proportional to the valve displacement, thus  $p = cy$ , where  $c$  is the constant of proportionality.

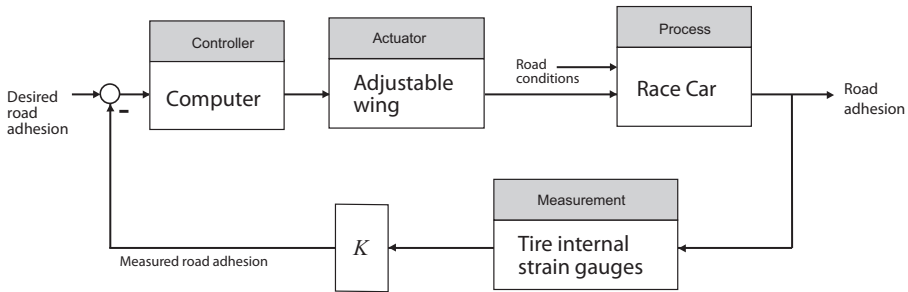




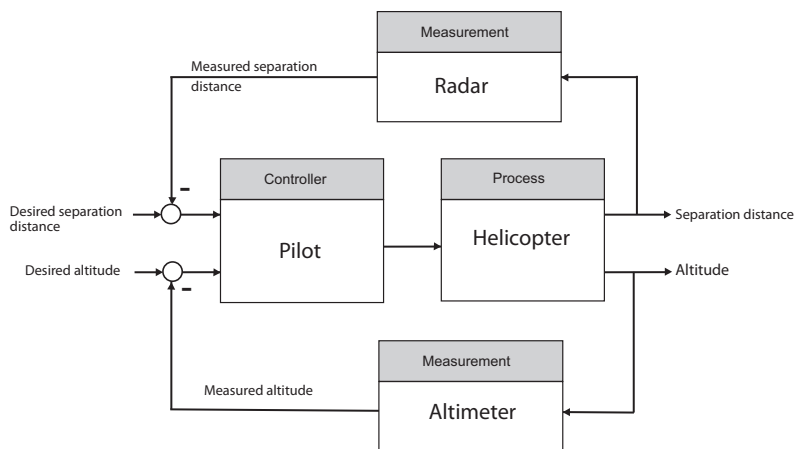
**P1.19** A control system to keep a car at a given relative position offset from a lead car:



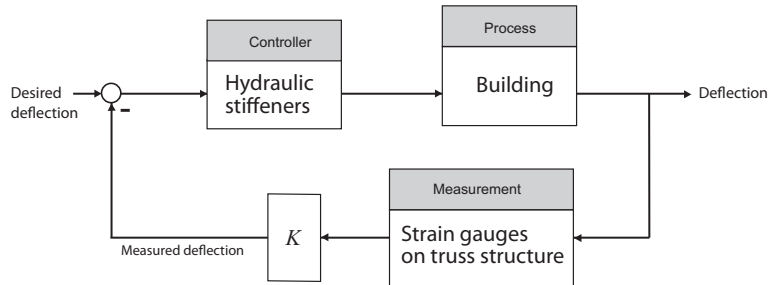
**P1.20** A control system for a high-performance car with an adjustable wing:



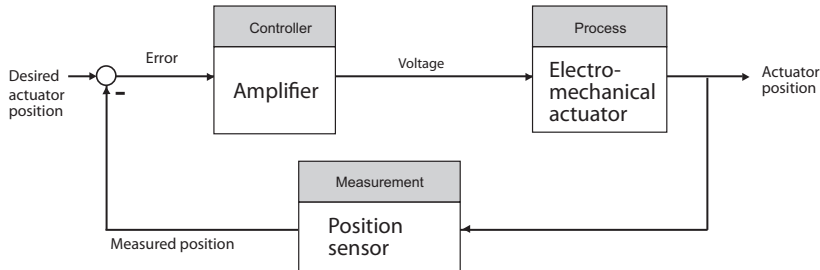
**P1.21** A control system for a twin-lift helicopter system:



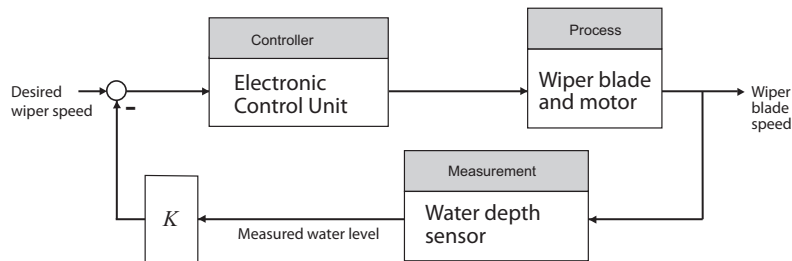
**P1.22** The desired building deflection would not necessarily be zero. Rather it would be prescribed so that the building is allowed moderate movement up to a point, and then active control is applied if the movement is larger than some predetermined amount.



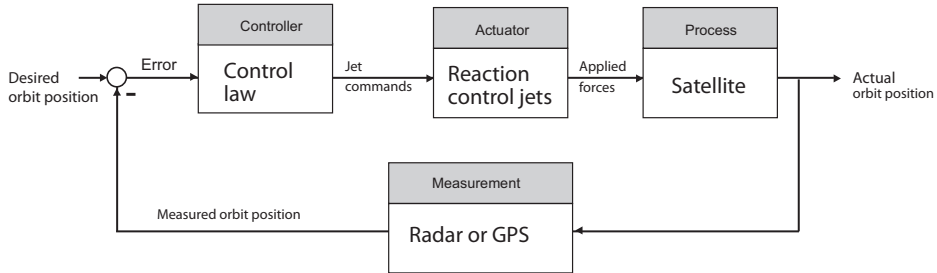
**P1.23** The human-like face of the robot might have micro-actuators placed at strategic points on the interior of the malleable facial structure. Cooperative control of the micro-actuators would then enable the robot to achieve various facial expressions.



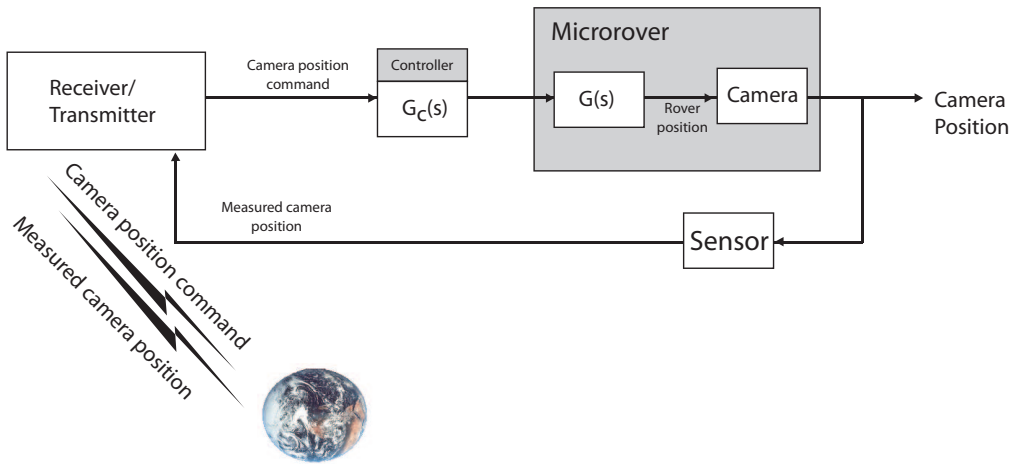
**P1.24** We might envision a sensor embedded in a “gutter” at the base of the windshield which measures water levels—higher water levels corresponds to higher intensity rain. This information would be used to modulate the wiper blade speed.



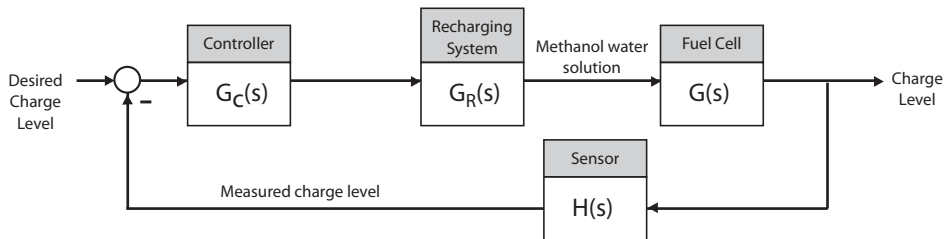
**P1.25** A feedback control system for the space traffic control:



**P1.26** Earth-based control of a microrover to point the camera:

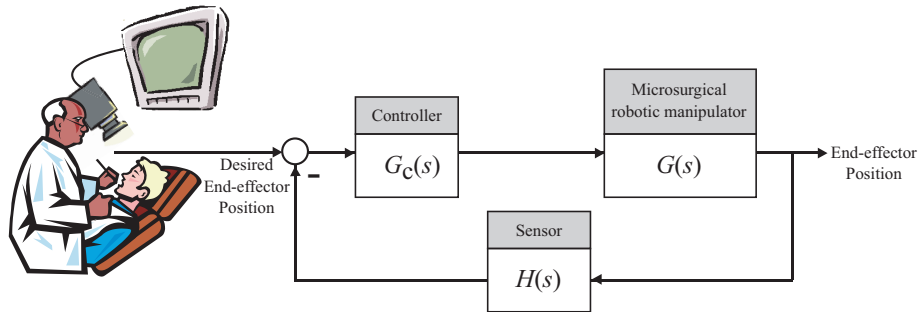


**P1.27** Control of a methanol fuel cell:

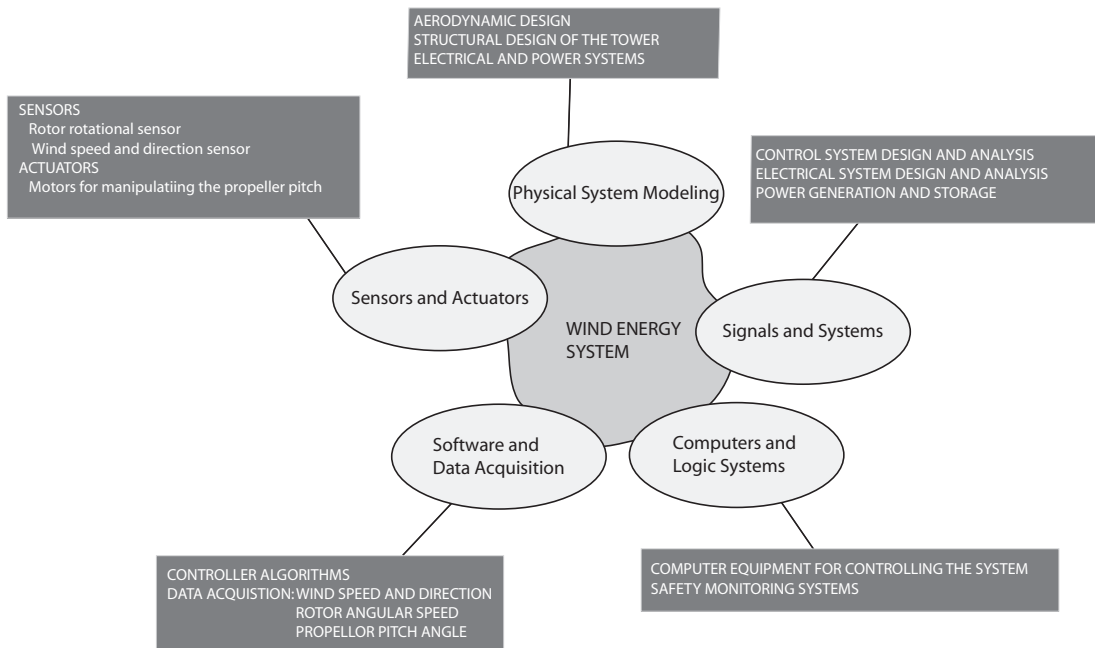


# Advanced Problems

AP1.1 Control of a robotic microsurgical device:

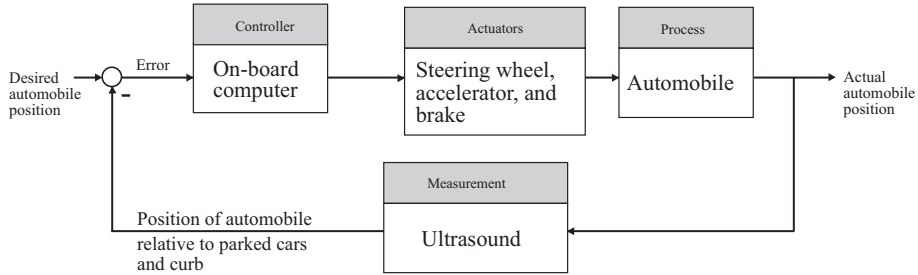


AP1.2 An advanced wind energy system viewed as a mechatronic system:

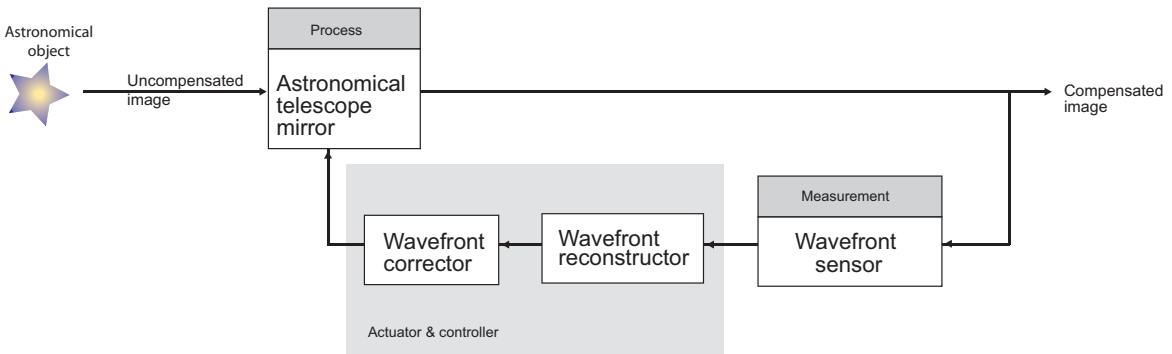


AP1.3 The automatic parallel parking system might use multiple ultrasound sensors to measure distances to the parked automobiles and the curb. The sensor measurements would be processed by an on-board computer to determine the steering wheel, accelerator, and brake inputs to avoid collision and to properly align the vehicle in the desired space.

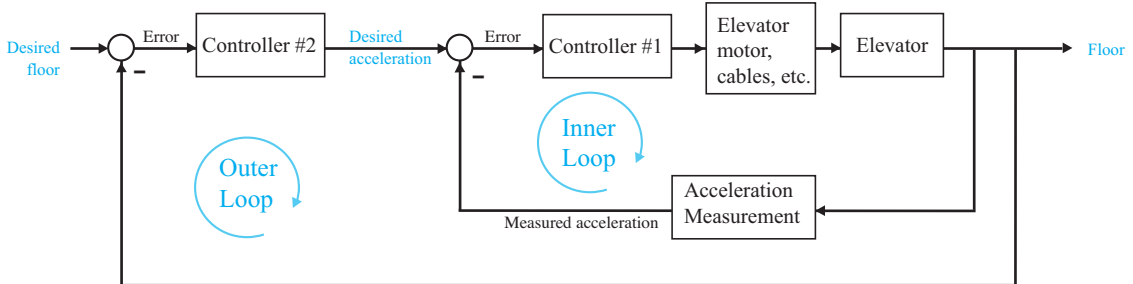
Even though the sensors may accurately measure the distance between the two parked vehicles, there will be a problem if the available space is not big enough to accommodate the parking car.



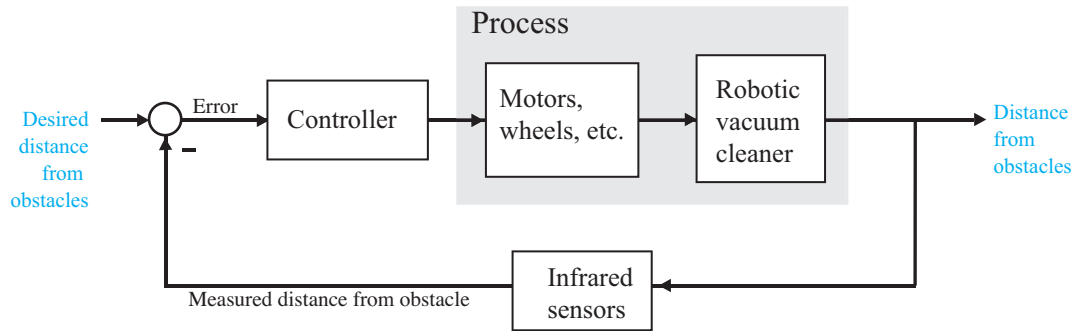
**AP1.4** There are various control methods that can be considered, including placing the controller in the feedforward loop (as in Figure 1.3). The adaptive optics block diagram below shows the controller in the feedback loop, as an alternative control system architecture.



**AP1.5** The control system might have an inner loop for controlling the acceleration and an outer loop to reach the desired floor level precisely.

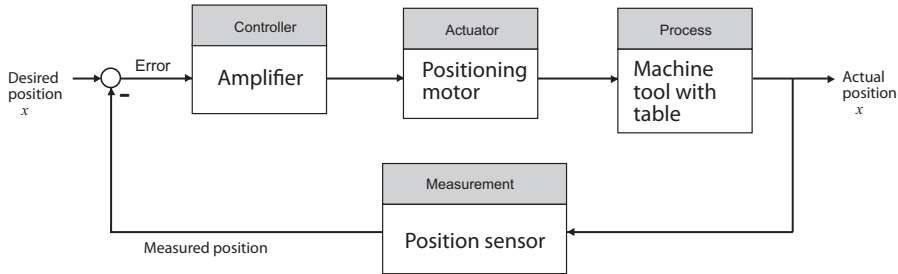


**AP1.6** An obstacle avoidance control system would keep the robotic vacuum cleaner from colliding with furniture but it would not necessarily put the vacuum cleaner on an optimal path to reach the entire floor. This would require another sensor to measure position in the room, a digital map of the room layout, and a control system in the outer loop.

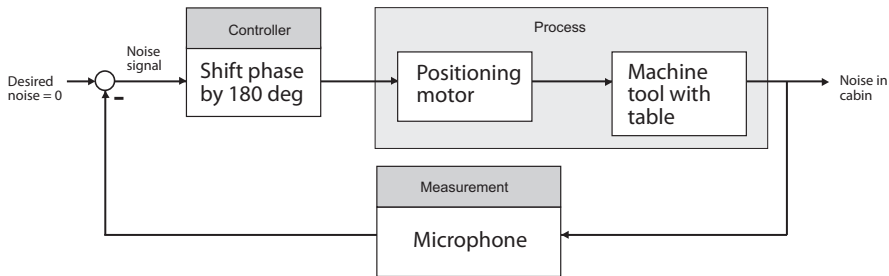


# Design Problems

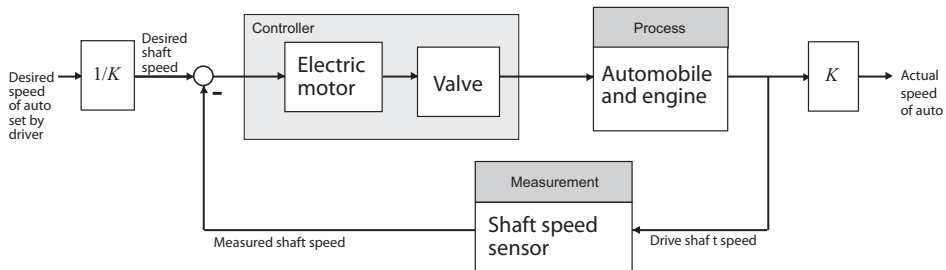
**CDP1.1** The machine tool with the movable table in a feedback control configuration:



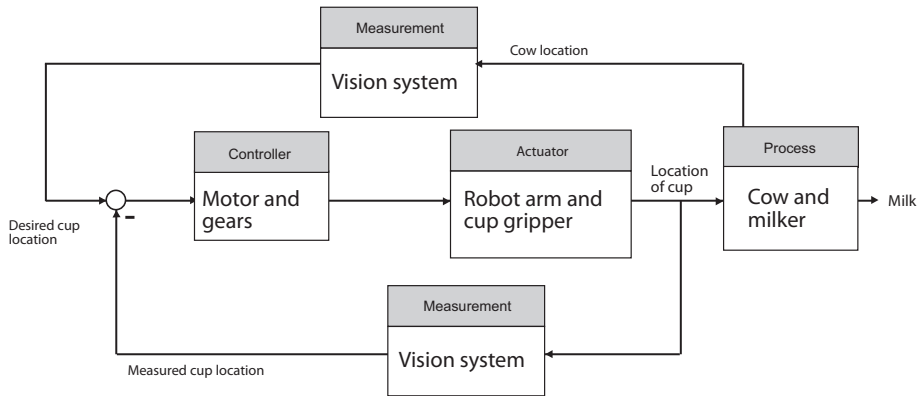
**DP1.1** Use the stereo system and amplifiers to cancel out the noise by emitting signals 180° out of phase with the noise.



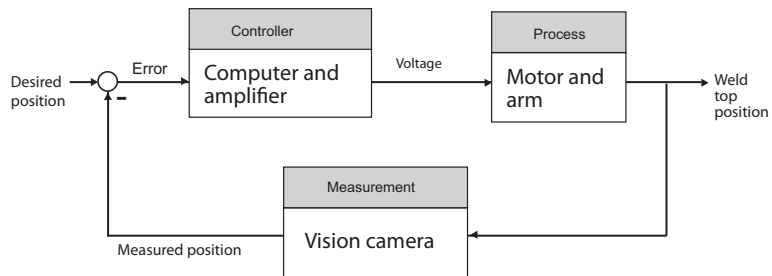
**DP1.2** An automobile cruise control system:



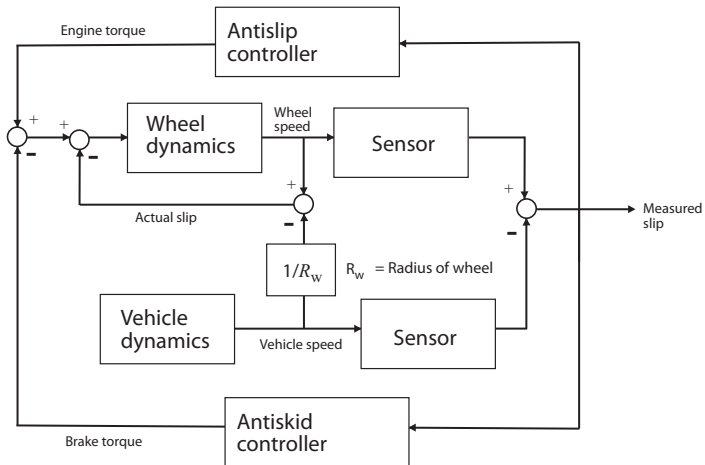
DP1.3 An automated cow milking system:



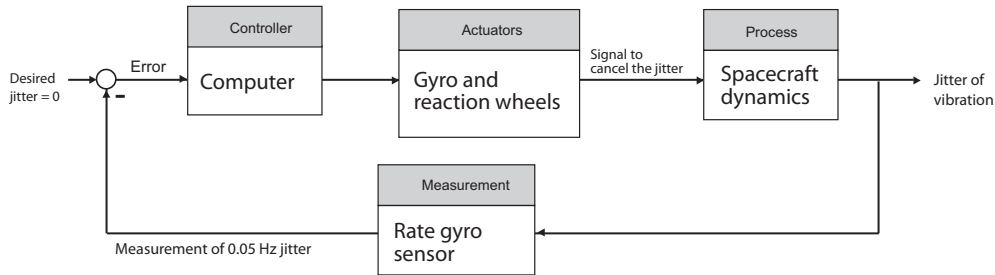
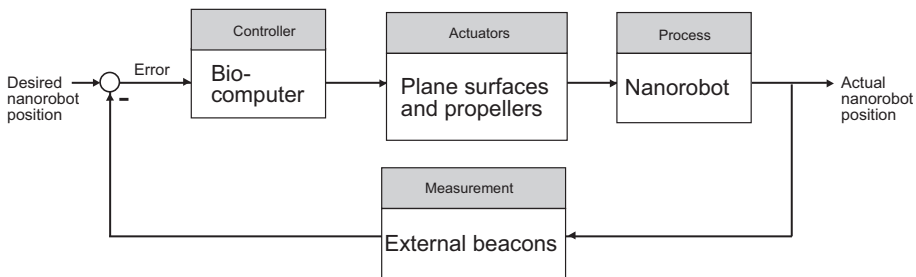
DP1.4 A feedback control system for a robot welder:



DP1.5 A control system for one wheel of a traction control system:



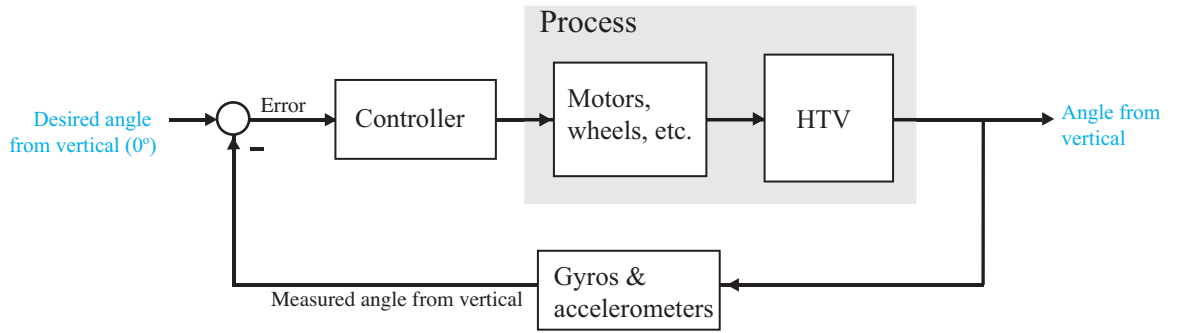


**DP1.6** A vibration damping system for the Hubble Space Telescope:**DP1.7** A control system for a nanorobot:

Many concepts from underwater robotics can be applied to nanorobotics within the bloodstream. For example, plane surfaces and propellers can provide the required actuation with screw drives providing the propulsion. The nanorobots can use signals from beacons located outside the skin as sensors to determine their position. The nanorobots use energy from the chemical reaction of oxygen and glucose available in the human body. The control system requires a bio-computer—an innovation that is not yet available.

For further reading, see A. Cavalcanti, L. Rosen, L. C. Kretly, M. Rosenfeld, and S. Einav, “Nanorobotic Challenges n Biomedical Application, Design, and Control,” *IEEE ICECS Intl Conf. on Electronics, Circuits and Systems*, Tel-Aviv, Israel, December 2004.

**DP1.8** The feedback control system might use gyros and/or accelerometers to measure angle change and assuming the HTV was originally in the vertical position, the feedback would retain the vertical position using commands to motors and other actuators that produced torques and could move the HTV forward and backward.



# Mathematical Models of Systems

## Exercises

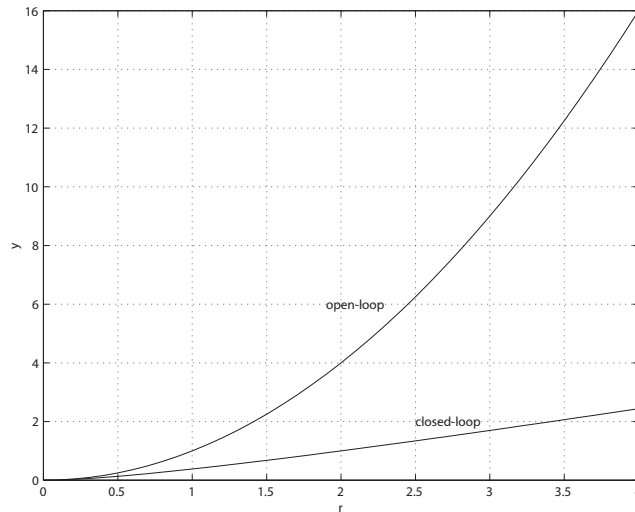
**E2.1** We have for the open-loop

$$y = r^2$$

and for the closed-loop

$$e = r - y \quad \text{and} \quad y = e^2 .$$

So,  $e = r - e^2$  and  $e^2 + e - r = 0$  .



**FIGURE E2.1**  
Plot of open-loop versus closed-loop.

For example, if  $r = 1$ , then  $e^2 + e - 1 = 0$  implies that  $e = 0.618$ . Thus,  $y = 0.382$ . A plot  $y$  versus  $r$  is shown in Figure E2.1.

**E2.2** Define

$$f(T) = R = R_0 e^{-0.1T}$$

and

$$\Delta R = f(T) - f(T_0), \quad \Delta T = T - T_0.$$

Then,

$$\Delta R = f(T) - f(T_0) = \left. \frac{\partial f}{\partial T} \right|_{T=T_0=20^\circ} \Delta T + \dots$$

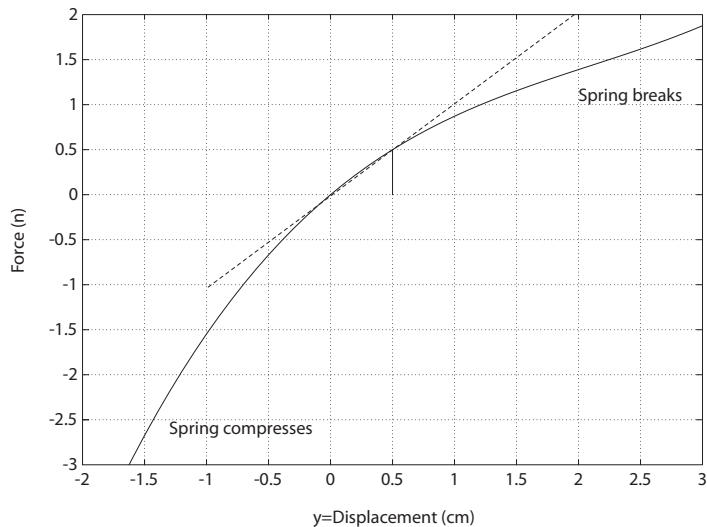
where

$$\left. \frac{\partial f}{\partial T} \right|_{T=T_0=20^\circ} = -0.1R_0 e^{-0.1T_0} = -135,$$

when  $R_0 = 10,000\Omega$ . Thus, the linear approximation is computed by considering only the first-order terms in the Taylor series expansion, and is given by

$$\Delta R = -135\Delta T.$$

**E2.3** The spring constant for the equilibrium point is found graphically by estimating the slope of a line tangent to the force versus displacement curve at the point  $y = 0.5\text{cm}$ , see Figure E2.3. The slope of the line is  $K \approx 1$ .



**FIGURE E2.3**  
Spring force as a function of displacement.

**E2.4** Since

$$R(s) = \frac{1}{s}$$

we have

$$Y(s) = \frac{4(s + 50)}{s(s + 20)(s + 10)} .$$

The partial fraction expansion of  $Y(s)$  is given by

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s + 20} + \frac{A_3}{s + 10}$$

where

$$A_1 = 1 , \quad A_2 = 0.6 \text{ and } A_3 = -1.6 .$$

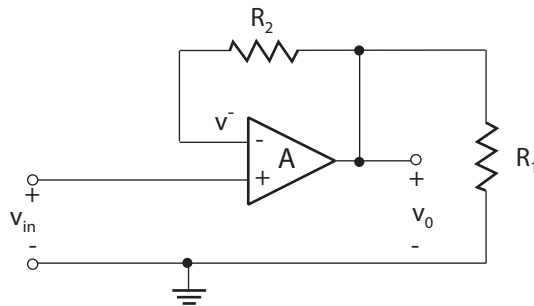
Using the Laplace transform table, we find that

$$y(t) = 1 + 0.6e^{-20t} - 1.6e^{-10t} .$$

The final value is computed using the final value theorem:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s \left[ \frac{4(s + 50)}{s(s^2 + 30s + 200)} \right] = 1 .$$

**E2.5** The circuit diagram is shown in Figure E2.5.



**FIGURE E2.5**  
Noninverting op-amp circuit.

With an ideal op-amp, we have

$$v_o = A(v_{in} - v^-),$$

where  $A$  is very large. We have the relationship

$$v^- = \frac{R_1}{R_1 + R_2} v_o.$$

Therefore,

$$v_o = A \left( v_{in} - \frac{R_1}{R_1 + R_2} v_o \right),$$

and solving for  $v_o$  yields

$$v_o = \frac{A}{1 + \frac{AR_1}{R_1 + R_2}} v_{in}.$$

Since  $A \gg 1$ , it follows that  $1 + \frac{AR_1}{R_1 + R_2} \approx \frac{AR_1}{R_1 + R_2}$ . Then the expression for  $v_o$  simplifies to

$$v_o = \frac{R_1 + R_2}{R_1} v_{in}.$$

**E2.6** Given

$$y = f(x) = e^x$$

and the operating point  $x_o = 1$ , we have the linear approximation

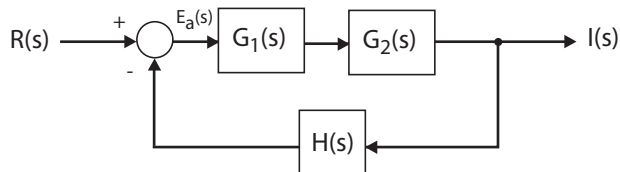
$$y = f(x) = f(x_o) + \left. \frac{\partial f}{\partial x} \right|_{x=x_o} (x - x_o) + \dots$$

where

$$f(x_o) = e, \quad \left. \frac{df}{dx} \right|_{x=x_o=1} = e, \quad \text{and} \quad x - x_o = x - 1.$$

Therefore, we obtain the linear approximation  $y = ex$ .

**E2.7** The block diagram is shown in Figure E2.7.



**FIGURE E2.7**  
Block diagram model.

Starting at the output we obtain

$$I(s) = G_1(s)G_2(s)E(s).$$

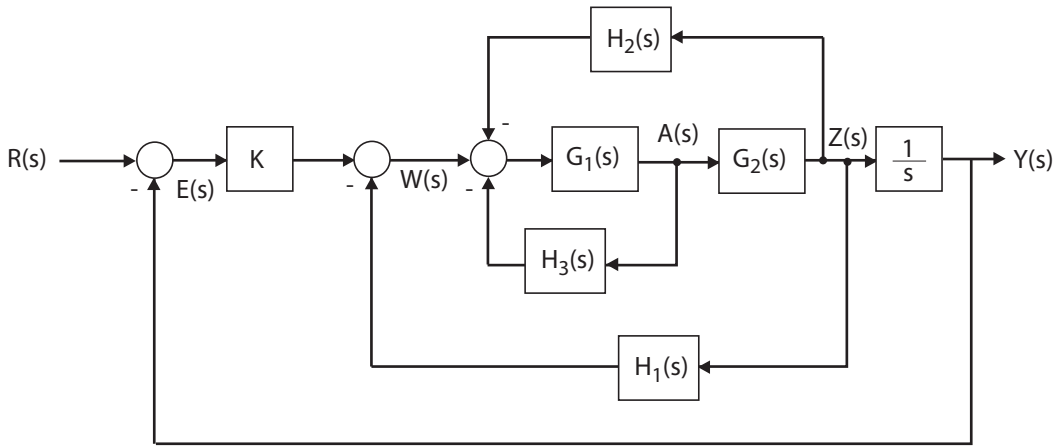
But  $E(s) = R(s) - H(s)I(s)$ , so

$$I(s) = G_1(s)G_2(s) [R(s) - H(s)I(s)].$$

Solving for  $I(s)$  yields the closed-loop transfer function

$$\frac{I(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}.$$

**E2.8** The block diagram is shown in Figure E2.8.



**FIGURE E2.8**  
Block diagram model.

Starting at the output we obtain

$$Y(s) = \frac{1}{s}Z(s) = \frac{1}{s}G_2(s)A(s).$$

But  $A(s) = G_1(s) [-H_2(s)Z(s) - H_3(s)A(s) + W(s)]$  and  $Z(s) = sY(s)$ , so

$$Y(s) = -G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) + \frac{1}{s}G_1(s)G_2(s)W(s).$$

Substituting  $W(s) = KE(s) - H_1(s)Z(s)$  into the above equation yields

$$Y(s) = -G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) + \frac{1}{s}G_1(s)G_2(s) [KE(s) - H_1(s)Z(s)]$$

and with  $E(s) = R(s) - Y(s)$  and  $Z(s) = sY(s)$  this reduces to

$$Y(s) = [-G_1(s)G_2(s)(H_2(s) + H_1(s)) - G_1(s)H_3(s) - \frac{1}{s}G_1(s)G_2(s)K]Y(s) + \frac{1}{s}G_1(s)G_2(s)KR(s).$$

Solving for  $Y(s)$  yields the transfer function

$$Y(s) = T(s)R(s),$$

where

$$T(s) = \frac{KG_1(s)G_2(s)/s}{1 + G_1(s)G_2(s)[(H_2(s) + H_1(s)) + G_1(s)H_3(s) + KG_1(s)G_2(s)/s]}.$$

**E2.9** From Figure E2.9, we observe that

$$F_f(s) = G_2(s)U(s)$$

and

$$F_R(s) = G_3(s)U(s).$$

Then, solving for  $U(s)$  yields

$$U(s) = \frac{1}{G_2(s)}F_f(s)$$

and it follows that

$$F_R(s) = \frac{G_3(s)}{G_2(s)}U(s).$$

Again, considering the block diagram in Figure E2.9 we determine

$$F_f(s) = G_1(s)G_2(s)[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)].$$

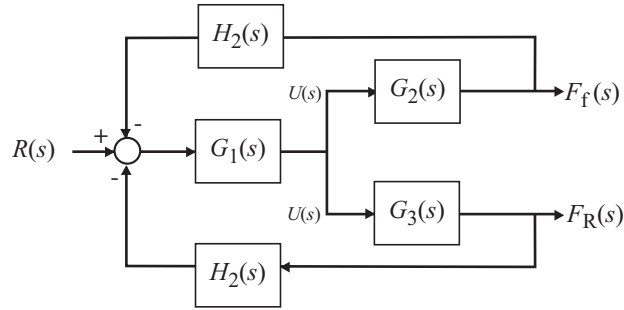
But, from the previous result, we substitute for  $F_R(s)$  resulting in

$$F_f(s) = G_1(s)G_2(s)R(s) - G_1(s)G_2(s)H_2(s)F_f(s) - G_1(s)H_2(s)G_3(s)F_f(s).$$

Solving for  $F_f(s)$  yields

$$F_f(s) = \left[ \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_2(s) + G_1(s)G_3(s)H_2(s)} \right] R(s).$$

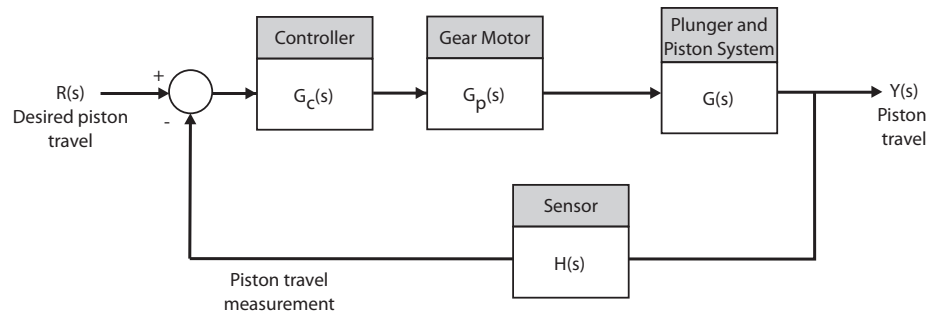




**FIGURE E2.9**  
Block diagram model.

**E2.10** The shock absorber block diagram is shown in Figure E2.10. The closed-loop transfer function model is

$$T(s) = \frac{G_c(s)G_p(s)G(s)}{1 + H(s)G_c(s)G_p(s)G(s)} .$$



**FIGURE E2.10**  
Shock absorber block diagram.

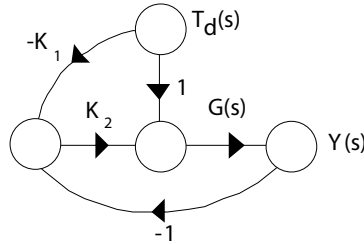
**E2.11** Let  $f$  denote the spring force (n) and  $x$  denote the deflection (m). Then

$$K = \frac{\Delta f}{\Delta x} .$$

Computing the slope from the graph yields:

- (a)  $x_o = -0.14\text{m} \rightarrow K = \Delta f / \Delta x = 10 \text{ n} / 0.04 \text{ m} = 250 \text{ n/m}$
- (b)  $x_o = 0\text{m} \rightarrow K = \Delta f / \Delta x = 10 \text{ n} / 0.05 \text{ m} = 200 \text{ n/m}$
- (c)  $x_o = 0.35\text{m} \rightarrow K = \Delta f / \Delta x = 3\text{n} / 0.05 \text{ m} = 60 \text{ n/m}$

**E2.12** The signal flow graph is shown in Fig. E2.12. Find  $Y(s)$  when  $R(s) = 0$ .



**FIGURE E2.12**  
Signal flow graph.

The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$Y(s) = \frac{G(s)T_d(s) - K_1K_2G(s)T_d(s)}{1 - (-K_2G(s))} = \frac{G(s)(1 - K_1K_2)T_d(s)}{1 + K_2G(s)}.$$

If we set

$$K_1K_2 = 1,$$

then  $Y(s) = 0$  for any  $T_d(s)$ .

**E2.13** The transfer function from  $R(s)$ ,  $T_d(s)$ , and  $N(s)$  to  $Y(s)$  is

$$Y(s) = \left[ \frac{K}{s^2 + 10s + K} \right] R(s) + \left[ \frac{1}{s^2 + 10s + K} \right] T_d(s) - \left[ \frac{K}{s^2 + 10s + K} \right] N(s)$$

Therefore, we find that

$$Y(s)/T_d(s) = \frac{1}{s^2 + 10s + K} \quad \text{and} \quad Y(s)/N(s) = -\frac{K}{s^2 + 10s + K}$$

**E2.14** Since we want to compute the transfer function from  $R_2(s)$  to  $Y_1(s)$ , we can assume that  $R_1 = 0$  (application of the principle of superposition). Then, starting at the output  $Y_1(s)$  we obtain

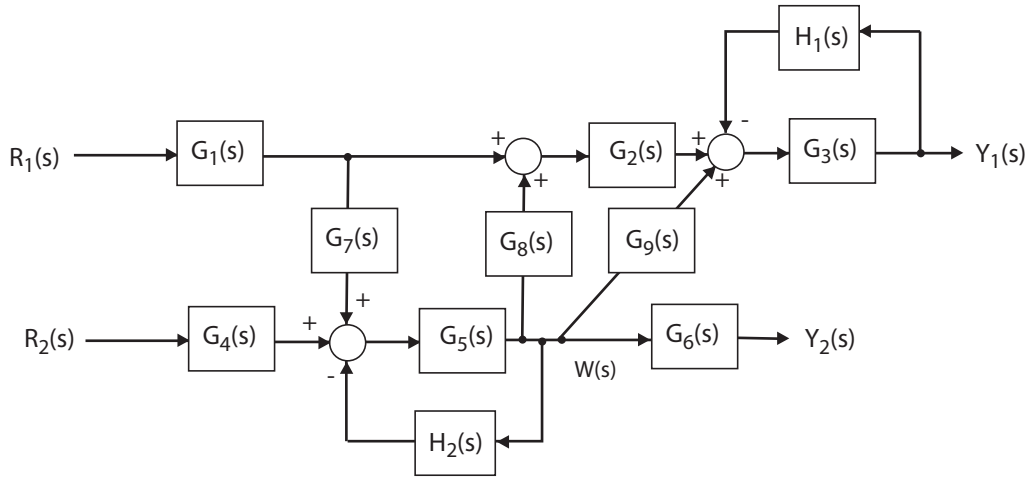
$$Y_1(s) = G_3(s) [-H_1(s)Y_1(s) + G_2(s)G_8(s)W(s) + G_9(s)W(s)],$$

or

$$[1 + G_3(s)H_1(s)] Y_1(s) = [G_3(s)G_2(s)G_8(s)W(s) + G_3(s)G_9(s)] W(s).$$

Considering the signal  $W(s)$  (see Figure E2.14), we determine that

$$W(s) = G_5(s) [G_4(s)R_2(s) - H_2(s)W(s)],$$



**FIGURE E2.14**  
Block diagram model.

or

$$[1 + G_5(s)H_2(s)] W(s) = G_5(s)G_4(s)R_2(s).$$

Substituting the expression for  $W(s)$  into the above equation for  $Y_1(s)$  yields

$$\frac{Y_1(s)}{R_2(s)} = \frac{G_2(s)G_3(s)G_4(s)G_5(s)G_8(s) + G_3(s)G_4(s)G_5(s)G_9(s)}{1 + G_3(s)H_1(s) + G_5(s)H_2(s) + G_3(s)G_5(s)H_1(s)H_2(s)}.$$

**E2.15** For loop 1, we have

$$R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1} \int (i_1 - i_2) dt + R_2 (i_1 - i_2) = v(t) .$$

And for loop 2, we have

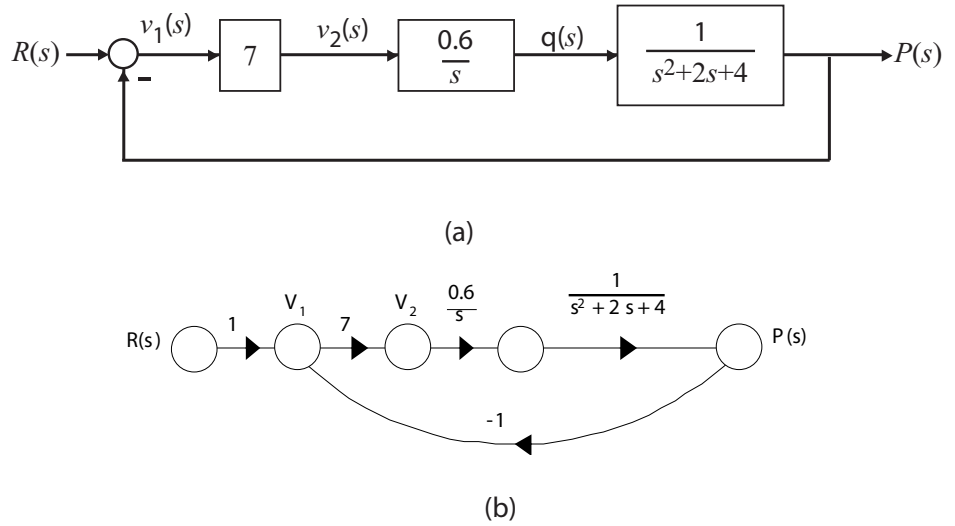
$$\frac{1}{C_2} \int i_2 dt + L_2 \frac{di_2}{dt} + R_2 (i_2 - i_1) + \frac{1}{C_1} \int (i_2 - i_1) dt = 0 .$$

**E2.16** The transfer function from  $R(s)$  to  $P(s)$  is

$$\frac{P(s)}{R(s)} = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} .$$

The block diagram is shown in Figure E2.16a. The corresponding signal flow graph is shown in Figure E2.16b for

$$P(s)/R(s) = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} .$$

**FIGURE E2.16**

(a) Block diagram, (b) Signal flow graph.

**E2.17** A linear approximation for  $f$  is given by

$$\Delta f = \left. \frac{\partial f}{\partial x} \right|_{x=x_o} \Delta x = 2kx_o \Delta x = k \Delta x$$

where  $x_o = 1/2$ ,  $\Delta f = f(x) - f(x_o)$ , and  $\Delta x = x - x_o$ .**E2.18** The linear approximation is given by

$$\Delta y = m \Delta x$$

where

$$m = \left. \frac{\partial y}{\partial x} \right|_{x=x_o} .$$

(a) When  $x_o = 1$ , we find that  $y_o = 2.4$ , and  $y_o = 13.2$  when  $x_o = 2$ .(b) The slope  $m$  is computed as follows:

$$m = \left. \frac{\partial y}{\partial x} \right|_{x=x_o} = 1 + 4.2x_o^2 .$$

Therefore,  $m = 5.2$  at  $x_o = 1$ , and  $m = 18.8$  at  $x_o = 2$ .

**E2.19** The output (with a step input) is

$$Y(s) = \frac{15(s+1)}{s(s+7)(s+2)}.$$

The partial fraction expansion is

$$Y(s) = \frac{15}{14s} - \frac{18}{7} \frac{1}{s+7} + \frac{3}{2} \frac{1}{s+2}.$$

Taking the inverse Laplace transform yields

$$y(t) = \frac{15}{14} - \frac{18}{7}e^{-7t} + \frac{3}{2}e^{-2t}.$$

**E2.20** The input-output relationship is

$$\frac{V_o}{V} = \frac{A(K-1)}{1+AK}$$

where

$$K = \frac{Z_1}{Z_1 + Z_2}.$$

Assume  $A \gg 1$ . Then,

$$\frac{V_o}{V} = \frac{K-1}{K} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1 C_1 s + 1} \quad \text{and} \quad Z_2 = \frac{R_2}{R_2 C_2 s + 1}.$$

Therefore,

$$\frac{V_o(s)}{V(s)} = -\frac{R_2(R_1 C_1 s + 1)}{R_1(R_2 C_2 s + 1)} = -\frac{2(s+1)}{s+2}.$$

**E2.21** The equation of motion of the mass  $m_c$  is

$$m_c \ddot{x}_p + (b_d + b_s) \dot{x}_p + k_d x_p = b_d \dot{x}_{in} + k_d x_{in}.$$

Taking the Laplace transform with zero initial conditions yields

$$[m_c s^2 + (b_d + b_s)s + k_d]X_p(s) = [b_d s + k_d]X_{in}(s).$$

So, the transfer function is

$$\frac{X_p(s)}{X_{in}(s)} = \frac{b_d s + k_d}{m_c s^2 + (b_d + b_s)s + k_d} = \frac{0.7s + 2}{s^2 + 2.8s + 2}.$$

**E2.22** The rotational velocity is

$$\omega(s) = \frac{2(s+4)}{(s+5)(s+1)^2} \frac{1}{s}.$$

Expanding in a partial fraction expansion yields

$$\omega(s) = \frac{8}{5} \frac{1}{s} + \frac{1}{40} \frac{1}{s+5} - \frac{3}{2} \frac{1}{(s+1)^2} - \frac{13}{8} \frac{1}{s+1}.$$

Taking the inverse Laplace transform yields

$$\omega(t) = \frac{8}{5} + \frac{1}{40} e^{-5t} - \frac{3}{2} t e^{-t} - \frac{13}{8} e^{-t}.$$

**E2.23** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{K_1 K_2}{s^2 + (K_1 + K_2 K_3 + K_1 K_2)s + K_1 K_2 K_3}.$$

**E2.24** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{10}{s^2 + 21s + 10}.$$

**E2.25** Let  $x = 0.6$  and  $y = 0.8$ . Then, with  $y = ax^3$ , we have

$$0.8 = a(0.6)^3.$$

Solving for  $a$  yields  $a = 3.704$ . A linear approximation is

$$y - y_o = 3ax_o^2(x - x_o)$$

or  $y = 4x - 1.6$ , where  $y_o = 0.8$  and  $x_o = 0.6$ .

**E2.26** The equations of motion are

$$\begin{aligned} m_1 \ddot{x}_1 + k(x_1 - x_2) &= F \\ m_2 \ddot{x}_2 + k(x_2 - x_1) &= 0. \end{aligned}$$

Taking the Laplace transform (with zero initial conditions) and solving for  $X_2(s)$  yields

$$X_2(s) = \frac{k}{(m_2 s^2 + k)(m_1 s^2 + k) - k^2} F(s).$$

Then, with  $m_1 = m_2 = k = 1$ , we have

$$X_2(s)/F(s) = \frac{1}{s^2(s^2 + 2)}.$$

**E2.27** The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$Y(s)/T_d(s) = \frac{G_2(s)}{1 + G_1G_2H(s)} .$$

**E2.28** The transfer function is

$$\frac{V_o(s)}{V(s)} = \frac{R_2R_4C}{R_3}s + \frac{R_2R_4}{R_1R_3} = 24s + 144 .$$

**E2.29** (a) If

$$G(s) = \frac{1}{s^2 + 15s + 50} \quad \text{and} \quad H(s) = 2s + 15 ,$$

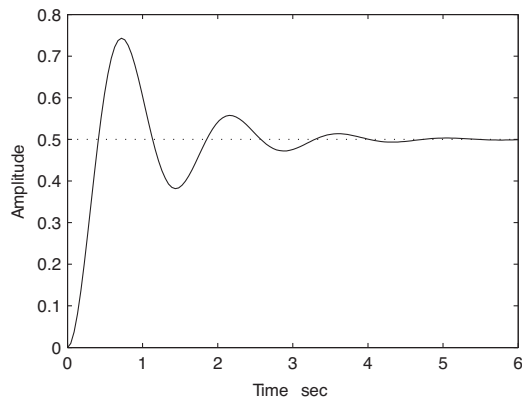
then the closed-loop transfer function of Figure E2.28(a) and (b) (in Dorf & Bishop) are equivalent.

(b) The closed-loop transfer function is

$$T(s) = \frac{1}{s^2 + 17s + 65} .$$

**E2.30** (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} \frac{1}{s} = \frac{10}{s(s^2 + 2s + 20)} \quad \text{where} \quad G(s) = \frac{10}{s^2 + 2s + 10} .$$



**FIGURE E2.30**  
Step response.

(b) The output  $Y(s)$  (when  $R(s) = 1/s$ ) is

$$Y(s) = \frac{0.5}{s} - \frac{-0.25 + 0.0573j}{s + 1 - 4.3589j} + \frac{-0.25 - 0.0573j}{s + 1 + 4.3589j} .$$

(c) The plot of  $y(t)$  is shown in Figure E2.30. The output is given by

$$y(t) = \frac{1}{2} \left[ 1 - e^{-t} \left( \cos \sqrt{19}t - \frac{1}{\sqrt{19}} \sin \sqrt{19}t \right) \right]$$

**E2.31** The partial fraction expansion is

$$V(s) = \frac{a}{s + p_1} + \frac{b}{s + p_2}$$

where  $p_1 = 4 - 19.6j$  and  $p_2 = 4 + 19.6j$ . Then, the residues are

$$a = -10.2j \quad b = 10.2j .$$

The inverse Laplace transform is

$$v(t) = -10.2je^{(-4+19.6j)t} + 10.2je^{(-4-19.6j)t} = 20.4e^{-4t} \sin 19.6t .$$



## Problems

- P2.1** The integrodifferential equations, obtained by Kirchoff's voltage law to each loop, are as follows:

$$R_1 i_1 + \frac{1}{C_1} \int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_2(i_1 - i_2) = v(t) \quad (\text{loop 1})$$

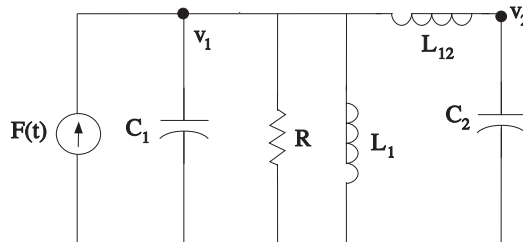
and

$$R_3 i_2 + \frac{1}{C_2} \int i_2 dt + R_2(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 \quad (\text{loop 2}) .$$

- P2.2** The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$\begin{aligned} M_1 \ddot{y}_1 + k_{12}(y_1 - y_2) + b \dot{y}_1 + k_1 y_1 &= F(t) \\ M_2 \ddot{y}_2 + k_{12}(y_2 - y_1) &= 0 . \end{aligned}$$

Using a force-current analogy, the analogous electric circuit is shown in Figure P2.2, where  $C_i \rightarrow M_i$ ,  $L_1 \rightarrow 1/k_1$ ,  $L_{12} \rightarrow 1/k_{12}$ , and  $R \rightarrow 1/b$ .



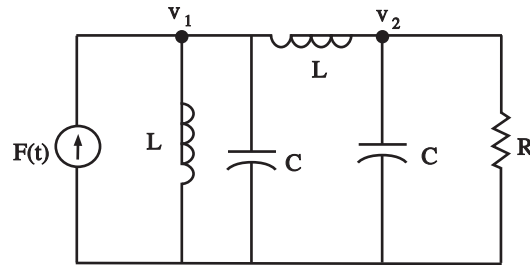
**FIGURE P2.2**  
Analogous electric circuit.

- P2.3** The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$\begin{aligned} M \ddot{x}_1 + k x_1 + k(x_1 - x_2) &= F(t) \\ M \ddot{x}_2 + k(x_2 - x_1) + b \dot{x}_2 &= 0 . \end{aligned}$$

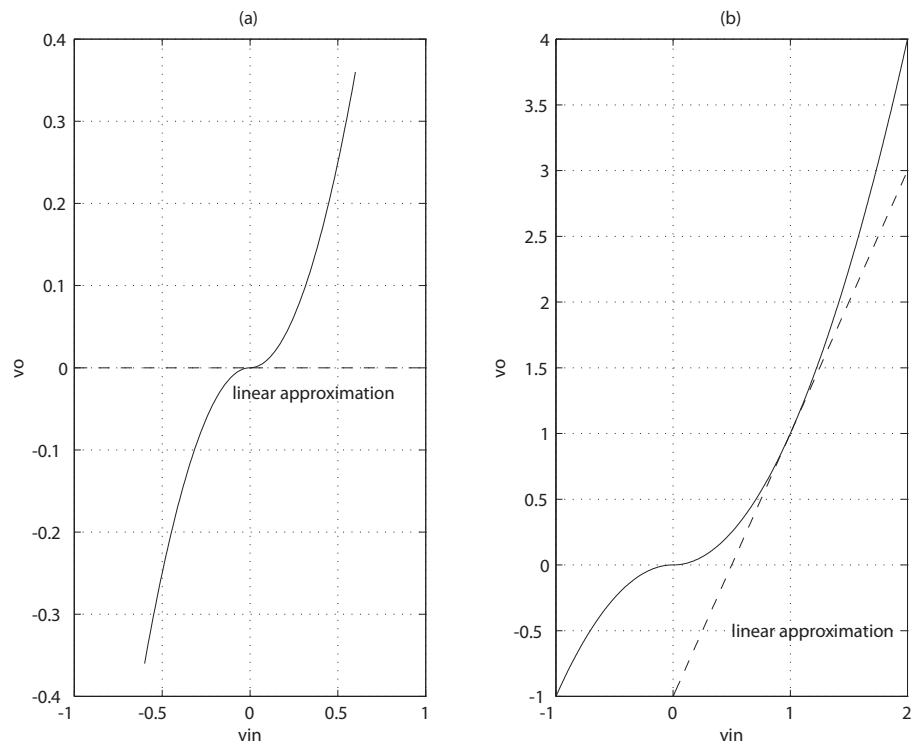
Using a force-current analogy, the analogous electric circuit is shown in Figure P2.3, where

$$C \rightarrow M \quad L \rightarrow 1/k \quad R \rightarrow 1/b .$$



**FIGURE P2.3**  
Analogous electric circuit.

- P2.4** (a) The linear approximation around  $v_{in} = 0$  is  $v_o = 0v_{in}$ , see Figure P2.4(a).  
 (b) The linear approximation around  $v_{in} = 1$  is  $v_o = 2v_{in} - 1$ , see Figure P2.4(b).



**FIGURE P2.4**  
Nonlinear functions and approximations.

**P2.5** Given

$$Q = K(P_1 - P_2)^{1/2} .$$

Let  $\delta P = P_1 - P_2$  and  $\delta P_o =$  operating point. Using a Taylor series expansion of  $Q$ , we have

$$Q = Q_o + \left. \frac{\partial Q}{\partial \delta P} \right|_{\delta P = \delta P_o} (\delta P - \delta P_o) + \dots$$

where

$$Q_o = K\delta P_o^{1/2} \quad \text{and} \quad \left. \frac{\partial Q}{\partial \delta P} \right|_{\delta P = \delta P_o} = \frac{K}{2}\delta P_o^{-1/2} .$$

Define  $\Delta Q = Q - Q_o$  and  $\Delta P = \delta P - \delta P_o$ . Then, dropping higher-order terms in the Taylor series expansion yields

$$\Delta Q = m\Delta P$$

where

$$m = \frac{K}{2\delta P_o^{1/2}} .$$

**P2.6** From P2.1 we have

$$R_1 i_1 + \frac{1}{C_1} \int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_2(i_1 - i_2) = v(t)$$

and

$$R_3 i_2 + \frac{1}{C_2} \int i_2 dt + R_2(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 .$$

Taking the Laplace transform and using the fact that the initial voltage across  $C_2$  is 10v yields

$$\left[ R_1 + \frac{1}{C_1 s} + L_1 s + R_2 \right] I_1(s) + [-R_2 - L_1 s] I_2(s) = 0$$

and

$$[-R_2 - L_1 s] I_1(s) + \left[ L_1 s + R_3 + \frac{1}{C_2 s} + R_2 \right] I_2(s) = -\frac{10}{s} .$$

Rewriting in matrix form we have

$$\begin{bmatrix} R_1 + \frac{1}{C_1 s} + L_1 s + R_2 & -R_2 - L_1 s \\ -R_2 - L_1 s & L_1 s + R_3 + \frac{1}{C_2 s} + R_2 \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -10/s \end{pmatrix} .$$

Solving for  $I_2$  yields

$$\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} L_1s + R_3 + \frac{1}{C_2s} + R_2 & R_2 + L_1s \\ R_2 + L_1s & R_1 + \frac{1}{C_1s} + L_1s + R_2 \end{bmatrix} \begin{pmatrix} 0 \\ -10/s \end{pmatrix}.$$

or

$$I_2(s) = \frac{-10(R_1 + 1/C_1s + L_1s + R_2)}{s\Delta}$$

where

$$\Delta = (R_1 + \frac{1}{C_1s} + L_1s + R_2)(L_1s + R_3 + \frac{1}{C_2s} + R_2) - (R_2 + L_1s)^2.$$

**P2.7** Consider the differentiating op-amp circuit in Figure P2.7. For an ideal op-amp, the voltage gain (as a function of frequency) is

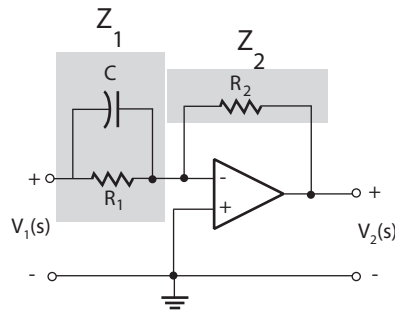
$$V_2(s) = -\frac{Z_2(s)}{Z_1(s)}V_1(s),$$

where

$$Z_1 = \frac{R_1}{1 + R_1Cs}$$

and  $Z_2 = R_2$  are the respective circuit impedances. Therefore, we obtain

$$V_2(s) = -\left[\frac{R_2(1 + R_1Cs)}{R_1}\right]V_1(s).$$



**FIGURE P2.7**  
Differentiating op-amp circuit.

**P2.8** Let

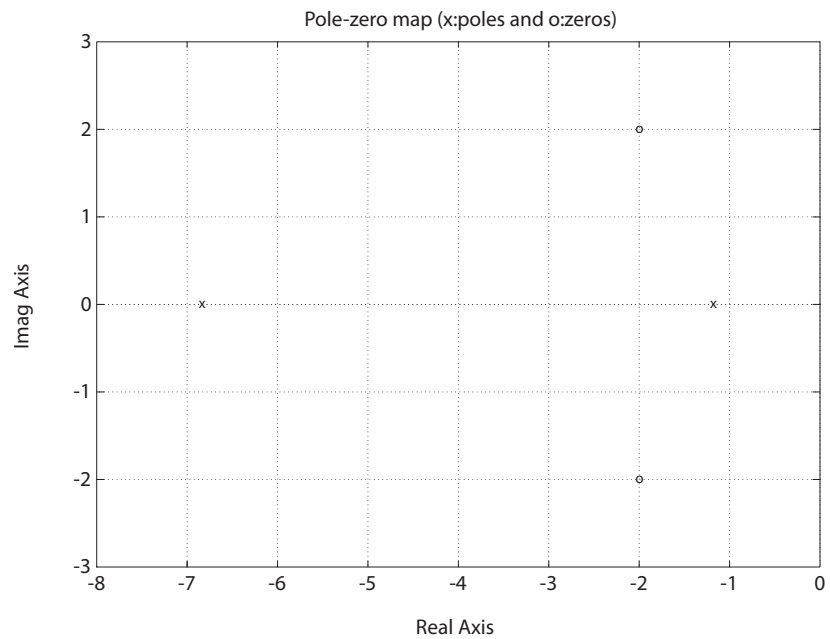
$$\Delta = \begin{vmatrix} G_2 + Cs & -Cs & -G_2 \\ -Cs & G_1 + 2Cs & -Cs \\ -G_2 & -Cs & Cs + G_2 \end{vmatrix}.$$

Then,

$$V_j = \frac{\Delta_{ij}}{\Delta} I_1 \quad \text{or} \quad \text{or } \frac{V_3}{V_1} = \frac{\Delta_{13} I_1 / \Delta}{\Delta_{11} I_1 / \Delta}.$$

Therefore, the transfer function is

$$T(s) = \frac{V_3}{V_1} = \frac{\Delta_{13}}{\Delta_{11}} = \frac{\begin{vmatrix} -Cs & 2Cs + G_1 \\ -G_2 & -Cs \end{vmatrix}}{\begin{vmatrix} 2Cs + G_1 & -Cs \\ -Cs & Cs + G_2 \end{vmatrix}}$$



**FIGURE P2.8**  
Pole-zero map.

$$= \frac{C^2 R_1 R_2 s^2 + 2C R_1 s + 1}{C^2 R_1 R_2 s^2 + (2R_1 + R_2)C s + 1} .$$

Using  $R_1 = 0.5$ ,  $R_2 = 1$ , and  $C = 0.5$ , we have

$$T(s) = \frac{s^2 + 4s + 8}{s^2 + 8s + 8} = \frac{(s + 2 + 2j)(s + 2 - 2j)}{(s + 4 + \sqrt{8})(s + 4 - \sqrt{8})} .$$

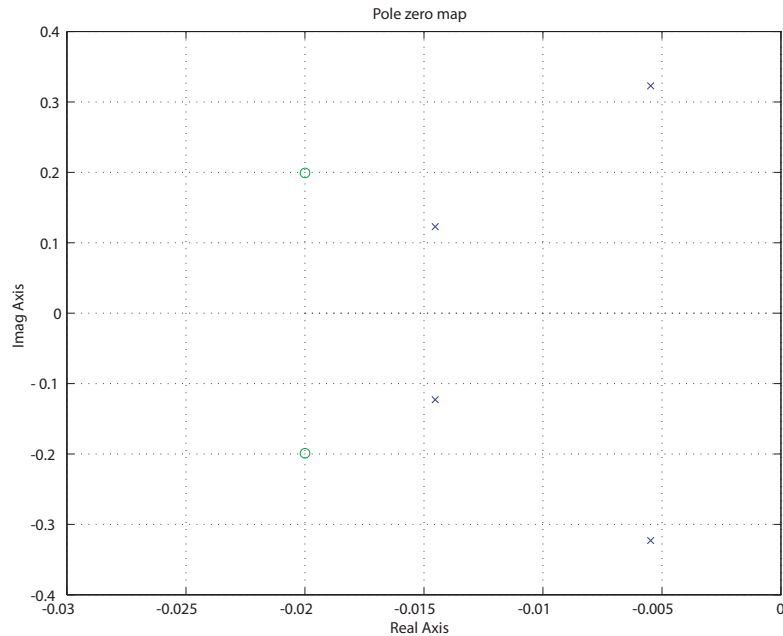
The pole-zero map is shown in Figure P2.8.

**P2.9** From P2.3 we have

$$\begin{aligned} M\ddot{x}_1 + kx_1 + k(x_1 - x_2) &= F(t) \\ M\ddot{x}_2 + k(x_2 - x_1) + b\dot{x}_2 &= 0 . \end{aligned}$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} Ms^2 + 2k & -k \\ -k & Ms^2 + bs + k \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} F(s) \\ 0 \end{pmatrix} ,$$



**FIGURE P2.9**  
Pole-zero map.

or

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} Ms^2 + bs + k & k \\ k & Ms^2 + 2k \end{bmatrix} \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

where  $\Delta = (Ms^2 + bs + k)(Ms^2 + 2k) - k^2$ . So,

$$G(s) = \frac{X_1(s)}{F(s)} = \frac{Ms^2 + bs + k}{\Delta}.$$

When  $b/k = 1$ ,  $M = 1$ ,  $b^2/Mk = 0.04$ , we have

$$G(s) = \frac{s^2 + 0.04s + 0.04}{s^4 + 0.04s^3 + 0.12s^2 + 0.0032s + 0.0016}.$$

The pole-zero map is shown in Figure P2.9.

**P2.10** From P2.2 we have

$$\begin{aligned} M_1\ddot{y}_1 + k_{12}(y_1 - y_2) + b\dot{y}_1 + k_1y_1 &= F(t) \\ M_2\ddot{y}_2 + k_{12}(y_2 - y_1) &= 0. \end{aligned}$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} M_1s^2 + bs + k_1 + k_{12} & -k_{12} \\ -k_{12} & M_2s^2 + k_{12} \end{bmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} M_2s^2 + k_{12} & k_{12} \\ k_{12} & M_1s^2 + bs + k_1 + k_{12} \end{bmatrix} \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

where

$$\Delta = (M_2s^2 + k_{12})(M_1s^2 + bs + k_1 + k_{12}) - k_{12}^2.$$

So, when  $f(t) = a \sin \omega_o t$ , we have that  $Y_1(s)$  is given by

$$Y_1(s) = \frac{aM_2\omega_o(s^2 + k_{12}/M_2)}{(s^2 + \omega_o^2)\Delta(s)}.$$

For motionless response (in the steady-state), set the zero of the transfer function so that

$$\left(s^2 + \frac{k_{12}}{M_2}\right) = s^2 + \omega_o^2 \quad \text{or} \quad \omega_o^2 = \frac{k_{12}}{M_2}.$$

**P2.11** The transfer functions from  $V_c(s)$  to  $V_d(s)$  and from  $V_d(s)$  to  $\theta(s)$  are:

$$V_d(s)/V_c(s) = \frac{K_1 K_2}{(L_q s + R_q)(L_c s + R_c)}, \text{ and}$$

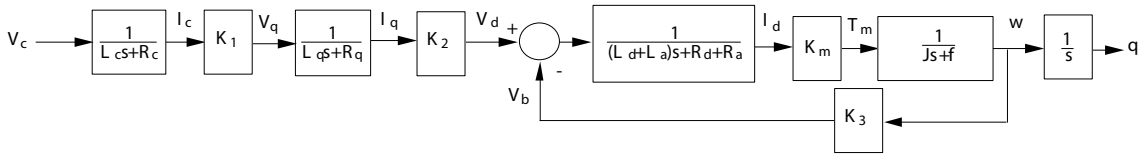
$$\theta(s)/V_d(s) = \frac{K_m}{(J s^2 + f s)((L_d + L_a) s + R_d + R_a) + K_3 K_m s}.$$

The block diagram for  $\theta(s)/V_c(s)$  is shown in Figure P2.11, where

$$\theta(s)/V_c(s) = \frac{\theta(s)}{V_d(s)} \frac{V_d(s)}{V_c(s)} = \frac{K_1 K_2 K_m}{\Delta(s)},$$

where

$$\Delta(s) = s(L_c s + R_c)(L_q s + R_q)((J s + b)((L_d + L_a) s + R_d + R_a) + K_m K_3).$$



**FIGURE P2.11**  
Block diagram.

**P2.12** The open-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K}{s + 20}.$$

With  $R(s) = 1/s$ , we have

$$Y(s) = \frac{K}{s(s + 20)}.$$

The partial fraction expansion is

$$Y(s) = \frac{K}{20} \left( \frac{1}{s} - \frac{1}{s + 20} \right),$$

and the inverse Laplace transform is

$$y(t) = \frac{K}{20} (1 - e^{-20t}),$$

As  $t \rightarrow \infty$ , it follows that  $y(t) \rightarrow K/20$ . So we choose  $K = 20$  so that  $y(t)$



approaches 1. Alternatively we can use the final value theorem to obtain

$$y(t)_{t \rightarrow \infty} = \lim_{s \rightarrow 0} sY(s) = \frac{K}{20} = 1 .$$

It follows that choosing  $K = 20$  leads to  $y(t) \rightarrow 1$  as  $t \rightarrow \infty$ .

**P2.13** The motor torque is given by

$$\begin{aligned} T_m(s) &= (J_m s^2 + b_m s)\theta_m(s) + (J_L s^2 + b_L s)n\theta_L(s) \\ &= n((J_m s^2 + b_m s)/n^2 + J_L s^2 + b_L s)\theta_L(s) \end{aligned}$$

where

$$n = \theta_L(s)/\theta_m(s) = \text{gear ratio} .$$

But

$$T_m(s) = K_m I_g(s)$$

and

$$I_g(s) = \frac{1}{(L_g + L_f)s + R_g + R_f} V_g(s) ,$$

and

$$V_g(s) = K_g I_f(s) = \frac{K_g}{R_f + L_f s} V_f(s) .$$

Combining the above expressions yields

$$\frac{\theta_L(s)}{V_f(s)} = \frac{K_g K_m}{n \Delta_1(s) \Delta_2(s)} .$$

where

$$\Delta_1(s) = J_L s^2 + b_L s + \frac{J_m s^2 + b_m s}{n^2}$$

and

$$\Delta_2(s) = (L_g s + L_f s + R_g + R_f)(R_f + L_f s) .$$

**P2.14** For a field-controlled dc electric motor we have

$$\omega(s)/V_f(s) = \frac{K_m/R_f}{J s + b} .$$

With a step input of  $V_f(s) = 80/s$ , the final value of  $\omega(t)$  is

$$\omega(t)_{t \rightarrow \infty} = \lim_{s \rightarrow 0} s\omega(s) = \frac{80K_m}{R_f b} = 2.4 \quad \text{or} \quad \frac{K_m}{R_f b} = 0.03 .$$

Solving for  $\omega(t)$  yields

$$\omega(t) = \frac{80K_m}{R_f J} \mathcal{L}^{-1} \left\{ \frac{1}{s(s + b/J)} \right\} = \frac{80K_m}{R_f b} (1 - e^{-(b/J)t}) = 2.4(1 - e^{-(b/J)t}) .$$

At  $t = 1/2$ ,  $\omega(t) = 1$ , so

$$\omega(1/2) = 2.4(1 - e^{-(b/J)t}) = 1 \quad \text{implies} \quad b/J = 1.08 \text{ sec} .$$

Therefore,

$$\omega(s)/V_f(s) = \frac{0.0324}{s + 1.08} .$$

**P2.15** Summing the forces in the vertical direction and using Newton's Second Law we obtain

$$\ddot{x} + \frac{k}{m}x = 0 .$$

The system has no damping and no external inputs. Taking the Laplace transform yields

$$X(s) = \frac{x_0 s}{s^2 + k/m} ,$$

where we used the fact that  $x(0) = x_0$  and  $\dot{x}(0) = 0$ . Then taking the inverse Laplace transform yields

$$x(t) = x_0 \cos \sqrt{\frac{k}{m}} t .$$

**P2.16** Using Cramer's rule, we have

$$\begin{bmatrix} 1 & 1.5 \\ 2 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

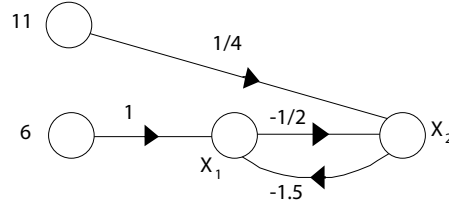
or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} 4 & -1.5 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

where  $\Delta = 4(1) - 2(1.5) = 1$  . Therefore,

$$x_1 = \frac{4(6) - 1.5(11)}{1} = 7.5 \quad \text{and} \quad x_2 = \frac{-2(6) + 1(11)}{1} = -1 .$$

The signal flow graph is shown in Figure P2.16.



**FIGURE P2.16**  
Signal flow graph.

So,

$$x_1 = \frac{6(1) - 1.5(\frac{11}{4})}{1 - \frac{3}{4}} = 7.5 \quad \text{and} \quad x_2 = \frac{11(\frac{1}{4}) + \frac{-1}{2}(6)}{1 - \frac{3}{4}} = -1 .$$

**P2.17** (a) For mass 1 and 2, we have

$$\begin{aligned} M_1 \ddot{x}_1 + K_1(x_1 - x_2) + b_1(\dot{x}_3 - \dot{x}_1) &= 0 \\ M_2 \ddot{x}_2 + K_2(x_2 - x_3) + b_2(\dot{x}_3 - \dot{x}_2) + K_1(x_2 - x_1) &= 0 . \end{aligned}$$

(b) Taking the Laplace transform yields

$$\begin{aligned} (M_1 s^2 + b_1 s + K_1)X_1(s) - K_1 X_2(s) &= b_1 s X_3(s) \\ -K_1 X_1(s) + (M_2 s^2 + b_2 s + K_1 + K_2)X_2(s) &= (b_2 s + K_2)X_3(s) . \end{aligned}$$

(c) Let

$$\begin{aligned} G_1(s) &= K_2 + b_2 s \\ G_2(s) &= 1/p(s) \\ G_3(s) &= 1/q(s) \\ G_4(s) &= s b_1 , \end{aligned}$$

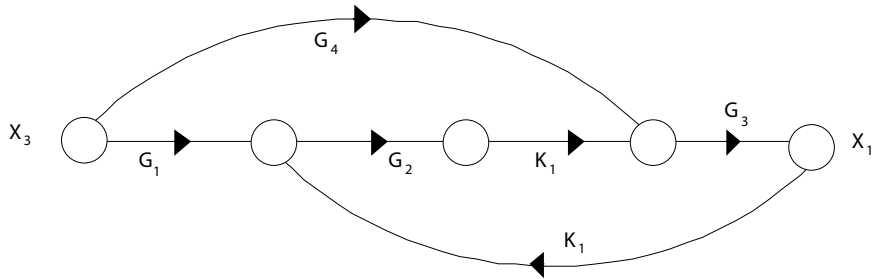
where

$$p(s) = s^2 M_2 + s b_2 + K_1 + K_2$$

and

$$q(s) = s^2 M_1 + s b_1 + K_1 .$$

The signal flow graph is shown in Figure P2.17.

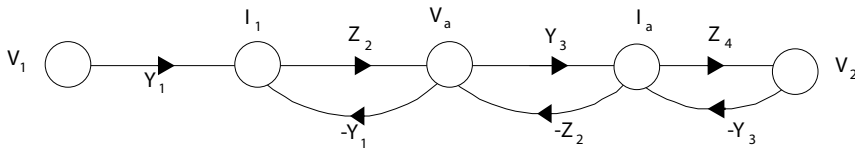


**FIGURE P2.17**  
Signal flow graph.

(d) The transfer function from  $X_3(s)$  to  $X_1(s)$  is

$$\frac{X_1(s)}{X_3(s)} = \frac{K_1 G_1(s) G_2(s) G_3(s) + G_4(s) G_3(s)}{1 - K_1^2 G_2(s) G_3(s)}$$

**P2.18** The signal flow graph is shown in Figure P2.18.



**FIGURE P2.18**  
Signal flow graph.

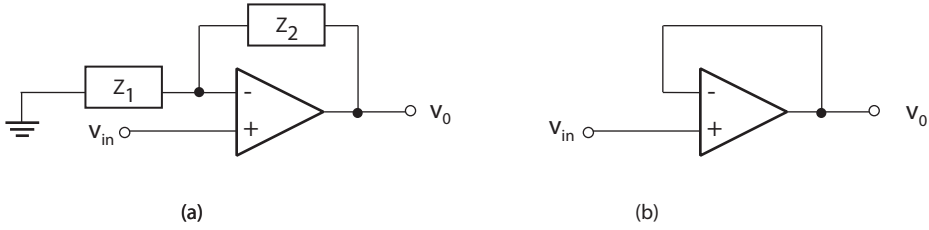
The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{Y_1 Z_2 Y_3 Z_4}{1 + Y_1 Z_2 + Y_3 Z_2 + Y_3 Z_4 + Y_1 Z_2 Z_4 Y_3}$$

**P2.19** For a noninverting op-amp circuit, depicted in Figure P2.19a, the voltage gain (as a function of frequency) is

$$V_o(s) = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} V_{in}(s),$$

where  $Z_1(s)$  and  $Z_2(s)$  are the impedances of the respective circuits. In the case of the voltage follower circuit, shown in Figure P2.19b, we have



**FIGURE P2.19**  
 (a) Noninverting op-amp circuit. (b) Voltage follower circuit.

$Z_1 = \infty$  (open circuit) and  $Z_2 = 0$ . Therefore, the transfer function is

$$\frac{V_o(s)}{V_{in}(s)} = \frac{Z_1}{Z_1} = 1.$$

**P2.20** (a) Assume  $R_g \gg R_s$  and  $R_s \gg R_1$ . Then  $R_s = R_1 + R_2 \approx R_2$ , and

$$v_{gs} = v_{in} - v_o,$$

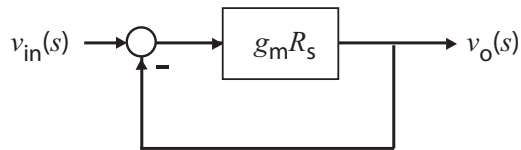
where we neglect  $i_{in}$ , since  $R_g \gg R_s$ . At node S, we have

$$\frac{v_o}{R_s} = g_m v_{gs} = g_m (v_{in} - v_o) \quad \text{or} \quad \frac{v_o}{v_{in}} = \frac{g_m R_s}{1 + g_m R_s}.$$

(b) With  $g_m R_s = 20$ , we have

$$\frac{v_o}{v_{in}} = \frac{20}{21} = 0.95.$$

(c) The block diagram is shown in Figure P2.20.



**FIGURE P2.20**  
 Block diagram model.

**P2.21** From the geometry we find that

$$\Delta z = k \frac{l_1 - l_2}{l_1} (x - y) - \frac{l_2}{l_1} y.$$

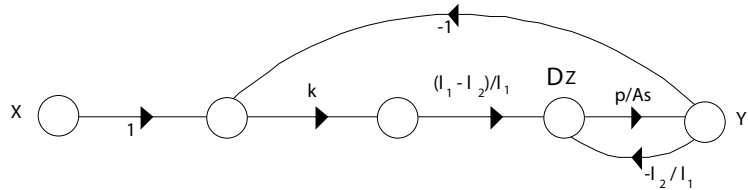
The flow rate balance yields

$$A \frac{dy}{dt} = p \Delta z \quad \text{which implies} \quad Y(s) = \frac{p \Delta Z(s)}{As} .$$

By combining the above results it follows that

$$Y(s) = \frac{p}{As} \left[ k \left( \frac{l_1 - l_2}{l_1} \right) (X(s) - Y(s)) - \frac{l_2}{l_1} Y(s) \right] .$$

Therefore, the signal flow graph is shown in Figure P2.21. Using Mason's



**FIGURE P2.21**  
Signal flow graph.

gain formula we find that the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{\frac{k(l_1 - l_2)p}{l_1 As}}{1 + \frac{l_2 p}{l_1 As} + \frac{k(l_1 - l_2)p}{l_1 As}} = \frac{K_1}{s + K_2 + K_1} ,$$

where

$$K_1 = \frac{k(l_1 - l_2)p}{l_1 A} p \quad \text{and} \quad K_2 = \frac{l_2 p}{l_1 A} .$$

**P2.22** (a) The equations of motion for the two masses are

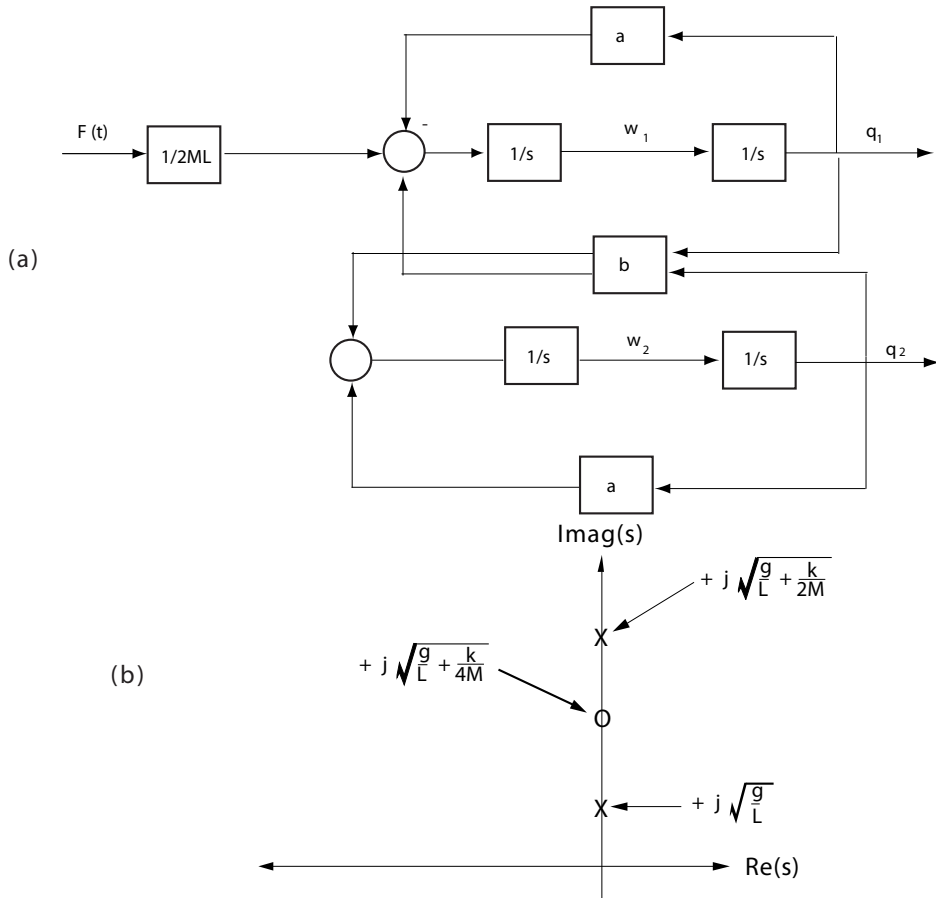
$$ML^2 \ddot{\theta}_1 + MgL\theta_1 + k \left( \frac{L}{2} \right)^2 (\theta_1 - \theta_2) = \frac{L}{2} f(t)$$

$$ML^2 \ddot{\theta}_2 + MgL\theta_2 + k \left( \frac{L}{2} \right)^2 (\theta_2 - \theta_1) = 0 .$$

With  $\dot{\theta}_1 = \omega_1$  and  $\dot{\theta}_2 = \omega_2$ , we have

$$\dot{\omega}_1 = - \left( \frac{g}{L} + \frac{k}{4M} \right) \theta_1 + \frac{k}{4M} \theta_2 + \frac{f(t)}{2ML}$$

$$\dot{\omega}_2 = \frac{k}{4M} \theta_1 - \left( \frac{g}{L} + \frac{k}{4M} \right) \theta_2 .$$



**FIGURE P2.22**  
 (a) Block diagram. (b) Pole-zero map.

(b) Define  $a = g/L + k/4M$  and  $b = k/4M$ . Then

$$\frac{\theta_1(s)}{F(s)} = \frac{1}{2ML} \frac{s^2 + a}{(s^2 + a)^2 - b^2}.$$

(c) The block diagram and pole-zero map are shown in Figure P2.22.

**P2.23** The input-output ratio,  $V_{ce}/V_{in}$ , is found to be

$$\frac{V_{ce}}{V_{in}} = \frac{\beta(R - 1) + h_{ie}R_f}{-\beta h_{re} + h_{ie}(-h_{oe} + R_f)}.$$

**P2.24** (a) The voltage gain is given by

$$\frac{v_o}{v_{in}} = \frac{R_L \beta_1 \beta_2 (R_1 + R_2)}{(R_1 + R_2)(R_g + h_{ie1}) + R_1(R_1 + R_2)(1 + \beta_1) + R_1 R_L \beta_1 \beta_2}.$$

(b) The current gain is found to be

$$\frac{i_{c2}}{i_{b1}} = \beta_1 \beta_2 .$$

(c) The input impedance is

$$\frac{v_{in}}{i_{b1}} = \frac{(R_1 + R_2)(R_g + h_{ie1}) + R_1(R_1 + R_2)(1 + \beta_1) + R_1 R_L \beta_1 \beta_2}{R_1 + R_2} ,$$

and when  $\beta_1 \beta_2$  is very large, we have the approximation

$$\frac{v_{in}}{i_{b1}} \approx \frac{R_L R_1 \beta_1 \beta_2}{R_1 + R_2} .$$

**P2.25** The transfer function from  $R(s)$  and  $T_d(s)$  to  $Y(s)$  is given by

$$\begin{aligned} Y(s) &= G(s) \left( R(s) - \frac{1}{G(s)} (G(s)R(s) + T_d(s)) \right) + T_d(s) + G(s)R(s) \\ &= G(s)R(s) . \end{aligned}$$

Thus,

$$Y(s)/R(s) = G(s) .$$

Also, we have that

$$Y(s) = 0 .$$

when  $R(s) = 0$ . Therefore, the effect of the disturbance,  $T_d(s)$ , is eliminated.

**P2.26** The equations of motion for the two mass model of the robot are

$$\begin{aligned} M\ddot{x} + b(\dot{x} - \dot{y}) + k(x - y) &= F(t) \\ m\ddot{y} + b(\dot{y} - \dot{x}) + k(y - x) &= 0 . \end{aligned}$$

Taking the Laplace transform and writing the result in matrix form yields

$$\begin{bmatrix} Ms^2 + bs + k & -(bs + k) \\ -(bs + k) & ms^2 + bs + k \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} .$$

Solving for  $Y(s)$  we find that

$$\frac{Y(s)}{F(s)} = \frac{\frac{1}{mM}(bs + k)}{s^2[s^2 + (1 + \frac{m}{M})(\frac{b}{m}s + \frac{k}{m})]} .$$



**P2.27** The describing equation of motion is

$$m\ddot{z} = mg - k\frac{i^2}{z^2} .$$

Defining

$$f(z, i) = g - \frac{ki^2}{mz^2}$$

leads to

$$\ddot{z} = f(z, i) .$$

The equilibrium condition for  $i_o$  and  $z_o$ , found by solving the equation of motion when

$$\dot{z} = \ddot{z} = 0 ,$$

is

$$\frac{ki_o^2}{mg} = z_o^2 .$$

We linearize the equation of motion using a Taylor series approximation. With the definitions

$$\Delta z = z - z_o \quad \text{and} \quad \Delta i = i - i_o ,$$

we have  $\dot{\Delta}z = \dot{z}$  and  $\ddot{\Delta}z = \ddot{z}$ . Therefore,

$$\ddot{\Delta}z = f(z, i) = f(z_o, i_o) + \left. \frac{\partial f}{\partial z} \right|_{\substack{z=z_o \\ i=i_o}} \Delta z + \left. \frac{\partial f}{\partial i} \right|_{\substack{z=z_o \\ i=i_o}} \Delta i + \dots$$

But  $f(z_o, i_o) = 0$ , and neglecting higher-order terms in the expansion yields

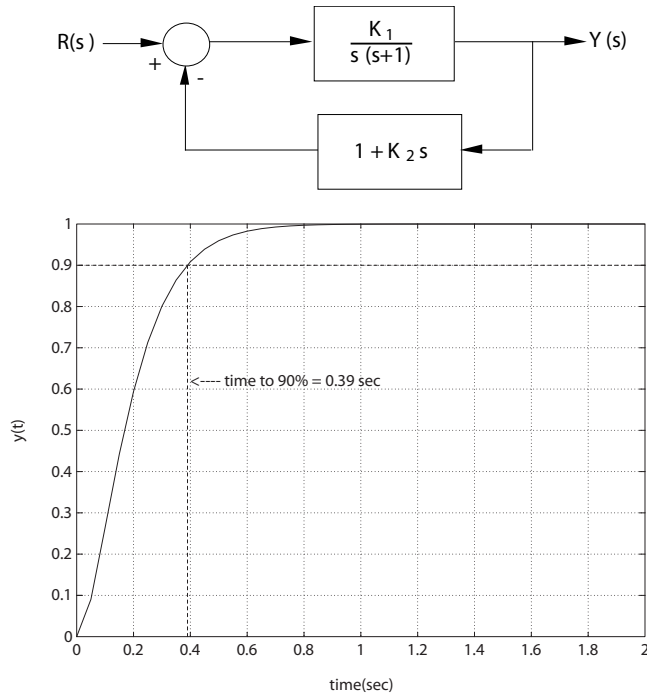
$$\ddot{\Delta}z = \frac{2ki_o^2}{mz_o^3} \Delta z - \frac{2ki_o}{mz_o^2} \Delta i .$$

Using the equilibrium condition which relates  $z_o$  to  $i_o$ , we determine that

$$\ddot{\Delta}z = \frac{2g}{z_o} \Delta z - \frac{g}{i_o} \Delta i .$$

Taking the Laplace transform yields the transfer function (valid around the equilibrium point)

$$\frac{\Delta Z(s)}{\Delta I(s)} = \frac{-g/i_o}{s^2 - 2g/z_o} .$$

**FIGURE P2.35**

The equivalent block diagram and the system step response.

**P2.36** (a) Given  $R(s) = 1/s^2$ , the partial fraction expansion is

$$Y(s) = \frac{24}{s^2(s+2)(s+3)(s+4)} = \frac{3}{s+2} - \frac{8/3}{s+3} + \frac{3/4}{s+4} + \frac{1}{s^2} - \frac{13/12}{s}.$$

Therefore, using the Laplace transform table, we determine that the ramp response is

$$y(t) = 3e^{-2t} - \frac{8}{3}e^{-3t} + \frac{3}{4}e^{-4t} + t - \frac{13}{12}, \quad t \geq 0.$$

(b) For the ramp input,  $y(t) \approx 0.21$  at  $t = 1$ . second (see Figure P2.36a).

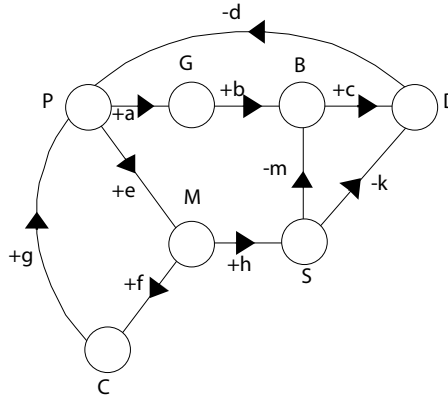
(c) Given  $R(s) = 1$ , the partial fraction expansion is

$$Y(s) = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4}.$$

Therefore, using the Laplace transform table, we determine that the impulse response is

$$y(t) = 12e^{-2t} - 24e^{-3t} + 12e^{-4t}, \quad t \geq 0.$$

**P2.28** The signal flow graph is shown in Figure P2.28.



**FIGURE P2.28**  
Signal flow graph.

- The PGBDP loop gain is equal to  $-abcd$ . This is a **negative** transmission since the population produces garbage which increases bacteria and leads to diseases, thus reducing the population.
- The PMCP loop gain is equal to  $+efg$ . This is a **positive** transmission since the population leads to modernization which encourages immigration, thus increasing the population.
- The PMSDP loop gain is equal to  $+ehkd$ . This is a **positive** transmission since the population leads to modernization and an increase in sanitation facilities which reduces diseases, thus reducing the rate of decreasing population.
- The PMSBDP loop gain is equal to  $+ehmcd$ . This is a **positive** transmission by similar argument as in (3).

**P2.29** Assume the motor torque is proportional to the input current

$$T_m = ki .$$

Then, the equation of motion of the beam is

$$J\ddot{\phi} = ki ,$$

where  $J$  is the moment of inertia of the beam and shaft (neglecting the inertia of the ball). We assume that forces acting on the ball are due to gravity and friction. Hence, the motion of the ball is described by

$$m\ddot{x} = mg\phi - b\dot{x}$$

where  $m$  is the mass of the ball,  $b$  is the coefficient of friction, and we have assumed small angles, so that  $\sin \phi \approx \phi$ . Taking the Laplace transform of both equations of motion and solving for  $X(s)$  yields

$$X(s)/I(s) = \frac{gk/J}{s^2(s^2 + b/m)} .$$

**P2.30** Given

$$H(s) = \frac{k}{\tau s + 1}$$

where  $\tau = 4\mu s = 4 \times 10^{-6}$  seconds and  $0.999 \leq k < 1.001$ . The step response is

$$Y(s) = \frac{k}{\tau s + 1} \cdot \frac{1}{s} = \frac{k}{s} - \frac{k}{s + 1/\tau} .$$

Taking the inverse Laplace transform yields

$$y(t) = k - ke^{-t/\tau} = k(1 - e^{-t/\tau}) .$$

The final value is  $k$ . The time it takes to reach 98% of the final value is  $t = 15.6\mu s$  independent of  $k$ .

**P2.31** From the block diagram we have

$$\begin{aligned} Y_1(s) &= G_2(s)[G_1(s)E_1(s) + G_3(s)E_2(s)] \\ &= G_2(s)G_1(s)[R_1(s) - H_1(s)Y_1(s)] + G_2(s)G_3(s)E_2(s) . \end{aligned}$$

Therefore,

$$Y_1(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}R_1(s) + \frac{G_2(s)G_3(s)}{1 + G_1(s)G_2(s)H_1(s)}E_2(s) .$$

And, computing  $E_2(s)$  (with  $R_2(s) = 0$ ) we find

$$E_2(s) = H_2(s)Y_2(s) = H_2(s)G_6(s) \left[ \frac{G_4(s)}{G_2(s)}Y_1(s) + G_5(s)E_2(s) \right]$$

or

$$E_2(s) = \frac{G_4(s)G_6(s)H_2(s)}{G_2(s)(1 - G_5(s)G_6(s)H_2(s))}Y_1(s) .$$

Substituting  $E_2(s)$  into equation for  $Y_1(s)$  yields

$$Y_1(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}R_1(s)$$

$$+ \frac{G_3(s)G_4(s)G_6(s)H_2(s)}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s))} Y_1(s) .$$

Finally, solving for  $Y_1(s)$  yields

$$Y_1(s) = T_1(s)R_1(s)$$

where

$$T_1(s) = \left[ \frac{G_1(s)G_2(s)(1 - G_5(s)G_6(s)H_2(s))}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)} \right] .$$

Similarly, for  $Y_2(s)$  we obtain

$$Y_2(s) = T_2(s)R_1(s) .$$

where

$$T_2(s) = \left[ \frac{G_1(s)G_4(s)G_6(s)}{(1 + G_1(s)G_2(s)H_1(s))(1 - G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)} \right] .$$

**P2.32** The signal flow graph shows three loops:

$$\begin{aligned} L_1 &= -G_1G_3G_4H_2 \\ L_2 &= -G_2G_5G_6H_1 \\ L_3 &= -H_1G_8G_6G_2G_7G_4H_2G_1 . \end{aligned}$$

The transfer function  $Y_2/R_1$  is found to be

$$\frac{Y_2(s)}{R_1(s)} = \frac{G_1G_8G_6\Delta_1 - G_2G_5G_6\Delta_2}{1 - (L_1 + L_2 + L_3) + (L_1L_2)} ,$$

where for path 1

$$\Delta_1 = 1$$

and for path 2

$$\Delta_2 = 1 - L_1 .$$

Since we want  $Y_2$  to be independent of  $R_1$ , we need  $Y_2/R_1 = 0$ . Therefore, we require

$$G_1G_8G_6 - G_2G_5G_6(1 + G_1G_3G_4H_2) = 0 .$$

**P2.33** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_3(s)G_1(s)(G_2(s) + K_5K_6)}{1 - G_3(s)(H_1(s) + K_6) + G_3(s)G_1(s)(G_2(s) + K_5K_6)(H_2(s) + K_4)} .$$

**P2.34** The equations of motion are

$$\begin{aligned} m_1\ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) &= 0 \\ m_2\ddot{y}_2 + b(\dot{y}_2 - \dot{y}_1) + k_1(y_2 - y_1) + k_2y_2 &= k_2x \end{aligned}$$

Taking the Laplace transform yields

$$\begin{aligned} (m_1s^2 + bs + k_1)Y_1(s) - (bs + k_1)Y_2(s) &= 0 \\ (m_2s^2 + bs + k_1 + k_2)Y_2(s) - (bs + k_1)Y_1(s) &= k_2X(s) \end{aligned}$$

Therefore, after solving for  $Y_1(s)/X(s)$ , we have

$$\frac{Y_2(s)}{X(s)} = \frac{k_2(bs + k_1)}{(m_1s^2 + bs + k_1)(m_2s^2 + bs + k_1 + k_2) - (bs + k_1)^2} .$$

**P2.35** (a) We can redraw the block diagram as shown in Figure P2.35. Then,

$$T(s) = \frac{K_1/s(s+1)}{1 + K_1(1 + K_2s)/s(s+1)} = \frac{K_1}{s^2 + (1 + K_2K_1)s + K_2} .$$

(b) The signal flow graph reveals two loops (both touching):

$$L_1 = \frac{-K_1}{s(s+1)} \quad \text{and} \quad L_2 = \frac{-K_1K_2}{s+1} .$$

Therefore,

$$T(s) = \frac{K_1/s(s+1)}{1 + K_1/s(s+1) + K_1K_2/(s+1)} = \frac{K_1}{s^2 + (1 + K_2K_1)s + K_1} .$$

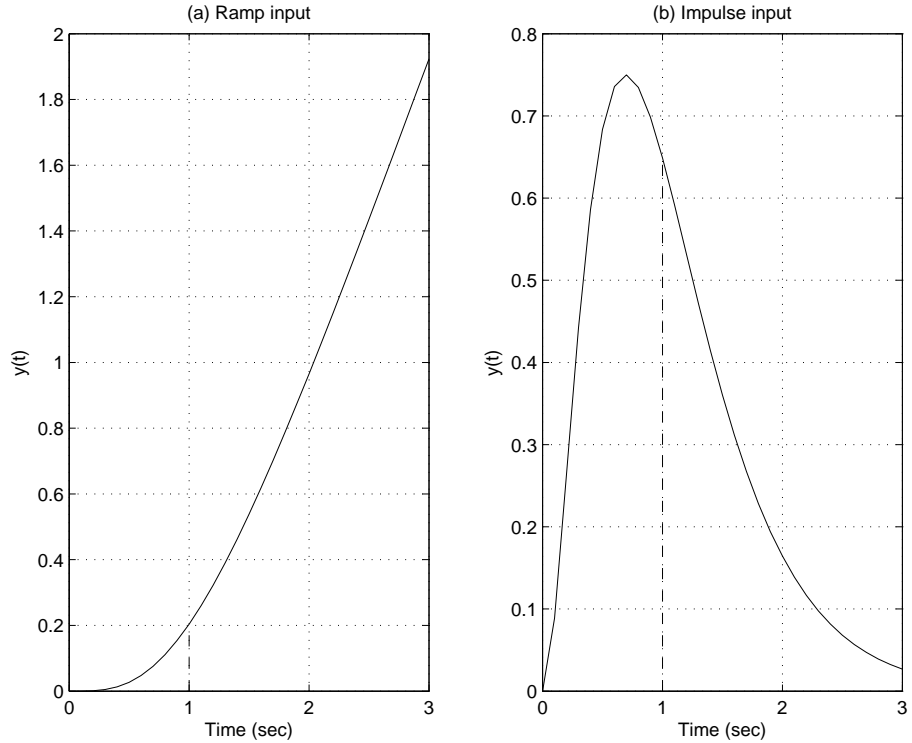
(c) We want to choose  $K_1$  and  $K_2$  such that

$$s^2 + (1 + K_2K_1)s + K_1 = s^2 + 20s + 100 = (s + 10)^2 .$$

Therefore,  $K_1 = 100$  and  $1 + K_2K_1 = 20$  or  $K_2 = 0.19$ .

(d) The step response is shown in Figure P2.35.

(d) For the impulse input,  $y(t) \approx 0.65$  at  $t = 1$  seconds (see Figure P2.36b).



**FIGURE P2.36**

(a) Ramp input response. (b) Impulse input response.

**P2.37** The equations of motion are

$$m_1 \frac{d^2 x}{dt^2} = -(k_1 + k_2)x + k_2 y \quad \text{and} \quad m_2 \frac{d^2 y}{dt^2} = k_2(x - y) + u .$$

When  $m_1 = m_2 = 1$  and  $k_1 = k_2 = 1$ , we have

$$\frac{d^2 x}{dt^2} = -2x + y \quad \text{and} \quad \frac{d^2 y}{dt^2} = x - y + u .$$

**P2.38** The equation of motion for the system is

$$J \frac{d^2 \theta}{dt^2} + b \frac{d\theta}{dt} + k\theta = 0 ,$$

where  $k$  is the rotational spring constant and  $b$  is the viscous friction coefficient. The initial conditions are  $\theta(0) = \theta_o$  and  $\dot{\theta}(0) = 0$ . Taking the

Laplace transform yields

$$J(s^2\theta(s) - s\theta_o) + b(s\theta(s) - \theta_o) + k\theta(s) = 0 .$$

Therefore,

$$\theta(s) = \frac{(s + \frac{b}{J}\theta_o)}{(s^2 + \frac{b}{J}s + \frac{K}{J})} = \frac{(s + 2\zeta\omega_n)\theta_o}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Neglecting the mass of the rod, the moment of inertia is determined to be

$$J = 2Mr^2 = 0.5 \text{ kg} \cdot \text{m}^2 .$$

Also,

$$\omega_n = \sqrt{\frac{k}{J}} = 0.02 \text{ rad/s} \quad \text{and} \quad \zeta = \frac{b}{2J\omega_n} = 0.01 .$$

Solving for  $\theta(t)$ , we find that

$$\theta(t) = \frac{\theta_o}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \phi) ,$$

where  $\tan \phi = \sqrt{1 - \zeta^2}/\zeta$ . Therefore, the envelope decay is

$$\theta_e = \frac{\theta_o}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} .$$

So, with  $\zeta\omega_n = 2 \times 10^{-4}$ ,  $\theta_o = 4000^\circ$  and  $\theta_f = 10^\circ$ , the elapsed time is computed as

$$t = \frac{1}{\zeta\omega_n} \ln \frac{\theta_o}{\sqrt{1 - \zeta^2} \theta_f} = 8.32 \text{ hours} .$$

**P2.39** When  $t < 0$ , we have the steady-state conditions

$$i_1(0) = 1A \quad , \quad v_a(0) = 2V \quad \text{and} \quad v_c(0) = 5V ,$$

where  $v_c(0)$  is associated with the 1F capacitor. After  $t \geq 0$ , we have

$$2 \frac{di_1}{dt} + 2i_1 + 4(i_1 - i_2) = 10e^{-2t}$$

and

$$\int i_2 dt + 10i_2 + 4(i_2 - i_1) - i_1 = 0 .$$



Taking the Laplace transform (using the initial conditions) yields

$$2(sI_1 - i_1(0)) + 2I_1 + 4I_1 - 4I_2 = \frac{10}{s+2} \quad \text{or} \quad (s+3)I_1(s) - 2I_2(s) = \frac{s+7}{s+2}$$

and

$$\left[\frac{1}{s}I_2 - v_c(0)\right] + 10I_2 + 4(I_2 - I_1) = I_1(s) \quad \text{or} \quad -5sI_1(s) + (14s+1)I_2(s) = 5s.$$

Solving for  $I_2(s)$  yields

$$I_2 = \frac{5s(s^2 + 6s + 13)}{14(s+2)\Delta(s)},$$

where

$$\Delta(s) = \begin{vmatrix} s+3 & -2 \\ -5s & 14s+1 \end{vmatrix} = 14s^2 + 33s + 3.$$

Then,

$$V_o(s) = 10I_2(s).$$

**P2.40** The equations of motion are

$$J_1\ddot{\theta}_1 = K(\theta_2 - \theta_1) - b(\dot{\theta}_1 - \dot{\theta}_2) + T \quad \text{and} \quad J_2\ddot{\theta}_2 = b(\dot{\theta}_1 - \dot{\theta}_2).$$

Taking the Laplace transform yields

$$(J_1s^2 + bs + K)\theta_1(s) - bs\theta_2(s) = K\theta_2(s) + T(s)$$

and

$$(J_2s^2 + bs)\theta_2(s) - bs\theta_1(s) = 0.$$

Solving for  $\theta_1(s)$  and  $\theta_2(s)$ , we find that

$$\theta_1(s) = \frac{(K\theta_2(s) + T(s))(J_2s + b)}{\Delta(s)} \quad \text{and} \quad \theta_2(s) = \frac{b(K\theta_2(s) + T(s))}{\Delta(s)},$$

where

$$\Delta(s) = J_1J_2s^3 + b(J_1 + J_2)s^2 + J_2Ks + bK.$$

**P2.41** Assume that the only external torques acting on the rocket are control torques,  $T_c$  and disturbance torques,  $T_d$ , and assume small angles,  $\theta(t)$ . Using the small angle approximation, we have

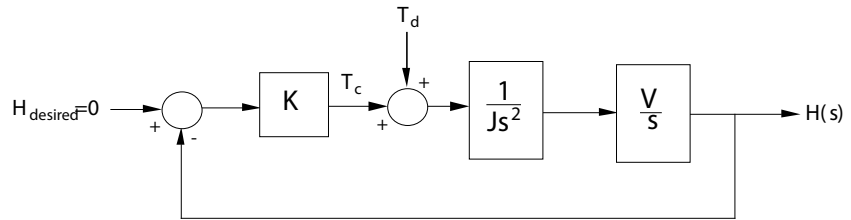
$$\dot{h} = V\theta$$

$$J\ddot{\theta} = T_c + T_d ,$$

where  $J$  is the moment of inertia of the rocket and  $V$  is the rocket velocity (assumed constant). Now, suppose that the control torque is proportional to the lateral displacement, as

$$T_c(s) = -KH(s) ,$$

where the negative sign denotes a negative feedback system. The corresponding block diagram is shown in Figure P2.41.



**FIGURE P2.41**  
Block diagram.

- P2.42** (a) The equation of motion of the motor is

$$J \frac{d\omega}{dt} = T_m - b\omega ,$$

where  $J = 0.1$ ,  $b = 0.06$ , and  $T_m$  is the motor input torque.

- (b) Given  $T_m(s) = 1/s$ , and  $\omega(0) = 0.7$ , we take the Laplace transform of the equation of motion yielding

$$s\omega(s) - \omega(0) + 0.6\omega(s) = 10T_m$$

or

$$\omega(s) = \frac{0.7s + 10}{s(s + 0.6)} .$$

Then, computing the partial fraction expansion, we find that

$$\omega(s) = \frac{A}{s} + \frac{B}{s + 0.6} = \frac{16.67}{s} - \frac{15.97}{s + 0.6} .$$

The step response, determined by taking the inverse Laplace transform, is

$$\omega(t) = 16.67 - 15.97e^{-0.6t} , \quad t \geq 0 .$$

**P2.43** The work done by each gear is equal to that of the other, therefore

$$T_m \theta_m = T_L \theta_L .$$

Also, the travel distance is the same for each gear, so

$$r_1 \theta_m = r_2 \theta_L .$$

The number of teeth on each gear is proportional to the radius, or

$$r_1 N_2 = r_2 N_1 .$$

So,

$$\frac{\theta_m}{\theta_L} = \frac{r_2}{r_1} = \frac{N_2}{N_1} ,$$

and

$$\begin{aligned} N_1 \theta_m &= N_2 \theta_L \\ \theta_L &= \frac{N_1}{N_2} \theta_m = n \theta_m , \end{aligned}$$

where

$$n = N_1 / N_2 .$$

Finally,

$$\frac{T_m}{T_L} = \frac{\theta_L}{\theta_m} = \frac{N_1}{N_2} = n .$$

**P2.44** The inertia of the load is

$$J_L = \frac{\pi \rho L r^4}{2} .$$

Also, from the dynamics we have

$$T_2 = J_L \dot{\omega}_2 + b_L \omega_2$$

and

$$T_1 = n T_2 = n (J_L \dot{\omega}_2 + b_L \omega_2) .$$

So,

$$T_1 = n^2 (J_L \dot{\omega}_1 + b_L \omega_1) ,$$

since

$$\omega_2 = n\omega_1 .$$

Therefore, the torque at the motor shaft is

$$T = T_1 + T_m = n^2(J_L\dot{\omega}_1 + b_L\omega_1) + J_m\dot{\omega}_1 + b_m\omega_1 .$$

**P2.45** Let  $U(s)$  denote the human input and  $F(s)$  the load input. The transfer function is

$$P(s) = \frac{G(s) + KG_1(s)}{\Delta(s)}U(s) + \frac{G_c(s) + KG_1(s)}{\Delta(s)}F(s) ,$$

where

$$\Delta = 1 + GH(s) + G_1KBH(s) + G_cE(s) + G_1KE(s) .$$

**P2.46** Consider the application of Newton's law ( $\sum F = m\ddot{x}$ ). From the mass  $m_v$  we obtain

$$m_v\ddot{x}_1 = F - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for  $X_1(s)$  yields

$$X_1(s) = \frac{1}{\Delta_1(s)}F(s) + \frac{b_1s + k_1}{\Delta_1(s)}X_2(s),$$

where

$$\Delta_1 := m_v s^2 + b_1 s + k_1.$$

From the mass  $m_t$  we obtain

$$m_t\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for  $X_2(s)$  yields

$$X_2(s) = \frac{b_1s + k_1}{\Delta_2(s)}X_1(s),$$

where

$$\Delta_2 := m_t s^2 + (b_1 + b_2)s + k_1 + k_2.$$

Substituting  $X_2(s)$  above into the relationship for  $X_1(s)$  yields the transfer function

$$\frac{X_1(s)}{F(s)} = \frac{\Delta_2(s)}{\Delta_1(s)\Delta_2(s) - (b_1s + k_1)^2}.$$

**P2.47** Using the following relationships

$$\begin{aligned}h(t) &= \int (1.6\theta(t) - h(t))dt \\ \omega(t) &= \dot{\theta}(t) \\ J\dot{\omega}(t) &= K_m i_a(t) \\ v_a(t) = 50v_i(t) &= 10i_a(t) + v_b(t) \\ \dot{\theta} &= K v_b\end{aligned}$$

we find the differential equation is

$$\frac{d^3h}{dt^3} + \left(1 + \frac{K_m}{10JK}\right) \frac{d^2h}{dt^2} + \frac{K_m}{10JK} \frac{dh}{dt} = \frac{8K_m}{J} v_i .$$

**P2.48** (a) The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + sR_1C_1)(1 + sR_2C_2)}{R_1C_2s} .$$

(b) When  $R_1 = 100 \text{ k}\Omega$ ,  $R_2 = 200 \text{ k}\Omega$ ,  $C_1 = 1 \text{ }\mu\text{F}$  and  $C_2 = 0.1 \text{ }\mu\text{F}$ , we have

$$\frac{V_2(s)}{V_1(s)} = \frac{0.2(s + 10)(s + 50)}{s} .$$

**P2.49** (a) The closed-loop transfer function is

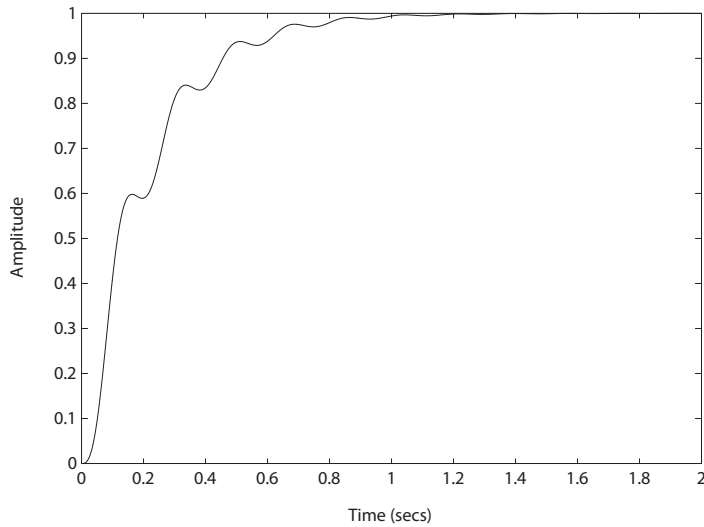
$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{6205}{s^3 + 13s^2 + 1281s + 6205} .$$

(b) The poles of  $T(s)$  are  $s_1 = -5$  and  $s_{2,3} = -4 \pm j35$ .

(c) The partial fraction expansion (with a step input) is

$$Y(s) = 1 - \frac{1.0122}{s + 5} + \frac{0.0061 + 0.0716j}{s + 4 + j35} + \frac{0.0061 - 0.0716j}{s + 4 - j35} .$$

(d) The step response is shown in Figure P2.49. The real and complex roots are close together and by looking at the poles in the s-plane we have difficulty deciding which is dominant. However, the residue at the real pole is much larger and thus dominates the response.



**FIGURE P2.49**  
Step response.

**P2.50** (a) The closed-loop transfer function is

$$T(s) = \frac{14000}{s^3 + 45s^2 + 3100s + 14500} .$$

(b) The poles of  $T(s)$  are

$$s_1 = -5 \quad \text{and} \quad s_{2,3} = -20 \pm j50 .$$

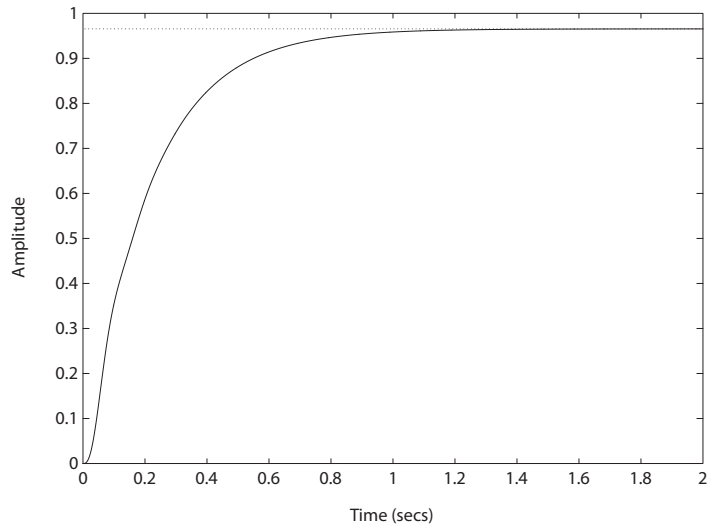
(c) The partial fraction expansion (with a step input) is

$$Y(s) = \frac{0.9655}{s} - \frac{1.0275}{s+5} + \frac{0.0310 - 0.0390j}{s+20+j50} + \frac{0.0310 + 0.0390j}{s+20-j50} .$$

(d) The step response is shown in Figure P2.50. The real root dominates the response.

(e) The final value of  $y(t)$  is

$$y_{ss} = \lim_{s \rightarrow 0} sY(s) = 0.9655 .$$

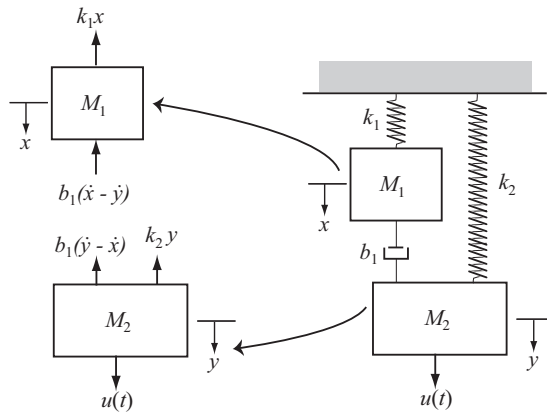


**FIGURE P2.50**  
Step response.

**P2.51** Consider the free body diagram in Figure P2.51. Using Newton’s Law and summing the forces on the two masses yields

$$M_1\ddot{x}(t) + b_1\dot{x}(t) + k_1x(t) = b_1\dot{y}(t)$$

$$M_2\ddot{y}(t) + b_1\dot{y}(t) + k_2y(t) = b_1\dot{x}(t) + u(t)$$



**FIGURE P2.51**  
Free body diagram.

## Advanced Problems

**AP2.1** The transfer function from  $V(s)$  to  $\omega(s)$  has the form

$$\frac{\omega(s)}{V(s)} = \frac{K_m}{\tau_m s + 1} .$$

In the steady-state,

$$\omega_{ss} = \lim_{s \rightarrow 0} s \left[ \frac{K_m}{\tau_m s + 1} \right] \frac{5}{s} = 5K_m .$$

So,

$$K_m = 70/5 = 14 .$$

Also,

$$\omega(t) = V_m K_m (1 - e^{-t/\tau_m})$$

where  $V(s) = V_m/s$ . Solving for  $\tau_m$  yields

$$\tau_m = \frac{-t}{\ln(1 - \omega(t)/\omega_{ss})} .$$

When  $t = 2$ , we have

$$\tau_m = \frac{-2}{\ln(1 - 30/70)} = 3.57 .$$

Therefore, the transfer function is

$$\frac{\omega(s)}{V(s)} = \frac{14}{3.57s + 1} .$$

**AP2.2** The closed-loop transfer function from  $R_1(s)$  to  $Y_2(s)$  is

$$\frac{Y_2(s)}{R_1(s)} = \frac{G_1 G_4 G_5(s) + G_1 G_2 G_3 G_4 G_6(s)}{\Delta}$$

where

$$\Delta = [1 + G_3 G_4 H_2(s)][1 + G_1 G_2 H_3(s)] .$$

If we select

$$G_5(s) = -G_2 G_3 G_6(s)$$

then the numerator is zero, and  $Y_2(s)/R_1(s) = 0$ . The system is now decoupled.



**AP2.3** (a) Computing the closed-loop transfer function:

$$Y(s) = \left[ \frac{G(s)G_c(s)}{1 + G_c(s)G(s)H(s)} \right] R(s) .$$

Then, with  $E(s) = R(s) - Y(s)$  we obtain

$$E(s) = \left[ \frac{1 + G_c(s)G(s)(H(s) - 1)}{1 + G_c(s)G(s)H(s)} \right] R(s) .$$

If we require that  $E(s) \equiv 0$  for any input, we need  $1 + G_c(s)G(s)(H(s) - 1) = 0$  or

$$H(s) = \frac{G_c(s)G(s) - 1}{G_c(s)G(s)} = \frac{n(s)}{d(s)} .$$

Since we require  $H(s)$  to be a causal system, the order of the numerator polynomial,  $n(s)$ , must be less than or equal to the order of the denominator polynomial,  $d(s)$ . This will be true, in general, only if both  $G_c(s)$  and  $G(s)$  are proper rational functions (that is, the numerator and denominator polynomials have the same order). Therefore, making  $E \equiv 0$  for any input  $R(s)$  is possible only in certain circumstances.

(b) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$Y(s) = \left[ \frac{G_d(s)G(s)}{1 + G_c(s)G(s)H(s)} \right] T_d(s) .$$

With  $H(s)$  as in part (a) we have

$$Y(s) = \left[ \frac{G_d(s)}{G_c(s)} \right] T_d(s) .$$

(c) No. Since

$$Y(s) = \left[ \frac{G_d(s)G(s)}{1 + G_c(s)G(s)H(s)} \right] T_d(s) = T(s)T_d(s) ,$$

the only way to have  $Y(s) \equiv 0$  for any  $T_d(s)$  is for the transfer function  $T(s) \equiv 0$  which is not possible in general (since  $G(s) \neq 0$ ).

**AP2.4** (a) With  $q(s) = 1/s$  we obtain

$$\tau(s) = \frac{1/C_t}{s + \frac{QS+1/R}{C_t}} \cdot \frac{1}{s} .$$

Define

$$\alpha := \frac{QS + 1/R}{C_t} \quad \text{and} \quad \beta := 1/C_t .$$

Then, it follows that

$$\tau(s) = \frac{\beta}{s + \alpha} \cdot \frac{1}{s} = \frac{-\beta/\alpha}{s + \alpha} + \frac{\beta/\alpha}{s}.$$

Taking the inverse Laplace transform yields

$$\tau(t) = \frac{-\beta}{\alpha} e^{-\alpha t} + \frac{\beta}{\alpha} = \frac{\beta}{\alpha} [1 - e^{-\alpha t}].$$

(b) As  $t \rightarrow \infty$ ,  $\tau(t) \rightarrow \frac{\beta}{\alpha} = \frac{1}{Qs+1/R}$ .

(c) To increase the speed of response, you want to choose  $C_t$ ,  $Q$ ,  $S$  and  $R$  such that

$$\alpha := \frac{Qs + 1/R}{C_t}$$

is "large."

**AP2.5** Considering the motion of each mass, we have

$$\begin{aligned} M_3 \ddot{x}_3 + b_3 \dot{x}_3 + k_3 x_3 &= u_3 + b_3 \dot{x}_2 + k_3 x_2 \\ M_2 \ddot{x}_2 + (b_2 + b_3) \dot{x}_2 + (k_2 + k_3) x_2 &= u_2 + b_3 \dot{x}_3 + k_3 x_3 + b_2 \dot{x}_1 + k_2 x_1 \\ M_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 + (k_1 + k_2) x_1 &= u_1 + b_2 \dot{x}_2 + k_2 x_2 \end{aligned}$$

In matrix form the three equations can be written as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{bmatrix} b_1 + b_2 & -b_2 & 0 \\ -b_2 & b_2 + b_3 & -b_3 \\ 0 & -b_3 & b_3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

**AP2.6** Considering the cart mass and using Newton's Law we obtain

$$M\ddot{x} = u - b\dot{x} - F \sin \varphi$$

where  $F$  is the reaction force between the cart and the pendulum. Considering the pendulum we obtain

$$m \frac{d^2(x + L \sin \varphi)}{dt^2} = F \sin \varphi$$

$$m \frac{d^2(L \cos \varphi)}{dt^2} = F \cos \varphi + mg$$

Eliminating the reaction force  $F$  yields the two equations

$$\begin{aligned}(m + M)\ddot{x} + b\dot{x} + mL\ddot{\varphi} \cos \varphi - mL\dot{\varphi}^2 \sin \varphi &= u \\ mL^2\ddot{\varphi} + mgL \sin \varphi + mL\ddot{x} \cos \varphi &= 0\end{aligned}$$

If we assume that the angle  $\varphi \approx 0$ , then we have the linear model

$$\begin{aligned}(m + M)\ddot{x} + b\dot{x} + mL\ddot{\varphi} &= u \\ mL^2\ddot{\varphi} + mgL\varphi &= -mL\ddot{x}\end{aligned}$$

**AP2.7** The transfer function from the disturbance input to the output is

$$Y(s) = \frac{1}{s + 20 + K} T_d(s) .$$

When  $T_d(s) = 1$ , we obtain

$$y(t) = e^{-(20+K)t} .$$

Solving for  $t$  when  $y(t) < 0.1$  yields

$$t > \frac{2.3}{20 + K} .$$

When  $t = 0.05$  and  $y(0.05) = 0.1$ , we find  $K = 26.05$ .

**AP2.8** The closed-loop transfer function is

$$T(s) = \frac{200K(0.25s + 1)}{(0.25s + 1)(s + 1)(s + 8) + 200K}$$

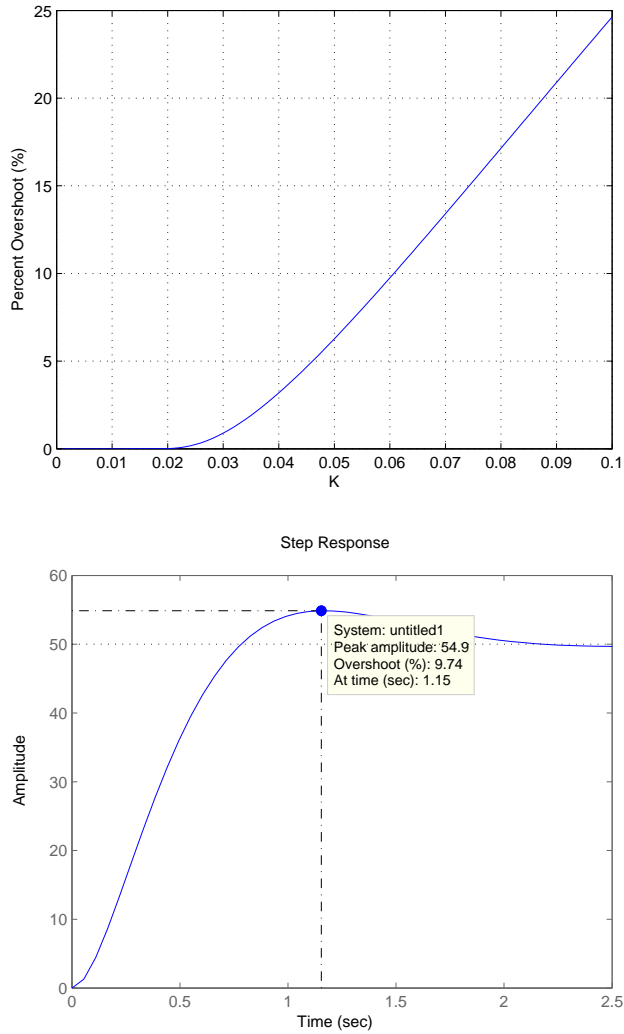
The final value due to a step input of  $R(s) = A/s$  is

$$v(t) \rightarrow A \frac{200K}{200K + 8} .$$

We need to select  $K$  so that  $v(t) \rightarrow 50$ . However, to keep the percent overshoot to less than 10%, we need to limit the magnitude of  $K$ . Figure AP2.8a shows the percent overshoot as a function of  $K$ . Let  $K = 0.06$  and select the magnitude of the input to be  $A = 83.3$ . The inverse Laplace transform of the closed-loop response with  $R(s) = 83.3/s$  is

$$v(t) = 50 + 9.85e^{-9.15t} - e^{-1.93t} (59.85 \cos(2.24t) + 11.27 \sin(2.24t))$$

The result is  $P.O. = 9.74\%$  and the steady-state value of the output is approximately 50 m/s, as shown in Figure AP2.8b.



**FIGURE AP2.8** (a) Percent overshoot versus the gain  $K$ . (b) Step response.

**AP2.9** The transfer function is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)},$$

where

$$Z_1(s) = \frac{R_1}{R_1 C_1 s + 1} \quad \text{and} \quad Z_2(s) = \frac{R_2 C_2 s + 1}{C_2 s} .$$

Then we can write

$$\frac{V_o(s)}{V_i(s)} = K_P + \frac{K_I}{s} + K_D s$$

where

$$K_P = -\left(\frac{R_1 C_1}{R_2 C_2} + 1\right), \quad K_I = -\frac{1}{R_1 C_2}, \quad K_D = -R_2 C_1 .$$

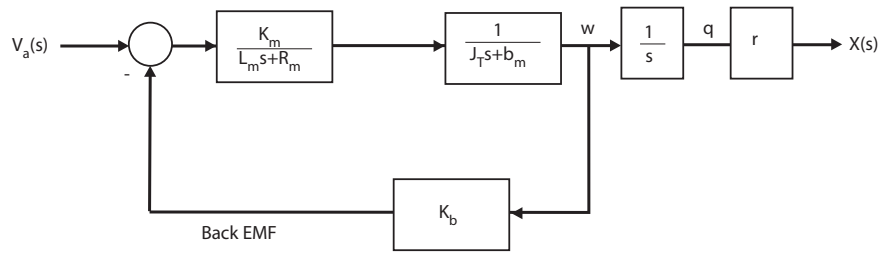
## Design Problems

**CDP2.1** The model of the traction drive, capstan roller, and linear slide follows closely the armature-controlled dc motor model depicted in Figure 2.18 in Dorf and Bishop. The transfer function is

$$T(s) = \frac{rK_m}{s[(L_m s + R_m)(J_T s + b_m) + K_b K_m]} ,$$

where

$$J_T = J_m + r^2(M_s + M_b) .$$



**DP2.1** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)H_1(s) - G_2(s)H_2(s)} .$$

When  $G_1 H_1 = G_2 H_2$  and  $G_1 G_2 = 1$ , then  $Y(s)/R(s) = 1$ . Therefore, select

$$G_1(s) = \frac{1}{G_2(s)} \quad \text{and} \quad H_1(s) = \frac{G_2(s)H_2(s)}{G_1(s)} = G_2^2(s)H_2(s) .$$

**DP2.2** At the lower node we have

$$v \left( \frac{1}{4} + \frac{1}{3} + G \right) + 2i_2 - 20 = 0 .$$

Also, we have  $v = 24$  and  $i_2 = Gv$  . So

$$v \left( \frac{1}{4} + \frac{1}{3} + G \right) + 2Gv - 20 = 0$$

and

$$G = \frac{20 - v \left( \frac{1}{4} + \frac{1}{3} \right)}{3v} = \frac{1}{12} S .$$

**DP2.3** Taking the Laplace transform of

$$y(t) = e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4} + \frac{1}{2}t$$

yields

$$Y(s) = \frac{1}{s+1} - \frac{1}{4(s+2)} - \frac{3}{4s} + \frac{1}{2s^2} .$$

Similarly, taking the Laplace transform of the ramp input yields

$$R(s) = \frac{1}{s^2} .$$

Therefore

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{(s+1)(s+2)} .$$

**DP2.4** For an ideal op-amp, at node a we have

$$\frac{v_{in} - v_a}{R_1} + \frac{v_o - v_a}{R_1} = 0 ,$$

and at node b

$$\frac{v_{in} - v_b}{R_2} = C\dot{v}_b ,$$

from it follows that

$$\left[ \frac{1}{R_2} + Cs \right] V_b = \frac{1}{R_2} V_{in} .$$

Also, for an ideal op-amp,  $V_b - V_a = 0$ . Then solving for  $V_b$  in the above equation and substituting the result into the node a equation for  $V_a$  yields

$$\frac{V_o}{V_{in}} = \frac{2}{\frac{1}{R_2} + Cs} \left[ \frac{1}{R_2} - \frac{\frac{1}{R_2} + Cs}{2} \right]$$

or

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} .$$

For  $v_{in}(t) = At$ , we have  $V_{in}(s) = A/s^2$ , therefore

$$v_o(t) = A \left[ \frac{2}{\beta} e^{-\beta t} + t - \frac{2}{\beta} \right]$$

where  $\beta = 1/R_2C$ .

**DP2.5** The equation of motion describing the motion of the inverted pendulum (assuming small angles) is

$$\ddot{\varphi} + \frac{g}{L}\varphi = 0 .$$

Assuming a solution of the form  $\varphi = k \cos \varphi$ , taking the appropriate derivatives and substituting the result into the equation of motion yields the relationship

$$\dot{\varphi} = \sqrt{\frac{g}{L}} .$$

If the period is  $T = 2$  seconds, we compute  $\dot{\varphi} = 2\pi/T$ . Then solving for  $L$  yields  $L = 0.99$  meters when  $g = 9.81$  m/s<sup>2</sup>. So, to fit the pendulum into the grandfather clock, the dimensions are generally about 1.5 meters or more.



## Computer Problems

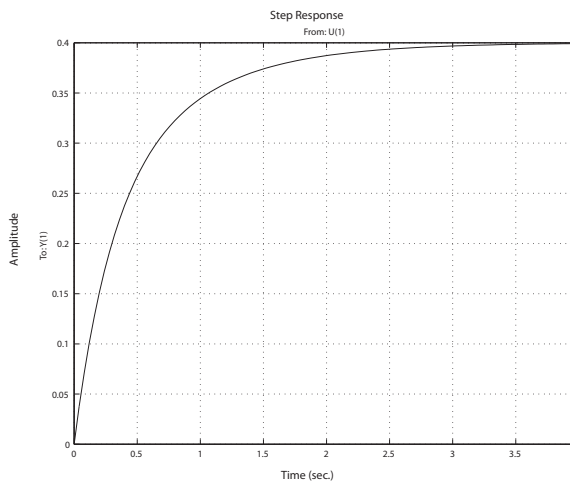
**CP2.1** The m-file script is shown in Figure CP2.1.

<pre>p=[1 7 10]; q=[1 2]; % Part (a) pq=conv(p,q) % Part (b) P=roots(p), Z=roots(q) % Part (c) value=polyval(p,-1)</pre>	<pre> pq =     1    9   24   20 P =    -5    -2 Z =    -2    -2 value =     4</pre>
--	---

**FIGURE CP2.1**  
Script for various polynomial evaluations.

**CP2.2** The m-file script and step response is shown in Figure CP2.2.

<pre>numc = [1]; denc = [1 1]; sysc = tf(numc,denc) numg = [1 2]; deng = [1 3]; sysg = tf(numg,deng) % part (a) sys_s = series(sysc,sysg); sys_cl = feedback(sys_s,[1]) % part (b) step(sys_cl); grid on</pre>	<pre> Transfer function:     s + 2 ----- s^2 + 5 s + 5</pre>
--	--



**FIGURE CP2.2**  
Step response.

**CP2.3** Given

$$\ddot{y} + 4\dot{y} + 3y = u$$

with  $y(0) = \dot{y} = 0$  and  $U(s) = 1/s$ , we obtain (via Laplace transform)

$$Y(s) = \frac{1}{s(s^2 + 4s + 3)} = \frac{1}{s(s+3)(s+1)} .$$

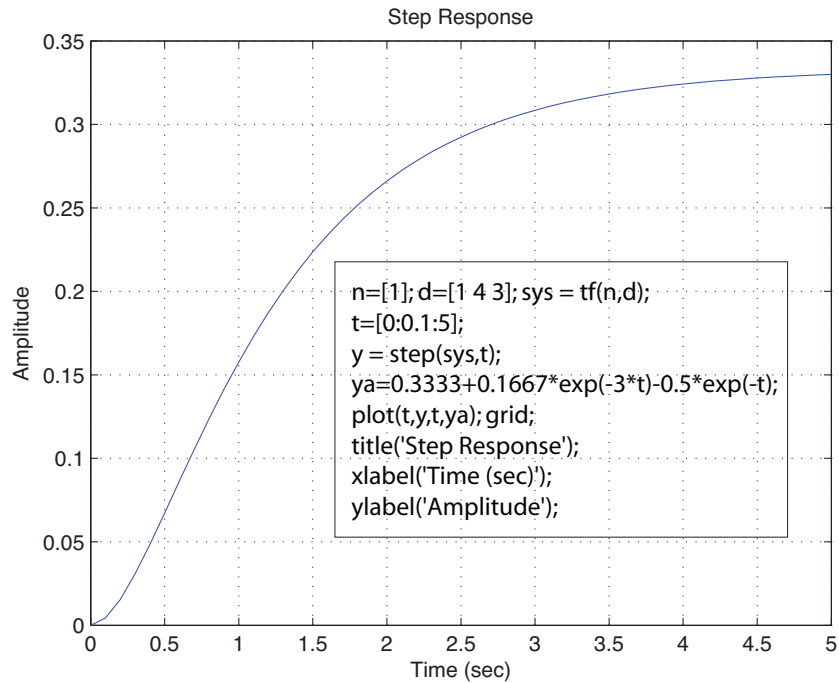
Expanding in a partial fraction expansion yields

$$Y(s) = \frac{1}{3s} - \frac{1}{6(s+3)} - \frac{1}{2(s+1)} .$$

Taking the inverse Laplace transform we obtain the solution

$$y(t) = 0.3333 + 0.1667e^{-3t} - 0.5e^{-t} .$$

The m-file script and step response is shown in Figure CP2.3.



**FIGURE CP2.3**  
Step response.

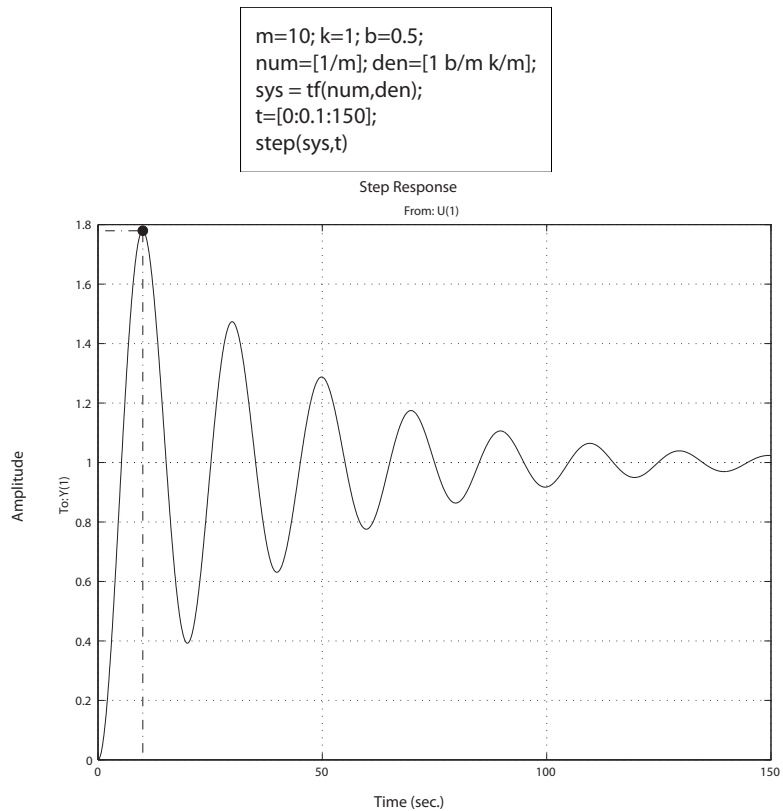
**CP2.4** The mass-spring-damper system is represented by

$$m\ddot{x} + b\dot{x} + kx = f .$$

Taking the Laplace transform (with zero initial conditions) yields the transfer function

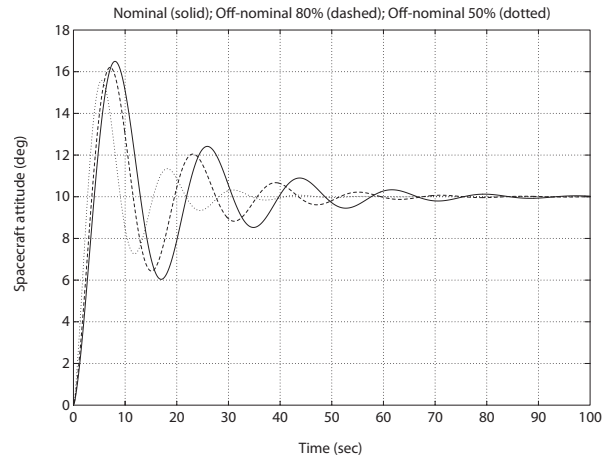
$$X(s)/F(s) = \frac{1/m}{s^2 + bs/m + k/m} .$$

The m-file script and step response is shown in Figure CP2.4.



**FIGURE CP2.4**  
Step response.

**CP2.5** The spacecraft simulations are shown in Figure CP2.5. We see that as  $J$  is decreased, the time to settle down decreases. Also, the overshoot from  $10^o$  decreases as  $J$  decreases. Thus, the performance seems to get better (in some sense) as  $J$  decreases.



```

%Part (a)
a=1; b=8; k=10.8e+08; J=10.8e+08;
num=k*[1 a];
den=J*[1 b 0 0]; sys=tf(num,den);
sys_cl=feedback(sys,[1]);
%
% Part (b) and (c)
t=[0:0.1:100];
%
% Nominal case
f=10*pi/180; sysf=sys_cl*f;
y=step(sysf,t);
%
% Off-nominal case 80%
J=10.8e+08*0.8; den=J*[1 b 0 0];
sys=tf(num,den); sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y1=step(sysf,t);
%
% Off-nominal case 50%
J=10.8e+08*0.5; den=J*[1 b 0 0];
sys=tf(num,den); sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y2=step(sysf,t);
%
plot(t,y*180/pi,t,y1*180/pi,'--',t,y2*180/pi,':'),grid
xlabel('Time (sec)')
ylabel('Spacecraft attitude (deg)')
title('Nominal (solid); Off-nominal 80% (dashed); Off-nominal 50% (dotted)')

```

FIGURE CP2.5

Step responses for the nominal and off-nominal spacecraft parameters.

**CP2.6** The closed-loop transfer function is

$$T(s) = \frac{4s^6 + 8s^5 + 4s^4 + 56s^3 + 112s^2 + 56s}{\Delta(s)},$$

```

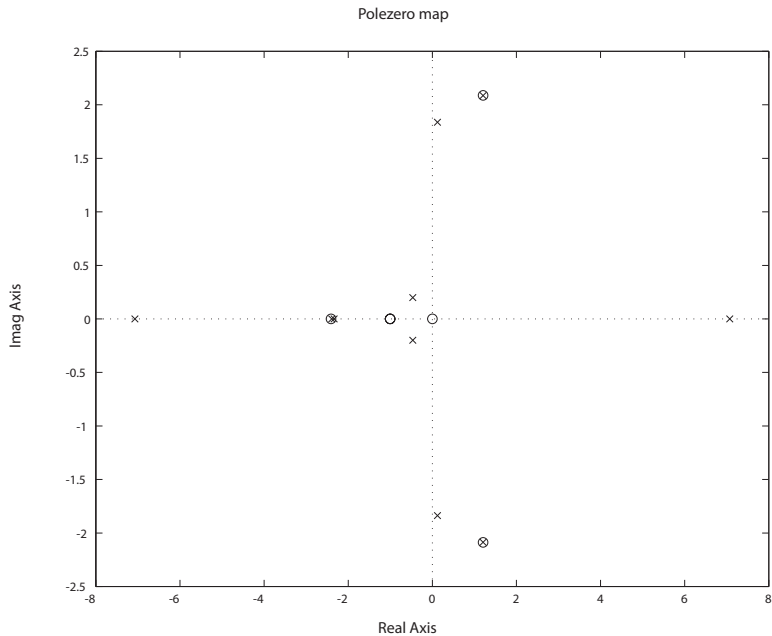
num1=[4]; den1=[1]; sys1 = tf(num1,den1);
num2=[1]; den2=[1 1]; sys2 = tf(num2,den2);
num3=[1 0]; den3=[1 0 2]; sys3 = tf(num3,den3);
num4=[1]; den4=[1 0 0]; sys4 = tf(num4,den4);
num5=[4 2]; den5=[1 2 1]; sys5 = tf(num5,den5);
num6=[50]; den6=[1]; sys6 = tf(num6,den6);
num7=[1 0 2]; den7=[1 0 0 14]; sys7 = tf(num7,den7);
sysa = feedback(sys4,sys6,+1);
sysb = series(sys2,sys3);
sysc = feedback(sysb,sys5);
sysd = series(sysc,sysa);
syse = feedback(sysd,sys7);
sys = series(sys1,syse)
%
pzmap(sys)
%
p=pole(sys)
z=zero(sys)
    
```

p =

7.0709  
-7.0713  
1.2051 + 2.0863i  
1.2051 - 2.0863i  
0.1219 + 1.8374i  
0.1219 - 1.8374i  
-2.3933  
-2.3333  
-0.4635 + 0.1997i  
-0.4635 - 0.1997i

z =

0  
1.2051 + 2.0872i  
1.2051 - 2.0872i  
-2.4101  
-1.0000 + 0.0000i  
-1.0000 - 0.0000i



**FIGURE CP2.6**  
Pole-zero map.

where

$$\Delta(s) = s^{10} + 3s^9 - 45s^8 - 125s^7 - 200s^6 - 1177s^5 - 2344s^4 - 3485s^3 - 7668s^2 - 5598s - 1400 .$$

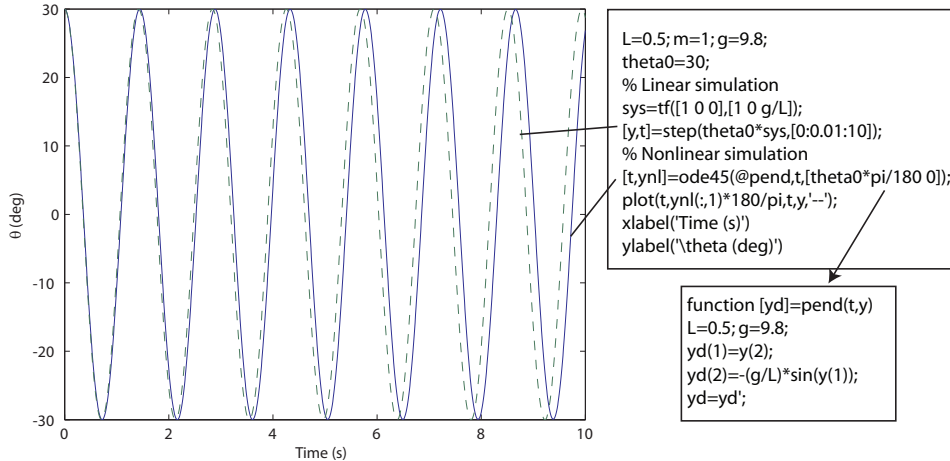
**CP2.7** The m-file script and plot of the pendulum angle is shown in Figure CP2.7. With the initial conditions, the Laplace transform of the linear system is

$$\theta(s) = \frac{\theta_0 s}{s^2 + g/L}.$$

To use the step function with the m-file, we can multiply the transfer function as follows:

$$\theta(s) = \frac{s^2}{s^2 + g/L} \frac{\theta_0}{s},$$

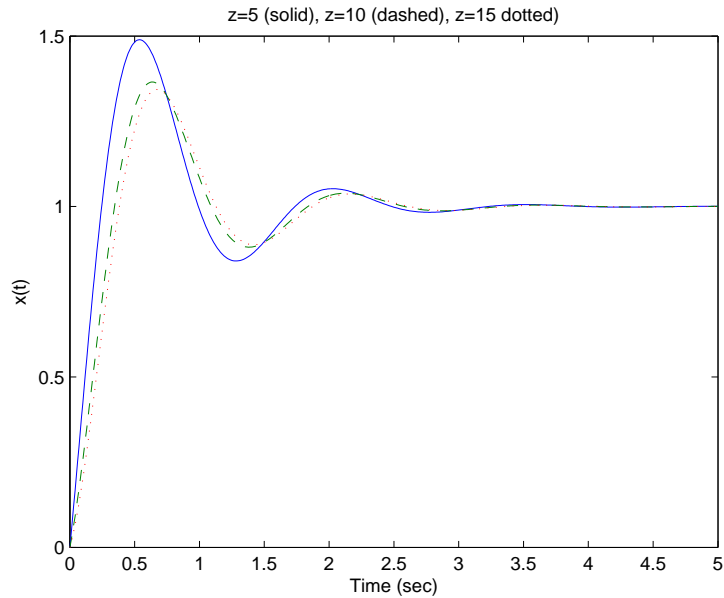
which is equivalent to the original transfer function except that we can use the step function input with magnitude  $\theta_0$ . The nonlinear response is shown as the solid line and the linear response is shown as the dashed line. The difference between the two responses is not great since the initial condition of  $\theta_0 = 30^\circ$  is not that large.



**FIGURE CP2.7**

Plot of  $\theta$  versus  $xt$  when  $\theta_0 = 30^\circ$ .

**CP2.8** The system step responses for  $z = 5, 10,$  and  $15$  are shown in Figure CP2.8.



**FIGURE CP2.8**  
The system response.

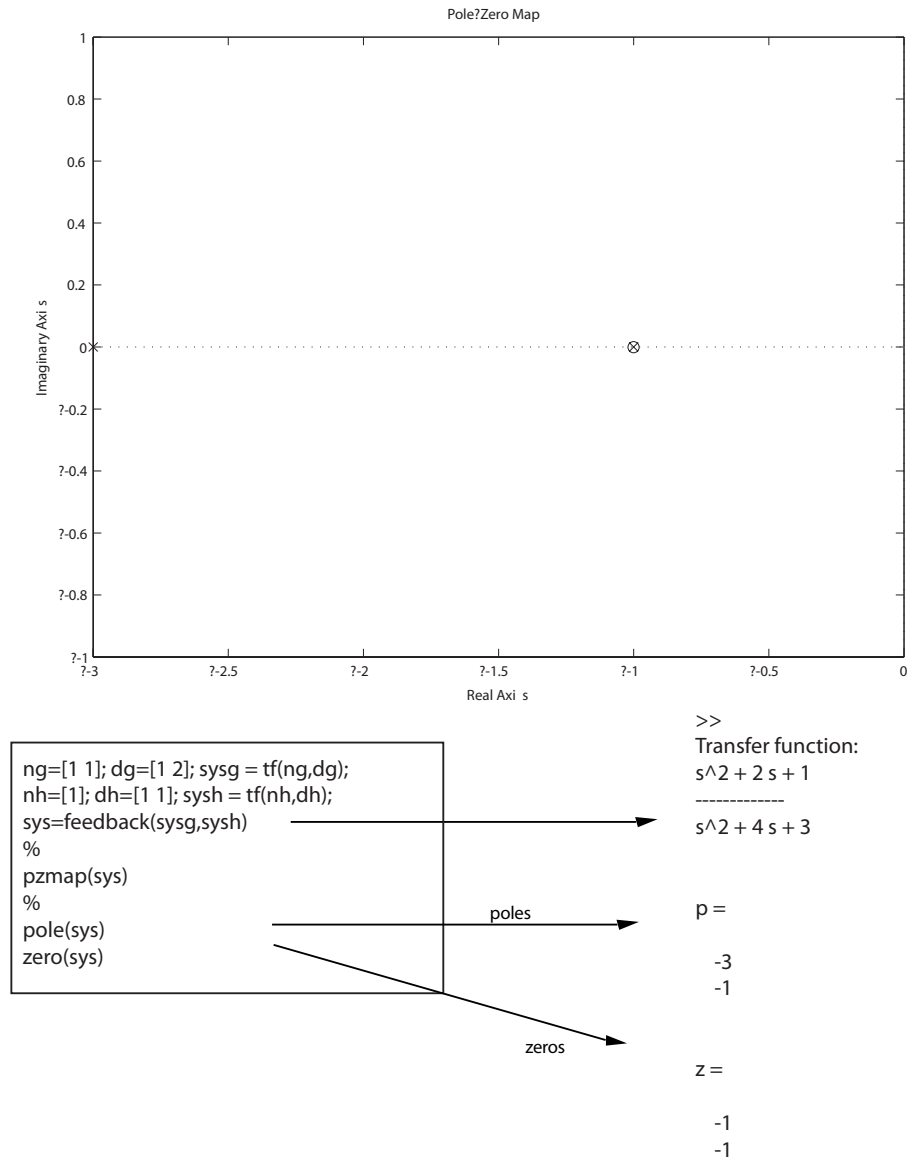
**CP2.9** (a,b) Computing the closed-loop transfer function yields

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s^2 + 2s + 1}{s^2 + 4s + 3}.$$

The poles are  $s = -3, -1$  and the zeros are  $s = -1, -1$ .

(c) Yes, there is one pole-zero cancellation. The transfer function (after pole-zero cancellation) is

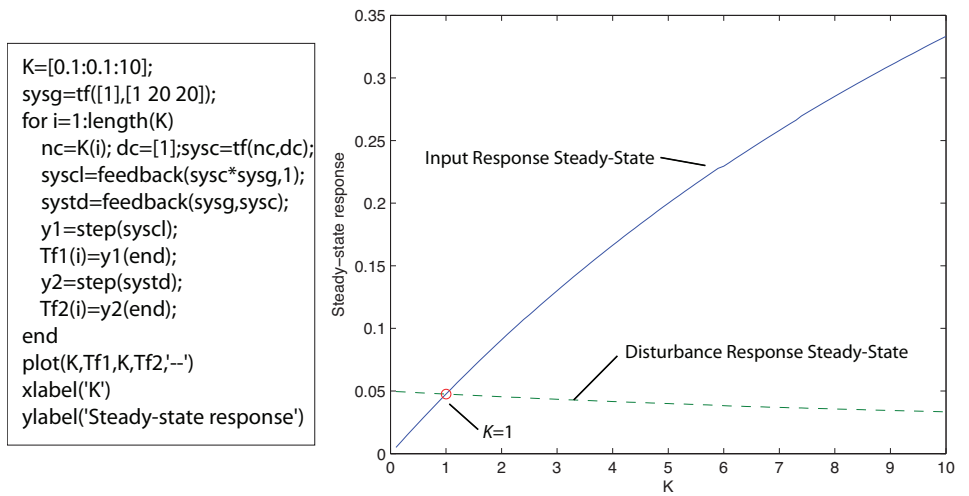
$$T(s) = \frac{s + 1}{s + 3}.$$



**FIGURE CP2.9**  
Pole-zero map.

**CP2.10** Figure CP2.10 shows the steady-state response to a unit step input and a unit step disturbance. We see that  $K = 1$  leads to the same steady-state response.



**FIGURE CP2.10**Gain  $K$  versus steady-state value.

# State Variable Models

## Exercises

**E3.1** One possible set of state variables is

- (a) the current  $i_{L_2}$  through  $L_2$ ,
- (b) the voltage  $v_{C_2}$  across  $C_2$ , and
- (c) the current  $i_{L_1}$  through  $L_1$ .

We can also choose  $v_{C_1}$ , the voltage across  $C_1$  as the third state variable, in place of the current through  $L_1$ .

**E3.2** We know that the velocity is the derivative of the position, therefore we have

$$\frac{dy}{dt} = v ,$$

and from the problem statement

$$\frac{dv}{dt} = -k_1v(t) - k_2y(t) + k_3i(t) .$$

This can be written in matrix form as

$$\frac{d}{dt} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ k_3 \end{bmatrix} i .$$

Define  $u = i$ , and let  $k_1 = k_2 = 1$ . Then,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 0 \\ k_3 \end{bmatrix} , \text{ and } \mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix} .$$

**E3.3** The characteristic roots, denoted by  $\lambda$ , are the solutions of  $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ . For this problem we have

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \left( \begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix} \right) = \lambda(\lambda + 2) + 1 = \lambda^2 + 2\lambda + 1 = 0 .$$

Therefore, the characteristic roots are

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -1 .$$

**E3.4** The system in phase variable form is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -6 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}, \quad \mathbf{C} = [1 \ 0 \ 0] .$$

**E3.5** From the block diagram we determine that the state equations are

$$\begin{aligned} \dot{x}_2 &= -(fk + d)x_2 + ax_1 + fu \\ \dot{x}_1 &= -kx_2 + u \end{aligned}$$

and the output equation is

$$y = bx_2 .$$

Therefore,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}u , \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -k \\ a & -(fk + d) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ f \end{bmatrix}, \quad \mathbf{C} = [0 \ b] \quad \text{and} \quad \mathbf{D} = [0] .$$

**E3.6** (a) The state transition matrix is

$$\Phi(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \dots$$

But  $\mathbf{A}^2 = 0$ , thus  $\mathbf{A}^3 = \mathbf{A}^4 = \dots = 0$ . So,

$$\Phi(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} .$$

(b) The state at any time  $t \geq 0$  is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0)$$

and since  $x_1(0) = x_2(0) = 1$ , we determine that

$$\mathbf{x}_1(t) = \mathbf{x}_1(0) + t\mathbf{x}_2(0) = 1 + t$$

$$\mathbf{x}_2(t) = \mathbf{x}_2(0) = 1 .$$

**E3.7** The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -100x_1 - 20x_2 + u$$

or, in matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u .$$

So, the characteristic equation is determined to be

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1 \\ 100 & \lambda + 20 \end{bmatrix} = \lambda^2 + 20\lambda + 100 = (\lambda + 10)^2 = 0 .$$

Thus, the roots of the characteristic equation are

$$\lambda_1 = \lambda_2 = -10 .$$

**E3.8** The characteristic equation is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 6 & \lambda + 3 \end{bmatrix} = \lambda(\lambda^2 + 3\lambda + 6) = 0 .$$

Thus, the roots of the characteristic equation are

$$\lambda_1 = 0 , \quad \lambda_2 = -1.5 + j1.9365 \text{ and } \lambda_3 = -1.5 - j1.9365 .$$

**E3.9** Analyzing the block diagram yields

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{1}{2}x_2 + r \\ \dot{x}_2 &= x_1 - \frac{3}{2}x_2 - r \\ y &= x_1 - \frac{3}{2}x_2 - r.\end{aligned}$$

In state-variable form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & -\frac{3}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} r, \quad y = \begin{bmatrix} 1 & -\frac{3}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \end{bmatrix} r.$$

The characteristic equation is

$$s^2 + \frac{5}{2}s + 1 = (s + 2)(s + \frac{1}{2}) = 0.$$

**E3.10** (a) The characteristic equation is

$$\det[\lambda \mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} \lambda & -6 \\ 1 & (\lambda + 5) \end{bmatrix} = \lambda(\lambda + 5) + 6 = (\lambda + 2)(\lambda + 3) = 0.$$

So, the roots are  $\lambda_1 = -2$  and  $\lambda_2 = -3$ .

(b) We note that

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} s & -6 \\ 1 & s + 5 \end{bmatrix}^{-1} = \frac{1}{(s + 2)(s + 3)} \begin{bmatrix} s + 5 & 6 \\ -1 & s \end{bmatrix}.$$

Taking the inverse Laplace transform yields the transition matrix

$$\Phi(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 6e^{-2t} - 6e^{-3t} \\ -e^{-2t} + e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}.$$

**E3.11** A state variable representation is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}r \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

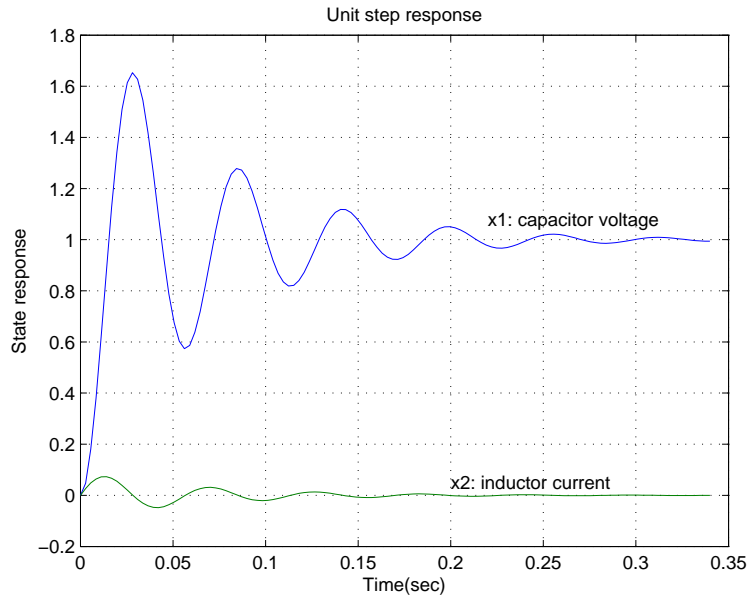
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 12 & 4 \end{bmatrix}.$$

**E3.12** The equation of motion is

$$L \frac{di}{dt} + Ri + v_c = v_{in}$$

where

$$v_c = \frac{1}{C} \int i dt .$$



**FIGURE E3.12**

State variable time history for a unit step input.

Selecting the state variables  $x_1 = v_c$  and  $x_2 = i$ , we have

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C} x_2 \\ \dot{x}_2 &= -\frac{R}{L} x_2 - \frac{1}{L} x_1 + \frac{1}{L} v_{in} . \end{aligned}$$

This can be written in matrix form as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_{in} .$$

When  $C = 0.001F$ ,  $R = 4\Omega$ , and  $L = 0.1H$ , we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1000 \\ -10 & -40 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} v_{in} .$$

The step response is shown in Figure E3.12.

- E3.13** (a) Select the state variables as  $x_1 = y$  and  $x_2 = \omega$ .  
 (b) The corresponding state equation is

$$\begin{aligned} \dot{x}_1 &= -x_1 - ax_2 + 2u \\ \dot{x}_2 &= bx_1 - 4u \end{aligned}$$

or, in matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -a \\ b & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

- (c) The characteristic equation is

$$\det[\lambda \mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} \lambda + 1 & a \\ -b & \lambda \end{bmatrix} = \lambda^2 + \lambda + ab = 0 .$$

So, the roots are

$$\lambda = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4ab} .$$

- E3.14** Assume that the mass decay is proportional to the mass present, so that

$$\dot{M} = -qM + Ku$$

where  $q$  is the constant of proportionality. Select the state variable,  $x$ , to be the mass,  $M$ . Then, the state equation is

$$\dot{x} = -qx + Ku .$$

- E3.15** The equations of motion are

$$\begin{aligned} m\ddot{x} + kx + k_1(x - q) + b\dot{x} &= 0 \\ m\ddot{q} + kq + b\dot{q} + k_1(q - x) &= 0 . \end{aligned}$$

In state variable form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k+k_1)}{m} & -\frac{b}{m} & \frac{k_1}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & 0 & -\frac{(k+k_1)}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}$$

where  $x_1 = x, x_2 = \dot{x}, x_3 = q$  and  $x_4 = \dot{q}$ .

**E3.16** The governing equations of motion are

$$\begin{aligned} m_1 \ddot{x} + k_1(x - q) + b_1(\dot{x} - \dot{q}) &= u(t) \\ m_2 \ddot{q} + k_2 q + b_2 \dot{q} + b_1(\dot{q} - \dot{x}) + k_1(q - x) &= 0 . \end{aligned}$$

Let  $x_1 = x, x_2 = \dot{x}, x_3 = q$  and  $x_4 = \dot{q}$ . Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{k_1}{m_1} & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & -\frac{(b_1+b_2)}{m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t) .$$

Since the output is  $y(t) = q(t)$ , then

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} .$$

**E3.17** At node 1 we have

$$C_1 \dot{v}_1 = \frac{v_a - v_1}{R_1} + \frac{v_2 - v_1}{R_2}$$

and at node 2 we have

$$C_2 \dot{v}_2 = \frac{v_b - v_2}{R_3} + \frac{v_1 - v_2}{R_2} .$$

Let

$$x_1 = v_1$$

and

$$x_2 = v_2 .$$



Then, in matrix form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ -\frac{1}{R_2C_2} & -\left(\frac{1}{R_3C_2} + \frac{1}{R_2C_2}\right) \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ 0 & \frac{1}{R_3C_2} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}.$$

**E3.18** The governing equations of motion are

$$\begin{aligned} Ri_1 + L_1 \frac{di_1}{dt} + v &= v_a \\ L_2 \frac{di_2}{dt} + v &= v_b \\ i_L = i_1 + i_2 &= C \frac{dv}{dt}. \end{aligned}$$

Let  $x_1 = i_1, x_2 = i_2, x_3 = v, u_1 = v_a$  and  $u_2 = v_b$ . Then,

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -\frac{R}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \mathbf{u} \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} + [0] \mathbf{u}. \end{aligned}$$

**E3.19** First, compute the matrix

$$sI - \mathbf{A} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}.$$

Then,  $\Phi(s)$  is

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

where  $\Delta(s) = s^2 + 4s + 3$ , and

$$G(s) = \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+4}{\Delta(s)} & \frac{1}{\Delta(s)} \\ -\frac{3}{\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{10}{s^2 + 4s + 3}.$$

**E3.20** The linearized equation can be derived from the observation that  $\sin \theta \approx \theta$  when  $\theta \approx 0$ . In this case, the linearized equations are

$$\ddot{\theta} + \frac{g}{L}\theta + \frac{k}{m}\dot{\theta} = 0.$$

Let  $x_1 = \theta$  and  $x_2 = \dot{\theta}$ . Then in state variable form we have

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -g/L & -k/m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}(0) = \begin{bmatrix} \theta(0) \\ \dot{\theta}(0) \end{bmatrix}.$$

**E3.21** The transfer function is

$$G(s) = \mathbf{C} [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B} + \mathbf{D} = \frac{-1}{s^2 + 2s + 1}.$$

The unit step response is

$$y(t) = -1 + e^{-t} + te^{-t}.$$

**E3.22** The transfer function is

$$G(s) = \frac{s - 6}{s^2 - 7s + 6}.$$

The poles are at  $s_1 = 1$  and  $s_2 = 6$ . The zero is at  $s = 6$ . So, we see that there is a pole-zero cancellation. We can write the system in state variable form as

$$\begin{aligned}\dot{x} &= x - \sqrt{2}u \\ y &= -\frac{\sqrt{2}}{2}x\end{aligned}$$

and the transfer function is

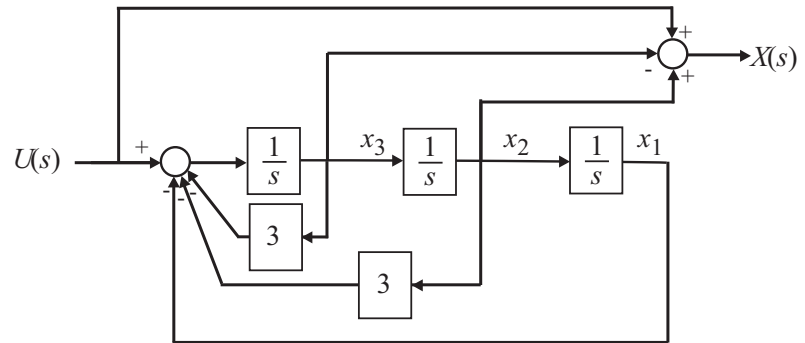
$$G(s) = \frac{1}{s - 1}.$$

**E3.23** The system in state variable form can be represented by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}u\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 \end{bmatrix}.$$



**FIGURE E3.23**  
Block diagram.

## Problems

**P3.1** The loop equation, derived from Kirchoff's voltage law, is

$$\frac{di}{dt} = \frac{1}{L} v - \frac{R}{L} i - \frac{1}{L} v_c$$

where

$$v_c = \frac{1}{C} \int i dt .$$

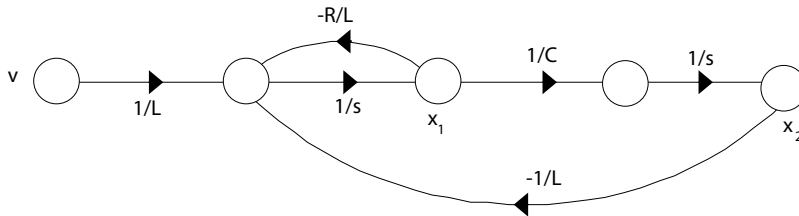
(a) Select the state variables as  $x_1 = i$  and  $x_2 = v_c$ .

(b) The corresponding state equations are

$$\begin{aligned} \dot{x}_1 &= \frac{1}{L} v - \frac{R}{L} x_1 - \frac{1}{L} x_2 \\ \dot{x}_2 &= \frac{1}{C} x_1 . \end{aligned}$$

(c) Let the input  $u = v$ . Then, in matrix form, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u .$$

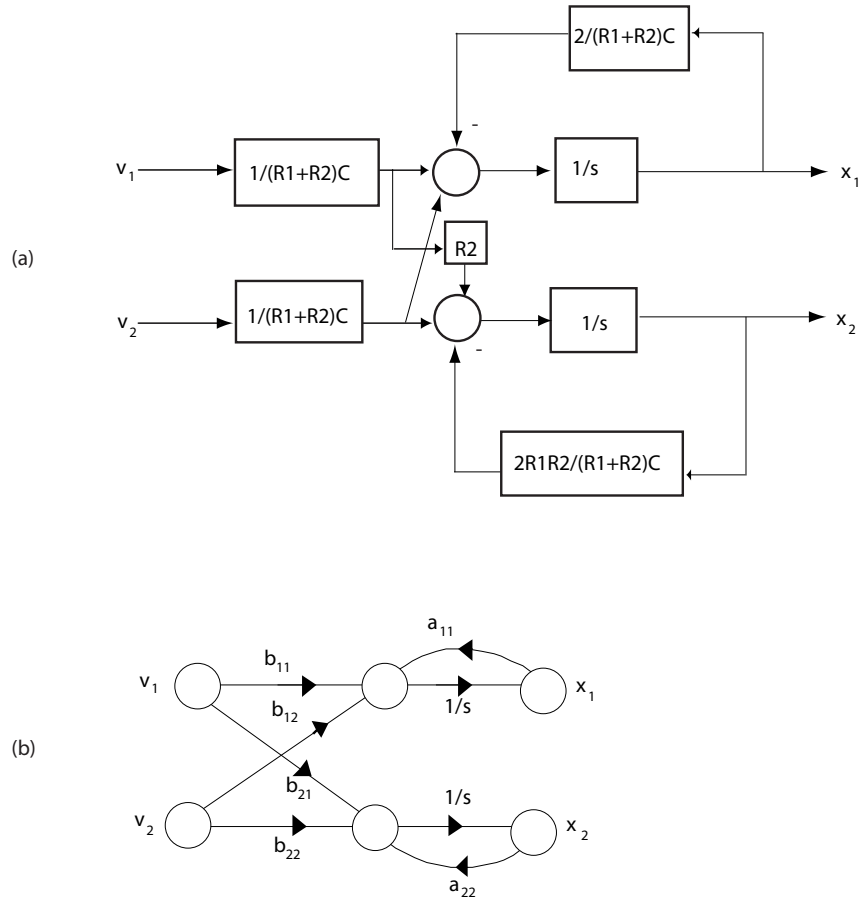


**FIGURE P3.1**  
Signal flow graph.

**P3.2** Let

$$\begin{aligned} a_{11} &= \frac{-2}{(R_1 + R_2)C} , & a_{22} &= \frac{-2R_1R_2}{(R_1 + R_2)L} , \\ b_{11} = b_{12} &= \frac{1}{(R_1 + R_2)C} , & b_{21} = -b_{22} &= \frac{R_2}{(R_1 + R_2)L} . \end{aligned}$$

The corresponding block diagram is shown in Figure P3.2.



**FIGURE P3.2**  
 (a) Block diagram. (b) Signal flow graph.

**P3.3** Using Kirchoff’s voltage law around the outer loop, we have

$$L \frac{di_L}{dt} - v_c + v_2 - v_1 = 0 .$$

Then, using Kirchoff’s current law at the node, we determine that

$$C \frac{dv_c}{dt} = -i_L + i_R ,$$

where  $i_R$  is the current through the resistor  $R$ . Considering the right loop we have

$$i_R R - v_2 + v_c = 0 \quad \text{or} \quad i_R = -\frac{v_c}{R} + \frac{v_2}{R} .$$

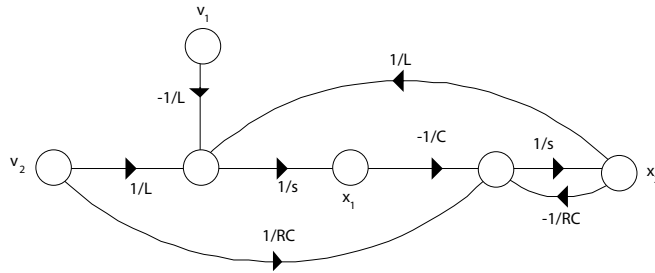
Thus,

$$\frac{dv_c}{dt} = -\frac{v_c}{RC} - \frac{i_L}{C} + \frac{v_2}{RC} \quad \text{and} \quad \frac{di_L}{dt} = \frac{v_c}{L} + \frac{v_1}{L} - \frac{v_2}{L}.$$

In matrix form, the state equations are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 1/L & -1/L \\ 0 & 1/RC \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where  $x_1 = i_L$  and  $x_2 = v_c$ . The signal flow graph is shown in Figure P3.3.



**FIGURE P3.3**  
Signal flow graph.

- P3.4** (a) The block diagram model for phase variable form is shown in Figure P3.4a. The phase variable form is given by

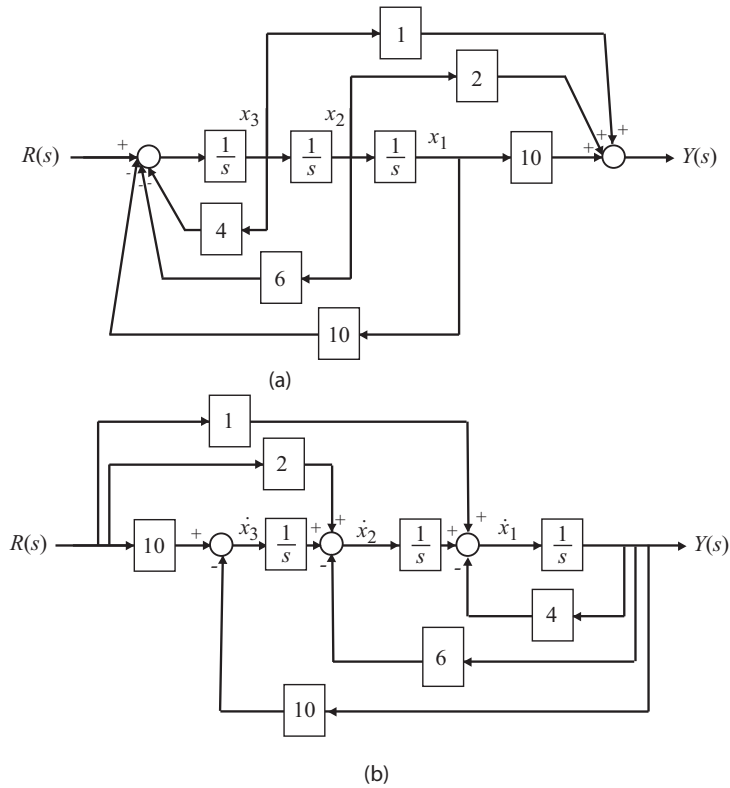
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -6 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [ 10 \quad 2 \quad 1 ] \mathbf{x}.$$

- (b) The block diagram in input feedforward form is shown in Figure P3.4b. The input feedforward form is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ -6 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix} r(t)$$

$$y = [ 1 \quad 0 \quad 0 ] \mathbf{x}.$$



**FIGURE P3.4** (a) Block diagram model for phase variable form. (b) Block diagram model for input feedforward form.

**P3.5** (a) The closed-loop transfer function is

$$T(s) = \frac{s + 1}{s^3 + 4s^2 - 11s + 1} .$$

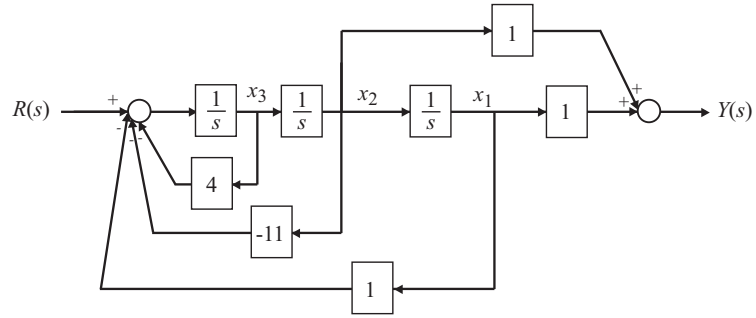
(b) A matrix differential equation is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 11 & -4 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \quad \mathbf{C} = [ 1 \quad 1 \quad 0 ] .$$

The block diagram is shown in Figure P3.5.



**FIGURE P3.5**  
Block diagram model.

**P3.6** The node equations are

$$\begin{aligned} 0.00025 \frac{dv_1}{dt} + i_L - \frac{v_i - v_1}{4000} &= 0 \\ 0.0005 \frac{dv_2}{dt} - i_L + \frac{v_2}{1000} - i_3 &= 0 \\ 0.002 \frac{di_L}{dt} + v_2 - v_1 &= 0. \end{aligned}$$

Define the state variables

$$x_1 = v_1 \quad x_2 = v_2 \quad x_3 = i_L.$$

Then,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -4000 \\ 0 & -2 & 2000 \\ 500 & -500 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2000 \\ 0 & 0 \end{bmatrix}$$

**P3.7** Given  $K = 1$ , we have

$$KG(s) \cdot \frac{1}{s} = \frac{(s+1)^2}{s(s^2+1)}.$$

We then compute the closed-loop transfer function as

$$T(s) = \frac{s^2 + 2s + 1}{3s^3 + 5s^2 + 5s + 1} = \frac{s^{-1} + 2s^{-2} + s^{-3}}{3 + 5s^{-1} + 5s^{-2} + s^{-3}}.$$



A state variable model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/3 & -5/3 & -5/3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/3 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \mathbf{x} .$$

**P3.8** The state-space equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{ku}{x_3} - g \\ \dot{x}_3 &= u . \end{aligned}$$

This is a set of **nonlinear equations**.

**P3.9** (a) The closed-loop transfer function is

$$T(s) = \frac{10}{Js^3 + (b + 10J)s^2 + 10bs + 10K_1} = \frac{10s^{-3}}{1 + 10.1s^{-1} + s^{-2} + 5s^{-3}} ,$$

where  $K_1 = 0.5$ ,  $J = 1$ , and  $b = 0.1$ .

(b) A state-space model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & -10.1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r$$

$$\omega = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} .$$

(c) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 5 & 1 & s + 10.1 \end{bmatrix} = s^3 + 10.1s^2 + s + 5 = 0 .$$

The roots of the characteristic equation are

$$s_1 = -10.05 \quad \text{and} \quad s_{2,3} = -0.0250 \pm 0.7049j .$$

All roots lie in the left hand-plane, therefore, the system is stable.

- P3.10** (a) From the signal flow diagram, we determine that a state-space model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} K_1 & -K_2 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}.$$

- (b) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + (K_2 + K_1)s + 2K_1K_2 = 0.$$

- (c) When  $K_1 = K_2 = 1$ , then

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The state transition matrix associated with  $\mathbf{A}$  is

$$\Phi = \mathcal{L}^{-1} \left\{ [s\mathbf{I} - \mathbf{A}]^{-1} \right\} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

- P3.11** The state transition matrix is

$$\Phi(t) = \begin{bmatrix} (2t-1)e^{-t} & -2te^{-t} \\ 2te^{-t} & (-2t+1)e^{-t} \end{bmatrix}.$$

So, when  $x_1(0) = x_2(0) = 10$ , we have

$$\mathbf{x}(t) = \Phi(t)x(0)$$

or

$$\begin{aligned} x_1(t) &= 10e^{-t} \\ x_2(t) &= 10e^{-t} \end{aligned}$$

- P3.12** (a) A state variable representation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [40 \ 8 \ 0]\mathbf{x} .$$

(b) The state transition matrix is

$$\Phi(t) = \left[ \begin{array}{c} \Phi_1(t) \\ \Phi_2(t) \\ \Phi_3(t) \end{array} \right] ,$$

where

$$\Phi_1(t) = \left[ \begin{array}{c} e^{-6t} - 3e^{-4t} + 3e^{-2t} \\ -6e^{-6t} + 12e^{-4t} - 6e^{-2t} \\ 36e^{-6t} - 48e^{-4t} + 12e^{-2t} \end{array} \right] \quad \Phi_2(t) = \left[ \begin{array}{c} \frac{3}{4}e^{-6t} - 2e^{-4t} + \frac{5}{4}e^{-2t} \\ -\frac{9}{2}e^{-6t} + 8e^{-4t} - \frac{5}{2}e^{-2t} \\ 27e^{-6t} - 32e^{-4t} + 5e^{-2t} \end{array} \right]$$

$$\Phi_3(t) = \left[ \begin{array}{c} \frac{1}{8}e^{-6t} - \frac{1}{4}e^{-4t} + \frac{1}{8}e^{-2t} \\ -\frac{3}{4}e^{-6t} + e^{-4t} - \frac{1}{4}e^{-2t} \\ \frac{9}{2}e^{-6t} - 4e^{-4t} + \frac{1}{2}e^{-2t} \end{array} \right] .$$

**P3.13** (a) The RLC circuit state variable representation is

$$\dot{\mathbf{x}} = \left[ \begin{array}{cc} -10 & -4 \\ 6 & 0 \end{array} \right] \mathbf{x} + \left[ \begin{array}{c} 4 \\ 0 \end{array} \right] u .$$

The characteristic equation is

$$s^2 + 10s + 24 = 0 .$$

All roots of the characteristic equation (that is,  $s_1 = -4$  and  $s_2 = -6$ ) are in the left half-plane; therefore the system is stable.

(b) The state transition matrix is

$$\Phi(t) = \left[ \begin{array}{cc} 3e^{-6t} - 2e^{-4t} & 2e^{-6t} + 2e^{-4t} \\ -3e^{-6t} + 3e^{-4t} & -2e^{-6t} + 3e^{-4t} \end{array} \right] .$$

(c) Given

$$x_1(0) = 0.1 , \quad x_2(0) = 0 \quad \text{and} \quad e(t) = 0 ,$$

we have

$$\begin{aligned} i(t) &= x_1(t) = 0.3e^{-6t} - 0.2e^{-4t} \\ v_c(t) &= x_2(t) = -0.3e^{-6t} + 0.3e^{-4t} . \end{aligned}$$

(d) When  $\mathbf{x}(0) = 0$  and  $u(t) = E$ , we have

$$\mathbf{x}(t) = \int_0^t \Phi(t - \tau) \mathbf{B}u(\tau) d\tau ,$$

where

$$\mathbf{B}u(t) = \begin{bmatrix} 4E \\ 0 \end{bmatrix} .$$

Integrating yields

$$\begin{aligned} x_1(t) &= (-2e^{-6t} + 2e^{-4t})E \\ x_2(t) &= (1 + 2e^{-6t} - 3e^{-4t})E . \end{aligned}$$

**P3.14** A state space representation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}r , \quad y = \mathbf{C}\mathbf{x}$$

where

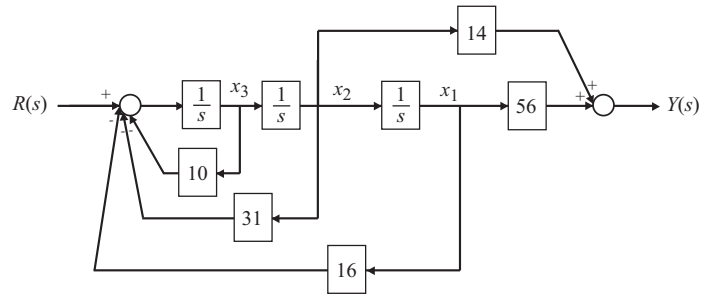
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -34 & -10 & -12 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} , \quad \mathbf{C} = [50 \ 1 \ 0 \ 0] .$$

**P3.15** A state variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -31 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

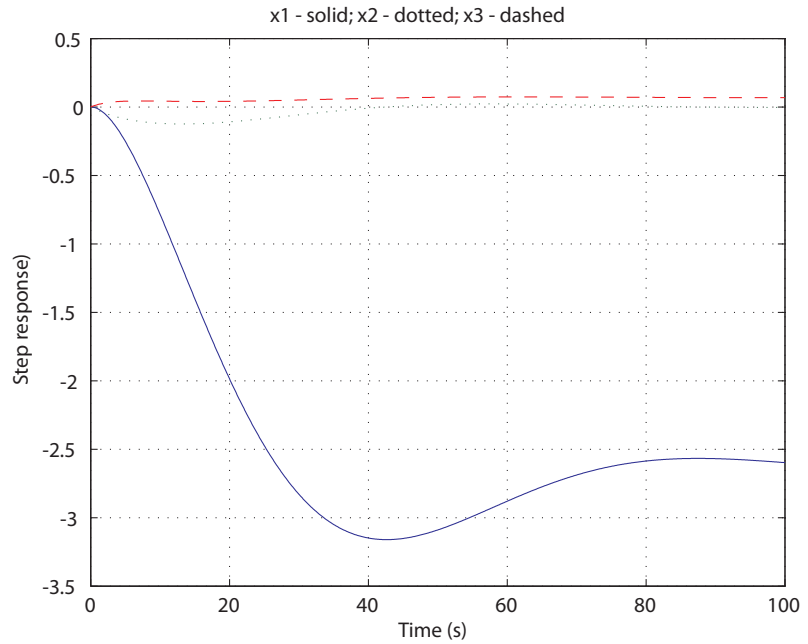
$$y = [56 \ 14 \ 0] \mathbf{x} .$$

The block diagram is shown in Figure P3.15.



**FIGURE P3.15**  
Block diagram model.

**P3.16** (a) The characteristic equation is



**FIGURE P3.16**  
Step response of magnitude  $0.285^\circ$ .

$$\det(s\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} s & -1 & 0 \\ 0.0071 & s + 0.111 & -0.12 \\ 0 & -0.07 & s + 0.3 \end{bmatrix}$$

$$= s^3 + 0.411s^2 + 0.032s + 0.00213 = 0 .$$

The roots are

$$s_1 = -0.3343 \quad \text{and} \quad s_{2,3} = -0.0383 \pm 0.0700j .$$

All the poles lie in the left half-plane, therefore, the system is stable.

(b) The solution of the system to a step of magnitude  $0.285^\circ$  is given by

$$\begin{aligned} x_1(t) &= -2.66 - 0.11e^{-0.33t} + e^{-0.038t} (2.77 \cos 0.07t + 0.99 \sin 0.07t) \\ x_2(t) &= 0.037e^{-0.33t} - e^{-0.038t} (0.037 \cos 0.07t + 0.23 \sin 0.07t) \\ x_3(t) &= 0.069 - 0.075e^{-0.33t} + e^{-0.038t} (0.006 \cos 0.07t - 0.06 \sin 0.07t) \end{aligned}$$

**P3.17** The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{-4s + 12}{s^3 - 14s^2 + 37s + 20} .$$

**P3.18** Define the state variables as

$$\begin{aligned} x_1 &= \phi_1 - \phi_2 \\ x_2 &= \frac{\omega_1}{\omega_o} \\ x_3 &= \frac{\omega_2}{\omega_o} . \end{aligned}$$

Then, the state equations of the robot are

$$\begin{aligned} \dot{x}_1 &= \omega_o x_2 - \omega_o x_3 \\ \dot{x}_2 &= \frac{-J_2 \omega_o}{J_1 + J_2} x_1 - \frac{b}{J_1} x_2 + \frac{b}{J_1} x_3 + \frac{K_m}{J_1 \omega_o} i \\ \dot{x}_3 &= \frac{J_1 \omega_o}{J_1 + J_2} x_2 + \frac{b}{J_2} x_2 - \frac{b}{J_2} x_3 \end{aligned}$$

or, in matrix form

$$\dot{\mathbf{x}} = \omega_o \begin{bmatrix} 0 & 1 & -1 \\ a - 1 & -b_1 & b_1 \\ a & b_2 & -b_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix} i$$

where

$$a = \frac{J_1}{(J_1 + J_2)} , \quad b_1 = \frac{b}{J_1 \omega_o} , \quad b_2 = \frac{b}{J_2 \omega_o} \quad \text{and} \quad d = \frac{K_m}{J_1 \omega_o} .$$

**P3.19** The state equation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

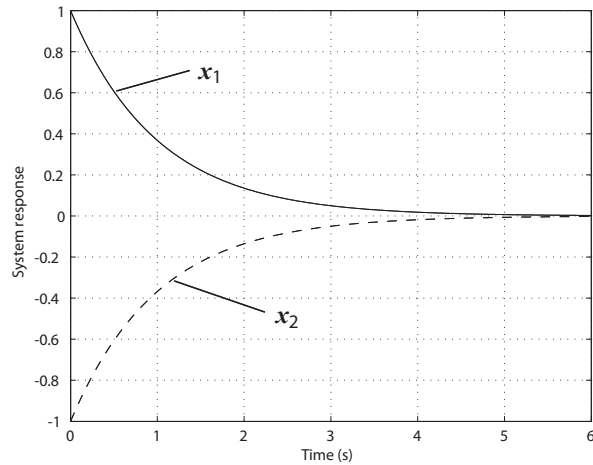
where  $x_1(0) = 1$  and  $x_2(0) = -1$ . The state transition matrix is

$$\Phi(t) = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}.$$

The system response is

$$\begin{aligned} x_1(t) &= (-e^{-2t} + 2e^{-t}) x_1(0) + (-e^{-2t} + e^{-t}) x_2(0) \\ x_2(t) &= (2e^{-2t} - 2e^{-t}) x_1(0) + (2e^{-2t} - e^{-t}) x_2(0). \end{aligned}$$

The state response is shown in Figure P3.19.



**FIGURE P3.19**

Response with  $x_1(0) = 1$  and  $x_2(0) = -1$ .

**P3.20** The state equation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{0.693}{6.7} & 0 \\ -1 & -\frac{0.693}{9.2} \end{bmatrix} \mathbf{x} \quad \text{where} \quad \mathbf{x}(0) = \begin{bmatrix} 0.3 \times 10^{16} \\ 7 \times 10^{16} \end{bmatrix}.$$

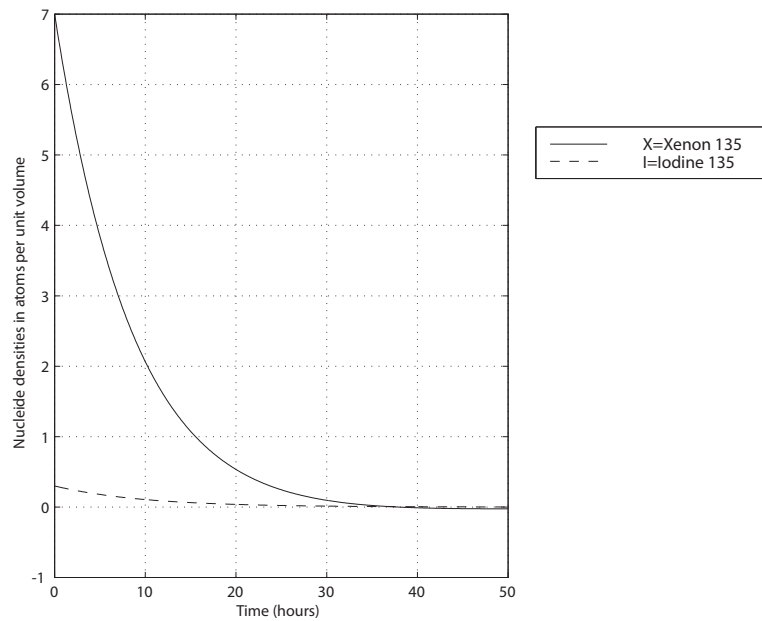
The state transition matrix is

$$\Phi(t) = \begin{bmatrix} e^{-0.103433t} & 0 \\ 35.5786(e^{-0.103433t} - e^{-0.0753261t}) & e^{-0.0753261t} \end{bmatrix}.$$

The system response is

$$\begin{aligned} x_1(t) &= e^{-0.103433t} x_1(0) \\ x_2(t) &= 35.5786 \left[ e^{-0.103433t} - e^{-0.0753261t} \right] x_1(0) + e^{-0.0753261t} x_2(0). \end{aligned}$$

The state response is shown in Figure P3.20.



**FIGURE P3.20**

Nuclear reactor state response to initial conditions.



**P3.21** Referring to Figure P3.21 we have

$$\begin{aligned} Y(s) &= \frac{1}{s}W(s) = \frac{1}{s} \left[ h_1U(s) + \frac{1}{s}Q(s) \right] \\ &= \frac{h_1}{s}U(s) + \frac{1}{s^2} [h_0U(s) - a_0Y(s) - a_1sY(s) + a_1h_1U(s)]. \end{aligned}$$

Gathering like terms and re-arranging yields

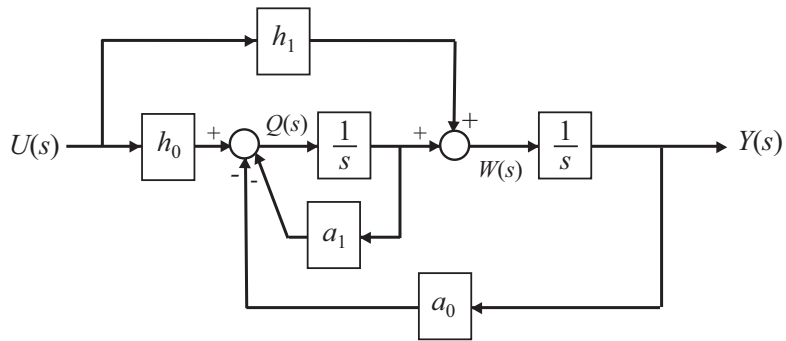
$$\left( 1 + \frac{a_1}{s} + \frac{a_0}{s^2} \right) Y(s) = \left( \frac{h_1}{s} + \frac{h_0}{s^2} + \frac{a_1h_1}{s^2} \right) U(s)$$

or

$$Y(s) = \left[ \frac{h_1s + h_0 + a_1h_1}{s^2 + a_1s + a_0} \right] U(s).$$

Computing the transfer function from the state variable representation yields

$$\begin{aligned} G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+a_1}{s^2+a_1s+a_0} & \frac{1}{s^2+a_1s+a_0} \\ \frac{-a_0}{s^2+a_1s+a_0} & \frac{s}{s^2+a_1s+a_0} \end{bmatrix} \begin{bmatrix} h_1 \\ h_0 \end{bmatrix} = \frac{h_1s + h_0 + a_1h_1}{s^2 + a_1s + a_0}. \end{aligned}$$



**FIGURE P3.21**  
Block diagram with labeled signals.

**P3.22** The governing equations are

$$\begin{aligned} L \frac{di}{dt} &= v_2 \\ C_1 \frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v) + \frac{1}{R_2}(v_1 - v_2) &= 0 \end{aligned}$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_2}(v_2 - v_1) + i + \frac{v_2}{R_3} = 0 .$$

Let  $u = v$ ,  $x_1 = i$ ,  $x_2 = v_1$  and  $x_3 = v_2$ . Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & \frac{1}{L} \\ 0 & -\frac{1}{a} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) & \frac{1}{C_1 R_2} \\ -\frac{1}{C_2} & \frac{1}{R_2 C_2} & -\left( \frac{1}{R_2 C_2} + \frac{1}{R_3 C_2} \right) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{R_1 C_1} \\ 0 \end{bmatrix} u$$

$$y = [0 \ 0 \ 1] \mathbf{x} .$$

**P3.23** A state variable representation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -31 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [1 \ 0 \ 0] \mathbf{x} .$$

Other representations include the input feedforward representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 & 0 \\ -31 & 0 & 1 \\ -30 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [1 \ 0 \ 0] \mathbf{x} ,$$

the physical variable representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [1 \ 0 \ 0] \mathbf{x} ,$$

and the decoupled representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \mathbf{x} .$$

**P3.24** The matrix representation of the state equations is

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d .$$

When  $u_1 = 0$  and  $u_2 = d = 1$ , we have

$$\begin{aligned} \dot{x}_1 &= 3x_1 + u_2 \\ \dot{x}_2 &= 2x_2 + 2u_2 \end{aligned}$$

So we see that we have two independent equations for  $x_1$  and  $x_2$ . With  $U_2(s) = 1/s$  and zero initial conditions, the solution for  $x_1$  is found to be

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1} \{X_1(s)\} = \mathcal{L}^{-1} \left\{ \frac{1}{s(s-3)} \right\} \\ &= \mathcal{L}^{-1} \left\{ -\frac{1}{3s} + \frac{1}{3} \frac{1}{s-3} \right\} = -\frac{1}{3} (1 - e^{3t}) \end{aligned}$$

and the solution for  $x_2$  is

$$x_2(t) = \mathcal{L}^{-1} \{X_2(s)\} = \mathcal{L}^{-1} \left\{ \frac{2}{s(s-2)} \right\} = \mathcal{L}^{-1} \left\{ -\frac{1}{s} + \frac{1}{s-2} \right\} = -1 + e^{2t} .$$

**P3.25** Since  $\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ , we have

$$\Phi(s) = \begin{bmatrix} s+1 & 0 \\ -2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} s+3 & 0 \\ 2 & s+1 \end{bmatrix} \frac{1}{\Delta(s)}$$

where  $\Delta(s) = (s+1)(s+3)$ . The state transition matrix is

$$\Phi(t) = \mathcal{L}^{-1} \{ \Phi(s) \} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-3t} & e^{-3t} \end{bmatrix} .$$

**P3.26** The state variable differential equation is

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} r \\ y &= [1 \ 0] \mathbf{x} . \end{aligned}$$

and

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+6 & 1 \\ -25 & s \end{bmatrix} \frac{1}{\Delta(s)}$$

where  $\Delta(s) = s^2 + 6s + 25$ .

