Approximation by product means (N,p,q) (E,s) of the Fourier and its Conjugate series in $W(L_r,\xi(t))$ $(r\geq1)$ class.

A major project report submitted in partial fulfillment of the requirements for the degree of

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By

SHAKSHI TYAGI

(Roll no. 18SBAS2040002)

Under the supervision of

Dr. ARADHANA DUTT JUHARI

To

The Department of Mathematics

School of Basic and Applied Sciences

GALGOTIAS UNIVERSITY

GREATER NOIDA -201310, INDIA

School of Basic and Applied Science

CERTIFICATE

This is to Certify that to certify that the work contained in this report entitled **"Approximation by product means (N,p,q)(E,s) of the Fourier and its conjugate series in W(L_r,ξ(t)) (r≥1) class"** submitted by Shakshi Tyagi Roll no. 18SBAS2040002,Department of Mathematics, Galgotias university , Greater Noida towards the requirement of the course MSCM9999 Project has been carrying out by her under my guidance and supervision. The final project MSCM9999 is completed satisfactorily towards fulfilling the requirements for submission of Final Project in the 4th semester ,2020.The results obtained ,in this project report have not been submitted in part or full , to any other university or institution for degree or Diploma.

(Signature)
Prof.(Dr.) Dhabalendu Samanta
(Name of HOD/Division Chair)
Name of Division
School of Basic & Applied Sciences
(Galgotias University)

(Signature)
Prof. (Dr.)Aradhana Dutt Jauhari
(Name of Supervisor with Designation)
Name of Division

(Signature)
Prof.(Dr.) Arvind Kumar Jain
Dean
School of Basic & Applied Sciences
(Galgotias University)

CANDIDATE DECLARATION

I hereby declare that the dissertation entitled **"Approximation by product means (N,p,q)(E,s) of the Fourier and its conjugate series in W(L_r,ξ(t)) (r≥1) class"** submitted by me in partial fulfillment for the degree of M.Sc. in Mathematics to the Division of Mathematics, School of Basic and Applied Science, Galgotias University, Greater Noida , Uttar Pradesh, India is my original work. It has not been submitted in part or full to this University of any other Universities for the award of diploma or degree.

(Signature)
Shakshi Tyagi

Name of student
Enroll. No. 18032040005
M.Sc. Mathematics (IVth Semester)
Division of Mathematics
School of Basic and Applied Science
Galgotias University
Greater Noida
Uttar Pradesh, India

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(Signature)
Shakshi Tyagi
Name of student

SYNOPSIS

In this present work, we proved two theorems belonging to generalised $W(L_r,\xi(t))$ $(r\geq 1)$-class by (N,p,q)(E,s) product mean for Fourier and its Conjugate Fourier series. To prove these theorems, we used Lemmas and after proving this we are presenting application in form of corollaries. In the last we are giving application of our work in different field of mathematics as well as Engineering.

In 2016 PRADHAN,T. et al.[11]worked on Approximation of signals belonging to generalised Lipschitz class using $(\overline{N},p_n,q_n)(E,s)$ – summability mean of Fourier series and In 2012 MISHRA,V.N. et al.[8] worked on Product Summability Transform of conjugate series of Fourier series. We advanced our work in new direction by using (N,p,q)(E,s) product mean for Fourier and its conjugate Fourier series.

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1. INTRODUCTION

In most historical accounts of the theory of divergent series considerable stress is laid on the dictum of Abel and Cauchy with regard to their use and its impact on the study of such series while not wishing to deny the importance of this influence, this has doubtless arisen from the fact that the excellent historical discussion given in Borel's "Lecons Sur Les Series Divergentes" has generally been accepted as having broad scope than it actually possesses.

The extent to which the study of divergent series was continued, both in England and Germany, during the first sixty years of the 19th century has been clearly pointed out by Burkhardt in his paper of 1911 in the Mathematische Annalen, Uber den Gebraruch divergenter Reihen in der Zeit von 1750-1860. There was a gradually development of notion of rigor during that same period, one may naturally inquire why the rigorous treatment of divergent series did not get an earlier start.

I think that the reason is not far to seek. From the point of view of relative frequency and relative simplicity, convergent series bear somewhat the same relationship to series in general that analytic do to general functions. Just as it was not feasible for mathematicians to undertake the study of very general types of functions until they had a considerable understanding of the highly special but particularly important class of analytics functions, so it was exceedingly difficult to build a rigorous theory of divergent series before the theory of convergent series had reached some degree of completeness.