**P3.27** Equating the change in angular momentum to the sum of the external torques yields

$$J\ddot{\theta} - H\omega \cos \theta = -b\dot{\theta} - k\theta$$

where  $b$  is the damping coefficient,  $k$  is the spring constant, and  $J$  is the wheel moment of inertia. Defining the state variables  $x_1 = \theta$  and  $x_2 = \dot{\theta}$  and the input  $u = \omega$ , we can write the equations of motion as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{J}x_1 - \frac{b}{J}x_2 + \frac{H}{J}u \cos x_1 \end{aligned}$$

With a small angle assumption (that is,  $\cos x_1 \approx 1$ ) we have

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -k/J & -b/J \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ H/J \end{bmatrix} u \\ y = \theta &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} . \end{aligned}$$

**P3.28** The governing equations of motion are

$$\begin{aligned} m_1\ddot{y}_1 + k(y_1 - y_2) + b\dot{y}_1 &= u \\ m_2\ddot{y}_2 + k(y_2 - y_1) + b\dot{y}_2 &= 0 \\ y &= y_2 . \end{aligned}$$

Let  $x_1 = y_1$ ,  $x_2 = \dot{y}_1$ ,  $x_3 = y_2$  and  $x_4 = \dot{y}_2$ . Then

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & -\frac{b}{m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} . \end{aligned}$$

**P3.29** The equations of motion are

$$\begin{aligned} I\ddot{q}_1 + MgL \sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - k(q_1 - q_2) &= u . \end{aligned}$$

Let  $x_1 = q_1$ ,  $x_2 = \dot{q}_1$ ,  $x_3 = q_2$ , and  $x_4 = \dot{q}_2$  and linearize the equations using small angle assumptions (i.e.  $\sin q_1 \approx q_1$ ). Then, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}x_1 - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u . \end{aligned}$$

**P3.30** Using Kirchoff's current law, we find that

$$C \frac{dv_c}{dt} = i_2 + i_3$$

where  $i_3 =$  current in  $R_3$ . Let  $i_1 =$  current in  $R_1$ . Using Kirchoff's voltage law, we have

$$L \frac{di_L}{dt} = v_1 - R_1 i_1$$

and

$$R_1 i_1 + R_2 i_2 + v_c = v_1 .$$

But

$$i_2 = i_1 - i_L ,$$

so

$$(R_1 + R_2)i_1 = v_1 - v_c + R_2 i_L .$$

Using Kirchoff's voltage law once again, we calculate  $i_3$  as

$$i_3 = \frac{v_2 - v_c}{R_3} .$$

Utilizing the above equations, we can solve for  $di_L/dt$  and  $dv_c/dt$ , as follows:

$$\begin{aligned} \frac{di_L}{dt} &= \frac{R_2}{L(R_1 + R_2)}v_1 + \frac{R_1}{L(R_1 + R_2)}v_c - \frac{R_1 R_2}{L(R_1 + R_2)}i_L \\ \frac{v_c}{dt} &= \frac{v_1}{C(R_1 + R_2)} - \frac{v_c}{C(R_1 + R_2)} - \frac{v_c}{CR_3} - \frac{R_1 i_L}{C(R_1 + R_2)} + \frac{v_2}{CR_3} \end{aligned}$$

Define the state variables  $x_1 = v_c$  and  $x_2 = i_L$ . Then, in matrix form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{(R_1+R_2+R_3)}{CR_3(R_1+R_2)} & -\frac{R_1}{C(R_1+R_2)} \\ \frac{R_1}{L(R_1+R_2)} & -\frac{R_1R_2}{L(R_1+R_2)} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{C(R_1+R_2)} & \frac{1}{CR_3} \\ \frac{R_2}{L(R_1+R_2)} & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$y = i_2 = \begin{bmatrix} -\frac{1}{(R_1+R_2)} & -\frac{R_1}{(R_1+R_2)} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{(R_1+R_2)} & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

**P3.31** A state variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 30 \end{bmatrix} u .$$

The state transition matrix can be computed as follows:

$$\begin{aligned} \Phi &= \mathcal{L}^{-1} \{ [s\mathbf{I} - \mathbf{A}]^{-1} \} = \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} \right\} \\ &= \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix} \end{aligned}$$

where

$$\Delta(s) = s^2 + 4s + 3 = (s+1)(s+3) .$$

**P3.32** A state variable representation is

$$\begin{aligned} \dot{m}_1 &= -k_1 m_1 + r \\ \dot{m}_2 &= k_1 m_1 - k_2 m_2 \end{aligned}$$

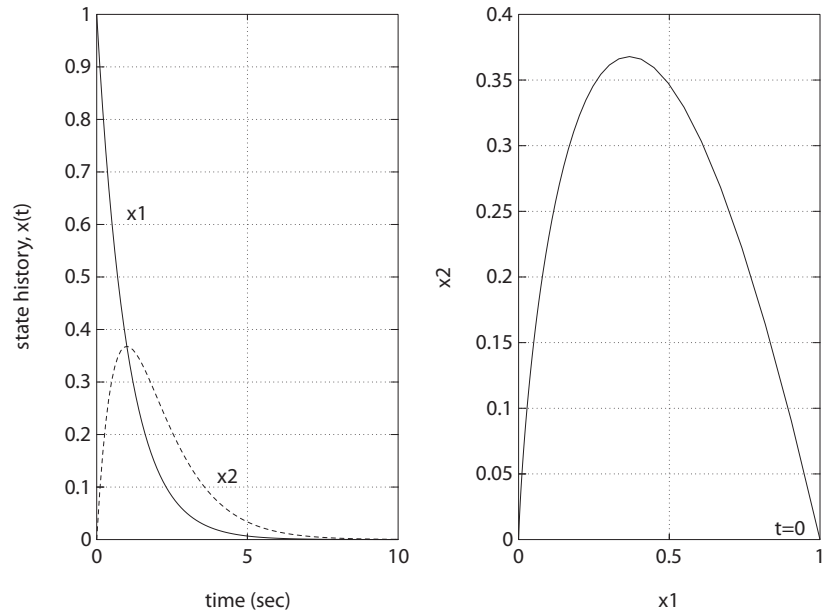
where  $k_1$  and  $k_2$  are constants of proportionality. In matrix form, we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}r = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

where  $x_1 = m_1$  and  $x_2 = m_2$ . Let  $k_1 = k_2 = 1$  and assume that  $r(t) = 0$  and  $x_1 = 1$  and  $x_2 = 0$ . Then

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix} .$$

The simulation is shown in Figure P3.32.



**FIGURE P3.32**  
Actual versus approximate state response.

**P3.33** The system (including the feedback) is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \end{bmatrix} \mathbf{x}.$$

The characteristic equation is

$$\det[\lambda\mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} \lambda & -1 \\ 1/2 & \lambda + 1 \end{bmatrix} = \lambda^2 + \lambda + \frac{1}{2} = 0.$$

The roots of the characteristic equation are

$$\lambda_{1,2} = -\frac{1}{2} \pm j\frac{1}{2}.$$

The system response is

$$\begin{aligned} \mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) &= \begin{bmatrix} e^{-t/2} \cos \frac{t}{2} + e^{-t/2} \sin \frac{t}{2} & 2e^{-t/2} \sin \frac{t}{2} \\ -e^{-t/2} \sin \frac{t}{2} & e^{-t/2} \cos \frac{t}{2} - e^{-t/2} \sin \frac{t}{2} \end{bmatrix} \mathbf{x}(0) \\ &= e^{-t/2} \begin{bmatrix} 2 \sin \frac{t}{2} \\ \cos \frac{t}{2} - \sin \frac{t}{2} \end{bmatrix} \end{aligned}$$

where  $x_1(0) = 0$  and  $x_2(0) = 1$ .

**P3.34** (a) The state space representation is

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \\ y &= [6 \ 0 \ 0] \mathbf{x} . \end{aligned}$$

(b) The element  $\phi_{11}(t)$  of the state transition matrix is

$$\phi_{11}(t) = e^{-3t} - 3e^{-2t} + 3e^{-t} .$$

**P3.35** The state equations are

$$\begin{aligned} \dot{h} = \dot{x}_1 &= \frac{1}{50}[80\theta - 50h] = -x_1 + \frac{8}{5}x_2 \\ \dot{\theta} = \dot{x}_2 &= \omega = x_3 \\ \dot{\omega} = \dot{x}_3 &= \frac{K_m}{J}i_a = -\frac{K_m K_b}{JR_a}\omega + \frac{K_m K_a}{JR_a}v_i = -\frac{353}{30}x_3 + \frac{25000}{3}v_i . \end{aligned}$$

In state variable form, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{8}{5} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{353}{30} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{25000}{3} \end{bmatrix} v_i .$$

**P3.36** Using Newton's Law and summing the forces on the two masses yields

$$\begin{aligned} M_1 \ddot{x}(t) + b_1 \dot{x}(t) + k_1 x(t) &= b_1 \dot{y}(t) \\ M_2 \ddot{y}(t) + b_1 \dot{y}(t) + k_2 y(t) &= b_1 \dot{x}(t) + u(t) \end{aligned}$$

Let

$$z_1 = x, \quad z_2 = \dot{x}, \quad z_3 = y, \quad \text{and} \quad z_4 = \dot{y} .$$



Then we write the system in state variable form as

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{M_1} & -\frac{b_1}{M_1} & 0 & \frac{b_1}{M_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_1}{M_2} & -\frac{k_2}{M_2} & -\frac{b_1}{M_2} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{z} .$$

**P3.37** From the block diagram in Figure P3.37, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -10x_1 - 4x_2 - 3x_3 + u \\ y &= x_1 + 12x_2 + 5x_3 \end{aligned}$$

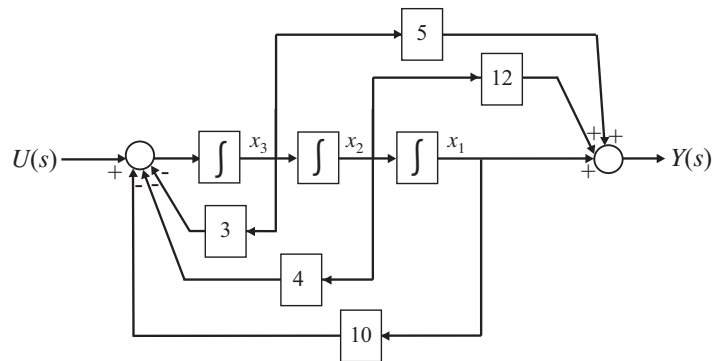
or

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -4 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ 12 \ 5] \mathbf{x} .$$

The third-order differential equation model is

$$\ddot{y} + 3\dot{y} + 4y + 10y = 5\ddot{u} + 12\dot{u} + u .$$



**FIGURE P3.37**  
Block diagram with states labeled.

## Advanced Problems

**AP3.1** With the state variables are defined as

$$\mathbf{z} = \begin{bmatrix} x \\ \dot{x} \\ i \end{bmatrix},$$

the nonlinear equations of motion are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ g - \frac{K}{m} \frac{(I_o + z_3)^2}{(X_o + z_1)^2} \\ \frac{1}{L}(v - Rz_3) \end{bmatrix},$$

where the control is the voltage  $v$ . We assume that  $z_1 = x$  is measurable. The linearized equations of motion are

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} + \mathbf{B}v \\ y &= \mathbf{C}\mathbf{z} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2K}{m} \frac{I_o^2}{X_o^3} & 0 & -\frac{2K}{m} \frac{I_o}{X_o^2} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

With the constants

$$\begin{aligned} R &= 23.2 \\ L &= 0.508 \\ m &= 1.75 \\ K &= 2.9 \times 10^{-4} \\ I_o &= 1.06 \\ X_o &= 4.36 \times 10^{-3} \end{aligned}$$

the transfer function is

$$G(s) = \frac{-36.38}{s^3 + 45.67s^2 + 4493s + 205195} .$$

**AP3.2** The differential equation describing the motion of  $y$  is

$$m\ddot{y} + b\dot{y} + ky = bu + ku .$$

Taking Laplace transforms (with zero initial conditions) yields the transfer function

$$\frac{Y(s)}{U(s)} = \frac{(b/m)s + (k/m)}{s^2 + (b/m)s + (k/m)} .$$

In state space form, we have

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} k/m & b/m \end{bmatrix} \mathbf{x} . \end{aligned}$$

**AP3.3** The transfer function is

$$\frac{Y(s)}{R(s)} = \frac{2s^2 + 6s + 5}{s^3 + 4s^2 + 5s + 2} .$$

In (nearly) diagonal form, we have

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} , \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} .$$

The matrix  $\mathbf{A}$  is not exactly diagonal due to the repeated roots in the denominator of the transfer function.

**AP3.4** The differential equations describing the motion of  $y$  and  $q$  are

$$\begin{aligned} m\ddot{y} + k_2\dot{y} + k_1(y - q) &= f \\ -b\dot{q} + k_1(y - q) &= f \end{aligned}$$

where  $k_1 = 2$  and  $k_2 = 1$ . Assume the mass  $m = 1$ . Then with the state

variables defined as  $\mathbf{z} = [y \ \dot{y} \ q]^T$ , we have the state variable model

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & 2 \\ 2/b & 0 & -2/b \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ -1/b \end{bmatrix} f$$

$$y = [1 \ 0 \ 0] \mathbf{z}$$

If we model a large bump at high speeds as an impulse and a small bump at low speeds as a step, then  $b = 0.8$  provides good performance. In both cases, the ride settles out completely in about 10 seconds.

**AP3.5** The differential equations describing the motion of  $x$  and  $\theta$  are

$$(M + m)\ddot{x} + ML \cos \theta \ddot{\theta} - ML \sin \theta \dot{\theta}^2 = -kx$$

$$g \sin \theta + \cos \theta \ddot{x} + L\ddot{\theta} = 0$$

Assuming  $\theta$  and  $\dot{\theta}$  are small, it follows that

$$(M + m)\ddot{x} + ML\ddot{\theta} = -kx$$

$$\ddot{x} + L\ddot{\theta} = -g\theta$$

Define the state variables as  $\mathbf{z} = [x \ \dot{x} \ \theta \ \dot{\theta}]^T$ . Then, the state variable model is

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m & 0 & gM/m & 0 \\ 0 & 0 & 0 & 1 \\ k/(Lm) & 0 & -g(M + m)/(Lm) & 0 \end{bmatrix} \mathbf{z}$$

**AP3.6**

**AP3.7** Computing the closed-loop system yields

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} -1 & 1 \\ -K_1 & -K_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = [2 \ 1].$$

The characteristic polynomial is

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = s^2 + (K_2 + 1)s + K_1 + K_2 = 0.$$

The roots are in the left-half plane whenever  $K_2 + 1 > 0$  and  $K_1 + K_2 > 0$ .

**AP3.8** (a) A state variable representation is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -Kx_1 - 12x_2 - 6x_3 + Kr \\ y &= x_1 \end{aligned}$$

or, in matrix form

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K & -12 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} r \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

(b) The characteristic roots are found by solving

$$\det[\lambda\mathbf{I} - \mathbf{A}] = 0$$

or

$$\lambda^3 + 6\lambda^2 + 12\lambda + K = 0$$

When  $K = 8$ , we have characteristic roots at  $\lambda_1 = -2$ ,  $\lambda_2 = -2$ , and  $\lambda_3 = -2$ , as desired.

(c) The unit step response is given by

$$y(t) = 1 - e^{-2t} - 2te^{-2t} - 2t^2e^{-2t} .$$

## Design Problems

**CDP3.1** The transfer model of the traction drive, capstan roller, and linear slide was given in CDP2.1 as

$$\frac{X(s)}{V_a(s)} = \frac{rK_m}{s[(L_ms + R_m)(J_Ts + b_m) + K_bK_m]},$$

where

$$J_T = J_m + r^2(M_s + M_b).$$

Define  $x_1 = x$ ,  $x_2 = \dot{x}$ , and  $x_3 = \ddot{x}$ . Then, a state variable representation is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}v_a \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{R_m b_m + K_b K_m}{L_m J_T} & -\frac{L_m b_m + R_m J_T}{L_m J_T} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{rK_m}{L_m J_T} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

**DP3.1** (a) The equation of motion of the spring-mass-damper is

$$m\ddot{y} + b\dot{y} + ky = u$$

or

$$\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u.$$

Select the state variables

$$x_1 = y \quad \text{and} \quad x_2 = \dot{y}.$$

Then, we have

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

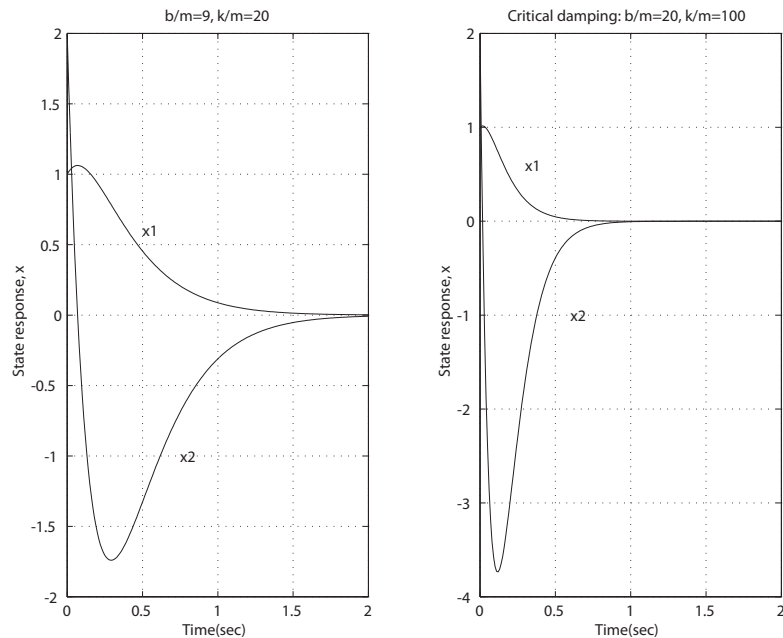
$\mathbf{A}$  is the system matrix. The characteristic equation is

$$\det[\lambda\mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} s & -1 \\ 20 & s+9 \end{bmatrix} = s^2 + 9s + 20 = 0.$$

The roots of the characteristic equation are  $s_1 = -4$  and  $s_2 = -5$ , and the transition matrix is

$$\Phi(t) = \begin{bmatrix} 5e^{-4t} - 4e^{-5t} & e^{-4t} - e^{-5t} \\ -20e^{-4t} + 20e^{-5t} & -4e^{-4t} + 5e^{-5t} \end{bmatrix}.$$

- (b) Assume the initial conditions are  $x_1(0) = 1$  and  $x_2(0) = 2$ . The zero-input response is shown in Figure DP3.1.
- (c) Suppose we redesign the system by choosing  $b$  and  $k$  to quickly damp out  $x_2$  and  $x_1$ . We can select  $b$  and  $k$  to achieve critical damping.



**FIGURE DP3.1**  
Zero input state response.

If we desire the characteristic polynomial to be  $p_d(s) = (s + 10)^2 = s^2 + 20s + 100$ , then we need  $b = 20$  and  $k = 100$ .

**DP3.2** The desired transfer function is

$$\frac{Y(s)}{U(s)} = \frac{6}{s^2 + 7s + 10} .$$

The transfer function derived from the phase variable representation is

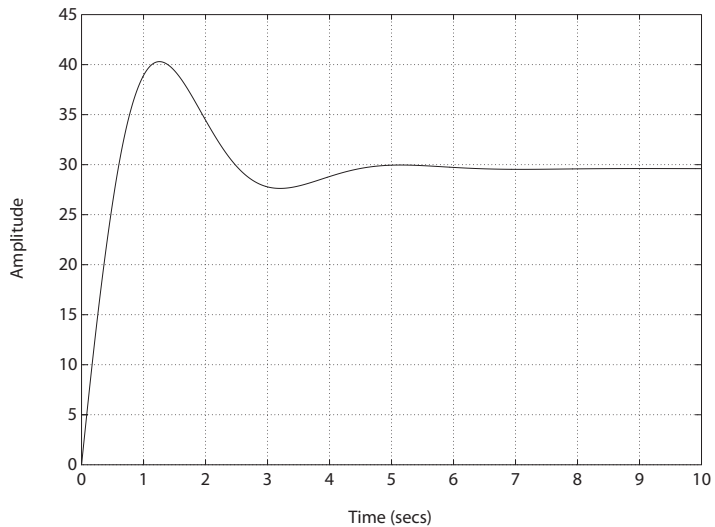
$$\frac{Y(s)}{U(s)} = \frac{d}{s^2 + bs + a} .$$

Therefore, we select  $d = 6$ ,  $a = 10$  and  $b = 7$ .

**DP3.3** Assume the aircraft lands precisely on the centerline. The linearized equations of motion are

$$\begin{aligned} m_3 \ddot{x}_3 + K_D \dot{x}_3 + K_2(x_3 - x_2) &= 0 \\ m_2 \ddot{x}_2 + K_2(x_2 - x_3) + K_1(x_2 - x_1) &= 0 \\ m_1 \ddot{x}_1 &= -\frac{2}{\sqrt{2}} K_2(x_1 - x_2) \end{aligned}$$

where  $x_1(0) = x_2(0) = \dot{x}_2(0) = \dot{x}_3 = 0$  and  $\dot{x}_1(0) = 60$ . The system response is shown in Figure DP3.3 where  $K_D = 215$ . The aircraft settles out at 30 m, although initially it overshoots by about 10 m at 1 second.



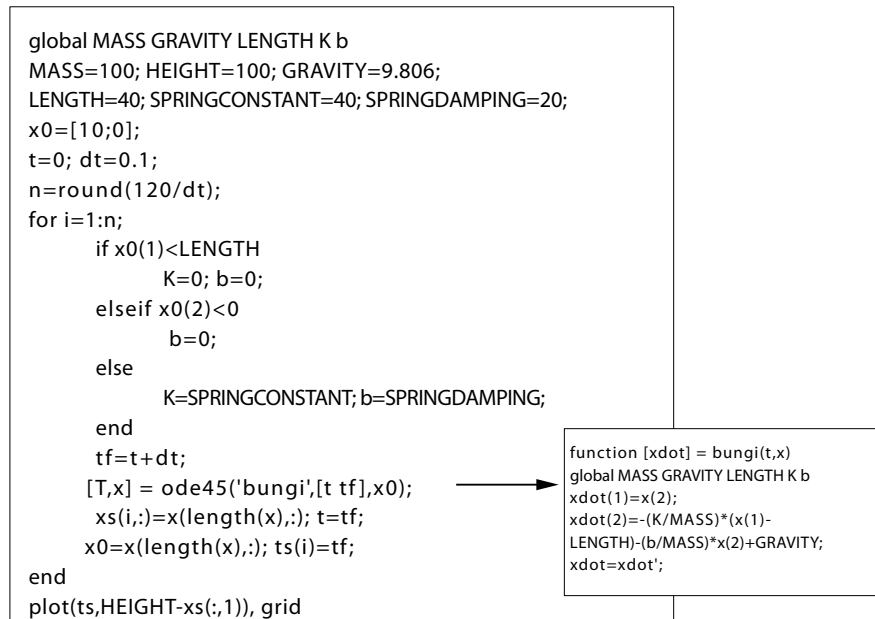
**FIGURE DP3.3**  
Aircraft arresting gear response.



**DP3.4** We can model the bungi cord system as a mass-spring-damper. This is actually an over-simplification because the bungi cord cannot “push” the jumper down as a spring would—it can only exert a restoring force when the cord is stretched (that is, when the jumper exceeds the length,  $L$ , of the cord). The problem is **nonlinear**! When the distance of the jumper from the platform is less than  $L$  we should model the cord spring constant and damping as  $K = 0$  and  $b = 0$ , respectively. Only gravity acts on the jumper. Also, when  $\dot{x}$  (the jumper velocity) is negative (where we define positive towards the ground), then we should model  $b = 0$ . A reasonable set of equations of motion are

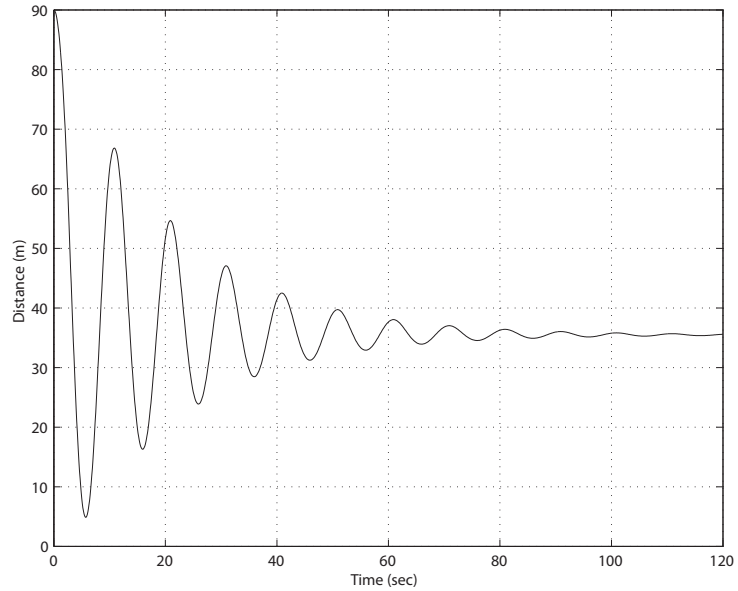
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{m}x_1 - \frac{b}{m}x_2 + g\end{aligned}$$

where  $x_1$  is the distance measured from the top of the platform and  $x_2$  is the jumper velocity. For the initial conditions we have  $x_1(0) = 10$  and  $x_2(0) = 0$ . A reasonable set of parameters for the bungi cord are  $L = 40$  m,  $K = 40$  N/m and  $b = 20$  kg/m. The system response is shown in Figure DP3.4 for a person with  $m = 100$  kg. The accelerations experienced by the jumper never exceed 1.5 g.



**FIGURE DP3.4**

(a) Bungi cord system response m-file script.



**FIGURE DP3.4**  
CONTINUED: Bungee cord system time history response.

**DP3.5** Computing the closed-loop system yields

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -2 - K_1 & 3 - K_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = s^2 + (K_2 - 3)s + K_1 + 2 = 0.$$

Suppose that the desired poles are in the left-half plane and are denoted by  $-p_1$  and  $-p_2$ . Then the desired characteristic polynomial is

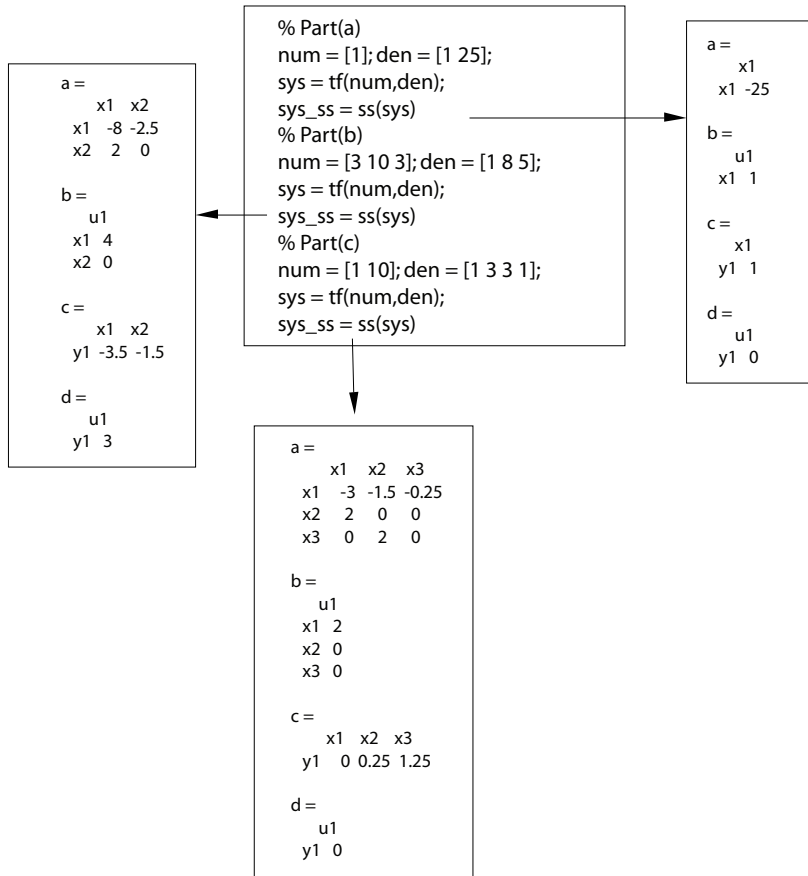
$$(s + p_1)(s + p_2) = s^2 + (p_1 + p_2)s + p_1p_2 = 0.$$

Equating coefficients and solving for  $\mathbf{K} = [K_1 \ K_2]$  yields

$$\begin{aligned} K_1 &= p_1p_2 - 2 \\ K_2 &= p_1 + p_2 + 3. \end{aligned}$$

## Computer Problems

**CP3.1** The m-file script to compute the state-space models using the `ss` function is shown in Figure CP3.1.



**FIGURE CP3.1**  
Script to compute state-space models from transfer functions.

For example, in part (c) the state-space model is

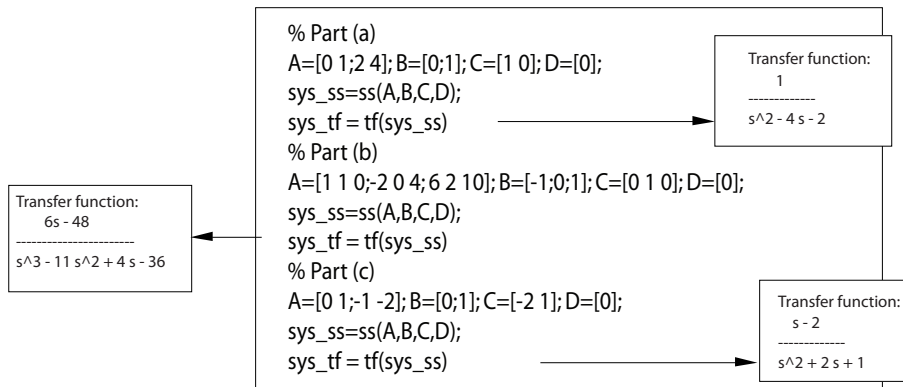
$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx} + \mathbf{Du} ,$$

where  $\mathbf{D} = [0]$  and

$$\mathbf{A} = \begin{bmatrix} -3 & -1.5 & -0.25 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0.25 & 1.25 \end{bmatrix}.$$

**CP3.2** The m-file script to compute the transfer function models using the `tf` function is shown in Figure CP3.2.



**FIGURE CP3.2**

Script to compute transfer function models from the state-space models.

**CP3.3** For an ideal op-amp, the voltage gain (as a function of frequency) is

$$V_o(s) = -\frac{Z_2(s)}{Z_1(s)}V_{in}(s),$$

where

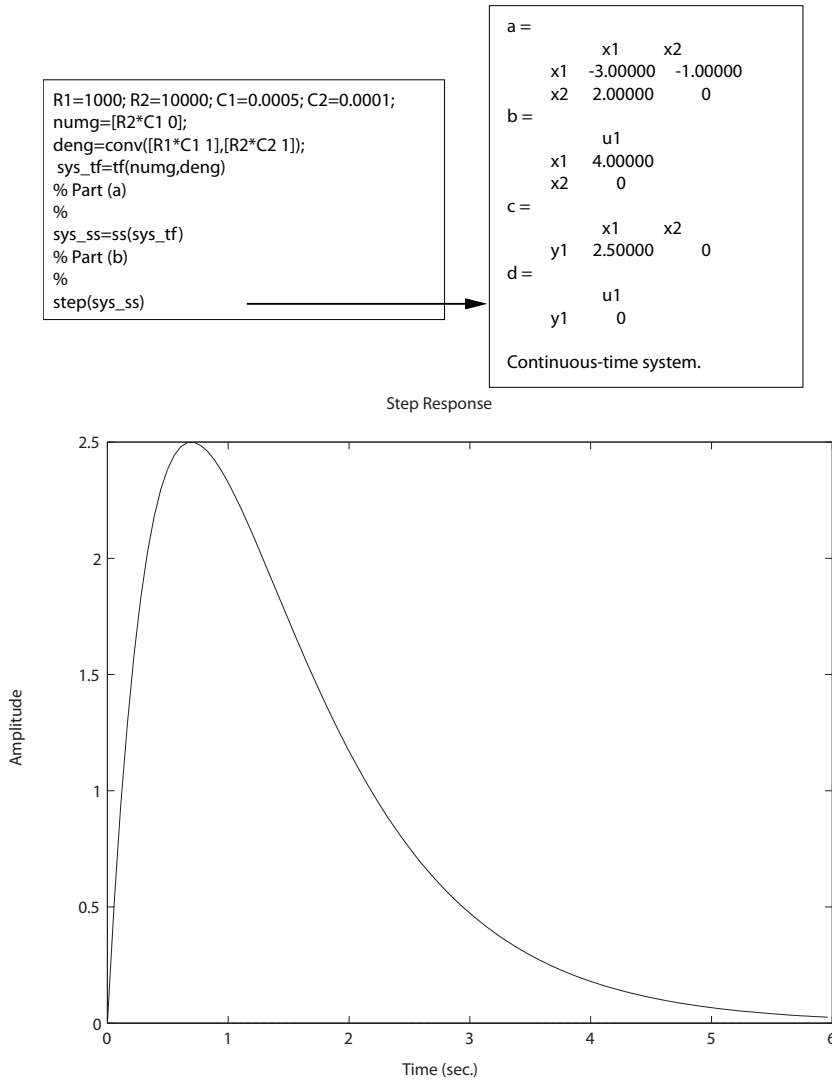
$$Z_1 = R_1 + \frac{1}{C_1 s}$$

$$Z_2 = \frac{R_2}{1 + R_2 C_2 s}$$

are the respective circuit impedances. Therefore, we obtain

$$V_o(s) = -\left[ \frac{R_2 C_1 s}{(1 + R_1 C_1 s)(1 + R_2 C_2 s)} \right] V_{in}(s).$$

The m-file script and step response is shown in Figure CP3.3.

**FIGURE CP3.3**

The m-file script using the **step** function to determine the step response.

**CP3.4** The m-file script and state history is shown in Figure CP3.4. The transfer function equivalent is

$$G(s) = \frac{1}{s^3 + 5s^2 + 2s + 3} .$$

The computed state vector at  $t = 10$  is the same using the simulation and the state transition matrix.

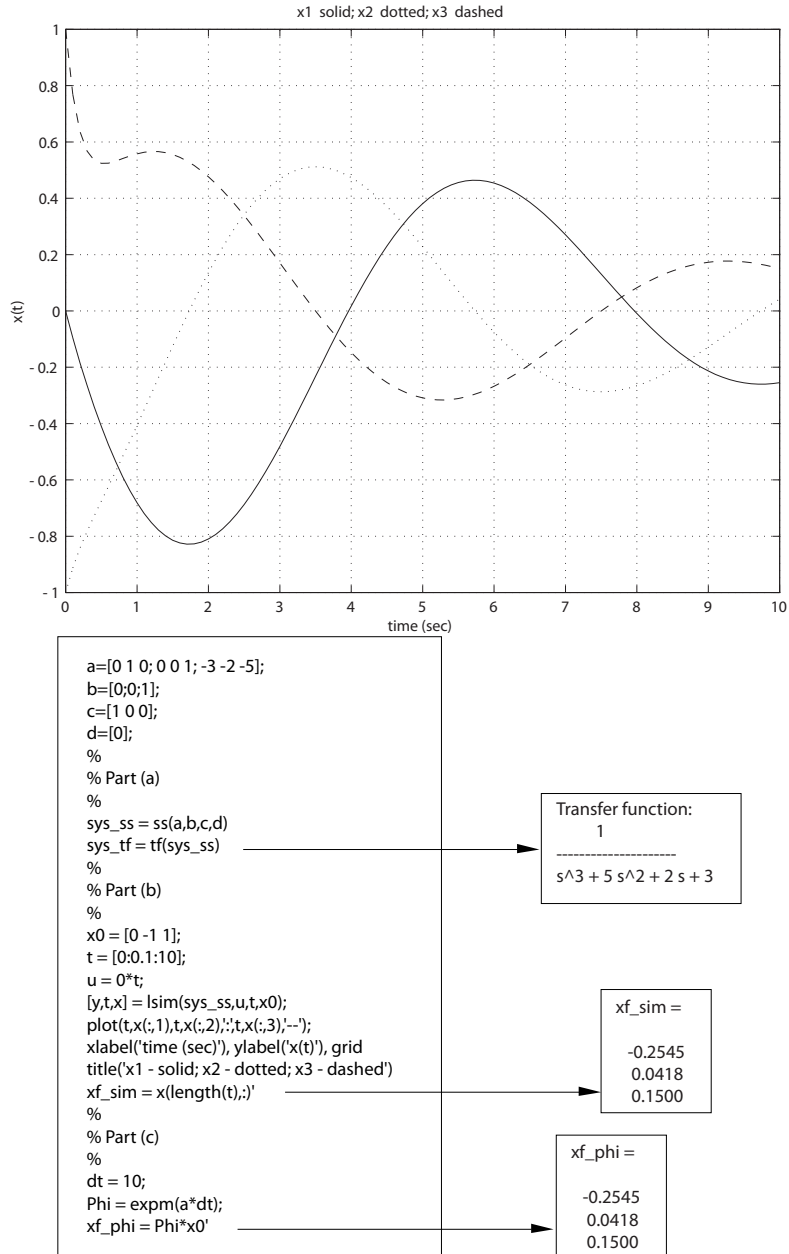


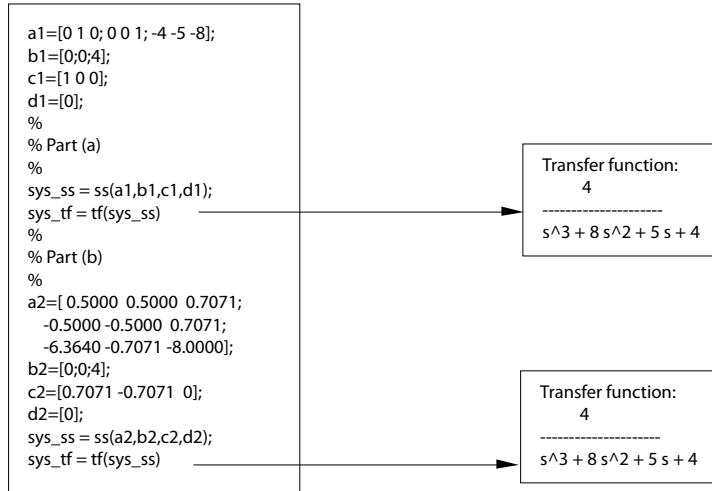
FIGURE CP3.4

The m-file script using the **lsim** function to determine the step response.

**CP3.5** The two state-space models represent the same transfer function, as shown in Figure CP3.5. The transfer function in both cases is

$$G(s) = \frac{4}{s^3 + 8s^2 + 5s + 4}.$$

We see that a state-space representation of a transfer function is not unique.



**FIGURE CP3.5**  
Comparison of the transfer functions of two state-space models.

**CP3.6** The m-file script and impulse response are shown in Figure CP3.6. The controller state-space representation is

$$\begin{aligned}\dot{x} &= -2x + u \\ y &= x\end{aligned}$$

and the plant state-space representation is

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -2 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}\end{aligned}$$

The closed-loop system state variable representation is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}u,\end{aligned}$$

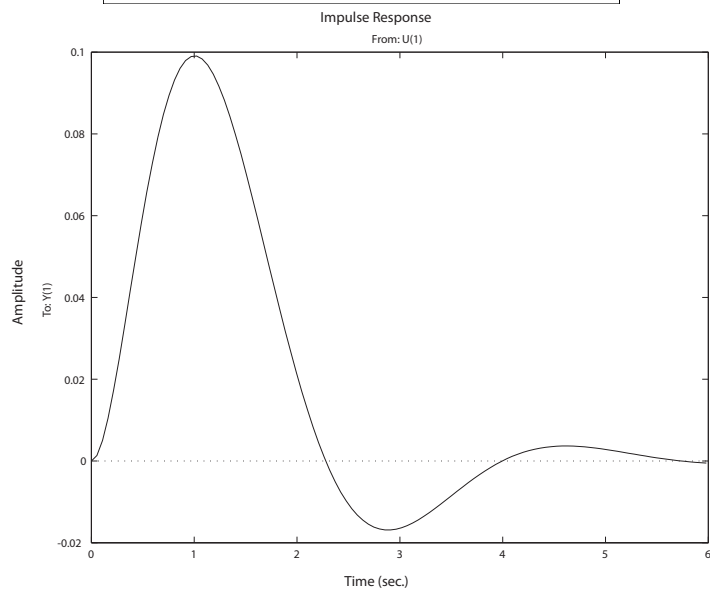
where  $\mathbf{D} = [0]$  and

$$\mathbf{A} = \begin{bmatrix} -2 & -2 & 0.5 \\ 2 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

```
numc=[1]; denc=[1 2]; sys_tfc = tf(numc,denc)
numg=[1]; deng=[1 2 4]; sys_tfg = tf(numg,deng)

%
% Part (a)
%
sys_ssc = ss(sys_tfc)
%
% Part (b)
%
sys_ssg = ss(sys_tfg)
%
% Part (c)
%
sys_s = series(sys_ssc,sys_ssg);
sys_cl = feedback(sys_s,[1]);

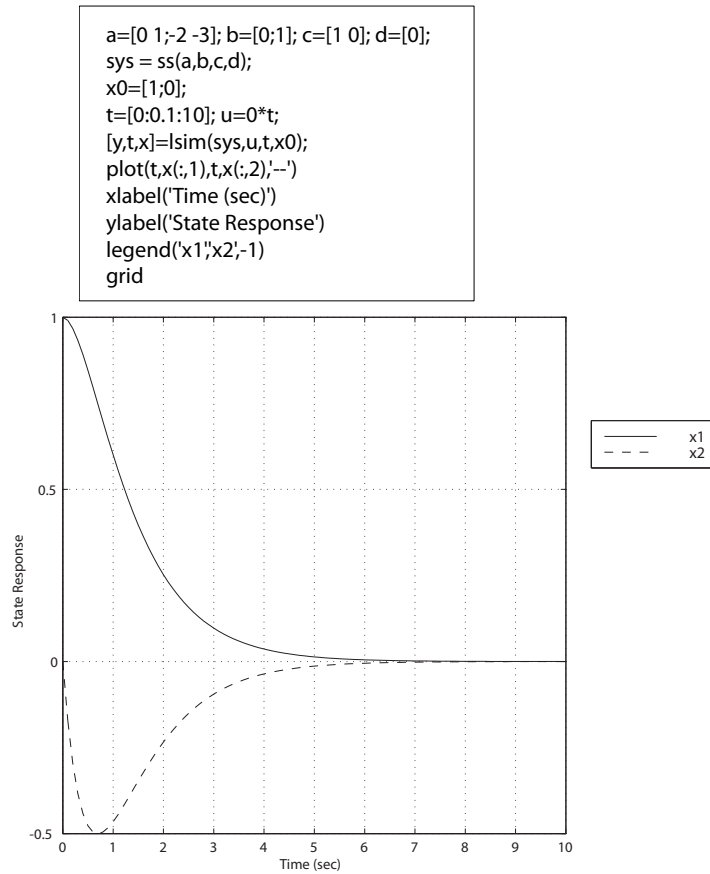
impulse(sys_cl)
```



**FIGURE CP3.6**  
Computing the state-space representations and the impulse response.



**CP3.7** The m-file script and system response is shown in Figure CP3.7.



**FIGURE CP3.7**

Using the `lsim` function to compute the zero input response.

# Feedback Control System Characteristics

## Exercises

**E4.1** (a) The system sensitivity to  $\tau$  is given by

$$S_{\tau}^T = S_G^T S_{\tau}^G .$$

In this case, we have

$$S_G^T = \frac{1}{1 + GH(s)} = \frac{1}{1 + \frac{100}{3s+1}} = \frac{3s + 1}{3s + 101}$$

and

$$S_{\tau}^G = \frac{-\tau s}{\tau s + 1} = \frac{-3s}{3s + 1} ,$$

where  $\tau = 3$ . Therefore,

$$S_{\tau}^T = \frac{-3s}{3s + 101} .$$

(b) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + GH(s)} = \frac{100}{3s + 101} = \frac{100/101}{\frac{3}{101}s + 1} = \frac{0.99}{\tau_c s + 1} ,$$

where the time-constant  $\tau_c = 3/101 = 0.0297$  second.

**E4.2** (a) The system sensitivity to  $K_2$  is

$$S_{K_2}^T = \frac{\partial T}{\partial K_2} \frac{K_2}{T} = \frac{1}{1 + K_1 K_2} .$$

(b) The transfer function from  $T_d(s)$  to  $V_o(s)$  is

$$V_o(s) = \frac{K_2}{1 + K_1 K_2} T_d(s) .$$

(c) We would select  $K_1 \gg 1$ , so that the transfer function from  $T_d(s)$  to  $V_o(s)$  is **small**.

**E4.3** (a) The tracking error,  $E(s) = R(s) - Y(s)$ , is given by

$$E(s) = \frac{R(s)}{1 + G(s)} = \frac{A/s}{1 + K/(s+5)^2} .$$

The steady-state error (computed using the final value theorem) is

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \left[ \frac{A}{1 + \frac{K}{(s+5)^2}} \right] = \frac{A}{1 + K/25} .$$

(b) A disturbance would be the wind shaking the robot arm.

**E4.4** (a) The tracking error,  $E(s) = R(s) - Y(s)$ , is given by

$$E(s) = \frac{R(s)}{1 + KG(s)} .$$

The steady-state position error is computed (using the final value theorem) to be

$$e_{ss} = \lim_{s \rightarrow 0} s \left[ \frac{A/s}{1 + KG(s)} \right] = \lim_{s \rightarrow 0} \left[ \frac{A}{1 + \frac{10K}{s(\tau s + 1)}} \right] = 0 .$$

(b) The ramp input of 0.1 m/sec is given by

$$R(s) = \frac{0.1}{s^2} .$$

Then, using the final value theorem, we have

$$e_{ss} = \lim_{s \rightarrow 0} s \left[ \frac{0.1/s^2}{1 + \frac{10K}{s(\tau s + 1)}} \right] = \lim_{s \rightarrow 0} \left[ \frac{0.1}{s + \frac{10K}{\tau s + 1}} \right] ,$$

or

$$e_{ss} = \frac{0.1}{10K} = \frac{0.01}{K} .$$

We desire  $e_{ss} \leq 0.0001$  m, so

$$K \geq \frac{0.01}{0.0001} = 100 .$$

**E4.5** (a) The sensitivity is

$$S_p^T = \frac{\partial T}{\partial p} \frac{p}{T} = p \left[ \frac{-s^4 - 15s^2 + 3s + 10}{(s^2 + ps + 10)^2} \right] T(s).$$

(b) The tracking error is

$$E(s) = [1 - T(s)] R(s) = \frac{s^3 + (2p - 1)s^2 + (4 - p)s - 7 - p}{s^3 + 2ps^2 + 4s + 3 - p} R(s)$$

Using the final value theorem with  $R(s) = 1/s$  we obtain the steady-state tracking error as

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{-7 - p}{3 - p} .$$

**E4.6** The closed-loop transfer function is

$$T(s) = \frac{10K}{s^2 + bs + 10K} .$$

The tracking error is

$$E(s) = [1 - T(s)] R(s) = \frac{s(s + b)}{s^2 + bs + 10K} \frac{1}{s^2},$$

where we let  $R(s) = 1/s^2$ . Using the final value theorem we obtain the steady-state tracking error as

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{b}{10K} .$$

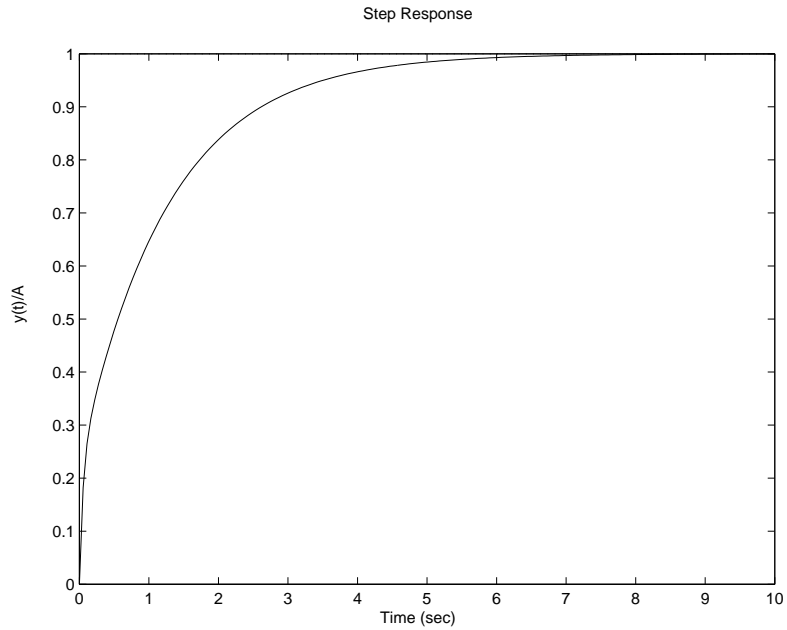
If we require that  $b < K$  then the steady-state error is less than 0.1 to the ramp input.

**E4.7** The light bounces off the surface of the slide and into a detector. If the light fails to hit the detector dead center, the unbalanced electric signal causes the motor to adjust the position of the light source, and simultaneously the lens.

**E4.8** The closed-loop transfer function is

$$T(s) = \frac{5(s + 3)}{s^2 + 20s + 15} .$$

The step response is shown in Figure E4.8.



**FIGURE E4.8**  
Step response.

**E4.9** (a) The closed-loop transfer function is

$$T(s) = \frac{KK_1}{s + K_1(K + K_2)}.$$

(b) The sensitivities are

$$S_K^T = \frac{\partial T/T}{\partial K/K} = \frac{s + K_1K_2}{s + K_1(K + K_2)}$$

and

$$S_{K_1}^T = \frac{s}{s + K_1(K + K_2)}.$$

(c) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = \frac{-1}{s + K_1(K_2 + K)}.$$

Therefore, since  $E(s) = -KY(s)$  (when  $R(s) = 0$ ), we have

$$E(s) = \frac{K}{s + K_1(K_2 + K)}T_d(s)$$

and

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{K}{K_1(K + K_2)} .$$

(d) With  $K = K_2 = 1$ , we have

$$T(s) = \frac{K_1}{s + 2K_1} .$$

Then,

$$Y(s) = \frac{K_1}{s + 2K_1} \frac{1}{s}$$

and

$$y(t) = \frac{1}{2} [1 - e^{-2K_1 t}] u(t) ,$$

where  $u(t)$  is the unit step function. Therefore, select  $K_1 = 10$  for the fastest response.

**E4.10** The closed-loop transfer function is

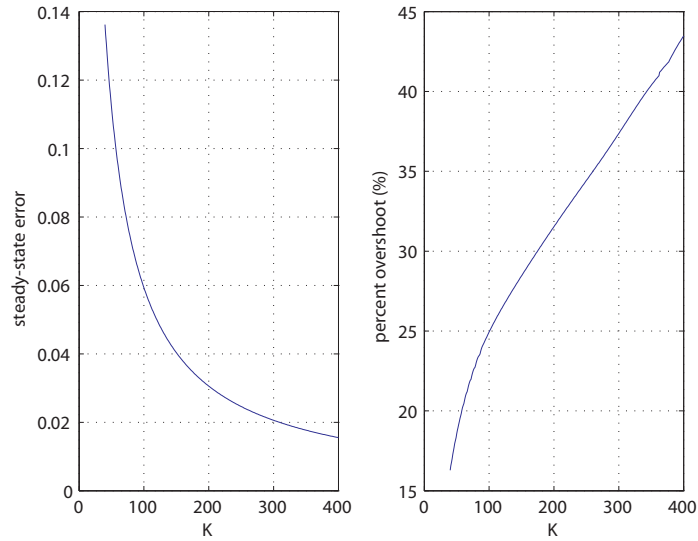
$$T(s) = \frac{46.24K(s + 50)(s + 425)}{(s + 200)(s + 425)(s^2 + 16.7s + 72.9) + 19652K(s + 50)} .$$

The steady-state error is determined to be

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{1}{s} \\ &= 1 - \lim_{s \rightarrow 0} T(s) = 1 - T(0) \\ &= \frac{6.3}{6.3 + K} . \end{aligned}$$

The plots of the steady-state error versus  $K$  and the percent overshoot  $P.O.$  versus  $K$  are shown in Figure E4.10 for

$$40 \leq K \leq 400 .$$

**FIGURE E4.10**

(a) Steady-state error. (b) Percent overshoot.

**E4.11** (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K(s^2 + 5s + 6)}{s^3 + 15s^2 + 56s + 60 + 14K}$$

(b) With  $E(s) = R(s) - Y(s)$  we obtain

$$\begin{aligned} E(s) &= \left[ 1 - \frac{G(s)}{1 + G(s)H(s)} \right] R(s) = \frac{1 - G(s)(1 - H(s))}{1 + G(s)H(s)} R(s) \\ &= \frac{s^3 + (15 - K)s^2 + (56 - 5K)s + (60 + 8K)}{s^3 + 15s^2 + 56s + 60 + 14K} \cdot \frac{1}{s} \end{aligned}$$

Then, using the final value theorem we find

$$\lim_{s \rightarrow 0} sE(s) = \frac{(60 + 8K)}{60 + 14K}.$$

(c) The transfer function from the disturbance  $T_d(s)$  to the output is

$$Y(s) = \frac{1}{1 + G(s)H(s)} T_d(s) = \frac{s^3 + 15s^2 + 56s + 60}{s^3 + 15s^2 + 56s + 60 + 14K} T_d(s).$$

The steady-state error to a unit step disturbance is

$$\lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \frac{s^3 + 15s^2 + 56s + 60}{s^3 + 15s^2 + 56s + 60 + 14K} \cdot \frac{1}{s} = \frac{60}{60 + 14K}.$$

(d) The sensitivity is

$$\begin{aligned} S_K^T &= \frac{\partial T}{\partial K} \frac{K}{T} = \frac{\partial T}{\partial G} \frac{\partial G}{\partial K} \frac{K}{T} \\ &= \frac{1}{(1 + G(s)H(s))^2} \left( \frac{K}{s + 10} \right) \frac{1 + G(s)H(s)}{G(s)} = \frac{1}{1 + G(s)H(s)}. \end{aligned}$$

**E4.12** (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{100K_1(s + 5)}{s^2 + 105s + (500 + 100K_1K_2)}.$$

The steady-state tracking error is

$$\begin{aligned} E(s) &= R(s) - Y(s) = \left[ \frac{1 - G_c(s)G(s)(1 - H(s))}{1 + G_c(s)G(s)H(s)} \right] R(s) \\ &= \frac{s^2 + (105 - 100K_1)s + 500 - 100K_1(5 - K_2)}{s^2 + 105s + 500 + 100K_1K_2} \cdot \frac{1}{s} \end{aligned}$$

and

$$\lim_{s \rightarrow 0} sE(s) = \frac{5 - K_1(5 - K_2)}{5 + K_1K_2}.$$

(b) The transfer function from the noise disturbance  $N(s)$  to the output  $Y(s)$  is

$$Y(s) = \left[ \frac{-G_c(s)G(s)H(s)}{1 + G_c(s)G(s)H(s)} \right] N(s) = \left[ \frac{-100K_1K_2}{s^2 + 105s + (500 + 100K_1K_2)} \right] N(s).$$

The steady-state error to a unit step  $N(s) = 1/s$  is

$$\lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \left[ \frac{-100K_1K_2}{s^2 + 105s + (500 + 100K_1K_2)} \right] \cdot \frac{1}{s} = \frac{-K_1K_2}{5 + K_1K_2}.$$

(c) The design trade-off would be to make  $K_1K_2$  as large as possible to improve tracking performance while keeping  $K_1K_2$  as small as possible to reject the noise.

**E4.13** The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + 20s + K}.$$

The sensitivity is

$$S_K^T = \frac{\partial T/T}{\partial K/K} = \frac{s^2 + 20s}{s^2 + 20s + K}.$$



**E4.14** (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{K}{s^2 + K_1s + K} .$$

The sensitivity is

$$S_{K_1}^T = \frac{\partial T/T}{\partial K_1/K_1} = -\frac{sK_1}{s^2 + K_1s + K} .$$

(b) You would make  $K$  as large as possible to reduce the sensitivity to changes in  $K_1$ . But the design trade-off would be to keep  $K$  as small as possible to reject measurement noise.

**E4.15** (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)} = \frac{120}{s^2 + 10s + 120} .$$

The steady-state tracking error is

$$\begin{aligned} E(s) &= R(s) - Y(s) = \left[ \frac{1}{1 + G_c(s)G(s)} \right] R(s) \\ &= \frac{s^2 + 10s}{s^2 + 10s + 120} \cdot \frac{1}{s} \end{aligned}$$

and

$$\lim_{s \rightarrow 0} sE(s) = 0 .$$

(b) The transfer function from the disturbance  $T_d(s)$  to the output  $Y(s)$  is

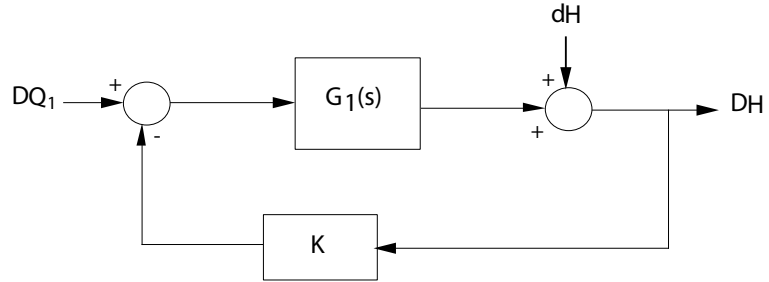
$$Y(s) = \left[ \frac{1}{s^2 + 10s + 120} \right] T_d(s) .$$

The steady-state error to a unit step  $T_d(s) = 1/s$  is

$$\lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \left[ \frac{1}{s^2 + 10s + 120} \right] \cdot \frac{1}{s} = \frac{1}{120} .$$

## Problems

**P4.1** The tank level control block diagram is shown in Figure P4.1.



**FIGURE P4.1**  
Tank level control block diagram.

(a) For the open-loop system the transfer function is

$$G_1(s) = \frac{R}{RCs + 1}.$$

Thus,

$$S_R^{G_1} = \frac{\partial G_1}{\partial R} \cdot \frac{R}{G_1} = \frac{1}{RCs + 1}.$$

For the closed-loop system, the transfer function is

$$T(s) = \frac{G_1}{1 + KG_1} = \frac{R}{RCs + 1 + KR}.$$

Thus,

$$S_R^T = \frac{\partial T}{\partial R} \cdot \frac{R}{T} = \frac{1}{RCs + 1 + KR},$$

and

$$S_K^T = \frac{\partial T}{\partial K} \cdot \frac{K}{T} = \frac{-KR}{RCs + 1 + KR}.$$

(b) For the open-loop system

$$\frac{\Delta H(s)}{\delta H(s)} = 1.$$

All disturbances show up directly in the output, thus the open-loop system has no capability to reject disturbances. On the other hand,

for the closed-loop system we have

$$\frac{\Delta H(s)}{\delta H(s)} = \frac{1}{1 + KG_1(s)} = \frac{RCs + 1}{RCs + 1 + KR} .$$

By selecting  $K$  large, we reduce the effects of any disturbances. For example, consider a step disturbance. The steady-state error due to the disturbance is

$$e_{ss} = \lim_{s \rightarrow 0} s \left( \frac{-(RCs + 1)}{RCs + 1 + KR} \right) \frac{A}{s} = \frac{-A}{1 + KR} .$$

As  $K$  gets larger, the steady-state error magnitude gets smaller, as desired.

(c) Consider the step input

$$\Delta Q_1(s) = \frac{A}{s} .$$

Then, for the open-loop system we have

$$e_{ss} = \lim_{s \rightarrow 0} s (1 - G_1) \frac{A}{s} = (1 - R)A .$$

The steady-state error is zero when  $R = 1$ , but is sensitive to changes in  $R$ . For the closed-loop system we have

$$e_{ss} = \lim_{s \rightarrow 0} s \left( \frac{1}{1 + KG_1} \right) \frac{A}{s} = \frac{A}{1 + KR} .$$

By selecting  $K$  large, the effect of the disturbance is reduced and is relatively insensitive to changes in  $R$ .

**P4.2** (a) The open-loop transfer function is

$$T(s) = K_a G(s) .$$

Therefore,  $S_{K_1}^T$  is undefined and

$$S_{K_a}^T = 1 .$$

The closed-loop transfer function is

$$T(s) = \frac{K_a G(s)}{1 + K_a K_1 G(s)} .$$

Therefore,

$$S_{K_1}^T = \frac{\partial T}{\partial K_1} \cdot \frac{K_1}{T} = \frac{-K_a K_1 G(s)}{1 + K_a K_1 G(s)}$$

and

$$S_{K_a}^T = \frac{1}{1 + K_1 K_a G(s)} .$$

- (b) The tracking error,  $E(s) = \theta_d(s) - \theta(s) = -\theta(s)$ , since  $\theta_d(s) = 0$ . The transfer function from the wave disturbance to the output  $\theta(s)$  is

$$\theta(s) = \frac{G(s)}{1 + K_1 K_a G(s)} T_d(s) .$$

Consider a step disturbance input for the open- and closed-loop systems. For the open-loop system, we have

$$e_{ss} = - \lim_{s \rightarrow 0} s G(s) \frac{A}{s} = -A .$$

Thus, we see that the open-loop system does not have the capability to reduce the effect of disturbances. For the closed-loop system, we have

$$e_{ss} = \lim_{s \rightarrow 0} s \left( \frac{-G(s)}{1 + K_1 K_a G(s)} \right) \frac{A}{s} = \frac{-A \omega_n^2}{1 + K_1 K_a \omega_n^2} .$$

We see that the larger we make  $K_1 K_a$ , that smaller the effect of the wave disturbance on the output in steady-state.

- P4.3** (a) The open-loop transfer function is

$$G(s) = \frac{K}{\tau s + 1}$$

where  $K = k_1 k_a E_b$ . Then, computing the sensitivity yields

$$S_K^G = 1 .$$

The closed-loop system transfer function is

$$T(s) = \frac{K}{\tau s + K K_{th} + 1} .$$

Similarly, computing the sensitivity yields

$$S_K^T = \frac{1}{1 + K_{th} G(s)} = \frac{\tau s + 1}{\tau s + 1 + K K_{th}} .$$

- (b) For the closed-loop system

$$\mathcal{T}(s) = \frac{1/(\tau s + 1)}{1 + K K_{th}/(\tau s + 1)} \mathcal{T}_e(s) \approx \frac{\mathcal{T}_e(s)}{K K_{th}}$$

when  $KK_{th} \gg 1$ . So, by choosing  $KK_{th}$  large, we can reduce the effect of the disturbance. This cannot be done with the open-loop system.

(c) Consider the step input

$$E_{des}(s) = \frac{A}{s} .$$

The tracking error for the open-loop system is

$$E(s) = E_{des}(s) - \mathcal{T}(s) .$$

Thus,

$$e_{ss} = \lim_{s \rightarrow 0} s \left[ 1 - \frac{K}{\tau s + 1} \right] \left( \frac{A}{s} \right) = (1 - K)A .$$

So,  $e_{ss} = 0$  when  $K = 1$ , but is sensitive to changes in  $K$ . The tracking error for the closed-loop system is

$$E(s) = \frac{\tau s + 1 + K(K_{th} - 1)}{\tau s + 1 + KK_{th}} E_{des}(s)$$

and

$$e_{ss} = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s \left( \frac{\tau s + 1 + K(K_{th} - 1)}{\tau s + 1 + KK_{th}} \right) \frac{A}{s} = \frac{A(1 + K(K_{th} - 1))}{1 + KK_{th}} .$$

Selecting  $K_{th} = 1$  and  $K \gg 1$  reduces the steady-state error.

**P4.4** (a) The overall transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{MG(s) + UQG(s)}{1 + QG(s)} .$$

(b) From Eq. (4.16) in Dorf & Bishop, we have

$$S_G^T = S_G^N - S_G^D .$$

In our case, we find that

$$S_G^N = 1 ,$$

and

$$S_G^D = \frac{QG(s)}{1 + QG(s)} .$$

Thus,

$$S_G^T = 1 - \frac{QG(s)}{1 + QG(s)} = \frac{1}{1 + QG(s)} .$$

(c) The sensitivity does not depend upon  $U(s)$  or  $M(s)$ .

**P4.5** The closed-loop transfer function is

$$T(s) = \frac{G_1G(s)}{1 + G_1G(s)} .$$

(a) The sensitivity of  $T(s)$  to changes in  $k_a$  is

$$S_{k_a}^T = \frac{1}{1 + G_1G(s)} .$$

(b) The transfer function from  $T_d(s)$  to  $\theta(s)$  is

$$\theta(s) = \frac{G(s)}{1 + G_1G(s)} T_d(s) .$$

Since we want  $\theta(s)$  due to a disturbance,  $E(s) = -\theta(s)$  and

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \left( \frac{-G(s)}{1 + G_1G(s)} \right) \frac{10}{s} = \frac{-10}{k_a} .$$

Since our maximum desired error magnitude is

$$e_{ss} = \frac{0.10^\circ \cdot \pi}{180} = 0.001745 \text{ rad} ,$$

we select

$$k_a \geq 5730 .$$

(c) The open-loop transfer function is

$$\theta(s) = G(s)T_d(s) .$$

So,

$$e_{ss} = -\lim_{s \rightarrow 0} sG(s) \left( \frac{10}{s} \right) \rightarrow \infty .$$

**P4.6** The closed-loop transfer function is

$$T(s) = \frac{G_1G(s)}{1 + G_1G(s)} .$$

(a) The sensitivity is

$$S_{K_e}^T = \frac{1}{1 + G_1 G(s)} = \frac{(\tau_1 s + 1)(\tau_e s + 1)}{(\tau_1 s + 1)(\tau_e s + 1) + K_1 K_e} .$$

(b) The speed is affected by the load torque through the transfer function

$$V(s) = \frac{-K_g G(s)}{1 + G G_1(s)} \Delta T_d(s) .$$

(c) Let  $R(s) = 30/s$ , and  $K_e K_1 \gg 1$ . When the car stalls,  $V(s) = 0$ . Using the final value theorem, we find

$$\begin{aligned} \lim_{s \rightarrow 0} s \left( \frac{-K_g G(s)}{1 + G G_1(s)} \right) \frac{\Delta d}{s} + \lim_{s \rightarrow 0} s \left( \frac{G_1 G(s)}{1 + G G_1(s)} \right) \frac{30}{s} \\ = -\Delta d \left( \frac{+K_g K_e}{1 + K_e K_1} \right) + 30 \left( \frac{K_1 K_e}{1 + K_1 K_e} \right) . \end{aligned}$$

Since  $K_e K_1 \gg 1$ , we have

$$V_{ss} = -\Delta d \left( \frac{K_g}{K_1} \right) + 30 .$$

When  $V_{ss} = 0$ , we have

$$\Delta d = \frac{30 K_1}{K_g} .$$

Thus, if

$$\frac{K_g}{K_1} = 2 ,$$

then  $\Delta d = 15$  percent grade ( i.e.  $\Delta d = 15$  ft rise per 100 ft horizontally) will stall the car.

**P4.7** (a) Let

$$G_1(s) = k_1 , \quad G_2(s) = \frac{k_2}{s(\tau s + 1)} , \quad \text{and} \quad H(s) = k_3 + k_4 s .$$

Then the transfer function from  $T_L(s)$  to  $Y(s)$  is

$$Y(s) = -\frac{G_2(s)}{1 + G_1 G_2 H(s)} T_L(s) = -\frac{k_2}{s(\tau s + 1) + k_1 k_2 (k_3 + k_4 s)} T_L(s) .$$

(b) The sensitivity of the closed-loop system to  $k_2$  is

$$S_{k_2}^T = \frac{1}{1 + G_1 G_2 H(s)} ,$$

where  $T(s)$  is the closed loop transfer function

$$T(s) = \frac{G_1 G_2(s)}{1 + G_1 G_2 H(s)} = \frac{k_1 k_2}{s(\tau s + 1) + k_1 k_2 (k_3 + k_4 s)} .$$

(c) The error is given by

$$E(s) = R(s) - T(s)R(s) .$$

With

$$R(s) = \frac{1}{s} ,$$

we have

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{1}{s} = 1 - T(0) = 1 - \frac{1}{k_3} .$$

**P4.8** (a) The sensitivity is

$$S_K^T = \frac{1}{1 + G_c G(s)} = \frac{(0.1s + 1)(s^2 + 20s + 180)}{(0.1s + 1)(s^2 + 20s + 180) + 180K} .$$

(b) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$Y(s) = \frac{G(s)}{1 + G_c G(s)} T_d(s) = \frac{180(0.1s + 1)}{(0.1s + 1)(s^2 + 20s + 180) + 180K} T_d(s) .$$

**P4.9** (a) Computing the derivative of  $R$  with respect to  $i$  yields

$$\frac{dR}{di} = \frac{-0.201R}{(i - 0.005)^{3/2}} .$$

When  $v_{out} = 35$  volts, we have

$$i = \frac{35}{5000} = 7\text{ma} .$$

At the operating point  $i = 7$  ma, we find from Figure P4.9(b) in Dorf & Bishop that  $R \approx 20K$  (note: If we use the given formula, we find that  $R \approx 8.2K$  when  $i = 7$  ma, thus we see that the formula is just an approximation to the plot). Using  $R = 20K$ , we have

$$\frac{dR}{di} = \frac{-0.402 \times 10^4}{0.896 \times 10^{-4}} = -45 \text{ kohms/ma} .$$

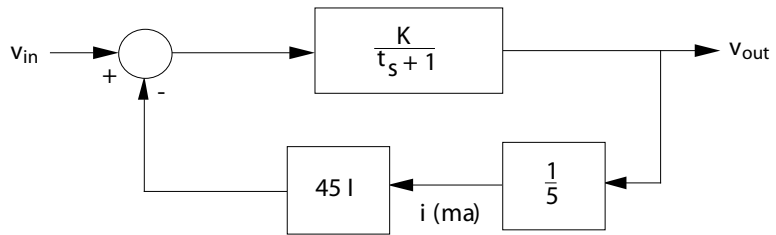


The transfer function (valid around the operating point) is

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{(s\tau + 1) + K\left(\frac{45I}{5}\right)}$$

$$= \frac{K}{s\tau + 1 + 9KI} .$$

The photosensor block diagram is shown in Figure P4.9.

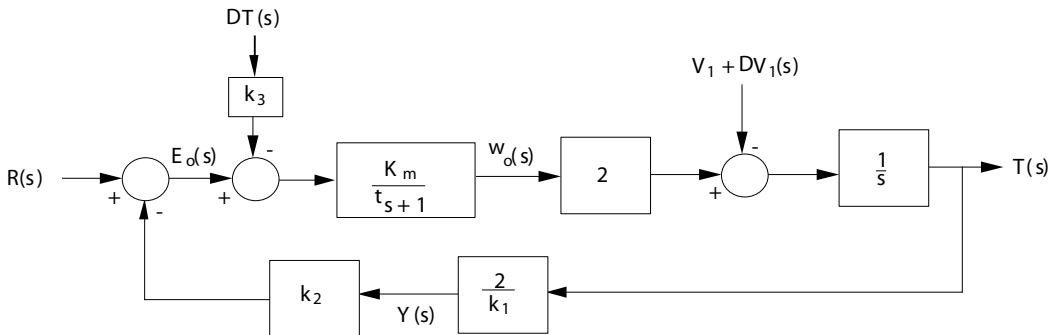


**FIGURE P4.9**  
Photosensor block diagram.

(b) The sensitivity of the system to changes in  $K$  is

$$S_K^T = \frac{\tau s + 1}{\tau s + 1 + 9KI} .$$

**P4.10** (a) and (b) The paper tension control block diagram is shown in Figure P4.10.



**FIGURE P4.10**  
Paper tension control block diagram.

(c) The closed-loop transfer function,  $T_c$ , is given by

$$T_c(s) = \frac{T(s)}{R(s)} = \frac{\frac{2K_m}{s(\tau s+1)}}{1 + \frac{4K_m k_2}{k_1 s(\tau s+1)}} = \frac{2K_m}{\tau s^2 + s + \frac{4K_m k_2}{k_1}}.$$

The sensitivity of  $T_c$  to changes in  $K_m$  is

$$S_{K_m}^{T_c} = \frac{1}{1 + \frac{4K_m k_2}{k_1 s(\tau s+1)}} = \frac{s(\tau s+1)}{\tau s^2 + s + \frac{4K_m k_2}{k_1}}.$$

(d) The transfer function from  $\Delta V_1(s)$  to  $T(s)$  is

$$T(s) = \frac{\frac{-1}{s}}{1 + \frac{4K_m k_2}{k_1 s(\tau s+1)}} \Delta V_1(s) = \frac{-1}{s + \frac{4K_m k_2}{k_1(\tau s+1)}} \Delta V_1(s).$$

When  $\Delta V_1(s) = A/s$ , we have

$$T(s) = \frac{-(\tau s+1)}{\tau s^2 + s + \frac{4K_m k_2}{k_1}} \frac{A}{s}.$$

and

$$\lim_{t \rightarrow \infty} T(t) = \lim_{s \rightarrow 0} sT(s) = \frac{-Ak_1}{4K_m k_2}.$$

**P4.11** (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{(8s+1)(3s+1) + K} = \frac{K}{24s^2 + 11s + 1 + K}.$$

(b) The sensitivity  $S_K^T$  is

$$S_K^T = \frac{\partial T}{\partial K} \cdot \frac{K}{T} = \frac{24s^2 + 11s + 1}{24s^2 + 11s + 1 + K}.$$

(c) Define  $E(s) = R(s) - Y(s)$ . Then

$$E(s) = \frac{R(s)}{1 + G_c(s)G(s)} = \left[ \frac{24s^2 + 11s + 1}{24s^2 + 11s + K + 1} \right] R(s).$$

With

$$R(s) = \frac{A}{s},$$

we have

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{A}{1 + K}.$$

(d) We want  $|e(t)| \leq 0.02A$  as  $t \rightarrow \infty$ . So,

$$0.02A \geq \frac{A}{K+1}$$

implies

$$K \geq 49.$$

**P4.12** (a) The two transfer functions are

$$T_1(s) = \frac{K_1 K_2}{s^2 + 3s - 4 + 6K_1 K_2}$$

and

$$T_2(s) = \frac{K_1 K_2}{s^2 + (3 - 2K_1 + 2K_2)s - 4 + 8K_2 + 2K_1 - 4K_1 K_2}.$$

When  $K_1 = K_2 = 1$ , we find that

$$T_1(s) = T_2(s) = \frac{1}{s^2 + 3s + 2}.$$

(b) The sensitivity  $S_{K_1}^{T_1}$  is

$$S_{K_1}^{T_1} = \frac{\partial T_1}{\partial K_1} \cdot \frac{K_1}{T_1} = \frac{s^2 + 3s - 4}{s^2 + 3s - 4 + 6K_1 K_2} = \frac{(s+4)(s-1)}{(s+2)(s+1)},$$

when  $K_1 = K_2 = 1$ . The sensitivity  $S_{K_1}^{T_2}$  is

$$S_{K_1}^{T_2} = \frac{\partial T_2}{\partial K_1} \cdot \frac{K_1}{T_2} = \frac{s+4}{s+4-2K_1} = \frac{s+4}{s+2},$$

when  $K_1 = 1$ . Thus,

$$S_{K_1}^{T_1} = S_{K_1}^{T_2} \frac{s-1}{s+1}.$$

**P4.13** (a) Let  $N(s) = G_1(s) + kG_2(s)$  and  $T_d(s) = G_3(s) + kG_4(s)$ . Then

$$\begin{aligned} S_k^T &= \frac{\partial N}{\partial k} \cdot \frac{k}{N} - \frac{\partial D}{\partial k} \cdot \frac{k}{D} = \frac{G_2 k}{G_1 + kG_2} - \frac{G_4 k}{G_3 + kG_4} \\ &= \frac{k(G_2 G_3 - G_1 G_4)}{(G_1 + kG_2)(G_3 + kG_4)}. \end{aligned}$$

(b) The closed-loop transfer function is

$$T(s) = \frac{MG(s) + kUG(s)}{1 + kGH(s)} = \frac{G_1(s) + kG_2(s)}{G_3(s) + kG_4(s)}.$$

Then using result from (a), we have

$$S_k^T = \frac{k(UG(s) - MG^2H(s))}{(MG(s) + kUG(s))(1 + kGH(s))} .$$

**P4.14** The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{10(s + 4)}{s(s + a)(s + 1) + 10(s + 4)} .$$

Then

$$S_a^T = S_a^N - S_a^D ,$$

where  $N$  is the numerator and  $D$  is the denominator. We have

$$S_a^N = 0 .$$

Let

$$G(s) = \frac{p(s)}{q(s)(s + a)} ,$$

where  $p(s) = 10(s + 4)$  and  $q(s) = s(s + 1)$ . Then

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{p(s)}{q(s)(s + a) + p(s)} ,$$

and

$$S_a^T = -S_a^D = -\frac{dD}{da} \cdot \frac{a}{D} = \frac{-aq(s)}{q(s)(s + a) + p(s)} = -\frac{a}{s + a} \cdot \frac{1}{1 + G(s)} .$$

**P4.15** (a) The closed-loop transfer function for the disturbance to the output is

$$\frac{Y(s)}{T_d(s)} = \frac{G(s)}{1 + KG(s)} ,$$

with  $R = 0$ . The steady-state deviation is

$$y_{ss} = \lim_{s \rightarrow 0} s \left( \frac{G(s)}{1 + KG(s)} \right) \frac{1}{s} = \frac{G(0)}{1 + KG(0)} = \frac{1}{1 + K} .$$

So, with  $K = 10$  we have  $y_{ss} = 1/11$ , and with  $K = 25$  we have  $y_{ss} = 1/26$ .

(b) Considering the rudder input, we have

$$Y(s) = \frac{G(s)T_d(s) + KG(s)R(s)}{1 + KG(s)} = \frac{G(s)(T_d(s) + KR(s))}{1 + KG(s)} .$$

Setting  $R(s) = -\frac{T_d(s)}{K}$  yields  $Y(s) = 0$ .

**P4.16** (a) Let

$$G_1(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad \text{and} \quad G_2(s) = \frac{G_1(s)}{100}.$$

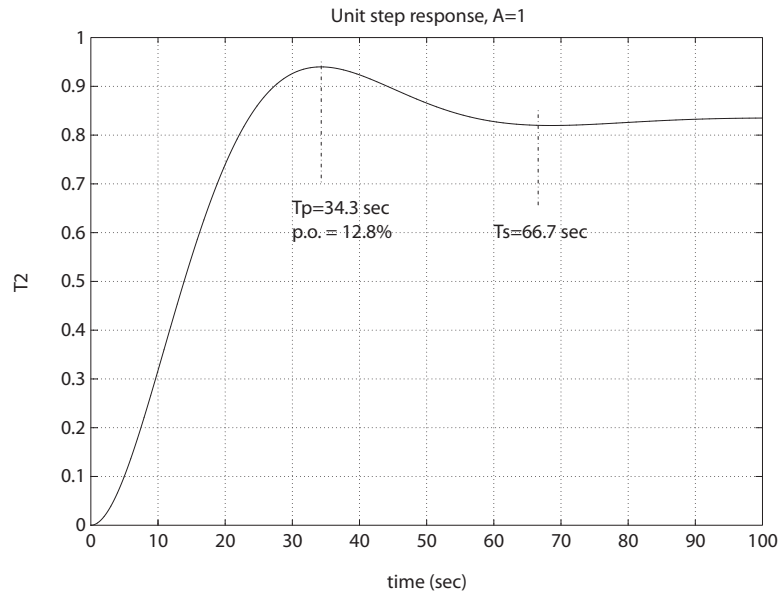
Then

$$T_2(s) = \frac{G_1(s)}{1 + G_2 G_c(s)} T_o(s) + \frac{G_2 G_c(s)}{1 + G_2 G_c(s)} T_{2d}(s).$$

(b) We can equivalently consider the case of a step input,  $T_{2d} = A/s$ ,  $T_o = 0$ , and zero initial conditions. Thus,

$$T_2(s) = \frac{G_c G_2}{1 + G_c G_2} T_{2d} = \frac{5}{500s^2 + 60s + 6} \frac{A}{s},$$

where  $G_c(s) = 500$ . The transient response is shown in the Figure P4.16 for a unit step input ( $A = 1$ ).



**FIGURE P4.16**

Two tank temperature control system response.

(c) With

$$E(s) = T_{2d}(s) - T_2(s),$$

we have

$$E(s) = T_{2d}(s) - M(s)T_{2d}(s)$$

where

$$M(s) = \frac{G_2 G_c(s)}{1 + G_2 G_c(s)} .$$

Then

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - M(s)) \frac{A}{s} = (1 - M(0))A = (1 - \frac{5}{6})A = \frac{A}{6} .$$

**P4.17** (a) The closed-loop transfer function is

$$\frac{\theta(s)}{\theta_d(s)} = \frac{600}{0.1s^2 + s + 600} = \frac{6000}{s^2 + 10s + 6000} .$$

The solution for a step input is

$$\theta(t) = 1 - 1.0021e^{-5.0349t} \sin(77.2962t + 1.5058).$$

(b) The transfer function from the disturbance to the output is

$$\frac{\theta(s)}{T_d(s)} = \frac{-1}{0.1s^2 + s + 600} .$$

Thus,

$$\theta_{ss} = - \lim_{s \rightarrow 0} s\theta(s) = \frac{A}{600} .$$

Therefore, the disturbance input magnitude reduced by 600 at the output.

(c) Using the final value theorem we have (for  $\theta_d(s) = 1/s^2$  )

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s(1 - T(s))\theta_d(s) \\ &= \lim_{s \rightarrow 0} s \left( \frac{0.1s^2 + s}{0.1s^2 + s + 600} \right) \frac{1}{s^2} = \frac{1}{600} . \end{aligned}$$

## Advanced Problems

**AP4.1** The plant transfer function is

$$G_p(s) = \frac{R}{RCs + 1}.$$

The closed-loop output is given by

$$H(s) = \frac{1}{1 + GG_p(s)}Q_3(s) + \frac{GG_p(s)}{1 + GG_p(s)}H_d(s).$$

Therefore, with  $E(s) = H_d(s) - H(s)$ , we have

$$E(s) = \frac{-1}{1 + GG_p(s)}Q_3(s),$$

since  $H_d(s) = 0$ .

(a) When  $G(s) = K$ , we have

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{-1}{1 + KR}.$$

(b) When  $G(s) = K/s$ , we have

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = 0.$$

**AP4.2** Define

$$G(s) = \frac{K_m G_c(s)}{s(L_a s + R_a)(Js + f) + K_m K_b G_c(s)}.$$

Then,

$$\theta(s)/\theta_d(s) = \frac{nG(s)}{1 + nG(s)}$$

and

$$E(s) = \frac{1}{1 + nG(s)}\theta_d(s).$$

So,

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + nG(s)} \frac{A}{s} = \frac{A}{1 + nG(0)} = \frac{AK_b}{K_b + n}.$$

When  $\theta_d(s) = 0$  and  $T_d(s) = M/s$ , we have

$$\theta(s)/T_d(s) = \frac{n(L_a s + R_a)}{s(L_a s + R_a)(Js + f) + K_m K_b + K_m G_c n}.$$

If  $G_c(s) = K$ , then

$$e_{ss} = \frac{-nMR_a}{K_m(K_b + nK)}$$

and if  $G_c(s) = K/s$ , we determine that  $e_{ss} = 0$ .

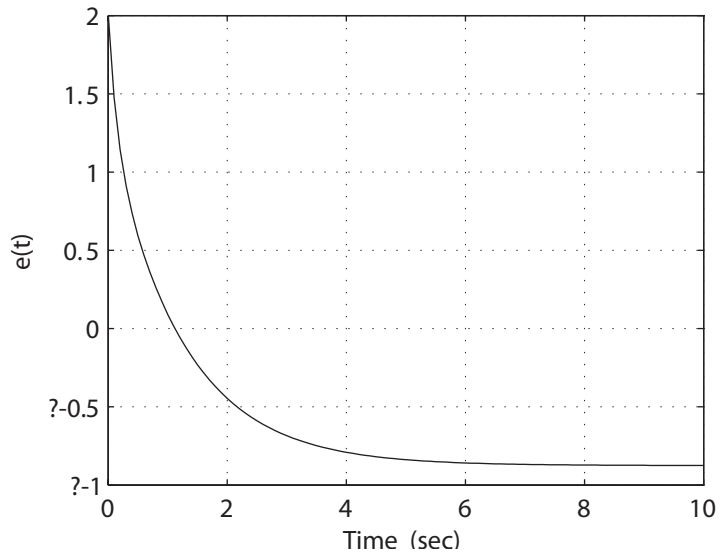
**AP4.3** (a) The input  $R(s)$  is

$$R(s) = \frac{1}{s} - \frac{1}{s^2}$$

and the disturbance is  $T_d(s) = 0$ . So,

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + G(s)} R(s) = \lim_{s \rightarrow 0} \frac{1 - \frac{1}{s}}{1 + \frac{10(2s+4)}{7s(s+5)}} = -0.8750.$$

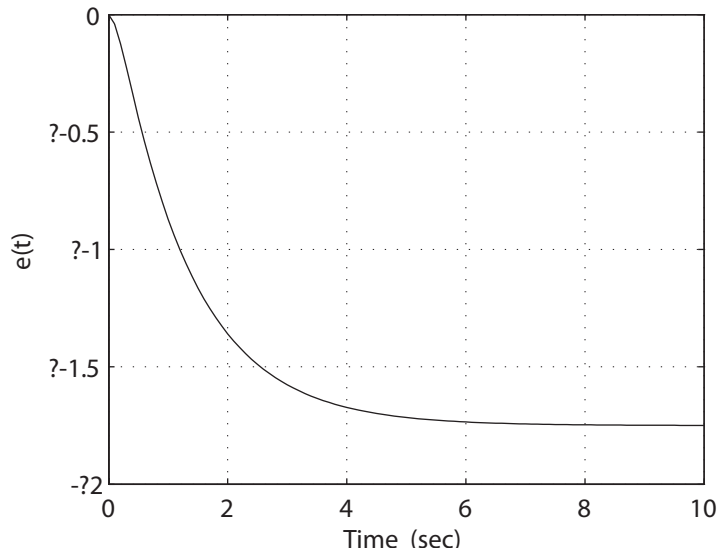
(b) The error plot is shown in Figure AP4.3a.



**FIGURE AP4.3**

(a) Error plot with  $d(t)=0$ .





**FIGURE AP4.3**  
CONTINUED: (b) Error plot with  $r(t)=0$ .

(c) The transfer function from  $T_d(s)$  to  $Y(s)$  (with  $R(s) = 0$ ) is

$$Y(s)/T_d(s) = \frac{-70}{7s^2 + 55s + 40} .$$

The steady-state error due to a disturbance  $T_d(s) = 1/s$  is

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{-70}{7s^2 + 55s + 40} \frac{1}{s} = -1.75 .$$

(d) The error  $e(t)$  is shown in Figure AP4.3b.

**AP4.4** (a) The closed-loop transfer function is

$$\omega(s)/V(s) = \frac{K_m}{R_a J s^2 + K_b K_m s + K_m K K_t} .$$

With  $v(t) = t$ , we have  $V(s) = 1/s^2$ , and  $T_d(s) = 0$ . Using the final value theorem yields

$$e_{ss} = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} \frac{1}{s + \frac{K K_m}{R_a J s + K_m K_b}} = \frac{K_b}{K} = \frac{0.1}{K} .$$

We desire that

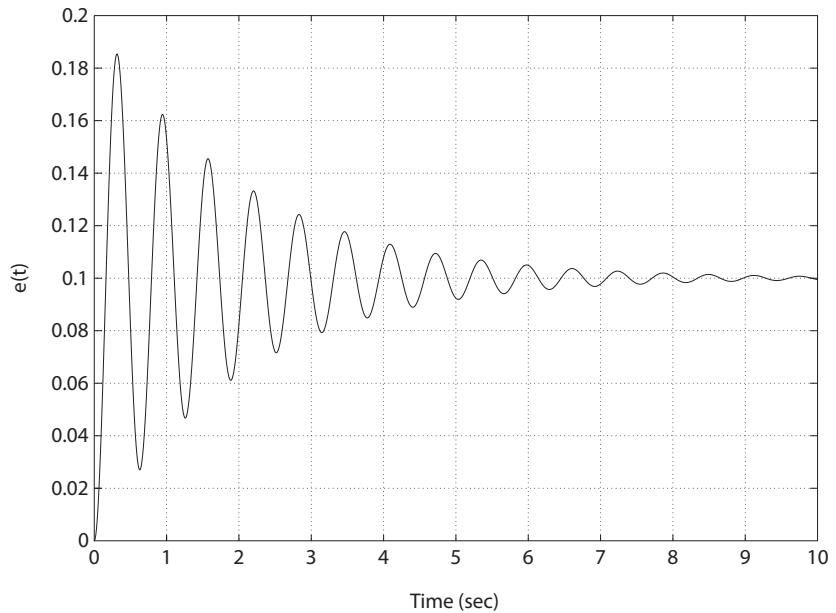
$$e_{ss} = \frac{0.1}{K} < 0.1 .$$

Therefore, we should select  $K > 1$ . For example, we can take  $K = 8$ .

(b) The transfer function from  $T_d(s)$  to  $\omega(s)$  is given by

$$\frac{\omega(s)}{T_d(s)} = \frac{-10s}{s^2 + 10s + 100} .$$

The error plot is shown in Figure AP4.4, where  $e(s) = -\omega(s)$  ( $V(s) = 0$ .)



**FIGURE AP4.4**  
Error plot with a ramp disturbance input.

**AP4.5** (a) The transfer function from the disturbance  $T_d(s)$  to the output  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = \frac{-s}{s^3 + 4s^2 + 4s + K} .$$

The steady-state error (when  $T_d(s) = 1/s$ ) is

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{s}{s^3 + 4s^2 + 4s + K} \frac{1}{s} = 0 .$$

(b) The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K}{s^3 + 4s^2 + 4s + K}.$$

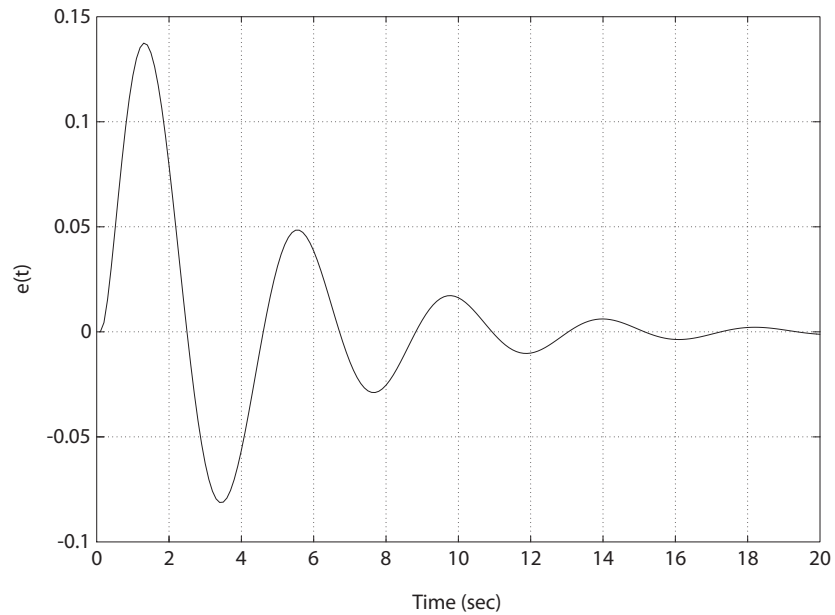
The steady-state error (when  $R(s) = 1/s^2$ ) is

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{s^3 + 4s^2 + 4s}{s(s^3 + 4s^2 + 4s + K)} = \frac{4}{K}.$$

(c) Let  $K = 8$ . Then,

$$\frac{Y(s)}{T_d(s)} = \frac{-s}{s^3 + 4s^2 + 4s + 8}.$$

The error plot is shown in Figure AP4.5, for  $r(t) = 0$ .



**FIGURE AP4.5**

Error plot with a step disturbance input and  $K=8$ .

**AP4.6** (a) The transfer function is

$$\frac{V_o(s)}{V(s)} = \frac{1 + RCs}{2 + RCs}.$$

(b) The system sensitivity is defined as

$$S_C^G = \frac{\partial G/G}{\partial C/C}.$$

Therefore, the sensitivity is determined to be

$$S_C^G = \frac{RCs}{(2 + RCs)(1 + RCs)} = \frac{1}{\left(1 + \frac{2}{RCs}\right)\left(1 + \frac{1}{RCs}\right)}.$$

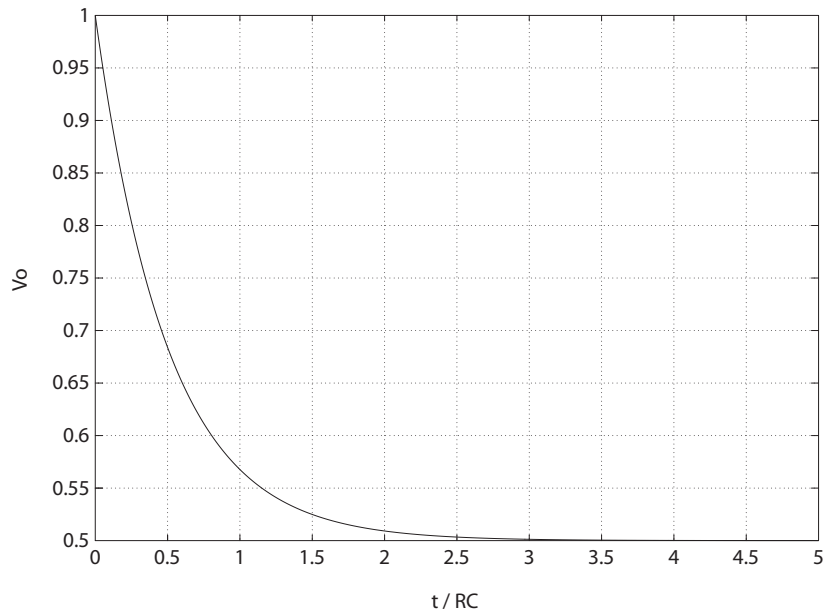
(c) Let  $V(s) = 1/s$ . Then

$$V_o(s) = \frac{1 + RCs}{2 + RCs} \frac{1}{s} = \frac{0.5}{s} + \frac{0.5RC}{RCs + 2}.$$

Taking the inverse Laplace transform yields

$$v_o(t) = 0.5(1 + e^{-2t/RC})u(t)$$

where  $u(t)$  is the unit step function. A plot of  $v_o(t)$  versus  $t/RC$  is shown in Figure AP4.6.



**FIGURE AP4.6**  
Step response.

**AP4.7** (a) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = -\frac{s}{s(s+1)+K}.$$

(b) The transfer function from  $N(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{N(s)} = \frac{K}{s(s+1)+K}.$$

(c) Let  $T_d(s) = A/s$  and  $N(s) = B/s$ . Then,

$$e_{ss} = -y_{ss} = \lim_{s \rightarrow 0} s \frac{s}{s(s+1)+K} \frac{A}{s} - \lim_{s \rightarrow 0} s \frac{K}{s(s+1)+K} \frac{B}{s} = -B.$$

So,  $K$  has no effect on the steady-state errors. However, choosing  $K = 100$  will minimize the effects of the disturbance  $T_d(s)$  during the transient period.

**AP4.8** (a) The closed-loop transfer function is

$$T(s) = \frac{Kb}{s + Kb + 2}.$$

(b) The sensitivity is determined to be

$$S_b^T = \frac{\partial T/T}{\partial b/b} = \frac{s+2}{s+Kb+2}.$$

(c) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = \frac{b}{s + Kb + 2}.$$

So, choose  $K$  as large as possible, to make  $Y(s)/T_d(s)$  as “small” as possible. Thus, select

$$K = 50.$$

This also minimizes  $S_b^T$  at low frequencies.

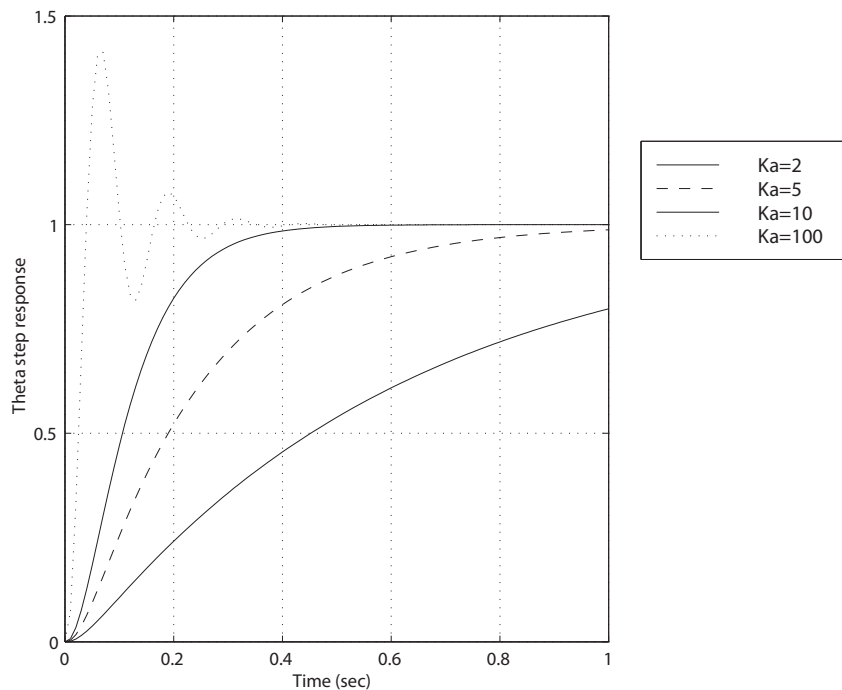
## Design Problems

**CDP4.1** The model of the traction drive, capstan roller, and linear slide was developed in CDP2.1:

$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s [(L_m s + R_m)(J_T s + b_m) + K_b K_m]}$$

The step response for the closed-loop system (with the tachometer not in the loop) and various values of the controller gain  $K_a$  is shown below.

```
% System parameters
Ms=5.693; Mb=6.96; Jm=10.91e-03; r=31.75e-03;
bm=0.268; Km=0.8379; Kb=0.838; Rm=1.36; Lm=3.6e-03; Lm=0;
% Controller gain
Ka=100;
% Motor and slide model
Jt=Jm+r^2*(Ms+Mb);
num=[Km];
den=[Lm*Jt Rm*Jt+Lm*bm Kb*Km+Rm*bm 0];
sys=tf(num,den);
%Closed-loop tf and step response
sys_cl=feedback(Ka*sys,[1]);
step(sys_cl)
```



- DP4.1** (a) The transfer function from the load disturbance to the output speed is

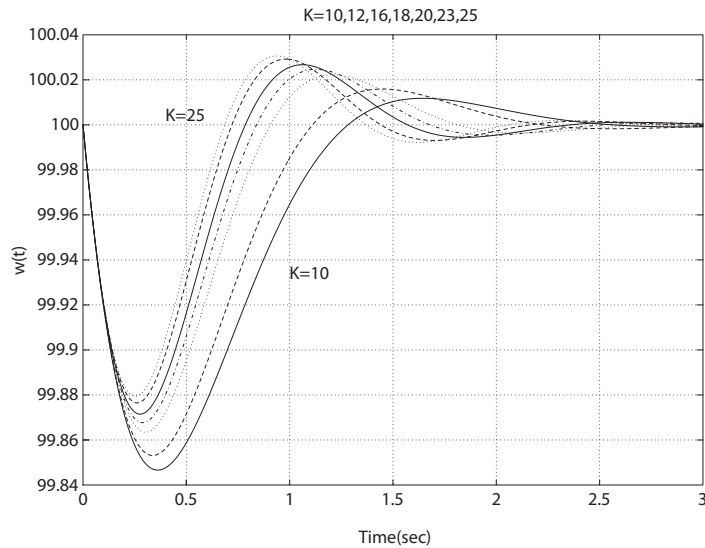
$$\frac{\omega(s)}{T_d(s)} = \frac{-G(s)}{1 + G_c G(s)} = \frac{-s}{s^2 + 4s + K} .$$

Thus, the effect on  $\omega(s)$  (of a unit step disturbance) at steady-state is

$$\lim_{t \rightarrow \infty} \omega(t) = \lim_{s \rightarrow 0} s \left( \frac{-s}{s^2 + 4s + K} \right) \frac{1}{s} = 0 .$$

We see that the load disturbance has no effect on the output at steady-state.

- (b) The system response for  $10 \leq K \leq 25$  is shown in Figure DP4.1.



**FIGURE DP4.1**  
Speed control system response.

For example, if we select  $K = 16$ , then  $\omega_n = 4$ ,  $\zeta = \frac{1}{2}$ , and the response due to a unit step disturbance is

$$\omega(s) = \frac{-s}{s^2 + 4s + 16} \left( \frac{1}{s} \right) = \frac{-1}{(s + 2)^2 + 12} .$$

Hence, if we are originally at  $\omega(t) = 100$  for  $t < \tau$ , we have

$$\omega(t) = 100 - \frac{1}{\sqrt{12}} e^{-2t} \sin \sqrt{12}t \quad t \geq \tau .$$

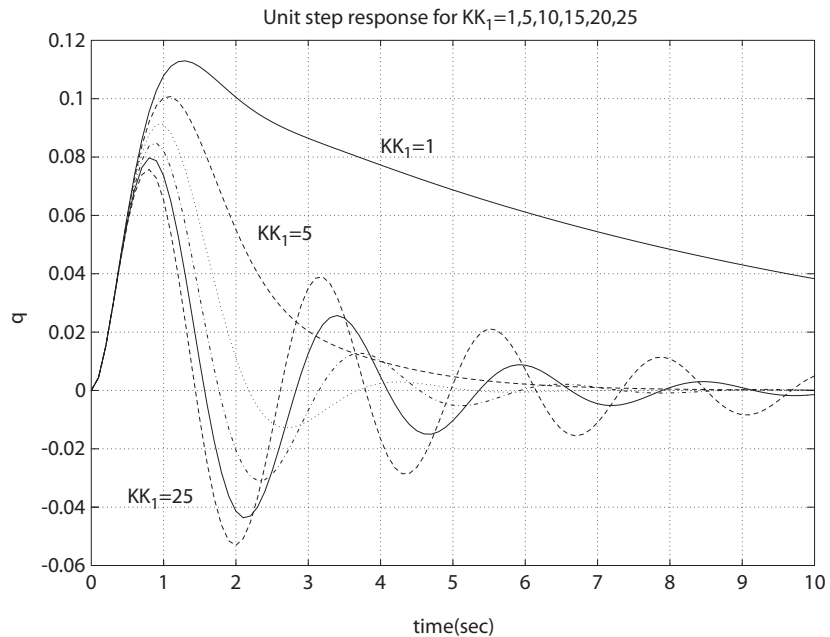
**DP4.2** With  $\theta_d = 0$ , we have

$$\theta(s) = \frac{G(s)}{1 + G(s)\frac{KK_1}{s}}T_d(s) = \frac{s}{s^3 + 4s^2 + 9s + KK_1}T_d(s) .$$

For  $T_d = A/s$ , we have

$$\theta(s) = \frac{A}{s^3 + 4s^2 + 9s + KK_1} .$$

The system response to a unit step disturbance for various values of  $KK_1$  are shown in Figure DP4.2. From the plot we see that when  $KK_1$  is small the response is slow but not oscillatory. On the other hand, when  $KK_1$  is large the response is fast but highly oscillatory. In fact, if  $KK_1 > 35$ , the system is unstable. Thus, we might select  $KK_1 = 10$  as a reasonable trade-off between fast performance and stability.



**FIGURE DP4.2**

Aircraft roll angle control system response to a disturbance.

**DP4.3** (a) The closed-loop transfer function is

$$T(s) = \frac{\omega(s)}{\omega_d(s)} = \frac{K}{s^2 + 5s + KK_1} .$$



Then,

$$E(s) = (1 - T(s))\omega_d(s) = \frac{s^2 + 5s + K(K_1 - 1)}{s^2 + 5s + KK_1} \frac{1}{s} .$$

So, if

$$0.99 < K_1 < 1.01 ,$$

then

$$|e_{ss}| < 0.01 .$$

(b) The transfer function from  $T_d(s)$  to  $\omega(s)$  is

$$\omega(s) = \frac{-s}{s^2 + 5s + KK_1} T_d(s) .$$

So, with  $E(s) = -\omega(s)$  and  $T_d(s) = 2/s^2$ , we have

$$\lim_{s \rightarrow 0} sE(s) = \frac{2}{KK_1} .$$

Therefore, we select  $KK_1 > 20$  to obtain  $e_{ss} < 0.1$ .

**DP4.4** The steady-state error for a step input command is zero for any  $K_1$ . The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = \frac{G(s)}{1 + KG(s)} = \frac{2}{s^3 + 5s^2 + 4s + 2K} .$$

Thus, the output at steady-state due to a step disturbance  $T_d(s) = A/s$  is

$$\lim_{s \rightarrow 0} sY(s) = \frac{A}{K} .$$

We want to maximize  $K$  to reduce the effect of the disturbance. As we will see in Chapter 6, we cannot select  $K$  too high or the system will become unstable. That is why the problem statement suggests a maximum gain of  $K = 10$ . For the design we choose

$$K = 10 .$$

**DP4.5** The transfer function from  $V(s)$  to  $V_o(s)$  is

$$V_o(s)/V(s) = \frac{ks}{s+a}$$

where

$$k = \frac{R_2 + R_3}{R_2} \quad \text{and} \quad a = \frac{1}{R_1 C} .$$

Computing the step response, we find that

$$v_o(t) = ke^{-at} = 5e^{-100t} .$$

Solving for  $R_1, R_2, R_3$  and  $C$  yields

$$R_1 C = 0.01 \quad \text{and} \quad \frac{R_2}{R_3} = 4 .$$

**DP4.6** (a) The closed-loop transfer function is

$$\theta(s) = \frac{K/J}{s^2 + K/J} \theta_d(s) .$$

Since  $J > 0$ , the system is unstable when  $K < 0$  and marginally stable when  $K > 0$ .

(b) Since the system is marginally stable, the system response does not have a steady-state value—it oscillates indefinitely.

(c) The closed-loop transfer function is

$$\theta(s) = \frac{K_D s + K_P}{J s^2 + K_D s + K_P} \theta_d(s) .$$

The system is stable for all  $K_D > 0$  and  $K_P > 0$ , given that  $J > 0$ .

(d) The tracking error  $E(s) = \theta_d(s) - \theta(s)$  is

$$E(s) = \frac{J s^2}{J s^2 + K_D s + K_P} .$$

Therefore, using the final value theorem we obtain the steady-state value

$$\lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s \frac{J s^2}{J s^2 + K_D s + K_P} \cdot \frac{1}{s} = 0 .$$

**DP4.7** (a) The closed-loop transfer function is

$$Y(s) = \frac{s}{s^2 + Ks + 2K} T_d(s) = \frac{1}{s^2 + Ks + 2K}$$

where the disturbance is a unit step  $T_d(s) = 1/s$ . Considering the poles of the closed-loop system, we find that when  $K > 8$  the system

has two real poles. In that case the disturbance step response is

$$y(t) = \frac{1}{\sqrt{K^2 - 8K}} \left( e^{-\alpha t} - e^{-\beta t} \right),$$

where

$$\alpha = \frac{K - \sqrt{K^2 - 8K}}{2} \quad \text{and} \quad \beta = \frac{K + \sqrt{K^2 - 8K}}{2}$$

Bounding the maximum  $y(t)$  yields the inequality

$$|y(t)| = \frac{1}{\sqrt{K^2 - 8K}} \left| e^{-\alpha t} - e^{-\beta t} \right| \leq 0.05.$$

We know that

$$\left| e^{-\alpha t} - e^{-\beta t} \right| \leq 1,$$

for any  $\alpha$  and  $\beta$  computed as shown above where  $K > 8$ . So, if we choose  $K$  such that

$$\frac{1}{\sqrt{K^2 - 8K}} \leq 0.05.$$

we will guarantee that the maximum bound of 0.05 is not exceeded. Solving for  $K$  yields

$$K > 24.4.$$

For any  $K > 24.4$  we know that the maximum value of the disturbance step response will be less than 0.05. When  $K = 24.4$  the maximum unit step disturbance response is 0.035. Solving explicitly for  $K$  so that the maximum is 0.05 we find that  $K = 16.3$  (this was found numerically since it is very difficult to obtain analytically).

- (b) Since the system is type 2, we know that the steady-state value of the disturbance step response is zero for a unit step disturbance.

**DP4.8** (a) The sensitivities are

$$S_{\tau_1}^T = \frac{\partial T/T}{\partial \tau_1/\tau_1} = \left[ \frac{-s^2 (\tau_2 s + 1) \tau_1}{K} \right] T(s)$$

and

$$S_{\tau_2}^T = \frac{\partial T/T}{\partial \tau_2/\tau_2} = \left[ \frac{-s^2 (\tau_1 s + 1) \tau_2}{K} \right] T(s)$$

where we assume that  $K \neq 0$ .

(b) Computing the closed-loop transfer function yields

$$Y(s) = \frac{s(\tau_1 s + 1)}{s(\tau_1 s + 1)(\tau_2 s + 1) + K} T_d(s)$$

When  $T_d(s) = 1/s$ , using the final value theorem we find that

$$\lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{s(\tau_1 s + 1)}{s(\tau_1 s + 1)(\tau_2 s + 1) + K} = 0$$

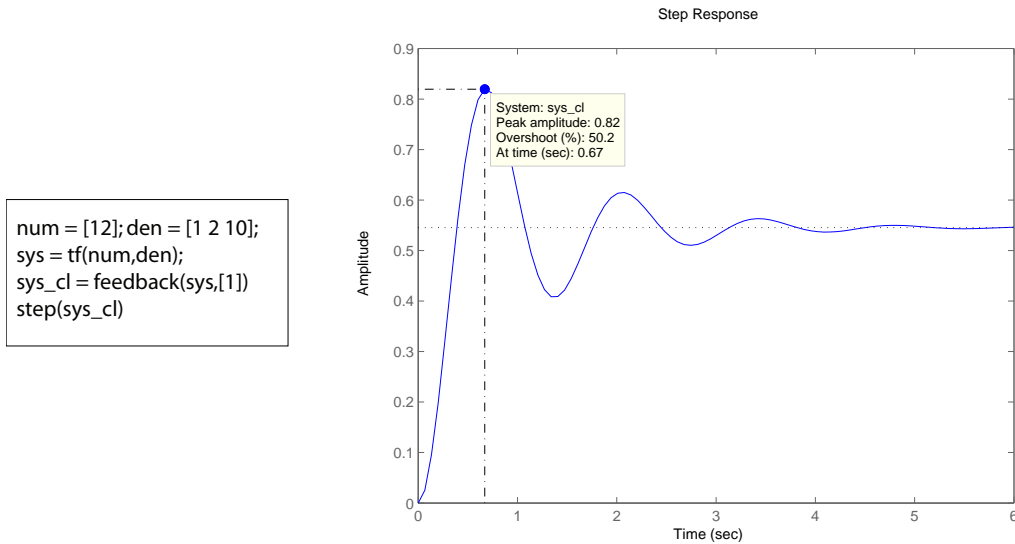
as long as  $K \neq 0$ . We assume here that final value theorem applies (i.e., the system is stable, more on this in Chapter 6).

## Computer Problems

**CP4.1** The step response and an m-file script which generates the step response is shown in Figure CP4.1. The closed-loop transfer function is

$$T(s) = \frac{12}{s^2 + 2s + 22}.$$

The percent overshoot is  $P.O. = 50.2\%$  and the steady-state error is  $e_{ss} = 0.45$ .



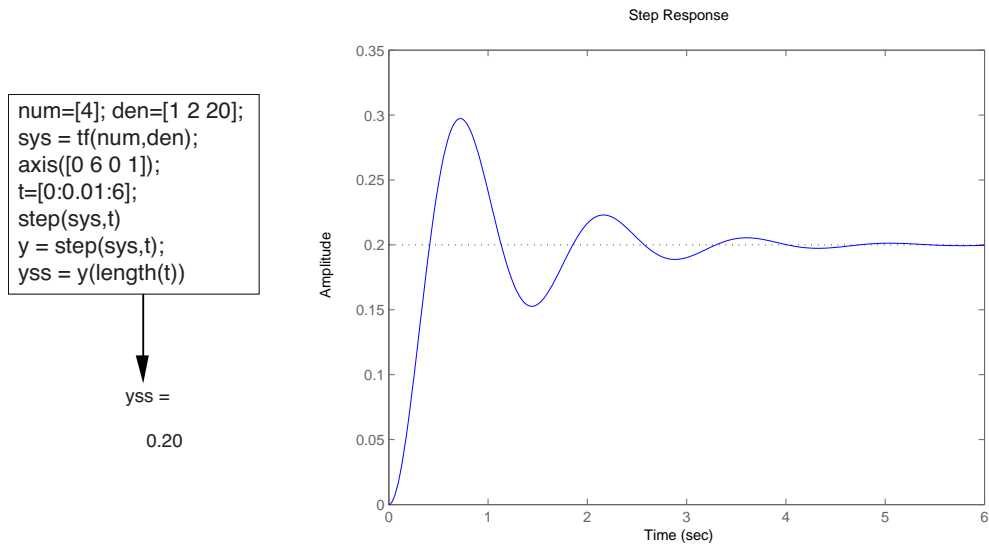
**FIGURE CP4.1**  
Step response.

**CP4.2** The transfer function is

$$G(s) = \frac{4}{s^2 + 2s + 20}.$$

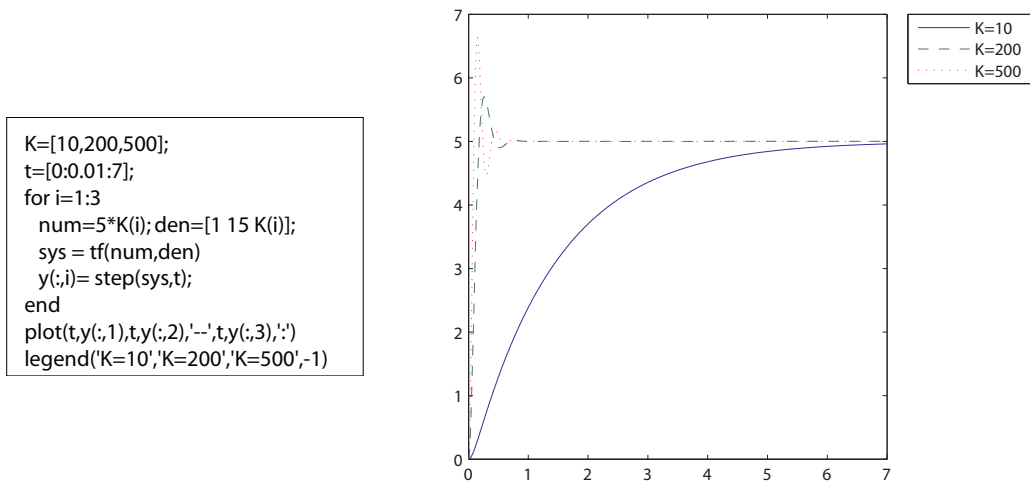
An m-file script which generates the step response is shown in Figure CP4.2. The step response is also shown in Figure CP4.2. The step response is generated using the **step** function. In the script, the transfer function numerator is represented by **num** and the denominator is represented by **den**. The steady-state value is  $y_{ss} = 0.2$  and the desired value is 1.0. Therefore, the steady-state error is

$$e_{ss} = 0.8.$$



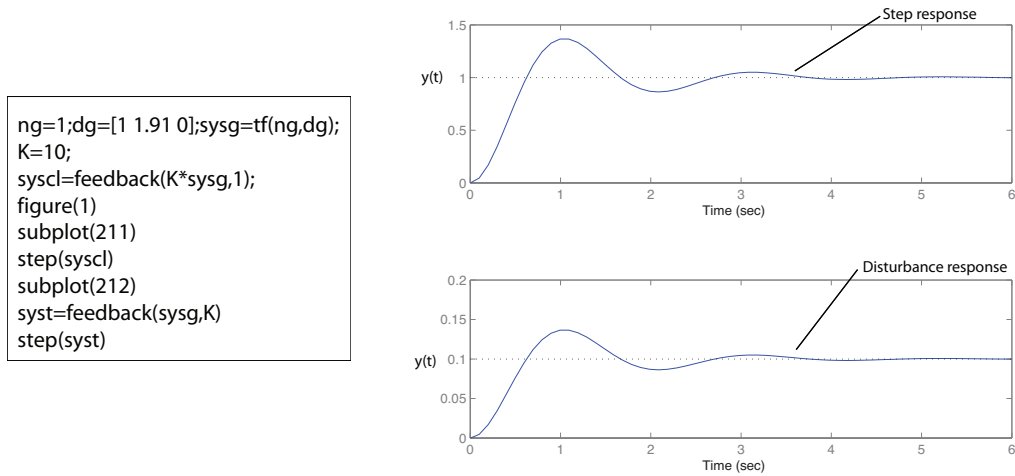
**FIGURE CP4.2**  
Step response.

**CP4.3** The step responses and the m-file script which generates the step responses is shown in Figure CP4.3.



**FIGURE CP4.3**  
Step responses for  $K = 10, 100, 500$ .

**CP4.4** (a,b) The m-file and plots are shown in Figure CP4.4.



**FIGURE CP4.4**  
Step response and disturbance response.

- (c) The estimated steady-state tracking error due to a unit step input is zero, and the estimated steady-state tracking error to a unit disturbance is 0.1.
- (d) The estimated maximum tracking error due to a unit step input is 0.4, and the estimated maximum tracking error to a unit disturbance is 0.14. The maximum occurs at approximately  $t = 1$  s.

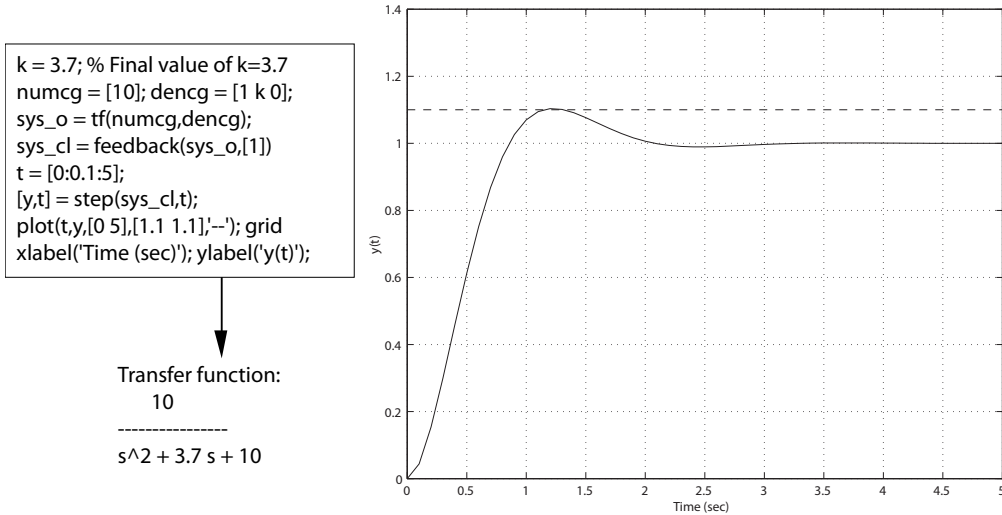
**CP4.5** The step response and the m-file script which generates the step response is shown in Figure CP4.5. The closed-loop transfer function is determined to be

$$T(s) = \frac{10}{s^2 + 3.7s + 10} .$$

Using the m-file script, a trial-and-error search on  $k$  yields

$$k = 3.7 .$$

The percent overshoot  $P.O. = 10\%$  and the steady-state value is 1, as expected.



**FIGURE CP4.5**  
Step response.

**CP4.6** The closed-loop transfer function is

$$T(s) = \frac{K}{s - a + K}$$

where  $K = 2$ . When  $a = 1$  and  $R(s) = 1/s$ , the final value is

$$\lim_{s \rightarrow 0} sT(s)R(s) = \lim_{s \rightarrow 0} T(s) = \frac{K}{K - a} = 2 .$$

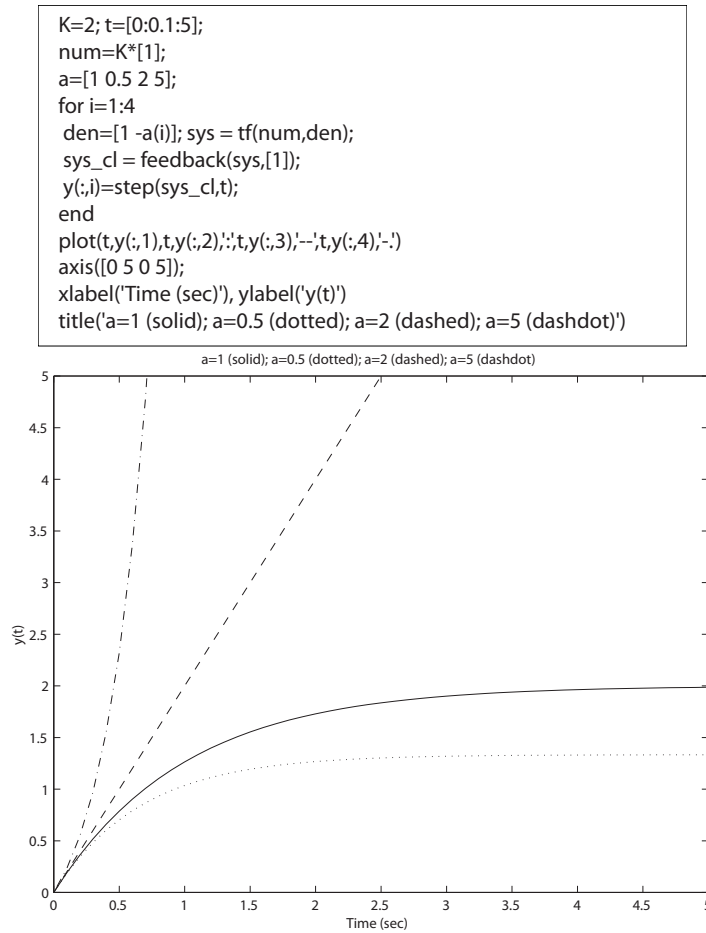
The output is within 2% of the final value at around  $t = 4.6$  seconds. The plot of the step responses for

$$a = 1, 0.5, 2, 5$$

is shown in Figure CP4.6. The output is unstable for

$$a > 2.$$



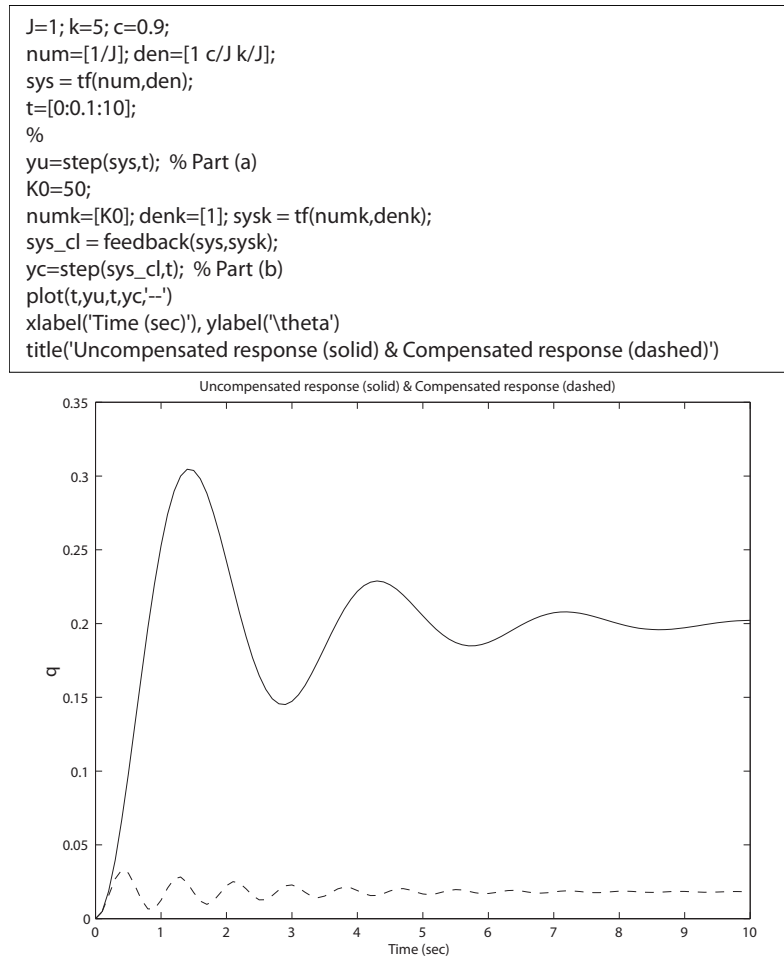


**FIGURE CP4.6**  
Step response for  $a=1, 0.5, 2,$  and  $5.$

**CP4.7** The transfer function from the disturbance to the output is

$$T(s) = \frac{G(s)}{1 + K_0 G(s)} = \frac{1}{Js^2 + bs + k + K_0}.$$

The disturbance response is shown in Figure CP4.7. The compensated system response is significantly reduced from the uncompensated system response. The compensated system output is about 11 times less than the uncompensated system output. So, closed-loop feedback has the advantage of reducing the effect of unwanted disturbances on the output.

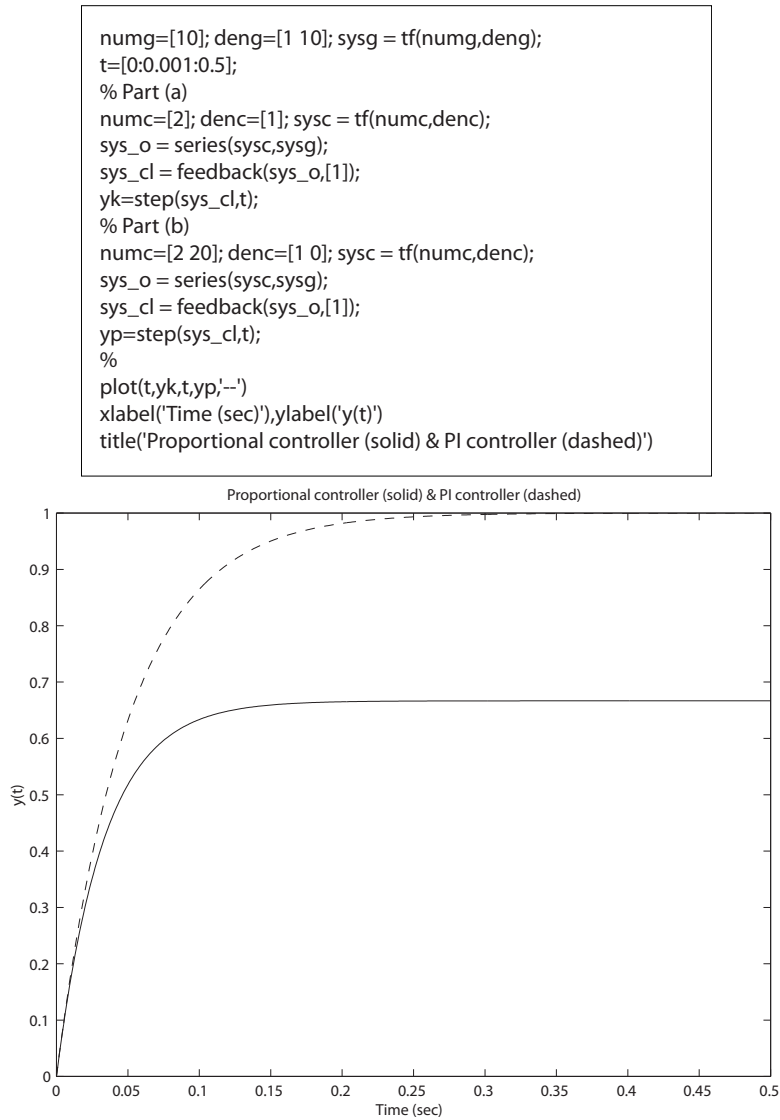
**FIGURE CP4.7**

Disturbance responses for both the uncompensated and compensated systems.

**CP4.8** The step responses for the proportional and PI controller are shown in Figure CP4.8. The steady-state tracking error for the proportional controller is

$$e_{ss} = 0.33 .$$

Increasing the complexity of the controller from a proportional controller to a proportional plus integral (PI) controller allows the closed-loop system to track the unit step response with zero steady-state error. The cost is controller complexity, which translates into higher costs (\$).



**FIGURE CP4.8** Step response for proportional controller and PI controller.

**CP4.9** (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} R(s) = \frac{10s^2 + 500s}{s^2 + 200s + 5000} R(s) .$$

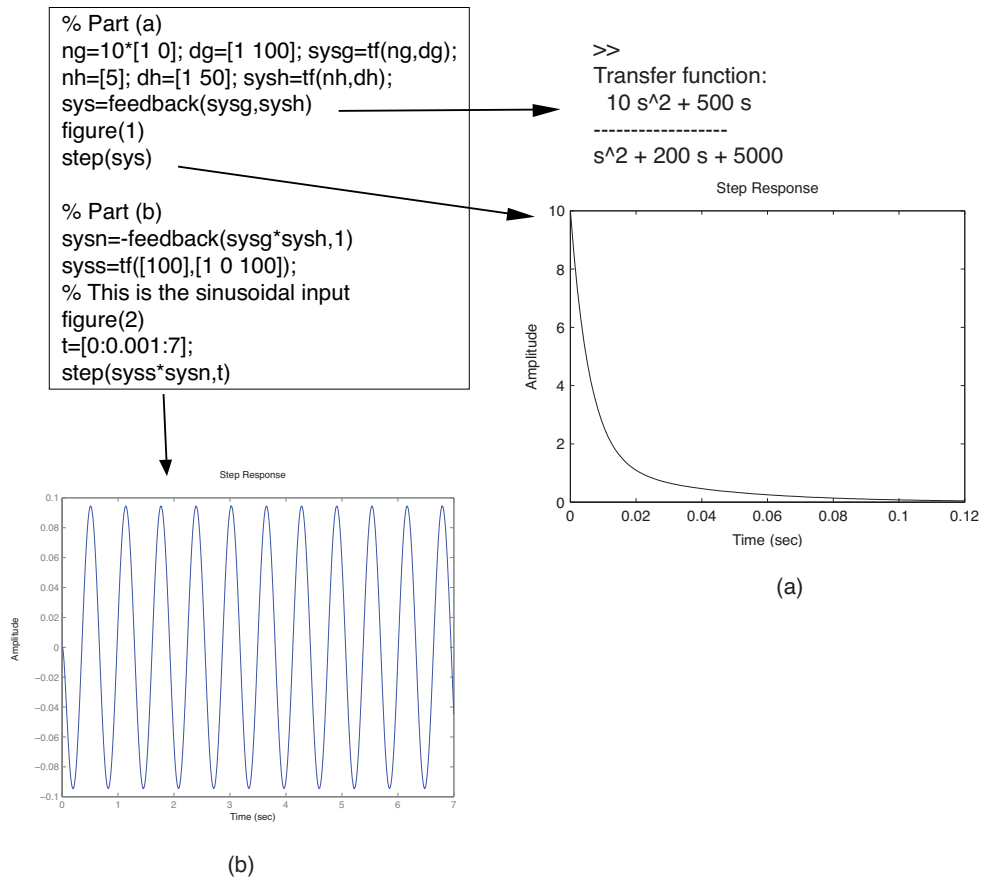
The step response is shown in Figure CP4.9a.

(b) The response of the system to the sinusoidal disturbance

$$N(s) = \frac{100}{s^2 + 100}$$

is shown in Figure CP4.9b.

(c) In the steady-state, the magnitude of the peak response is 0.095 and the frequency is 10 rad/sec (see Figure CP4.9b).



**FIGURE CP4.9**

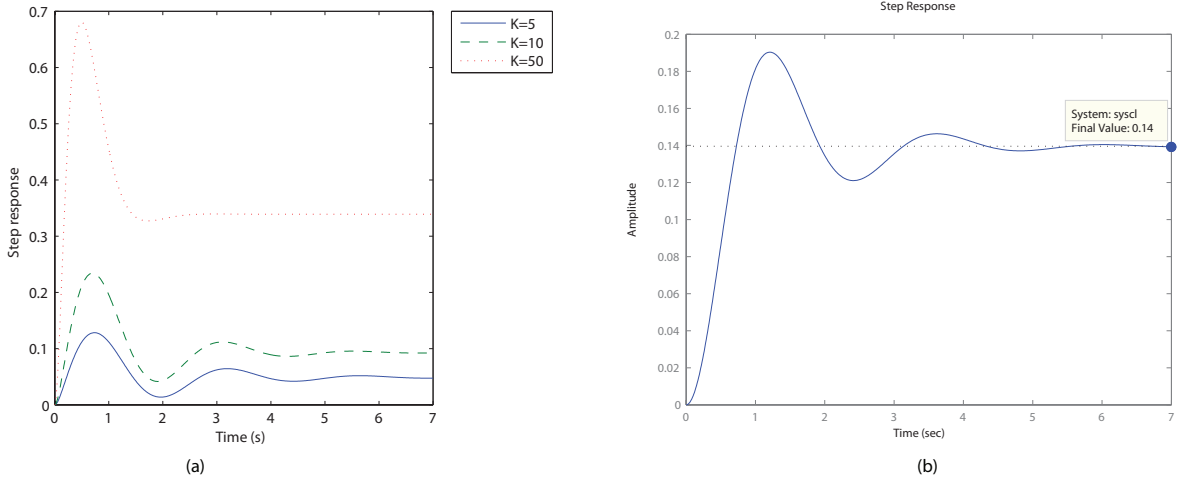
(a) Unit step response. (b) Response to sinusoidal noise input at  $\omega = 10$  rad/sec.

**CP4.10** (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G(s)G_c(s)}R(s) = \frac{K(s + 1)}{(s + 15)(s^2 + s + 6.5) + K(s + 1)}R(s).$$

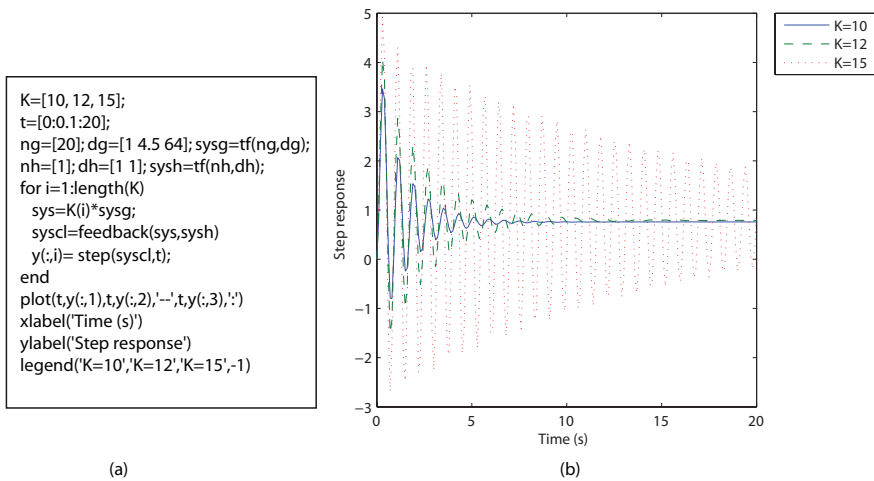
(b) The step responses are shown in Figure CP4.10a.

(c) The unit disturbance response of the system is shown in Figure CP4.10b. The steady-state value is 0.14.



**FIGURE CP4.10** (a) Unit step responses for  $K = [5, 10, 50]$ . (b) Disturbance unit step response.

**CP4.11** The m-file is shown in Figure CP4.11a and the step responses in Figure CP4.11b.



**FIGURE CP4.11** (a) M-file script. (b) Unit step responses for  $K = [10, 12, 15]$ .

# The Performance of Feedback Control Systems

## Exercises

**E5.1** For a zero steady-state error, when the input is a step we need one integration, or a type 1 system. A type 2 system is required for  $e_{ss} = 0$  for a ramp input.

**E5.2** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{240}{(s + 4)(s + 6) + 240} = \frac{240}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The steady-state error is given by

$$e_{ss} = \frac{A}{1 + K_p},$$

where  $R(s) = A/s$  and

$$K_p = \lim_{s \rightarrow 0} G(s) = \frac{240}{24} = 10.$$

Therefore,

$$e_{ss} = \frac{A}{11}.$$

(b) The closed-loop system is a second-order system with natural frequency

$$\omega_n = \sqrt{264},$$

and damping ratio

$$\zeta = \frac{10}{2\sqrt{264}} = 0.31 .$$

The percent overshoot is thus computed to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 36\% .$$

**E5.3** The closed-loop transfer function is

$$\frac{Y(s)}{I(s)} = \frac{G(s)}{1 + G(s)} = \frac{K}{s(s + 14) + K} = \frac{K}{s^2 + 14s + K} .$$

Utilizing Table 5.6 in Dorf & Bishop, we find that the optimum coefficients are given by

$$s^2 + 1.4\omega_n s + \omega_n^2 .$$

We have

$$s^2 + 14s + K ,$$

so equating coefficients yields  $\omega_n = 10$  and  $K = \omega_n^2 = 100$  . We can also compute the damping ratio as

$$\zeta = \frac{14}{2\omega_n} = 0.7 .$$

Then, using Figure 5.8 in Dorf & Bishop, we find that  $P.O. \approx 5\%$ .

**E5.4** (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{2(s + 8)}{s^2 + 6s + 16} .$$

(b) We can expand  $Y(s)$  in a partial fraction expansion as

$$Y(s) = \frac{2(s + 8)}{(s^2 + 6s + 16)} \frac{A}{s} = A \left( \frac{1}{s} - \frac{s + 4}{s^2 + 6s + 16} \right) .$$

Taking the inverse Laplace transform (using the Laplace transform tables), we find

$$y(t) = A[1 - 1.07e^{-3t} \sin(\sqrt{7}t + 1.21)] .$$

(c) Using the closed-loop transfer function, we compute  $\zeta = 0.75$  and

$\omega_n = 4$ . Thus,

$$\frac{a}{\zeta\omega_n} = \frac{8}{3} = 2.67 ,$$

where  $a = 8$ . From Figure 5.13(a) in Dorf & Bishop, we find (approximately) that  $P.O. = 4\%$ .

(d) This is a type 1 system, thus the steady-state error is zero and  $y(t) \rightarrow A$  as  $t \rightarrow \infty$ .

**E5.5** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 4s + K} .$$

Utilizing Table 5.6 in Dorf & Bishop, we find that the optimum coefficients are given by

$$s^2 + 1.4\omega_n s + \omega_n^2 .$$

We have

$$s^2 + 4s + K ,$$

so equating coefficients yields  $\omega_n = 2.86$  and  $K = \omega_n^2 = 8.16$ . We can also compute the damping ratio as

$$\zeta = \frac{4}{2\omega_n} = 0.7 .$$

Then, using Figure 5.8 in Dorf & Bishop, we find that  $P.O. \approx 5\%$ .

**E5.6** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + GH(s)} = \frac{100}{s^2 + 100Ks + 100} ,$$

where  $H(s) = 1 + Ks$  and  $G(s) = 100/s^2$ . The steady-state error is computed as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s[R(s) - Y(s)] = \lim_{s \rightarrow 0} s[1 - T(s)] \frac{A}{s^2} \\ &= \lim_{s \rightarrow 0} \left[ 1 - \frac{\frac{100}{s^2}}{1 + \frac{100}{s^2}(1 + Ks)} \right] \frac{A}{s} = KA . \end{aligned}$$

(b) From the closed-loop transfer function,  $T(s)$ , we determine that  $\omega_n =$



10 and

$$\zeta = \frac{100K}{2(10)} = 5K .$$

We want to choose  $K$  so that the system is critically damped, or  $\zeta = 1.0$ . Thus,

$$K = \frac{1}{5} = 0.20 .$$

The closed-loop system has no zeros and the poles are at

$$s_{1,2} = -50K \pm 10\sqrt{25K^2 - 1} .$$

The percent overshoot to a step input is

$$P.O. = 100e^{\frac{-5\pi K}{\sqrt{1-25K^2}}} \quad \text{for } 0 < K < 0.2$$

and  $P.O. = 0$  for  $K > 0.2$ .

**E5.7** The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{K(s+2)}{s(s+1) + K(s+2)} = \frac{K(s+2)}{s^2 + s(K+1) + 2K} .$$

Therefore,  $\omega_n = \sqrt{2K}$  and  $\zeta = \frac{K+1}{2\sqrt{2K}}$ . So,

$$\frac{a}{\zeta\omega_n} = \frac{4}{K+1} .$$

From Figure 5.13a in Dorf & Bishop, we determine that

$$\frac{a}{\zeta\omega_n} \approx 1.5$$

when  $\zeta = 0.707$ . Thus, solving for  $K$  yields

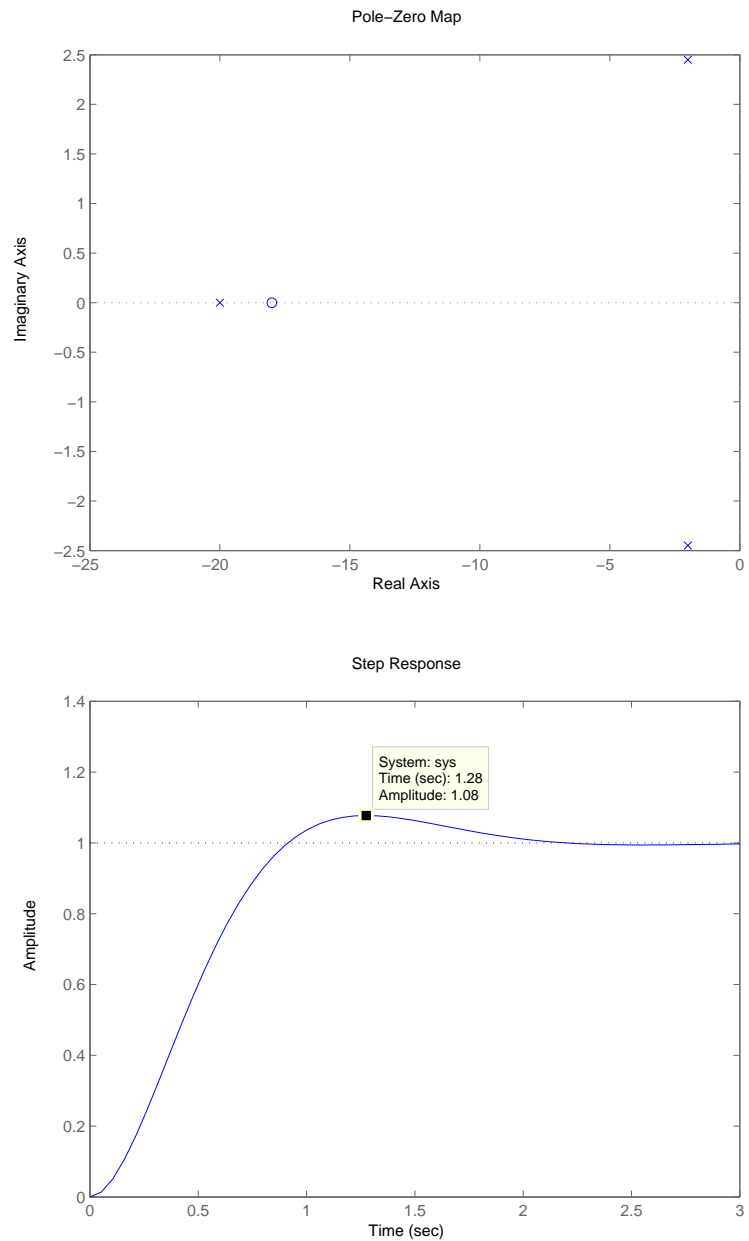
$$\frac{4}{K+1} = 1.5$$

or  $K = 1.67$ .

**E5.8** The pole-zero map is shown in Figure E5.8. Since the dominant poles are at  $s = -2 \pm 2.45i$  we have a damping ratio  $\zeta = 0.63$ . We estimate the percent overshoot to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 7.69\%$$

The step response is shown in Figure E5.8b. The actual overshoot is 8%.

**FIGURE E5.8**

(a) Pole-zero map. (b) Unit step response.

**E5.9** (a) The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + \sqrt{2K}s + K} .$$

The damping ratio is  $\zeta = \sqrt{2}/2$  and the natural frequency is  $\omega_n = \sqrt{K}$ . Therefore, we compute the percent overshoot to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 4.3\%$$

for  $\zeta = 0.707$ . The settling time is estimated via

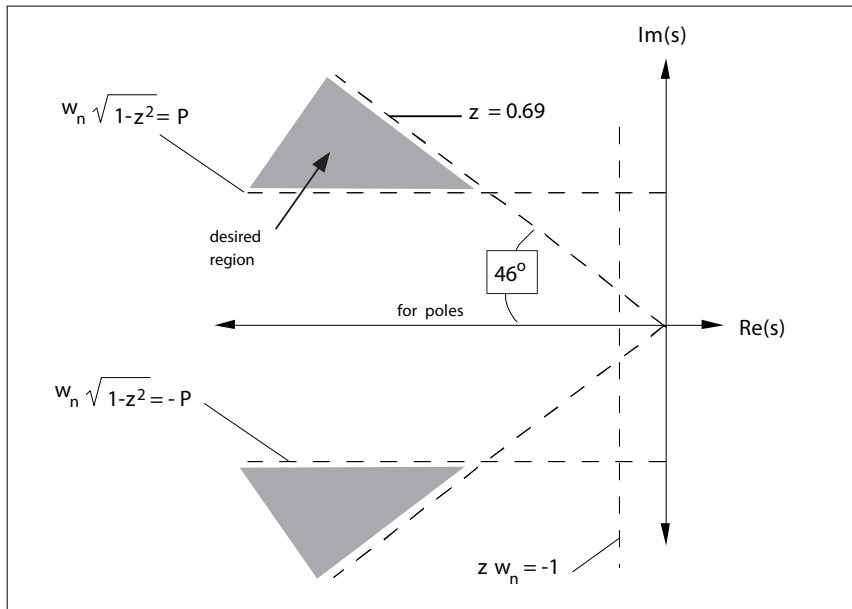
$$T_s = \frac{4}{\zeta\omega_n} = \frac{8}{\sqrt{2K}}.$$

(b) The settling time is less than 1 second whenever  $K > 32$ .

**E5.10** The second-order closed-loop transfer function is given by

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

From the percent overshoot specification, we determine that  $P.O. \leq 5\%$  implies  $\zeta \geq 0.69$ . From the settling time specification, we find that  $T_s < 4$  implies  $\omega_n\zeta > 1$ . And finally, from the peak time specification we have  $T_p < 1$  implies  $\omega_n\sqrt{1-\zeta^2} > \pi$ . The constraints imposed on  $\zeta$  and  $\omega_n$  by the performance specifications define the permissible area for the poles of  $T(s)$ , as shown in Figure E5.10.



**FIGURE E5.10**  
Permissible area for poles of  $T(s)$ .

**E5.11** The system is a **type 1**. The error constants are

$$K_p = \infty \quad \text{and} \quad K_v = 1.0 .$$

Therefore, the steady-state error to a step input is 0; the steady-state error to a ramp input is  $1.0A_0$ , where  $A_0$  is the magnitude (slope) of the ramp input.

**E5.12** (a) The tracking error is given by

$$E(s) = \frac{R(s)}{1 + G_c G(s)} = \frac{(s+9)(s+2)(s+4)}{(s+9)(s+2)(s+4) + K(s+6)} R(s) .$$

The steady-state tracking error (with  $R(s) = 1/s$ ) is

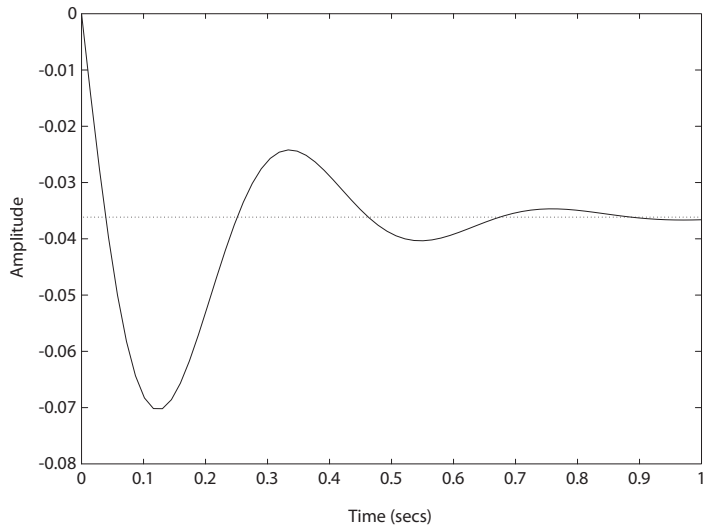
$$\lim_{s \rightarrow 0} sE(s) = \frac{72}{72 + 6K} .$$

We require  $e_{ss} < 0.05$ , so solving for  $K$  yields  $K > 228$ .

(b) The tracking error due to the disturbance is

$$E(s) = \frac{-G(s)}{1 + G_c G(s)} T_d(s) = \frac{-(s+9)(s+6)}{(s+9)(s+2)(s+4) + K(s+6)} T_d(s) .$$

The tracking error is shown in Figure E5.12.



**FIGURE E5.12**  
Tracking error due a step disturbance.

**E5.13** The system is a **type 0**. The error constants are  $K_p = 0.4$  and  $K_v = 0$ . The steady-state error to a ramp input is  $\infty$ . The steady-state error to a step input is

$$e_{ss} = \frac{1}{1 + K_p} = 0.71.$$

**E5.14** (a) The tracking error is given by

$$E(s) = [1 - T(s)] R(s).$$

The steady-state tracking error (with  $R(s) = 1/s$ ) is

$$e_{ss} = \lim_{s \rightarrow 0} s [1 - T(s)] R(s) = \lim_{s \rightarrow 0} [1 - T(s)] = 1 - T(0).$$

The closed-loop transfer function is

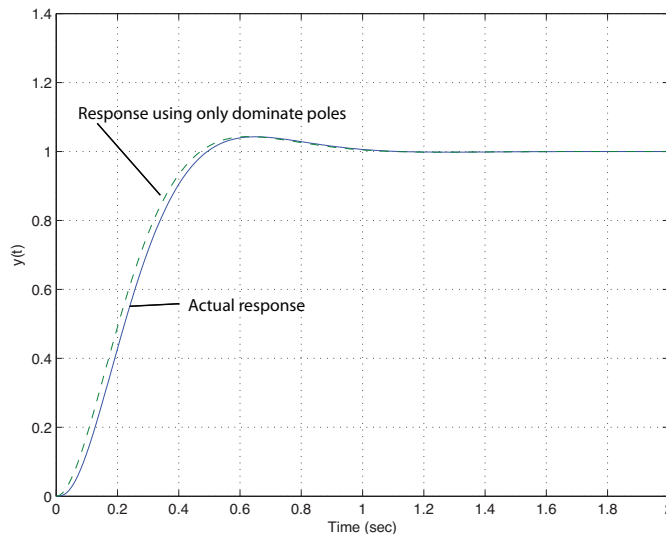
$$T(s) = \frac{K(s + 0.1)}{s(s + 0.1)(s + 2) + K(s + 3)},$$

and  $T(0) = 0.033$ . Therefore,  $e_{ss} = 1 - T(0) = 0.967$ .

(b) Use  $G_p(s) = 30$ . Then,

$$e_{ss} = \lim_{s \rightarrow 0} s [1 - T(s)G_p(s)] R(s) = 1 - \lim_{s \rightarrow 0} T(s)G_p(s) = 1 - 30 T(0) = 0.$$

**E5.15** The plot of  $y(t)$  is shown in Figure E5.15.



**FIGURE E5.15**

Plot of  $y(t)$  with  $T(s)$  (solid line) and approximate  $T_a(s)$  (dashed line).

Using the dominant poles, the approximate closed-loop transfer function is

$$T_a(s) = \frac{50}{s^2 + 10s + 50} .$$

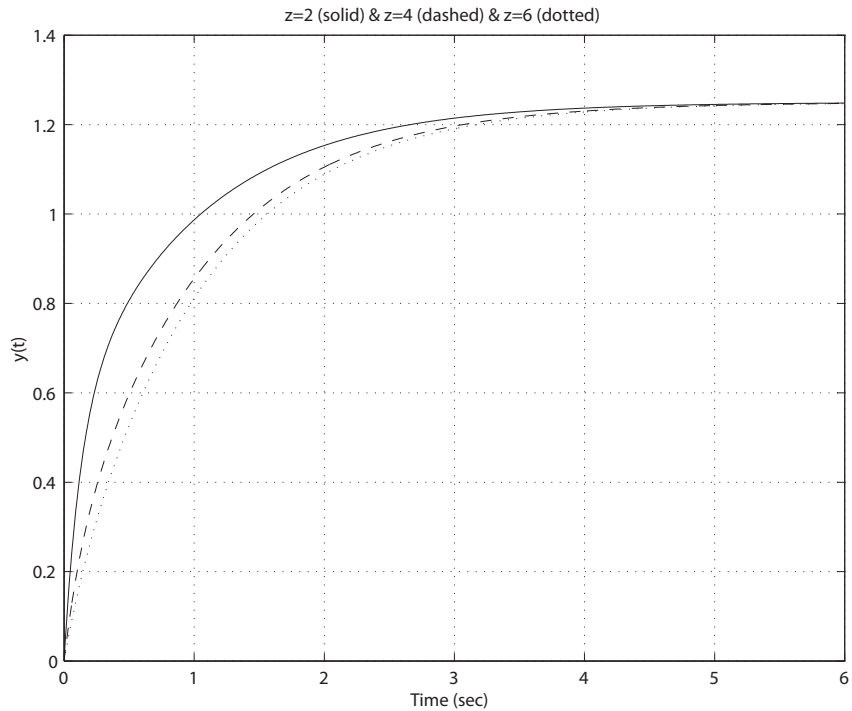
The actual transfer function is

$$T(s) = \frac{500}{(s + 10)(s^2 + 10s + 50)} .$$

**E5.16** The partial fraction expansion is

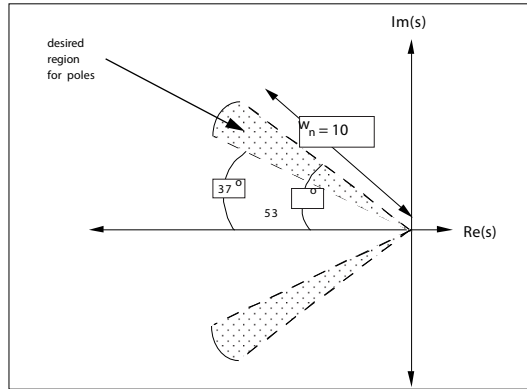
$$y(t) = -\frac{10(z-1)}{7z} e^{-t} + \frac{10(z-8)}{56z} e^{-8t} + 1.25 .$$

The plot of  $y(t)$  for  $z = 2, 4, 6$  is shown in Figure E5.16.

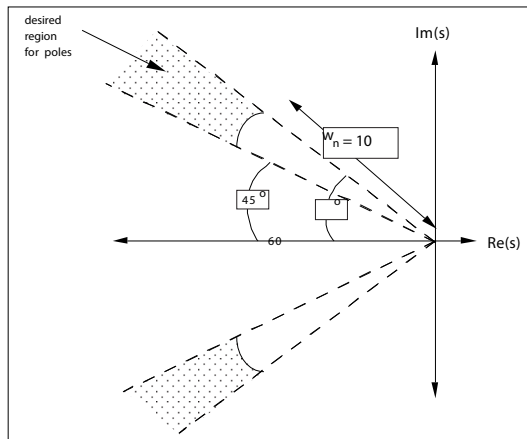


**FIGURE E5.16**  
Plot of  $y(t)$  for  $z=2, 4, 6$ .

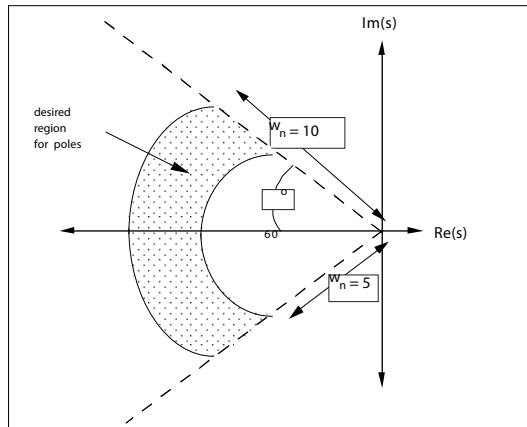
**E5.17** The desired pole locations for the 5 different cases are shown in Figure E5.17.



(a)  $0.6 < z < 0.8$  and  $w_n < 10$

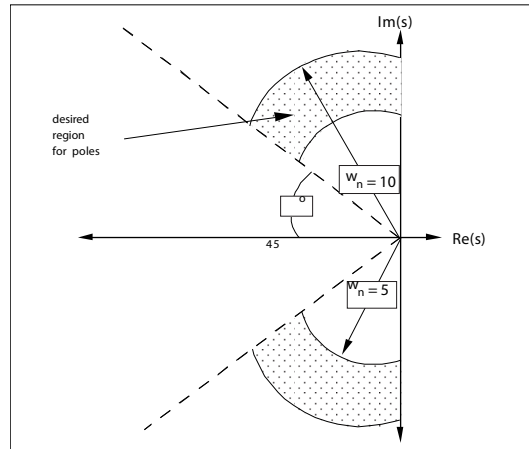
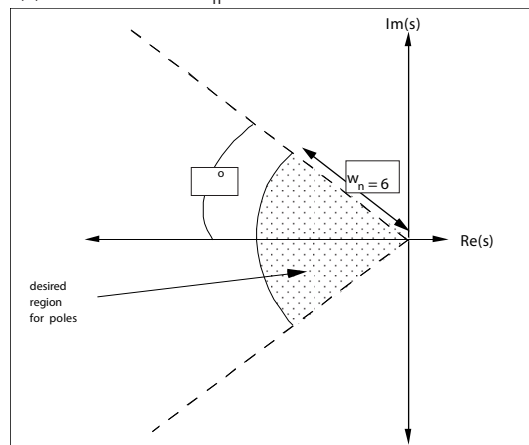


(b)  $0.5 < z < 0.707$  and  $w_n > 10$



(c)  $0.5 < z$  and  $5 < w_n < 10$

**FIGURE E5.17**  
Desired pole locations.

(d)  $0.707 > z$  and  $5 < w_n < 10$ (e)  $0.6 < z$  and  $w_n < 6$ 

**FIGURE E5.17**  
CONTINUED: Desired pole locations.

**E5.18** The output is given by

$$Y(s) = T(s)R(s) = K \frac{G(s)}{1 + G(s)} R(s) .$$

When  $K = 1$ , the steady-state error is

$$e_{ss} = 0.2$$

which implies that

$$\lim_{s \rightarrow 0} sY(s) = 0.8 .$$



Since we want  $e_{ss} = 0$ , it follows that

$$\lim_{s \rightarrow 0} sY(s) = 1 ,$$

or

$$0.8K = 1 .$$

Therefore,  $K = 1.25$ .

**E5.19** (a) The characteristic equation is

$$s^2 = 2\zeta\omega_n s + \omega_n^2 = s^2 + 3.17s + 7 = 0 ,$$

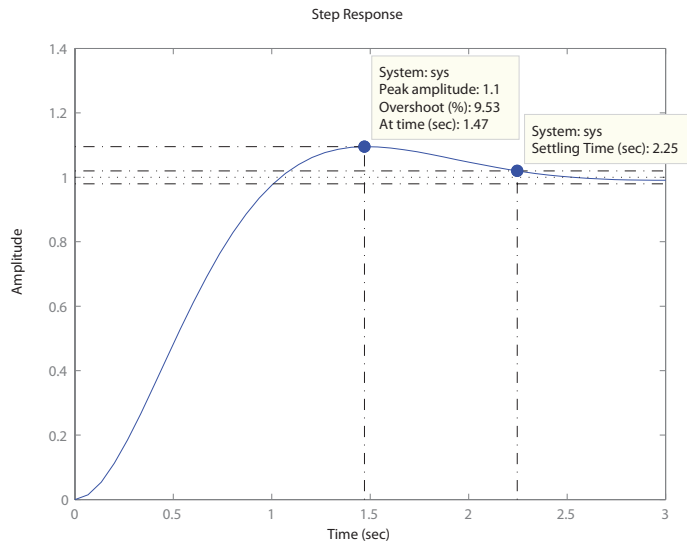
from which it follows that

$$\omega_n = \sqrt{7} = 2.65, \quad \zeta = \frac{3.17}{2\omega_n} = 0.6 .$$

Therefore, we compute the percent overshoot and the estimated settling time to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 9.53\% \quad \text{and} \quad T_s = \frac{4}{\zeta\omega_n} = 2.5 \text{ s} .$$

(b) The unit step response is shown in Figure E5.19.



**FIGURE E5.19**  
Unit step response.

**E5.20** (a) The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + \sqrt{2K}s + K} .$$

The damping ratio is

$$\zeta = \frac{\sqrt{2}}{2}$$

and the natural frequency is  $\omega_n = \sqrt{K}$ . Therefore, we compute the percent overshoot to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 4.3\%$$

for  $\zeta = 0.707$ . The settling time is estimated via

$$T_s = \frac{4}{\zeta\omega_n} = \frac{8}{\sqrt{2K}} .$$

(b) The settling time is less than 1 second whenever  $K > 32$ .

## Problems

**P5.1** (a) The system error is

$$E(s) = \frac{1}{1 + \frac{K_a K_m}{s\tau_m + 1}} R(s)$$

where

$$R(s) = \frac{25^\circ/sec}{s} .$$

So,

$$\lim_{t \rightarrow 0} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{25}{1 + K_a K_m} .$$

(b) If we desire  $e_{ss} \leq 1^\circ/sec$ , then

$$\frac{25^\circ/s}{1 + K_a K_m} \leq 1^\circ/sec ,$$

and solving for  $K_a K_m$  yields

$$K_a K_m \geq 24 .$$

(c) The closed-loop transfer function is

$$T(s) = \frac{V_b(s)}{V_c(s)} = \frac{K_a K_m}{s\tau_m + 1 + K_a K_m} .$$

The step response of the system (i.e.  $v_c(t) = A$ ) is

$$v_b(t) = \frac{AK_a K_m}{1 + K_a K_m} \left( 1 - e^{-\frac{(K_a K_m + 1)t}{\tau_m}} \right) .$$

So, at settling time, we have

$$1 - e^{-\frac{(1 + K_a K_m)t}{\tau_m}} \geq 0.98 ,$$

where  $\tau_m = 0.4$ . Setting  $t = 0.03$  and solving for  $K_a K_m$  yields

$$K_a K_m \geq 52 .$$

**P5.2** (a) The settling time specification

$$T_s = \frac{4}{\zeta\omega_n} < 0.6$$

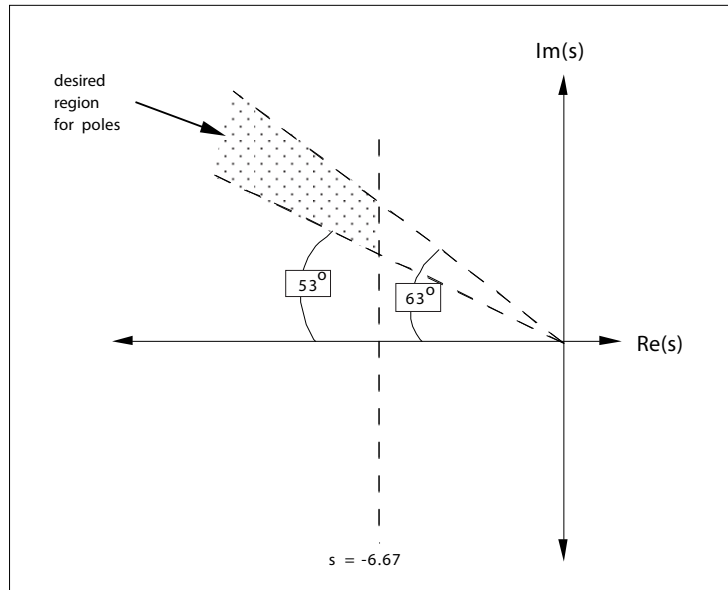
is used to determine that  $\zeta\omega_n > 6.67$ . The  $P.O. < 20\%$  requirement is used to determine

$$\zeta < 0.45 \quad \text{which implies} \quad \theta < 63^\circ$$

and the  $P.O. > 10\%$  requirement is used to determine

$$\zeta > 0.60 \quad \text{which implies} \quad \theta > 53^\circ,$$

since  $\cos\theta = \zeta$ . The desired region for the poles is shown in Figure P5.2.



**FIGURE P5.2**

Desired region for pole placement.

- (b) The third root should be at least 10 times farther in the left half-plane, so

$$|r_3| \geq 10|\zeta\omega_n| = 66.7 .$$

- (c) We select the third pole such that  $r_3 = -66.7$ . Then, with  $\zeta = 0.45$  and  $\zeta\omega_n = 6.67$ , we determine that  $\omega_n = 14.8$ . So, the closed-loop transfer function is

$$T(s) = \frac{66.7(219.7)}{(s + 66.7)(s^2 + 13.3s + 219.7)} ,$$

where the gain  $K = (66.7)(219.7)$  is chosen so that the steady-state

tracking error due to a step input is zero. Then,

$$T(s) = \frac{G(s)}{1 + G(s)},$$

or

$$G(s) = \frac{T(s)}{1 - T(s)}.$$

**P5.3** Given the input

$$R(s) = \frac{1}{s^3},$$

we compute the steady-state error as

$$e_{ss} = \lim_{s \rightarrow 0} s \left( \frac{1}{1 + G(s)} \right) \frac{1}{s^3} = \lim_{s \rightarrow 0} \left( \frac{1}{s^2 G(s)} \right) = \lim_{s \rightarrow 0} \left( \frac{1}{s^2 \left( \frac{K}{s^2} \right)} \right) = \frac{1}{K}.$$

Since we require that  $e_{ss} \leq 0.5$  cm, we determine

$$K \geq 2.$$

**P5.4** (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{s^2 + 2s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Thus,

$$\omega_n = \sqrt{K} \quad \text{and} \quad \zeta = 1/\omega_n = 1/\sqrt{K}.$$

Our percent overshoot requirement of 5% implies that  $\zeta = 1/\sqrt{2}$ , which in turn implies that  $\omega_n = \sqrt{2}$ . However, the corresponding time to peak would be

$$T_p = \frac{4.4}{\sqrt{2}} = 3.15.$$

Our desired  $T_p = 1.1$ —we cannot meet both specification simultaneously.

(b) Let  $T_p = 1.1\Delta$  and  $P.O. = 0.05\Delta$ , where  $\Delta$  is the relaxation factor to be determined. We have that  $K = \omega_n^2$  and  $\zeta\omega_n = 1$ , so

$$\zeta = \frac{1}{\sqrt{K}}.$$

Thus,

$$P.O. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi/\sqrt{K-1}} .$$

Also,

$$T_p = \frac{\pi}{\sqrt{K-1}} = 1.1\Delta .$$

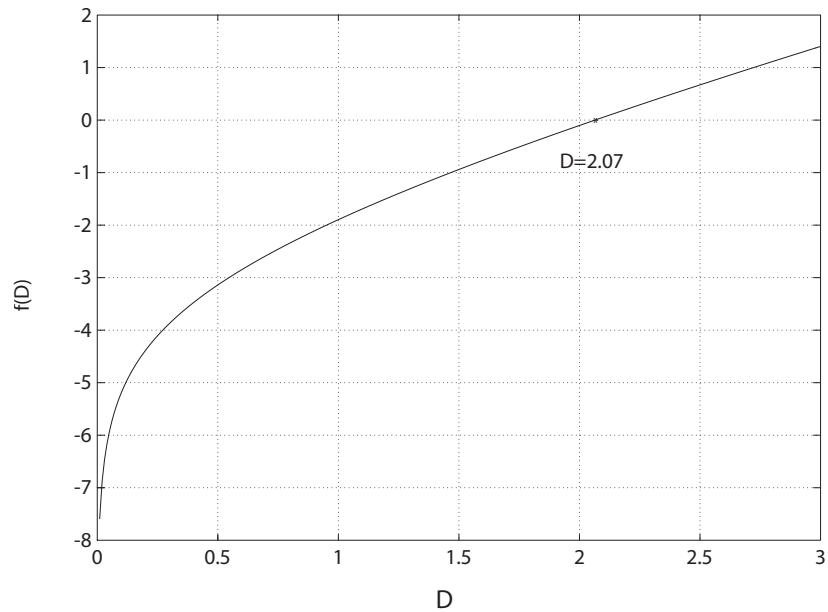
Therefore, from the proceeding two equations we determine that

$$P.O. = 0.05\Delta = e^{-1.1\Delta} .$$

Solving for  $\Delta$  yields

$$f(\Delta) = \ln \Delta + \ln(0.05) + 1.1\Delta = 0 .$$

The plot of  $f(\Delta)$  versus  $\Delta$  is shown in Figure P5.4. From the plot we



**FIGURE P5.4**  
Solving for the zeros of  $f$ .

see that  $\Delta = 2.07$  results in  $f(\Delta) = 0$ . Thus,

$$\begin{aligned} P.O. &= 0.05\Delta = 10\% \\ T_p &= 1.1\Delta = 2.3 \text{ sec.} \end{aligned}$$

So, we can meet the specifications if they are relaxed by a factor of about 2 (i.e.  $\Delta = 2.07$ ).

**P5.5** (a) The closed-loop transfer function is

$$T(s) = \frac{K_1 K_2 (s + 1)}{s^2 + K_1 K_2 s + K_1 K_2} .$$

A percent overshoot less than 5% implies  $\zeta \geq 0.69$ . So, choose  $\zeta = 0.69$ . Then set  $2\zeta\omega_n = K_1 K_2$  and  $\omega_n^2 = K_1 K_2$ . Then

$$2(0.69)\omega_n = \omega_n^2 ;$$

and solving for  $\omega_n$  yields

$$\omega_n = 1.38 .$$

Therefore  $K_1 K_2 = \omega_n^2 = 1.9$ . When  $K_1 K_2 \geq 1.9$  it follows that  $\zeta \geq 0.69$ .

(b) We have a type 2 system, so the steady-state tracking error to both a step and ramp input is zero.

(c) For a step input, the optimum ITAE characteristic equation is

$$s^2 + 1.4\omega_n s + \omega_n^2 = 0 .$$

For a ramp input, the optimum ITAE characteristic equation is

$$s^2 + 3.2\omega_n s + \omega_n^2 = 0 .$$

Thus,  $K_1 K_2 = \omega_n^2 = 3.2\omega_n$ . So,  $\omega_n = 3.2$  and  $K_1 K_2 = 10.24$ .

**P5.6** We have a ramp input,  $r(t) = t$ . So

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \left[ \frac{75(s + 1)}{s(s + 5)(s + 25)} \right] = \frac{75}{125} = 0.6 ,$$

and

$$e_{ss} = \frac{|R|}{K_v} = \frac{1}{0.6} = 1.67 .$$

**P5.7** (a) The closed-loop transfer function is

$$T(s) = \frac{K_1 K_2}{I s^2 + K_1 K_2 K_3 s + K_1 K_2} .$$

The steady-state tracking error for a ramp input is

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s(1 - T(s))R(s) = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{1}{s^2}$$

$$= \lim_{s \rightarrow 0} \frac{Is + K_1 K_2 K_3}{Is^2 + K_1 K_2 K_3 s + K_1 K_2} = K_3 .$$

But we desire  $e_{ss} = 0.01$  m, so  $K_3 = 0.01$ .

(b) For  $P.O. = 10\%$ , we have  $\zeta = 0.6$ . Also,

$$2\zeta\omega_n = \frac{0.01K_1K_2}{25}$$

and

$$\omega_n^2 = \frac{K_1K_2}{25} .$$

Thus, solving for  $K_1K_2$  yields  $K_1K_2 = 36 \times 10^4$ .

**P5.8** (a) The closed-loop transfer function is

$$T(s) = \frac{P(s)}{R(s)} = \frac{G(s)/s}{1 + G(s)H(s)/s} = \frac{20}{s(s + 40)} .$$

Therefore, the closed-loop system time constant is  $\tau = 1/40$  sec.

(b) The transfer function from  $T_d(s)$  to the output  $P(s)$  is

$$\frac{P(s)}{T_d(s)} = \frac{-G(s)}{1 + G(s)H(s)/s} = \frac{-20}{s + 40} .$$

The response to a unit step disturbance is

$$p(t) = -\frac{1}{2}(1 - e^{-40t}) .$$

At settling time,  $p(t) = 0.98p_{ss} = -0.49$ . Thus, solving for  $t(= T_s)$  we determine that  $T_s = 0.098$  sec.

**P5.9** We need to track at the rate

$$\omega = \frac{v}{r} = \frac{16000}{2500} = 1.78 \times 10^{-3} \text{ radians/sec} .$$

The desired steady-state tracking error is

$$e_{ss} \leq \frac{1}{10} \text{ degree} = 0.1754 \times 10^{-2} \text{ rad} .$$

Therefore, with

$$e_{ss} = \frac{|\omega|}{K_v} ,$$

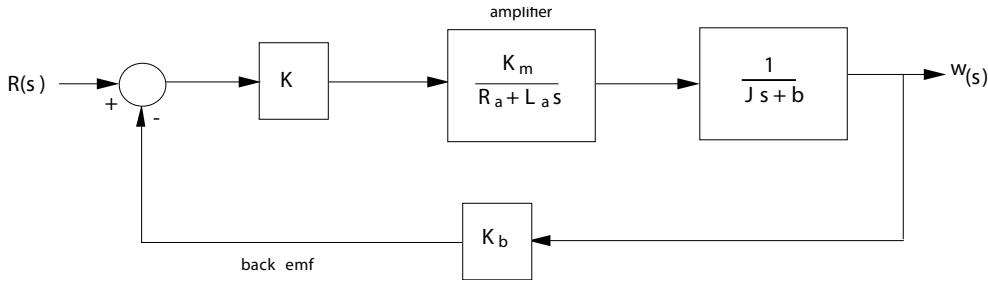


we compute  $K_v$  as

$$K_v = \frac{1.78 \times 10^{-3}}{0.1754 \times 10^{-2}} = 1.02 .$$

This assumes that the system is type 1.

**P5.10** (a) The armature controlled DC motor block diagram is shown in Figure P5.10.



**FIGURE P5.10**  
Armature controlled DC motor block diagram.

(b) The closed-loop transfer function is

$$T(s) = \frac{\omega(s)}{R(s)} = \frac{KG(s)}{1 + KK_bG(s)} ,$$

where

$$G(s) = \frac{K_m}{(R_a + L_a s)(J s + b)} .$$

Thus,

$$T(s) = \frac{K}{s^2 + 2s + 1 + K} ,$$

where  $R_a = L_a = J = b = K_b = K_m = 1$ . The steady-state tracking error is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s(R(s) - Y(s)) = \lim_{s \rightarrow 0} s \left( \frac{A}{s} \right) (1 - T(s)) \\ &= A(1 - T(0)) = \left( 1 - \frac{K}{1 + K} \right) = \frac{A}{1 + K} . \end{aligned}$$

(c) For a percent overshoot of 15%, we determine that  $\zeta = 0.5$ . From our characteristic polynomial we have  $2\zeta\omega_n = 2$  and  $\omega_n = \sqrt{1 + K}$ . Solving for  $\omega_n$  yields  $\omega_n = 2$ , thus  $K = 3$ .

**P5.11** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s + K} .$$

To include the initial condition, we obtain the differential equation:

$$\dot{y}(t) + Ky(t) = Kr(t) .$$

Taking the Laplace transform yields:

$$sY(s) - y(t_o) + KY(s) = K \left( \frac{A}{s} \right) ,$$

where  $y(t_o) = Q$ . Computing the inverse Laplace transform,  $L^{-1}\{Y(s)\}$  yields

$$y(t) = A(1 - e^{-Kt}) + Qe^{-Kt} .$$

Also, the tracking error is given by

$$e(t) = A - y(t) = e^{-Kt}(A - Q) .$$

Thus, the performance index,  $I$  is determined to be (for  $K > 0$ )

$$\begin{aligned} I &= \int_0^{\infty} (A - Q)^2 e^{-2Kt} dt = (A - Q)^2 \left( \frac{1}{-2K} \right) e^{-2Kt} \Big|_0^{\infty} \\ &= \frac{(A - Q)^2}{2K} . \end{aligned}$$

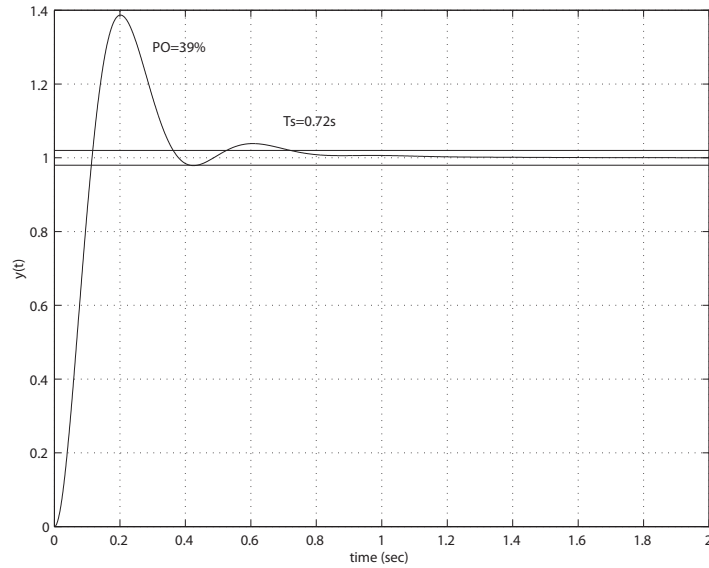
- (b) The minimum  $I$  is obtained when  $K = \infty$ , which is not practical.  
 (c) Set  $K$  at the maximum value allowable such that the process does not saturate. For example, if  $K = 50$ , then

$$I = \frac{(A - Q)^2}{100} .$$

**P5.12** The optimum ITAE transfer function for a ramp input is

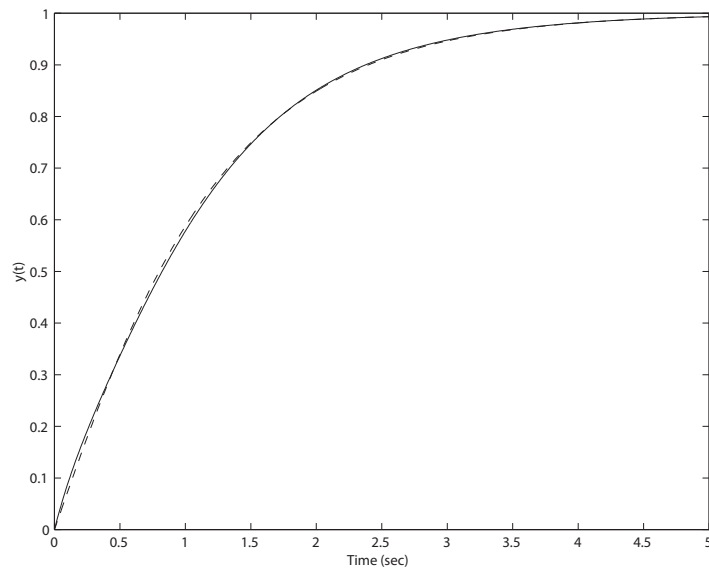
$$T(s) = \frac{3.25\omega_n^2 s + \omega_n^3}{s^3 + 1.75\omega_n s + 3.25\omega_n^2 s + \omega_n^3} .$$

The steady-state tracking error,  $e_{ss} = 0$ , for a ramp input. The step response is shown in Figure P5.12 for  $\omega_n = 10$ . The percent overshoot is  $P.O. = 39\%$ , and the settling time is  $T_s = 0.72$  s .



**FIGURE P5.12**  
Step input system response.

**P5.13** The step responses for the actual system and the approximate system are shown in Figure P5.13. It can be seen that the responses are nearly identical.



**FIGURE P5.13**  
Closed-loop system step response: Actual  $T(s)$  (solid line) and second-order approximation (dashed line).

**P5.14** Consider

$$L(s) = \frac{2(c_1s + 1)}{(s + 1)(s + 2)} .$$

After cancellation of like factors, we compute  $H(s)/L(s)$ ,

$$\frac{H(s)}{L(s)} = \frac{s^3 + 7s^2 + 24s + 24}{(s + 3)(s + 4)2(c_1s + 1)} .$$

Therefore,

$$\begin{aligned} M(s) &= s^3 + 7s^2 + 24s + 24 , \quad \text{and} \\ \Delta(s) &= 2[c_1s^3 + (7c_1 + 1)s^2 + (12c_1 + 7)s + 12] . \end{aligned}$$

Then, following the procedure outlined in Section 5.10, we have

$$M^0(0) = 24, \quad M^1(0) = 24, \quad M^2(0) = 14, \quad M^3(0) = 6, \quad \text{and}$$

$$\Delta^0(0) = 24, \quad \Delta^1(0) = (12c_1 + 7)2, \quad \Delta^2(0) = 2(2 \cdot (7c_1 + 1)), \quad \Delta^3(0) = 12c_1 .$$

For  $q = 1$ :

$$\begin{aligned} M_2 &= 240, \quad \text{and} \\ \Delta_2 &= 4[144c_1^2 + 25] . \end{aligned}$$

Then, equating  $\Delta_2$  and  $M_2$ , we find  $c_1$ ,

$$c_1 = 0.493 .$$

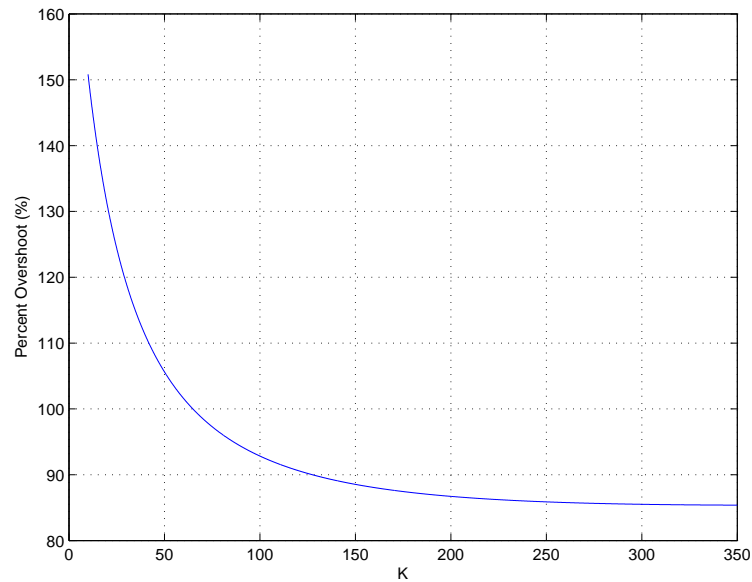
So,

$$L(s) = \frac{2(0.493s + 1)}{(s + 1)(s + 2)} = \frac{0.986s + 2}{s^2 + 3s + 2} = \frac{0.986(s + 2.028)}{(s + 1)(s + 2)} .$$

**P5.15** The closed-loop transfer function is

$$T(s) = \frac{K(s + 1)}{(s + 4)(s^2 + s + 10) + K(s + 1)} .$$

The percent overshoot as function of the gain,  $K$ , is shown in Figure P5.15. It can be seen that the percent overshoot decreases as the gain increases approaching a minimum around 85%. The larger the gain, the smaller the percent overshoot. For a gain  $K \approx 250$ , we have essentially minimized the percent overshoot.



**FIGURE P5.15**  
Percent overshoot versus the gain,  $K$ .

**P5.16** The open-loop transfer function is

$$G(s) = \frac{10}{(s+1)(50Cs+1)}.$$

Define  $\tau = 50C$ . Then, the closed-loop transfer function is

$$\frac{V_o(s)}{V_{in}(s)} = \frac{10}{(s+1)(\tau s+1)+10} = \frac{10/\tau}{s^2 + \left(\frac{\tau+1}{\tau}\right)s + \frac{11}{\tau}}.$$

With

$$\omega_n^2 = \frac{11}{\tau} \quad \text{and} \quad \zeta = \frac{1}{\sqrt{2}} = \frac{\tau+1}{2\tau\omega_n},$$

we can solve for  $\tau$ , yielding

$$\tau^2 - 20\tau + 1 = 0.$$

Therefore,  $\tau = 19.95$  and  $0.05$ . For each value of  $\tau$  we determine  $C$  as follows:  $\tau = 19.95 = 50C$ , implies  $C = 0.399F$ , and  $\tau = 0.05 = 50C$ , implies  $C = 1mF$ .

**P5.17** (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{12K}{s^2 + 12s + 12K} .$$

The percent overshoot specification  $P.O. \leq 10\%$  implies  $\zeta \geq 0.59$ . From the characteristic equation we find that

$$\omega_n^2 = 12K \quad \text{and} \quad \zeta\omega_n = 6 .$$

Solving for  $K$  yields

$$2(0.59)\sqrt{12K} = 12 \quad \text{which implies that} \quad K = 8.6 .$$

So, any gain in the interval  $0 < K < 8.6$  is valid. The settling time is  $T_s = 4/\zeta\omega_n = 4/6$  seconds and satisfies the requirement. Notice that  $T_s$  is not a function of  $K$ .

(b) The sensitivity is

$$S_K^T(s) = \frac{1}{1 + G(s)} = \frac{s(s + 12)}{s^2 + 12s + 120}$$

when  $K = 10$ .

(c) The sensitivity at DC ( $s = 0$ ) is

$$S_K^T(0) = 0 .$$

(d) In this case,  $s = j2\pi \cdot 1 \text{ beat/sec} = j2\pi$ . So, the sensitivity at  $s = 2\pi j$  is

$$|S_K^T(j2\pi)| = \frac{85.1084}{110.31} = 0.77 .$$

**P5.18** We select  $L(s)$  as

$$L(s) = \frac{1}{\alpha s + 1} ,$$

then

$$\frac{H(s)}{L(s)} = \frac{6\alpha s + 6}{s^3 + 6s^2 + 11s + 6} .$$

Therefore,

$$M(s) = 6\alpha s + 6 ,$$

and  $M^o(0) = 6$ ,  $M^1(0) = 6\alpha$ ,  $M^2(0) = 0$ . Also,

$$\Delta(s) = s^3 + 6s^2 + 11s + 6 ,$$

and  $\Delta^o(0) = 6$  ,  $\Delta^1(0) = 11$  ,  $\Delta^2(0) = 12$ . So, computing  $M_2$  and  $\Delta_2$  yields

$$M_2 = 36\alpha^2 , \quad \text{and}$$

$$\Delta_2 = 49 .$$

Finally, equating  $M_2 = \Delta_2$  yields  $36\alpha^2 = 49$  , or

$$\alpha = 1.167 .$$

Thus,

$$L(s) = \frac{1}{1.167s + 1} = \frac{0.857}{s + 0.857} .$$

**P5.19** (a) The closed-loop transfer function is

$$T(s) = \frac{8}{s^3 + 6s^2 + 12s + 8} .$$

(b) The second-order approximation is

$$L(s) = \frac{1}{d_2s^2 + d_1s + 1} ,$$

where  $d_1$  and  $d_2$  are to be determined using the methods of Section 5.10 in Dorf & Bishop. Given

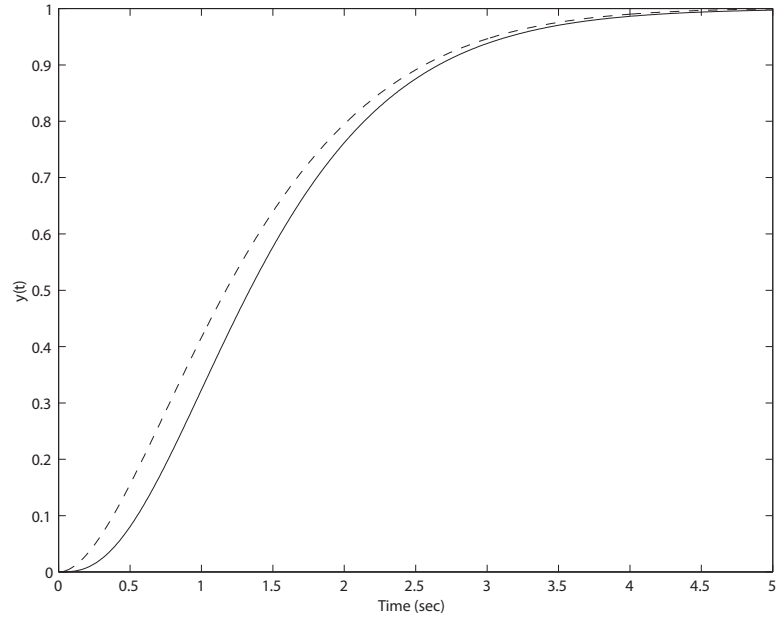
$$\begin{aligned} M(s) &= 8d_2s^2 + 8d_1s + 8 \\ \Delta(s) &= s^3 + 6s^2 + 12s + 8 \end{aligned}$$

we determine that

$$\begin{aligned} M_2 &= -128d_2 + 64d_1^2 \\ M_4 &= 64d_2^2 \\ \Delta_2 &= 48 \\ \Delta_4 &= 18 . \end{aligned}$$

Equating  $M_2 = \Delta_2$  and  $M_4 = \Delta_4$ , and solving for  $d_1$  and  $d_2$  yields

$$d_1 = 1.35 \quad \text{and} \quad d_2 = 0.53 .$$

**FIGURE P5.19**

Closed-loop system step response: Actual  $T(s)$  (solid line) and second-order approximation (dashed line).

Thus, the second-order approximation is

$$L(s) = \frac{1}{0.53s^2 + 1.35s + 1} .$$

(c) The plot of the step response for the actual system and the approximate system is shown in Figure P5.19.

**P5.20** The steady-state error is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{(s+5)(s+11) + K(1-K_1)}{(s+5)(s+11) + K} = \frac{55 + K(1-K_1)}{55 + K} .$$

To achieve a zero steady-state tracking error, select  $K_1$  as follows:

$$K_1 = 1 + \frac{55}{K} .$$

**P5.21** The closed-loop transfer function is

$$T(s) = \frac{s+a}{s^2 + (2k+a)s + 2ak+1} .$$



(a) If  $R(s) = 1/s$ , we have the tracking error

$$E(s) = R(s) - Y(s) = [1 - T(s)]R(s)$$

or

$$E(s) = \frac{s^2 + (2k + a - 1)s + 2ak + 1 - a}{s^2 + (2k + a)s + 2ak + 1} \cdot \frac{1}{s}.$$

From the final value theorem we obtain

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \frac{2ak + 1 - a}{2ak + 1}.$$

Selecting  $k = (a - 1)/(2a)$  leads to a zero steady-state error due to a unit step input.

(b) To meet the percent overshoot specification we desire  $\zeta \geq 0.69$ . From  $T(s)$  we find  $\omega_n^2 = 2ak + 1$  and  $2\zeta\omega_n = 2k + a$ . Therefore, solving for  $a$  and  $k$  yields

$$a = 1.5978 \quad \text{and} \quad k = 0.1871$$

when we select  $\zeta = 0.78$ . We select  $\zeta > 0.69$  to account for the zero in the closed-loop transfer function which will impact the percent overshoot. With  $a$  and  $k$ , as chosen, we have

$$T(s) = \frac{s + 1.598}{s^2 + 1.972s + 1.598}$$

and the step response yields  $P.O. \approx 4\%$ .

**P5.22** The closed-loop transfer function is

$$T(s) = \frac{2(2s + \tau)}{(s + 0.2K)(2s + \tau) + 4}.$$

(a) If  $R(s) = 1/s$ , we have the unit step response

$$Y(s) = \frac{2(2s + \tau)}{(s + 0.2K)(2s + \tau) + 4} \frac{1}{s}.$$

From the final value theorem we obtain

$$y_{ss} = \lim_{s \rightarrow 0} sY(s) = \frac{2\tau}{0.2K\tau + 4}.$$

Selecting  $K = 10 - 20/\tau$  leads to  $y_{ss} = 1$  and a zero steady-state error due to a unit step input.

(b) The characteristic equation is

$$(s + 0.2K)(2s + \tau) + 4 = 2s^2 + (0.4K + \tau)s + 0.2K\tau + 4 = 0 .$$

So, with  $K = 10 - 20/\tau$ , the natural frequency and damping ratio are:

$$\omega_n = \sqrt{\tau} \quad \text{and} \quad \zeta = \frac{\tau^2 + 4\tau - 8}{4\tau^{3/2}} .$$

The settling time and percent overshoot are found using the standard design formulas

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad \text{and} \quad P.O. = 100e^{-\zeta\pi\sqrt{1-\zeta^2}}$$

with  $\omega_n$  and  $\zeta$  given above (as a function of  $\tau$ ). Since the closed-loop system has a zero at  $s = -\tau/2$ , the formulas for  $T_p$  and  $P.O.$  will only be approximate. Also, note that for the closed-loop system poles to be in the left half-plane (that is, all the poles have negative real parts), we require that  $\tau > 2\sqrt{3} - 2 \approx 1.4642$ . As seen in the next chapter, this is the condition for stability. Having  $\tau > 2\sqrt{3} - 2$  insures that the damping ratio  $\zeta$  is positive.

## Advanced Problems

**AP5.1** (a) The steady-state error is

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - T(s))R(s) = 1 - T(0) = 1 - \frac{108(3)}{9(36)} = 0 .$$

(b) Assume the complex poles are dominant. Then, we compute

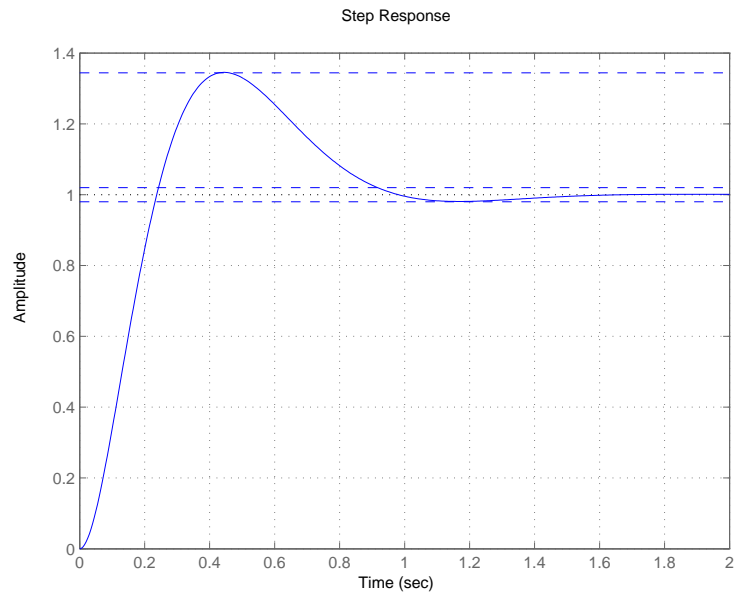
$$\frac{a}{\zeta\omega_n} = 0.75 ,$$

since  $a = 3$ ,  $\zeta = 0.67$  and  $\omega_n = 6$ . Using Figure 5.13 in Dorf & Bishop, we estimate the settling time and percent overshoot to be

$$P.O. = 45\% \quad \text{and} \quad T_s = \frac{4}{\zeta\omega_n} = 1 \text{ second} .$$

(c) The step response is shown in Figure AP5.1. The actual settling time and percent overshoot are

$$P.O. = 34.4\% \quad \text{and} \quad T_s = 1.18 \text{ second} .$$

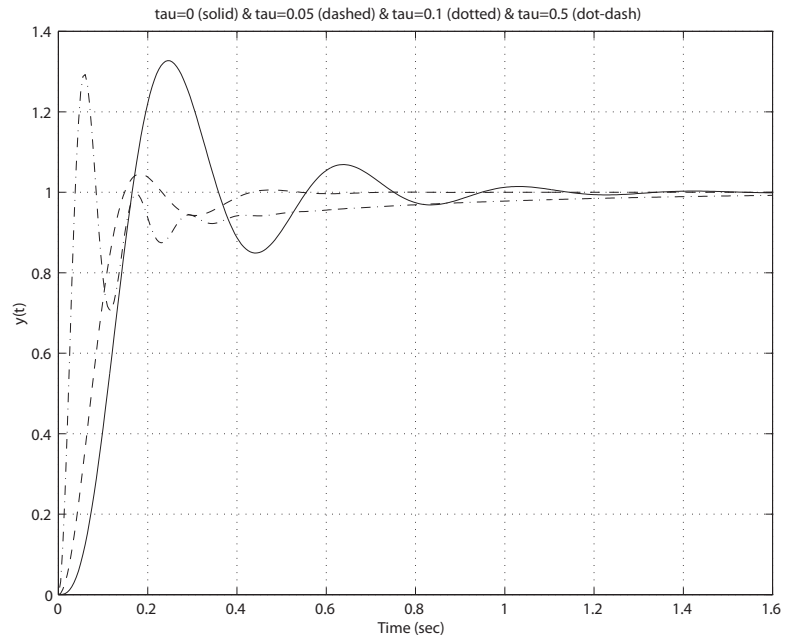


**FIGURE AP5.1**  
Closed-loop system step response.

**AP5.2** The closed-loop transfer function is

$$T(s) = \frac{5440(\tau_z s + 1)}{s^3 + 28s^2 + (432 + 5440\tau_z)s + 5440} .$$

The closed-loop step responses are shown in Figure AP5.2. The performance results are summarized in Table AP5.2.



**FIGURE AP5.2**  
Closed-loop system step responses.

$\tau_z$	$T_r$	$T_s$	$P.O.$	closed-loop poles
0	0.16	0.89	32.7%	$p = -20, -4 \pm 16j$
0.05	0.14	0.39	4.5%	$p = -10.4, -8.77 \pm 21.06j$
0.1	0.10	0.49	0%	$p = -6.5, -10.74 \pm 26.84j$
0.5	0.04	1.05	29.2%	$p = -1.75, -13.12 \pm 54.16j$

**TABLE AP5.2** Performance summary.

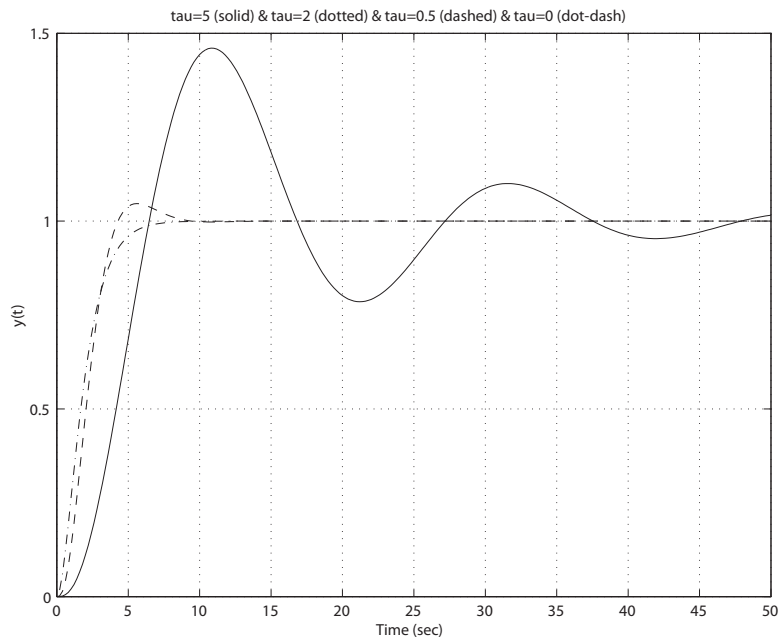
As  $\tau_z$  increases from 0 to 0.1, the  $P.O.$  decreases and the response is faster

and more stable. However, as  $\tau_z$  is increased beyond 0.1, the  $P.O.$  and  $T_s$  increase, although  $T_r$  continues to decrease.

**AP5.3** The closed-loop transfer function is

$$T(s) = \frac{1}{\tau_p s^3 + (1 + 2\tau_p)s^2 + 2s + 1} .$$

The closed-loop step responses for  $\tau_p = 0, 0.5, 2, 5$  are shown in Figure AP5.3. The performance results are summarized in Table AP5.3.



**FIGURE AP5.3** Closed-loop system step responses.

$\tau_p$	$T_r$	$T_s$	$P.O.$	closed-loop poles
0	4	5.8	0%	$p = -1, -1$
0.5	3.6	7.4	4.75%	$p = -2.84, -0.58 \pm 0.6j$
2	4.6	22.4	27.7%	$p = -2.14, -0.18 \pm 0.45j$
5	6	45.8	46%	$p = -2.05, -0.07 \pm 0.3j$

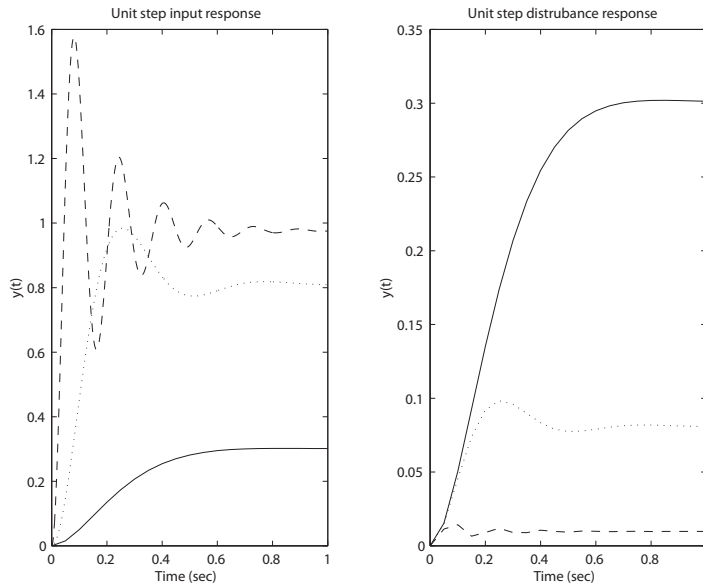
**TABLE AP5.3** Performance summary.

As  $\tau_p$  increases, the  $P.O.$ ,  $T_r$  and  $T_s$  also increase; adding the pole makes the system less stable with more overshoot.

**AP5.4** The system transfer function is

$$Y(s) = \frac{15K}{(s + 5)(s + 7) + 15K}R(s) + \frac{15}{(s + 5)(s + 7) + 15K}T_d(s) .$$

When considering the input response, we set  $T_d(s) = 0$ , and similarly, when considering the disturbance response, we set  $R(s) = 0$ . The closed-loop step input and disturbance responses for  $K = 1, 10, 100$  are shown in Figure AP5.4. The performance results are summarized in Table AP5.4.



**FIGURE AP5.4** Closed-loop system input and disturbance responses ( $K=1$ : solid line,  $K=10$ : dotted line, and  $K=100$ :dashed line).

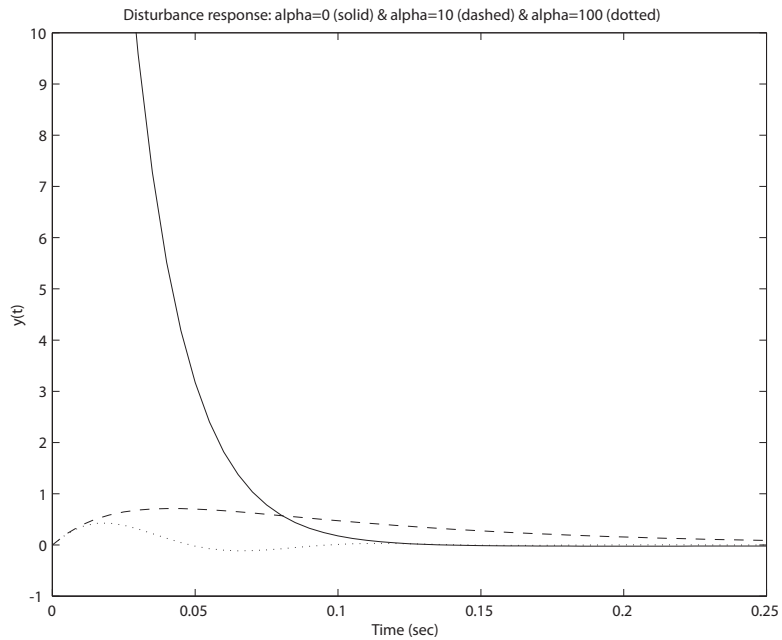
$K$	$e_{ss}$	$T_s$	$P.O.$	$ y/d _{max}$
1	0.7	0.45	0%	0.3
10	0.19	0.6	17.3%	0.1
100	0.023	0.59	60.0%	0.01

**TABLE AP5.4** Performance summary.

The best value of the gain is  $K = 10$ , which is compromise between (i) percent overshoot, and (ii) disturbance rejection and tracking error.

**AP5.5** The system transfer function is

$$Y(s) = \frac{50(s + \alpha)(s + 2)}{s(s + 3)(s + 4) + 50(s + \alpha)(s + 2)}R(s) + \frac{50s(s + 2)}{s(s + 3)(s + 4) + 50(s + \alpha)(s + 2)}T_d(s) .$$



**FIGURE AP5.5**  
Closed-loop system disturbance response.

When considering the input response, we set  $T_d(s) = 0$ , and similarly, when considering the disturbance response, we set  $R(s) = 0$ . The steady-state tracking error is

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - T(s))R(s) = \lim_{s \rightarrow 0} 1 - \frac{50(s + \alpha)(s + 2)}{s(s + 3)(s + 4) + 50(s + \alpha)(s + 2)} .$$

When  $\alpha = 0$ , we have

$$e_{ss} = 1 - \frac{100}{100 + 12} = 0.11 ,$$

and, for  $\alpha \neq 0$

$$e_{ss} = 0 .$$

The closed-loop step input and disturbance responses for  $\alpha = 0, 10, 100$  are shown in Figure AP5.5. For disturbance rejection and steady-state tracking error the best value of the parameter is

$$\alpha = 100 .$$

However, when considering both the disturbance and input response we would select the parameter

$$\alpha = 10 ,$$

since it offers a good compromise between input response overshoot (about 5% for  $\alpha = 10$ ) and disturbance rejection/tracking error.

**AP5.6** (a) The closed-loop transfer function is

$$T(s) = \frac{KK_m}{KK_m + s(s + K_mK_b + 0.01)} .$$

The steady-state tracking error for a ramp input  $R(s) = 1/s^2$  is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s(1 - T(s))R(s) \\ &= \lim_{s \rightarrow 0} \frac{s + K_mK_b + 0.01}{KK_m + s(s + K_mK_b + 0.01)} \\ &= \frac{K_mK_b + 0.01}{KK_m} . \end{aligned}$$

(b) With

$$K_m = 10$$

and

$$K_b = 0.05 ,$$

we have

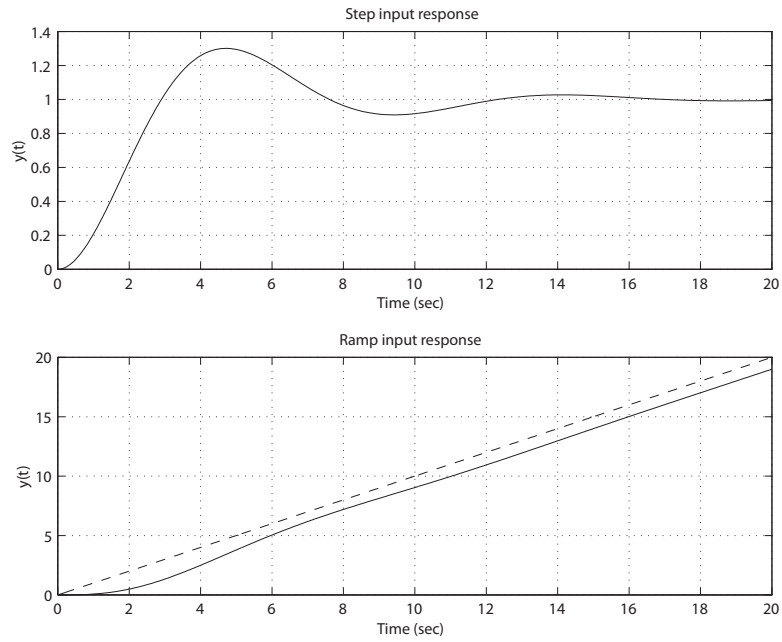
$$\frac{K_mK_b + 0.01}{KK_m} = \frac{10(0.05) + 0.01}{10K} = 1 .$$

Solving for  $K$  yields

$$K = 0.051 .$$



- (c) The plot of the step and ramp responses are shown in Figure AP5.6. The responses are acceptable.

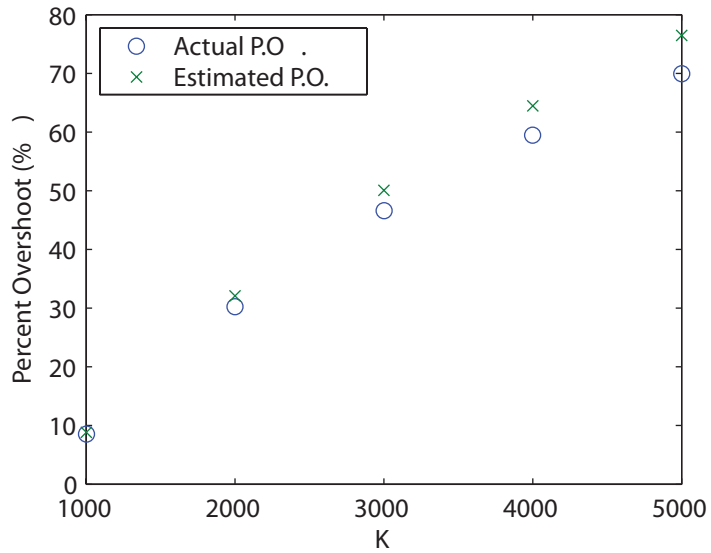


**FIGURE AP5.6**  
Closed-loop system step and ramp responses.

- AP5.7** The performance is summarized in Table AP5.7 and shown in graphical form in Fig. AP5.7.

$K$	Estimated Percent Overshoot	Actual Percent Overshoot
1000	8.8 %	8.5 %
2000	32.1 %	30.2 %
3000	50.0 %	46.6 %
4000	64.4 %	59.4 %
5000	76.4 %	69.9 %

**TABLE AP5.7** Performance summary.



**FIGURE AP5.7**  
Percent overshoot versus  $K$ .

The closed-loop transfer function is

$$T(s) = \frac{100K}{s(s+50)(s+100) + 100K}.$$

The impact of the third pole is more evident as  $K$  gets larger as the estimated and actual percent overshoot deviate in the range 0.3% at  $K = 1000$  to 6.5% at  $K = 5000$ .

**AP5.8** The closed-loop transfer function is

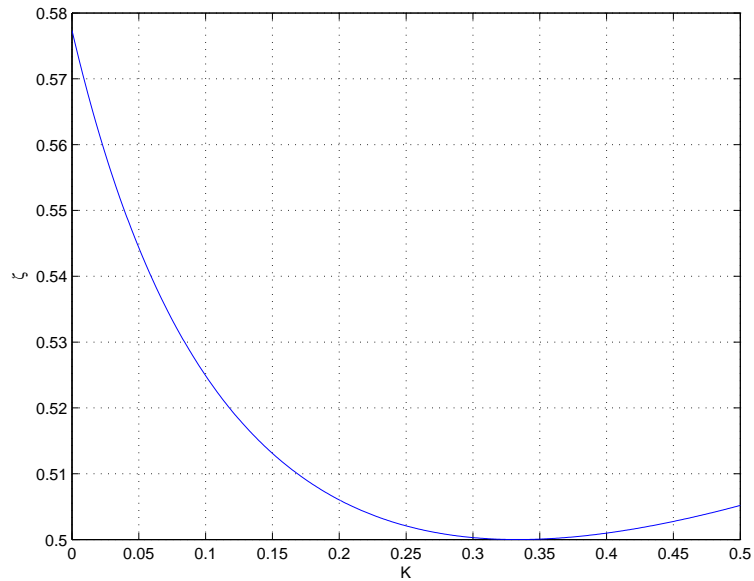
$$T(s) = \frac{K(s+2)}{s^2 + (\frac{2}{3} + K)s + \frac{1}{3} + 2K}.$$

Comparing  $T(s)$  to a second-order system we have

$$\omega_n = \sqrt{1/3 + 2K}$$

$$\zeta = \frac{2/3 + K}{2\sqrt{1/3 + 2K}}$$

For the closed-loop transfer function to have complex roots, we require  $K^2 - (20/3)K - (8/9) < 0$ . This occurs when  $-0.13 \leq K \leq 6.8$ . When  $K = 1/3$ , we have the minimum  $\zeta = 0.5$ , as shown in Figure AP5.8.



**FIGURE AP5.8**  
Damping ratio,  $\zeta$ , versus  $K$ .

**AP5.9** The closed-loop characteristic equation is

$$s^4 + 40s^3 + 375s^2 + K_P s + K_I = 0.$$

The desired characteristic equation is

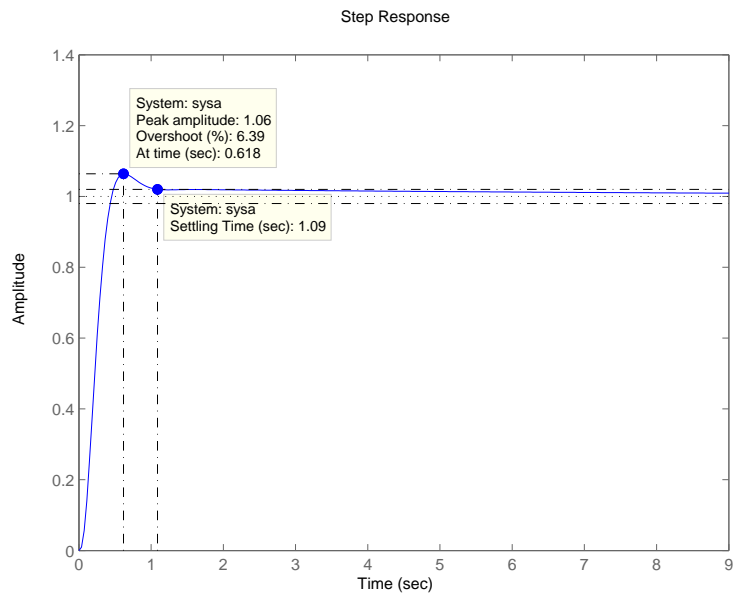
$$(s + a)(s + b)(s^2 + \sqrt{2}\omega_n s + \omega_n^2) = 0.$$

Expanding the desired characteristic equation and equating terms to the actual characteristic equation yields

$$ab\omega_n^2 = K_I, \quad \omega_n^2(a + b) + \sqrt{2}ab\omega_n = K_P$$

$$\sqrt{2}(a + b)\omega_n + ab = 375, \quad \sqrt{2}\omega_n + a + b = 40$$

This represents 4 equations with 5 unknowns ( $a$ ,  $b$ ,  $K_P$ ,  $K_I$ , and  $\omega_n$ ). We can choose one variable as part of the controller design. Let  $K_I = 0.1K_P$ . Then, solving the 4 equations for the remaining 4 variables yields  $a = 29.15$ ,  $b = 0.1$ ,  $K_P = 1720$ ,  $K_I = 172$ , and  $\omega_n = 7.6$ . The resulting  $T_s = 1.1\text{s}$  and  $P.O. = 6.4\%$ , as shown in Figure AP5.9.

**FIGURE AP5.9**

Step response for  $K_P = 1720$  and  $K_I = 172$ .

## Design Problems

**CDP5.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$\frac{\theta(s)}{V_a(s)} = \frac{26.035}{s(s + 33.142)},$$

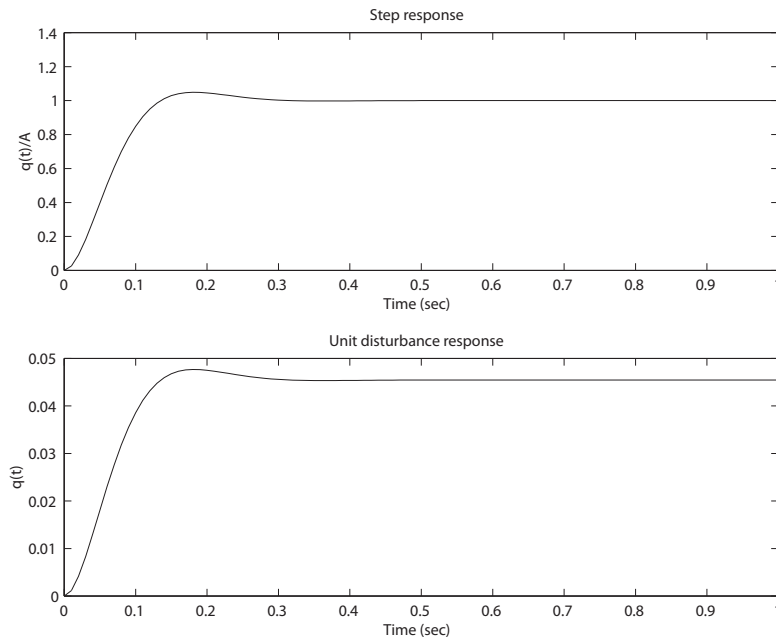
where we neglect the motor inductance  $L_m$ . The closed-loop transfer function from the disturbance to the output is

$$\frac{\theta(s)}{T_d(s)} = \frac{26.035}{s^2 + 33.142s + 26.035K_a}.$$

For a unit step disturbance input the steady-state response is

$$\theta_{ss} = \frac{1}{K_a}.$$

Therefore, we want to use the maximum  $K_a$  while keeping the percent overshoot less than 5%. The step response for the closed-loop system (with the tachometer not in the loop) and  $K_a = 22$  is shown below. Values of  $K_a$  greater than 22 lead to overshoots greater than 5%.



**DP5.1** (a) The closed-loop transfer function is

$$\frac{\phi(s)}{\phi_d(s)} = \frac{12.2K}{s(s+2.2)(s+7) + 12.2K} = \frac{12.2K}{s^3 + 9.2s^2 + 15.4s + 12.2K} .$$

(b) For  $K = 0.7$ , we have the characteristic equation

$$s^3 + 9.2s^2 + 15.4s + 8.54 = 0 ,$$

with roots  $s_1 = -7.23$  and  $s_{2,3} = -0.98 \pm 0.46j$ . For  $K = 3$ , we have the characteristic equation

$$s^3 + 9.2s^2 + 15.4s + 36.6 = 0 ,$$

with roots  $s_1 = -7.83$  and  $s_{2,3} = -0.68 \pm 2.05j$ . And for  $K = 6$ , we have the characteristic equation

$$s^3 + 9.2s^2 + 15.4s + 73.2 = 0 ,$$

with roots  $s_1 = -8.4$  and  $s_{2,3} = -0.4 \pm 2.9j$ .

(c) Assuming the complex conjugate pair are the dominant roots, we expect the following:

(i) for  $K = 0.7$ : P.O.=0.13% and  $T_p = 6.8$  sec

(ii) for  $K = 3$ : P.O.=35.0% and  $T_p = 1.5$  sec

(iii) for  $K = 6$ : P.O.=65.2% and  $T_p = 1.1$  sec

(d),(e) We select

$$K = 1.71$$

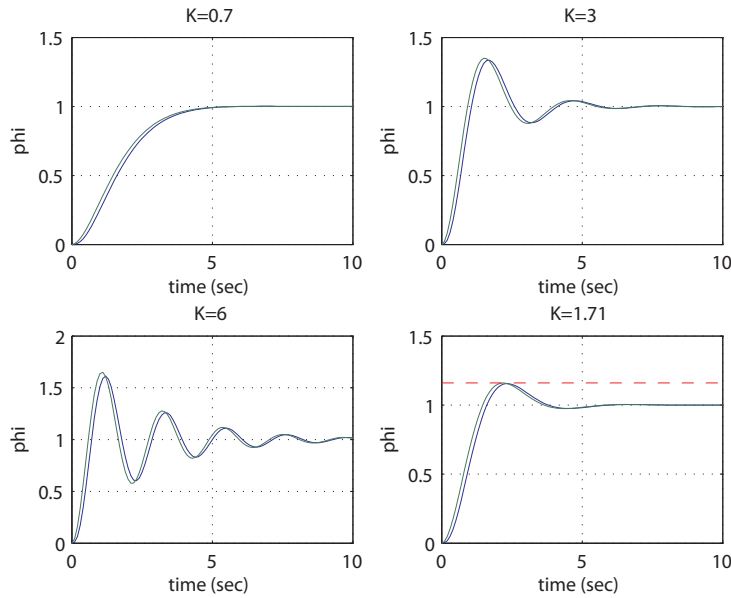
to have a  $P.O. = 16\%$  and  $T_p = 2.18$ sec. All four cases ( $K = 0.7, 3, 6, 1.71$ ) are shown in Figure DP5.1. In each case, the approximate transfer function is derived by neglecting the non-dominant real pole and adjusting the gain for zero steady-state error. The approximate transfer functions are

$$T_{K=0.7}(s) = \frac{1.18}{s^2 + 1.965s + 1.18} = \frac{0.7908}{(s + 0.98 + 0.46j)(s + 0.98 - 0.46j)}$$

$$T_{K=3}(s) = \frac{4.67}{s^2 + 1.37s + 4.67} = \frac{3.299}{(s + 0.68 + 2.05j)(s + 0.68 - 2.05j)}$$

$$T_{K=6}(s) = \frac{8.71}{s^2 + 0.796s + 8.71} = \frac{6.399}{(s + 0.4 + 2.9j)(s + 0.4 - 2.9j)}$$

$$T_{K=1.71}(s) = \frac{2.77}{s^2 + 1.679s + 2.77} = \frac{1.458}{(s + 0.83 + 1.43j)(s + 0.83 - 1.43j)}$$



**FIGURE DP5.1** Step responses (actual response:solid lines; approximate response: dotted lines).

**DP5.2** The closed-loop transfer function is

$$T(s) = \frac{K\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K\omega_n^2},$$

where  $\zeta = 0.6$ . From the second-order system approximation, we have

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}.$$

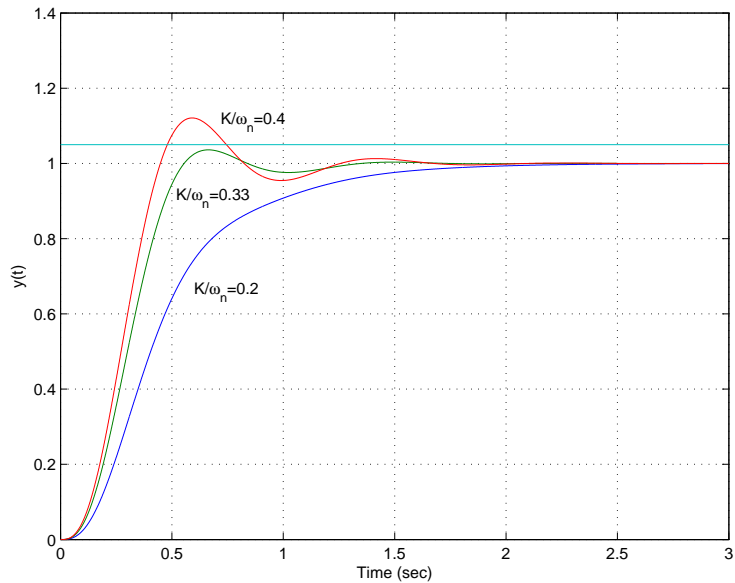
So, with  $\zeta = 0.6$  given, we should select  $\omega_n$  “large” to make  $T_p$  “small.” Also, from the problem hint, let

$$0.2 < K/\omega_n < 0.4 .$$

As a first attempt, we can select  $\omega_n = 10$ . See Figure DP5.8 for various values of  $K/\omega_n$ . Our final selection is

$$K = 3.33 \quad \text{and} \quad \omega_n = 10 .$$

This results in  $P.O. = 3.6\%$  and  $T_p = 0.66$  second.



**FIGURE DP5.2**  
Closed-loop system response.

**DP5.3** The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + qs + K} .$$

From the ITAE specification, we desire

$$T(s) = \frac{\omega_n^2}{s^2 + 1.4\omega_n s + \omega_n^2} .$$

But

$$2\zeta\omega_n = 1.4\omega_n \quad \text{which implies} \quad \zeta = 0.7 .$$

Since we want  $T_s \leq 0.5$ , we require  $\zeta\omega_n \geq 8$ . So,

$$\omega_n \geq \frac{8}{0.7} = 11.4 .$$

We can select  $\omega_n = 12$ . Then,

$$T(s) = \frac{144}{s^2 + 16.8s + 144} .$$

Therefore,  $K = 144$  and  $q = 16.8$ . The predicted percent overshoot is



$P.O. = 4.5\%$ .

**DP5.4** The loop transfer function is

$$G_c(s)G(s) = \frac{10K}{(s+70)(s+3)(s+7)} = \frac{10K/70}{(s/70+1)(s+3)(s+7)} .$$

The second-order approximation is obtained by neglecting the fastest first-order pole. Thus,

$$G_c(s)G(s) \approx \frac{K/7}{(s+3)(s+7)} .$$

The closed-loop transfer function is

$$T(s) = \frac{K/7}{s^2 + 10s + 21 + K/7} .$$

When  $\zeta \geq 0.52$ , we have less than 15% overshoot. So, we have

$$2\zeta\omega_n = 10$$

and

$$\omega_n = \sqrt{21 + K/7} .$$

Eliminating  $\omega_n$  and solving for  $K$  (with  $P.O. \leq 15\%$ ) yields

$$K \leq 500.19 .$$

Also,

$$K_p = \lim_{s \rightarrow 0} GG_c(s) = \frac{K}{7(21)}$$

and

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \frac{K}{147}} < 0.12$$

implies

$$K \geq 1078 .$$

Therefore, we have an inconsistency. We require  $1078 \leq K$  to meet the steady-state requirement and  $K \leq 500.18$  to meet the percent overshoot requirement. It is not possible to meet both specifications.

**DP5.5** The closed-loop characteristic equation is

$$1 + K_1 G_1(s) + K_2 G_1 G_2(s) = 1 + \frac{K_1}{s(s+1)} - \frac{2K_2}{s(s+1)(s+2)} = 0$$

or

$$s^3 + 3s^2 + (2 + K_1)s + 2(K_1 - K_2) = 0 .$$

Assuming that  $K_1 > 0$  and  $K_2 > 0$ , the range of the gains for stability is

$$0 < K_2 < K_1 .$$

**DP5.6** The closed-loop transfer function is

$$T(s) = \frac{K_1}{s^2 + (K_1 K_2 + 1)s + K_1} .$$

The percent overshoot specification  $P.O. \leq 2\%$  is satisfied when  $\zeta > 0.78$ . The peak time specification  $T_p \leq 0.5$  s is satisfied when  $\omega_n = 10$  and  $\zeta = 0.78$ . So, given

$$K_1 = \omega_n^2 \quad \text{and} \quad K_1 K_2 + 1 = 2\zeta\omega_n ,$$

we determine that the specifications are satisfied when

$$K_1 = 100$$

and

$$K_2 = 0.15 .$$

**DP5.7** The plant is

$$G(s) = \frac{2}{s(s+1)(s+4)}$$

and the PD controller is

$$G_c(s) = K_D s + K_P .$$

The characteristic equation is

$$s^3 + 6s^2 + (8 + 2K_D)s + 2K_P = 0 .$$

The desired characteristic equation is

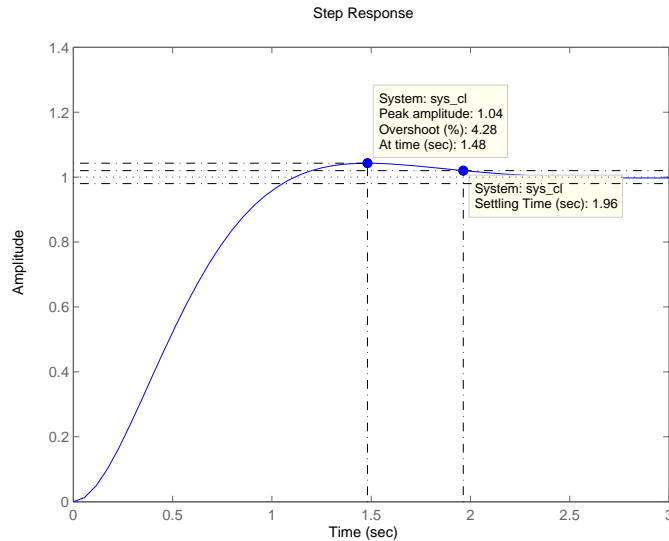
$$(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2) = s^3 + (2\zeta\omega_n + a)s^2 + (\omega_n^2 + 2\zeta\omega_n a)s + a\omega_n^2 = 0 .$$

Equating the desired characteristic equation to the actual characteristic

equation yields

$$2\zeta\omega_n + a = 6, \quad \omega_n^2 + 2\zeta\omega_n a = 8 + 2K_D, \quad a\omega_n^2 = 2K_P,$$

where  $\zeta = 0.69$  and  $\omega_n = 3$  to meet the design specifications. This represents 3 equations in 3 unknowns ( $a$ ,  $K_D$ , and  $K_P$ ). Solving yields  $a = 1.86$ ,  $K_D = 4.35$  and  $K_P = 8.37$ . The step response is shown in Figure DP5.7.



**FIGURE DP5.7**

Step response with  $K_D = 4.35$  and  $K_P = 8.37$ .

**DP5.8** The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + 6s + 5 + K}$$

The damping ratio and natural frequency is

$$\zeta = \frac{3}{\sqrt{K+5}} \quad \text{and} \quad \omega_n = \sqrt{K+5}$$

Using the design formulas for second-order systems we have

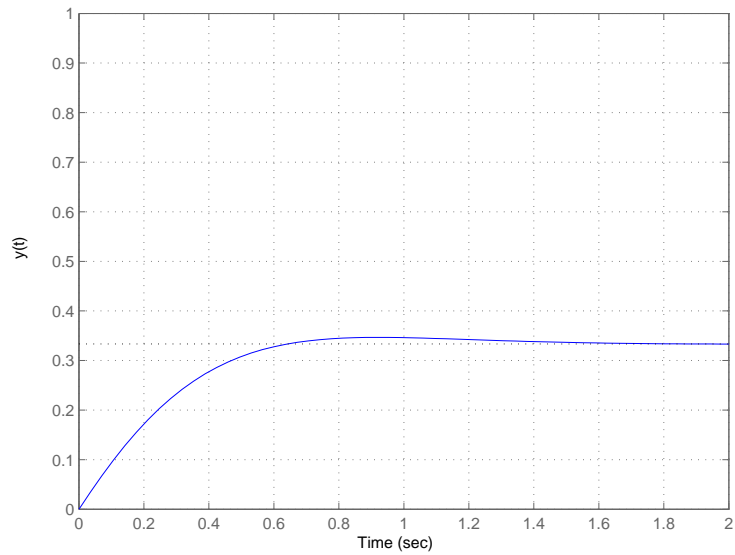
$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \text{and} \quad T_s = \frac{4}{\zeta\omega_n}.$$

We know that the formula for  $T_s$  is approximate and that the formulas apply only to systems with  $\zeta < 1$ . For  $K = 1$  the closed-loop poles are

both real, so there is no overshoot and the design formula for settling time does not apply. Thus we obtain the results shown in Table DP5.8. We can choose  $K = 10$  as a good trade-off between percent overshoot, settling time, and steady-state tracking error. The disturbance response is shown in Figure DP5.8.

**TABLE DP5.8** Step response for  $K=1, 10,$  and  $20$ .

K	% <i>P.O.</i>	$T_s$ , sec	$T_s$ , sec	$e_{ss}$
		Estimated	Actual	
1	0	-	3.24	0.83
10	2.13	1.33	1.38	0.33
20	9.48	1.33	1.19	0.20



**FIGURE DP5.8**

Closed-loop system disturbance response for  $K = 10$ .

## Computer Problems

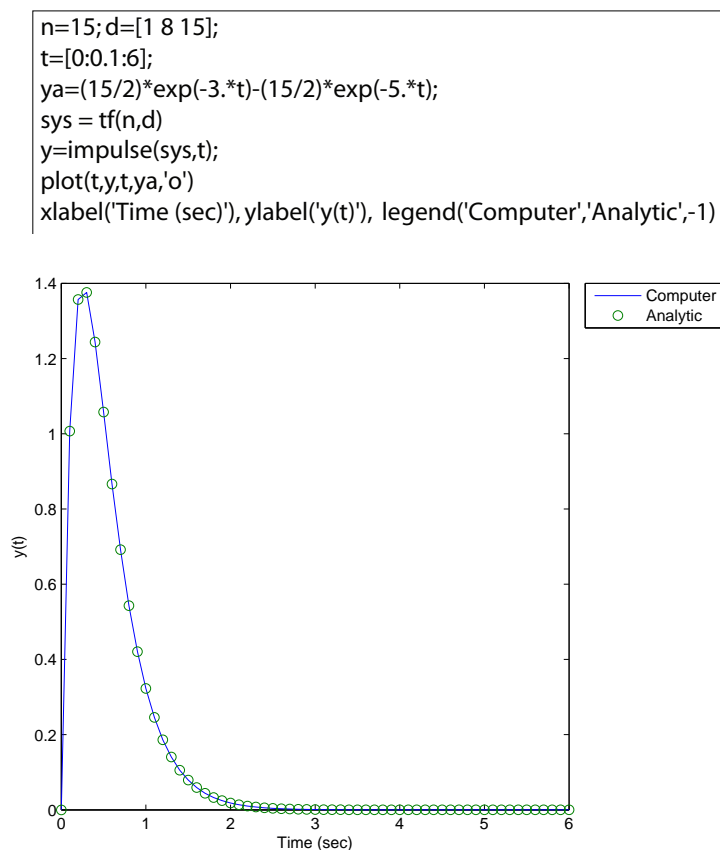
**CP5.1** With the impulse input we have  $R(s) = 1$ . The transfer function is

$$Y(s) = \frac{15}{(s+3)(s+5)} R(s) = \frac{15}{(s+3)(s+5)} .$$

Therefore, taking the inverse Laplace transforms yields the output response:

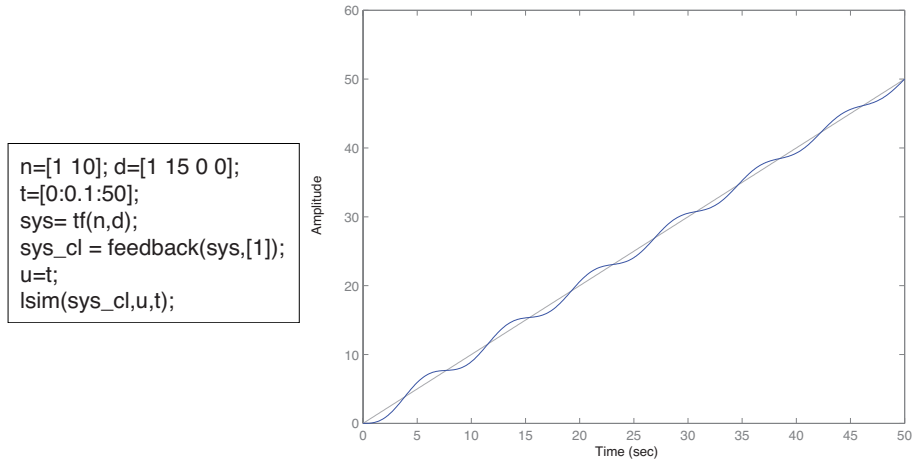
$$y(t) = \frac{15}{2}e^{-3t} - \frac{15}{2}e^{-5t} .$$

The impulse response and the analytic response is shown in Figure CP5.1.



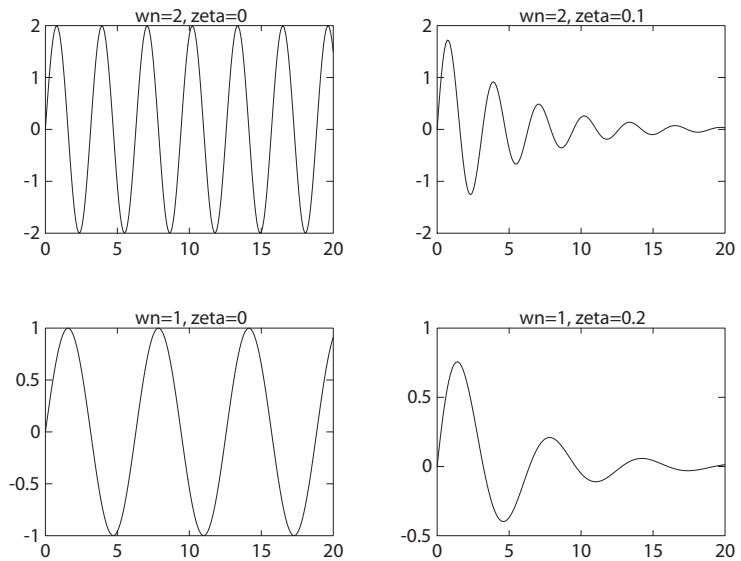
**FIGURE CP5.1**  
Impulse responses.

**CP5.2** The ramp response is shown in Figure CP5.2. The unity feedback system is type 2, so that the steady-state tracking error is  $\lim_{t \rightarrow \infty} e_{ss} \rightarrow 0$ .



**FIGURE CP5.2**  
Ramp responses.

**CP5.3** The m-file script and the four plots are shown in Figure CP5.3. The plots can be compared to Figure 5.17 in Dorf & Bishop.



**FIGURE CP5.3**  
Impulse responses.

```

w1=2; z1=0;
w2=2; z2=0.1;
w3=1; z3=0;
w4=1; z4=0.2;
t=[0:0.1:20];
%
num1=[w1^2]; den1=[1 2*z1*w1 w1^2];
sys1 = tf(num1,den1);
[y1,x1]=impulse(sys1,t);
%
num2=[w2^2]; den2=[1 2*z2*w2 w2^2];
sys2 = tf(num2,den2);
[y2,x2]=impulse(sys2,t);
%
num3=[w3^2]; den3=[1 2*z3*w3 w3^2];
sys3 = tf(num3,den3);
[y3,x3]=impulse(sys3,t);
%
num4=[w4^2]; den4=[1 2*z4*w4 w4^2];
sys4 = tf(num4,den4);
[y4,x4]=impulse(sys4,t);
%
clf
subplot(221),plot(t,y1),title('wn=2, zeta=0')
subplot(222),plot(t,y2),title('wn=2, zeta=0.1')
subplot(223),plot(t,y3),title('wn=1, zeta=0')
subplot(224),plot(t,y4),title('wn=1, zeta=0.2')

```

**FIGURE CP5.3**  
CONTINUED: Impulse response m-file script.

**CP5.4** The closed-loop system is

$$T(s) = \frac{21}{s^2 + 2s + 21} .$$

Therefore, the natural frequency is

$$\omega_m = \sqrt{21} = 4.58$$

and the damping ratio is computed as

$$2\zeta\omega_n = 2 ,$$

which implies

$$\zeta = 0.218 .$$

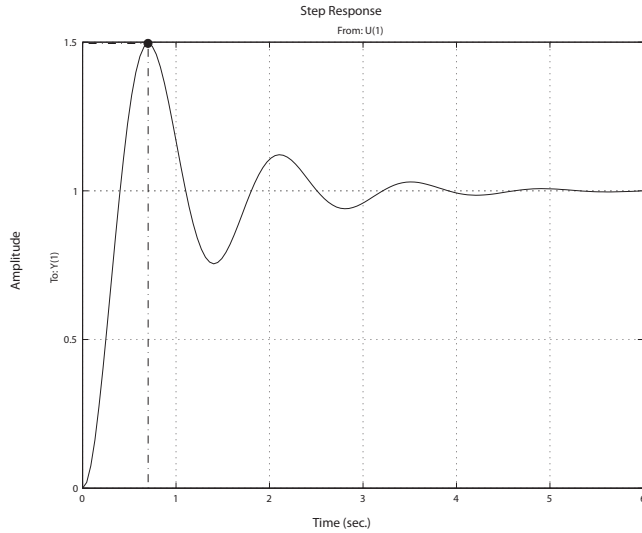
The percent overshoot is estimated to be

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 50\% ,$$

since  $\zeta = 0.218$ . The actual overshoot is shown in Figure CP5.4.

```

numc=[21]; denc=[1 0];
sysc = tf(numc,denc);
numg=[1]; deng=[1 2];
sysg = tf(numg,deng);
sys_o = series(sysc,sysg);
sys_cl = feedback(sys_o,1)
step(sys_cl)
    
```



**FIGURE CP5.4**  
Impulse responses.

**CP5.5** For a step input, the optimum ITAE characteristic equation is

$$s^3 + 1.75\omega s^2 + 2.15\omega^2 s + \omega^3 = 0 .$$

Examining Figure 5.30 for  $n=3$  in Dorf & Bishop, we estimate that

$$\omega T_s = 8.$$

So, once we decide on the desired  $T_s$  we can estimate  $\omega$ . For this design we let  $T_s=8$  so that  $\omega = 1$ . Computing the desired characteristic equation and the actual characteristic equation and comparing the coefficients leads to the following relationships:

$$p = 1.75\omega - 2\zeta\omega_n$$

$$K = (2.15\omega^2 - \omega_n^2 - 2\zeta\omega_n p)/\omega_n^2$$

$$z = (\omega^3 - p\omega_n^2)/(K\omega_n^2)$$

where  $\zeta = 0.59$  and  $\omega_n = 0.45$ . The controller and prefilter are

$$G_c(s) = 6.42 \frac{s + 0.58}{s + 1.22} \quad \text{and} \quad G_p(s) = \frac{1}{1.3s + 0.75} .$$

The unit step response is shown in Figure CP5.5.



```

wn=0.45; zeta=0.59
ng=wn^2; dg=[1 2*zeta*wn wn^2];
sysg=tf(ng,dg);
Ts=8; w=8/Ts;
p=1.75*w-2*zeta*wn;
K=(2.15*w^2-wn^2-2*zeta*wn*p)/wn^2;
z=(w^3-p*wn^2)/(K*wn^2);
nc=K*[1 z]; dc=[1 p]; sysc=tf(nc,dc);
sys=series(sysc,sysg);
syscl=feedback(sys,1);
[num,den]=tfdata(syscl,'v');
syp=tf([den(end)],num);
step(syscl*syp)

```

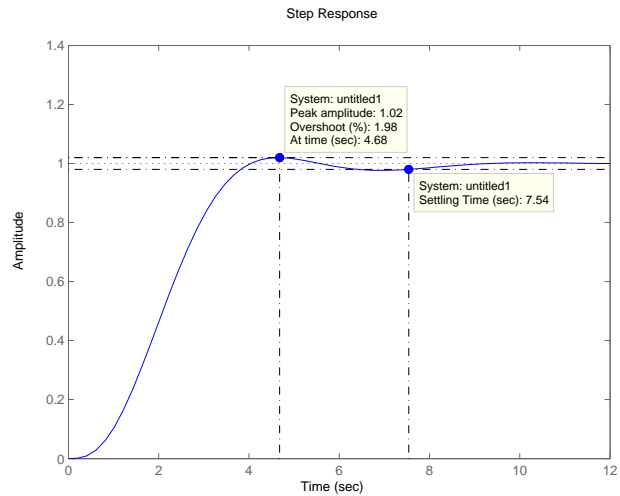


FIGURE CP5.5

Closed-loop system step response m-file script.

**CP5.6** The unit step response is shown in Figure CP5.6. The performance numbers are as follows:  $M_p = 1.16$ ,  $T_p = 0.73$ , and  $T_s = 1.62$ .

```

numg=[25]; deng=[1 5 0];
sys = tf(numg,deng);
sys_cl = feedback(sys,[1]);
t=[0:0.01:2];
step(sys_cl,t);

```

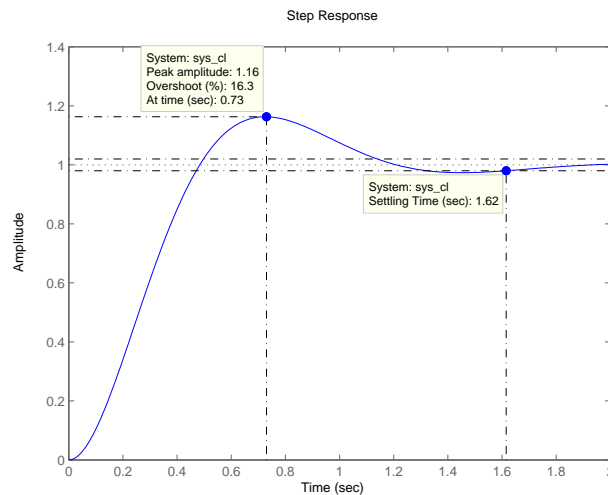
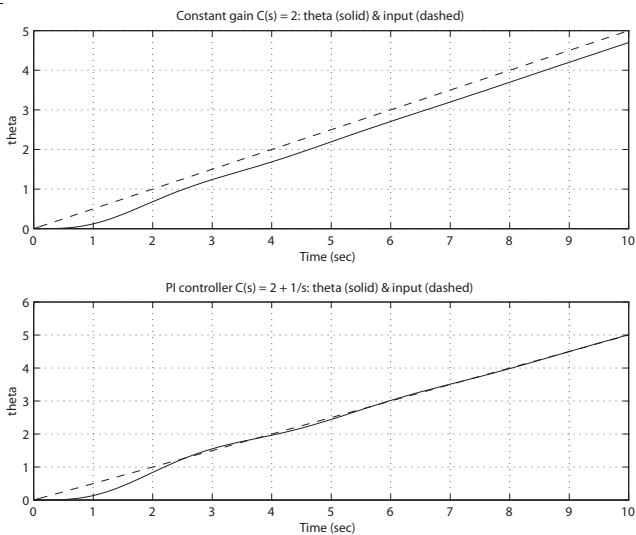


FIGURE CP5.6

Closed-loop system step response m-file script.

**CP5.7** The m-file script and the simulations are shown in Figure CP5.7.

```
% Part (a)
numc=[2]; denc=[1]; sys_c = tf(numc,denc);
nums=[-10]; dens=[1 10]; sys_s = tf(nums,dens);
numg=[-1 -5]; deng=[1 3.5 6 0]; sys_g = tf(numg,deng);
sysa = series(sys_c,sys_s);
sysb = series(sysa,sys_g);
sys = feedback(sysb,[1]);
f=0.5*pi/180; % Convert to rad/sec
t=[0:0.1:10]; u=f*t;
[y,x]=lsim(sys,u,t);(y(length(t),1)-u(1,length(t)))*180/pi
subplot(211)
plot(t,y*180/pi,t,u*180/pi,'--'), grid
xlabel('Time (sec)'),ylabel('theta')
title('Constant gain C(s) = 2: theta (solid) & input (dashed)')
% Part (b)
numc=[2 1]; denc=[1 0]; sys_c = tf(numc,denc);
[numa,dena]=series(numc,denc,nums,dens);
sysa = series(sys_c,sys_s);
sysb = series(sysa,sys_g);
sys = feedback(sysb,[1]);
[y,x]=lsim(sys,u,t);(y(length(t),1)-u(1,length(t)))*180/pi
subplot(212), plot(t,y*180/pi,t,u*180/pi,'--'), grid
xlabel('Time (sec)'),ylabel('theta')
title('PI controller C(s) = 2 + 1/s: theta (solid) & input (dashed)')
```



**FIGURE CP5.7**

Closed-loop system response to a ramp input for two controllers.

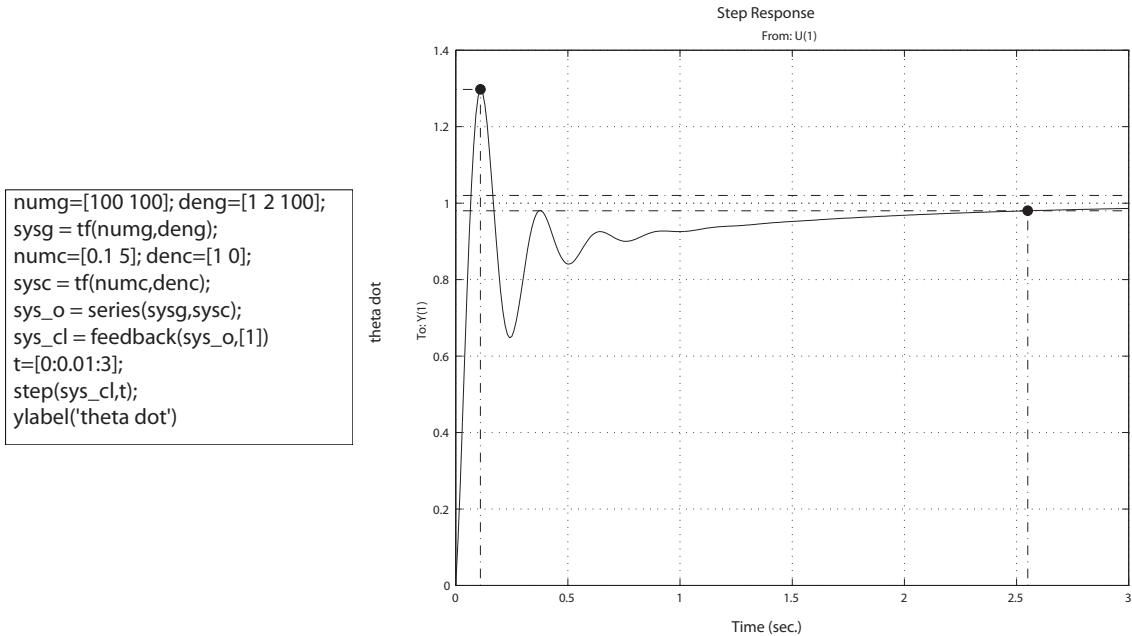
For the constant gain controller, the attitude error after 10 seconds is  $e_{ss} = -0.3$  deg. On the other hand, the PI controller has a zero steady-state error  $e_{ss} = 0$  deg. So, we can decrease the steady-state error by

using a more sophisticated controller, in this case a PI controller versus a constant gain controller.

**CP5.8** The closed-loop characteristic equation is

$$s^3 + 12s^2 + 610s + 500 = (s + 0.8324)(s^2 + 11.1676s + 600.7027) = 0 .$$

The natural frequency and damping ratio of the complex roots are  $\omega_n = 24.5$  and  $\zeta = 0.23$ . From this we predict  $M_p = 1.48$ ,  $T_s = 0.72$ , and  $T_p = 0.13$ . The actual response is shown in Figure CP5.8. The differences



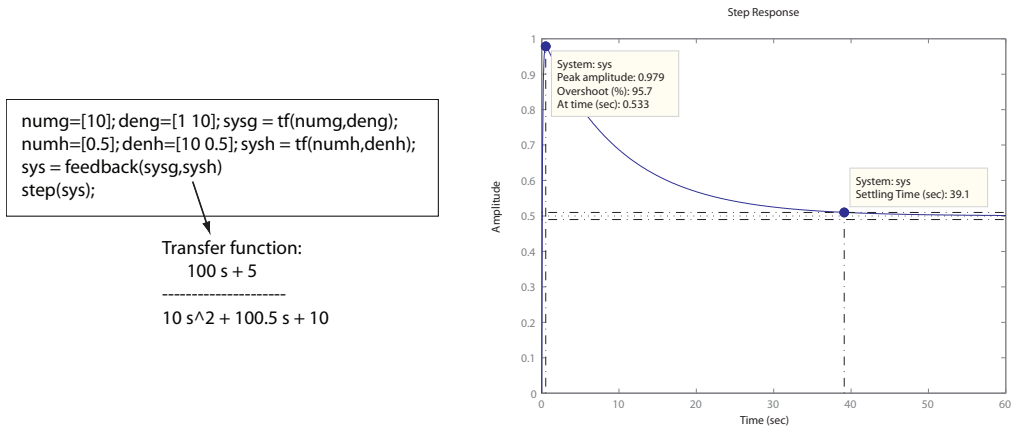
**FIGURE CP5.8**  
Missile rate loop autopilot simulation.

can be explained by realizing that the system is not a second-order system. The closed-loop system actually has two zeros, one real pole, and two complex-conjugate poles:

$$T(s) = \frac{(s + 50)(s + 1)}{(s + 0.8324)(s^2 + 11.1676s + 600.7027)} .$$

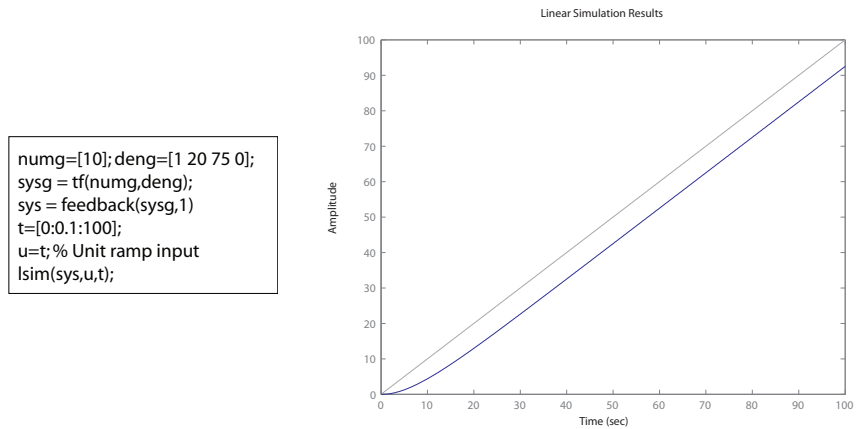
The effect of the pole at  $s = -0.8324$  is diminished by the zero at  $s = -1$ . The third pole and the zeros affect the overall response such that the analytic formulas for second-order systems are not exact predictors of the transient response.

**CP5.9** Figure CP5.9 shows an m-file to compute the closed-loop transfer function and to simulate and plot the step response.



**FIGURE CP5.9** M-file to compute the transfer function and to simulate the step response.

**CP5.10** Figure CP5.10 shows an m-file to compute the closed-loop transfer function and to simulate and plot the ramp response. The steady-state error



**FIGURE CP5.10** M-file to compute the transfer function and to simulate the ramp response.

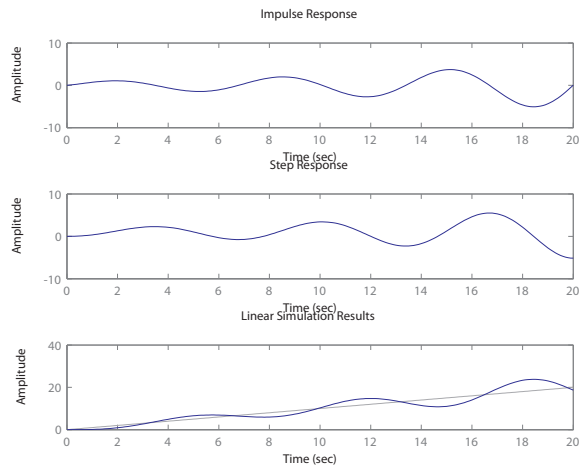
is 7.5.

**CP5.11** Figure CP5.11 shows an m-file to compute the closed-loop transfer function and to simulate and plot the impulse, step, and ramp responses. Notice that the closed-loop system is unstable.

```

numg=[1]; deng=[1 2 0]; sysg = tf(numg,deng);
numc=[0.5 2]; denc=[1 0]; sysc = tf(numc,denc);
sysss=series(sysg,sysc);
sys = feedback(sysss,1)
t=[0:0.1:20];
subplot(311)
impz(sys,t);
subplot(312)
step(sys,t);
subplot(313)
u=t; % Unit ramp input
lsim(sys,u,t);

```

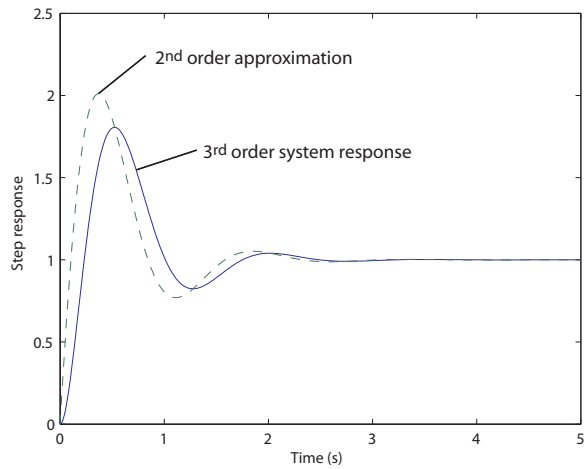


**FIGURE CP5.11**

M-file to compute the transfer function and to simulate the ramp response.

**CP5.12** Figure CP5.12 shows an m-file to simulate and plot the step response for the original system and the 2nd-order approximation. For the original system, we find  $T_s = 2.28$  and  $P.O. = 80.6\%$ . For the 2nd-order approximation we find  $T_s = 2.16$  and  $P.O. = 101\%$

```
num=77*[1 2];den=conv([1 7],[1 4 22]);  
sys = tf(num,den)  
na=(77/7)*[1 2];da=[1 4 22];sysa=tf(na,da);  
t=[0:0.01:5];  
y=step(sys,t);  
ya=step(sysa,t);  
plot(t,y,t,ya,'-')  
xlabel('Time (s)'),ylabel('Step response')
```

**FIGURE CP5.12**

Step response.

# *The Stability of Linear Feedback Systems*

## Exercises

**E6.1** The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & K+1 \\ s^2 & K & 6 \\ s^1 & b & 0 \\ s^0 & 6 & \end{array}$$

where

$$b = \frac{K(K+1) - 6}{K}.$$

For stability, we require  $K > 0$  and  $b > 0$ . Therefore, using the condition that  $b > 0$ , we obtain

$$K^2 + K - 6 > 0,$$

and solving for  $K$  yields  $K > 2$  and  $K < -3$ . We select  $K > 2$ , since we also have the condition that  $K > 0$ .

**E6.2** The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 10 & 30 \\ s^1 & -1 & 0 \\ s^0 & 30 & \end{array}$$

The system is unstable since the first column shows two sign changes.

**E6.3** The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 32 & 20 \\ s^3 & 10 & 37 & \\ s^2 & 28.3 & 20 & \\ s^1 & 29.9 & & \\ s^0 & 20 & & \end{array}$$

By the Routh-Hurwitz criterion, the system is stable (i.e., all the numbers in the first column are positive).

**E6.4** The closed-loop transfer function is

$$T(s) = \frac{-K(s-2)}{s^3 + 5s^2 + (4-K)s + 2K} .$$

Therefore, the characteristic equation is

$$s^3 + 5s^2 + (4-K)s + 2K = 0 .$$

The corresponding Routh array is given by

$$\begin{array}{c|cc} s^3 & 1 & 4-K \\ s^2 & 5 & 2K \\ s^1 & b & 0 \\ s^0 & 2K & \end{array}$$

where

$$b = \frac{5(4-K) - 2K}{5} = \frac{20 - 7K}{5} .$$

For stability we require  $K > 0$  and  $b > 0$ . Thus, the range of  $K$  for stability is  $0 < K < 20/7$ .

**E6.5** The closed-loop transfer function is

$$T(s) = \frac{K}{s^3 + 10s^2 + 27s + 18 + K} .$$

When  $K = 20$ , the roots of the characteristic polynomial are

$$s_{1,2} = -1.56 \pm j1.76$$



and

$$s_3 = -6.88 .$$

**E6.6** When  $K = 252$ , the roots of the characteristic equation are on the imaginary axis. The roots are

$$s_{1,2} = \pm j5.2 \quad \text{and} \quad s_3 = -10 .$$

**E6.7** (a) The closed-loop system characteristic equation is

$$1 + GH(s) = 1 + \frac{K(s+2)}{s(s-1)} = 0 ,$$

or

$$s^2 + (K-1)s + 2K = 0 .$$

We have the relationships  $\omega_n = \sqrt{2K}$  and  $2\zeta\omega_n = K-1$ , where  $\zeta = 0.707$ . Thus,

$$2 \left( \frac{1}{\sqrt{2}} \right) \sqrt{2K} = K-1 ,$$

or

$$\left( \frac{2}{\sqrt{2}} \right)^2 = \left( \frac{K-1}{\sqrt{2K}} \right)^2 ,$$

and

$$K^2 - 6K + 1 = 0 .$$

Solving for  $K$  yields  $K = 5.83$  and  $K = 0.17$ . However, for stability we require  $K > 1$  (from the Routh array), so we select  $K = 5.83$ .

(b) The two roots on the imaginary axis when  $K = 1$  are  $s_{1,2} = \pm j\sqrt{2}$ .

**E6.8** The closed-loop system characteristic equation is

$$s^3 + 20s^2 + (100 + K)s + 20K = 0 .$$

The corresponding Routh array is

$$\begin{array}{c|cc} s^3 & 1 & (100 + K) \\ s^2 & 20 & 20K \\ s^1 & b & 0 \\ s^0 & 20K & \end{array}$$

where

$$b = \frac{20(100 + K) - 20K}{20} = \frac{20(100)}{20} = 100 .$$

Therefore, the system is stable for all  $K > 0$ .

**E6.9** The characteristic equation is

$$s^3 + 2s^2 + (K + 1)s + 8 = 0 ,$$

and the Routh array is given by

$$\begin{array}{c|cc} s^3 & 1 & K + 1 \\ s^2 & 2 & 8 \\ s^1 & b & 0 \\ s^0 & 8 & \end{array}$$

where

$$b = \frac{2(K + 1) - 8}{2} = K - 3 .$$

Setting  $b = 0$ , yields

$$K - 3 = 0 \quad \text{or} \quad K > 3 .$$

**E6.10** Stable with your eyes open and (generally) unstable with your eyes closed.

**E6.11** The system is unstable. The poles are  $s_1 = -5.66$ ,  $s_2 = -0.90$  and  $s_{3,4} = 0.28 \pm j0.714$ .

**E6.12** The characteristic equation is

$$s^2 + as + b = 0,$$

so, the Routh array is

$$\begin{array}{c|cc} s^2 & 1 & b \\ s^1 & a & 0 \\ s^0 & b & \end{array}$$

The system is stable if and only if  $a > 0$  and  $b > 0$ . For a second-order system, a necessary and sufficient condition for stability is that all the coefficients have the same sign.

**E6.13** The characteristic equation is

$$s^2 + (K_D + 2)s + 4K_P = 0.$$

The Routh array is

$$\begin{array}{c|cc} s^2 & 1 & 4K_P \\ s^1 & K_D + 2 & 0 \\ s^0 & 4K_P & \end{array}$$

The system is stable if and only if  $K_P > 0$  and  $K_D > -2$ .

**E6.14** The characteristic equation associated with the system matrix is

$$s^3 + 3s^2 + 5s + 6 = 0 .$$

The roots of the characteristic equation are  $s_1 = -2$  and  $s_{2,3} = -5 \pm j1.66$ . The system is stable.

**E6.15** The roots of  $q(s)$  are  $s_1 = -4$ ,  $s_2 = -3$ ,  $s_{3,4} = -1 \pm j2$  and  $s_{5,6} = \pm j0.5$ . The system is marginally stable. The Routh array is

$$\begin{array}{c|cccc} s^6 & 1 & 31.25 & 67.75 & 15 \\ s^5 & 9 & 61.25 & 14.75 & \\ s^4 & 24.44 & 66.11 & 15 & \\ s^3 & 31.909 & 9.2273 & 0 & \\ s^2 & 60 & 15 & & \\ s^1 & 0 & 0 & & \\ s^0 & & & & \end{array}$$

The auxillary equation is

$$60s^2 + 15 = 0 .$$

Solving the auxillary equation yields two roots at  $s_{1,2} = \pm j0.5$ . After accounting for the row of zeros, the completed Routh array verifies that the system has no poles in the right half-plane.

**E6.16** The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 45 & 50 \\ s^3 & 9 & 87 & \\ s^2 & 35.33 & 50 & \\ s^1 & 74.26 & 0 & \\ s^0 & 50 & & \end{array}$$

The system is stable. The roots of  $q(s)$  are  $s_{1,2} = -3 \pm j4$ ,  $s_3 = -2$  and  $s_4 = -1$ .

**E6.17** The characteristic equation is

$$s^3 + 7s^2 + 36s + 24 = 0 .$$

The system is stable. The roots of the characteristic equation are  $s_1 = -0.77$ ,  $s_{2,3} = -3.12 \pm 4.64i$ .

**E6.18** The roots of  $q(s)$  are  $s_1 = -20$  and  $s_{2,3} = \pm j2.24$ . The system is marginally stable. The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 20 & 100 \\ s^1 & 0 & 0 \\ s^0 & & \end{array}$$

The auxillary equation is

$$20s^2 + 100 = 0 .$$

The roots are  $s = \pm j2.24$ . So, the system has roots at  $s = \pm j2.24$ . Completing the Routh array (after accounting for the row of zeros) verifies that no poles lie in the right half-plane.

- E6.19** (a) Unstable.  
 (b) Stable.  
 (c) Stable.

- E6.20** (a) The roots are  $s_{1,2} = -2$  and  $s_3 = -1$ .  
 (b) The roots are  $s_{1,2,3} = -3$ .

**E6.21** The characteristic equation is

$$(s_n - 2)^3 + 10(s_n - 2)^2 + 29(s_n - 2) + K = 0$$

or

$$s_n^3 + 4s_n^2 + s_n - 26 + K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 4 & K - 26 \\ s^1 & \frac{30-K}{4} & 0 \\ s^0 & K - 26 & \end{array}$$

If  $K = 30$ , then the auxiliary equation is  $4s_n^2 + 4 = 0$  or  $s_n = \pm j$ . Therefore,  $s = s_n - 2$  implies  $s = -2 \pm j$ .

**E6.22** This system is not stable. The output response to a step input is a ramp  $y(t) = kt$ .

**E6.23** The characteristic polynomial is

$$s^3 + 4s^2 + ks + 8 = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & k \\ s^2 & 4 & 8 \\ s^1 & \frac{4k-8}{4} & \\ s^0 & 8 & \end{array}$$

So,  $k > 2$  for stability.

**E6.24** The transfer function is

$$\begin{aligned} G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ &= [1 \ 0 \ 0] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k & k & s+k \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= [1 \ 0 \ 0] \begin{bmatrix} s^2 + ks + k & s+k & 1 \\ -k & s^2 + ks & s \\ -ks & -ks - k & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\Delta(s)} \end{aligned}$$

where  $\Delta(s) = s^3 + ks^2 + ks + k$ . Thus, the transfer function is

$$G(s) = \frac{1}{s^3 + ks^2 + ks + k} .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & k \\ s^2 & k & k \\ s^1 & k-1 & \\ s^0 & k & \end{array}$$

For stability  $k > 1$ .

**E6.25** The closed-loop transfer function is

$$T(s) = \frac{Ks + 1}{s^2(s + p) + Ks + 1}.$$

Therefore, the characteristic equation is

$$s^3 + ps^2 + Ks + 1 = 0.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & K \\ s^2 & p & 1 \\ s^1 & (pK - 1)/p & \\ s^0 & 1 & \end{array}$$

We see that the system is stable for any value of  $p > 0$  and  $pK - 1 > 0$ .

**E6.26** The closed-loop transfer function is

$$T(s) = \frac{10}{2s^2 + (K - 20)s + 10 - 10K}.$$

Therefore, the characteristic equation is

$$2s^2 + (K - 20)s + 10 - 10K = 0.$$

The Routh array is

$$\begin{array}{c|cc} s^2 & 2 & 10-10K \\ s^1 & K-20 & \\ s^0 & 10-10K & \end{array}$$

We see that the system is stable for any value of  $K > 20$  and any  $K < 1$ . Therefore, the system is unstable for all  $K > 0$  since the gain  $K$  cannot be simultaneously greater than 20 and less than 1.

## Problems

P6.1 (a) Given

$$s^2 + 5s + 2 ,$$

we have the Routh array

$$\begin{array}{c|cc} s^2 & 1 & 2 \\ s^1 & 5 & 0 \\ s^0 & 2 & \end{array}$$

Each element in the first column is positive, thus the system is stable.

(b) Given

$$s^3 + 4s^2 + 8s + 4 ,$$

we have the Routh array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 4 & 4 \\ s^1 & 7 & 0 \\ s^0 & 4 & \end{array}$$

Each element in the first column is positive, thus the system is stable.

(c) Given

$$s^3 + 2s^2 - 6s + 20 ,$$

we determine by inspection that the system is unstable, since it is necessary that all coefficients have the same sign. There are two roots in the right half-plane.

(d) Given

$$s^4 + s^3 + 2s^2 + 12s + 10 ,$$

we have the Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 10 \\ s^3 & 1 & 12 & 0 \\ s^2 & -10 & 10 & 0 \\ s^1 & 13 & 0 & \\ s^0 & 10 & & \end{array}$$

There are two sign changes in the first column, thus the system is unstable with two roots in the right half-plane.

(e) Given

$$s^4 + s^3 + 3s^2 + 2s + K ,$$

we have the Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 3 & K \\ s^3 & 1 & 2 & 0 \\ s^2 & 1 & K & \\ s^1 & 2 - K & 0 & \\ s^0 & K & & \end{array}$$

Examining the first column, we determine that the system is stable for  $0 < K < 2$ .

(f) Given

$$s^5 + s^4 + 2s^3 + s + 6 ,$$

we know the system is unstable since the coefficient of the  $s^2$  term is missing. There are two roots in the right half-plane.

(g) Given

$$s^5 + s^4 + 2s^3 + s^2 + s + K ,$$

we have the Routh array

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 1 & K \\ s^3 & 1 & 1 - K & \\ s^2 & K & K & \\ s^1 & -K & 0 & \\ s^0 & K & & \end{array}$$

Examining the first column, we determine that for stability we need  $K > 0$  and  $K < 0$ . Therefore the system is unstable for all  $K$ .

**P6.2** (a) The closed-loop characteristic polynomial is

$$s^4 + 27.88s^3 + 366.4s^2 + 1500s + 1500k_a = 0 .$$

The Routh array is



$$\begin{array}{c|ccc}
 s^4 & 1 & 366.4 & 1500k_a \\
 s^3 & 27.88 & 1500 & \\
 s^2 & 312.6 & 1500k_a & \\
 s^1 & b & & \\
 s^o & 1500k_a & & 
 \end{array}$$

where

$$b = 1500 - 133.78k_a .$$

Examining the first column of the Routh array, we find that  $b > 0$  and  $1500k_a > 0$  for stability. Thus,

$$0 < k_a < 11.21 .$$

(b) With

$$T_s = 1.5 = \frac{4}{\zeta\omega_n} ,$$

we determine that

$$\zeta\omega_n = 2.67 .$$

So, shift the axis by  $s = s_o - 2.67$ , and

$$\begin{aligned}
 (s_o - 2.67)^4 + 27.88(s_o - 2.67)^3 + 366.4(s_o - 2.67)^2 + 1500(s_o - 2.67) + \\
 1500k_a = s_o^4 + 17.2s_o^3 + 185.85s_o^2 + 63.55s_o - 1872.8 + 1500k_a .
 \end{aligned}$$

The Routh array is

$$\begin{array}{c|ccc}
 s^4 & 1 & 185.85 & 1500k_a-1872.8 \\
 s^3 & 17.2 & 63.55 & \\
 s^2 & 182.16 & 1500k_a-1872.8 & \\
 s^1 & b & & \\
 s^o & 1500k_a-1872.8 & & 
 \end{array}$$

where

$$b = 240.38 - 141.63k_a .$$

Examining the first column of the Routh array, we find that  $b > 0$  and  $1500k_a - 1872.8 > 0$ . Thus,  $1.25 < k_a < 1.69$ .

**P6.3** (a) Given

$$G(s) = \frac{K}{(s+1)(s+2)(0.5s+1)},$$

and

$$H(s) = \frac{1}{0.005s+1},$$

the closed-loop transfer function is

$$T(s) = \frac{K(0.005s+1)}{0.0025s^4 + 0.5125s^3 + 2.52s^2 + 4.01s + 2 + K}.$$

Therefore, the characteristic equation is

$$0.0025s^4 + 0.5125s^3 + 2.52s^2 + 4.01s + (2 + K) = 0.$$

The Routh array is given by

$s^4$	0.0025	2.52	2 + K
$s^3$	0.5125	4.01	0
$s^2$	2.50	2 + K	
$s^1$	3.6 - 0.205K	0	
$s^0$	2 + K		

Examining the first column, we determine that for stability we require

$$-2 < K < 17.6.$$

(b) Using  $K = 9$ , the roots of the characteristic equation are

$$s_1 = -200, \quad s_{2,3} = -0.33 \pm 2.23j, \quad \text{and} \quad s_4 = -4.35.$$

Assuming the complex roots are dominant, we compute the damping ratio  $\zeta = 0.15$ . Therefore, we estimate the percent overshoot as

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 62\%.$$

The actual overshoot is 27%, so we see that assuming that the complex poles are dominant does not lead to accurate predictions of the system response.

**P6.4** (a) The closed-loop characteristic equation is

$$1 + GH(s) = 1 + \frac{K(s+40)}{s(s+10)(s+20)} = 0,$$

or

$$s^3 + 30s^2 + 200s + Ks + 40K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 200 + K \\ s^2 & 30 & 40K \\ s^1 & 200 - \frac{K}{3} & 0 \\ s^0 & 40K & \end{array}$$

Therefore, for stability we require  $200 - K/3 > 0$  and  $40K > 0$ . So, the range of  $K$  for stability is

$$0 < K < 600 .$$

(b) At  $K = 600$ , the auxiliary equation is

$$30s^2 + 40(600) = 0 \quad \text{or} \quad s^2 + 800 = 0 .$$

The roots of the auxiliary equation are

$$s = \pm j28.3 .$$

(c) Let  $K = 600/2 = 300$ . Then, to shift the axis, first define  $s_o = s + 1$ . Substituting  $s = s_o - 1$  into the characteristic equation yields

$$(s_o - 1)^3 + 30(s_o - 1)^2 + 500(s_o - 1) + 12000 = s_o^3 + 27s_o^2 + 443s_o + 11529 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 443 \\ s^2 & 27 & 11529 \\ s^1 & 16 & 0 \\ s^0 & 11529 & \end{array}$$

All the elements of the first column are positive, therefore all the roots lie to left of  $s = -1$ . We repeat the procedure for  $s = s_o - 2$  and obtain

$$s_o^3 + 24s_o^2 + 392s_o + 10992 = 0 .$$

The Routh array is

$$\begin{array}{r|ll}
 s^3 & 1 & 392 \\
 s^2 & 24 & 10992 \\
 s^1 & -66 & 0 \\
 s^o & 10992 & 
 \end{array}$$

There are two sign changes in the first column indicating two roots to right of  $s = -2$ . Combining the results, we determine that there are two roots located between  $s = -1$  and  $s = -2$ . The roots of the characteristic equation are

$$s_1 = -27.6250 \quad \text{and} \quad s_{2,3} = -1.1875 \pm 20.8082j .$$

We see that indeed the two roots  $s_{2,3} = -1.1875 \pm 20.8082j$  lie between -1 and -2.

**P6.5** (a) Given the characteristic equation,

$$s^3 + 3s^2 + 4s + 2 = 0 ,$$

we compute the roots  $s_1 = -1$ , and  $s_{2,3} = -1 \pm j$ .

(b) The roots of the characteristic equation

$$s^4 + 9s^3 + 30s^2 + 42s + 20 = 0$$

are  $s_1 = -1$ ,  $s_2 = -2$ , and  $s_{3,4} = -3 \pm j1$ .

(c) The roots of the characteristic equation

$$s^3 + 19s^2 + 110s + 200 = 0$$

are  $s_1 = -4$ ,  $s_2 = -5$ , and  $s_3 = -10$ .

**P6.6** (a) The characteristic equation is

$$1 + G(s) = 0 ,$$

or

$$s^3 + s^2 + 10s + 2 = 0 .$$

The roots are:  $s_1 = -0.2033$ , and  $s_{2,3} = -0.3984 \pm j3.1112$ .

(b) The characteristic equation is

$$s^4 + 10s^3 + 35s^2 + 50s + 24 = 0 .$$

The roots are  $s_1 = -1$ ,  $s_2 = -2$ ,  $s_3 = -3$ , and  $s_4 = -4$ .

(c) The characteristic equation is

$$s^3 + 11s^2 + 29s + 6 = 0 .$$

The roots are  $s_1 = -0.2258$ ,  $s_2 = -3.8206$  and  $s_3 = -6.9536$ .

**P6.7** (a) The closed-loop characteristic equation is

$$s^3 + 101s^2 + (100 + 10KK_a)s + 100KK_a = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 100 + 10KK_a \\ s^2 & 101 & 100KK_a \\ s^1 & b & \\ s^0 & 100KK_a & \end{array}$$

where

$$b = 100 + \frac{910}{101}KK_a > 0 .$$

Thus, examining the first column, we determine that  $KK_a > 0$  stabilizes the system.

(b) The tracking error is

$$e(s) = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{100}{s^2} = \frac{100}{KK_a} .$$

We require  $E(s) < 1^\circ = 0.01745$ . So,

$$KK_a > \frac{100}{0.01745} = 5729 .$$

When  $KK_a = 5729$ , the roots of the characteristic polynomial are

$$s_1 = -10.15 \quad \text{and} \quad s_{2,3} = -45.43 \pm j233.25 .$$

**P6.8** (a) The closed-loop characteristic equation is

$$1 + \frac{K}{(0.5s + 1)(s + 1)(\frac{1}{4}s + 1)} = 0 ,$$

or

$$s^3 + 7s^2 + 14s + 8(1 + K) = 0 .$$

The Routh array is

$$\begin{array}{l|ll} s^3 & 1 & 14 \\ s^2 & 7 & 8(1+K) \\ s^1 & b & \\ s^0 & 8(1+K) & \end{array}$$

where

$$b = \frac{7(14) - 8(1+K)}{7} .$$

For stability, we require  $b > 0$  and  $8(1+K) > 0$ . Therefore, the range of  $K$  for stability is

$$-1 < K < 11.25 .$$

(b) Let  $K = 11.25/3 = 3.75$ . Then, the closed-loop transfer function is

$$T(s) = \frac{3.37}{s^3 + 7s^2 + 14s + 38} .$$

The settling time to a step input is  $T_s \approx 6$  seconds.

(c) We want  $T_s = 4$  sec, so

$$T_s = 4 = \frac{4}{\zeta\omega_n} \quad \text{implies} \quad \zeta\omega_n = 1 .$$

Our desired characteristic polynomial is

$$(s + b)(s^2 + 2\zeta\omega_n s + \omega_n^2) = s^3 + (2 + b)s^2 + (\omega_n^2 + 2b)s + b\omega_n^2$$

where we have used the fact that  $\zeta\omega_n = 1$  and  $\omega_n$  and  $b$  are to be determined. Our actual characteristic polynomial is

$$s^3 + 7s^2 + 14s + 8(1+K) = 0 .$$

Comparing the coefficients of the actual and desired characteristic polynomials, we find the following relationships:

$$\begin{aligned} 2 + b &= 7 \\ \omega_n^2 + 2b &= 14 \\ b\omega_n^2 &= 8(1+K) . \end{aligned}$$

Solving these three equations yields

$$b = 5 , \quad \omega_n = 2 \quad \text{and} \quad K = 1.5 .$$

The actual settling time is  $T_s = 4.17$  sec. This is not exactly our

desired  $T_s$  since we have the contribution of the additional pole at  $s = -5$ . The closed-loop poles are

$$s_1 = -5 \quad \text{and} \quad s_{2,3} = -1 \pm 1.73j .$$

**P6.9** (a) The closed-loop characteristic equation is

$$1 + GH(s) = 1 + \frac{10K}{(s + 100)(s + 20)^2} ,$$

or

$$s^3 + 140s^2 + 4400s + 40000 + 10K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 4400 \\ s^2 & 140 & 40000 + 10K \\ s^1 & b & \\ s^0 & 40000 + 10K & \end{array}$$

where

$$b = \frac{140(4400) - (40000 + 10K)}{140} .$$

Examining the first column and requiring all the terms to be positive, we determine that the system is stable if

$$-4000 < K < 57600 .$$

(b) The desired characteristic polynomial is

$$(s + b)(s^2 + 1.38\omega_n s + \omega_n^2) = s^3 + (1.38\omega_n + b)s^2 + (\omega_n^2 + 1.38\omega_n b)s + b\omega_n^2$$

where we have used the fact that  $\zeta = 0.69$  to achieve a 5% overshoot, and  $\omega_n$  and  $b$  are to be determined. The actual characteristic polynomial is

$$s^3 + 140s^2 + 4400s + 40000 + 10K = 0 .$$

Equating the coefficients of the actual and desired characteristic polynomials, and solving for  $K$ ,  $b$ , and  $\omega_n$  yields

$$b = 104.2 , \quad \omega_n = 25.9 \quad \text{and} \quad K = 3003 .$$

So, a suitable gain is  $K = 3003$ .

**P6.10** (a) The closed-loop characteristic equation is

$$s^4 + 7s^3 + 20s^2 + (24 + K)s + 10K = 0 .$$

The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 20 & 10K \\ s^3 & 7 & 24 + K & 0 \\ s^2 & \frac{116-K}{7} & 10K & \\ s^1 & b & & \\ s^0 & 10K & & \end{array}$$

where

$$b = \frac{\left(\frac{116-K}{7}\right)(24 + K) - 70K}{\left(\frac{116-K}{7}\right)} .$$

Setting  $b > 0$  yields

$$2784 - 398K - K^2 > 0 ,$$

which holds when

$$-404.88 < K < 6.876 .$$

Examining the first column, we also find that  $K < 116$  and  $K > 0$  for stability. Combining all the stability regions, we determine that for stability

$$0 < K < 6.876 .$$

(b) When  $K = 6.876$ , the roots are

$$s_{1,2} = -3.5 \pm 1.63j , \quad \text{and} \quad s_{3,4} = \pm 2.1j .$$

**P6.11** Given

$$s^3 + (1 + K)s^2 + 10s + (5 + 15K) = 0 ,$$

the Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 1 + K & 5 + 15K \\ s^1 & b & \\ s^0 & 5 + 15K & \end{array}$$



where

$$b = \frac{(1+K)10 - (5+15K)}{1+K} = \frac{5-5K}{1+K}.$$

Given that  $K > 0$ , we determine that the system is stable when  $5-5K > 0$  or

$$0 < K < 1.$$

When  $K = 1$ , the  $s^2$  row yields the auxiliary equation

$$2s^2 + 20 = 0.$$

The roots are  $s = \pm j\sqrt{10}$ . So, the system frequency of oscillation is  $\sqrt{10}$  rads/sec.

**P6.12** The characteristic equation is

$$s^3 + as^2 + bs + c = 0,$$

so, the Routh array is

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c \\ s^1 & \frac{ab-c}{a} & \\ s^0 & c & \end{array}$$

For the system to be stable, we require that  $a > 0$ ,  $ab - c > 0$  and  $c > 0$ . When  $a > 0$  and  $c > 0$ , we know that  $b > 0$ . So, a necessary condition for stability is that all coefficients  $a$ ,  $b$ , and  $c$  be positive. The necessary and sufficient conditions for stability also require that  $b > c/a$ , in addition to  $a > 0$  and  $c > 0$ .

**P6.13** The characteristic equation is

$$s^3 + (p + 2\zeta\omega_n)s^2 + (2\zeta\omega_np + K\omega_n^2)s + K\omega_n^2z = 0.$$

The conditions for stability (see P6.12) are  $p + 2\zeta\omega_n > 0$ ,  $2\zeta\omega_np + K\omega_n^2 > (K\omega_n^2z)/(p + 2\zeta\omega_n)$ , and  $K\omega_n^2z > 0$ . Since we know that  $K > 0$ ,  $\zeta > 0$ , and  $\omega_n > 0$ , it follows that for stability  $z > 0$ ,  $p > -2\zeta\omega_n$ , and

$$2\zeta\omega_np + K\omega_n^2 > \frac{K\omega_n^2z}{p + 2\zeta\omega_n}.$$

**P6.14** The system has the roots

$$s_{1,2} = \pm j, \quad s_{3,4} = \pm j, \quad \text{and} \quad s_{5,6} = -1 \pm 3j,$$

Therefore, the system is not stable since there are repeated roots on the  $j\omega$ -axis.

- P6.15** (a) Neglecting the zeros and poles, we have the characteristic equation

$$s^4 + 30s^3 + 325s^2 + 2500s + K = 0 .$$

The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 325 & K \\ s^3 & 30 & 2500 & 0 \\ s^2 & 241.67 & K & \\ s^1 & b & & \\ s^0 & K & & \end{array}$$

where

$$b = \frac{604166.67 - 30K}{241.67} .$$

Therefore, the system is stable for  $0 < K < 20139$ .

- (b) Without neglecting the zeros and poles, the closed-loop characteristic equation is

$$\begin{aligned} s^6 + 90s^5 + 5525s^4 + 12400s^3 + (1255000 + K)s^2 \\ + (8500000 + 30K)s + 1125K = 0 . \end{aligned}$$

This is stable for

$$0 < K < 61818 .$$

We see that the additional poles and zero makes the system stable for a much larger gain  $K$ .

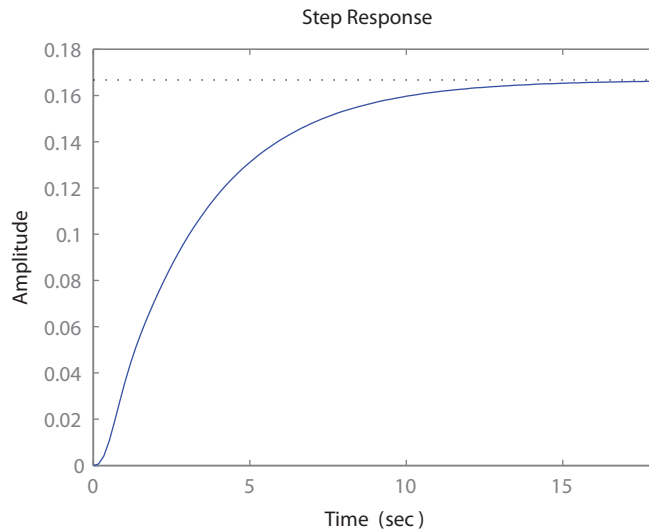
- P6.16** (a) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 5 & 6 \\ s^1 & 3.8 & \\ s^0 & 6 & \end{array}$$

Examining the first column of the Routh array, we see no sign changes. So, the system is stable.

- (b) The roots of the system are  $s_1 = -0.3246$  and  $s_{2,3} = -2.3377 \pm 3.6080j$ .

(c) The step response is shown in Figure P6.16.



**FIGURE P6.16**  
Unit step response.

**P6.17** The closed-loop transfer function is

$$T(s) = \frac{K + 1}{s^3 + 3s^2 + 3s + K + 1} .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & K + 1 \\ s^1 & \frac{8-K}{3} & \\ s^0 & K + 1 & \end{array}$$

So, for stability we require  $-1 < K < 8$ .

**P6.18** The system characteristic equation is

$$s^2 + (h - k)s + ab - kh = 0 .$$

For stability we require  $h > k$  and  $ab > kh$ . If  $k > h$ , the system is unstable.

**P6.19** (a) The characteristic equation is

$$s^3 + 9s^2 + (K - 10)s + 2K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & K - 10 \\ s^2 & 9 & 2K \\ s^1 & \frac{7K-90}{9} & \\ s^0 & 2K & \end{array}$$

For stability

$$K > 90/7 .$$

(b) When  $K = 90/7$ , the system is marginally stable. The roots are

$$s_{1,2} = \pm j\sqrt{20/7} ,$$

at the  $j\omega$ -axis crossing.

**P6.20** The closed-loop characteristic equation is

$$q(s) = s^5 + s^4 + 9s^3 + Ks^2 + 2Ks + K .$$

The range of stability for the vertical-liftoff vehicle is

$$5.177 < K < 7.823 .$$

Therefore, for  $K = 6$ , the system is stable. When  $K = 6$  we have

$$q(s) = s^5 + s^4 + 9s^3 + 6s^2 + 12s + 6$$

The Routh array is

$$\begin{array}{c|ccc} s^5 & 1 & 9 & 12 \\ s^4 & 1 & 6 & 6 \\ s^3 & 3 & 6 & \\ s^2 & 4 & 6 & \\ s^1 & 1.5 & & \\ s^0 & 6 & & \end{array}$$

All entries in the first column are positive, so the system is stable.

**P6.21** The state transition matrix is

$$\Phi(t, 0) = \frac{1}{p_2 - p_1} \begin{bmatrix} (k_2 - p_1)e^{-p_1 t} - (k_2 - p_2)e^{-p_2 t} & e^{-p_1 t} - e^{-p_2 t} \\ -k_1 e^{-p_1 t} + k_1 e^{-p_2 t} & -p_1 e^{-p_1 t} + p_2 e^{-p_2 t} \end{bmatrix}$$

where  $p_1 p_2 = k_1$  and  $p_1 + p_2 = k_2$ . We assume that  $p_1 \neq p_2$ . In the case when  $p_1 = p_2$ , the state transition matrix will change, but the factors  $e^{-p_1 t}$  and  $e^{-p_2 t}$  will remain. The eigenvalues of  $\mathbf{A}$  are given by the solution to

$$\det |\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 + k_2 \lambda + k_1 = 0 .$$

Therefore, the eigenvalues are  $\lambda_{1,2} = -k_2/2 \pm \sqrt{k_2^2 - 4k_1}$ . If  $k_2 > 0$  and  $k_1 > 0$ , then the eigenvalues are in the left half-plane, and the system is stable. The transfer function is given by

$$G(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = -\frac{s - 1}{s^2 + k_2 s + k_1} .$$

Therefore the characteristic equation is  $s^2 + k_2 s + k_1 = 0$  and the poles are  $s_{1,2} = -k_2/2 \pm \sqrt{k_2^2 - 4k_1}$ . If  $k_2 > 0$  and  $k_1 > 0$ , then the poles are in the left half-plane, and the system is stable. Notice that the values of  $\lambda_{1,2}$  and  $s_{1,2}$  are the same. Also, the eigenvalues are the same as the values of  $-p_1$  and  $-p_2$ . So, if the eigenvalues are negative, then the elements of the state transition matrix will decay exponentially.

## Advanced Problems

AP6.1 The Routh array is

$$\begin{array}{c|ccc}
 s^4 & 1 & K_1 & K_2 \\
 s^3 & 20 & 4 & \\
 s^2 & \frac{20K_1-4}{20} & K_2 & \\
 s^1 & b & 0 & \\
 s^0 & K_2 & & 
 \end{array}$$

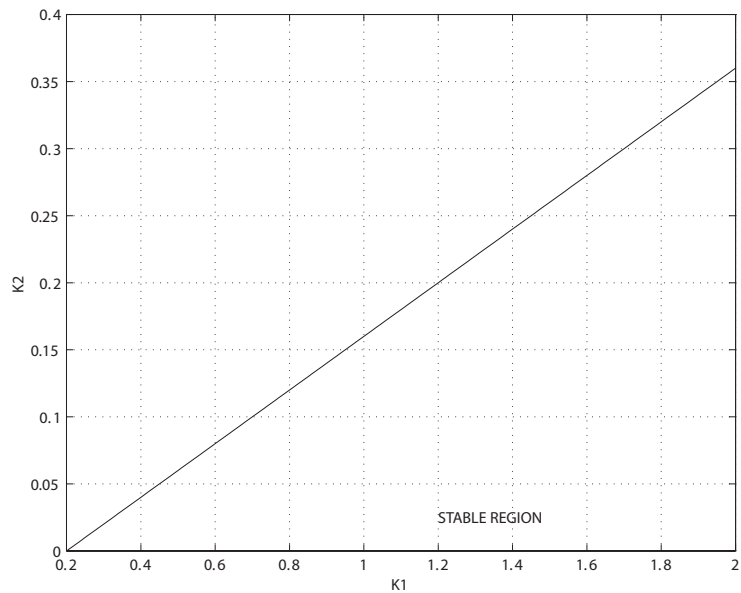
where

$$b = \frac{20K_1 - 4 - 100K_2}{5K_1 - 1}.$$

For stability, we require  $K_2 > 0$ ,  $K_1 > 0.2$ , and  $b > 0$ . Therefore, using the condition that  $b > 0$ , we obtain

$$K_2 < 0.2K_1 - 0.04.$$

The stability region is shown in Figure AP6.1.



**FIGURE AP6.1**  
Stability region.

**AP6.2** The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 30 & K \\ s^3 & 9 & K - 40 & \\ s^2 & \frac{310-K}{9} & K & \\ s^1 & b & 0 & \\ s^0 & K & & \end{array}$$

where

$$b = \frac{(310 - K)(K - 40) - 81K}{310 - K} .$$

Therefore, using the condition that  $b > 0$ , we obtain the stability range for  $K$ :

$$59.07 < K < 209.94 .$$

**AP6.3** (a) The steady-state tracking error to a step input is

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - T(s))R(s) = 1 - T(0) = 1 - \alpha .$$

We want

$$|1 - \alpha| < 0.05 .$$

This yields the bounds for  $\alpha$

$$0.95 < \alpha < 1.05 .$$

(b) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & \alpha \\ s^2 & 1 + \alpha & 1 \\ s^1 & b & 0 \\ s^0 & 1 & \end{array}$$

where

$$b = \frac{\alpha^2 + \alpha - 1}{1 + \alpha} .$$

Therefore, using the condition that  $b > 0$ , we obtain the stability range for  $\alpha$ :

$$\alpha > 0.618 .$$

(c) Choosing  $\alpha = 1$  satisfies both the steady-state tracking requirement and the stability requirement.

**AP6.4** The closed-loop transfer function is

$$T(s) = \frac{K}{s^3 + (p+1)s^2 + ps + K}.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & p \\ s^2 & 1+p & K \\ s^1 & b & 0 \\ s^0 & K & \end{array}$$

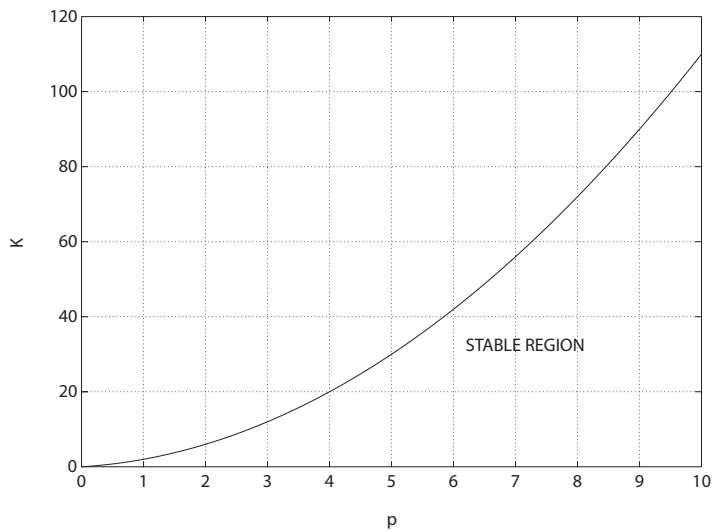
where

$$b = \frac{p^2 + p - K}{1 + p}.$$

Therefore, using the condition that  $b > 0$ , we obtain the relationship

$$K < p^2 + p.$$

The plot of  $K$  as a function of  $p$  is shown in Figure AP6.4.



**FIGURE AP6.4**  
Stability region.



**AP6.5** The closed-loop transfer function is

$$T(s) = \frac{30K_1K_2}{(s + 1 + K_1K_3)(s - 10)(2s + K_2K_3 - 4) + 30K_1K_2K_4} .$$

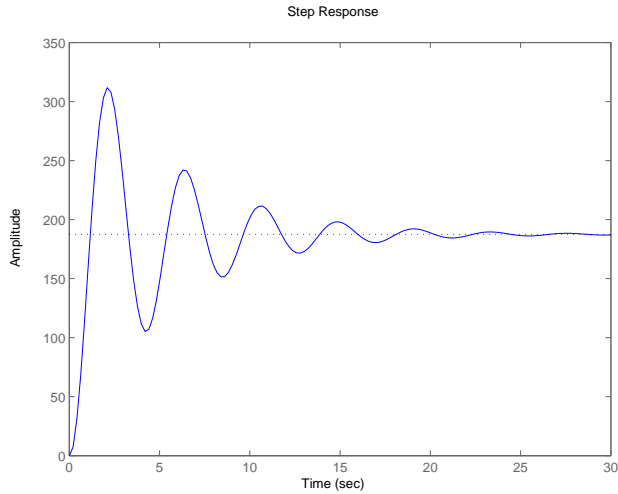
The Routh array is

$$\begin{array}{c|cc} s^3 & 2 & a \\ s^2 & b & c \\ s^1 & d & 0 \\ s^0 & c & \end{array}$$

where  $a = -9K_2K_3 + 16 + K_1K_2K_3^2 - 24K_1K_3$ ,  $b = 2K_1K_3 + K_2K_3 - 22$ , and  $c = -10K_2K_3 + 40 - 10K_1K_2K_3^2 + 40K_1K_3$  and  $d = (ab - 2c)/b$ . The conditions for stability are

$$\begin{aligned} 2K_1K_3 + K_2K_3 - 22 &> 0 \\ -10K_2K_3 + 40 - 10K_1K_2K_3^2 + 40K_1K_3 &> 0 \\ -2(-10K_2K_3 + 40 - 10K_1K_2K_3^2 + 40K_1K_3) + (9K_2K_3 \\ &+ 16 + K_1K_2K_3^2 - 24K_1K_3)(2K_1K_3 + K_2K_3 - 22) > 0 \end{aligned}$$

Valid values for the various gains are:  $K_1 = 50$ ,  $K_2 = 30$ ,  $K_3 = 1$ , and  $K_4 = 0.3$ . The step response is shown in Figure AP6.5.



**FIGURE AP6.5**  
Stability region.

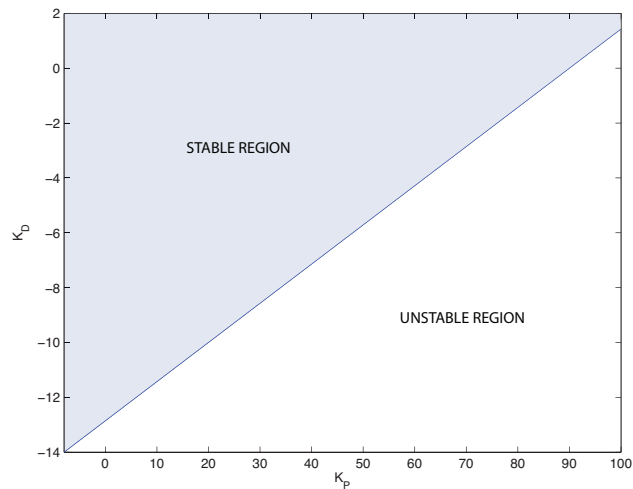
**AP6.6** The characteristic equation is

$$s^3 + 7s^2 + (K_D + 14)s + K_P + 8 = 0.$$

For stability we require that  $K_P > -8$  and

$$K_D > \frac{K_P + 8}{7} - 14.$$

The relationship between  $K_D$  and  $K_P$  is shown in Figure AP6.6.



**FIGURE AP6.6**  
Stability region.

**AP6.7** The characteristic equation is

$$0.1s^4 + 2.05s^3 + s^2 + 8K_P s + 8K_I = 0.$$

From the Routh array we find the conditions for stability are

$$0 < K_I < 0.3125$$

$$1.2812 - \sqrt{1.6416 - 5.2531K_I} < K_P < 1.2812 + \sqrt{1.6416 - 5.2531K_I}$$

## Design Problems

**CDP6.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$\frac{\theta(s)}{V_a(s)} = \frac{26.035}{s(s + 33.142)} .$$

In the above transfer function we have neglected the motor inductance  $L_m$ . The closed-loop transfer function from the input to the output is

$$\frac{\theta(s)}{R(s)} = \frac{26.035K_a}{s^2 + 33.142s + 26.035K_a} .$$

The Routh array is

$$\begin{array}{c|cc} s^2 & 1 & 26.035K_a \\ s^1 & 33.142 & 0 \\ s^0 & 26.035K_a & \end{array}$$

Stability is achieved for any  $0 < K_a < \infty$ .

**DP6.1** The closed-loop characteristic polynomial is

$$s^3 + s^2(5 + p + \frac{1}{5}K) + s(\frac{1}{5}Kp + \frac{1}{5}K + 5p) + K = 0 .$$

(i) When  $p = 2$ , we have

$$s^3 + s^2(7 + \frac{1}{5}K) + s(10 + \frac{3}{5}K) + K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 10 + \frac{3}{5}K \\ s^2 & 7 + \frac{K}{5} & K \\ s^1 & b & \\ s^0 & K & \end{array}$$

where

$$b = \frac{(7 + K/5)(10 + 3K/5) - K}{7 + \frac{1}{5}K} .$$

When  $-32.98 < K < -17.69$ , we find that  $b > 0$ . Examining the other terms in the first column of the array, we find that the system

is stable for any  $K > 0$ .

(ii) When  $p = 0$ , we have

$$s^3 + s^2\left(5 + \frac{1}{5}K\right) + s\left(\frac{1}{5}K\right) + K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & \frac{1}{5}K \\ s^2 & 5 + \frac{1}{5}K & K \\ s^1 & b & \\ s^0 & K & \end{array}$$

where

$$b = \frac{\left(5 + \frac{1}{5}K\right)\frac{1}{5}K - K}{\left(5 + K/5\right)} = \frac{K^2/25}{\left(5 + K/5\right)} .$$

Again, examination of the first column reveals that any  $K > 0$  results in a stable system. So, we just need to select any  $K > 0$ ; e.g.  $K = 10$ .

**DP6.2** (a) The closed-loop characteristic equation is

$$1 + \frac{20(Ks + 1)}{s^2(s + 20)} = 0 ,$$

or

$$s^3 + 20s^2 + 20Ks + 20 = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 20K \\ s^2 & 20 & 20 \\ s^1 & b & \\ s^0 & 1 & \end{array}$$

where

$$b = \frac{20K - 1}{1} .$$

For stability, we require  $K > 0.05$ .

(b) The desired characteristic polynomial is

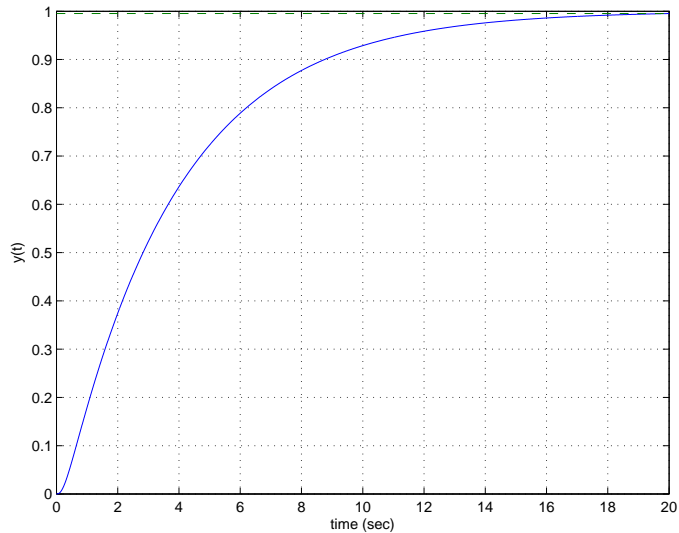
$$(s^2 + as + b)(s + 5) = s^3 + s^2(a + 5) + s(5a + b) + 5b = 0 .$$

Equating coefficients with the actual characteristic equation we can

solve for  $a$ ,  $b$  and  $K$ , yielding  $b = 4$ ,  $a = 15$ , and

$$K = \frac{5a + b}{20} = \frac{79}{20} .$$

- (c) The remaining two poles are  $s_1 = -14.73$  and  $s_2 = -0.27$ .  
 (d) The step response is shown in Figure DP6.2.



**FIGURE DP6.2**  
Mars guided vehicle step response.

- DP6.3** (a) The closed-loop characteristic equation is

$$2\tau s^3 + (\tau + 2)s^2 + (K + 1)s + 2K = 0 .$$

The Routh array is

$$\begin{array}{l|ll} s^3 & 2\tau & K + 1 \\ s^2 & \tau + 2 & 2K \\ s^1 & b & \\ s^0 & 2K & \end{array}$$

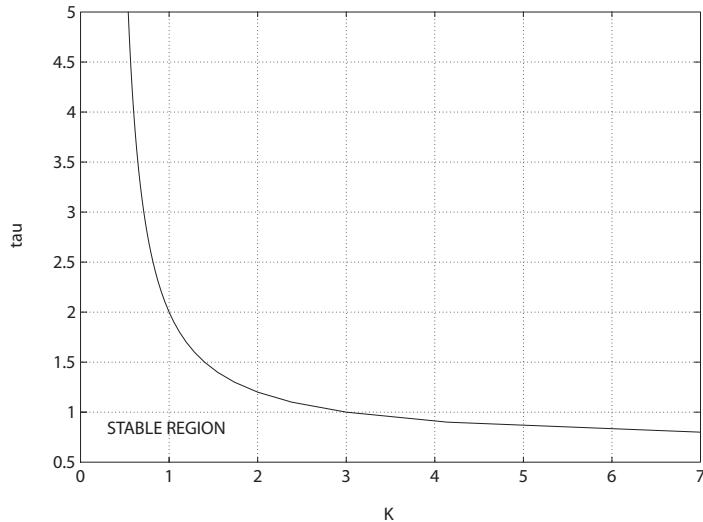
where

$$b = \frac{(\tau + 2)(K + 1) - 4K\tau}{(\tau + 2)} .$$

Examining the first column of the Routh array, we determine that for stability  $\tau > 0$ ,  $K > 0$  and setting  $b > 0$  yields the relationships:

$$(1) K < \frac{\tau + 2}{3\tau - 2} \quad \text{when } \tau > \frac{2}{3} \quad (2) K > 0 \quad \text{when } 0 < \tau \leq \frac{2}{3} .$$

The plot of  $\tau$  versus  $K$  is shown in Figure DP6.3a.



**FIGURE DP6.3**

(a) The plot of  $\tau$  versus  $K$ .

(b) The steady-state error is

$$e_{ss} = \frac{A}{K_v}, \quad \text{where } K_v = 2K .$$

So,

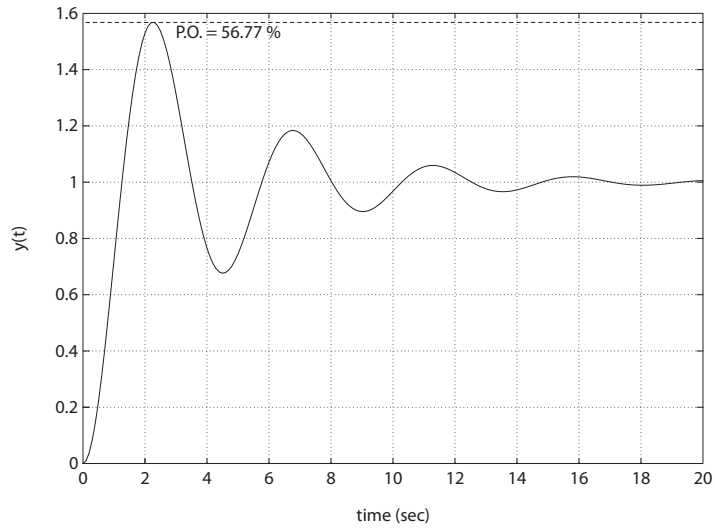
$$\frac{e_{ss}}{A} = \frac{1}{2K} .$$

We require that  $e_{ss} \leq 0.25A$ , therefore

$$K \geq 2 .$$

One solution is to select  $\tau = 0.5$ , then we remain within the stable region.

(c) The step response is shown in Figure DP6.3b. The percent overshoot is  $P.O. = 57\%$ .



**FIGURE DP6.3**  
CONTINUED: (b) Closed-loop system step response.

**DP6.4** (a) The closed-loop characteristic polynomial is

$$s^3 + Ks^2 + [(2 + m)K - 1]s + 2mK = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2K + mK - 1 \\ s^2 & K & 2mK \\ s^1 & b & \\ s^0 & 2mK & \end{array}$$

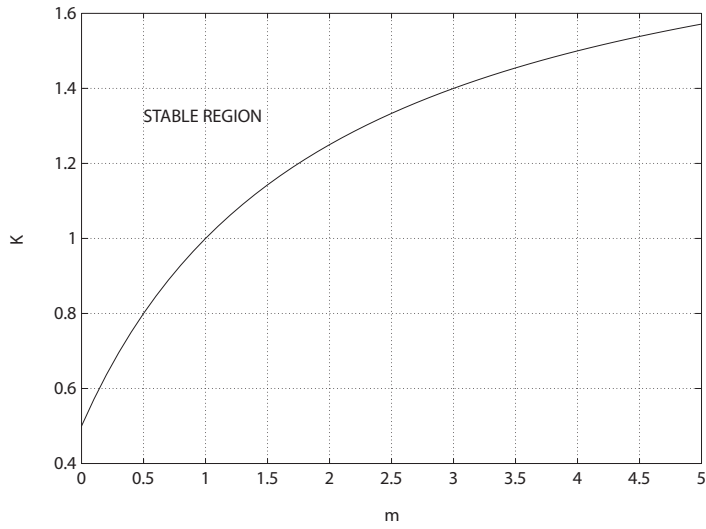
Examining the first column of the Routh array, we see that for stability we require  $m > 0, K > 0$ , and  $b > 0$ , where

$$b = \frac{(2K + mK - 1)K - 2mK}{K} = (2 + m)K - (1 + 2m) > 0 ,$$

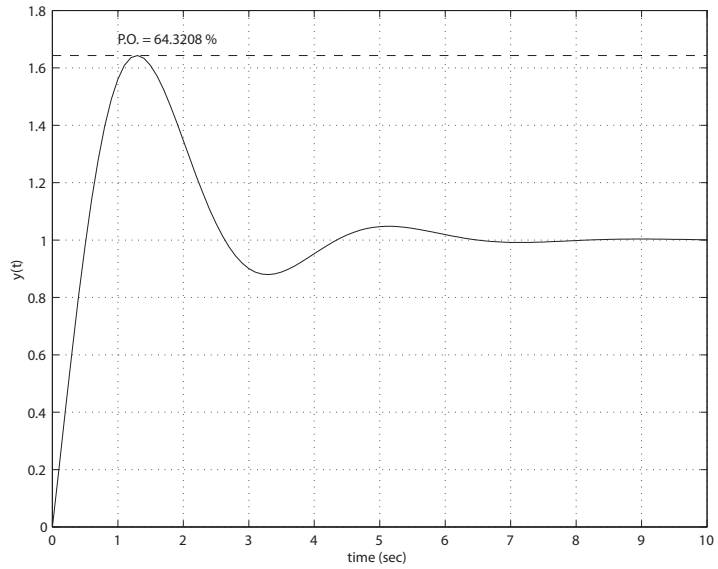
or

$$K > \frac{1 + 2m}{2 + m} .$$

The plot of  $K$  vs  $m$  is shown in Figure DP6.4a.



**FIGURE DP6.4**  
 (a) The plot of  $K$  versus  $m$ .



**FIGURE DP6.4**  
 CONTINUED: (b) Shuttle attitude control step response.

(b) The steady-state error is

$$\frac{e_{ss}}{A} = \frac{1}{K_v} = \frac{1}{2mK} < 0.10 ,$$



or  $mK > 5$ . For example, we can select  $m = 0.5$  and  $K = 2$ .

(c) See Figure DP6.4b for the step response where  $P.O. = 64.3\%$ .

**DP6.5** The closed-loop transfer function is

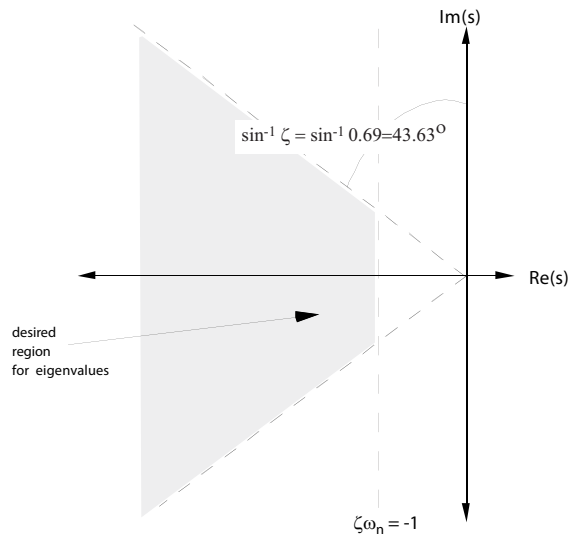
$$T(s) = \frac{K}{s^3 + 10s^2 + 20s + K}.$$

The range of  $K$  for stability is  $0 < K < 200$ . If we let  $K = K_m/N$  where  $K_m = 200$ , then  $N = 6.25$  results in a step response with  $P.O. = 15.7\%$  and  $T_s = 1.96$  seconds.

**DP6.6** The closed-loop system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 2 - K_1 & -2 - K_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

The characteristic polynomial is  $s^2 + (2 + K_2)s + K_1 - 2 = 0$ . So the system is stable for  $K_1 > 2$  and  $K_2 > -2$ . Selecting  $K = \begin{bmatrix} 10 & 2 \end{bmatrix}$  results in closed-loop eigenvalues at  $s = -2 \pm 2j$ . The closed-loop step response has a settling time of 2.11 s and a percent overshoot of 4.32%.



**FIGURE DP6.6**  
Performance region.

**DP6.7** (a) The inner loop closed-loop transfer function is

$$\frac{Y(s)}{U(s)} = \frac{20s}{s^3 + 10s^2 + 20s + 20K_1} .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 20 \\ s^2 & \omega & 20K_1 \\ s^1 & \frac{200-20K_1}{10} & \\ s^0 & 20K_1 & \end{array} .$$

For stability  $0 < K_1 < 10$ .

(b) The fastest response (that is, the quickest settling time) occurs when  $K_1 = 2.2$

(c) With  $K_1 = 2.2$ , the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{20K_2s}{s^3 + 10s^2 + (20 + 20K_2)s + 44} .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 20(K_2 + 1) \\ s^2 & 10 & 44 \\ s^1 & \frac{200K_2+156}{10} & \\ s^0 & 44 & \end{array} .$$

For stability, we require

$$200K_2 + 156 > 0 .$$

Therefore,  $K_2 > -0.78$ .

**DP6.8** The closed-loop characteristic equation is

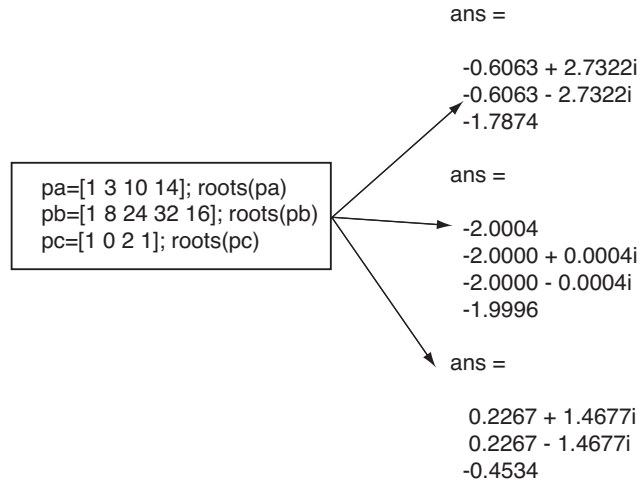
$$s^2 + 4K_Ds + 4(K_P + 1) = 0 .$$

So, it is possible to find  $K_P$  and  $K_D$  to stabilize the system. For example, any  $K_P > 0$  and  $K_D > 0$  leads to stability. Choosing  $K_P \geq 9$  results in a steady-state tracking error less than 0.1 due to a unit step input. Then, the damping ratio  $\zeta = \sqrt{2}/2$  is achieved by selecting

$$K_D = \frac{\sqrt{2}\sqrt{K_P + 1}}{2} .$$

## Computer Problems

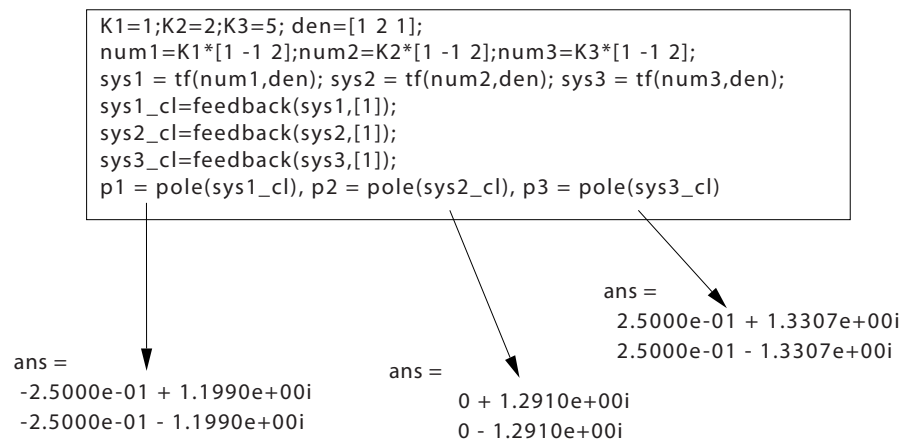
**CP6.1** The m-file script is shown in Figure CP6.1.



**FIGURE CP6.1**

Computing the polynomial roots with the **roots** function.

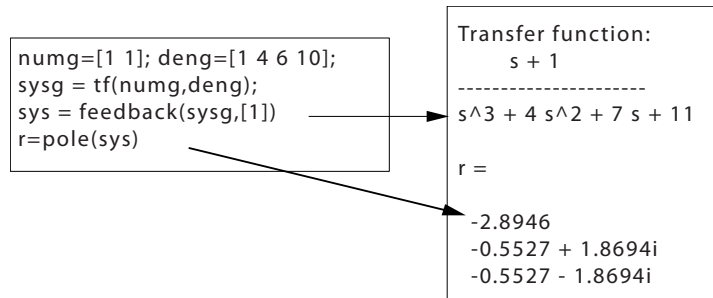
**CP6.2** The m-file script is shown in Figure CP6.2.



**FIGURE CP6.2**

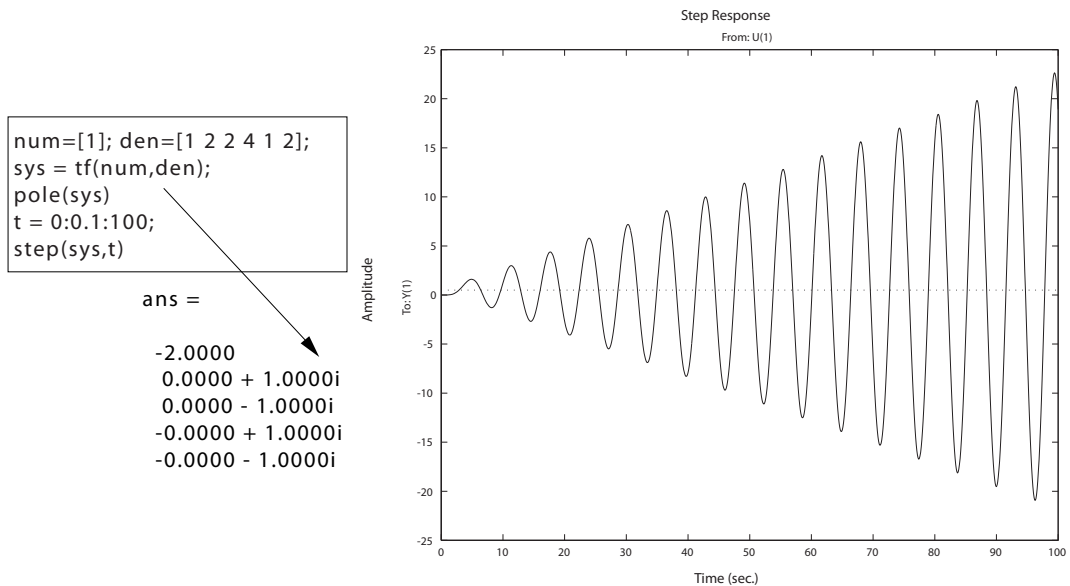
$K = 1$  is stable;  $K = 2$  is marginally stable; and  $K = 5$  is unstable.

**CP6.3** The closed-loop transfer function and the roots of the characteristic equation are shown in Figure CP6.3.



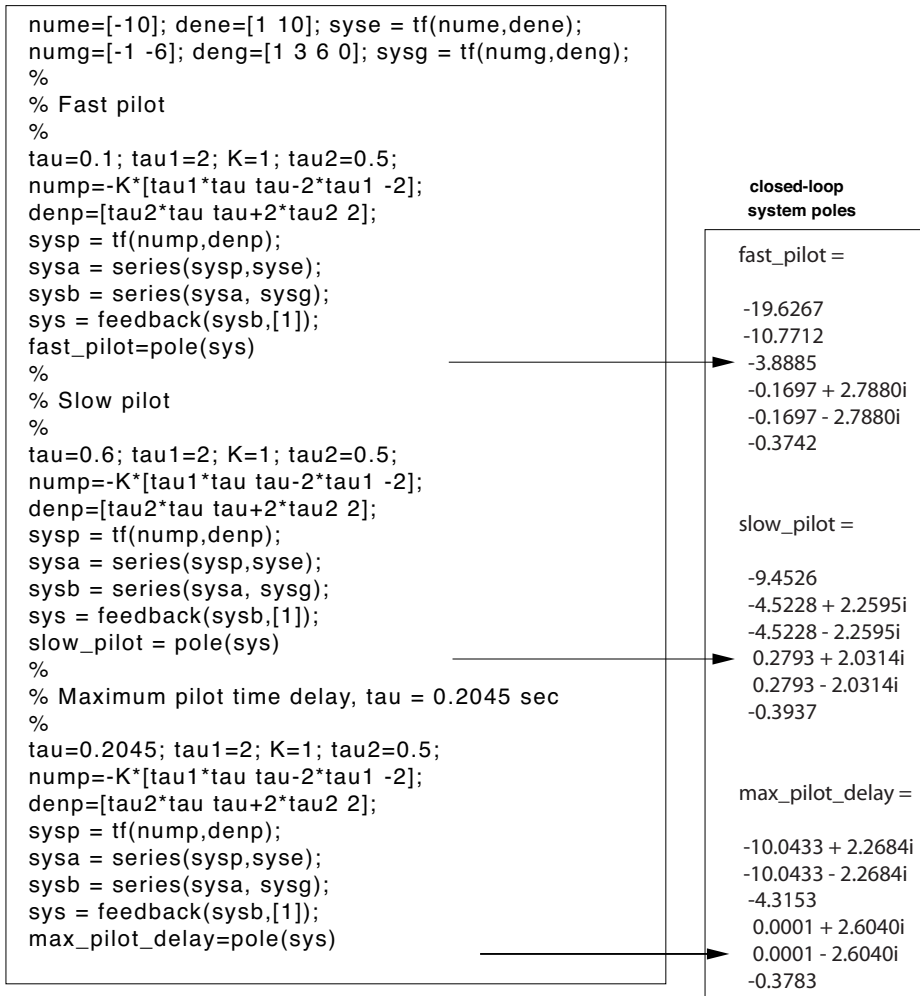
**FIGURE CP6.3**  
 Closed-loop transfer function and roots.

**CP6.4** There are no poles in the right half-plane, but the system is unstable since there are multiple poles on the  $j\omega$ -axis at  $s = \pm j$  and  $s = \pm j$  (see Figure CP6.4).



**FIGURE CP6.4**  
 Unstable system step response.

**CP6.5** The closed-loop system poles for the slow/fast pilots are shown in Figure CP6.5. The maximum allowable time delay is 0.2045seconds. At the maximum allowable time delay, the system has roots on the  $j\omega$ -axis at  $s = \pm 2.6j$ . The slow pilot destabilizes the aircraft.



**FIGURE CP6.5** Closed-loop system poles for an aircraft with a pilot in-the-loop.

**CP6.6** The closed-loop transfer function is

$$T(s) = \frac{1}{s^3 + 5s^2 + (K - 3)s + K + 1} .$$

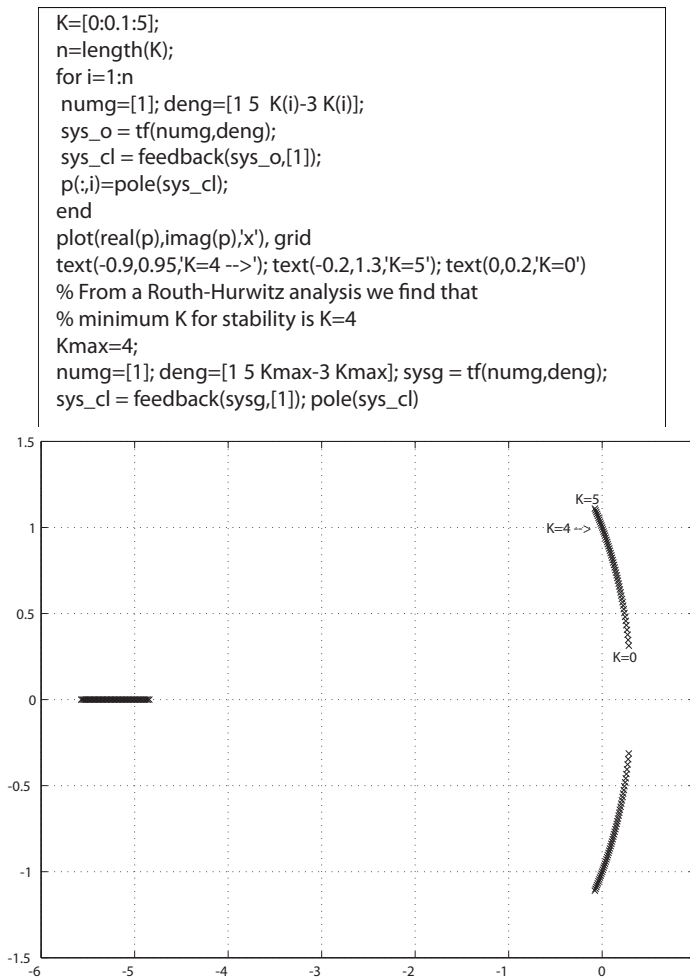
Utilizing the Routh-Hurwitz approach, for stability we determine that

$$K > 4 .$$

When  $K = 4$ , the roots of the characteristic equation are

$$s_1 = -5 \quad \text{and} \quad s_{2,3} = \pm j .$$

The m-file script which generates a plot of the roots of the characteristic equation as a function of  $K$  is shown in Figure CP6.6.

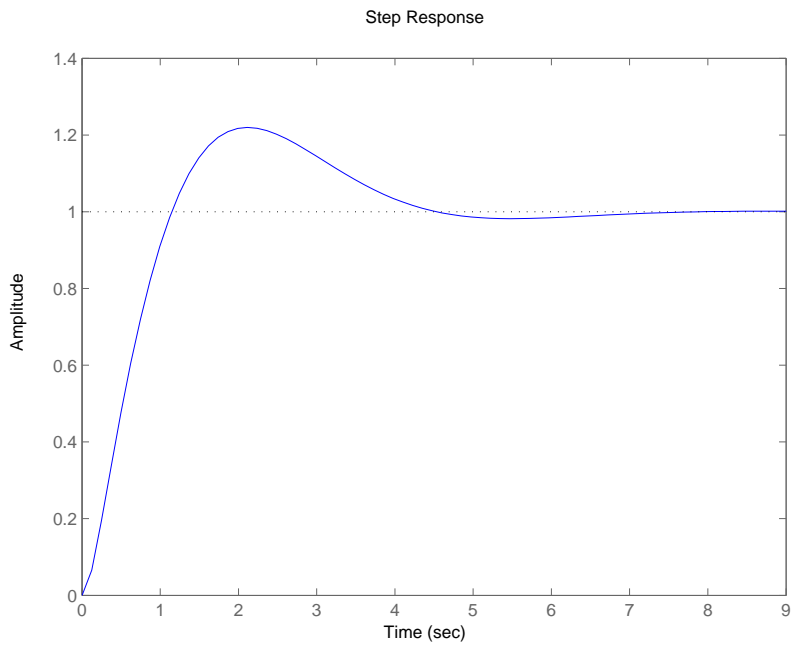
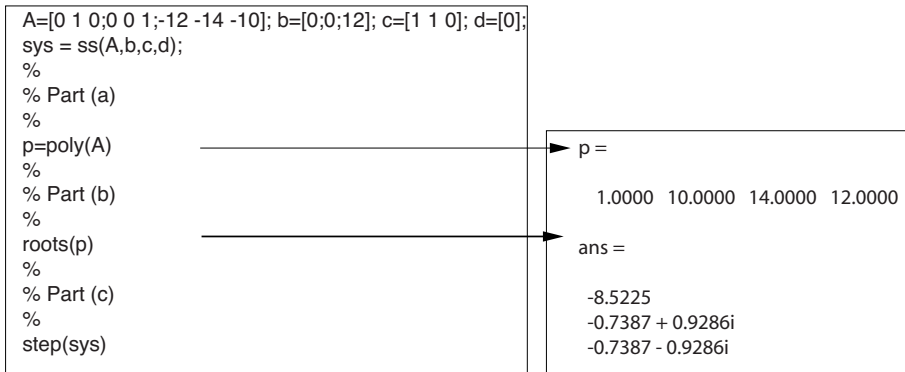


**FIGURE CP6.6**

Roots of the characteristic equation as a function of  $K$ , where  $0 < K < 5$ .

**CP6.7** The characteristic equation is

$$p(s) = s^3 + 10s^2 + 14s + 12 .$$



**FIGURE CP6.7** Characteristic equation from the state-space representation using the **poly** function.

The roots of the characteristic equation are

$$s_1 = -8.5225 \quad \text{and} \quad s_{2,3} = -0.7387 \pm 0.9286j .$$

The system is stable since all roots of the characteristic equation are in the left half-plane. The unit step response and associated m-file script are shown in Figure CP6.7.

**CP6.8** The characteristic equation is

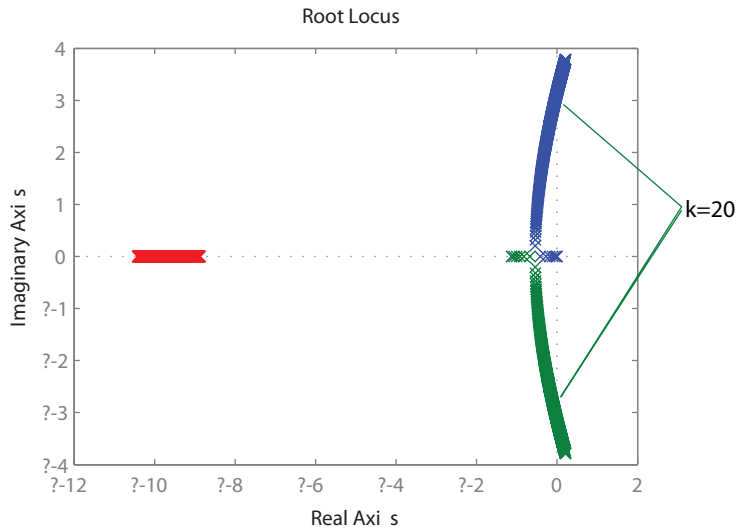
$$s^3 + 10s^2 + 10s + 5K_1 = 0 .$$

(a) The Routh array is

$s^3$	1	10
$s^2$	10	$5K_1$
$s^1$	$\frac{100-5K_1}{10}$	
$s^0$	$5K_1$	

From the Routh-Hurwitz criterion, we obtain the limits  $0 < K_1 < 20$  for stability.

(b) The plot of the pole locations is  $0 < K_1 < 30$  is shown in Figure CP6.8. As seen in Figure CP6.8, when  $K_1 > 20$ , the pole locations move into the right half-plane.



**FIGURE CP6.8**  
Pole locations for  $0 < K_1 < 30$ .

**CP6.9** (a) The characteristic equation is

$$s^3 + 2s^2 + s + k - 4 = 0 .$$

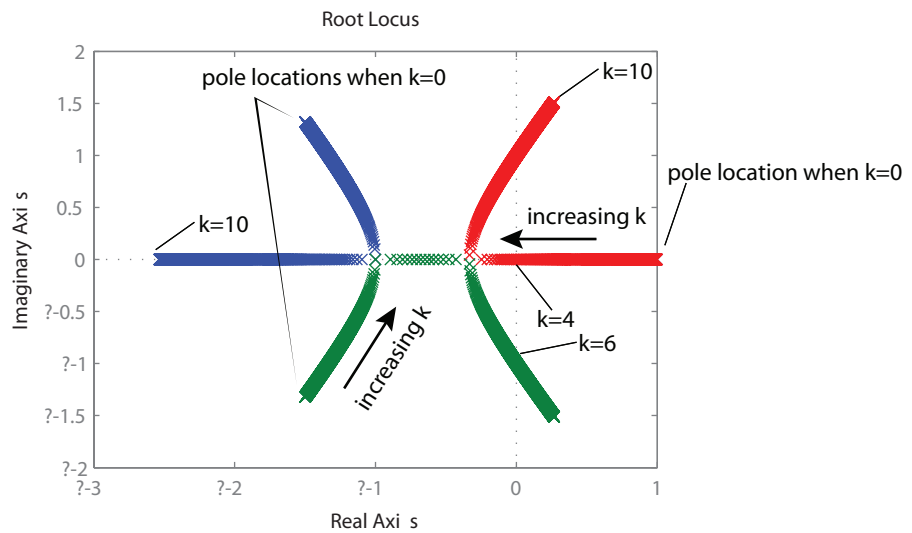


The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & k-4 \\ s^1 & \frac{6-k}{2} & \\ s^0 & k-4 & \end{array}$$

For stability, we obtain  $4 < k < 6$ .

- (b) The pole locations for  $0 < k < 10$  are shown in Figure CP6.9. We see that for  $0 < k < 4$  the system is unstable. Similarly, for  $6 < k < 10$ , the system is unstable.



**FIGURE CP6.9**  
Pole locations for  $0 < k < 10$ .

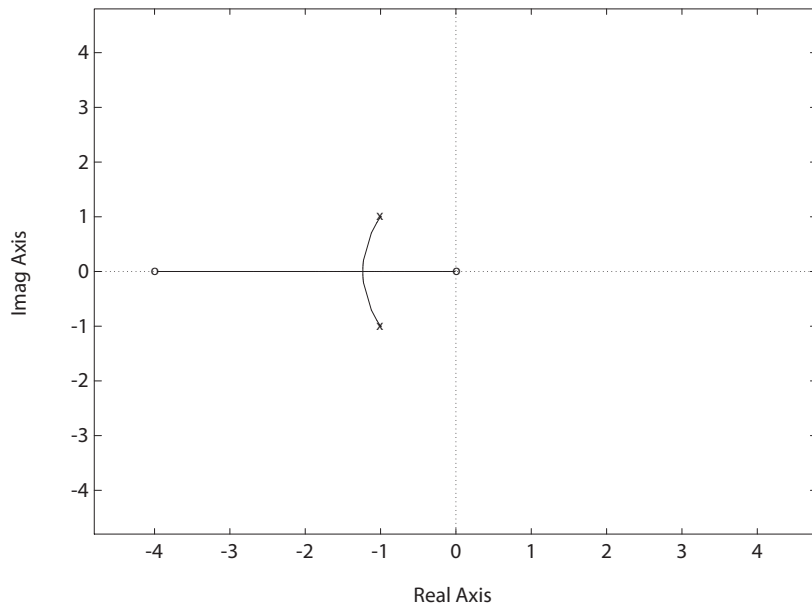
# The Root Locus Method

## Exercises

E7.1 (a) For the characteristic equation

$$1 + K \frac{s(s+4)}{s^2 + 2s + 2} = 0,$$

the root locus is shown in Figure E7.1.



**FIGURE E7.1**  
Root locus for  $1 + K \frac{s(s+4)}{s^2+2s+2} = 0$ .

(b) The system characteristic equation can be written as

$$(1 + K)s^2 + (2 + 4K)s + 2 = 0 .$$

Solving for  $s$  yields

$$s = \frac{-(1 + 2K)}{(1 + K)} \pm \frac{\sqrt{(2 + 4K)^2 - 8(1 + K)}}{2(1 + K)} .$$

When

$$(2 + 4K)^2 - 8(1 + K) = 0 ,$$

then we have two roots at  $s_{1,2} = -\frac{(1+2K)}{1+K}$ . Solving for  $K$  yields  $K = 0.31$ .

(c) When  $K = 0.31$ , the roots are

$$s_{1,2} = \frac{-(1 + 0.62)}{(1.31)} = -1.24 .$$

(d) When  $K = 0.31$ , the characteristic equation is

$$s^2 + 2.472s + 1.528 = (s + 1.24)^2 = 0 .$$

Thus,  $\omega_n = 1.24$  and  $\zeta = 1$ , the system is critically damped. The settling time is  $T_s \approx 4$  sec.

**E7.2** (a) The root locus is shown in Figure E7.2. When  $K = 6.5$ , the roots of the characteristic equation are

$$s_{1,2} = -2.65 \pm j1.23 \quad \text{and} \quad s_{3,4} = -0.35 \pm j0.8 .$$

The real part of the dominant root is 8 times smaller than the other two roots.

(b) The dominant roots are

$$(s + 0.35 + j0.8)(s + 0.35 - j0.8) = s^2 + 0.7s + 0.7625 .$$

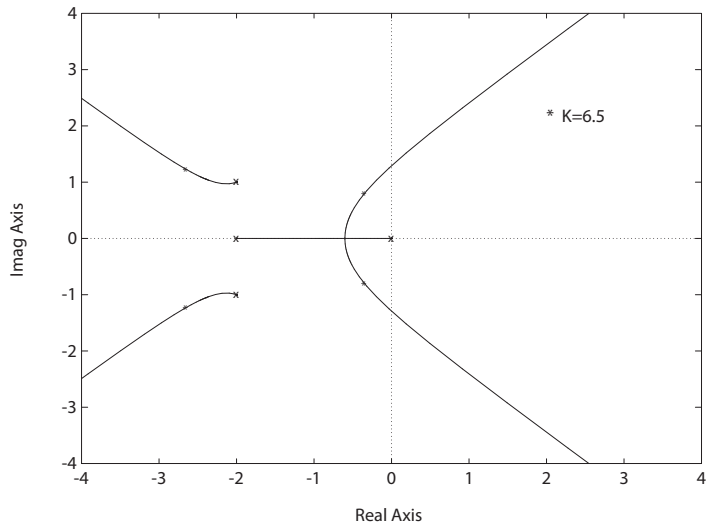
From this we determine that

$$\omega_n = 0.873 \quad \text{and} \quad \zeta = \frac{0.7}{2(0.873)} = 0.40 .$$

Thus, the settling time is

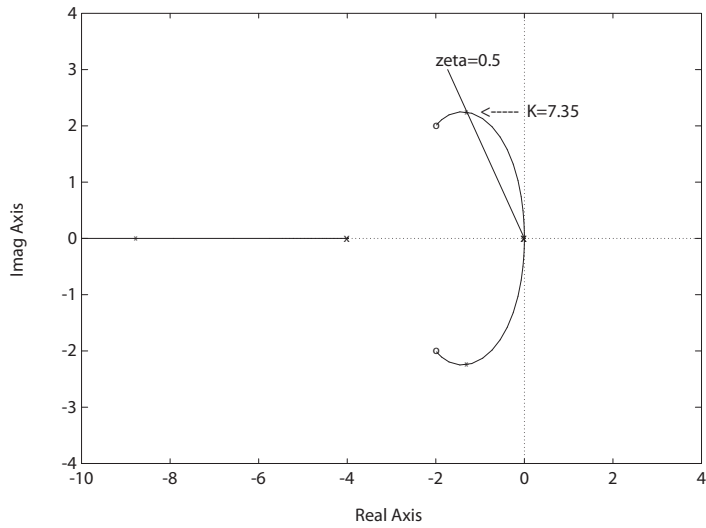
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.35} = 11.43 \text{ sec} .$$

The percent overshoot is  $P.O. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 25.4\%$ .



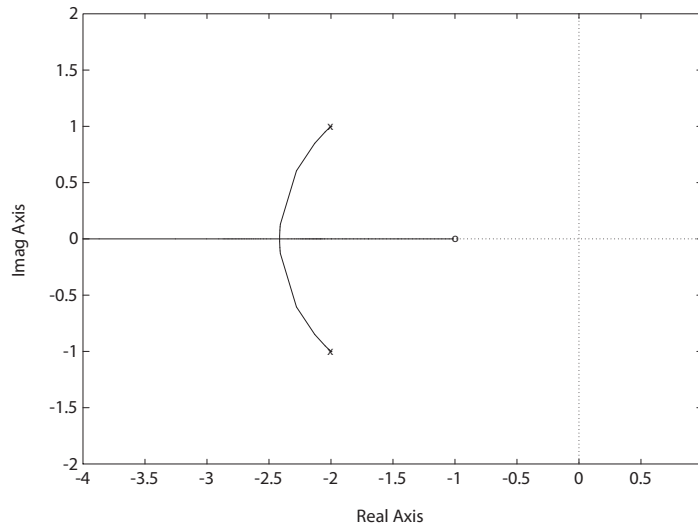
**FIGURE E7.2**  
 Root locus for  $1 + K \frac{1}{s(s+2)(s^2+4s+5)} = 0$ .

**E7.3** The root locus is shown in Figure E7.3. The roots are  $s_1 = -8.7$ ,  $s_{2,3} = -1.3 \pm j2.2$  when  $K = 7.35$  and  $\zeta = 0.5$ .



**FIGURE E7.3**  
 Root locus for  $1 + K \frac{s^2+4s+8}{s^2(s+4)} = 0$ .

**E7.4** The root locus is shown in Figure E7.4.



**FIGURE E7.4**

Root locus for  $1 + K \frac{s+1}{s^2+4s+5} = 0$ .

The departure angles and entry points are

$$\theta_d = 225^\circ, \quad -225^\circ$$

and

$$\sigma_b = -2.4.$$

**E7.5** (a) The root locus is in Figure E7.5. The breakaway points are

$$\sigma_{b1} = -13.0, \quad \sigma_{b2} = -5.89.$$

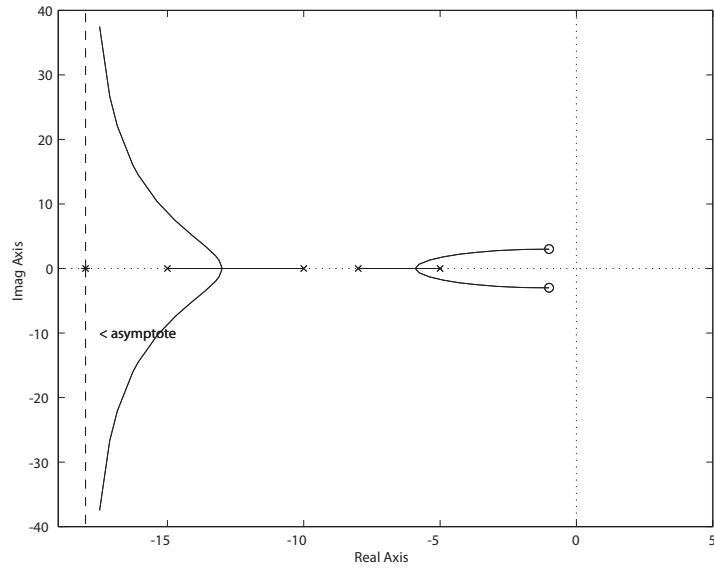
(b) The asymptote centroid is

$$\sigma_{cent} = -18,$$

and

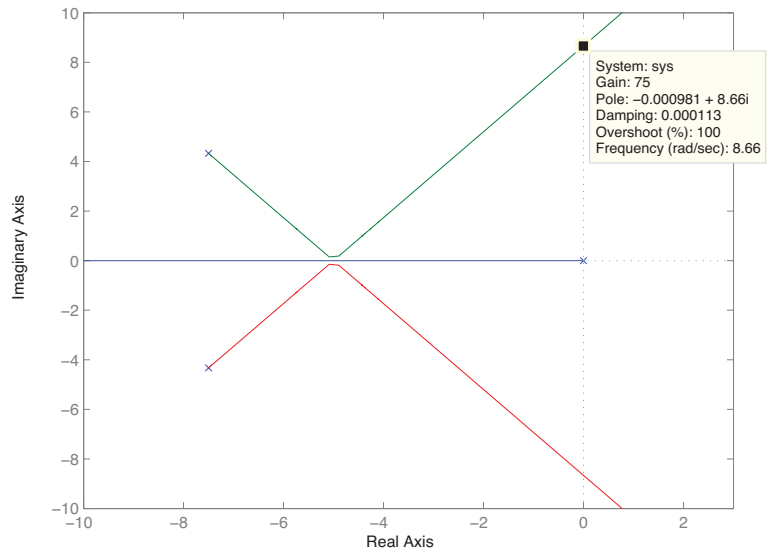
$$\phi_{asym} = \pm 90^\circ.$$

(c) The gains are  $K_1 = 1.57$  and  $K_2 = 2.14$  at the breakaway points.



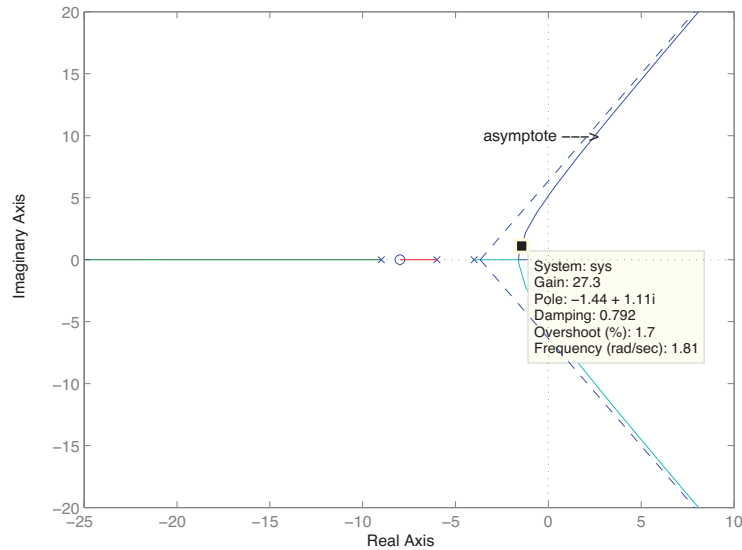
**FIGURE E7.5**  
 Root locus for  $1 + K \frac{s^2 + 2s + 10}{(s^4 + 38s^3 + 515s^2 + 2950s + 6000)} = 0$ .

**E7.6** The system is unstable for  $K > 75$ .



**FIGURE E7.6**  
 Root locus for  $1 + K \frac{15K}{s(s^2 + 15s + 75)} = 0$ .

**E7.7** The root locus is shown in Figure E7.7. The characteristic equation has



**FIGURE E7.7**  
 Root locus for  $1 + K \frac{s+8}{s(s+4)(s+6)(s+9)} = 0$ .

4 poles and 1 zero. The asymptote angles are  $\phi = +60^\circ, -60^\circ, -180^\circ$  centered at  $\sigma_{cent} = -3.7$ . When  $K = 27.35$  then  $\zeta = 0.8$  for the complex roots.

**E7.8** The characteristic equation is

$$1 + K \frac{(s + 1)}{s^2(s + 9)} = 0 ,$$

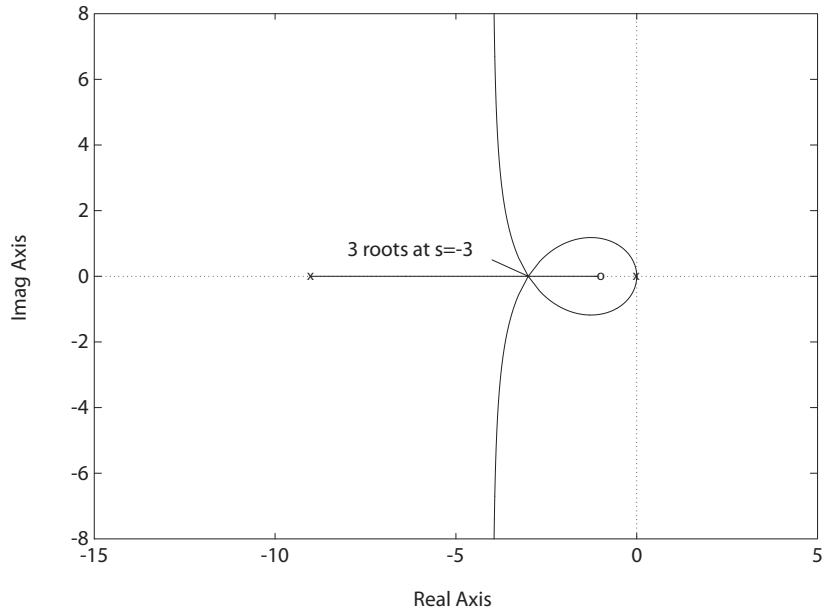
or

$$s^3 + 9s^2 + Ks + K = 0 .$$

For all the roots to be equal and real, we require

$$(s + r)^3 = s^3 + 3rs^2 + 3r^2s + r^3 = 0 .$$

Equating terms and solving for  $K$  yields  $K = 27$ . All three roots are equal at  $s = -3$ , when  $K = 27$ . The root locus is shown in Figure E7.8.

**FIGURE E7.8**

Root locus for  $1 + K \frac{s+1}{s^2(s+9)} = 0$ .

**E7.9** The characteristic equation is

$$1 + K \frac{1}{s(s^2 + 2s + 5)} = 0$$

or

$$s^3 + 2s^2 + 5s + K = 0.$$

- (a) The system has three poles at  $s = 0$  and  $-1 \pm j2$ . The number of asymptotes is  $n_p - n_z = 3$  centered at  $\sigma_{cent} = -2/3$ , and the angles are  $\phi_{asympt}$  at  $\pm 60^\circ, 180^\circ$ .
- (b) The angle of departure,  $\theta_d$ , is  $90^\circ + \theta_d + 116.6^\circ = 180^\circ$ , so  $\theta_d = -26.6^\circ$ .
- (c) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 2 & K \\ s^1 & b & \\ s^0 & K & \end{array}$$

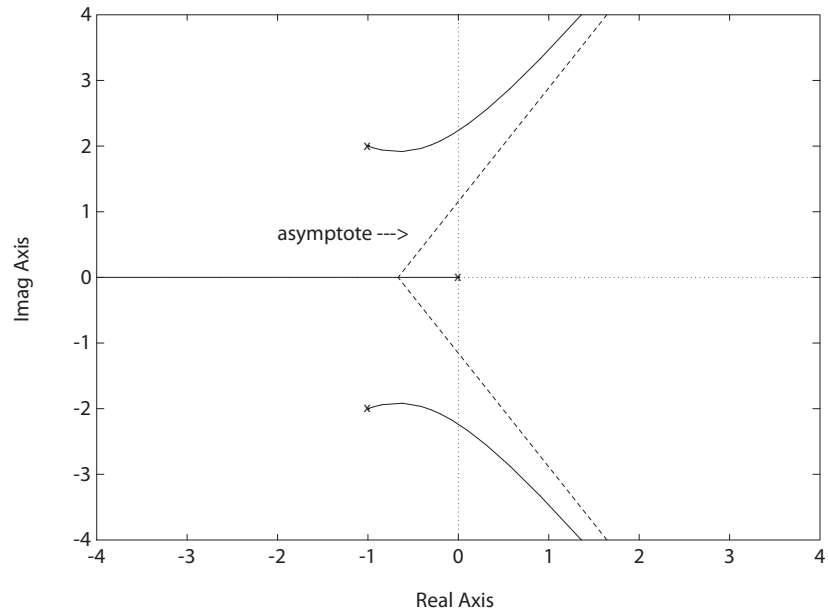
where  $b = 5 - K/2$ . So, when  $K = 10$  the roots lie on the imaginary



axis. The auxiliary equation is

$$2s^2 + 10 = 0 \quad \text{which implies} \quad s_{1,2} = \pm j\sqrt{5}.$$

(d) The root locus is shown in Figure E7.9.



**FIGURE E7.9**

Root locus for  $1 + K \frac{1}{s(s^2+2s+5)} = 0$ .

**E7.10** (a) The characteristic equation is

$$1 + \frac{K(s+2)}{s(s+1)} = 0.$$

Therefore,

$$K = -\frac{(s^2 + s)}{(s+2)},$$

and

$$\frac{dK}{ds} = -\frac{s^2 + 4s + 2}{(s+2)^2} = 0.$$

Solving  $s^2 + 4s + 2 = 0$  yields  $s = -0.586$  and  $-3.414$ . Thus, the system

breakaway and entry points are at  $s = -0.586$  and  $s = -3.414$ .

(b) The desired characteristic polynomial is

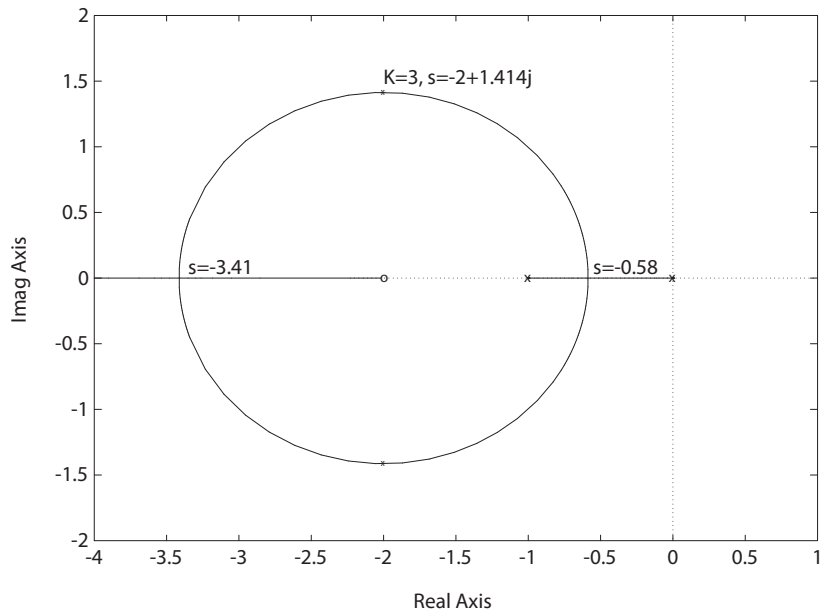
$$(s + 2 + aj)(s + 2 - aj) = s^2 + 4s + 4 + a^2 = 0 ,$$

where  $a$  is not specified. The actual characteristic polynomial is

$$s^2 + (1 + K)s + 2K = 0 .$$

Equating coefficients and solving for  $K$  yields  $K = 3$  and  $a = \sqrt{2}$ . Thus, when  $K = 3$ , the roots are  $s_{1,2} = -2 \pm \sqrt{2}j$ .

(c) The root locus is shown in Figure E7.10.



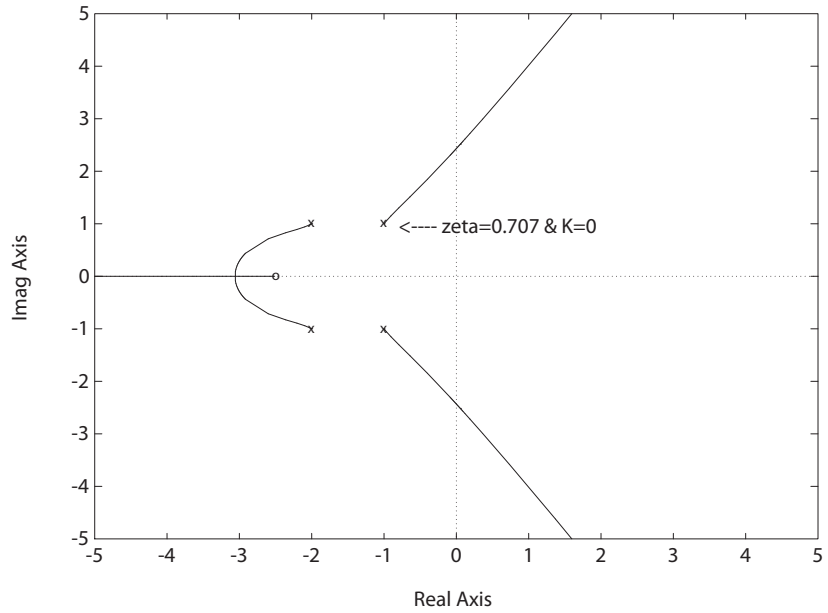
**FIGURE E7.10**

Root locus for  $1 + K \frac{s+2}{s(s+1)} = 0$ .

**E7.11** The root locus is shown in Figure E7.11 for the characteristic equation

$$1 + \frac{K(s + 2.5)}{(s^2 + 2s + 2)(s^2 + 4s + 5)} = 0 .$$

From the root locus we see that we can only achieve  $\zeta = 0.707$  when  $K = 0$ .



**FIGURE E7.11**  
 Root locus for  $1 + \frac{K(s+2.5)}{(s^2+2s+2)(s^2+4s+5)} = 0$ .

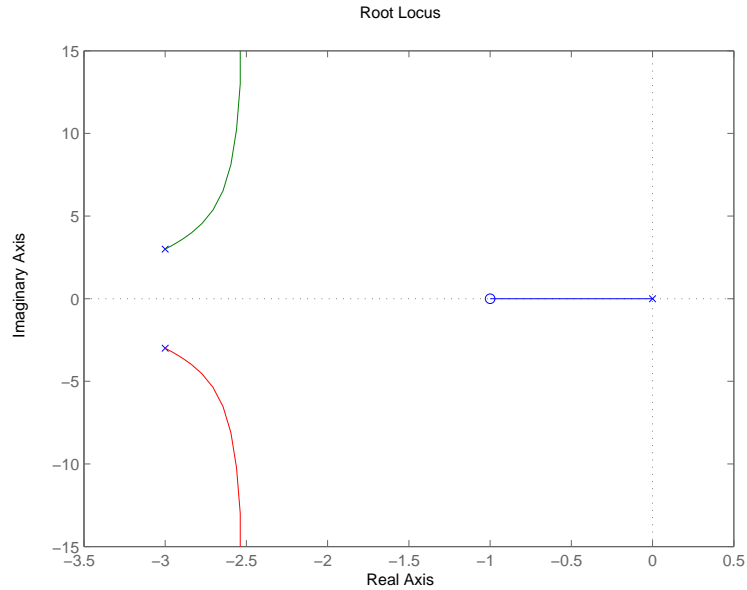
**E7.12** (a) The root locus is shown in Figure E7.12 for the characteristic equation

$$1 + \frac{K(s + 1)}{s(s^2 + 6s + 18)} = 0 .$$

- (b) The roots of the characteristic equation are
- (i)  $K = 10$ :  $s_{1,2} = -2.8064 \pm 4.2368j$  and  $s_3 = -0.3872$
  - (ii)  $K = 20$ :  $s_{1,2} = -2.7134 \pm 5.2466j$  and  $s_3 = -0.5732$
- (c) The step response performance of the system is summarized in Table E7.12.

$K$	10	20
$T_s$ (sec)	9.0	5.5
$P.O.$	0	0
$T_r$ (sec)	4.8	2.6

**TABLE E7.12** System performance when  $K = 10$  and  $K = 20$ .



**FIGURE E7.12**  
Root locus for  $1 + \frac{K(s+1)}{s(s^2+6s+18)} = 0$ .

**E7.13** (a) The characteristic equation is

$$s(s+1)(s+3) + 4s + 4z = 0.$$

Rewriting with  $z$  as the parameter of interest yields

$$1 + z \frac{4}{s(s+1)(s+3) + 4s} = 0.$$

The root locus is shown in Figure E7.13a.

(b) The root locations for

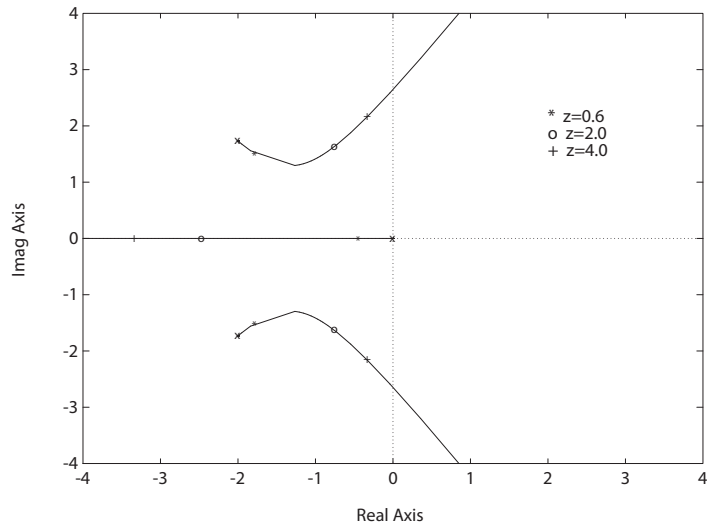
$$z = 0.6, \quad 2.0, \quad \text{and} \quad 4.0$$

are shown in Figure E7.13a. When  $z = 0.6$ , we have  $\zeta = 0.76$  and  $\omega_n = 2.33$ . Therefore, the predicted step response is

$$P.O. = 2.4\% \quad \text{and} \quad T_s = 2.3 \text{ sec} \quad (\zeta = 0.6).$$

When  $z = 2.0$ , we have  $\zeta = 0.42$  and  $\omega_n = 1.79$ . Therefore, the predicted step response is

$$P.O. = 23\% \quad \text{and} \quad T_s = 5.3 \text{ sec} \quad (\zeta = 2.0).$$

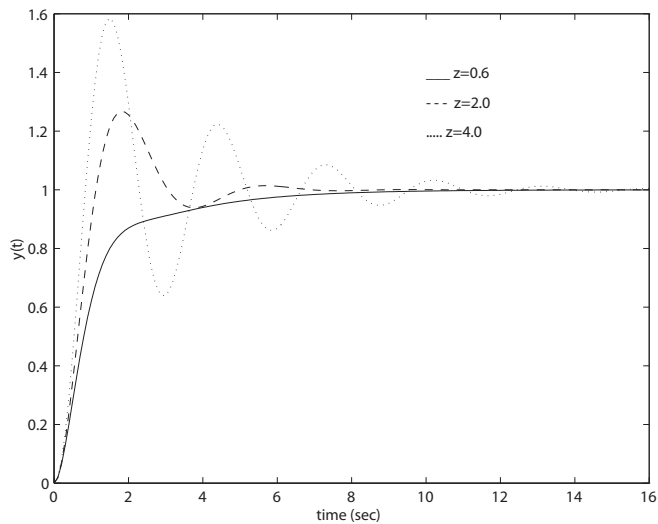


**FIGURE E7.13**

(a) Root locus for  $1 + z \frac{4}{s(s+1)(s+3)+4s} = 0$ .

Finally, when  $z = 4.0$ , we have  $\zeta = 0.15$  and  $\omega_n = 2.19$ . Therefore, the predicted step response is  $P.O. = 62\%$  and  $T_s = 12$  sec.

(c) The actual step responses are shown in Figure E7.13b.



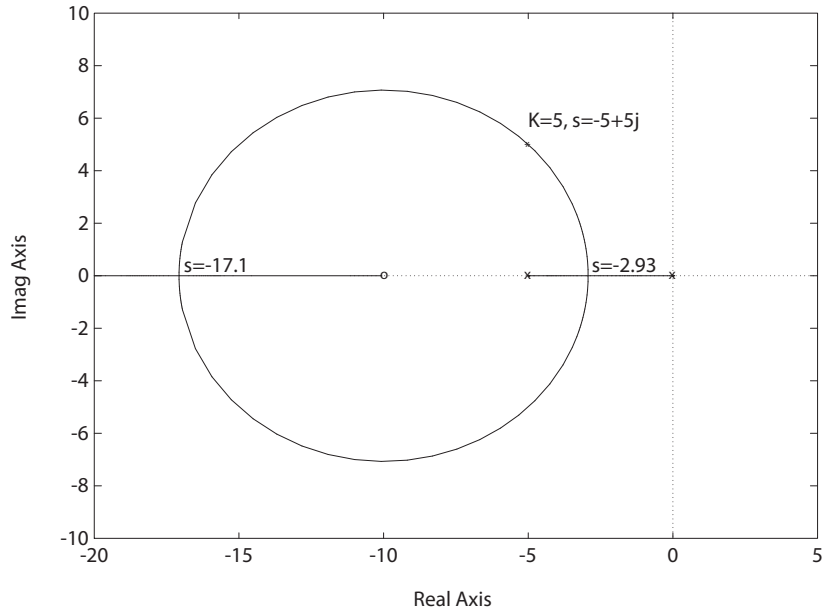
**FIGURE E7.13**

CONTINUED: (b) Step Responses for  $z = 0.6, 2.0,$  and  $4.0$ .

**E7.14** (a) The root locus is shown in Figure E7.14 for the characteristic equation

$$1 + \frac{K(s+10)}{s(s+5)} = 0 .$$

The breakaway point is  $s_b = -2.93$ ; the entry point is  $s_e = -17.1$ .



**FIGURE E7.14**  
Root locus for  $1 + \frac{K(s+10)}{s(s+5)} = 0$ .

(b) We desire  $\zeta = 1/\sqrt{2} = 0.707$ . So, the desired characteristic polynomial is

$$s^2 + 2\left(\frac{1}{\sqrt{2}}\right)\omega_n s + \omega_n^2 = 0 .$$

Comparing the desired characteristic polynomial to the actual we find the relationships

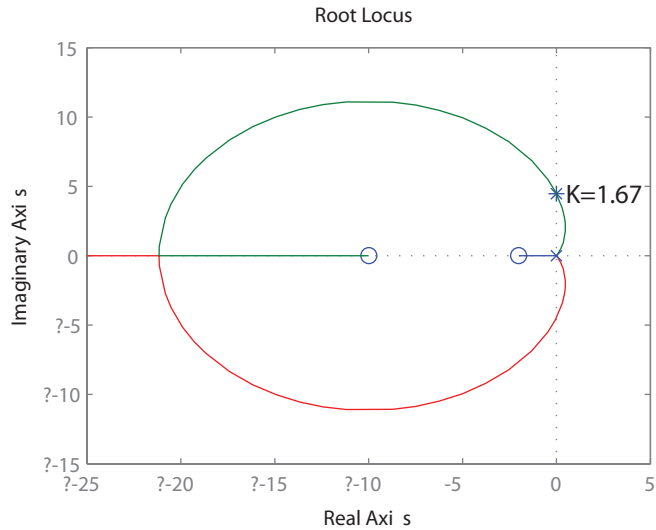
$$\omega_n^2 = 10K \quad \text{and} \quad \sqrt{2}\omega_n = 5 + K .$$

Solving for  $K$  and  $\omega_n$  yields  $K = 5$  and  $\omega_n = 7.07$ . The roots are  $s_{1,2} = -5 \pm j5$  when  $K = 5$ .

**E7.15** (a) The characteristic equation

$$1 + K \frac{(s + 10)(s + 2)}{s^3} = 0$$

has the root locus in Figure E7.15.



**FIGURE E7.15**  
Root locus for  $1 + \frac{K(s+10)(s+2)}{s^3} = 0$ .

(b) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 12K \\ s^2 & K & 20K \\ s^1 & b & \\ s^0 & 20K & \end{array}$$

when  $b = 12K - 20$ . For stability, we require all elements in the first column to be positive. Therefore,

$$K > 1.67 .$$

(c) When  $K > 3/4$ , we have

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + GH(s)} \cdot \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{s^2}{s^3 + K(s + 1)(s + 3)} = 0 .$$

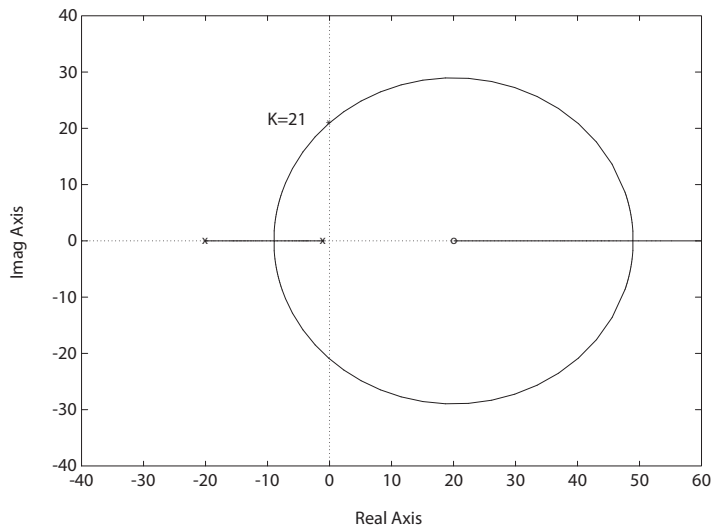
**E7.16** The expansion for  $e^{-Ts}$  is

$$e^{-Ts} = 1 - Ts + \frac{(Ts)^2}{2!} - \dots$$

If  $(Ts) \ll 1$ , then

$$e^{-Ts} \approx 1 - Ts = \frac{a + bs}{c + ds},$$

where  $a, b, c$  and  $d$  are constants to be determined. Using long division,



**FIGURE E7.16**  
Root locus for  $1 + \frac{K(20-s)}{(s+1)(20+s)} = 0$ .

we expand  $(a + bs)/(c + ds)$  and match as many coefficients as possible. In this case, we determine that  $a = c = (2/T)$  and also that  $b = -d = -1$ . In this case, with  $T = 0.1$ , we have

$$e^{-Ts} = \frac{20 - s}{20 + s} = \frac{-(s - 20)}{(s + 20)}.$$

So, the characteristic equation is

$$1 + \frac{-K(s - 20)}{(s + 1)(s + 20)},$$

and the root locus is shown in Figure E7.16. Using a Routh-Hurwitz

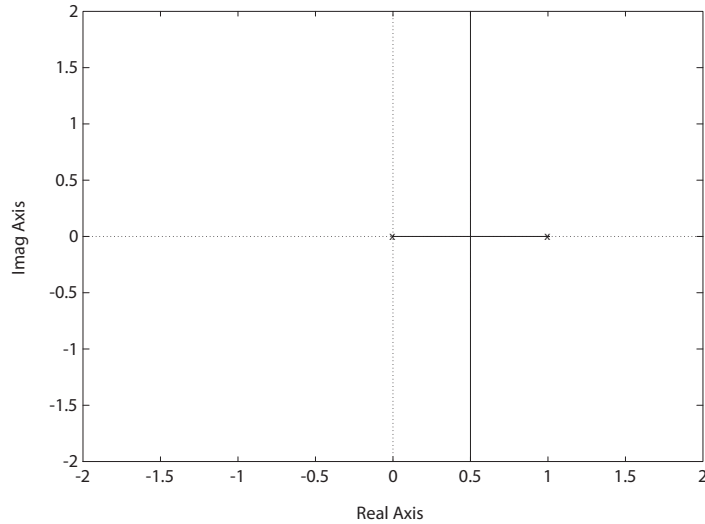


analysis with the characteristic polynomial

$$s^2 + (21 - K)s + 20 + 20K = 0 ,$$

we determine that the system is stable for  $-1 < K < 21$ .

**E7.17** (a) The root locus is in Figure E7.17a.



**FIGURE E7.17**

(a) Root locus for  $1 + \frac{K}{s(s-1)} = 0$ .

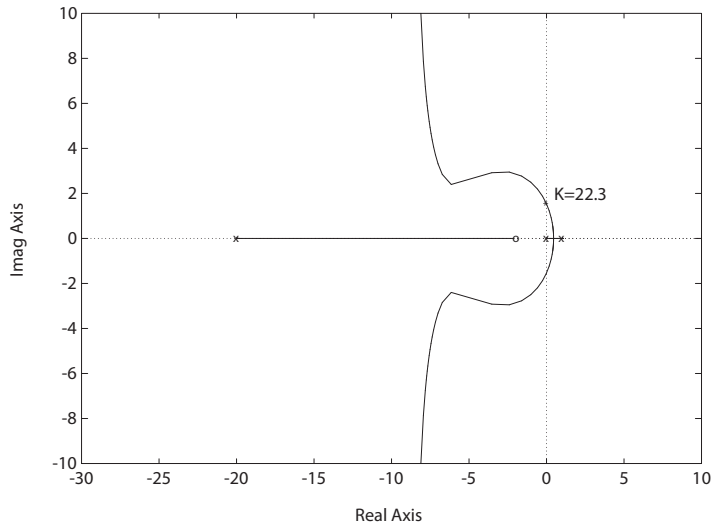
The root locus is always in the right half-plane; the system is unstable for  $K > 0$ .

(b) The characteristic equation is

$$1 + \frac{K(s+2)}{s(s-1)(s+20)} = 0 ,$$

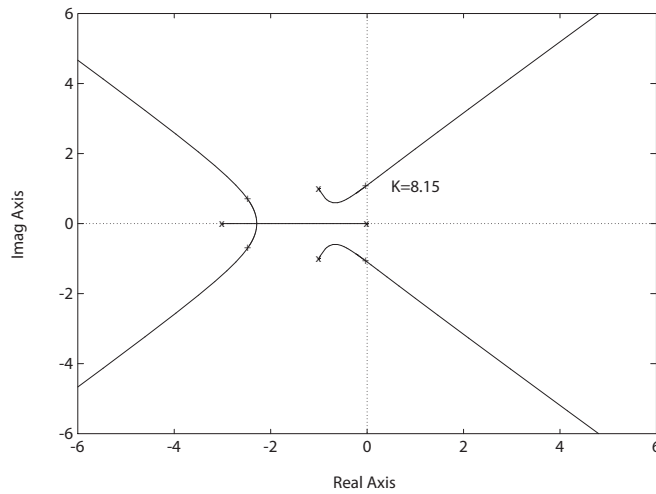
and the root locus is shown in Figure E7.17b. The system is stable for  $K > 22.3$  and when  $K = 22.3$ , the roots are

$$s_{1,2} = \pm j1.53 \quad \text{and} \quad s_3 = -19 .$$



**FIGURE E7.17**  
CONTINUED: (b) Root locus for  $1 + \frac{K(s+2)}{s(s+20)(s-1)} = 0$ .

**E7.18** The root locus is shown in Figure E7.18.



**FIGURE E7.18**  
Root locus for  $1 + \frac{K}{s(s+3)(s^2+2s+2)} = 0$ .

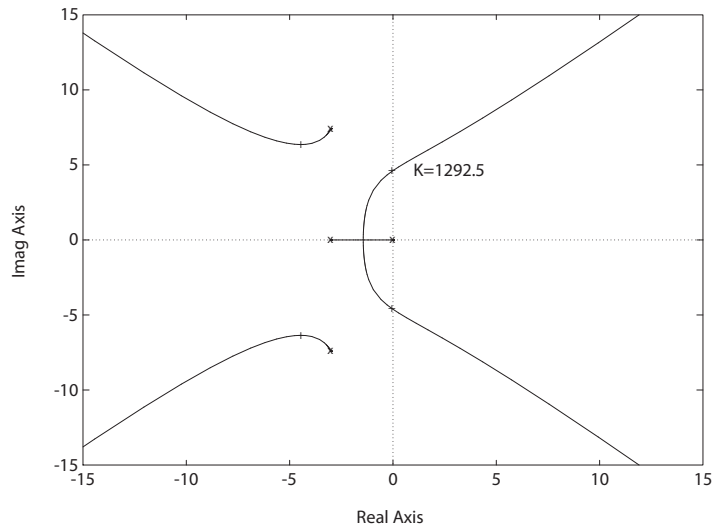
When  $K = 8.15$ , the roots are  $s_{1,2} = \pm j1.095$  and  $s_{3,4} = -2.5 \pm j0.74$ .

**E7.19** The characteristic equation is

$$1 + \frac{K}{s(s+3)(s^2+6s+64)} = 0 ,$$

and the root locus is shown in Figure E7.19. When  $K = 1292.5$ , the roots are

$$s_{1,2} = \pm j4.62 \quad \text{and} \quad s_{3,4} = -4.49 \pm j6.36 .$$



**FIGURE E7.19**  
Root locus for  $1 + \frac{K}{s(s+3)(s^2+6s+64)} = 0$ .

**E7.20** The characteristic equation is

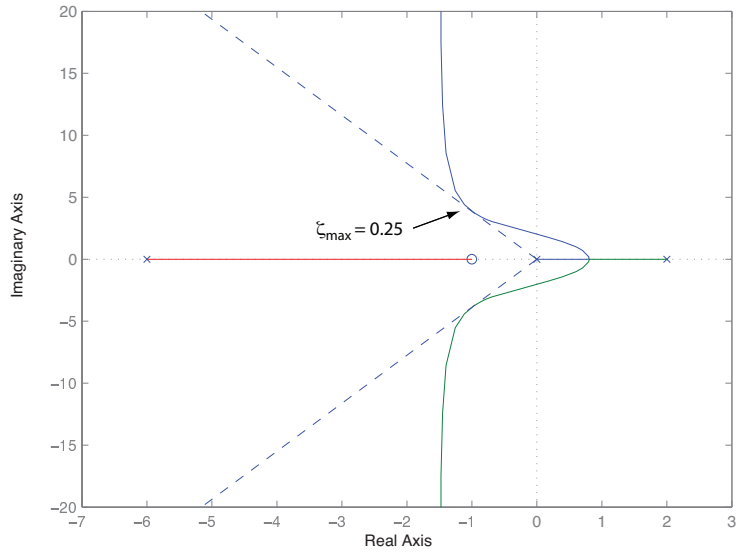
$$1 + \frac{K(s+1)}{s(s-2)(s+6)} = 0 ,$$

and the root locus is shown in Figure E7.20. The system is stable for

$$K > 16 .$$

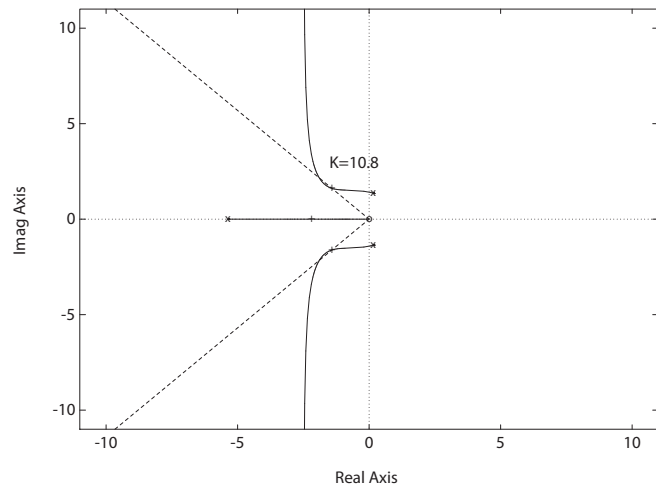
The maximum damping ratio of the stable complex roots is

$$\zeta = 0.25 .$$



**FIGURE E7.20**  
 Root locus for  $1 + \frac{K(s+1)}{s(s-2)(s+6)} = 0$ .

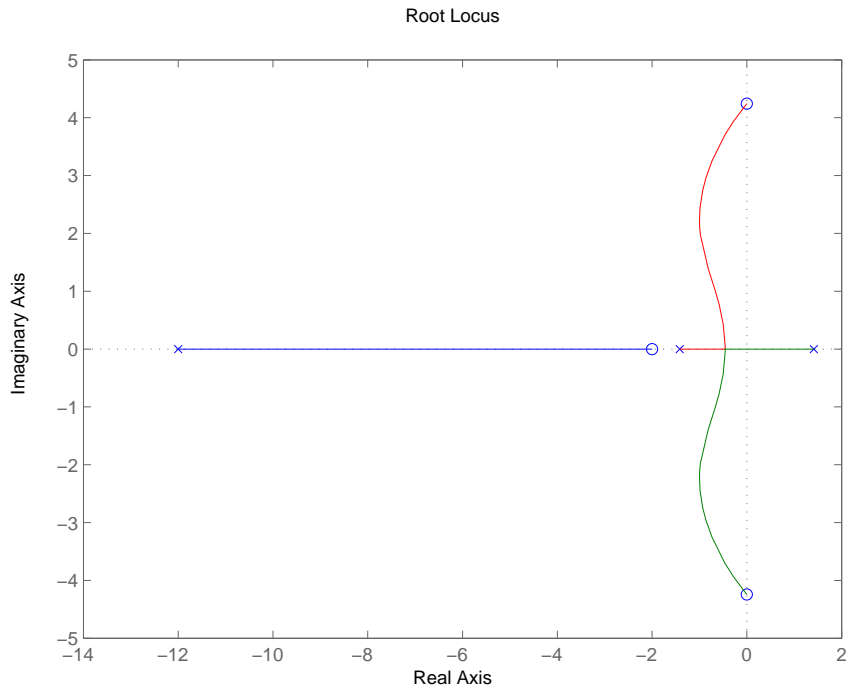
**E7.21** The gain is  $K = 10.8$  when the complex roots have  $\zeta = 0.66$ .



**FIGURE E7.21**  
 Root locus for  $1 + \frac{Ks}{s^3 + 5s^2 + 10} = 0$ .

**E7.22** The root locus is shown in Figure E7.22. The characteristic equation is

$$1 + \frac{K(s^2 + 18)(s + 2)}{(s^2 - 2)(s + 12)} = 0 .$$



**FIGURE E7.22** Root locus for  $1 + \frac{K(s^2+18)(s+2)}{(s^2-2)(s+12)} = 0$ .

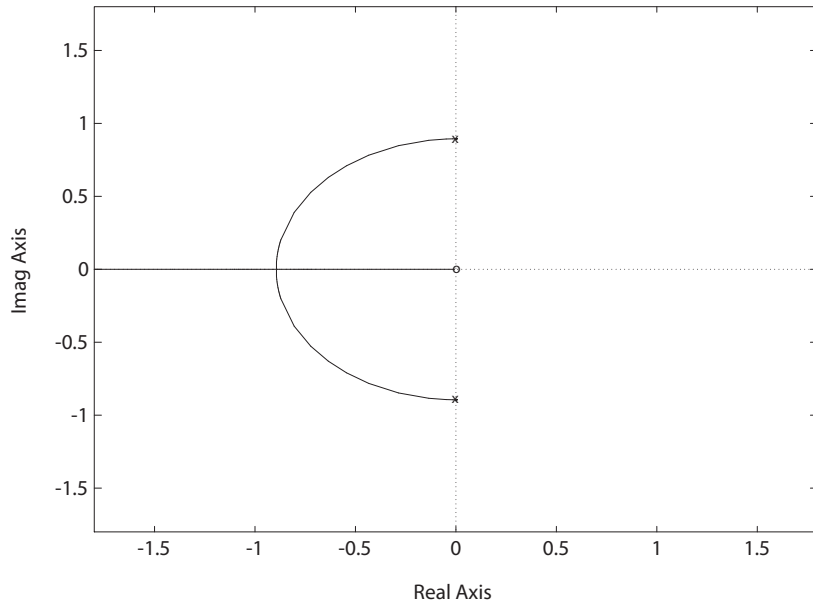
**E7.23** The characteristic equation is

$$5s^2 + as + 4 = 0 ,$$

which can be rewritten as

$$1 + \frac{as}{5s^2 + 4} = 0 .$$

The root locus (with  $a$  as the parameter) is shown in Figure E7.23.



**FIGURE E7.23**  
Root locus for  $1 + \frac{as}{5s^2+4} = 0$ .

**E7.24** The transfer function is

$$\begin{aligned} G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ &= [1 \ 0] \begin{bmatrix} s & -1 \\ 4 & s+k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 + ks + 4}. \end{aligned}$$

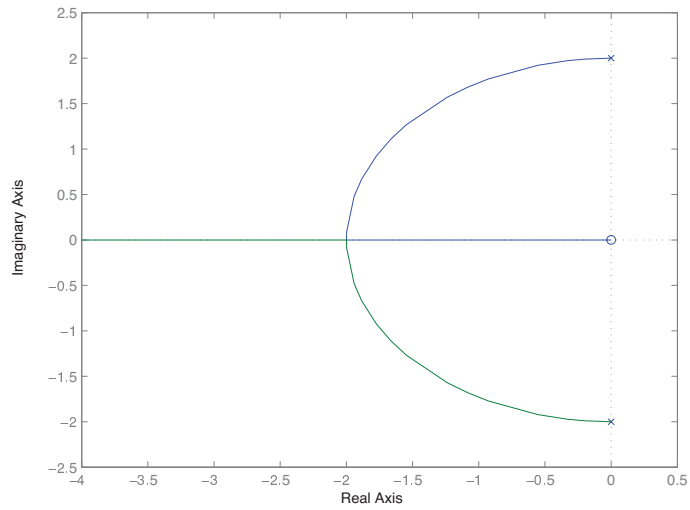
Therefore, the characteristic equation is

$$s^2 + ks + 4 = 0,$$

or

$$1 + k \frac{s}{s^2 + 4} = 0.$$

The root locus for  $0 < k < \infty$  is shown in Figure E7.24. The closed-loop system is stable for all  $0 < k < \infty$ .

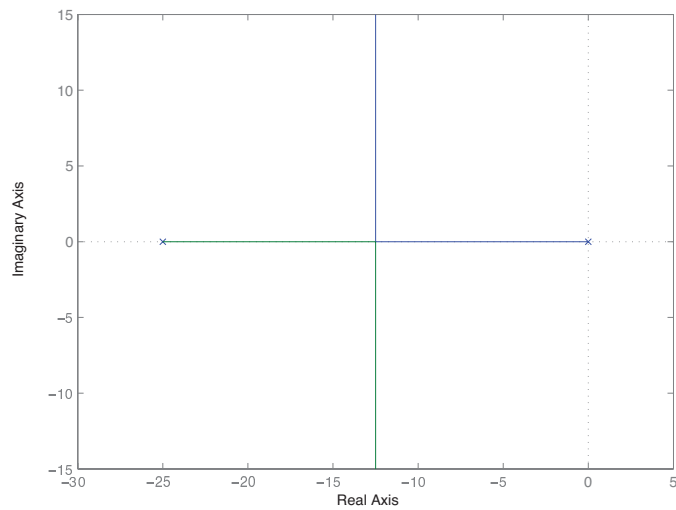


**FIGURE E7.24**  
Root locus for  $1 + k\frac{s}{s^2+4} = 0$ .

**E7.25** The characteristic equation is

$$1 + K\frac{10}{s(s+25)} = 0.$$

The root locus shown in Figure E7.25 is stable for all  $0 < K < \infty$ .



**FIGURE E7.25**  
Root locus for  $1 + K\frac{10}{s(s+25)} = 0$ .

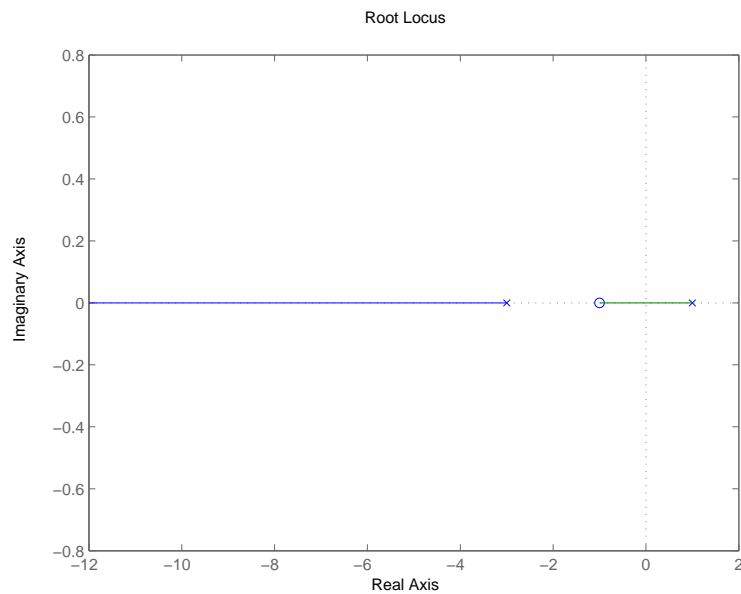
**E7.26** The characteristic polynomial is

$$\det \begin{bmatrix} s & -1 \\ s + K - 3 & s + K + 2 \end{bmatrix} = 0$$

or

$$1 + K \frac{s + 1}{s^2 + 2s - 3} = 0 .$$

The root locus shown in Figure E7.26 is stable for all  $0 < K < 3$ .



**FIGURE E7.26**

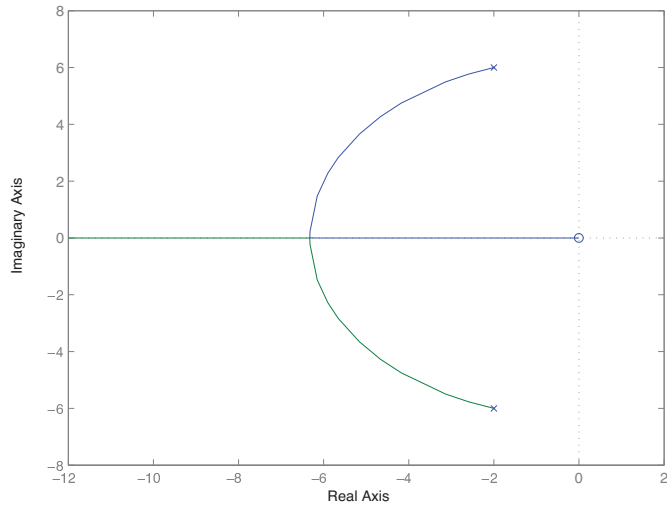
Root locus for  $1 + K \frac{s+1}{s^2+2s-3} = 0$ .

**E7.27** The characteristic equation is

$$1 + p \frac{s}{s^2 + 4s + 40} = 0 .$$

The root locus shown in Figure E7.27 is stable for all  $0 < p < \infty$ .



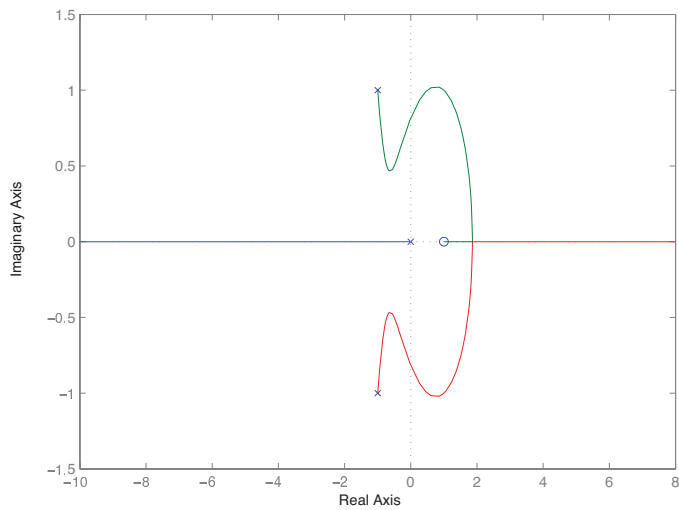
**FIGURE E7.27**

Root locus for  $1 + p \frac{s}{s^2 + 4s + 40} = 0$ .

**E7.28** The characteristic equation is

$$1 + K \frac{s - 1}{s(s^2 + 2s + 2)} = 0 .$$

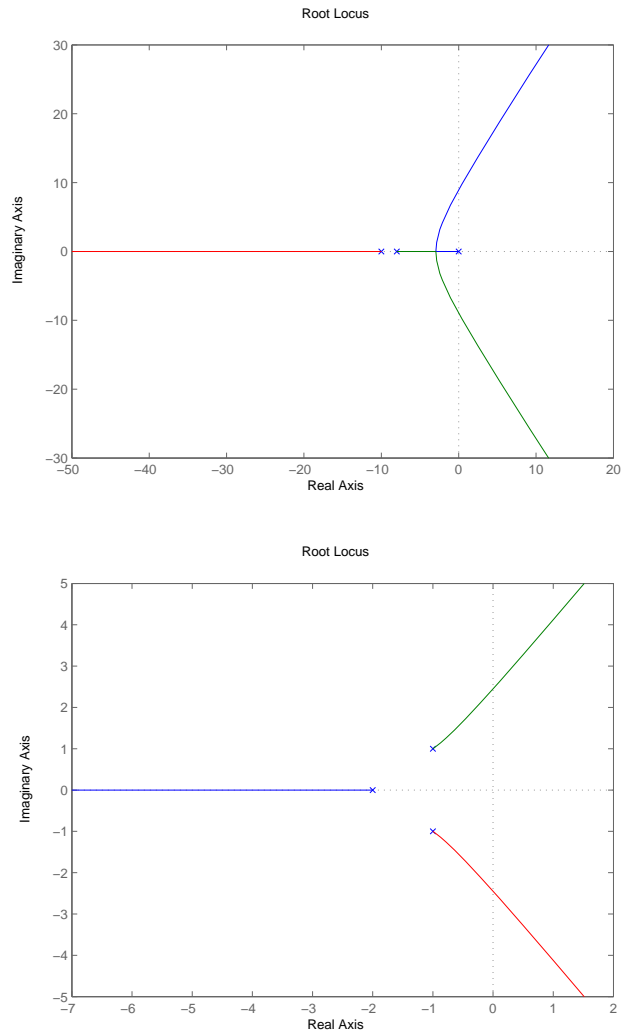
The system is stable for  $-1.33 < K < 0$ .

**FIGURE E7.28**

Root locus for  $1 + K \frac{s-1}{s(s^2 + 2s + 2)} = 0$ .

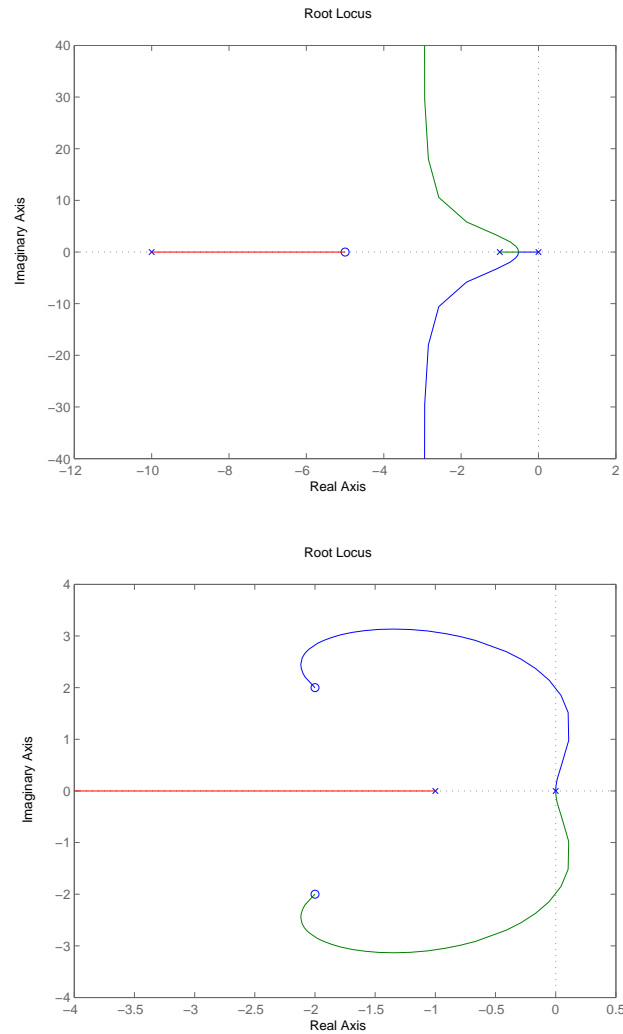
# Problems

## P7.1



**FIGURE P7.1**

(a) Root locus for  $1 + \frac{K}{s(s+10)(s+8)} = 0$ , and (b)  $1 + \frac{K}{(s^2+2s+2)(s+1)} = 0$ .

**FIGURE P7.1**

CONTINUED: (c) Root locus for  $1 + \frac{K(s+5)}{s(s+2)(s+7)} = 0$ , and (d)  $1 + \frac{K(s^2+4s+8)}{s^2(s+7)} = 0$ .

**P7.2** The root locus is shown in Figure P7.2 for the characteristic equation

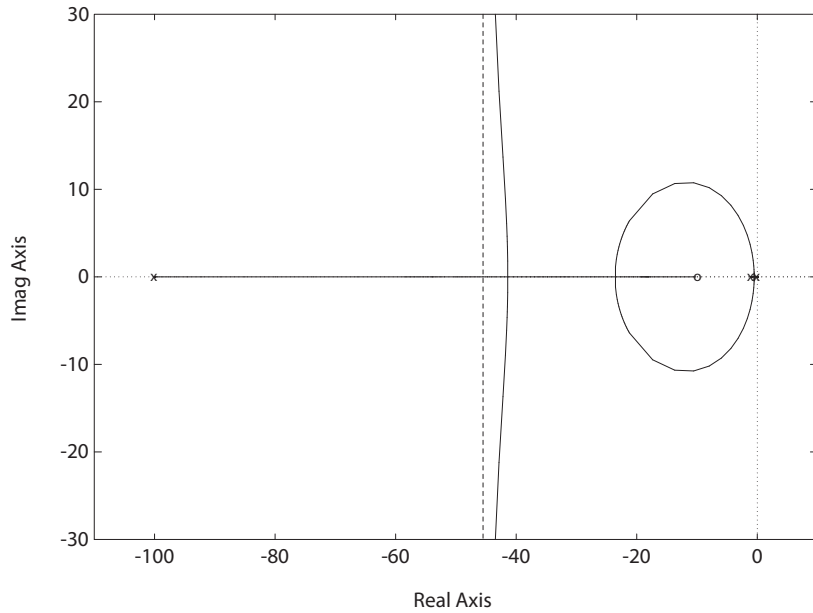
$$1 + \frac{10K_v(s+10)}{s(s+1)(s+100)} = 0.$$

The damping ratio is  $\zeta = 0.6$  when  $K_v = 0.8, 135$  and  $648$ . The roots of the characteristic equation are:

(a)  $K_v = 0.8 : s_1 = -99.9, s_{2,3} = -0.54 \pm j0.71$

(b)  $K_v = 135 : s_1 = -85.9, s_{2,3} = -7.5 \pm j10$

(c)  $K_v = 648 : s_1 = -11.7, s_{2,3} = -44.6 \pm j59.5$



**FIGURE P7.2**

Root locus for  $1 + \frac{10K_v(s+10)}{s(s+1)(s+100)} = 0$ .