The study of the theory of trigonometric approximation is of great practical importance. Mainly speaking, signals are treated as function of one variable and images are represented by functions of two variables.

The study of the concept is directly linked to the emerging area of information technology. Approximation by trigonometric polynomial is at the heart of approximation theory. The most important trigonometric polynomial used in the area approximation theory obtained by linear summation method of Fourier series of 2 – periodic function on real lines.

In **1975** KHAN, H.H. [7] have worked on the degree of approximation to a function by triangular matrix of its Fourier series 1; In **1987** CHANDRA, P. [2] have worked on the degree of approximation of continuous functions.

In **1988** CHANDRA,P.[2]have worked on the degree of approximation of continuous functions; In **2012** MISHRA,V.N. et.al.[8] worked on product Summability Transform of conjugate series of Fourier series; In **2013** MISHRA, V.N. etal.[9]have worked on L_r-Approximation of signals (functions) belonging to weighted $W(L_r, \xi(t))$-class by $C1.N_p$ summability method of conjugate series of its Fourier series. In **2014** DEEPMALA et al.[4]have worked on Trigonometric Approximation of signals (functions)belonging to the $W(L^r,\xi(t))$ $(r\geq1)$- class by $(E,q)(q\geq0)$-means of the conjugate series of its Fourier series.

In **2014** SHARMA,K et al.[13] worked on degree of approximation of function belonging to $W(L^r,\xi(t))$ $(r\geq1)$- class product summability transform. In **2016** ACAR,T. and

MOHIUDDINE ,S.A.[1] have worked on Statistical (C,1)(E,1) summability and Korovkin's theorem; In **2016** PRADHAN ,P. et al.[11] have worked on approximation of signals belonging to generalised Lipchitz class using $(\overline{N}, p_n, q_n)(E,s)$ – summability mean of Fourier series. In 2017 SINGH,M.V et al.[12] have worked on approximation of functions in the generalized Zygmund class using Hausdorff means.
In **2017** RAY, S. et al.[12] worked on some sequences spaces and matrix transformation with Vedic relations; In **2018** SONKER ,S. and MUNJAL,A. [14]have worked on generalized absolute Riese Summability factor of infinite series; In **2019** SONKER, S., MUNJAL, A.[13] worked on absolute $|C,1|k$ summability factor of improper integrals .
In **2019**SONKER, S., MUNJAL, A. [16] have worked on sufficient conditions for absolute Cesáro Summable factor.
In **2020** JAUHARI, A.D.[5] have worked on a study on degree of approximation by product means of the Fourier series in a $W(L_r, \xi(t))$ class.

2. DEFINITION AND NOTATIONS

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex.

Such that

$$P_n=p_1+p_2+\ldots\ldots\ldots+p_n=\sum_{r=0}^{n} p_r\to \infty,\text{as } n\to \infty$$

$$Q_n=q_1+q_2+\ldots\ldots\ldots+q_n=\sum_{r=0}^{n} q_r\to \infty,\text{as } n\to \infty$$

$$R_n=p_0q_n+p_1q_{n-1}+\ldots\ldots+p_nq_0=\sum_{r=0}^{n} p_r q_{n-r} \to \infty,\text{as } n\to \infty \qquad (2.1)$$

Given two sequences $\{p_n\}$ and $\{q_n\}$ convolution $(p*q)$ is defined as

$$R_n=(p_n=q_n)= \sum_{r=0}^{n} p_{n-r}q_r \qquad (2.2)$$

Definition 2.1: Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

We write,

$$T_N^{p,q}=\frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v \qquad (2.3)$$

If $R_n\neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_N^{p,q}\}$.

If $t_N^{p,q}\to S$,as $n\to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ or sequence $\{s_n\}$ is summable to S by

$$S_n\to S\ (N, p, q) \qquad (2.4)$$

Definition 2. 2. The (E, s) transform is defined as the n^{th} partial sum of (E, s) Summability is given by

$$E^s=(E,s)=\frac{1}{(1+s)^n}\sum_{v=0}^{n}\binom{n}{v}s^{n-v}S_v \text{ as } n\to \infty \qquad (2.5)$$

Definition 2.3: The (E, s) transform of the (N, p, q) (E, s) product transform.

$$T_N^{p,q,E} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k E_k^s$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{n-v}\, S_v\right\} \qquad (2.6)$$

Then the series Σa_n is said to be summable by (N,p,q)(E,q)means.