- P7.3** (a) The breakaway point is  $s = -0.88$  at  $K = 4.06$ .  
 (b) The characteristic equation can be written as

$$s(s + 2)(s + 5) + K = 0 .$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 7 & K \\ s^1 & b & 0 \\ s^0 & K & \end{array}$$

where

$$b = \frac{70 - K}{7} .$$

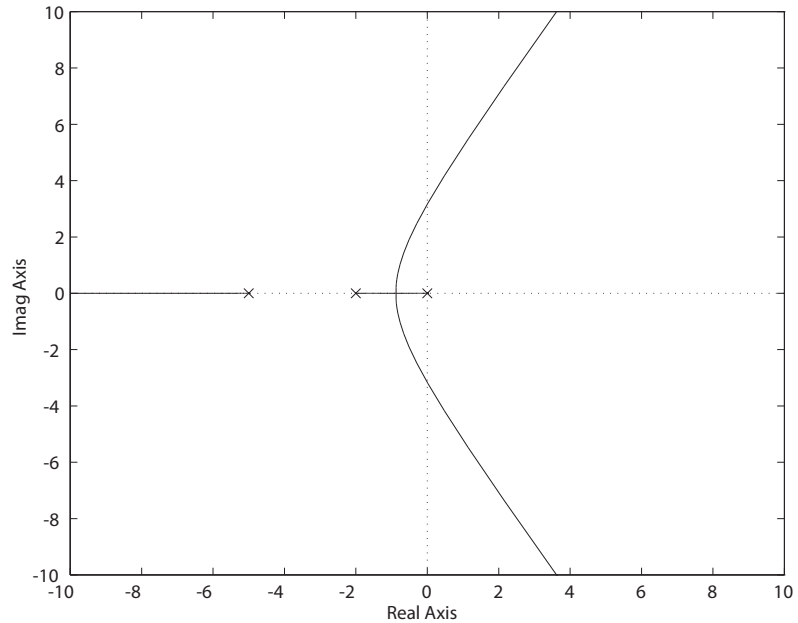
When  $K = 70$ , the system has roots on  $j\omega$ -axis at  $s = \pm j\sqrt{10}$ .

(c) When  $K = 6$ , the roots are  $s_{1,2} = -0.83 \pm j0.66$ ,  $s_3 = -5.34$ .

(d) The characteristic equation

$$1 + \frac{K}{s(s+2)(s+5)} = 0$$

has the root locus shown in Figure P7.3.



**FIGURE P7.3**

Root locus for  $1 + \frac{K}{s(s+2)(s+5)} = 0$ .

**P7.4** The characteristic equation for the large antenna is

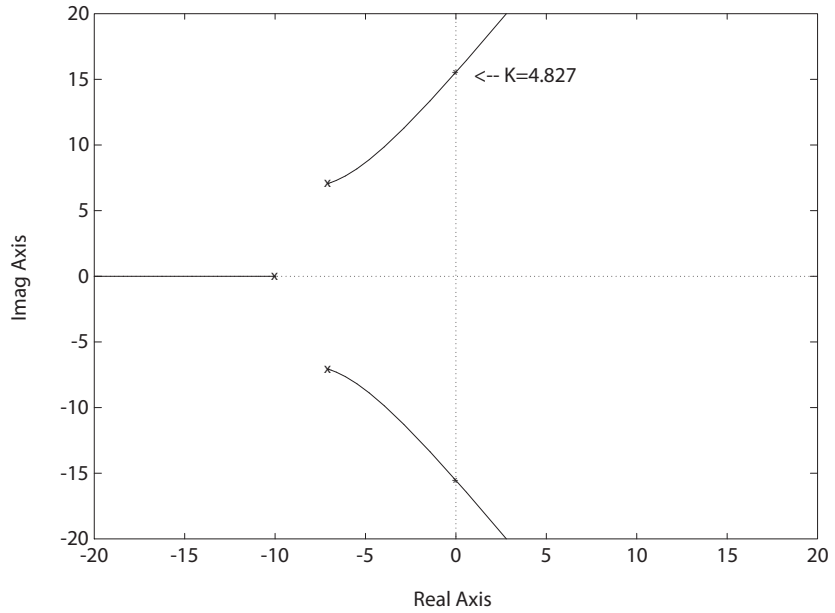
$$1 + G_1G(s) = 1 + \frac{100k_a}{(0.1s+1)(s^2+14.4s+100)} = 0,$$

or

$$1 + \frac{1000k_a}{(s+10)(s^2+14.4s+100)} = 0.$$

The root locus is shown in Figure P7.4. Using Routh's criteria, we find that the system is stable for

$$-1 < k_a < 4.83.$$



**FIGURE P7.4**  
 Root locus for  $1 + \frac{1000k_a}{(s^2 + 14.14s + 100)(s + 10)} = 0$ .

When  $k_a = 4.83$ , we have  $s_{1,2} = \pm j15.53$ .

**P7.5** (a) The characteristic equation for hands-off control is

$$1 + \frac{25K_2(s + 0.03)(s + 1)}{(s + 0.4)(s^2 - 0.36s + 0.16)(s + 9)} = 0 .$$

The root locus is shown in Figure P7.5a. The damping ratio is  $\zeta = 0.707$  when  $K_2 = 1.6$  or  $K_2 = 0.74$ .

(b) The transfer function from  $T_d(s)$  to  $Y(s)$  is

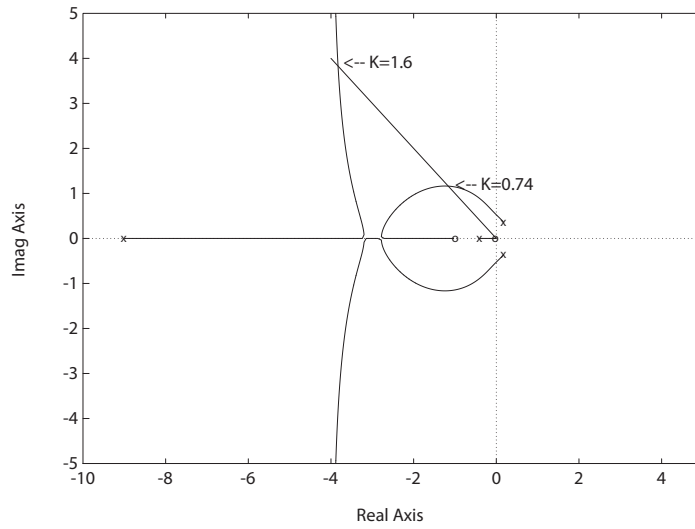
$$Y(s) = \frac{G_2(s)T_d(s)}{1 + G_2(s)G_f(s)} ,$$

where

$$G_f(s) = \frac{K_2(s + 1)}{s + 9} .$$

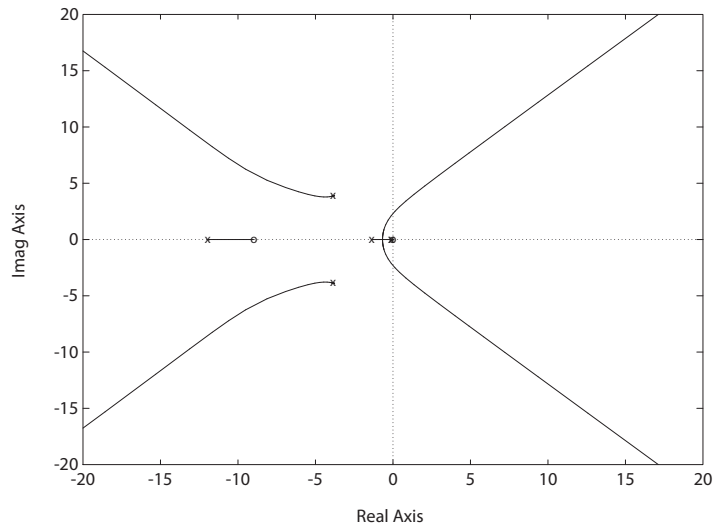
Using the final value theorem, we determine that

$$y_{ss} = \lim_{s \rightarrow 0} s \frac{G_2(s)}{1 + G_2(s)G_f(s)} \frac{1}{s} = \frac{11.7}{1 + 11.7 \left( \frac{K_2}{9} \right)} = 3.8 ,$$



**FIGURE P7.5**

(a) Root locus for  $1 + \frac{25K_2(s+0.03)(s+1)}{(s+9)(s^2-0.36s+0.16)(s+0.4)} = 0$ .



**FIGURE P7.5**

CONTINUED: (b) Root locus for  $1 + \frac{25K_1(s+0.03)(s+9)}{(s+0.045)(s^2+12s+1)(s+1.33)(s^2+7.66s+29.78)} = 0$ .

where we have selected  $K_2 = 1.6$ . For  $K_2 = 0.74$ , we find that  $y_{ss} = 5.96$ .

(c) The closed-loop characteristic equation with the pilot loop added is

$$1 + \frac{25K_1(s + 0.03)(s + 9)}{(s + 0.045)(s + 1.33)(s^2 + 7.66s + 29.78)(s^2 + 12s + 1)} = 0 .$$

The root locus is shown in Figure P7.5b.

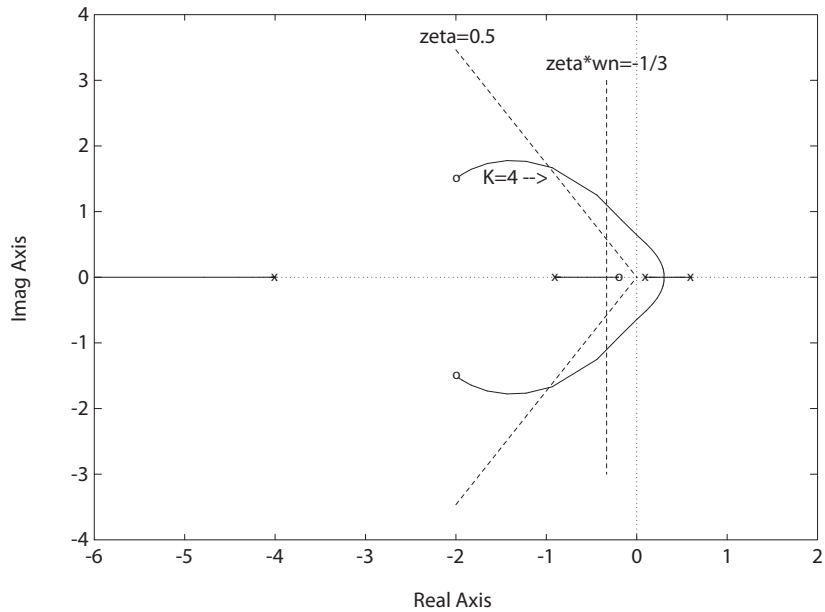
(d) Using  $K_1 = 2$ , we determine that

$$e_{ss} = 0.44 .$$

**P7.6** (a) The characteristic equation is

$$1 + \frac{K(s + 0.20)(s^2 + 4s + 6.25)}{(s + 0.9)(s - 0.6)(s - 0.1)(s + 4)} = 0 .$$

The root locus is shown in Figure P7.6.



**FIGURE P7.6**

Root locus for  $1 + \frac{K(s+0.2)(s^2+4s+6.25)}{(s+0.9)(s-0.6)(s-0.1)(s+4)} = 0$ .

(b) For  $T_s < 12$  sec, we require  $\zeta\omega_n > 1/3$ . Also, we want  $\zeta > 0.5$ . So, we seek roots for a stable system with  $\zeta\omega_n > 1/3$  and  $\zeta > 0.5$ . This occurs when  $K > 4$ .



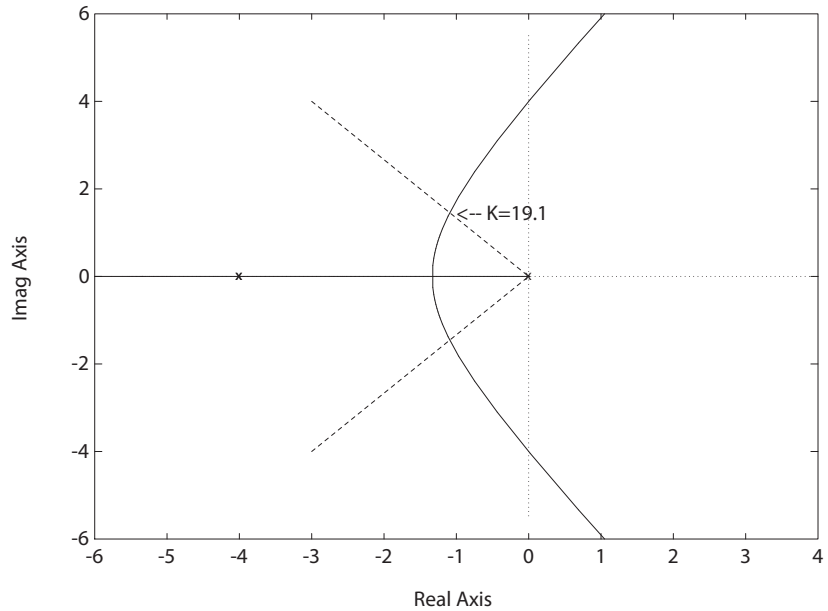
**P7.7** (a) The characteristic equation for the speed control system is

$$1 + \frac{K}{(s + 4)^2(s + \delta)} = 0 ,$$

where

$$K = \frac{0.004}{R} \quad \text{and} \quad \delta = \frac{0.75}{4000} = 0.0001875 .$$

The root locus is shown in Figure P7.7. At  $\zeta = 0.6$ , we have  $K = 19.1$ ,



**FIGURE P7.7**  
Root locus for  $1 + \frac{K}{(s+4)^2(s+1.875e-04)} = 0$ .

therefore

$$R = 0.00021 .$$

When  $K = 19.1$  the roots are

$$s_{1,2} = -1.1 \pm j1.43 \quad \text{and} \quad s_3 = -5.80 .$$

(b) The steady-state error is

$$\lim_{s \rightarrow 0} s \Delta \omega(s) = \lim_{s \rightarrow 0} s \frac{(0.25s + 1)^2}{(0.25s + 1)^2 (Js + b) + 1/R} \Delta L(s)$$

$$= \frac{1}{b + 1/R} \Delta L \approx \Delta LR ,$$

when  $R < 0.1$ .

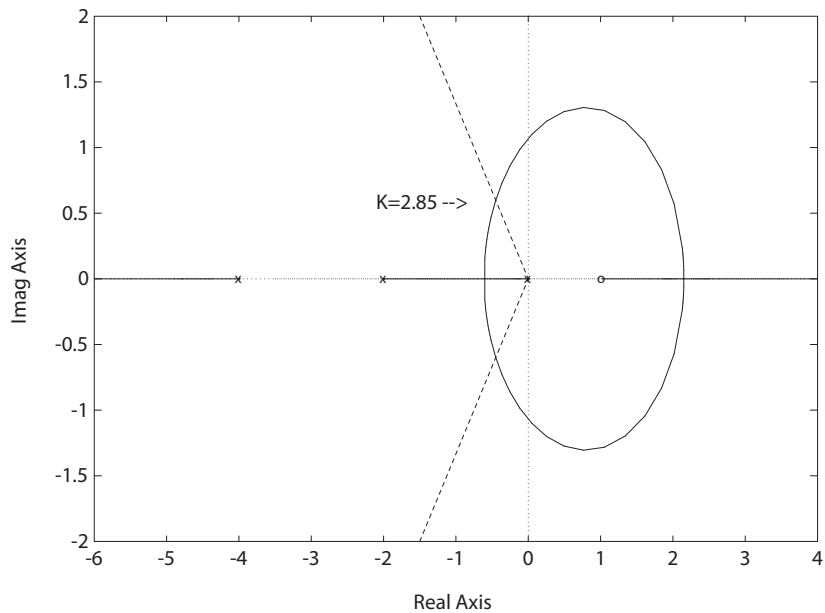
- P7.8** (a) The characteristic equation for the speed control system with the hydroturbine is

$$1 + \frac{K(-s + 1)}{(s + 4)(s + 2)(s + \delta)} = 0 ,$$

where

$$K = \frac{0.002}{R} \quad \text{and} \quad \delta = \frac{0.75}{4000} = 0.0001875 .$$

The root locus is shown in Figure P7.8. At  $\zeta = 0.6$ , we have  $K = 2.85$ ,



**FIGURE P7.8**

Root locus for  $1 + \frac{K(-s+1)}{(s+4)(s+2)(s+\delta)} = 0$ .

therefore

$$R = 0.0007 .$$

When  $K = 2.85$  the roots are  $-0.45 \pm j0.60$ , and  $-5.1$ .

(b) The steady-state error is

$$\begin{aligned}\lim_{s \rightarrow 0} s \Delta \omega(s) &= \lim_{s \rightarrow 0} s \frac{(0.25s + 1)(0.5s + 1)}{(0.25s + 1)(0.5s + 1)(Js + f) + (-s + 1)/R} \Delta L(s) \\ &= \frac{1}{f + 1/R} \Delta L \approx \Delta LR ,\end{aligned}$$

when  $R < 0.1$ .

**P7.9** The characteristic equation is

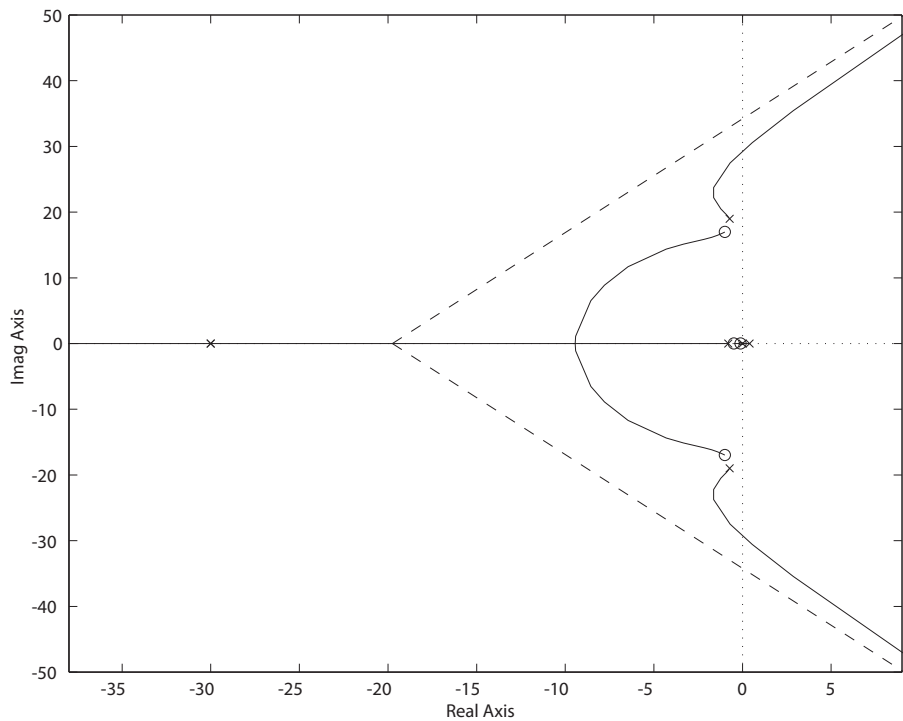
$$1 + K \frac{(s + 0.5)(s + 0.1)(s^2 + 2s + 289)}{s(s + 30)^2(s - 0.4)(s + 0.8)(s^2 + 1.45s + 361)} = 0$$

where  $K = K_1 K_2$ . The root locus is shown in Figure P7.9. When

$$K = 4000 ,$$

the roots are

$$s_{1,2} = -0.82 \pm j19.4$$



**FIGURE P7.9**

Root locus for  $1 + \frac{K(s+0.5)(s+0.1)(s^2+2s+289)}{s(s+30)^2(s-0.4)(s+0.8)(s^2+1.45s+361)} = 0$ .

$$s_3 = -39.8$$

$$s_4 = -14.9$$

$$s_5 = -5.0$$

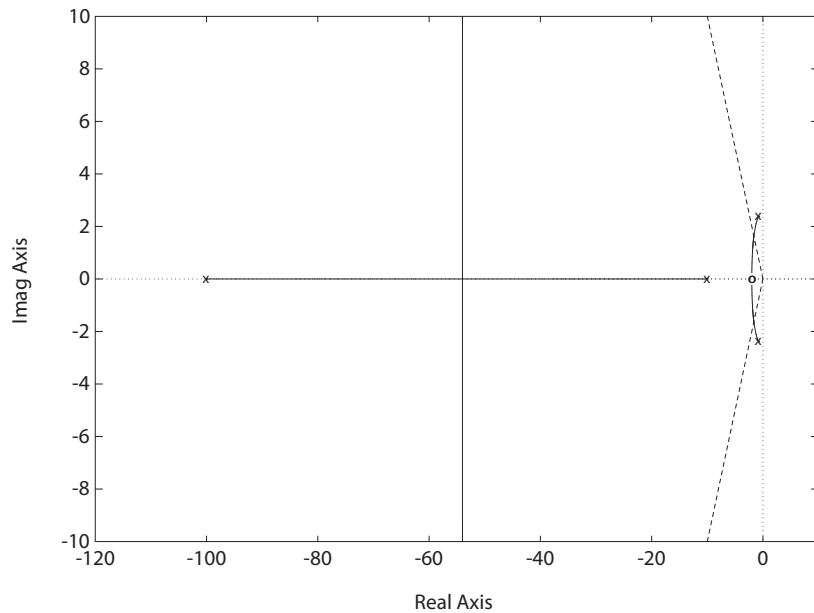
$$s_6 = -0.38$$

$$s_7 = -0.14 .$$

**P7.10** (a) The characteristic equation is

$$1 + \frac{K_1 K_2 (s + 2)^2}{(s + 10)(s + 100)(s^2 + 1.5s + 6.25)} = 0 .$$

The root locus is shown in Figure P7.10.



**FIGURE P7.10**

Root locus for  $1 + \frac{K_1 K_2 (s+2)^2}{(s+10)(s+100)(s^2+1.5s+6.25)} = 0$ .

(b) The gain

$$K_1 K_2 = 1620$$

when  $\zeta = 0.707$ . Therefore,

$$K_2 = 81000 ,$$

since  $K_1 = 0.02$  at medium weight cruise condition.

(c) At lightweight cruise condition

$$K_1 = 0.2 .$$

Using  $K_2 = 81000$ , we find the roots are

$$s_{1,2} = -54 \pm j119$$

$$s_{3,4} = -2 \pm j0.6 .$$

The roots  $s_{3,4}$  become negligible and the roots at  $s_{1,2}$  become highly oscillatory. Hence, in this case

$$\zeta = 0.41 .$$

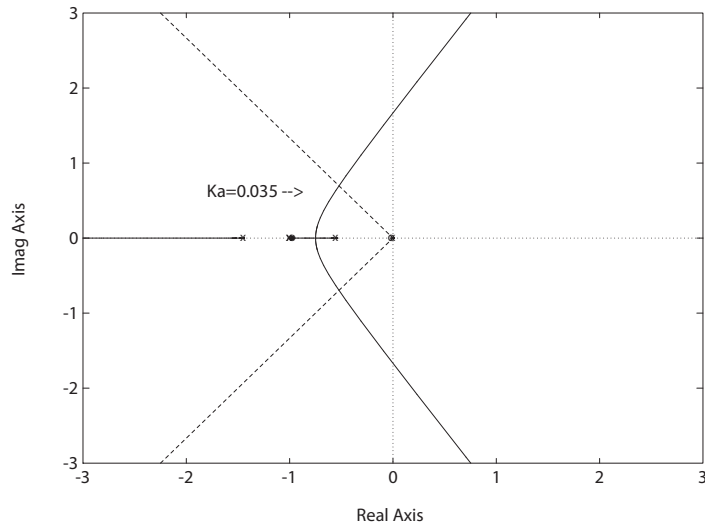
**P7.11** (a) The closed-loop characteristic equation is

$$1 + \frac{20K_a(s^2 + s + 0.02)}{s(s+1)^2(s^2 + 2s + 0.8)} = 0 ,$$

where

$$K_2 = 10 .$$

Then, the root locus is shown in Figure P7.11a.



**FIGURE P7.11**

(a) Root locus for  $1 + K_a \frac{20s^2 + 20s + 0.4}{s(s+1)^2(s^2 + 2s + 0.8)} = 0$ , where  $K_2 = 10$ .

(b) When

$$K_a < 0.035 ,$$

all the roots have a damping greater than or equal to 0.60.

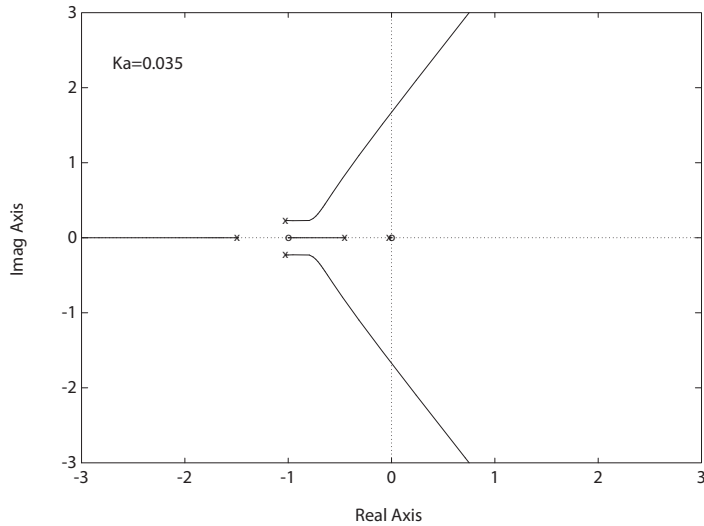
(c) Select

$$K_a = 0.035 .$$

Then, the characteristic equation with  $K_2$  as the parameter is

$$1 + K_2 \frac{0.07(s^2 + s)}{s^5 + 4s^4 + 5.8s^3 + 3.6s^2 + 0.8s + 0.014} = 0 .$$

The root locus is shown in Figure P7.11b.



**FIGURE P7.11** CONTINUED: (b) Root locus for  $1 + K_2 \frac{0.07s(s+1)}{s(s+1)^2(s^2+2s+0.8)+0.014} = 0$ , where  $K_a = 0.035$ .

**P7.12** (a) The closed-loop transfer function is

$$T(s) = \frac{K_a K_m (s + 25)(s + 15)}{1.8s^2(s + 2) + K_a K_m (s + 25)(s + 15) + 1.6K_m s(s + 2)} .$$

So, with  $E(s) = R(s) - Y(s)$ , we have  $E(s) = (1 - T(s))R(s)$  and

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = 1 - T(0) = 0 .$$

Therefore, when the system is stable, it has zero steady-state error.

(b) The characteristic equation is

$$s^3 + (3.6 + K_a)s^2 + (3.2 + 40K_a)s + 375K_a .$$

The Routh array is

$s^3$	1	$3.2 + 40K_a$
$s^2$	$3.6 + K_a$	$375K_a$
$s^1$	$b$	
$s^0$	$375K$	

Solving for  $b > 0$  leads to  $0 < K_a < 0.05$  or  $K_a > 5.64$  for stability.

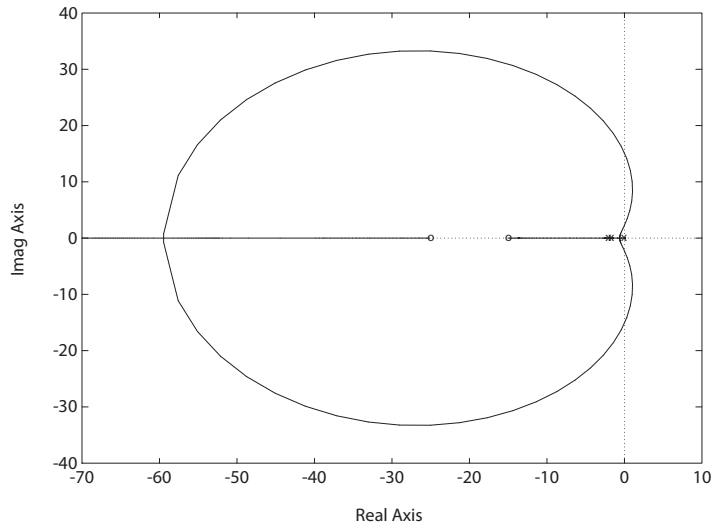
(c) The characteristic equation can be written as

$$1 + \frac{K_a(s + 25)(s + 15)}{s(s + 2)(s + 1.6)} = 0 .$$

The root locus is shown in Figure P7.12.

(d) When

$$K > 40 ,$$



**FIGURE P7.12**  
 Root locus for  $1 + K_a \frac{(s+25)(s+15)}{s(s+2)(s+1.6)} = 0$ , where  $K_m = 1.8$ .

the roots are

$$s_1 = -123 \quad \text{and} \quad s_{2,3} = -15.6 \pm j31.2 .$$

From the step response we find

$$P.O. = 5\%$$

$$T_p = 0.67 \text{ sec}$$

$$T_s = 0.25 \text{ sec} .$$

**P7.13** (a) The characteristic equation is

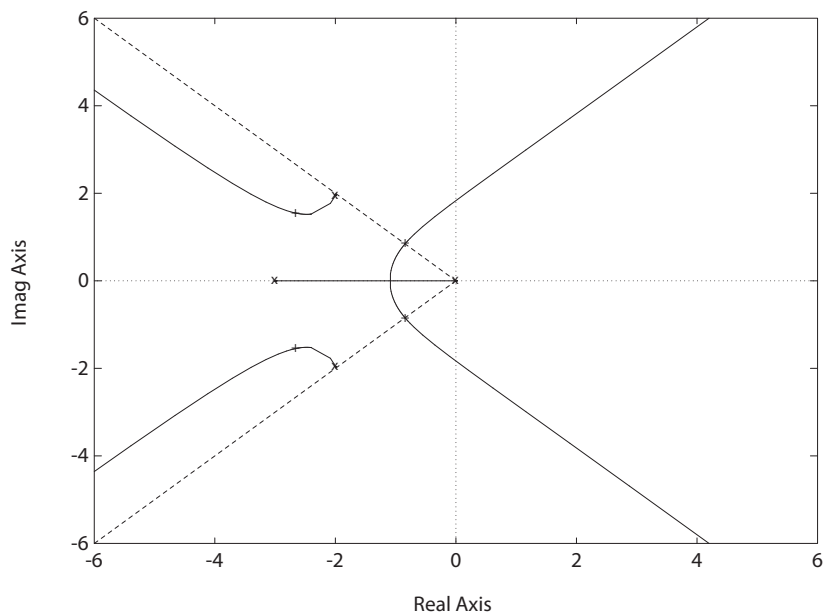
$$1 + \frac{K}{s(s+3)(s^2+4s+7.84)} = 0 .$$

The root locus is shown in Figure P7.13. The breakaway point is  $s = -1.09$  at  $K = 9.72$ .

(b) When  $K = 13.5$ , the roots are

$$s_{1,2} = -0.84 \pm j0.84$$

$$s_{3,4} = -2.66 \pm j1.55 .$$



**FIGURE P7.13**

Root locus for  $1 + \frac{K}{s(s+3)(s^2+4s+7.84)} = 0$ .



(c) The roots

$$s = -0.84 \pm j0.84$$

are dominant roots.

(d) For the dominant roots, we determine that  $\zeta = 0.7$  and  $\omega_n = 1.19$ . Therefore, the settling time is

$$T_s = \frac{4}{\zeta\omega_n} = 4.8 \text{ sec} .$$

**P7.14** The characteristic equation is

$$1 + \frac{K(s + 2.5)(s + 3.2)}{s^2(s + 1)(s + 10)(s + 30)} = 0 .$$

The root locus is shown in Figure P7.14. When  $K = 559.3$ , the roots are

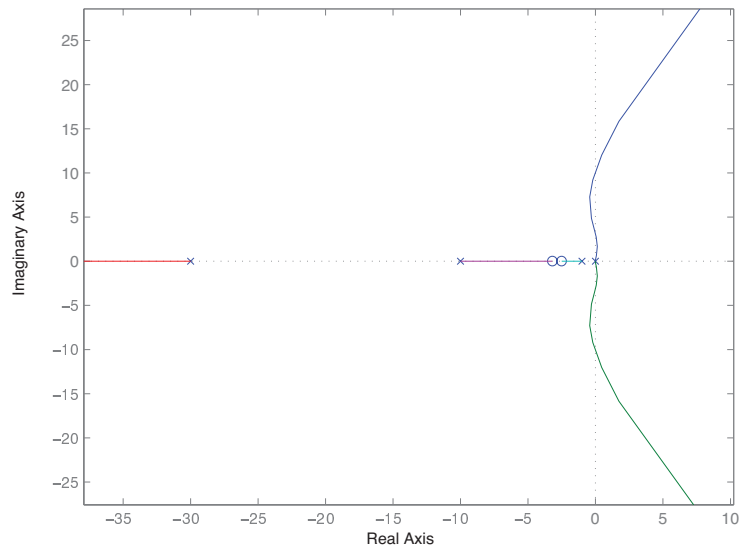
$$s_1 = -30.75 \quad s_2 = -8.48 \quad s_3 = -1.78 \quad s_{4,5} = \pm j3.11 .$$

When  $K = 4321$ , the roots are

$$s_1 = -34.45 \quad s_2 = -4.35 \quad s_3 = -2.21 \quad s_{4,5} = \pm j10.23 .$$

The crossover points are

$$s = \pm j3.11 \quad \text{and} \quad s = \pm j10.23 .$$



**FIGURE P7.14**

Root locus for  $1 + K \frac{(s+2.5)(s+3.5)}{s^2(s+1)(s+10)(s+30)} = 0$ .

Therefore, the system is stable for

$$559.3 < K < 4321 .$$

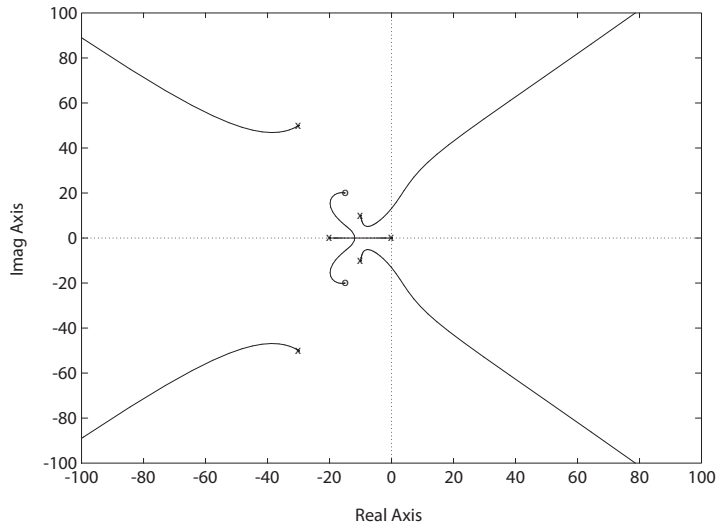
**P7.15** The characteristic equation is

$$1 + \frac{K(s^2 + 30s + 625)}{s(s + 20)(s^2 + 20s + 200)(s^2 + 60s + 3400)} .$$

The root locus is shown in Figure P7.15. When  $K = 30000$ , the roots are

$$s_1 = -18.5 \quad s_2 = -1.69 \quad s_{3,4} = -9.8 \pm j8.9 \quad s_{5,6} = -30.1 \pm j49.9.$$

The real root near the origin dominates, and the step response is over-damped.



**FIGURE P7.15**

Root locus for  $1 + K \frac{s^2 + 30s + 625}{s(s + 20)(s^2 + 20s + 200)(s^2 + 60s + 3400)} = 0$ .

- P7.16** (a) Let  $\tau = 0$ . Then, first reduce the motor and rolls to an equivalent  $G(s)$  as follows:

$$G(s) = \frac{\frac{0.25}{s(s+1)}}{1 + \frac{0.25}{s(s+1)}} = \frac{0.25}{s(s+1) + 0.25} = \frac{0.25}{(s+0.5)^2}.$$

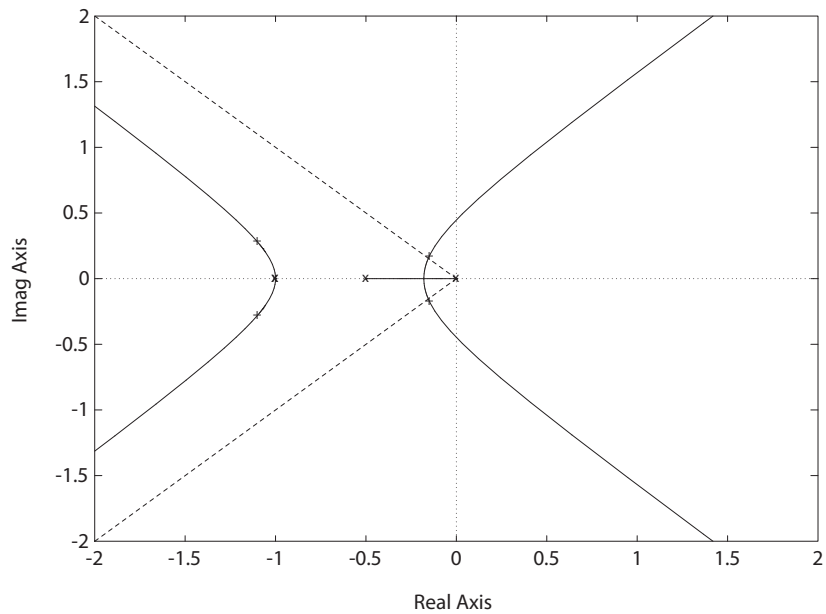
The loop transfer function is then

$$L(s) = \frac{2(s+0.5)K_a(0.25)}{s(s+1)^2(s+0.5)^2} = \frac{0.5K_a}{s(s+1)^2(s+0.5)}.$$

The characteristic equation is

$$1 + K_a \frac{0.5}{s(s+1)^2(s+0.5)} = 0.$$

The root locus is shown in Figure P7.16.



**FIGURE P7.16**

Root locus for  $1 + \frac{0.5K_a}{s(s+1)^2(s+0.5)} = 0$ .

- (b) When  $K = 0.123$ , the roots of the characteristic equation are

$$\begin{aligned} s_{1,2} &= -1.1 \pm j0.27 \\ s_{3,4} &= -0.15 \pm j0.15. \end{aligned}$$

The roots at  $s = -0.15 \pm j0.15$  have a damping ratio of  $\zeta = 0.707$ .

(c) When  $\tau$  becomes nonnegligible, the root locus will have an additional pole, and the root locus will change accordingly.

**P7.17** The characteristic equation is

$$(M_1 s^2 + bs + k_1 + k_{12})(M_2 s^2 + k_{12}) - k_{12}^2 = 0 .$$

If we let  $M_1 = k_1 = b = 1$ , and assume  $k_{12} < 1$  so that  $k_{12}^2$  is negligible and  $k_1 + k_{12} \approx k_1$ , then the characteristic equation is

$$(s^2 + s + 1)(M_2 s^2 + k_{12}) = 0 \quad \text{or} \quad 1 + \frac{k}{s^2} = 0 ,$$

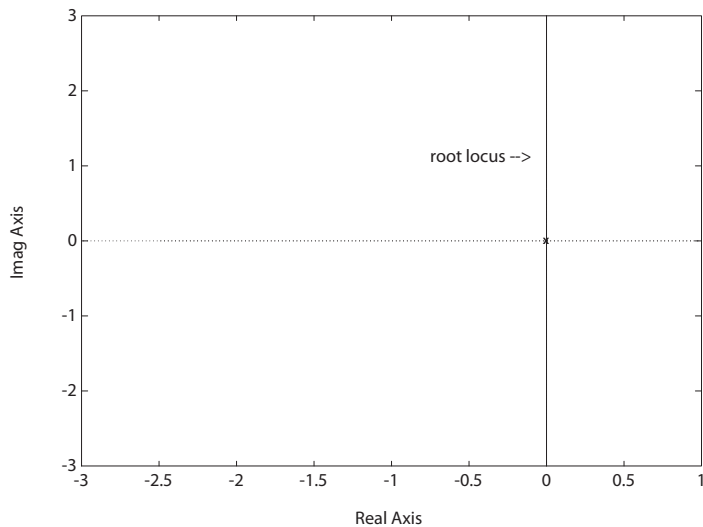
where

$$k = \frac{k_{12}}{M_2} .$$

The root locus is shown in Figure P7.17. All the roots lie on the  $j\omega$  axis. If we select

$$\sqrt{\frac{k_{12}}{M_2}} = \omega_o ,$$

then we cancel the vibration.



**FIGURE P7.17**  
Root locus for  $1 + \frac{k}{s^2} = 0$ .

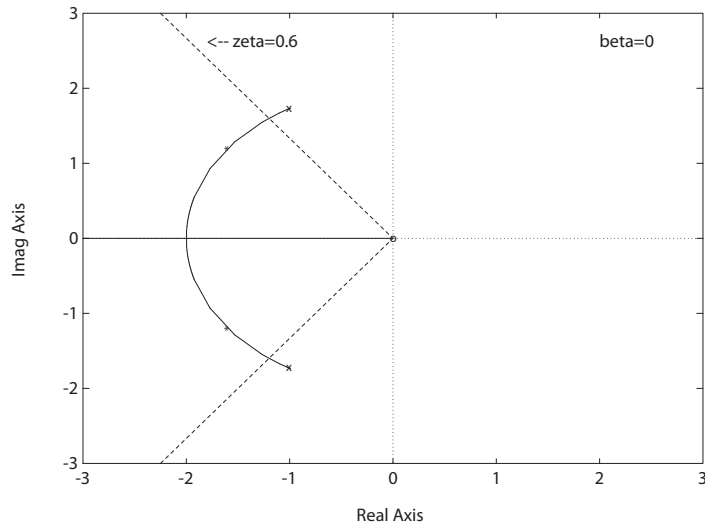
**P7.18** The characteristic equation is

$$\beta s^3 + (1 + 2\beta)s^2 + (2 + 4\alpha)s + 4 = 0 .$$

When  $\beta = 0$  we have

$$1 + \frac{4\alpha s}{s^2 + 2s + 4} = 0 .$$

The root locus for  $\beta = 0$  is shown in Figure P7.18.



**FIGURE P7.18**

Root locus for  $1 + \alpha \frac{4s}{s^2 + 2s + 4} = 0$ , where  $\beta = 0$ .

For  $\alpha = 0.3$ , the poles are

$$s = -1.6 \pm j1.2 .$$

Then, we have

$$1 + \frac{\beta(s + 2)s^2}{s^2 + (2 + 4\alpha)s + 4} = 0 .$$

When  $\beta = 0.121$

$$\begin{aligned} s_{1,2} &= -1.51 \pm j1.51 \\ s_3 &= -7.24 . \end{aligned}$$



where

$$K = \frac{4\alpha}{\beta} \quad \alpha = 0.3 \quad \beta = 0.121 .$$

The root sensitivity to changes in  $K$  is found to be

$$S_K^{r_1} \cong \frac{\Delta r_1}{\Delta K/K} = 1.18\angle -149.75^\circ .$$

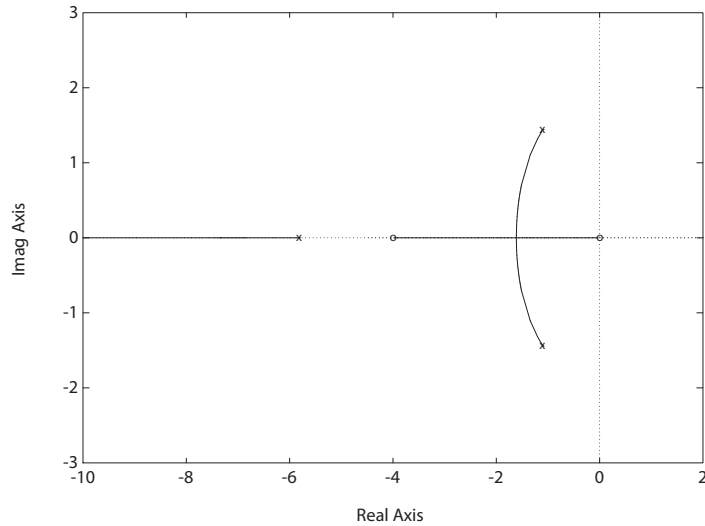
The root sensitivity to changes in the pole at  $s = -2$  is found to be

$$S_\Delta^{r_1} \cong \frac{\Delta r_1}{\Delta/2} = 1.65\angle -137^\circ , \quad \text{where the pole is } s + 2 + \Delta .$$

- P7.21** (a) Let the pole be  $(s + 4 + \Delta)$  and neglect  $\Delta^2$  terms. Then, the characteristic equation is

$$1 + \Delta \frac{2s^2 + (8 + 2\delta)s + 8\delta}{s^3 + (8 + \delta)s^2 + (16 + 8\delta)s + 16\delta + K} = 0$$

where  $\delta = 0.000788$  and  $K = 19.1$ .



**FIGURE P7.21**

Root locus for  $1 + \Delta \frac{2s^2 + (8 + 2\delta)s + 8\delta}{s^3 + (8 + \delta)s^2 + (16 + 8\delta)s + 16\delta + K} = 0$ , ( $\delta = 0.000788$  and  $K = 19.1$ ).

The root sensitivity is determined to be

$$S_\Delta^{r_1} \cong \frac{\Delta r_1}{\Delta/4} = 3.314\angle -132^\circ .$$

(b) Let  $R = R_o + \Delta R$ , where  $R = 0.00021$ . Then,

$$S_R^{r_1} \cong \frac{\Delta r_1}{\Delta R/R} = 1.31 \angle -107^\circ .$$

**P7.22** The characteristic equation is

$$s^3 + 2s^2 + s + K ,$$

where  $K = 0.24$  for  $\zeta = 0.707$ . The root sensitivity to changes in the pole at  $s = -1$  is found to be

$$S_{\Delta}^{r_1} \cong \frac{\Delta r_1}{\Delta} = 0.95 \angle -126^\circ ,$$

where the pole is  $s + 1 + \Delta$ .

**P7.23** The characteristic equation is

$$s^3 + 5s^2 + (6 + K)s + K ,$$

where  $K = 6.3$  for  $\zeta = 0.707$ . The root sensitivity to changes in the pole at  $s = -2$  is found to be

$$S_{\Delta}^{r_1} \cong \frac{\Delta r_1}{\Delta/2} = 1.25 \angle -169.4^\circ ,$$

where the pole is  $s + 2 + \Delta$ . The root sensitivity to changes in the zero at  $s = -1$  is found to be

$$S_{\Delta}^{r_1} \cong \frac{\Delta r_1}{\Delta} = 0.55 \angle 34.4^\circ ,$$

where the zero is  $s + 1 + \Delta$ .

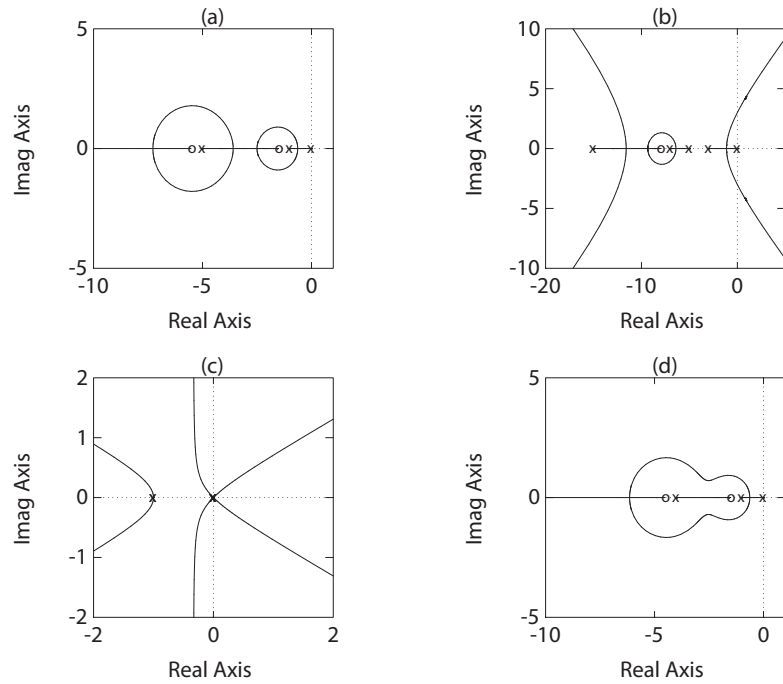
**P7.24** The root locus for each of the four cases shown is shown in Figure P7.24. The four open-loop transfer functions are

$$(a) \quad KF(s) = \frac{s^2 + 7s + 8.25}{s^3 + 6s^2 + 5s}$$

$$(b) \quad KF(s) = \frac{s + 8}{s^5 + 30s^4 + 296s^3 + 1170s^2 + 1575s}$$

$$(c) \quad KF(s) = \frac{1}{s^6 + 2s^5 + s^4} \quad (d) \quad KF(s) = \frac{s^2 + 6s + 6.75}{s^3 + 5s^2 + 4s}$$





**FIGURE P7.24**  
Root locus for the four cases.

**P7.25** The characteristic equation is

$$1 + KG_c(s)G(s) = 0 ,$$

therefore,

$$KG_c(s)G(s) = -1 .$$

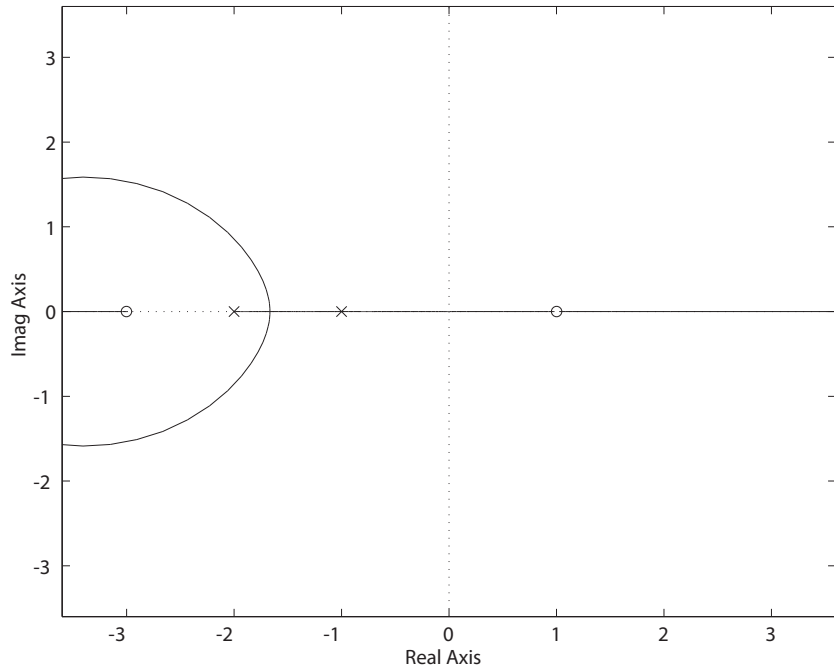
Squaring both sides yields  $K^2G_c^2(s)G^2(s) = 1$  and

$$1 - K^2G_c^2(s)G^2(s) = 0 .$$

The root locus with  $0 < K^2 < \infty$  is shown in Figure P7.25. The value of  $K^2$  for which the locus crosses the imaginary axis is

$$K^2 = 2/3 ,$$

therefore  $K = \sqrt{2/3} = 0.8165$  corresponds to the  $j\omega$ -axis crossing (at  $s = 0$ ). You can check that  $1 + KG_c(s)G(s) = 0$  for  $K = 0.8165$  and  $s = 0$ .

**FIGURE P7.25**

Root locus for the equation  $1 - K^2 G_c^2(s) G^2(s) = 0$ .

- P7.26** (a) The characteristic equation is

$$1 + \frac{K(s+2)^2}{s(s^2+1)(s+8)} = 0.$$

The root locus is shown in Figure P7.26.

- (b) Using Routh's criteria, we determine that

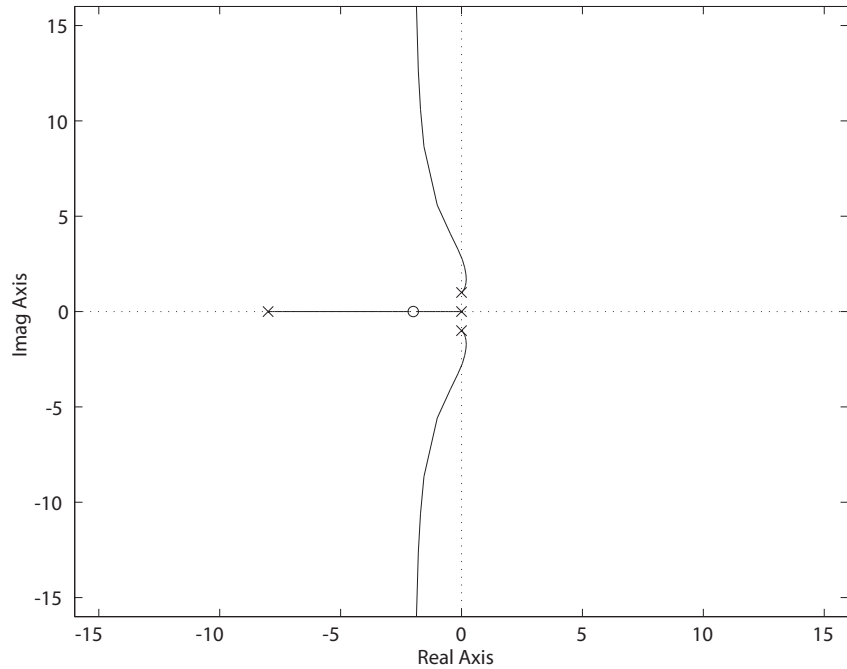
$$K > 14$$

for stability.

- (c) From the Routh array, we determine that for  $K = 14$ , we have two purely imaginary poles at

$$s = \pm j\sqrt{8}.$$

- (d) When  $K > 50$ , the real part of the complex roots is approximately equal to the real part of the two real roots and therefore the complex roots are not dominant roots.

**FIGURE P7.26**

Root locus for  $1 + K \frac{(s+2)^2}{s(s^2+1)(s+8)} = 0$ .

**P7.27** The characteristic equation is

$$1 + \frac{K(s^2 + 0.1)}{s(s^2 + 2)} = 0 .$$

The root locus is shown in Figure P7.27a. The locus enters the axis at

$$s = -1.26$$

and leaves the axis at

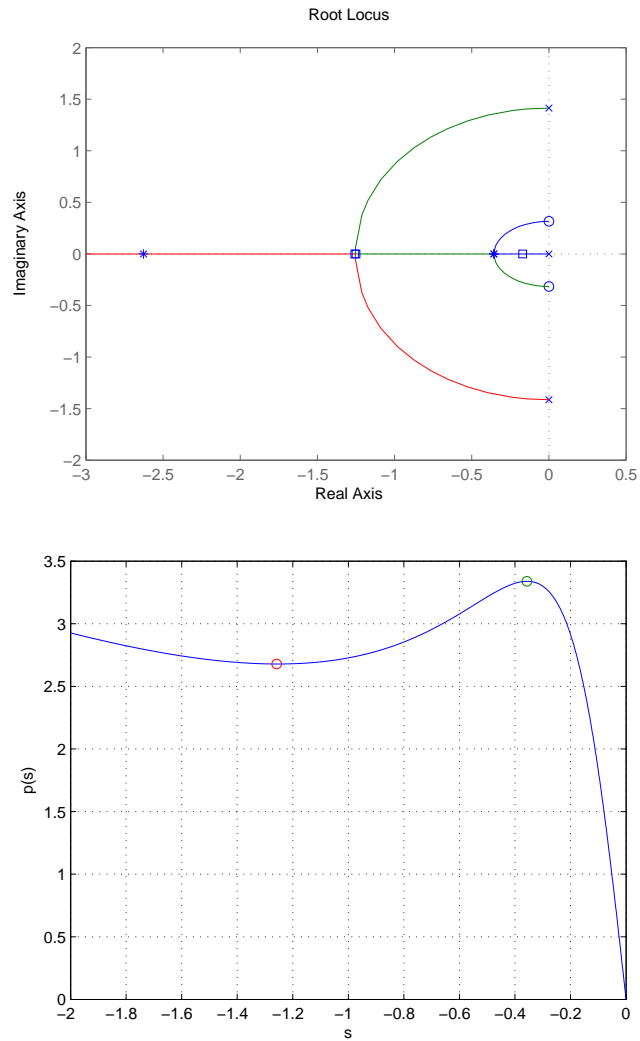
$$s = -0.36 .$$

Define

$$p(s) = K = \frac{-(s^3 + 2s)}{s^2 + 0.1} .$$

Then, a plot of  $p(s)$  vs  $s$  is shown in Figure P7.27b, where it can be seen that  $p(s)$  has two inflection points at

$$s = -1.28 \quad \text{and} \quad s = -0.36 .$$

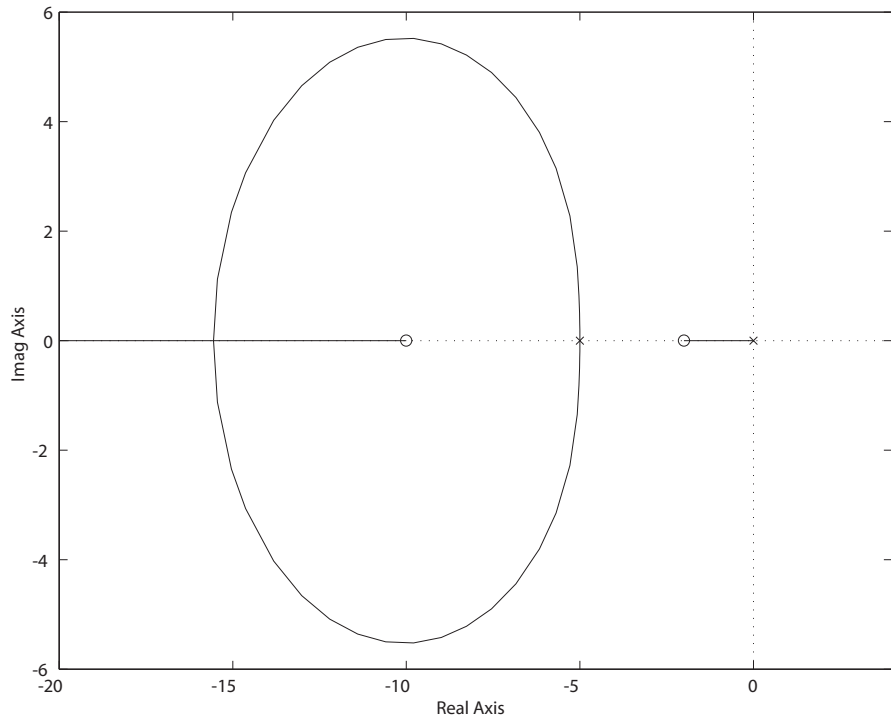


**FIGURE P7.27** (a) Root locus for  $1 + K \frac{s^2+0.1}{s(s^2+2)} = 0$ . (b) Plot of  $p(s) = -\frac{s^3+2s}{s^2+0.1}$  versus  $s$ .

**P7.28** The characteristic equation is

$$1 + L(s) = 1 + \frac{K(s^2 + 12s + 20)}{s^3 + 10s^2 + 25s} = 0 .$$

The root locus is shown in Figure P7.28. The breakaway point is  $s = -5.0$

**FIGURE P7.28**

Root locus for  $1 + K \frac{(s^2 + 12s + 20)}{s^3 + 10s^2 + 25s} = 0$ .

and the entry point is  $s = -15.6$ . When  $K = 2$ , the roots are

$$\begin{aligned} s_1 &= -1.07 \\ s_{2,3} &= -5.46 \pm j2.75 . \end{aligned}$$

When  $K = 2$ , the roots are

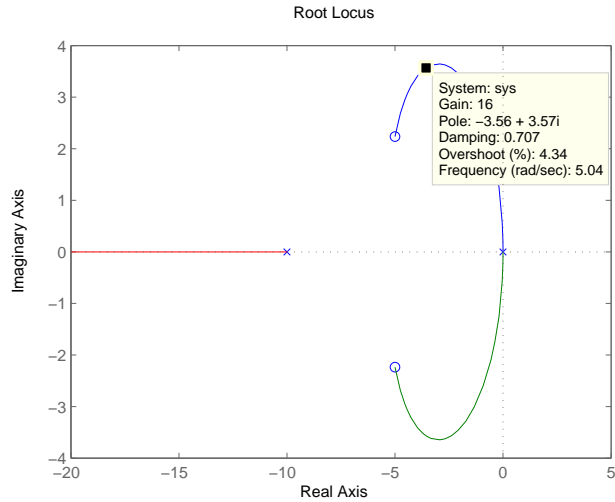
$$\begin{aligned} s_1 &= -1.07 \\ s_{2,3} &= -4.36 \pm j1.68 . \end{aligned}$$

The predicted step response when  $K = 2$  is  $T_s = 9$  sec and  $PO \approx 0\%$ .

**P7.29** The characteristic equation is

$$1 + K \frac{s^2 + 10s + 30}{s^2(s + 10)} = 0 .$$

The root locus is shown in Figure P7.29. When  $\zeta = 0.707$ , the necessary gain is  $K = 16$ . The corresponding roots are  $s_1 = -18.87$  and  $s_{2,3} = -3.56 \pm j3.56$ .

**FIGURE P7.29**

Root locus for  $1 + K \frac{s^2 + 10s + 30}{s^2(s + 10)} = 0$ .

**P7.30** The transfer function is

$$Z(s) = \frac{LCRs^2 + Ls}{LCs^2 + CRs + 1} = \frac{Rs^2 + s}{s^2 + Rs + 1}.$$

So,

$$r_1 = -\frac{R}{2} + \left(\frac{R^2}{4} - 1\right)^{\frac{1}{2}}.$$

Thus, the nominal  $r_{1o} = -\frac{1}{2}$ . Simultaneously,

$$r_2 = -\frac{R}{2} - \left(\frac{R^2}{4} - 1\right)^{\frac{1}{2}}.$$

Thus, the nominal  $r_{2o} = -2$ . We see that there is a difference by a factor of 4. Also,

$$S_R^{r_1} = \left. \frac{\partial r_1}{\partial R} \right|_{R_o} \cdot R_o = -\frac{R_o}{2} + \frac{R_o^2}{4} \left(\frac{R_o^2}{4} - 1\right)^{-\frac{1}{2}} = \frac{5}{6},$$

where  $R_o = 2.5$ . And

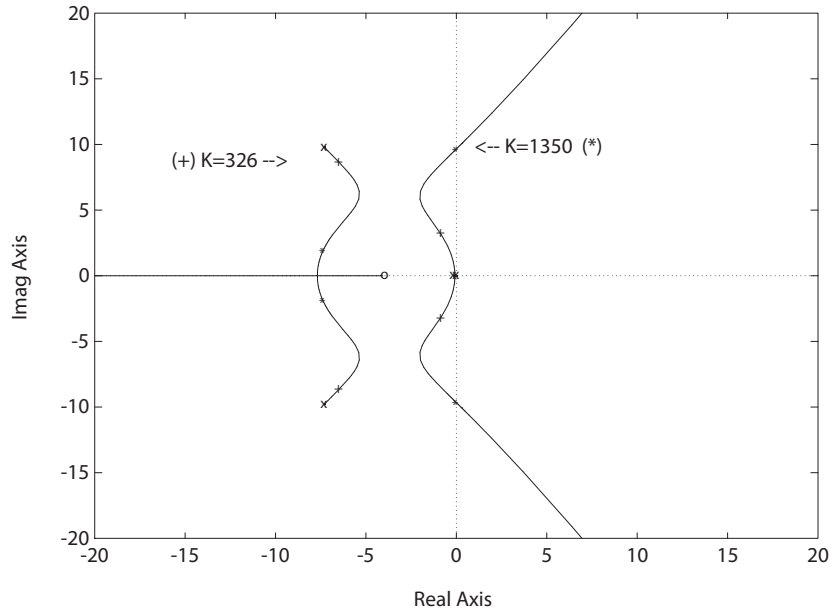
$$S_R^{r_2} = \left. \frac{\partial r_2}{\partial R} \right|_{R_o} \cdot R_o = -\frac{R_o}{2} - \frac{R_o^2}{4} \left(\frac{R_o^2}{4} - 1\right)^{-\frac{1}{2}} = \frac{-10}{3}.$$

So, the magnitude of  $|S_R^{r2}| = 4|S_R^{r1}|$ .

**P7.31** The characteristic equation is

$$1 + K \frac{s + 4}{s(s + 0.16)(s^2 + 14.6s + 148.999)} = 0 .$$

The root locus is shown in Figure P7.31. When  $K = 1350$ , the roots are



**FIGURE P7.31**

Root locus for  $1 + K \frac{s+4}{s(s+0.16)(s^2+14.6s+148.999)} = 0$ .

$$s_{1,2} = \pm j9.6$$

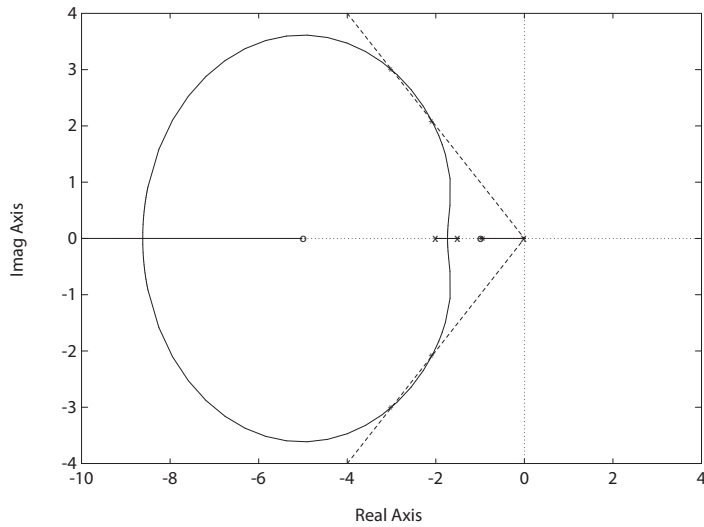
$$s_{3,4} = -7.4 \pm j1.9 .$$

When  $K = 326$ , the roots are

$$s_{1,2} = -6.5 \pm j8.7 \quad s_{3,4} = -0.9 \pm j3.2 .$$

**P7.32** The characteristic equation is

$$1 + \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)} = 0 .$$



**FIGURE P7.32**  
 Root locus for  $1 + K \frac{(s+1)(s+5)}{s(s+1.5)(s+2)} = 0$ .

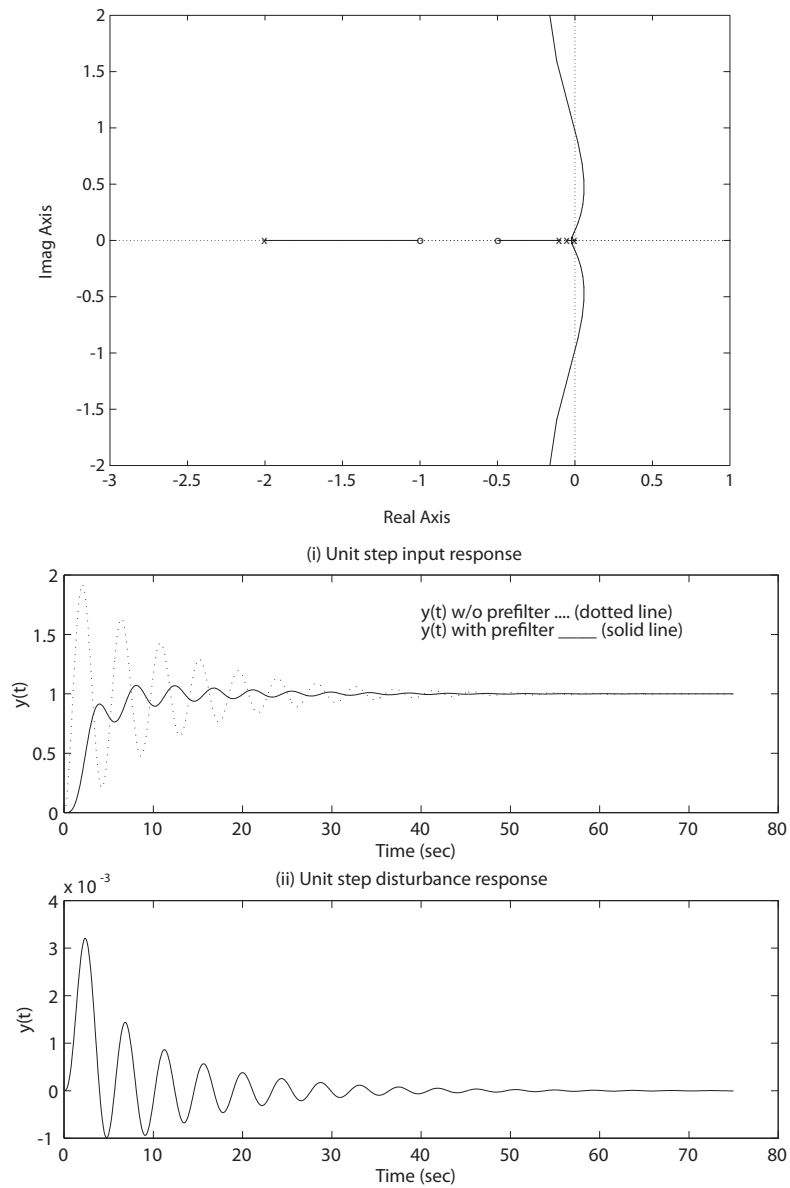
$K$	$\zeta$	$T_s$ (sec)	$P.O.$ (%)
1.57	0.707	0.98	1.4
3.48	0.707	1.1	5.8
2.35	0.69	1.3	4.0

**TABLE P7.32** Step Response Results for  $K = 1.57$ ,  $K = 3.48$ , and  $K = 2.35$ .

- (a) The breakaway point is  $s = -1.73$ ; the entry point is  $s = -8.62$ .
- (b) The damping ratio  $\zeta = 0.707$  when  $K = 1.57$  and again when  $K = 3.46$ .
- (c) The minimum damping ratio  $\zeta = 0.69$  is achieved when  $K = 2.35$ .
- (d) The results are summarized in Table P7.32. The best choice of gain is  $K = 1.57$ .

- P7.33**
- (a) The root locus for the V-22 is shown in Figure P7.33a. The system is stable when  $0 < K < 0.48$  and  $K > 136.5$ .
  - (b) The unit step input response (for  $K = 280$ ) is shown in Figure P7.33b. The step response has a  $P.O. = 90\%$  and  $T_s \approx 50$  sec.
  - (c) The plot of  $y(t)$  for a unit step disturbance is shown in Figure P7.33b.



**FIGURE P7.33**

(a) Root locus for  $1 + K \frac{s^2 + 1.5s + 0.5}{s(20s + 1)(10s + 1)(0.5s + 1)} = 0$ . (b) (i) Unit step input response with and without prefilter; (ii) Unit step disturbance response.

The response to the disturbance is oscillatory, but the maximum value of oscillation is about 0.003; so it is negligible.

(d) The effect of adding a prefilter can be seen in Figure P7.33b. With

the prefilter we find  $PO = 7\%$  and  $T_s \approx 40$  sec.

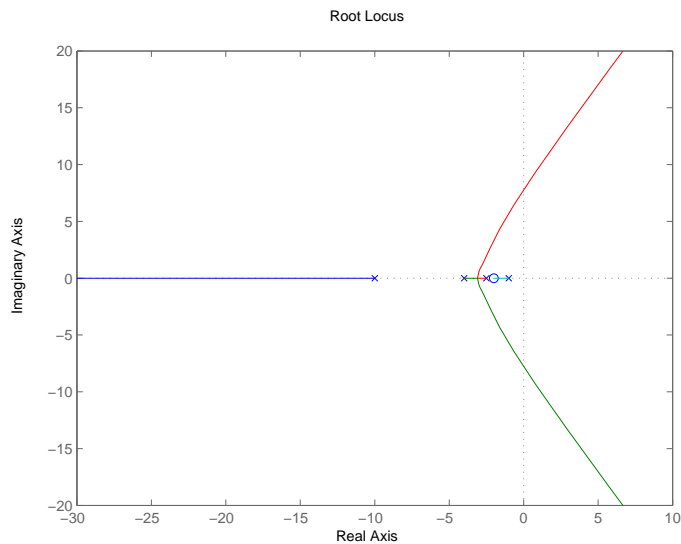
**P7.34** The characteristic equation is

$$1 + \frac{K(s+2)}{(s+1)(s+2.5)(s+4)(s+10)} = 0.$$

The root locus is shown in Figure P7.34a. The roots, predicted and actual percent overshoot for  $K = 400, 500,$  and  $600$  are summarized in Table P7.34. The actual unit step input responses are shown in Figure P7.34b.

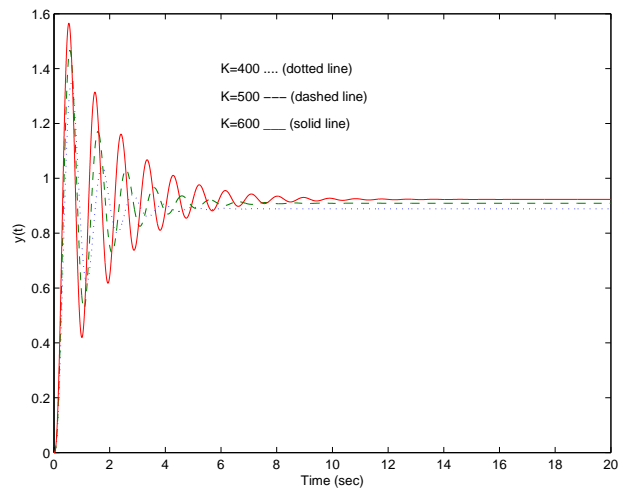
$K$	roots	$\zeta$	predicted $P.O.$ (%)	actual $P.O.$ (%)
400	-13.5, $-1.00 \pm 5.71j$ , -1.98	0.173	57.6	51.6
500	-14.0, $-0.75 \pm 6.24j$ , -1.98	0.120	68.4	61.2
600	-14.4, $-0.53 \pm 6.71j$ , -1.98	0.079	77.9	69.6

**TABLE P7.34** Summary for  $K = 400, 500, 600$ .



**FIGURE P7.34**

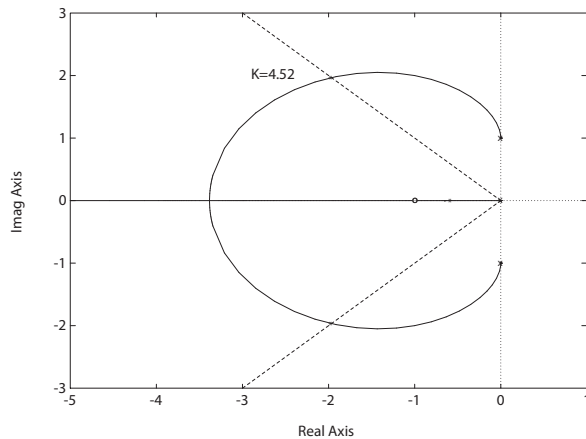
(a) Root locus for  $1 + K \frac{s+2}{(s+1)(s+2.5)(s+4)(s+10)} = 0$ .



**FIGURE P7.34**  
CONTINUED (b) Unit step input responses for  $K = 400, 500, 600$ .

**P7.35** (a) The root locus is shown in Figure P7.35 for the characteristic equation

$$1 + \frac{K(s+1)^2}{s(s^2+1)} = 0.$$



**FIGURE P7.35**  
Root locus for  $1 + K \frac{(s+1)^2}{s(s^2+1)} = 0$ .

(b) When  $K = 4.52$ , the roots are

$$s_1 = -0.58$$

$$s_{2,3} = -1.96 \pm j1.96 .$$

The complex roots have  $\zeta = 0.707$ .

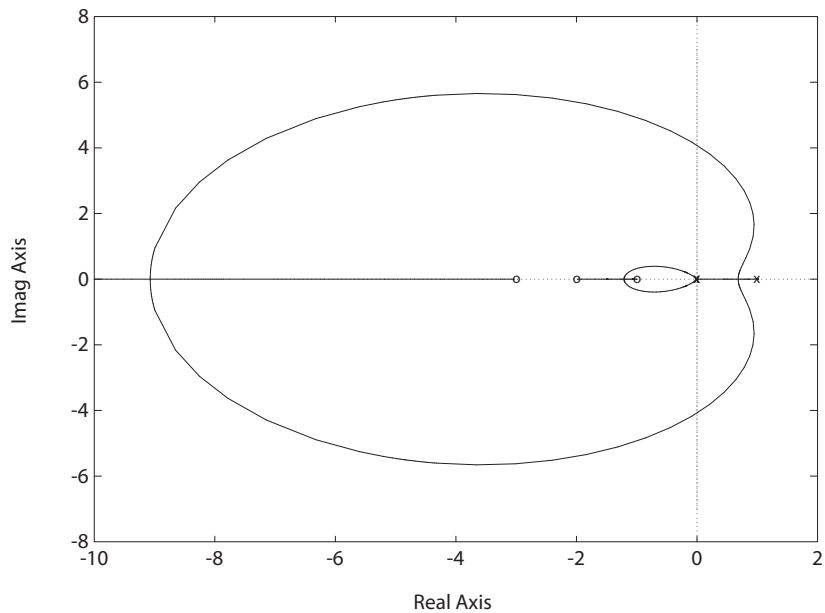
(c) The entry point is  $s = -3.38$  when  $K = 7.41$ .

(d) The predicted  $P.O. = 4.5\%$  ( $\zeta = 0.707$ ) and the actual  $P.O. = 17\%$ .

**P7.36** The characteristic equation is

$$1 + \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)} = 0 .$$

(a) The root locus is shown in Figure P7.36.



**FIGURE P7.36**

Root locus for  $1 + K \frac{(s+1)(s+2)(s+3)}{s^3(s-1)} = 0$ .

(b) When  $K = 2.96$ , the roots are

$$s_{1,2} = \pm j4.08$$

$$s_{3,4} = -0.98 \pm j0.33 .$$

(c) When  $K = 20$ , the roots are

$$\begin{aligned}s_1 &= -1.46 \\s_2 &= -1.07 \\s_{3,4} &= -8.23 \pm j2.99 .\end{aligned}$$

When  $K = 100$ , the roots are

$$\begin{aligned}s_1 &= -92.65 \\s_2 &= -3.51 \\s_3 &= -1.82 \\s_4 &= -1.01 .\end{aligned}$$

(d) When  $K = 20$ , the damping ratio is  $\zeta = 0.94$ . Therefore, the predicted  $P.O. = 0.02\%$ . The actual overshoot is  $P.O. = 23\%$ .

**P7.37** Since we know that  $e_{ss} = 0$  for a step input, we know that  $a = 0$  or  $b = 0$ . Select  $a = 0$ . Also,  $\omega_n = 2\pi/T = 20$  rad/sec. The desired characteristic polynomial is

$$(s + r_1)(s + j20)(s - j20) = s^3 + r_1s^2 + 400s + 400r_1 = 0 .$$

The actual characteristic polynomial is

$$1 + \frac{2K}{s(s+b)(s+40)} = 0 , \quad \text{or} \quad s^3 + (40+b)s^2 + 40bs + 2K = 0 .$$

Comparing the coefficients in the desired and actual characteristic polynomials, we determine that  $b = 10$ ,  $r_1 = 50$ , and  $K = 10000$ .

**P7.38** (a) The characteristic equation is

$$1 + \frac{K(s+1)}{s(s-3)} = 0 .$$

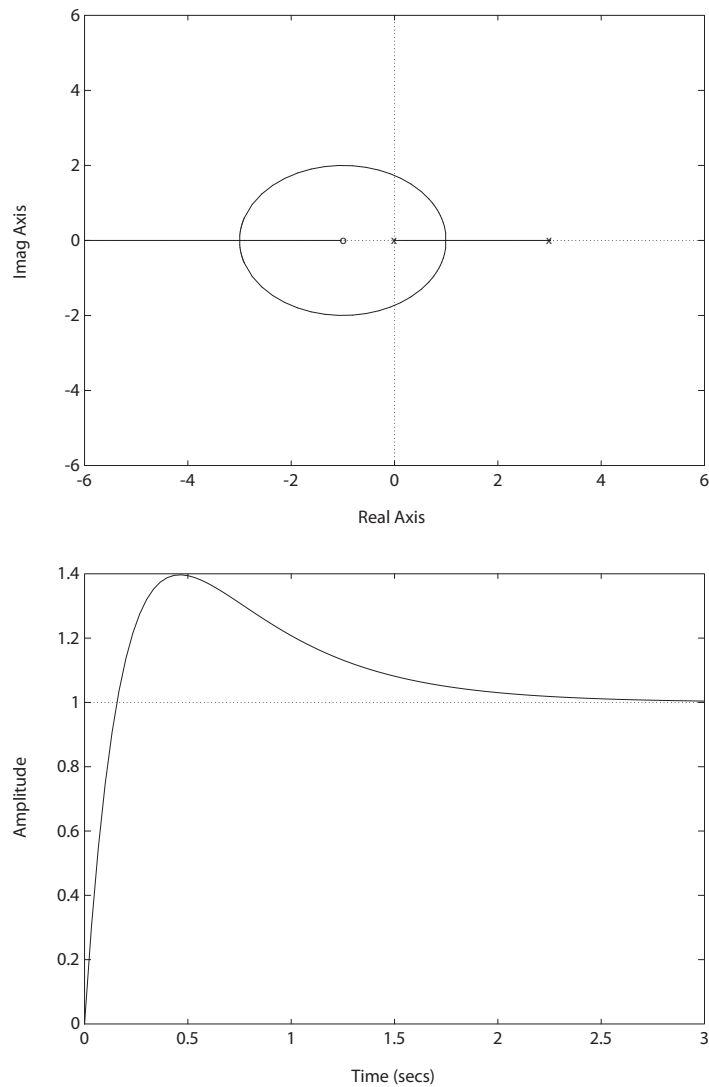
The system is stable for  $K > 3$ . When  $K = 3$ , the roots are  $s = \pm j\sqrt{3}$ .

(b) The root locus is shown in Figure P7.38a.

(c) When  $K = 10$ , the roots are

$$\begin{aligned}s_1 &= -2 \\s_2 &= -5 .\end{aligned}$$

Since both roots are real and stable, we expect that there will be zero overshoot. The actual response has a 40% overshoot, as seen in Figure P7.38b.

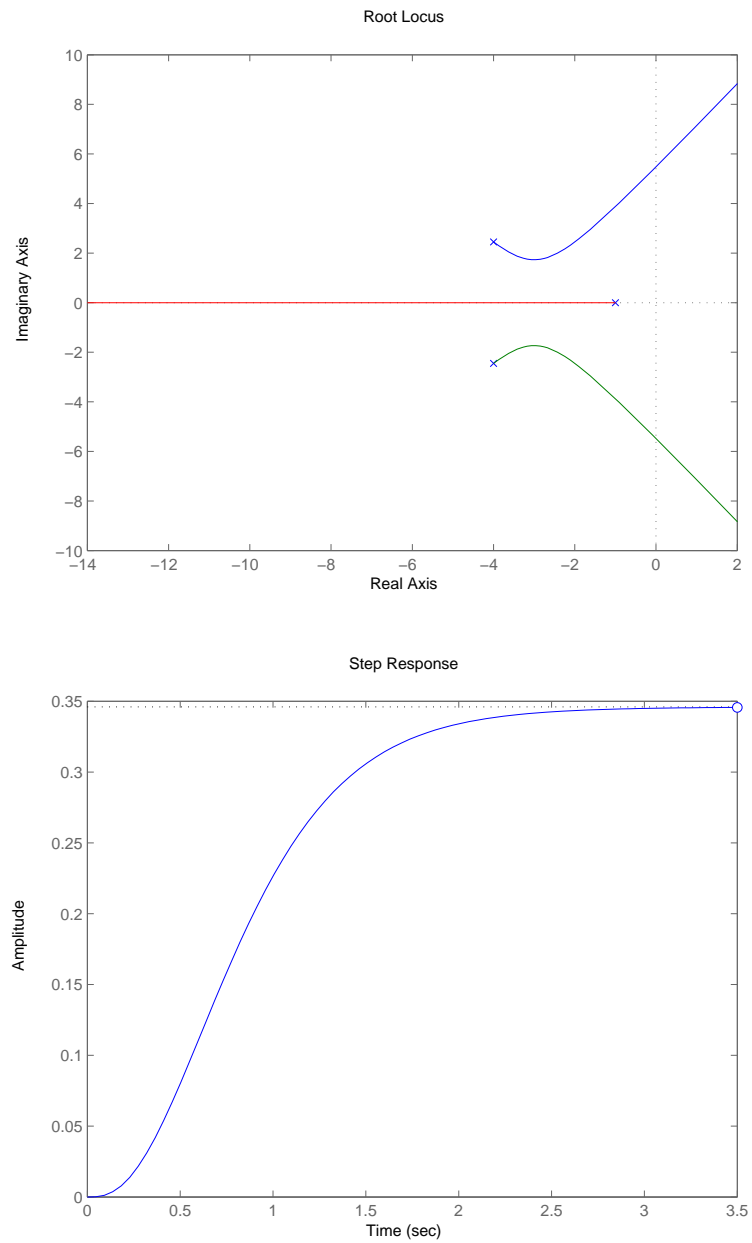
**FIGURE P7.38**

(a) Root locus for  $1 + K \frac{s+1}{s(s-3)} = 0$ . (b) Unit step response.

**P7.39** The loop transfer function is

$$G_c(s)G(s) = \frac{22K}{(s+1)(s^2+8s+22)}.$$

When  $K = 0.529$ , the closed-loop poles are  $s_{1,2} = -3.34 \pm 1.83j$  and  $s_3 = -2.32$  and have the maximum damping  $\zeta = 0.877$ . The root locus is shown in Figure P7.39a. The step response is shown in Figure P7.39b.



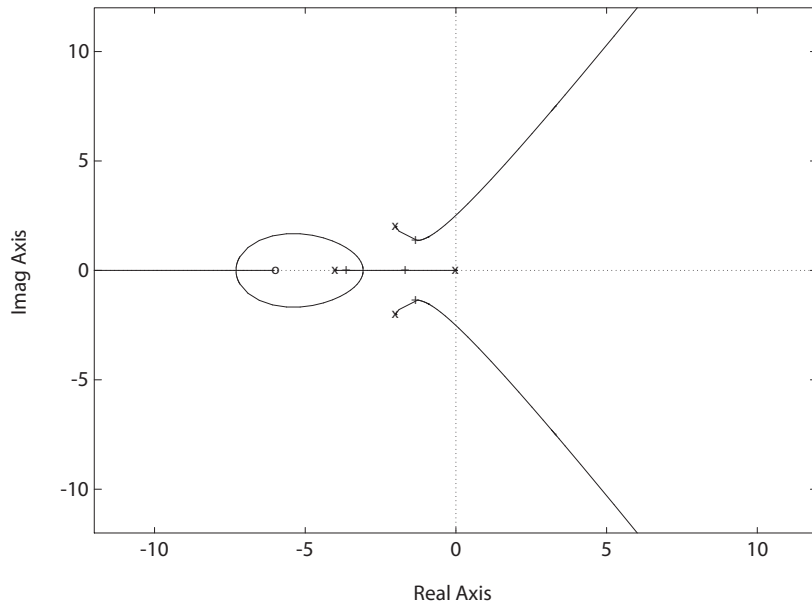
**FIGURE P7.39**  
(a) Root locus for  $\frac{22K}{(s+1)(s^2+8s+22)} = 0$ . (b) Unit step response.

## Advanced Problems

AP7.1 The characteristic equation is

$$1 + K \frac{s + 6}{s(s + 4)(s^2 + 4s + 8)} = 0 .$$

The root locus is shown in Figure AP7.1. The gain at maximum  $\zeta$  is



**FIGURE AP7.1**

Root locus for  $1 + K \frac{s(s+4)}{s^2+2s+2} = 0$ .

$$K = 3.7 .$$

The roots at  $K = 3.7$  are

$$s_1 = -3.6424 \quad s_{2,3} = -1.3395 \pm +1.3553j \quad s_4 = -1.6786 .$$

Using Figure 5.13 in Dorf & Bishop, the predicted percent overshoot and settling time are

$$P.O. = 5\% \quad \text{and} \quad T_s = 3 \text{ sec} ,$$



since  $\zeta = 0.7$  and

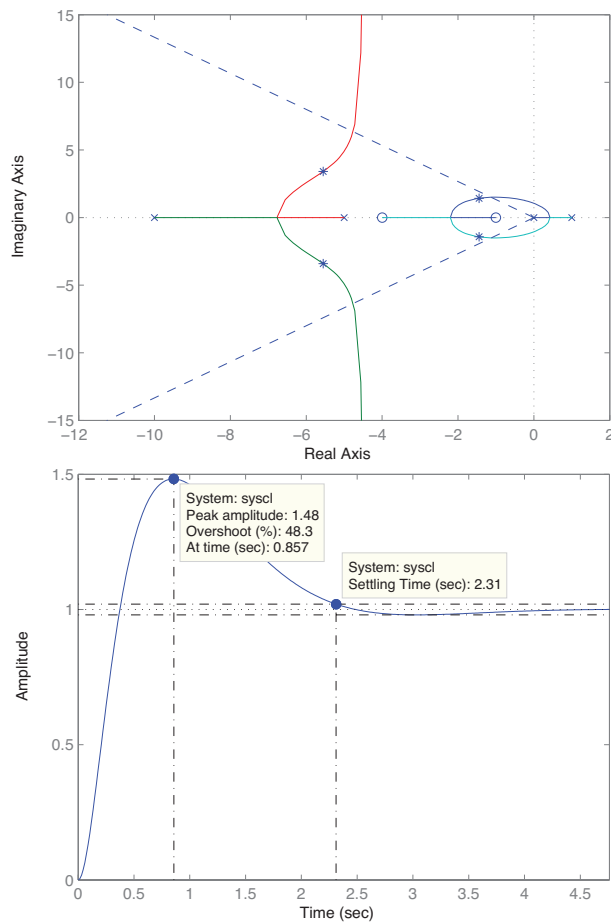
$$\frac{a}{\omega_n \zeta} = \frac{6}{1.9(0.7)} = 4.5 .$$

The actual percent overshoot and settling time are  $P.O. = 1\%$  and  $T_s = 2.8$  sec.

**AP7.2** The characteristic equation is

$$1 + K \frac{(s + 1)(s + 4)}{s(s - 1)(s + 5)(s + 10)} = 0 .$$

The root locus is shown in Figure AP7.2a. The selected gain is  $K = 43.7$ .



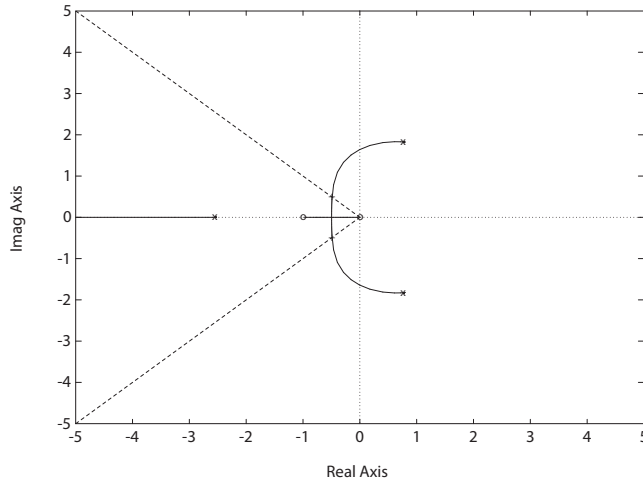
**FIGURE AP7.2**  
 (a) Root locus for  $1 + K \frac{(s+1)(s+4)}{s(s-1)(s+5)(s+10)} = 0$ ; (b) Step response for  $K = 43.7$ .

The actual percent overshoot (see Figure AP7.2b) is  $P.O. = 48.3\%$ .

**AP7.3** The characteristic equation (with  $p$  as the parameter) is

$$1 + p \frac{s(s+1)}{s^3 + s^2 + 10} = 0 .$$

The root locus is shown in Figure AP7.3.



**FIGURE AP7.3**  
Root locus for  $1 + p \frac{s(s+1)}{s^3 + s^2 + 10} = 0$ .

When  $p = 21$  the dominant roots have a damping ratio of  $\zeta = 0.707$ .

**AP7.4** The characteristic equation (with  $\alpha$  as the parameter) is

$$1 + \alpha \frac{s(s+1)}{s^3 + s^2 + 1} = 0 .$$

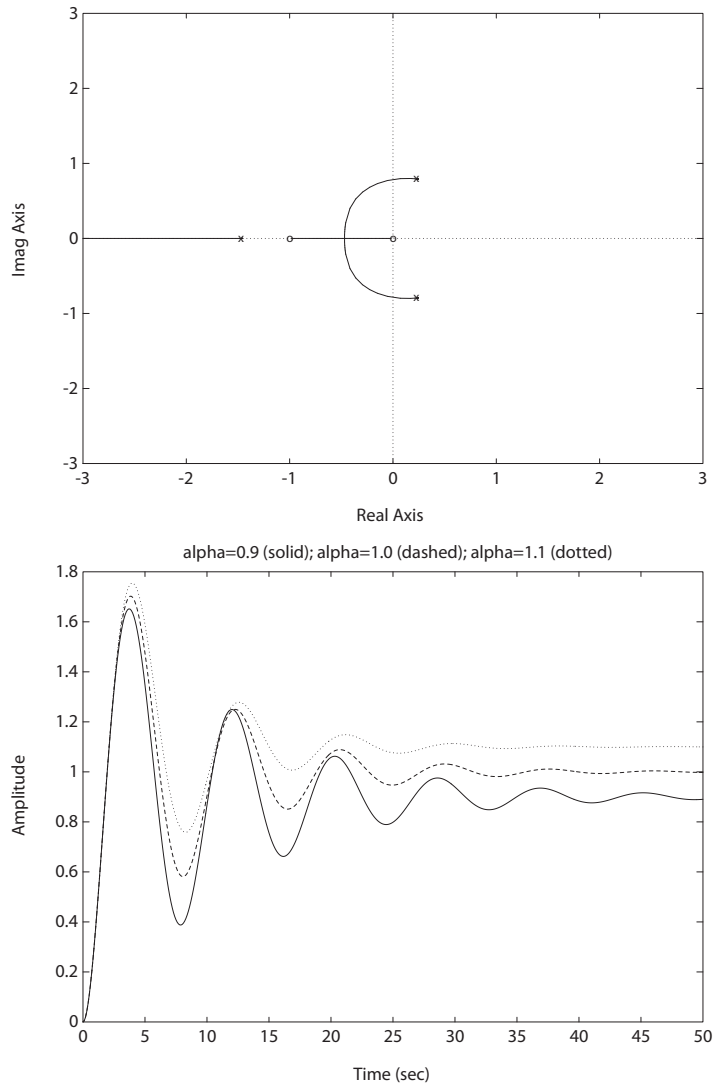
The root locus is shown in Figure AP7.4a. The steady-state error is

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{1}{1 + G(s)} = 1 - \alpha .$$

To meet the steady-state error specification, we require

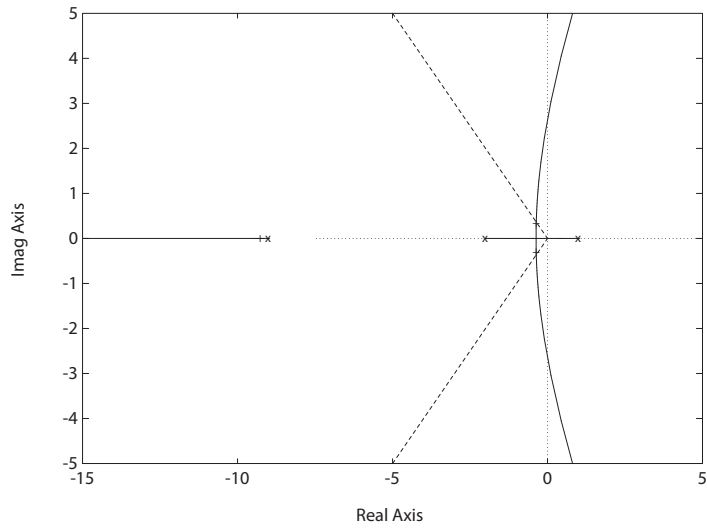
$$0.9 < \alpha < 1.1 .$$

The step responses for  $\alpha = 0.9, 1$  and  $1.1$  are shown in Figure AP7.4b.



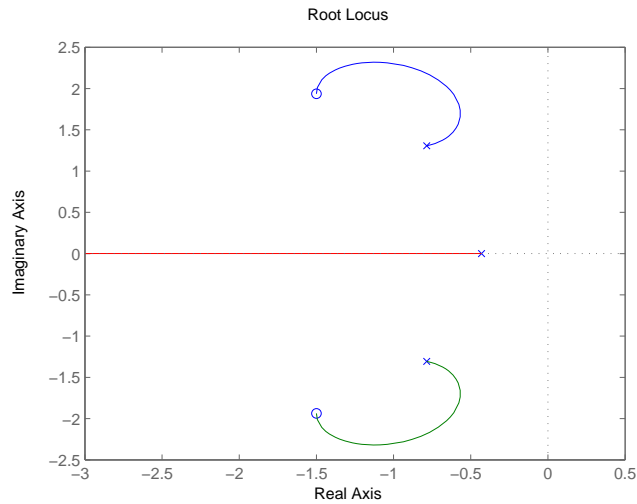
**FIGURE AP7.4** (a) Root locus for  $1 + p \frac{s(s+1)}{s^3 + s^2 + 10} = 0$ . (b) Step responses for  $\alpha = 0.9, 1$  and  $1.1$ .

**AP7.5** The root locus is shown in Figure AP7.5. When  $K = 20.45$ ,  $\zeta = 0.707$ . The root sensitivity is  $S_K^{r_1} \cong \Delta r_1 / (\Delta K / 20.45) = 3.15 \angle 87.76^\circ$ . When  $K = 88$ , the complex roots lie on the  $j\omega$ -axis—a 330% **increase** in the gain.



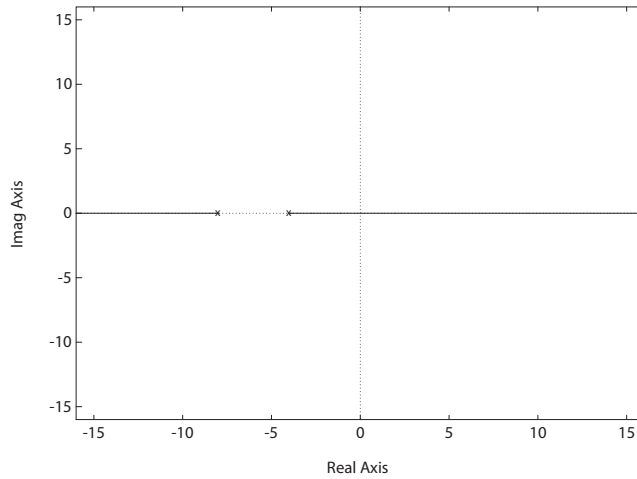
**FIGURE AP7.5**  
 Root locus for  $1 + K \frac{1}{s^3 + 10s^2 + 7s - 18} = 0$ .

**AP7.6** A gain of  $K = 13$  provides an acceptable response of  $T_s < 1$  and  $P.O. < 7.5\%$ . The root locus is shown in Figure AP7.6.



**FIGURE AP7.6**  
 Root locus for  $1 + K \frac{s^2 + 3s + 6}{s^3 + 2s^2 + 3s + 1} = 0$ .

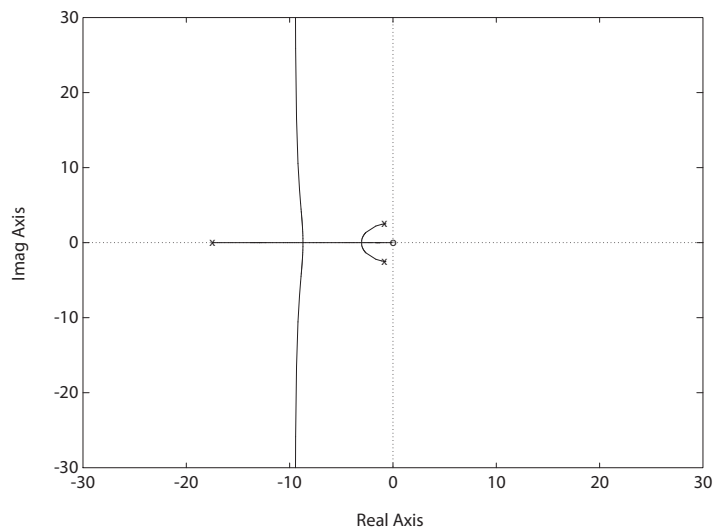
**AP7.7** The root locus for the positive feedback system is shown in Figure AP7.7.



**FIGURE AP7.7**

Root locus for  $1 + K \frac{-1}{s^2 + 12s + 32} = 0$ .

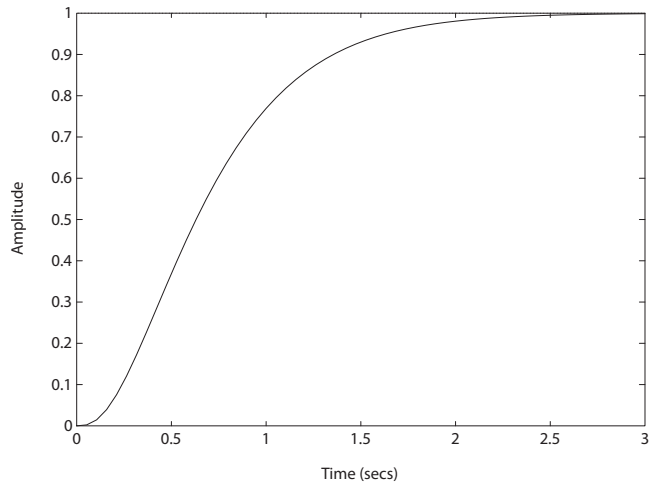
**AP7.8** The root locus is shown in Figure AP7.8a. When  $k = 0.448$ , all the roots



**FIGURE AP7.8**

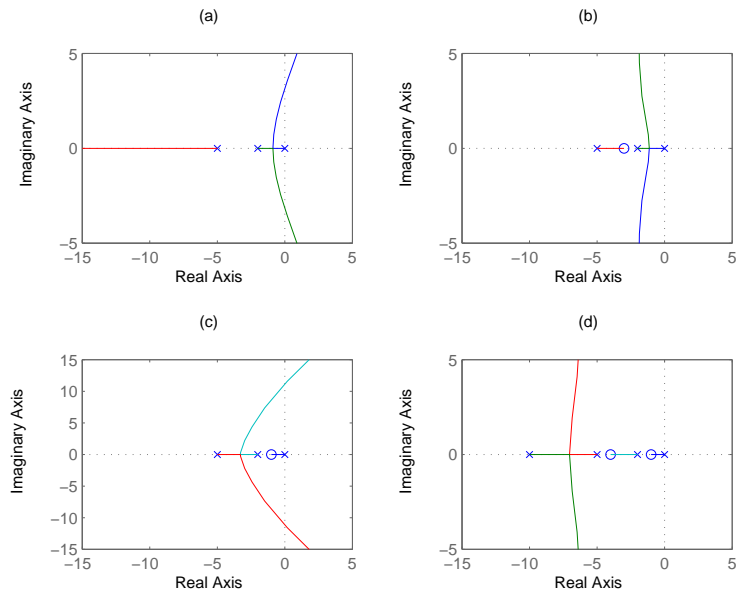
(a) Root locus for  $1 + k \frac{120s}{s^3 + 19s^2 + 34s + 120} = 0$ .

of the characteristic equation are real—the step response is shown in Figure AP7.8b.



**FIGURE AP7.8**  
CONTINUED (b) Step response with  $k = 0.448$ .

**AP7.9** The root locus for each controller is shown in Figure AP7.9.

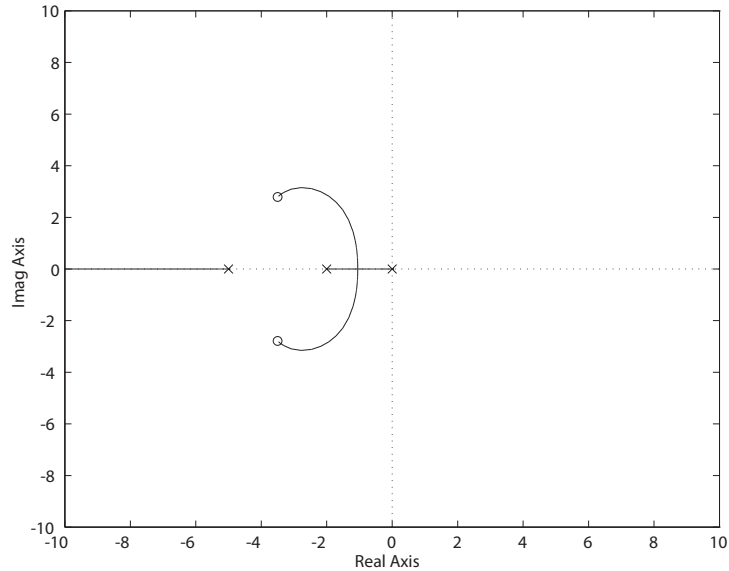


**FIGURE AP7.9**  
Root locus for the various controllers.

**AP7.10** The characteristic equation (with  $K$  as the parameter) is

$$1 + K \frac{s^2 + 7s + 20}{s(s^2 + 7s + 10)} = 0 .$$

The root locus is shown in Figure AP7.10. The steady-state value of the



**FIGURE AP7.10**

Root locus for  $1 + K \frac{s^2 + 7s + 20}{s(s^2 + 7s + 10)} = 0$ .

step response for any  $K$  is 0.5. With  $K = 15$  the closed-loop transfer function is

$$T(s) = \frac{10s + 150}{s^3 + 22s^2 + 115s + 300} .$$

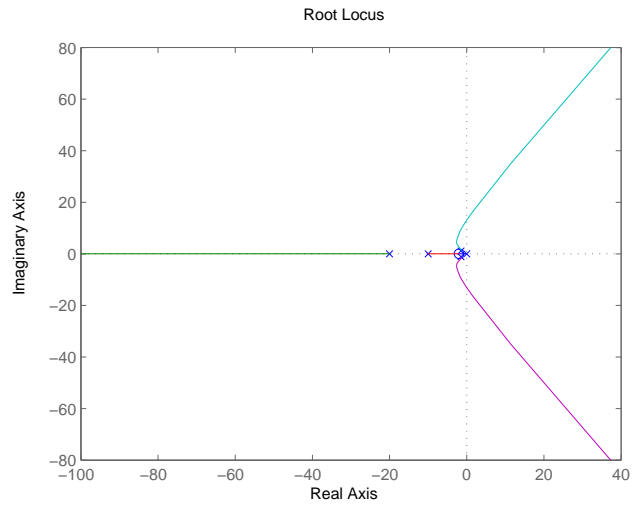
The step response has the following characteristics:

$$P.O. = 4.8\% \quad \text{and} \quad T_s = 2 \text{ seconds} .$$

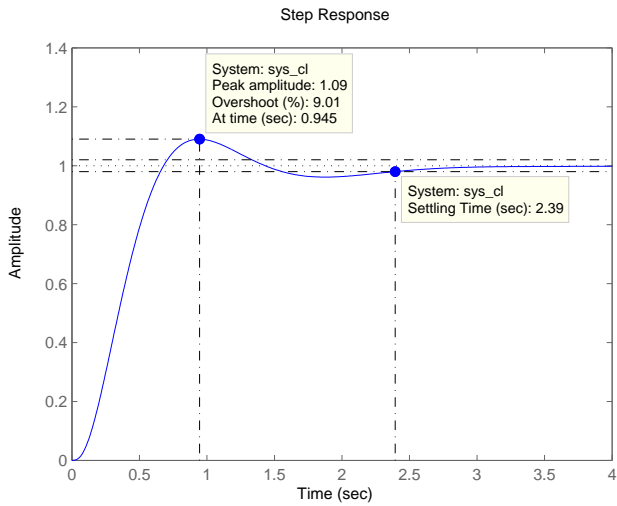
**AP7.11** The root locus is shown in Figure AP7.11a. A suitable gain is

$$K = 500 .$$

The step response is shown in Figure AP7.11b.



**FIGURE AP7.11**  
 (a) Root locus for  $1 + K \frac{(s+2)^2}{s(s+10)(s+20)(s^2+3s+3.5)} = 0$ .

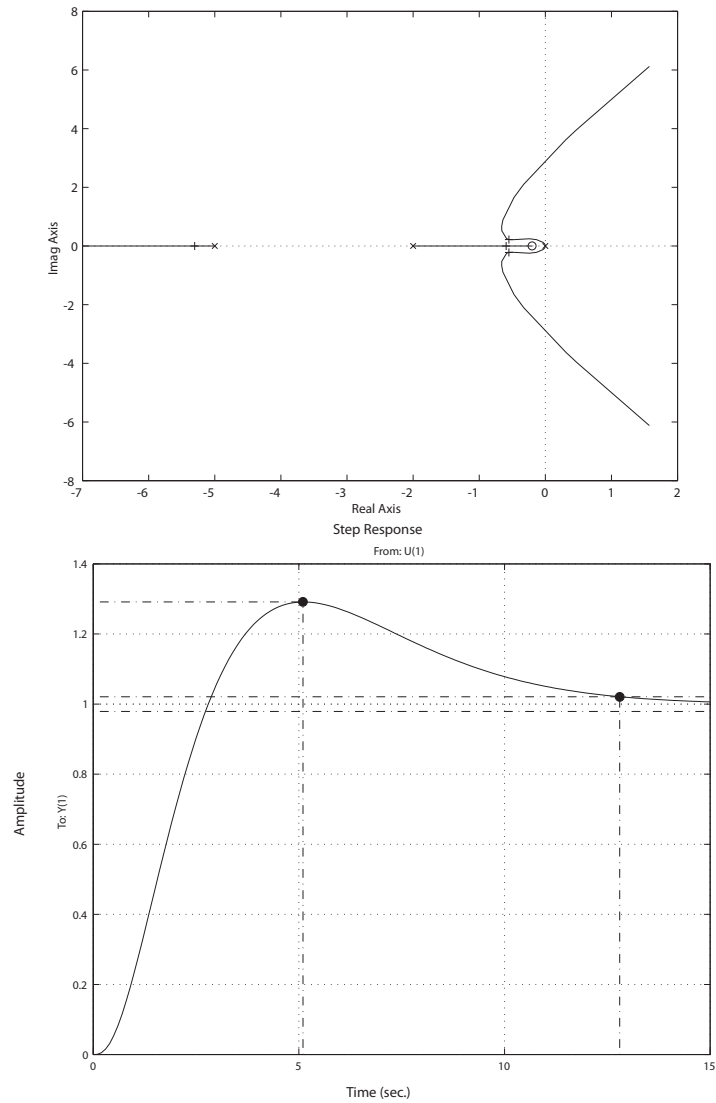


**FIGURE AP7.11**  
 CONTINUED: (b) Step response with  $K = 500$ .

**AP7.12** The root locus is shown in Figure AP7.12a. The PI controller can be written as

$$G_c(s) = \frac{K_p s + K_I}{s}$$





**FIGURE AP7.12**  
 (a) Root locus for  $1 + K_p \frac{(s+0.2)}{s(s^2+7s+10)} = 0$ . (b) Step response with  $K_p = 5.54$ .

and setting  $K_I = 0.2K_p$ , the characteristic equation can be written as

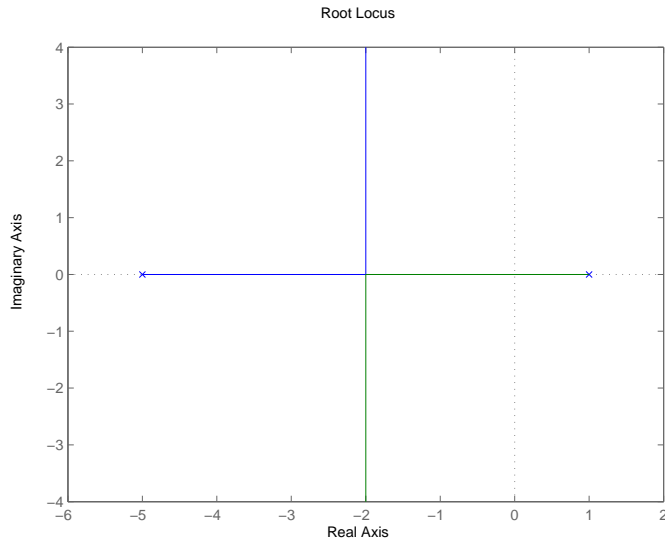
$$1 + K_p \frac{(s+0.2)}{s(s^2+7s+10)} = 0$$

A suitable gain is  $K_p = 5.55$ . The step response is shown in Figure AP7.12b.

**AP7.13** The characteristic equation is

$$1 + K_1 K_2 \frac{1}{(s+5)(s-1)} = 0.$$

The root locus is shown in Figure AP7.12a. The fastest expected settling

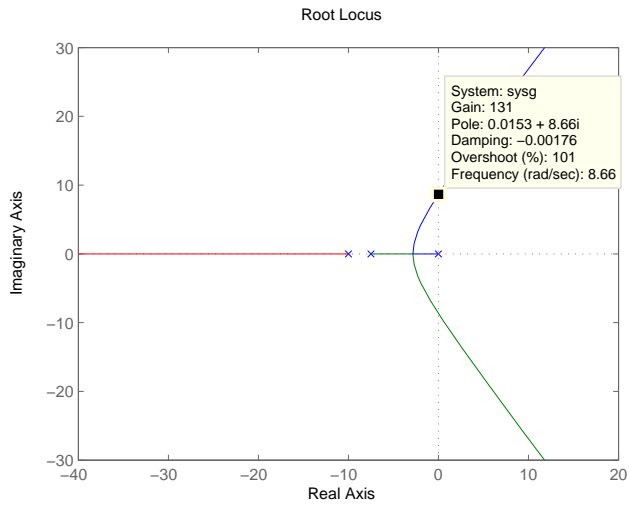


**FIGURE AP7.13**

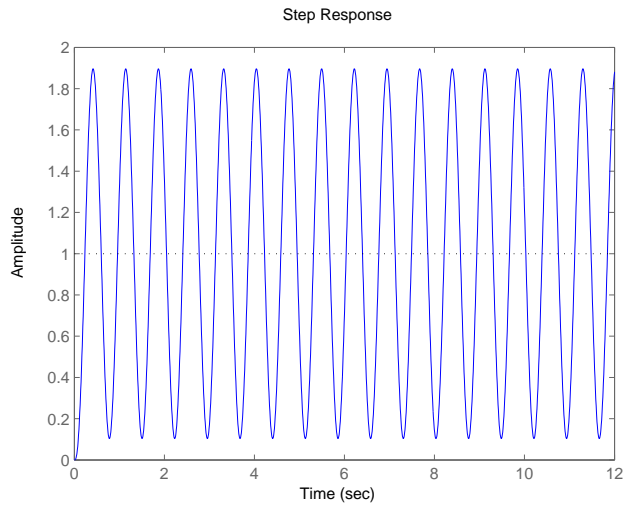
Root locus for  $1 + K_1 K_2 \frac{1}{(s+5)(s-1)} = 0$ .

time is  $T_s = 4/\omega_n \zeta = 2$  seconds since maximum  $|\omega_n \zeta| = 2$ .

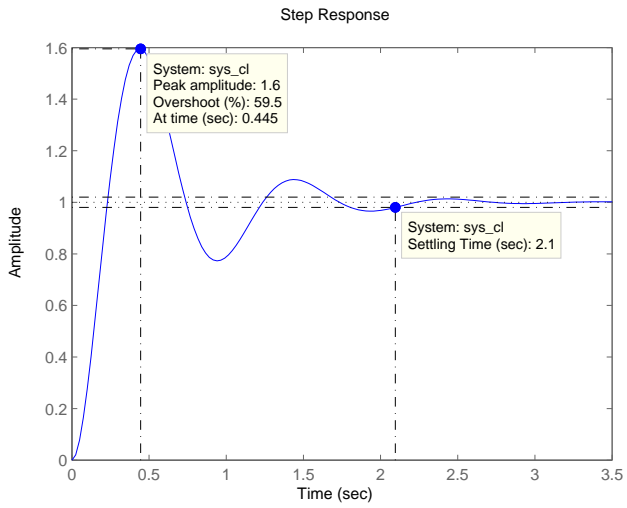
**AP7.14** The root locus of the uncompensated transfer function is shown in Figure AP7.14a. It can be seen that the system is unstable for  $K_u = 131.25$  with a period of  $T_u = 0.72$ , as illustrated in Figure AP7.14b. Using the Ziegler-Nichols design formulas yields  $K_P = 0.6K_u = 78.75$ ,  $K_I = 1.2K_u/T_u = 218.75$ , and  $K_D = 0.6K_u T_u = 7.0875$  where



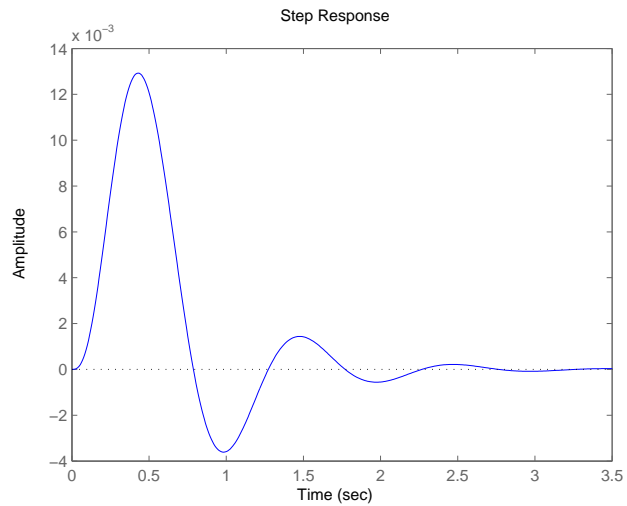
**FIGURE AP7.14**  
 (a) Root locus for  $1 + K_u \frac{10}{s(s+10)(s+7.5)} = 0$ .



**FIGURE AP7.14**  
 CONTIUED: (b) Step response at the ultimate gain  $K_u = 131$ .



**FIGURE AP7.14**  
CONTINUED: (c) Step response with the Ziegler-Nichols tuned PID controller.



**FIGURE AP7.14**  
CONTINUED: (d) Disturbance response with the Ziegler-Nichols tuned PID controller.

## Design Problems

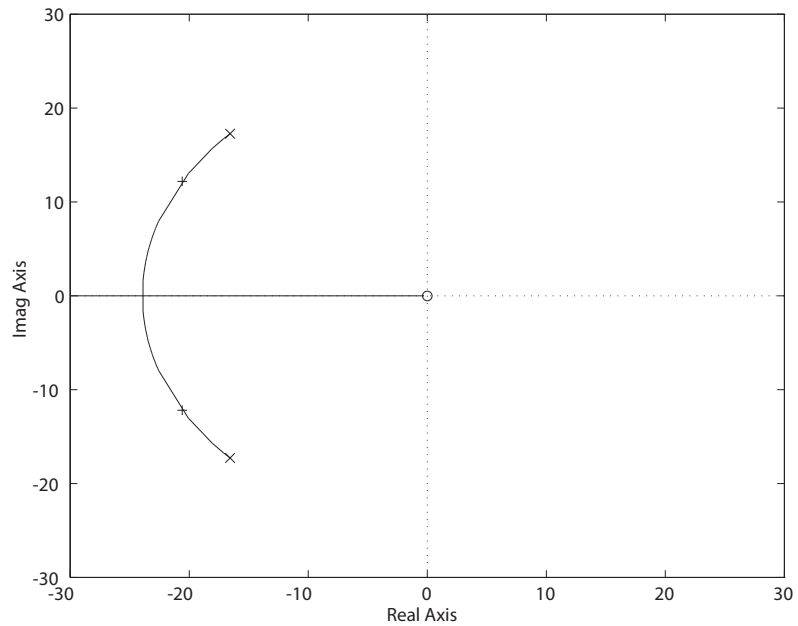
**CDP7.1** The closed-loop transfer function from the input to the output is

$$\frac{\theta(s)}{R(s)} = \frac{26.035K_a}{s^2 + (33.1415 + 26.035K_aK_1)s + 26.035K_a},$$

where we consider for the first time the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). The characteristic equation is

$$1 + K_1 \frac{26.035K_a s}{s^2 + 33.1415s + 26.035K_a} = 0.$$

The root locus is shown below. In accordance with the discussion in Chap-

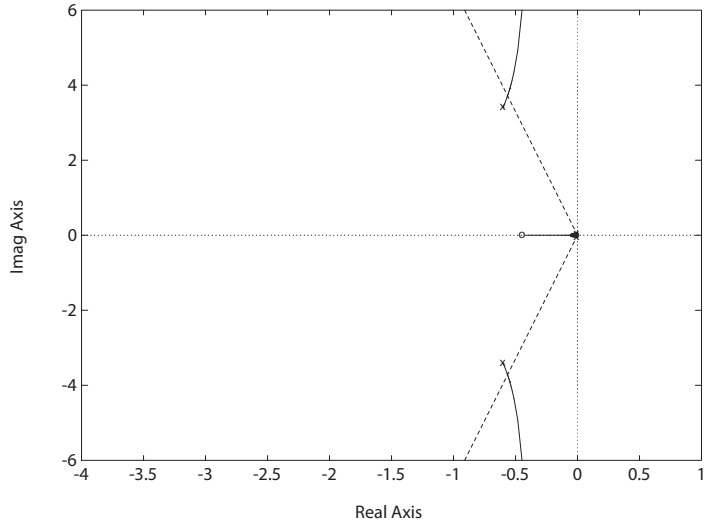


ter 5, we continue to use  $K_a = 22$ . This allows us to meet the overshoot specification ( $P.O. < 5\%$ ) without the tachometer feedback and to provide good steady-state tracking errors to a step input. To meet the design specifications of both  $P.O.$  and  $T_s$  we want the closed-loop poles to the left of  $-\zeta\omega = -4/0.3 = -13.33$  and  $\zeta > 0.69$ . A reasonable selection is  $K_1 = 0.012$ . This places the closed-loop poles at  $s = -20 \pm j13$ .

**DP7.1** (a) The characteristic equation is

$$1 + \frac{18K(s + 0.015)(s + 0.45)}{(s^2 + 1.2s + 12)(s^2 + 0.01s + 0.0025)} = 0.$$

Since we want a negative feedback system, we have  $G_c(s) = -K$ . When  $\omega_n > 2$  and  $\zeta = 0.15$ , the gain  $K = 0.12$ . The root locus is shown in Figure DP7.1a.



**FIGURE DP7.1**

(a) Root locus for  $1 + K \frac{18(s+0.015)(s+0.45)}{(s^2+1.2s+12)(s^2+0.01s+0.0025)} = 0$ .

(b) The unit step response is shown in Figure DP7.1b. The percent overshoot is

$$P.O. = 100\% .$$

(c) The characteristic equation with the anticipatory controller is

$$1 + \frac{18K(s+2)(s+0.015)(s+0.45)}{(s^2+1.2s+12)(s^2+0.01s+0.002s)} = 0 .$$

The root locus is shown in Figure DP7.1c. If we select

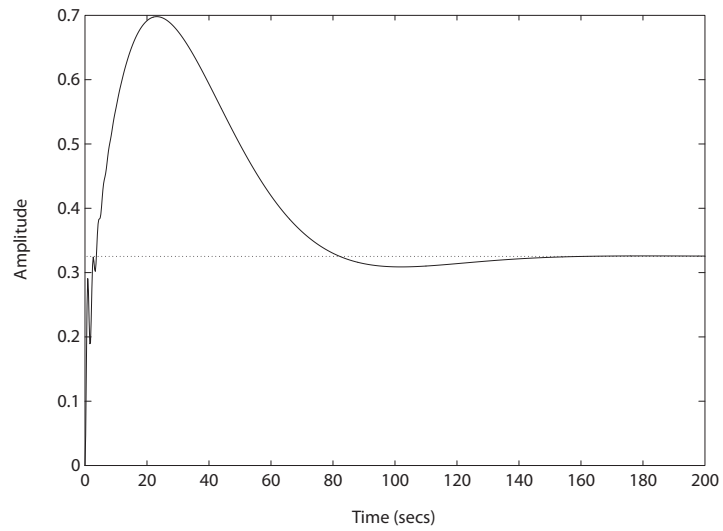
$$K = 9.2/18 ,$$

then the complex roots have a damping  $\zeta = 0.90$ . The roots are at

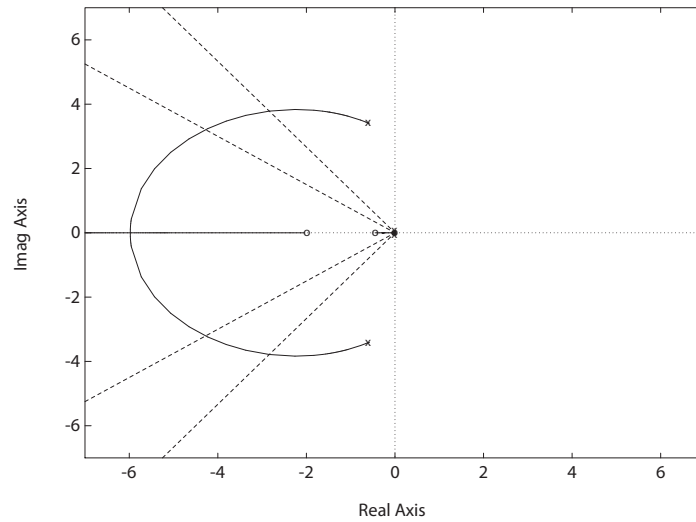
$$s_1 = -0.253$$

$$s_2 = -0.019$$

$$s_{3,4} = -5.07 \pm j2.50 .$$

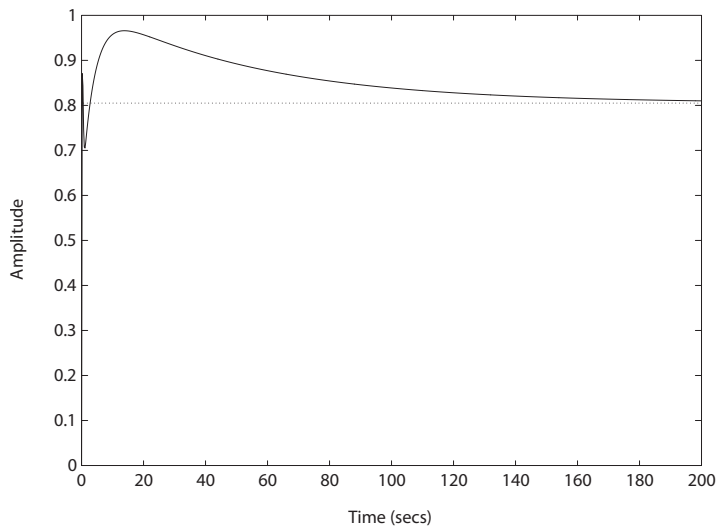


**FIGURE DP7.1**  
CONTINUED: (b) Unit step response for gain controller.



**FIGURE DP7.1**  
CONTINUED: (c) Root locus for  $1 + K \frac{18(s+2)(s+0.015)(s+0.45)}{(s^2+1.2s+12)(s^2+0.01s+0.0025)} = 0$ .

(d) The unit step response for the system with the anticipatory controller is shown in Figure DP7.1d.

**FIGURE DP7.1**

CONTINUED: (d) Unit step response for anticipatory controller.

**DP7.2** The characteristic equation is

$$1 + \frac{10K(s+1)}{s(s^2 + 4.5s + 9)} = 0 .$$

(a) The root locus is shown in Figure DP7.2a. When  $K = 0.435$ , we have  $\zeta = 0.6$  and the roots are

$$\begin{aligned} s_1 &= -0.368 \\ s_{2,3} &= -2.1 \pm j2.75 . \end{aligned}$$

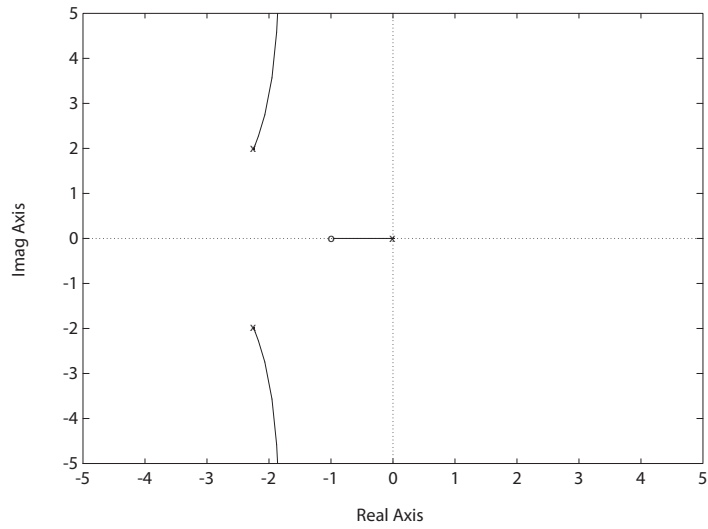
(b) The response to a step input is shown in Figure DP7.2b. The performance results are

$$\begin{aligned} P.O. &= 0\% \\ T_{ss} &= 10 \text{ sec} \\ e_{ss} &= 0 . \end{aligned}$$

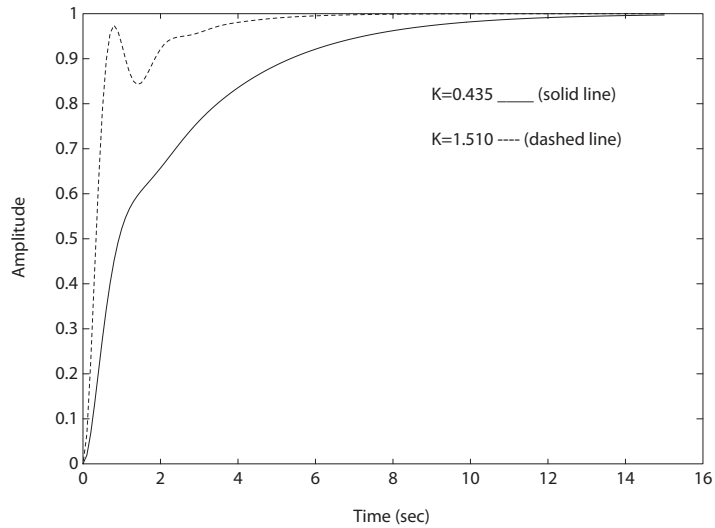
(c) We have  $\zeta = 0.41$  when  $K = 1.51$ . The step response is shown in Figure DP7.2b. The performance results to the step input are

$$\begin{aligned} P.O. &= 0\% \\ T_s &= 4 \text{ sec} \\ e_{ss} &= 0 . \end{aligned}$$





**FIGURE DP7.2**  
 (a) Root locus for  $1 + K \frac{10(s+1)}{s(s^2+4.5s+9)} = 0$ .

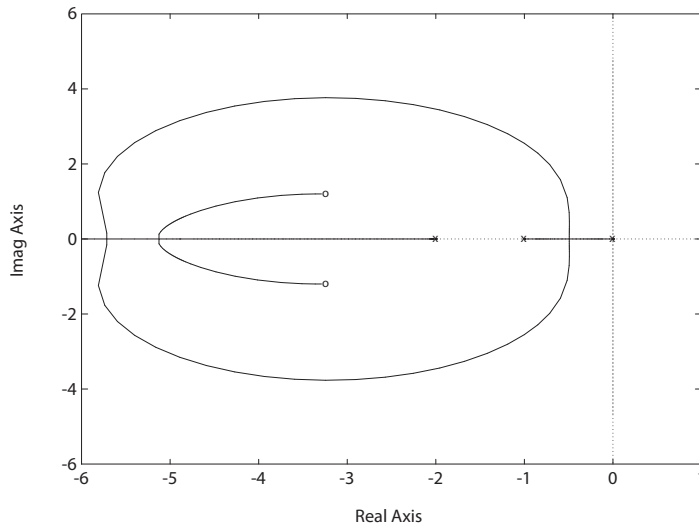


**FIGURE DP7.2**  
 CONTINUED: (b) Unit step responses for  $K = 0.425, 1.51$ .

**DP7.3** The characteristic equation is

$$1 + \frac{K(s^2 + 6.5s + 12)}{s(s + 1)(s + 2)} = 0 .$$

(a) The root locus is shown in Figure DP7.3.



**FIGURE DP7.3**  
 Root locus for  $1 + K \frac{s^2 + 6.5s + 12}{s(s+1)(s+2)} = 0$ .

When  $K = 41$ , the roots are  $s_1 = -37.12$  and  $s_{2,3} = -3.44 \pm j1.19$ .

- (b) The percent overshoot is  $P.O. \approx 1\%$  when  $\zeta = 0.82$  at  $K = 0.062$ .
- (c) Select  $K > 300$ .

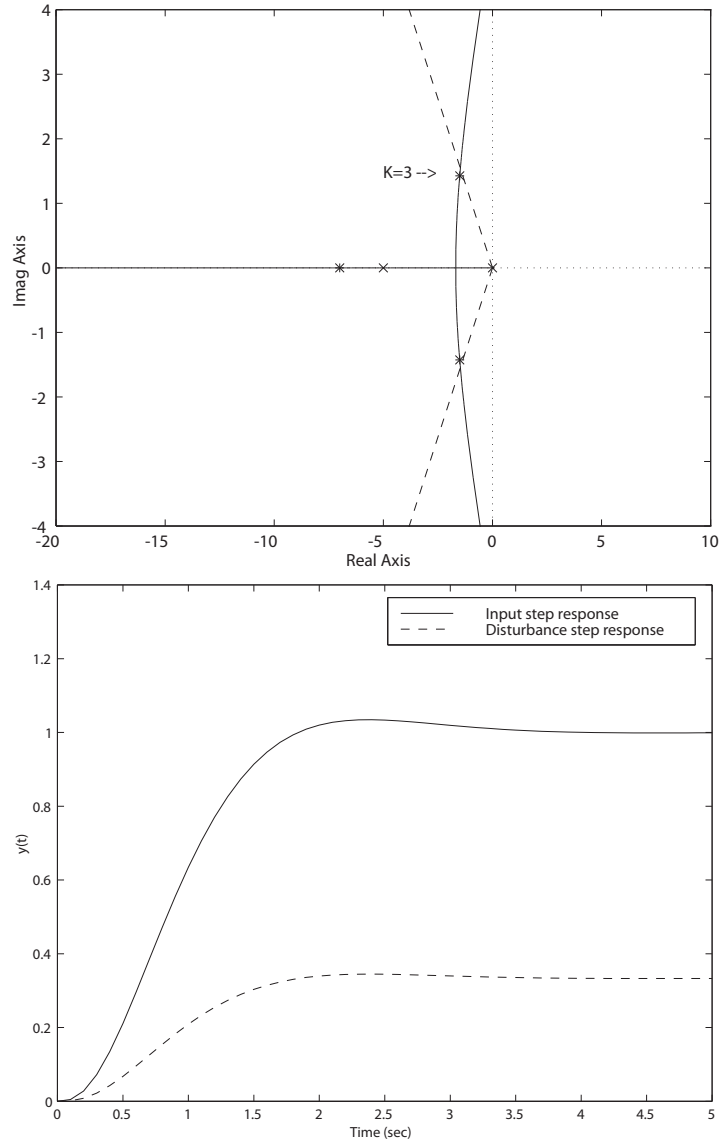
**DP7.4** The characteristic equation is

$$1 + K \frac{10(0.01s + 1)}{s(s^2 + 10s + 10K_1)} = 0 .$$

If we choose  $K_1 = 2.5$ , then the root locus will start at  $s = 0, -5$  and  $-5$ . This is shown in Figure DP7.4. The root locus then has a nice shape so that we can select  $K$  to place the complex poles where desired and the one real root will be farther in the left half-plane; thus the notion of dominant poles will be valid. So, if we desire a  $P.O. < 5\%$ , we want  $\zeta > 0.69$ . This occurs when  $K \approx 3$ . Thus, our design is

$$K_1 = 2.5 \quad \text{and} \quad K = 3 .$$

The unit step response is shown in Figure DP7.4. The settling time is less than 3.5 sec and the  $PO < 4\%$ . The response to a unit step disturbance is also shown in Figure DP7.4. The steady-state error magnitude to the disturbance is 0.33.



**FIGURE DP7.4** (a) Root locus for  $1 + K \frac{10(0.01s+1)}{s(s^2+10s+25)} = 0$ . (b) System response to step input and disturbance.

**DP7.5** The characteristic equation is

$$1 + K \frac{s + 1}{s(s - 0.1)(s^2 + 10s + 41)} = 0 .$$

The root locus is shown in Figure DP7.5a.

The system is stable for  $5 < K < 300$ . The step response with  $K =$

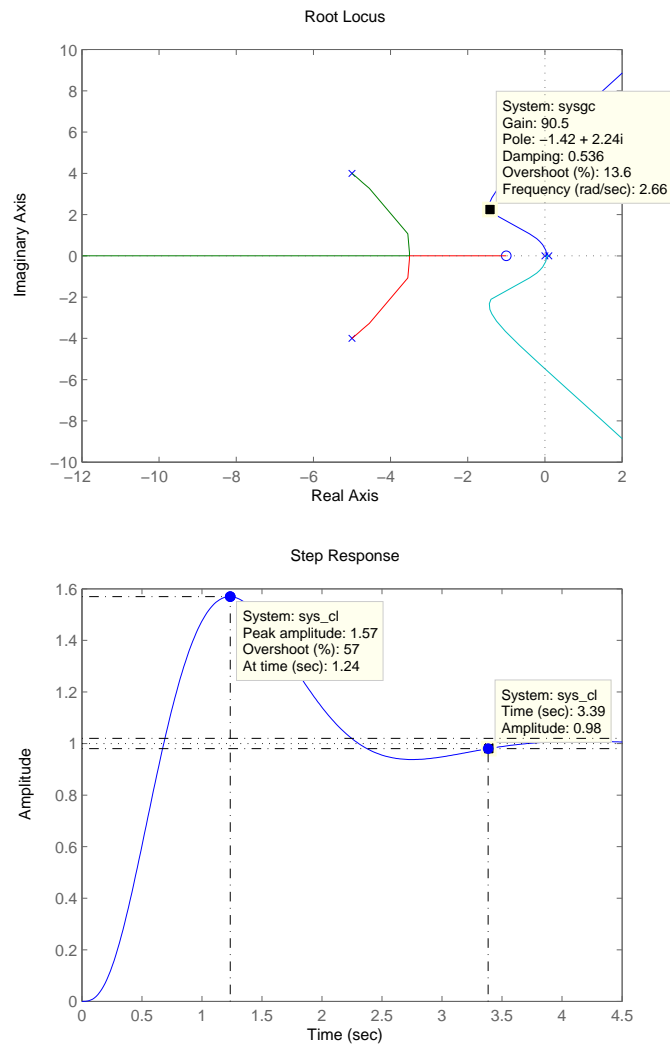


FIGURE DP7.5

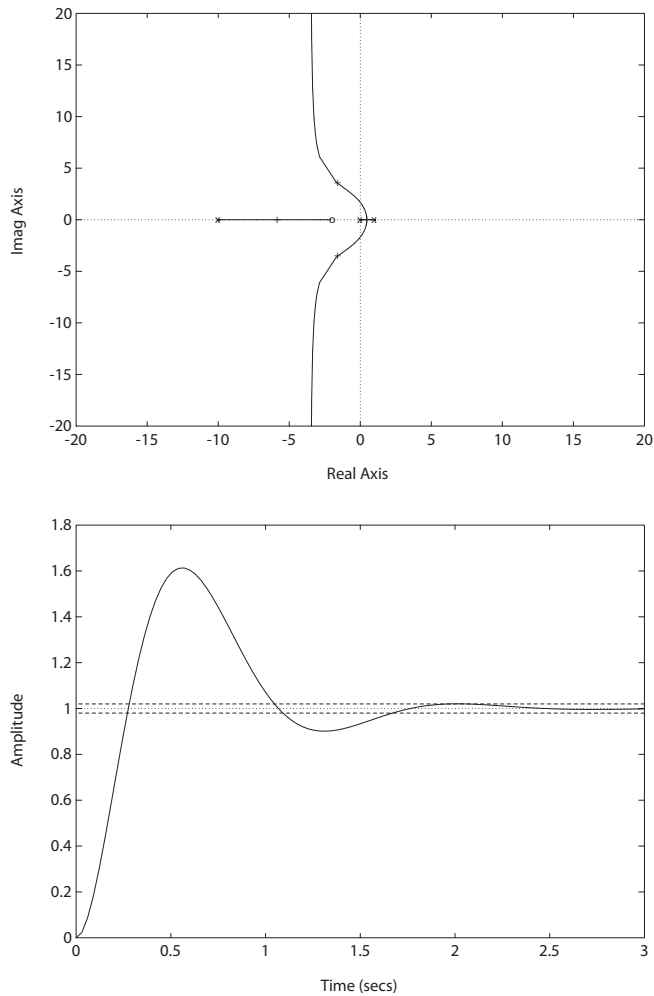
(a) Root locus for  $1 + K \frac{s+1}{s(s-0.1)(s^2+10s+41)} = 0$ . (b) Step response with  $K = 875$ .

90.5 is shown in Figure DP7.5b. We choose  $K = 90.5$  to minimize the settling time. The damping of the dominant poles is  $\zeta = 0.54$ , so that the estimated percent overshoot is  $P.O. = 13\%$ . The actual percent overshoot and settling time are  $P.O. = 57\%$  and  $T_s = 3.4$  seconds. The match between the actual and predicted percent overshoot can be improved by selecting a much higher gain  $K$ , but then the step response becomes overly oscillatory and the settling time increases too much for a typical high-performance aircraft.

**DP7.6** The characteristic equation is

$$1 + K \frac{s + 2}{s(s + 10)(s - 1)} = 0 .$$

The maximum damping is  $\zeta = 0.46$  at  $K = 55$ . The root locus is shown in Figure DP7.6a; the step response is shown in Figure DP7.6b. The percent overshoot and settling time are  $P.O. = 61.3\%$  and  $T_s = 2$  seconds.



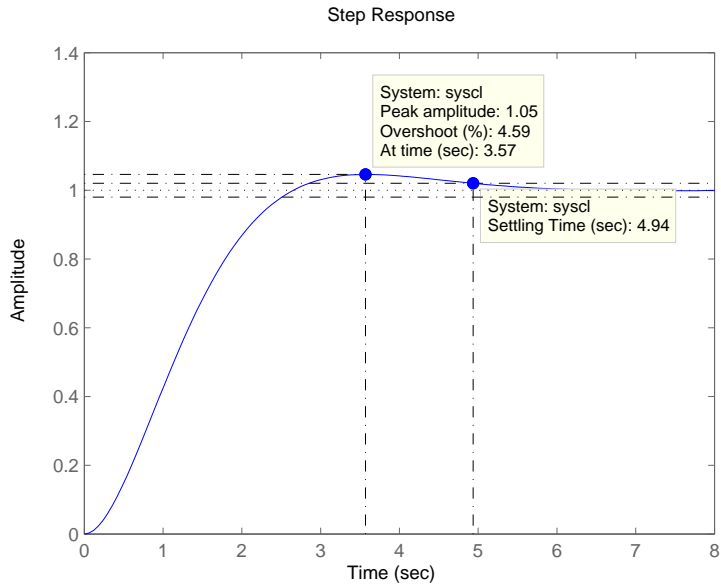
**FIGURE DP7.6**

(a) Root locus for  $1 + K \frac{s+2}{s(s+10)(s-1)} = 0$ . (b) Step response with  $K = 55$ .

**DP7.7** The loop transfer function is

$$G_c(s)G(s) = \frac{K_P s + K_I}{s(s+1)(0.5s+1)}.$$

One possible set of PI controller gains are  $K_P = 0.82$  and  $K_I = 0.9$ . The step response is shown in Figure DP7.7.



**FIGURE DP7.7**

Step response for with PI controller  $G_c(s) = (0.82s + 0.9)/s$ .

**DP7.8** The closed-loop transfer function is

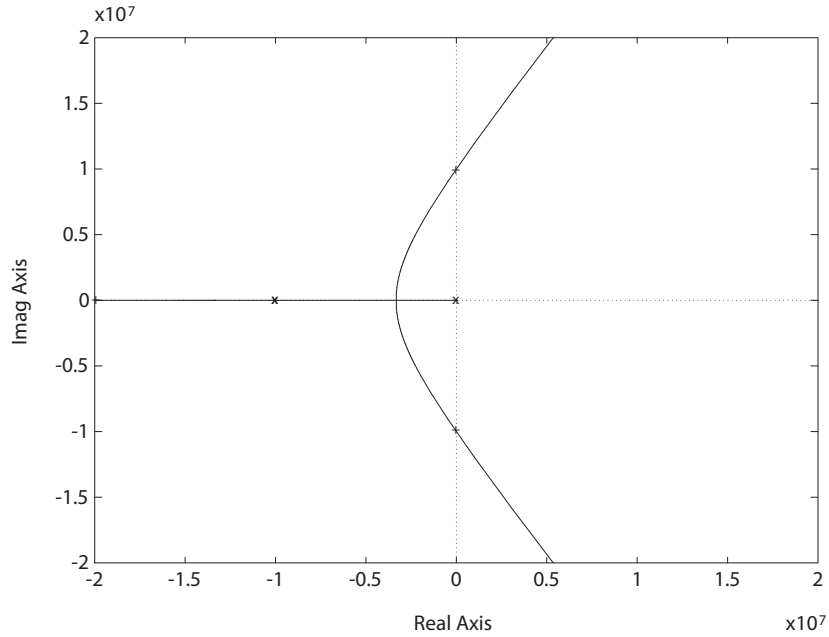
$$T(s) = \frac{V_o(s)}{V(s)} = \frac{G(s)}{1 + KG(s)}.$$

The dc gain is

$$T(0) = \frac{G(0)}{1 + KG(0)} \approx \frac{1}{K}.$$

The root locus is shown in Figure DP7.8. The maximum value of  $K$  for stability is

$$K = 0.062.$$



**FIGURE DP7.8**  
 Root locus for  $1 + K \frac{3.142K_1 \times 10^{17}}{(s+3142)(s+10^7)^2} = 0$ .

Therefore, the minimum dc gain is about  $1/0.062=16$ . Selecting

$$K = 0.05 \quad \text{and} \quad R_1 = 10 K$$

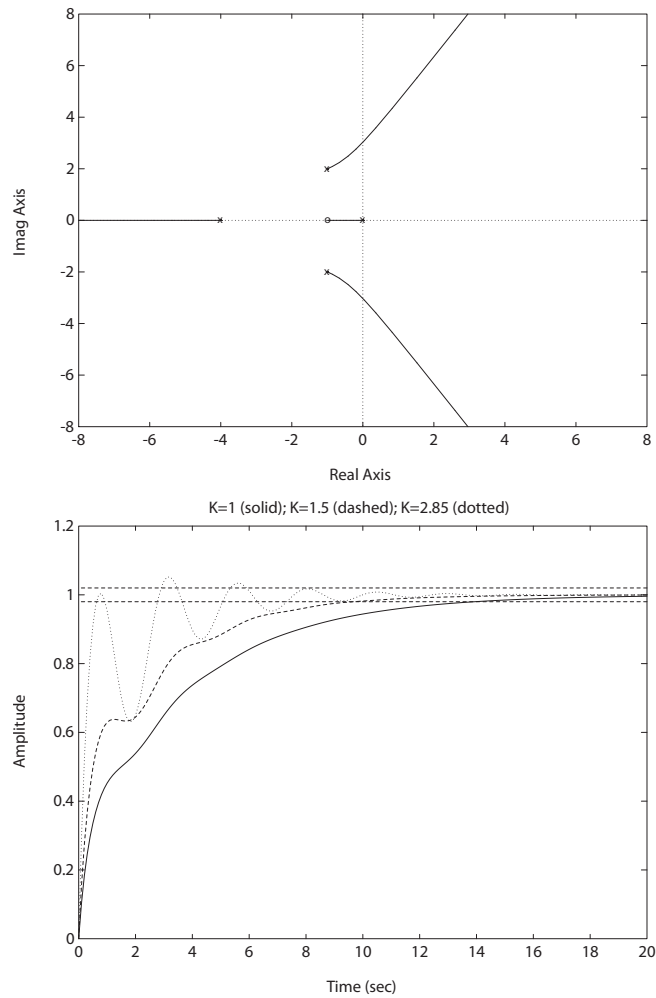
yields

$$R_2 = 19R_1 = 190 K .$$

**DP7.9** The closed-loop transfer function (with  $G_p(s) = 1$  and  $K = 1$ ) is

$$T(s) = \frac{2s^3 + 6s^2 + 14s + 10}{s^4 + 6s^3 + 13s^2 + 26s + 6} .$$

So, if we select  $G_p(s) = 1/T(0) = 0.6$ , the step response (with  $K = 1$ ) will have a zero steady-state tracking error. The root locus is shown in Figure DP7.9a. The step responses for  $K = 1, 1.5$  and  $2.85$  are shown in Figure DP7.9b. For  $K = 1$ , we have  $P.O. = 0\%$ ,  $T_r = 7.8$  and  $T_s = 13.9$ ; for  $K = 1.5$ , we have  $P.O. = 0\%$ ,  $T_r = 5.4$  and  $T_s = 9.6$ ; and for  $K = 2.85$ , we have  $P.O. = 5.2\%$ ,  $T_r = 0.5$  and  $T_s = 7.3$ . The best gain selection is  $K = 2.85$ .



**FIGURE DP7.9**  
 (a) Root locus for  $1 + K \frac{6(s+1)}{s(s+4)(s^2+2s+5)} = 0$ . (b) Step responses with  $K = 1, 1.5, 2.85$ .

**DP7.10** A suitable selection of the various parameters is

$$\zeta = 0.5 \quad \text{and} \quad q = 3/5 .$$

With  $q = 3/5$ , the open-loop zeros are real and equal. Then, it follows that

$$\lambda = \frac{2q}{1-q} = 3 .$$

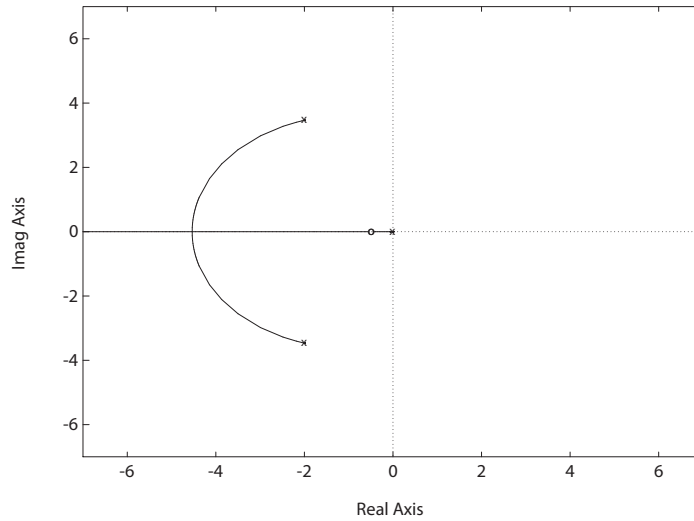


The root locus is shown in Figure DP7.10. A reasonable choice of gain is

$$K = 30 .$$

The resulting step response is extremely fast with no overshoot. The closed-loop transfer function is approximately given by

$$T(s) \approx \frac{1923}{s + 1923} .$$



**FIGURE DP7.10**

Root locus for  $1 + K \frac{4s^2 + 4s + 1}{0.0625s^3 + 0.25s^2 + s} = 0$ .

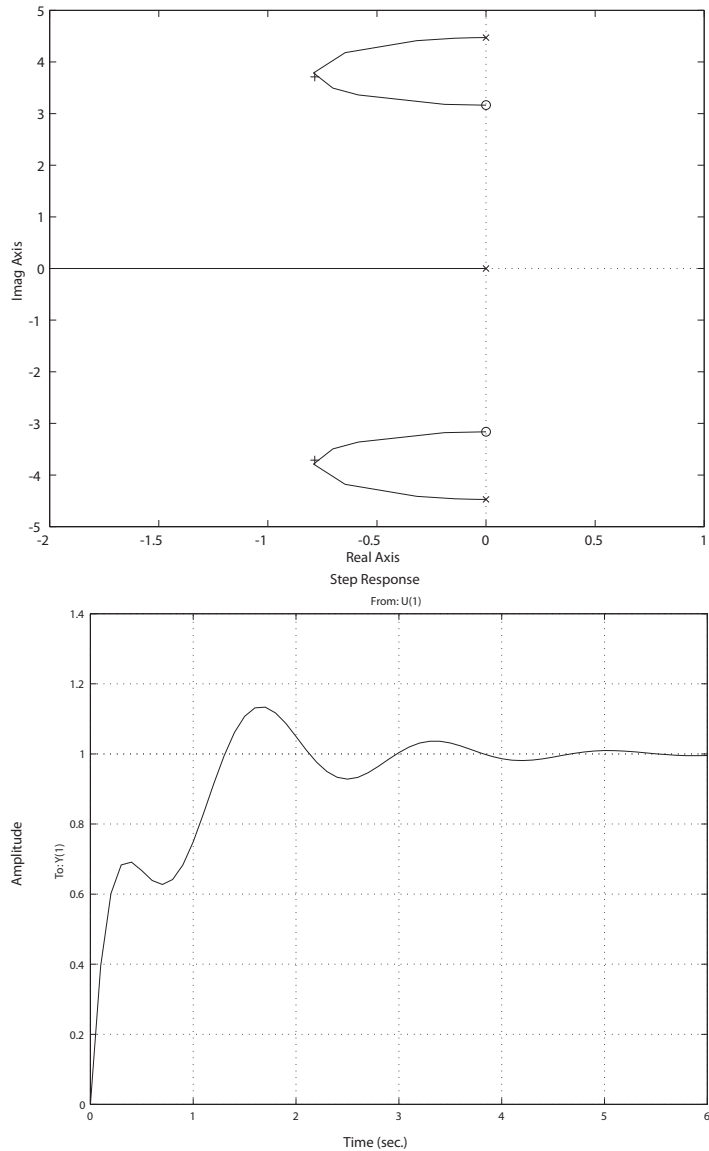
**DP7.11** The characteristic equation (with  $K$  as the parameter) is

$$1 + K \frac{10(s^2 + 10)}{s^3 + 20s} = 0 .$$

The root locus is shown in Figure DP7.11a. To maximize the closed-loop system damping we choose

$$K = 0.513.$$

The step response is shown in Figure DP7.11b.

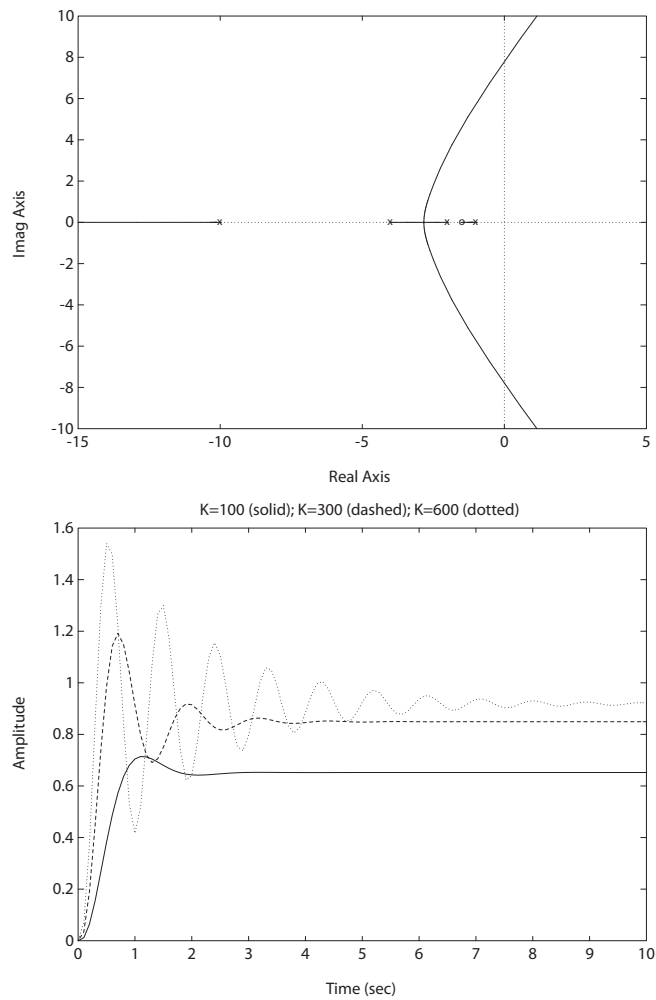


**FIGURE DP7.11**  
 (a) Root locus for  $1 + K \frac{10(s^2+10)}{s^3+20s} = 0$ . (b) Step response with  $K = 0.513$ .

**DP7.12** The characteristic equation is

$$1 + K \frac{s + 1.5}{(s + 1)(s + 2)(s + 4)(s + 10)} = 0 .$$

The root locus is shown in Figure DP7.12a.

**FIGURE DP7.12**

(a) Root locus for  $1 + K \frac{s+1.5}{(s+1)(s+2)(s+4)(s+10)} = 0$ . (b) Step response with  $K = 100, 300, 600$ .

The closed-loop system roots are:

$$\begin{aligned}
 K = 100 : \quad s_1 &= -11.38 & s_{2,3} &= -2.09 \pm 3.10j & s_4 &= -1.45 \\
 K = 300 : \quad s_1 &= -12.94 & s_{2,3} &= -1.29 \pm 5.10j & s_4 &= -1.48 \\
 K = 600 : \quad s_1 &= -14.44 & s_{2,3} &= -0.53 \pm 6.72j & s_4 &= -1.49
 \end{aligned}$$

The step responses are shown in Figure DP7.12b.

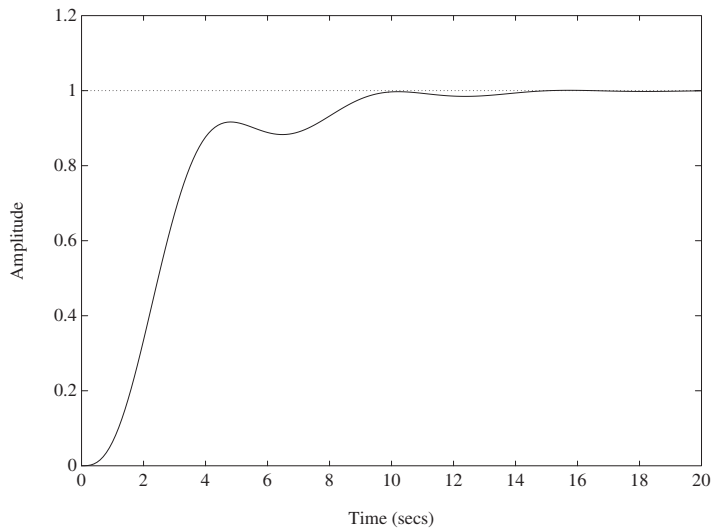
**DP7.13** The closed-loop transfer function is

$$T(s) = \frac{K_a}{s^3 + s^2 + K_a K_2 s + K_a} .$$

A suitable choice of gains is

$$K_a = 0.52 \quad \text{and} \quad K_2 = 3 .$$

The step response is shown in Figure DP7.13.



**FIGURE DP7.13**  
Step response with  $K_a = 0.52$  and  $K_2 = 3$ .

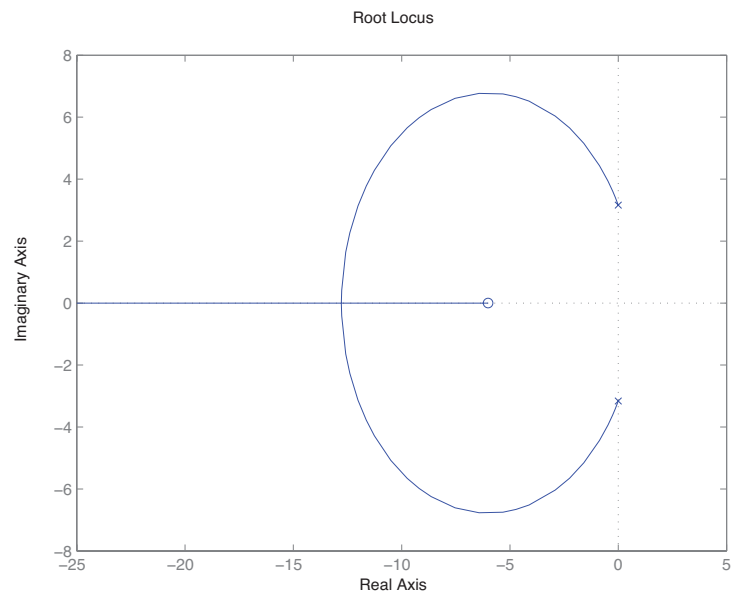
**DP7.14** The characteristic equation is

$$s^2 + 10K_D s + 10(K_P + 1) = 0 .$$

In the Evans form we have

$$1 + K_D \frac{10(s + \tau)}{s^2 + 10} = 0 .$$

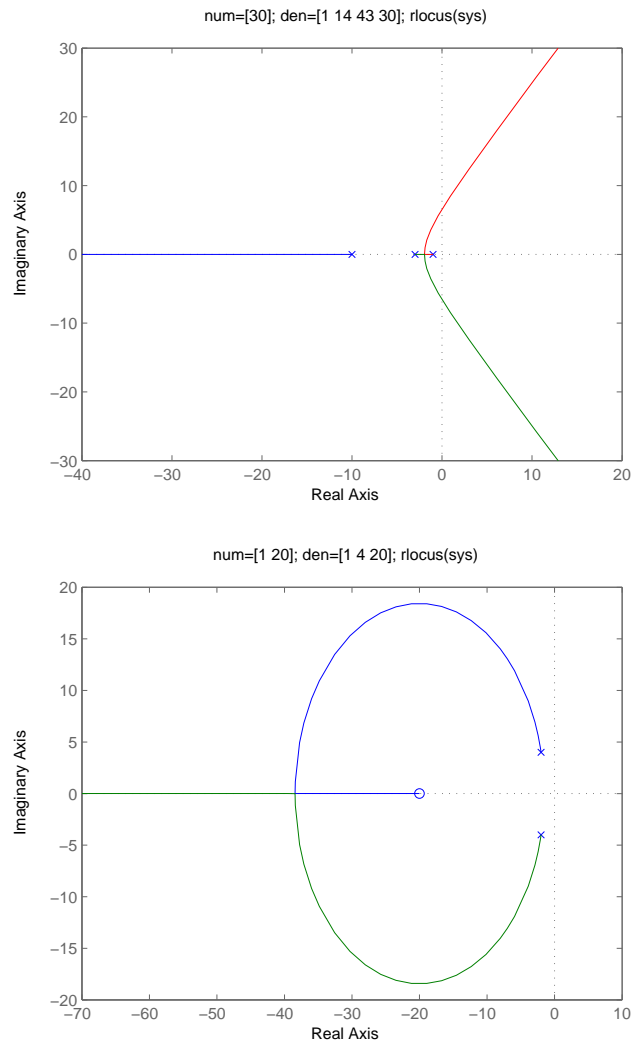
The root locus is shown in Figure DP7.14 for  $\tau = 6$ . As  $\tau \rightarrow 0$ , the dominant closed-loop pole approaches  $s = 0$  as  $K_D \rightarrow \infty$ . As  $\tau \rightarrow \sqrt{10}$ , the dominant closed-loop pole approaches  $s = -\sqrt{10}$  as  $K_D \rightarrow \infty$ . A viable controller is  $K_P = 72$  and  $K_D = 12$  when  $\tau = 6$ .



**FIGURE DP7.14**  
Root locus when  $\tau = 6$ .

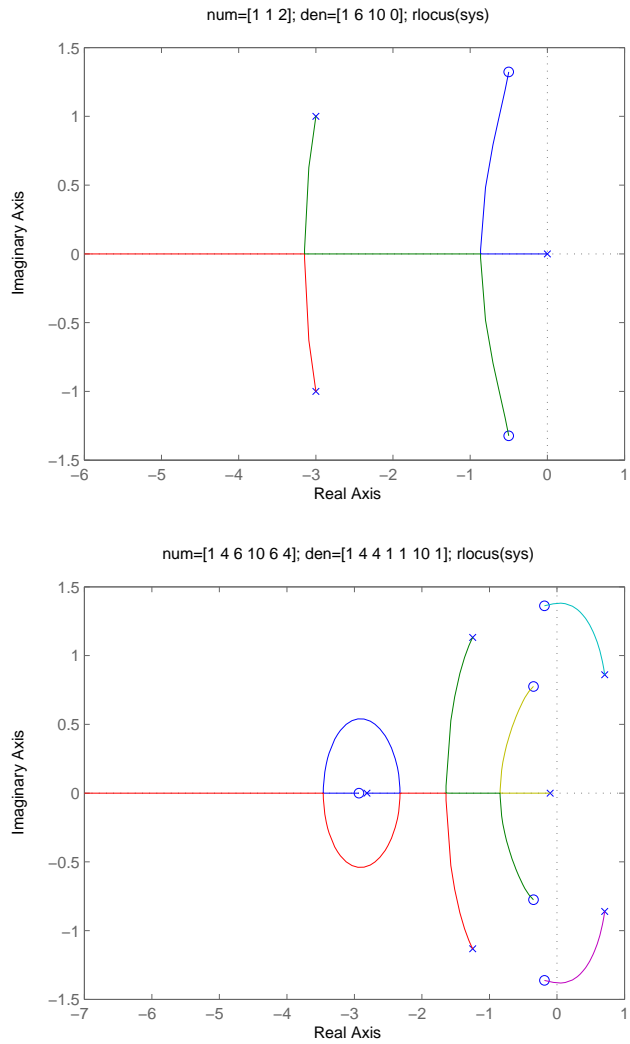
# Computer Problems

CP7.1 The root locus for parts (a)-(d) are shown in Figures CP7.1a - CP7.1d.



**FIGURE CP7.1**

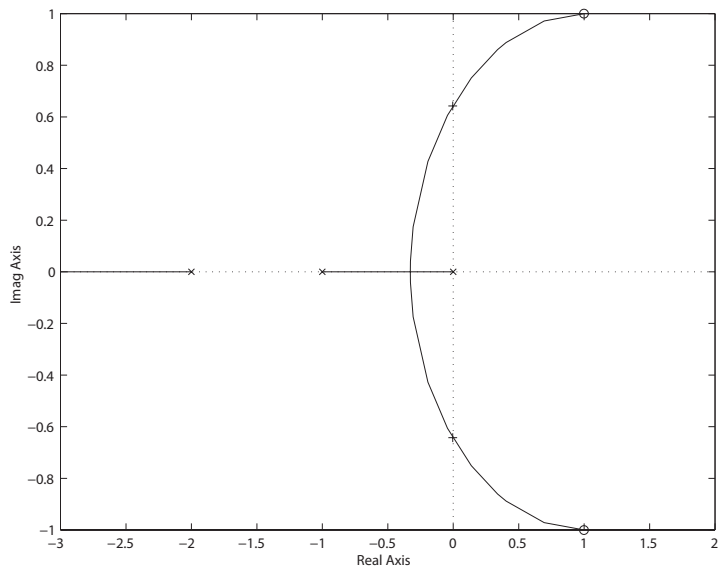
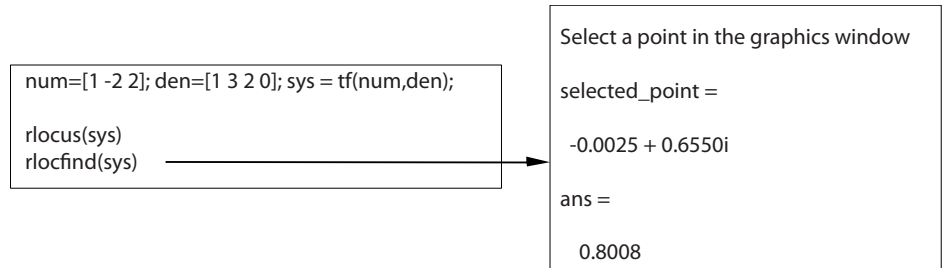
(a) Root locus for  $1 + k \frac{30}{s^3 + 14s^2 + 43s + 30} = 0$ . (b) Root locus for  $1 + k \frac{s+20}{s^2 + 4s + 20} = 0$ .



**FIGURE CP7.1**

CONTINUED: (c) Root locus for  $1 + k \frac{s^2 + s + 2}{s(s^2 + 6s + 10)} = 0$ . (d) Root locus for  $1 + k \frac{s^5 + 4s^4 + 6s^3 + 10s^2 + 6s + 4}{s^6 + 4s^5 + 4s^4 + s^3 + s^2 + 10s + 1} = 0$ .

**CP7.2** The maximum value of the gain for stability is  $k = 0.791$ . The m-file script and root locus is shown in Figure CP7.2.



**FIGURE CP7.2**  
Using the **rlocfind** function.

The value of  $k = 0.8008$  selected by the **rlocfind** function is not exact since you cannot select the  $j\omega$ -axis crossing precisely. The actual value is determined using Routh-Hurwitz analysis.

**CP7.3** The partial fraction expansion of  $Y(s)$  is

$$Y(s) = \frac{s + 6}{s(s^2 + 5s + 4)} = \frac{0.1667}{s + 4} - \frac{1.6667}{s + 1} + \frac{1.5}{s}.$$

The m-file script and output is shown in Figure CP7.3.



```

r =
    0.1667
   -1.6667
    1.5000

num=[1 6]; den=[1 5 4 0];
[r,p,k]=residue(num,den) → p =
   -4
   -1
    0

k =
    []

```

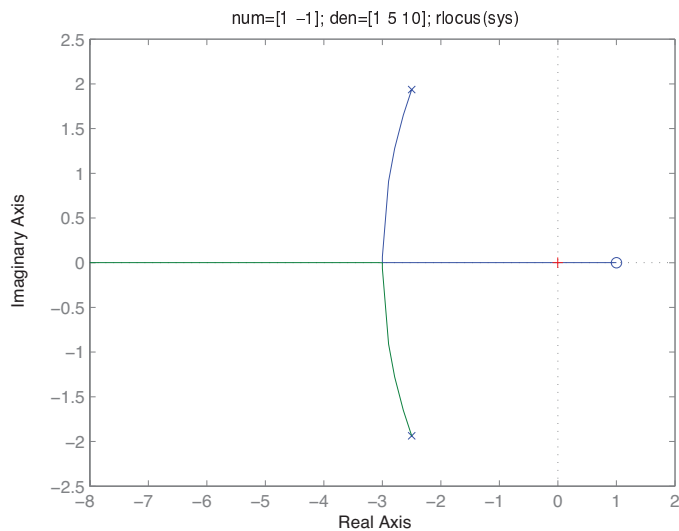
**FIGURE CP7.3**  
Using the **residue** function.

**CP7.4** The characteristic equation is

$$1 + p \frac{s-1}{s^2 + 5s + 10} = 0 .$$

The root locus is shown in Figure CP7.4. The closed-loop system is stable for

$$0 < p < 10 .$$

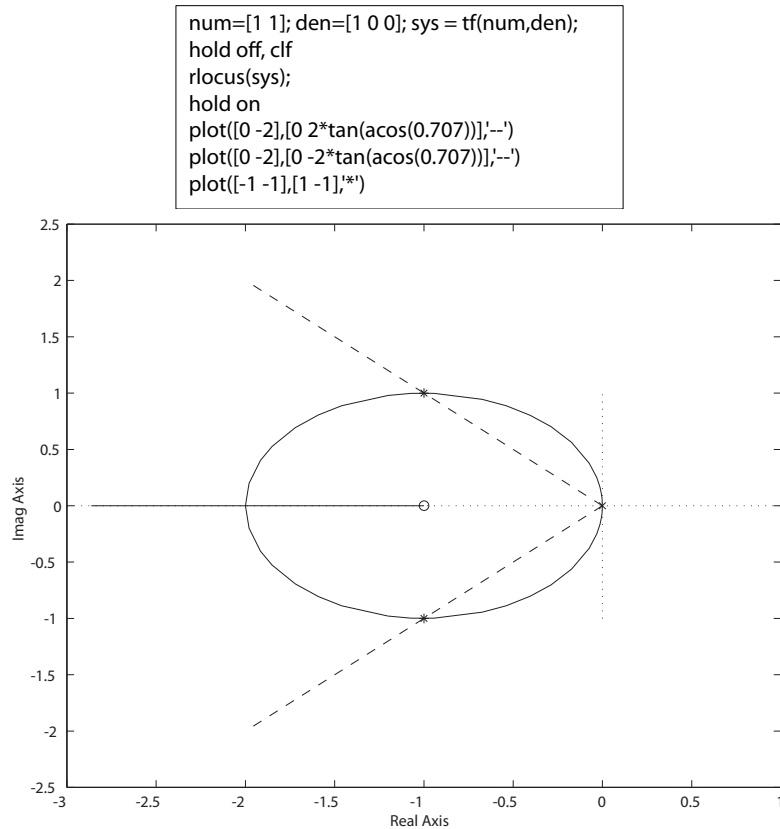


**FIGURE CP7.4**  
Root locus for  $1 + p \frac{s-1}{s^2+5s+10} = 0$ .

**CP7.5** The characteristic equation is

$$1 + k \frac{s+1}{s^2} = 0 .$$

The root locus is shown in Figure CP7.5. For  $k = 2$  we obtain  $s_{1,2} = -1 \pm j$ , that is, we have  $\zeta = 0.707$ .



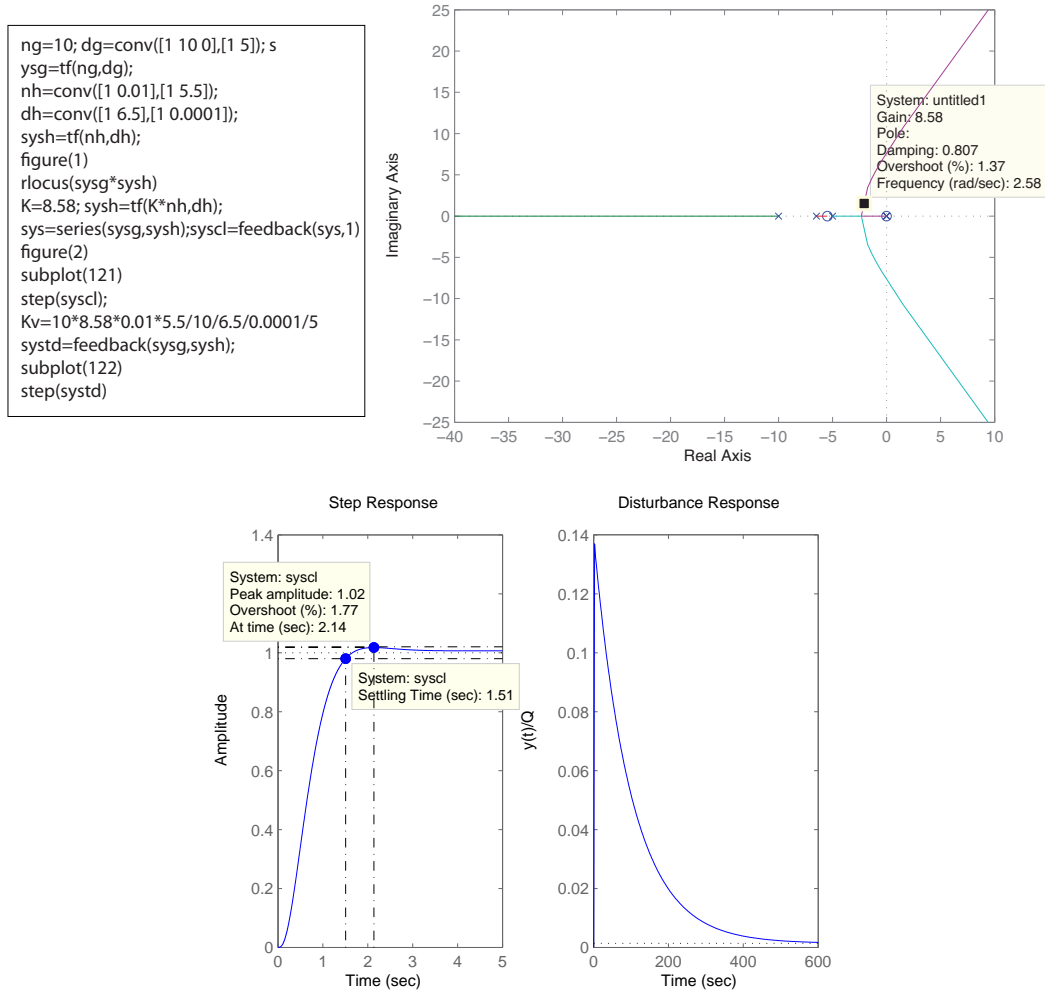
**FIGURE CP7.5**  
Root locus for  $1 + k \frac{s+1}{s^2} = 0$ .

**CP7.6** We choose a controller with two real poles and two real zeros selected to meet the steady-state specification. The characteristic equation is

$$1 + K \frac{(s+5.5)(s+0.01)}{(s+6.5)(s+0.0001)} \frac{10}{s^3 + 15s^2 + 50s} = 0 .$$

The m-file and root locus is shown in Figure CP7.4a. From the root locus we can select the value of the gain  $K$  that results in an estimated  $P.O. \leq 5\%$  and a  $\zeta\omega_n \leq -2$  to meet the settling time specification. We

select  $K = 8.58$ . The step response is shown in Figure CP7.6b showing the percent overshoot and settling time specifications are satisfied. The velocity constant is  $k_v = 145.2$  which implies a steady-state error to a ramp input of  $e_{ss} = 1/k_v = 0.0069$ .



**FIGURE CP7.6**

(a) Root locus. (b) Step response and disturbance response.

**CP7.7** The m-file script to generate the root locus for each controller in parts (a)-(c) is shown in Figure CP7.7. The performance region is indicated on each root locus in Figures CP7.7b - CP7.7d. For part (a), the controller gain is found to be  $G_c(s) = 11.3920$ . The integral controller in part (b)

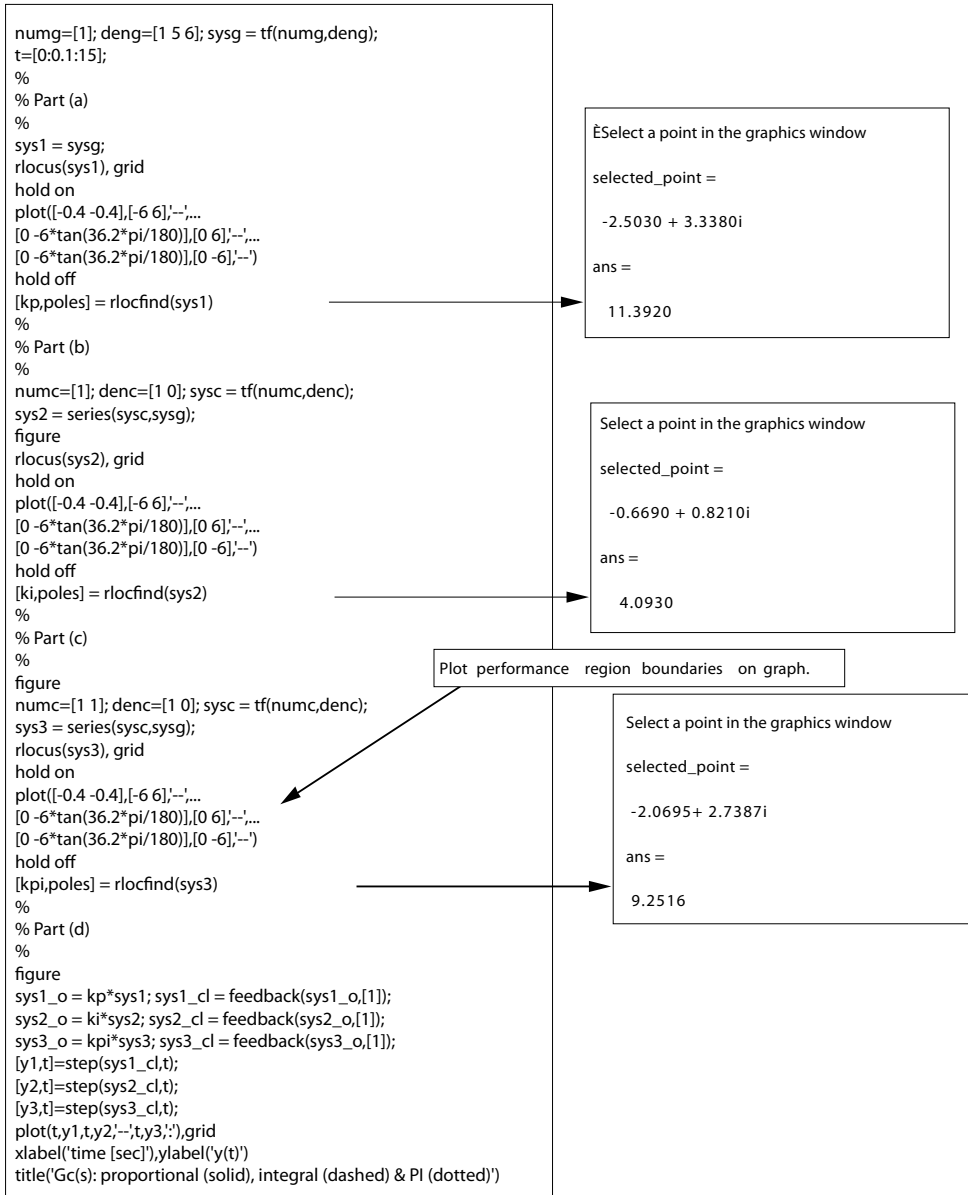
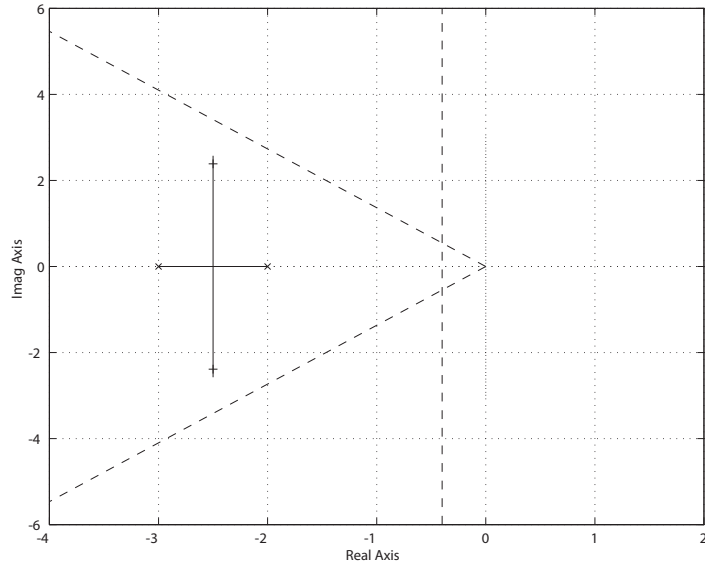


FIGURE CP7.7

(a) Script to generate the root locus for each controller.

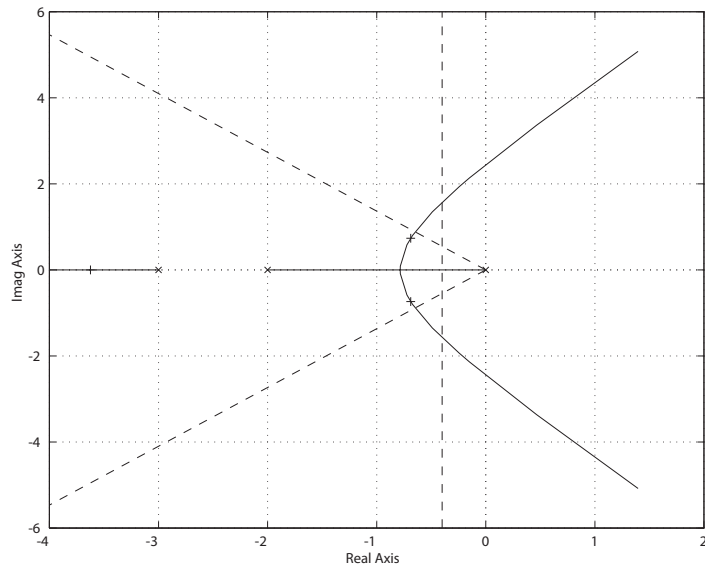
is determined to be

$$G_c(s) = \frac{4.093}{s} .$$

**FIGURE CP7.7**

CONTINUED: (b) Root locus for proportional controller with selected  $K = 11.3920$ .

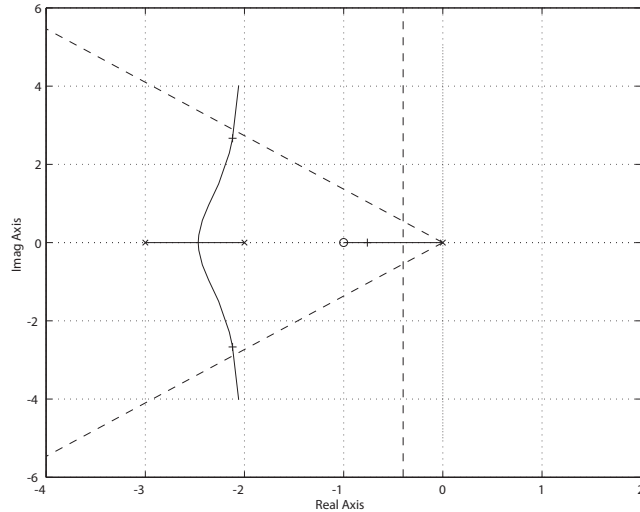
The proportional integral (PI) controller in part (c) is

**FIGURE CP7.7**

CONTINUED: (c) Root locus for integral controller with selected  $K = 4.0930$ .

$$G_c(s) = \frac{9.2516(s + 1)}{s}.$$

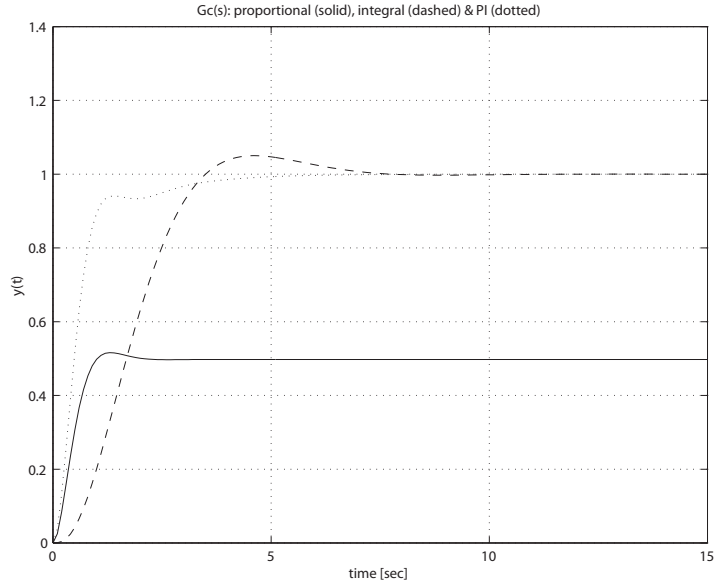
The **proportional controller** is stable for all  $K > 0$  but has a significant



**FIGURE CP7.7**

CONTINUED: (d) Root locus for PI controller with selected  $K = 9.2516$ .

steady-state error. The **integral controller** has no steady-state error, but is stable only for  $K < 30$ . The **PI controller** has zero steady-state error and is stable for all  $K > 0$ . Additionally, the PI controller has a fast transient response. The step responses for each controller is shown in Figure CP7.7e.



**FIGURE CP7.7**  
CONTINUED: (e) Step responses for each controller.

**CP7.8** The loop transfer function can be written as

$$G_c(s)G(s) = \frac{K_1 + K_2s}{Js^2} = \bar{K}_2 \frac{s + 5}{s^2}$$

where

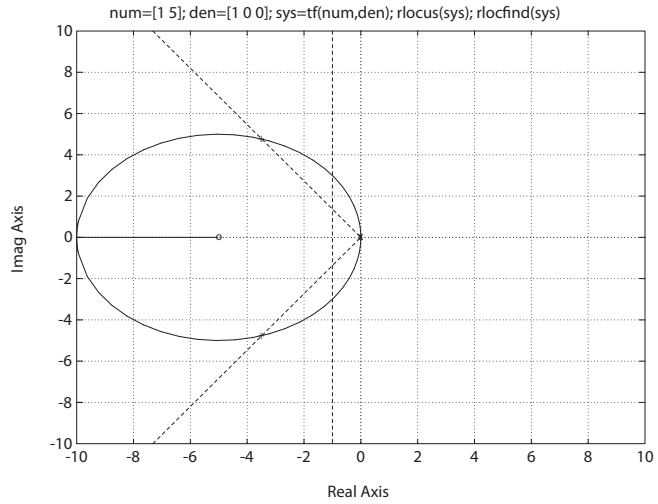
$$\bar{K}_2 = K_2/J .$$

The parameter of interest for the root locus is  $\bar{K}_2$ . The root locus is shown in Figure CP7.8. The selected value of

$$\bar{K}_2 = 7.1075 .$$

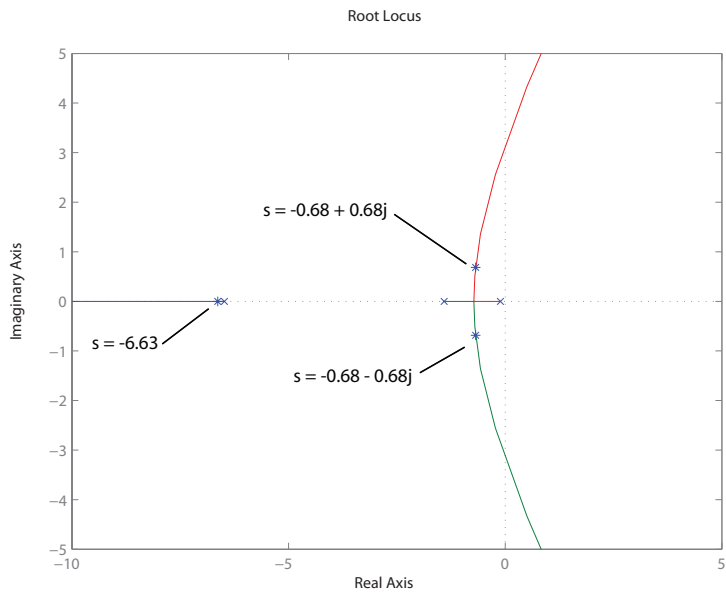
Therefore,

$$\frac{K_2}{J} = 7.1075 \quad \text{and} \quad \frac{K_1}{J} = 35.5375 .$$



**FIGURE CP7.8**  
Root locus to determine  $\bar{K}_2$ .

**CP7.9** The value of  $K$  that results in a damping ratio of  $\zeta = 0.707$  is  $K = 5.2$ . The root locus is shown in Figure CP7.9.



**FIGURE CP7.9**  
Root locus for  $1 + K \frac{1}{s^3 + 8s^2 + 10s + 1} = 0$ .

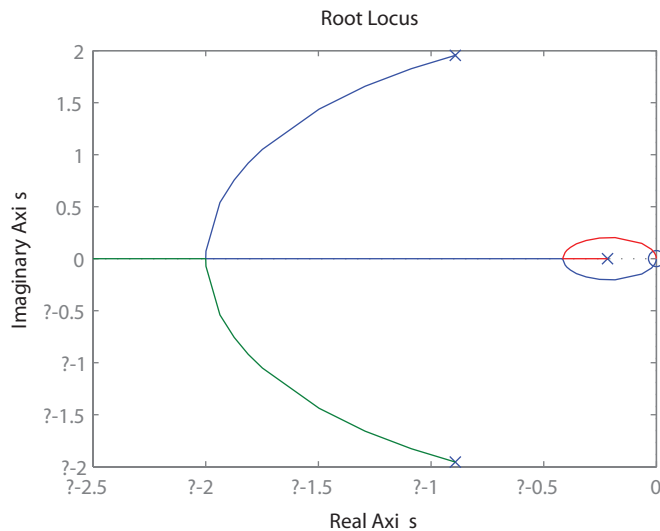


**CP7.10** (a) The characteristic equation is

$$s^3 + (2 + k)s^2 + 5s + 1 = 0 .$$

(b) The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 2 + k & 1 \\ s^1 & \frac{5k+9}{2+k} & \\ s^0 & 1 & \end{array}$$



**FIGURE CP7.10**  
Root locus for  $1 + k \frac{s^2}{s^3 + 2s^2 + 5s + 1} = 0$ .

For stability we require

$$2 + k > 0 \quad \text{or} \quad k > -2$$

and

$$5k + 9 > 0 \quad \text{or} \quad k > -9/5 .$$

Therefore, the stability region is defined by

$$k > -1.8 .$$

(c) Rearranging the characteristic equation yields

$$1 + k \frac{s^2}{s^3 + 2s^2 + 5s + 1} = 0 .$$

The root locus is shown in Figure CP7.10. We see that the system is stable for all  $k > 0$ .

# Frequency Response Methods

## Exercises

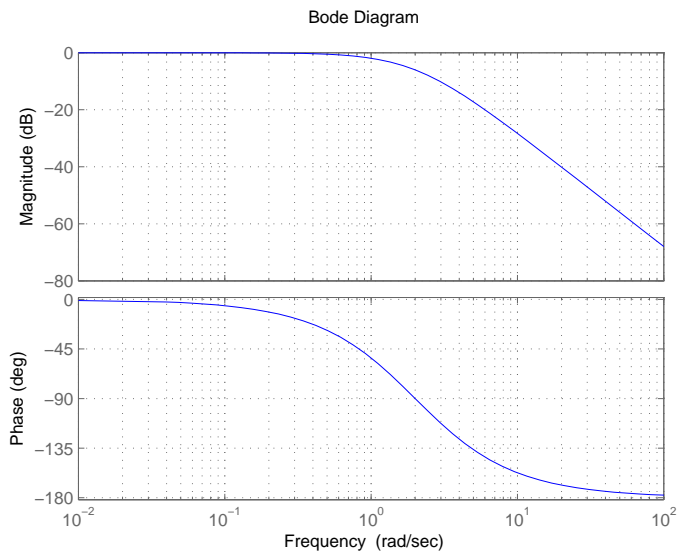
E8.1 Given the loop transfer function

$$L(s) = \frac{4}{(s+2)^2},$$

we determine that

$$|L(j\omega)| = \frac{4}{4 + \omega^2} \quad \text{and} \quad \phi(\omega) = -2 \tan^{-1} \omega/2.$$

The frequency response is shown in Figure E8.1.



**FIGURE E8.1**  
Frequency response for  $L(s) = \frac{4}{(s+2)^2}$ .

The magnitude and phase angle for  $\omega = 0, 0.5, 1, 2, 4, \infty$  are summarized in Table E8.1.

$\omega$	0	0.5	1	2	4	$\infty$
$ L(j\omega) $	1	0.94	0.80	0.50	0.20	0
$\phi$ (deg)	0	-28.07	-53.13	-90	-126.87	-180

**TABLE E8.1** Magnitude and phase for  $L(s) = \frac{4}{(s+2)^2}$ .

**E8.2** The transfer function is

$$G(s) = \frac{5000}{(s+70)(s+500)}.$$

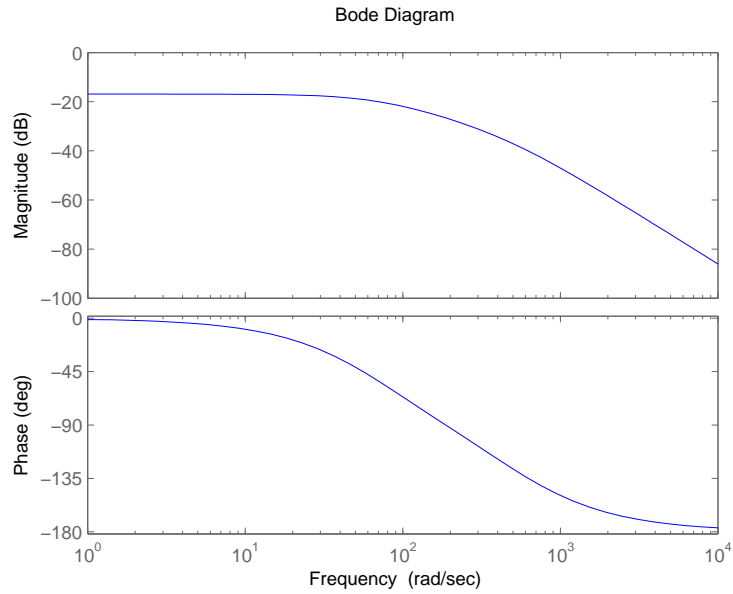
The frequency response plot is shown in Figure E8.2. The phase angle is computed from

$$\phi = -\tan^{-1} \frac{\omega}{70} - \tan^{-1} \frac{\omega}{500}.$$

The phase angles for  $\omega = 10, 100$  and  $700$  are summarized in Table E8.2.

$\omega$	10	200	700
$ G(j\omega) $	-16.99	-27.17	-41.66
$\phi$ (deg)	-9.28	-92.51	-138.75

**TABLE E8.2** Magnitude and phase for  $G(s) = \frac{5000}{(s+70)(s+500)}$ .



**FIGURE E8.2**  
Frequency response for  $G(s) = \frac{5000}{(s+70)(s+500)}$ .

**E8.3** The loop transfer function is

$$L(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)}.$$

The phase angle is computed via

$$\phi(\omega) = -90^\circ - \tan^{-1} \frac{\omega}{10} - \tan^{-1} \frac{\omega}{40} + \tan^{-1} \frac{\omega}{100}.$$

At  $\omega = 28.3$ , we determine that

$$\phi = -90^\circ - 70.5^\circ - 35.3^\circ + 15.8^\circ = -180^\circ.$$

Computing the magnitude yields

$$|L(j\omega)| = \frac{300(100)(1 + (\frac{\omega}{100})^2)^{\frac{1}{2}}}{\omega 10(1 + (\frac{\omega}{10})^2)^{\frac{1}{2}} 40(1 + (\frac{\omega}{40})^2)^{\frac{1}{2}}} = 0.75,$$

when  $\omega = 28.3$ . We can also rewrite  $L(s)$  as

$$L(s) = \frac{75(\frac{s}{100} + 1)}{s(\frac{s}{10} + 1)(\frac{s}{40} + 1)}.$$

Then, the magnitude in dB is

$$20 \log_{10} |L| = 20 \log_{10}(75) + 10 \log_{10}\left(1 + \left(\frac{\omega}{100}\right)^2\right) - 10 \log_{10}\left(1 + \left(\frac{\omega}{10}\right)^2\right) \\ - 10 \log_{10}\left(1 + \left(\frac{\omega}{40}\right)^2\right) - 20 \log_{10} \omega = -2.5 \text{ dB} ,$$

at  $\omega = 28.3$ .

**E8.4** The transfer function is

$$G(s) = \frac{Ks}{(s+a)(s+10)^2} .$$

Note that  $\phi = 0^\circ$  at  $\omega = 3$ , and that

$$\phi = +90^\circ - \tan^{-1} \frac{\omega}{a} - 2 \tan^{-1} \frac{\omega}{10} .$$

Substituting  $\omega = 3$  and solving for  $a$  yields

$$a = 2 .$$

Similarly, from the magnitude relationship we determine that

$$K = 400 .$$

**E8.5** The lower portion for  $\omega < 2$  is

$$20 \log \frac{K}{\omega} = 0 \text{ dB} ,$$

at  $\omega = 8$ . Therefore,

$$20 \log \frac{K}{8} = 0 \text{ dB}$$

which occurs when

$$K = 8 .$$

We have a zero at  $\omega = 2$  and another zero at  $\omega = 4$ . The zero at  $\omega = 4$  yields

$$a = 0.25 .$$

We also have a pole at  $\omega = 8$ , and a second pole at  $\omega = 24$ . The pole at  $\omega = 24$  yields

$$b = 1/24 .$$

Therefore,

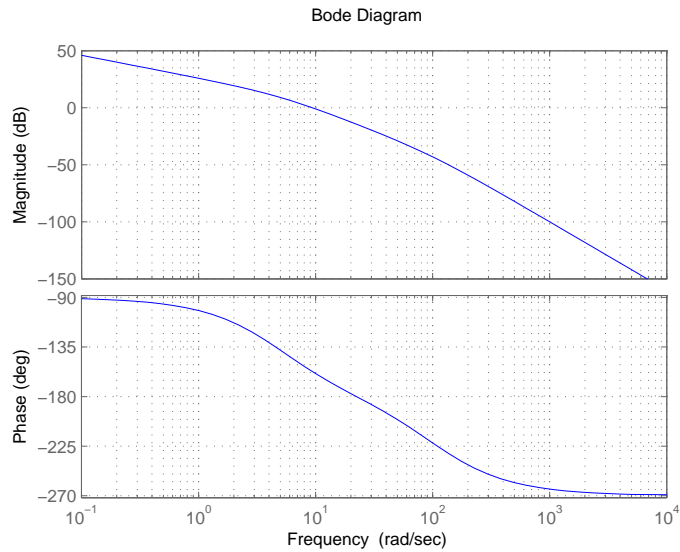
$$G(s) = \frac{8(1 + s/2)(1 + s/4)}{s(1 + s/8)(1 + s/24)(1 + s/36)} .$$

**E8.6** The loop transfer function is

$$L(s) = \frac{10}{s(s/5 + 1)(s/100 + 1)} .$$

The Bode diagram is shown in Figure E8.6. When  $20 \log_{10} |L(j\omega)| = 0 \text{ dB}$ , we have

$$\omega = 9.4 \text{ rad/sec} .$$



**FIGURE E8.6**

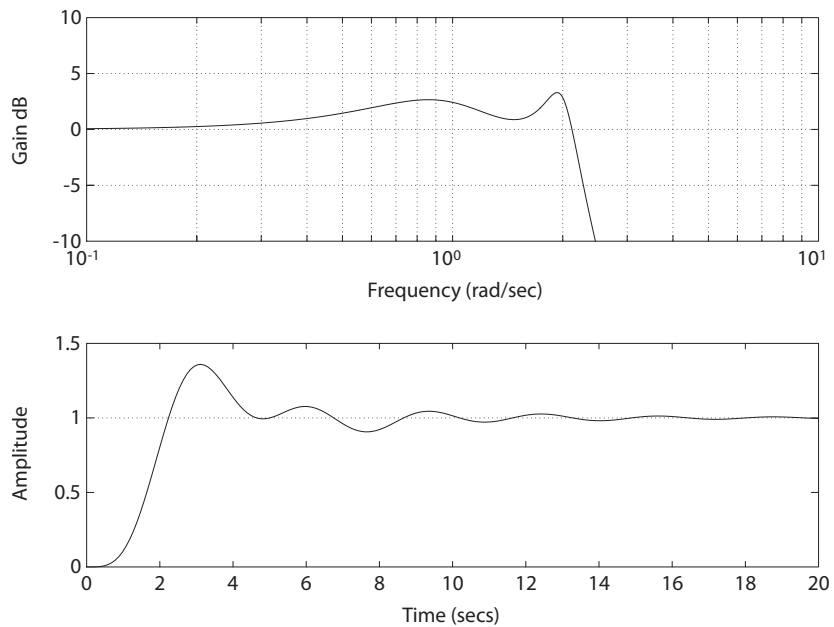
Bode Diagram for  $L(s) = \frac{10}{s(s/5+1)(s/100+1)}$ .

**E8.7** The transfer function is

$$T(s) = \frac{4}{(s^2 + s + 1)(s^2 + 0.4s + 4)} .$$

(a) The frequency response magnitude is shown in Figure E8.7. The frequency response has two resonant peaks at

$$\omega_{r1} = 0.8 \text{ rad/sec} \quad \text{and} \quad \omega_{r2} = 1.9 \text{ rad/sec} .$$

**FIGURE E8.7**

(a) Bode Diagram for  $T(s) = \frac{4}{(s^2 + s + 1)(s^2 + 0.4s + 4)}$ . (b) Unit step response.

(b) The percent overshoot is

$$P.O. = 35\% ,$$

and the settling time is

$$T_s \approx 16 \text{ sec} .$$

(c) The step response is shown in Figure E8.7.

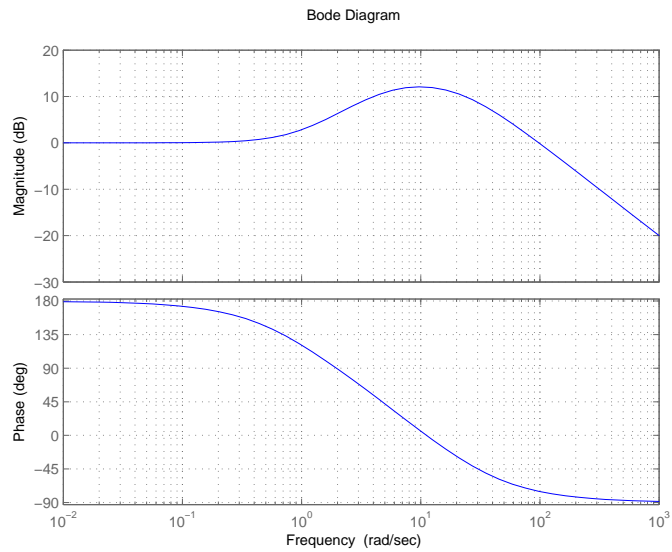
**E8.8** (a) The break frequencies are

$$\omega_1 = 1 \text{ rad/sec}, \quad \omega_2 = 5 \text{ rad/sec}, \quad \text{and} \quad \omega_3 = 20 \text{ rad/sec} .$$

(b) The slope of the asymptotic plot at low frequencies is 0 dB/decade.  
And at high frequencies the slope of the asymptotic plot is -20 dB/decade.

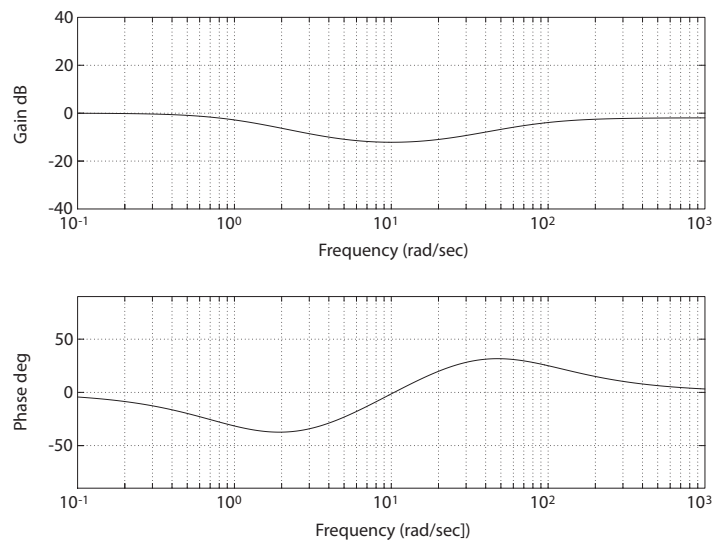
(c) The Bode plot is shown in Figure E8.8.





**FIGURE E8.8**  
Bode Diagram for  $G_c(s)G(s) = \frac{100(s-1)}{s^2+25s+100}$ .

**E8.9** The Bode diagram for  $G(s)$  is shown in Figure E8.9.



**FIGURE E8.9**  
Bode Diagram for  $G(s) = \frac{(s/5+1)(s/20+1)}{(s+1)(s/80+1)}$ .

- E8.10** The frequency response has two peaks; the first peak at  $f \approx 1.8$  and the second peak at  $f \approx 3.1$ . One possible  $G(j\omega)$  is

$$G(j\omega) = \frac{1}{(j\omega\tau + 1) \left( 1 + \left( \frac{2\zeta_1}{\omega_{n_1}} \right) j\omega + \left( \frac{j\omega}{\omega_{n_1}} \right)^2 \right) \left( 1 + \left( \frac{2\zeta_2}{\omega_{n_2}} \right) j\omega + \left( \frac{j\omega}{\omega_{n_2}} \right)^2 \right)},$$

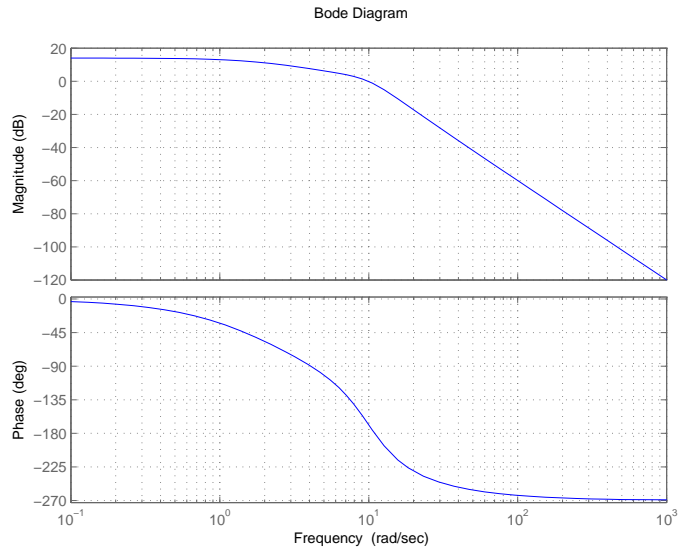
where

$$\tau = \frac{1}{2\pi(0.2)},$$

$$\zeta_1 = 0.15; \quad \omega_{n_1} = 2\pi(1.8 \times 10^3) \quad \zeta_2 = 0.15; \quad \omega_{n_2} = 2\pi(3.1 \times 10^3).$$

The damping ratios are estimated using Figure 8.10 in Dorf & Bishop.

- E8.11** The Bode plot is shown in Figure E8.11. The frequency when  $20 \log_{10} |G_C G(j\omega)| = 0$  is  $\omega = 9.9$  rad/sec.



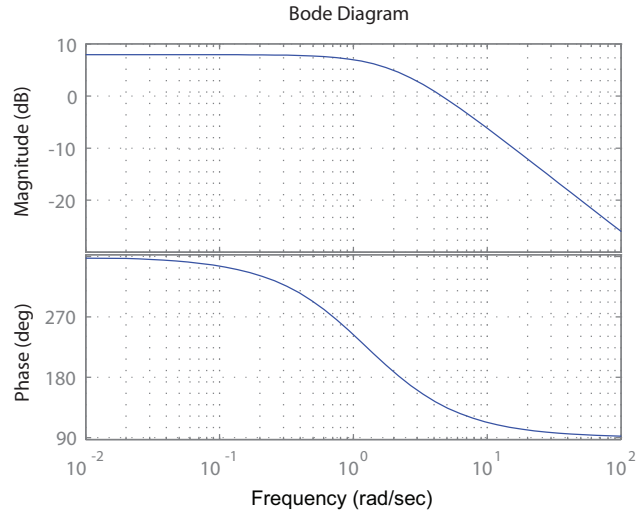
**FIGURE E8.11**

Bode Diagram for  $G_c(s)G(s) = \frac{1000}{(s^2 + 10s + 100)(s + 2)}$ .

- E8.12** (a) The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{-5(s - 1)}{s^2 + 3s + 2}.$$

- (b) The Bode plot is shown in Figure E8.12.

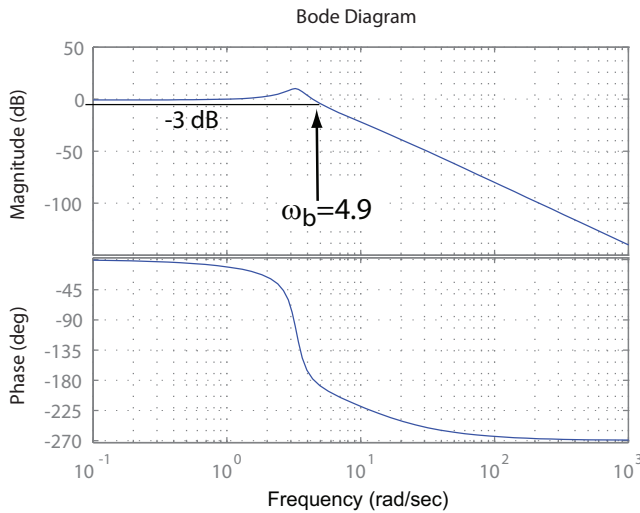


**FIGURE E8.12**  
Bode Diagram for  $G(s) = \frac{-5(s-1)}{s^2+3s+2}$ .

**E8.13** The closed-loop transfer function is

$$T(s) = \frac{100}{s^3 + 11s^2 + 20s + 110} .$$

The Bode plot of  $T(s)$  is shown in Figure E8.13, where  $\omega_B = 4.9$  rad/sec.

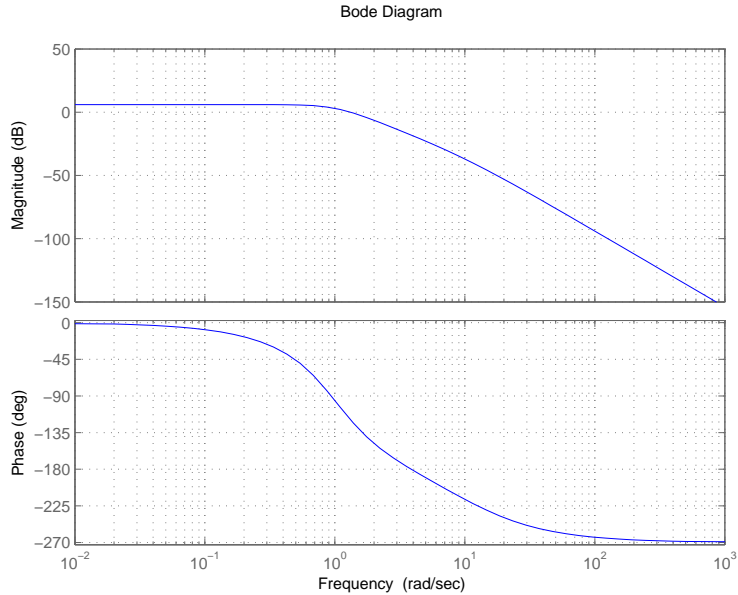


**FIGURE E8.13**  
Bode Diagram for  $T(s) = \frac{100}{s^3+11s^2+20s+110}$ .

**E8.14** The loop transfer function is

$$L(s) = \frac{20}{(s^2 + 1.4s + 1)(s + 10)}.$$

The Bode plot of  $L(s)$  is shown in Figure E8.14. The frequency when  $20 \log_{10} |L(j\omega)| = 0$  is  $\omega = 1.32$  rad/sec.



**FIGURE E8.14**

Bode Diagram for  $L(s) = \frac{20}{(s^2 + 1.4s + 1)(s + 10)}$ .

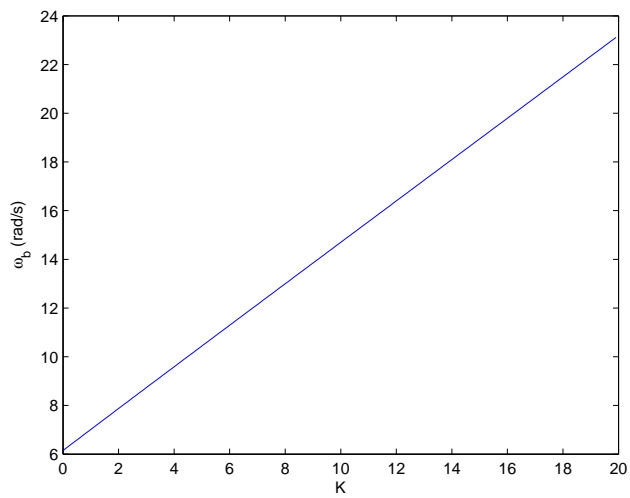
**E8.15** The closed-loop transfer function is

$$T(s) = \frac{3s + 5}{s^2 + s + K + 6}.$$

The bandwidth as a function of  $K$  is shown in Figure E8.15. The bandwidth as a function of  $K$  is:

- (a)  $K = 1$  and  $\omega_b = 7.0$  rad/sec.
- (b)  $K = 2$  and  $\omega_b = 7.9$  rad/sec.
- (c)  $K = 10$  and  $\omega_b = 14.7$  rad/sec.

The bandwidth increases as  $K$  increases.

**FIGURE E8.15**

Bandwidth of  $T(s) = \frac{3s+5}{s^2+s+K+6}$ .

# Problems

P8.1 (a) The transfer function is

$$G_c(s)G(s) = \frac{1}{(1 + 0.25s)(1 + 3s)},$$

and

$$G_c(j\omega)G(j\omega) = \frac{1}{(1 - 0.75\omega^2) + j3.25\omega}.$$

The polar plot is shown in Figure P8.1a. A summary of the magnitude and phase angles for  $\omega = 0, 0.5, 1, 2, 5$  and  $\infty$  can be found in Table P8.1a.

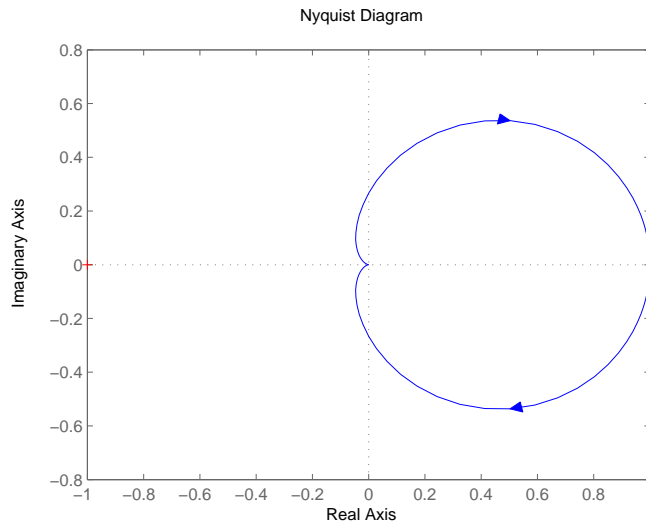


FIGURE P8.1

(a) Polar plot for  $G_c(s)G(s) = \frac{1}{(1+0.25s)(1+3s)}$ .

$\omega$	0	0.5	1	2	5	$\infty$
$ G_c(j\omega)G(j\omega) $ (dB)	1.00	0.55	0.31	0.15	0.04	0
$\phi$ (deg)	0	-63.4	-85.6	-107.1	-137.51	-180

TABLE P8.1 (a) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{1}{(1+0.25s)(1+3s)}$ .

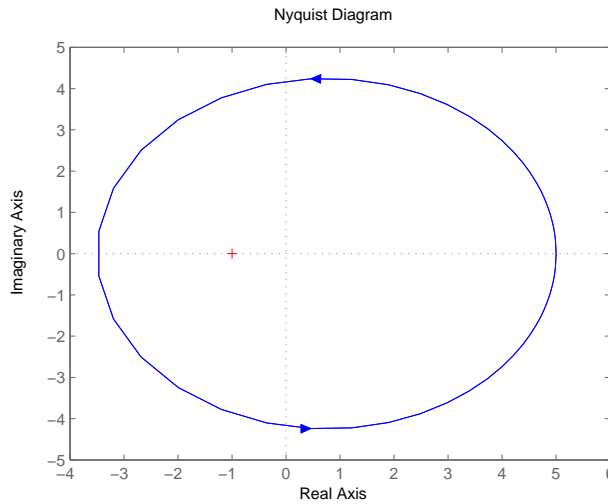
(b) The transfer function is

$$G_c(s)G(s) = \frac{5(s^2 + 1.4s + 1)}{(s - 1)^2}$$

and

$$G_c(j\omega)G(j\omega) = \frac{5((1 - \omega^2) + 1.4j\omega)}{(1 - \omega^2) - 2j\omega}.$$

The polar plot is shown in Figure P8.1b. A summary of the magnitude and phase angles for  $\omega = 0, 0.25, 0.5, 1, 2, 8, 16$  and  $\infty$  can be found in Table P8.1b.



**FIGURE P8.1**  
CONTINUED: (b) Polar plot for  $G_c(s)G(s) = \frac{5(s^2+1.4s+1)}{(s-1)^2}$ .

$\omega$	0	0.25	0.5	1	2	8	16	$\infty$
$ G_c(j\omega)G(j\omega) $ (dB)	5.00	4.71	4.10	3.50	4.10	4.92	4.98	5.00
$\phi$ (deg)	0	48.5	96.1	180.0	-96.2	-24.3	-12.2	0

**TABLE P8.1** CONTINUED: (b) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{5(s^2+1.4s+1)}{(s-1)^2}$ .

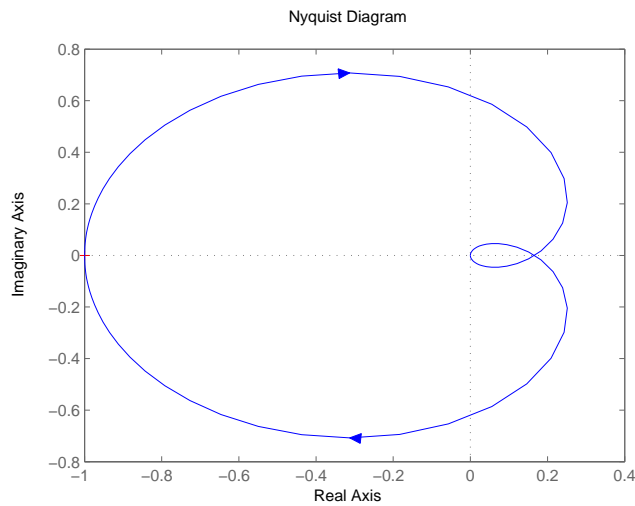
(c) The transfer function is

$$G_c(s)G(s) = \frac{(s - 8)}{(s^2 + 6s + 8)} .$$

The polar plot is shown in Figure P8.1c. A summary of the magnitude and phase angles for

$$\omega = 0, 1, 2, 3, 4, 5, 6, \infty$$

can be found in Table P8.1c.



**FIGURE P8.1**  
CONTINUED: (c) Polar plot for  $G_c(s)G(s) = \frac{s-8}{s^2+6s+8}$ .

$\omega$	0	1	2	3	4	5	6	$\infty$
$ G_c(j\omega)G(j\omega) $ (dB)	1.00	0.87	0.65	0.47	0.35	0.27	0.22	0.00
$\phi$ (deg)	180.0	132.3	94.4	66.3	45.0	28.5	15.3	-90.0

**TABLE P8.1** CONTINUED: (c) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{s-8}{s^2+6s+8}$ .



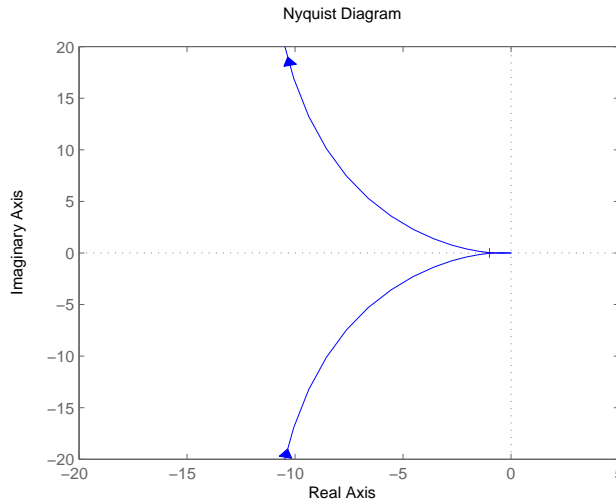
(d) The transfer function is

$$G_c(s)G(s) = \frac{20(s + 8)}{s(s + 2)(s + 4)} .$$

The polar plot is shown in Figure P8.1d. A summary of the magnitude and phase angles for

$$\omega = 1, 0.1, 0.8, 1.6, 3.2, 12.8, \infty$$

can be found in Table P8.1d.



**FIGURE P8.1**

CONTINUED: (d) Polar plot for  $G_c(s)G(s) = \frac{20(s+8)}{s(s+2)(s+4)}$ .

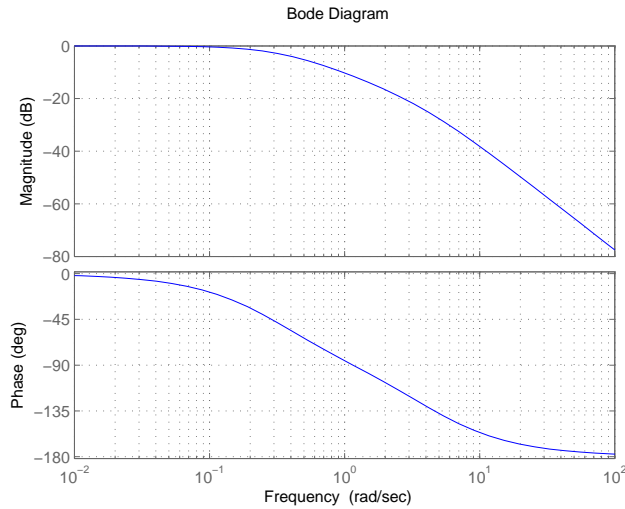
$\omega$	0	0.1	0.8	1.6	3.2	12.8	$\infty$
$ G_c(j\omega)G(j\omega) $ (dB)	$\infty$	199.70	22.87	9.24	2.79	0.14	0.00
$\phi$ (deg)	0	-93.6	-117.4	-139.1	-164.8	174.3	180.0

**TABLE P8.1** CONTINUED: (d) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{20(s+8)}{s(s+2)(s+4)}$ .

**P8.2** (a) The Bode plot is shown in Figure P8.2a. A summary of the magnitude and phase angles for

$$\omega = 0.25, 0.5, 1, 2, 4, 8, 16$$

can be found in Table P8.2a.



**FIGURE P8.2**

(a) Bode plot for  $G_c(s)G(s) = \frac{1}{(1+0.25s)(1+3s)}$ .

$\omega$	0.25	0.5	1.0	2.0	4.0	8.0	16.0
$ G_c(j\omega)G(j\omega) $ (dB)	-1.95	-5.19	-10.26	-16.65	-24.62	-34.60	-45.93
$\phi$ (deg)	-40.5	-63.4	-85.6	-107.1	-130.2	-151.0	-164.8

**TABLE P8.2** (a) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{1}{(1+0.25s)(1+3s)}$ .

(b) The transfer function is

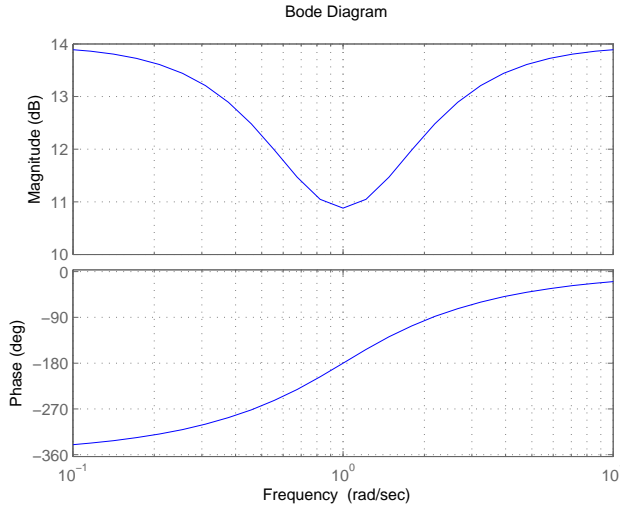
$$G_c(s)G(s) = \frac{5(s^2 + 1.4s + 1)}{(s - 1)^2}$$

The Bode plot is shown in Figure P8.2b. A summary of the magnitude

and phase angles for

$$\omega = 0.25, 0.5, 1, 2, 4, 8, 16$$

can be found in Table P8.2b.



**FIGURE P8.2**  
CONTINUED: (b) Bode plot for  $G_c(s)G(s) = \frac{5(s^2+1.4s+1)}{(s-1)^2}$ .

$\omega$	0.25	0.5	1.0	2.0	4.0	8.0	16.0
$ G_c(j\omega)G(j\omega) $ (dB)	13.46	12.26	10.88	12.26	13.46	13.84	13.95
$\phi$ (deg)	48.5	96.2	180.0	-96.2	-48.5	-24.3	-12.2

**TABLE P8.2** CONTINUED: (b) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{5(s^2+1.4s+1)}{(s-1)^2}$ .

(c) The transfer function is

$$G_c(s)G(s) = \frac{(s - 8)}{(s^2 + 6s + 8)} .$$

The Bode plot is shown in Figure P8.2c. A summary of the magnitude and phase angles for  $\omega = 0.6, 1, 2, 3, 4, 5, 6, \infty$  can be found in Table P8.2c.

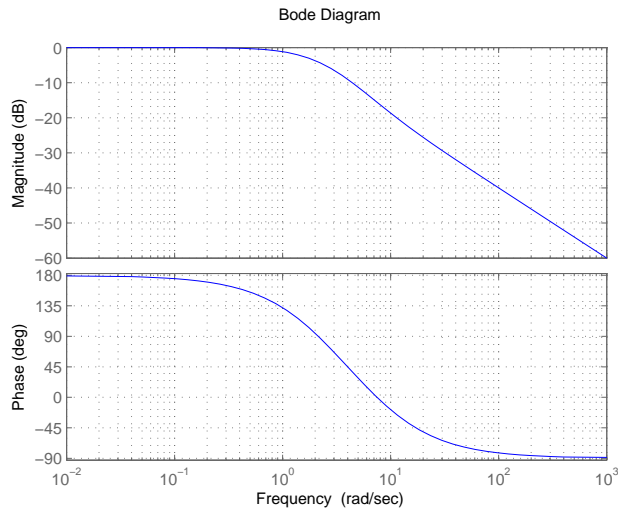


FIGURE P8.2

CONTINUED: (c) Bode plot for  $G_c(s)G(s) = \frac{s-8}{s^2+6s+8}$ .

$\omega$	0.6	1	2	3	4	5	6	$\infty$
$ G_c(j\omega)G(j\omega) $ (dB)	-0.45	-1.17	-3.72	-6.49	-9.03	-11.26	-13.18	-120.00
$\phi$ (deg)	150.5	132.3	94.4	66.3	45.0	28.5	15.3	-90.0

TABLE P8.2 CONTINUED: (c) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{s-8}{s^2+6s+8}$ .

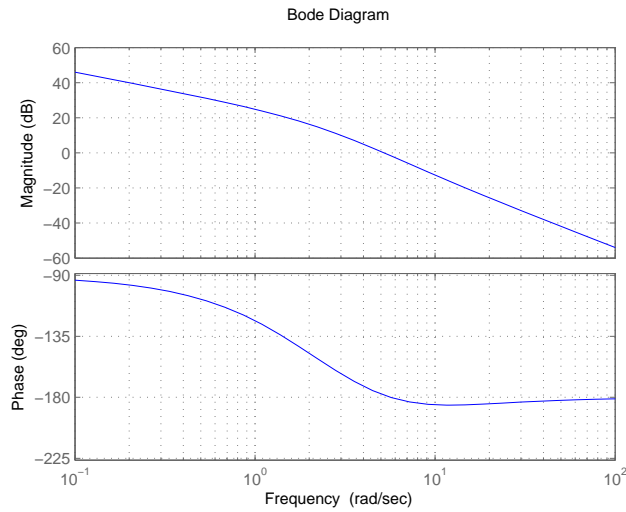
(d) A summary of the magnitude and phase angles for

$$\omega = 0.2, 0.8, 3.2, 6.4, 12.8, 25.6, 51.2$$

can be found in Table P8.2d. The Bode plot is shown in Figure P8.2d.

$\omega$	0.2	0.8	3.2	6.4	12.8	25.6	51.2
$ G_c(j\omega)G(j\omega) $ (dB)	39.95	27.19	8.90	-3.98	-17.35	-30.0355	-42.28
$\phi$ (deg)	-97.1	-117.4	-164.8	178.0	174.2	176.0	177.8

TABLE P8.2 CONTINUED: (d) Magnitudes and phase angles for  $G_c(s)G(s) = \frac{20(s+8)}{s(s+2)(s+4)}$ .

**FIGURE P8.2**

CONTINUED: (d) Bode plot for  $G_c(s)G(s) = \frac{20(s+8)}{s(s+2)(s+4)}$ .

**P8.3** (a) The bridged-T network we found has zeros at

$$s = \pm j\omega_n$$

and poles at

$$s = -\frac{\omega_n}{Q} \pm \omega_n \sqrt{1/Q^2 - 1}.$$

The frequency response is shown in Figure P8.3 for  $Q = 10$ .

(b) For the twin-T network, we evaluate the magnitude at

$$\omega = 1.1\omega_n$$

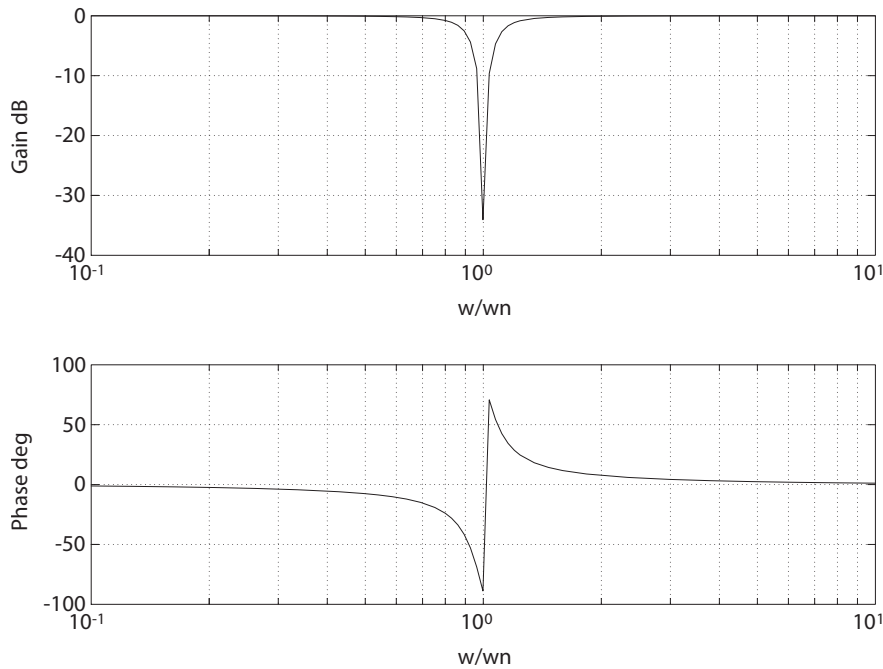
or 10% from the center frequency (see Example 8.4 in Dorf & Bishop). This yields

$$|G| \approx 2.1 \times \left(\frac{0.1}{3.9}\right) \times 1.1 = 0.05.$$

Similarly, for the bridged-T network

$$|G| = 2.1 \times \left(\frac{0.1}{2.1}\right) \times 0.14 = 0.707.$$

The bridged-T network possesses a narrower band than the twin-T network.



**FIGURE P8.3**  
 Bode plot for  $G(s) = \frac{s^2 + \omega_n^2}{s^2 + (2\omega_n/Q)s + \omega_n^2}$ , where  $\zeta = 1/Q = 0.1$ .

**P8.4** The transfer function is

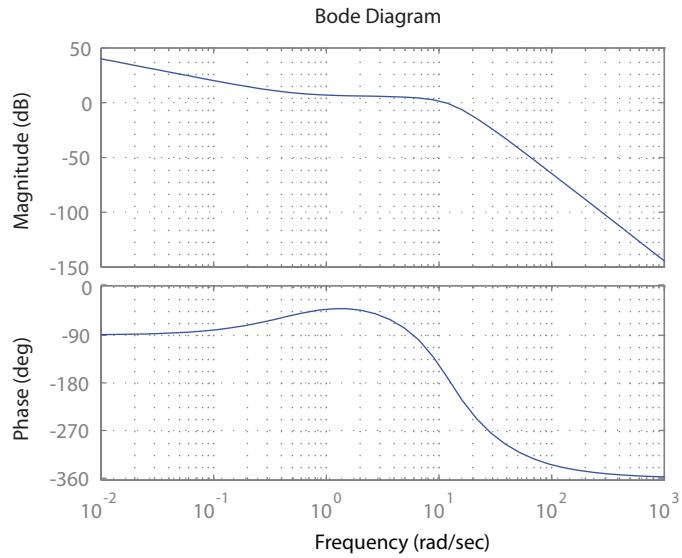
$$G(s) = G_c G_1 H(s) \left[ \frac{1}{s} \right]$$

$$= \frac{30000(2s + 1)}{s(s + 10)(s + 20)(s^2 + 15s + 150)}$$

A summary of the magnitude and phase angles can be found in Table P8.4. The Bode plot is shown in Figure P8.4.

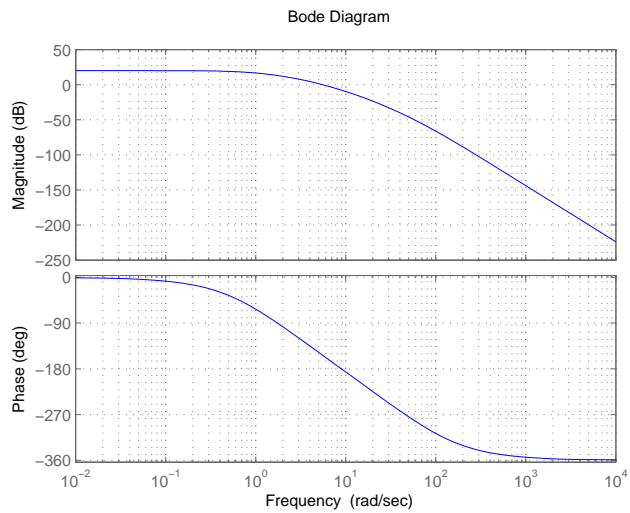
$\omega$	1	3	5	8	10	15	24
$ G(j\omega) $ dB	6.95	5.78	5.08	3.38	1.59	-5.01	-17.56
$\phi(deg)$	$-40.89^\circ$	$-52.39^\circ$	$-77.28^\circ$	$-118.41^\circ$	$-145.99^\circ$	$-203.52^\circ$	$-258.57^\circ$

**TABLE P8.4** Magnitudes and phase angles for  $GH(s) = \frac{30000(2s+1)}{s(s+10)(s+20)(s^2+15s+150)}$ .

**FIGURE P8.4**

Bode plot for  $GH(s) = \frac{30000(2s+1)}{s(s+10)(s+20)(s^2+15s+150)}$ .

**P8.5** The Bode plot is shown in Figure P8.5.

**FIGURE P8.5**

Bode plot for  $G(s) = \frac{10}{(s/4+1)(s+1)(s/20+1)(s/80+1)}$ .

**P8.6** (a) The transfer function is

$$GH(s) = \frac{3.98(1 + s/1)}{s(1 + s/10)^2}.$$

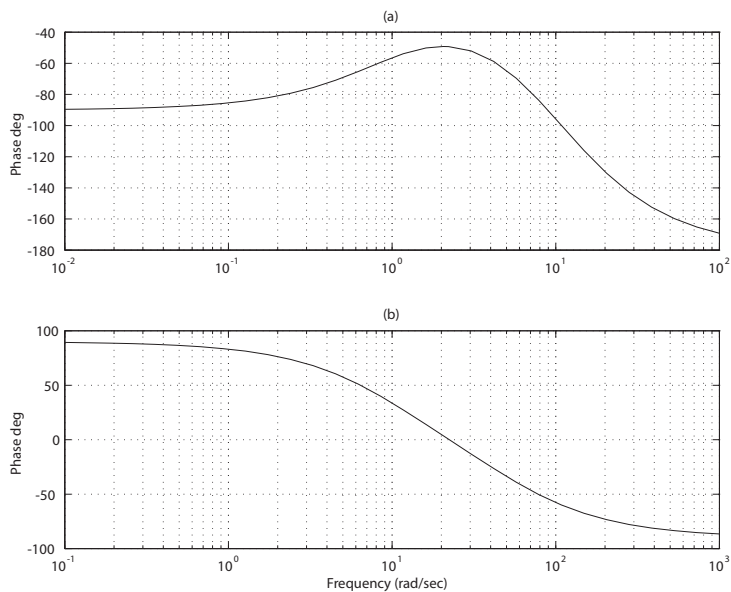
We have a zero at  $\omega = 1$  and two poles at  $\omega = 10.0$ . The low frequency approximation is  $K/s$  and at  $\omega = 1$  we have

$$20 \log \left( \frac{K}{\omega} \right) = 12 \text{dB}.$$

Therefore,

$$K = 3.98$$

at  $\omega = 1$  (an approximation). The phase plot is shown in Figure P8.6a.



**FIGURE P8.6**

Phase plots for (a)  $G(s) = \frac{3.98(s/1+1)}{s(s/10+1)^2}$ . (b)  $G(s) = \frac{s}{(s/10+1)(s/50+1)}$ .

(b) The transfer function is

$$GH(s) = \frac{s}{(1 + s/10)(1 + s/50)}.$$

The poles are located by noting that the slope is  $\pm 20$  dB/dec. The



low frequency approximation is  $Ks$ , so

$$20 \log K\omega = 0 \text{ dB} .$$

At  $\omega = 1$  we determine that

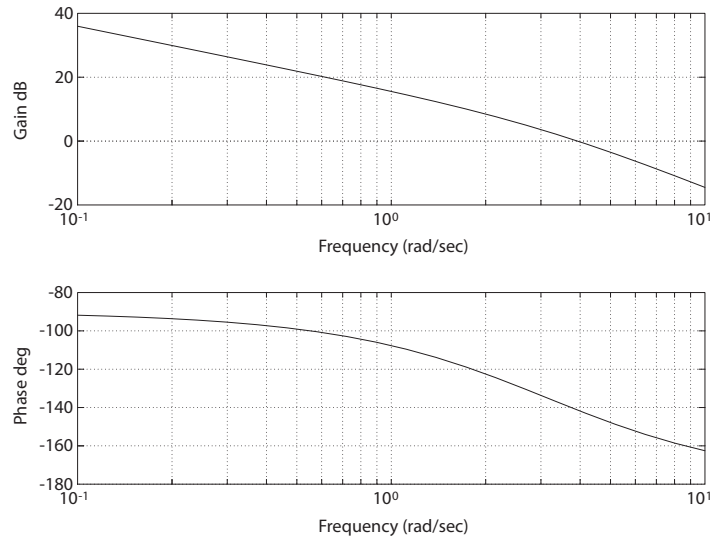
$$K = 1 .$$

The phase plot is shown in Figure P8.6b.

**P8.7** The loop transfer function is

$$L(s) = \frac{K_v}{s(s/\pi + 1)^2} .$$

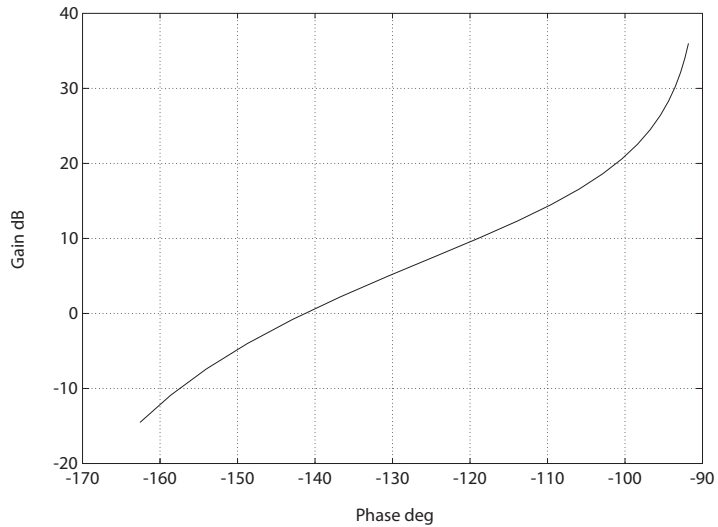
(a) Set  $K_v = 2\pi$ . The Bode plot is shown in Figure P8.7a.



**FIGURE P8.7**

(a) Bode plot for  $L(s) = \frac{K_v}{s(s/\pi+1)}$ , where  $K_v = 2\pi$ .

(b) The logarithmic magnitude versus the phase angle is shown in Figure P8.7b.

**FIGURE P8.7**CONTINUED: (b) Log-magnitude-phase curve for  $L(j\omega)$ .**P8.8** The transfer function is

$$T(s) = \frac{K}{s^2 + 10s + K} .$$

(a) When  $P.O. = 15\%$ , we determine that  $\zeta = 0.517$  by solving

$$15 = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} .$$

So,  $2\zeta\omega_n = 10$  implies that  $\omega_n = 9.67$ , hence  $K = \omega_n^2 = 93.53$ . Also,

$$M_{p\omega} = (2\zeta\sqrt{1-\zeta^2})^{-1} = 1.13 .$$

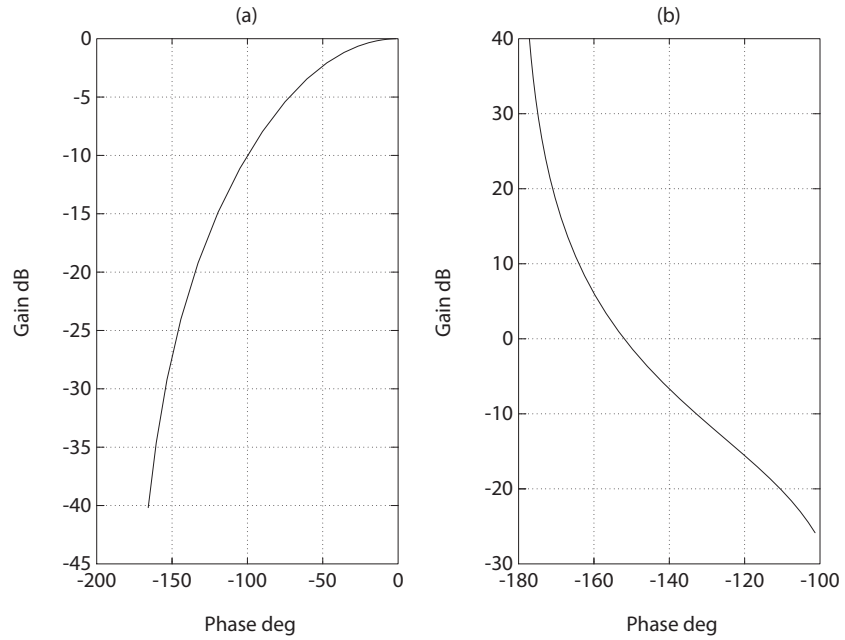
(b) For second-order systems we have

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} = 6.59$$

when  $\zeta = 0.517$  and  $\omega_n = 9.67$ .(c) We estimate  $\omega_B$  to be

$$\omega_B \approx (-1.19\zeta + 1.85)\omega_n = 11.94 \text{ rad/s} .$$

**P8.9** The log-magnitude phase curves are shown in Figure P8.9.



**FIGURE P8.9**

Log-magnitude-phase curve for (a)  $G_c(s)G(s) = \frac{1}{(1+0.5s)(1+2s)}$  and (b)  $G_c(s)G(s) = \frac{1+0.5s}{s^2}$ .

**P8.10** The governing equations of motion are

$$F(s) = K_f I_f(s) \quad \text{and} \quad I_f(s) = \frac{V_f(s)}{R_f + L_f s} .$$

Without loss of generality we can let  $K_f = 1.0$ . Also, we have

$$F(s) = (Ms^2 + bs + K)Y(s) .$$

Therefore, the transfer function is

$$GH(s) = \frac{KK_f}{(R_f + L_f s)(Ms^2 + bs + K)} = \frac{50K}{(s + 0.5)(s^2 + 2s + 4)} .$$

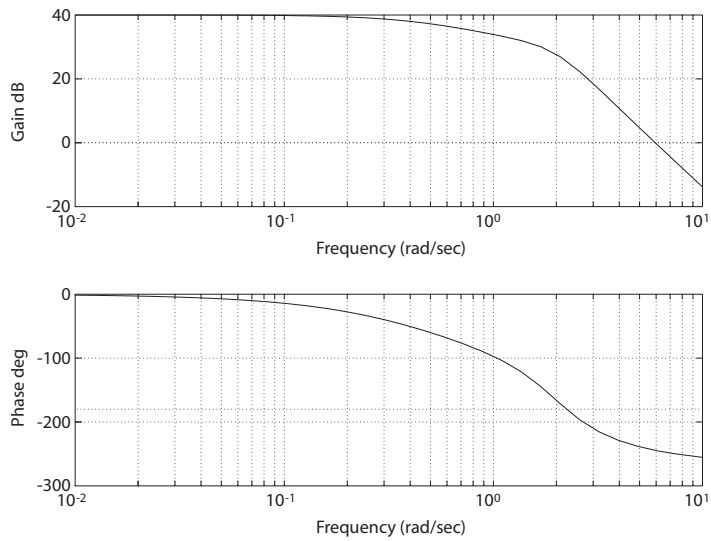
This is a type 0 system, therefore  $K_p = 25K$ .

(a) If we allow a 1% error, we have  $e_{ss} = |R|/(1 + K_p) = 0.01|R|$ . Thus  $K_p = 25K = 99$ . Select

$$K = 4 .$$

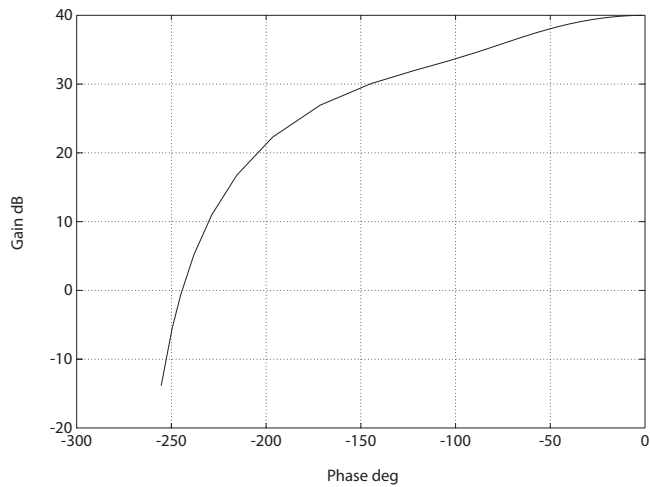
(b) The Bode plot is shown in Figure P8.10a.

(c) The log-magnitude phase curve is shown in Figure P8.10b.



**FIGURE P8.10**

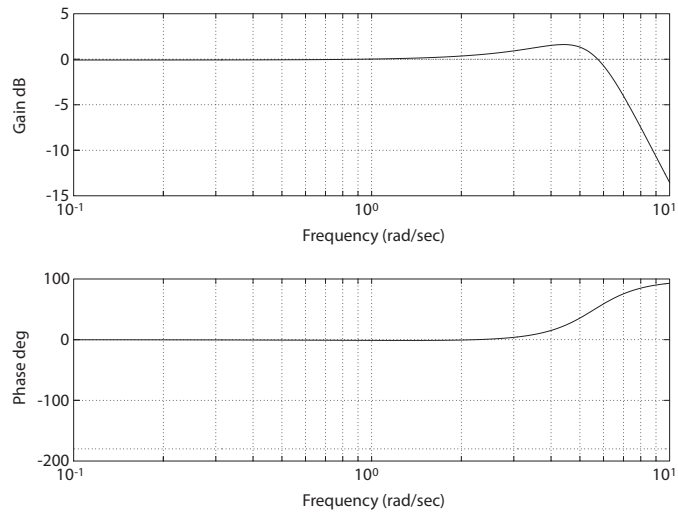
(a) Bode plot for  $GH(s) = \frac{200}{(s^2+2s+4)(s+0.5)}$ .



**FIGURE P8.10**

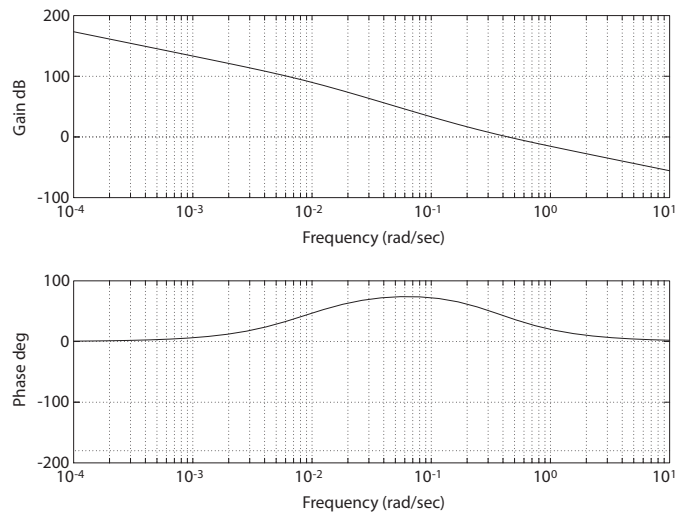
CONTINUED: (b) Log-magnitude-phase curve for  $GH(s) = \frac{200}{(s^2+2s+4)(s+0.5)}$ .

(d) The closed-loop transfer function Bode plot is shown in Figure P8.10c. We determine from the plot that  $M_{p\omega} = 1.6$ ,  $\omega_r = 4.4$  and  $\omega_B = 6.8$ .



**FIGURE P8.10**  
CONTINUED: (c) Bode plot for closed-loop  $T(s) = Y(s)/R(s)$ .

**P8.11** The Bode plot is shown in Figure P8.11.



**FIGURE P8.11**  
Bode plot for  $G(s) = \frac{0.164(s+0.2)(-s+0.32)}{s^2(s+0.25)(s-0.009)}$ .

**P8.12** The three transfer functions are

$$G_1(s) = 10 \quad G_2(s) = \frac{1}{s(s/0.6 + 1)} \quad G_3(s) = 3s .$$

(a) When  $G_3(s)$  is out of the loop, the characteristic equation is

$$1 + G_1G_2(s) = 1 + \frac{10}{s(s/0.6 + 1)} = 0$$

or  $s^2 + 0.6s + 6 = 0$ . Thus,  $\zeta = 0.6/(2\sqrt{6}) = 0.12$ .

(b) With  $G_3(s)$ , the characteristic equation is

$$1 + G_1G_2(s) + G_2G_3(s) = 1 + \frac{6}{s(s + 0.6)} + \frac{1.85}{s(s + 0.6)} = 0 ,$$

or

$$s^2 + 2.4s + 6 = 0 .$$

Thus,  $\zeta = 2.4/(2\sqrt{6}) = 0.49$ .

**P8.13** By inspection of the frequency response, we determine

$$L(s) = G_c(s)G(s)H(s) = \frac{K}{s(s/100 + 1)(s/1000 + 1)^2} .$$

For small  $\omega$ , we have  $20 \log K/\omega = 40$  dB at  $\omega = 10$ . So,  $K = 1000$ .

**P8.14** The data we have are  $R_1 = R_2 = 1000\Omega$ ,  $c_1 = 10^{-7}$  farad and  $c_2 = 10^{-6}$  farad. The governing equations are

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{C_1s}}{R_1 + \frac{1}{C_1s}} ,$$

and

$$\frac{V_o(s)}{V_2(s)} = \frac{KR_2}{R_2 + \frac{1}{C_2s}} .$$

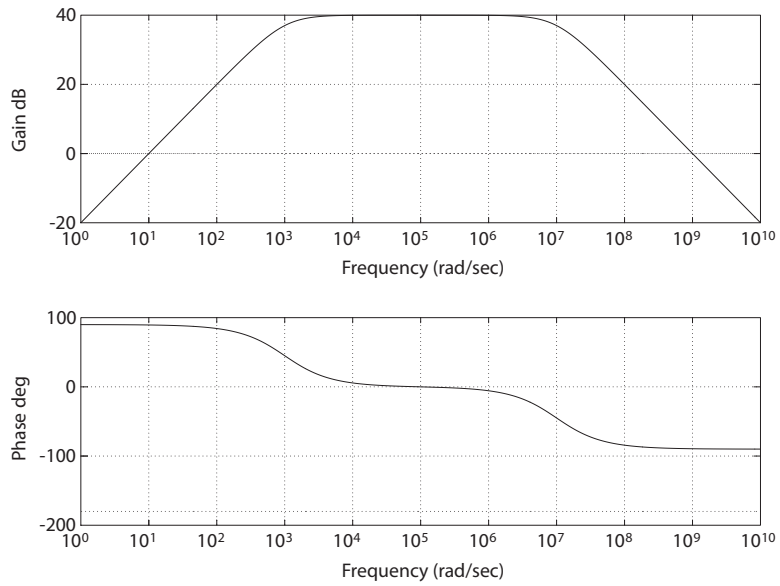
So

$$\frac{V_o(s)}{V_1(s)} = \frac{KR_2C_2s}{(R_1C_1s + 1)(R_2C_2s + 1)} = \frac{10^9s}{(s + 10^7)(s + 1000)} .$$

(a) The Bode plot is shown in Figure P8.14.

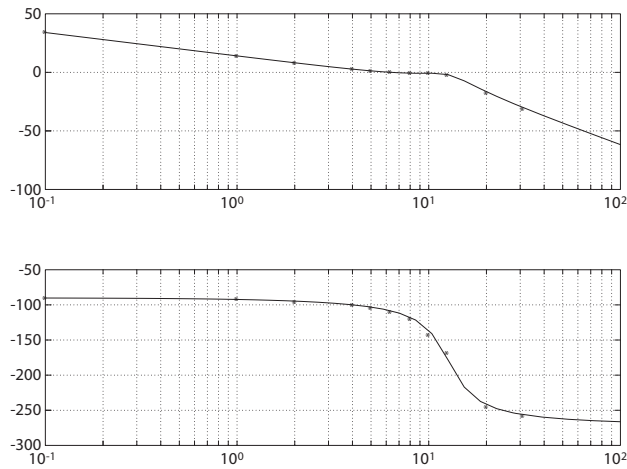
(b) The mid-band gain is = 40 dB.

(c) The -3 dB points are (rad/sec):  $\omega_{low} \approx 7$  and  $\omega_{high} \approx 1.5 \times 10^9$  .



**FIGURE P8.14**  
Bode plot for  $G(s) = \frac{10^9 s}{(s+10^7)(s+10^3)}$ .

**P8.15** The data are plotted in Figure P8.15, denoted by an asterisk (\*).



**FIGURE P8.15**  
Bode plot for  $G(s) = \frac{809.7}{s(s^2+6.35s+161.3)}$ ; tabular data is indicated by an asterick (\*).

The low frequency slope is -20 dB/dec and the initial low frequency  $\phi$  is

$-90^\circ$ , so we have an integrator of the form  $K/s$ . The initial phase is  $-90^\circ$  and the final phase  $-270^\circ$ , so we have a minimum phase  $G(s)$ . Now,  $|G|$  is 0.97 at  $\omega = 8$  and  $\omega = 10$  indicating two complex poles. We postulate a transfer function of the form

$$G(s) = \frac{K}{s \left( \frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1 \right)}.$$

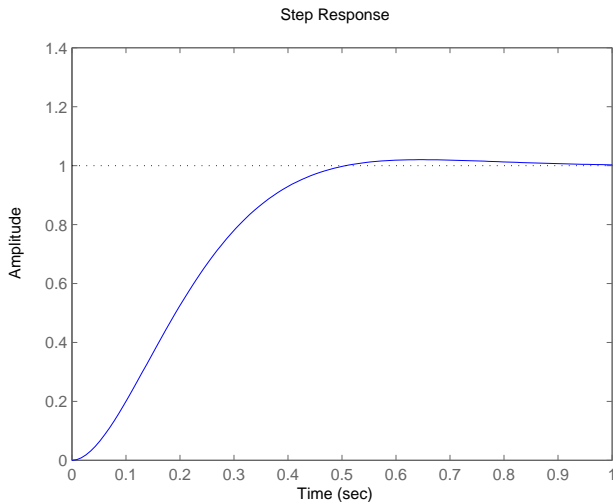
The phase angle  $\phi = -180^\circ$  at  $\omega = \omega_n$ . Then, from Figure 8.10 in Dorf & Bishop, we determine that  $\omega_n = 12.7$ . At  $\omega = 8$ ,  $\frac{\omega}{\omega_n} = 0.63$  and  $\phi$ , due to the complex poles is  $-30^\circ$  (subtract  $-90^\circ$  due to the integrator). Again, from Figure 8.10 in Dorf & Bishop, we estimate  $\zeta = 0.25$ . To determine  $K$ , note that when  $\frac{\omega}{\omega_n} \leq 0.1$ , the effect of the complex poles on magnitude is negligible, so at  $\omega = 1$  we have

$$|G| = 5.02 \cong \left| \frac{K}{j1} \right|.$$

So  $K = 5.02$ . Therefore,

$$G(s) = \frac{5.02}{s \left( \frac{s^2}{161.3} + \frac{0.5s}{12.7} + 1 \right)} = \frac{809.7}{s(s^2 + 6.35s + 161.3)}.$$

**P8.16** (a) The unit step input response is shown in Figure P8.16. The step



**FIGURE P8.16**

Unit step input response for  $T(s) = \frac{60.2}{s^2 + 12.1s + 60.2}$ .



response is given by

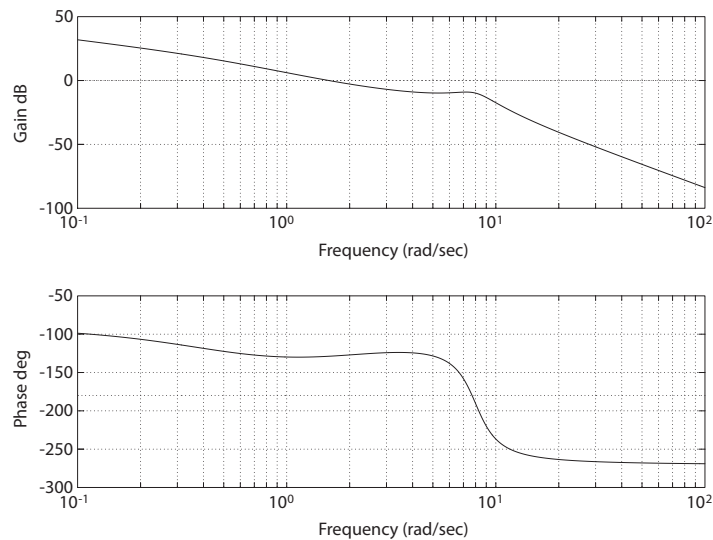
$$y(t) = 1 - e^{-6.05t} (\cos 4.85t + 1.25 \sin 4.85t) .$$

(b) The system bandwidth is  $\omega_B = 4.95$  rad/sec.

**P8.17** The transfer function is

$$G_c(s)G(s) = \frac{4(0.5s + 1)}{s(2s + 1)(s^2/64 + s/20 + 1)} .$$

(a) The Bode plot is shown in Figure P8.17.



**FIGURE P8.17**

Bode plot for  $G_s(s)G(s) = \frac{4(0.5s+1)}{s(2s+1)(s^2/64+s/20+1)}$ .

(b) When the magnitude is 0 dB, we have

$$\omega_1 = 1.6 \text{ rad/sec}$$

and when  $\phi = -180^\circ$ , we have

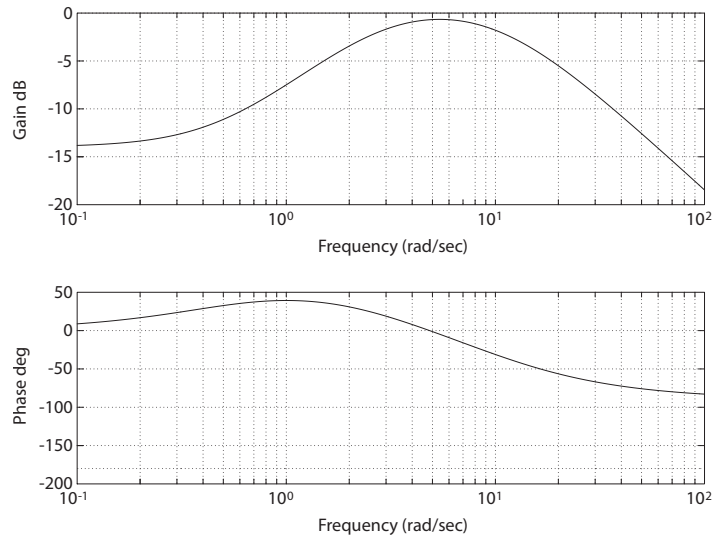
$$\omega_2 = 7.7 \text{ rad/sec} .$$

**P8.18** The transfer function is

$$G_c(s)G(s) = \frac{12(s + 0.5)}{(s + 3)(s + 10)} = \frac{0.2(2s + 1)}{(s/3 + 1)(s/10 + 1)} .$$

The Bode plot is shown in Figure P8.18. Near 0 dB, the frequency is

$\omega = 5.4$  rad/sec.



**FIGURE P8.18**

Bode plot for  $G_c(s)G(s) = \frac{12(s+0.5)}{s^2+13s+30}$ .

- P8.19** Examining the frequency response, we postulate a second-order transfer function

$$\frac{\theta(s)}{I(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

From the data we see that  $\phi = -90^\circ$  at  $\omega = 2$ . Using Figure 8.10 in Dorf & Bishop, we determine that  $\omega_n = \omega = 2$ . We also estimate  $\zeta = 0.4$  from Figure 8.10. Thus,

$$\frac{\theta(s)}{I(s)} = \frac{4}{s^2 + 1.6s + 4}.$$

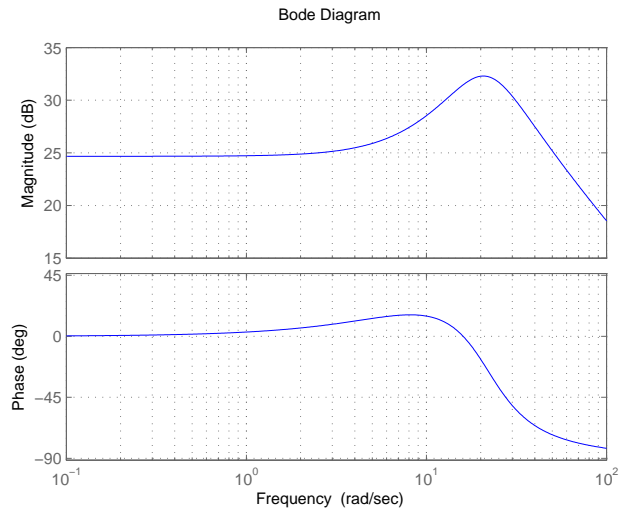
- P8.20** The transfer function is

$$G_c(s)G(s) = \frac{823(s + 9.8)}{s^2 + 22s + 471}.$$

The Bode plot is shown in Figure P8.20. The maximum value of

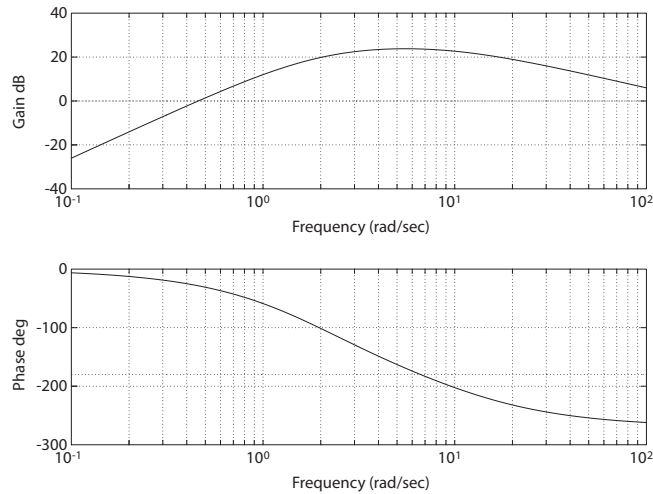
$$20 \log_{10} |G_c(j\omega)G(j\omega)| = 32.3 \text{ dB}$$

occurs at  $\omega = 20.6$  rad/sec and the corresponding phase is  $\phi = -19.6^\circ$ .

**FIGURE P8.20**

Bode plot for  $G_c(s)G(s) = \frac{832(s+9.8)}{s^2+22s+471}$ .

**P8.21** The Bode plot is shown in Figure P8.21. The gain is 24 dB when  $\phi = -180^\circ$

**FIGURE P8.21**

Bode plot for  $G_c(s)G(s) = \frac{-200s^2}{s^3+14s^2+44s+40}$ .

**P8.22** The transfer function is

$$G(s) = \frac{10000(s+1)(s+80)}{s(s+300)(s+9000)} .$$

**P8.23** The transfer function is

$$G(s) = \frac{100(s+20)(s+8000)}{(s+1)(s+80)(s+500)} .$$

The system is type 0 and the steady-state error to a unit step input is

$$e_{ss} = \frac{1}{1+K_p} = 0.0025$$

since

$$K_p = \lim_{s \rightarrow 0} G(s) = 400 .$$

**P8.24** (a) From the Bode plot we see that

$$20 \log_{10} M_{p\omega} = 12$$

or  $M_{p\omega} = 3.981$ . For a second-order system we know that

$$M_{p\omega} = (2\zeta\sqrt{1-\zeta^2})^{-1} .$$

Solving for  $\zeta$  (with  $M_{p\omega} = 3.981$ ) yields  $\zeta = 0.12$ . Also, from the Bode plot,

$$\omega_r = 0.9 \text{ rad/sec} .$$

So,

$$\omega_n = \frac{\omega_r}{\sqrt{1-2\zeta^2}} = 0.91 .$$

Therefore, the second-order approximate transfer function is

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.83}{s^2 + 0.22s + 0.83} .$$

(b) The predicted overshoot and settling time are  $P.O. = 68\%$  and  $T_s = 37$  sec.

**P8.25** The transfer function is

$$G(s) = \frac{100(s+10)}{s^2(s+100)} .$$

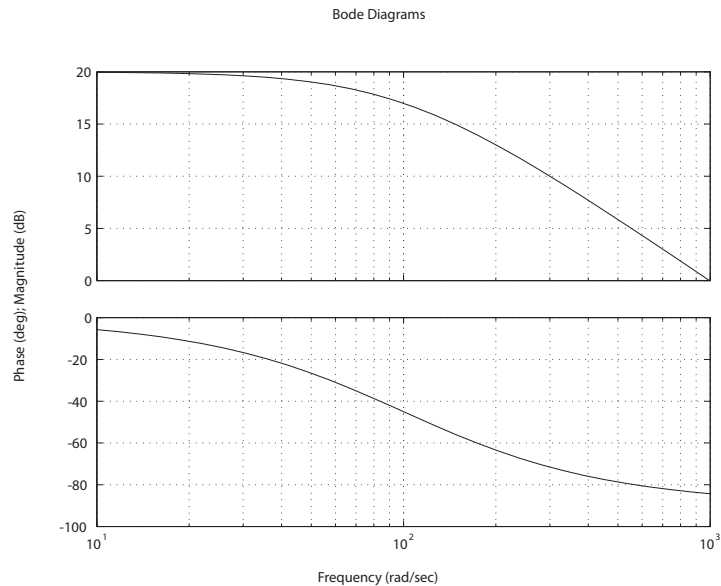
**P8.26** The transfer function is

$$T(s) = \frac{V_o(s)}{V(s)} = \frac{1 + R_1/R_2}{1 + RCs}.$$

Substituting  $R = 10k\Omega$ ,  $C = 1\mu F$ ,  $R_1 = 9k\Omega$ , and  $R_2 = 1k\Omega$  yields

$$T(s) = \frac{10}{1 + 0.01s}.$$

The frequency response is shown in Figure P8.26.

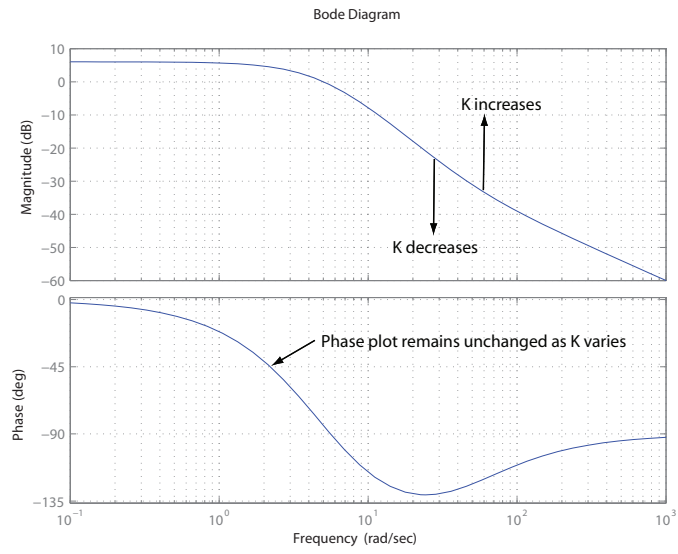


**FIGURE P8.26**  
Bode plot for  $T(s) = \frac{1+R_1/R_2}{1+RCs}$

**P8.27** The frequency response is shown in Figure P8.27.

$K$	0.75	1	10
$ L(j\omega) _{j\omega=0}$ , dB	3.52	12.04	26.02
$\omega_b$ , rad/s	8.3	14.0	33.4
$\omega_c$ , rad/s	3.5	8.7	22.9

**TABLE P8.27** System performance as  $K$  varies.



**FIGURE P8.27**  
Bode plot for  $K = 1$

## Advanced Problems

**AP8.1** The spring-mass-damper system is described by

$$m\ddot{x} + b\dot{x} + kx = p .$$

Taking the Laplace transform (with zero initial conditions) yields

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k} .$$

From Figure AP8.1(b) in Dorf & Bishop, we determine that

$$20 \log \left| \frac{X(j0)}{P(j0)} \right| = 20 \log \left| \frac{1}{k} \right| = -26 \text{dB} .$$

Solving for  $k$  yields

$$k = 19.96 \text{ N/m} .$$

Also,  $\omega_n^2 = k/m$  implies  $m = k/\omega_n^2$ , where  $\omega_n =$  corner frequency = 3.2 rad/sec. So,

$$m = 1.949 \text{ kg} .$$

Comparing Figure AP8.1(b) in Dorf & Bishop to the known standard Bode plot of a second-order system, we estimate  $\zeta \approx 0.32$ . Therefore,

$$b = 2m\zeta\omega_n = 2(1.949)(0.32)(3.2) = 3.992 \text{ N} - \text{s/m} .$$

**AP8.2** The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{Kb}{s + 1 + 0.2Kb} .$$

With  $K = 5$ , we have

$$T(s) = \frac{5b}{s + 1 + b} .$$

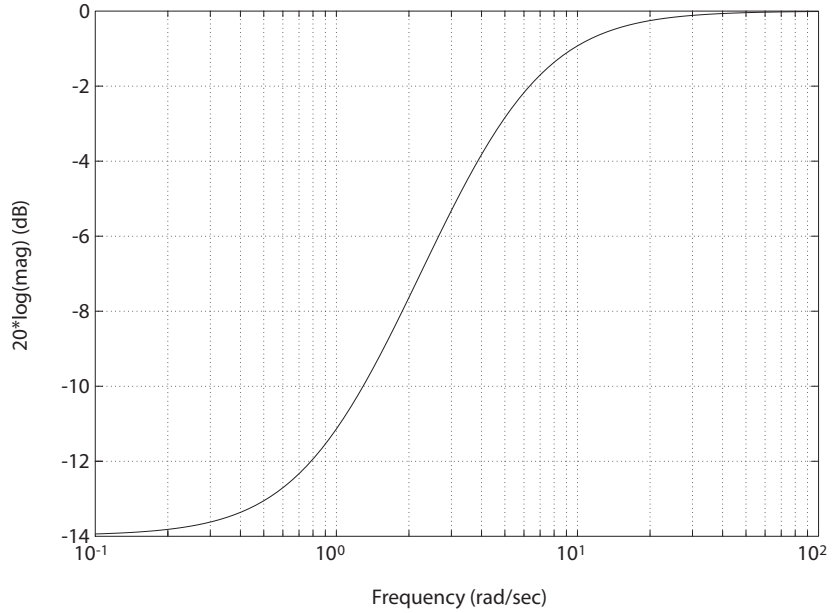
The sensitivity is

$$S_b^T = \frac{\partial T}{\partial b} \frac{b}{T} = \frac{s + 1}{s + 1 + b} .$$

With the nominal value of  $b = 4$ , we have

$$S_b^T = \frac{s + 1}{s + 5} .$$

The sensitivity plot is shown in Figure AP8.2.



**FIGURE AP8.2**  
Bode plot for  $S_b^T(s) = \frac{s+1}{s+5}$ .

**AP8.3** The equation of motion is

$$m\ddot{x} + b\dot{x} + Kx = b\dot{r} + Kr.$$

Taking Laplace transforms yields

$$\frac{X(s)}{R(s)} = \frac{bs + K}{ms^2 + bs + K}.$$

Then, given the various system parameters  $m = 1$  kg,  $b = 4$  Ns/m,  $K = 18$  N/m, we obtain the transfer function:

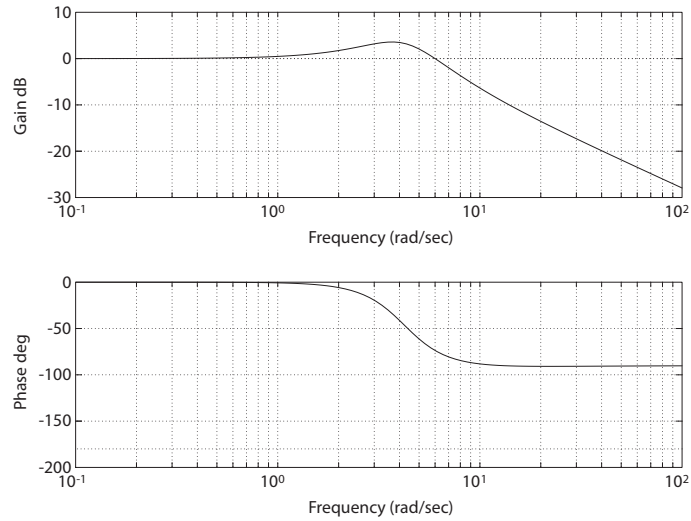
$$\frac{X(s)}{R(s)} = \frac{4s + 18}{s^2 + 4s + 18}.$$

Also,  $\omega_n =$  corner frequency  $= \sqrt{K/m} = \sqrt{18} = 4.243$  rad/s and

$$\zeta = \text{damping ratio} = \frac{b/m}{2\omega_n} = \frac{4}{2(4.243)} = 0.471.$$



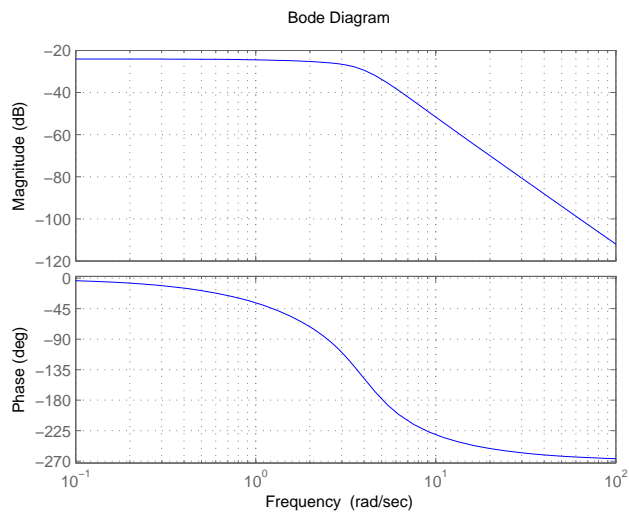
The Bode plot is shown in Figure AP8.3.



**FIGURE AP8.3**

Bode plot for  $G(s) = \frac{4s+18}{s^2+4s+18}$ .

**AP8.4** The Bode plot is shown in Figure AP8.4.



**FIGURE AP8.4**

Bode plot for  $L(s) = \frac{1}{(0.4s+1)(s^2+3.9s+15)}$ .

**AP8.5** The closed-loop transfer function with unity feedback is given by

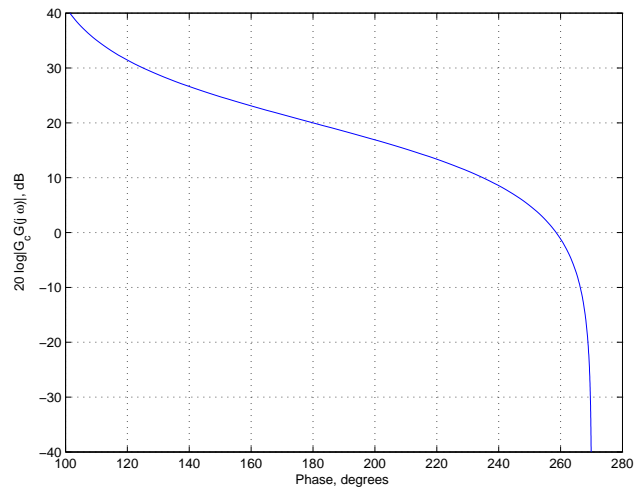
$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{10(s + 1)}{s^2 + 9s + 10} .$$

(a) Solving for  $G_c(s)G(s)$  yields

$$G_c(s)G(s) = \frac{10(s + 1)}{s(s - 1)} .$$

(b) A summary of the plot data (see Figure AP8.5) is presented in Table AP8.5.

(c) The open-loop system is unstable; the closed-loop system is stable.



**FIGURE AP8.5** Log-magnitude-phase curve for  $G_c G(j\omega)$ .

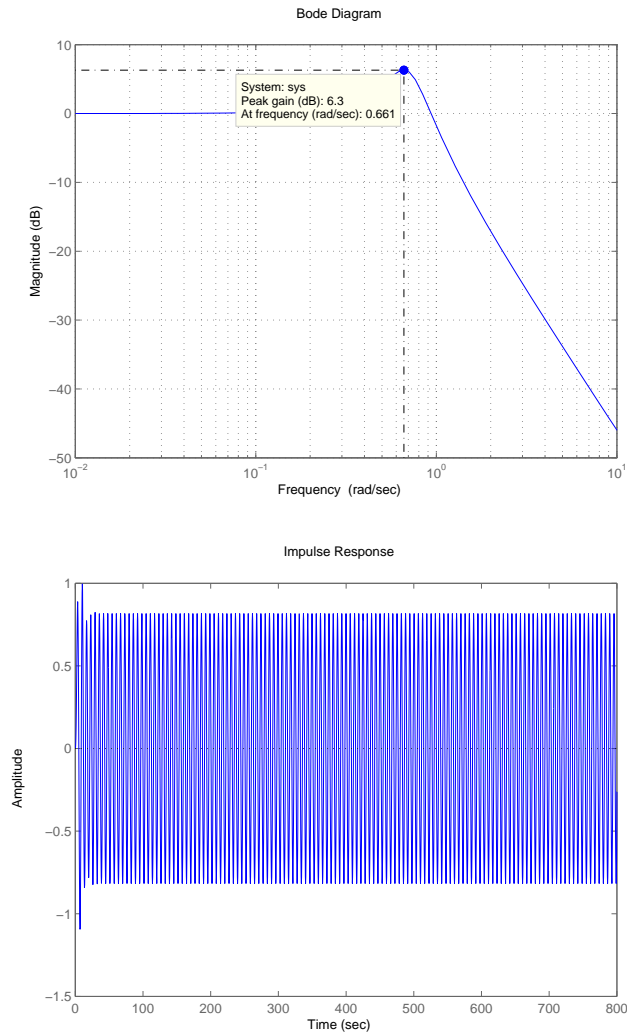
$\omega$	1	10	50	110	500
$20 \log  G_c G $	40	4.85	-13.33	-20.61	-33.94
phase (deg)	101.42	250.17	267.53	268.93	269.77

**TABLE AP8.5** Summary of magnitude and phase for  $\omega = 1, 10, 50, 110, 500$ .

**AP8.6** The transfer function is given by

$$T(s) = \frac{1/m}{s^2 + (b/m)s + (k/m)}.$$

Selecting  $k = 1$  and  $b = 2$  results in the Bode plot magnitude always less than 0 dB. Choosing  $b = \sqrt{2}/2$  leads to a peak response with a sinusoidal input at  $\omega = 0.66$  rad/s. Figure AP8.6a shows the Bode plot and Figure AP8.6b shows the response to a sinusoidal input with frequency  $\omega = 1$  rad/s is less than 1 in the steady-state, as desired.



**FIGURE AP8.6**

(a) Bode plot for  $b/m = 1$  and  $k/m = 1$ . (b) Response to a sinusoidal input.

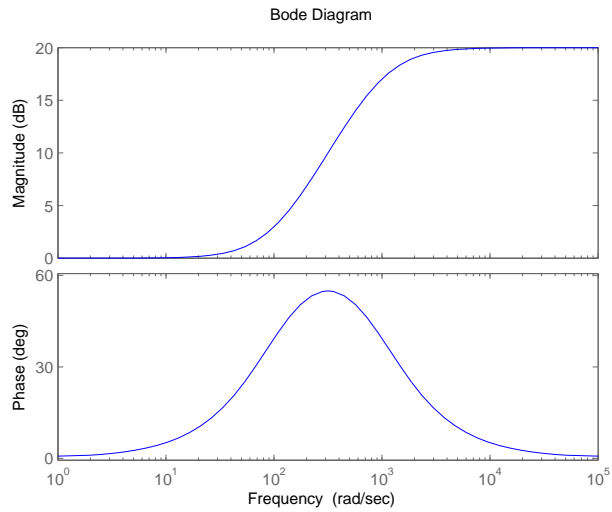
**AP8.7** The transfer function is

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1 + R_2 C_2 s}{1 + R_1 C_1 s} .$$

Substituting  $C_1 = 0.1 \mu F$ ,  $C_2 = 1 mF$ ,  $R_1 = 10 k\Omega$ , and  $R_2 = 10 \Omega$  yields

$$G(s) = \frac{0.01s + 1}{0.001s + 1} .$$

The frequency response is shown in Figure AP8.7.



**FIGURE AP8.7**

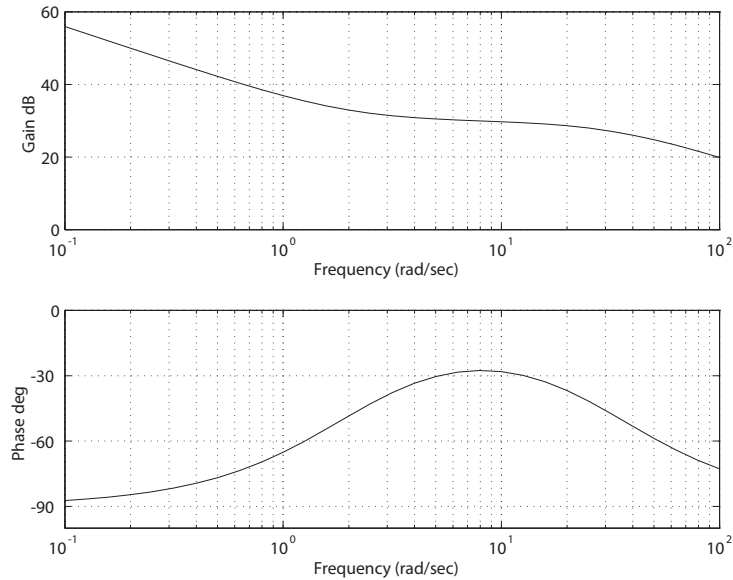
Bode plot for  $G(s) = \frac{0.01s+1}{0.001s+1}$

## Design Problems

**CDP8.1** With the PI controller in the loop, the closed-loop transfer function from the input to the output is

$$\frac{\theta(s)}{R(s)} = \frac{26.035K(s + 2)}{s^2 + (33.1415 + 26.035K)s + 52.07K} ,$$

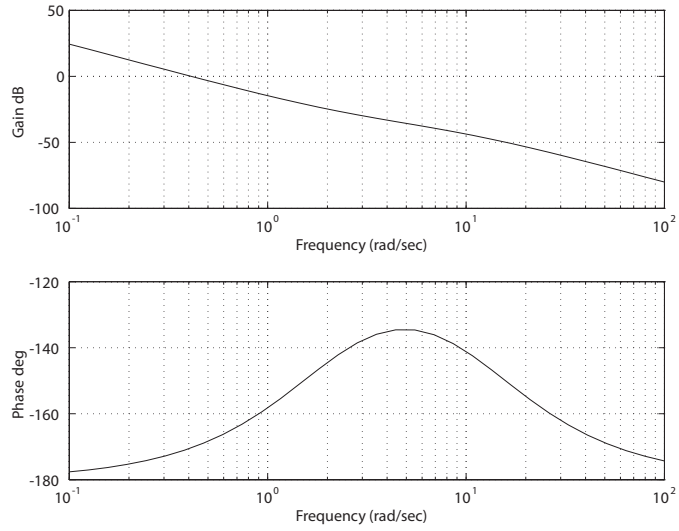
where we switch off the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). The Bode plot is shown below for  $K = 40$ . From the step response we determine that  $P.O. = 0$  and  $T_s = 0.19$ . With  $K = 40$ , the closed-loop poles are both real roots with values of  $s_1 = -1072.6$  and  $s_2 = -1.9$ .



**DP8.1** The loop transfer function is

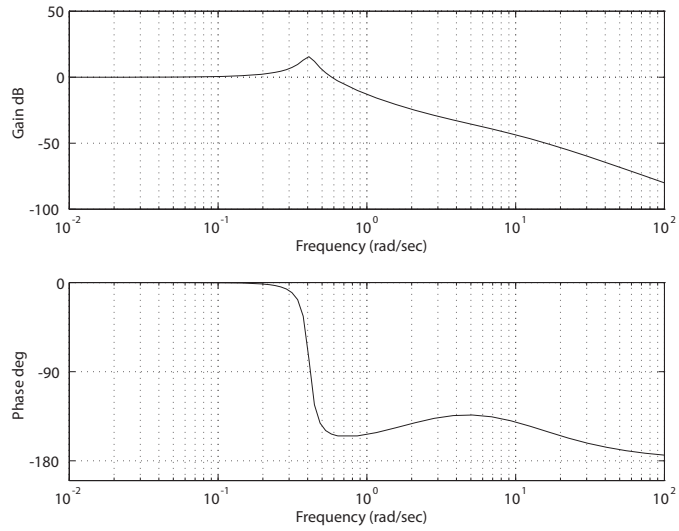
$$L(s) = G_c(s)G(s) = \frac{K(s + 2)}{s^2(s + 12)} .$$

- (a,b) Let  $K = 1$ . The Bode plot of the loop transfer function and the closed-loop transfer functions are shown in Figure DP8.1a and Figure DP8.1b, respectively.
- (c) Let  $K = 50$ . The Bode plot of the loop transfer function and the closed-loop transfer functions are shown in Figure DP8.1c and Figure DP8.1d, respectively.



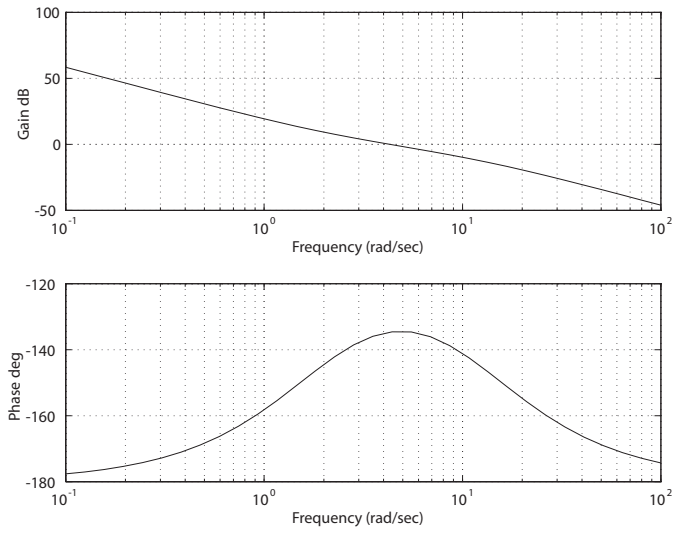
**FIGURE DP8.1**

(a) Bode plot for the loop transfer function  $G_c(s)G(s) = \frac{(s+2)}{s^2(s+12)}$ .

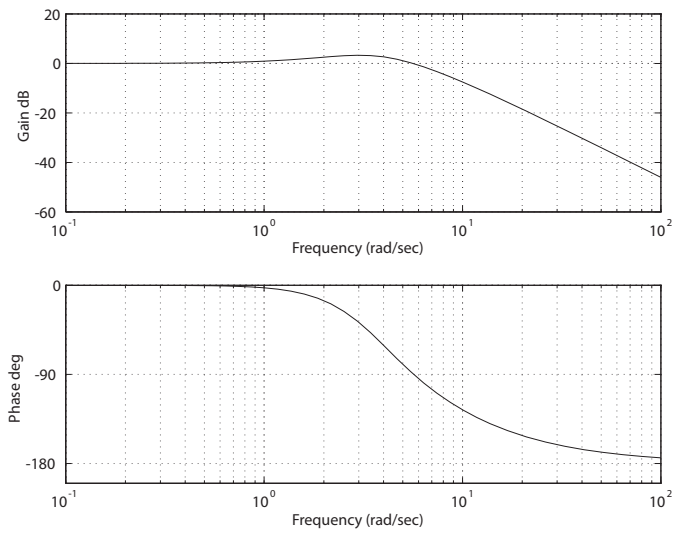


**FIGURE DP8.1**

CONTINUED: (b) Bode plot for the closed-loop  $T(s) = \frac{(s+2)}{s^3+12s^2+s+2}$ .



**FIGURE DP8.1**  
CONTINUED: (c) Bode plot for the loop transfer function  $G_c(s)G(s) = \frac{50(s+2)}{s^2(s+12)}$ .



**FIGURE DP8.1**  
CONTINUED: (d) Bode plot for the closed-loop  $T(s) = \frac{50(s+2)}{s^3+12s^2+50s+100}$ .

- (d) The peak value of  $M_p \leq 2$  occurs for  $14 \leq K \leq 350$ . The maximum bandwidth is achieved for the largest gain  $K$ . Thus, we select  $K = 350$  and the corresponding bandwidth is  $\omega_B = 29$  rad/sec.
- (e) The system is type 2—the steady-state error is zero for a ramp input.

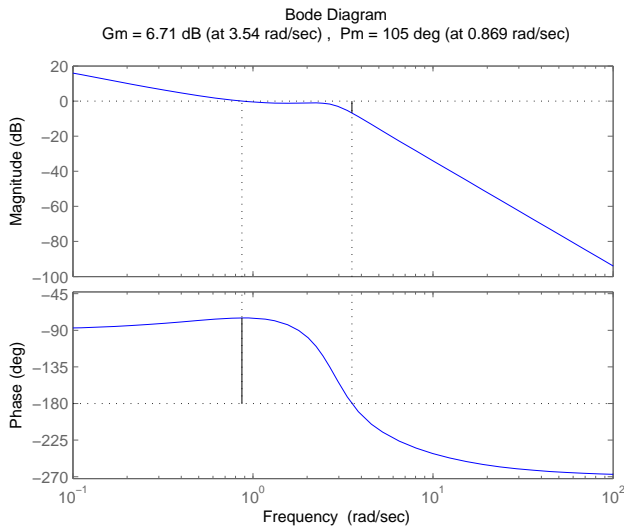
**DP8.2** The open-loop transfer function is

$$G_c(s)G(s) = \frac{20(s + 1)}{s(s + 4)(s^2 + 2s + 8)} .$$

- (a) The phase angle is  $\phi = -180^\circ$  when  $\omega = 3.54$  rad/sec. The magnitude is 0 dB when  $\omega = 0.87$  rad/sec.
- (b) The closed-loop transfer function is

$$T(s) = \frac{20(s + 1)}{s^4 + 6s^3 + 16s^2 + 52s + 20} .$$

The closed-loop Bode plot is shown in Figure DP8.2.



**FIGURE DP8.2**

Bode plot for closed-loop  $T(s) = \frac{20(s+1)}{s^4+6s^3+16s^2+52s+20}$ .

- (c) When  $K = 22$ , we have

$$M_{p\omega} = 4.84\text{dB} , \quad \omega_r = 3.11 , \quad \text{and} \quad \omega_B = 3.78 \text{ rad/sec} .$$

When  $K = 25$ , we have

$$M_{p\omega} = 7.18 \text{ dB} , \quad \omega_r = 3.18 \text{ rad/sec} , \quad \text{and} \quad \omega_B = 3.94 \text{ rad/sec} .$$



(d) Select  $K = 22$ .

**DP8.3** The closed-loop transfer function is

$$T(s) = \frac{K(s + 5)}{s^3 + 7s^2 + 12s + 10 + 5K} .$$

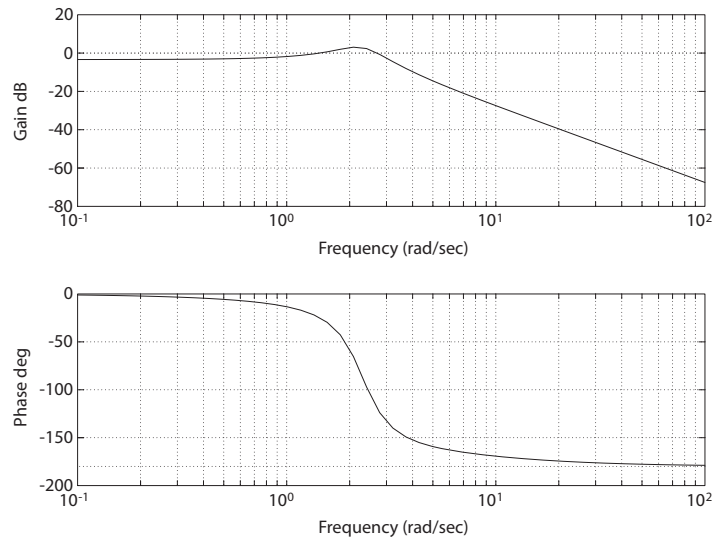
When  $K = 4.2$ , we have  $10 \log_{10} M_{p\omega} = 3$  dB. The system bandwidth is  $\omega_b = 3.7178$  rad/sec. The steady-state tracking error to a unit step input is

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} 1 - T(s) .$$

So,

$$e_{ss} = 1 - \frac{5K}{10 + 5K} = 0.322 ,$$

when  $K = 4.2$ . Since the system is unstable when  $K > 14.8$ , the steady-state error does not exist after  $K = 14.8$ . The Bode plot is shown in Figure DP8.3.



**FIGURE DP8.3**

Bode plot for closed-loop  $T(s) = \frac{K(s+5)}{s^3+7s^2+12s+10+5K}$ , where  $K = 4.2$ .

**DP8.4** We have a second-order loop transfer function

$$G_c(s)G(s) = \frac{K}{(0.3s + 1)(0.6s + 1)} .$$

With  $M_{p\omega} = 1.5$ , we determine that

$$M_{p\omega} = (2\zeta\sqrt{1 - \zeta^2})^{-1} \quad \text{or} \quad \zeta = 0.3568 .$$

Now the characteristic equation is

$$s^2 + 5s + 5.56(1 + K) = 0 .$$

So, solving  $2\zeta\omega_n = 5$  yields  $\omega_n = 7$ . Therefore,

$$K = 0.18\omega_n^2 - 1 = 7.82 .$$

The closed-loop transfer function is

$$T(s) = \frac{K}{K + 1} \frac{5.56(K + 1)}{s^2 + 5s + 5.56(K + 1)} .$$

So, the overall gain of the standard second-order system will be attenuated by the factor  $K/(K + 1)$ . To compensate, we amplify the gain by a small factor. Thus we choose  $K = 10$ . The bandwidth is  $\omega_b = 11.25$  rad/sec and the peak magnitude is  $M_{p\omega} = 1.5$ .

**DP8.5** From the Bode plot of  $G(s)$  we find that there exists two poles, at approximately  $\omega = 1$  rad/sec and  $\omega = 10$  rad/sec. Then, by examining the Bode plot we estimate

$$G(s) = \frac{10}{(s + 1)(s + 10)} .$$

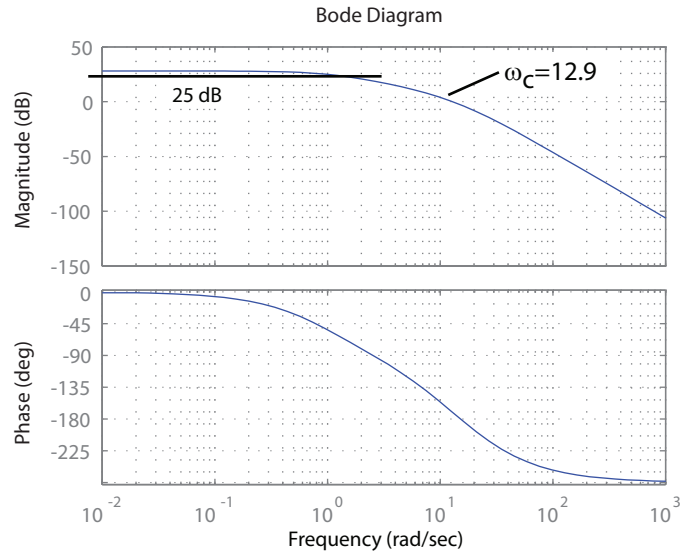
We use a scale factor of 10 because at low frequency the Bode plot has magnitude 0 dB (or a DC gain of 1). With  $G(s)$  as above, we can utilize the controller

$$G_c(s) = \frac{500}{s + 20}$$

yielding a crossover

$$\omega_c = 12.9 \text{ rad/sec}$$

and a magnitude of at least 25 dB for  $\omega < 0.1$  rad/sec. Figure DP8.5 shows the compensator Bode plot of  $G_c(s)G(s)$ .

**FIGURE DP8.5**

Bode Diagram for  $G(s)G_c(s) = \frac{5000}{(s+1)(s+10)(s+20)}$ .

- DP8.6** Let  $K = -1$  to meet the steady-state tracking error requirement and  $p = 2\zeta$ , where  $\zeta = 0.69$  to obtain a 5% overshoot. The system is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1.38 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

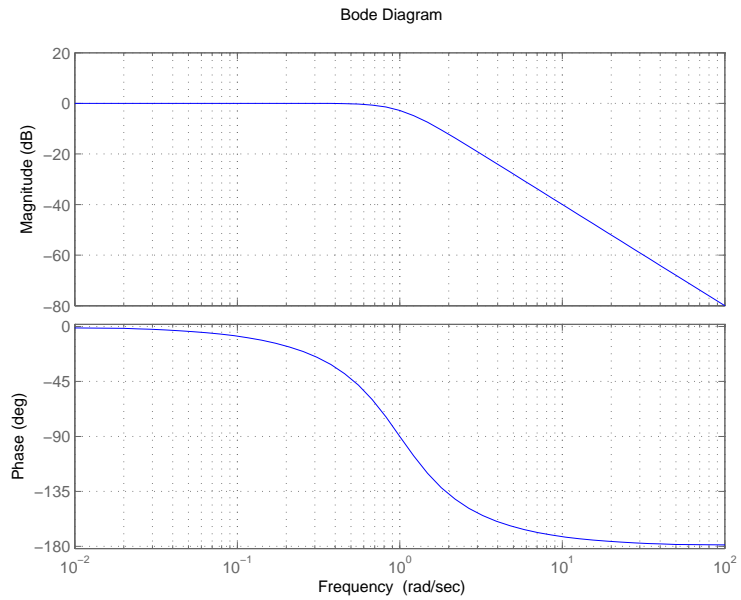
The characteristic polynomial is

$$s^2 + 1.38s + 1 = 0.$$

The associated damping ratio is  $\zeta = 0.69$  and the natural frequency is  $\omega_n = 1$  rad/s. Using the approximation

$$\omega_b = (-1.19\zeta + 1.85)\omega_n$$

we obtain  $\omega_b \approx 1.028$  rad/s. The Bode plot is shown in Figure DP8.6. The bandwidth is  $\omega_b = 1.023$  rad/s.



**FIGURE DP8.6**  
Bode diagram for  $K = -1$  and  $p = 1.38$ .

**DP8.7** A viable controller is

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = 5.5 + \frac{3.33}{s} + 3.5s.$$

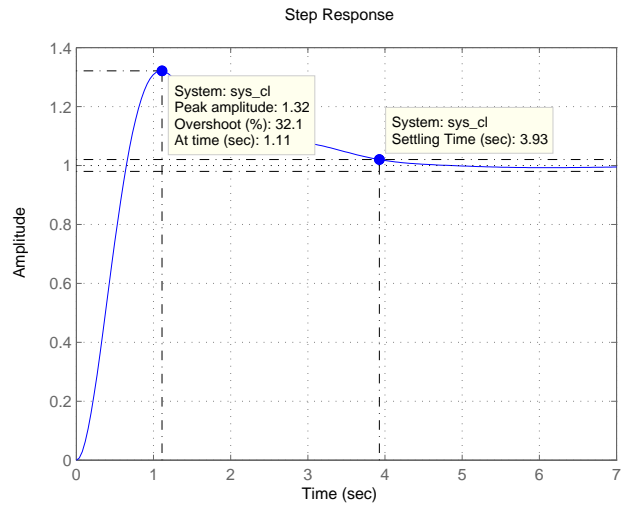
The loop transfer function is

$$G_c(s)G(s) = \frac{10.5s^2 + 16.5s + 10}{s^2(s^2 + 4s + 5)}$$

and computing  $K_a$  yields

$$K_a = \lim_{s \rightarrow 0} s^2 G_c(s)G(s) = \frac{10}{5} = 2,$$

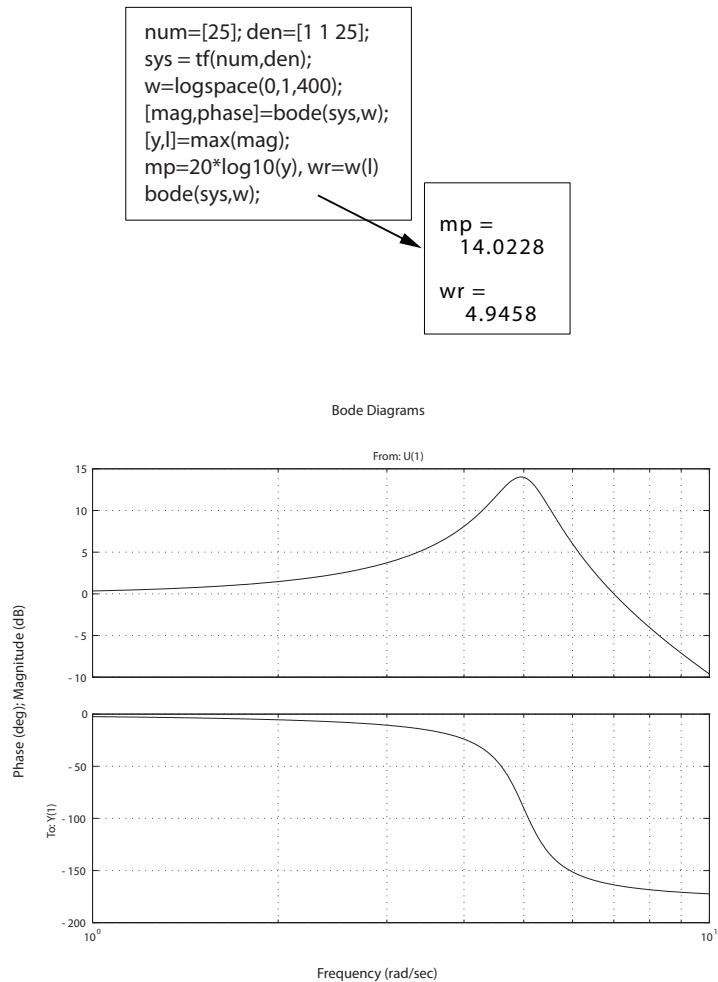
as desired. The phase margin is  $P.O. = 44.35^\circ$  and the bandwidth is  $\omega_b = 4.5$  rad/sec. The step response is shown in Figure DP8.7.

**FIGURE DP8.7**

Step response for  $K_P = 5.5$ ,  $K_I = 3.33$ , and  $K_D = 3.5$ .

## Computer Problems

**CP8.1** The m-file script and Bode plot are shown in Figure CP8.1. The script automatically computes  $M_{p\omega}$  and  $\omega_r$ .



**FIGURE CP8.1**

Generating a Bode plot with the **bode** function.

**CP8.2** The m-file script to generate the Bode plots is shown in Figure CP8.2a. The Bode plots are presented in Figures CP8.2b-CP8.2e. The transfer functions are

$$(a) : G(s) = \frac{1000}{(s + 10)(s + 100)} ; \quad (b) : G(s) = \frac{s + 100}{(s + 2)(s + 25)} ;$$

$$(c) : G(s) = \frac{100}{s^2 + 2s + 50} ; \quad (d) : G(s) = \frac{s - 6}{(s + 3)(s^2 + 12s + 50)} .$$

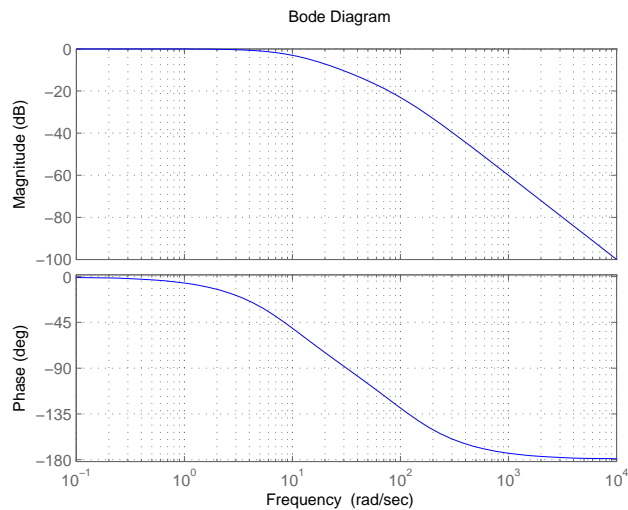
```

% Part (a)
num=[1000]; den=conv([1 10],[1 100]); sys1=tf(num,den);
sys = tf(num,den);
figure(1), bode(sys1), grid
% Part (b)
num=[1 100]; den=conv([1 2],[1 25]); sys2=tf(num,den);
sys = tf(num,den);
figure(2), bode(sys2), grid
% Part (c)
num=[100]; den=[1 2 50]; sys3=tf(num,den);
sys = tf(num,den);
figure(3), bode(sys3), grid
% Part (d)
num=[1 -6]; den=conv([1 3],[1 12 50]); sys4=tf(num,den);
sys = tf(sys);
figure(4), bode(sys4), grid

```

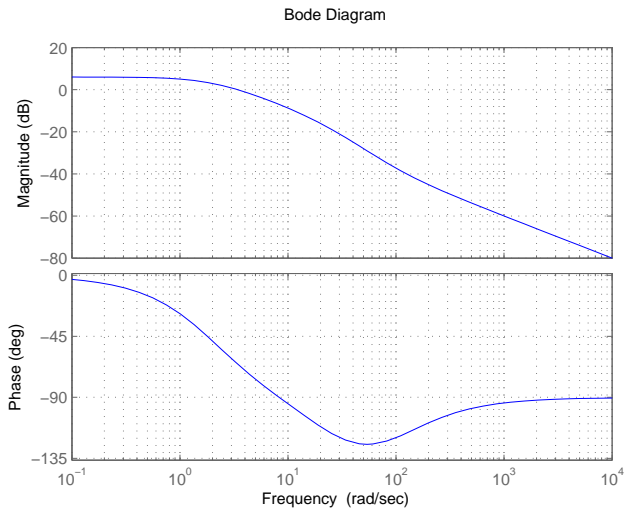
**FIGURE CP8.2**

(a) Script to generate the four Bode plots.



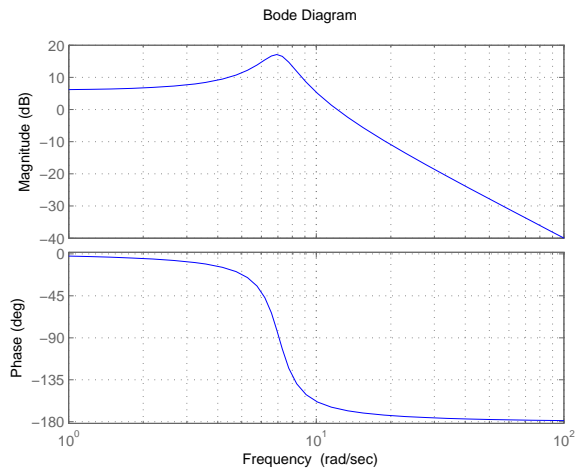
**FIGURE CP8.2**

CONTINUED: (b) Bode plot for  $G(s) = \frac{1000}{(s+10)(s+100)}$ .



**FIGURE CP8.2**

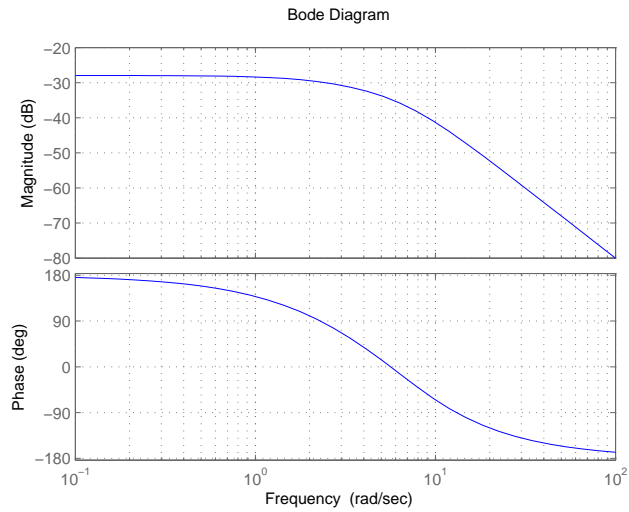
CONTINUED: (c) Bode plot for  $G(s) = \frac{s+100}{(s+2)(s+25)}$ .



**FIGURE CP8.2**

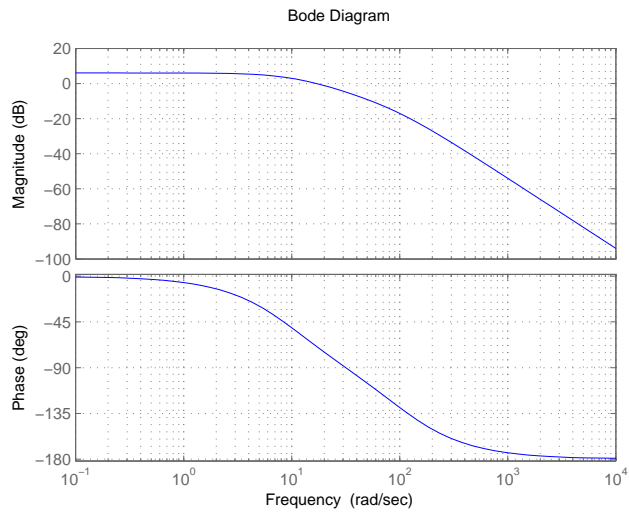
CONTINUED: (d) Bode plot for  $G(s) = \frac{100}{s^2+2s+50}$ .



**FIGURE CP8.2**

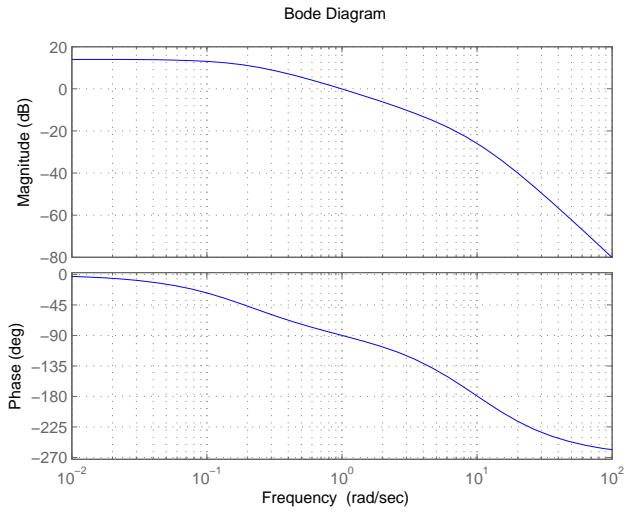
CONTINUED: (e) Bode plot for  $G(s) = \frac{s-6}{(s+3)(s^2+12s+50)}$ .

**CP8.3** The Bode plots are shown in Figure CP8.3(a-d) with the transfer functions listed in the caption. The crossover frequency for (a) is 17 rad/sec.

**FIGURE CP8.3**

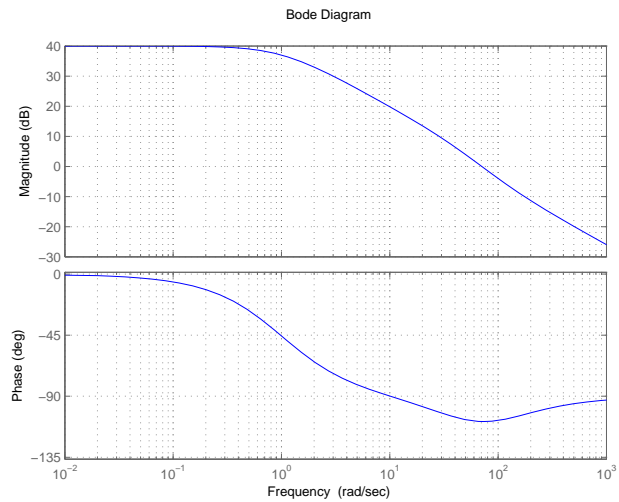
(a) Bode plot for  $G(s) = \frac{2000}{(s+10)(s+100)}$ .

The crossover frequency for (b) is 0.99 rad/sec.



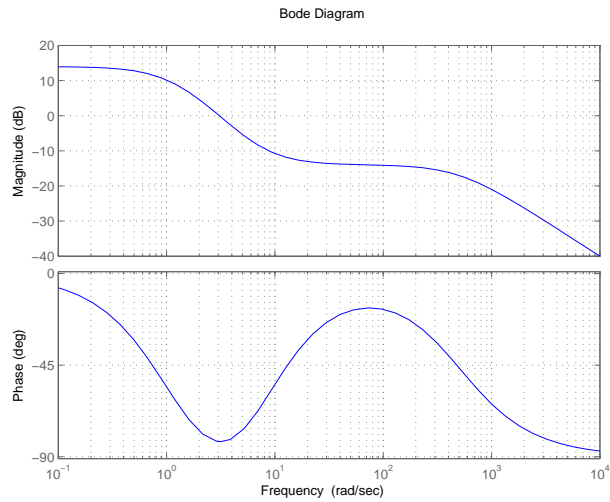
**FIGURE CP8.3**  
CONTINUED: (b) Bode plot for  $G(s) = \frac{100}{(s+1)(s^2+10s+2)}$ .

The crossover frequency for (c) is 70.7 rad/sec.



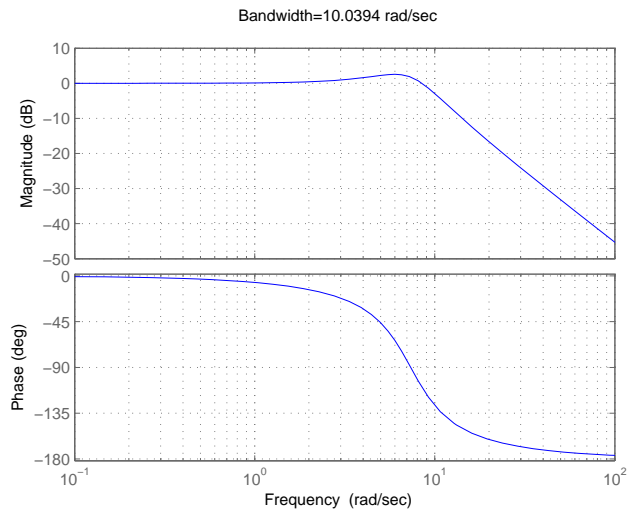
**FIGURE CP8.3**  
CONTINUED: (c) Bode plot for  $G(s) = \frac{50(s+100)}{(s+1)(s+50)}$ .

The crossover frequency for (d) is 3.1 rad/sec.

**FIGURE CP8.3**

CONTINUED: (d) Bode plot for  $G(s) = \frac{100(s^2 + 14s + 50)}{(s+1)(s+2)(s+500)}$ .

**CP8.4** The m-file script and Bode plot are shown in Figure CP8.4a and b. The bandwidth is  $\omega_b = 10$  rad/sec.

**FIGURE CP8.4**

(a) Bode plot for  $T(s) = \frac{54}{s^2 + 6s + 54}$ .

```

numg=[54]; deng=[1 6 0];
sys_o = tf(numg,deng);
sys_cl = feedback(sys_o,1)
wb=bandwidth(sys_cl)
bode(sys_cl), grid
titlename=strcat('Bandwidth= ', num2str(wb), ' rad/sec')
title(titlename)

```

**FIGURE CP8.4**

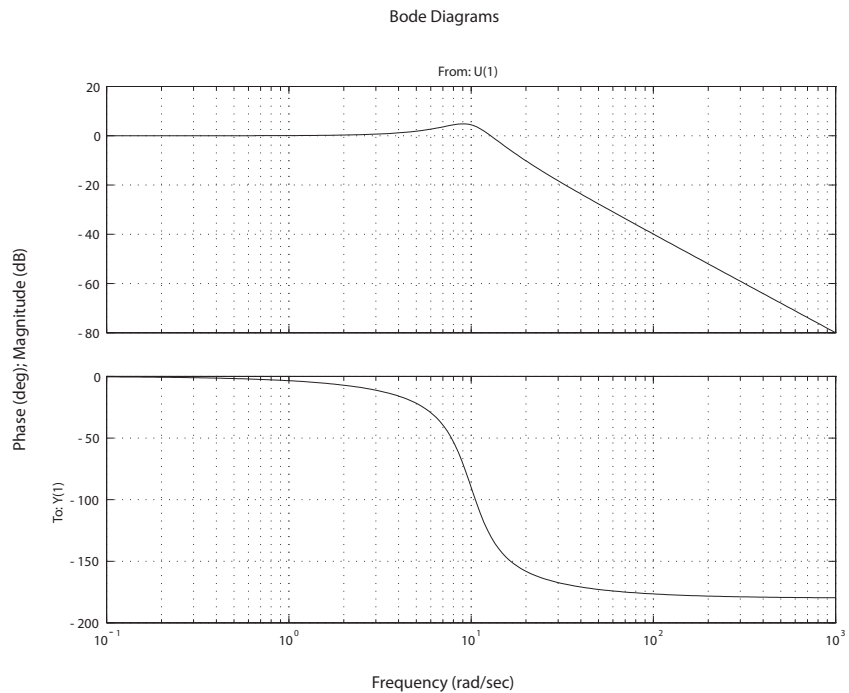
CONTINUED: (b) M-file script to obtain the closed-loop Bode plot.

**CP8.5** The Bode plot of the closed-loop system is shown in Figure CP8.5. The closed-loop transfer function is

$$T(s) = \frac{100}{s^2 + 6s + 100} .$$

(a) From the Bode plot we determine that

$$M_{p\omega} \approx 5 \text{ dB} \quad \text{and} \quad \omega_r \approx 9 \text{ rad/sec} .$$

**FIGURE CP8.5**

Closed-loop system Bode plot.

(b) From Equations (8.36) and (8.37) in Dorf & Bishop, we find that

$$\zeta \approx 0.28 \quad \text{and} \quad \omega_r/\omega_n \approx 0.92$$

which implies that

$$\omega_n = \omega_r/0.92 = 9.8 \text{ rad/sec} .$$

(c) From  $T(s)$  we find that

$$\omega_n = 10 \text{ rad/sec} \quad \text{and} \quad \zeta = 0.3 .$$

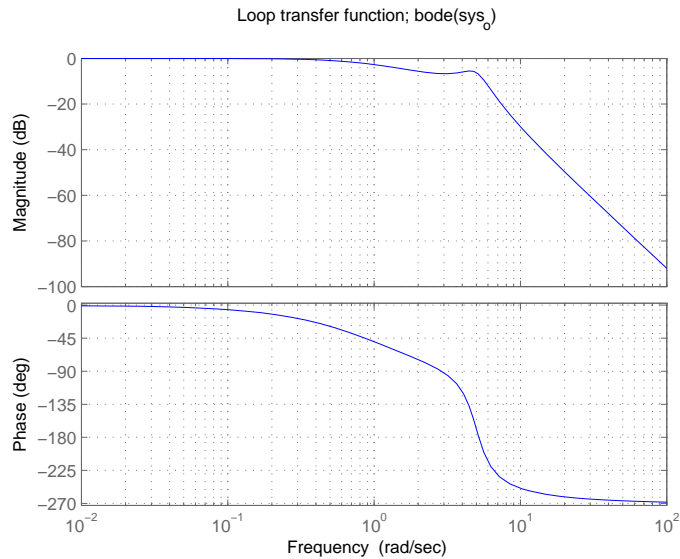
The actual values and the estimated values compare very well.

**CP8.6** The open-loop and closed-loop Bode plots are shown in Figure CP8.6a and b. The open-loop and closed-loop transfers functions are

$$G_c(s)G(s) = \frac{25}{s^3 + 3s^2 + 27s + 25}$$

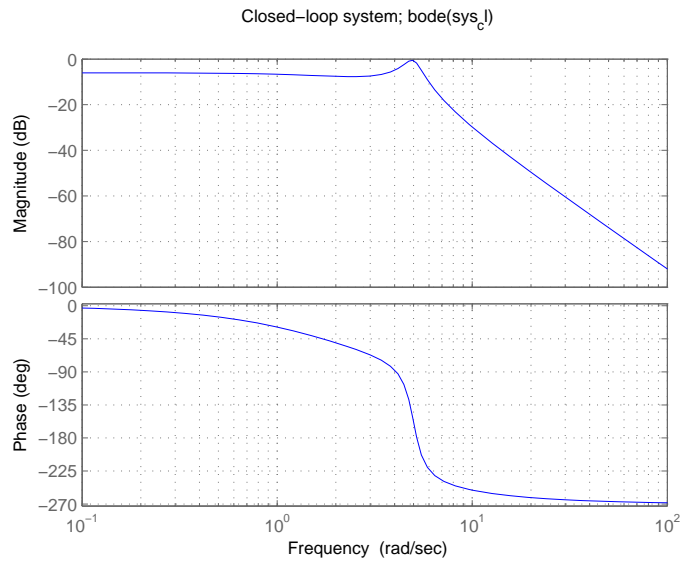
and

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{25}{s^3 + 3s^2 + 27s + 50} .$$



**FIGURE CP8.6**

(a) Open-loop system Bode plot for  $G_c(s)G(s) = \frac{25}{s^3 + 3s^2 + 27s + 25}$ .

**FIGURE CP8.6**

CONTINUED: (b) Closed-loop system Bode plot  $T(s) = \frac{25}{s^3 + 3s^2 + 27s + 50}$ .

**CP8.7** The m-file script and plot of  $\omega_b$  versus  $p$  are shown in Figure CP8.7a and b.

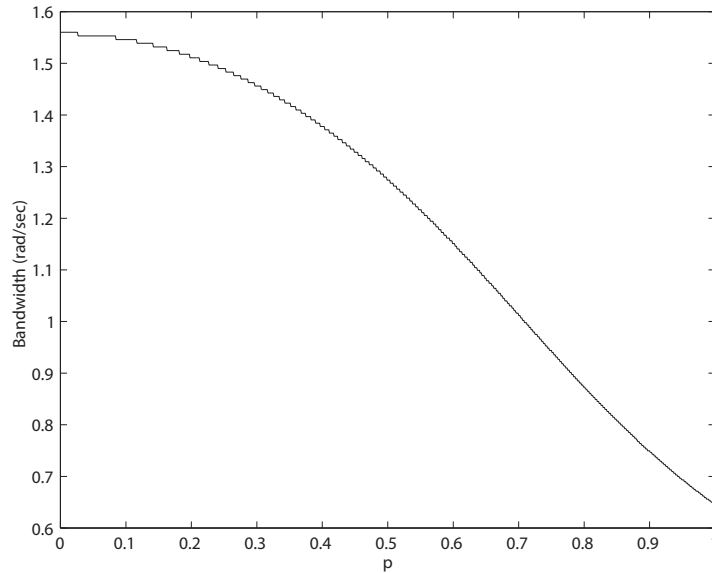
```

p=[0:0.001:1];
w=logspace(-1,1,1000);
n=length(p);
for i=1:n
    num=[1]; den=[1 2*p(i) 0];
    sys = tf(num,den);
    sys_cl = feedback(sys,1);
    [mag,phase,w]=bode(sys_cl,w);
    a=find(mag<0.707); wb(i)=w(a(1));
end
plot(p,wb)
xlabel('p'), ylabel('Bandwidth (rad/sec)')

```

**FIGURE CP8.7**

(a) M-file script to generate plot of  $\omega_b$  versus  $p$ .



**FIGURE CP8.7**  
CONTINUED: (b) Plot of  $\omega_b$  versus  $p$ .

**CP8.8** The transfer function from  $T_d(s)$  to  $\theta(s)$  is

$$\theta(s)/T_d(s) = \frac{-0.01(s + 10)}{s^3 + 10s^2 + (0.01K - 10.791)s - 107.91 + 0.05K} .$$

Using the final value theorem and  $T_d(s) = 1/s$ , we determine that

$$\lim_{s \rightarrow 0} s\theta(s) = \frac{-0.1}{-107.91 + 0.05K} .$$

The design specifications require that

$$|e_{ss}| < 0.1^\circ .$$

So, solving for  $K$  yields

$$K > 3300 .$$

We can select

$$K = 3300$$

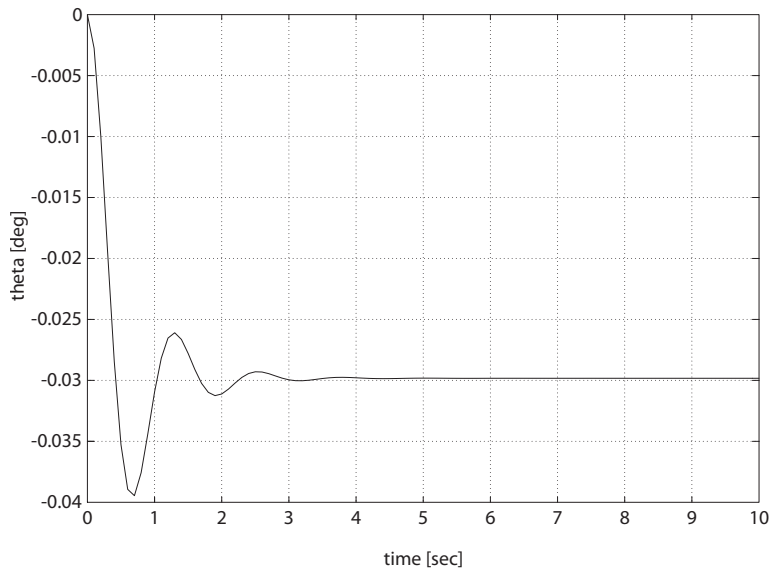
as the initial value of  $K$  for the design. The m-file script is shown in Figure CP8.8a. For the design shown, the final selection for the gain is  $K = 6000$ . The disturbance response is shown in Figure CP8.8b.

```

Mb=100; Ms=10; L=1; g=9.81; a=5; b=10;
%
K=6000; % Final design value of K
%
numg=[-1/Mb/L]; deng=[1 0 -(Mb+Ms)*g/Mb/L];
sysg = tf(numg,deng);
numc=-K*[1 a]; denc=[1 b];
sysc = tf(numc,denc);
%
% Part (a)
%
sys = feedback(sysg,sysc);
w=logspace(0,1,400);
bode(sys,w)
[mag,phase]=bode(sys,w);
[M,l]=max(mag);
MpDb=20*log10(M)-20*log10(mag(1)) % Mpw in decibels
wr=w(l) % Mpw and peak frequency
%
% Part (b)
%
% From Eqs. (8.35) and (8.37)
Mpw=10^(MpDb/20);zeta=sqrt((1-sqrt(1-(1/Mpw^2)))/2);
wn=wr/sqrt(1-2*zeta^2);
ts=4/zeta/wn
po=100*exp(-zeta*pi/sqrt(1-zeta^2))
%
% Part (c)
%
t=[0:0.1:10];
[y,x]=step(sys,t);
plot(t,y*180/pi)
xlabel('time [sec]')
ylabel('theta [deg]')
grid
    
```

MpDb =  
4.0003  
wr =  
4.7226

meets specs  
ts =  
2.23  
po =  
32.75



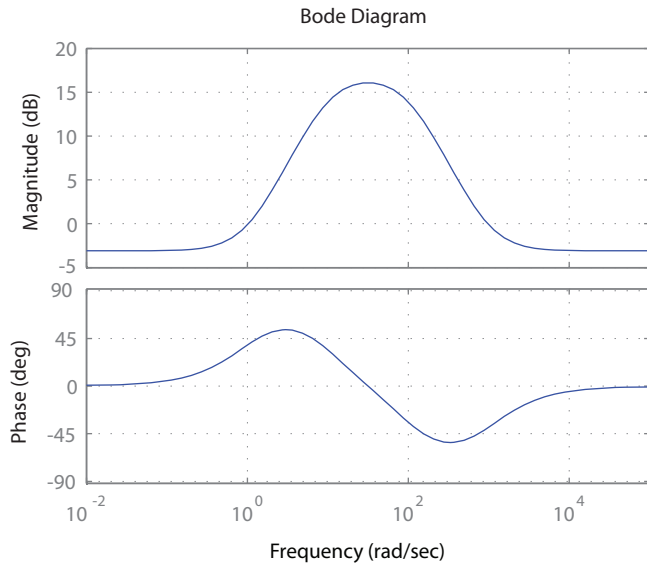
**FIGURE CP8.8**  
(a) Design script. (b) Disturbance response - meets all specs!



**CP8.9** A viable filter is

$$G(s) = 0.7 \frac{(s + 1000)(s + 1)}{(s + 100)(s + 10)}.$$

The Bode plot is shown in Figure CP8.9



**FIGURE CP8.9**  
Bode plot for  $G(s) = 0.7 \frac{(s+1000)(s+1)}{(s+100)(s+10)}$

# Stability in the Frequency Domain

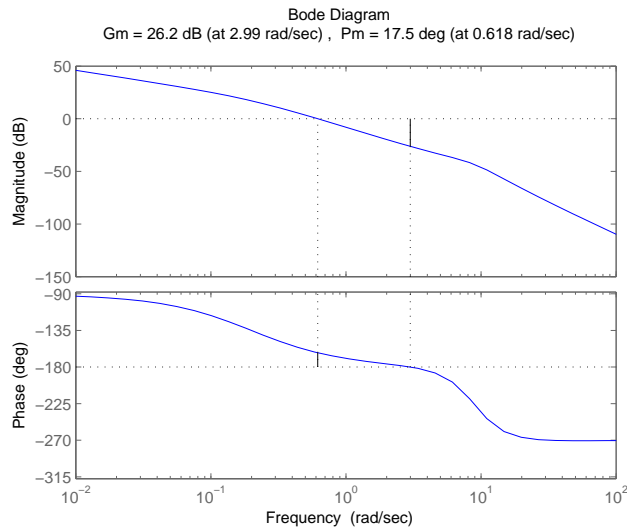
## Exercises

- E9.1** The Bode plot for the transfer function  $G_c(s)G(s)$  is shown in Figure E9.1, where

$$G_c(s)G(s) = \frac{2(1 + s/10)}{s(1 + 5s)(1 + s/9 + s^2/81)} .$$

The gain and phase margins are

$$G.M. = 26.2 \text{ dB and } P.M. = 17.5^\circ .$$



**FIGURE E9.1**  
Bode Diagram for  $G_c(s)G(s) = \frac{2(1+s/10)}{s(1+5s)(1+s/9+s^2/81)}$ .

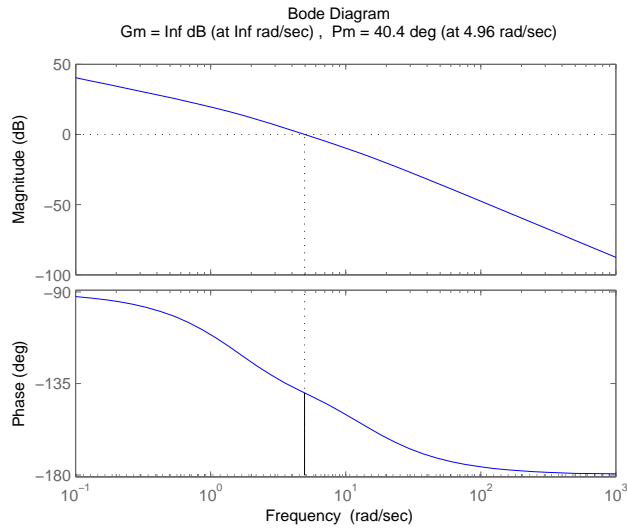
**E9.2** The loop transfer function is

$$G_c(s)G(s) = \frac{10.5(1 + s/5)}{s(1 + s/2)(1 + s/10)} .$$

The Bode plot is shown in Figure E9.2. The phase margin is

$$P.M. = 40.4^\circ$$

at  $\omega_c = 4.96$  rad/sec.



**FIGURE E9.2**

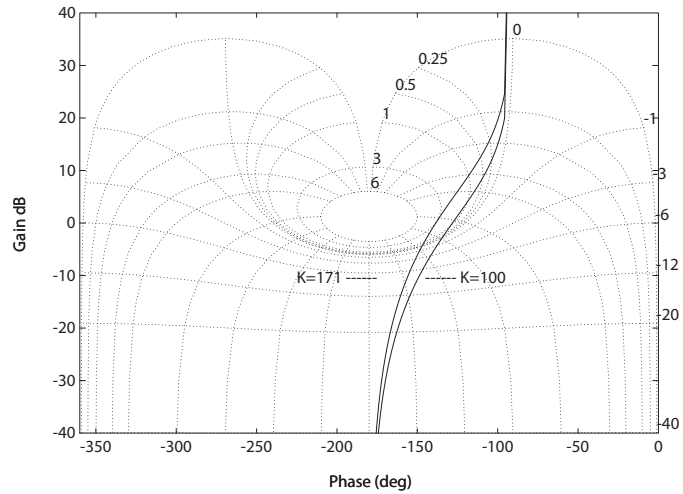
Bode Diagram for  $G_c(s)G(s) = \frac{10.5(1+s/5)}{s(1+s/2)(1+s/10)}$ .

**E9.3** The phase margin  $P.M. \approx 75^\circ$  at 200 kHz. We estimate the  $-180^\circ$  phase angle at 2 MHz, so the gain margin is  $G.M. \approx 25$  dB.

**E9.4** The loop transfer function is

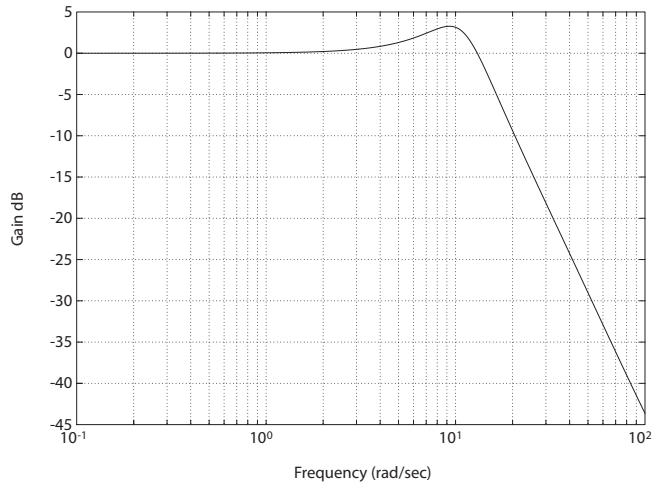
$$G_c(s)G(s) = \frac{100}{s(s + 10)} .$$

The Nichols diagram is shown in Figure E9.4. When the gain is raised by 4.6 dB,  $M_{p_\omega} = 3$  and the resonant frequency is  $\omega_R = 11$  rad/sec.



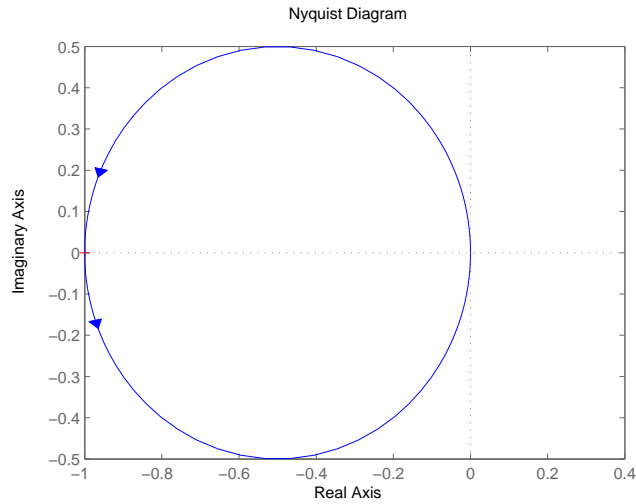
**FIGURE E9.4** Nichols Diagram for  $G_c(s)G(s) = \frac{K}{s(s+10)}$ , where  $K = 100$  and  $K = 171$ .

- E9.5** (a) The  $G.M. \approx 5$  dB and the  $P.M. \approx 10^\circ$ . (b) Lower the gain by 10 dB to obtain  $P.M. \approx 60^\circ$ .
- E9.6** The Bode plot of the closed-loop transfer function is shown in Figure E9.6. The value of  $M_{p\omega} = 3$  dB. The phase margin is  $P.M. = 40^\circ$  when  $K = 50$ .



**FIGURE E9.6** Closed-loop Bode Diagram for  $T(s) = \frac{50(s+100)}{s^3+50s^2+450s+5000}$ .

**E9.7** The Nyquist plot is shown in Figure E9.7 for  $K = 5$ ; the plot is a circle with diameter =  $K/5$ . For  $K > 5$ , we have  $P = 1$  and  $N = -1$  (ccw as

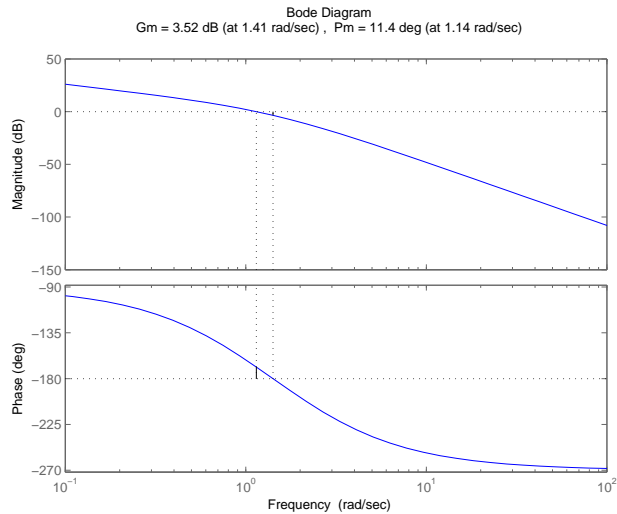


**FIGURE E9.7**

Nyquist Diagram for  $G_c(s)G(s) = \frac{K}{s-5}$ , where  $K = 5$ .

shown). So  $Z = N + P = -1 + 1 = 0$  and the system is stable for  $K > 5$ .

**E9.8** (a) When  $K = 4$ , the  $G.M. = 3.5$  dB. This is illustrated in Figure E9.8.

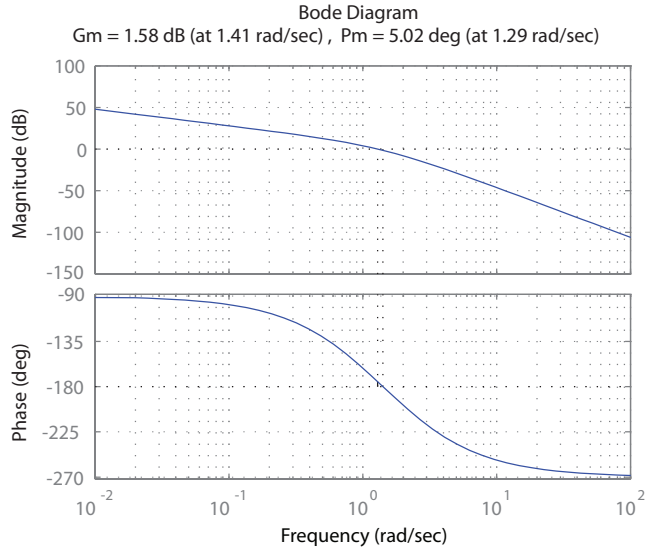


**FIGURE E9.8**

Bode Diagram for  $G_c(s)G(s) = \frac{K}{s(s+1)(s+2)}$ , where  $K = 4$ .

(b) The new gain should be  $K = 1$  for a gain margin  $G.M. = 16$  dB.

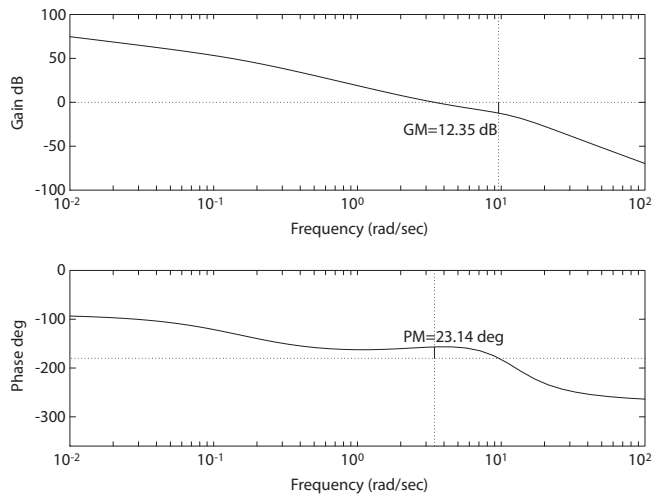
**E9.9** For  $K = 5$ , the phase margin  $P.M. = 5^\circ$  as shown in Figure E9.9.



**FIGURE E9.9**

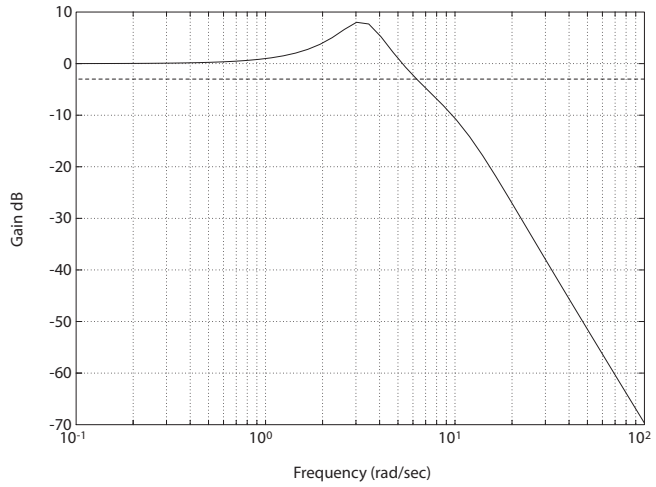
Bode Diagram for  $G_c(s)G(s) = \frac{K}{s(s+1)(s+2)}$ , where  $K = 5$ .

**E9.10** The Bode plot is shown in Figure E9.10a. The closed-loop frequency



**FIGURE E9.10**

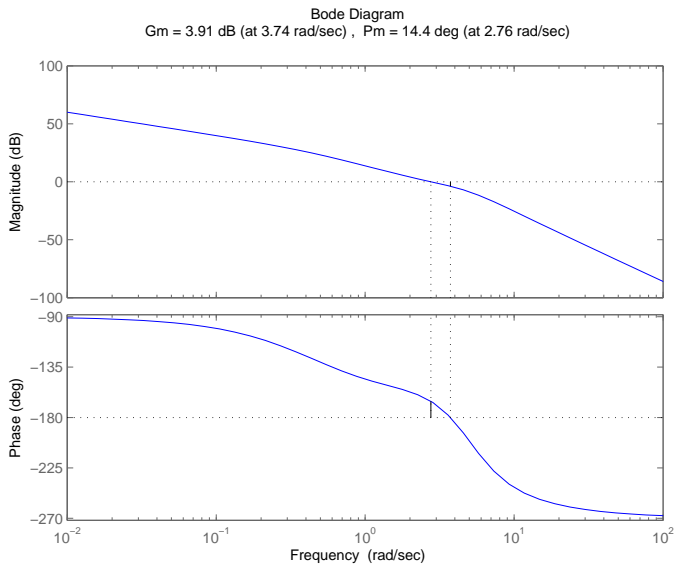
(a) Bode Diagram for  $G_c(s)G(s) = \frac{326s+1304}{s^4+14.76s^3+151.3s^2+23.84s}$ .



**FIGURE E9.10**  
CONTINUED: (b) Closed-loop frequency response:  $\omega_B = 6$  rad/sec.

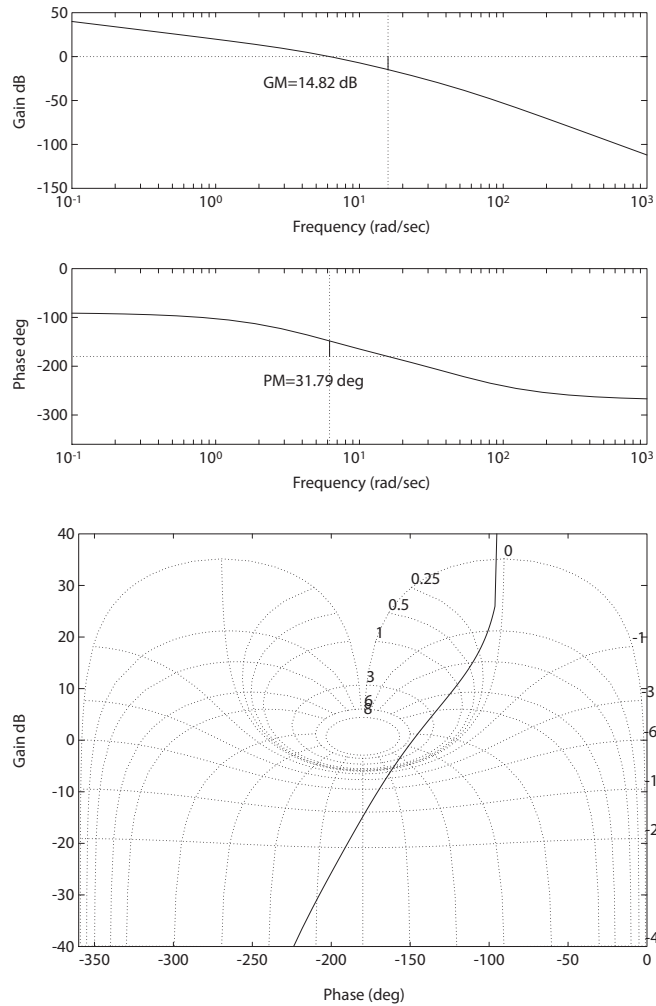
response is shown in Figure E9.10b. The bandwidth is  $\omega_B = 6$  rad/sec.

**E9.11** The Bode plot is shown in Figure E9.11. The system is **stable**.



**FIGURE E9.11**  
Bode Diagram for  $G_c(s)G(s) = \frac{10(1+0.4s)}{s(1+2s)(1+0.24s+0.04s^2)}$ .

- E9.12** We select the gain  $K = 10$  to meet the 10% steady-state tracking error specification for a ramp input. The Bode plot and Nichols chart are shown in Figures E9.12a and E9.12b, respectively.

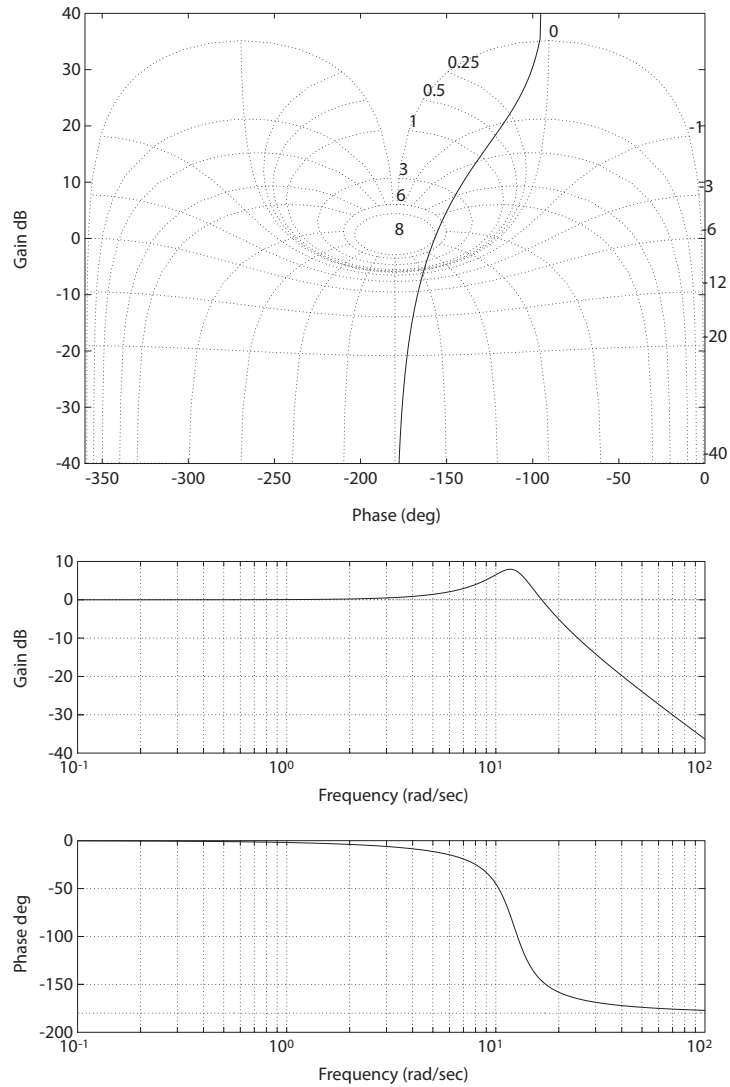


**FIGURE E9.12**

(a) Bode Diagram for  $G_c(s)G(s) = \frac{10}{s(0.02s+1)(0.2s+1)}$ . (b) Nichols chart for  $G_c(s)G(s) = \frac{10}{s(0.02s+1)(0.2s+1)}$ .

- E9.13** (a) The Nichols diagram is shown in Figure E9.13a and  $M_{p\omega} = 7.97$  dB.  
 (b) The closed-loop Bode plot is shown in Figure E9.13b. The bandwidth  $\omega_B = 18.65$  rad/sec and the resonant frequency is  $\omega_r = 11.69$  rad/sec.



**FIGURE E9.13**

(a) Nichols Diagram for  $G_c(s)G(s) = \frac{150}{s(s+5)}$ . (b) Closed-loop Bode Diagram for  $T(s) = \frac{150}{s^2+5s+150}$ .

(c) From  $M_{p\omega} = 8$  dB, we estimate  $\zeta = 0.2$ , so the expected  $P.O. = 52\%$ .

- E9.14** (a) The peak resonance  $M_{p\omega} = 6$  dB.  
 (b) The resonant frequency is  $\omega_r = \omega_2 = 3$  rad/sec.  
 (c) The bandwidth is  $\omega_B = \omega_4 = 10$  rad/sec.  
 (d) The phase margin is  $P.M. = 30^\circ$ .

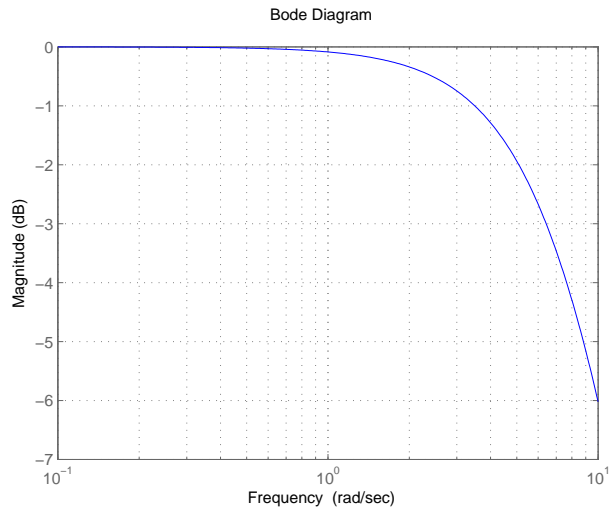
**E9.15** The loop transfer function is

$$G_c(s)G(s) = \frac{100}{s(s+20)},$$

and the closed-loop transfer function is

$$T(s) = \frac{100}{s^2 + 20s + 100}.$$

The magnitude plot for the closed-loop system is shown in Figure E9.15. With bandwidth defined as frequency at which the magnitude is reduced



**FIGURE E9.15**

Magnitude plot for the closed-loop  $T(s) = \frac{100}{s^2 + 20s + 100}$ .

-3 dB from the dc value, we determine that  $\omega_B = 6.4$  rad/sec.

**E9.16** The transfer function of the approximation is

$$G(j\omega) = \frac{1 - j\omega/10}{1 + j\omega/10},$$

and the magnitude is

$$|G(j\omega)| = \left| \frac{1 - j\omega/10}{1 + j\omega/10} \right| = 1,$$

which is equivalent to the actual time delay magnitude. The phase ap-

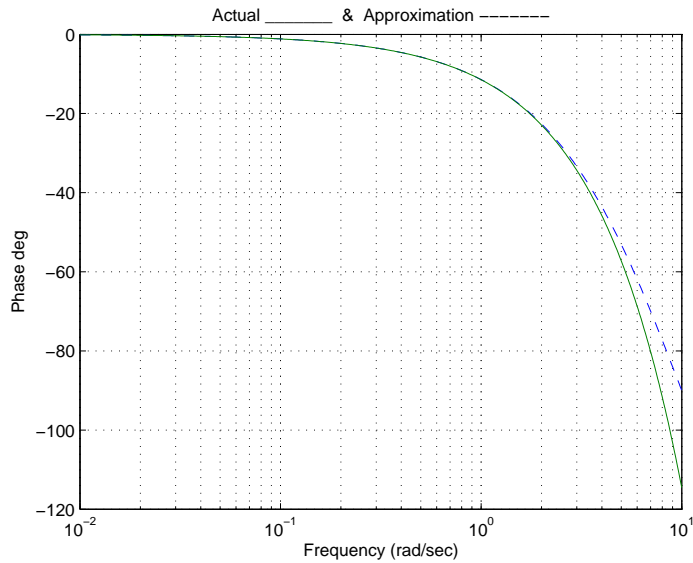
proximation is

$$\phi = -\tan^{-1} \omega/10 + \tan^{-1}(-\omega/10) = -2 \tan^{-1} \omega/10$$

and the actual phase is

$$\phi = -0.2\omega .$$

The phase plots are shown in Figure E9.16. The approximation is accurate for  $\omega < 3$  rad/sec.



**FIGURE E9.16**  
Phase plots for time delay actual vs approximation.

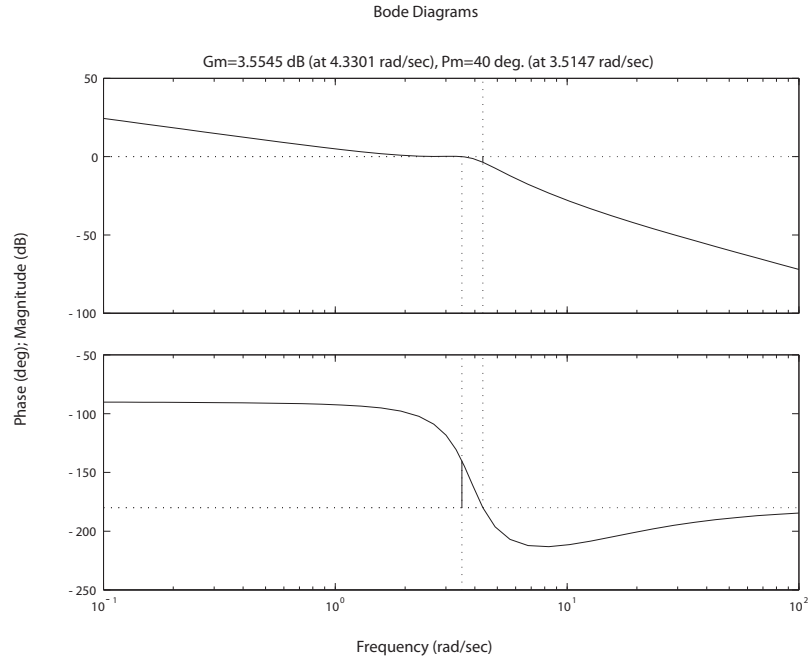
**E9.17** (a,b) The phase angle for  $P.M. = 30$  is

$$\phi = -90^\circ + \tan^{-1} \frac{\omega}{2} - \tan^{-1} \frac{2\omega}{15 - \omega^2} = -150^\circ .$$

Solving for  $\omega$  yields  $\omega = 4.7$ . Then, at  $\omega = 4.7$ , we have  $K = 10.82$  when

$$|G_c G(j\omega)| = \frac{K(\omega^2 + 4)^{\frac{1}{2}}}{\omega((2\omega^2)^2 + (15 - \omega^2)^2)^{\frac{1}{2}}} = 1 .$$

The Bode plot is shown in Figure E9.17.

**FIGURE E9.17**

Bode Diagram for  $G_c(s)G(s) = \frac{K(s+2)}{s^3+2s^2+15s}$ , where  $K = 10.82$ .

(c) The steady-state error for a ramp is

$$e_{ss} = \frac{A}{K_v} = \frac{A}{\frac{10K}{15}} = 0.60A ,$$

where  $R(s) = A/s^2$ .

**E9.18** (a) The gain crossover is at  $\omega_c = 486$  Hz, and the phase margin  $P.O. = 36.2^\circ$ . So,  $\zeta \approx 0.36$ . Then, the expected percent overshoot to a step input is

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 30\% , \quad \text{where } \zeta = 0.36 .$$

(b) The estimated bandwidth is  $\omega_B \approx 2\pi(600)$ .

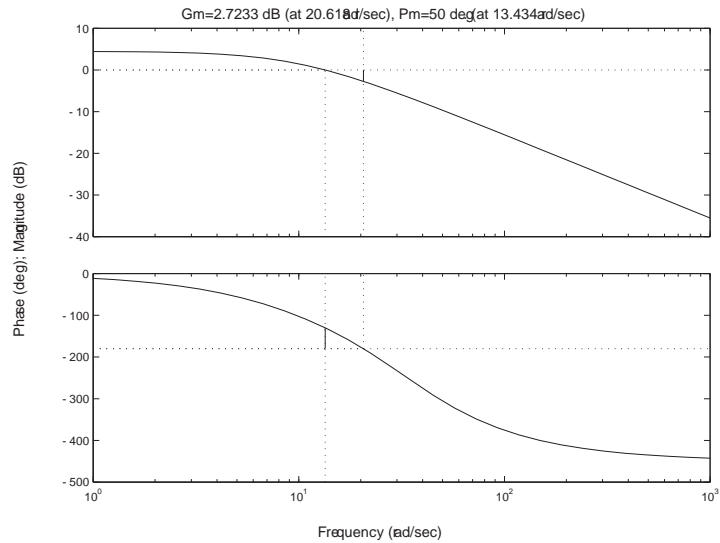
(c) Approximate

$$\omega_n \approx \omega_r = 2\pi(480) .$$

Then,

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.36)2\pi(480)} \approx 4 \text{ ms} .$$

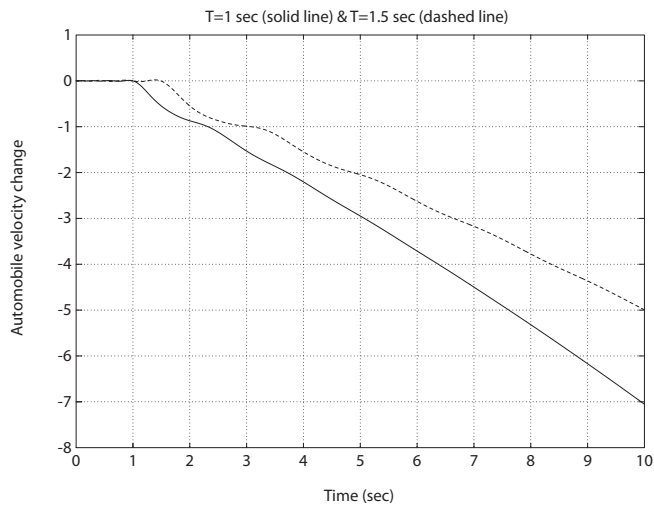
**E9.19** The Bode plot is shown in Figure E9.19 for  $K = 16.75$ . The phase and gain margins are  $PM = 50.0^\circ$  and  $GM = 2.72$  dB.



**FIGURE E9.19**

Bode Diagram for  $G_c(s)G(s) = K \frac{e^{-0.1s}}{s+10}$ , where  $K = 16.75$ .

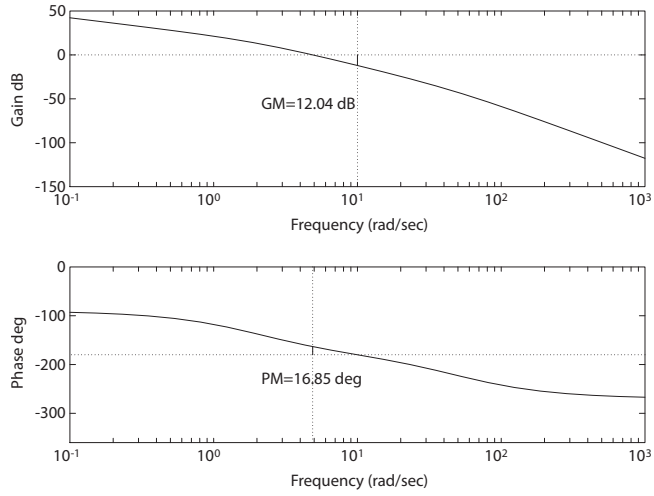
**E9.20** The system response for both drivers is shown in Figure E9.20.



**FIGURE E9.20**

Change in automobile velocity due to braking for two drivers.

**E9.21** The Bode plot is shown in Figure E9.21.

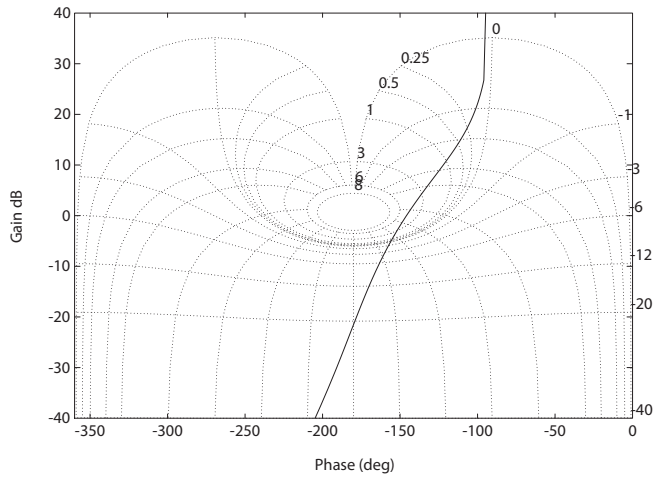


**FIGURE E9.21**

Bode Diagram for  $G_c(s)G(s) = \frac{1300}{s(s+2)(s+50)}$ .

**E9.22** When  $K = 10$ , the  $P.M. = 36.9^\circ$ ; the system is stable. Decreasing the gain to  $K = 4$  results in a  $P.M. = 60^\circ$ .

**E9.23** The Nichols chart is shown in Figure E9.23.



**FIGURE E9.23**

Nichols chart for  $G_c(s)G(s) = \frac{438}{s(s+2)(s+50)}$ .

The actual values are

$$M_{p\omega} = 1.6598 \text{ (4.4 dB)} \quad \omega_r = 2.4228 \text{ rad/sec} \quad \omega_B = 4.5834 \text{ rad/sec} .$$

**E9.24** Using the Nyquist criterion, we have

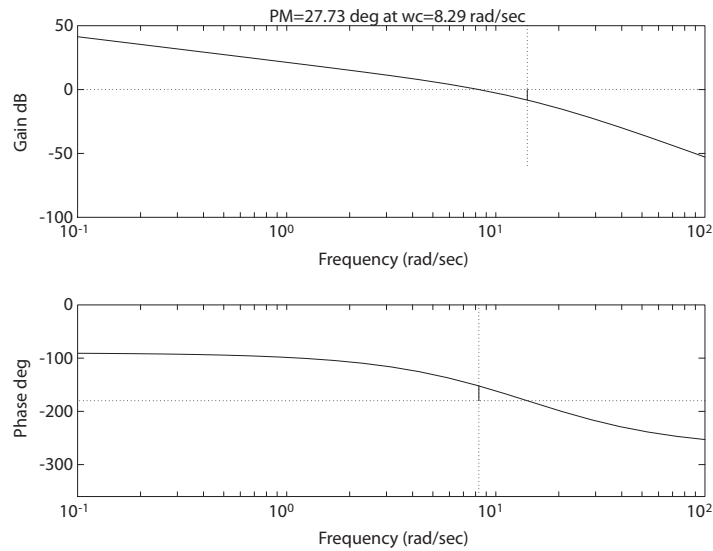
$$P = 1 \quad \text{and} \quad N = 0$$

which implies

$$Z = N + P = 1 .$$

Therefore, the system has one root in the right half-plane.

**E9.25** The Bode plot is shown in Figure E9.25.



**FIGURE E9.25**

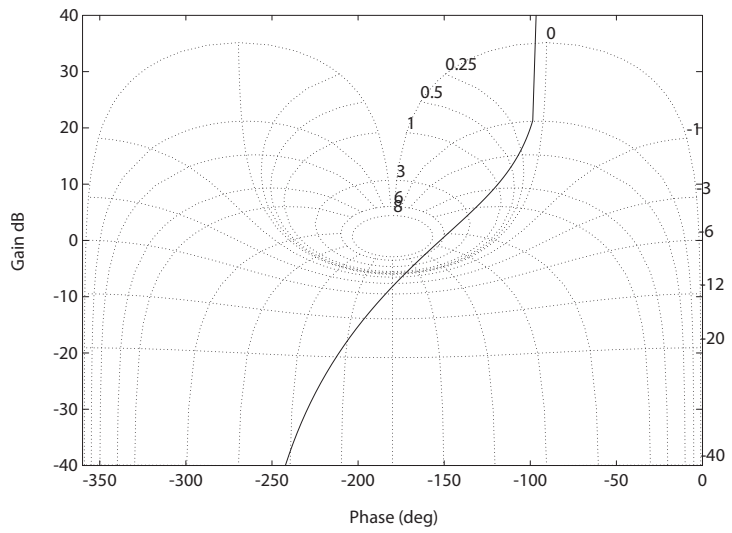
Bode plot for  $G_c(s)G(s) = \frac{11.7}{s(0.05s+1)(0.1s+1)}$ .

**E9.26** The Nichols chart for

$$G_c(s)G(s) = \frac{11.7}{s(0.05s + 1)(0.1s + 1)}$$

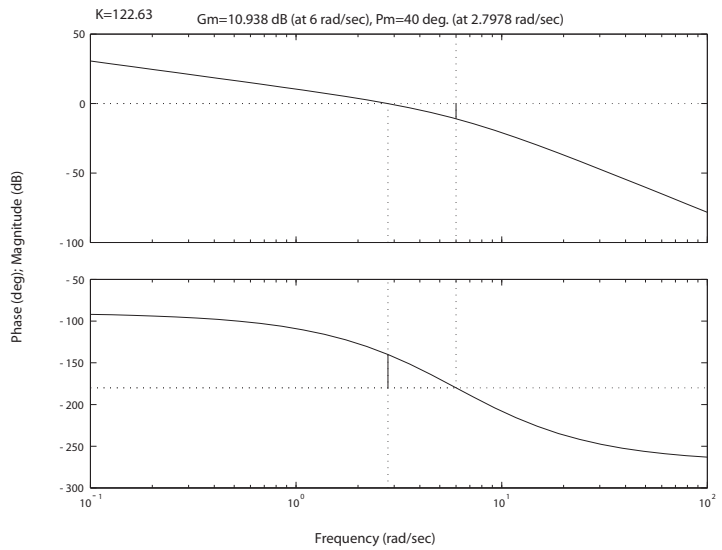
is shown in Figure E9.26, where we find that

$$M_{p\omega} = 6.76 \text{ dB} \quad \omega_r = 8.96 \text{ rad/sec} \quad \omega_B = 13.73 \text{ rad/sec} .$$



**FIGURE E9.26**  
 Nichols chart for  $G_c(s)G(s) = \frac{11.7}{s(0.05s+1)(0.1s+1)}$ .

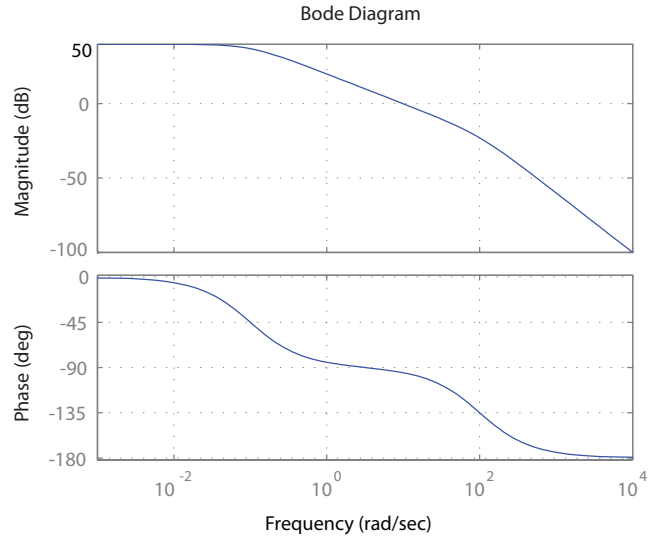
**E9.27** The Bode plot for  $G(s)$  with  $K = 122.62$  is shown in Figure E9.27.



**FIGURE E9.27**  
 Bode plot for  $G_c(s)G(s) = \frac{K}{s(s+6)^2}$ , with  $K = 122.62$ .

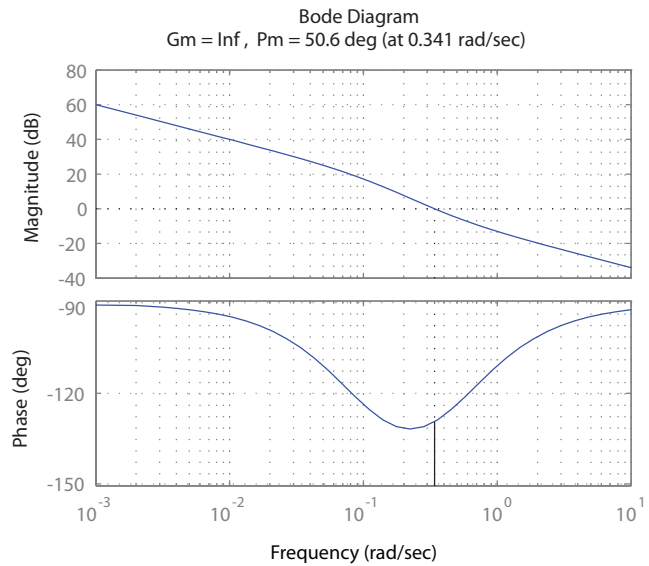






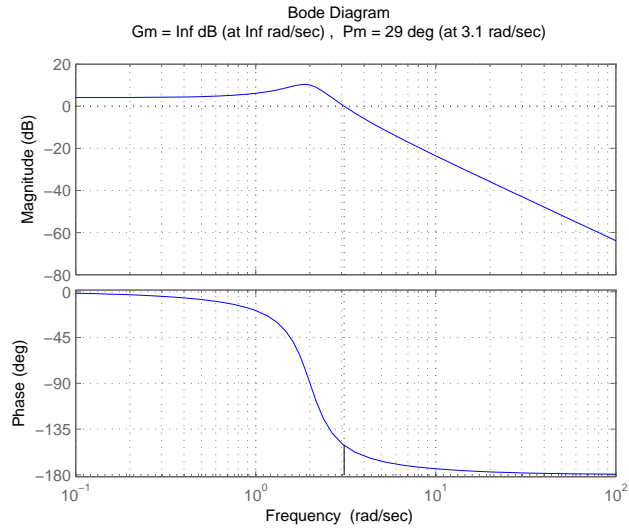
**FIGURE E9.30**  
 Bode plot for  $G(s) = C[sI - A]^{-1}B + D = \frac{1000}{s^2 + 100s + 10}$ .

**E9.31** The Bode plot is shown in Figure E9.31. The phase margin is  $P.M. = 50.6$  deg.



**FIGURE E9.31**  
 Bode plot for  $L(s) = G(s)H(s) = \frac{2s+1}{10s^2+s}$ .

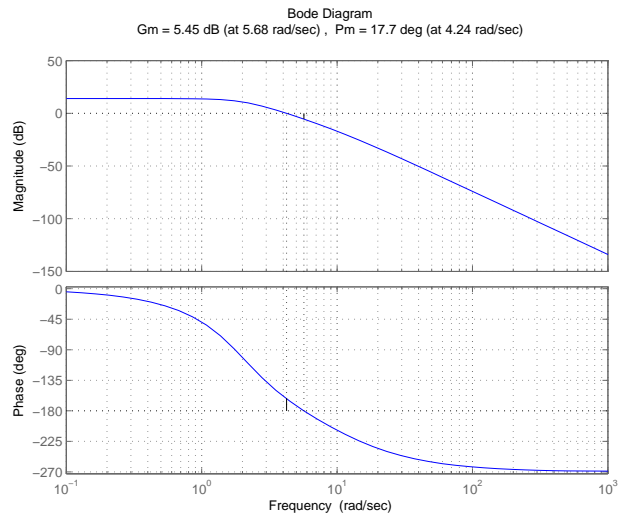
**E9.32** The Bode plot is shown in Figure E9.32. The phase margin is  $P.M. = 29^\circ$ .



**FIGURE E9.32**

Bode plot for  $G(s) = \mathbf{C} [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B} + \mathbf{D} = \frac{6.4}{s^2 + s + 4}$ .

**E9.33** The Bode plot is shown in Figure E9.33. The phase margin is  $P.M. = 17.7^\circ$  and the gain margin is  $G.M. = 5.45$  dB.



**FIGURE E9.33**

Bode plot for  $L(s) = \frac{200}{(s^2 + 2.83s + 4)(s + 10)}$ .

## Problems

**P9.1** (a) The loop transfer function is

$$G_c(s)G(s) = \frac{1}{(1 + 0.5s)(1 + 2s)} .$$

$P = 0, N = 0$ ; therefore  $Z = N + P = 0$ . The system is stable. (**Note:** See P8.1 for the polar plots.)

(b) The loop transfer function is

$$\frac{1 + 0.5s}{s^2} .$$

$P = 0, N = 0$ , therefore  $Z = N + P = 0$ . The system is stable.

(c) The loop transfer function is

$$\frac{s + 4}{s^2 + 5s + 25} .$$

$P = 0, N = 0, Z = N + P = 0$ . Therefore, the system is stable.

(d) The loop transfer function is

$$\frac{30(s + 8)}{s(s + 2)(s + 4)} .$$

$P = 0, N = 2$  therefore  $Z = P + N = 2$ . Therefore, the system has two roots in the right half-plane, and is unstable.

**P9.2** (a) The loop transfer function is

$$G_c(s)G(s) = \frac{K}{s(s^2 + s + 6)} ,$$

and

$$G_c(j\omega)G(j\omega) = \frac{K}{j\omega(-\omega^2 + j\omega + 6)} = \frac{K[-\omega^2 - j\omega(6 - \omega^2)]}{[(6 - \omega^2)^2\omega^2 + \omega^4]} .$$

To determine the real axis crossing, we let

$$\text{Im}\{G_c(j\omega)G(j\omega)\} = 0 = -K\omega(6 - \omega^2)$$

or

$$\omega = \sqrt{6} .$$

Then,

$$\operatorname{Re}\{G_c(j\omega)G(j\omega)\}_{\omega=\sqrt{6}} = \frac{-K\omega^2}{\omega^4} \Big|_{\omega=\sqrt{6}} = \frac{-K}{6} .$$

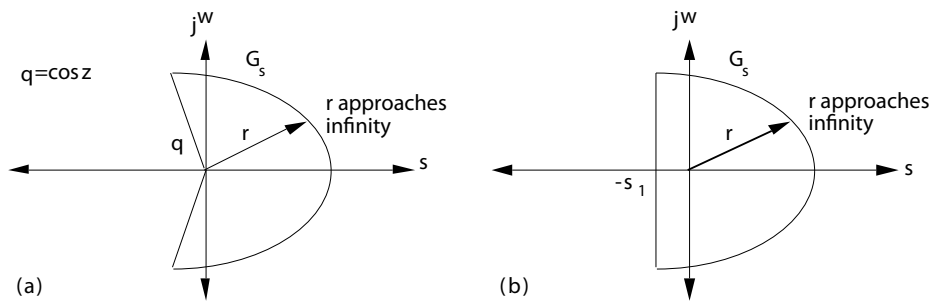
So,  $-K/6 > -1$  for stability. Thus  $K < 6$  for a stable system.

(b) The loop transfer function is

$$G_c(s)G(s) = \frac{K(s+1)}{s^2(s+6)} .$$

The polar plot never encircles the -1 point, so the system is stable for all gains  $K$  (See Figure 10 in Table 9.6 in Dorf & Bishop).

**P9.3** (a,b) The suitable contours are shown in Figure P9.3.



**FIGURE P9.3**  
Suitable contours  $\Gamma_s$  for (a) and (b).

(c) Rewrite the characteristic equation as

$$1 + \frac{96}{s(s^2 + 11s + 56)} = 0 .$$

In this case,  $-\sigma_1 = -1$ . Therefore, we have one pole inside the contour at  $s = 0$ , so  $P = 1$ . The polar plot yields  $N = -1$ , so  $Z = N + P = 0$ . Therefore, all three roots have real parts less than -1. In fact, the roots are  $s_1 = -3$ , and  $s_{2,3} = -4 \pm j4$ .

- P9.4** (a)  $P = 0, N = 2$ , therefore  $Z = 2$ . The system has two roots in the right hand s-plane.  
 (b) In this case,  $N = +1 - 1 = 0$ , so  $Z = 0$ . Therefore the system is stable.

**P9.5** (a) The loop transfer function is

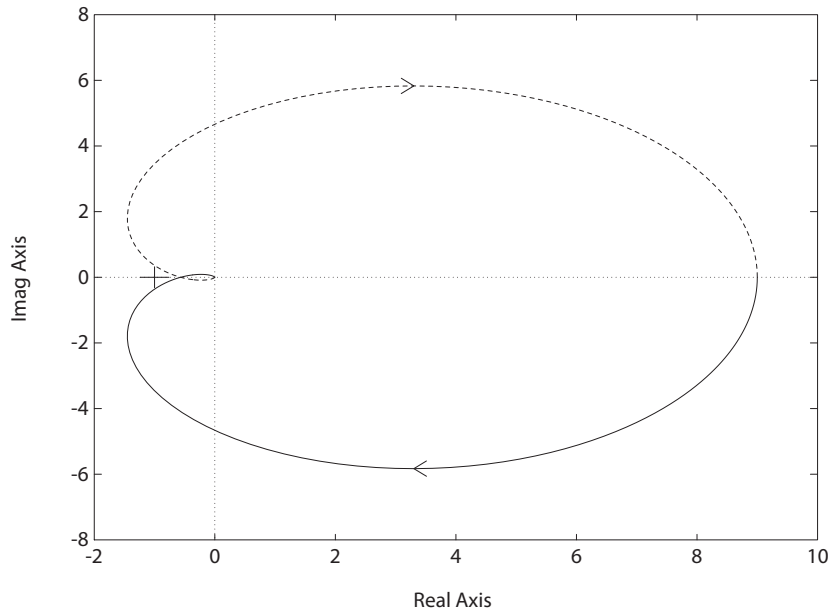
$$L(s) = G_c(s)G(s)H(s) = \frac{K}{(s+1)(3s+1)(0.4s+1)}.$$

The steady-state error is

$$e_{ss} = \frac{|R|}{1+K}.$$

We require  $e_{ss} = 0.1|R|$ , so  $K > 9$ .

(b) Use  $K = 9$ . The Nyquist plot is shown in Figure P9.5. We determine that  $P = 0$  and  $N = 0$ . Therefore,  $Z = 0$  and the system is stable.



**FIGURE P9.5**

Nyquist Diagram for  $L(s) = G_c(s)G(s)H(s) = \frac{9}{(s+1)(3s+1)(0.4s+1)}$ .

(c) The phase and gain margins are  $P.M. = 18^\circ$  and  $G.M. = 5$  dB.

**P9.6** The rotational velocity transfer function is

$$\frac{\omega(s)}{R(s)} = G(s) = \frac{K}{\left(1 + \frac{s}{3.7(2\pi)}\right) \left(\frac{s}{68(2\pi)} + 1\right)}.$$

At low frequency, we have the magnitude near 35 dB, so  $20 \log K = 35$  dB

and  $K = 56$ . Since the frequency response plot is for rotational velocity  $\omega(s)$ , and we are interested in position control, we add an integrator. The characteristic equation is

$$1 + G(s)\frac{1}{s} = 1 + \frac{56(23)(427)}{s(s+23)(s+427)} = 0 .$$

The roots are

$$s_1 = -430 \quad \text{and} \quad s_{2,3} = -10 \pm j35 .$$

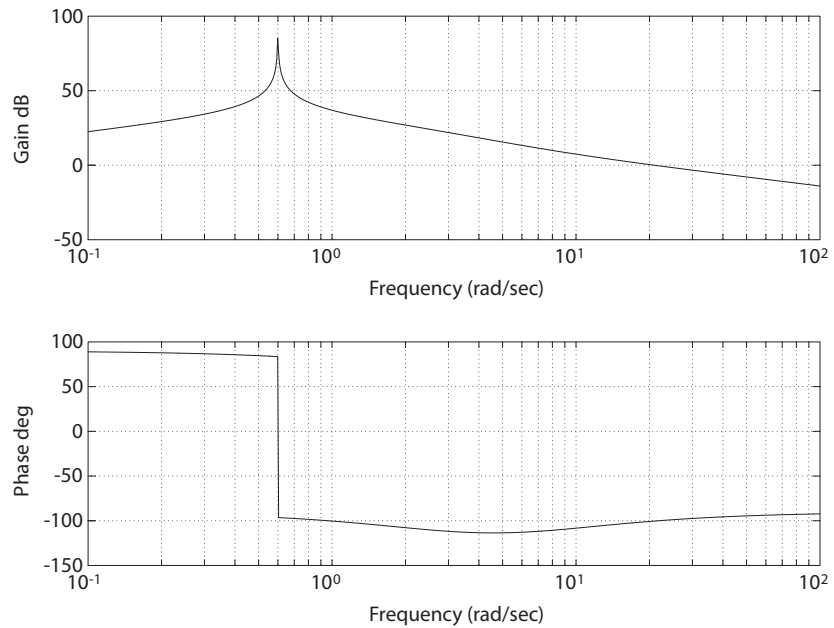
Thus,  $\omega_n = 36$  and  $\zeta = 0.28$ . The time constant of the closed-loop system is

$$\tau = \frac{1}{\zeta\omega_n} = 99.6 \text{ msec} .$$

**P9.7** The loop transfer function is

$$L(s) = G_c(s)G(s)H(s) = \frac{10K_1s(s+7)}{(s+3)(s^2+0.36)} .$$

(a) The Bode plot is shown in Figure P9.7 for  $K_1 = 2$ .



**FIGURE P9.7**

Bode Diagram for  $G_c(s)G(s)H(s) = \frac{10K_1s(s+7)}{(s+3)(s^2+0.36)}$ , where  $K_1 = 2$ .

- (b) The phase margin  $P.M. = 80^\circ$  and the gain margin  $G.M. = \infty$ , since  $\phi$  never crosses  $= -180^\circ$ .
- (c) The transfer function from  $T_d(s)$  to  $\theta(s)$  is

$$\theta(s) = \frac{G(s)}{1 + G_c(s)G(s)H(s)}T_d(s) .$$

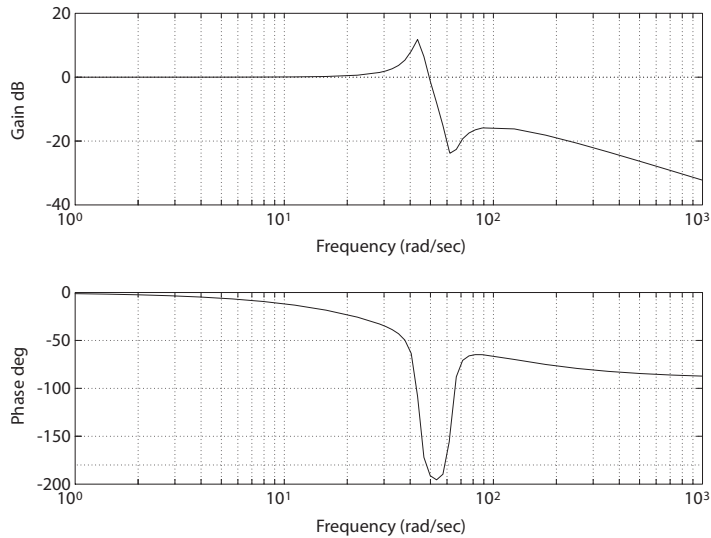
Then, for a step disturbance  $\theta(\infty) = \lim_{s \rightarrow 0} s\theta(s) = G(0) = 10/0.36 = 27.8$ , since  $H(0) = 0$ .

- (d) The system is so highly damped, there is very little resonant peak.
- (e) The estimated  $\zeta = P.M./100 = 0.80$ . The actual  $\zeta = 0.97$ .

**P9.8** (a) The loop transfer function is

$$G_c(s)G(s)H(s) = \frac{\left(\frac{s^2}{\omega_1^2} + \frac{2\zeta_1 s}{\omega_1} + 1\right)}{(0.02s + 1)\left(\frac{s^2}{\omega_2^2} + \frac{2\zeta_2 s}{\omega_2} + 1\right)} ,$$

where  $\omega_1 = 20\pi = 62.8$ ,  $\omega_2 = 14\pi = 43.9$ ,  $\zeta_1 = 0.05$  and  $\zeta_2 = 0.05$ . The Bode plot is shown in Figure P9.8a. The phase margin is  $P.M. = -9^\circ$ . Therefore, the system is unstable.

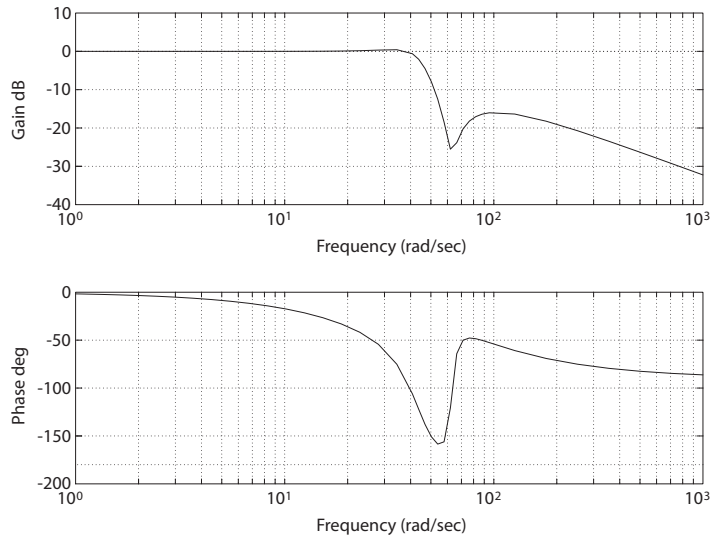


**FIGURE P9.8**

(a) Bode Diagram for  $G_c(s)G(s)H(s) = \frac{s^2/\omega_1^2 + (0.1/\omega_1)s + 1}{(0.02s + 1)(s^2/\omega_2^2 + (0.1/\omega_2)s + 1)}$ , where  $\omega_1 = 20\pi$  and  $\omega_2 = 14\pi$ .

(b) In this case  $\zeta_2 = 0.25$ , with all other parameters the same as before.





**FIGURE P9.8**

CONTINUED: (b) Bode Diagram for  $G_c(s)G(s)H(s) = \frac{s^2/\omega_1^2 + (0.1/\omega_1)s + 1}{(0.02s + 1)(s^2/\omega_2^2 + (0.5/\omega_2)s + 1)}$ , where  $\omega_1 = 20\pi$  and  $\omega_2 = 14\pi$ .

The Bode plot is shown in Figure P9.8b. The phase margin is  $P.M. = 86^\circ$ . Therefore, the system is now stable.

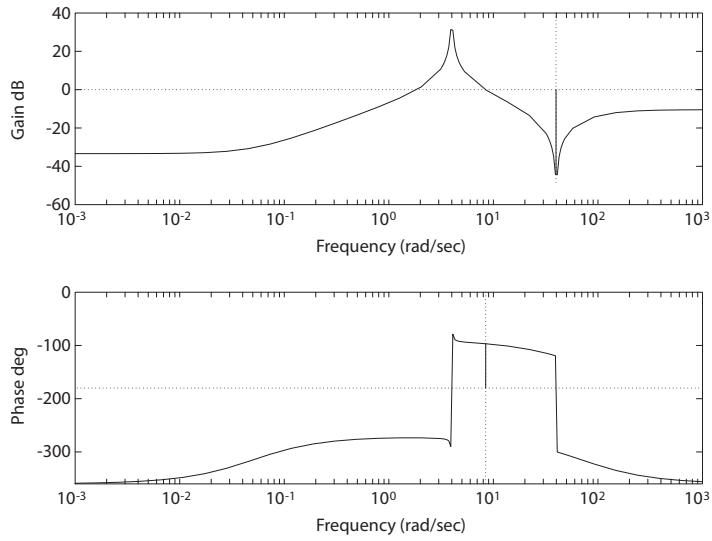
- P9.9** (a) The Bode plot is shown in Figure P9.9a  
 The phase margin is  $P.M. = 83^\circ$  and the gain margin is  $G.M. = \infty$ .  
 (b) With the compensator, the loop transfer function is

$$G_c(s)G(s)H(s) = K_1 \frac{0.30(s + 0.05)(s^2 + 1600)(s + 0.5)}{s(s^2 + 0.05s + 16)(s + 70)},$$

where

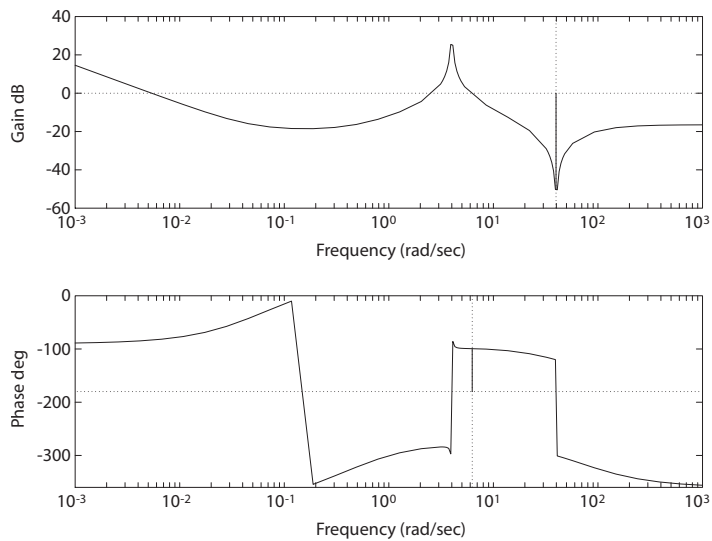
$$K_2/K_1 = 0.5 .$$

Let  $K_1 = 1$ . The Bode plot is shown in Figure P9.9b. The phase margin is  $P.M. = 80^\circ$  and the gain margin is  $G.M. = \infty$ , essentially the same as in (a). But the system in (b) is a type one, so that  $e_{ss} = 0$  to a step input or disturbance. We cannot achieve a  $G.M. = 10$  dB by increasing or decreasing  $K_1$ .



**FIGURE P9.9**

(a) Bode Diagram for  $G_c(s)G(s)H(s) = \frac{0.3(s+0.05)(s^2+1600)}{(s+70)(s^2+0.05s+16)}$ .



**FIGURE P9.9**

CONTINUED: (b) Bode Diagram for  $G_s(s)G(s)H(s) = \frac{0.15K_1(s+0.05)(s^2+1600)(s+0.5)}{(s+70)(s^2+0.05s+16)}$ ,  $K_1 = 1$ .

**P9.10** The equations of motion are

$$F(s) = 3I(s) \quad \text{and} \quad I(s) = \frac{E_o(s)}{R + Ls} = \frac{E_o(s)}{0.1 + 0.2s} .$$

So,

$$F(s) = \frac{30}{(2s + 1)} E_o(s) .$$

The actuator without the spring (see Table 2.7, Number 9 in Dorf & Bishop) is modeled via

$$\frac{X(s)}{Y(s)} = \frac{1}{Ms^2 + Bs} = \frac{K_a}{\tau_a s^2 + s} .$$

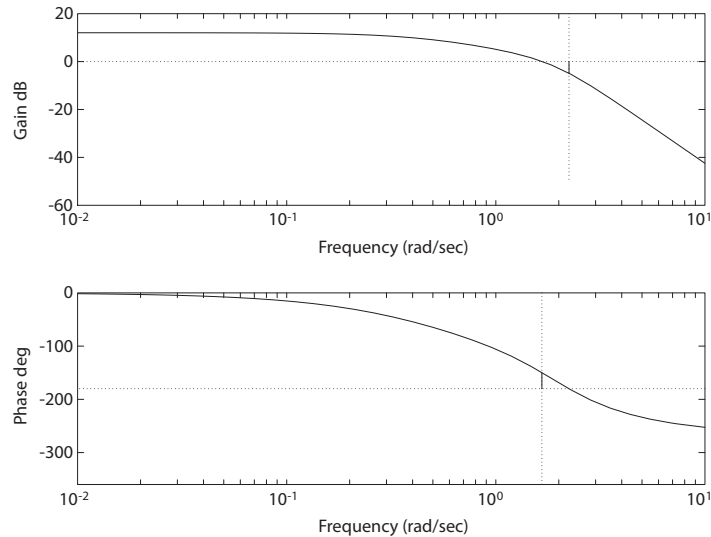
With the spring, we have

$$\frac{X(s)}{Y(s)} = \frac{K_a}{\tau_a s^2 + s + K_s} \quad \text{or} \quad G_A(s) = \frac{1}{0.4s^2 + s + 1.5} .$$

Then, the loop transfer function is

$$L(s) = \frac{30K_1}{(2s + 1)(0.4s^2 + s + 1.5)} .$$

(a) The Bode plot for  $K_1 = 0.2$  in Fig. P9.10 shows the  $P.M. = 30^\circ$ .