The generalized weighted $W(L_r,\xi(t))$,($r\geq 1$)- class is generalization of Lipα, Lip (α,r) and Lip ($\xi(t)$,r) classes. Therefore, in the present paper, a theorem on degree of approximation of conjugate of signals belonging to the generalized weighted $W(L_r,\xi(t))$, ($r\geq 1$) class by (N,p,q)(E,s) product summability means of conjugate series of Fourier series.

Let f(x) be a 2π- periodic function and integrable in the Lebesgue sense. The Fourier series of f(x) at any point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\equiv \sum_{n=1}^{\infty} A_n(x) \qquad (2.7)$$

and the conjugate series of the Fourier series (2.7) is given by

$$\overline{f(x)} \sim \sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \qquad (2.8)$$

The L_∞ norm of a function f: R→ R is

$$\| f \|_\infty = \sup\{|f(x)|; x \in R\}$$

and the L_α norm is defined by-

$$\| f \|_\alpha = \left(\int_0^{2\pi} |f(x)|^\alpha\right)^{\frac{1}{\alpha}} dx \ , \ \alpha \geq 1 \qquad (2.9)$$

And $f\epsilon Lip(\alpha,r)$,for $0\leq x\leq 2\pi$, if

$$[\int_{[0,2\pi]}|f(x+t)-f(x)|^r dx]^{1/r}=O(|t|^{\alpha}) \text{ for } 0<\alpha\leq 1,\ t>0, r\geq 1.$$

Again, $f\epsilon Lip(\xi(t),r)$, if

$$\|f(x+t)-f(x)\|_r= [\int_{[0,2\pi]}|f(x+t)-f(x)|^r dx]^{1/r}=O(\xi(t)),\ r\geq 1\ ,\ t>0,$$

here $\xi(t)$ is a positive increasing function.

Similarly, $f\epsilon W(L_r,\xi(t))$, if

$$\|[f(x+t)-f(x)]\sin^{\beta}x\|_r=[\int_{[0,2\pi]}|[f(x+t)-f(x)]\sin^{\beta}x|^r dx]^{1/r}$$

$$=O(\xi(t)),\quad \beta\geq 0.$$

Further as regards to the norm in L_∞ and L_r spaces, we may recall that L_∞ norm of a function $f:R\rightarrow R$ is defined by

$$\|f\|_\infty=\sup\{|f(x)|:x\epsilon R\}$$

And L_r norm of a function $f:R\rightarrow R$ is defined by

$$\|f\|_r=(\int_{[0,2\pi]}|f(x)|^r dx)^{1/r},\ r\geq 1.$$

Next the degree of approximation of a function $f: R\rightarrow R$ by a trigonometric polynomial t_n of order n under $\|\ .\ \|_\infty$ is defined by

$$\|\ t_n - f(x)\ \|_\infty=\sup\{|t_n(x)-f(x)|:x\epsilon R\}$$

and the degree of approximation of $E_n(f)$ of a function $f\epsilon L_r$

is given by $$E_n(f)=\min\|\ t_n-f\|_r.$$

$$\phi(t) = f(x+t)+f(x-t)-2f(x)$$

$$\varphi(t)=f(x+t)+f(x-t)-2f(x)$$

$$Q_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}\, q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\sin\left(v+\frac{1}{2}\right)t}{sin\frac{t}{2}}\}.$$

$$\bar{Q}_n(t)=\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k} q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\cos(v+\frac{1}{2})t}{sin\frac{t}{2}}\}.$$

3. THEOREMS

Various Mathematician MUKHERJEE,S and KHAN ,A.J.[10], JUHARI, A.D.[5],MISHRA,V.N. et al.[8],SAXENA,K. and VERMA,S.[17] and PRADHAN, T.et al.[11] worked on different topic of Product summability.

In 2016 PRADHAN, T.et al. [11] have worked on "Approximation of signals belonging to generalized Lipchitz class using ($\overline{N}$,p_n,q_n)(E,s) – summability mean of Fourier series."In **2012** MISHRA, V.N. et.al. [8] worked on "product Summability Transform of conjugate series of Fourier series."