**FIGURE P9.10**

Bode Diagram for  $L(s) = \frac{30K_1}{(2s+1)(0.4s^2+s+1.5)}$ , where  $K_1 = 0.2$ .

- (b) For  $K_1 = 0.2$ , we determine that  $M_{p\omega} = 7.8$  dB,  $\omega_r = 1.9$  rad/sec, and  $\omega_B = 2.8$  rad/sec.
- (c) The estimated percent overshoot is  $P.O. = 51\%$  and the estimated settling time is  $T_s = 10$  sec. This is based on  $\zeta = 0.21$  and  $\omega_n \approx \omega_r = 1.9$  rad/sec.

**P9.11** The loop transfer function is

$$G_c(s)G(s) = \frac{5(K_1s + K_2)e^{-1.5s}}{s(5s + 1)} .$$

- (a) Let  $K_1 = K_2 = 1$ . Then

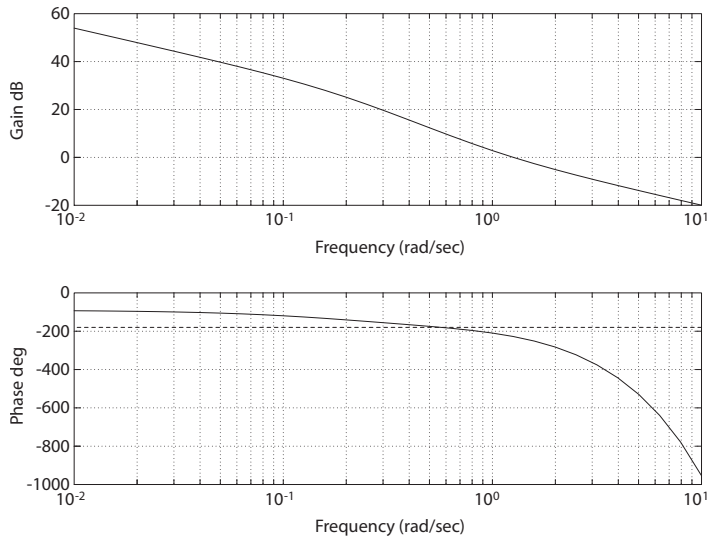
$$G_c(s)G(s) = \frac{5(s + 1)}{s(5s + 1)} e^{-1.5s} .$$

The Bode plot is shown in Figure P9.11a. The phase margin is  $P.M. = -48^\circ$ . The system is unstable.

- (b) Let  $K_1 = 0.1$  and  $K_2 = 0.04$ . Then, the loop transfer function is

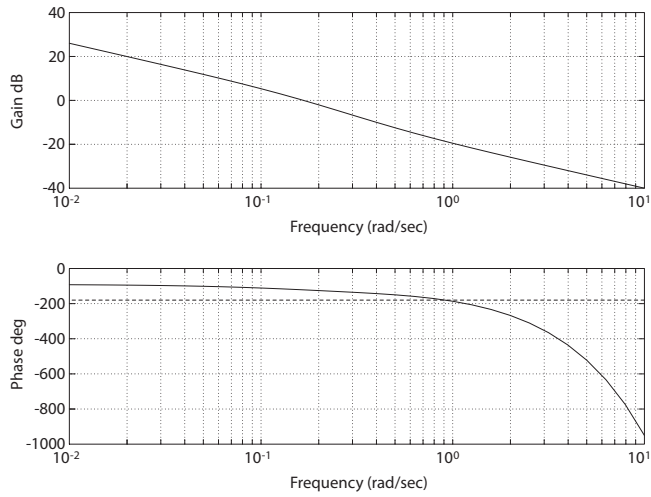
$$G_c(s)G(s) = \frac{5(0.1s + 0.04)e^{-1.5s}}{s(5s + 1)} .$$

The Bode plot shown in Figure P9.11b shows  $P.M. = 45^\circ$ . Thus, the system is stable.



**FIGURE P9.11**

(a) Bode Diagram for  $G_c(s)G(s) = \frac{5(s+1)e^{-sT}}{s(5s+1)}$ , where  $T = 1.5$ .

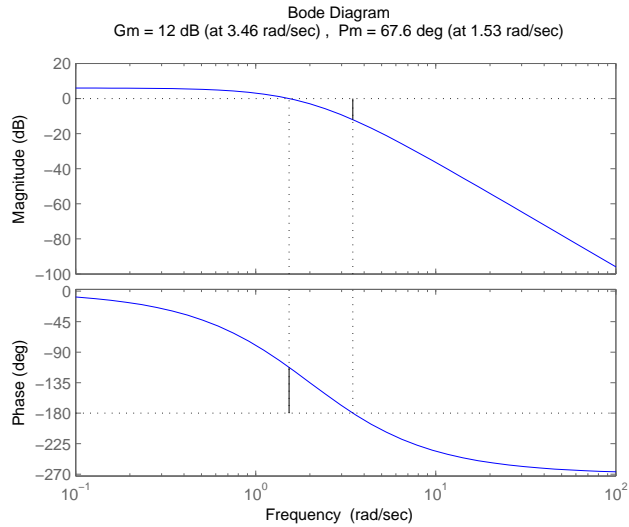


**FIGURE P9.11**

CONTINUED: (b) Bode Diagram for  $G_c(s)G(s) = \frac{5(0.1s+0.04)e^{-sT}}{s(5s+1)}$ , where  $T = 1.5$ .

(c) When  $K_2 = 0.1394$ , the phase margin is  $P.M. = 0^\circ$  and  $G.M. = 0$  dB. So, for stability we require  $K_2 \leq 0.1394$  when  $K_1 = 0$ .

**P9.12** (a) The Bode plot is shown in Figure P9.12.



**FIGURE P9.12**

Bode Diagram for  $G_c(s)G(s) = \frac{2}{(0.5s+1)^3}$ .

The loop transfer function (without the time delay) is

$$G_c(s)G(s) = \frac{2}{(0.5s + 1)^3}.$$

The phase margin is  $P.M. = 67.6^\circ$ .

(b) With the delay, the loop transfer function is

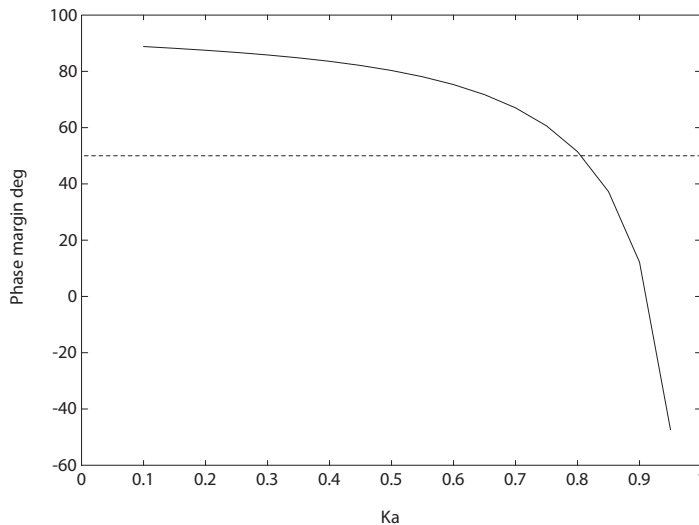
$$G_c(s)G(s)H(s) = \frac{2e^{-0.5s}}{(0.5s + 1)^3}.$$

The phase margin is now  $P.M. = 23.7^\circ$ . So the 0.5 sec time delay has reduced the phase margin by  $43.9^\circ$ .

**P9.13** The loop transfer function is

$$G_c(s)G(s) = \frac{K_a(Ks + 1)}{s}e^{-1.2s}.$$

- (a) Let  $K_a = K = 1$ . Without the time delay, the system has infinite phase and gain margin. However, with the time delay, the system has a negative gain margin, hence it is unstable.
- (b) A plot of phase margin versus  $K_a$  is shown in Figure P9.13.



**FIGURE P9.13**

Phase margin as a function of  $K_a$  for  $G_c(s)G(s) = \frac{K_a(s+1)e^{-1.2s}}{s}$ .

Let  $K = 1$ , and find  $K_a$  for a stable system. Then,

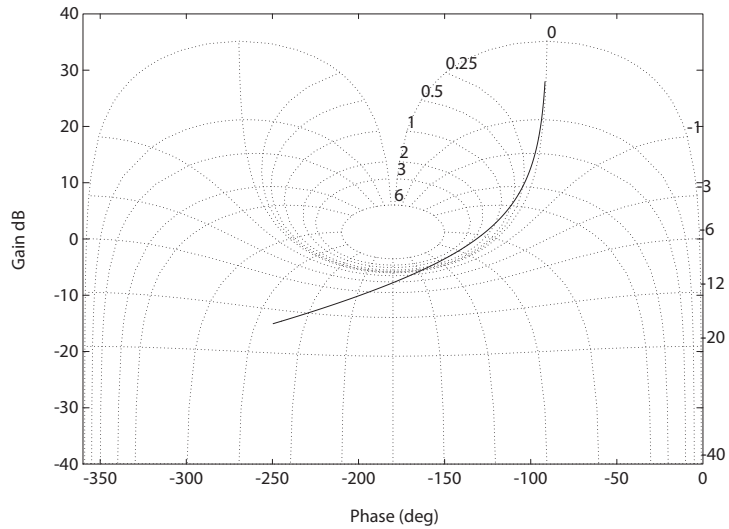
$$G_c(s)G(s) = \frac{K_a(s + 1)e^{-1.2s}}{s} .$$

If  $K_a = 0.8$ , then the phase margin is  $P.M. = 50^\circ$ .

**P9.14** The loop transfer function is

$$G_c(s)G(s) = \frac{Ke^{-0.2s}}{s(0.1s + 1)} .$$

(a) The Nichols diagram is shown in Figure P9.14 for  $K = 2.5$ .



**FIGURE P9.14**  
Nichols diagram for  $G_c(s)G(s) = \frac{Ke^{-0.2s}}{s(0.1s+1)}$ , for  $K = 2.5$ .

It can be seen that

$$M_{p\omega} = 2.0 \text{ dB} .$$

The phase and gain margins are  $P.M. = 48.5^\circ$  and  $G.M. = 7.77 \text{ dB}$ .

- (b) We determine that  $\zeta = 0.43$  (based on  $M_{p\omega} = 2 \text{ dB}$ ) and  $\zeta = 0.48$  (based on the phase margin  $P.M. = 48.5^\circ$ ).
- (c) The bandwidth is

$$\omega_B = 5.4 \text{ rad/sec} .$$

**P9.15** (a) The ship transfer function is

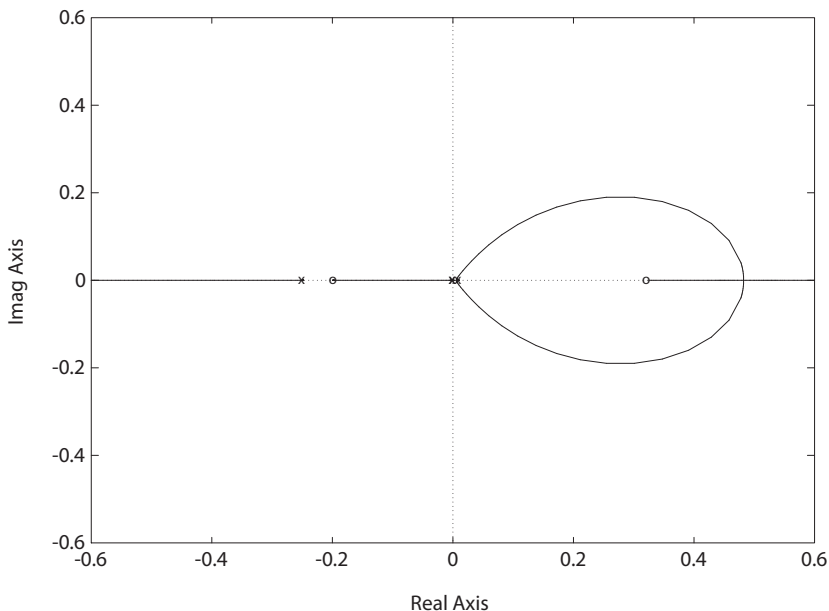
$$G(s) = \frac{-0.164(s + 0.2)(s - 0.32)}{s^2(s + 0.25)(s - 0.009)}.$$

The closed-loop system is unstable; the roots are

$$\begin{aligned} s_1 &= -0.5467 \\ s_{2,3} &= 0.2503 \pm 0.1893j \\ s_4 &= -0.1949 \end{aligned}$$

Therefore the ship will not track the straight track.

- (b) The system cannot be stabilized by lowering the gain; this is verified in the root locus in Figure P9.15, where it is seen that the locus has a branch in the right half-plane for all  $K > 0$ .
- (c) Yes, the system can be stabilized.
- (d) When the switch is closed, we have a derivative feedback, which adds  $90^\circ$  phase lead. This is not enough to stabilize the system. Additional lead networks are necessary.



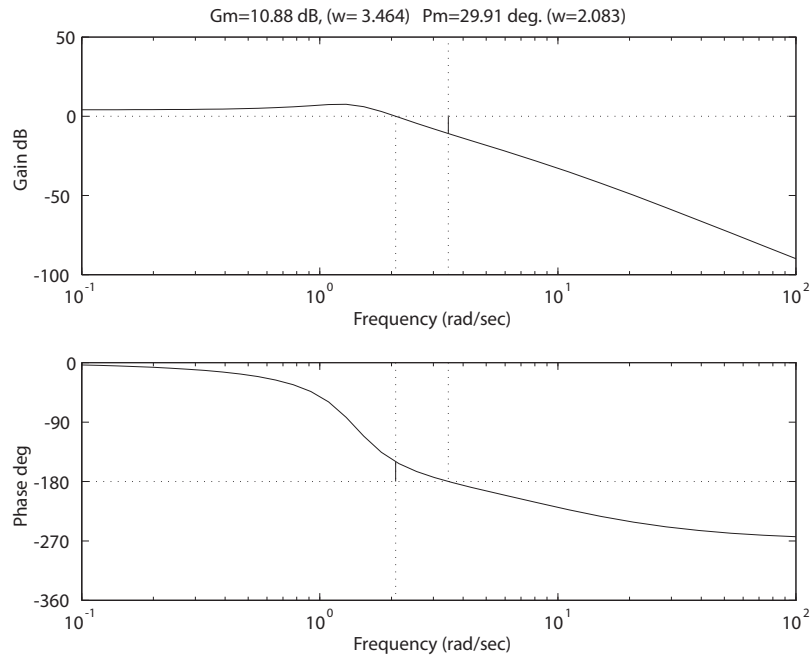
**FIGURE P9.15**  
 Root locus for  $1 + GH(s) = 1 + K \frac{-0.164(s+0.2)(s-0.32)}{s^2(s+0.25)(s-0.009)} = 0$ .



**P9.16** The loop transfer function is

$$G_c(s)G(s) = \frac{K}{(s/10 + 1)(s^2 + s + 2)}.$$

When  $K = 3.2$ , the phase margin is  $P.M. \approx 30^\circ$ . The Bode plot is shown in Figure P9.16.



**FIGURE P9.16**

Bode plot for  $G_c(s)G(s) = \frac{K}{(s/10+1)(s^2+s+2)}$ , where  $K = 3.2$ .

**P9.17** (a) We require  $e_{ss} \leq 0.05A$ , and we have

$$e_{ss} = \frac{A}{1 + K_p} < 0.05A$$

or  $K_p > 19$ . But

$$K_p = \lim_{s \rightarrow 0} G_1(s)G_2(s)G_3(s)G_4(s) = \lim_{s \rightarrow 0} \frac{20K_1}{(0.5s + 1)} \left( \frac{0.1}{1 + 4s} \right)^2 = 0.2K_1.$$

So,  $K_p = 0.2K_1 > 19$ , or  $K_1 > 95$ .

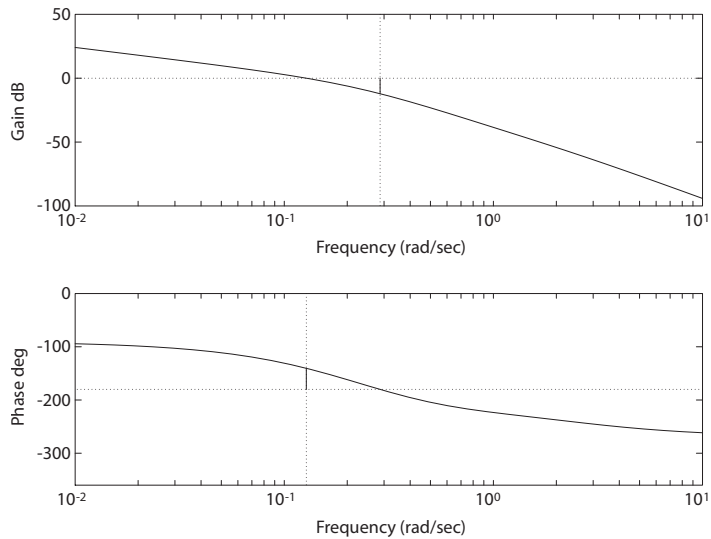
(b) Given

$$G_1(s) = K_1 \left(1 + \frac{1}{s}\right) = K_1 \left(\frac{s+1}{s}\right) ,$$

we require  $1.05 < M_{P_t} < 1.30$ , or  $0.70 > \zeta > 0.36$ , or  $70^\circ > P.M. > 36^\circ$ . Then,

$$G_1(s)G_2(s)G_3(s)G_4(s) = \frac{0.2K_1(s+1)}{s(0.5s+1)(4s+1)^2} .$$

When  $K_1 = 0.8$ , the  $P.M. = 40^\circ$ . The Bode plot is shown in Figure P9.17a.



**FIGURE P9.17**

(a) Bode plot for  $G_1(s)G_2(s)G_3(s)G_4(s) = \frac{0.2K_1(s+1)}{s(0.5s+1)(4s+1)^2}$ , where  $K_1 = 0.8$  and  $P.M. = 40^\circ$ .

(c) For part (a), we had

$$G_1(s)G_2(s)G_3(s)G_4(s) = \frac{2.375}{(s+2)(s+0.25)^2} .$$

The characteristic equation is

$$s^3 + 2.5s^2 + 1.06s + 2.50 = (s + 2.48)(s^2 + 0.02s + 1.013) .$$

The dominant complex roots are lightly damped since  $\zeta = 0.01$  and  $\zeta\omega_n = 0.01$ .

Thus,

$$T_s = \frac{4}{\zeta\omega_n} = 400 \text{ sec} .$$

For part (b), we had

$$G_1(s)G_2(s)G_3(s)G_4(s) = \frac{(0.2)(0.8)(s + 1)}{s(0.5s + 1)(4s + 1)^2} .$$

The characteristic equation is

$$8s^4 + 20s^3 + 8.5s^2 + 1.16s + 0.16 = 0 .$$

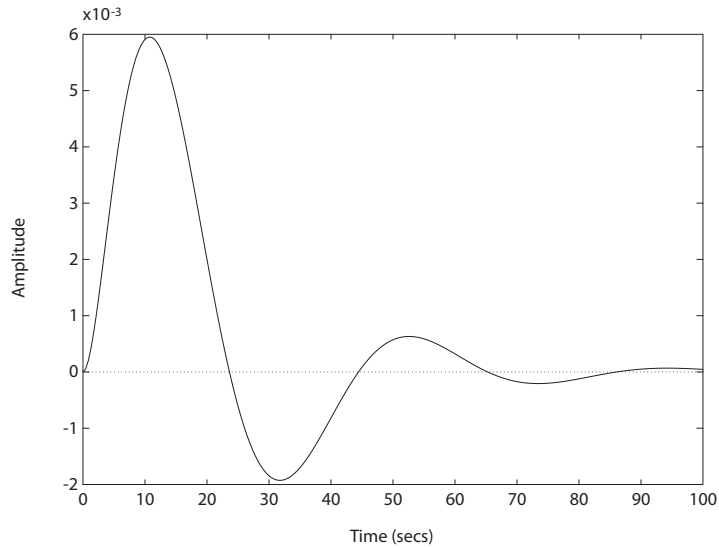
The roots are  $s_1 = -2, s_2 = -0.4$  and  $s_{3,4} = -0.05 \pm j0.15$ . Thus  $\zeta = 0.16$  and  $\zeta\omega_n = 0.05$ . So,

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.05} = 75 \text{ sec} .$$

(d) Let  $U(s)$  be a unit step disturbance and  $R(s) = 0$ . Then

$$\frac{Y(s)}{U(s)} = \frac{G_3(s)G_4(s)}{1 + G_1(s)G_2(s)G_3(s)G_4(s)} = \frac{\left(\frac{0.1}{1+4s}\right)^2}{1 + \frac{20K_1(s+1)}{s(0.5s+1)(4s+1)^2}} .$$

The disturbance response is shown in Figure P9.17b.

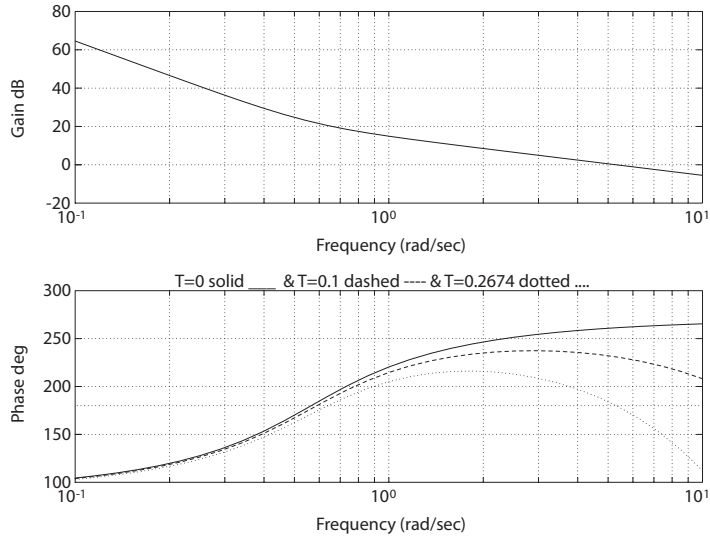


**FIGURE P9.17**  
CONTINUED: (b) System response to a unit disturbance  $U(s)$ .

**P9.18** The transfer function is

$$G_c(s)G(s)H(s) = \frac{5.3(s^2 + 0.8s + 0.32)e^{-Ts}}{s^3}$$

The Bode plot is shown in Figure P9.18.



**FIGURE P9.18**

Bode diagram for  $G_c(s)G(s)H(s) = \frac{K(s^2+0.8s+0.32)e^{-sT}}{s^3}$ , where  $T = 0$  (solid line),  $T = 0.1$  (dashed line), and  $T = 0.2674$  (dotted line).

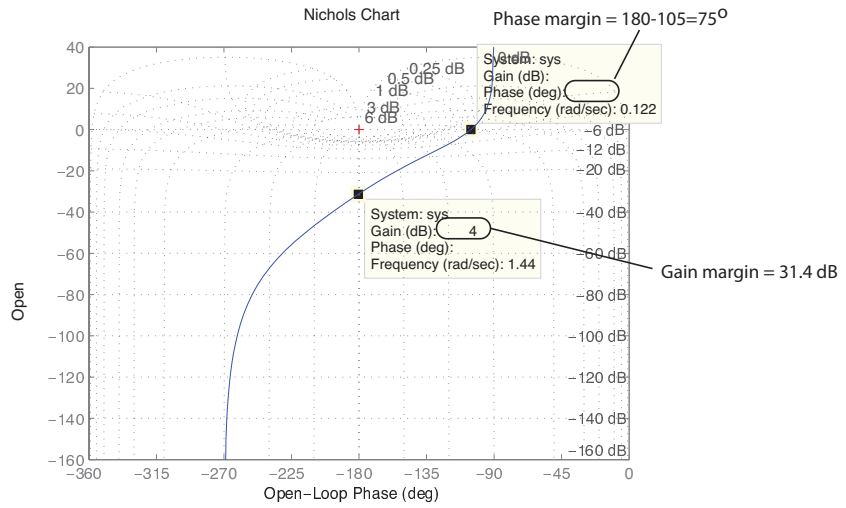
The following results are verified in the figure.

- The phase margin is  $P.M. = 81^\circ$  at  $\omega = 5.3$  when  $T = 0$ .
- For  $T = 0.1$ , the added phase is  $\phi = -T\omega$  (in radians). The phase margin is  $P.M. = 51^\circ$  at  $\omega = 5.3$  when  $T = 0.1$ .
- The system is borderline stable when  $T = 0.2674$  sec. The phase margin is  $P.M. = 0^\circ$  at  $\omega = 5.3$ .

**P9.19** The transfer function is

$$G_c(s)G(s) = \frac{0.5}{s(1+2s)(4+s)}$$

- The Nichols diagram is shown in Figure P9.19. The gain margin is  $G.M. = 31.4$  dB.
- The phase margin is  $P.M. = 75^\circ$  and  $M_{p\omega} = 0$  dB. The bandwidth is  $0.17$  rad/sec.

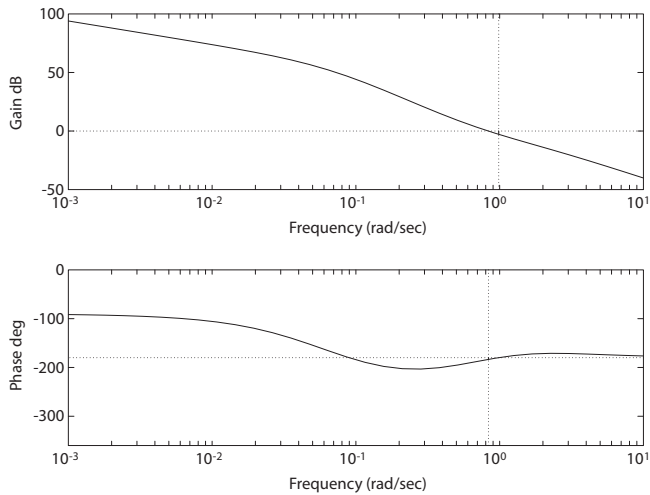


**FIGURE P9.19**

Nichols diagram for  $G_c(s)G(s) = \frac{0.5}{s(2s+1)(s+4)}$ .

**P9.20** (a) Let  $K = 100$ . The Bode plot is shown in Figure P9.20a. The loop transfer function is

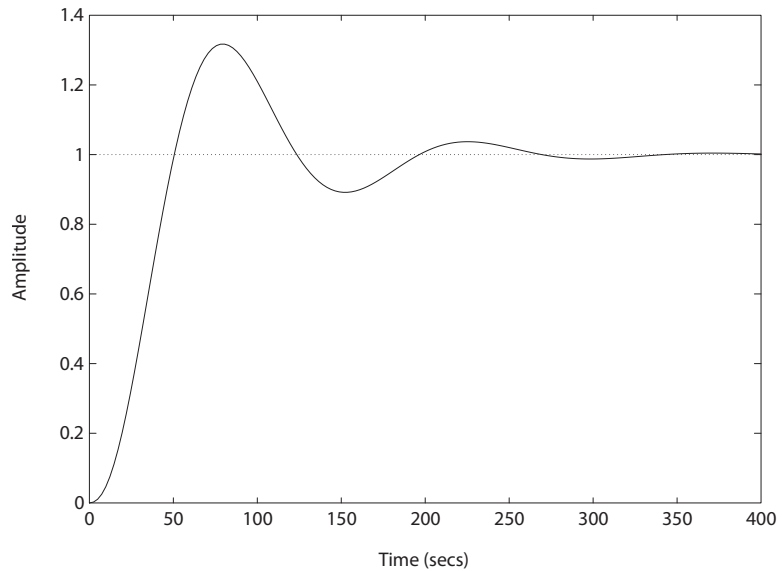
$$G_c(s)G(s) = \frac{K(s^2 + 1.5s + 0.5)}{s(20s + 1)(10s + 1)(0.5s + 1)}$$



**FIGURE P9.20**

(a) Bode diagram for  $G_c(s)G(s) = \frac{K(s^2+1.5s+0.5)}{s(20s+1)(10s+1)(0.5s+1)}$ , where  $K = 100$ .

- (b) The phase margin is  $P.M. = -3.5^\circ$  and the gain margin is  $G.M. = 2.7$  dB.
- (c) You must decrease  $K$  below 100 to achieve a  $P.M. = 40^\circ$ . For  $K = 0.1$ , the phase margin  $P.M. = 37.9^\circ$ .
- (d) The step response is shown in Figure P9.20b for  $K = 0.1$ .



**FIGURE P9.20**

CONTINUED: (b) Unit step response  $K = 0.1$ .

**P9.21** The loop transfer function is

$$G_c(s)G(s) = \frac{K}{s(s+1)(s+4)} .$$

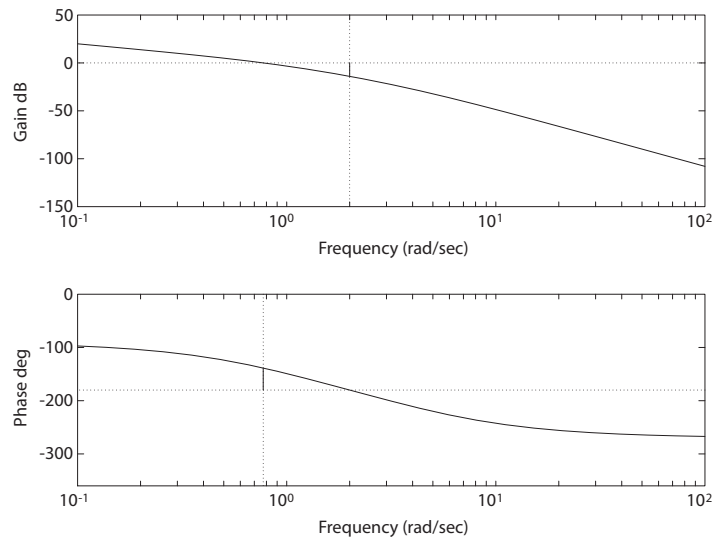
- (a) The Bode plot is shown in Figure P9.21 for  $K = 4$ .
- (b) The gain margin is

$$G.M. = 14 \text{ dB} .$$

- (c) When  $K = 5$ , the gain margin is

$$G.M. = 12 \text{ dB} .$$

- (d) We require  $K_v > 3$ , but  $K_v = \frac{K}{4}$ . So, we need  $K > 12$ . This gain can be utilized since  $K < 20$  is required for stability.

**FIGURE P9.21**

Bode diagram for  $G_c(s)G(s) = \frac{K}{s(s+1)(s+4)}$ , where  $K = 4$ .

- P9.22** (a) The resonant frequency  $\omega_r = 5.2$  rad/sec is point 6 on the Nichol's chart.
- (b) The bandwidth is between points 8 and 9. We estimate the bandwidth to be  $\omega_B = 7.5$  rad/sec.
- (c) The phase margin  $P.M. = 30^\circ$ .
- (d) The gain margin  $G.M. = 8$  dB.
- (e) Since we have  $P.M. = 30^\circ$ , then we estimate  $\zeta = 0.3$ . We can also approximate

$$\omega_n \approx \omega_r = 5.2 \text{ .ap9.1}$$

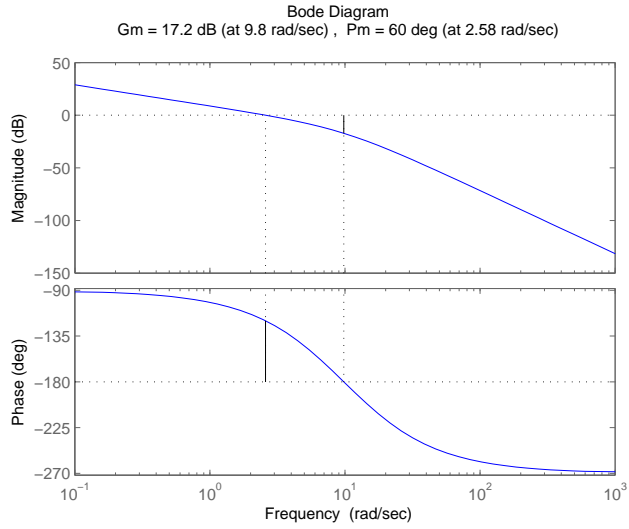
Thus,

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.56} = 2.5 \text{ sec .}$$

- P9.23** The phase margin is  $P.M. = 60$  deg when  $K = 266$ . The gain margin is

$$G.M. = 17.2 \text{ dB .}$$

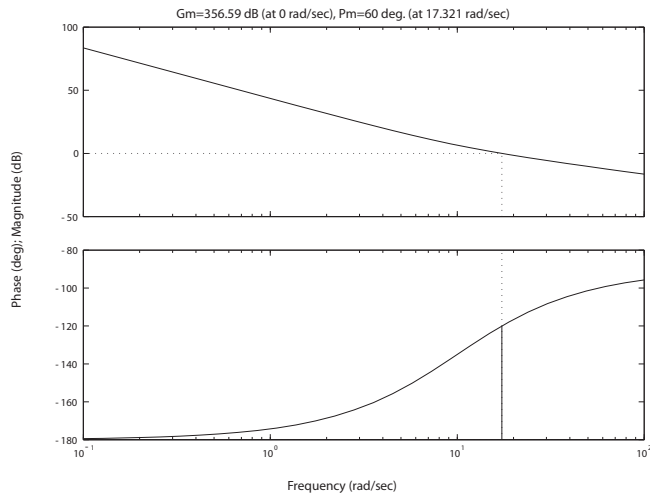
The Bode plot is shown in Figure P9.23.



**FIGURE P9.23**

Bode diagram for  $G_c(s)G(s) = \frac{K}{s(s+8)(s+12)}$ , where  $K = 266$ .

**P9.24** When  $K = 14.1$ , then  $P.M. = 45$  deg,  $G.M. = \infty$  dB and  $\omega_B = 29.3$  rad/sec.

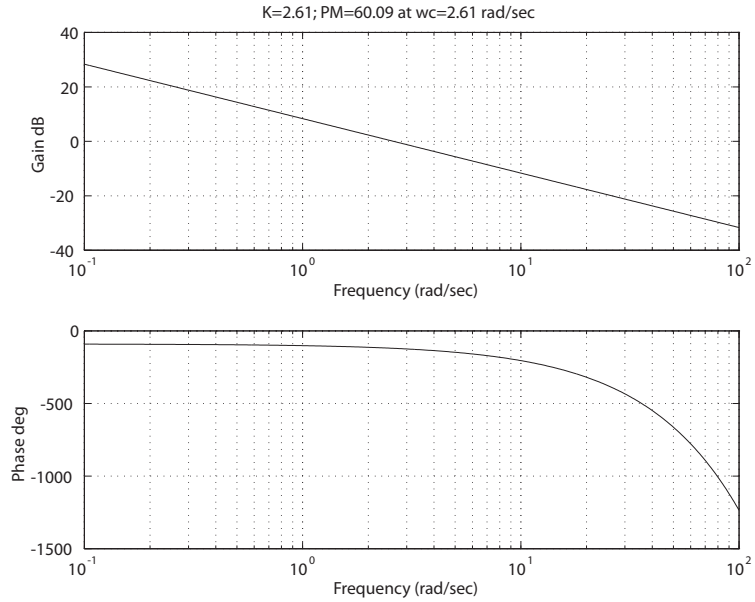


**FIGURE P9.24**

Bode diagram for  $G(s) = \frac{K(s+20)}{s^2}$ , where  $K = 14.1$ .

**P9.25** The phase margin is  $P.M. = 60$  deg when  $K = 2.61$  and  $T = 0.2$  second. The Bode plot is shown in Figure P9.25.





**FIGURE P9.25**  
 Bode diagram for  $G_c(s)G(s) = \frac{K e^{-0.2s}}{s}$ , where  $K = 2.61$ .

**P9.26** The loop transfer function is

$$G_c(s)G(s) = \frac{K}{s(0.25s + 1)(0.1s + 1)} .$$

The Bode plot is shown in Figure P9.26a for  $K = 10$ . The Nichols chart is shown in Figure P9.26b. The phase and gain margins are

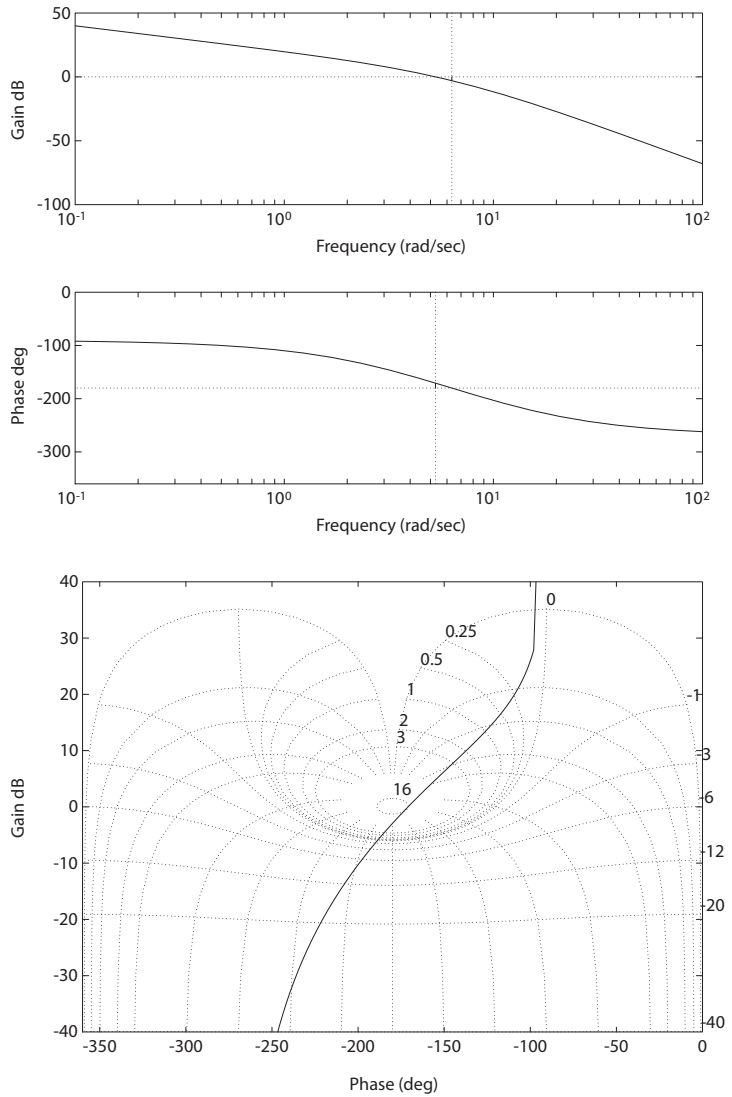
$$P.M. = 9^\circ \quad \text{and} \quad G.M. = 3 \text{ dB} .$$

The system bandwidth is  $\omega_B = 8 \text{ rad/sec}$ . From the  $P.M. = 9^\circ$ , we estimate  $\zeta = 0.09$ . Therefore, the predicted overshoot is

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 75\% , \quad \text{where } \zeta = 0.09 .$$

The resonant peak occurs at  $\omega_r = 5.5 \text{ rad/sec}$ . If we estimate  $\omega_n \approx \omega_r = 5.5 \text{ rad/sec}$ , then the settling time is

$$T_s = \frac{4}{\zeta\omega_n} = 8 \text{ sec} .$$



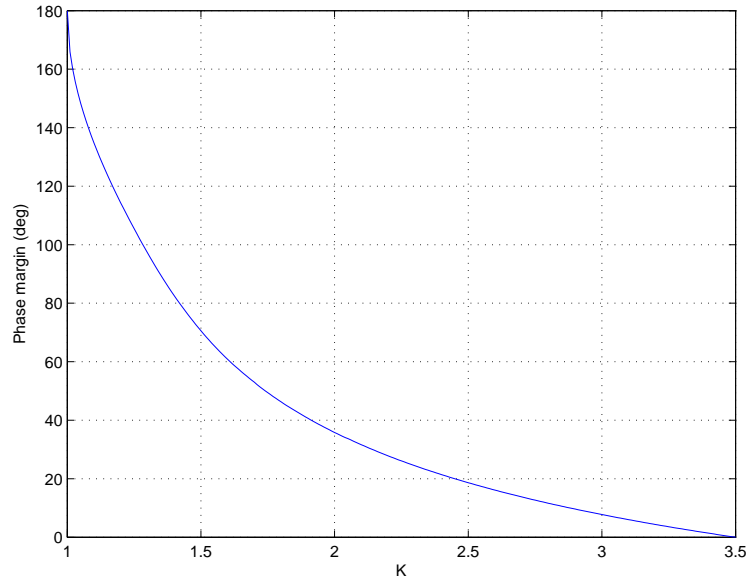
**FIGURE P9.26** (a) Bode diagram for  $G_c(s)G(s) = \frac{K}{s(0.25s+1)(0.1s+1)}$ , where  $K = 10$ . (b) Nichols chart for  $G_c(s)G(s) = \frac{K}{s(0.25s+1)(0.1s+1)}$ , where  $K = 10$ .

**P9.27** The loop transfer function is

$$L(s) = G_c(s)G(s)H(s) = \frac{4K}{(s^2 + 2s + 4)(s + 1)} .$$

The plot of the phase margin versus the gain  $K$  is shown in Figure P9.27. As the gain increases towards  $K_{max} = 3.5$ , the phase margin decreases

towards zero.



**FIGURE P9.27**  
Phase margin versus the gain  $K$ .

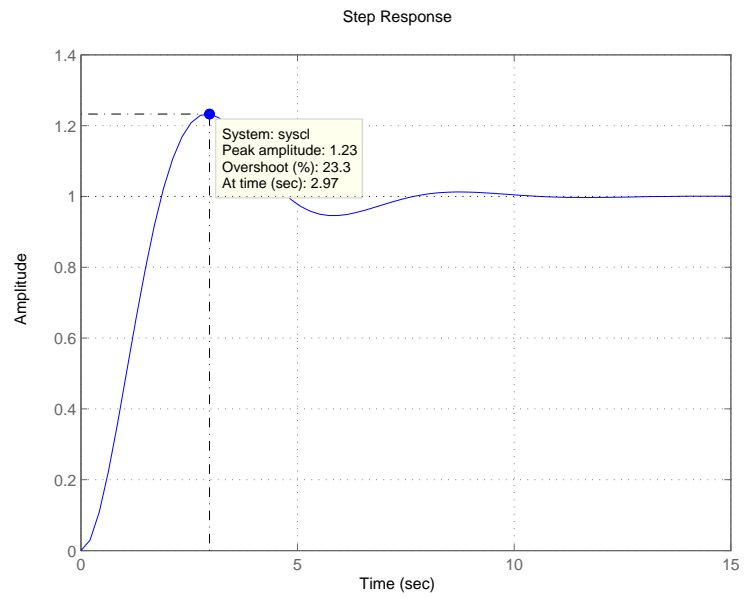
**P9.28** The loop transfer function is

$$G_c(s)G(s) = \frac{K_P}{s(s+1)} .$$

When  $K_P = 1.414$ , we have  $P.M. \approx 45^\circ$ . Using the approximation that  $\zeta \approx P.M./100$  we estimate that  $\zeta = 0.45$ . Then using the design formula

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 20.5\% .$$

The actual overshoot is 23.4%. The step input response is shown in Figure P9.28. The actual damping ratio is  $\zeta = 0.42$ . This shows that the approximation  $\zeta \approx P.M./100$  is quite applicable and useful in predicting the percent overshoot.

**FIGURE P9.28**

Step response showing a 23.3% overshoot.

## Advanced Problems

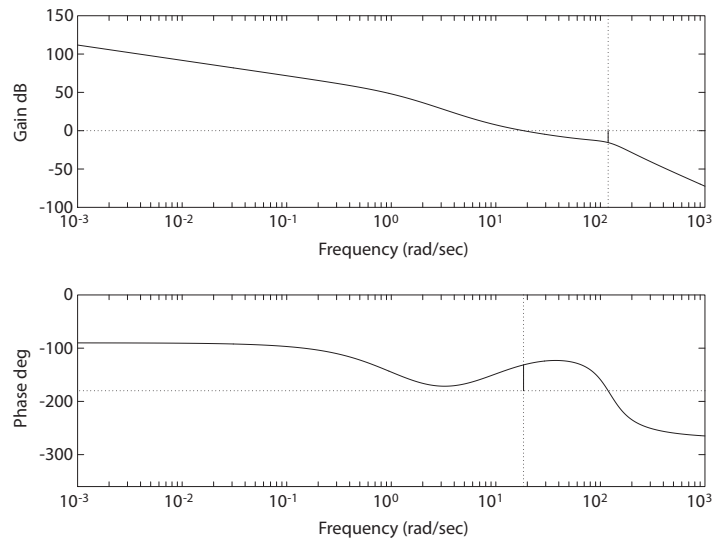
**AP9.1** The loop transfer function is

$$L(s) = G_c(s)G(s)H(s) = \frac{236607.5(s + 10)(s + 5)}{s(s + 2)(s^2 + 100s + \omega_n^2)(s + 1)}$$

(a) The Bode plot for

$$\omega_n^2 = 15267$$

is shown in Figure AP9.1a.



**FIGURE AP9.1**

(a) Bode Diagram for  $L(s) = \frac{236607.5(s+10)(s+5)}{s(s+2)(s^2+100s+\omega_n^2)(s+1)}$ , where  $\omega_n^2 = 15267$ .

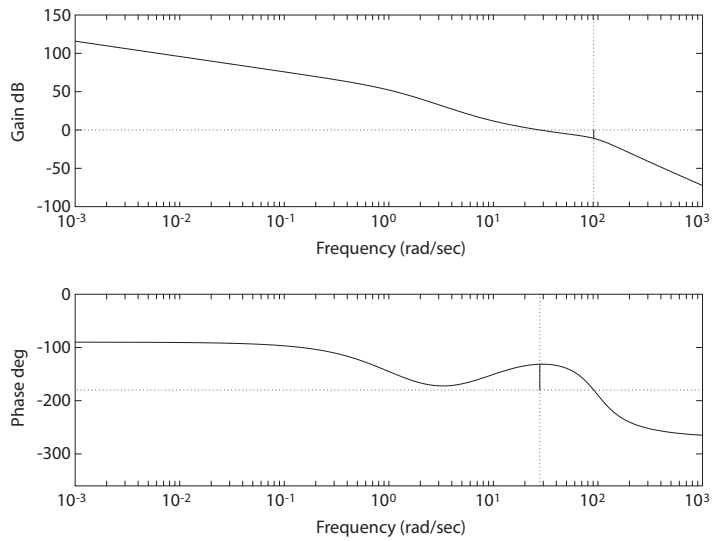
The phase and gain margins are

$$P.M. = 48.6^\circ \quad \text{and} \quad G.M. = 15.5 \text{ dB} .$$

(b) The Bode plot for  $\omega_n^2 = 9500$  is shown in Figure AP9.1b. The gain and phase margins are

$$P.M. = 48.5^\circ \quad \text{and} \quad G.M. = 10.9 \text{ dB} .$$

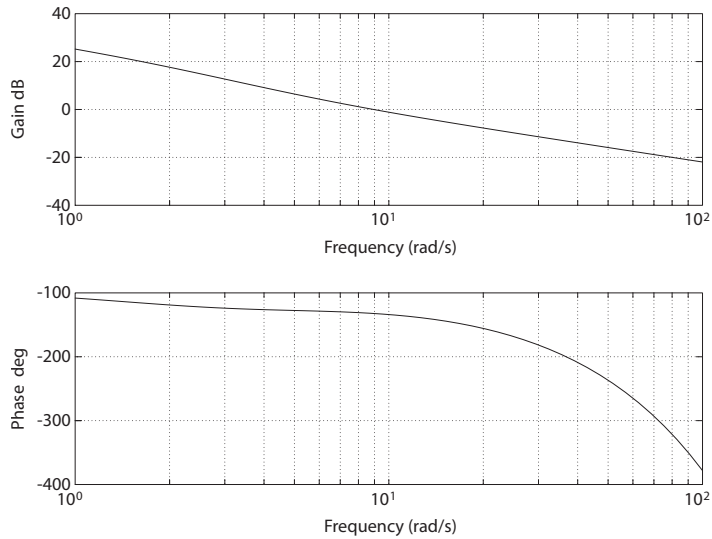
Reducing the natural frequency by 38% has the effect of reducing the gain margin by 30%.



**FIGURE AP9.1**

CONTINUED: (b) Bode Diagram for  $L(s) = \frac{236607.5(s+10)(s+5)}{s(s+2)(s^2+100s+\omega_n^2)(s+1)}$ , where  $\omega_n^2 = 9500$ .

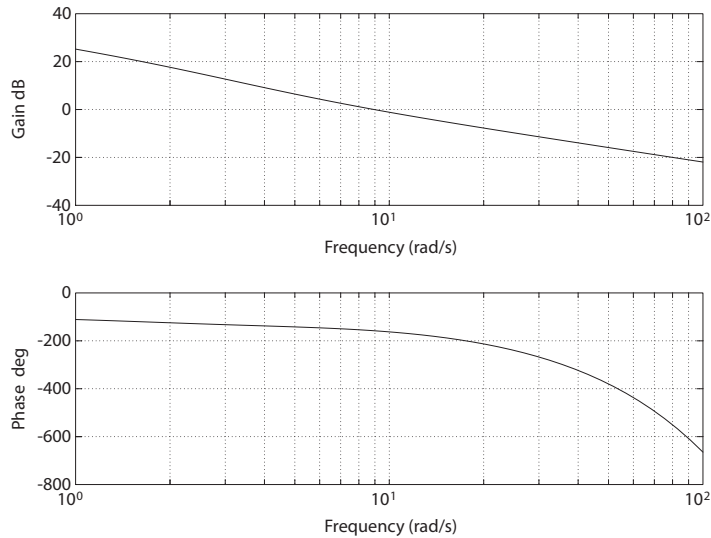
**AP9.2** (a) The Bode plot with  $T = 0.05$  sec is shown in Figure AP9.2a. The phase margin is  $P.M. = 47.7^\circ$  and the gain margin is  $G.M. = 11.2$  dB.



**FIGURE AP9.2**

(a) Bode Diagram for  $G_c(s)G(s)H(s) = 8\frac{(s+5)}{s(s+2)}e^{-sT}$ , where  $T = 0.05$ s.

(b) The Bode plot with  $T = 0.1$  sec is shown in Figure AP9.2b. The



**FIGURE AP9.2**  
CONTINUED: (b) Bode Diagram for  $G_c(s)G(s)H(s) = 8 \frac{(s+5)}{s(s+2)} e^{-sT}$ , where  $T = 0.1s$ .

phase margin is  $P.M. = 22.1^\circ$  and the gain margin is  $G.M. = 4.18$  dB. A 100% increase in time delay  $T$  leads to a 50% decrease in phase and gain margins.

(c) The damping ratio  $\zeta \approx P.M./100$  and

$$P.O. \approx 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} .$$

So, for  $T = 0.05$  sec,  $\zeta \approx 0.47$  and  $P.O. \approx 18.7\%$ . Also, for  $T = 0.1$  sec,  $\zeta \approx 0.22$  and  $P.O. \approx 49.2\%$ .

**AP9.3** The loop transfer function is

$$L(s) = G_c(s)G(s)H(s) = \frac{66K(1 + 0.1s)}{(1 + 0.01s)(1 + 0.01s)(1 + 1.5s)(1 + 0.2s)} .$$

(a) When  $K = 1$ , the gain and phase margins are

$$G.M. = 18.4 \text{ dB} \quad \text{and} \quad P.M. = 55^\circ .$$

(b) When  $K = 1.5$ , the gain and phase margins are

$$G.M. = 14.9 \text{ dB} \quad \text{and} \quad P.M. = 47.8^\circ .$$

(c,d) The bandwidth and settling time with  $K = 1$  are  $\omega_B = 233.6$  rad/sec and  $T_s = 0.4$  second. When  $K = 1.5$ , we determine that  $\omega_B = 294.20$  rad/sec and  $T_s = 0.33$  second.

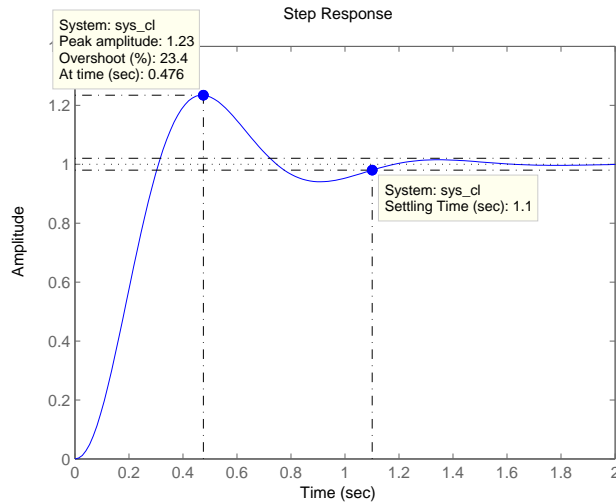
**AP9.4** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K(s + 40)}{s(s + 15)(s + 10)} .$$

The gain  $K = 28.8$  satisfies the specifications. The actual gain and phase margins are

$$G.M. = 18.8 \text{ dB} \quad \text{and} \quad P.M. = 45^\circ .$$

The system bandwidth is  $\omega_B = 10.3$  rad/sec. The step response is shown in Figure AP9.4.



**FIGURE AP9.4**  
Closed-loop system step response.

**AP9.5** The loop transfer function is

$$L(s) = G_c(s)G(s) = K \frac{s + 0.4}{s^4 + 9s^3 + 18s^2} .$$

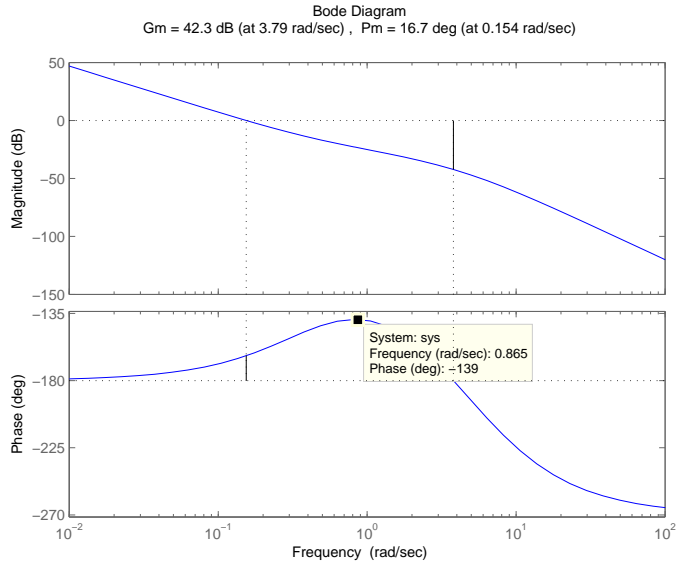
The Bode plot for  $K = 1$  is shown in Figure AP9.5. From the phase response, we determine that the maximum  $P.M. \approx 41^\circ$ . From the magnitude response (for  $K = 1$ ), we find that the gain needs to be raised to  $K = 14$  to achieve maximum phase margin at  $\omega = 0.826$  rad/sec. The



gain and phase margin with  $K = 14$  are

$$G.M. = 19.3 \text{ dB} \quad \text{and} \quad P.M. = 40.9^\circ .$$

Also, the overshoot is  $P.O. = 38.3\%$ .



**FIGURE AP9.5**

Bode plot for  $L(s) = K \frac{s+0.4}{s^4+9s^3+18s^2}$  with  $K = 1$ .

**AP9.6** With  $D > 2m$ , the gain can be increased up to  $K = 100$ , while still retaining stability.

**AP9.7** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K(s + 4)}{s^2} .$$

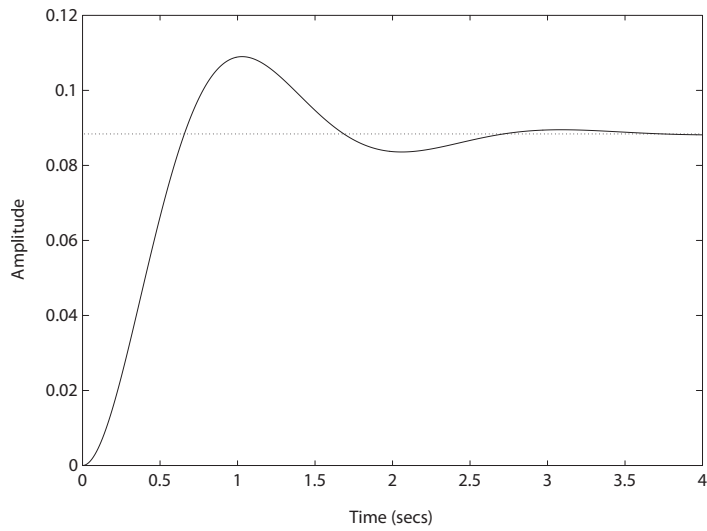
We select

$$K = 2\sqrt{2}$$

for  $P.M. = 45^\circ$ . The system bandwidth is

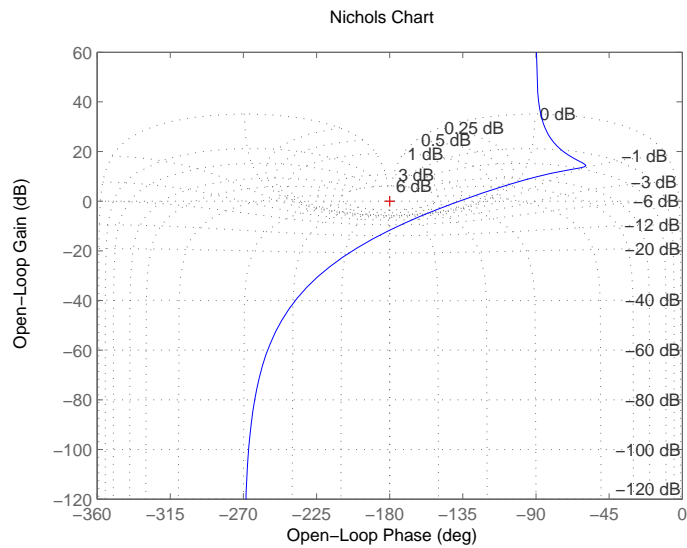
$$\omega_B = 5.88 \text{ rad/sec} .$$

The disturbance response is shown in Figure AP9.7. The maximum output due to a disturbance is  $y(t) = 0.11$ .



**FIGURE AP9.7**  
Closed-loop system disturbance response.

**AP9.8** A reasonable choice for the gain is  $K = 2680$ . The phase margin is  $P.M. = 42.8^\circ$  and the percent overshoot is  $P.O. = 18.9\%$ . The Nichols chart is shown in Figure AP9.8.

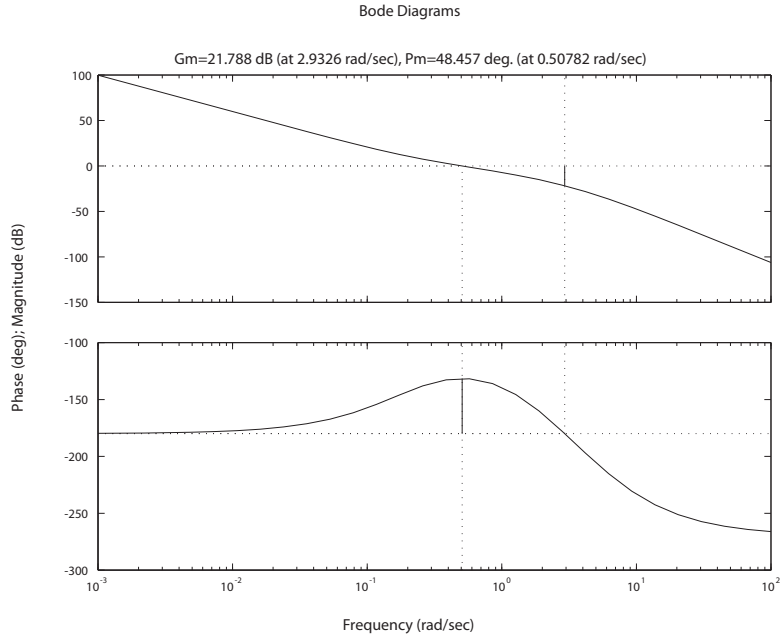


**FIGURE AP9.8**  
Nichols chart.

**AP9.9** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K_p(s + 0.2)}{s^2(s^2 + 7s + 10)} .$$

At the maximum phase margin,  $K_p = 4.9$  for  $P.M. = 48.6^\circ$ . The Bode diagram is shown in Figure AP9.9.



**FIGURE AP9.9** Phase and gain margin.

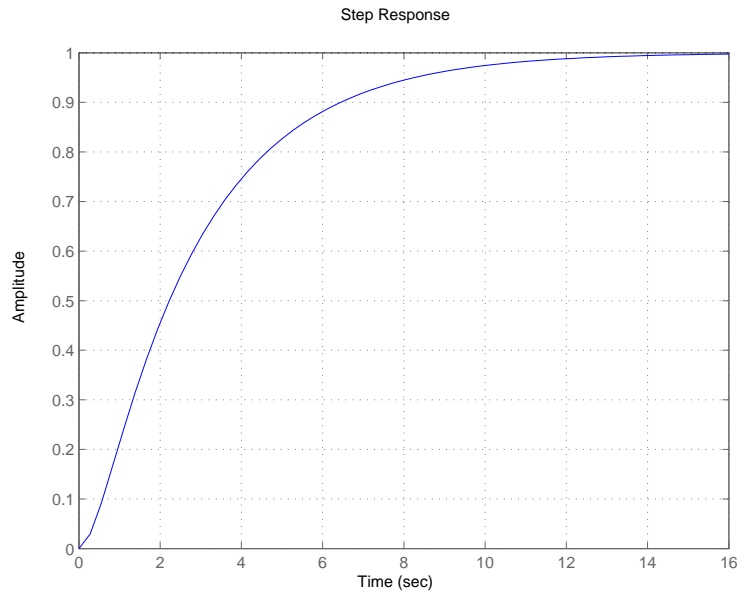
**AP9.10** The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + 3s + 1} .$$

We require  $K = 1$  a zero steady-state tracking error to a unit step. The step response is shown in Figure AP9.10. Computing  $T(j\omega) = 0.707$  it follows that

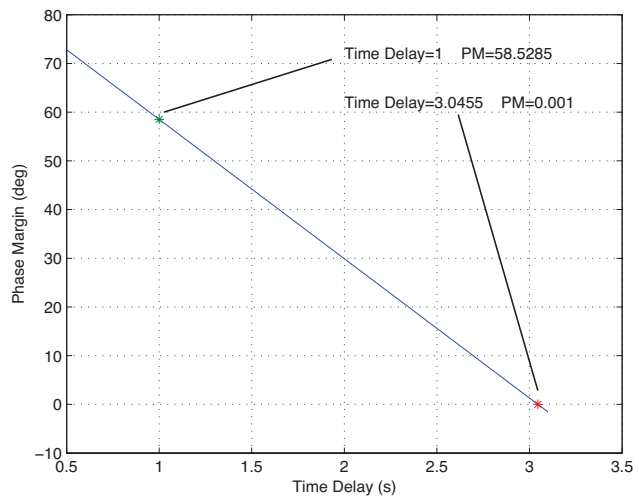
$$\left| \frac{1}{(j\omega)^2 + 3j\omega + 1} \right| = 0.707 \quad \text{or} \quad \omega^4 + 7\omega^2 - 1 = 0 .$$

Solving for  $\omega$  yields  $\omega = 0.37$  rad/s. This is the bandwidth of the system.



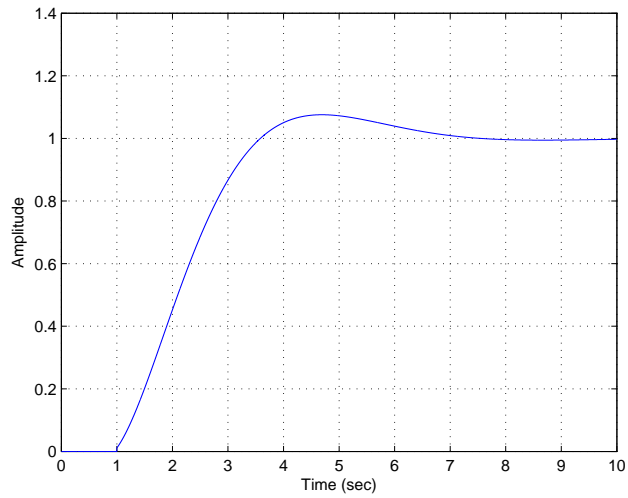
**FIGURE AP9.10**  
Unit step response.

**AP9.11** The phase margin versus time delay is shown in Figure AP9.11a.



**FIGURE AP9.11**  
Phase margin versus time delay.

The maximum time delay is  $T = 3.04$  s for stability. The step response is shown in Figure AP9.11b. The percent overshoot is  $P.O. = 7.6\%$ .



**FIGURE AP9.11**  
Unit step response.

## Design Problems

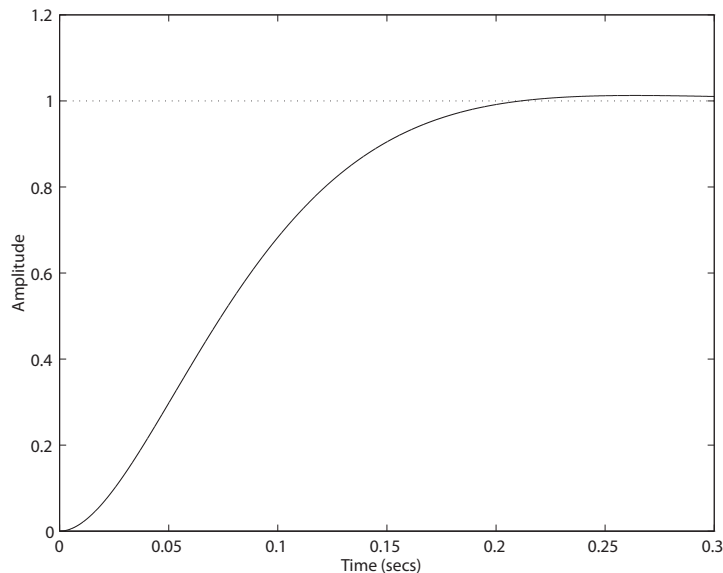
**CDP9.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$\frac{\theta(s)}{V_a(s)} = \frac{26.035}{s(s + 33.142)},$$

where we neglect the motor inductance  $L_m$  and where we switch off the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). The closed-loop system characteristic equation is

$$1 + \frac{26.035K_a}{s(s + 33.142)} = 0.$$

The phase margin is  $P.M. = 70.4^\circ$  when  $K_a = 16$ . The step response with  $K = 16$  is shown below.

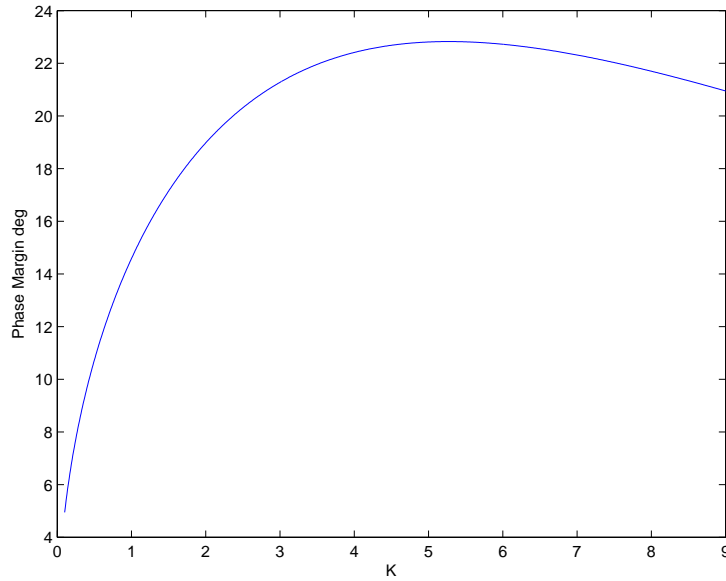


- DP9.1**
- The gain and phase margins are  $G.M. = 7$  dB and  $P.M. = 60^\circ$ .
  - The resonant peak and frequency are  $M_{p\omega} = 2$  dB and  $\omega_r = 5$  rad/sec.
  - We have  $\omega_B = 20$  rad/sec. From  $M_{p\omega} = 2$  dB we estimate  $\zeta = 0.45$  (Figure 8.11 in Dorf & Bishop). Also,  $\omega_r/\omega_n = 0.8$ , so  $\omega_n = 6.25$ . Thus,  $T_s = 1.4$ .
  - We need  $P.O. = 30^\circ$  or  $\zeta = 0.3$  or  $P.M. \approx 30^\circ$ . So, we need to raise the gain by 10 dB or  $K = 3.2$ .

**DP9.2** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K(s + 0.5)}{s^2(s^2 + 7.5s + 9)} .$$

When  $K = 6.25$ , we have the maximum phase margin. The phase margin maximum is  $P.M. = 23^\circ$ . The plot of  $P.M.$  versus  $K$  is shown in Figure DP9.2a.



**FIGURE DP9.2**

(a) Phase margin versus  $K$  for  $L(s) = \frac{K(s+0.5)}{s^2(s^2+7.5s+9)}$ .

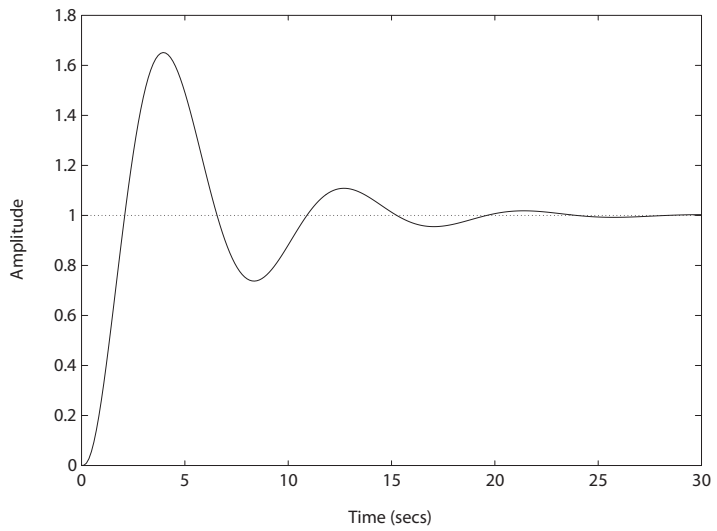
The predicted damping is  $\zeta = 0.23$ . It then follows that the predicted percent overshoot is

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 48\% .$$

The actual overshoot is 65%. The step input response is shown in Figure DP9.2b. The resonant peak occurs at  $\omega_r = 0.75$  rad/sec. Approximating  $\omega_n \approx \omega_r = 0.75$  rad/sec, we can estimate the settling time as

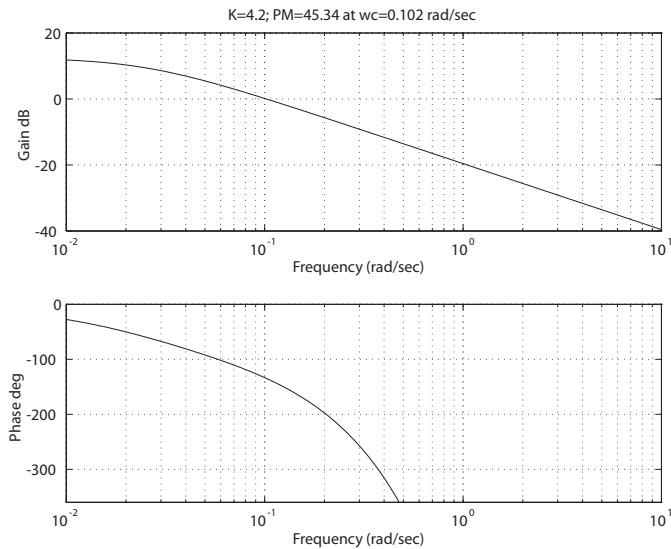
$$T_s = \frac{4}{\zeta\omega_n} = 23 \text{ sec} .$$

The actual settling time is 20 sec.



**FIGURE DP9.2**  
CONTINUE: (b) Closed-loop unit step response.

**DP9.3** We want to select the gain  $K$  as large as possible to reduce the steady-state error, but we want a minimum phase margin of  $P.M. = 45^\circ$  to achieve good dynamic response. A suitable gain is  $K = 4.2$ , see Figure DP9.3.



**FIGURE DP9.3**  
Bode plot for  $G(s) = \frac{K e^{-10s}}{40s + 1}$ .



**DP9.4** We are given the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s+1)(s+4)}$$

which can be written as

$$G_c(s)G(s) = \frac{K_v}{s(s+1)(0.25s+1)} .$$

The performance results are summarized in Table DP9.4.

$K_v$	$G.M.$ (dB)	$P.M.$ (deg)	$\omega_B$ (rad/sec)	$P.O.$ (%)	$T_s$ (sec)
0.40	21.9	64.2	0.62	4.4	9.8
0.75	16.5	49.0	1.09	19.0	10.1

**TABLE DP9.4** Summary for  $K_v = 0.40$  and  $K_v = 0.75$ .

When  $K_v = 0.40$ , we have

$$\frac{e_{ss}}{A} = \frac{1}{0.40} = 2.5 ,$$

or 2 1/2 times the magnitude of the ramp. This system would be acceptable for step inputs, but unacceptable for ramp inputs.

**DP9.5** (a) With a time delay of  $T = 0.8$  second, we determine that the proportional controller

$$G_c(s) = K = 7$$

provides a suitable response with

$$P.O. = 8.3 \% \quad e_{ss} = 12.5 \% \quad T_s = 4.38 \text{ sec} .$$

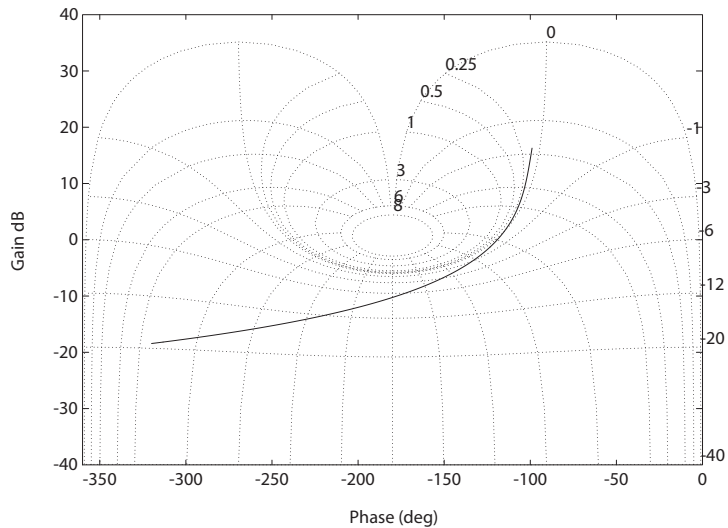
(b) A suitable proportional, integral controller is

$$G_c(s) = K_1 + K_2/s = 6 + 0.6/s .$$

The response to a unit step is

$$P.O. = 5.14 \% \quad e_{ss} = 0 \% \quad T_s = 6.37 \text{ sec} .$$

The Nichols chart is shown in Figure DP9.5.



**FIGURE DP9.5**  
Nichols chart for  $G_c(s)G(s) = \frac{(K_1s+K_2)e^{-0.8s}}{s(10s+1)}$ , where  $K_1 = 6$  and  $K_2 = 0.6$ .

**DP9.6** With  $K = 170$ , at the two extreme values of  $b$ , we have

$$\begin{aligned} b = 80 \quad P.M. &= 91.62^\circ \quad G.M. = 13.66 \text{ dB} \\ b = 300 \quad P.M. &= 75.23^\circ \quad G.M. = 25.67 \text{ dB} . \end{aligned}$$

Since reducing the value of  $K$  only increases the  $P.M.$  and  $G.M.$ , a value of

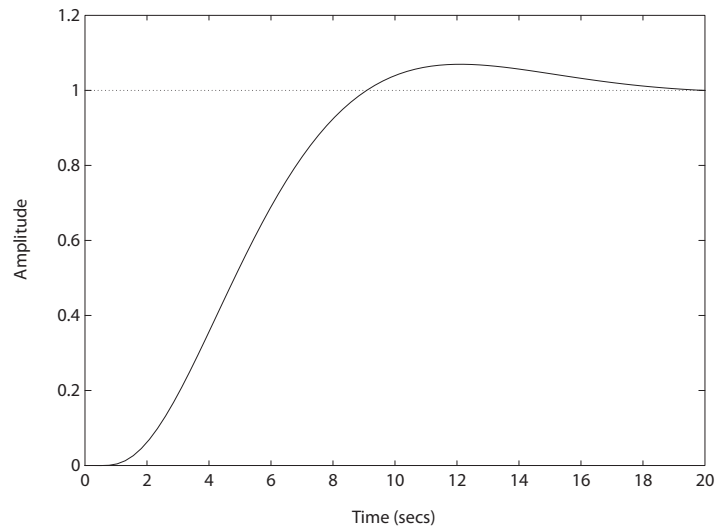
$$K = 170$$

is suitable to meet  $P.M. = 40^\circ$  and  $G.M. = 8 \text{ dB}$  for the range of  $b$ .

**DP9.7** A suitable gain is

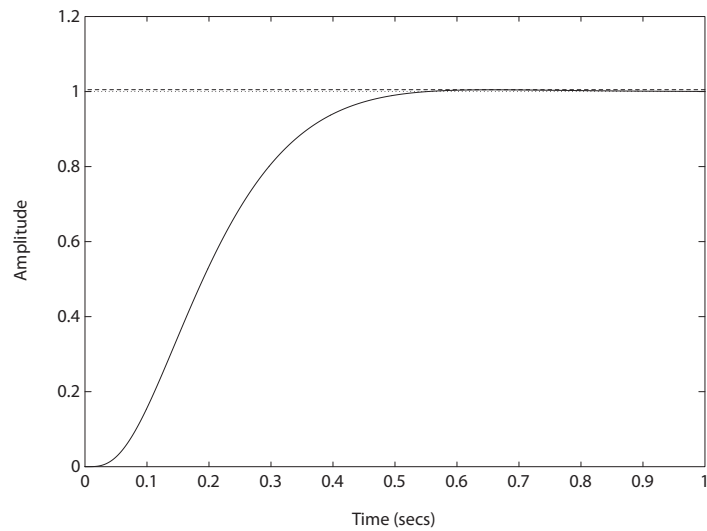
$$K = 0.22 .$$

This results in  $P.M. = 60.17^\circ$  and  $G.M. = 13.39 \text{ dB}$ . The step response is shown in Figure DP9.7.



**FIGURE DP9.7**  
Lunar vehicle step response.

**DP9.8** A gain of  $K = 315000$  will satisfy the *P.O.* specification, while giving the fastest response. The step response is shown in Figure DP9.8.



**FIGURE DP9.8**  
Steel rolling mill step response.

**DP9.9** The closed-loop transfer function is

$$T_s(2) = \frac{G_1(s)}{1 + G_c(s)G_2(s)}T_o(s) + \frac{G_c(s)G_2(s)}{1 + G_c(s)G_2(s)}T_{2d}(s) .$$

where

$$G_1(s) = \frac{1}{(10s + 1)(50s + 1)}$$

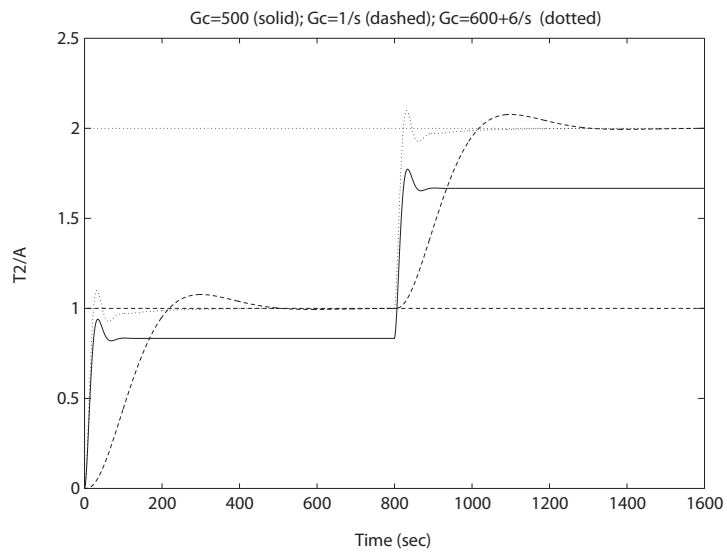
and

$$G_2(s) = \frac{0.01}{(10s + 1)(50s + 1)} .$$

The steady-state error (with  $G_c(s) = 500$ ) to a unit step  $2A$  (and after the system has settled out subsequent to a step of magnitude  $A$ ) is

$$e_{ss} = 2(0.167) = 0.33 .$$

The step response is shown in Figure DP9.9.



**FIGURE DP9.9**

Two tank temperature control step response.

A suitable integral controller is

$$G_c(s) = \frac{1}{s} .$$

In this case, the steady-state tracking error is zero, since the system is a type 1. The system response is shown in Figure DP9.9. With the integral controller, the settling time is about  $T_s = 438$  seconds and the  $P.O. = 7\%$ . A suitable PI controller is

$$G_c(s) = 600 + \frac{6}{s} .$$

With the PI controller, the settling time is about  $T_s = 150$  seconds and the  $P.O. = 10\%$ .

**DP9.10** The system is given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}r \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 - K_1 & 3 - K_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} .$$

The associated transfer function is

$$T(s) = \frac{1}{s^2 + (K_2 - 3)s + K_1 - 2} .$$

The characteristic polynomial is

$$s^2 + (K_2 - 3)s + K_1 - 2 = 0 .$$

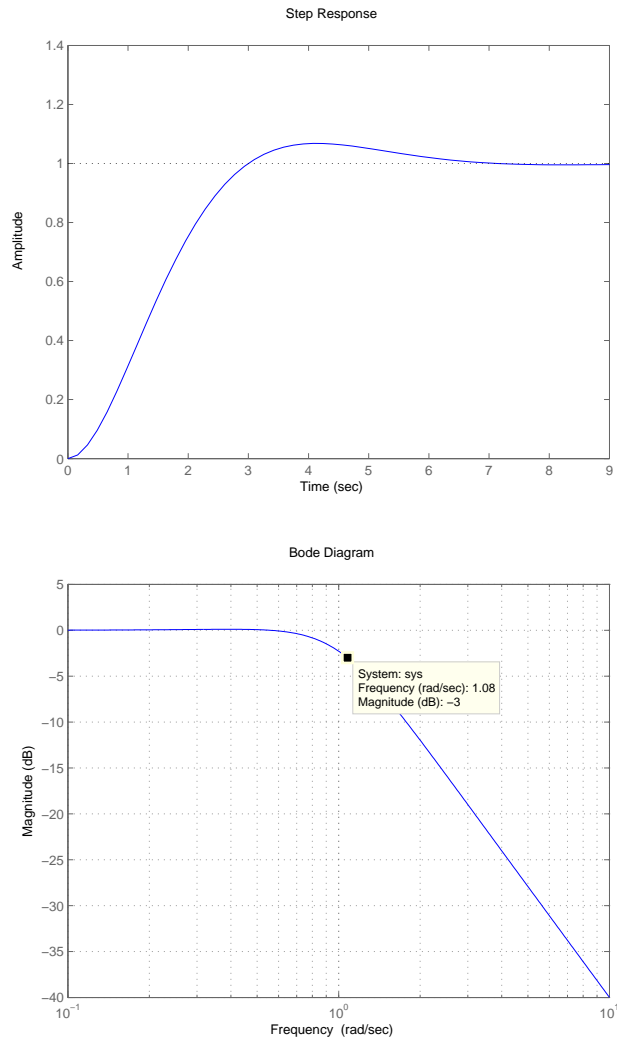
If we select  $K_1 = 3$ , then we have a zero-steady error to a unit step response  $R(s) = 1/s$ , since

$$\lim_{s \rightarrow 0} s [1 - T(s)] R(s) = \lim_{s \rightarrow 0} \frac{s^2 + (K_2 - 3)s}{s^2 + (K_2 - 3)s + K_1 - 2} = 0 .$$

Let

$$\mathbf{K} = \begin{pmatrix} 3 & 4.3 \end{pmatrix} .$$

The step response is shown in Figure DP9.10a. The bandwidth is  $\omega_b = 1.08$  rad/s, as seen in Figure DP9.10b.

**FIGURE DP9.10**

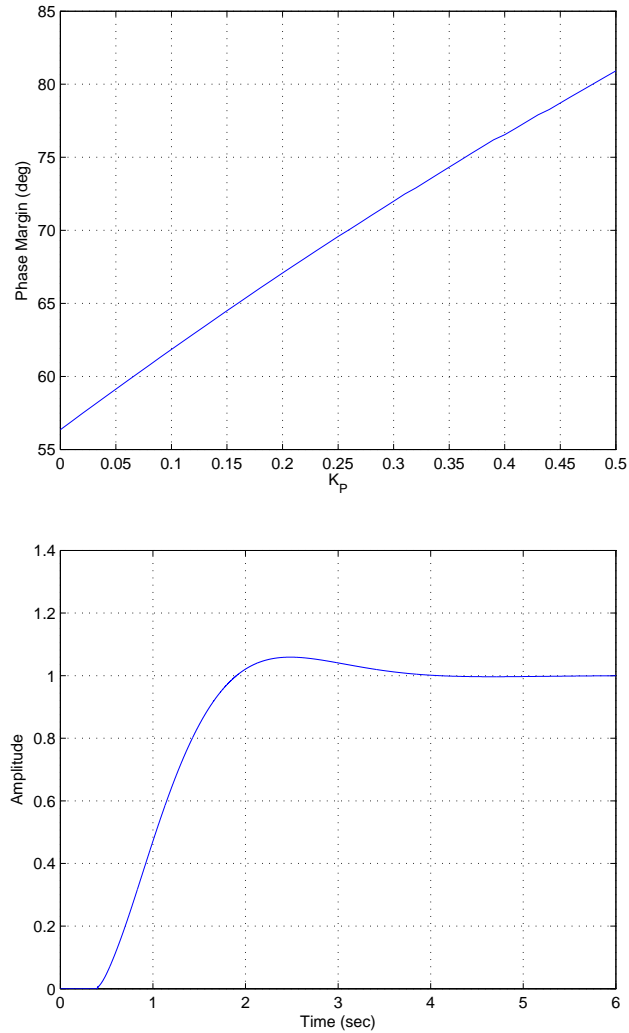
Step response with  $\mathbf{K} = [3 \ 4.3]$  and closed-loop Bode plot.

- DP9.11** A time domain step response specification  $P.O. > 10\%$  requires the dominant poles to have a damping ratio of  $\zeta = 0.6$ . This time domain specification can be transformed to a frequency response specification using the approximation  $P.M. \approx 100\zeta = 60^\circ$ . To keep the problem tractable, we consider the controller with the form

$$G_c(s) = K_P + \frac{K_I}{s} = K_P + \frac{1}{s},$$

where we let  $K_I = 1$ . The plot of the  $P.M.$  as a function of  $K_P$  is shown

in Figure DP9.11a. If we select  $K_P = 0.07$  we expect a phase margin of approximately  $60^\circ$ , hence a percent overshoot  $P.O. \leq 10\%$ . The step response is shown in Figure DP9.11b. The actual phase margin is  $P.M. = 60.2^\circ$ , the percent overshoot is  $P.O. = 5.9\%$  and the settling time is  $T_s = 3.4$  sec.



**FIGURE DP9.11**

(a) Phase margin versus controller gain  $K_P$  and  $K_I = 1$ . (b) Step response with  $K_P = 0.07$  and  $K_I = 1$ .

## Computer Problems

**CP9.1** The m-file script to generate the Bode plot (from which the gain and phase margin can be determined) is shown in Figure CP9.1. The transfer function is

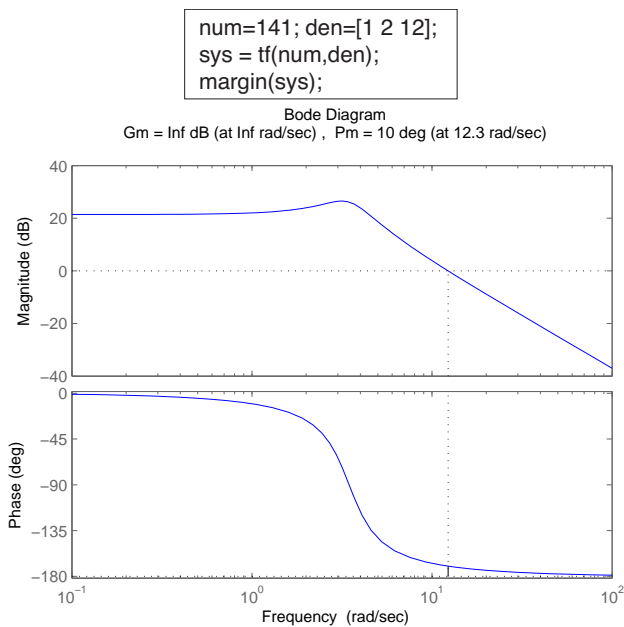
$$G(s) = \frac{141}{s^2 + 2s + 12} .$$

The gain margin is

$$G.M. = \infty$$

and the phase margin is

$$P.M. = 10^\circ .$$

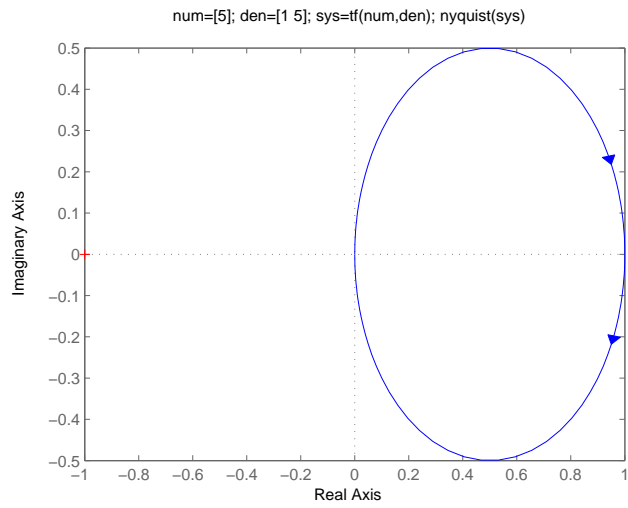


**FIGURE CP9.1**

Gain and phase margin with the `margin` function.

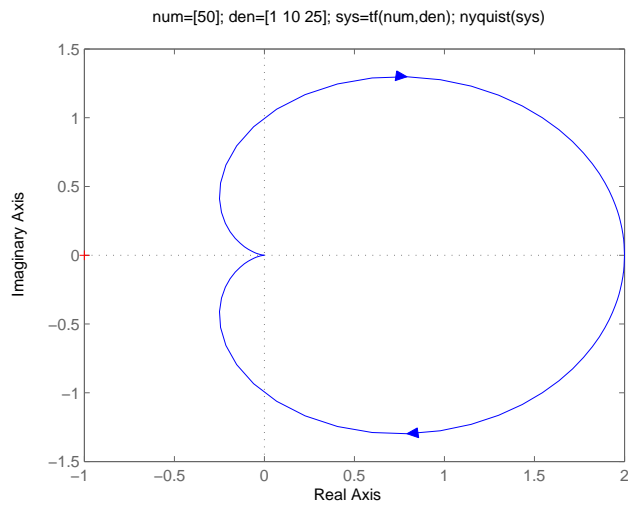


**CP9.2** The Nyquist plots are shown in Figures CP9.2a-c.



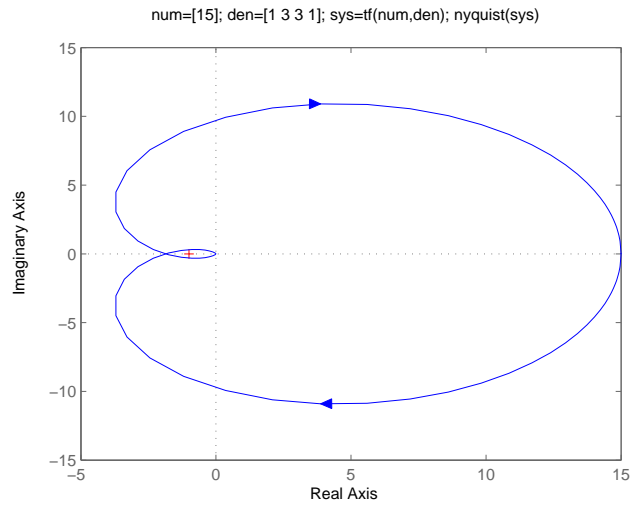
**FIGURE CP9.2**

(a) Nyquist plot for  $G(s) = \frac{5}{s+5}$ .



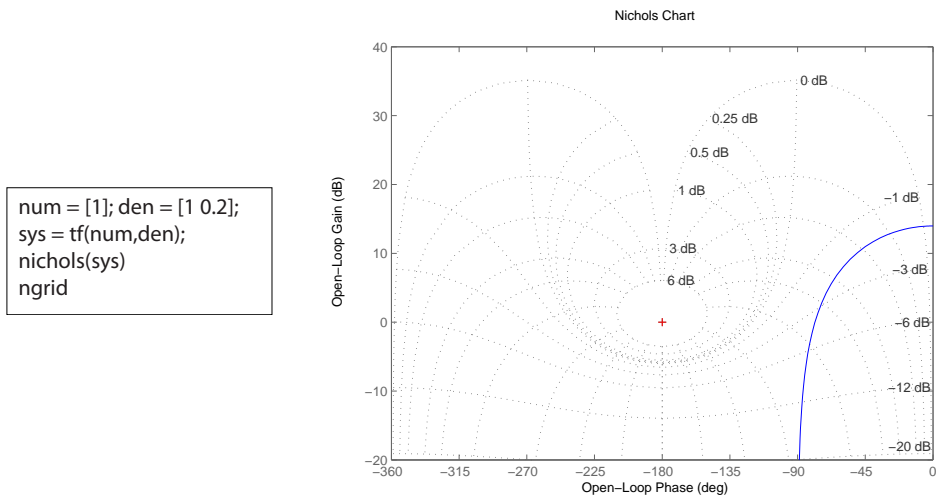
**FIGURE CP9.2**

CONTINUED: (b) Nyquist plot for  $G(s) = \frac{50}{s^2+10s+25}$ .



**FIGURE CP9.2**  
 CONTINUED: (c) Nyquist plot for  $G(s) = \frac{15}{s^3 + 3s^2 + 3s + 1}$ .

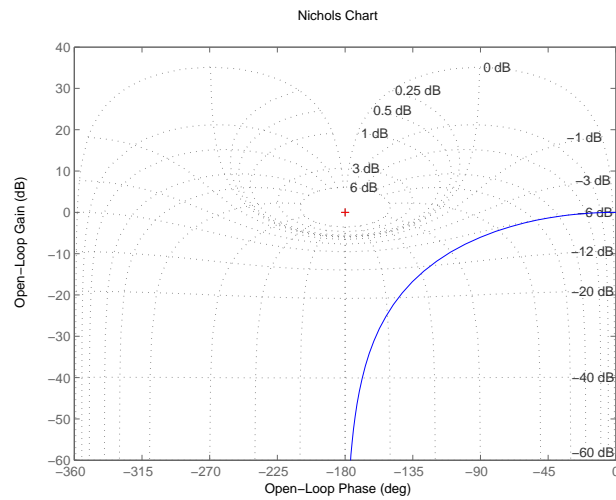
**CP9.3** The m-file script to generate the Nichols chart for part (a) is shown in Figure CP9.3a. The Nichols charts for (b) and (c) are similarly generated; all plots are in Figure CP9.3a-c.



**FIGURE CP9.3**  
 (a) M-file script and Nichols chart for  $G(s) = \frac{1}{s + 0.1}$ .

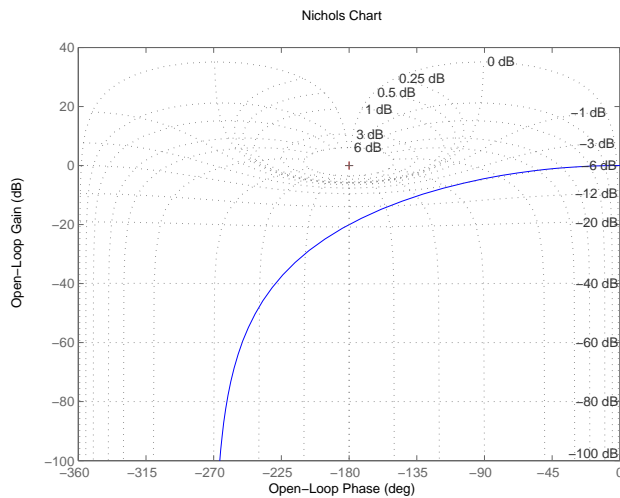
The gain and phase margin for each transfer function are as follows:

- (a)  $G.M. = \infty$  and  $P.M. = 102^\circ$
- (b)  $G.M. = \infty$  and  $P.M. = \infty$
- (c)  $G.M. = 20$  dB and  $P.M. = \infty$



**FIGURE CP9.3**

CONTINUED: (b) Nichols chart for  $G(s) = \frac{1}{s^2 + 2s + 1}$ .



**FIGURE CP9.3**

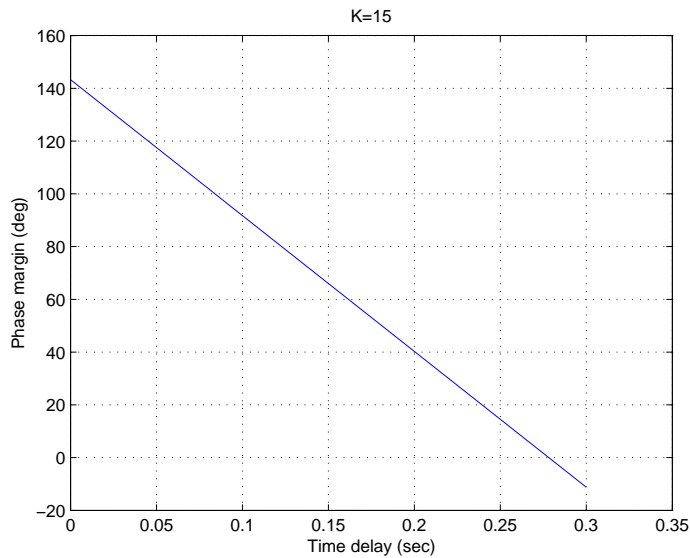
CONTINUED: (c) Nichols chart for  $G(s) = \frac{24}{s^3 + 9s^2 + 26s + 24}$ .

**CP9.4** To obtain a phase margin  $P.M. = 40^\circ$  we select

$$K = 15$$

when  $T = 0.2$  second. The variation in the phase margin for  $0 \leq T \leq 0.3$  is shown in Figure CP9.4.

```
T=[0:0.01:0.3]; K=15;
num=K;den=[1 12]; sys = tf(num,den);
% w=logspace(-2,1,400);
for i=1:length(T)
    [mag,phase,w]=bode(sys);
    ph(1:length(phase))=phase(1,1,:); ph=ph';
    ph2=ph-w*T(i)*180/pi;
    [Gm,Pm,Wcg,Wcp]=margin(mag,ph2,w);
    clear ph ph2
    PMo(i)=Pm;
end
plot(T,PMo), grid
xlabel('Time delay (sec)')
ylabel('Phase margin (deg)')
```



**FIGURE CP9.4**

Variation in the phase margin for  $0 \leq T \leq 0.3$  with  $K = 15$ .

**CP9.5** The loop transfer function is

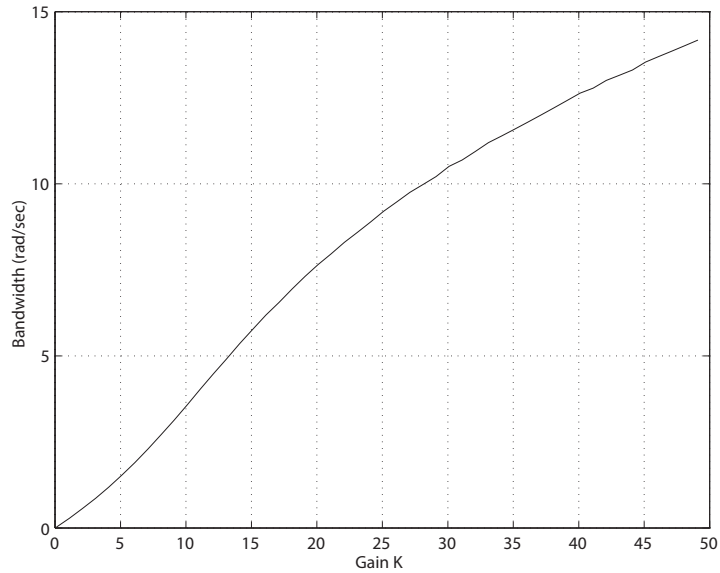
$$L(s) = G_c(s)G(s) = \frac{K(s + 50)}{s(s + 20)(s + 10)} .$$

The plot of system bandwidth versus the gain  $K$  is shown in Figure CP9.7.

```

K=[0.1:1:50];
w=logspace(-2,3,2000);
den=[1 30 200 0];
for i=1:length(K)
    num=K(i)*[1 50]; sys = tf(num,den);
    sys_cl = feedback(sys,[1]);
    [mag,phase,w]=bode(sys_cl,w);
    L=find(mag<0.707); wb(i)=w(L(1));
end
plot(K,wb), grid
xlabel('Gain K')
ylabel('Bandwidth (rad/sec)')

```



**FIGURE CP9.5**

Variation in the system bandwidth for  $0 \leq K \leq 50$ .

**CP9.6** The m-file script and Bode plot are shown in Figure CP9.6. The gain and phase margin and  $\omega_c$  are determined to be  $G.M. = 2.23$ ,  $P.M. = 26^\circ$  and  $\omega_c = 12.6$  rad/sec. So, the maximum value of  $b_o$  is found to be

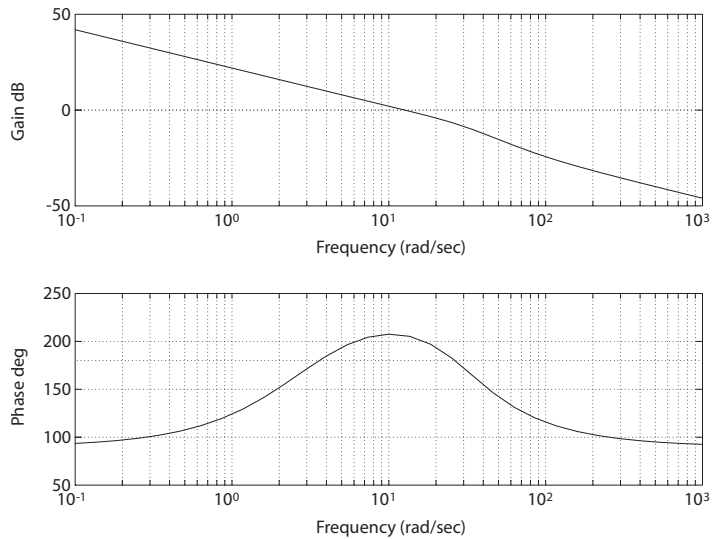
$$b_{o\max} = 2.13b_o = 1.11 .$$

In this problem, there is also a minimum value of  $b_o$ . Using the Routh-

```

numg = -0.5*[1 0 -2500]; deng = [1 47 850 -3000];
sysg = tf(numg,deng);
numc = 10*[1 3]; denc = [1 0]; sysc = tf(numc,denc);
sys_o = series(sysc,sysg);
bode(sys_o)
[mag,phase,w] = bode(sys_o);
[gm,pm,wg,wc] = margin(mag,phase,w)
    
```

gm =	2.2238
pm =	26.3187
wg =	26.1155
wc =	12.6487



**FIGURE CP9.6**  
Using the `margin` function to compute stability margins.

Hurwitz method, we determine that (for stability) the range of  $b_o$  is

$$0.14 < b_o < 1.11 .$$

**CP9.7** The m-file script is shown in Figure CP9.7a. Since we do not have a value for  $J$ , we write the loop transfer function as

$$G_c(s)G(s) = \frac{\bar{K}_1 + \bar{K}_2s}{s^2}$$

where  $\bar{K}_1 = K_1/J$  and  $\bar{K}_2 = K_2/J$ . We work with  $\bar{K}_1$  and  $\bar{K}_2$ , then we can always compute  $K_1$  and  $K_2$  whenever  $J$  is specified. A PD controller which meets the specs is given by

$$G_c(s) = 0.04 + 0.3s .$$

The step response is shown in Figure CP9.7b. The Bode plot is shown in

```

%
% Part (a)
%
numc = [0.3 0.04]; denc = [1]; sysc = tf(numc,denc);
numg = [1]; deng = [1 0 0]; sysg = tf(numg,deng);
sys_o = series(sysc,sysg);
sys_cl = feedback(sys_o,[1]);
step(sys_cl), pause
%
% Part (b)
%
w = logspace(-1,1,400);
[mag,phase] = bode(sys_o,w);
[gm,pm,w1,w2] = margin(mag,phase,w);
margin(mag,phase,w), pause
%
% Part (c)
%
T = [1:0.1:5];
for i = 1:length(T)
    [numd,dend] = pade(T(i),2); sysd = tf(numd,dend);
    sys_o1 = series(sysd,sys_o);
    sys_cl1 = feedback(sys_o1,sysd);
    p(:,i) = pole(sys_cl1);
end
plot(real(p),imag(p),'*');grid
xlabel('Real Axis');
ylabel('Imag Axis')

```

**FIGURE CP9.7**

Script to assist in all three parts of the problem.

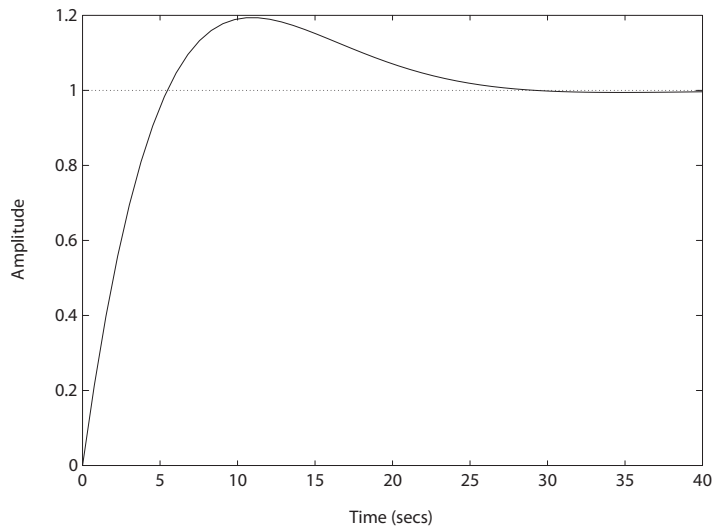
Figure CP9.7c. The phase margin is  $P.M. = 67.7^\circ$  at  $\omega = 0.32$  rad/sec. The loop transfer function is

$$G_c(s)G(s)H(s) = \frac{\bar{K}_1 + \bar{K}_2 s}{s^2} e^{-2Ts}$$

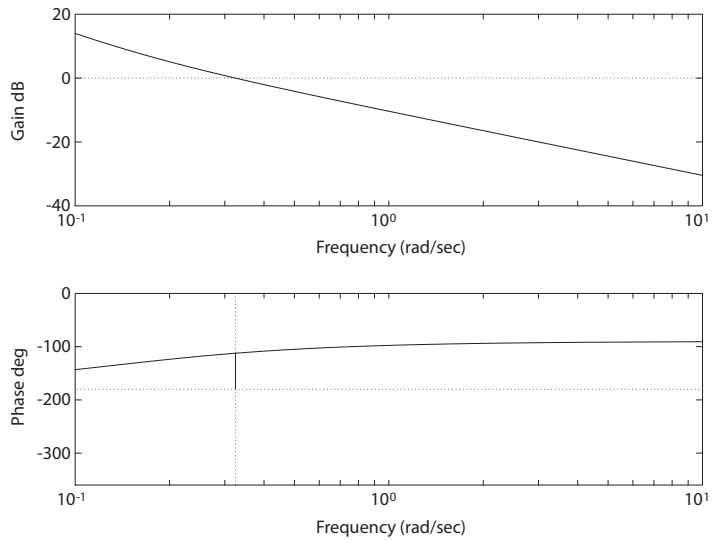
where  $T$  is the one-way time delay. If the phase lag introduced by the delay is greater than  $67.7^\circ$  at  $\omega = 0.32$  rad/sec, then the system will become unstable. So, since the phase lag due to the time delay  $\tilde{T}$  is

$$\phi(\omega) = \omega\tilde{T}$$

we have  $67.7^\circ\pi/180 = 0.32(2T)$  where  $\tilde{T} = 2T$ . Solving for  $T$  yields  $T = 1.82$  seconds. This is the maximum allowable one-way time delay. Executing the third part of the m-file script in Figure CP9.7a generates the plot illustrating the movement of the closed-loop system roots as the time delay is varied. The plot is shown in Figure CP9.7d. Examining the root locations, we find that when  $T = 1.9$ , the closed-loop roots



**FIGURE CP9.7**  
CONTINUED: (b) Step response without time delays meets specs.



**FIGURE CP9.7**  
CONTINUED: (c) System Bode plot shows  $P.M. = 67.7^\circ$ .

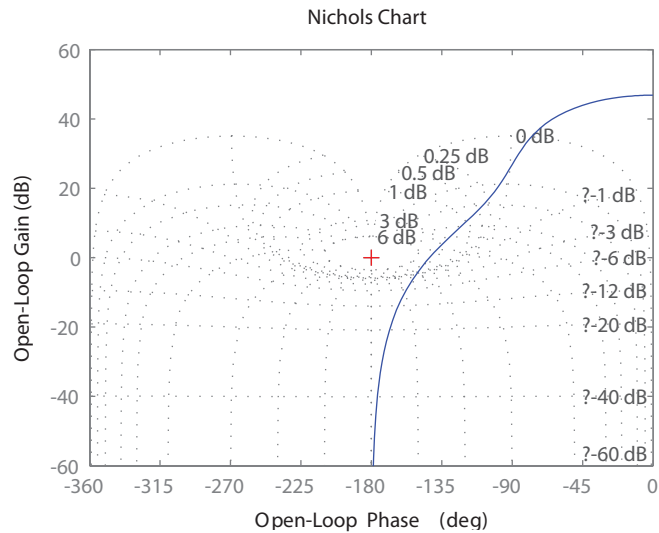
are  $s_1 = -4.56$ ,  $s_{2,3} = -0.94 \pm 2.02j$ ,  $s_4 = -0.19$ , and  $s_{5,6} = \pm 0.32j$ . Therefore, the system is marginally stable when  $T = 1.9$ , and is unstable as the time delay increases.





**CP9.9** The Nichols chart is shown in Figure CP9.9. The phase and gain margins are 37.1 degrees and  $\infty$  dB, respectively.

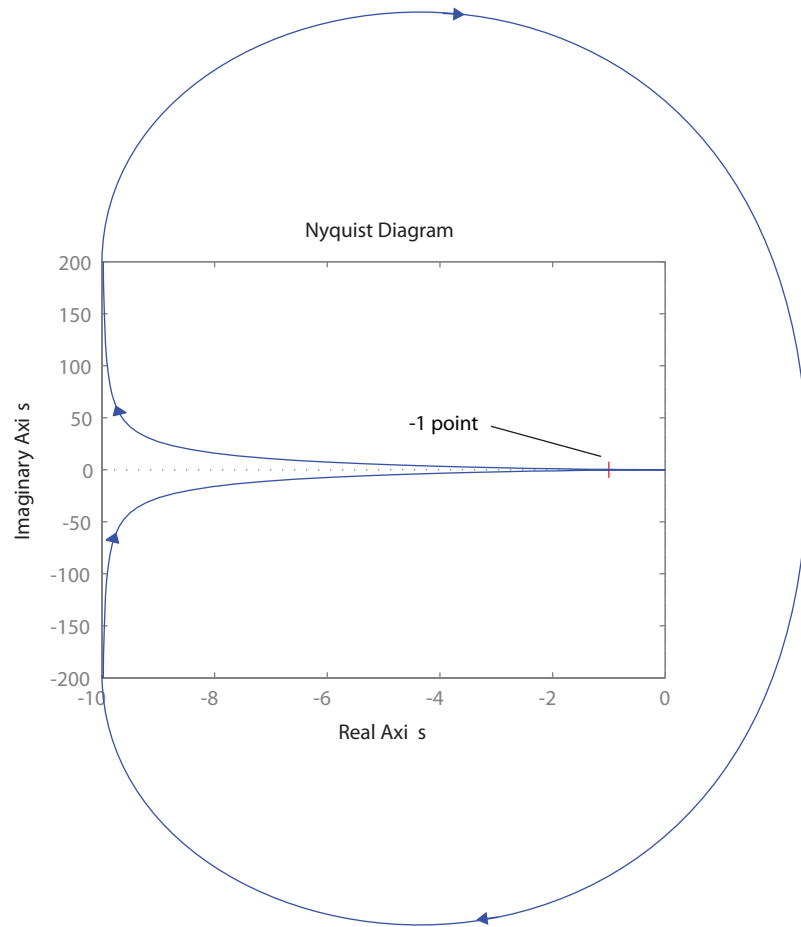
```
a=[0 1;-1 -10]; b=[0;22]; c=[10 0]; d=[0];
sys=ss(a,b,c,d);
nichols(sys)
ngrid
```



**FIGURE CP9.9**

The Nichols chart for the system in CP9.8.

- CP9.10**
- The Nyquist plot is shown in Figure CP9.10. The phase margin is  $P.M. = 18^\circ$ .
  - When the time delay is  $T = 0.05$  seconds, the phase margin is  $P.M. = 9^\circ$ .
  - When the time delay is  $T = 0.1$  seconds, the system is marginally stable. So, for  $T > 0.1$  seconds, the system is unstable.



**FIGURE CP9.10**  
Nyquist plot for  $G(s)H(s) = \frac{10}{s(s+1)}$ .

# The Design of Feedback Control Systems

## Exercises

**E10.1** From the design specifications, we determine that our desired  $\zeta = 0.69$  and  $\omega_n = 5.79$ . The characteristic equation is

$$1 + G_c(s)G(s) = 1 + \frac{K(s+a)}{s(s+2)} = 0 ,$$

or

$$s^2 + (2+K)s + Ka = 0 .$$

Our desired characteristic polynomial is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 8s + 33.6 = 0 .$$

Thus,  $K + 2 = 8$ , or

$$K = 6$$

and  $Ka = 33.6$ , so  $a = 5.6$ . The actual percent overshoot and settling time will be different from the predicted values due to the presence of the closed-loop system zero at  $s = -a$ . In fact, the actual percent overshoot and settling time are  $P.O. = 12.6\%$  and  $T_s = 0.87s$ , respectively.

**E10.2** The characteristic equation is

$$1 + G_c(s)G(s) = 1 + \frac{400}{s(s+40)} \left( K_1 + \frac{1}{s} \right) = 1 + \frac{400(K_1 s + 1)}{s^2(s+40)} = 0 ,$$

or

$$1 + K_1 \frac{400s}{s^3 + 40s^2 + 400} = 0 .$$

We desire  $\zeta = 0.45$  for an overshoot of 20%. The root locus is shown in Figure E10.2. We select a point slightly inside the performance region (defined by  $\zeta = 0.45$ ) to account for the zero. Thus,

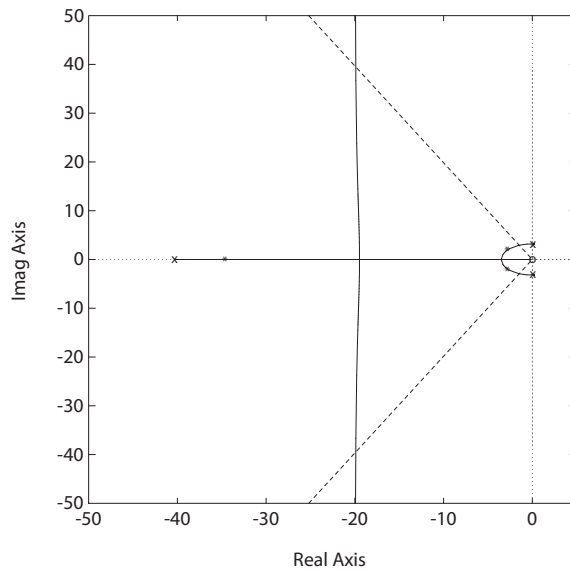
$$K_1 = 0.5$$

and the closed-loop poles are

$$\begin{aligned} s_1 &= -35 \\ s_{2,3} &= -2.7 \pm j2 \end{aligned}$$

The actual

$$P.O. = 20.7\% .$$



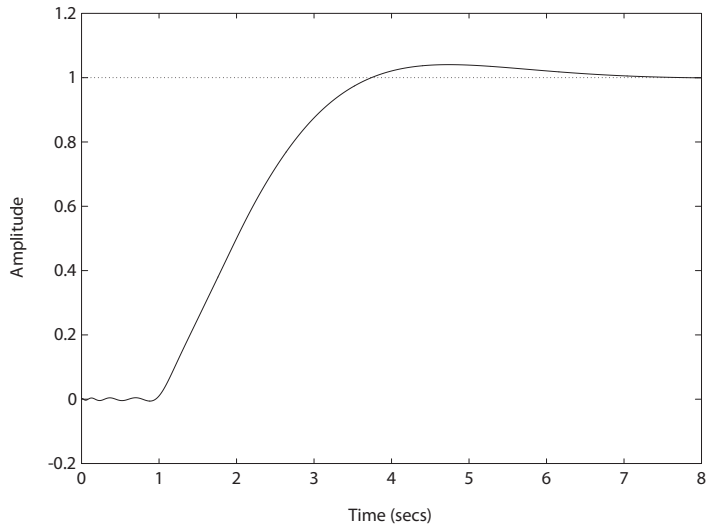
**FIGURE E10.2**

Root locus for  $1 + K_1 \frac{400s}{s^3 + 40s^2 + 400} = 0$ .

**E10.3** The step response is shown in Figure E10.3 for  $\tau = 1$  and  $K = 0.5$ . It can be seen that the

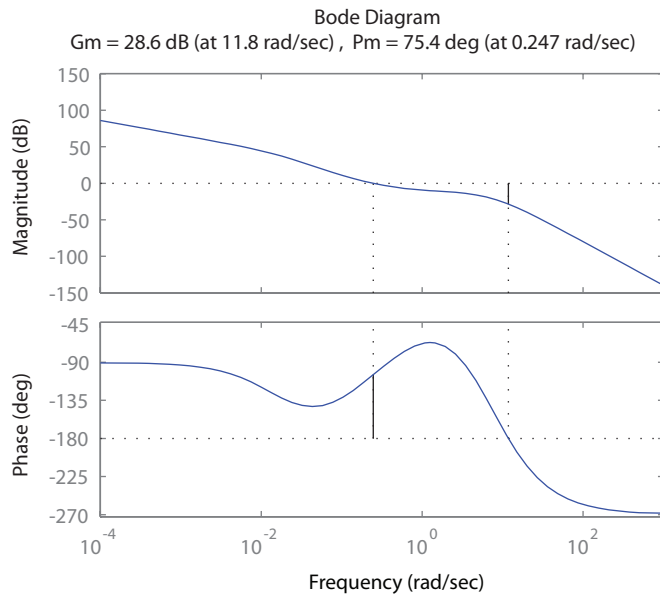
$$P.O. = 4\% ,$$

so this is a valid solution.

**FIGURE E10.3**

Step response for  $K = 0.5$  and  $\tau = 1$ .

- E10.4** The Bode plot is shown in Figure E10.4. The phase and gain margins are marked on the plot, where it can be seen that  $P.M. = 75.4^\circ$  and  $G.M. = 28.6$  dB.

**FIGURE E10.4**

Bode plot for  $G_c(s)G(s) = \frac{100(s+0.15)(s+0.7)}{s(s+5)(s+10)(s+0.015)(s+7)}$ .

- E10.5** We require that  $K_v \geq 2.7$ ,  $\zeta = 0.5$  and  $\omega_n = 3$  for the dominant roots. We want to place a zero to left of the pole at  $-2$ , so the complex roots will dominate. Set the zero at  $s = -2.2$ . Then for the desired roots find the location of pole  $p$  in compensator

$$G_c(s) = \frac{K_1(s + 2.2)}{(s + p)}$$

to satisfy  $180^\circ$  phase at the desired roots. This yields  $p = 16.4$ . Using root locus methods, we find that  $KK_1 = 165.7$ , so with  $K_1 = 7.53$ , we determine that  $K = 22$ , and

$$G_c(s) = \frac{7.46(s + 2.2)}{(s + 16.4)} .$$

Then

$$K_v = 2.78 .$$

- E10.6** The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{326(s + 4)}{s^4 + 14.76s^3 + 151.3s^2 + 349.8s + 1304} .$$

The roots are

$$\begin{aligned} s_{1,2} &= -0.87 \pm j3.2 \\ s_{3,4} &= -6.5 \pm j8.7 . \end{aligned}$$

Assuming  $s_{1,2}$  dominates, then we expect overshoot

$$P.O. = 43\% \quad \text{and} \quad T_s = 4.6 \text{ sec} .$$

The discrepancies with the actual  $P.O.$  and  $T_s$  are due to the poles  $s_{3,4}$  and the zero at  $s = -4$ .

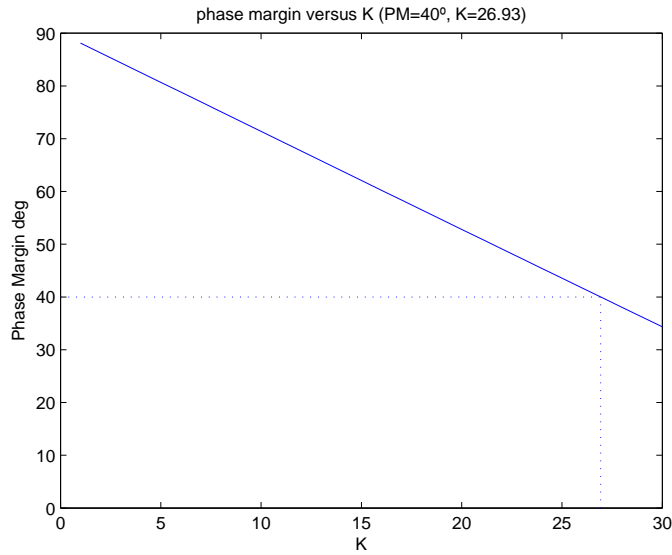
- E10.7** The loop transfer function is

$$L(s) = \frac{Ke^{-0.6s}}{s(s + 20)} .$$

A plot of  $P.M.$  as a function of  $K$  is shown in Figure E10.7. It can be seen that

$$P.M. = 40^\circ$$

when  $K = 26.93$ .



**FIGURE E10.7**  
Plot of phase margin versus  $K$ .

**E10.8** The open-loop transfer function is

$$G(s) = \frac{2257}{s(0.0028s + 1)} = \frac{806071.4}{s(s + 357.14)},$$

and the compensator is

$$G_c(s) = \frac{K_1(s + z)}{s},$$

where  $z = K_2/K_1$ . The characteristic equation is

$$s^3 + 357.14s^2 + K_1s + K_2 = 0.$$

Using Routh-Hurwitz methods, the system is stable for

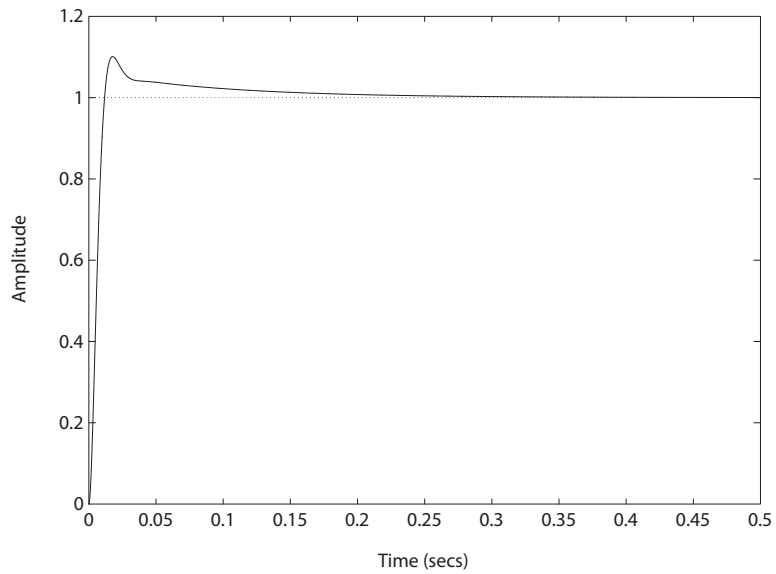
$$0 < K_2 < 357.14 K_1$$

or  $K_2/K_1 < 357.14$ . Select the zero  $z$  at  $s = -10$ , then using root locus methods we determine that  $K_1 = 0.08$  and  $K_2 = 0.8$ . The roots of the characteristic equation are

$$s_1 = -10.6 \quad \text{and} \quad s_{2,3} = -175 \pm j175,$$

and  $\zeta = 0.707$ , as desired. The step response is shown in Figure E10.8.



**FIGURE E10.8**

Step response with  $K_1 = 0.08$  and  $K_2 = 0.8$ .

**E10.9** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K_1(s + K_2/K_1)}{s(s + 1)},$$

and

$$K_v = \lim_{s \rightarrow 0} sG_c(s)G(s) = K_2.$$

Select  $K_2 = 5$ . The characteristic equation is

$$s^2 + (K_1 + 1)s + K_2 = 0,$$

and we want

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$$

Equating coefficients yields  $\omega_n = \sqrt{K_2} = \sqrt{5}$ . Also, since we want  $P.O. = 5\%$ , we require  $\zeta = 0.69$ . Thus,

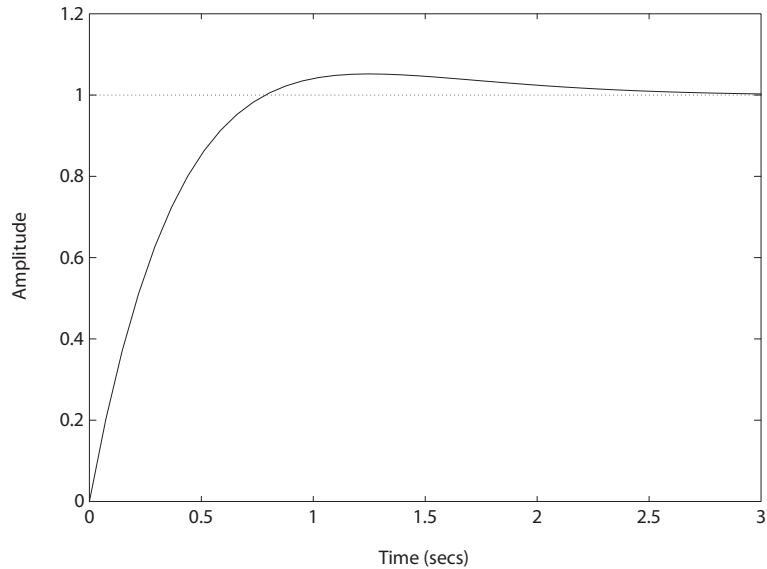
$$2\zeta\omega_n = K_1 + 1 \quad \text{implies} \quad K_1 = 2.08.$$

The step response with  $K_1 = 2.08$  and  $K_2 = \sqrt{5}$  yields a  $P.O. > 5\%$ . This

is due to the zero at

$$s = -1.08 .$$

So, we raise the gain  $K_1 = 3$  and then the  $P.O. = 5\%$ . The step response is shown in Figure E10.9.



**FIGURE E10.9**  
Step response with  $K_1 = 3$  and  $K_2 = 5$ .

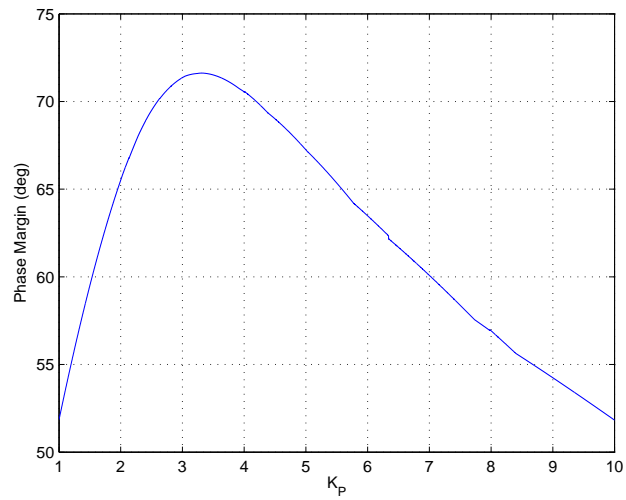
**E10.10** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{(K_P s + K_I)}{s(s+1)(s+2)} .$$

Let  $K_I = 2$ . Then, the plot of the phase margin as a function of  $K_P$  is shown in Figure E10.10, where it can be seen that

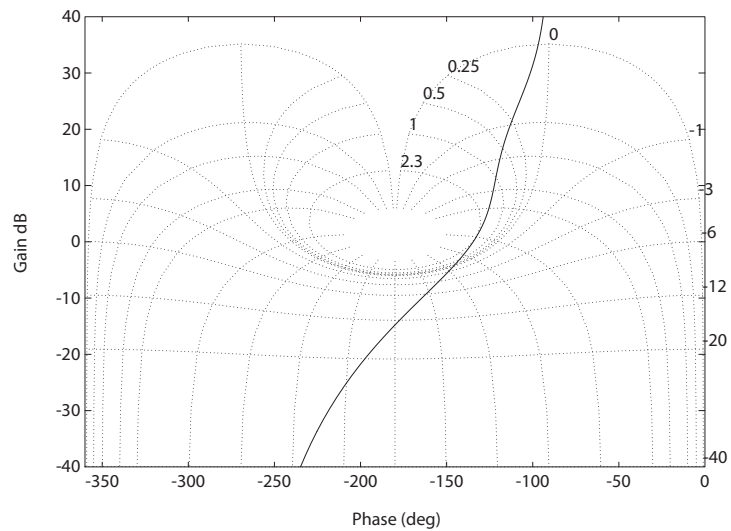
$$P.M. = 71.6^\circ$$

is the maximum achievable phase margin. When  $K_P = 1.54$  and  $K_I = 2$  we have  $P.M. = 60^\circ$ , as desired, and  $P.O. = 9\%$  and  $T_p = 3.4$  sec.

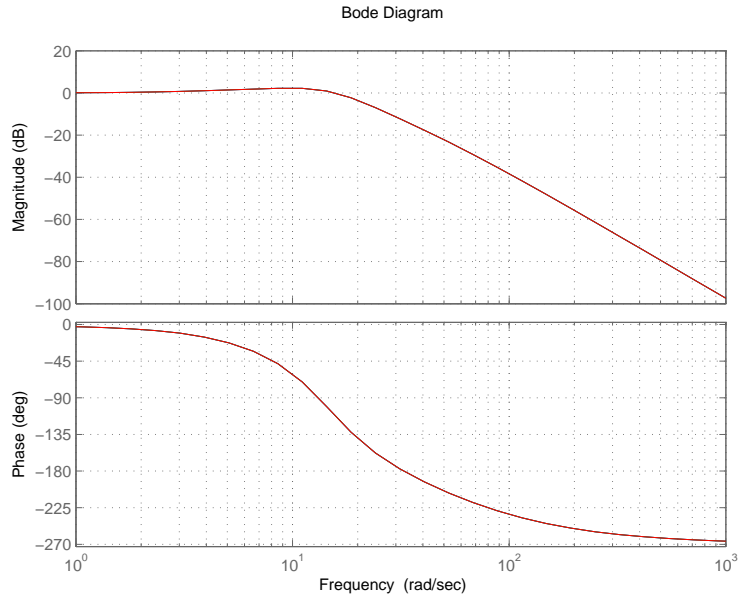


**FIGURE E10.10**  
Phase margin versus  $K_P$  with  $K_I = 2$ .

**E10.11** The Nichols diagram and the closed-loop Bode plot are shown in Figures E10.11a and E10.11b, respectively.



**FIGURE E10.11**  
(a) Nichols diagram for  $G_c(s)G(s) = \frac{1350(1+0.25s)}{s(s+2)(s+30)(1+0.025s)}$ .



**FIGURE E10.11**  
CONTINUED: (b) Closed-loop Bode plot.

**E10.12** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{KK_1 \left( s + \frac{1}{2} \right)}{s^2(s + 5)} .$$

When  $KK_1 = 5.12$ , the roots are

$$\begin{aligned} s_{1,2} &= -0.58 \pm j0.58 \\ s_3 &= -3.84 . \end{aligned}$$

The complex poles have  $\zeta = 0.707$  and the predicted settling time is

$$T_s = 4/0.58 = 6.89 \text{ sec} .$$

The actual settling time is  $T_s = 6.22$  s.

**E10.13** For the cascade compensator, we have

$$T_1(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{8.1(s + 1)}{(s + r_1)(s + \hat{r}_1)(s + r_2)} ,$$

where  $r_1 = -1 + j2$  and  $r_2 = -1.67$ . For the feedback compensator, we

have

$$T_2(s) = \frac{G(s)}{1 + G_c(s)G(s)} = \frac{8.1(s + 3.6)}{(s + r_1)(s + \hat{r}_1)(s + r_2)},$$

where

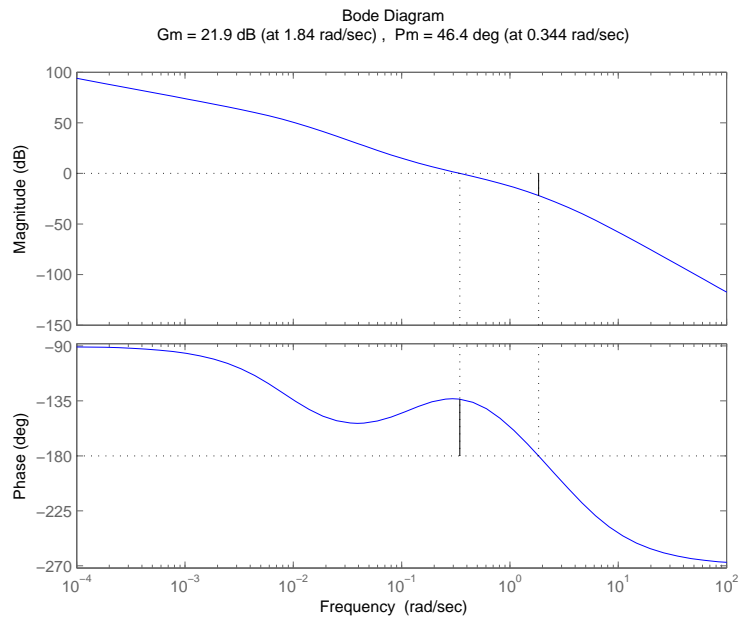
$$G(s) = \frac{8.1}{s^2}$$

and

$$G_c(s) = \frac{s + 1}{s + 3.6}.$$

The response of the two systems differ due to different value of the zero of  $T_1$  and  $T_2$ , however, both systems have the same characteristic equation.

**E10.14** The Bode plot (with the lag network) is shown in Figure E10.14; the phase margin is  $P.M. = 46^\circ$ .



**FIGURE E10.14**

Bode plot for  $G_c(s)G(s) = \frac{5(7.5s+1)}{s(s+1)(0.25s+1)(110s+1)} = 0$ .

**E10.15** At the desired crossover frequency  $\omega_c = 10$  rad/sec, we have

$$20 \log |G_c(j10)G(j10)| = -8.1 \text{ dB} \quad \text{and} \quad \angle G_c(j10)G(j10) = -169^\circ.$$

Therefore, the phase margin is  $P.M. = 11^\circ$ . So,

$$\phi = 30^\circ - 11^\circ = 19^\circ \quad \text{and} \quad M = 8.1 \text{ dB} .$$

Since  $\phi > 0$  and  $M > 0$ , a lead compensator is required.

**E10.16** At the desired crossover frequency  $\omega_c = 2$  rad/sec, we have

$$20 \log |G_c(j2)G(j2)| = 17 \text{ dB} \quad \text{and} \quad \angle G_c(j2)G(j2) = -134^\circ .$$

Therefore, the phase margin is  $P.M. = 46^\circ$ . So,

$$\begin{aligned} \phi &= 30^\circ - 46^\circ = -16^\circ \\ M &= -17 \text{ dB} . \end{aligned}$$

Since  $\phi < 0$  and  $M < 0$ , a lag compensator is required.

**E10.17** Using a prefilter

$$G_p(s) = \frac{K_I}{K_P s + K_I}$$

the closed-loop transfer function is

$$T(s) = \frac{K_I}{s^2 + (K_P + 1)s + K_I} .$$

The required coefficients for a deadbeat system are  $\alpha = 1.82$  and  $T_s = 4.82$ . Therefore,

$$\begin{aligned} K_I &= \omega_n^2 \\ K_P &= \alpha \omega_n - 1 . \end{aligned}$$

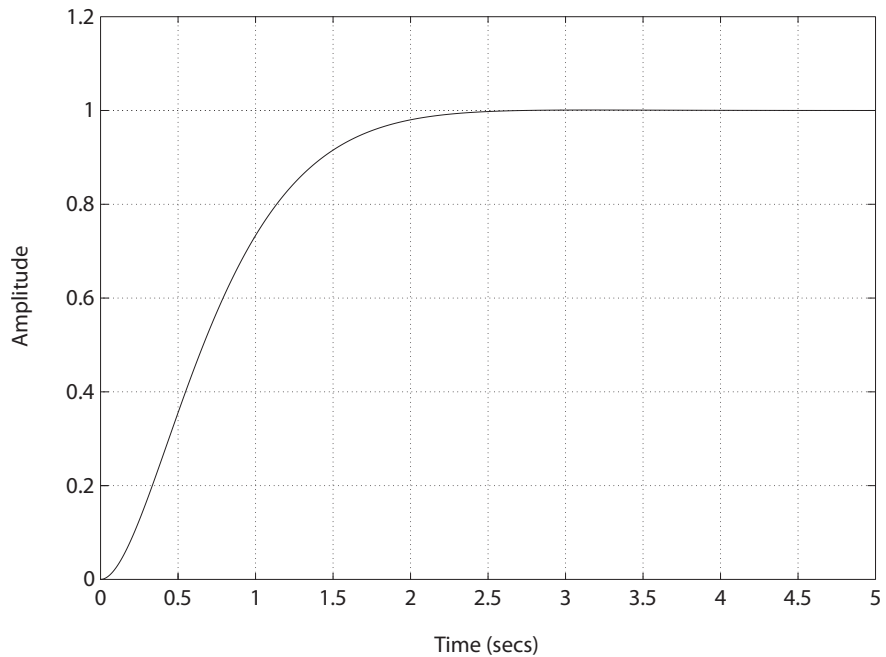
Since we desired a settling time less than 2 seconds, we determine that

$$\omega_n = T_s/2 = 4.82/2 = 2.41 .$$

Then, the gains are

$$\begin{aligned} K_P &= 3.39 \\ K_I &= 5.81 . \end{aligned}$$

The step response (with the prefilter) is shown in Figure E10.17. The percent overshoot is  $P.O. = 0.098\%$  and the settling time is  $T_s = 1.99$  seconds.



**FIGURE E10.17**  
Step response for the deadbeat system.

**E10.18** Consider the PI controller

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s} = \frac{30s + 300}{s}$$

and the prefilter

$$G_p(s) = 10 .$$

Then, the closed-loop system is

$$T(s) = \frac{300s + 3000}{s^2 + 280s + 3000} .$$

The percent overshoot is  $P.O. = 9.2\%$  and the settling time  $T_s = 0.16$  seconds. The steady-state tracking error to a unit step is zero, as desired.

**E10.19** Consider the PID controller

$$G_c(s) = 29 \frac{s^2 + 10s + 100}{s} .$$

The closed-loop transfer function is

$$T(s) = \frac{29(s^2 + 10s + 100)}{s^3 + 24s^2 + 290s + 2900} .$$

The settling time to a unit step is  $T_s = 0.94$  seconds.

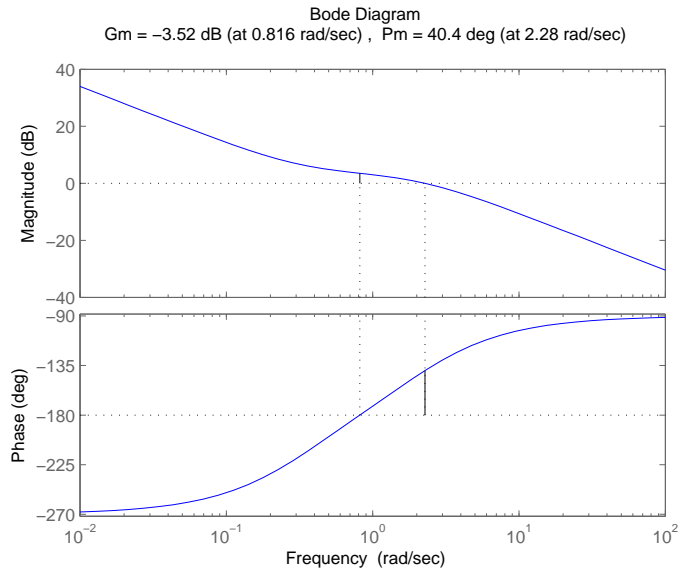
**E10.20** Consider the PD controller

$$G_c(s) = K_D s + K_p = 3s + 1 .$$

The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{3s + 1}{s(s - 2)} .$$

The Bode plot is shown in Figure E10.20. The phase margin is  $P.M. = 40.4^\circ$ . This is a situation where decreasing the gain leads to instability. The Bode plot shows a negative gain margin indicating that the system gain can be decreased up to -3.5 dB before the closed-loop becomes unstable.



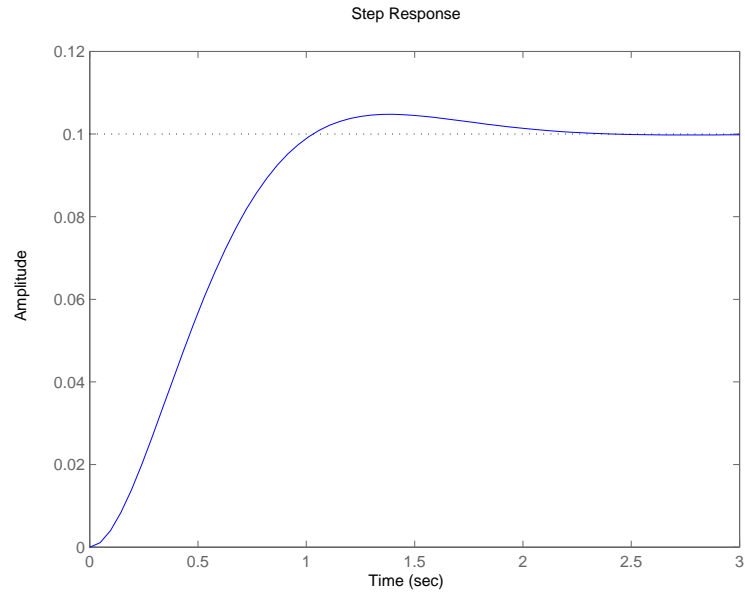
**FIGURE E10.20**  
Bode plot for the loop transfer function  $L(s) = \frac{3s+1}{s(s-2)}$ .

**E10.21** The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$T(s) = \frac{1}{s^2 + 4.4s + K} .$$



The tracking error is  $E(s) = R(s) - Y(s)$ . When  $R(s) = 0$ , then  $E(s) = -Y(s)$ . The final value of the output to a unit step disturbance is  $e_{ss} = 1/K$ . If we want the tracking error to be less than 0.1, then we require  $K > 10$ . When  $K = 10$ , we have the disturbance response shown in Figure E10.21.



**FIGURE E10.21**  
Disturbance response for  $K = 10$ .

## Problems

**P10.1** (a) The loop transfer function is

$$L(s) = G_c(s)G(s)H(s) = \frac{(1 + \alpha\tau s)K_1K_2}{\alpha(1 + \tau s)(Js^2)} .$$

We desire  $\zeta = 0.6$ ,  $T_s \leq 2.5$  or  $\zeta\omega_n \geq 1.6$ . The uncompensated closed-loop system is

$$T(s) = \frac{K}{s^2 + K} ,$$

where  $K = K_1K_2/J$  and  $K = \omega_n^2$ . We can select  $K = 20$ , and then  $\zeta\omega_n > 1.6$ . First, plot the Bode diagram for

$$G(s)H(s) = \frac{20}{s^2}$$

where  $K_1K_2/\alpha J = 20$ . The phase margin of the uncompensated system is  $0^\circ$ . We need to add phase at  $\omega_c$ . After several iterations, we choose to add  $40^\circ$  phase at  $\omega_c$ , so

$$\sin 40^\circ = \frac{\alpha - 1}{\alpha + 1} = 0.64 .$$

Therefore,  $\alpha = 4.6$ . Then,

$$10 \log \alpha = 10 \log 4.6 = 6.63 \text{ dB} .$$

We determine the frequency where magnitude is -6.63 dB to be  $\omega_m = 6.6$  rad/sec. Then,

$$p = \omega_n \sqrt{\alpha} = 14.1 \quad \text{and} \quad z = p/\alpha = 3.07 .$$

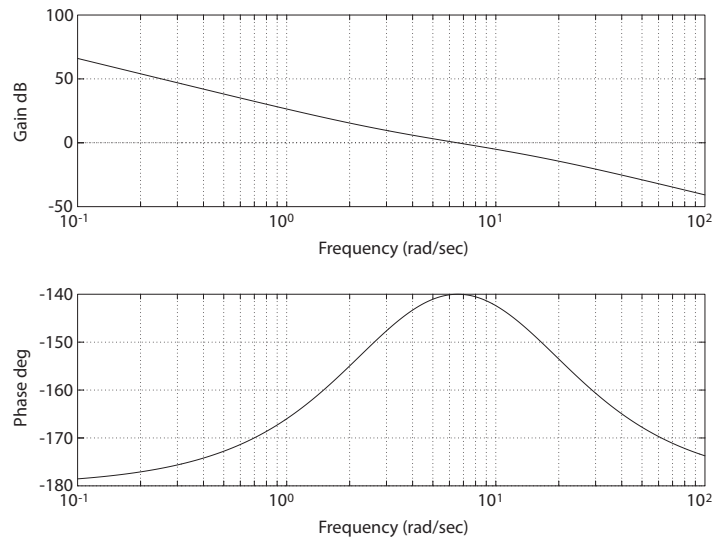
The compensated loop transfer function (see Figure P10.1a) is

$$G_c(s)G(s)H(s) = \frac{20 \left( \frac{s}{3.07} + 1 \right)}{s^2 \left( \frac{s}{14.1} + 1 \right)} .$$

(b) Since we desire  $\zeta\omega_n \geq 1.6$ , we place the compensator zero at  $z = 1.6$ . Then, we place the compensator pole far in the left half-plane; in this case, we selected  $p = 20$ . Thus, the compensator is

$$G_c(s) = \frac{s + 1.6}{s + 20} .$$

The root locus is shown in Figure P10.1b. To satisfy the  $\zeta = 0.6$  requirement, we find  $K = 250$ , and the compensated loop transfer

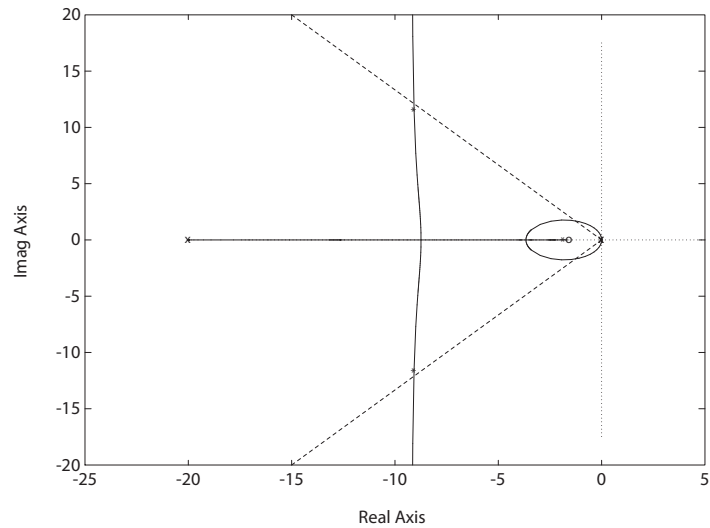


**FIGURE P10.1**

(a) Compensated Bode plot for  $G_c(s)G(s)H(s) = \frac{20(s/3.07+1)}{s^2(s/14.1+1)}$ .

function is

$$G_c(s)G(s)H(s) = \frac{250(s + 1.6)}{s^2(s + 20)} = \frac{20 \left( \frac{s}{1.6} + 1 \right)}{s^2 \left( \frac{s}{20} + 1 \right)}$$



**FIGURE P10.1**

CONTINUED: (b) Root locus for  $G_c(s)G(s)H(s) = 1 + K \frac{s+1.6}{s^2(s+20)}$ .

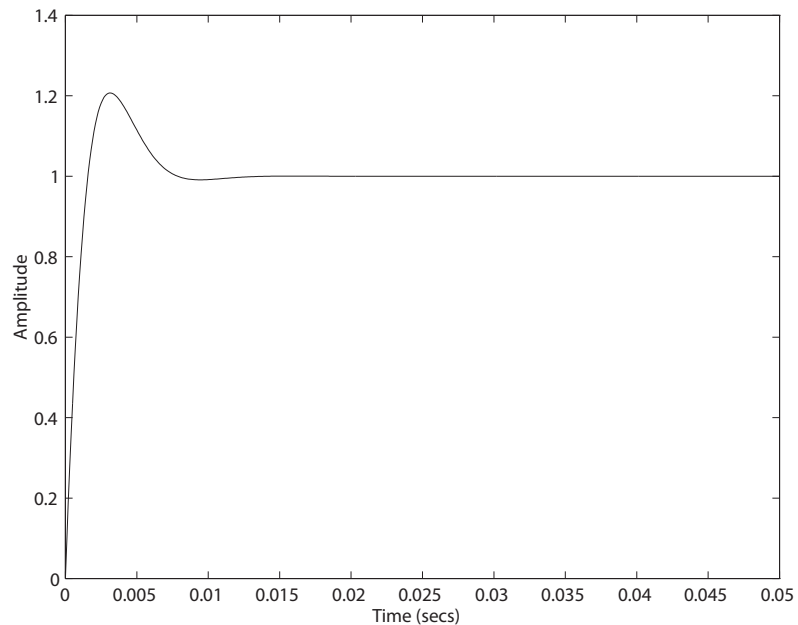
**P10.2** The transfer function of the system is

$$G(s) = \frac{1.0e + 14}{s^3 + 2000s^2 + 1e + 11s} ,$$

where we use the system parameters given in P7.11 with the following modifications:  $\tau_1 = \tau_1 = 0$  and  $K_1 = 1$ . Also we have scaled the transfer function so that the time units are seconds. The parameters in P7.11 are given for time in milliseconds. A suitable compensator is

$$G_c(s) = \frac{s + 500}{s + 1} .$$

The closed-loop system response is shown in Figure P10.2. The percent overshoot is  $P.O. \approx 20\%$  and the time to settle is  $T_s < 0.01$  second.



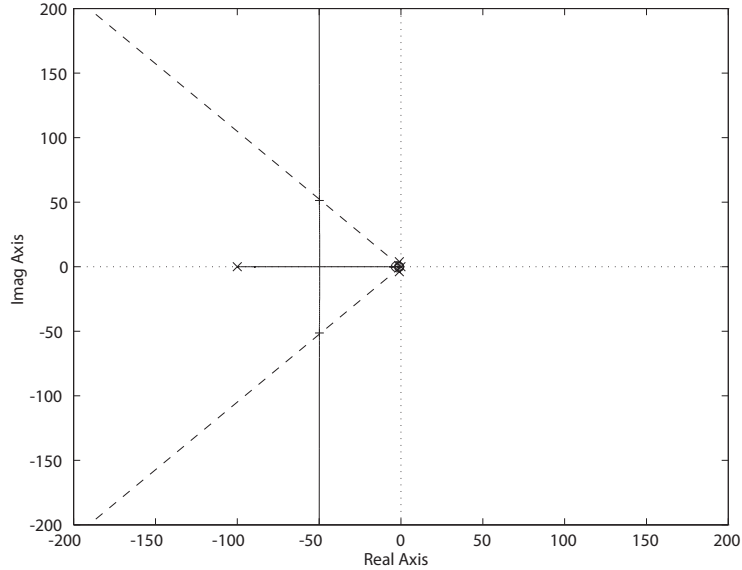
**FIGURE P10.2**  
Step response.

**P10.3** The loop transfer function is

$$G_c(s)G(s) = \frac{16(s + 1)}{s(s^2 + 2s + 16)} \frac{K(s + z)}{(s + p)} .$$

We desire dominant roots with  $T_s < 5$  sec and  $P.O. < 5\%$ , so use  $\zeta = 0.69$  and  $\zeta\omega_n = 0.8$ . One solution is to select  $z = 1.1$  (i.e. to the left of the

existing zero at  $s = -1$ ) and determine the pole  $p$  and gain  $K$  for dominant roots with  $\zeta = 0.69$ . After iteration, we can select  $p = 100$ , so that the root locus has the form shown in Figure P10.3. Then, we select  $K = 320$ ,



**FIGURE P10.3**  
 Root locus for  $1 + K \frac{16(s+1)(s+1.1)}{s(s^2+2s+16)(s+100)} = 0$ .

so that  $\zeta = 0.69$ . The final compensator is

$$G_c(s) = \frac{320(s + 1.1)}{s + 100} .$$

The design specifications are satisfied with this compensator.

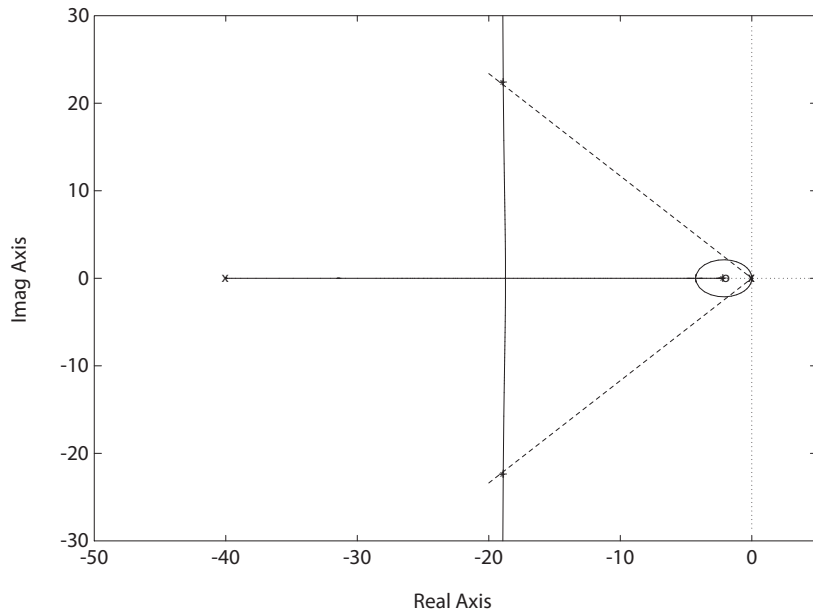
**P10.4** The uncompensated loop transfer function is

$$G(s) = \frac{1}{s^2(\frac{1}{40}s + 1)} = \frac{40}{s^2(s + 40)} .$$

We desire  $10\% < P.O. < 20\%$ , so  $0.58 < \zeta < 0.65$ , and  $T_s < 2$  implies  $\zeta\omega_n < 2$ . We will utilize a PD compensator  $K_a(s + a)$ . We select  $a = 2$ , to obtain the root locus shown in Figure P10.4. Then with  $K_a = 23.5$ , we have the desired root location, and

$$G_c(s) = 23.5(s + 2) .$$

The design specifications are satisfied with the PD compensator.

**FIGURE P10.4**

Root locus for  $1 + K_a \frac{40(s+2)}{s^2(s+40)} = 0$ .

- P10.5** We desire  $P.O. < 10\%$  and  $T_s < 1.5$  sec. The compensator is a PI-type, given by

$$G_c(s) = K_2 + \frac{K_3}{s} = \frac{K_2 s + K_3}{s} = \frac{K_2(s+a)}{s}$$

where  $a = K_3/K_2$ . So,  $e_{ss} = 0$  for a step input and

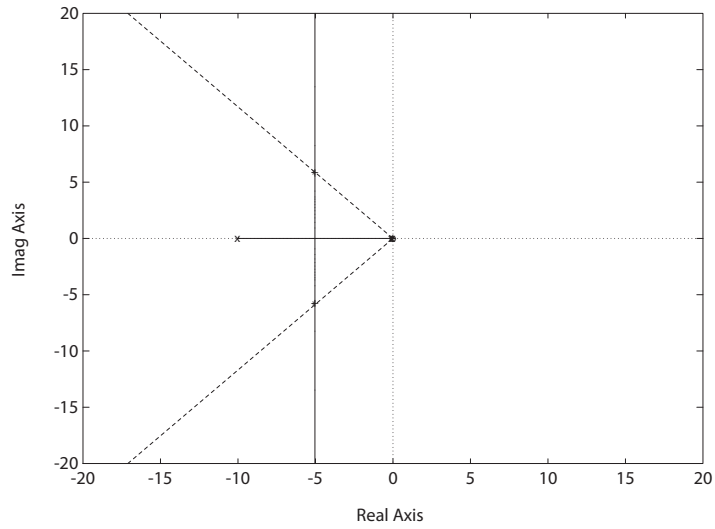
$$G(s) = \frac{3.75K_a}{(s+0.15)(0.15s+1)} = \frac{25K_a}{(s+0.15)(s+6.67)}.$$

The loop transfer function is

$$G_c(s)G(s) = \frac{25K_a K_2 (s+a)}{s(s+0.15)(s+6.67)}.$$

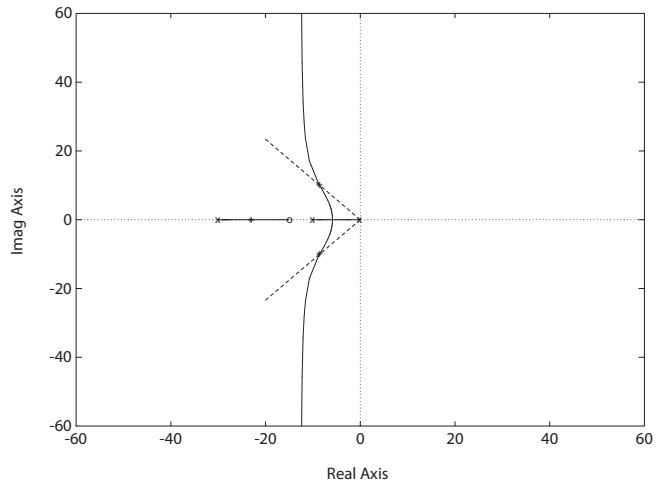
Using root locus methods, we select  $a = 0.2$  (after several iterations) and determine  $K_a K_2$  to yield  $\zeta = 0.65$ . This results in  $K_a K_2 = 1$ . The root locus is shown in Figure P10.5. The design specifications are met. The actual percent overshoot and settling time are  $P.O. = 7.4\%$  and  $T_s = 1.3$  s. The controller is

$$G_c(s) = 1 + \frac{0.2}{s}.$$

**FIGURE P10.5**

Root locus for  $1 + K_a K_2 \frac{25(s+0.2)}{s(s+0.15)(s+6.67)} = 0$ .

**P10.6** As in P10.5, using root locus we find that placing  $z = 15$  and  $p = 30$  yields a root locus shape (see Figure P10.6) where the loop transfer function is

**FIGURE P10.6**

Root locus for  $1 + K_a \frac{25(s+15)}{(s+0.15)(s+6.67)(s+30)} = 0$ .

$$G_c(s)G(s) = \frac{25K_a(s+z)}{(s+p)(s+0.15)(s+6.67)} .$$

and where  $z, p$  and  $K_a$  are the parameters to be determined. Properly choosing the parameter values allows us to increase  $\zeta\omega_n$  of the dominant roots (compared to the PI compensator of P10.5). Then, with  $K_a = 3.7$ , the dominant roots have  $\zeta = 0.65$ . The design specifications are met with the compensator.

**P10.7** The plant transfer function is

$$G(s) = \frac{e^{-50s}}{(40s+1)^2} .$$

The steady-state error is

$$e_{ss} = \frac{A}{1+K_p} < 0.1A .$$

Therefore,  $K_p > 9$ . Insert an amplifier with the compensator with a dc gain = 9, as follows

$$G_c(s)G(s) = \frac{9e^{-50s}(s+2)}{(40s+1)^2(s+p)} .$$

The system is unstable without compensation, and it is very difficult to compensate such a time delay system with a lead compensator. Consider a lag network

$$G_c(s) = \frac{s+z}{s+p}$$

where  $z > p$ . Let

$$z = 10p.$$

Then, a plot of the  $P.M.$  versus  $p$  is shown in Figure P10.8a. Suitable system performance can be obtained with  $P.M. > 45^\circ$ , so choose

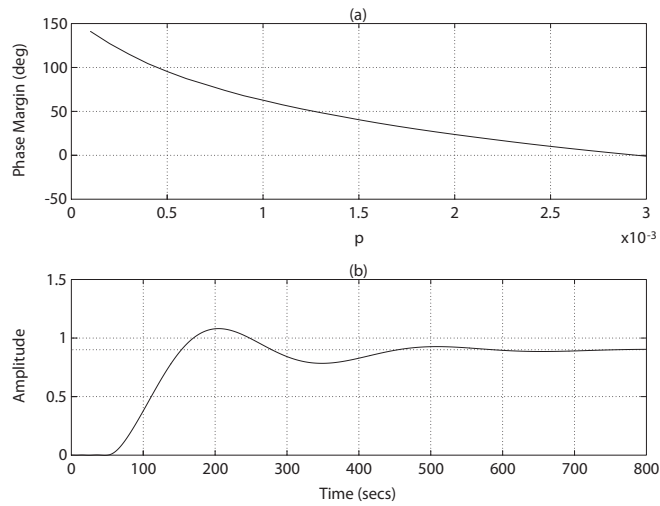
$$p = 0.0001.$$

The Bode plot of the compensated and uncompensated systems is shown in Figure P10.7c, where we have selected  $z = 0.001$  and  $p = 0.0001$ . The compensated system has

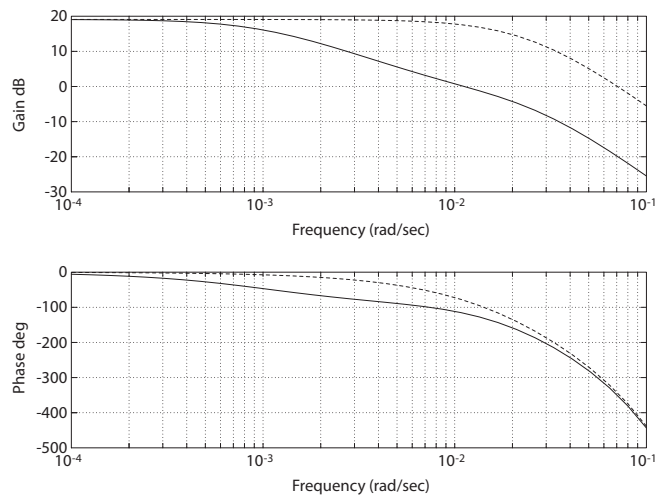
$$P.M. = 62^\circ \quad \text{and} \quad T_s = 9 \text{ minutes} .$$

The step response is shown in Figure P10.7b.



**FIGURE P10.7**

(a) Phase margin versus  $p$ . (b) Step response with  $p = 0.0001$  and  $z = 0.001$ .

**FIGURE P10.7**

CONTINUED: (c) Bode plot for the compensated system (solid lines) and the uncompensated system (dashed line).

**P10.8** The transfer function is

$$G(s) = \frac{5000}{s(s+10)^2}.$$

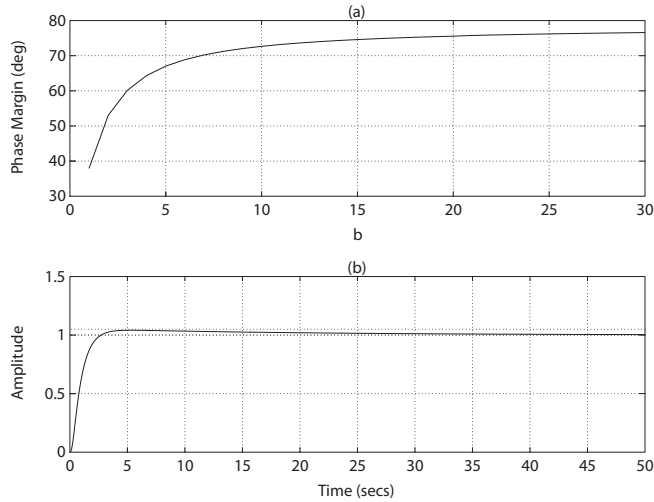
To meet the steady-state accuracy, we need  $K_v > 40$ . The uncompensated

$K_v = 50$ , so the steady-state accuracy can be met.

- (a) Using the Bode method, we need  $P.M. = 70\%$  (to meet  $P.O. < 5\%$  specification). Let

$$G_c(s) = \frac{bs + 1}{as + 1}.$$

The plot of  $P.M.$  versus  $b$  is shown in Figure P10.8a, where we set  $a = 50b$ . Choosing  $b = 20$  should satisfy the  $P.O.$  specification. The Bode plot is shown in Figure P10.8c. Thus,



**FIGURE P10.8**

(a) Phase margin versus  $b$ ; (b) Step response for lag compensator designed with Bode where  $a = 1000$  and  $b = 20$ .

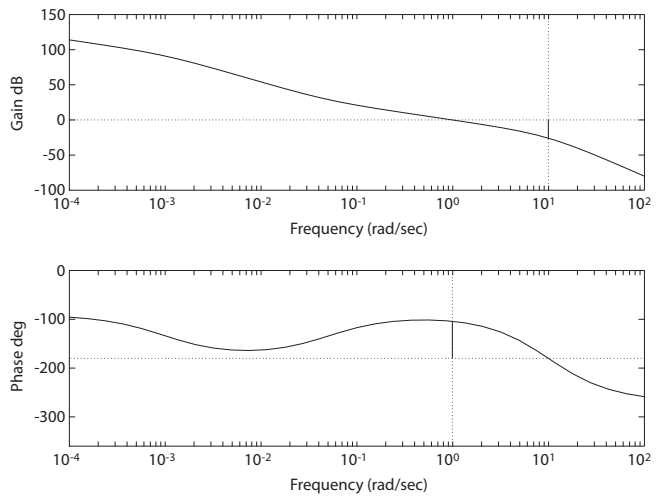
$$G_c(s)G(s) = \frac{5000(20s + 1)}{s(s + 10)^2(1000s + 1)}.$$

The step response is shown in Figure P10.8b.

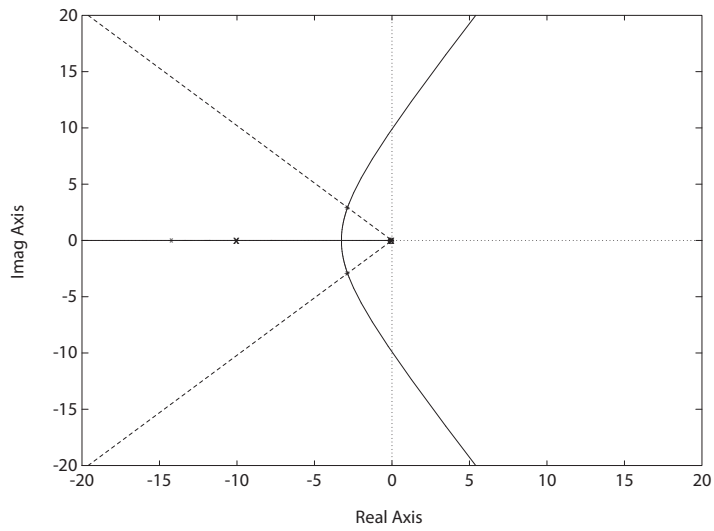
- (b) We require that  $\zeta = 0.7$  to meet the  $P.O.$  specifications. Let

$$G_c(s) = \frac{K(bs + 1)}{(as + 1)}.$$

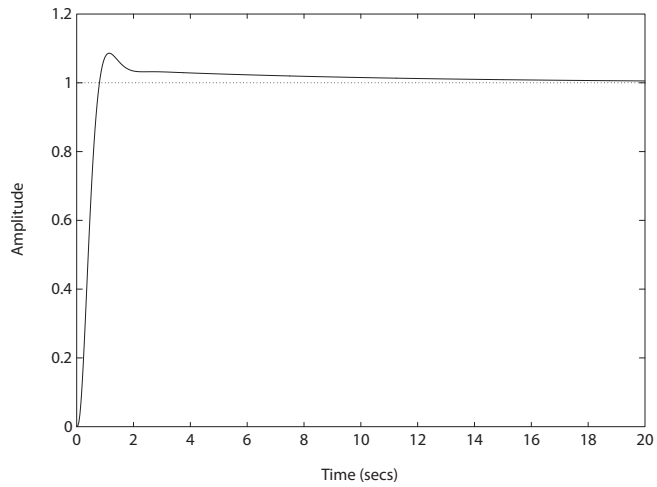
Using root locus methods, we fix  $a$  and  $b$ , and then determine  $K$  for  $\zeta = 0.7$ . Let  $a = 50b$  and select  $b = 10$  (other values will work). The root locus is shown in Figure P10.8d. We find  $K = 2.5$  when  $\zeta = 0.7$ . Now,  $K_v = 125$ , so the steady-state accuracy requirement is satisfied for the step response as shown in Figure P10.8e.



**FIGURE P10.8**  
 CONTINUED: (c) Bode plot for the compensated system with  $G_c(s) = \frac{20s+1}{1000s+1}$ .

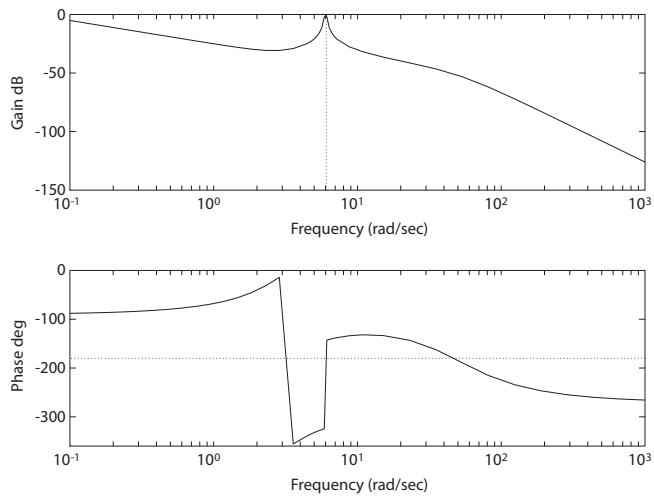


**FIGURE P10.8**  
 CONTINUED: (d) Root locus for  $1 + K \frac{5000(10s+1)}{s(s+10)^2(500s+1)}$ .

**FIGURE P10.8**

CONTINUED: (e) Step response for lag compensator designed with root locus methods, where  $K = 2.5$ .

**P10.9** We desire a small response for a disturbance at 6 rad/sec. The Bode plot of  $G_c(s)G(s)$  is shown in Figure P10.9a where we consider a compensator

**FIGURE P10.9**

(a) Bode plot for the compensated system with  $G_c(s) = \frac{10(s^2+4s+10)}{s^2+36}$ .

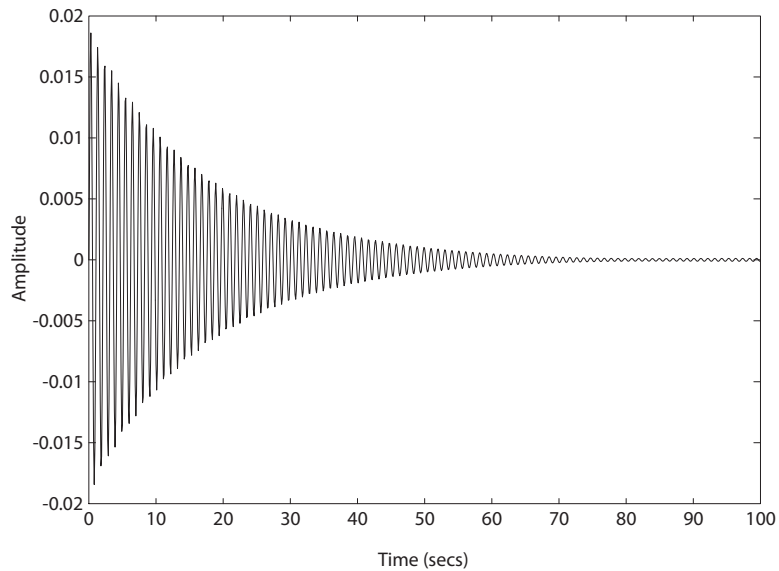
of the form

$$G_c(s) = \frac{K(s^2 + as + b)}{s^2 + 36} .$$

Notice that the magnitude is large at  $\omega = 6$ , as desired. We select

$$a = 4, \quad b = 10 \quad \text{and} \quad K = 10 .$$

The response to a sinusoidal disturbance at 6 rad/sec is shown in Figure P10.9b. Notice that the effect of the disturbance is virtually eliminated in steady-state.



**FIGURE P10.9**

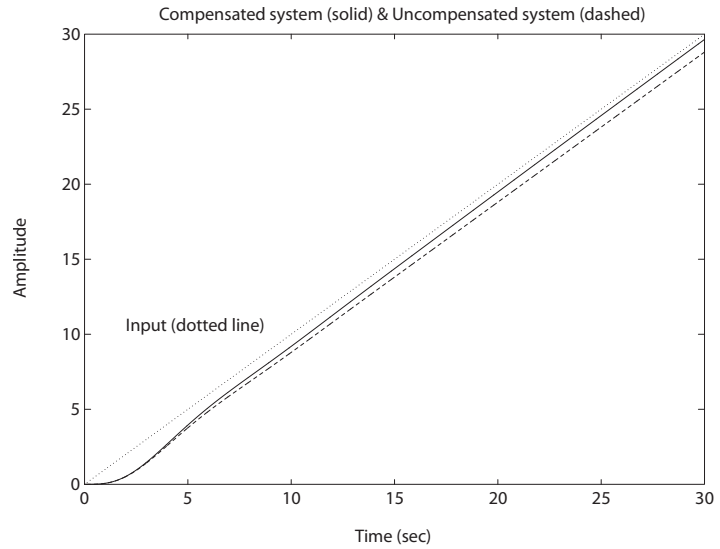
CONTINUED: (b) Disturbance response for a sinusoidal disturbance at 6 rad/sec.

- P10.10** The step response with  $G_c(s) = 1$  is shown in Figure P10.10. A suitable lag compensator is

$$G_c(s) = \frac{s + 0.05}{s + 0.005} .$$

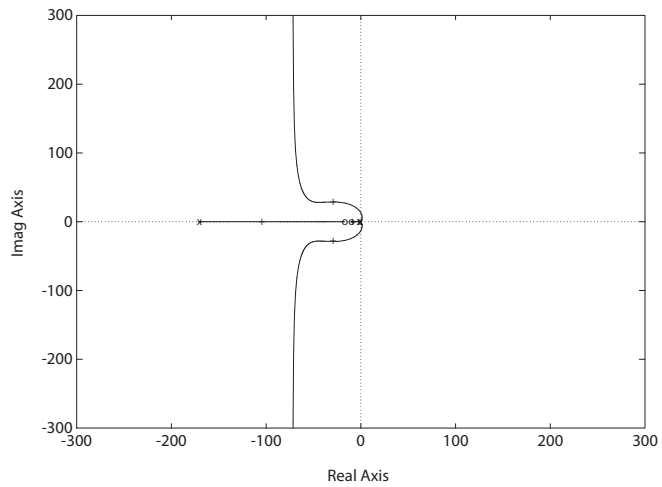
The step response of the compensated system is also shown in Figure P10.10. The settling time of the compensated system is

$$T_s = 28 \text{ seconds} .$$



**FIGURE P10.10**  
Step response of uncompensated and compensated systems.

**P10.11** The root locus is shown in Figure P10.11 where a suitable lead-lag com-



**FIGURE P10.11**  
Root locus for  $1 + K \frac{160(s+17)(s+10)}{s^2(s+170)(s+1)} = 0$ .

pensator is

$$G_c(s) = K \frac{s + 10}{s + 1} \frac{s + 17}{s + 170} .$$

The selected gain is  $K = 57$ , so that the damping of the complex roots is about  $\zeta = 0.7$ . For this particular design, the closed-loop system zeros will affect the system response and the percent overshoot specification may not be satisfied. Some design iteration may be necessary or a prefilter can be utilized. A suitable prefilter is

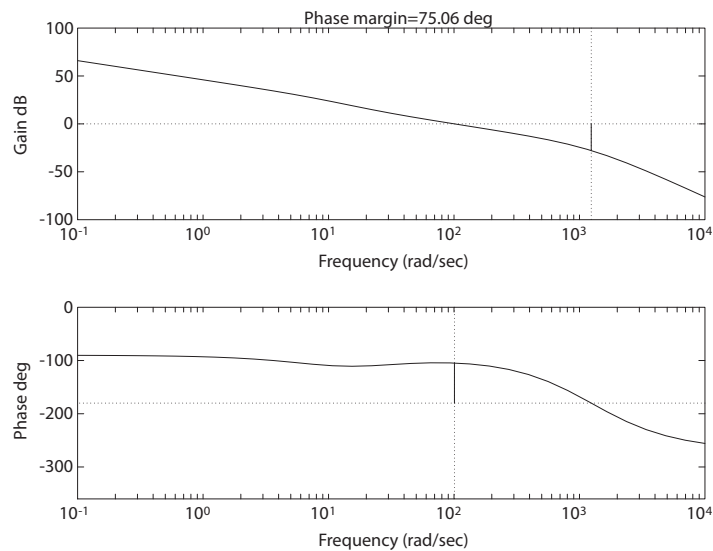
$$G_p(s) = \frac{17}{s + 17} .$$

The acceleration constant is  $K_a = 9120$ .

**P10.12** We choose  $K = 10$ . This yields a velocity constant  $K_v = 20K = 200$ , as desired. A suitable two-stage lead compensator is

$$G_c(s) = \frac{(0.05s + 1)(0.05s + 1)}{(0.0008s + 1)(0.0008s + 1)} .$$

The Bode plot is shown in Figure P10.12. The phase margin is  $P.M. = 75.06^\circ$ .



**FIGURE P10.12**

Bode plot for  $\frac{200(0.05s+1)^2}{s(0.1s+1)(0.05s+1)(0.0008s+1)^2}$ .

**P10.13** (a) When

$$G_c(s) = K = 0.288 ,$$

the phase margin is  $P.M. = 49.3^\circ$  and the bandwidth is  $\omega_B = 0.95$  rad/sec.

(b) A suitable lag compensator is

$$G_c(s) = \frac{25s + 1}{113.6s + 1} .$$

The compensated system phase margin is  $P.M. = 52.21^\circ$  and  $K_v = 2$ , as desired.

**P10.14** A suitable lead compensator is

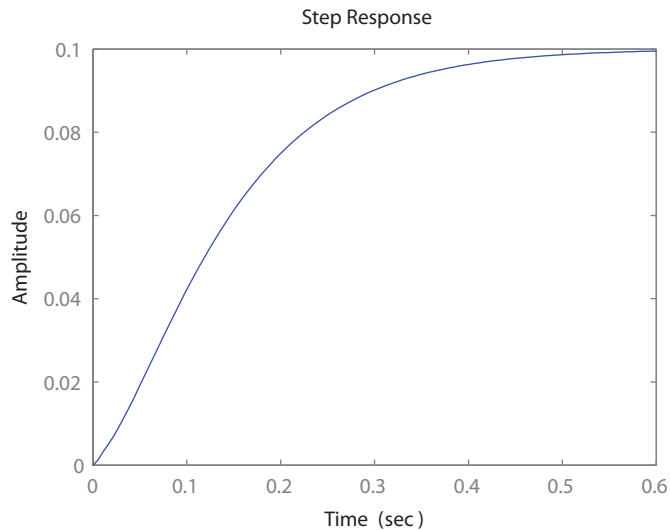
$$G_c(s) = \frac{1.155s + 1}{0.032s + 1} .$$

The compensated system phase margin is  $P.M. = 50^\circ$  and  $K_v = 2$ , as desired. The settling time is  $T_s = 3.82$  seconds.

**P10.15** One possible solution is

$$G_c(s) = K \frac{(s + 12)(s + 15)}{(s + 120)(s + 150)} ,$$

where  $K = 900$ . The disturbance response is shown in Figure P10.15.



**FIGURE P10.15**  
Compensated system disturbance response.



**P10.16** The PI controller is given by

$$G_c(s) = \frac{K(s+b)}{s},$$

where  $K$  and  $b$  are to be determined. To meet the design specifications, we need

$$\zeta = 0.6 \quad \text{and} \quad \omega_n = 6.67 \text{ rad/sec}.$$

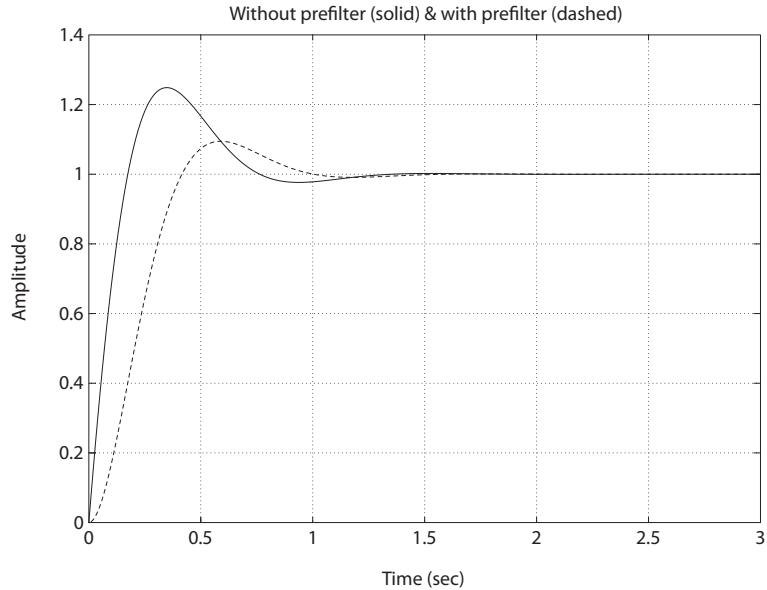
The closed-loop transfer function is

$$T(s) = \frac{K(s+b)}{s^2 + Ks + bK}.$$

Solving for the gains yields  $K = 2\zeta\omega_n = 8$  and  $b = \omega_n^2/K = 5.55$ . A suitable prefilter is

$$G_p(s) = \frac{5.55}{s + 5.55}.$$

The step response, with and without the prefilter, is shown in Figure P10.16.



**FIGURE P10.16**

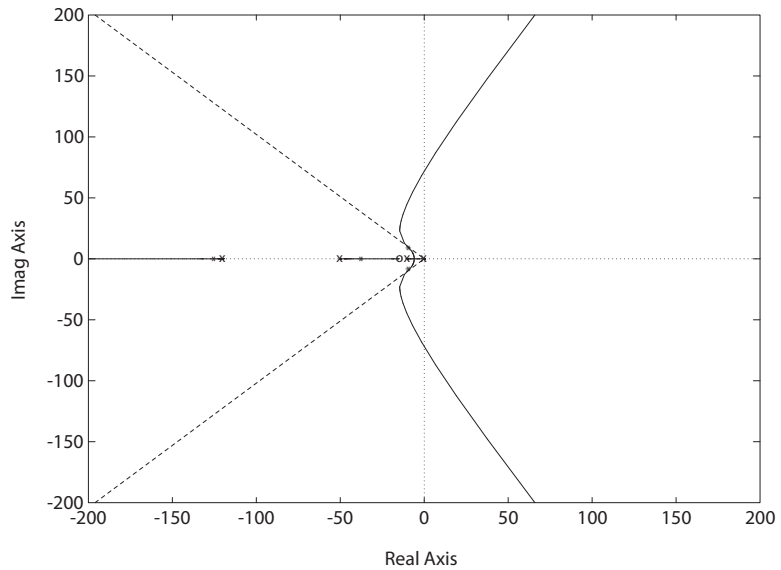
Compensated system response with and without a prefilter.

**P10.17** The plant transfer function is

$$G(s) = \frac{K}{s(s+10)(s+50)} .$$

We desire  $\zeta\omega_n > 10$  to meet  $T_s < 0.4$  sec and  $\zeta = 0.65$  to meet  $P.O. < 7.5\%$ . Try a pole at  $s = -120$ . The root locus is shown in Figure P10.17. The gain  $K = 6000$  for  $\zeta = 0.65$ . Thus,

$$G_c(s)G(s) = \frac{6000(s/15 + 1)}{s(s+10)(s+50)(s/120 + 1)} \quad \text{and} \quad K_v = \frac{6000}{500} = 12 .$$



**FIGURE P10.17**  
 Root locus for  $1 + K \frac{s/15+1}{s(s+10)(s+50)(s/120+1)}$ .

**P10.18** (a) The loop transfer function is

$$L(s) = \frac{K_1 e^{-2Ts}}{0.25s + 1}$$

where  $T = 1.28$ . The phase angle is

$$\phi = -2.56\omega - \tan 0.25\omega .$$

So,  $\omega = 1.12$  rad/sec when  $\phi = -180^\circ$ . However, the break frequency

is 4 rad/sec. Therefore, you cannot achieve  $P.M. = 30^\circ$  and have the system be stable for  $K_1 < 1$ . The steady-state error is

$$e_{ss} = \frac{A}{1 + K_p} = \frac{A}{1 + K_1}$$

since  $K_1 = K_p$ .

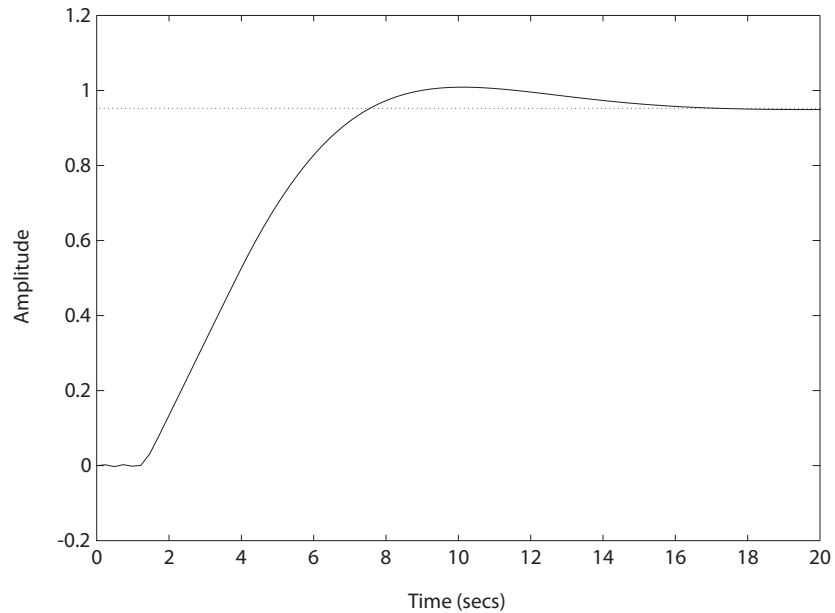
- (b) Set  $K_1 = 20$ , then  $K_p = 20$  and this yields a 5% steady-state error. Without compensation, the system is now unstable. Let

$$G_c(s) = \frac{s/b + 1}{s/a + 1}$$

where  $b = 5$  and  $a = 0.01$ . Then, the system is stable with

$$P.M. = 63^\circ .$$

The system response is shown in Figure P10.18.



**FIGURE P10.18**

Unit step response with  $G_c(s) = \frac{20(s/5+1)}{s/0.01+1}$ .

**P10.19** (a) The open-loop transfer function is

$$G(s) = \frac{Ke^{-sT}}{(s+1)(s+3)},$$

where  $T = 0.5$  sec. We desire  $P.O. < 30\%$ , thus  $\zeta > 0.36$ . We will design for  $\zeta = 0.4$ , which implies  $P.M. = 40^\circ$ . Then

$$\phi = -\tan^{-1} \omega - \tan^{-1} \frac{\omega}{3} - 0.5\omega(57.3^\circ).$$

At  $\omega_c = 1.75$ , the phase margin is  $P.M. = 40^\circ$ , and solving

$$|G(j\omega)| = \frac{K}{[(3 - \omega^2)^2 + (4\omega)^2]^{\frac{1}{2}}} = 1$$

at  $\omega = 1.75$  yields  $K = 7$ . Then  $e_{ss} = 0.3$ .

(b) We want  $e_{ss} < 0.12$ , so use  $e_{ss} = 0.10$  as the goal. Then

$$G_c(s)G(s) = \frac{Ke^{-0.5s}(s+2)}{(s+1)(s+3)(s+b)},$$

and

$$e_{ss} = \frac{1}{1 + K_p}$$

where  $K_p = \frac{2K}{3b}$ . If  $b = 0.1$  then  $K_p = 6.7K$  and

$$e_{ss} = \frac{1}{1 + 6.7K}.$$

So, we need  $6.7K = 9$ , or  $K = 1.35$ . We need a lag compensator (i.e.  $b < 2$ ) to meet  $e_{ss} < 12\%$  and have stability.

**P10.20** We desire  $K_v = 20$ ,  $P.M. = 45^\circ$  and  $\omega_B > 4$  rad/sec. Thus, we set  $K = 20$ , and

$$G(s) = \frac{20}{s\left(\frac{s}{2} + 1\right)\left(\frac{s}{6} + 1\right)}.$$

Then, the Bode plot yields  $P.M. = -21^\circ$  uncompensated at  $\omega_c = 5.2$  rad/sec. The phase lead compensator must add  $66^\circ$  plus phase lead to account for the shift of the crossover to a higher frequency with the phase lead compensator. Consider

$$G_c(s) = \left(\frac{1 + \alpha\tau s}{1 + \tau s}\right)^2.$$

One solution is to use

$$\begin{aligned}\alpha &= 10 \\ \tau &= 1/67 .\end{aligned}$$

Then

$$G_c(s) = \frac{100(s + 6.7)^2}{(s + 67)^2} .$$

The compensator has two zeros at  $\omega = 6.7$ , two poles at  $\omega = 67$  yielding  $P.M. = 47^\circ$ ,  $\omega_c = 7.3$  and  $\omega_B = 12$  rad/sec.

**P10.21** We desire  $K_v = 20$ ,  $P.M. = 45^\circ$  and  $\omega_B \geq 2$ . The lag compensator is

$$G_c(j\omega) = \frac{1 + j\omega\tau}{1 + j\omega\alpha\tau}$$

where  $\alpha > 1$ . From the Bode plot,  $\phi = -135^\circ$  at  $\omega \cong 1.3$ . So, at  $\omega = 1.3$ , we need to lower the magnitude by 22 dB to cause  $\omega = 1.3$  to be  $\omega'_c$ , the new crossover frequency. Thus, solving

$$22 = 20 \log \alpha$$

yields  $\alpha = 14$ . We select the zero one decade below  $\omega'_c$  or  $\frac{1}{\tau} = 0.13$ . Therefore,

$$\frac{1}{\alpha\tau} = \frac{0.13}{14} = 0.0093 .$$

Then, the lag compensator is given by

$$G_c(s) = \frac{1 + \frac{s}{0.13}}{1 + \frac{s}{0.0093}} = \frac{s + 0.13}{14(s + 0.0093)} .$$

The new crossover is  $\omega'_c = 1.3$ , and  $\omega_B = 2.14$  rad/sec.

**P10.22** We desire  $P.M. = 45^\circ$ ,  $K_v = 20$  and  $2 \leq \omega_B \leq 10$ . The lead-lag compensator is

$$G_c(s) = \frac{1 + \frac{s}{b}}{1 + \frac{s}{10b}} \cdot \frac{1 + \frac{s}{10a}}{1 + \frac{s}{a}} .$$

Since  $\omega_B \cong 1.5\omega_c$ , we design for a new crossover frequency  $\omega'_c$  so that

$$1.4 < \omega'_c < 7 .$$

Try for  $\omega'_c = 4$ . The phase  $\phi = -190^\circ$  at  $\omega = 4$ , so we need to add phase lead of  $55^\circ$  plus phase to account for lag part of network at  $\omega'_c$ . Use  $\alpha = 10$  and bracket  $\omega = 4$  with the lead network. Put the zero at  $\omega = 0.8 = b$

and the pole at  $\omega = 8$ . For the lag compensator, put the zero at a lower frequency than  $\omega'_c/10$ . So try a zero at  $\omega = 0.2 = 10a$  and a pole at  $\omega = 0.02 = a$ . Then, the lead-lag compensator is

$$G_c(s) = \frac{(1 + \frac{s}{0.8})(1 + \frac{s}{0.2})}{(1 + \frac{s}{8})(1 + \frac{s}{0.02})}.$$

The compensated Bode plot yields

$$\omega'_c = 3.5 \text{ rad/sec}, \quad P.M. = 50^\circ \quad \text{and} \quad \omega_B = 6.2 \text{ rad/sec}.$$

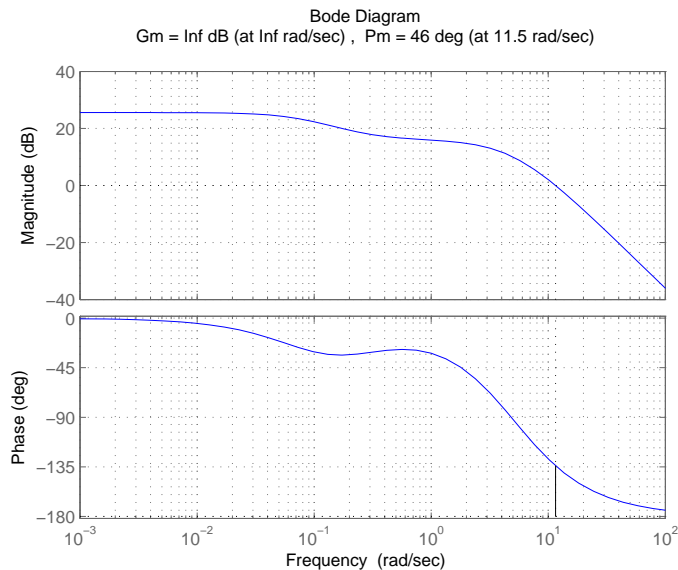
**P10.23** The steady-state error is

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + K/25} = 0.05.$$

So, we need  $K/25 \geq 19$  or  $K \geq 475$ . One possible solution is

$$G_c(s) = \frac{4s + 1}{12s + 1} \quad \text{and} \quad K = 475.$$

The compensated Bode plot is shown in Figure P10.23. The phase margin is  $P.M. = 46^\circ$ .



**FIGURE P10.23**

Bode plot for  $G_c(s)G(s) = \frac{475(4s+1)}{(s+5)^2(12s+1)}$ .

**P10.24** The arm-rotating dynamics are represented by

$$G(s) = \frac{80}{s \left( \frac{s^2}{4900} + \frac{s}{70} + 1 \right)} .$$

We desire  $K_v = 20$ , and  $P.O. < 10\%$ . One possible solution is the lead-lag compensator

$$G_c(s) = \frac{(s + 50)(s + 0.48)}{4(s + 400)(s + 0.06)} .$$

With this compensator, we have

$$P.O. = 9.5\% \quad \text{and} \quad K_v = 20 .$$

**P10.25** Neglect the pole of the airgap feedback loop at  $s = 200$ . The characteristic equation is

$$1 + \bar{K} \frac{(s + 20)(s + c)}{s^3} = 0 ,$$

where

$$\begin{aligned} \bar{K} &= \frac{K}{K_1 + K_2} \\ c &= \frac{K_2 b}{K_1 + K_2} . \end{aligned}$$

Choose

$$c = 10$$

to attain the root locus structure shown in Figure P10.25. The gain

$$\bar{K} = 38.87$$

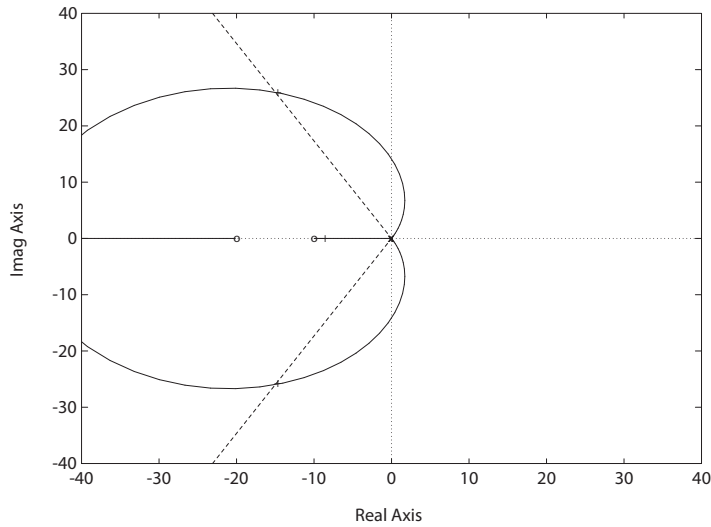
insures the damping ratio of  $\zeta = 0.5$ . Then, solving for  $K_1$  and  $b$  yields

$$K_1 = \frac{K}{38.87} - K_2$$

and

$$b = \frac{0.1K}{38.87K_2} .$$

For given values of  $K$  and  $K_2$  (unspecified in the problem), we can compute  $K_1$  and  $b$ .



**FIGURE P10.25**  
 Root locus for  $1 + \bar{K} \frac{(s+20)(s+10)}{s^3} = 0$ .

**P10.26** The loop transfer function is

$$G_c(s)G(s) = \frac{0.15K(10as + 1)}{s(s + 1)(5s + 1)(as + 1)},$$

where  $K$  and  $a$  are to be selected to meet the design specifications. Suitable values are

$$K = 6.25 \quad \text{and} \quad a = 0.15.$$

Then, the phase margin is  $P.M. = 30.79^\circ$  and the bandwidth is  $\omega_B = 0.746$  rad/sec. The lead compensator is

$$G_c(s) = 6.25 \frac{1.5s + 1}{0.15s + 1}.$$

**P10.27** (a) Let  $G_c(s) = K = 11$ . Then the phase margin is  $P.M. = 50^\circ$  and the performance summary is shown in Table P10.27.

(b) Let

$$G_c(s) = \frac{K(s + 12)}{(s + 20)},$$

where  $K = 32$ . Then, the phase margin is  $P.M. = 50^\circ$  and the performance summary is given in Table P10.27.



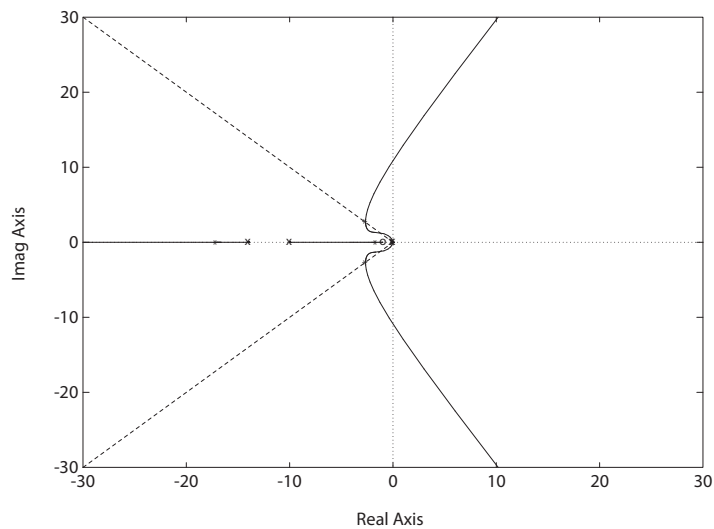
compensator	$P.M.$	$P.O.$	$T_p$	$T_s$	$M_{p\omega}$	$\omega_B$
$G_c(s) = K = 11$	$50^\circ$	18%	0.34 sec	0.78 sec	1.5 dB	13.9 rad/sec
$G_c(s) = \frac{32(s+12)}{s+20}$	$50^\circ$	18%	0.20 sec	0.47 sec	1.5 dB	26.3 rad/sec

TABLE P10.27 Performance Summary.

**P10.28** The loop transfer function is

$$G_c(s)G(s) = \frac{K(as + 1)}{s(s + 10)(s + 14)(10as + 1)},$$

where  $K$  and  $a$  are to be selected to meet the design specifications, and we have set  $\alpha = 10$ . The root locus is shown in Figure P10.28a. To satisfy



**FIGURE P10.28**  
 (a) Root locus for  $1 + K \frac{1400(s+1)}{s(s+10)(s+14)(10s+1)} = 0$ .

the steady-state tracking error we must select

$$K > 1400 .$$

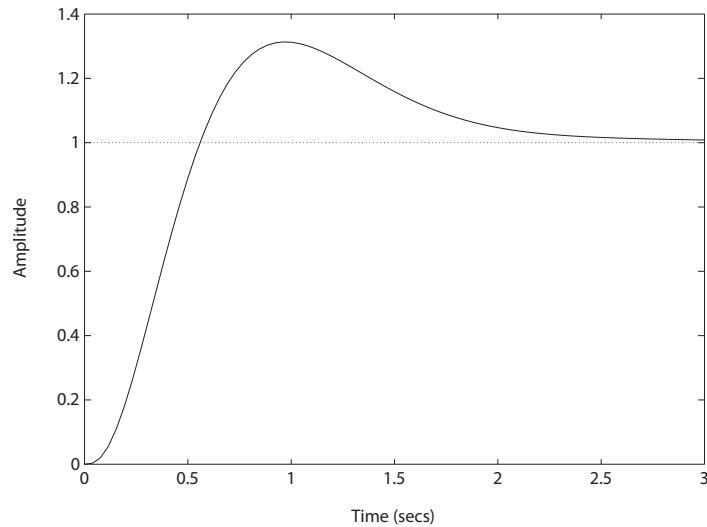
Suitable values for the lag compensator are

$$K = 4060 \quad \text{and} \quad a = 1 .$$

Then, the percent overshoot is  $P.O. = 31\%$  and the settling time is  $T_s = 2.34$  sec. The lag compensator is

$$G_c(s) = \frac{s + 1}{10s + 1} .$$

The step response is shown in Figure P10.28b.



**FIGURE P10.28**  
CONTINUED: (b) Step response.

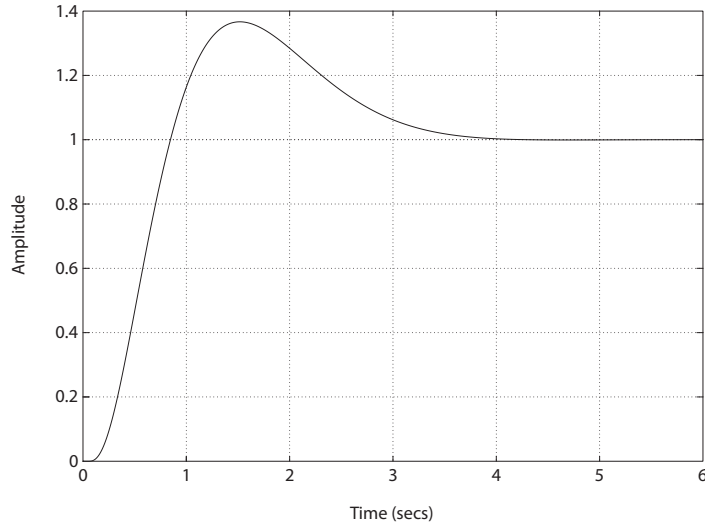
**P10.29** The plant transfer function is

$$G(s) = \frac{10e^{-0.05s}}{s^2(s + 10)} .$$

The lead network

$$G_c(s) = \frac{16(s + 0.7)}{(s + 9)}$$

provides  $M_{p\omega} = 3.4$  dB and  $\omega_r = 1.39$  rad/sec. The step response is shown in Figure P10.29. The overshoot is  $P.O. = 37\%$  and  $T_s = 3.5$  sec.



**FIGURE P10.29**  
Unit step response with  $G_c(s) = \frac{16(s+0.7)}{s+9}$ .

**P10.30** The vehicle is represented by

$$G(s) = \frac{K}{s(0.04s + 1)(0.001s + 1)} \approx \frac{K}{s(0.04s + 1)} .$$

For a ramp input, we want

$$\frac{e_{ss}}{A} = 0.01 = \frac{1}{K_v} .$$

So, let

$$G(s) = \frac{100}{s(0.04s + 1)} .$$

The uncompensated  $P.M. = 28^\circ$  at  $\omega_c = 47$  rad/sec. We need to add  $17^\circ$ .

Case (1) Phase lead compensation:

$$G_c(s) = \frac{1 + 0.021s}{1 + 0.01s} .$$

The phase margin is  $P.M. = 45^\circ$ .

Case (2) Phase lead compensation:

$$G_c(s) = \frac{1 + 0.04s}{1 + 0.005s} .$$

The phase margin is  $P.M. = 65^\circ$ .

For **Case 1**, we have

$$P.O. = 25\% , \quad T_s = 0.13 \text{ sec and } T_p = 0.05 \text{ sec} .$$

For **Case 2**, we have

$$P.O. = 4\% , \quad T_s = 0.04 \text{ sec and } T_p = 0.03 \text{ sec} .$$

**P10.31** As in P10.30, the plant is given by

$$G(s) = \frac{100}{s(0.04s + 1)} .$$

The uncompensated  $P.M. = 28^\circ$ . We need  $P.M. = 50^\circ$ . The phase lag compensator

$$G_c(s) = \frac{1 + 0.5s}{1 + 2.5s}$$

results in  $P.M. = 50^\circ$ . The  $P.O. = 21\%$ ,  $T_s = 0.72 \text{ sec}$  and  $T_p = 0.17 \text{ sec}$ .

**P10.32** (a) To obtain  $K_v = 100$ , we have

$$G_c(s)G(s) = \frac{43.33(s + 500)}{s(s + 0.0325)(s^2 + 2.57s + 6667)} .$$

With  $K = 43.33$ , we have

$$P.M. = 1.2^\circ , \quad M_{p\omega} = 26 \text{ dB} , \quad \omega_r = 1.8 \text{ rad/sec and } \omega_B = 3.7 \text{ rad/sec} .$$

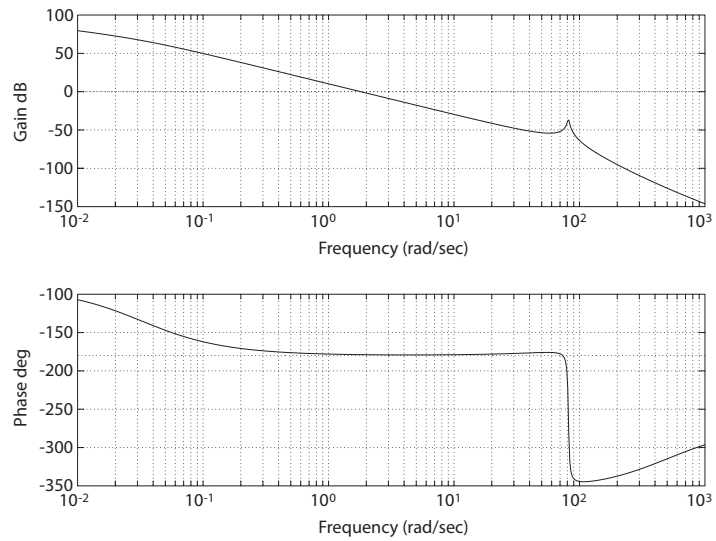
The Bode plot is shown in Figure P10.32.

(b) Let

$$G_c(s) = \frac{0.35s + 1}{0.001s + 1} ,$$

and  $K = 43.33$  (as before). Then,

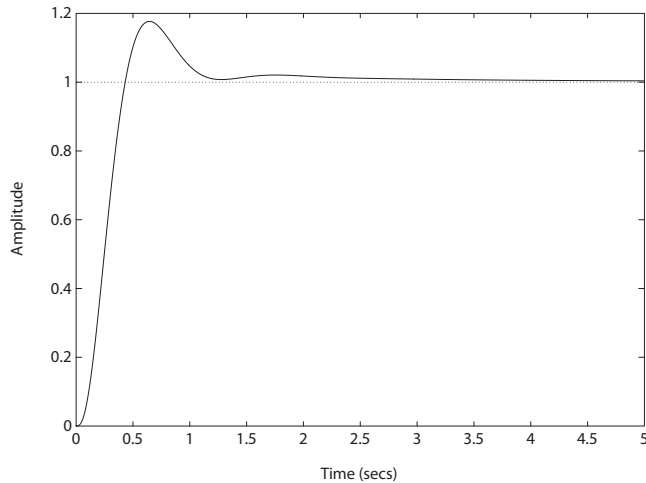
$$P.M. = 36^\circ , \quad M_{p\omega} = 5.4 \text{ dB} , \quad \omega_r = 1.7 \text{ rad/sec and } \omega_B = 3.0 \text{ rad/sec} .$$



**FIGURE P10.32**  
Bode plot with  $G_c(s) = K = 43.33$ .

**P10.33** The step response is shown in Figure P10.33, where

$$G_c(s) = \frac{10(s + 0.71)(s + 0.02)}{(s + 0.0017)(s + 10)}$$



**FIGURE P10.33**  
Step response with the lead-lag compensator  $G_c(s) = \frac{10(s+0.71)(s+0.02)}{(s+0.0017)(s+10)}$ .

Then,

$$K_v = 80$$

and  $P.O. = 17\%$ ,  $T_s = 1.8$  sec, and  $\zeta = 0.54$ .

**P10.34** The process model is

$$G(s) = \frac{1}{s^2(s + 10)},$$

and we consider the lead compensator

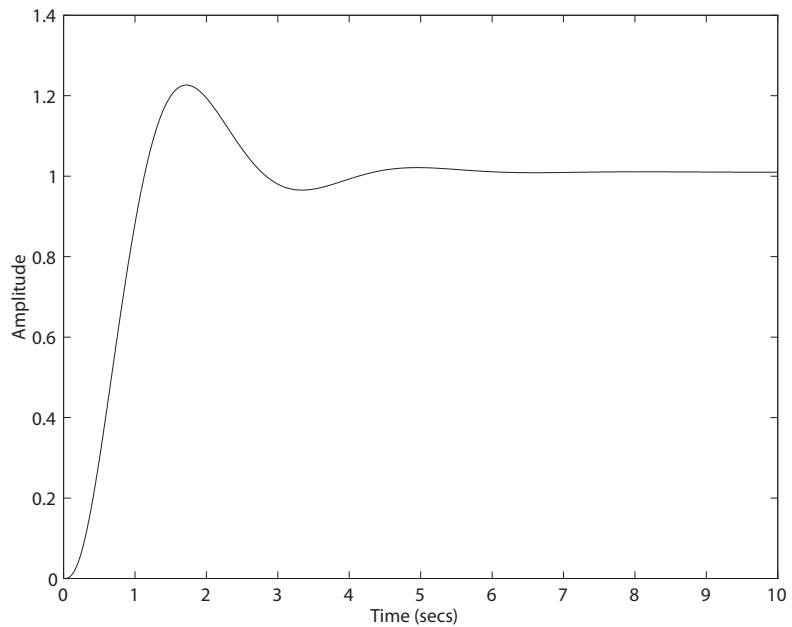
$$G_c(s) = K \frac{1 + s\alpha\tau}{1 + s\tau},$$

where  $\alpha = 100$ ,  $\tau = 0.4$  and  $K = 0.5$ . Then,  $P.M. = 46.4^\circ$ . The step response is shown in Figure P10.34. The system performance is

$$P.O. = 22.7\%$$

$$T_s = 5.2 \text{ sec}$$

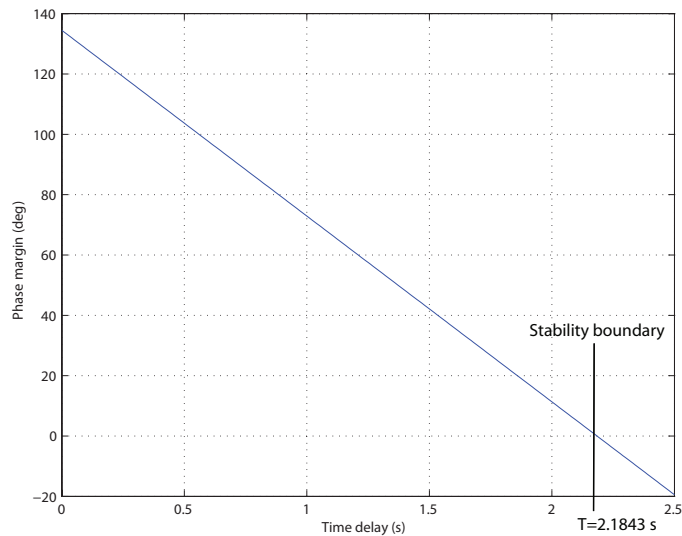
$$T_p = 1.72 \text{ sec}.$$



**FIGURE P10.34**

Step response with the lead compensator  $G_c(s) = 0.5 \frac{40s+1}{0.4s+1}$ .

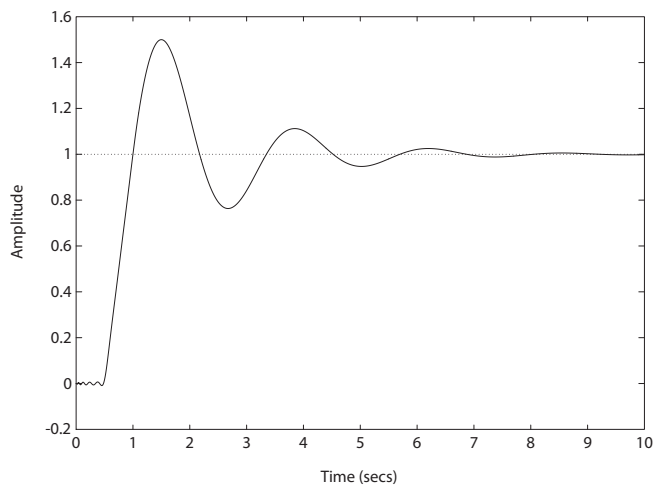
- P10.35** The phase margin is shown in Figure P10.35. As the time delay increases, the phase margin decreases. The system is unstable when  $T > 2.1843$  s.



**FIGURE P10.35**

Step response with  $G_c(s)G(s) = \frac{2s+0.54}{s(s+1.76)}e^{-Ts}$ , where  $0 \leq T \leq 2.5$ .

- P10.36** One possible solution is the integral controller  $G_c(s) = 2/s$ . The step response is shown in Figure P10.36. The steady-state tracking error to a



**FIGURE P10.36**

Step response with the integral controller  $G_c(s) = 2/s$ .

step input is zero since the system is type-1. The phase margin is  $P.M. = 32.8^\circ$  and the bandwidth is  $\omega_B = 4.3 \text{ rad/s}$ .

**P10.37** One possible solution is

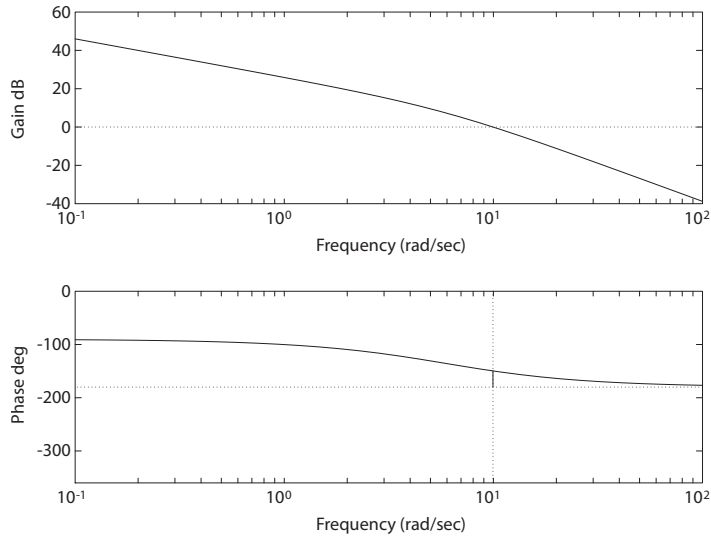
$$G_c(s) = \frac{1600(s+1)}{25s+1}.$$

The overshoot to a unit step is  $P.O. = 4.75\%$  and the steady-state error to a step input is  $e_{ss} = 1\%$ . The system bandwidth is  $\omega_B = 9.7 \text{ rad/sec}$ .

**P10.38** The lead compensator is

$$G_c(s) = \frac{2.88(s+2.04)}{s+5.88}.$$

The Bode plot is shown in Figure P10.38. The phase margin is  $P.M. = 30.4^\circ$  at  $\omega_c = 9.95 \text{ rad/sec}$  and the bandwidth is  $\omega_B = 17.43 \text{ rad/sec}$ .



**FIGURE P10.38**

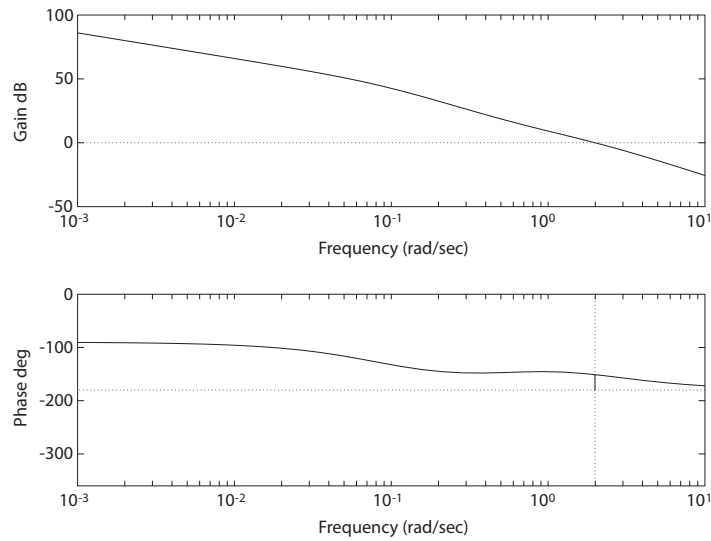
Bode plot for  $G_c(s)G(s) = \frac{115.29(s+2.04)}{s(s+2)(s+5.88)}$ .

**P10.39** The lag compensator is

$$G_c(s) = \frac{1+1.48s}{1+11.08s}.$$

The Bode plot is shown in Figure P10.39. The steady-state error specification is satisfied since  $K_v = 20$ . The phase margin is  $P.M. = 28.85^\circ$  at



**FIGURE P10.39**

Bode plot for  $G_c(s)G(s) = \frac{40(1+1.48s)}{s(s+2)(1+11.08s)}$ .

$\omega_c = 2$  rad/sec and the bandwidth is  $\omega_B = 3.57$  rad/sec.

**P10.40** The lag compensator is

$$G_c(s) = \frac{2.5(1 + 1.64s)}{1 + 30.5s}.$$

The Bode plot is shown in Figure P10.40. The steady-state error specification is satisfied since

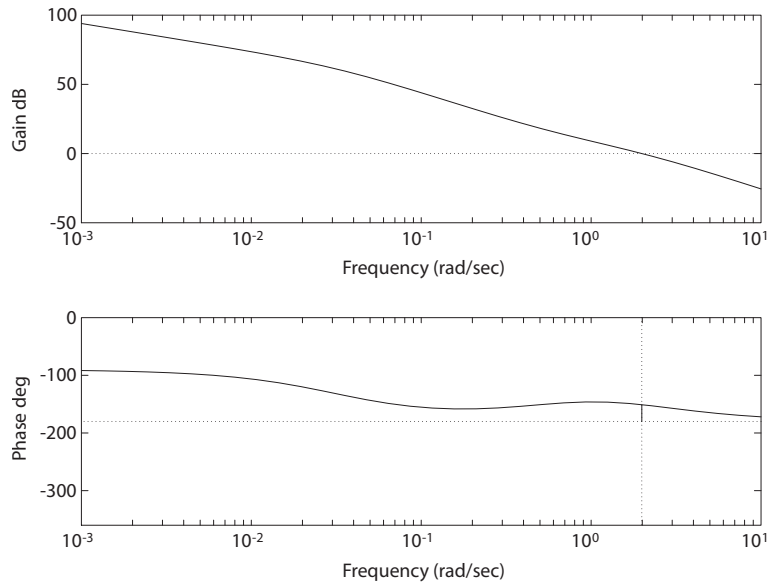
$$K_v = 50.$$

The phase margin is

$$P.M. = 28.93^\circ$$

at  $\omega_c = 1.98$  rad/sec and the bandwidth is

$$\omega_B = 3.59 \text{ rad/sec.}$$



**FIGURE P10.40**  
 Bode plot for  $G_c(s)G(s) = \frac{100(1+1.64s)}{s(s+2)(1+30.5s)}$ .

**P10.41** We use Table 10.2 in Dorf & Bishop to determine the required coefficients

$$\alpha = 1.9 \quad \text{and} \quad \beta = 2.2 .$$

Also,

$$\omega_n T_r = 4.32 \quad \text{implies} \quad \omega_n = 4.32,$$

since we require

$$T_r = 1 \text{ second}.$$

The characteristic equation is

$$s^3 + 8.21s^2 + 41.06s + 80.62 = s^3 + (1 + p)s^2 + (K + p)s + Kz = 0 .$$

Equating coefficients and solving yields

$$p = 7.21 \quad K = 33.85 \quad z = 2.38 .$$

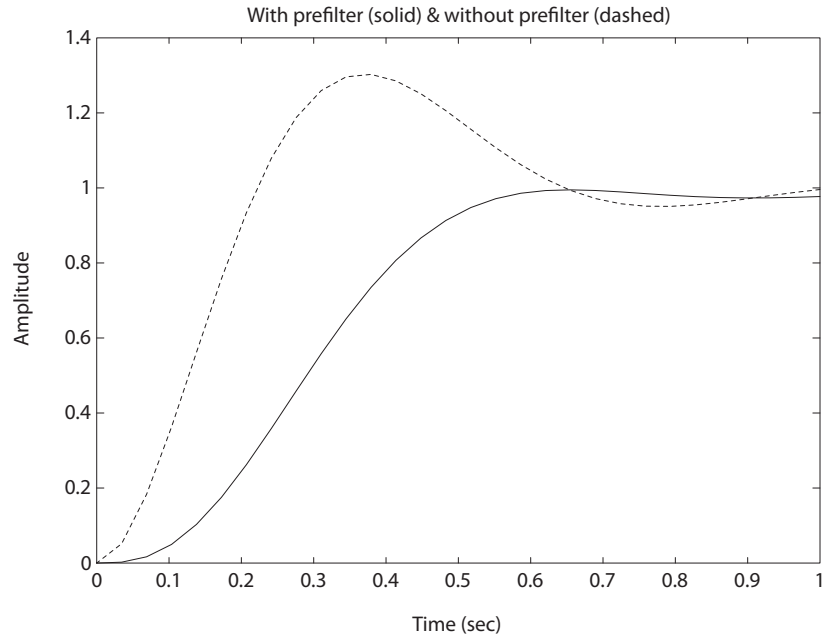
**P10.42** From Example 10.4 in Dorf & Bishop, we have the closed-loop transfer function

$$T(s) = \frac{96.5(s + 4)}{(s^2 + 8s + 80)(s + 4.83)} .$$

A suitable prefilter is

$$G_p(s) = \frac{4}{s + 4} .$$

The step response (with and without the prefilter) is shown in Figure P10.42.



**FIGURE P10.42**  
Step response with and without the prefilter.

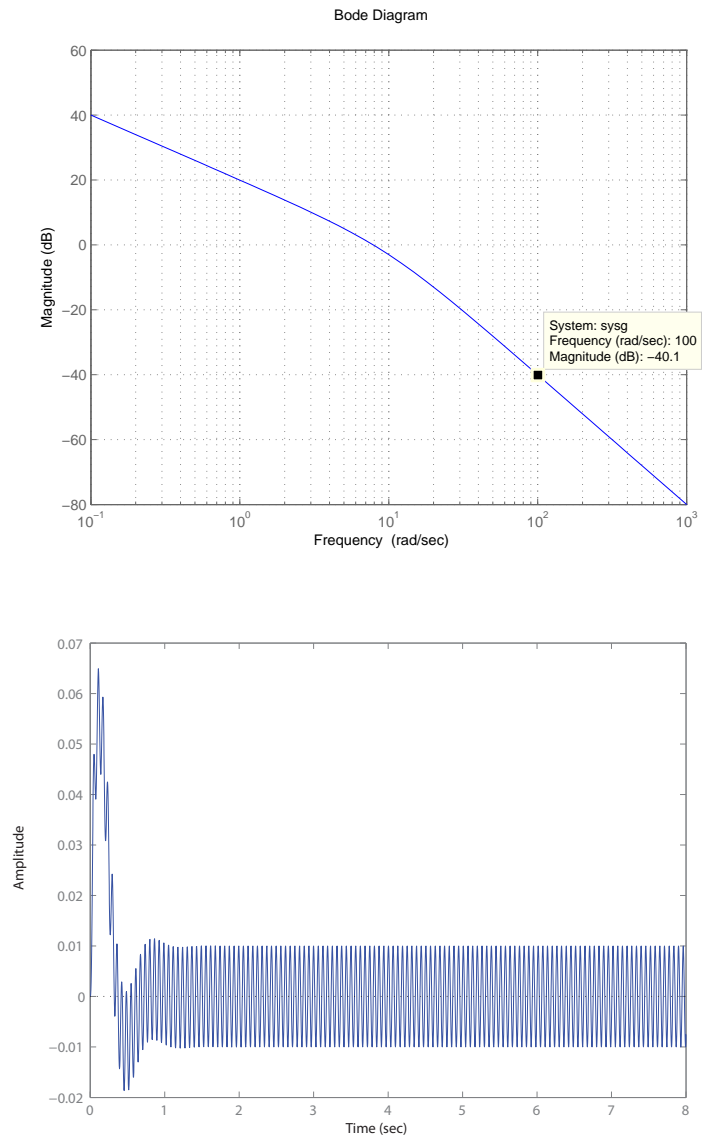
**P10.43** Let

$$K = 100.$$

The Bode plot is shown in Figure P10.43a and the response to a sinusoidal noise input with

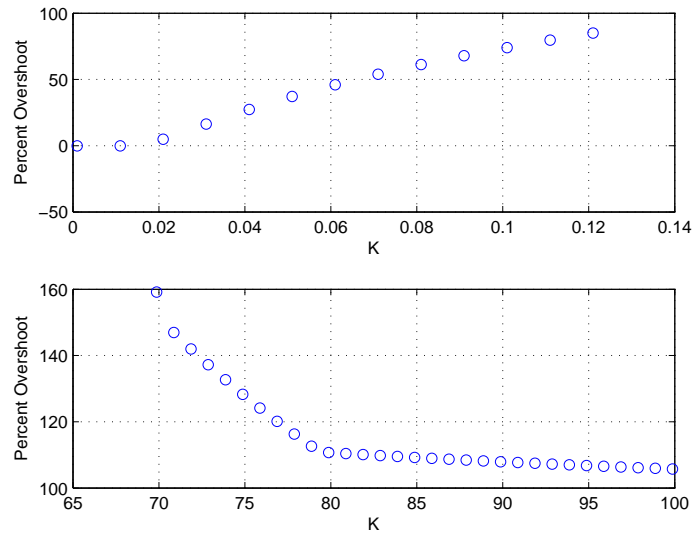
$$\omega = 100 \text{ rad/s}$$

is shown in Figure P10.43b.

**FIGURE P10.43**

(a) Bode magnitude plot. (b) Response to a noise input.

**P10.44** For  $0.129 < K \leq 69.87$ , the system is unstable. The percent overshoot is shown in Figure P10.44 .



**FIGURE P10.44**  
Percent overshoot.

## Advanced Problems

AP10.1 (a) With

$$G_c(s) = K ,$$

the closed-loop transfer function is

$$T(s) = \frac{K}{s^3 + 5s^2 + 4s + K} .$$

When  $K = 2.05$ , the characteristic equation is

$$s^3 + 5s^2 + 4s + 2.05 = 0$$

with poles at  $s = -4.1563$  and  $s = -0.4219 \pm j0.5615$ . Therefore  $\zeta = 0.6$ , and the predicted overshoot is

$$P.O. = 100e^{-\pi \cdot 0.6 / \sqrt{1-0.6^2}} = 9.5\% < 13\% .$$

The actual overshoot is  $P.O. = 9.3\%$  and  $T_s = 8.7$  seconds.

(b) When

$$G_c(s) = \frac{82.3(s + 1.114)}{s + 11.46}$$

the closed-loop transfer function is

$$\begin{aligned} T(s) &= \frac{82.3(s + 1.114)}{s^4 + 16.46s^3 + 61.3s^2 + 128.14s + 91.6822} \\ &= \frac{82.3(s + 1.114)}{(s + 1.196)(s + 12.26)(s + 1.5 \pm j2)} . \end{aligned}$$

Therefore  $\zeta = 0.6$  and the predicted overshoot is  $P.O. = 9.5\% < 13\%$ .

The actual overshoot is  $P.O. = 12\%$  and  $T_s = 2.5$  seconds.

AP10.2 The lag network is given by

$$G_c = \frac{K(s + a_1)}{s + a_2} .$$

The closed-loop transfer function is

$$T(s) = K \frac{s + a_1}{s^4 + (5 + a_2)s^3 + (4 + 5a_2)s^2 + (4a_2 + K)s + Ka_1} .$$

Computing the steady-state tracking error yields

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s^4 + (5 + a_2)s^3 + (4 + 5a_2)s^2 + 4a_2s}{s^5 + (5 + a_2)s^4 + (4 + 5a_2)s^3 + (4a_2 + K)s^2 + Ka_1s} \\ &= \frac{4a_2}{a_1K} < 0.125 . \end{aligned}$$

If we select  $K = 2.05$  (as in AP10.1), then

$$a_1 > 15.61a_2 .$$

So, take  $a_2 = a_1/16$ . The lag compensator can now be written as

$$G_c(s) = 2.05 \frac{s + a_1}{s + a_1/16} .$$

Select  $a_1 = 0.018$ . Then, the closed-loop transfer function is

$$T(s) = \frac{2.05(s + 0.018)}{s^4 + 5.0011s^3 + 4.0056s^2 + 2.0545s + 0.0369} .$$

The performance results are  $P.O. = 13\%$  and  $T_s = 29.6$  seconds for a step input, and  $e_{ss} = 0.12$  for a ramp input.

**AP10.3** The plant transfer function is

$$G(s) = \frac{1}{s(s+1)(s+4)}$$

and the PI controller is given by

$$G_c(s) = \frac{K_p s + K_I}{s} .$$

The closed-loop transfer function is

$$T(s) = \frac{K_p s + K_I}{s^4 + 5s^3 + 4s^2 + K_p s + K_I} .$$

For a unit ramp, the steady-state tracking error is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^4 + 5s^3 + 4s^2}{s^5 + 5s^4 + 4s^3 + K_p s^2 + K_I s} = 0 .$$

Any  $K_I > 0$  and  $K_p > 0$  (such that the system is stable) is suitable and will track a ramp with zero steady-state error. Since we want  $P.O. < 13\%$ , the damping of the dominant roots should be  $\zeta \approx 0.6$ . One suitable

solution is to place the zero at  $s = -0.01$  and select the PI controller

$$G_c(s) = \frac{2.05(s + 0.01)}{s} .$$

Therefore,  $K_p = 2.05$  and  $K_I = 0.0205$ . The closed-loop transfer function is

$$T(s) = \frac{2.05(s + 0.01)}{s^4 + 5s^3 + 4s^2 + 2.05s + 0.0205} .$$

The performance results are  $P.O. = 11.5\%$  and  $T_s = 9.8$  seconds for a step input, and  $e_{ss} = 0$  for a unit ramp.

**AP10.4** The closed-loop transfer function is

$$T(s) = \frac{10K_1}{s^2 + 10(1 + K_1K_2)s + 10K_1} .$$

From the performance specifications, we determine that the natural frequency and damping of the dominant poles should be  $\omega_n = 5.79$  and  $\zeta = 0.69$ . So,

$$s^2 + 8(1 + K_1K_2)s + 8K_1 = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 7.99s + 33.52 .$$

Solving for the gains yields  $K_1 = 4.19$  and  $K_2 = 0$ . The closed-loop transfer function is

$$T(s) = \frac{33.52}{s^2 + 8s + 33.52} .$$

The performance results are  $P.O. = 5\%$  and  $T_s = 1$  second.

**AP10.5** (a) From the overshoot specification  $P.O. = 10\%$ . The plant transfer function is

$$G(s) = \frac{1}{s(s+1)(s+10)} .$$

Let  $G_p = 1$ . A suitable compensator is

$$G_c = K \frac{s + 0.5}{s + 10} .$$

Using root locus methods, we determine that  $K = 45$  yields  $P.O. \approx 10\%$ . The closed-loop poles are  $s_{1,2} = -2.5 \pm j5.1$ ,  $s_3 = -15.48$ , and  $s_4 = -0.45$ .

(b) The closed-loop transfer function is

$$T(s) = \frac{450(s + 0.5)}{s^4 + 21s^3 + 120s^2 + 550s + 225} .$$



The step response is shown in Figure AP10.5. The overshoot and settling time are  $P.O. = 9.5\%$  and  $T_s = 3.4$  seconds.

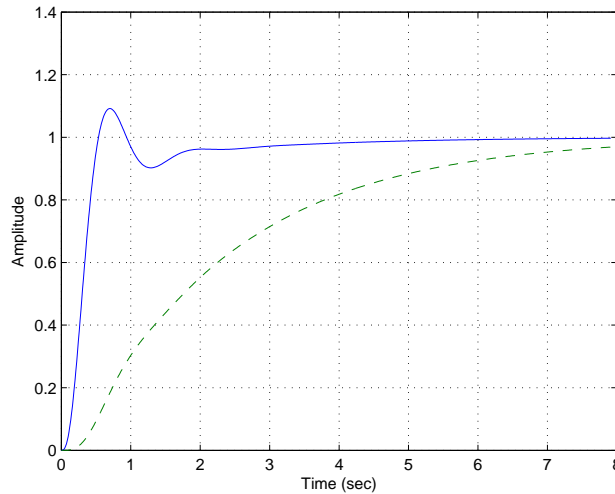
(c) A suitable prefilter is

$$G_p(s) = \frac{0.5}{s + 0.5} .$$

The closed-loop transfer function is

$$T(s) = \frac{225}{s^4 + 21s^3 + 120s^2 + 550s + 225} .$$

The step response is shown in Figure AP10.5. The overshoot and settling time are  $P.O. = 0\%$  and  $T_s = 6.85$  seconds.



**FIGURE AP10.5** Step response with prefilter (dashed line) and without prefilter (solid line).

**AP10.6** From Example 10.12 in Dorf & Bishop, we have the relationship

$$\omega_n T_s = 4.04 .$$

Therefore, minimizing  $T_s$  implies maximizing  $\omega_n$ . Using Table 10.2 in Dorf & Bishop, we equate the desired and actual characteristic polynomials

$$q(s) = s^3 + 1.9\omega_n s^2 + 2.2\omega_n^2 s + \omega_n^3 = s^3 + (1 + p)s^2 + (K + p)s + Kz .$$

Comparing coefficients yields

$$(1 + p) = 1.9\omega_n, \quad K + p = 2.2 \left( \frac{1 + p}{1.9} \right)^2, \quad Kz = \omega_n^3.$$

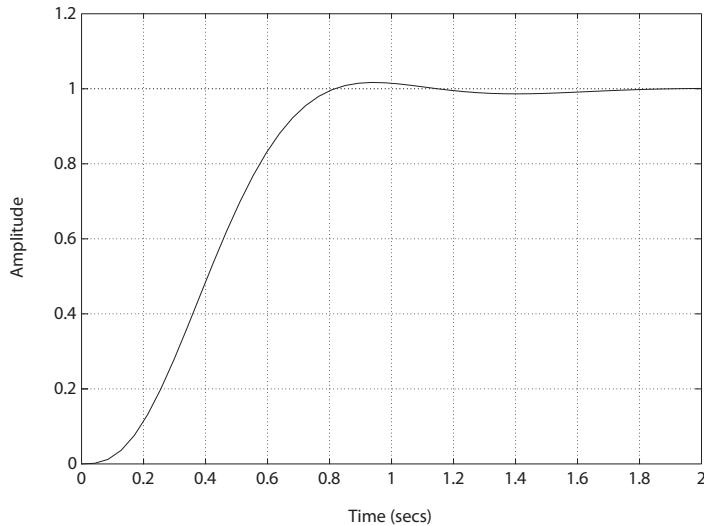
So, from the first relationship we see that maximizing  $\omega_n$  implies maximizing  $p$ . Solving for  $p$  while maintaining  $K < 52$

$$K = \frac{2.2}{3.61}(p^2 + 2p + 1) - p < 52$$

we determine that

$$-9.3643 < p < 9.005.$$

The largest  $p = 9$ . Therefore,  $K = 51.94$  and  $z = 2.81$ . The step response is shown in Figure AP10.6. The settling time is  $T_s = 0.77$  second.



**FIGURE AP10.6**  
Step response with minimum settling time.

**AP10.7** Let  $G_p = 1$ . The closed-loop transfer function is

$$T(s) = \frac{K(s + 3)}{s^4 + 38s^3 + 296s^2 + (K + 448)s + 3K}.$$

When  $K = 311$ , the characteristic equation

$$s^4 + 38s^3 + 296s^2 + 759s + 933 = 0$$

has poles at  $s = -1.619 \pm j1.617$  ( $\zeta = 1/\sqrt{2}$ ),  $s = -6.25$ , and  $s = -28.51$ .

(a) When  $G_p(s) = 1$  and  $K = 311$ , the closed-loop transfer function is

$$T(s) = \frac{311(s+3)}{s^4 + 38s^3 + 296s^2 + 759s + 933} .$$

The step input performance is  $P.O. = 6.5\%$ ,  $T_s = 2.5$  seconds, and  $T_r = 1.6$  seconds. With the prefilter

$$G_p(s) = \frac{3}{s+3}$$

and  $K = 311$ , the closed-loop transfer function is

$$T(s) = \frac{933}{s^4 + 38s^3 + 296s^2 + 759s + 933} .$$

In this case, the step response is  $P.O. = 3.9\%$ ,  $T_s = 2.8$  seconds, and  $T_r = 1.3$  seconds.

(b) Now, consider the prefilter

$$G_p(s) = \frac{1.8}{s+1.8}$$

and  $K = 311$ . The closed-loop transfer function is

$$T(s) = \frac{559.8(s+3)}{s^5 + 39.8s^4 + 364.4s^3 + 1291.8s^2 + 2299.2s + 1679.4} .$$

The step input response is  $P.O. = 0.7\%$ ,  $T_s = 2.14$  seconds and  $T_r = 1.3$  seconds.

**AP10.8** The plant transfer function is

$$G(s) = \frac{250}{s(s+2)(s+40)(s+45)} .$$

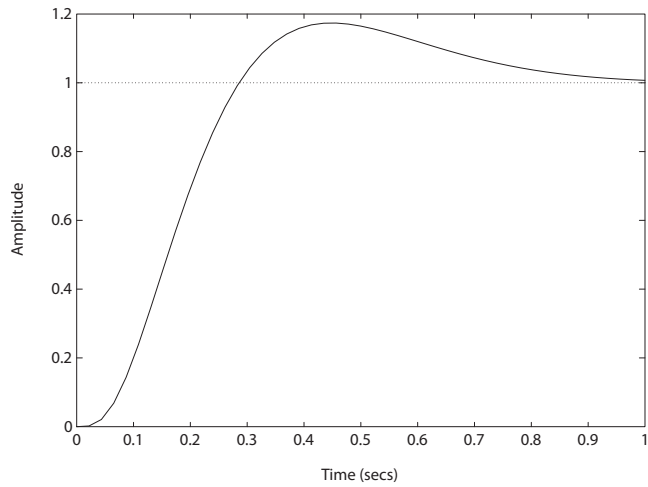
The performance specifications are  $P.O. < 20\%$ ,  $T_r < 0.5$  second,  $T_s < 1.2$  seconds and  $K_v \geq 10$ . A suitable lead compensator is

$$G_c = 1483.7 \frac{s+3.5}{s+33.75} .$$

The closed-loop transfer function is

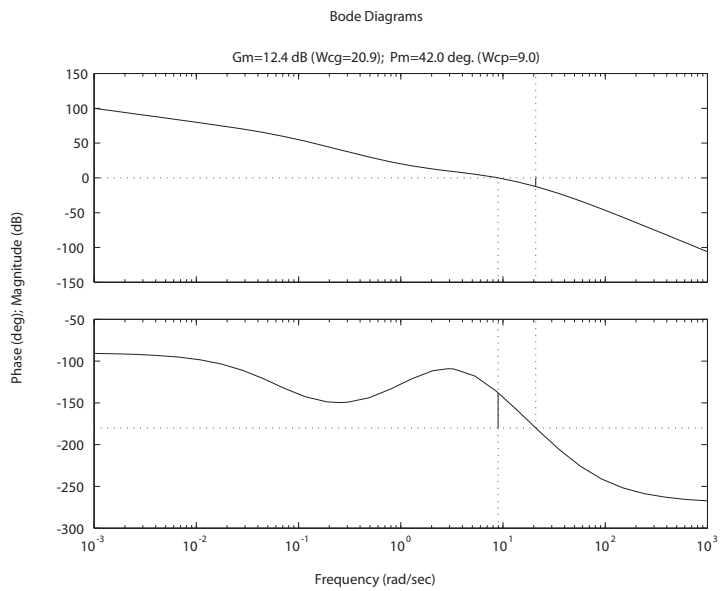
$$T(s) = \frac{250(1483.7)(s+35)}{s(s+2)(s+40)(s+45)(s+33.75) + 250(1483.7)(s+3.5)}$$

The actual step input performance (see Figure AP10.8) is  $P.O. = 18\%$ ,  $T_s = 0.88$  second,  $T_r = 0.18$  second, and  $K_v = 10.7$ .



**FIGURE AP10.8**  
Step response with lead compensator.

**AP10.9** The frequency response is shown in Figure AP10.9.



**FIGURE AP10.9**  
Bode plot with  $G_c(s) = \frac{(s+2.5)(s+0.9871)}{(s+36.54)(s+0.0675)}$

One lead-lag compensator that satisfies the specifications is

$$G_c(s) = \frac{(s + 2.5)(s + 0.9871)}{(s + 36.54)(s + 0.0675)} .$$

The gain and phase margins are  $Gm = 12.35$  dB and  $Pm = 41.8^\circ$ , respectively. The velocity error constant is  $K_v = 100$ . Therefore, all specifications are satisfied.

## Design Problems

**CDP10.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$\frac{\theta(s)}{V_a(s)} = \frac{26.035}{s(s + 33.142)} ,$$

where we neglect the motor inductance  $L_m$  and where we switch off the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). With a PD controller the closed-loop system characteristic equation is

$$s^2 + (33.142 + 26.035K_D)s + 26.035K_p = 0 .$$

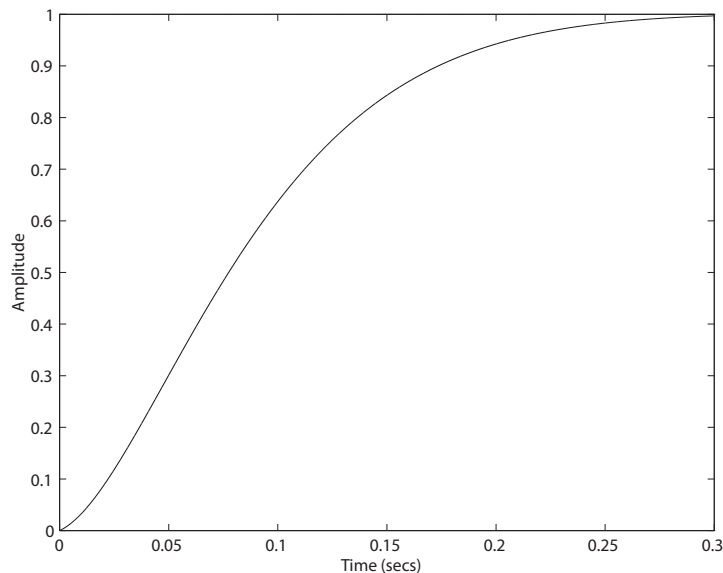
Using Table 10.2 in Dorf and Bishop we determine that for a second-order system with a deadbeat response we have  $\alpha = 1.82$  and  $\omega_n T_s = 4.82$ . Since we desire  $T_s < 0.25$  seconds, we choose  $\omega_n = 19.28$ . Equating the actual characteristic equation with the desired characteristic equation we obtain

$$s^2 + \omega_n \alpha s + \omega_n^2 = s^2 + (33.142 + 26.035K_D)s + 26.035K_p .$$

Solving for  $K_p$  and  $K_D$  yields the PD controller:

$$G_c(s) = 14.28 + 0.075s .$$

The step response is shown below. The settling time is  $T_s = 0.24$  second.



**DP10.1** The plant is given as

$$G(s) = \frac{20}{s(s+2)} .$$

One possible lead compensator is

$$G_{c_{lead}}(s) = \frac{50(s+1)}{s+20} .$$

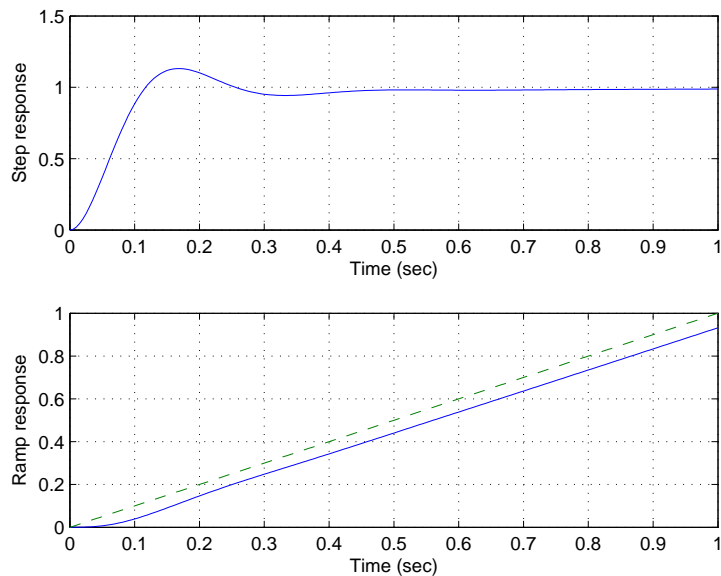
Similarly, a suitable lag compensator is

$$G_{c_{lag}}(s) = \frac{s+0.1}{s+0.022} .$$

The loop transfer function with the lead-lag compensator is

$$G_c(s)G(s) = \frac{1000(s+1)(s+0.1)}{s(s+2)(s+0.022)(s+20)} .$$

The step response and ramp response are shown in Figure DP10.1. The velocity constant is  $K_v = 50$ , so the steady-state error specification is satisfied.



**FIGURE DP10.1**

Step response and ramp response.

- DP10.2** (a) When  $G_c(s) = K$ , we require  $K > 20$  to meet the steady-state tracking specification of less than 5%.  
 (b) The system is unstable for  $K > 20$ .  
 (c) A single stage lead compensator is

$$G_{c1}(s) = \frac{1 + 0.49s}{1 + 0.0035s} .$$

With this compensator, the bandwidth is

$$\omega_B = 68.9 \text{ rad/sec}$$

and the phase margin is  $P.M. = 28.57^\circ$ .

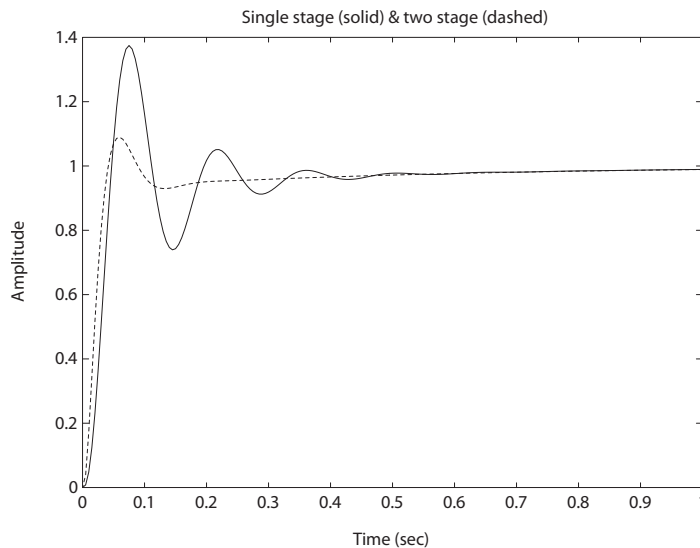
- (d) A two stage lead compensator is

$$G_{c2}(s) = \frac{(1 + 0.0185s)(1 + 0.49s)}{(1 + 0.00263s)(1 + 0.0035s)} .$$

With the two stage compensator, the bandwidth is

$$\omega_B = 83.6 \text{ rad/sec}$$

and the phase margin is  $P.M. = 56.79^\circ$ . The step response for the two compensators is shown in Figure DP10.2.



**FIGURE DP10.2**

Step response for one- and two-stage lead compensators.



**DP10.3** The mast flight system is modeled as

$$G(s) = \frac{6}{s(s + 1.5)(s + 3.9)} .$$

Consider the proportional controller

$$G_c(s) = K = 0.85 .$$

The system step response is shown in Figure DP10.3. The percent overshoot is

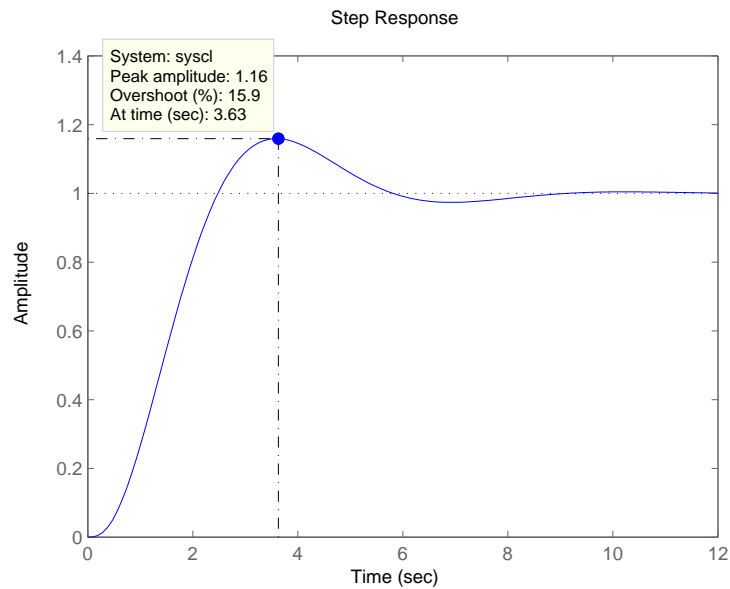
$$P.O. = 15.9\% ,$$

the rise time is

$$T_r = 3.63 \text{ seconds} ,$$

and the phase margin is

$$P.M. = 52^\circ .$$

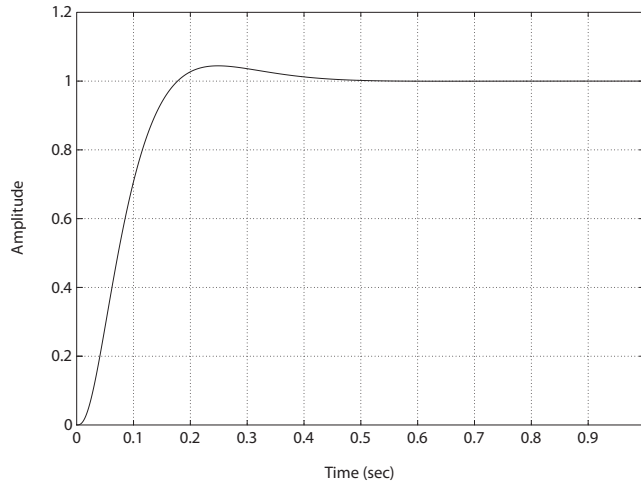


**FIGURE DP10.3**  
Step response for the mast flight system.

**DP10.4** One possible compensator is

$$G_c(s) = 5682 \frac{s + 12.6}{s + 87.3} .$$

The step response is shown in Figure DP10.4. The performance results



**FIGURE DP10.4**

Step response for the high speed train system.

are

$$P.O. = 4.44\% \quad T_s = 0.36 \text{ sec} \quad K_v = 14.1 .$$

**DP10.5** The design specifications are  $K_v > 200$ ;  $T_s < 12 \text{ ms}$  and percent overshoot  $P.O. < 10\%$ . The step response is shown in Figure DP10.5. A suitable compensator is

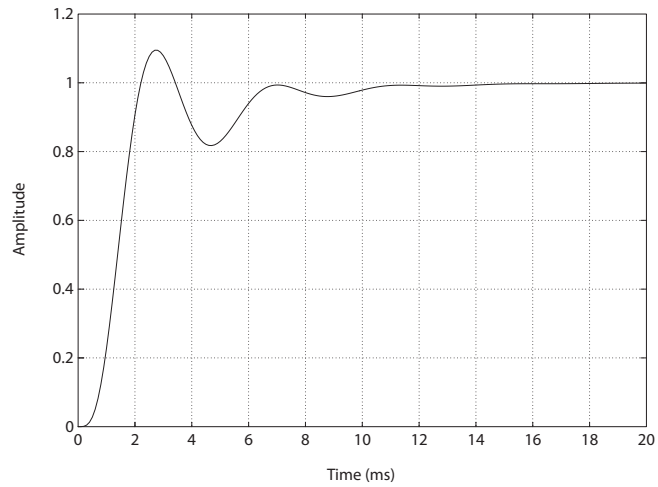
$$G_c(s) = K \frac{s + 403}{s + 2336} ,$$

where

$$K = 1.9476e + 13.$$

Then,

$$P.O. = 9.5\% \quad T_s = 10 \text{ ms} \quad K_v = 560 .$$

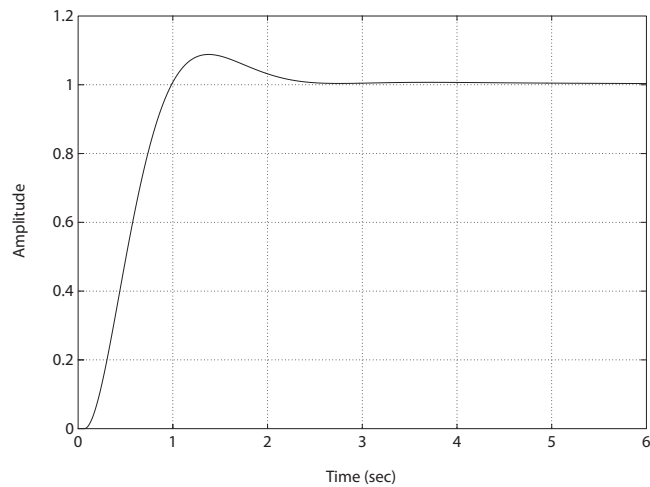


**FIGURE DP10.5**  
Step response for the tape transport system.

**DP10.6** A solution to the problem is the PI controller

$$G_c(s) = \frac{4.21s + 1.2}{s} .$$

The step response is shown in Figure DP10.6.



**FIGURE DP10.6**  
Step response for the engine control system.

The performance results are

$$P.O. = 8.8\% \quad \text{and} \quad T_s = 2.14 .$$

The system is a type-1, so the steady-state error for a step input is zero, as desired.

**DP10.7** The jet aircraft roll angle motion is represented by the transfer function

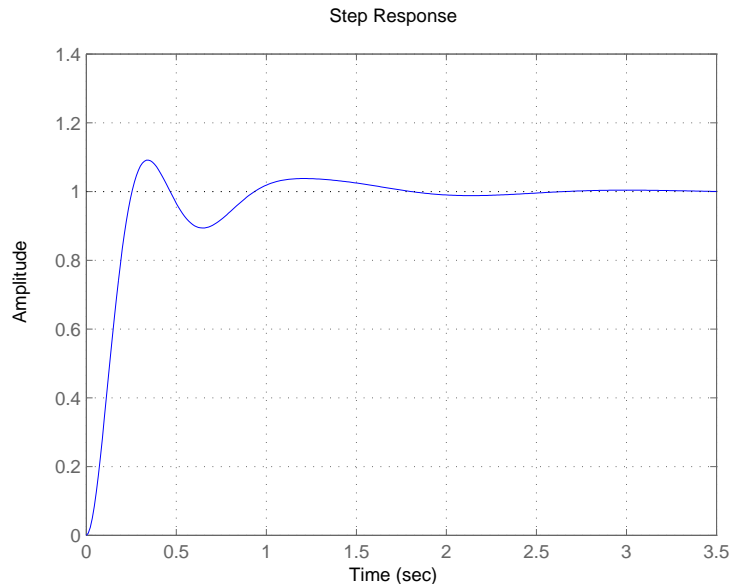
$$G(s) = \frac{10}{(s + 10)(s^2 + 2s + 20)} .$$

A good controls solution is obtained with a PID controller

$$G_c(s) = \frac{10s^2 + 20s + 150}{s} .$$

The system is type-1, so the steady-state tracking error is zero for a step input. The performance results are

$$P.O. = 9.13\% \quad \text{and} \quad T_s = 1.56 .$$



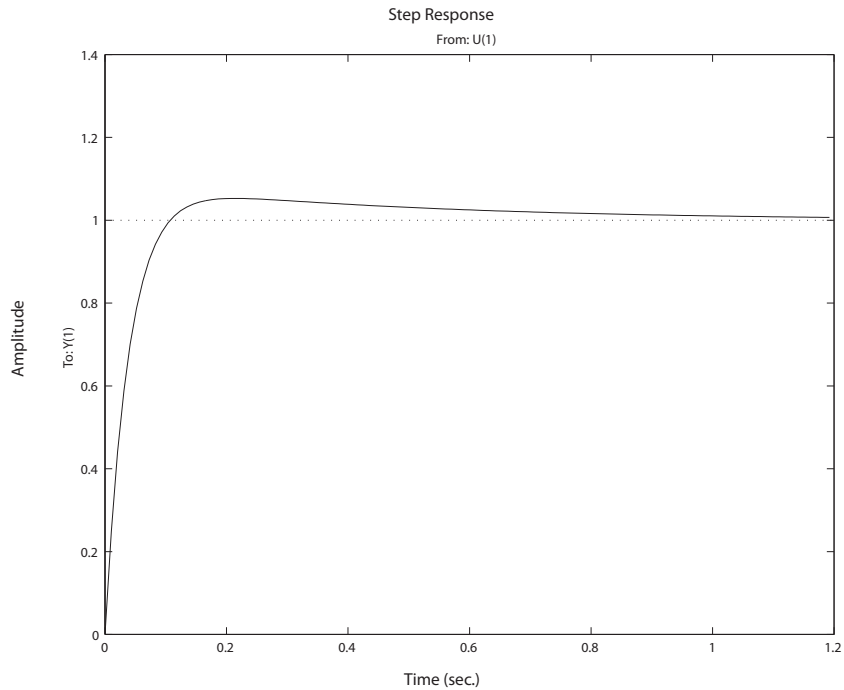
**FIGURE DP10.7**

Step response for the jet aircraft roll control system.

**DP10.8** One good solution is obtained with the following PI controller

$$G_c(s) = \frac{27.35(s + 2)}{s}.$$

The system is type-1, so the steady-state tracking error is zero for a step input. The step response is shown in Figure DP10.8.



**FIGURE DP10.8**  
Step response for the windmill radiometer.

**DP10.9** Consider the PID controller

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{1.554s^2 + 1.08s + 1}{s}$$

and the lead-lag controller

$$G_c(s) = K \left( \frac{s + a}{s + b} \right) \left( \frac{s + c}{s + d} \right) = 6.04 \frac{(s + 10)(s + 2)}{(s + 1)(s + 5)}.$$

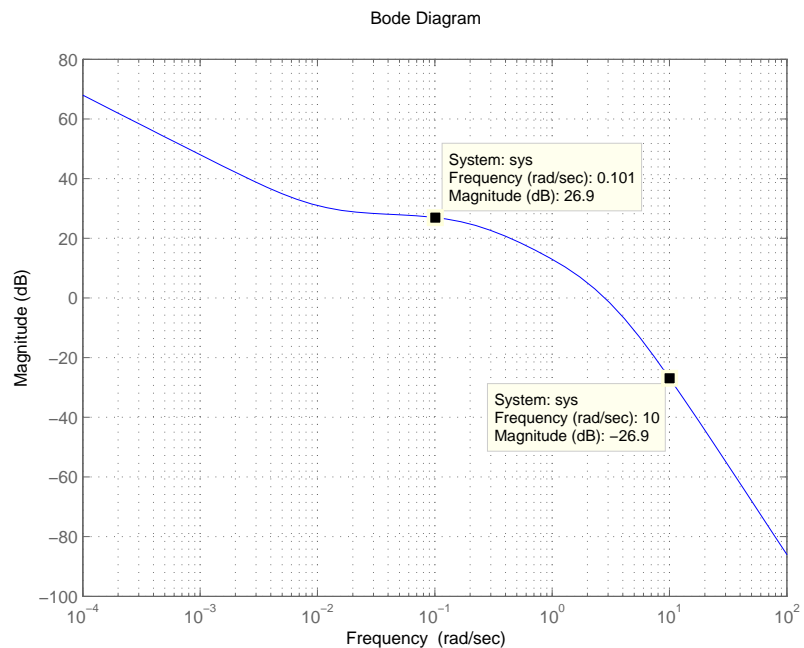
Both are stabilizing in the presence of a  $T = 0.1$  second time delay. For the PID controller the phase margin is  $P.M. = 40^\circ$ . For the lead-lag controller the phase margin is  $P.M. = 45^\circ$ . We find (for these particular designs)

that the lead-lag controller is more able to remain stable in the process of increasing time delay. For a time-delay of  $T = 0.2$  seconds, the lead-lag compensator has a phase margin of  $P.M. = 22^\circ$ , while the PID controller is unstable.

**DP10.10** One solution is

$$G_c(s) = \frac{50(s + 0.01)}{s + 2} .$$

The Bode magnitude is shown in Figure DP10.10. You want high gain at



**FIGURE DP10.10**

Step response for the windmill radiometer.

low frequency to improve disturbance rejection and decrease sensitivity to plant changes and low gain at high frequency to attenuate measurement noise.

**DP10.11** One solution is the PD controller

$$G_c(s) = 0008(s + 10) .$$

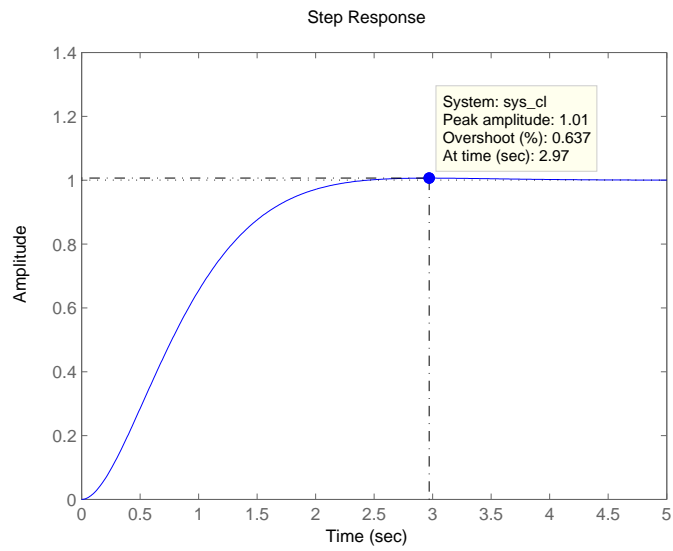
The step response is shown in Figure DP10.11. The closed-loop transfer

function is

$$T(s) = \frac{4}{s^2 + 3.4s + 4},$$

where we use the prefilter

$$G_p(s) = \frac{4}{0.36s + 3.6}.$$



**FIGURE DP10.11**

Step response for the polymerase chain reaction system.

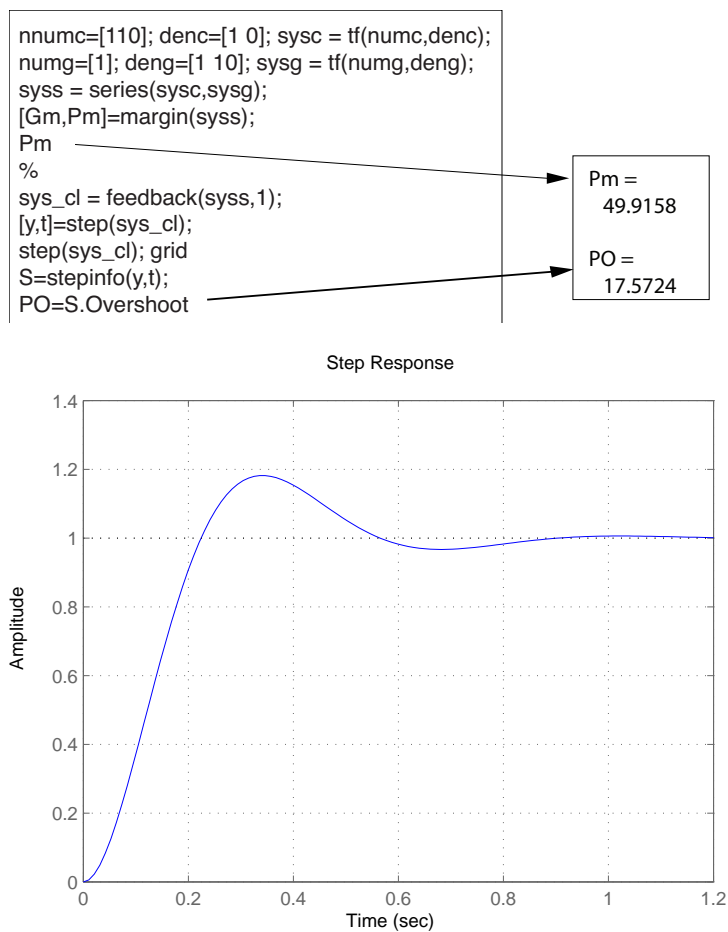
## Computer Problems

**CP10.1** The m-file script and step response is shown in Figure CP10.1. The phase margin and percent overshoot are

$$P.M. = 50^\circ$$

$$P.O. \approx 18\% ,$$

respectively.



**FIGURE CP10.1**

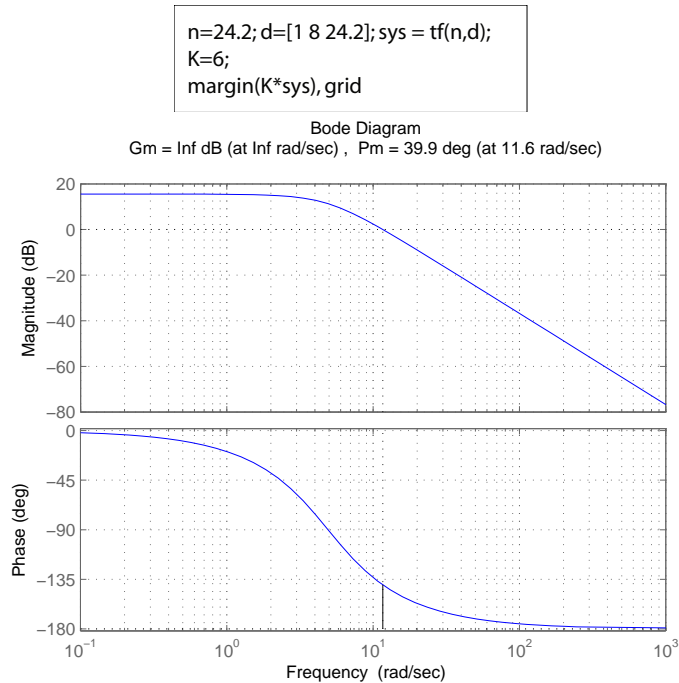
Phase margin and step response for the closed-loop system.



**CP10.2** Using a proportional controller the closed-loop characteristic equation is

$$1 + K \frac{24.2}{s^2 + 8s + 24.2} .$$

A simple m-file script which computes the  $P.M.$  as a function of the gain  $K$  yields the proportional controller gain  $K = 6$ . Checking the phase margin of the system reveals that  $P.M. \approx 40^\circ$ , as desired.



**FIGURE CP10.2**

Bode plot with a proportional controller  $K = 6$  in the loop.

**CP10.3** The uncompensated system is type-1. To realize a zero steady-state error to a ramp input we need to increase the system type by one. One controller that does this is the PI controller:

$$G_c(s) = \frac{K_P s + K_D}{s} .$$

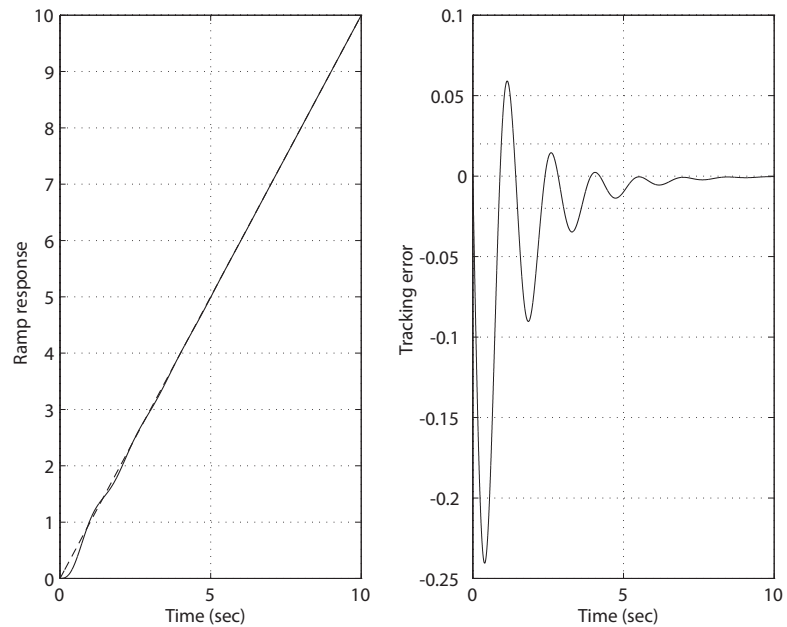
The step response is shown in Figure CP10.3 where it can be seen in the tracking error plot that the settling time is  $T_s < 5$  seconds. The actual settling time is

$$T_s = 3.6 \text{ seconds} .$$

```

KP=20; KD=10;
nc=[KP KD]; dc=[1 0]; sysc = tf(nc,dc);
n=1; d=[1 2 0]; sys = tf(n,d);
sys_o = series(sysc,sys);
sys_cl = feedback(sys_o,1);
t=[0:0.001:10];
sys1 = tf([1],[1 0]); sys_cl1 = series(sys_cl,sys1);
subplot(121)
y=step(sys_cl1,t);
plot(t,y,t,'--'), grid
xlabel('Time (sec)'), ylabel('Ramp response')
e=y-t; L=find(abs(e)>0.02);
Ts=t(L(length(L)))
subplot(122)
plot(t,e,[0 10],[0.02 0.02],':',[0 10],[-0.02 -0.02],':')
xlabel('Time (sec)'), ylabel('Tracking error')
grid

```



**FIGURE CP10.3**

Ramp response with a PI controller  $G_c(s) = \frac{20s+10}{s}$  in the loop.

- CP10.4** From the percent overshoot spec we determine that  $P.O. < 10\%$  implies  $\zeta > 0.6$ . So, we target a phase margin  $P.M. = 100\zeta = 60^\circ$ . The m-file script which generates the uncompensated Bode plot is shown in Figure CP10.4a.

```

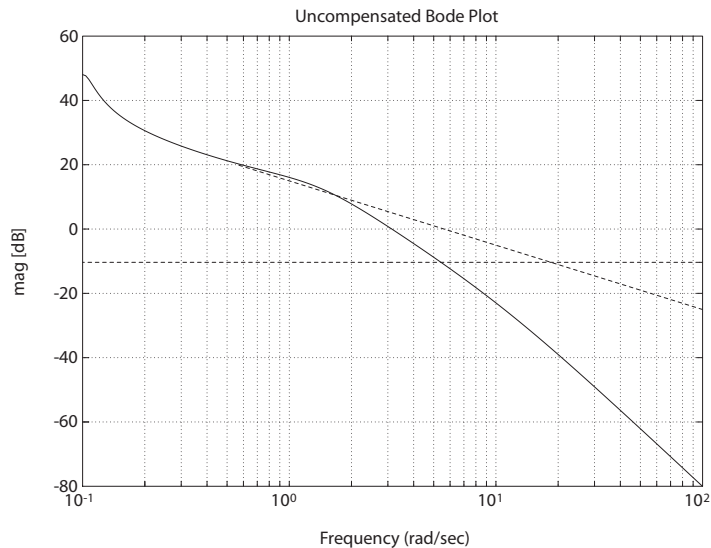
numg = 100*conv([1 1],[1 0.01]);
deng = conv([1 10],conv([1 2 2],[1 0.02 0.0101]));
sysg = tf(numg,deng)
w=logspace(-1,2,200);
[mag,phase,w]=bode(sysg,w);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
%
Phi=60-Pm
Pm
Phi=(60-Pm)*pi/180;
alpha=(1+sin(Phi))/(1-sin(Phi))
M=-10*log10(alpha)*ones(length(w),1);
[mag,phase,w]=bode(sysg,w);
for i = 1:length(w),
    magdB(i) = 20*log10(mag(1,1,i));
end
semilogx(w,magdB,w,M), grid
xlabel('Frequency (rad/sec)'), ylabel('mag [dB]')
title('Uncompensated Bode Plot')
hold on
semilogx([.56072 5.6072 56.072 560.72],[20 0 -20 -40], '--')

```

```

É
Phi =
    56.2111
Pm =
    3.7889
alpha =
    10.8408

```



**FIGURE CP10.4**

(a) Uncompensated Bode plot.

We assume that  $K = 1$  and raise the gain at a later step to meet settling time requirement. The uncompensated phase margin is  $P.M. = 3.7^\circ$ , so that the lead compensator needs to add  $\phi = 56.2^\circ$ . The script also calculates  $\alpha = 10.84$ . Following the design procedure outlined in Dorf & Bishop, we locate the compensator zero at  $\omega = 2$  rad/sec (see dashed line in Figure CP10.4a). Then,  $p = \alpha z$  implies  $p = 21.68$ . After several iter-

ations, we converge on  $K = 4$  as a “good” value. The lead compensator is

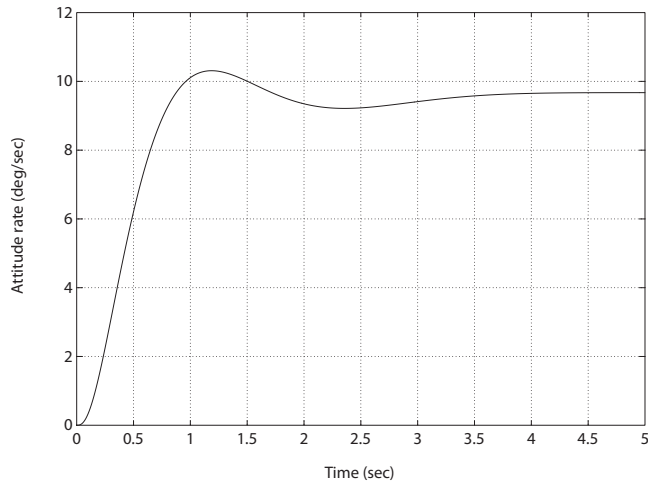
$$G_c(s) = 4 \frac{s + 2}{s + 22} .$$

The step response is shown in Figure CP10.4b. The compensated Bode is shown in Figure CP10.4c.

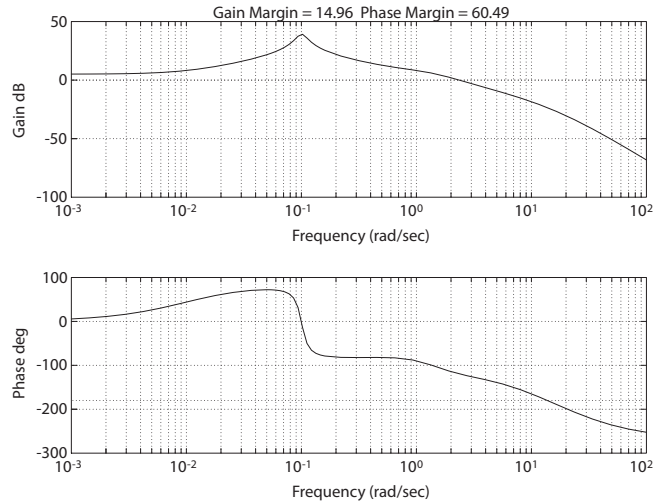
```

K=4;
numg = 100*conv([1 1],[1 0.01]);
deng = conv([1 10],conv([1 2 2],[1 0.02 0.0101]));
sysg = tf(numg,deng)
numc=K*[1 2]; denc=[1 22]; sysc = tf(numc,denc);
sys_o = series(sysc,sysg);
sys_cl = feedback(sys_o,[1]);
t=[0:0.01:5];
f=10*pi/180;
[y,t,x]=step(f*sys_cl,t);
plot(t,y*180/pi), grid
xlabel('Time (sec)')
ylabel('Attitude rate (deg/sec)'), pause
w=logspace(-1,2,200);
[mag,phase,w]=bode(sys_o,w);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
bode(sys_o)
title(['Gain Margin = ',num2str(Gm),' Phase Margin = ',num2str(Pm)])

```



**FIGURE CP10.4**  
CONTINUED: (b) Step response.

**FIGURE CP10.4**

CONTINUED: (c) Bode plot with lead compensator.

**CP10.5** The closed-loop transfer function is

$$\theta(s)/\theta_d(s) = \frac{\bar{K}_1 + \bar{K}_2 s}{s^2 + \bar{K}_2 s + \bar{K}_1}$$

where  $\bar{K}_1 = K_1/J$  and  $\bar{K}_2 = K_2/J$ . A percent overshoot  $P.O. \leq 20\%$  requires  $\zeta > 0.45$ . Select as the initial damping

$$\zeta = 0.7 \quad (\text{initial selection}) .$$

For a second-order system with  $\zeta = 0.7$ , we find that  $\omega/\omega_n \approx 0.9$  when  $|\theta(s)/\theta_d(s)| = 0.7$ . So, we select

$$\omega_n = \omega_B/0.9$$

as a starting choice. Therefore, since  $\omega_B = 10$ , we have

$$\omega_n = 11 .$$

The m-file script is shown in Figure CP10.5a. After several iterations, we find a set of “good” values for

$$\zeta = 0.8 \quad \text{and} \quad \omega_n = 4.5 \quad (\text{final selection}) .$$

The step response and closed-loop Bode plot are shown in Figures CP10.5b and CP10.5c.

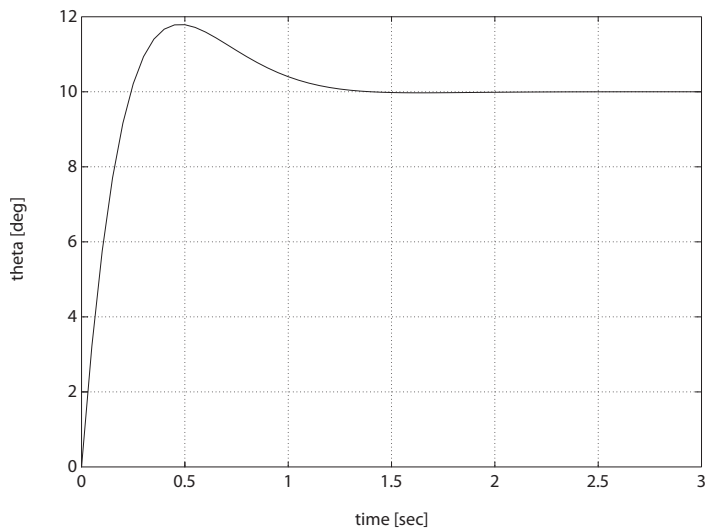
```

% Part (a)
wn=4.5; zeta=0.8; K2=2*zeta*wn; K1=wn^2;
% Part (b)
num=[K2 K1]; den=[1 0 0]; sys = tf(num,den);
sys_cl = feedback(sys,[1]);
f=10*pi/180; % set-up for 10 deg step input
t=[0:.05:3];
[y,t,x]=step(f*sys_cl,t);
plot(t,y*180/pi), xlabel('time [sec]'), ylabel('theta [deg]'),
grid, pause
% Part (c)
w=logspace(-1,2,400);
[mag,phase,w]=bode(sys_cl,w);
for i = 1:length(w),
    magdB(i) = 20*log10(mag(1,1,i));
end
semilogx(w,magdB,[w(1) w(length(w))],[ -3 -3]), grid
xlabel('Frequency (rad/sec)')
ylabel('Gain dB')

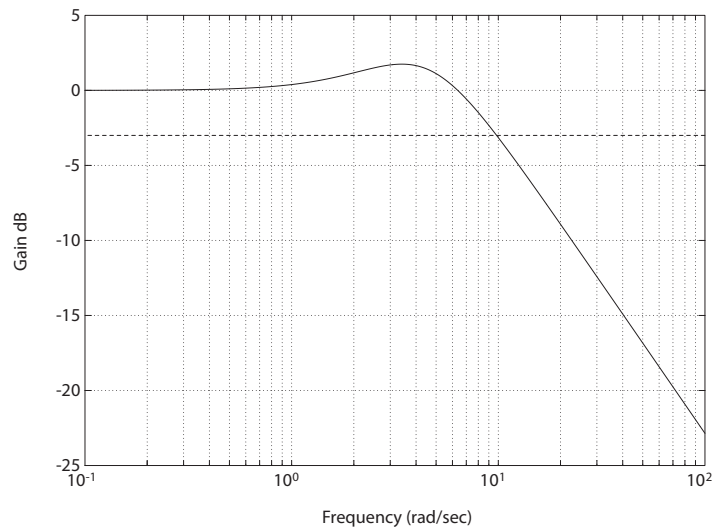
```

**FIGURE CP10.5**

(a) Script to generate the step response and the closed-loop Bode plot.

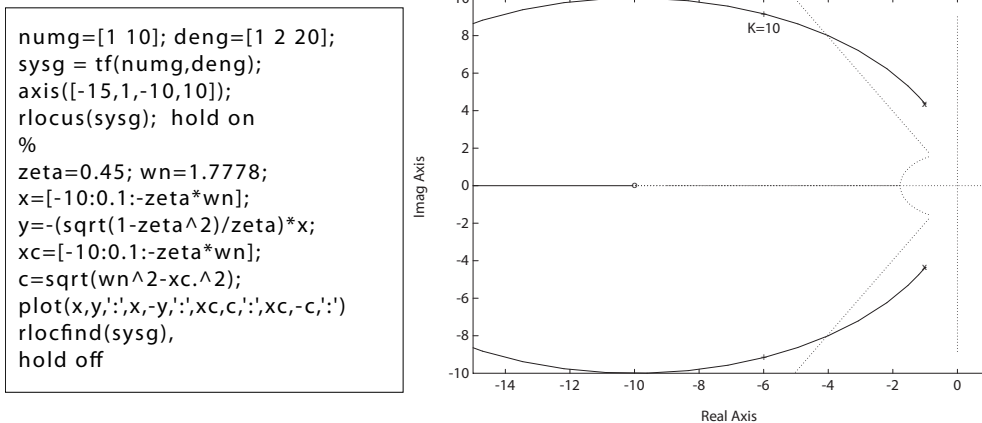
**FIGURE CP10.5**

CONTINUED: (b) Step response.



**FIGURE CP10.5**  
CONTINUED: (c) Closed-loop Bode plot.

**CP10.6** The settling time and phase margin specifications require that the dominant closed-loop poles have natural frequency and damping of  $\zeta \geq 0.45$  and  $\omega_n \geq 1.78$ . The uncompensated roots locus is shown in Figure CP10.6a.



**FIGURE CP10.6**  
(a) Uncompensated root locus.

From the final value theorem, we determine that

$$\lim_{s \rightarrow 0} sE(s) \leq 0.1A \quad \text{implies} \quad \frac{A}{1 + GG_c(s)} = 0.1A .$$

Therefore, the compensated  $K_{p_{comp}} \geq 9$ . With the compensator

$$G_c(s) = K \frac{s + z}{s + p}$$

we find that

$$K_{p_{comp}} = \frac{Kz}{p} K_{p_{uncomp}} .$$

But  $K_{p_{uncomp}} = 0.5$  and (from the uncompensated root locus) a gain of  $K = 10$  results in roots of the characteristic equation in the desired region. Solving for

$$\frac{z}{p} = \frac{1}{K} \frac{K_{p_{comp}}}{K_{p_{uncomp}}} \approx 2 .$$

Select  $z = 0.5$  to minimize changing the root locus. Then,  $p = 0.25$ , and the compensator is

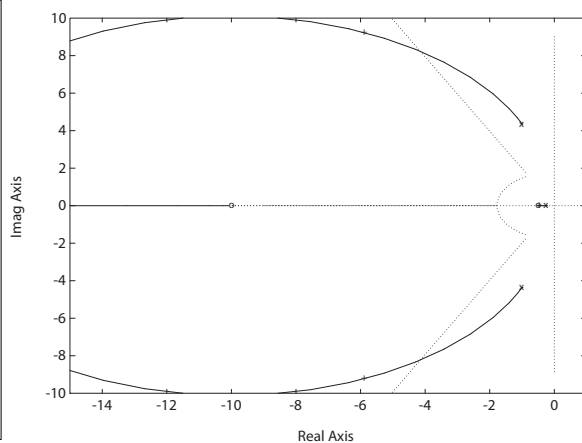
$$G_c(s) = 10 \frac{s + 0.5}{s + 0.25} .$$

The compensated root locus is shown in Figure CP10.6b and the step response is shown in Figure CP10.6c. The phase margin of the compensated

```

numg=[1 10]; deng=[1 2 20];
sysg = tf(numg,deng);
numc=[1 0.5]; denc=[1 0.25];
sysc = tf(numc,denc);
sys_o = series(sysc,sysg);
axis([-15,1,-10,10]);
rlocus(sys_o); hold on
%
zeta=0.45; wn=1.7778;
x=[-10:0.1:-zeta*wn];
y=-(sqrt(1-zeta^2)/zeta)*x;
xc=[-10:0.1:-zeta*wn];
c=sqrt(wn^2-xc.^2);
plot(x,y,':','x,-y',':','xc,c',':','xc,-c',':');
rlocfind(sys_o)
hold off

```

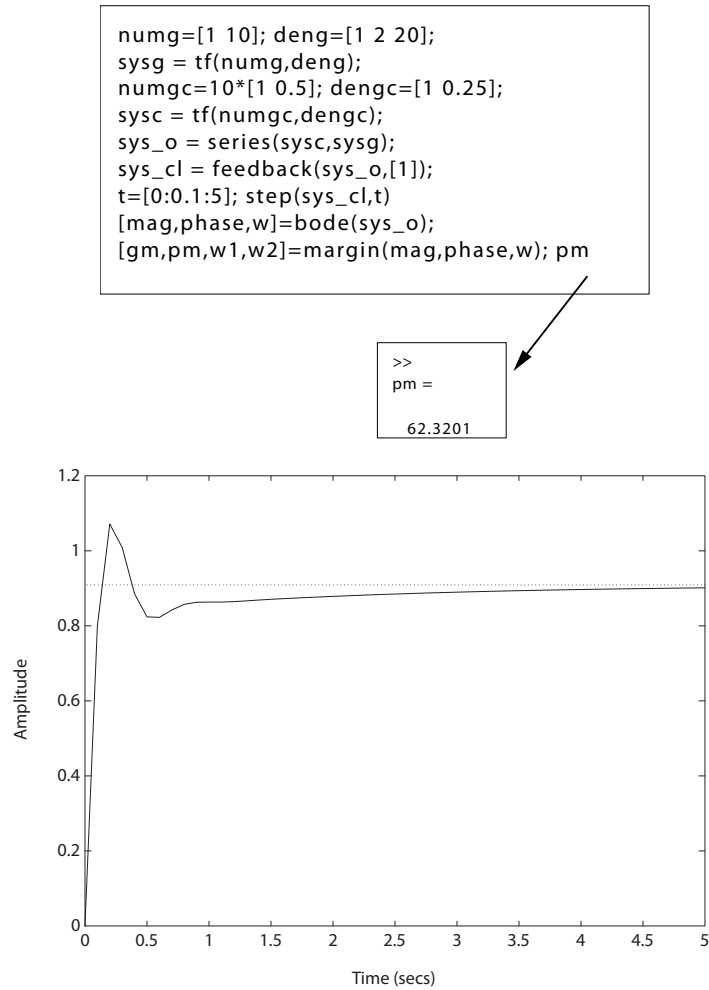


**FIGURE CP10.6**

CONTINUED: (b) Compensated root locus.



system is  $P.M. = 62.3^\circ$  and the settling time  $T_s < 5$  seconds.

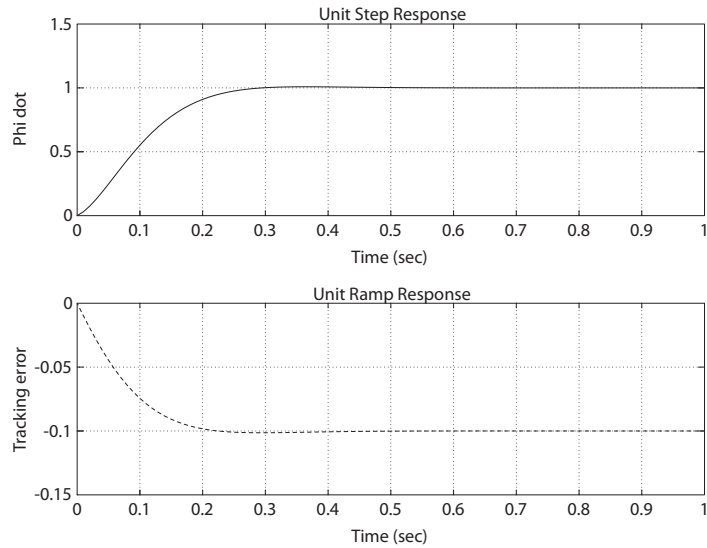


**FIGURE CP10.6**  
CONTINUED: (c) Step response and phase margin verification.

**CP10.7** Both design specifications can be satisfied with an integral controller

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{10}{s} .$$

The simulation results and m-file script are shown in Figures CP10.7a and b.



**FIGURE CP10.7**  
(a) Simulation results.

```

K1=0; K2=10;
numc=[K1 K2]; denc=[1 0]; sysc = tf(numc,denc);
numg=[23]; deng=[1 23]; sysg = tf(numg,deng);
sys_o = series(sysc,sysg);
sys_cl = feedback(sys_o,[1]);
t=[0:0.01:1];
ys=step(sys_cl,t);
subplot(211)
plot(t,ys), xlabel('Time (sec)'), ylabel('Phi dot')
title('Unit Step Response'), grid
u=t;
yr=lsim(sys_cl,u,t);
subplot(212)
plot(t,yr-u,'--')
xlabel('Time (sec)'), ylabel('Tracking error')
title('Unit Ramp Response'), grid

```

**FIGURE CP10.7**  
CONTINUED: (b) M-file design script.

**CP10.8** From Example 10.3, we have that the loop transfer function is

$$G_c(s)G(s) = \frac{8.1(s+z)}{s^2(s+3.6)},$$

where  $z = 1$ . We want to determine a value of  $z$  so the the percent overshoot is reduced from 46% to less than 32%. A valid design is

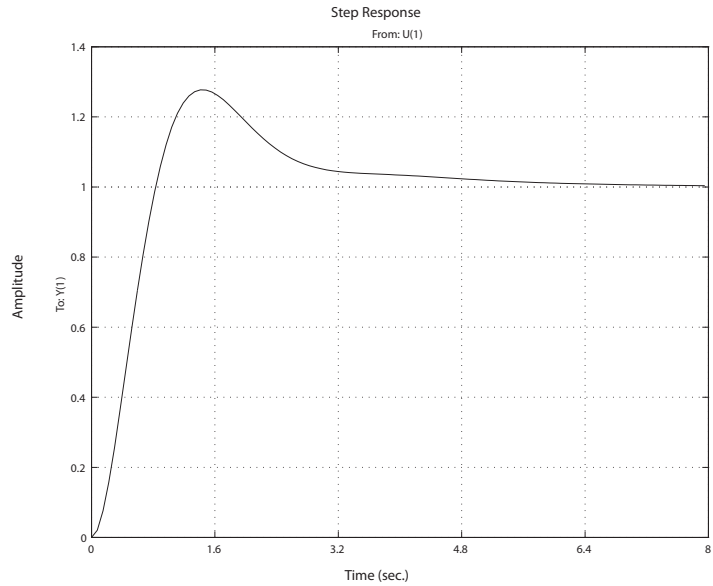
$$G_c(s)G(s) = \frac{8.1(s + 0.45)}{s^2(s + 3.6)}.$$

The m-file script and step response are shown in Figure CP10.8. The percent overshoot is P.O.=27.7 %.

```

K1 = 8.1;
numg = [K1]; deng = [1 0 0];
sysg = tf(numg,deng);
numc = [1 0.45]; denc = [1 3.6];
sysc = tf(numc,denc);
sys_o = series(sysc,sysg);
sys_cl = feedback(sys_o,[1]);
step(sys_cl)
y=step(sys_cl);
po=100*(max(y)-1)

```



**FIGURE CP10.8**

Response of system with new lead compensator design.

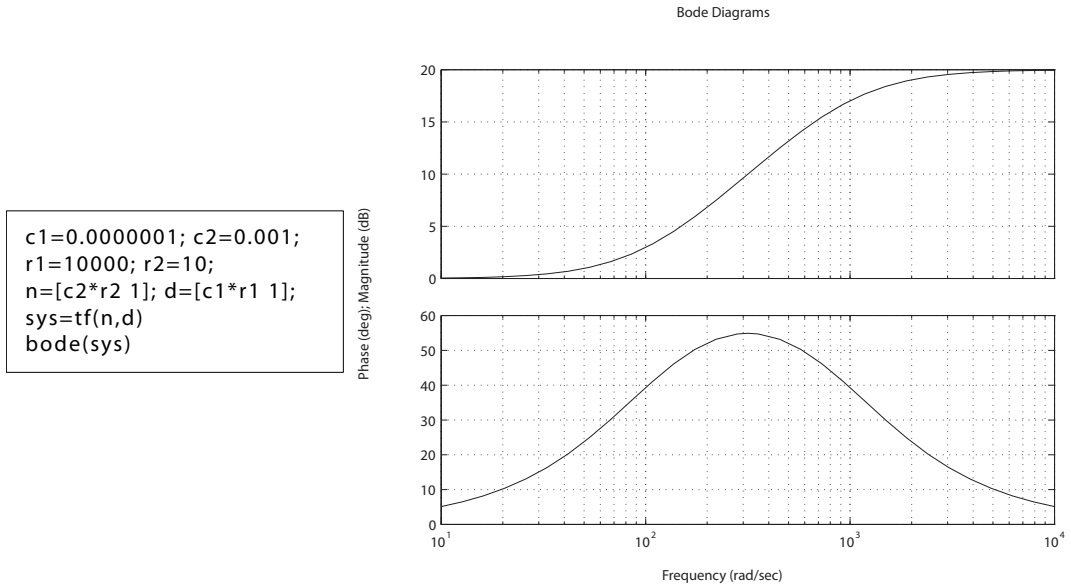
**CP10.9** From AP10.10, we have the transfer function is

$$\begin{aligned}
 T(s) &= \frac{V_o(s)}{V_i(s)} \\
 &= \frac{1 + R_2 C_2 s}{1 + R_1 C_1 s}.
 \end{aligned}$$

Substituting  $C_1 = 0.1 \mu F$ ,  $C_2 = 1 mF$ ,  $R_1 = 10 k\Omega$ , and  $R_2 = 10 \Omega$  yields

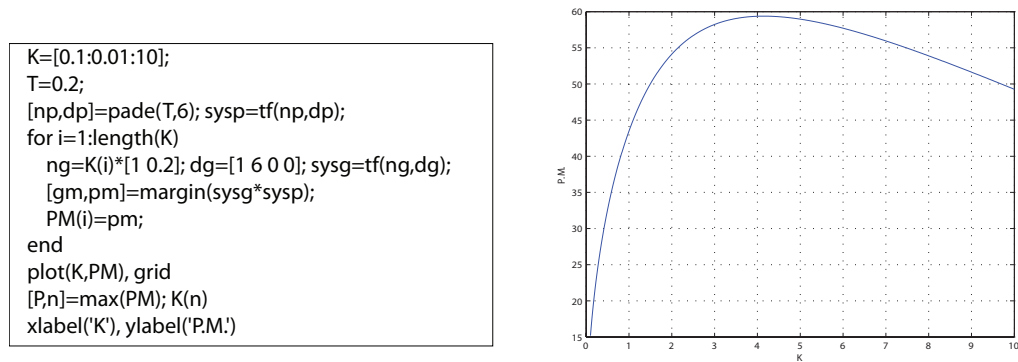
$$T(s) = \frac{1 + 0.01s}{1 + 0.001s}.$$

The frequency response is shown in Figure CP10.9.

**FIGURE CP10.9**

Op-amp circuit frequency response.

**CP10.10** The plot of  $K$  versus phase margin is shown in Figure CP10.10. The value of  $K$  that maximizes the phase margin is  $K = 4.15$ .

**FIGURE CP10.10**Plot of  $K$  versus phase margin.

# *The Design of State Variable Feedback Systems*

## Exercises

**E11.1** The system is given by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} &= \mathbf{K}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} -k & 0 \\ 0 & -2k \end{bmatrix}.$$

Then, with  $\mathbf{u} = \mathbf{K}\mathbf{x}$ , we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -k & 1 \\ -1 & -2k \end{bmatrix} \mathbf{x}.$$

The characteristic equation is

$$\begin{aligned}\det[s\mathbf{I} - \mathbf{A}] &= \det \begin{bmatrix} s+k & -1 \\ 1 & s+2k \end{bmatrix} = s^2 + 3ks + 2k^2 + 1 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.\end{aligned}$$

Solving for  $k$  where  $\omega_n^2 = 2k^2 + 1$  and  $\zeta = 1$  (critical damping) yields

$$k = 2.$$

**E11.2** Let

$$u = -k_1x_1 - k_2x_2 + r .$$

Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 9 - k_1 & -k_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r ,$$

and

$$\det(s\mathbf{I} - \mathbf{A}) = s^2 + k_2s + k_1 - 9 = 0 .$$

We want  $\zeta = 1$ , so the desired characteristic equation is

$$p_d(s) = (s + c_o)^2 ,$$

where  $c_o$  is to be determined to meet  $T_s = 4$  and where  $k_2 = 2c_o$  and  $k_1 = c_o^2 + 9$ . Solving for the state response of  $x_1(t)$  to a unit step input we find

$$x_1(t) = 1 - e^{-c_ot} - c_ot e^{-c_ot} .$$

When  $t \geq T_s = 4$  sec we want  $x_1(t) \geq 0.98$ . Solving for  $c_o$  at  $t = T_s$  yields

$$c_o = 1.459, \quad k_1 = 11.13, \quad \text{and} \quad k_2 = 2.92 .$$

**E11.3** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} ,$$

and  $\det \mathbf{P}_c \neq 0$ , therefore the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix} ,$$

and  $\det \mathbf{P}_o = 0$ ; therefore the system is unobservable.

**E11.4** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} ,$$

and the  $\det \mathbf{P}_c = 0$ ; therefore the system is uncontrollable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -10 & 0 \end{bmatrix},$$

and  $\det \mathbf{P}_o = 0$ ; therefore the system is also unobservable.

**E11.5** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix},$$

and  $\det \mathbf{P}_c = -1 \neq 0$ ; therefore the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and  $\det \mathbf{P}_o = 1 \neq 0$ ; therefore the system is observable.

**E11.6** The controllability matrix is

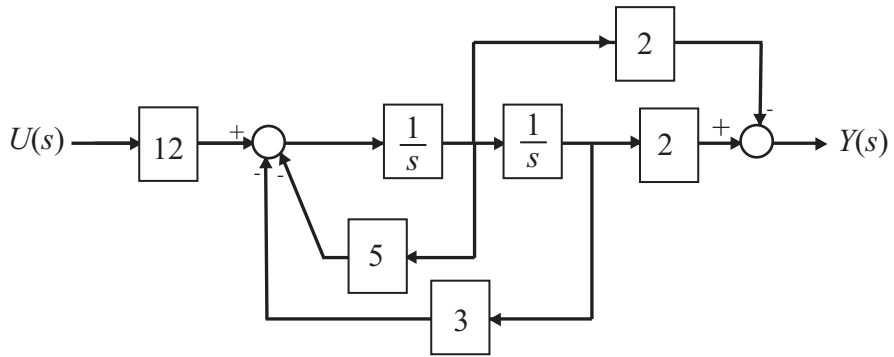
$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix},$$

and  $\det \mathbf{P}_c \neq 0$ ; therefore the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

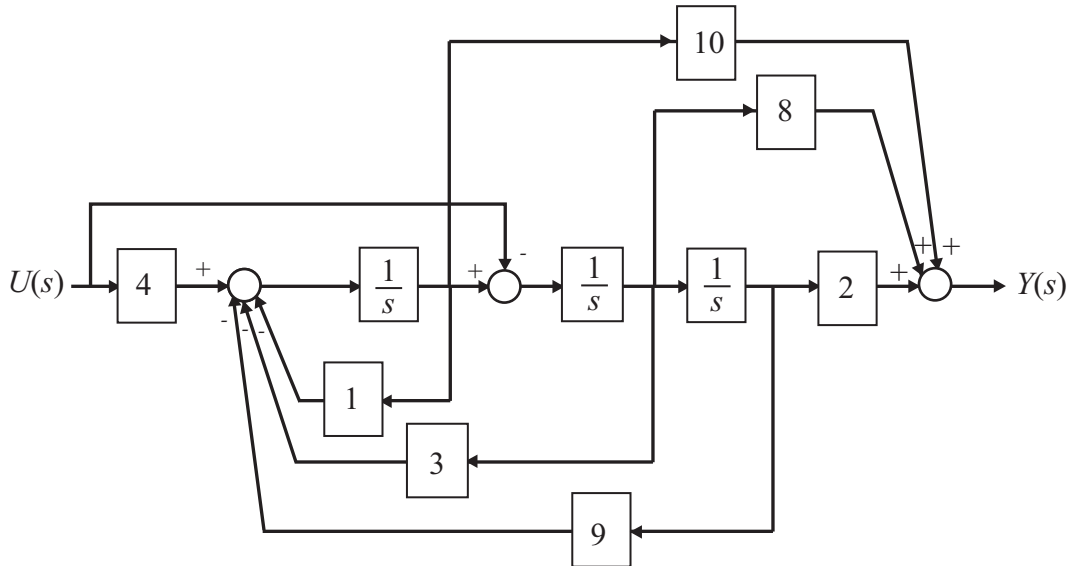
and  $\det \mathbf{P}_o \neq 0$ ; therefore the system is observable.

**E11.7** The block diagram is shown in Fig. E11.7.



**FIGURE E11.7**  
The block diagram for E11.7.

**E11.8** The block diagram is shown in Fig. E11.8.



**FIGURE E11.8**  
The block diagram for E11.8.



**E11.9** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} k_1 & k_1 - k_2 \\ k_2 & -k_1 + k_2 \end{bmatrix},$$

and  $\det \mathbf{P}_c = -k_1^2 + k_2^2$ . So, the condition for complete controllability is  $k_1^2 \neq k_2^2$ .

**E11.10** A matrix differential equation representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -6 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [-3 \quad 4 \quad 2]\mathbf{x} + [0]u.$$

**E11.11** The system is given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ y &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & -7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 2 \quad 0], \quad \text{and } \mathbf{D} = [1].$$

The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -7 \\ 1 & -7 & 49 \end{bmatrix},$$

and  $\det \mathbf{P}_c = -1 \neq 0$ ; therefore the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -4 & 0 & -13 \end{bmatrix},$$

and  $\det \mathbf{P}_o = -29 \neq 0$ ; therefore the system is observable.

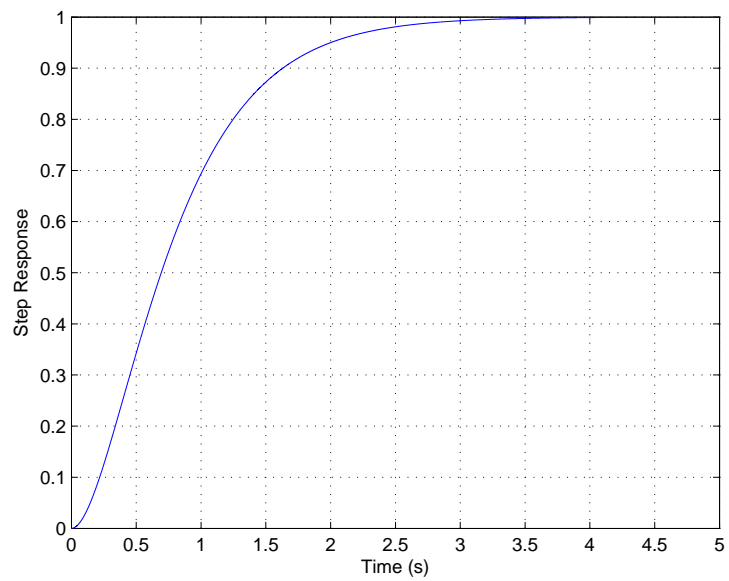
**E11.12** The transfer function is

$$G(s) = \frac{6}{s^2 + 5s + 6}.$$

The response of the system to a unit step is

$$y(t) = 1 - 3e^{-2t} + 2e^{-3t}.$$

The step response is shown in Figure E11.12



**FIGURE E11.12**  
Unit step response.

## Problems

**P11.1** Consider the system

$$\begin{aligned}\dot{x} &= x + u \\ u &= -kx .\end{aligned}$$

So,

$$\dot{x} = x - kx = (1 - k)x$$

and

$$x(t) = e^{(1-k)t}x(0) .$$

The system is stable if  $k > 1$ . Computing the value of  $J$  (assuming  $k > 1$ ) yields

$$J = \int_0^{\infty} e^{2(1-k)t}x^2(0)dt = \frac{1}{k-1} .$$

Thus,  $J$  is minimum when  $k \rightarrow \infty$ . This is not physically realizable. Select  $k = 35$ . Then, the value of the performance index  $J$  is

$$J = \frac{1}{34} .$$

The system is not stable without feedback.

**P11.2** (a) The performance index is given

$$J = \int_0^{\infty} (x^2 + \lambda u^2)dt .$$

The system is

$$\begin{aligned}\dot{x} &= x + u \\ u &= -kx .\end{aligned}$$

So,

$$J = \int_0^{\infty} (x^2 + \lambda k^2 x^2)dt = \int_0^{\infty} (1 + \lambda k^2)x^2 dt = (1 + \lambda k^2) \int_0^{\infty} x^2 dt .$$

Carrying out the integration (assuming  $k > 1$ ) yields

$$J = (1 + \lambda k^2) \frac{1}{k-1} .$$

We want to determine  $k$  that minimizes  $J$ . Taking the partial of  $J$

with respect to  $k$  and setting the result to zero yields

$$\frac{\partial J}{\partial k} = \frac{\lambda k^2 - 2\lambda k - 1}{(k-1)^2} = 0,$$

or

$$\lambda k^2 - 2\lambda k - 1 = 0.$$

Solving for  $k$  yields

$$k = 1 + \sqrt{1 + \frac{1}{\lambda}},$$

where we reject the solution  $k = 1 - \sqrt{1 + \frac{1}{\lambda}}$ , since we require  $k > 1$ .

(b) For  $\lambda = 2$ , we determine that  $k = 2.2$  and  $J_{min} = 8.9$ .

**P11.3** The system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$u = -k(x_1 + x_2) = -k[1 \ 1]\mathbf{x}.$$

Then, with feedback applied, the system is

$$\dot{\mathbf{x}} = \begin{bmatrix} (1-k) & -k \\ -(1+k) & (2-k) \end{bmatrix} \mathbf{x}.$$

Solving

$$\mathbf{H}^T \mathbf{P} + \mathbf{P} \mathbf{H} = -\mathbf{I}$$

yields

$$\begin{aligned} 2p_{11}(1-k) - 2p_{12}(k+1) &= -1 \\ p_{12}(3-2k) - p_{11}k - p_{22}(k+1) &= 0 \\ -2kp_{12} + 2p_{22}(2-k) &= -1. \end{aligned}$$

Solving for  $p_{11}$ ,  $p_{12}$  and  $p_{22}$  yields

$$p_{11} = \frac{-(2k^2 - 6k + 7)}{4(4k^2 - 8k + 3)}$$

$$p_{12} = \frac{2k^2 - 2k - 1}{4(4k^2 - 8k + 3)}$$

$$p_{22} = \frac{-(2k^2 - 6k + 3)}{4(4k^2 - 8k + 3)}.$$

The performance index is computed to be

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = p_{11} + 2p_{12} + p_{22} = \frac{1}{2k - 1},$$

when  $\mathbf{x}(0) = [1 \ 1]^T$ . So as  $k \rightarrow \infty$ ,  $J \rightarrow 0$ . The system is unstable without feedback.

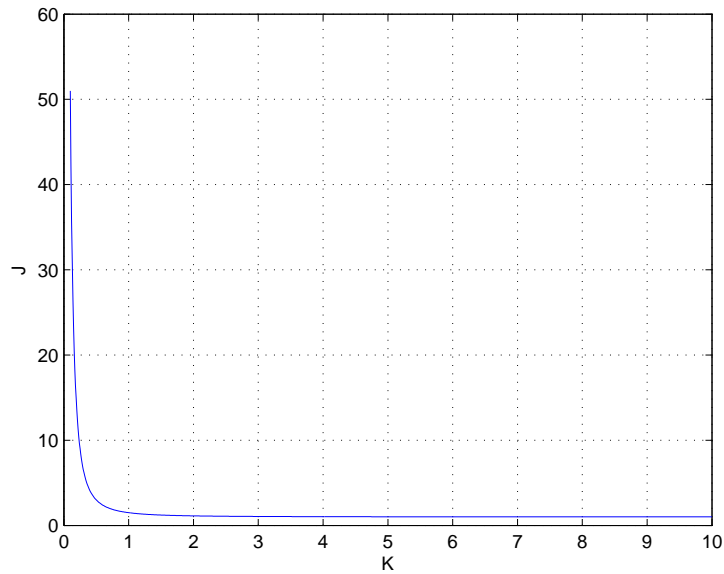
**P11.4** The performance index is

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = p_{11} - 2p_{12} + p_{22}.$$

From Example 11.12 in Dorf and Bishop, we determine that

$$J = \frac{2k^2 + 1}{2k^2}.$$

So, when  $k \rightarrow \infty$ , the performance index  $J \rightarrow 1$ . The plot of  $J$  versus  $k$  is shown in Figure P11.4.



**FIGURE P11.4**

The performance index  $J$  versus  $k$ .

**P11.5** The system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{u} = -k\mathbf{x} .$$

The performance index is

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{x} + \mathbf{u}^T \mathbf{u}) dt = \int_0^{\infty} (1 + k^2)(\mathbf{x}^T \mathbf{x}) dt .$$

First, we solve

$$\mathbf{H}^T \mathbf{P} + \mathbf{P} \mathbf{H} = -(1 + k^2) \mathbf{I} ,$$

yielding,

$$p_{12} = \frac{(1 + k^2)}{2k}$$

$$p_{22} = \frac{k^3 + k^2 + k + 1}{2k^2}$$

$$p_{11} = \frac{2k^3 + k^2 + 2k + 1}{2k} .$$

The performance index is then given by

$$J = p_{11} + 2p_{12} + p_{22} = \frac{2k^4 + 4k^3 + 3k^2 + 4k + 1}{2k^2} .$$

Taking the partial of  $J$  with respect to  $k$ , setting the result to zero and solving for  $k$  yields

$$\frac{\partial J}{\partial k} = \frac{2k^4 + 2k^3 - 2k - 1}{k^3} = 0$$

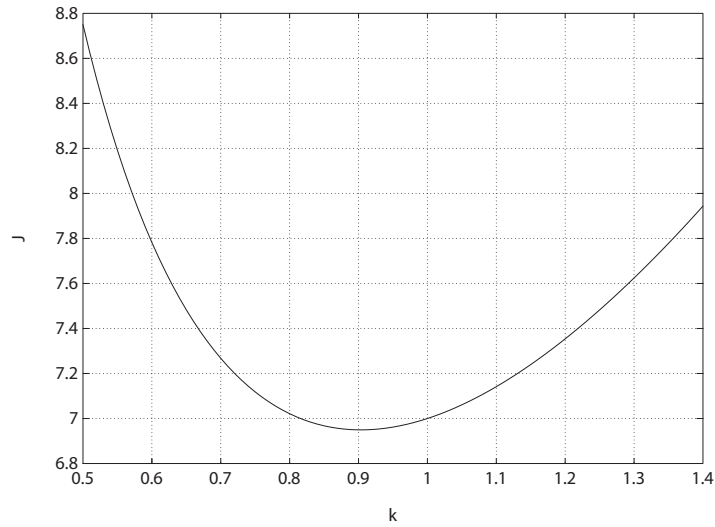
or

$$2k^4 + 2k^3 - 2k - 1 = 0 .$$

Solving for  $k$  yields  $k = 0.90$ . The plot of  $J$  versus  $k$  is shown in Figure P11.5. The value of the performance index is

$$J = 6.95$$

when  $k = 0.90$ .

**FIGURE P11.5**

The performance index  $J$  versus  $k$ .

**P11.6** (a) For P11.3, we have

$$J = \frac{1}{2k - 1} .$$

So, as  $k \rightarrow \infty$ , then  $J \rightarrow 0$ . But  $k = \infty$  is not a practical solution, so select  $k = 10$ . Then,  $J = 1/19$ , and

$$\dot{\mathbf{x}} = \begin{bmatrix} -9 & -10 \\ -11 & -8 \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x} .$$

The closed-loop system roots are determined by solving

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + 17s - 38 = 0 ,$$

which yields  $s_1 = -19$  and  $s_2 = 2$ . The system is unstable. The original system was unstable, and it remains unstable with the feedback. In general,

$$\dot{\mathbf{x}} = \begin{bmatrix} (1 - k) & -k \\ -(1 + k) & (2 - k) \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}$$

and  $\det[s\mathbf{I} - \mathbf{A}] = s^2 + s(2k - 3) + (2 - 4k) = 0$ . A Routh-Hurwitz analysis reveals that the system is unstable for all  $k$ .

(b) For P11.4, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x} ,$$

and

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + ks + k = 0 .$$

The performance index was found to be

$$J = 1 + \frac{4k + 1}{2k^2} .$$

As  $k \rightarrow \infty$ , we have  $J \rightarrow 0$ . But  $k = \infty$  is not a practical choice for  $k$ . Select  $k = 10$ . Then,

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + 10s + 10 = (s + 1.13)(s + 8.87) .$$

The closed-loop system is stable.

(c) In P11.5, we found that  $k = 0.90$  for  $J_{min}$ . We are given

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix} \mathbf{x}$$

and

$$\begin{aligned} \det[s\mathbf{I} - \mathbf{A}] &= s^2 + ks + k = s^2 + 0.9s + 0.9 \\ &= (s + 0.45 + j0.835)(s + 0.45 - j0.835) . \end{aligned}$$

**P11.7** The closed-loop system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \mathbf{x} = \mathbf{H}\mathbf{x} ,$$

and

$$\det[s\mathbf{I} - \mathbf{H}] = s^2 + k_2s + k_1 = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 .$$

We desire  $\omega_n = 2$ , so set  $k_1 = 4$ . With  $\mathbf{x}^T(0) = [1, 0]$ , we have  $J = p_{11}$ , and solving

$$\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I}$$



yields

$$\begin{aligned} \begin{bmatrix} 0 & -4 \\ 1 & -k_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -k_2 \end{bmatrix} \\ = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

and

$$p_{11} = \frac{k_2}{8} + \frac{20}{8k_2} = \frac{k_2^2 + 20}{8k_2}.$$

Select

$$k_2 = \sqrt{20}$$

for  $J_{min}$ , where  $J_{min} = \frac{\sqrt{5}}{2}$ . Then

$$\det[s\mathbf{I} - \mathbf{H}] = s^2 + \sqrt{20}s + 4 = 0,$$

and  $\omega_n = 2$  and  $\zeta = 1.12$ . The system is overdamped.

**P11.8** From Example 11.11 in Dorf and Bishop, we have

$$\mathbf{P} = \begin{bmatrix} \frac{k_2^2 + 2}{2k_2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{k_2} \end{bmatrix}.$$

So,

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = \frac{k_2^2 + 2}{2k_2}$$

when  $\mathbf{x}^T(0) = [1 \ 0]$ . Taking the partial of  $J$  with respect to  $k_2$  and setting the result to zero yields

$$\frac{\partial J}{\partial k_2} = 1 - \frac{k_2^2 + 2}{2k_2^2} = 0.$$

Solving for the optimum value of  $k_2$  yields

$$k_2 = \sqrt{2}.$$

**P11.9** Let  $x_1 = \phi$  and  $x_2 = \omega$ . We have that

$$\omega = \frac{d\phi}{dt}.$$

The state equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= Ku . \end{aligned}$$

Select a feedback such that

$$u = -x_1 - K_1 x_2 + r$$

when  $r(t)$  is the reference input. Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -K & -KK_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ K \end{bmatrix} r ,$$

and

$$\det[sI - A] = s^2 + K_1 K s + K .$$

We desire  $\zeta = \frac{1}{\sqrt{2}}$ , so that the overshoot is 4%. Since  $T_s = 1 = \frac{4}{\zeta\omega_n}$ , we require  $\zeta\omega_n = 4$  or  $\omega_n = 4\sqrt{2}$ . Then,  $s^2 + 8s + 32 = s^2 + K_1 K s + K$ , or  $K = 32$  and  $K_1 = \frac{8}{32} = \frac{1}{4}$ .

**P11.10** The system with feedback is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} -10 & -25 \\ 1 & 0 \end{bmatrix} \mathbf{x} ,$$

where  $x_1(0) = 1$ , and  $x_2(0) = -1$ . The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = \det \begin{bmatrix} s + 10 & 25 \\ -1 & s \end{bmatrix} = s(s + 10) + 25 = s^2 + 10s + 25 = 0 .$$

The roots are  $s_{1,2} = -5$ . The solution is

$$\mathbf{x}(t) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} \phi_{11} - \phi_{12} \\ \phi_{21} - \phi_{22} \end{bmatrix}$$

since  $x_1(0) = 1$  and  $x_2(0) = -1$ . We compute the elements of the state transition matrix as follows:

$$\phi_{22}(t) = (1 + 5t)e^{-5t} \quad \text{and} \quad \phi_{21}(t) = te^{-5t} ,$$

therefore

$$x_2(t) = -(1 + 4t)e^{-5t} .$$

Similarly,

$$\phi_{11}(t) = (1 - 5t)e^{-5t} \quad \text{and} \quad \phi_{12} = -25e^{-5t} .$$

Therefore,

$$x_1(t) = (1 + 20t)e^{-5t} .$$

**P11.11** Let

$$u = -k_1x_1 - k_2x_2 + \alpha r$$

where  $r(t)$  is the command input. A state variable representation of the plant is

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \end{bmatrix} u . \end{aligned}$$

The closed-loop transfer function is

$$T(s) = \frac{\alpha}{s^2 + (k_1/2 + 5)s + 4 + k_2} .$$

To meet the performance specifications we need  $\omega_n = 4.8$  and  $\zeta = 0.826$ . Therefore, the desired characteristic polynomial is

$$q(s) = s^2 + 2(0.826)4.8s + 23 = s^2 + 8s + 23 .$$

Equating coefficients and solving for  $k_1$  and  $k_2$  yields  $k_2 = 19$  and  $k_1 = 6$ . Select  $\alpha = 23$  to obtain zero steady-state error to a step input.

**P11.12** A state variable representation of the dc motor is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & -2 & -0.75 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 0 \ 0 \ 0 \ 2.75]\mathbf{x} .$$

The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} 1 & -3 & 3 & 4.5 & -18 \\ 0 & 3 & -9 & 9 & 13.5 \\ 0 & 0 & 6 & -18 & 18 \\ 0 & 0 & 0 & 6 & -18 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

and the  $\det \mathbf{P}_c \neq 0$ , so the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 2.75 \\ 0 & 0 & 0 & 5.5 & 0 \\ 0 & 0 & 5.5 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 \\ 33 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the  $\det \mathbf{P}_o \neq 0$ , so the system is observable.

**P11.13** To meet the  $K_v = 35$  specification, we need  $K = 2450$ . A state variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -70 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 2450 \end{bmatrix} u$$

$$y = [1 \ 0] \mathbf{x}.$$

Let

$$u = -k_1 x_1 - k_2 x_2.$$

Then, the closed-loop characteristic equation is

$$q(s) = s^2 + (2450k_2 + 70)s + 2450k_1 = 0.$$

The desired characteristic polynomial is

$$s^2 + 72.73s + 2644.63 = 0$$

where we select  $\zeta = 0.707$  and  $\omega_n = 51.42$  to meet the performance specifications. Equating coefficients and solving for the gains yields  $k_1 =$

1.08 and  $k_2 = 0.0011$ .

**P11.14** Let

$$u = -k_1x_1 - k_2x_2 - k_3r$$

where  $r(t)$  is the command input. Then, the closed-loop system in state variable form is

$$\dot{\mathbf{x}} = \begin{bmatrix} -10 - k_1 & -k_2 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$y = [0 \ 1] \mathbf{x} .$$

To meet the performance specifications, we want the closed-loop characteristic polynomial to be

$$q(s) = s^2 + 8s + 45.96 = 0$$

where  $\zeta = 0.59$  and  $\omega_n = 6.78$ . The actual characteristic polynomial is

$$\det(s\mathbf{I} - \mathbf{A}) = s^2 + (10 + k_1)s + k_2 = 0 .$$

Equating coefficients and solving for the gains yields  $k_2 = 45.96$  and  $k_1 = -2$ . Select  $k_3 = k_2 = 45.96$  to obtain a zero steady-state error to a step input. This results in a settling time of  $T_s = 0.87$  s and a percent overshoot of  $P.O. = 10\%$ .

**P11.15** The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{s + 1} .$$

The system is not controllable and not observable.

**P11.16** Let

$$u = -\mathbf{K}\mathbf{x} .$$

Then, Ackermann's formula is

$$\mathbf{K} = [0, 0, \dots, 1]\mathbf{P}_c^{-1}q(\mathbf{A})$$

where  $q(s)$  is the desired characteristic polynomial, which in this case is

$$q(s) = s^2 + 2s + 10 .$$

A state-space representation of the limb motion dynamics is

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u .$$

The controllability matrix is

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{P}_c^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} .$$

Also, we have

$$q(\mathbf{A}) = \mathbf{A}^2 + 2\mathbf{A} + 10\mathbf{I} = \begin{bmatrix} 18 & 0 \\ -3 & 9 \end{bmatrix} .$$

Using Ackermann's formula, we have

$$\mathbf{K} = [-3 \quad 9] .$$

**P11.17** The system is either uncontrollable or unobservable if  $a = 5$  or  $a = 8$ . Both of these values correspond to system real poles. So, if  $a$  takes on either value, a pole-zero cancellation occurs in the transfer function.

**P11.18** A matrix differential equation representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \mathbf{x} .$$

Let  $u(t) = -k_1x_1 - k_2x_2$ . Then, the closed-loop characteristic equation is

$$q(s) = s^2 + (2 + k_2)s + 1 + k_1 = 0 .$$

We desire the characteristic equation

$$s^2 + 2\sqrt{2}s + 2 = 0 .$$

Equating coefficients and solving for the gains yields  $k_1 = 1$  and  $k_2 = 2\sqrt{2} - 2 = 0.828$ .

**P11.19** A state space representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = [3 \quad 1]\mathbf{x} .$$

The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} ,$$

and  $\det \mathbf{P}_c \neq 0$ , so the system is controllable. The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} ,$$

and the  $\det \mathbf{P}_o = 0$ , so the system is not observable.

**P11.20** The characteristic equation associated with  $\mathbf{A}$  is

$$s^2(s^2 + 0.2s + 0.0015) = 0 .$$

There are two roots at the origin, so the system is unstable. The system can be stabilized with

$$\delta = -k_1x_1 - k_3x_3 = 20x_1 - 10x_3 .$$

**P11.21** (a) Let  $x_1 = i_1$ ,  $x_2 = i_2$  and  $u = v$ . Then, the state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{-(R_1+R_3)}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & \frac{-(R_3+R_2)}{L_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} \mathbf{u} .$$

Also,

$$y = v_o ,$$

but

$$y = [R_3 \quad -R_3]\mathbf{x} .$$

(b) The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} R_3 & -R_3 \\ -\frac{R_1 R_3}{L_1} - R_3^2 \left( \frac{1}{L_1} + \frac{1}{L_2} \right) & \frac{R_2 R_3}{L_2} + R_3^2 \left( \frac{1}{L_1} + \frac{1}{L_2} \right) \end{bmatrix}$$

and

$$\det \mathbf{P}_o = \left( \frac{R_2}{L_2} - \frac{R_1}{L_1} \right) R_3^2.$$

So, when

$$\frac{R_1}{L_1} = \frac{R_2}{L_2},$$

$\det \mathbf{P}_o = 0$  and the system is not observable.

(c) Let

$$a = \frac{R_1 + R_3}{L_1},$$

and

$$b = \frac{R_3 + R_2}{L_2}.$$

Then

$$\begin{aligned} \det[s\mathbf{I} - \mathbf{A}] &= \det \begin{bmatrix} (s + a) & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & (s + b) \end{bmatrix} \\ &= \left[ (s + a)(s + b) + \frac{R_3^2}{L_1 L_2} \right] = (s + r)^2 \\ &= s^2 + (a + b)s + ab + \frac{R_3^2}{L_1 L_2}. \end{aligned}$$

The system has two equal roots when

$$(a + b)^2 - 4 \left( ab + \frac{R_3^2}{L_1 L_2} \right)$$

or

$$\left( \frac{R_1 + R_3}{L_1} + \frac{R_3 + R_2}{L_2} \right)^2 - 4 \frac{(R_1 + R_3)(R_3 + R_2) + R_3^2}{L_1 L_2} = 0.$$

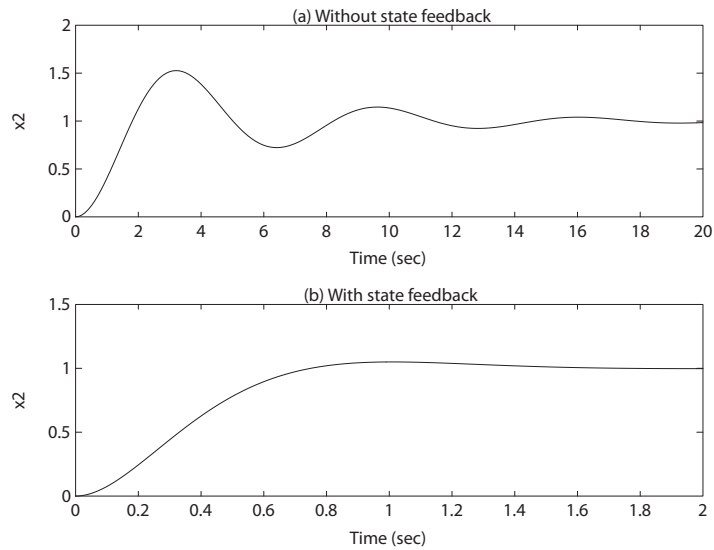


**P11.22** (a) Without state feedback the state differential equation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.4 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} .$$

The step response is shown in Figure P11.22a.



**FIGURE P11.22**

Step response (a) without state feedback, and (b) with state feedback.

(b) Consider state feedback

$$u = -K(ax_2 + bx_1) + cr$$

where  $r$  is the reference input and  $K$ ,  $a$ ,  $b$  and  $c$  are to be determined. Then, the state differential equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.4 - Kb & -1 - Ka \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} c \\ 0 \end{bmatrix} r$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x} ,$$

and  $\det(s\mathbf{I} - \mathbf{A}) = s^2 + (0.4 + Kb)s + (1 + Ka) = 0$ . Our specifications are  $P.O. = 5\%$  and  $T_s = 1.35$  sec. So,  $\zeta = 0.69$  and  $\omega_n = \frac{4}{\zeta 1.35} = 4.3$ .

Solving for  $K$ ,  $a$  and  $b$  yields

$$Ka = \omega_n^2 - 1$$

and

$$Kb = 2\zeta\omega_n - 0.4 .$$

Select  $K = 1$ . Then,  $a = 17.49$  and  $b = 5.53$ . Select  $c = 1 + Ka$  to achieve a zero steady-state tracking error.

(c) The step response is shown in Figure P11.22b for the system with state feedback.

**P11.23** Using the internal model design method for step inputs, we have

$$\begin{bmatrix} \dot{e} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w ,$$

where we choose

$$w = -K_1 e - \mathbf{K}_2 \mathbf{z} .$$

To place the poles at  $s = -10$  and  $s = -2 \pm j$  we use Ackermann's formula to compute

$$\begin{aligned} K_1 &= 50 \\ \mathbf{K}_2 &= [45 \quad 14] . \end{aligned}$$

The compensator has the form shown in Figure 11.14 in Dorf and Bishop.

**P11.24** Using the internal model design method for ramp inputs, we have

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w$$

where we choose

$$w = -K_1 e - K_2 \dot{e} - \mathbf{K}_3 \mathbf{z} .$$

To place the poles at  $s = -20$  and  $s = -2 \pm 2j$  we can use Ackermann's formula. We also need an additional pole (must be a stable pole); select

$s = -20$  as the fourth pole. Then,

$$\begin{aligned} K_1 &= 3200 \\ K_2 &= 1920 \\ \mathbf{K}_3 &= [568 \quad 44] . \end{aligned}$$

The compensator has the form shown in Figure 11.16 in Dorf and Bishop.

**P11.25** The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 21 & -36 \end{bmatrix} ,$$

and  $\det \mathbf{P}_o = 48 \neq 0$ ; therefore the system is completely observable. The desired poles of the observer are  $s_{1,2} = -1$ . This implies that the desired characteristic polynomial is

$$p_d(s) = s^2 + 2s + 1 .$$

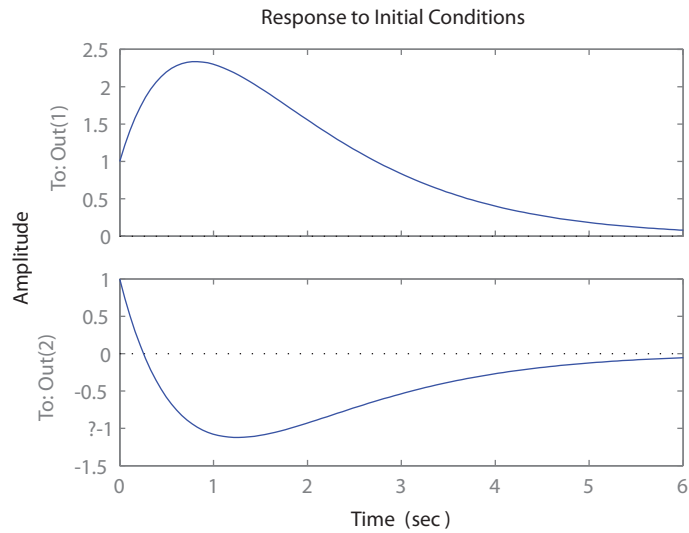
The actual characteristic polynomial is

$$\begin{aligned} \det |\lambda \mathbf{I} - (\mathbf{A} - \mathbf{LC})| &= \det \begin{vmatrix} \lambda - 1 + L_1 & -4 - 4L_1 \\ 5 + L_2 & \lambda - 10 - 4L_2 \end{vmatrix} \\ &= \lambda^2 + (L_1 - 4L_2 - 11)\lambda + 10L_1 + 8L_2 + 30 = 0 . \end{aligned}$$

Solving for  $L_1$  and  $L_2$  yields

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} -0.25 \\ -3.3125 \end{bmatrix} .$$

Checking we find that  $\det(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{LC})) = s^2 + 2s + 1$ . The response of the estimation error is shown in Figure P11.25, where  $\mathbf{e}(0) = [1 \quad 1]^T$ .



**FIGURE P11.25**  
Estimation error response to an initial condition.

**P11.26** The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 2 & -4 \\ 32 & 20 & 14 \end{bmatrix}.$$

The  $\det \mathbf{P}_o = 728 \neq 0$ , hence the system is observable. The gain matrix

$$\mathbf{L} = \begin{bmatrix} 0.14 \\ -0.93 \\ 0.79 \end{bmatrix}$$

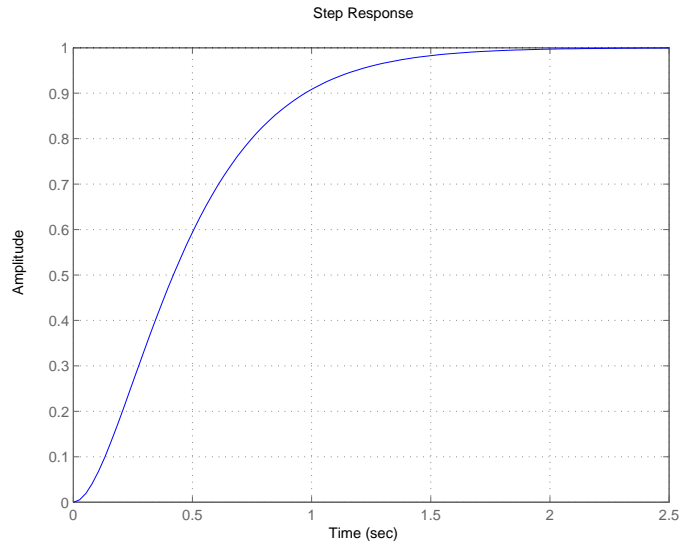
results in the observer poles at  $s_{1,2} = -1 \pm j$  and  $s_3 = -5$ , as desired.

**P11.27** The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

The  $\det \mathbf{P}_o = 0$ , hence the system is not completely observable. So, we cannot find an observer gain matrix that places the observer poles at the desired locations.

**P11.28** Selecting  $K = 16$  yields a zero steady-state error to a unit step input. The step response is shown in Figure P11.28.



**FIGURE P11.28**  
Estimation error response to an initial condition.

**P11.29** The system transfer function is

$$Y(s) = \frac{2}{s+3}U(s) .$$

The associated state variable model is

$$\begin{aligned}\dot{x} &= -3x + 2u \\ y &= x .\end{aligned}$$

## Advanced Problems

**AP11.1** The closed loop system in state-space form is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ -2KK_1 & -2KK_2 & -4 - 2KK_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2K \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

The closed-loop transfer function is

$$T(s) = \frac{4K}{s^3 + (2KK_3 + 5)s^2 + (4KK_2 + 2KK_3 + 4)s + 4KK_1} .$$

Setting the steady-state error to zero, we determine that

$$e_{ss} = 1 - T(0) = 1 - \frac{1}{K_1} .$$

Solving for  $K_1$  yields

$$K_1 = 0.5 .$$

Choosing

$$K_2 = 0.5 \quad \text{and} \quad K_3 = 1.5$$

results in a percent overshoot of  $P.O. = 2.82\%$ .

**AP11.2** A state variable representation is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A} = \begin{bmatrix} -3 & -1 & -1 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} .$$

Let

$$\mathbf{u} = -\mathbf{K}\mathbf{x} .$$

Then, with

$$\mathbf{K} = \begin{bmatrix} 4.00 & 24.33 & 39.67 \end{bmatrix} ,$$

the closed-loop system poles are  $s = -4, -5$ , and  $-6$ .

**AP11.3** Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} , \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} ,$$

we compute the determinant of the controllability matrix as

$$\det \mathbf{P}_c = \det[\mathbf{B} \quad \mathbf{AB}] = -(b_1 + b_2) .$$

The system is controllable if and only if the determinant is non-zero. So, for the system to be controllable, we require that  $b_2 \neq -b_1$ .

**AP11.4** Consider the state variable feedback law

$$\mathbf{u} = -\mathbf{K}\mathbf{x} .$$

Using Ackermann's formula, we determine that

$$\mathbf{K} = [-14.2045 \quad -17.0455 \quad -94.0045 \quad -31.0455]$$

results in the closed-loop system characteristic roots at  $s = -2 \pm j$ ,  $s = -5$  and  $s = -5$ .

**AP11.5** The closed-loop transfer function for the system is

$$T(s) = \frac{2K_p}{s^3 + (9 + 2K_3)s^2 + (26 + 2K_2 + 10K_3)s + (26 + 6K_2 + 12K_3)} .$$

Setting the steady-state error for a step input to zero yields

$$e_{ss} = 1 - \frac{2K_p}{26 + 6K_2 + 12K_3} = 0 .$$

Solving for  $K_p$  in terms of  $K_2$  and  $K_3$  yields

$$K_p = 13 + 3K_2 + 12K_3 .$$

Now, choosing

$$K_2 = 5$$

$$K_3 = 2$$

results in the closed-loop characteristic roots at

$$s_1 = -4 \quad s_2 = -4 \quad s_3 = -5 .$$

Also, the prefilter gain is

$$K_p = 52 .$$

**AP11.6** (a) A state variable representation is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} .$$

Since the determinant of the controllability matrix  $\det[\mathbf{B} \quad \mathbf{AB}] \neq 0$ , the system is controllable.

(b) The state variable representation is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} ,$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{u} .$$

The determinant of the controllability matrix

$$\det \mathbf{P}_c = \det[\mathbf{B} \quad \mathbf{AB}] = 0 .$$

Therefore, the system is uncontrollable.

**AP11.7** The closed-loop transfer function is

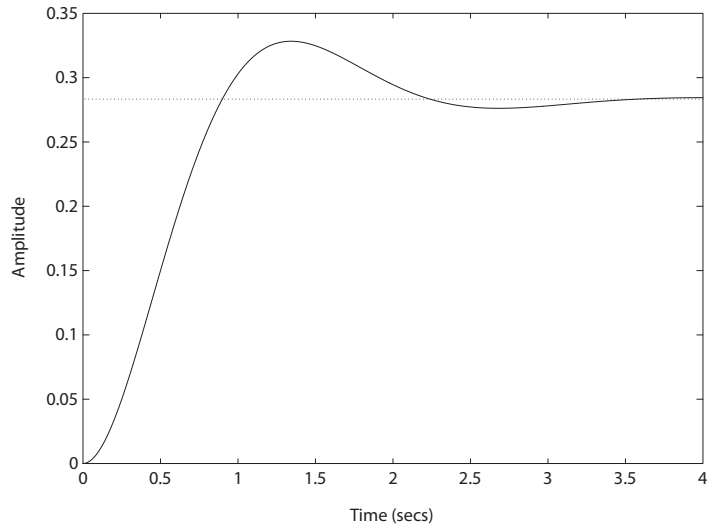
$$T(s) = \frac{120}{s^3 + (10 + 60K_3)s^2 + (16 + 120(K_3 + K_2))s + 120} .$$

The state feedback gains

$$K_2 = 0.283 \quad \text{and} \quad K_3 = 0.15$$

place the poles at the desired locations. The plot of the roll output for a unit step disturbance is shown in Figure AP11.7.





**FIGURE AP11.7**  
Roll angle response to a step disturbance.

**AP11.8** The state equations are (using the parameters of P3.36 in Dorf and Bishop)

$$\begin{aligned} \dot{h} = \dot{x}_1 &= \frac{1}{50}[80\theta - 50h] = -x_1 + \frac{8}{5}x_2 \\ \dot{\theta} = \dot{x}_2 &= \omega = x_3 \\ \dot{\omega} = \dot{x}_3 &= \frac{K_m i_a}{J} = -\frac{K_m K_b}{J R_a} \omega + \frac{K_m K_a}{J R_a} v_i = -\frac{353}{30} x_3 + \frac{25000}{3} v_i . \end{aligned}$$

In state variable form we have (without feedback)

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{8}{5} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{353}{30} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{25000}{3} \end{bmatrix} v_i .$$

(a) In this case we have  $v_i = -kh + ar = -kx_1 + ar$ , where  $k$  and  $a$  are the parameters to be determined and  $r$  is the reference input. With the feedback of  $h(t)$  we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{8}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{25000}{3}k & 0 & -\frac{353}{30} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ a\frac{25000}{3} \end{bmatrix} r .$$

Since we only have one parameter to adjust, namely  $k$ , we will probably not be able to simultaneously meet both design specifications, In fact with

$$k = 0.00056$$

we obtain the percent overshoot  $P.O. = 9.89\%$ . The settling time criterion cannot simultaneously be met—the best that can be obtained is  $T_s \approx 7.5$  seconds. In this case, we choose  $a = 0.00056$  to make the steady-state value of  $h(t) = 1$ .

- (b) In this case we have  $v_i = -k_1h - k_2\theta + ar = -k_1x_1 - k_2x_2 + ar$ , where  $k_1$ ,  $k_2$ , and  $a$  are the parameters to be determined and  $r$  is the reference input. Since we have two parameter to adjust, namely  $k_1$  and  $k_2$  we will probably be able to simultaneously meet both design specifications. In fact with

$$k_1 = 0.00056 \quad \text{and} \quad k_2 = 0.001$$

we obtain the percent overshoot  $P.O. = 4.35\%$ . The settling time criterion is easily met—  $T_s \approx 5$  seconds. In this case, we choose  $a = 0.0012$  to make the steady-state value of  $h(t) = 1$ .

- AP11.9** (a) The state vector differential equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u} ,$$

where  $x_1 = z$ ,  $x_2 = \dot{z}$ ,  $x_3 = y$  and  $x_4 = \dot{y}$ .

- (b) The characteristic equation is

$$s^4 + 3s^2 + 1 = (s + j0.618)(s - j0.618)(s + j1.618)(s - j1.618) = 0 .$$

So, the system is oscillatory.

- (c) Let  $u = -kx_4$ . Then characteristic equation is

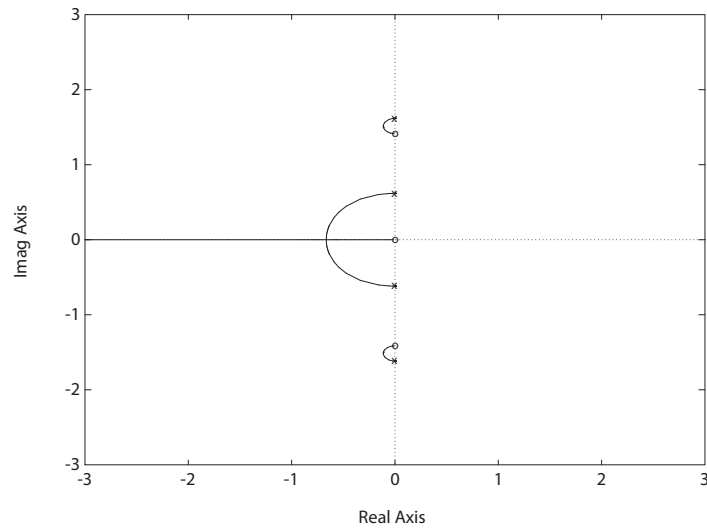
$$s^4 + ks^3 + 3s^2 + 2ks + 1 = 0$$

which is stable if  $k > 0$ .

- (d) Rewrite the characteristic equation as

$$1 + \frac{ks(s^2 + 2)}{s^4 + 3s^2 + 1} = 0 .$$

The root locus is shown in Figure AP11.9. A reasonable solution for  $k$  is  $k = 1.35$ .



**FIGURE AP11.9**  
Root locus for  $1 + k \frac{s(s^2+2)}{s^4+3s^2+1} = 0$ .

**AP11.10** The state differential equation is

$$\ddot{y} = ky + \alpha u$$

where  $k$  and  $\alpha$  depend on the system parameters, such as mass and length. The transfer function is

$$\frac{y}{u} = \frac{\alpha}{s^2 - k}$$

which is unstable at the top of the arc. Since we can only use  $\dot{y}$  for feedback, we have

$$\frac{\dot{y}}{u} = \frac{s\alpha}{s^2 - k}.$$

Let

$$G_c(s) = \frac{K_1 s + K_2}{s}.$$

Then

$$GG_c(s) = \frac{\alpha(K_1s + K_2)}{(s^2 - k)}$$

and the closed-loop characteristic equation is

$$\alpha K_1 s + \alpha K_2 + s^2 - k = 0$$

or

$$s^2 + \alpha K_1 s + \alpha K_2 - k = 0 .$$

Select  $\alpha K_2 - k > 0$  and  $\alpha K_1 > 0$  for stability.

**AP11.11** The state-space representation of the plant is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = [1 \ 0] .$$

With the intermediate variables defined as

$$\mathbf{z} = \dot{\mathbf{x}} \quad \text{and} \quad w = \dot{u}$$

we have

$$\begin{pmatrix} \dot{e} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{pmatrix} e \\ \mathbf{z} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

where

$$e = y - r .$$

To meet the design specifications, we require the closed-loop poles to lie to the left of the line in the complex plane defined by  $s = -0.8$ . We choose

$$\mathbf{K}_2 = [10 \ 3]$$

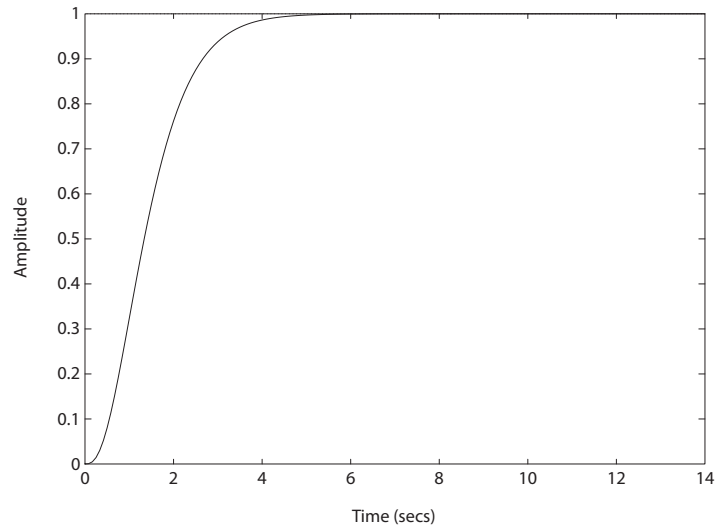
and

$$G_c(s) = \frac{8}{s} .$$

This places the closed-loop poles at  $s = -2, -2$  and  $-2$ . The closed-loop transfer function with the internal model controller is

$$T(s) = \frac{8}{s^3 + 6s^2 + 12s + 8}.$$

The step response is shown on Figure AP11.11.



**FIGURE AP11.11**  
Internal model controller step response.

**AP11.12** The state-space representation of the plant is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

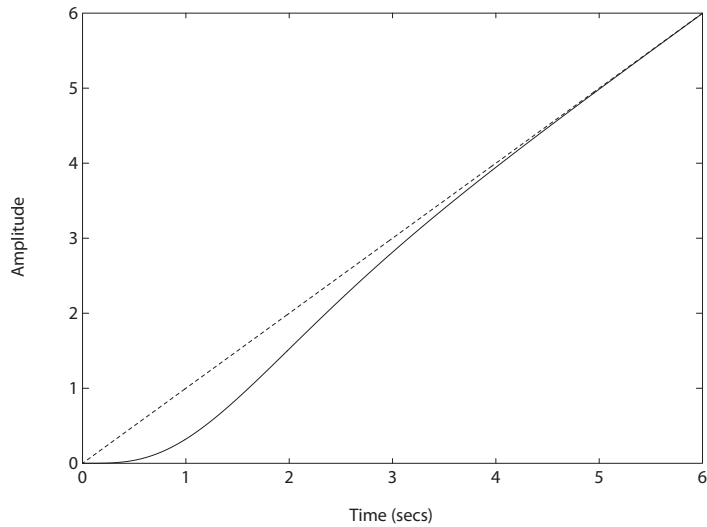
With the intermediate variables defined as

$$\mathbf{z} = \ddot{\mathbf{x}} \quad \text{and} \quad w = \ddot{u}$$

we have

$$\begin{pmatrix} \dot{e} \\ \ddot{e} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix} \begin{pmatrix} e \\ \dot{e} \\ \mathbf{z} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w$$

where  $e = y - r$ .



**FIGURE AP11.12**  
Internal model controller ramp response.

To meet the design specifications, we require the closed-loop poles to lie to the left of the line in the complex plane defined by  $s = -0.67$ . We choose

$$w = -[K_1 \quad K_2 \quad \mathbf{K}_3] \begin{pmatrix} e \\ \dot{e} \\ \mathbf{z} \end{pmatrix} = -[16 \quad 32 \quad 22 \quad 5] \begin{pmatrix} e \\ \dot{e} \\ \mathbf{z} \end{pmatrix} .$$

Then,

$$G_c(s) = \frac{K_1 + K_2 s}{s^2} = \frac{16 + 32s}{s^2} .$$

The closed-loop transfer function with the internal model controller is

$$T(s) = \frac{32s + 16}{s^4 + 8s^3 + 24s^2 + 32s + 16} .$$

This places the closed-loop poles at  $s = -2, -2, -2$  and  $-2$ . The ramp response is shown in Figure AP11.12.

**AP11.13** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} -5 & -3 \\ 1 & 18 \end{bmatrix}$$

and the observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} 4 & -3 \\ 22 & 44 \end{bmatrix} .$$

Computing the determinants yields

$$\det \mathbf{P}_c = -87 \neq 0 \quad \text{and} \quad \det \mathbf{P}_o = 242 \neq 0 ,$$

hence the system is controllable and observable. The controller gain matrix

$$\mathbf{K} = \begin{bmatrix} 3.02 & 6.11 \end{bmatrix}$$

places the closed-loop poles at the desired locations. Similarly, the observer gain matrix

$$\mathbf{L} = \begin{bmatrix} 2.38 \\ -1.16 \end{bmatrix}$$

places the observer poles at the desired locations.

**AP11.14** The controllability matrix is

$$\mathbf{P}_c = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & -12 \\ 4 & -12 & 24 \end{bmatrix}$$

and the observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} 2 & -9 & 2 \\ -16 & -4 & -15 \\ 120 & 29 & 41 \end{bmatrix}.$$

Computing the determinants yields

$$\det \mathbf{P}_c = -64 \neq 0 \quad \text{and} \quad \det \mathbf{P}_o = 10870 \neq 0,$$

hence the system is controllable and observable. The controller gain matrix

$$\mathbf{K} = \begin{bmatrix} -0.5 & 1.25 & 0.5 \end{bmatrix}$$

and the observer gain matrix

$$\mathbf{L} = \begin{bmatrix} 57.43 \\ -16.11 \\ -104.43 \end{bmatrix}$$

yields the desired closed-loop system poles and observer poles, respectively.

**AP11.15** The state-variable representation of the system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 4] \mathbf{x} + [0]u.$$

The observability matrix is

$$\mathbf{P}_o = \begin{bmatrix} 1 & 4 \\ -28 & -7 \end{bmatrix},$$

and  $\det \mathbf{P}_o = 105 \neq 0$ , hence the system is observable. The observer gain matrix

$$\mathbf{L} = \begin{bmatrix} -7.18 \\ 6.29 \end{bmatrix}$$

places the observer poles at  $s_{1,2} = -10 \pm 10j$ , as desired.



## Design Problems

**CDP11.1** A state variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -33.14 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0.827 \end{bmatrix} v_a$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

where  $x_1 = x$  and  $x_2 = \dot{x}$ . Note that we are neglecting the motor inductance and assuming that the position  $x(t)$  is the output. Assume that we have available for feedback the angle  $\theta$  and angle rate  $\dot{\theta}$  (see CDP4.1), so that

$$v_a = -\frac{k_1}{r}x_1 - \frac{k_2}{r}x_2 + au$$

where  $u(t)$  is the reference input (that is, the desired position  $x(t)$ ), the gains  $k_1$  and  $k_2$  and the scaling parameter  $a$  are to be determined. Recall that

$$x = r\theta = 0.03175\theta .$$

With the feedback in the loop we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -26.03k_1 & -33.14 - 26.03k_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0.827a \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Choosing  $k_1 = 50$ ,  $k_2 = 1$  and  $a = 1574.1$  results in

$$P.O. = 1.1\% \quad \text{and} \quad T_s = 0.11 \text{ second} .$$

The closed-loop poles are  $s_{1,2} = -29.59 \pm 20.65j$ .

**DP11.1** The governing differential equation is

$$\ddot{y} - 2000y = -20i .$$

In state variable form, the system is described by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 2000 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -20 \end{bmatrix} i .$$

Consider the state feedback

$$i = -k_1x_1 - k_2x_2 + \beta r$$

where  $r(t)$  is the reference input and  $k_1, k_2$  and  $\beta$  are to be determined. Then, the closed-loop system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 2000 - 20k_1 & -20k_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -20\beta \end{bmatrix} r .$$

The characteristic equation is

$$s^2 + 20k_2s - 2000 + 20k_1 = 0 .$$

For stability, let  $20k_1 - 2000 > 0$ . Select  $k_1 = 125$ . Then,  $\omega_n = 22.36$  rad/sec, and

$$k_2 = \frac{2\zeta\omega_n}{20} .$$

Let  $\zeta = 0.59$  to meet 10% overshoot specification. Thus,

$$k_2 = \frac{2(0.59)(22.36)}{20} = 1.32 .$$

The closed-loop transfer function is

$$T(s) = \frac{-20\beta}{s^2 + 26.4s + 500} .$$

Choose  $\beta = -25$  so that

$$T(s) = \frac{500}{s^2 + 26.4s + 500} .$$

The feedback law is

$$i = 125x_1 + 1.32x_2 - 25r .$$

**DP11.2** The automobile engine control system (see DP10.8 in Dorf and Bishop) is modeled as

$$G(s) = \frac{2e^{-sT}}{(0.21s + 1)(4s + 1)} .$$

In this case, we will assume the delay is negligible. Therefore,  $T = 0$ . A

state variable representation of the system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1.19 & -5.01 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1.19 \end{bmatrix} r$$

$$y = [1 \quad 0] \mathbf{x} .$$

Let

$$r(t) = -k_1 x_1 - k_2 x_2 + k_3 u$$

where  $u(t)$  is the command input. Using ITAE methods, our desired characteristic polynomial is

$$q(s) = s^2 + 1.4\omega_n s + \omega_n^2 = 0 .$$

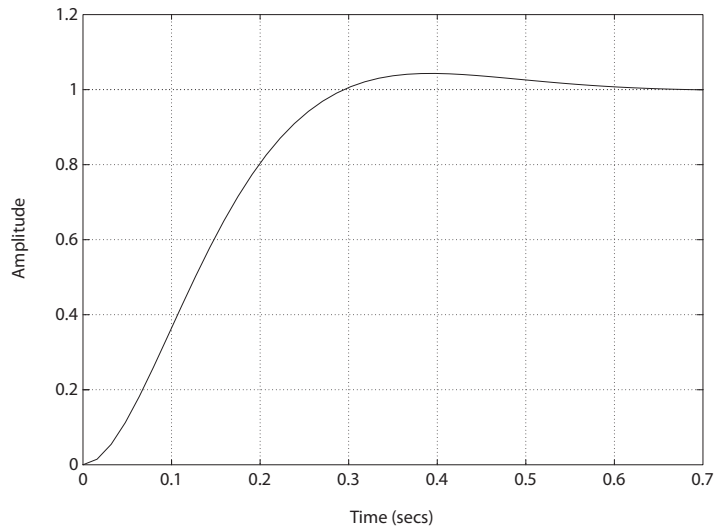
Select  $\omega_n = 11.315$  to obtain a settling time  $T_s < 0.5$  seconds. The characteristic polynomial of the closed-loop system is

$$s^2 + (5.01 + 1.19k_2)s + (1.19 + 1.19k_1) = 0 .$$

Equating coefficients and solving for the gains yields

$$k_1 = 106.59 \quad \text{and} \quad k_2 = 9.235 .$$

Select  $k_3 = 107.59$  to yield a zero steady-state error to a step input.



**FIGURE DP11.2**

The step response of the engine control system.

**DP11.3** The compensator is

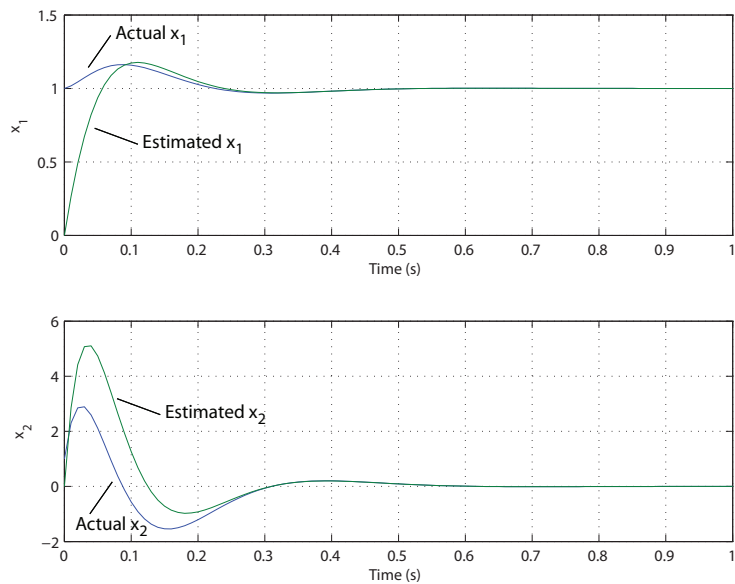
$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= [\mathbf{A} - \mathbf{BK} - \mathbf{LC}] \hat{\mathbf{x}} + \mathbf{L}y + \mathbf{M}r \\ \tilde{u} &= -\mathbf{K}\hat{\mathbf{x}}\end{aligned}$$

where

$$\mathbf{A} - \mathbf{BK} - \mathbf{LC} = \begin{bmatrix} -28.7 & 1 \\ -365.19 & -20 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 \\ 200 \end{bmatrix},$$

$$N = 363.64, \quad \mathbf{K} = \begin{bmatrix} 344.55 & 15.82 \end{bmatrix}, \quad \text{and} \quad \mathbf{L} = \begin{bmatrix} 28.7 \\ 165.19 \end{bmatrix}.$$

We selected the desired eigenvalues of  $\mathbf{A} - \mathbf{BK}$  at  $p = -10 \pm 10j$  and the desired eigenvalues of  $\mathbf{A} - \mathbf{LC}$  at  $q = -20 \pm 10j$ . For initial conditions we let  $\mathbf{x}(0) = [1 \ 1]$  and  $\hat{\mathbf{x}}(0) = [0 \ 0]$ .



**FIGURE DP11.3**

The step response showing the actual and estimated states.

**DP11.4** The design specifications are

- Percent overshoot  $< 20\%$
- $T_s < 1.5s$ , and

(c) steady-state error less than 20% of the input magnitude.

The state differential equation is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -\alpha_1 \\ g & -\alpha_2 & -\sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.415 & -0.0111 \\ 9.8 & -1.43 & -0.0198 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ n \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 6.27 \\ 9.8 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The transfer function is

$$\begin{aligned}\frac{\theta(s)}{\delta(s)} &= \frac{ns + n\sigma_2 - \alpha_1g}{s^3 + (\sigma_1 + \sigma_2)s^2 + (\sigma_1\sigma_2 - \alpha_1\alpha_2)s + \alpha_1g} \\ &= \frac{6.27s + 0.0154}{s^3 + 0.435s^2 - 0.0077 + 0.109}.\end{aligned}$$

Let  $u = -K_1x_1 - K_2x_2 - K_3x_3$ . Then the closed-loop system matrix is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ -nK_1 & -\sigma_1 - nK_2 & -\alpha_1 - nK_3 \\ g - gK_1 & -\alpha_2 - gK_2 & -\sigma_2 - gK_3 \end{bmatrix},$$

where  $\mathbf{K} = [K_1 \ K_2 \ K_3]$ . From the design specifications, we have the desired roots at

$$s^3 + a_2s^2 + a_1s + a_0 = s^3 + 36s^2 + 225s + 1350 = (s+30)(s+3+j6)(s+3-j6) = 0.$$

The actual characteristic equation is

$$\begin{aligned}s^3 + (gK_3 + K_2n + \sigma_1 + \sigma_2)s^2 + (-\alpha_1\alpha_2 - \alpha_1gK_2 + K_1n - \alpha_2nK_3 \\ + gK_3\sigma_1 + K_2n\sigma_2 + \sigma_1\sigma_2)s \\ + \alpha_1g - \alpha_1gK_1 + gK_3n + \sigma_2nK_1 = 0.\end{aligned}$$

Comparing coefficients yields

$$\begin{bmatrix} 0 & n & g \\ n & n\sigma_2 - \alpha_1 g & -\alpha_2 n + g\sigma_1 \\ -\alpha_1 g + \sigma_2 n & 0 & gn \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} a_2 - \sigma_1 - \sigma_2 \\ a_1 + \alpha_1 \alpha_2 - \sigma_1 \sigma_2 \\ a_0 - \alpha_1 g \end{bmatrix}$$

where

$$\begin{aligned} a_2 &= 36 \\ a_1 &= 225 \\ a_0 &= 1350 . \end{aligned}$$

The solution for  $\mathbf{K}$  is

$$\mathbf{K} = [53.11 \quad -28.64 \quad 21.96] .$$

**DP11.5** The controllability and observability matrices are

$$\mathbf{P}_c = \begin{bmatrix} 0.05 & -0.04 \\ 0.001 & -0.001 \end{bmatrix} \text{ and}$$

$$\mathbf{P}_o = \begin{bmatrix} 1 & 0 \\ -0.8 & 0.02 \end{bmatrix}, \text{ respectively.}$$

Computing the determinants yields

$$\det \mathbf{P}_c = -1.002e - 05 \neq 0 \quad \text{and} \quad \det \mathbf{P}_o = 0.02 \neq 0 ,$$

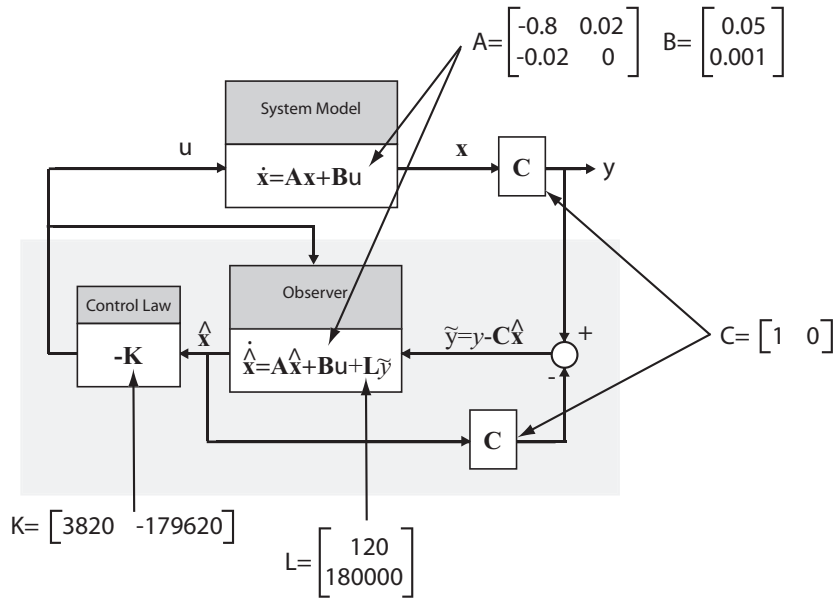
hence the system is controllable and observable. The feedback gain matrix

$$K = [ 3820 \quad -179620 ]$$

yields the desired closed-loop poles. The observer gain matrix

$$L = \begin{bmatrix} 120 \\ 180000 \end{bmatrix}$$

yields the desired observer poles. The integrated system is shown in Figure DP11.5.



**FIGURE DP11.5**  
Integrated controller and observer.

**DP11.6** (a) The characteristic equation associated with the system matrix is

$$q(s) = s^2 + (12 + K_2)s + (36 + K_1) = 0 ,$$

where we have assumed state feedback of the form

$$u = -K_1 x_1 - K_2 x_2 .$$

The deadbeat control characteristic equation is

$$s^2 + \alpha \omega_n s + \omega_n^2 = 0 ,$$

where  $\alpha = 1.82$  and we use  $\omega_n = 9.64$  to meet the settling time specification. Then, equating coefficients and solving for the gains yields

$$K_1 = 56.93 \quad \text{and} \quad K_2 = 5.54 .$$

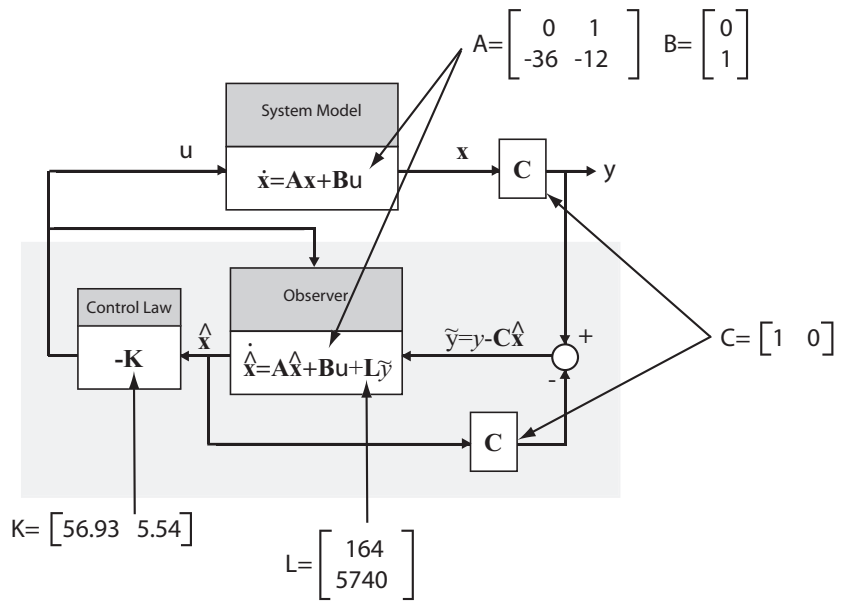
(b) Since the closed-loop poles are located at  $s_{1,2} = -8.77 \pm 4j$ , we can select the observer poles to be about ten times farther in the left-half plane, or

$$s_{1,2} = -88, -88 .$$

Then the observer gains are

$$L = \begin{bmatrix} 164 \\ 5740 \end{bmatrix}.$$

(c) The block diagram is shown in Figure DP11.6.



**FIGURE DP11.6** Block diagram for integrated controller and observer.

**DP11.7** The compensator is

$$\begin{aligned} \dot{\hat{x}} &= [A - LC] \hat{x} + Ly + Bu \\ u &= -K\hat{x} \end{aligned}$$

where

$$A - LC = \begin{bmatrix} -60 & 1 & 0 \\ -1095 & 0 & 1 \\ -3750 & -5 & -10 \end{bmatrix},$$

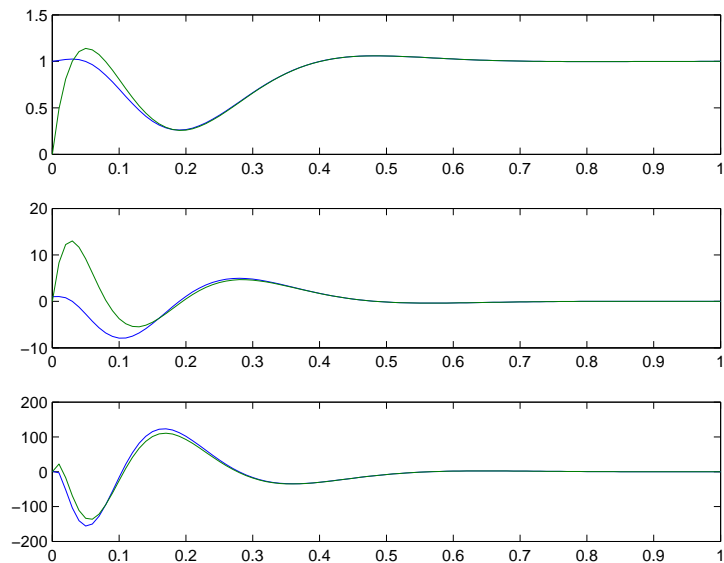


$$N = 4000, \quad \mathbf{K} = \begin{bmatrix} 3998 & 595 & 30 \end{bmatrix}, \quad \text{and} \quad \mathbf{L} = \begin{bmatrix} 60 \\ 1095 \\ 3748 \end{bmatrix}.$$

We selected the desired eigenvalues of  $\mathbf{A} - \mathbf{BK}$  at  $p_{1,2} = -10 \pm 10j$ ,  $p_3 = -20$  and the desired eigenvalues of  $\mathbf{A} - \mathbf{LC}$  at  $q_{1,2} = -20 \pm 10j$ ,  $q_3 = -30$ . For initial conditions we let  $\mathbf{x}(0) = [1 \ 1 \ 1]$  and  $\hat{\mathbf{x}}(0) = [0 \ 0 \ 0]$ . The transfer function from  $r$  to  $y$  is

$$T(s) = \frac{4000s^3 + 2.8e05s^2 + 6.8e06s + 6e07}{s^6 + 110s^5 + 5100s^4 + 1.29e05s^3 + 1.9e06s^2 + 1.58e07s + 6e07}.$$

The bandwidth is 11.7 rad/s.

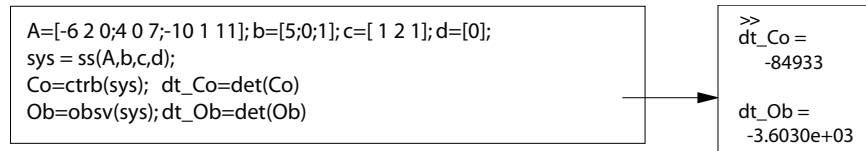


**FIGURE DP11.7**

The step response showing the actual and estimated states.

## Computer Problems

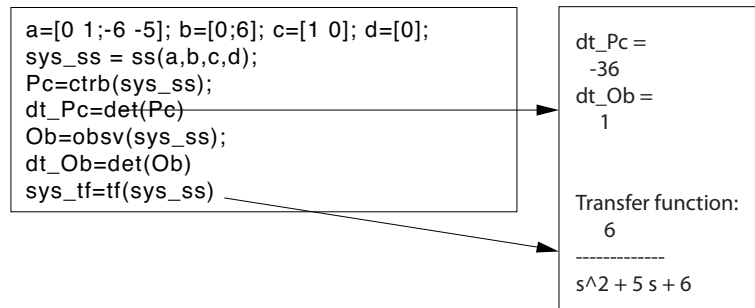
- CP11.1** The controllability and observability matrices have nonzero determinants, as shown in Figure CP11.1. Therefore, the system is observable and controllable.



**FIGURE CP11.1**

Determining controllability and observability.

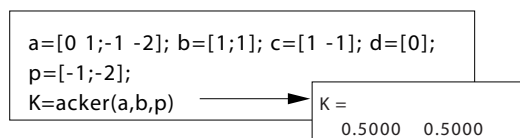
- CP11.2** The system is controllable since the determinant of the controllability matrix is nonzero, as shown in Figure CP11.2.



**FIGURE CP11.2**

M-file script to determine controllability and to compute equivalent transfer function model.

- CP11.3** The gain matrix (computed as shown in Figure CP11.3) is  $\mathbf{K} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$ .



**FIGURE CP11.3**

M-file script to place the closed-loop system poles using state feedback.

**CP11.4** The constant velocity guided missile is not controllable since the controllability matrix,  $C_o$ , has a zero determinant, as shown in Figure CP11.4. Using the `tf` function (see Figure CP11.4), we determine that the transfer function is

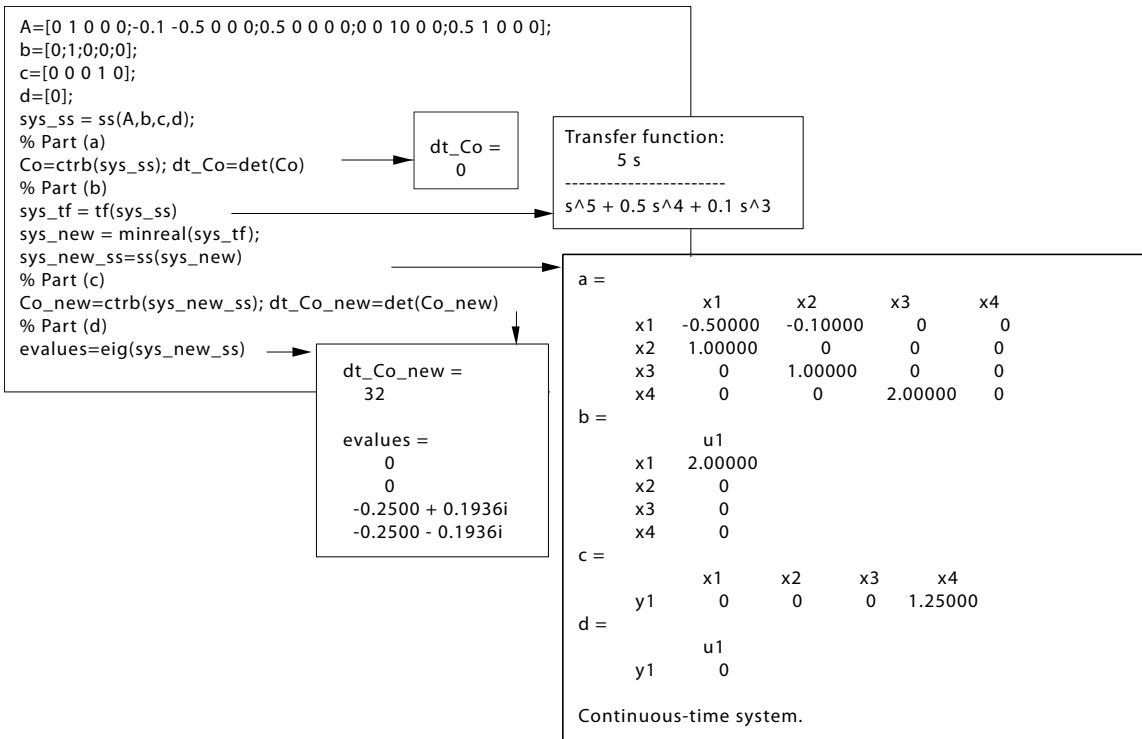
$$G(s) = \frac{5s}{s^5 + 0.5s^4 + 0.1s^3}.$$

Cancelling common terms in the transfer function yields

$$G(s) = \frac{5}{s^4 + 0.5s^3 + 0.1s^2}.$$

Then, using the `ss` function, we determine a state-space representation of  $G(s)$ . As shown in Figure CP11.4, the state-space representation is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$



**FIGURE CP11.4** Analysis of the constant velocity guided missile state-space model.

where

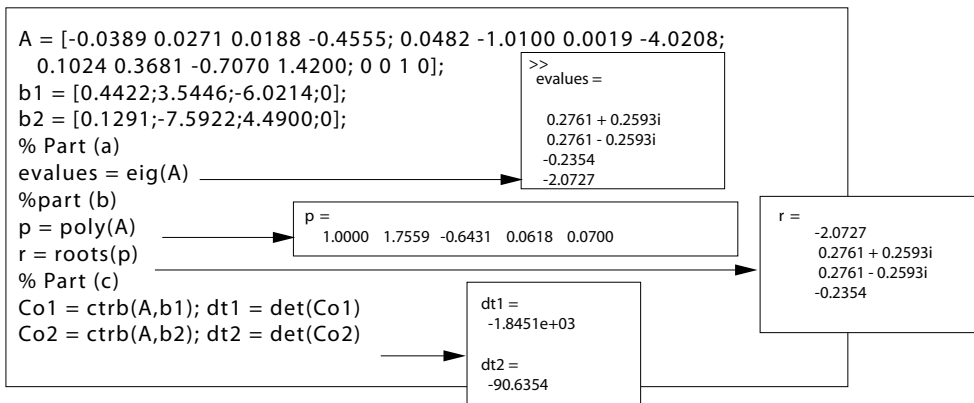
$$\mathbf{A} = \begin{bmatrix} -0.5 & -0.1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1.25 \end{bmatrix} .$$

The reduced system is controllable but not stable, since there are two poles at the origin. Systems that are not controllable have too many states. After eliminating unnecessary states, a controllable system of minimal complexity (i.e. states) is obtained. In this case, the number of states is reduced from five to four.

**CP11.5** The eigenvalues of  $\mathbf{A}$  are

$$\begin{aligned} e_1 &= -2.0727 \\ e_2 &= -0.2354 \\ e_{3,4} &= 0.2761 \pm 0.2593j \end{aligned}$$

The system is unstable since there are two eigenvalues in the right half-plane, see Figure CP11.5. The characteristic polynomial is



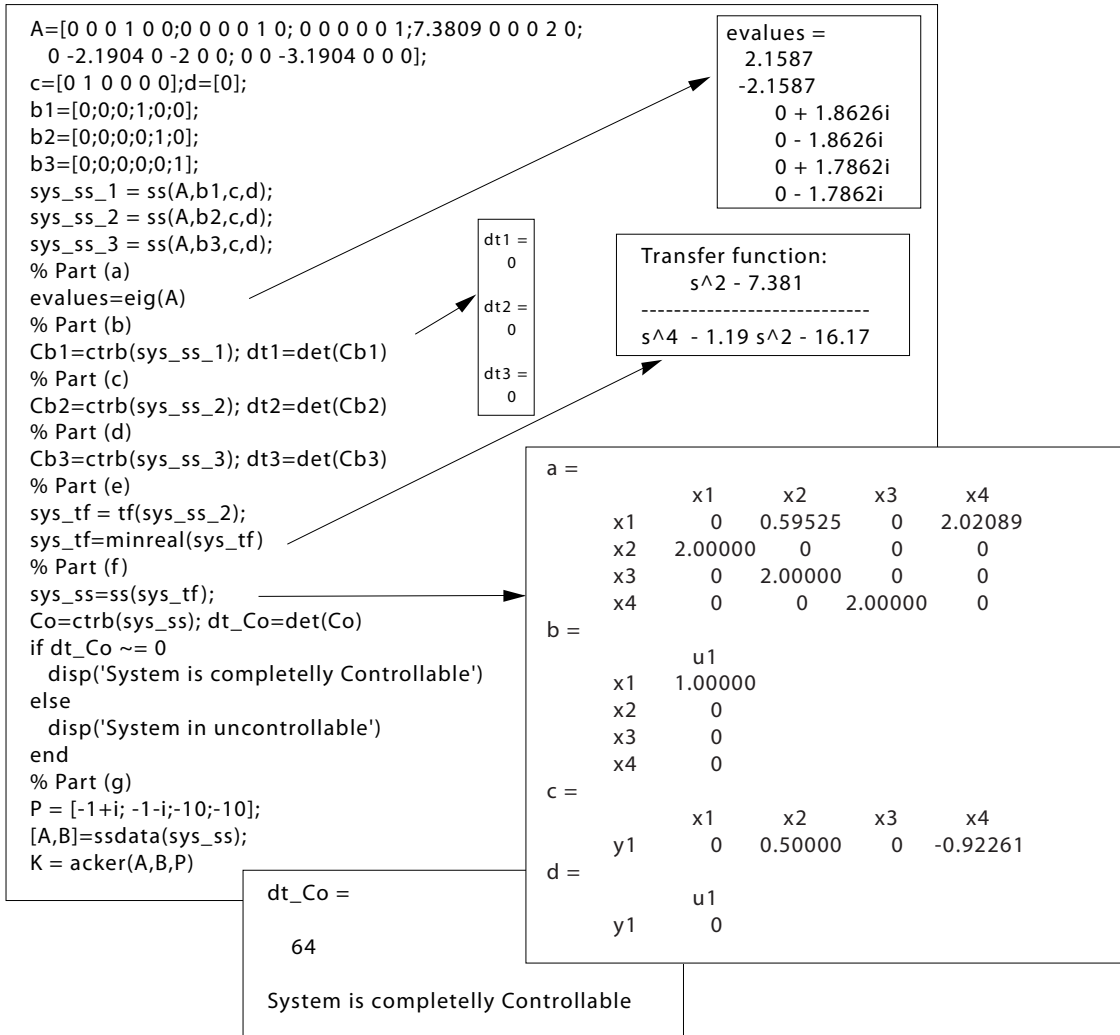
**FIGURE CP11.5** Analysis of the VTOL aircraft model.

$$p(s) = s^4 + 1.7559s^3 - 0.6431s^2 + 0.0618s + 0.0700 .$$

The roots of the characteristic equation are the same as the eigenvalues. Also, the system is controllable from either  $u_1$  or  $u_2$ . If the aircraft should lose the control of the vertical motion through  $u_1$ , then the control  $u_2$  can

be used to control both vertical and horizontal motion, and vice versa.

**CP11.6** The m-file script to analyze the translunar halo orbit problem is shown in Figure CP11.6. The translunar equilibrium point is not a stable point



**FIGURE CP11.6** Analysis of the translunar satellite halo orbit.

as evidenced by the eigenvalues of **A** in the right half-plane; the system is not completely controllable from any  $u_i$  individually. The transfer function

from  $u_2$  to  $\eta$  is

$$T(s) = \frac{s^4 - 4.191s^2 - 23.55}{s^6 + 2s^4 - 19.97s^2 - 51.58}.$$

A careful analysis reveals that  $T(s)$  can be reduced by eliminating common factors. The common factors are  $s^2 + 3.1834$ . The reduced transfer function is

$$T(s) = \frac{s^2 - 7.3815}{s^4 - 1.1837s^2 - 16.2030}.$$

Using state feedback

$$u_2 = -\mathbf{K}\mathbf{x}$$

the gain matrix  $K$  which places the desired poles (using Ackermann's formula) is

$$\mathbf{K} = \begin{bmatrix} 22 & 71.56 & 60 & 27.02 \end{bmatrix}.$$

**CP11.7** The m-file script to determine the initial state is shown in Figure CP11.7a. Given three data points at  $t = 0, 2, 4$ , we construct the three equations

```
A=[0 1 0;0 0 1;-2 -4 -6];
b=[0;0;0]; c=[1 0 0]; d=[0];
sys=ss(A,b,c,d);
%
% Part (b)
v1=c*expm(0*A); v2=c*expm(2*A); v3=c*expm(4*A);
V=[v1;v2;v3]; Vi=inv(V);
n=[1;-0.0256;-0.2522];
x0=Vi*n
%
% Part (c)
t=[0:0.1:4]; u=0.0*t;
[y,x]=lsim(sys,u,t,x0');
plot(t,y,[0 2 4],[1;-0.0256;-0.2522],'*'), grid
xlabel('Time (sec)'), ylabel('y(t)')
title('Data points denoted by *')
```

**FIGURE CP11.7**

(a) Script to determine the initial state from three observations.

$$y(0) = 1 = \mathbf{C}e^{0\mathbf{A}}\mathbf{x}_0$$

$$y(2) = -0.0256 = \mathbf{C}e^{2\mathbf{A}}\mathbf{x}_0$$

$$y(4) = -0.2522 = \mathbf{C}e^{4\mathbf{A}}\mathbf{x}_0$$

or, in matrix form

$$\begin{bmatrix} \mathbf{C}e^{0\mathbf{A}} \\ \mathbf{C}e^{2\mathbf{A}} \\ \mathbf{C}e^{4\mathbf{A}} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} 1 \\ -0.0256 \\ -0.2522 \end{bmatrix}.$$

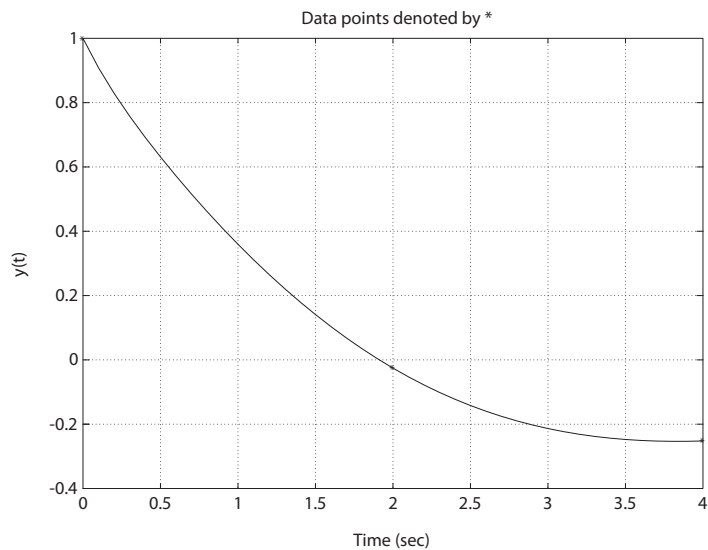
The problem is solvable if the matrix

$$\begin{bmatrix} \mathbf{C}e^{0\mathbf{A}} \\ \mathbf{C}e^{2\mathbf{A}} \\ \mathbf{C}e^{4\mathbf{A}} \end{bmatrix}$$

is invertible. In this case, the inverse does exist and the solution is

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ -1 \\ 1.9998 \end{bmatrix}.$$

The simulation is shown in Figure CP11.7b.



**FIGURE CP11.7**  
CONTINUED: (b) System simulation using computed initial state.

**CP11.8** Suppose we are given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the feedback

$$u = -\mathbf{K}\mathbf{x} = -[K_1 \ K_2]\mathbf{x} .$$

Solving  $\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I}$  for  $\mathbf{P}$  yields

$$p_{11} = \frac{K_2^2 + K_1^2 + 3K_1 + 2}{2(K_1 + 1)K_2}$$

$$p_{12} = \frac{1}{2(K_1 + 1)}$$

$$p_{22} = \frac{K_1 + 2}{2(K_1 + 1)K_2}$$

Then, with  $\mathbf{x}_o^T = [1, 0]$  we find that

$$J = \mathbf{x}_o^T \mathbf{P} \mathbf{x}_o = p_{11} .$$

Computing the partial of  $J$  with respect to  $K_2$  yields

$$\frac{\partial J}{\partial K_2} = \frac{1}{2} \left[ \frac{1}{K_1 + 1} - \frac{K_1 + 2}{K_2^2} \right] .$$

Setting

$$\frac{\partial J}{\partial K_2} = 0$$

and solving for  $K_2$ , we find that

$$K_2 = \sqrt{(K_1 + 2)(K_1 + 1)} .$$

For a given value of  $K_1$ , the value of  $K_2$  that minimizes  $J$  can be computed via the above equation. With  $K_2$  given as above, we can compute  $J$  to be

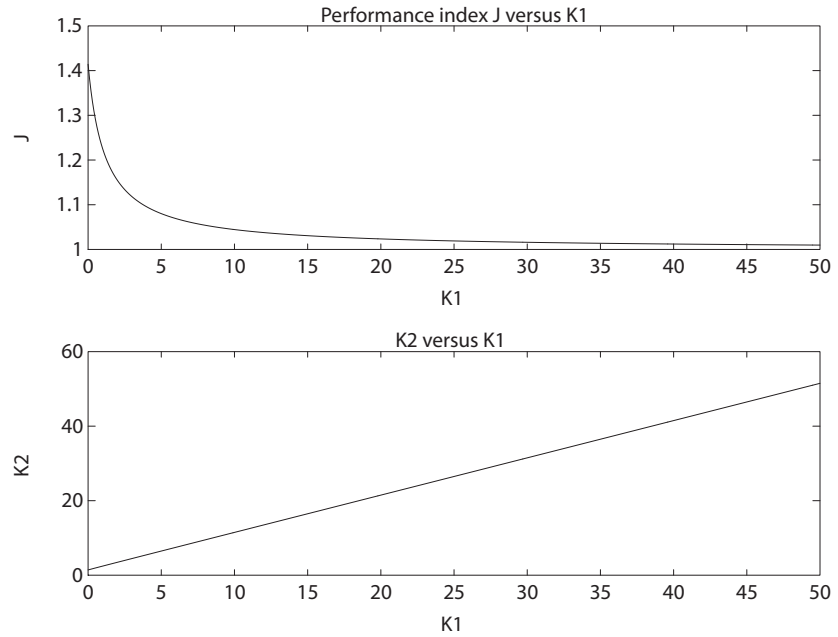
$$J = \sqrt{\frac{K_1 + 2}{K_1 + 1}} .$$

A plot of  $J$  versus  $K_1$  (with  $K_2$  equal to the minimizing value) is shown in Figure CP11.8. As  $K_1$  increases, the performance index  $J$  decreases. However, we see that the rate of decrease slows considerably after  $K_1 > 20$ . Also,  $K_2$  increases as  $K_1$  increases. We want to keep both gains as



small as possible, while still having a small  $J$ . A reasonable selection is

$$K_1 = 20 \quad \text{and} \quad K_2 = 21.5 .$$



**FIGURE CP11.8**

Performance index as a function of  $K_1$  and  $K_2$ .

**CP11.9** In this problem,  $A = -1$  and  $B = 1$ . Computing  $Q$  yields

$$Q = (1 + \lambda(-k)^2) = 1 + \lambda k^2 .$$

Define

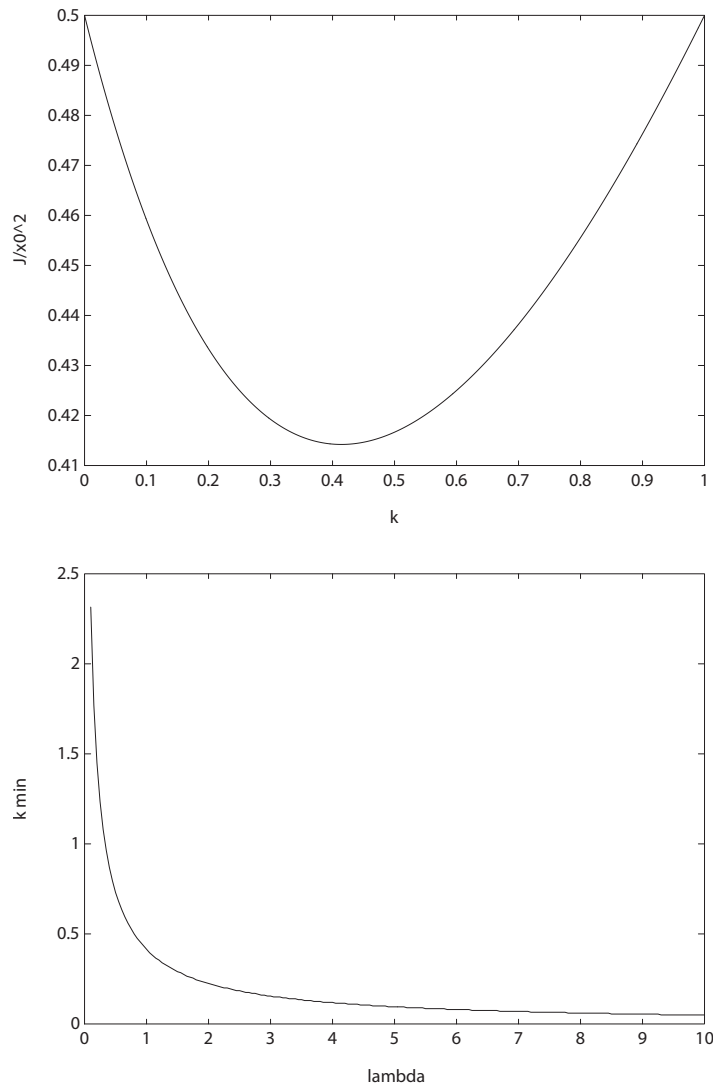
$$H = A - Bk = -1 - k .$$

Solving

$$H^T P + PH = -Q$$

yields

$$p = \frac{1 + \lambda k^2}{2(k + 1)} .$$



**FIGURE CP11.9**  
Plot of  $J/x_0^2$  versus  $k$  and the minimizing  $k$  versus  $\lambda$ .

The performance index is

$$J = x_0^2 p \quad \text{which implies} \quad J/x_0^2 = \frac{1 + \lambda k^2}{2(k+1)}.$$

The plot of  $J/x_0^2$  versus  $k$  is shown in Figure CP11.9. The minimum value is achieved when  $k = 0.41$ . To arrive at this result analytically, take the

partial of  $J/x_0^2$  with respect to  $k$ , set the result to zero and solve for  $k$ :

$$\frac{\partial J/x_0^2}{\partial k} = 0 \quad \text{when} \quad k^2 + 2K - 1/\lambda = 0 .$$

Solving for  $k$  yields  $k = -1 \pm \sqrt{1 + 1/\lambda}$ . So, when  $\lambda = 1$ ,  $k = 0.41$ . The plot of  $k_{min}$  versus  $\lambda$  is shown in Figure CP11.9.

**CP11.10** The m-file is shown in Figure CP11.10.

```

A=[0 1;-18.7 -10.4]; B=[10.1; 24.6]; C=[1 0]; D=[0];      >>
                                                              K =
% Controller Gains
p=[-2;-2];
K=acker(A,B,p)      -0.3081 -0.1337

% Observer Gains
q=[-20+4*j;-20-4*j];
L = acker(A',C',q); L=L
                                                              L =
                                                              29.6000
                                                              89.4600

```

**FIGURE CP11.10**

Using the acker function to compute the controller gains and the observer gains.

**CP11.11** The m-file is shown in Figure CP11.11(a). The compensator can be rep-

```

A=[0 1 0;0 0 1;-4.3 -1.7 -6.7]; B=[0;0;0.35]; C=[0 1 0]; D=[0];      >>
                                                              K =
% Controller Gains
p=[-1.4+1.4*j;-1.4-1.4*j;-2];
K=acker(A,B,p)      10.1143  22.3429  -5.4286

% Observer Gains
q=[-18+5*j;-18-5*j;-20];
L = acker(A',C',q); L=L'
                                                              L =
                                                              1.0e+003 *
                                                              -1.6223
                                                              0.0493
                                                              0.7370

% Simulation of closed-loop system with the observer
Ac=[A -B*K;L*C A-B*K-L*C];
Bc=[zeros(6,1)];
Cc=eye(6);
Dc=zeros(6,1);
sys=ss(Ac,Bc,Cc,Dc);
x0=[1;0;0;0.5;0.1;0.1]; t=[0:0.001:3.5];
[y,t]=initial(sys,x0,t);
subplot(311)
plot(t,y(:,1),t,y(:,4),'-'), grid
subplot(312)
plot(t,y(:,2),t,y(:,5),'-'), grid
subplot(313)
plot(t,y(:,3),t,y(:,6),'-'), grid

```

**FIGURE CP11.11**

(a) M-file.

resented as

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{L}y \quad \text{and} \quad u = -\mathbf{K}\hat{\mathbf{x}} .$$

Since  $y = Cx$ , we can write

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{LC}\mathbf{x} .$$

Similarly, with

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{and} \quad u = -\mathbf{K}\hat{\mathbf{x}}$$

we obtain

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{BK}\hat{\mathbf{x}} .$$

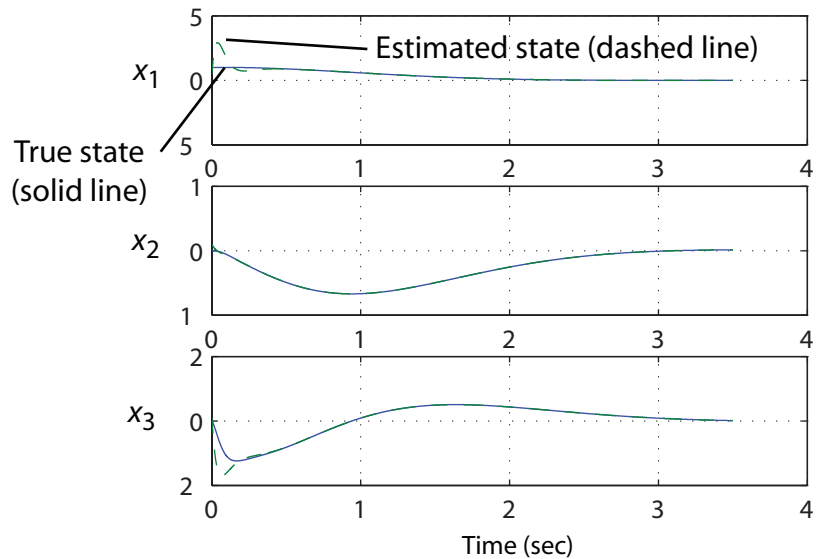
In matrix form, we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & \mathbf{A} - \mathbf{BK} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} ,$$

with initial conditions

$$\begin{bmatrix} \mathbf{x}(0)^T & \hat{\mathbf{x}}(0)^T \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0.1 & 0.1 \end{bmatrix}^T .$$

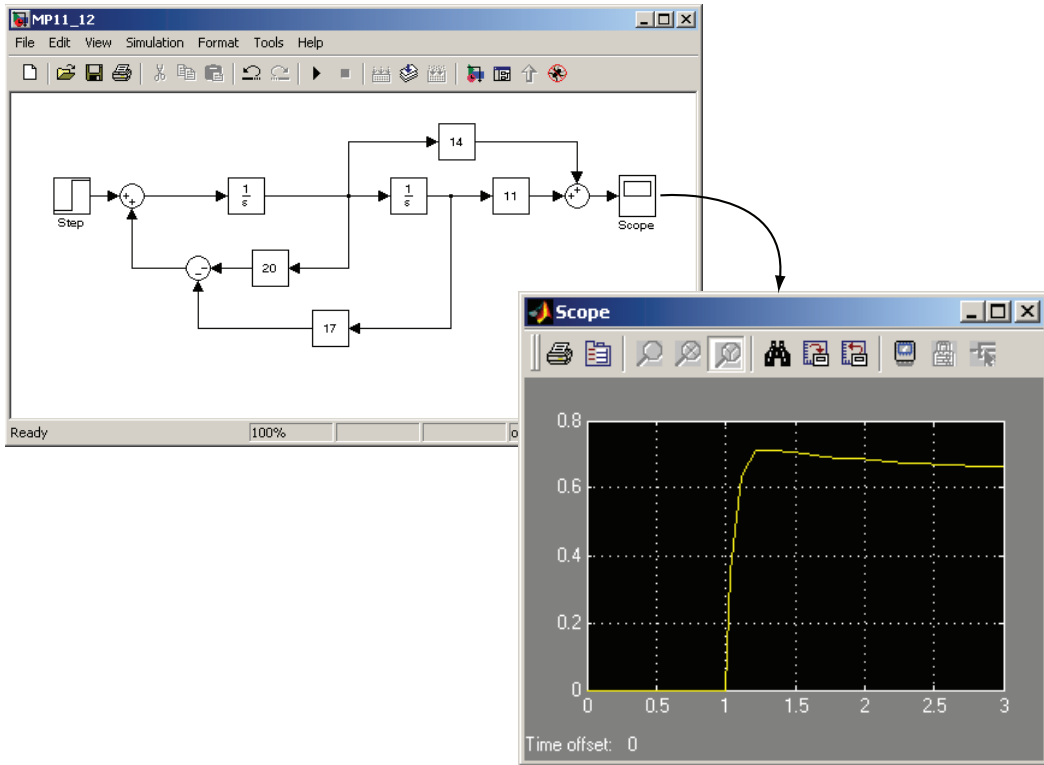
The response of the system is shown in Figure CP11.11(b).