Now we advance our study in new direction by taking product summability mean (N,p,q)(E,s) for Fourier and its conjugate series.

We prove two theorems on Fourier and its conjugate series.

THEOREM 3.1 Let 'f' be a 2π periodic function which is integrable in Lebesgue sense in [0,2π]. If fϵW(L_r,ξ(t)) class , then its degree of approximation is given by

$$\|T_N^{(E,s)}-f\|_r=O(n\xi(\frac{1}{n+1})(n+1)^{\beta+1/r}) \qquad (3.1.1)$$

Where ,$T_N^{E,s}$ is the (N,p,q)(E,s) transform of s_n provided ξ(t) satisfies the following conditions $\{\frac{\xi(t)}{t}\}$ be a decreasing sequence .

$$[\int_0^{\frac{1}{n+1}} (\frac{t|\phi(t)|}{\xi(t)})^r \sin^{\beta r}t\, dt]^{1/r}=O(\frac{1}{n+1}) \qquad (3.1.2)$$

And

$$[\int_{\frac{1}{n+1}}^{\pi}(t^{-\delta} \frac{|\phi(t)|}{\xi(t)})^r\, dt\,]^{1/r}=O(\frac{n}{(n+1)^{\delta}}) \qquad (3.1.3)$$

THEOREM 3.2 if $\overline{f(x)}$, conjugate to a 2π- periodic function f(x) , belongs to the generalized weighted W(L_r,ξ(t)) (r≥1) – class , then its degree of approximation by (N,p,q)(E,s) product summability means of conjugate series of Fourier series is given by

$$\| \overline{N^{p,q}} - \bar{f} \|_r = O\{ (n+1)^{v+1/r} \xi(\frac{1}{n+1}) \} \qquad (3.2.1)$$

Here ξ(t) satisfies the following conditions:

$$(\int_0^{\frac{\pi}{(n+1)}} \frac{(t|\varphi(t)|}{\xi(t)})^r \sin^{vr} (\frac{t}{2})\, dt)^{1/r} = O(\frac{1}{n+1}) \qquad (3.2.2)$$

$$(\int_{\frac{\pi}{(n+1)}}^{\pi} (\frac{t^{-\delta} \; |\varphi(t)|}{\xi(t)})^r dt)^{1/r} = O(n+1)^{\delta} \qquad (3.2.3)$$

$\{\frac{\xi(t)}{t}\}$is non increasing sequence in "t"

4. LEMMAS

To prove our theorems we required following Lemmas:

Lemma 4.1.$|Q_n(t)|=O(n)$, for $0\le t\le\frac{1}{n+1}$.

Proof: $|Q_n(t)|=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right|$

$$\le\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,(2v+1)\frac{\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}(2k+1)\left|\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\right|\right\}$$

$$\le\frac{1}{2\pi R_n}\left(\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}(1+s)^k\right\}2k+1\right)$$

$$\le\frac{(2n+1)}{2\pi Rn}\left(\sum_{k=0}^{n} p_{n-k}q_k\right)$$

$$\le\frac{(2n+1)}{2\pi}$$

$$\le\frac{1}{\pi}\left(n+\frac{1}{2}\right)$$

$$=O(n)$$

Lemma 4.2 $|Q_n(t)|=O\left(\frac{1}{t}\right)$, for $\frac{1}{n+1}<t\le\pi$.

Proof: $|Q_n(t)|=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right|$

$$\le\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\pi}{t}\right\}\right|$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\left|\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\right|\frac{\pi}{t}\}$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}(1+s)^k\}\frac{\pi}{t}$$

$$\le\frac{1}{2\pi R_n}\ O(\frac{\pi}{t}\sum_{k=0}^{n} p_{n-k}\ q_k)$$

$$=O\ (\frac{1}{t})$$

Lemma 4.3: $\overline{|Q}_n(t)|= O(\frac{1}{t})$, for $0<t\le\frac{\pi}{(n+1)}$.