**FIGURE CP11.11**

CONTINUED: (b) Response of system to an initial condition.

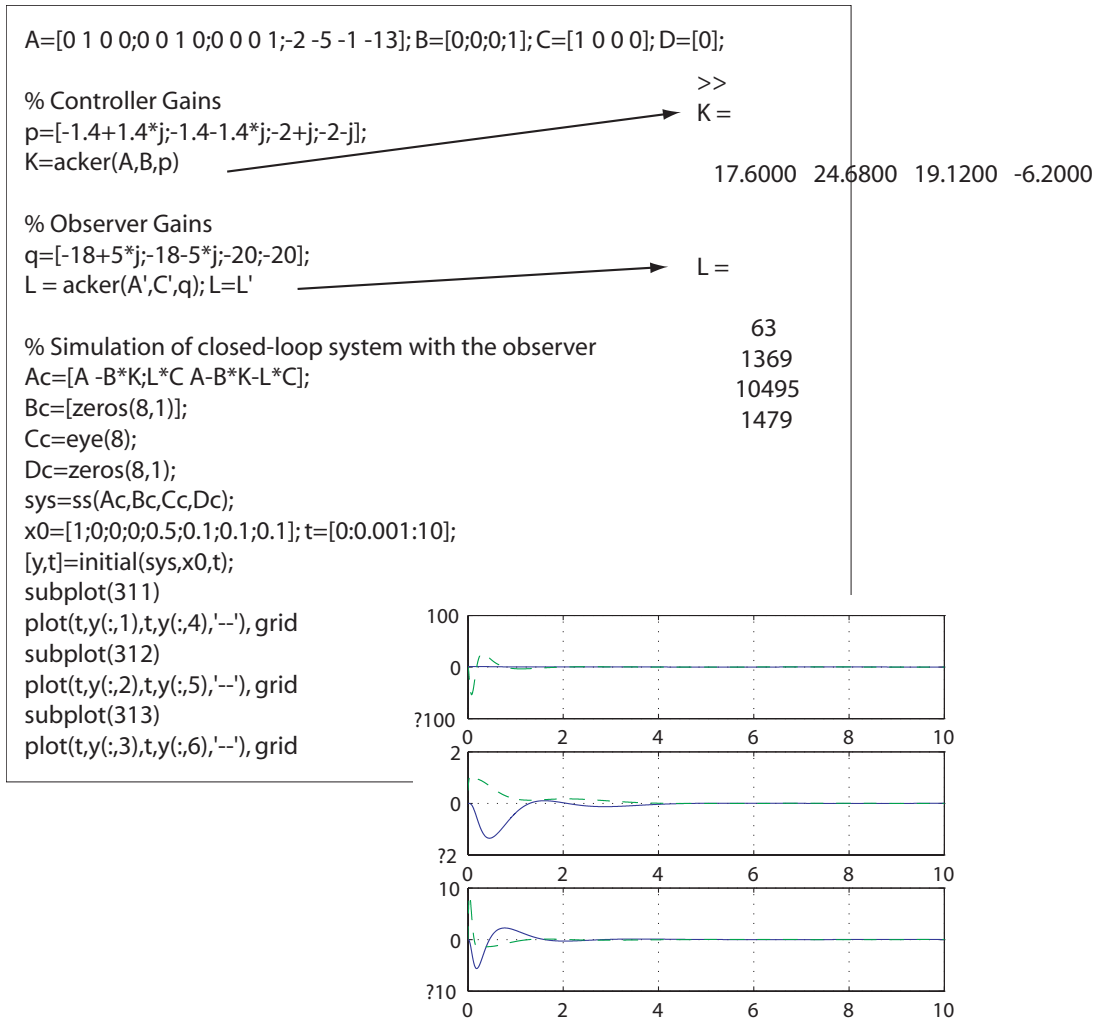
**CP11.12** The Simulink block diagram is shown in Figure CP11.12.



**FIGURE CP11.12**  
Simulink block diagram.

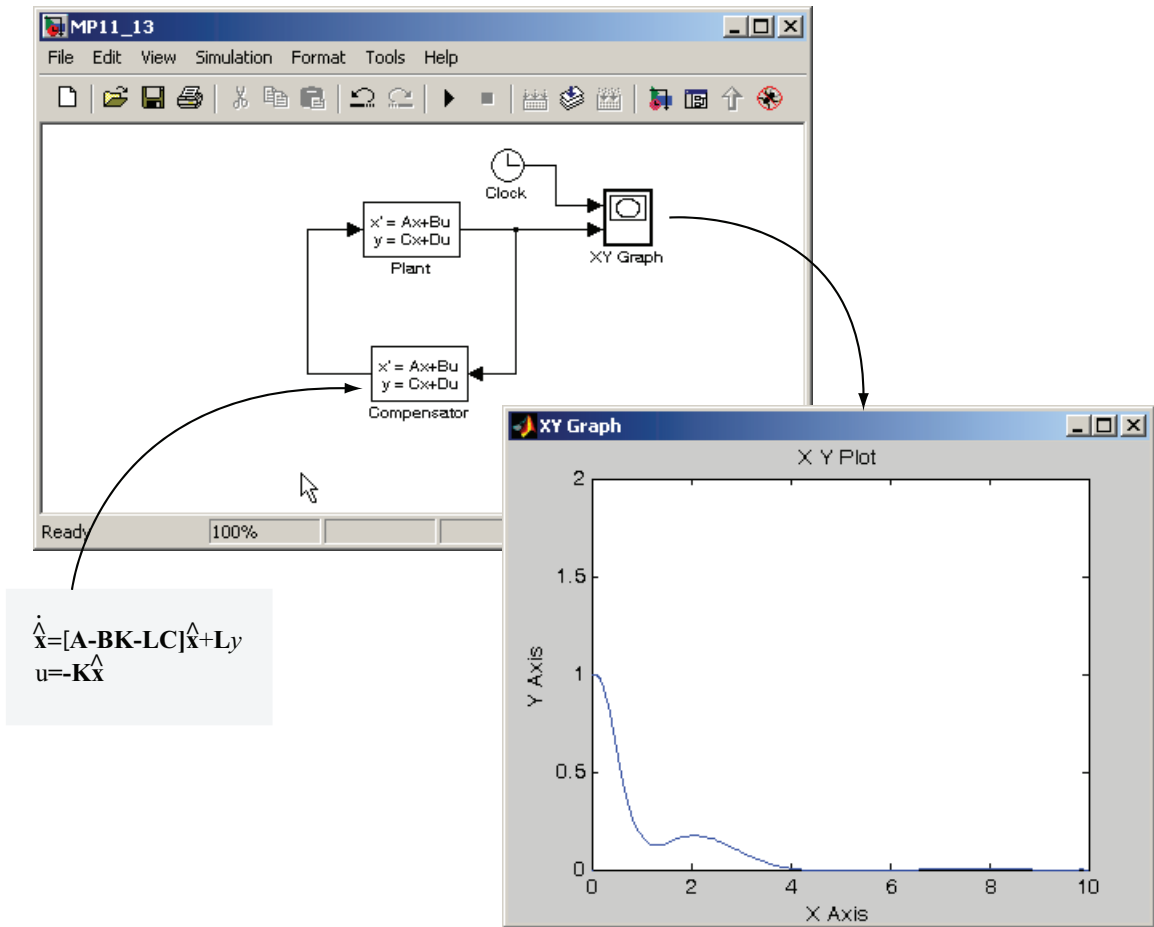
**CP11.13** The m-file to design the compensator is shown in Figure CP11.13(a). The Simulink simulation is shown in Figure CP11.13(b). The output shown on the x-y graph depicts the state  $x$  of the system. The initial conditions selected for the simulation are

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{x}}(0) = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} .$$



**FIGURE CP11.13**

(a) M-file to design the compensator, including the observer.



**FIGURE CP11.13**  
CONTINUED (b) The Simulink simulation.

# *Robust Control Systems*

## Exercises

**E12.1** The plant transfer function is

$$G(s) = \frac{3}{s + 3} .$$

Try a PI controller, given by

$$G_c = K_1 + \frac{K_2}{s} .$$

The ITAE characteristic equation is

$$s^2 + 1.4\omega_n s + \omega_n^2 ,$$

where  $\omega_n = 30$ . Then

$$K_1 = 13 \quad \text{and} \quad K_2 = 300 .$$

Without a prefilter, the closed-loop system is

$$\frac{Y(s)}{R(s)} = \frac{39s + 900}{s^2 + 42s + 900} ,$$

and with a prefilter, the closed-loop system is

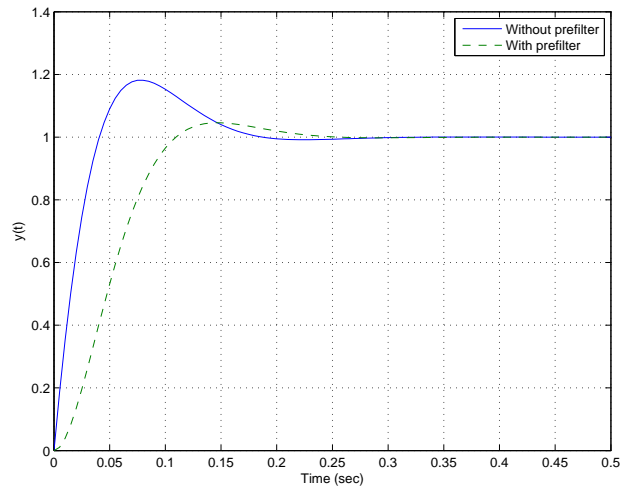
$$\frac{Y(s)}{R(s)} = \frac{900}{s^2 + 42s + 900} ,$$

where

$$G_p(s) = \frac{23.07}{s + 23.07} .$$

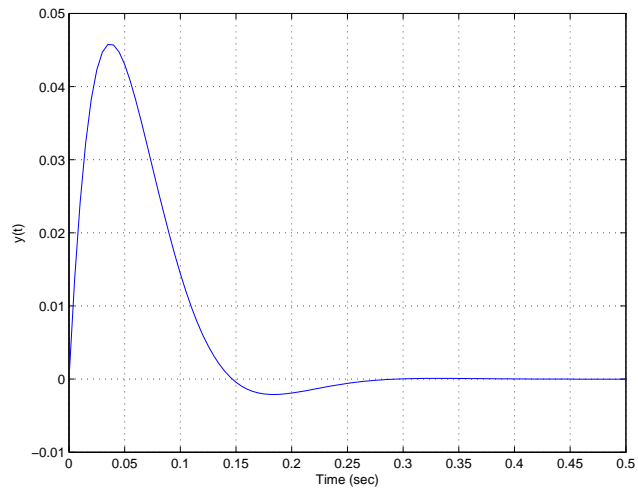
The step response, with and without the prefilter, is shown in Figure E12.1.



**FIGURE E12.1**

Step response: (a) w/o prefilter (solid line), and (b) w/prefilter (dashed line).

**E12.2** The disturbance response is shown in Figure E12.2.

**FIGURE E12.2**

Disturbance response for system in E12.1.

**E12.3** The closed-loop transfer function is

$$T(s) = \frac{25}{s^2 + bs + 25},$$

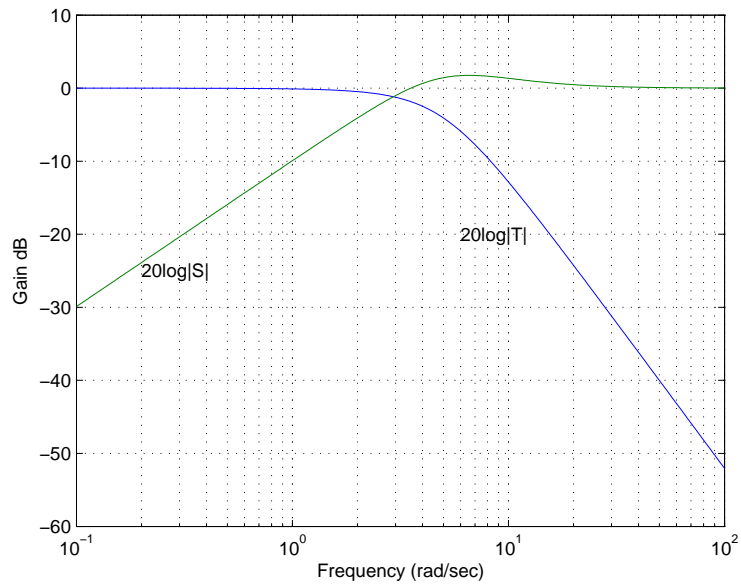
and the sensitivity function is

$$S(s) = \frac{s^2 + bs}{s^2 + bs + 25},$$

where  $b = 8$ , nominally. The sensitivity of  $T$  to changes in  $b$  is determined to be

$$S_b^T = \frac{\partial T}{\partial b} \frac{b}{T} = \frac{-bs}{s^2 + bs + 25}.$$

The plot of  $T(s)$  and  $S(s)$  is shown in Figure E12.3, where  $b = 8$ .



**FIGURE E12.3**

Plot of  $T(s)$  and the sensitivity function  $S(s)$ .

**E12.4** The plant transfer function is

$$G(s) = \frac{1}{(s + 20)(s + 36)},$$

and the PID controller is given by

$$G_c(s) = \frac{K_3(s+a)(s+b)}{s}.$$

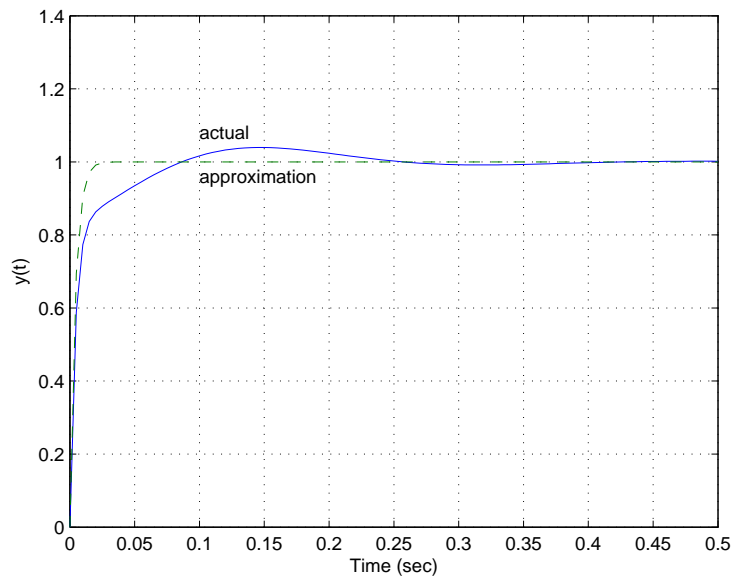
Let  $a=20$ ,  $b=500$ , and  $K_3 = 200$ . Then, the closed-loop system is

$$T(s) = \frac{200s^2 + 4000s + 100000}{s^3 + 256s^2 + 4720s + 100000}.$$

The closed-loop poles are  $s_1=-237.93$  and  $s_{2,3} = -9.04 \pm j18.5$  and the zeros are  $s_{1,2} = -10 \pm j20$ . Therefore, there is an approximate cancellation of the complex poles and zeros and the approximate system is

$$\hat{T}(s) = \frac{238}{s + 238}.$$

The actual response and approximation are shown in Figure E12.4.



**FIGURE E12.4**

Step response for closed-loop actual and approximate transfer functions.

**E12.5** The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{10K_D(s + K_P/K_D)}{s(s+3)(s+10)}.$$

Select  $K_P/K_D = 10$ . Then

$$L(s) = G_c(s)G(s) = \frac{10K_D}{s(s+3)},$$

and the closed-loop transfer function is

$$T(s) = \frac{10K_D}{s^2 + 3s + 10K_D}.$$

Let  $\zeta = 0.69$ , which implies  $P.O. < 5\%$ . Also,  $2\zeta\omega_n = 3$ , so  $\omega_n = 2.17$ . Thus,

$$10K_D = \omega_n^2 = 4.72.$$

Thus, the controller is  $G_c(s) = 0.47(s + 10)$ . The settling time is  $T_s = 2.8$  s and the percent overshoot is  $P.O. = 4.6\%$ . As  $K$  increases, the percent overshoot increases from 0% to 16% and the settling time generally decreases from 3.8 sec to 2.6 sec.

**E12.6** The loop transfer function with the PID controller is

$$G_c(s)G(s) = \frac{K_D s^2 + K_P s + K_I}{s} \frac{1}{(s+5)^2}.$$

The ITAE step response requires

$$s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3 = s^3 + (10 + K_D)s^2 + (25 + K_P)s + K_I.$$

For  $n = 3$  we estimate the normalized settling time to be

$$\omega_n T_s \approx 8 \text{ seconds.}$$

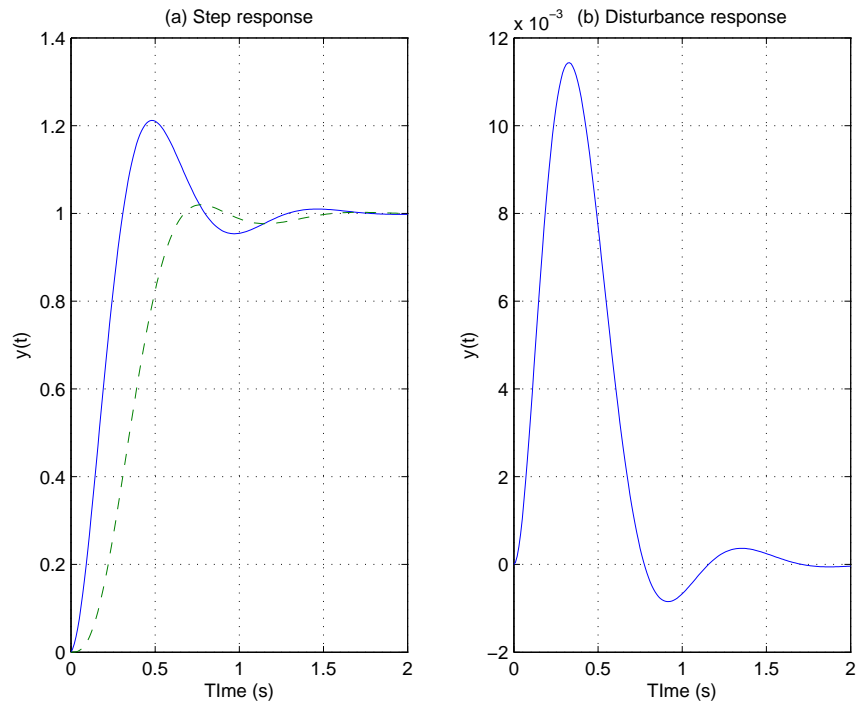
Thus,  $\omega_n \approx 6$ , and

$$K_D = 0.5, \quad K_P = 52.4, \quad \text{and} \quad K_I = 216.$$

The step response is shown in Figure E12.6. The transfer function from the disturbance to the output is

$$\frac{Y(s)}{T_d(s)} = \frac{G(s)}{1 + G_c(s)G(s)} = \frac{s}{s^3 + 10.5s^2 + 77.4s + 216}.$$

The disturbance response is shown in Figure E12.6. The system is effective in reducing the effects of the disturbance, and the maximum output is reduced by 1/100 for a step disturbance.

**FIGURE E12.6**

(a) Step response: w/o prefilter (solid line) and w/prefilter (dashed line); and (b) disturbance response.

**E12.7** The plant transfer function is

$$G(s) = \frac{1}{(s + 4)^2},$$

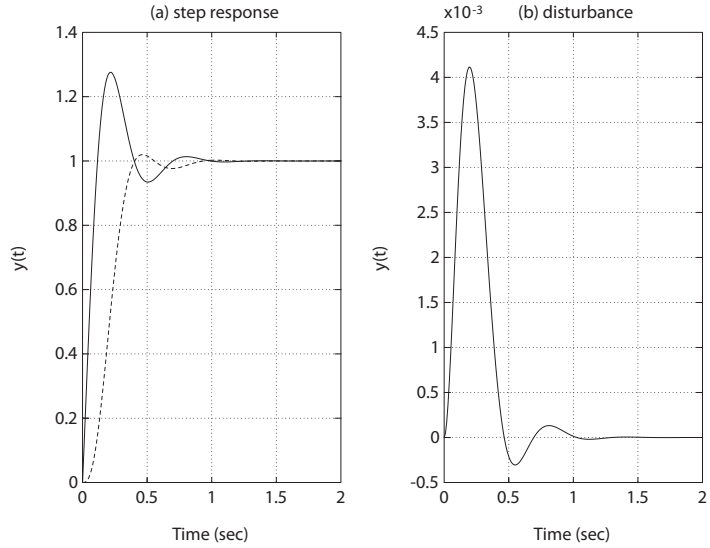
and the PID controller is

$$G_c(s) = \frac{K_1 s + K_2 + K_3 s^2}{s}.$$

Using the ITAE criteria and selecting  $\omega_n = 10$  yields

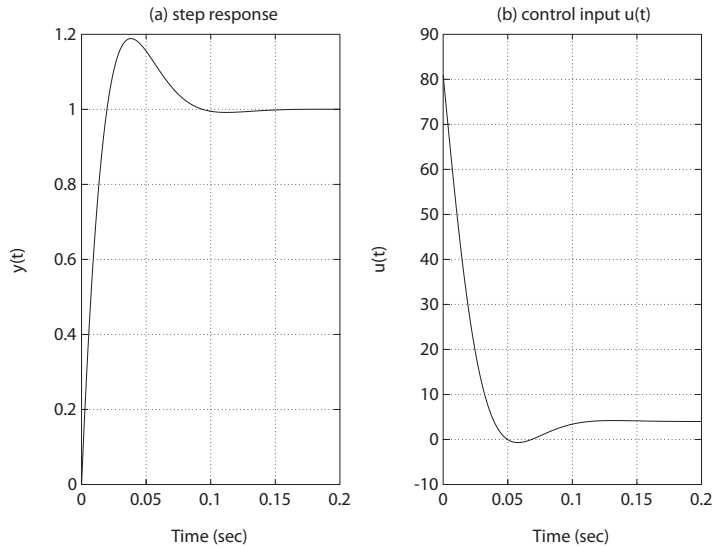
$$K_3 = 9.5 \quad K_2 = 1000 \quad \text{and} \quad K_1 = 199.$$

The step response is shown in Figure E12.7. The disturbance response is also shown in Figure E12.7. The maximum  $y(t) = 0.0041$ , so the system is effective in rejecting the step disturbance.

**FIGURE E12.7**

(a) Step response: w/o prefilter (solid line) and w/prefilter (dashed line); and (b) disturbance response.

**E12.8** The maximum  $\omega_n = 60$ . Then  $K_1 = 3600$  and  $K_2 = 80$ . The maximum control input is  $\max |u(t)| \approx 80$ . The plot of the step response and the control input  $u(t)$  is shown in Figure E12.8.

**FIGURE E12.8**

Step response w/o prefilter; and (b) control input  $u(t)$ .

**E12.9** One possible PD controller is

$$G_c(s) = 27.6s + 8.25s .$$

When

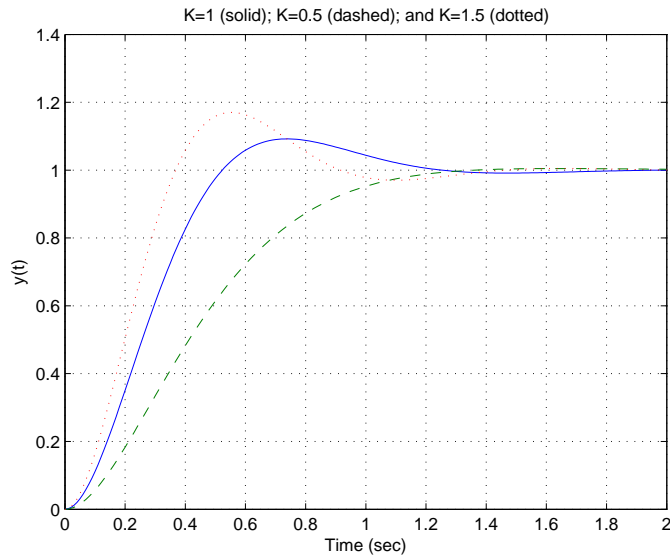
$$K = 1 ,$$

the system roots are

$$s_{1,2} = -3.2 \pm j4.3$$

$$s_3 = -9.5 .$$

The step response is shown in Figure E12.9 for  $K = 0.5, 1$ , and  $1.5$ .

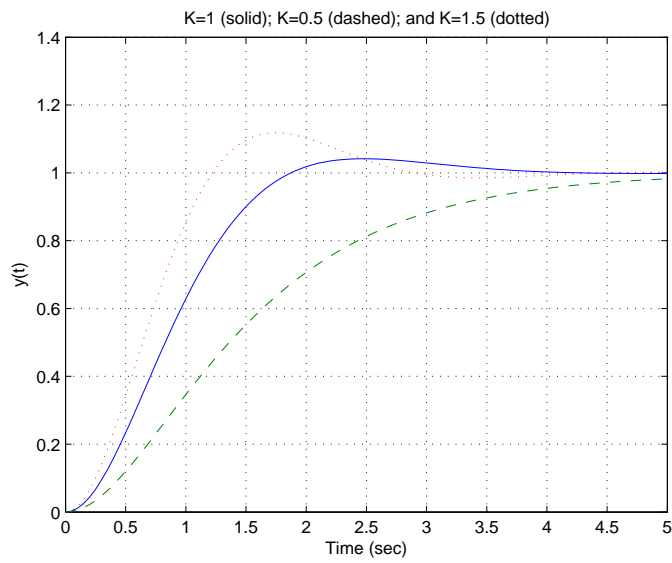


**FIGURE E12.9**  
Step response for  $K = 0.5, 1$ , and  $1.5$ .

**E12.10** One possible PI controller is

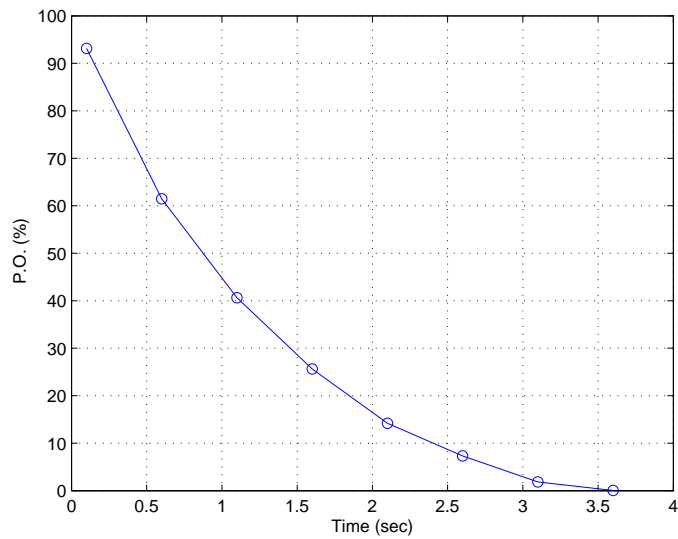
$$G_c(s) = \frac{2.2s + 22}{s} .$$

When  $K = 1$ , the system roots are  $s_{1,2} = -1.31 \pm j1.31$ , and  $s_3 = -6.37$ .  
The step response is shown in Figure E12.10.



**FIGURE E12.10**  
Step response for  $K = 0.5, 1,$  and  $1.5$ .

**E12.11** The plot is shown in Figure E12.11.



**FIGURE E12.11**  
Percent overshoot as a function of  $k$  in the interval  $0.1 \leq k \leq 4$ .



**E12.12** The controllability matrix is

$$P_c = \begin{bmatrix} c_1 & c_2 \\ c_2 & -ac_1 - bc_2 \end{bmatrix}$$

and

$$\det P_c = c_2^2 + [bc_1]c_2 + ac_1^2.$$

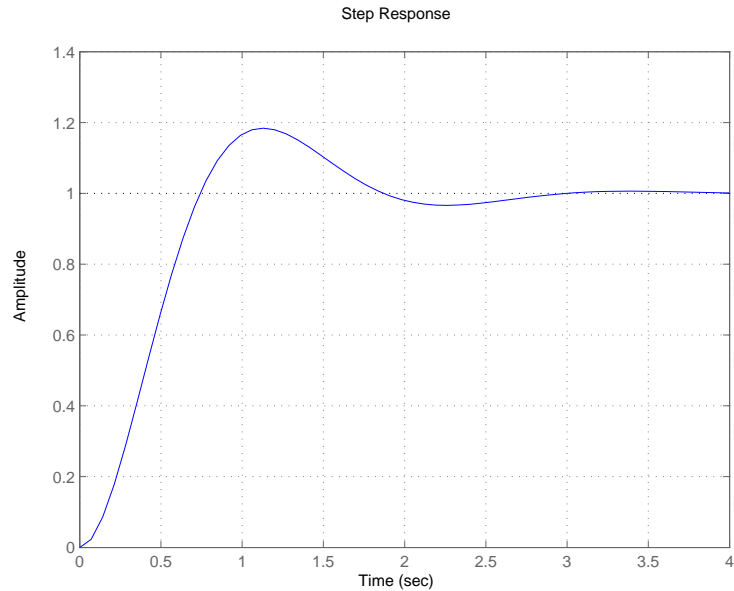
For controllability we require  $\det P_c \neq 0$ , hence

$$c_2^2 + [bc_1]c_2 + ac_1^2 \neq 0$$

implies

$$\frac{c_2}{c_1} \neq -\frac{b}{2} \pm \sqrt{(b/2)^2 - a}$$

where  $(b/2)^2 - a \geq 0$ . For real-valued  $c_1$  and  $c_2$ , if  $(b/2)^2 - a < 0$ , all real values of  $c_1$  and  $c_2$  are valid. Valid values of the constants are  $c_1 = 0$ ,  $c_2 = 10$ ,  $a = 10$ , and  $b = 3$ . The step response is shown in Figure E12.12.



**FIGURE E12.12**

Step response with  $c_1 = 0$ ,  $c_2 = 10$ ,  $a = 10$ , and  $b = 3$ .

# Problems

**P12.1** The closed-loop transfer function is

$$T(s) = \frac{4(s+2)}{s^2 + 4s + 8}$$

and the sensitivity function is

$$S(s) = \frac{s^2}{s^2 + 4s + 8} .$$

The plot of  $20 \log |T|$  and  $20 \log |S|$  is shown in Figure P12.1. The bandwidth is

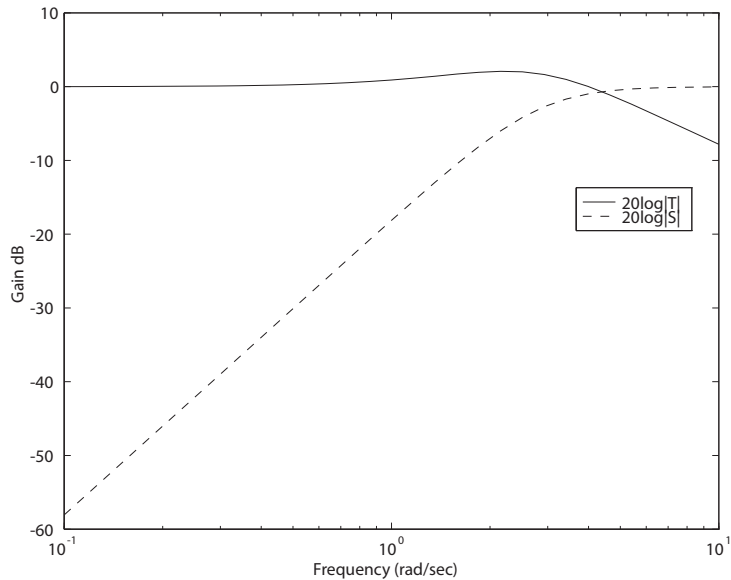
$$\omega_B = 6.31 \text{ rad/sec} .$$

Then

$$|S_K^T|_{\omega_B} = 0.98$$

$$|S_K^T|_{\frac{\omega_B}{2}} = 0.78$$

$$|S_K^T|_{\frac{\omega_B}{4}} = 0.30 .$$



**FIGURE P12.1**

Plot of  $T(s)$  and the sensitivity function  $S(s)$ .

**P12.2** (a) The loop transfer function is given by

$$G_c(s)G(s) = \frac{K}{s(0.02s + 1)(0.002s + 1)} .$$

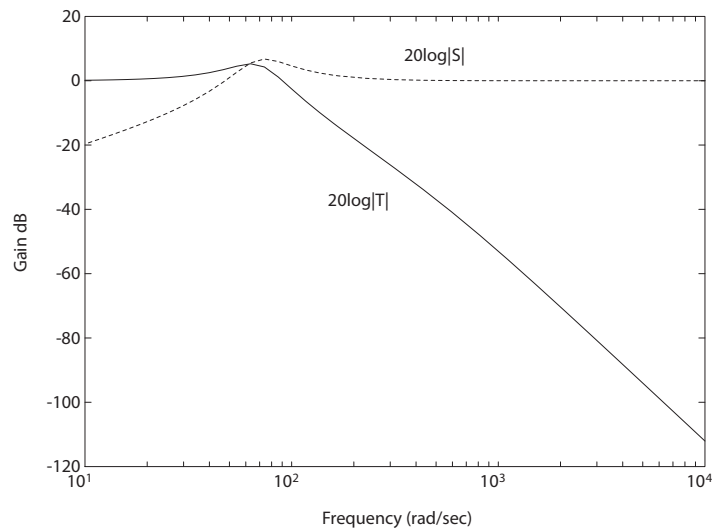
When

$$K = 100 ,$$

the peak magnitude is

$$M_{p\omega} = 1.84 .$$

(b) The plot of  $20 \log |T|$  and  $20 \log |S|$  is shown in Figure P12.2a.



**FIGURE P12.2**

(a) Plot of  $T(s)$  and the sensitivity function  $S(s)$ .

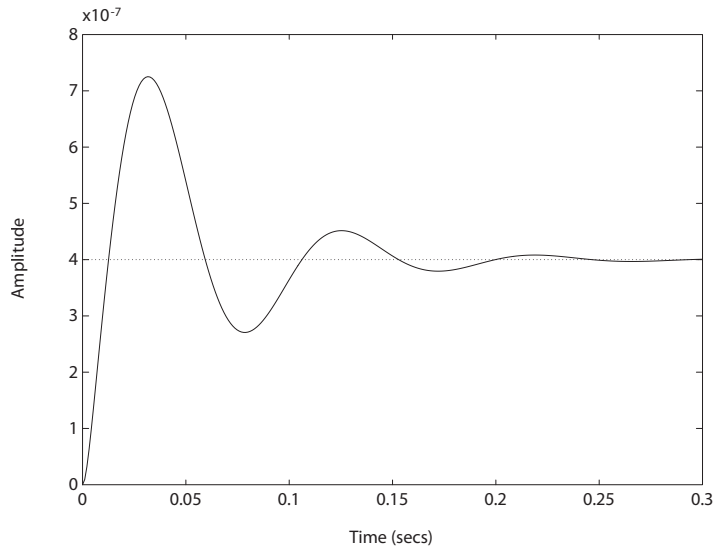
(c) The bandwidth is

$$\omega_B = 117 \text{ rad/sec} ,$$

and

$$\begin{aligned} |S_K^T|_{\omega_B} &= 1.47 \\ |S_K^T|_{\frac{\omega_B}{4}} &= 0.39 \\ |S_K^T|_{\frac{\omega_B}{2}} &= 1.62 . \end{aligned}$$

(c) The disturbance response is shown in Figure P12.2b.



**FIGURE P12.2**  
CONTINUED: (b) Disturbance response for  $K = 100$ .

**P12.3** (a) The loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K(s-4)(s-1)}{(s+0.02)(s+2)^2} .$$

The characteristic equation is

$$1 + G_c(s)G(s) = 1 + K \frac{(s-4)(s-1)}{(s+0.02)(s+2)^2} = 0$$

or

$$s^3 + (4.02 + K)s^2 + (4.08 - 5K)s + 0.08 + 4K = 0 .$$

Using Routh-Hurwitz we find that the system is stable for

$$-4.6987 < K < 0.6947 .$$

(b) The steady-state error is

$$e_{ss} = \frac{1}{1 + 50K} .$$

Select  $K = 0.18$  to obtain a steady-state error to a unit step of 0.1.

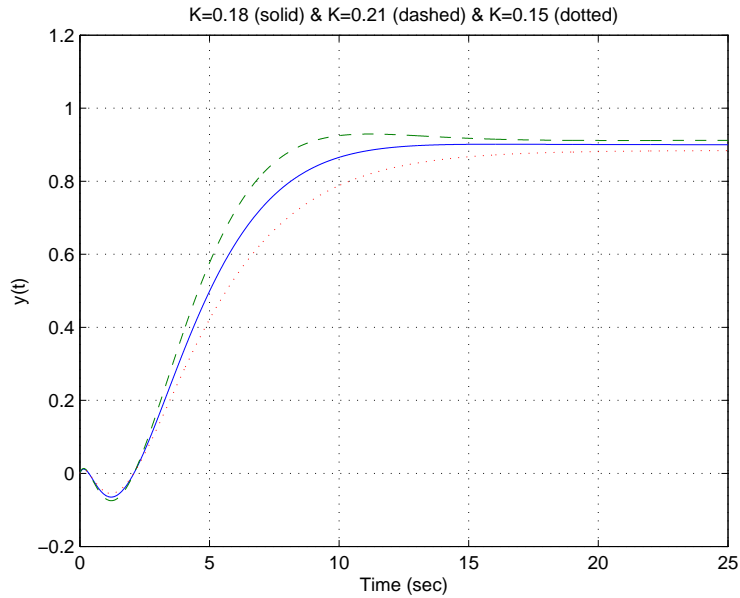
(c,d) The plots of  $y(t)$  for

$$K = 0.18 \quad (\text{nominal})$$

$$K = 0.21 \quad (+15\%)$$

$$K = 0.15 \quad (-15\%)$$

are shown in Figure P12.3.



**FIGURE P12.3**

Step input response for  $K = 0.18$ ,  $K = 0.21$  and  $K = 0.15$ .

**P12.4** (a) The plant is given by

$$G = \frac{1}{s \left( \frac{s}{25} + 1 \right)} .$$

We desire  $P.O. < 10\%$  and  $T_s < 100$  ms. Using a PD controller

$$G_c(s) = 100 + 2.2s ,$$

we determine that  $P.O. = 7\%$ ,  $T_s < 100$  ms and  $e_{ss} = \frac{A}{100}$  for a ramp input. The plot of  $y(t)$  is shown in Figure P12.4.

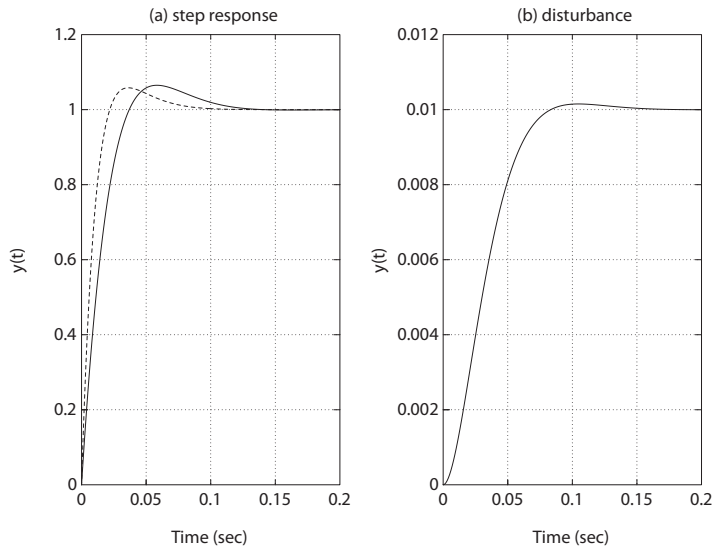
(b) The sensitivity is

$$|S_{K_1}^r| = 27.95$$

when  $K_1 = 1$ .

(c) The plot of  $y(t)$  when  $K_1 = 2$  (the compensator  $G_c(s)$  is unchanged) is shown in Figure P12.4.

(d) The disturbance response is shown in Figure P12.4.



**FIGURE P12.4**

(a) Step response:  $K_1 = 1$  (solid line) and  $K_1 = 2$  (dashed line); and (b) disturbance response.

**P12.5** (a) The plant is given by

$$G(s) = \frac{1}{s(s + p)}$$

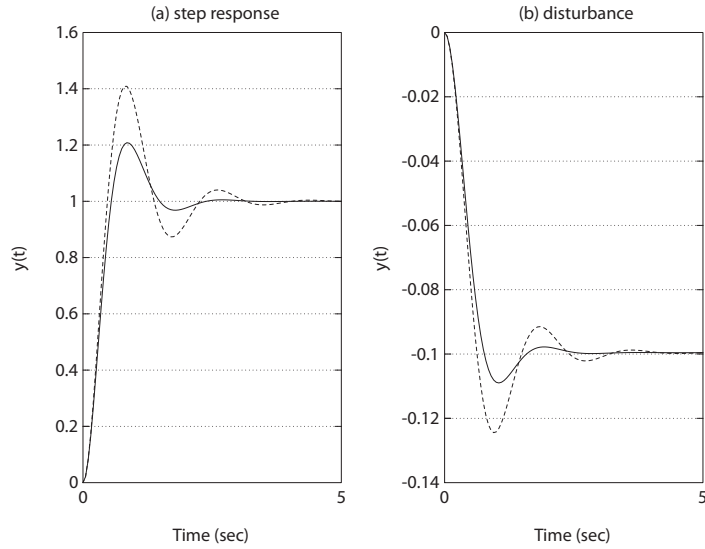
where  $p = 2$ , nominally. One solution is

$$G_c(s) = \frac{18.7(s + 2.9)}{(s + 5.4)} .$$

Then,

$$T(s) = \frac{18.7(s + 2.9)}{(s + 3.41)(s + 2 + 2\sqrt{3}j)(s + 2 - 2\sqrt{3}j)} .$$

- (b,d) The step responses are shown in Figure P12.5 for  $p = 2$  and  $p = 1$ .  
 (c,d) The disturbance responses are shown in Figure P12.5 for  $p = 2$  and  $p = 1$ .



**FIGURE P12.5**

(a) Step response:  $p = 2$  (solid line) and  $p = 1$  (dashed line); and (b) disturbance response:  $p = 2$  (solid line) and  $p = 1$  (dashed line).

**P12.6** (a) The plant is given by

$$G(s) = \frac{1}{s(s^2 + 4s + 5)},$$

and the PID controller is

$$G_c(s) = \frac{K(s+z)^2}{s}.$$

When

$$z = 1.25$$

and

$$K = 4,$$

all roots are

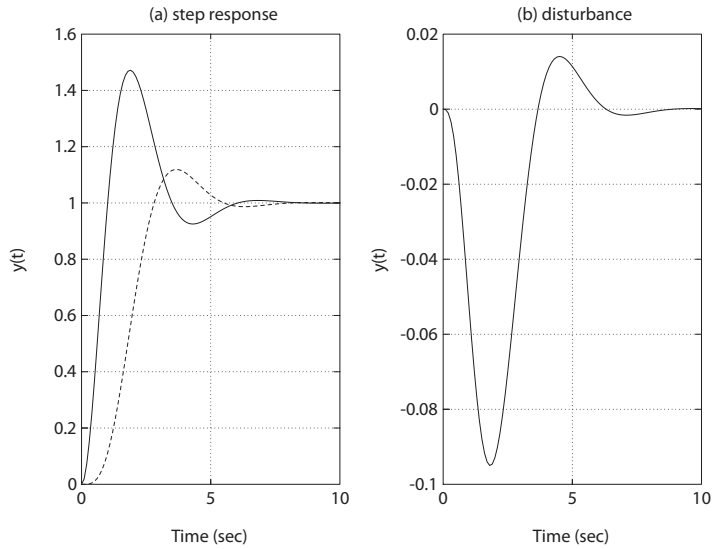
$$s = -1 \pm j1.22.$$

Then, the closed-loop transfer function is

$$T(s) = \frac{4(s + 1.25)^2}{s^4 + 4s^3 + 9s^2 + 10s + 6.25} .$$

(b,c) The step responses with and without a prefilter are shown in Figure P12.6.

(d) The disturbance response is shown in Figure P12.6.



**FIGURE P12.6**

(a) Step response: w/o prefilter (solid line) and w/prefilter (dashed line); and (b) disturbance response.

**P12.7** (a) The loop transfer function is

$$G_c(s)G(s) = \frac{10K_a(5s + 500 + 0.0475s^2)}{s^3} .$$

When

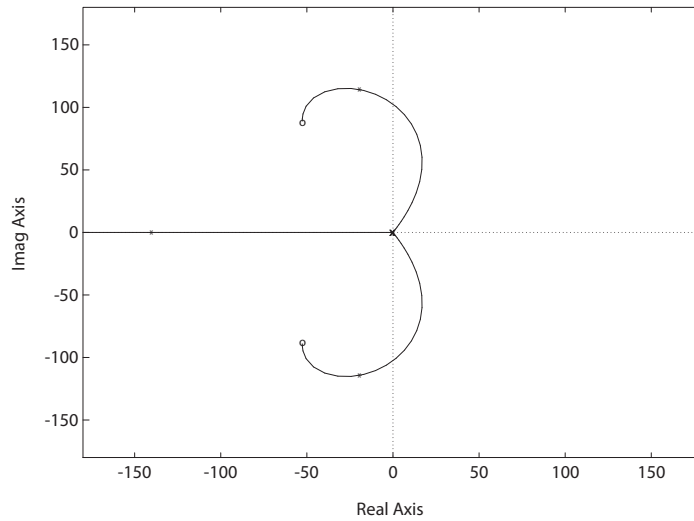
$$K_a = 374.5 ,$$

the phase margin is

$$P.M. = 40^\circ .$$

(b) The root locus is shown in Figure P12.7a.



**FIGURE P12.7**

(a) Root locus for  $1 + K_a \frac{10(0.0475s^2 + 5s + 500)}{s^3} = 0$ .

When

$$K_a = 374.5 ,$$

the roots are

$$\begin{aligned} s_1 &= -139.8 \\ s_{2,3} &= -19.1 \pm j114.2 . \end{aligned}$$

(c) The transfer function from  $T_d(s)$  to  $Y(s)$  is

$$\frac{Y(s)}{T_d(s)} = \frac{-s}{s^3 + 182s^2 + 19150s + 1915000} .$$

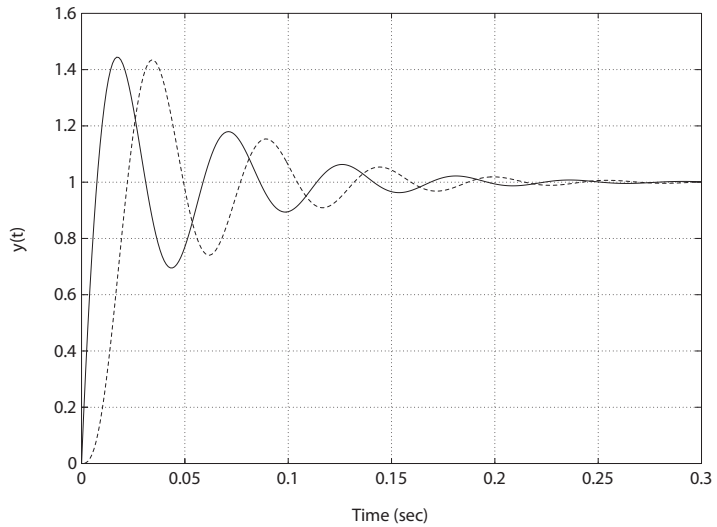
The maximum is

$$\max |y(t)| = 0.0000389 .$$

(d) The step responses, with and without a prefilter, are shown in Figure P12.7b.

**P12.8** The polynomial under investigation is

$$s^3 + 3s^2 + 3s + 4 = 0 .$$

**FIGURE P12.7**

CONTINUED: (b) Step response: w/o prefilter (solid line) and w/prefilter (dashed line).

From the uncertainty bounds on the coefficients, we define

$$\alpha_0 = 4 \quad \beta_0 = 5$$

$$\alpha_1 = 1 \quad \beta_1 = 4$$

$$\alpha_2 = 2 \quad \beta_2 = 4$$

Then, we must examine the four polynomials:

$$s^3 + 2s^2 + 4s + 5 = 0$$

$$s^3 + 4s^2 + s + 4 = 0$$

$$s^3 + 4s^2 + 4s + 4 = 0$$

$$s^3 + 2s^2 + s + 5 = 0$$

The fourth polynomial is not stable—therefore, the system is not stable for the uncertain parameters.

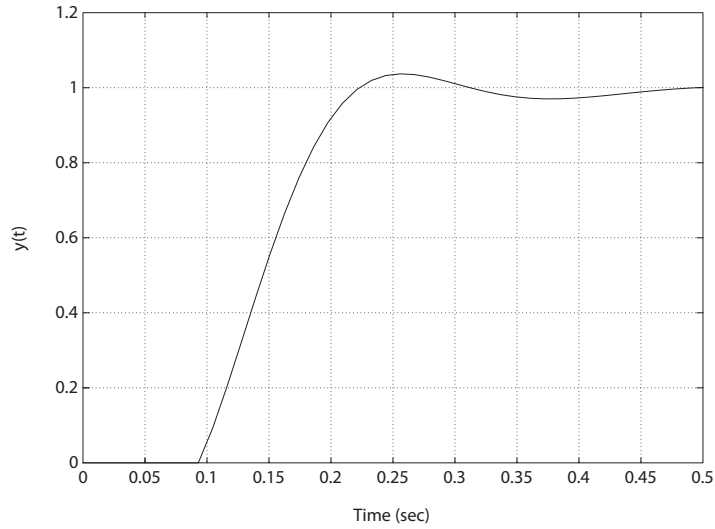
**P12.9** One possible PID controller is

$$G_c(s) = \frac{0.058s^2 + 2.17s + 16.95}{s} .$$

A first-order Pade approximation was used in the design to account for the delay system. The step input response is shown in Figure P12.9. A prefilter should also be used with the PID controller. A suitable prefilter

is

$$G_p(s) = \frac{K_2}{K_3s^2 + K_1s + K_2} .$$



**FIGURE P12.9** Step response with the PID controller and prefilter.

**P12.10** The PID controller is given by

$$G_c(s) = \frac{K_Ds^2 + K_Ps + K_I}{s} .$$

Using the ITAE method, we desire the characteristic polynomial to be

$$q(s) = s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3 = 0 ,$$

where we select  $\omega_n = 4$  to obtain a peak time of  $T_p = 1$  second. Here we use the approximation for ITAE third-order systems that  $\omega_n T_p \approx 4$  from Figure 5.30(c) in Dorf and Bishop. The actual characteristic equation is

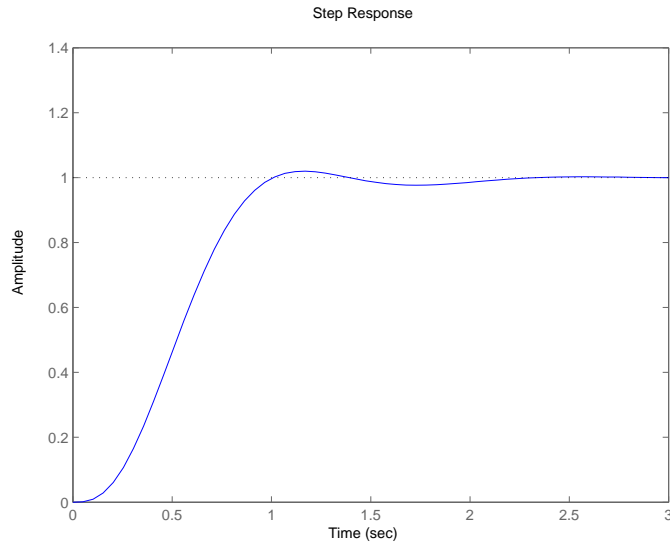
$$s^3 + 25K_Ds^2 + 25K_Ps + 25K_I = 0 .$$

Equating coefficients and solving for the gains yields

$$K_P = 1.376 , \quad K_D = 0.28 , \quad \text{and} \quad K_I = 2.56 .$$

The step response is shown in Figure P12.10, with the prefilter

$$G_p(s) = \frac{K_I}{K_D s^2 + K_P s + K_I} .$$



**FIGURE P12.10**  
Step response with the PID controller and prefilter.

**P12.11** We will design for the case where  $K = 1$  and  $p = 1$ . The design plant is

$$G(s) = \frac{1}{s(s+1)(s+4)} .$$

The nominal plant is given by

$$G(s) = \frac{2.5}{s(s+2)(s+4)} ,$$

and the PID controller is

$$G_c(s) = \frac{K_D s^2 + K_P s + K_I}{s} .$$

Using the ITAE method, we desire the characteristic polynomial to be

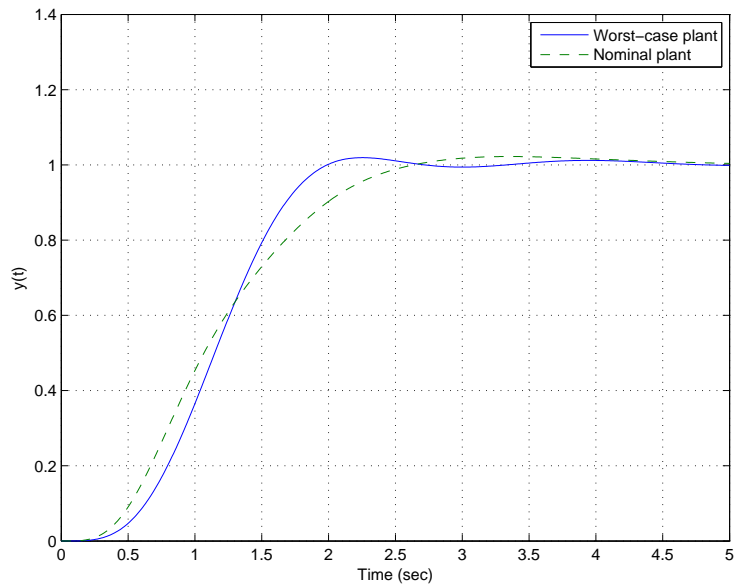
$$q(s) = s^4 + 2.1\omega_n s^3 + 3.4\omega_n^2 s^2 + 2.7\omega_n^3 s + \omega_n^4 = 0 ,$$

where we select  $\omega_n = 2.38$  to obtain a peak time around  $T_p = 3$  seconds. The actual characteristic equation (with the worst-case plant) is

$$s^4 + 5s^3 + (4 + K_D)s^2 + K_Ps + K_I = 0 .$$

Equating coefficients and solving for the gains yields  $K_P = 36.40$ ,  $K_I = 32.08$ , and  $K_D = 15.26$ . The step response is shown in Figure P12.11, with the prefilter

$$G_p(s) = \frac{K_I}{K_Ds^2 + K_Ps + K_I} .$$



**FIGURE P12.11** Step response with the prefilter: nominal plant (dashed line) & worst-case plant (solid line).

**P12.12** The transfer function is

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s & -3 \\ 5 & s + K \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{-6}{s^2 + Ks + 5} .$$

The sensitivity is

$$S_K^G = \frac{\partial G}{\partial K} \frac{K}{G} = \frac{-Ks}{s^2 + Ks + 5} .$$

## Advanced Problems

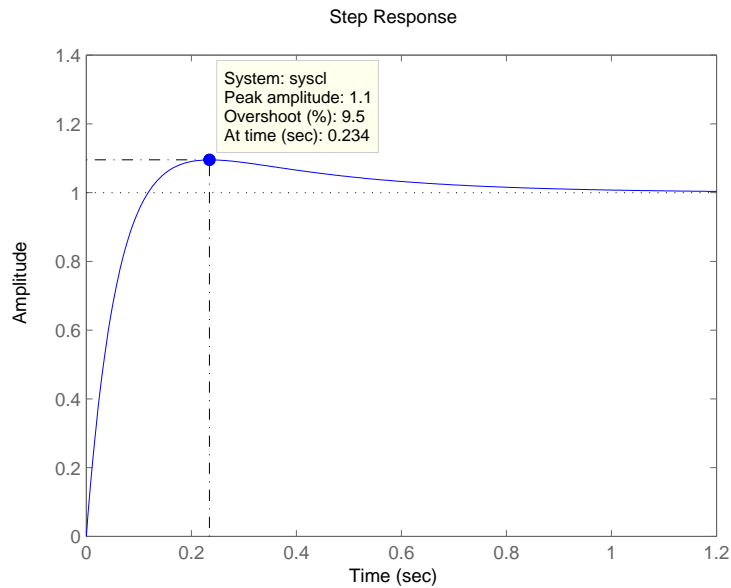
**AP12.1** Let  $G_p(s) = 1$ . A viable PID controller is

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{1000s^2 + 3000s + 100}{s} .$$

The loop transfer function is

$$G_c(s)G(s) = \frac{1000s^2 + 3000s + 100}{s(50s^2 + 1)} .$$

We can check that  $K_v = 100$ , as desired. The step response is shown in Figure AP12.1.



**FIGURE AP12.1**  
Step response with PID controller.

**AP12.2** For all three controllers, choose  $K = 1$  as the design value. Also, use as the nominal points  $a = 2$  and  $b = 5$  for each design. ITAE methods were employed in all designs, although this did not work well for the PI controller.

(a) **PI controller:** Let

$$G_p(s) = 1 .$$

Not all specifications could be met simultaneously with a PI con-

troller. The best over-all results are achieved when using  $a = 3$  and  $b = 4.5$  as the design values. An acceptable PI controller is

$$G_c(s) = 1.2 + \frac{3.96}{s} .$$

Controller	<i>P.O.</i>	$T_s$	$T_p$	$ u(t) _{max}$
PI	0%	2.29s	n.a.	4.43
PD	4.6%	1.72s	1.26s	12.25
PID	1.97%	0.65s	0.47s	37.25

TABLE AP12.2 PI, PD, and PID controller performance summary.

The final design is based on root locus methods since the ITAE methods did not produce an effective controller. The closed-loop transfer function is

$$T(s) = \frac{1.2s + 3.96}{s^3 + 3s^2 + 5.7s + 3.96} .$$

(b) **PD controller:** Let

$$G_p(s) = \frac{12.25}{7.25 + 2.9s} .$$

The closed-loop transfer function is

$$T(s) = \frac{7.25 + 2.9s}{s^2 + 4.9s + 12.25} ,$$

where the PD controller (based on ITAE methods) is

$$G_c(s) = 7.25 + 2.9s .$$

(c) **PID controller:** Let

$$G_p(s) = \frac{1000}{15.5s^2 + 210s + 1000} .$$

The closed-loop transfer function is

$$T(s) = \frac{15.5s^2 + 210s + 1000}{s^3 + 17.5s^2 + 215s + 1000} .$$

And the PID controller (based on ITAE methods) is

$$G_c(s) = \frac{15.5s^2 + 210s + 1000}{s} .$$

The performance of each controller is summarized in Table AP12.2.

**AP12.3** (a) The PID controller is

$$G_c(s) = \frac{K_D \left( s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D} \right)}{s} .$$

Since we want  $P.O. < 4\%$  and  $T_s < 1s$ , we choose the dominant closed-loop poles to have  $\omega_n = 6$  and  $\zeta = 0.8$ . Therefore, we place the zeros at

$$s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D} = s^2 + 10s + 36 .$$

Solving for the constants yields,

$$\frac{K_P}{K_I} = 10 , \quad \frac{K_I}{K_D} = 36 .$$

Then, using root locus methods, we choose  $K_D = 91$  to place the roots near the zeros. The PID controller gains are computed to be  $K_P = 910$ ,  $K_I = 3276$  and  $K_D = 91$ .

(b) The loop transfer function is

$$G_c(s)G(s) = \frac{K_D s^2 + K_P s + K_I}{s^2(s^2 + 5s + 4)} .$$

The closed-loop system characteristic equation is

$$s^3 + 5s^2 + 4s + K_D s^2 + K_P s + K_I = 0 .$$

Solving for the PID gains yields  $K_P = 73.4$ ,  $K_I = 216$  and  $K_D = 5.5$ .



Therefore, the controller is

$$G_c(s) = \frac{5.5(s^2 + 13.35s + 39.3)}{s}.$$

Using the prefilter

$$G_p(s) = \frac{39.3}{s^2 + 13.35s + 39.3},$$

we obtain the closed-loop transfer function

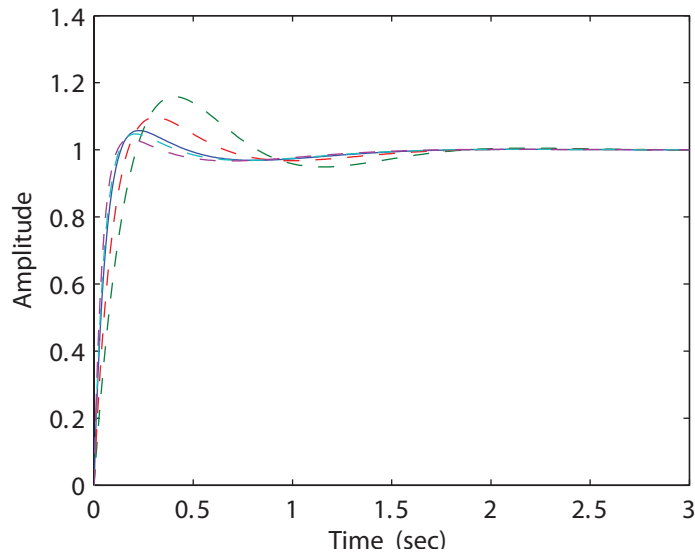
$$T(s) = \frac{216}{s^3 + 10.5s^2 + 77.4s + 216}.$$

The percent overshoot is  $P.O. \approx 3.5\%$  and the settling time is  $T_s \approx 1.67$  sec.

**AP12.4** The PID controller is

$$G_c(s) = \frac{K_D \left( s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D} \right)}{s}.$$

The bounds  $1 \leq a \leq 2$  and  $4 \leq b \leq 12$  imply that  $2 \leq \omega_n \leq 3.46$  and  $0.5 \leq \zeta\omega_n \leq 1$ . One solution is to place the PID controller zeros at



**FIGURE AP12.4**

Family of step response with PID controller with nominal case  $(a, b) = (1.5, 9)$  denoted by the solid line.

$s = -1 \pm j\sqrt{8}$  (i.e.  $\zeta\omega_n = 1$  and  $\omega_n = 3$ ). So,

$$s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D} = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 4s + 9 .$$

The **nominal case** for design is chosen to be  $a = 1.5$  and  $b = 9$ . Using root locus, we select  $K_D = 2.1$  to place the closed-loop characteristic roots near the zeros. Then, the PID controller gains are computed to be  $K_P = 8.4$ ,  $K_I = 18.9$ , and  $K_D = 2.1$ . The plot of the response to a step input is shown in Figure AP12.4. The **off-nominal cases** shown in the simulations are  $(a, b) = (1.2, 4)$ ,  $(1.4, 6)$ ,  $(1.6, 10)$ , and  $(1.8, 12)$ .

**AP12.5** To obtain a phase margin of  $P.M. = 49.77^\circ$ , select  $K = 1.5$ ,  $b = 36$  and choose  $G_p(s) = 1$ . The PID controller is

$$G_c(s) = \frac{1.5(s^2 + 20s + 36)}{s} .$$

When  $K_1 = 0.75$ , the phase margin is reduced to  $P.M. = 45.45^\circ$ ; and when  $K_1 = 1.25$ , the phase margin is increased to  $P.M. = 52.75^\circ$ .

**AP12.6** With the settling time  $T_s = 1$  and percent overshoot  $P.O. < 10\%$  specifications, we target for dominant closed-loop poles with  $\omega_n = 10$ . Here we estimate  $\omega_n T_s \approx 10$  associated with the ITAE performance. The closed-loop transfer function is

$$T(s) = G_p(s) \frac{1.5(K_D s^2 + K_P s + K_I)}{(1 + 1.5K_D)s^2 + 1.5K_P s + 1.5K_I} ,$$

where we have neglected  $\tau$ . Using the ITAE method, the desired characteristic polynomial is

$$s^2 + \sqrt{2}\omega_n s + \omega_n^2 = s^2 + \frac{1.5K_p}{1 + 1.5K_D}s + \frac{1.5K_I}{1 + 1.5K_D} .$$

Let  $K_D = 0.25$ . Then solving for the remaining PID gains yields  $K_P = 12.96$  and  $K_I = 91.67$ . The pre-filter is

$$G_p(s) = \frac{137.5}{0.375s^2 + 19.45s + 137.5} .$$

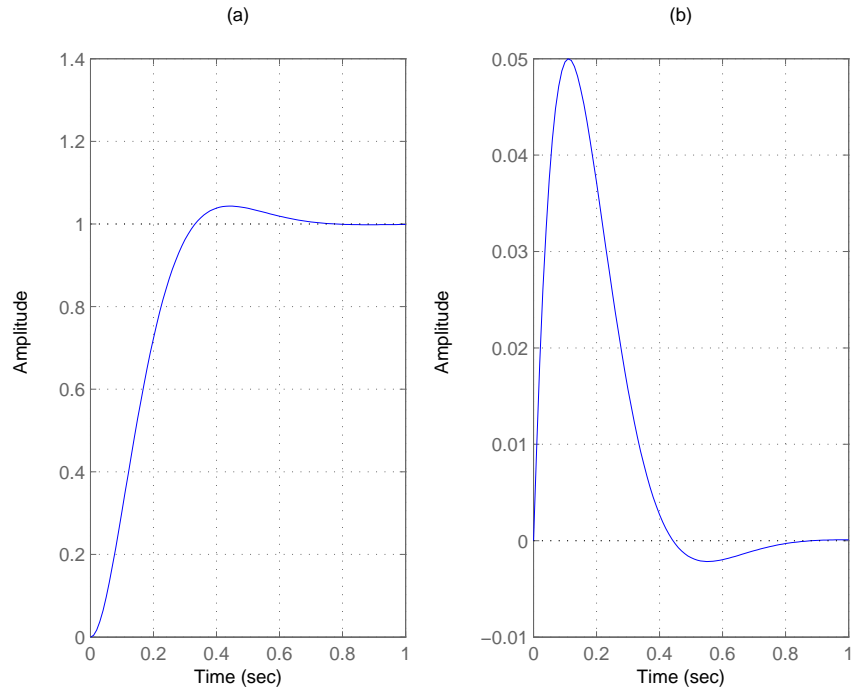
Then the closed-loop transfer function (with  $\tau = 0.001$ ) is

$$T(s) = \frac{137.5}{0.001s^3 + 1.375s^2 + 19.45s + 137.5} .$$

The transfer function from the disturbance to the output is

$$Y(s)/T_d(s) = \frac{1.5s}{0.001s^3 + 1.375s^2 + 19.45s + 137.5} .$$

The step input response and disturbance response are shown in Figure AP12.6.



**FIGURE AP12.6**  
(a) Input response; (b) Disturbance response.

**AP12.7** The PI controller is given by

$$G_c(s) = \frac{K_P s + K_I}{s} .$$

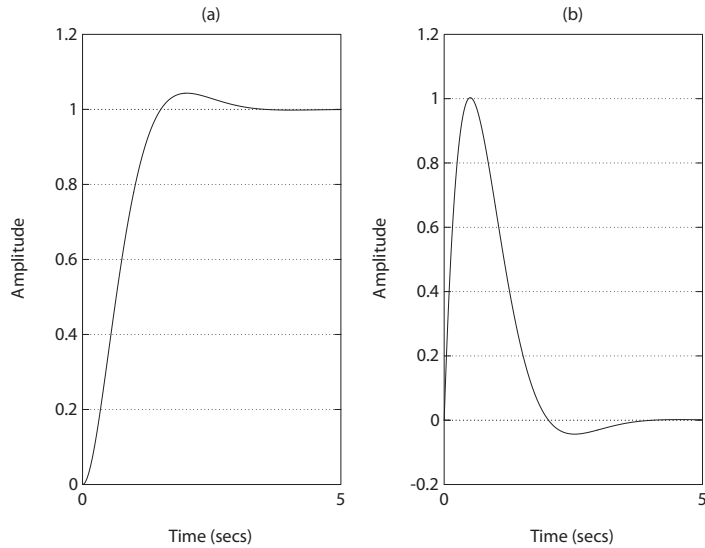
We will also use the prefilter

$$G_p(s) = \frac{K_I}{K_P s + K_I} .$$

Using the ITAE method, we determine that

$$K_P = \sqrt{2}\omega_n \quad \text{and} \quad K_I = \omega_n^2 .$$

Let  $\omega_n = 2.2$ . Then  $K_P = 3.11$  and  $K_I = 4.8$ . The step response and control  $u(t)$  are shown in Figure AP12.7.

**FIGURE AP12.7**

(a) Input response; (b) Control history  $u(t)$ .

**AP12.8** (a) A suitable **PD controller** is given by

$$G_c(s) = 0.6 + 0.4s .$$

The percent overshoot is  $P.O. = 18.8\%$  and the peak time is  $T_p = 2.4$  sec.

(b) A suitable **PI controller** is given by

$$G_c(s) = 0.15 + \frac{0.01}{s} .$$

The percent overshoot is  $P.O. = 23.7\%$  and the peak time is  $T_p = 7.8$  sec.

(c) A suitable **PID controller** is given by

$$G_c(s) = 0.6 + \frac{0.01}{s} + 0.4s .$$

The percent overshoot is  $P.O. = 19.9\%$  and the peak time is  $T_p = 2.5$  sec.

(d) The PD or PID controllers are the best choices.

**AP12.9** A robust PID controller designed with ITAE methods will be a suitable controller. From the settling time specification we select  $\omega_n = 10$ , where

we have used  $\zeta = 0.8$ . The worst case is

$$a = 1 \quad \text{and} \quad K = 2 .$$

The desired closed-loop transfer function is

$$T(s) = \frac{\omega_n^3}{s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3}$$

and the actual characteristic equation is

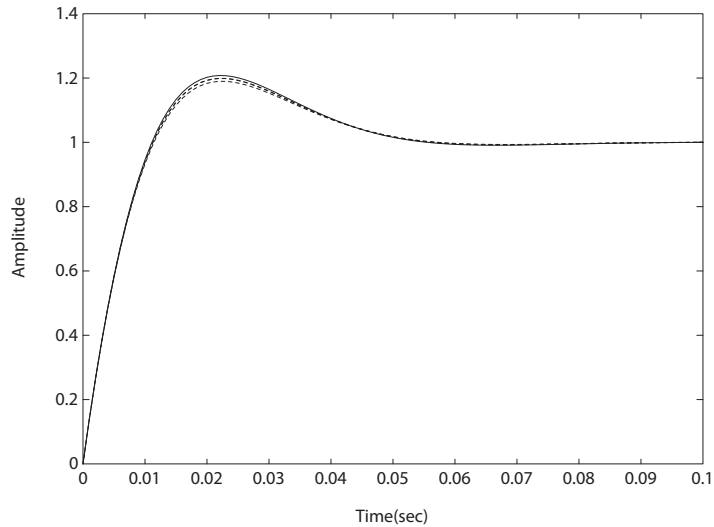
$$q(s) = s^3 + (2a + KK_D)s^2 + (a^2 + KK_P)s + KK_I .$$

Equating like terms, we find that

$$K_P = 107 \quad K_I = 500 \quad K_D = 7.75 .$$

**AP12.10** We use as the design plant

$$G(s) = \frac{s + 2}{s(s + 3)} .$$



**FIGURE AP12.10**

Family of step responses with the design plant  $(p, q, r) = (3, 0, 2)$  denoted by the solid line.

Select

$$p_1 = 2 \quad \text{and} \quad z_1 = 3$$

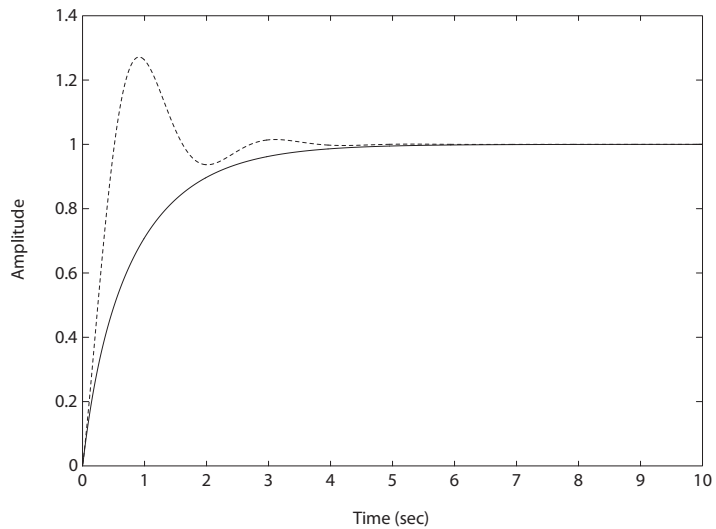
to cancel a design plant pole and zero. Then, choose  $p_2 = 0$  to have zero steady-state error to a unit step. The remaining variables  $K$  and  $z_2$  are selected based on ITAE methods, where  $\omega_n = 100$ . A suitable compensator is

$$G_c(s) = \frac{141.42(s + 3)(s + 70.71)}{s(s + 2)} .$$

A plot of the step responses for various values of  $p, q$  and  $r$  is shown in Figure AP12.10.

**AP12.11** A suitable compensator is

$$G_c(s) = \frac{1000(s + 1.8)(s + 3.5)(s + 5.5)}{s(s + 600)} .$$



**FIGURE AP12.11**

Step responses with nominal plant (solid line) and off-nominal plant with all poles reduced by 50% (dashed line).

## Design Problems

**CDP12.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$\frac{\theta(s)}{V_a(s)} = \frac{26.035}{s(s + 33.142)},$$

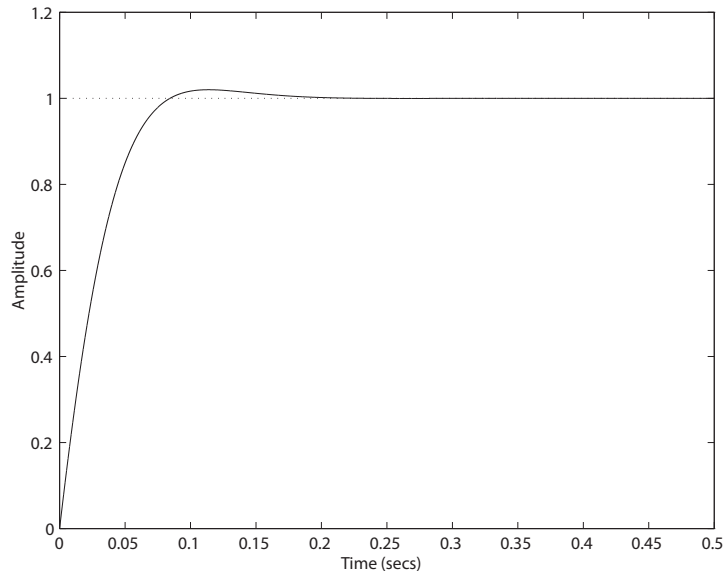
where we neglect the motor inductance  $L_m$  and where we switch off the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). With a PID controller, the closed-loop system characteristic equation is

$$s^3 + (33.142 + 26.035K_D)s^2 + 26.035K_Ps + 26.035K_I = 0.$$

A suitable PID controller is

$$G_c(s) = 50 + s + \frac{0.1}{s}.$$

This PID controller places the closed-loop system poles to the left of the  $-\zeta\omega_n$  line necessary to meet the settling time requirement. The step response is shown below. The settling time is  $T_s = 0.12$  second. In the steady-state the error due to a step disturbance is zero.



**DP12.1** The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K_m G_c(s)}{s^2 + (2 + K_m K_1)s + G_c(s)K_m} .$$

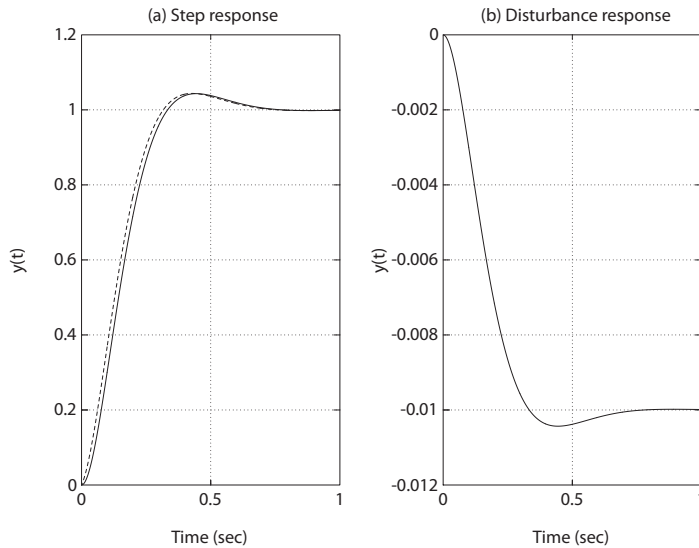
(a) When  $G_c = K$ , we have

$$T(s) = \frac{15K}{s^2 + (2 + 15K_1)s + 15K} ,$$

where  $K_m = 15$ . Using ITAE criteria and  $\omega_n = 10$ , we determine that  $K_1 = 0.81$  and  $K = 6.67$ . For the disturbance, we have

$$\frac{Y(s)}{T_L(s)} = \frac{-1}{s^2 + 14.14s + 100} .$$

The input and disturbance responses are shown in Figure DP12.1, without prefilters.



**FIGURE DP12.1**

(a) Step response:  $G_c(s) = K$  (solid line) and  $G_c(s) = K_P + K_D s$  (dashed line); and (b) disturbance response (same for both compensators).

(b) When  $G_c = K_P + K_D s$ , we have

$$\frac{Y(s)}{R(s)} = \frac{15(K_P + K_D s)}{s^2 + (2 + 15K_1 + 15K_D)s + 15K_P} .$$

For  $\omega_n = 10$  and with the ITAE criteria, we determine that (with



$$K_D = 0.1)$$

$$\frac{Y(s)}{R(s)} = \frac{15(6.67 + 0.1s)}{s^2 + 14.14s + 100} .$$

**DP12.2** The nominal plant is given by

$$G(s) = \frac{1}{s(s + 5)} .$$

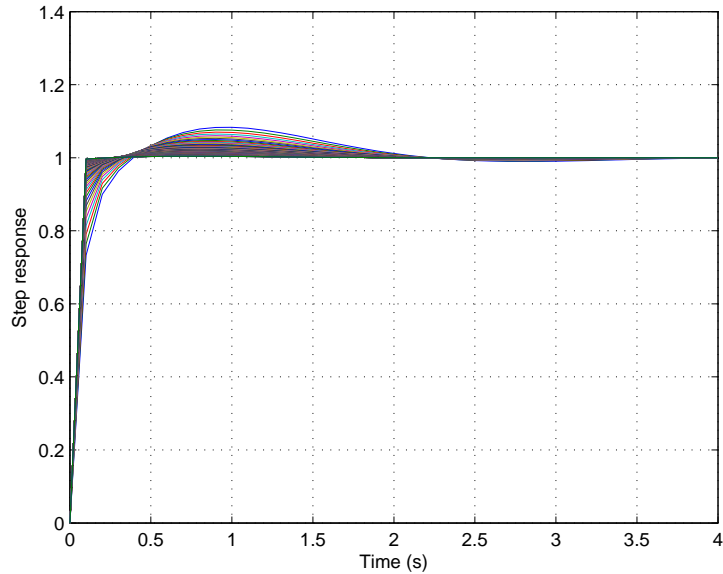
The closed-loop transfer function is

$$T(s) = \frac{K(K_D s^2 + K_P s + K_I)}{s^3 + (5 + K K_D) s^2 + K K_P s + K K_I} .$$

Let

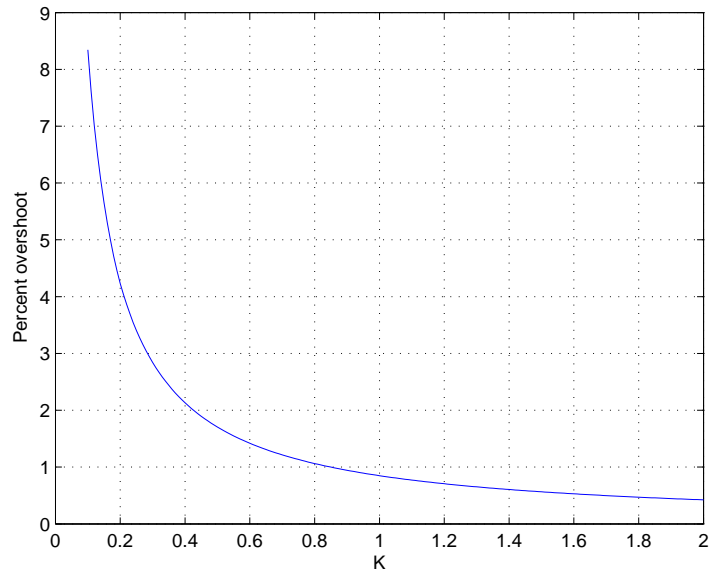
$$K_P = 450 , \quad K_I = 750 , \quad \text{and} \quad K_D = 150 .$$

A family of responses is shown in Figure DP12.2 a for various values of  $K$ . The percent overshoot for  $0.1 \leq K \leq 2$  is shown in Figure DP12.2b.



**FIGURE DP12.2**

(a) Family of step responses for various values of  $K$ .

**FIGURE DP12.2**CONTINUED: (b) Percent overshoot for various values of  $K$ .

**DP12.3** (a) The dexterous hand model is given by

$$G(s) = \frac{K_m}{s(s+5)(s+10)},$$

where  $K_m = 1$ , nominally. The PID controller is

$$G_c(s) = \frac{K_D(s^2 + 6s + 18)}{s}.$$

The root locus is shown in Figure DP12.3a. If we select

$$K_D = 90,$$

the roots are

$$s_{1,2} = -5.47 \pm j6.6$$

$$s_{3,4} = -2.03 \pm j4.23.$$

Thus, all roots have

$$\zeta\omega_n > 4/3$$

to meet the design specification

$$T_s < 3 \text{ sec} .$$

(b) The step responses for

$$K_m = 1$$

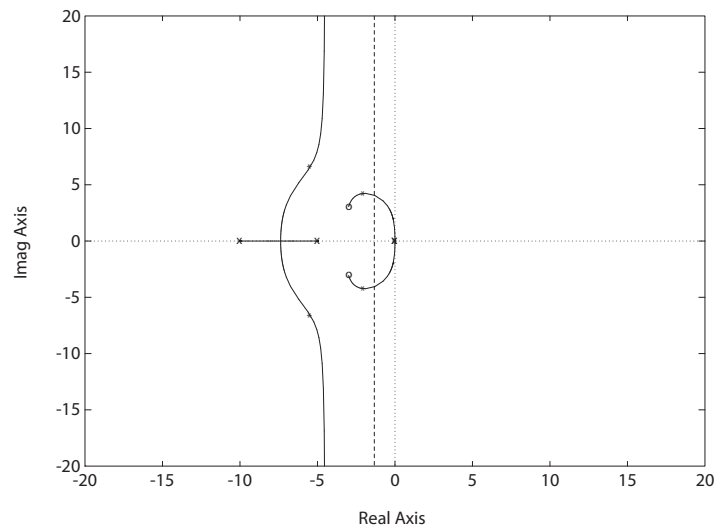
and

$$K_m = 1/2$$

are shown in Figure DP12.3b. When

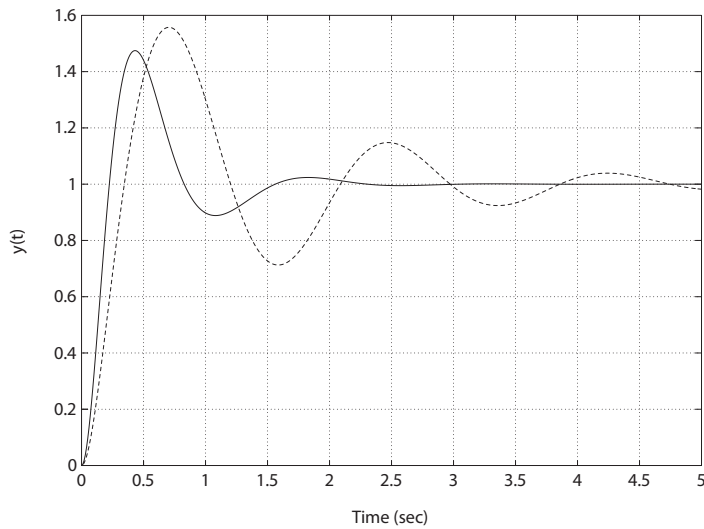
$$K = 1/2 ,$$

an off-nominal value, the settling time specification is no longer satisfied.



**FIGURE DP12.3**

(a) Root locus for  $1 + K_D \frac{s^2 + 6s + 18}{s^2(s+5)(s+10)} = 0$ .

**FIGURE DP12.3**

CONTINUED: (b) Step response (without prefilters): PID with  $K_3 = 90$  and  $K_m = 1$  (solid line) and PID with  $K_3 = 90$  and  $K_m = 0.5$  (dashed line).

**DP12.4** The nominal plant is

$$G(s) = \frac{17640}{s(s^2 + 59.4s + 1764)},$$

and the PID controller is

$$G_c(s) = \frac{K_I(\tau_1 s + 1)(\tau_2 s + 1)}{s}.$$

(a) Using ITAE methods, we determine that  $\omega_n = 28.29$ ,  $K_I = 36.28$ ,  $\tau_1 + \tau_2 = 0.0954$  and  $\tau_1\tau_2 = 0.00149$ . So,

$$G_c(s) = \frac{36.28(0.00149s^2 + 0.0954s + 1)}{s}.$$

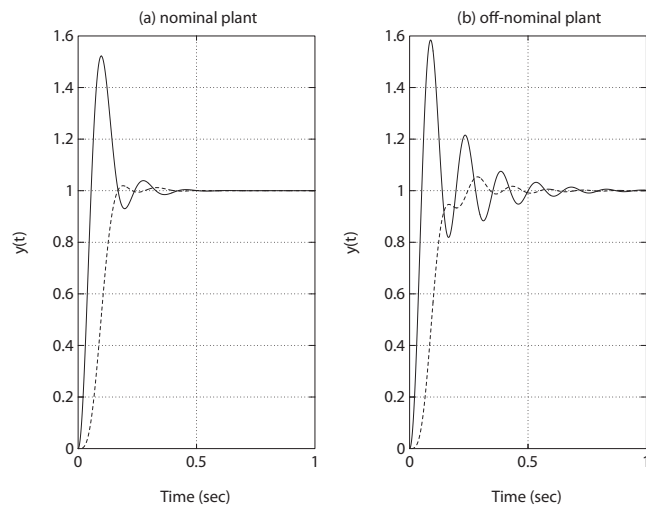
(b) The step response for the nominal plant and the PID controller is shown in Figure DP12.4a, with and without a prefilter.

(c) The disturbance response is shown in Figure DP12.4b.

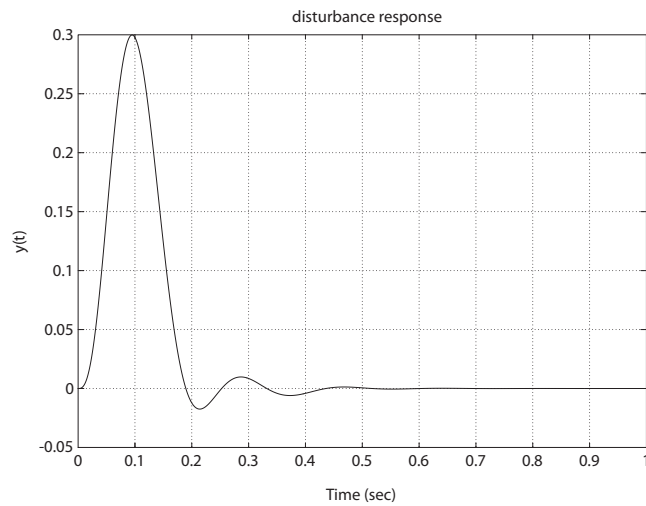
(d) The off-nominal plant is

$$G(s) = \frac{16000}{s(s^2 + 40s + 1600)}.$$

The step response for the off-nominal plant is shown in Figure DP12.4a.

**FIGURE DP12.4**

(a) Step response for (i) nominal plant: w/o prefilter (solid line) and w/prefilter (dashed line); and (ii) for off-nominal plant: w/o prefilter (solid line) and w/prefilter (dashed line).

**FIGURE DP12.4**

CONTINUED: (b) Disturbance response for the nominal plant.

**DP12.5** One possible solution is

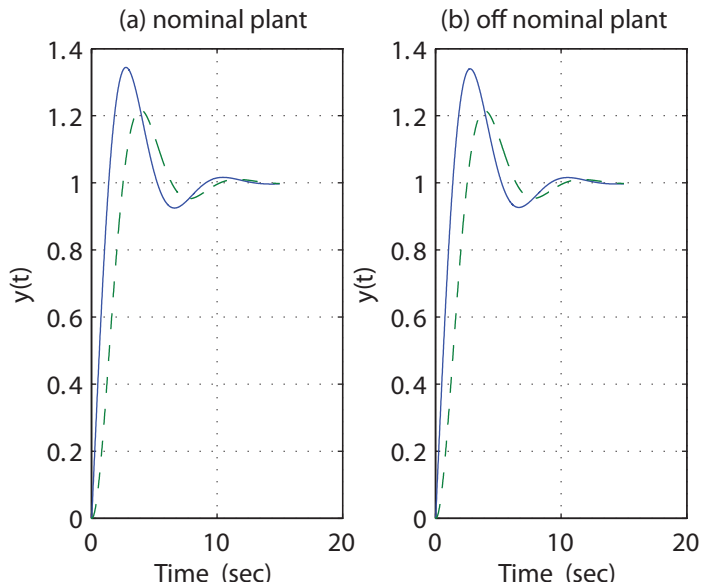
$$G_c(s) = 0.08 \frac{(0.01s + 1)(0.99s + 1)}{s} .$$

The phase margin with this controller is

$$P.M. = 45.5^\circ .$$

The step response is shown in Figure DP12.5 for the nominal plant (with and without a prefilter); the step response for the off-nominal plant is also shown in Figure DP12.5. The prefilter is

$$G_p(s) = \frac{1411}{13.97s^2 + 1411s + 1411} .$$



**FIGURE DP12.5**

(a) Step response for nominal plant: w/o prefilter (solid line) and w/prefilter (dashed line); and (b) for off-nominal plant: w/o prefilter (solid line) and w/prefilter (dashed line).

**DP12.6** Using ITAE methods, three controllers are designed for the nominal plant:

(i) PID controller:

$$G_c(s) = \frac{0.225s^2 + 0.535s + 34.3}{s}$$

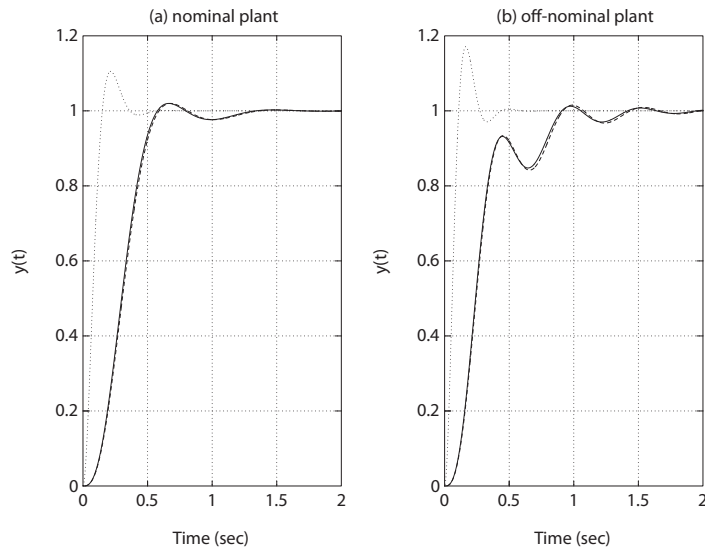
(ii) PI controller:

$$G_c(s) = \frac{0.9s + 22.5}{s}$$

(iii) PD controller:

$$G_c(s) = 0.9s + 22.5$$

The step responses for each controller is shown in Figure DP12.6. The responses for the PID and PI controller are the same since the gains were selected to obtain the same ITAE characteristic equation. An appropriate prefilter is used in all cases.



**FIGURE DP12.6**

(a) Step response for **nominal** plant: PID (solid line); PI (dashed line); and PD (dotted line); (b) for **off-nominal** plant: PID (solid line); PI (dashed line); and PD (dotted line).

**DP12.7** The loop transfer function is

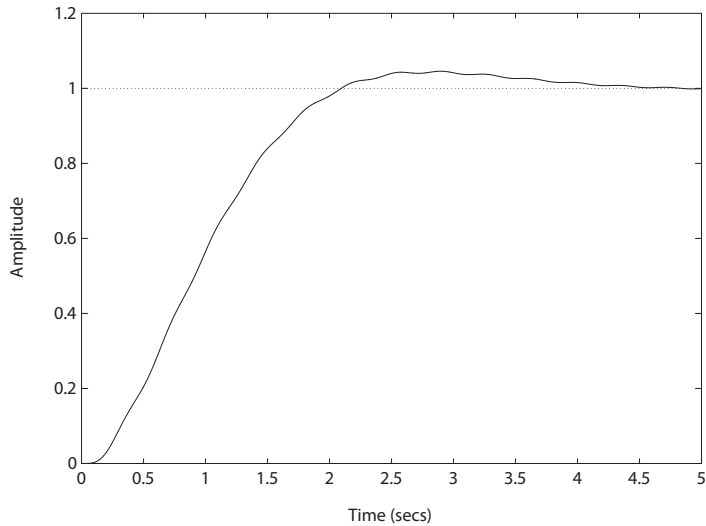
$$G(s) = \frac{K_a K_m}{(0.5s + 1)(\tau_f s + 1)s(s + 1)} = \frac{K}{s(s + 2)(s + 1)}$$

since  $\tau_f$  is negligible. A suitable PID controller is

$$G_c(s) = \frac{K K_D (s^2 + as + b)}{s} = \frac{300(s^2 + 2.236s + 2.5)}{s}$$

The step response is shown in Figure DP12.7. The percent overshoot is

$P.O. = 4.6\%$  and the settling time is  $T_s = 3.74$  seconds.



**FIGURE DP12.7**

Step response for the elevator position control.

**DP12.8** The system transfer function is

$$Y(s) = \left[ \frac{G(s)G_c(s)G_p(s)}{1 + G(s)G_c(s)} \right] R(s) .$$

We are given

$$G(s) = e^{-sT} \quad \text{where } T = 1 \text{ second} .$$

Using a second-order Pade approximation yields

$$G(s) \approx \frac{s^2 - 6s + 12}{s^2 + 6s + 12} .$$

Three controllers that meet the specifications are

$$G_{c1}(s) = \frac{0.5}{s} \quad (\text{Integral controller})$$

$$G_{c2}(s) = \frac{0.04s + 0.4}{s} \quad (\text{PI controller})$$

$$G_{c3}(s) = \frac{0.01s^2 + 0.04s + 0.4}{s} \quad (\text{PID controller}) .$$



In all cases, the steady-state error is zero.

	Integral	PI	PID
$P.O.(\%)$	4.05	0	0
$T_s$ (sec)	6.03	6.12	6.02
$T_p$ (sec)	4.75	N/A	N/A
$ V(t) _{max}$ (volts)	1.04	1	1

The prefilter  $G_p(s) = 1$  is used in all designs. To compute the voltage, the transfer function is

$$V(s) = \frac{G_p(s)G_c(s)}{1 + G_c(s)G(s)}R(s) .$$

**DP12.9** The space robot transfer function is

$$G(s) = \frac{1}{s(s+10)} .$$

(a) Consider  $G_c(s) = K$ . Then

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{s^2 + 10s + K} .$$

We determine that  $K = 50.73$  for  $\zeta = 0.702$ . Thus, we expect  $P.O. < 4.5\%$ . So,

$$G_c(s) = 50.73 .$$

(b) Consider the PD controller

$$G_c(s) = K_P + K_D s .$$

Then

$$T(s) = \frac{K_P + K_D s}{s^2 + (10 + K_D)s + K_P} .$$

Using the ITAE method, we compute

$$K_P = 100 \quad \text{and} \quad K_D = 4 .$$

Thus,

$$G_c(s) = 4s + 100 ,$$

and the prefilter is

$$G_p(s) = \frac{100}{4s + 100} .$$

(c) Consider the PI controller

$$G_c(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} .$$

Then,

$$T(s) = \frac{K_P s + K_I}{s^3 + 10s^2 + K_P s + K_I} .$$

Using the ITAE method, we have

$$\omega_n = 5.7 \quad K_P = 70.2 \quad \text{and} \quad K_I = 186.59 .$$

Thus,

$$G_c(s) = 70.2 + 186.59/s ,$$

and the prefilter is

$$G_p(s) = \frac{186.59}{70.2s + 186.59} .$$

(d) Consider the PID controller

$$G_c(s) = \frac{K_D s^2 + K_P s + K_I}{s} .$$

Then,

$$T(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + 10s^2 + K_D s^2 + K_P s + K_I} .$$

Using the ITAE method with  $\omega_n = 10$ , we have

$$K_D = 7.5 \quad K_P = 215 \quad \text{and} \quad K_I = 1000 .$$

Thus,

$$G_c(s) = \frac{7.5s^2 + 215s + 1000}{s} ,$$

and the prefilter is

$$G_p(s) = \frac{1000}{7.5s^2 + 215s + 1000} .$$

A summary of the performance is given in Table DP12.9.

$G_c(s)$	$P.O.$	$t_p$	$t_s$	$y_{ss}$	$\max y(t) $
$K$	4.5%	0.62 s	0.84 s	0	0.026
PD	5.2%	0.39 s	0.56s	0	0.010
PI	1.98%	0.81 s	1.32s	0	0.013
PID	1.98%	0.46 s	0.75 s	0	0.004

TABLE DP12.9 A summary of performance to a disturbance input.

## Computer Problems

CP12.1 The closed-loop transfer function is

$$T(s) = \frac{8K}{s^2 + 2s + 8K},$$

and the sensitivity function,  $S_K^T$ , is

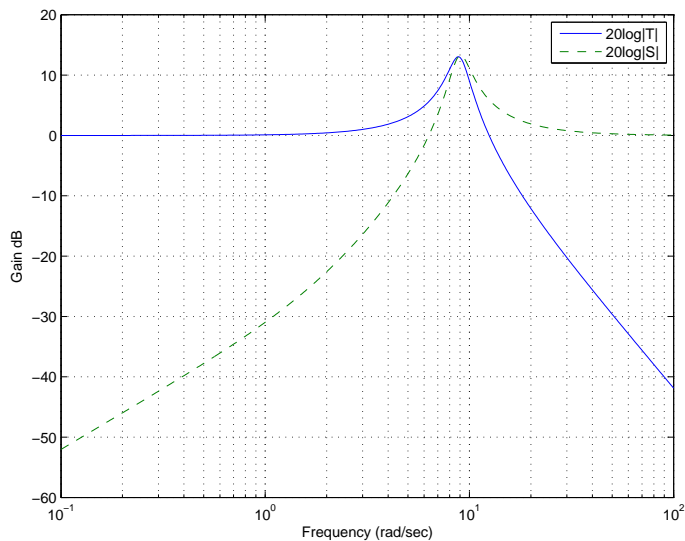
$$S(s) = \frac{s^2 + s}{s^2 + 2s + 8K}.$$

The plot of  $T(s)$  and  $S(s)$  is shown in Figure CP12.1, where  $K = 10$ .

```

nt=[80]; dt=[1 2 80]; syst = tf(nt,dt);
ns=[1 2 0]; ds=[1 2 80]; syss = tf(ns,ds);
w=logspace(-1,2,400);
[magt,phaset]=bode(syst,w);magtdB(1,:)= 20*log10(magt(1,1,:));
[mags,phases]=bode(syss,w); magsdB(1,:)= 20*log10(mags(1,1,:));
semilogx(w,magtdB,w,magsdB,'-')
legend('20log|T|','20log|S|')
xlabel('Frequency (rad/sec)')
ylabel('Gain dB')
grid

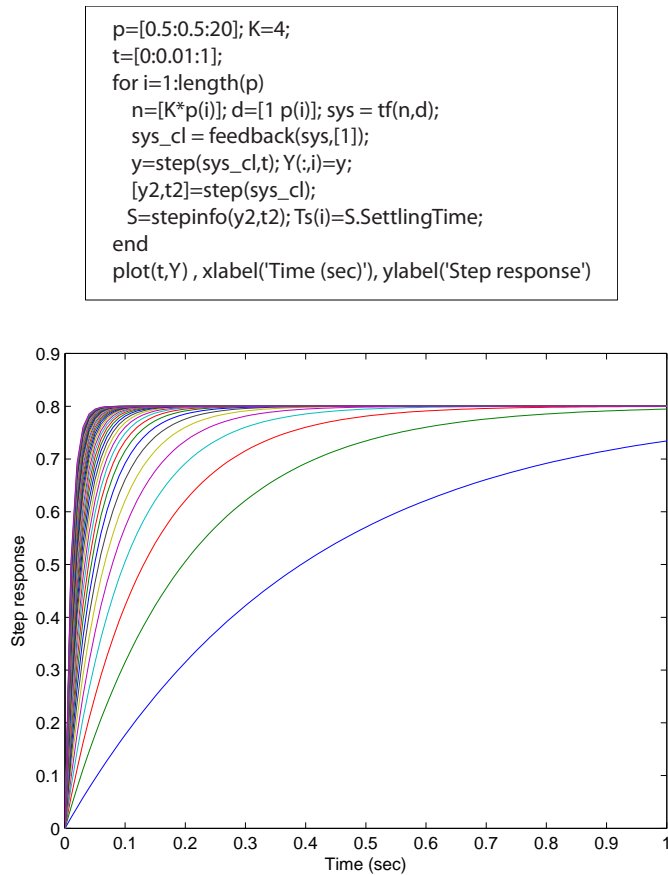
```



**FIGURE CP12.1**

Plot of  $T(s)$  and the sensitivity function  $S(s)$ .

**CP12.2** A reasonable value of the gain  $K = 4$ . The family of step responses is shown in Figure CP12.2.



**FIGURE CP12.2**  
Family of step responses for  $0.5 < p < 20$ .

**CP12.3** The closed-loop characteristic equation is

$$1 + K_D \frac{s^2 + as + b}{Js^3} = 0$$

where

$$a = K_P/K_D$$

$$b = K_I/K_D .$$

We select

$$a = 1 \quad \text{and} \quad b = 2$$

to move the root locus into the left hand-plane (see Figure CP12.3a). Then, we choose

$$K_D = 71$$

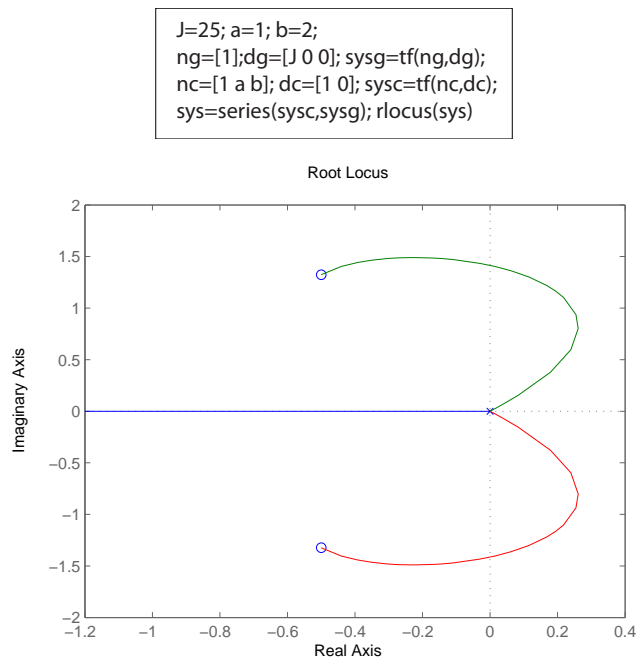
from the root locus using the **rlocfind** function. The closed-loop Bode plot in Figure CP12.3b verifies that the bandwidth

$$\omega_B < 5 \text{ rad/sec.}$$

Also, the phase margin is

$$P.M. = 45.7^\circ ,$$

which meets the design specification. The plot of phase margin versus  $J$  is shown in Figure CP12.3c. We see that as  $J$  increases, the phase margin decreases.



**FIGURE CP12.3**

(a) Root locus for  $1 + K_D \frac{s^2 + s + 2}{10s^3} = 0$ .

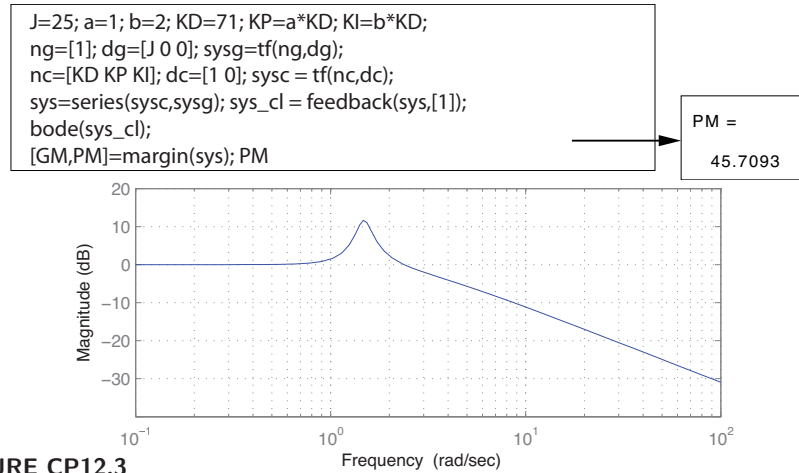


FIGURE CP12.3

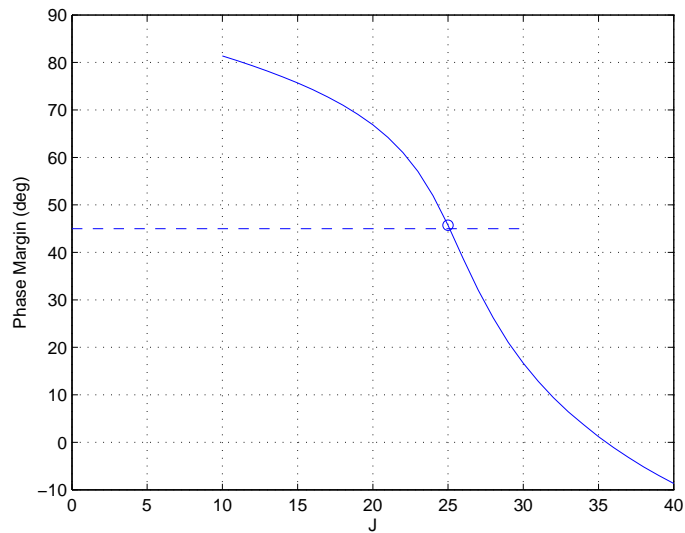
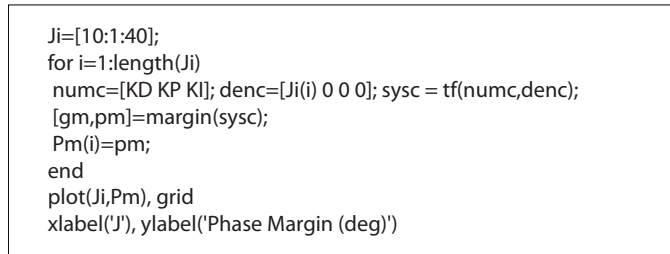
CONTINUED: (b) Closed-loop Bode plot with  $\omega_B < 5$  rad/sec.

FIGURE CP12.3

CONTINUED: (c) Phase margin versus  $J$ .

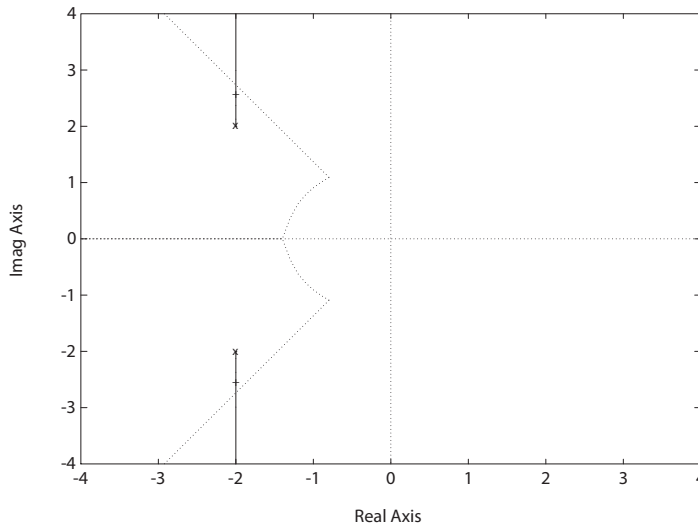
**CP12.4** The closed-loop characteristic equation is

$$1 + K \frac{1}{s^2 + bs + a} = 0$$

where  $a = 8$  and the nominal value of  $b = 4$ . The root locus is shown in Figure CP12.4a.

```
clf, hold off
a=8; b=4;
num=[1]; den=[1 b a]; sys = tf(num,den);
rlocus(sys), hold on
zeta=0.59; wn=1.35;
x=[-10:0.1:-zeta*wn]; y=-(sqrt(1-zeta^2)/zeta)*x;
xc=[-10:0.1:-zeta*wn]; c=sqrt(wn^2-xc.^2);
plot(x,y,':x,-y',':xc,c',':xc,-c',':')
rlocfind(sys)
```

Select a point in the graphics window  
 selected\_point =  
 -2.0165 + 2.5426i  
 ans =  
 2.4659 ← K



**FIGURE CP12.4**

(a) Root locus for  $1 + K \frac{1}{s^2 + 4s + 8}$ .

The performance region is specified by

$$\zeta = 0.59 \quad \text{and} \quad \omega_n = 1.35 ,$$

which derives from the design specifications

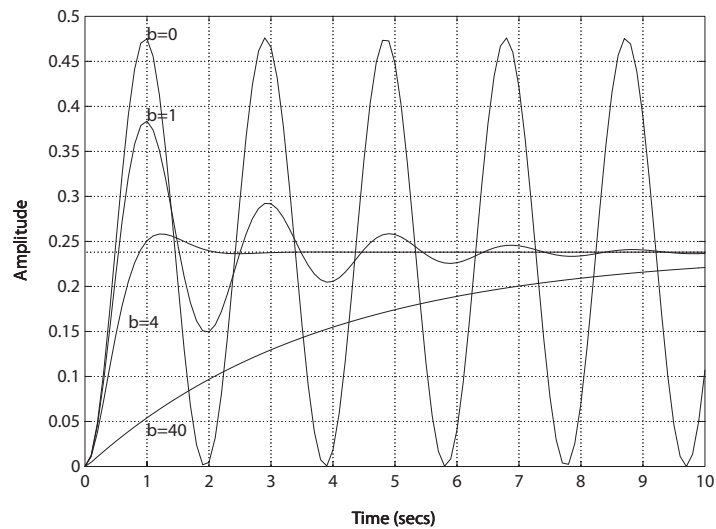
$$T_s < 5 \text{ sec} \quad \text{and} \quad P.O. < 10\% .$$



Using an m-file, the value of

$$K = 2.5$$

is selected with the **rlocfind** function. The step responses for  $b = 0, 1, 4$  and  $b = 40$  are shown in Figure CP12.4b. When  $b = 0$ , the system is marginally stable;  $b = 1$  results in a stable system with unsatisfactory performance. The nominal case  $b = 4$  is stable and all performance specs are satisfied. When  $b = 40$ , the system is heavily damped: the percent overshoot specification is satisfied, but the settling time is too long.



**FIGURE CP12.4**  
CONTINUED: (b) Step responses for  $b = 0, 1, 4$  and  $40$ .

- CP12.5** (a) An acceptable lead compensator (designed with root locus methods) is

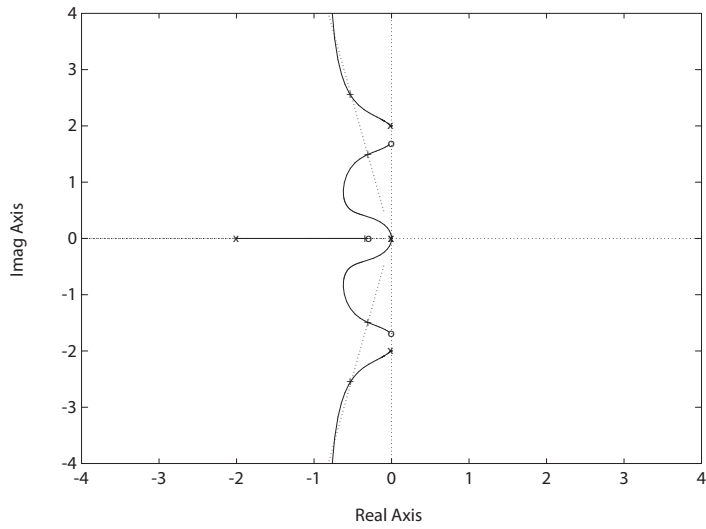
$$G_c(s) = K \frac{s + a}{s + b} = 5 \frac{s + 0.3}{s + 2} .$$

The compensated root locus is shown in Figure CP12.5a, where

$$K = 5$$

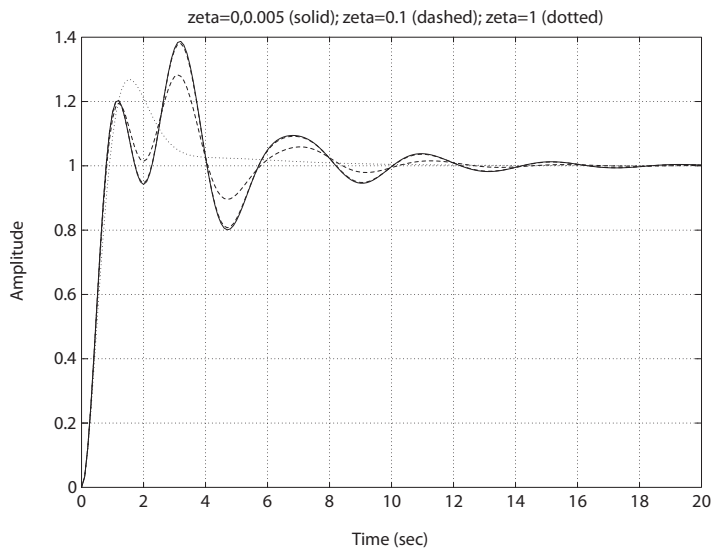
is selected to place the closed-loop poles in the performance region.

- (b) The step responses for  $\zeta = 0, 0.005, 0.1$  and  $1$  are shown in Figure CP12.5b.



**FIGURE CP12.5**  
(a) Compensated root locus.

(c) You would like the actual structural damping to be greater than the design value, if it must be different at all.



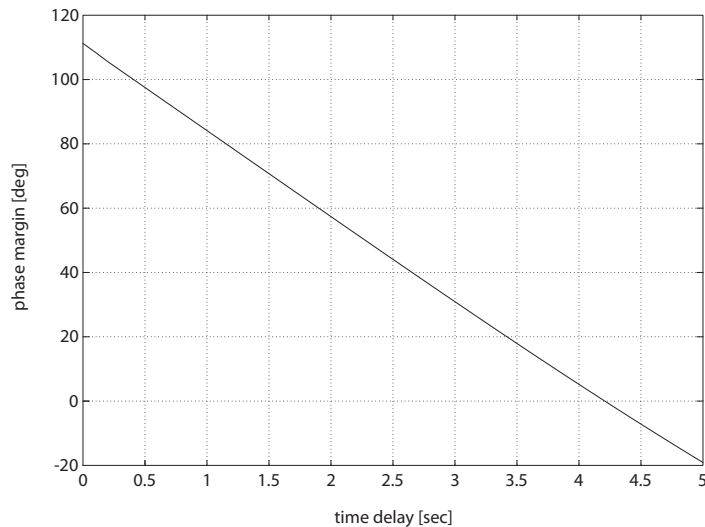
**FIGURE CP12.5**  
CONTINUED: (b) Step responses for  $\zeta = 0, 0.005, 0.1$  and  $1$ .

**CP12.6** The m-file script which computes the phase margin as a function of the time delay (using the **pade** function) is shown in Figure CP12.6. The maximum time delay (for stability) is  $t_d = 4.3$  seconds.

```

K=5;
numg=K*[1]; deng=[1 10 2]; sysg = tf(numg,deng);
td=[0:0.1:5]; ← time delay vector
for i=1:length(td)
[ndelay,ddelay]=pade(td(i),2); sysd = tf(ndelay,ddelay);
sys = series(sysg,sysd);
[mag,phase,w]=bode(sys);
[gm,pm,w1,w2]=margin(mag,phase,w);
pmv(i)=pm;
end
plot(td,pmv), grid
xlabel('time delay [sec]')
ylabel('phase margin [deg]')

```



**FIGURE CP12.6**  
Phase margin versus time delay.

**CP12.7** The m-file script is shown in Figure CP12.7a. The steady-state error (shown in Figure CP12.7b) is zero when

$$a = 0.5$$

and increases rapidly as  $a$  increases past  $a = 0.5$ . The maximum initial undershoot is shown in Figure CP12.7c. As  $a$  increases, the initial undershoot increases linearly. The gain margin is shown in Figure CP12.7d. It

can be seen that as  $a$  increases, the gain margin decreases very rapidly.

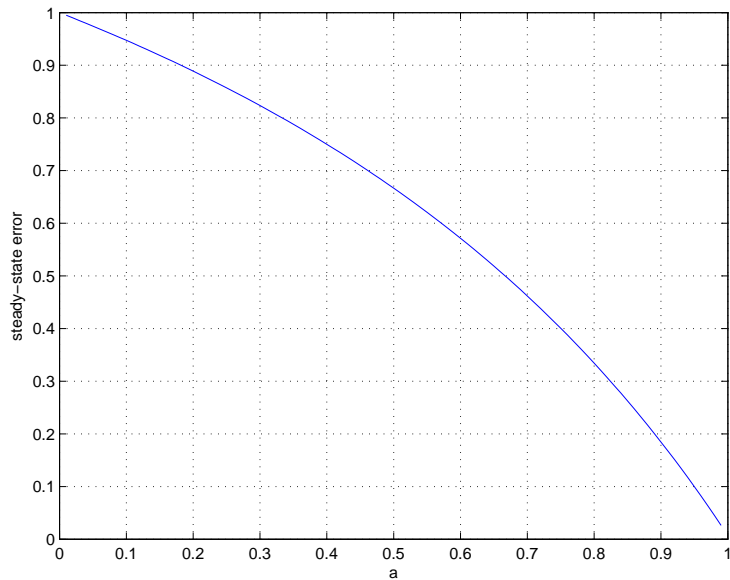
```

a=[0.01:0.01:0.99];
t=[0:0.1:30];
for i=1:length(a)
    num=a(i)*[1 -0.5]; den=[1 2 1]; sys_o = tf(num,den);
    [mag,phase,w]=bode(sys_o);
    [gm,pm,w1,w2]=margin(mag,phase,w);
    gmv(i)=gm; ← gain margin
    sys_cl = feedback(sys_o,[1]);
    [y,x]=step(-sys_cl,t); ← negative unit step input
    yf(i)=1-y(length(t)); ← steady-state tracking error
    ym(i)=-min(y)*100; ← max initial undershoot
end
figure(1), plot(a,gmv), grid, xlabel('a'), ylabel('gm')
figure(2), plot(a,yf), grid, xlabel('a'), ylabel('steady-state error')
figure(3), plot(a,ym), grid, xlabel('a'), ylabel('maximum initial undershoot [%]')

```

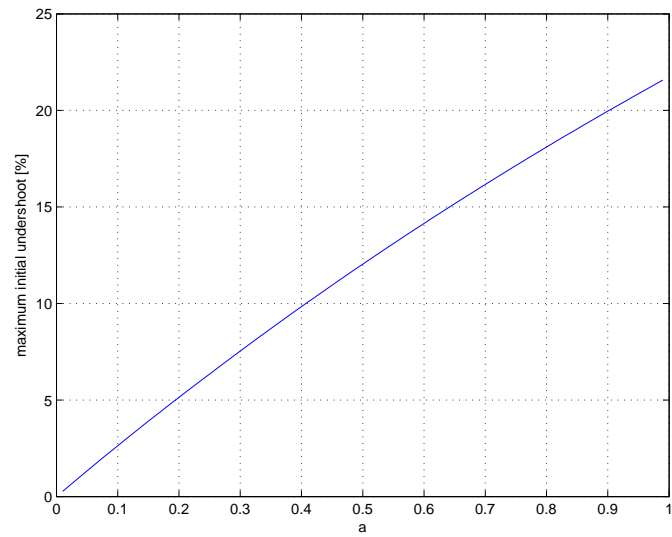
**FIGURE CP12.7**

Script to generate all the plots.

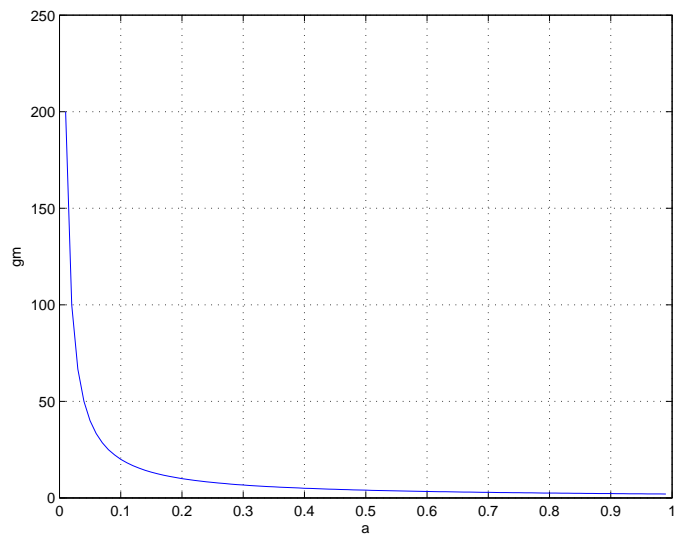


**FIGURE CP12.7**

CONTINUED: (b) Steady-state tracking error.



**FIGURE CP12.7**  
CONTINUED: (c) Maximum initial undershoot.



**FIGURE CP12.7**  
CONTINUED: (d) Gain margin.

**CP12.8** The plant (balloon and canister dynamics plus motor) is given by

$$G(s) = \frac{1}{(s+2)(s+4)(s+10)},$$

and the PID controller is

$$G_c(s) = \frac{K_D(s^2 + as + b)}{s}.$$

Let  $a = 6$ . Then using the root locus methods, we determine that with

$$K_D = 12.5 \quad \text{and} \quad b = 10$$

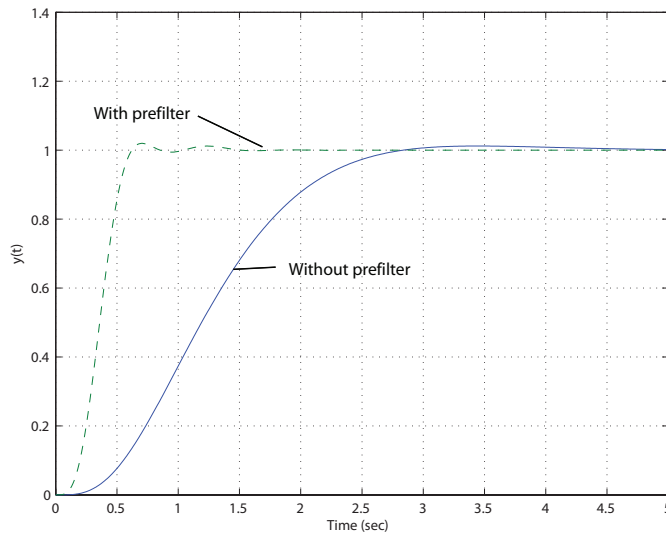
we have the roots

$$s_1 = -8.4$$

$$s_2 = -4.7$$

$$s_{3,4} = -1.43 \pm j1.05.$$

Thus,  $\zeta = 0.8$ . The plot of  $y(t)$  is shown in Figure CP12.8. The percent overshoot is less than 3%, as desired.



**FIGURE CP12.8**  
Simulation of the GRID device.

# Digital Control Systems

## Exercises

**E13.1** (a) Elevation contours on a map are **discrete** signals. (b) Temperature in a room is a **continuous** signal. (c) A digital clock display is a **discrete** signal. (d) The score of a basketball game is a **discrete** signal. (e) The output of a loudspeaker is a **continuous** signal.

**E13.2** (a) Using long-division we determine that

$$Y(z) = z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \dots$$

Therefore, with

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

we have

$$y(0) = 0 \quad y(T) = 1 \quad y(2T) = 3 \quad y(3T) = 7 \quad y(4T) = 15 .$$

(b) The exact solution is

$$y(kT) = e^{k \ln 2} - 1 .$$

**E13.3** For the system response

$$y(kT) = kT$$

where  $k \geq 0$ , we have

$$Y(z) = \frac{Tz}{(z-1)^2} .$$

**E13.4** The partial fraction expansion of  $Y(s)$  is

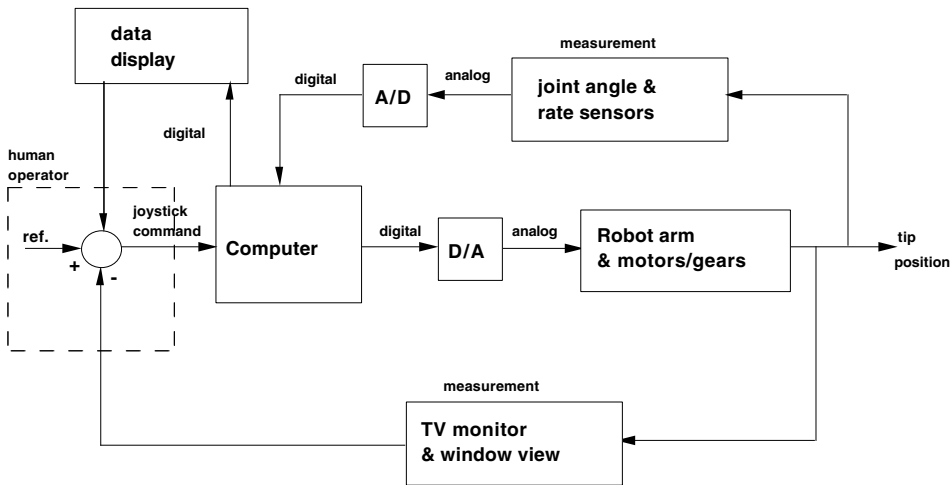
$$Y(s) = \frac{5}{s(s+2)(s+10)} = \frac{0.25}{s} + \frac{0.0625}{s+10} - \frac{0.3125}{s+2} .$$

Then, using Table 13.1 in Dorf and Bishop, we determine that

$$\begin{aligned} Y(z) &= 0.25 \frac{z}{z-1} + 0.0625 \frac{z}{z-e^{-10T}} - 0.3125 \frac{z}{z-e^{-2T}} \\ &= 0.25 \frac{z}{z-1} + 0.0625 \frac{z}{z-0.135} - 0.3125 \frac{z}{z-0.670}, \end{aligned}$$

where  $T = 0.1$ .

**E13.5** The Space Shuttle and robot arm control block diagram is shown in Figure E13.5. The human operator uses information from the computer generated data display and visual sensory data from the TV monitor and by looking out the window. He/she commands the robot arm via a joystick command to the computer.



**FIGURE E13.5**  
The Space Shuttle/robot arm control block diagram.

**E13.6** From Section 10.8 in Dorf and Bishop, we find that the design resulted in the compensator

$$G_c(s) = \frac{6.66s + 1}{66.6s + 1} = 0.1 \frac{s + 0.15}{s + 0.015}.$$

Using the relationships

$$A = e^{-aT}, \quad B = e^{-bT}, \quad \text{and} \quad C \frac{1-A}{1-B} = K \frac{a}{b},$$

we compute

$$A = e^{-0.15(0.001)} = 0.99985, \quad B = e^{-0.015(0.001)} = 0.999985, \quad \text{and} \quad C = 0.1.$$



Therefore,

$$D(z) = C \frac{z - A}{z - B} = 0.1 \frac{z - 0.99985}{z - 0.99985} .$$

**E13.7** Using long-division, we determine that

$$Y(z) = 1 + 3.5z^{-1} + 5.75z^{-2} + 6.875z^{-3} + \dots$$

Therefore, with

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

we have

$$y(0) = 1 \quad y(T) = 3.5 \quad y(2T) = 5.75 \quad y(3T) = 6.875 .$$

**E13.8** The closed-loop system with

$$T(z) = \frac{z}{z^2 + 0.2z - 1.0}$$

is **unstable** since one of the poles of the transfer function ( $z = -1.1$  and  $z = 0.90$ ) lies outside the unit circle in the  $z$ -plane.

**E13.9** (a) Using long-division we determine that

$$Y(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$

Therefore, with

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

we have

$$y(0) = 0 \quad y(T) = 1 \quad y(2T) = 1 \quad y(3T) = 1 \quad y(4T) = 1 .$$

(b) The exact solution is

$$y(kT) = 1 - \delta(k)$$

where  $\delta(k) = 1$  when  $k = 0$  and  $\delta(k) = 0$  when  $k \neq 0$ .

**E13.10** We compute  $T/\tau = 1.25$ .

(a) Using Figure 13.19 in Dorf and Bishop, we determine that  $K\tau = 0.8$  which implies  $K = 100$ .

(b) Using Figure 13.21 in Dorf and Bishop, we determine that  $e_{ss} = 0.75$ .

- (c) Using Figure 13.20 in Dorf and Bishop, we determine that  $K\tau = 0.7$  which implies  $K = 88$ .

**E13.11** (a) The transfer function (including the zero-order hold) is

$$G_o(s)G_p(s) = \frac{100(1 - e^{-sT})}{s(s^2 + 100)} .$$

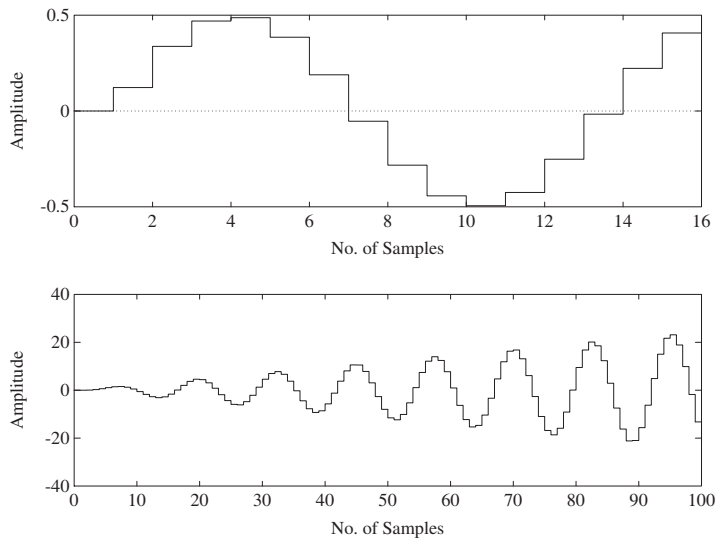
Expanding into partial fractions yields

$$\begin{aligned} G(z) &= (1 - z^{-1})\mathcal{Z} \left\{ \frac{1}{s} - \frac{s}{s^2 + 100} \right\} \\ &= (1 - z^{-1}) \left( \frac{z}{z - 1} - \frac{z(z - \cos 10T)}{z^2 - 2 \cos 10Tz + 1} \right) . \end{aligned}$$

When  $T = 0.05$  we have

$$G(z) = \frac{0.1224(z + 1)}{z^2 - 1.7552z + 1} .$$

- (b) The system is **marginally stable** since the system poles,  $z = -0.8776 \pm 0.4794j$ , are on the unit circle.
- (c) The impulse response and sinusoidal input response are shown in Figure E13.11.



**FIGURE E13.11**

Impulse and sinusoidal (natural frequency) input response.

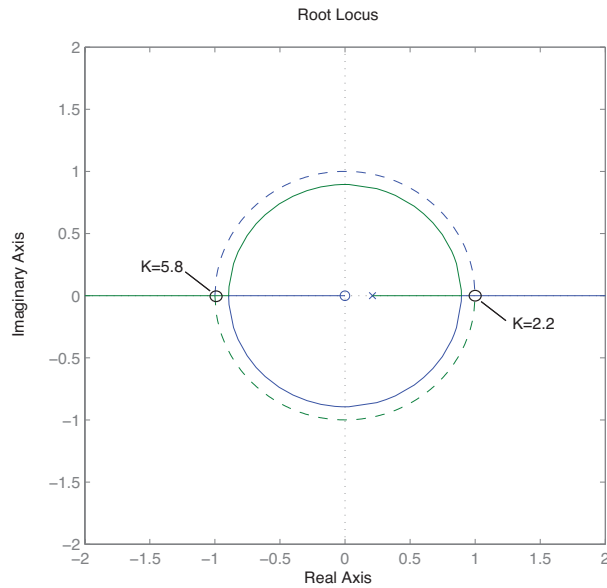
**E13.12** The partial fraction expansion of  $X(s)$  is

$$X(s) = \frac{s+1}{s^2+5s+6} = \frac{2}{s+3} - \frac{1}{s+2}.$$

Then, with  $T = 1$ , we have

$$X(z) = \frac{2z}{z-e^{-3}} - \frac{z}{z-e^{-2}} = \frac{2z}{z-0.0498} - \frac{z}{z-1353}.$$

**E13.13** The root locus is shown in Figure E13.13. For stability:  $2.2 < K < 5.8$ .



**FIGURE E13.13**

Root locus with unit circle (dashed curve).

**E13.14** Given  $G_p(s)$ , we determine that (with  $K = 5$ )

$$G(z) = \frac{5(1-e^{-1})z}{z(z-e^{-1})}.$$

The closed-loop characteristic equation is

$$z^2 + 1.792z + 0.368 = 0$$

and the system is **unstable**, since there is a pole at  $z = -1.55$ . The system is stable for

$$0 < K < 4.32.$$

**E13.15** The transfer function  $G(z)$  is

$$G(z) = \frac{0.1289z + 0.02624}{z^2 - 0.3862z + 0.006738} .$$

The sampling time is  $T = 1$  s.

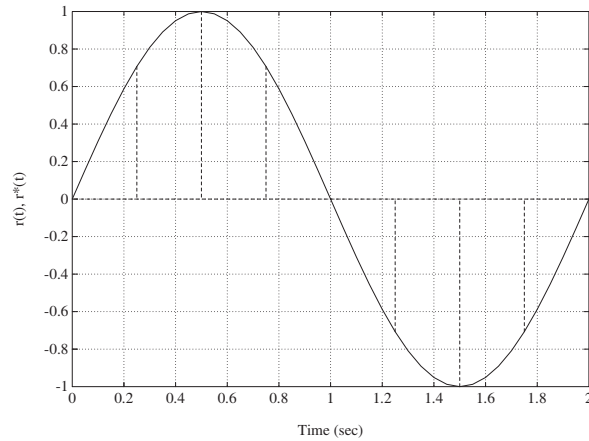
**E13.16** The transfer function  $G(z)$  is

$$G(z) = \frac{0.2759z + 0.1982}{z^2 - 1.368z + 0.3679} .$$

The sampling time is  $T = 0.5$  s.

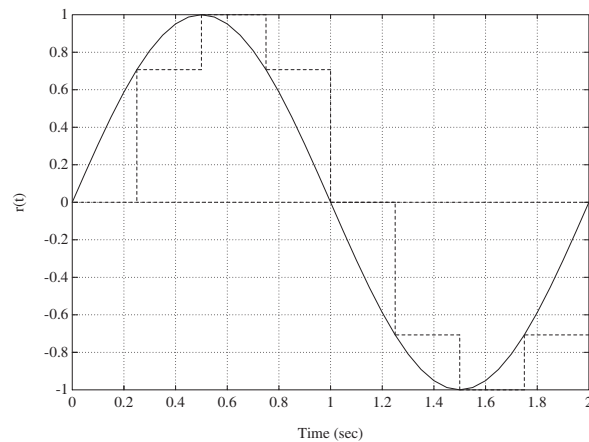
## Problems

- P13.1** The plot of the input to the sampler and the output  $r^*(t)$  is shown in Figure P13.1.



**FIGURE P13.1**  
Plot of  $r(t) = \sin(\omega t)$  and  $r^*(t)$ .

- P13.2** The plot of the input and the output is shown in Figure P13.2.



**FIGURE P13.2**  
Plot of  $r(t) = \sin(\omega t)$  and output of sample and hold.

**P13.3** The transfer function

$$Y(z)/R^*(z) = G(z) = \frac{z}{z - e^{-T}} .$$

The ramp input is represented by

$$R(z) = \frac{Tz}{(z - 1)^2} .$$

The output  $Y(z) = G(z)R(z)$  is obtained by long division:

$$\begin{aligned} Y(z) &= Tz^{-1} + T(2 + e^{-T})z^{-2} - T \left[ (1 + 2e^{-T}) - (2 + e^{-T})^2 \right] z^{-3} \\ &\quad + T \left[ e^{-T} + (1 + 2e^{-T})(2 + e^{-T}) \right. \\ &\quad \left. - (2 + e^{-T}) \left( (1 + 2e^{-T}) - (2 + e^{-T})^2 \right) \right] z^{-4} + \dots \end{aligned}$$

**P13.4** The transfer function

$$Y(s)/R^*(s) = \frac{1 - e^{-sT}}{s(s + 2)} .$$

The partial fraction expansion (with  $T = 1$ ) yields

$$\begin{aligned} G(z) &= (1 - z^{-1})\mathcal{Z} \left\{ \frac{0.5}{s} - \frac{0.5}{s + 2} \right\} = (1 - z^{-1}) \left\{ \frac{0.5z}{z - 1} - \frac{0.5z}{z - 0.1353} \right\} \\ &= \frac{0.4323}{z - 0.1353} . \end{aligned}$$

**P13.5** The step input is

$$R(z) = \frac{z}{z - 1} .$$

Also,

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.6321}{z + 0.2643} .$$

So,

$$Y(z) = T(z)R(z) = \frac{0.6321}{z + 0.2643} \frac{z}{z - 1} = \frac{0.6321z}{z^2 - 0.7357z - 0.2643} .$$

Using long-division we determine that

$$Y(z) = 0.6321z^{-1} + 0.4650z^{-2} + 0.5092z^{-3} + 0.4975z^{-4} + 0.5006z^{-5} + \dots$$

Therefore, with

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

we have  $y(0) = 0$ ,  $y(T) = 0.6321$ ,  $y(2T) = 0.4650$ ,  $y(3T) = 0.5092$ ,  $y(4T) = 0.4975$ , and  $y(5T) = 0.5006$ .

**P13.6** Using the final value theorem (see Table 13.1 in Dorf and Bishop), we determine that (for a step input)

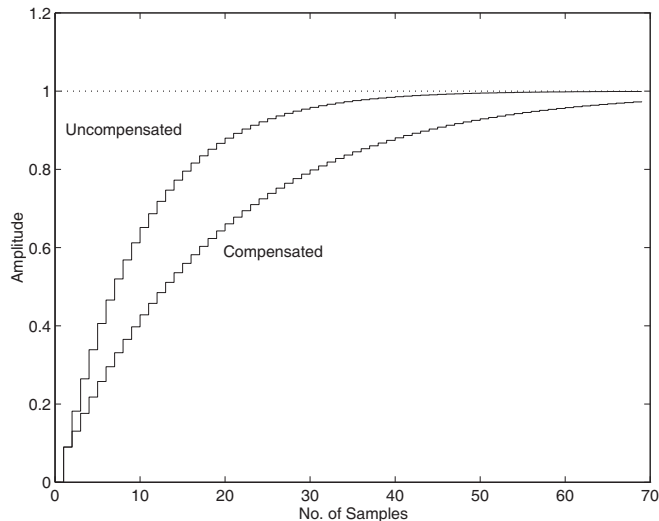
$$Y_{ss} = \lim_{z \rightarrow 1} (z-1)Y(z) = \lim_{z \rightarrow 1} (z-1) \frac{0.6321}{z+0.2643} \frac{z}{z-1} = \frac{0.6321}{1.2643} = 0.5 .$$

And using the initial value theorem, we compute

$$Y_o = \lim_{z \rightarrow \infty} Y(z) = \lim_{z \rightarrow \infty} \frac{0.6321}{z+0.2643} \frac{z}{z-1} = 0 .$$

**P13.7** Using Figures 13.19 and 13.21 in Dorf and Bishop, we determine that the performance specifications are satisfied when  $K\tau = 0.5$  and  $\frac{T}{\tau} = 2$ . Computing  $K$  and  $T$  (with  $\tau = 0.5$ ) yields  $K = 1$  and  $T = 1$ .

**P13.8** We can select  $K = 1$  and  $r = 0.2$ . The step responses for the compensated and uncompensated systems are shown in Figure P13.8.



**FIGURE P13.8**

Plot of compensated and uncompensated systems.

**P13.9** Consider the compensator

$$G_c(s) = K \frac{s + a}{s + b} .$$

Then, using Bode methods we can select  $a = 1$ ,  $b = 4$ , and  $K = 1$ . The compensated system phase margin is  $P.M. = 50^\circ$  and the gain margin is  $G.M. = 15\text{dB}$ . The crossover frequency is  $\omega_c = 2.15$  rad/sec. Utilizing the  $G_c(s)$ -to- $D(z)$  method and selecting  $T = 0.01$  second, we determine

$$D(z) = C \frac{z - A}{z - B} = \frac{z - 0.99}{z - 0.96} .$$

We use the relationships

$$A = e^{-aT} , \quad B = e^{-bT} , \quad \text{and} \quad C \frac{1 - A}{1 - B} = K \frac{a}{b} ,$$

to compute  $A = e^{-0.01} = 0.99$ ,  $B = e^{-0.04} = 0.96$ , and  $C = 1$ .

**P13.10** (a) The transfer function  $G(z)D(z)$  is

$$G(z)D(z) = K \frac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679} .$$

(b) The closed-loop system characteristic equation is

$$1 + K \frac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679} = 0 .$$

(c) Using root locus methods, the maximum value of  $K$  is found to be  $K_{max} = 239$ .

(d) Using Figure 13.19 in Dorf and Bishop for  $T/\tau = 1$  and a maximum overshoot of 0.3, we find that  $K = 75$ . (e) The closed-loop transfer function (with  $K = 75$ ) is

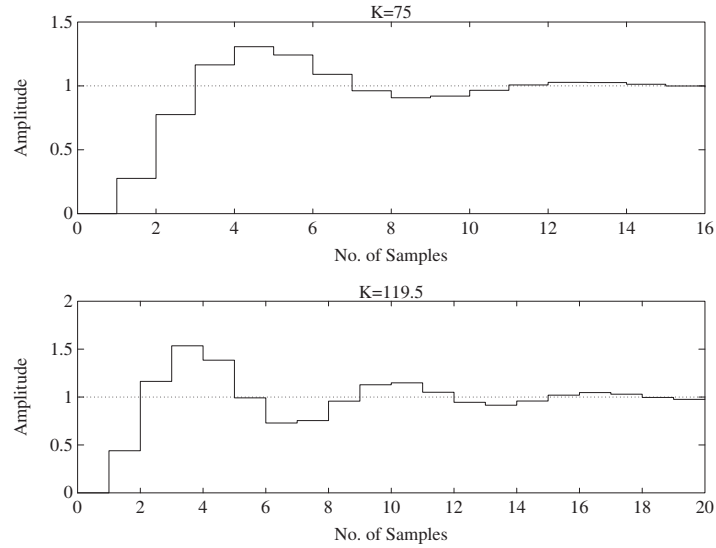
$$T(z) = \frac{0.2759z + 0.1982}{z^2 - 1.092z + 0.5661} .$$

The step response is shown in Figure P13.10.

(f) The closed-loop poles with  $K = 119.5$  are  $z = 0.4641 \pm 0.6843j$ . The overshoot is 0.55.

(g) The step response is shown in Figure P13.10 (for  $K = 119.5$ ).





**FIGURE P13.10**  
Step response for  $K = 75$  and  $K = 119.5$ .

**P13.11** (a) Consider the compensator

$$G_c(s) = K \frac{s + a}{s + b}.$$

Then, using Bode methods we can select  $a = 0.7$ ,  $b = 0.1$ , and  $K = 150$ . The compensated system overshoot and steady-state tracking error (for a ramp input) are  $P.O. = 30\%$  and  $e_{ss} < 0.01$ .

(b) Utilizing the  $G_c(s)$ -to- $D(z)$  method (with  $T = 0.1$  second), we determine

$$D(z) = C \frac{z - A}{z - B} = 155.3 \frac{z - 0.9324}{z - 0.99}.$$

We use the relationships

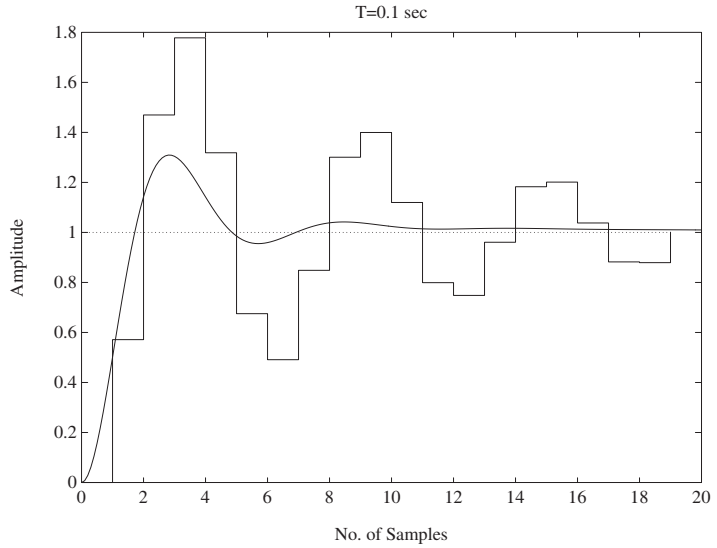
$$A = e^{-aT}, \quad B = e^{-bT}, \quad \text{and} \quad C \frac{1 - A}{1 - B} = K \frac{a}{b},$$

to compute

$$A = e^{-0.007} = 0.9324, \quad B = e^{-0.01} = 0.99, \quad \text{and} \quad C = 155.3.$$

(c) The step response for the continuous system with  $G_c(s)$  in part(a) and for the discrete system with  $D(z)$  in part (b) is shown in Figure P13.11a.

(d) Utilizing the  $G_c(s)$ -to- $D(z)$  method (with  $T = 0.01$  second), we de-



**FIGURE P13.11**  
 (a) Step response for continuous and discrete systems ( $T=0.1s$ ) in Parts (a) and (b).

termine

$$D(z) = C \frac{z - A}{z - B} = 150 \frac{z - 0.993}{z - 0.999} .$$

We use the relationships

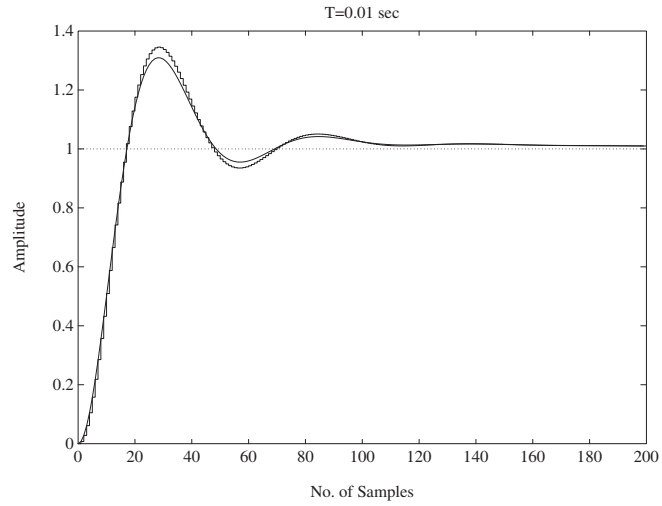
$$\begin{aligned} A &= e^{-aT} \\ B &= e^{-bT} \\ C \frac{1 - A}{1 - B} &= K \frac{a}{b} \end{aligned}$$

to compute

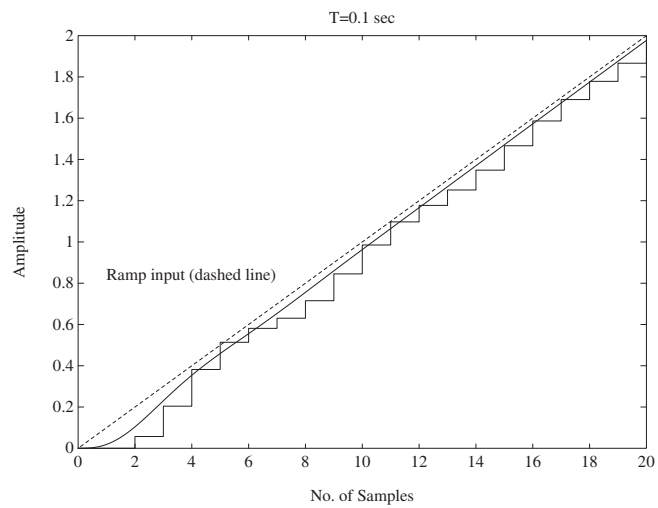
$$\begin{aligned} A &= e^{-0.07} = 0.993 \\ B &= e^{-0.001} = 0.999 \\ C &= 150 . \end{aligned}$$

The step response for the continuous system with  $G_c(s)$  in part(a) and for the discrete system with  $D(z)$  in part (d) is shown in Figure P13.11b.

- (e) The ramp response for the continuous system with  $G_c(s)$  in part(a) and for the discrete system with  $D(z)$  in part (b) is shown in Figure P13.11c.

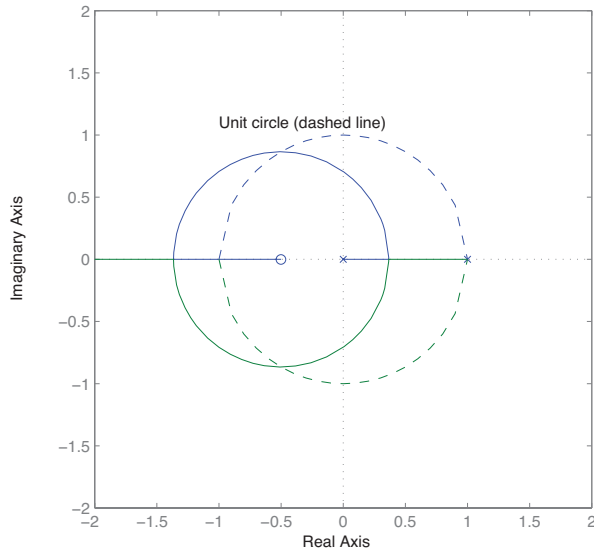
**FIGURE P13.11**

CONTINUED: (b) Step response for continuous and discrete systems ( $T=0.01s$ ) in Parts (a) and (d).

**FIGURE P13.11**

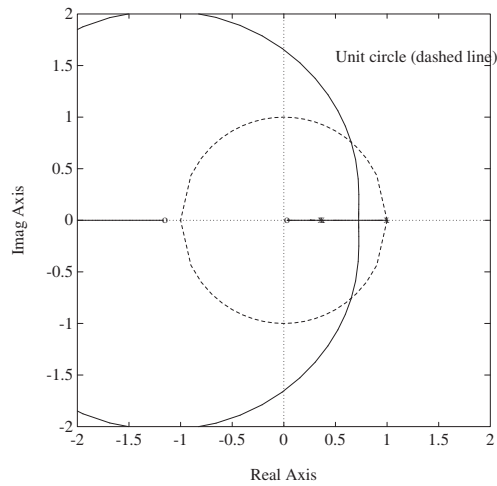
CONTINUED: (c) Ramp response for continuous and discrete systems ( $T=0.1s$ ) in Parts (a) and (b).

**P13.12** The root locus is shown in Figure P13.12. For stability:  $0 < K < 2$ .



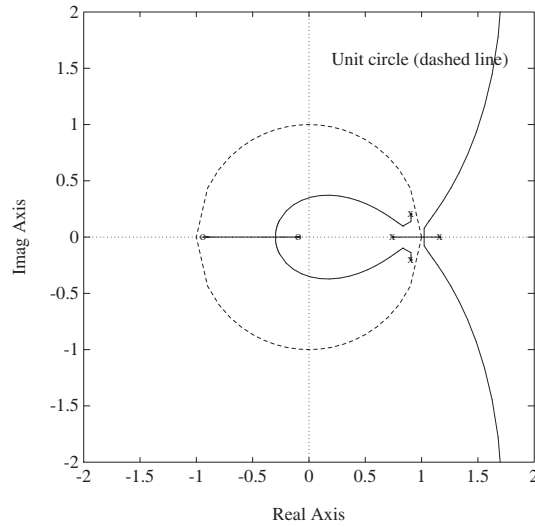
**FIGURE P13.12**  
 Root locus for  $1 + K \frac{z+0.5}{z(z-1)} = 0$ .

**P13.13** The root locus is shown in Figure P13.13. When  $K = 0.027$ , the characteristic equation has two equal roots:  $z_{1,2} = 0.7247$  and  $z_3 = 0.2593$ .



**FIGURE P13.13**  
 Root locus for  $1 + K \frac{z^2 + 1.1206z - 0.0364}{z^3 - 1.7358z^2 + 0.8711z - 0.1353} = 0$ .

- P13.14** The root locus is shown in Figure P13.14. When  $K = 9.5655 \times 10^{-5}$ , the two real roots break away from the real axis at  $z = 0.99$ . For stability:  $K < 9.7 \times 10^{-5}$ .



**FIGURE P13.14**

Root locus for  $1 + K \frac{z^3 + 10.3614z^2 + 9.758z + 0.8353}{z^4 - 3.7123z^3 + 5.1644z^2 - 3.195z + 0.7408} = 0$ .

- P13.15** Given

$$G_p(s) = \frac{20}{s - 5}$$

and the sample and hold ( $T=0.1s$ ) as shown in Figure 13.18 in Dorf and Bishop, we determine that

$$G(z) = \frac{2.595}{z - 1.649}.$$

Then, with  $R(z) = z/(z - 1)$ , we have

$$Y(z) = \frac{2.595z}{(z - 1)(z + 0.9462)}.$$

Therefore,  $Y(z) = 2.59z^{-1} + 0.14z^{-2} + 2.46z^{-3} + 0.26z^{-4} + \dots$ .

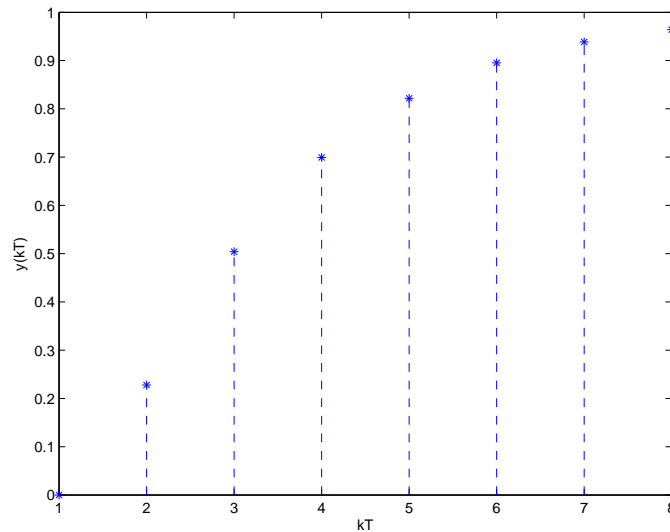
**P13.16** Given  $G_p(s)$  and the sample and hold ( $T=1s$ ) as shown in Figure 13.18 in Dorf and Bishop, we determine that

$$G(z) = \frac{0.22775z + 0.088984}{z^2 - 1.0498z + 0.049787} .$$

Then, with  $R(z) = z/(z - 1)$ , we have

$$Y(z) = \frac{0.22775z + 0.088984}{z^2 - 0.82203z + 0.13877} \frac{z}{z - 1} .$$

The plot of  $y(kT)$  is shown in Figure P13.16.

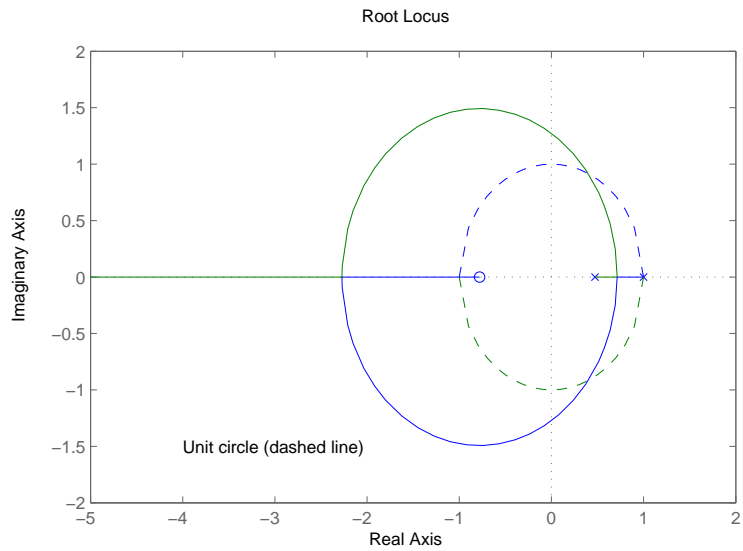


**FIGURE P13.16**  
Plot of  $y(kT)$  for a step input.

**P13.17** The root locus is shown in Figure P13.17 for

$$1 + K \frac{0.39532z + 0.30819}{z^2 - 1.4724z + 0.47237} = 0 .$$

The limiting value of the gain for stability is  $K = 1.71$ .

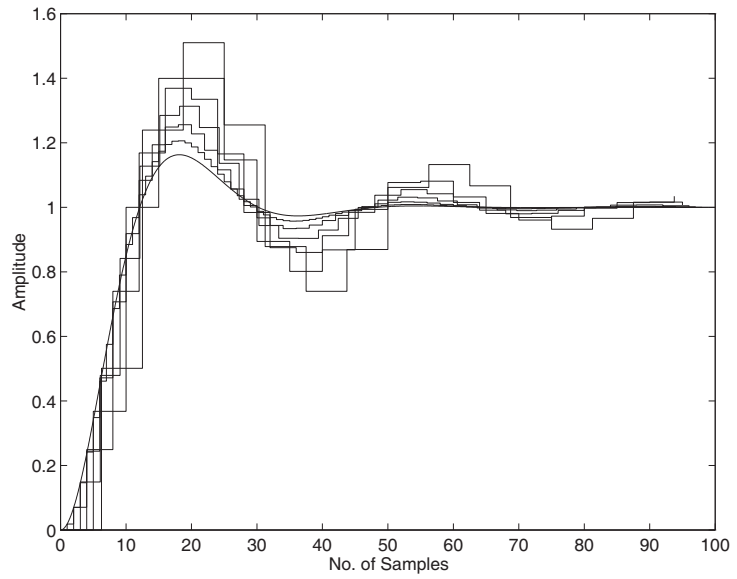
**FIGURE P13.17**

Root locus for  $1 + K \frac{0.39532z + 0.30819}{z^2 - 1.4724z + 0.47237} = 0$ .

- P13.18** The plot of the step responses for  $0 \leq T \leq 1.2$  is shown in Figure P13.18. The overshoot and settling time summary is given in Table P13.18.

$T$	0	0.2	0.4	0.6	0.8	1.0	1.2
$P.O.$	16.3%	20.6%	25.6%	31.3%	36.9%	40.0%	51.0%
$T_s$	8.1	8.4	8.8	11.4	14.4	16.0	19.2

**TABLE P13.18** Performance summary.



**FIGURE P13.18**  
Step responses for  $0 \leq T \leq 1.2$ .

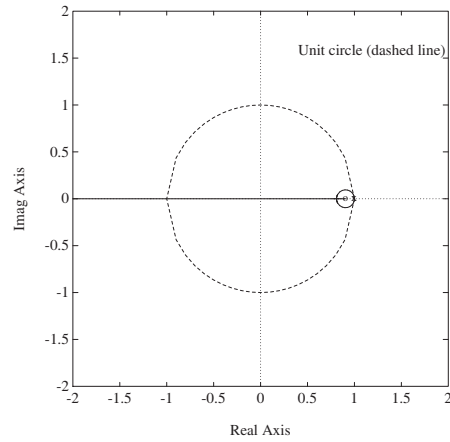


## Advanced Problems

**AP13.1** Given the sample and hold with  $G_p(s)$ , we determine that

$$G(z) = \frac{10.5K(z - 0.9048)}{(z - 1)^2}.$$

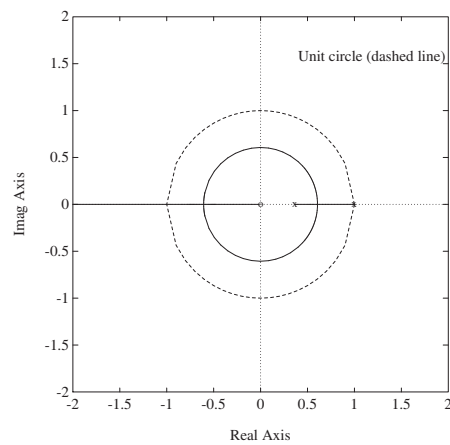
The root locus is shown in Figure AP13.1. For stability:  $0 < K < 0.2$ .



**FIGURE AP13.1**

Root locus for  $1 + K \frac{10.5(z - 0.9048)}{(z - 1)^2} = 0$  with unit circle (dashed line).

**AP13.2** The root locus is shown in Figure AP13.2a. The loop transfer function is

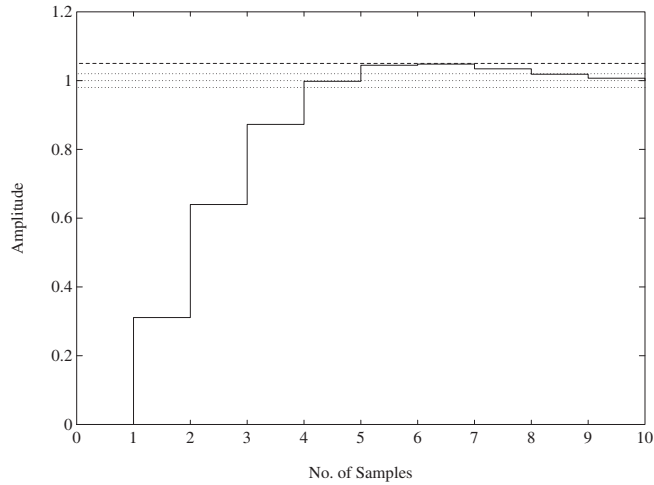


**FIGURE AP13.2**

(a) Root locus for  $1 + K \frac{0.0379z}{(z - 1)(z - 0.368)} = 0$ .

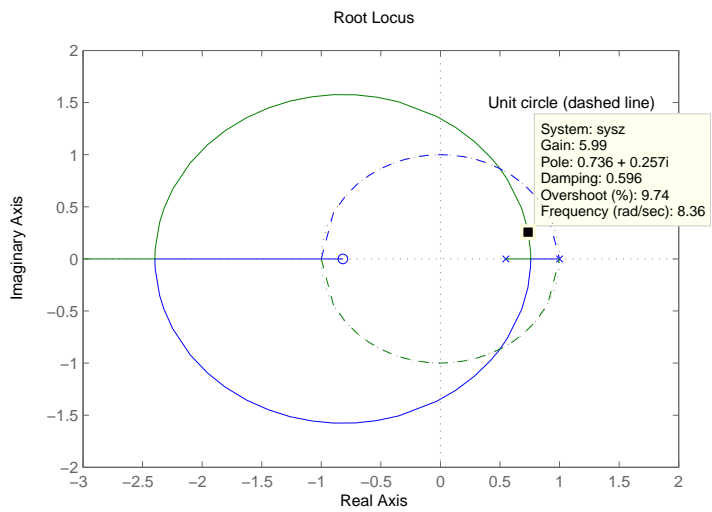
$$G(z)D(z) = K \frac{0.0379z}{(z - 1)(z - 0.368)} .$$

For stability:  $K_{max} = 72$ . We select  $K = 8.2$ . The step response is shown in Figure AP13.2b.



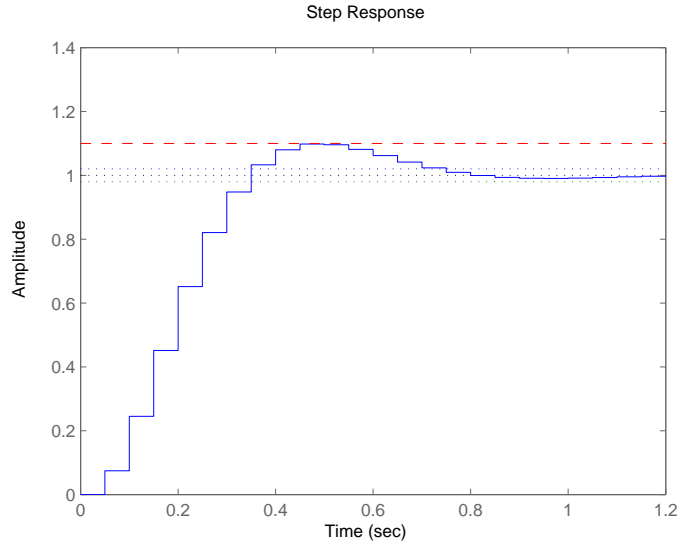
**FIGURE AP13.2**  
CONTINUED: (b) Step response with  $K = 8.2$ .

**AP13.3** The root locus is shown in Figure AP13.3a. The maximum gain for



**FIGURE AP13.3**  
(a) Root locus for  $1 + K \frac{0.07441z + 0.06095}{z^2 - 1.474z + 0.6098} = 0$ .

stability is  $K_{max} = 44.3$ . We select  $K = 6$ . The step response is shown in Figure AP13.3b.



**FIGURE AP13.3**  
CONTINUED: (b) Step response with  $K = 6$ .

**AP13.4** The loop transfer function is

$$G(z) = \frac{10(1 - e^{-T})}{z - e^{-T}},$$

and the closed-loop transfer function is

$$T(z) = \frac{10(1 - e^{-T})}{z - (11e^{-T} - 10)}.$$

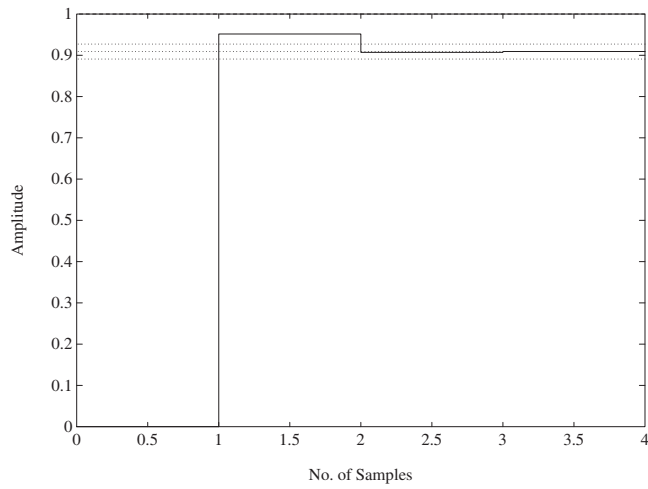
For stability, we require

$$|11e^{-T} - 10| < 1.$$

Solving for  $T$  yields

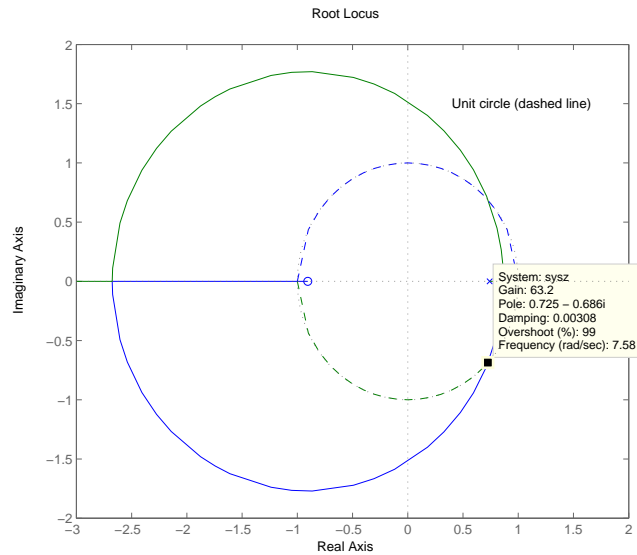
$$0 < T < 0.2.$$

Selecting  $T = 0.1$ s provides a stable system with rapid response; the settling time is  $T_s = 0.2$ s. The step response is shown in Figure AP13.4.



**FIGURE AP13.4**  
Step response with  $T = 0.1s$ .

**AP13.5** The maximum gain for stability is  $K_{max} = 63.15$ .



**FIGURE AP13.5**  
Root locus for  $1 + K \frac{0.004535z + 0.004104}{z^2 - 1.741z + 0.7408} = 0$ .

## Design Problems

**CDP13.1** The plant model with parameters given in Table CDP2.1 in Dorf and Bishop is given by:

$$G_p(s) = \frac{26.035}{s(s + 33.142)},$$

where we neglect the motor inductance  $L_m$  and where we switch off the tachometer feedback (see Figure CDP4.1 in Dorf and Bishop). Letting

$$G(z) = \mathcal{Z} \left\{ \mathcal{G}_i(f) \mathcal{G}_v(f) \right\}$$

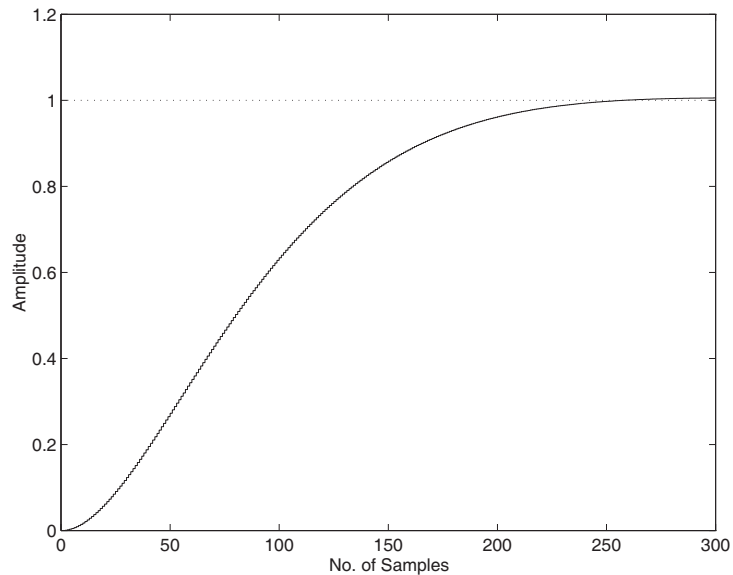
we obtain

$$G(z) = \frac{1.2875e - 05(z + 0.989)}{(z - 1)(z - 0.9674)}.$$

A suitable controller is

$$D(z) = \frac{20(z - 0.5)}{z + 0.25}.$$

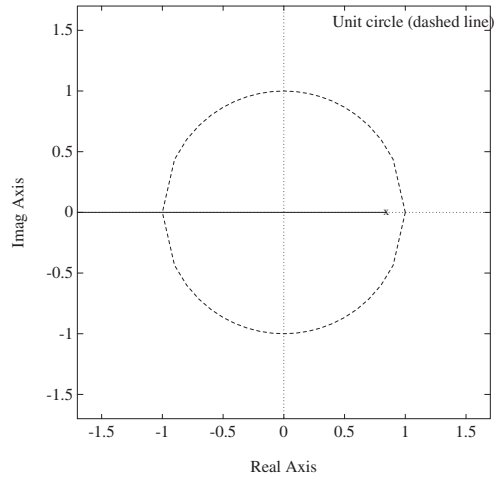
The step response is shown below. The settling time is under 250 samples. With each sample being 1 ms this means that  $T_s < 250$  ms, as desired. Also, the percent overshoot is  $P.O. < 5\%$ .



**DP13.1** (a) Given the sample and hold with  $G_p(s)$ , we determine that

$$KG(z) = K \frac{0.1228}{z - 0.8465} .$$

The root locus is shown in Figure DP13.1a. For stability:  $0 \leq K < 15$ .



**FIGURE DP13.1**

(a) Root locus for  $1 + K \frac{0.1228}{z - 0.8465} = 0$  with unit circle (dashed line).

(b) A suitable compensator is

$$G_c(s) = \frac{15(s + 0.5)}{s + 5} .$$

Utilizing the  $G_c(s)$ -to- $D(z)$  method (with  $T = 0.5$  second), we determine

$$D(z) = C \frac{z - A}{z - B} = 6.22 \frac{z - 0.7788}{z - 0.0821} .$$

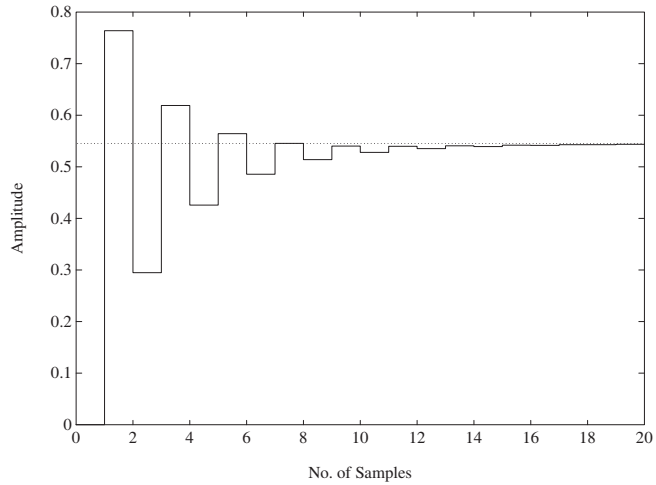
We use the relationships

$$A = e^{-aT} , \quad B = e^{-bT} , \quad \text{and} \quad C \frac{1 - A}{1 - B} = K \frac{a}{b} ,$$

to compute

$$A = e^{-0.5(0.5)} = 0.7788 , \quad B = e^{-0.5(5)} = 0.0821 , \quad \text{and} \quad C = 6.22 .$$

(c) The step response is shown in Figure DP13.1b.



**FIGURE DP13.1**  
CONTINUED: (b) Closed-loop system step response.

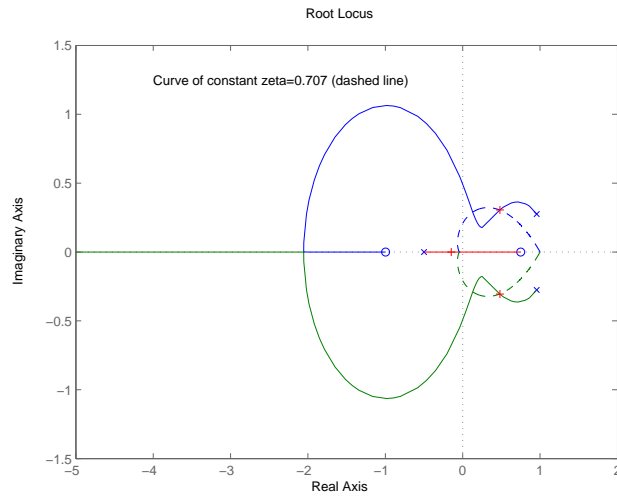
**DP13.2** With the sample and hold ( $T=10\text{ms}$ ), we have

$$G(z) = \frac{0.00044579z + 0.00044453}{z^2 - 1.9136z + 0.99154}.$$

A suitable compensator is

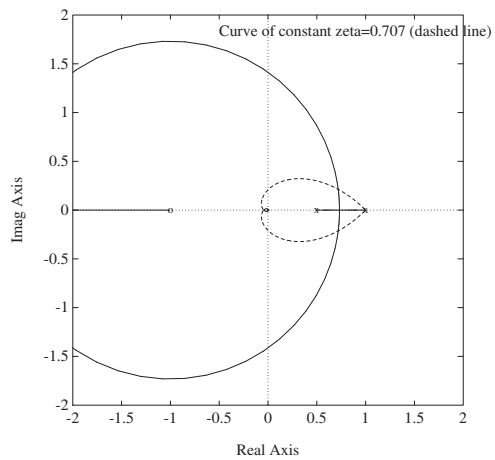
$$D(z) = K \frac{z - 0.75}{z + 0.5},$$

where  $K$  is determined so that  $\zeta$  of the system is  $1/\sqrt{2}$ . The root locus is shown in Figure DP13.2. We choose  $K = 1400$ .



**FIGURE DP13.2**  
 Root locus for  $1 + K \frac{z-0.75}{z+0.5} \frac{0.00044579z+0.00044453}{z^2-1.9136z+0.99154} = 0$ .

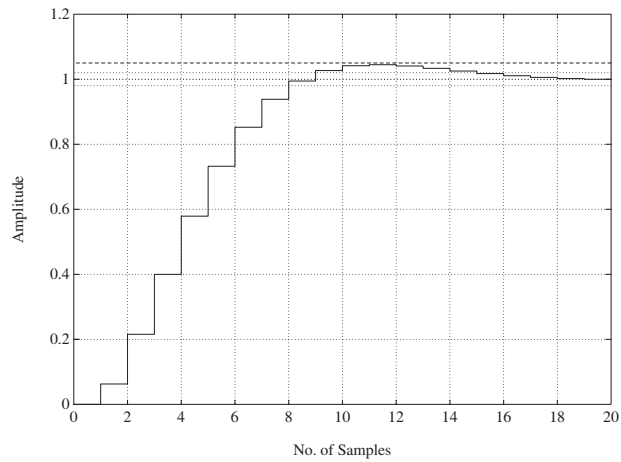
**DP13.3** The root locus is shown in Figure DP13.3a.



**FIGURE DP13.3**  
 (a) Root locus for  $1 + K \frac{z+1}{(z-1)(z-0.5)} = 0$ .

The gain for  $\zeta = 0.707$  is  $K = 0.0627$ . The step response is shown in Figure DP13.3b. The settling time is  $T_s = 14T = 1.4s$  and  $P.O. = 5\%$ .

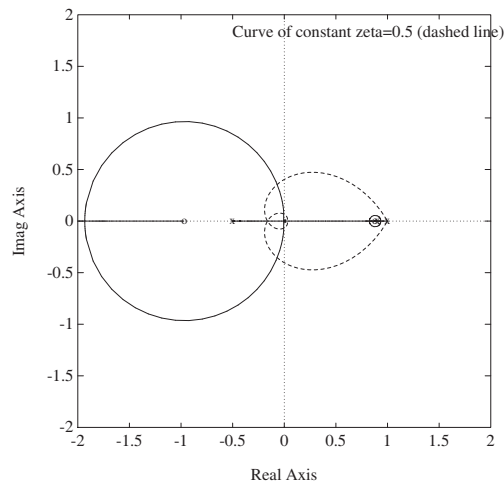




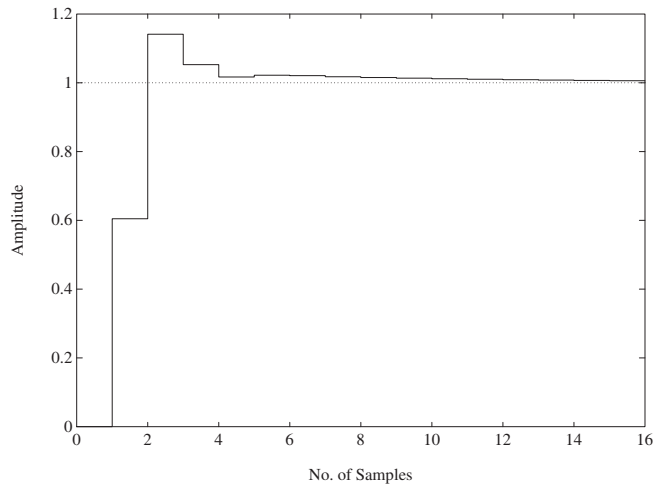
**FIGURE DP13.3**  
CONTINUED: (b) Step response with  $K = 0.0627$ .

**DP13.4** With the sample and hold ( $T=1s$ ), we have

$$G(z) = \frac{0.484(z + 0.9672)}{(z - 1)(z - 0.9048)}.$$



**FIGURE DP13.4**  
(a) Root locus for  $1 + K \frac{z-0.88}{z+0.5} \frac{0.484(z+0.9672)}{(z-1)(z-0.9048)} = 0$ .

**FIGURE DP13.4**CONTINUED: (b) Step response for  $K = 12.5$ .

A suitable compensator is

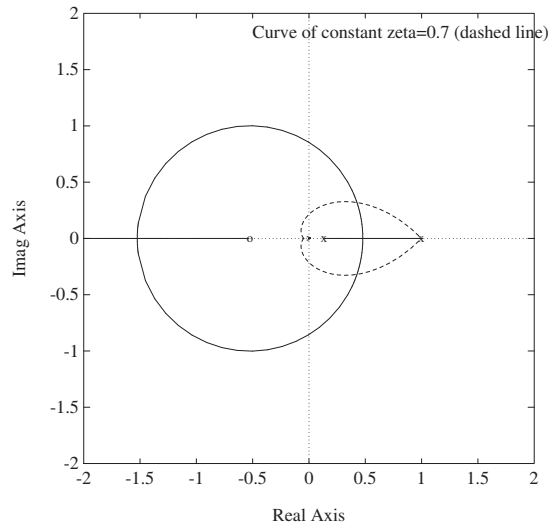
$$D(z) = K \frac{z - 0.88}{z + 0.5},$$

where  $K$  is determined so that  $\zeta$  of the system is 0.5. The root locus is shown in Figure DP13.4a. We choose  $K = 12.5$ . The step response is shown in Figure DP13.4b. Also,  $K_v = 1$ , so the steady-state error specification is satisfied.

**DP13.5** Select  $T = 1$  second. With the sample and hold, we have

$$G(z) = \frac{0.2838z + 0.1485}{z^2 - 1.135z + 0.1353}.$$

The root locus is shown in Figure DP13.5. To meet the percent overshoot specification, we choose  $K$  so that  $\zeta$  of the system is 0.7. This results in  $K = 1$ . The step response has an overshoot of  $P.O. = 4.6\%$ . Also, from Figure 13.21 in Dorf and Bishop, we determine that the steady-state error to a ramp input is  $e_{ss} = 2$  (since  $T/\tau = 2$ , and  $K\tau = 0.3$ ).

**FIGURE DP13.5**

Root locus for  $1 + K \frac{0.2838z + 0.1485}{z^2 - 1.135z + 0.1353} = 0$ .

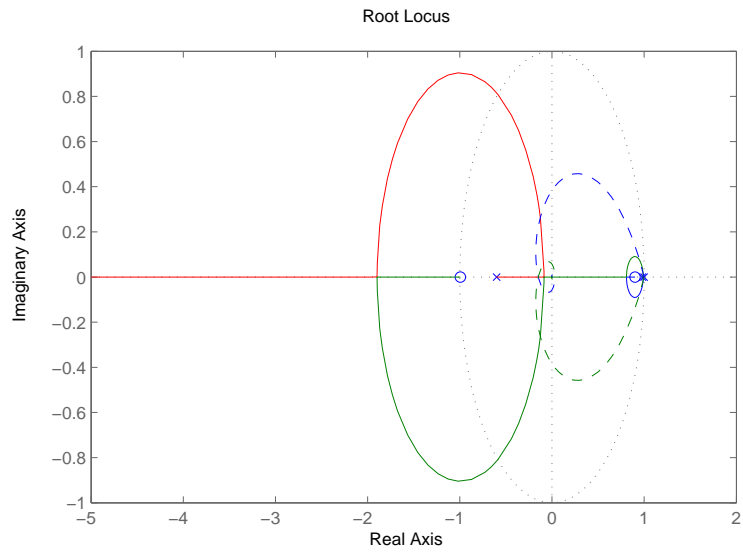
**DP13.6** With the sample and hold at  $T = 1$ , we have

$$G(z) = \frac{0.298z + 0.296}{z^2 - 1.98z + 0.9802}.$$

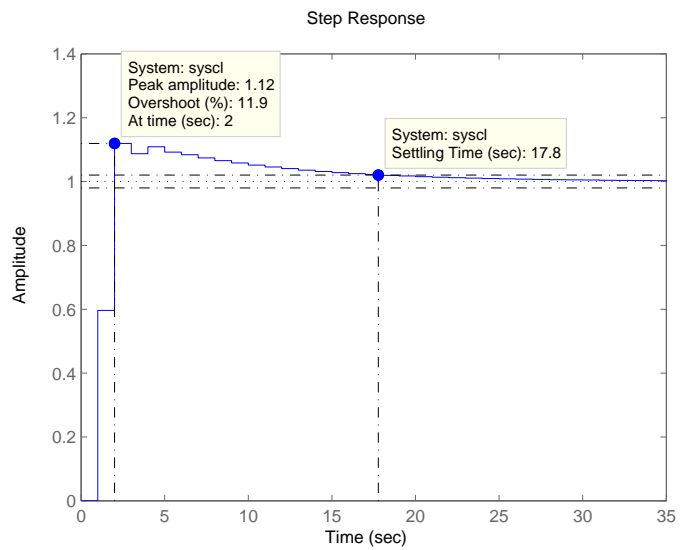
Consider the digital controller

$$D(z) = K \frac{z - 0.9}{z + 0.6}.$$

The root locus is shown in Figure DP13.6. To meet the percent overshoot specification, we choose  $K$  so that  $\zeta$  of the system is greater than 0.52. We select  $K = 2$ . The step response has an overshoot of  $P.O. = 11.9\%$  and the settling time is  $T_s = 17.8s$ .



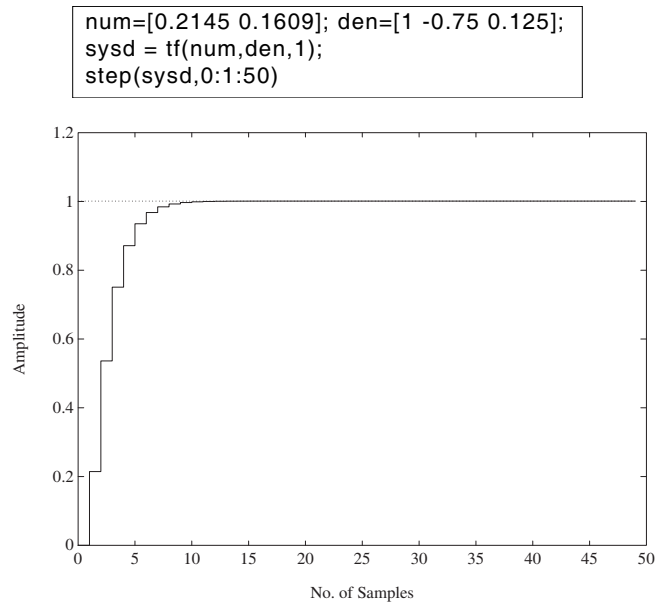
**FIGURE DP13.6**  
 Root locus for  $1 + K \frac{z-0.9}{z+0.6} \frac{0.298z+0.296}{z^2-1.98z+0.9802} = 0$ .



**FIGURE DP13.6**  
 CONTINUED: (b) Step response for  $K = 2$ .

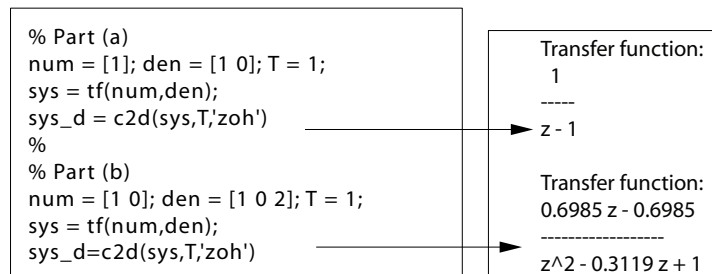
## Computer Problems

**CP13.1** The m-file script and unit step response are shown in Figure CP13.1.

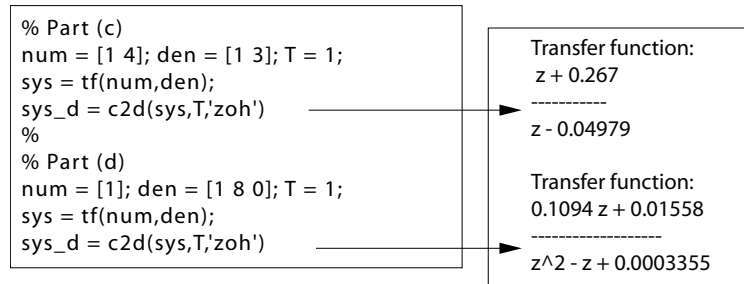


**FIGURE CP13.1**  
Step response.

**CP13.2** The m-file script utilizing the **c2d** function is shown in Figure CP13.2.

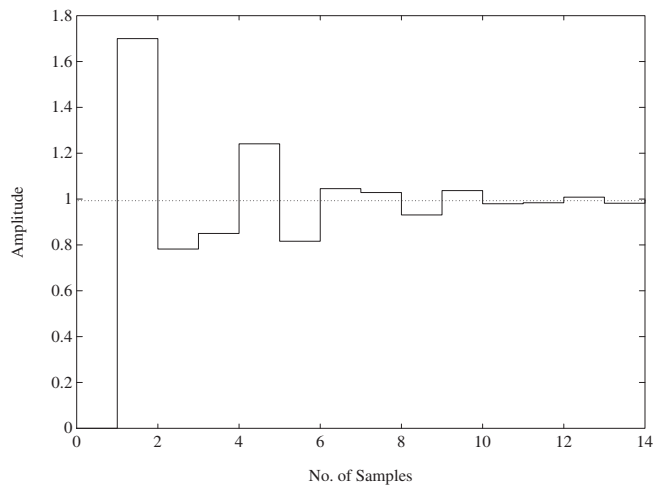


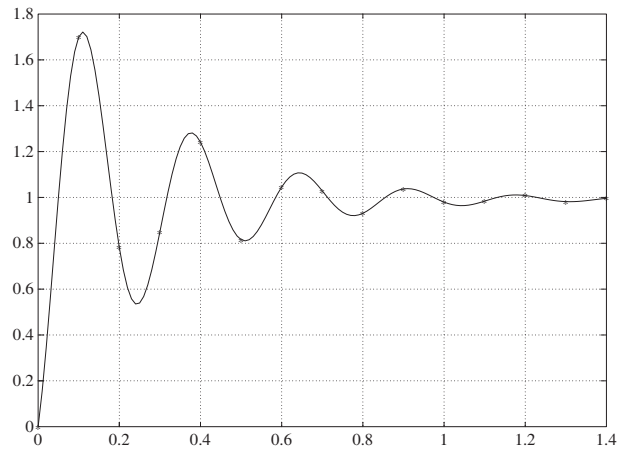
**FIGURE CP13.2**  
Script utilizing the **c2d** function for (a) and (b).

**FIGURE CP13.2**CONTINUED: Script utilizing the **c2d** function for (c) and (d).**CP13.3** The continuous system transfer function (with  $T = 0.1$  sec) is

$$T(s) = \frac{13.37s + 563.1}{s^2 + 6.931s + 567.2}.$$

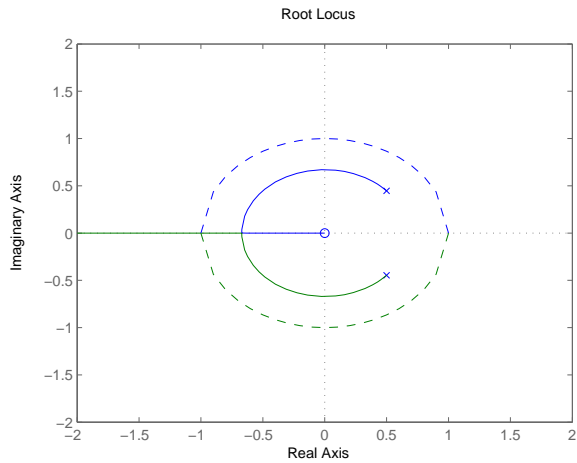
The step response using the **dstep** function is shown in Figure CP13.3a. The continuous system step response is shown in Figure CP13.3b.

**FIGURE CP13.3**(a) Unit step response using the **dstep** function.



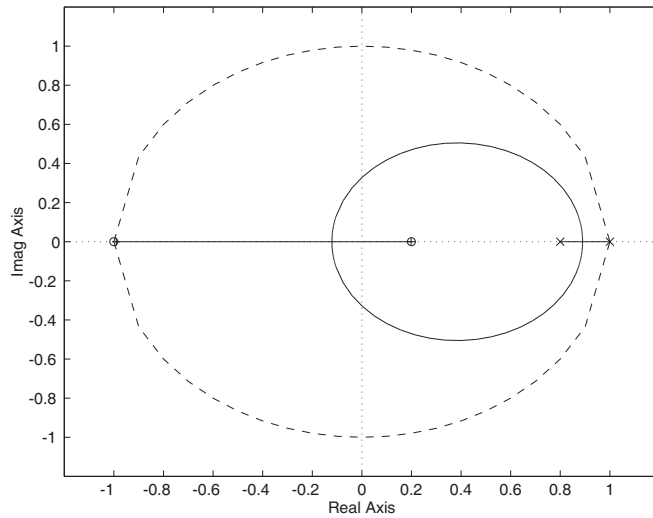
**FIGURE CP13.3**  
CONTINUED: (b) Continuous system step response (\* denote sampled-data step response).

**CP13.4** The root locus is shown in Figure CP13.4. For stability:  $0 < K < 2.45$ .



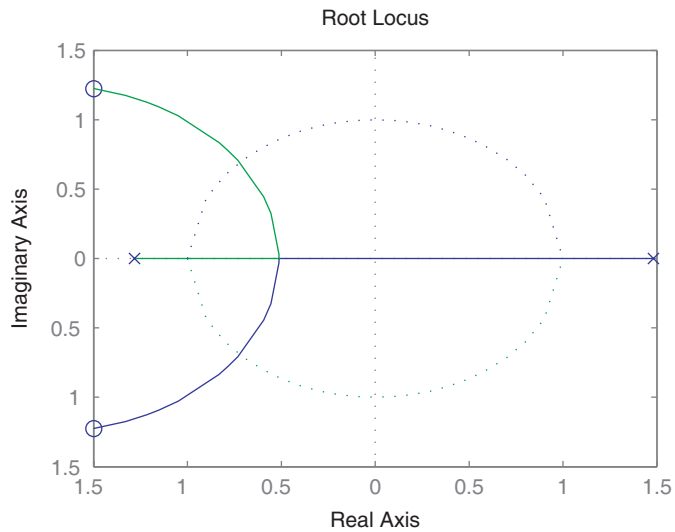
**FIGURE CP13.4**  
Root locus for  $1 + K \frac{z}{z^2 - z + 0.45} = 0$ .

**CP13.5** The root locus is shown in Figure CP13.5. For stability:  $0 < K < \infty$ .



**FIGURE CP13.5**  
 Root locus for  $1 + K \frac{(z-0.2)(z+1)}{(z-0.08)(z-1)} = 0$

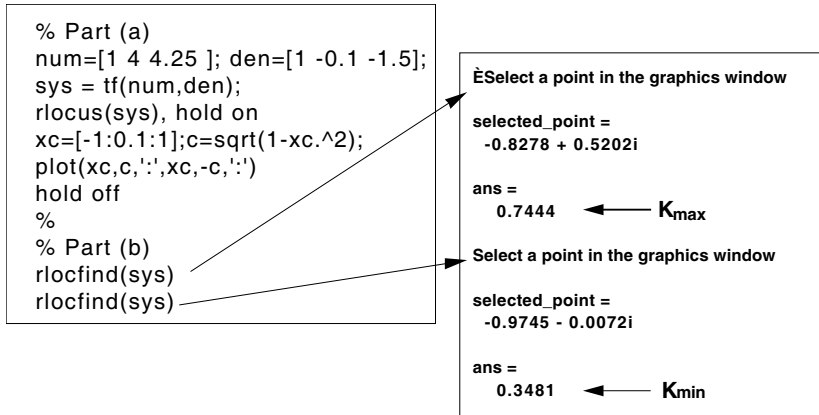
**CP13.6** The root locus is shown in Figure CP13.6.



**FIGURE CP13.6**  
 Root locus.



We determine the range of  $K$  for stability is  $0.4 < K < 1.06$ .

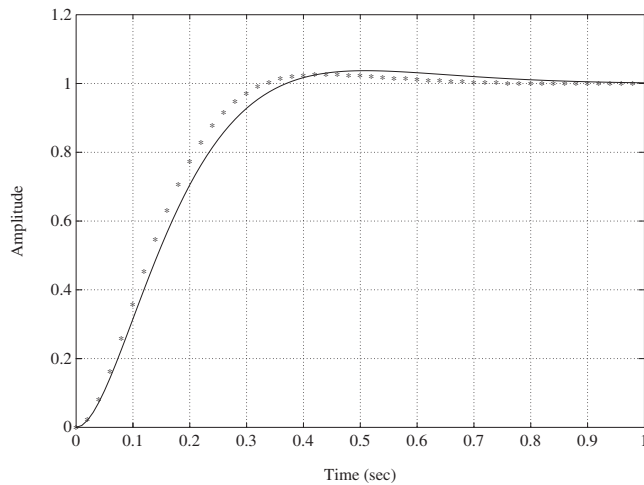


**FIGURE CP13.6**  
CONTINUED: Using the **rlocus** and **rlocfind** functions.

**CP13.7** Using root locus methods, we determine that an acceptable compensator is

$$G_c(s) = 11.7 \frac{s + 6}{s + 20} .$$

With a zero-order hold and  $T = 0.02$  sec, we find that



**FIGURE CP13.7**  
System step response (\* denotes sampled-data response).

$$D(z) = \frac{11.7z - 10.54}{z - 0.6703}.$$

The closed-loop step response is shown in Figure CP13.7.