Proof. For $0<t\le\frac{\pi}{(n+1)}$, $\sin(\frac{t}{2})\ge(\frac{t}{\pi})$ and $|\cos nt|\le 1$.

$$\overline{|Q}_n(t)|= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\frac{\cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\}\right|$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\left|\frac{\cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right|\}$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\frac{|\cos(v+\frac{1}{2})t|}{|\sin(\frac{t}{2})|}\}$$

$$\le\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\frac{1}{\frac{t}{\pi}}\}$$

$$=\frac{1}{2tR_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\}$$

$$=\frac{1}{2tR_n}\sum_{k=0}^{n} p_{n-k}q_k\{\frac{1}{(1+s)^k}(1+s)^k\}$$

$$=\frac{1}{2tR_n}O(\sum_{k=0}^{n} p_{n-k}q_k)$$

$$=O(\frac{1}{t})$$

Lemma 4.4: $\overline{|Q}_n(t)|=O(\frac{1}{t})$, for $0<t\le\pi$.

Proof. For $0<\frac{\pi}{(n+1)}\leq t\leq\pi$, $\sin(\frac{t}{2})\geq\frac{t}{\pi}$.

$$\overline{|Q}_n(t)| = \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,e^{i\left(v+\frac{1}{2}\right)t}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,e^{ivt}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}S^{k-v}\,e^{ivt}\right\}\right|+\frac{1}{2tR_n}\left|\sum_{k=\delta}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}S^{k-v}\,e^{ivt}\right\}\right|$$

$$=K_1+K_2 \tag{4.4.1}$$

$$K_1 \leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,e^{ivt}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\right\}\right|\left|e^{ivt}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\frac{1}{(1+s)^k}(1+s)^k\right|$$

$$\leq\frac{1}{2tR_n}\left|\sum_{k=0}^{\delta-1} p_{n-k}q_k\right| \tag{4.4.2}$$

Using Abel's lemma

$$|K_2|\leq\frac{1}{2tR_n}\left|\sum_{k=\delta}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,e^{ivt}\right\}\right|$$

$$\leq\frac{1}{2tR_n}\sum_{k=\delta}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\max\left|\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\,e^{ivt}\right|$$

$$\leq\frac{1}{2tR_n}\sum_{k=\delta}^{n} p_{n-k}q_k\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\max\left|e^{ivt}\right|$$

$\leq \frac{1}{2tR_n} \sum_{k=\delta}^{n} p_{n-k} q_k \frac{1}{(1+s)^k} (1+s)^k$

$$\leq \frac{1}{2tR_n} \sum_{k=\delta}^{n} p_{n-k} q_k \tag{4.4.3}$$

using (4.4.2) and 4.4.3) in (4.4.1)

$\overline{|Q}_n(t)| \leq \frac{1}{2tR_n} \left|\sum_{k=0}^{\delta-1} p_{n-k} q_k\right| + \frac{1}{2tR_n} \sum_{k=\delta}^{n} p_{n-k} q_k$

$\leq \frac{1}{2tR_n} O\left(\sum_{k=\delta}^{n} p_{n-k} q_k\right)$

$= O\left(\frac{1}{t}\right)$

5. PROOF OF THE THEOREMS

5.1 We shall prove the theorem 3.1 for Fourier series.

Proof: $S_n(f)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+\frac{1}{2})}{sin\frac{t}{2}}dt$

$$T_N\text{-}f(x)=\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q\int_0^{\pi}\frac{\phi(t)}{sin\frac{t}{2}}\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}\sin(v+\frac{1}{2})t\,dt\}$$

$$=[\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}]\phi(t)Q_n(t)\,dt$$

$$=I_1+I_2 \qquad \text{(Say)} \qquad (5.1.1)$$

Now $|I_1|\leq\int_0^{\frac{1}{n+1}}|\phi(t)||Qn(t)|dt$

$$=\int_0^{\frac{1}{n+1}}|\frac{t\phi(t)sin^{\beta}t}{\xi(t)}\frac{\xi(t)Qn(t)}{tsin^{\beta}t}|dt$$

Apply Hölder's inequality and Lemma 4.1

$$I_1\leq[\int_0^{\frac{1}{n+1}}|\frac{t\phi(t)sin^{\beta}t}{\xi(t)}|^r dt]^{1/r}[\lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{n+1}}|\frac{\xi(t)Qn(t)}{tsin^{\beta}t}|^s dt]^{1/s} \text{ for some } 0<\epsilon<\frac{1}{n+1}$$

$$=O(\frac{1}{n+1})[\lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{n+1}}|\frac{\xi(t)o(n)}{tsin^{\beta}t}|^s dt]^{1/s} \quad \{\text{using Lemma 4.1}\}$$

$$=O(\frac{n}{n+1})[\lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{n+1}}|\frac{\xi(t)}{tsin^{\beta}t}|^s dt]^{1/s}$$

By using second mean value theorem, we have

$$|I_1|=O(n\xi(\frac{1}{n+1})\{\int_{\varepsilon}^{\frac{1}{n+1}}(\frac{1}{t^{1+\beta}})^s dt\}^{1/s}$$

$$=O(n\xi(\frac{1}{n+1}))\ \{\int_0^{\frac{1}{n+1}}\frac{dt}{t^{s(1+\beta)}}\}^{1/s}$$

$$=O(n\xi(\frac{1}{n+1}))\ \{\int_0^{\frac{1}{n+1}} t^{-(1+\beta)s}dt\ \}^{1/s}$$

$$=O(n\xi\ (\frac{1}{n+1}))\ \{[t^{-(1+\beta)s+1}\frac{1}{-(1+\beta)s+1}\ ]_0^{1/n+1}\}^{1/s}$$

$$= O(n\xi\ (\frac{1}{n+1}))\ \{(\frac{1}{n+1})^{1-s-\beta s}/(1-s-\beta s)\}^{1/s}$$

$$=O(n\xi\ (\frac{1}{n+1}))\ \{(\frac{1}{n+1})^{1/s-1-\beta}\frac{1}{\frac{1}{(s}-1-\beta)}\}$$

$$=O(n\xi\ (\frac{1}{n+1}))[\frac{1}{n+1}]^{-1/r-\beta})$$

$$|I_1|=O\ (n\xi\ (\frac{1}{n+1})\ (n+1)^{1/r+\beta}). \qquad (5.1.2)$$

Now, $|I_2|\leq [\int_{\frac{1}{n+1}}^{\pi}|t^{-\delta}\ \frac{|\phi(t)|}{\xi(t)}\sin^{\beta}t\ \frac{\xi(t)Qn(t)}{t^{-\delta}sin^{\beta}\ t}|dt$

Using Hölder’s inequality and lemma 4.2.

$$|I_2|\leq [\int_{\frac{1}{n+1}}^{\pi}|t^{-\delta}\ \frac{|\phi(t)|}{\xi(t)}\sin^{\beta}t)^{r}dt\ ]^{1/r}\ [\int_{\frac{1}{n+1}}^{\pi}|\ \frac{\xi(t)Qn(t)}{t^{-\delta}sin^{\beta}\ t}\ |^{s}dt]^{1/s}$$

$$=O(n\ (n+1)^{-\delta})(\int_{\frac{1}{n+1}}^{\pi}(\xi(t)\ \frac{1}{t^{1-\delta+\beta}})^{s}dt\ )^{1/s}$$

$$I_2=\ \{\int_{\frac{1}{\pi}}^{n+1}(\xi(\frac{1}{y})\frac{1}{y^{\delta-1-\beta}})^{s}\frac{1}{y^2}dy\ \}^{1/s}$$

Using second mean value theorem

$$=\xi\ (\frac{1}{n+1})\ \{\frac{1}{y^{(\delta-\beta)S+s}}dy\ \}^{1/s}$$

$$=O(n\ (n+1)^{-\delta})\xi\ (\frac{1}{n+1})\ \{\frac{1}{y^{\delta S+s-\beta S}}dy\ \}^{1/s}$$

$$=O(n\,(n+1)^{-\delta})\xi\,(\frac{1}{n+1})\,\{[y^{-\delta s-s+\beta s+1}\,\frac{1}{\delta s+s-\beta s+1}]_{1/\pi}^{n+1}\}^{1/s}$$

$$=O(n\,(n+1)^{-\delta})\xi\,(\frac{1}{n+1})\,[\frac{1}{\delta s+s-\beta s+1}[(n+1)^{-\delta s-s+\beta s+1}-(\frac{1}{\pi})^{-\delta s-s+\beta s+1}]^{1/s}$$

$$=O(n\,(n+1)^{-\delta})\xi\,(\frac{1}{n+1})\,[\frac{1}{\delta s+s-\beta s+1}[(n+1)^{-\delta-1+\beta+1/s}-(\frac{1}{\pi})^{-\delta-1+\beta+1/s}]$$

$$=O(n\,(n+1)^{-\delta})\xi\,(\frac{1}{n+1})\,[\frac{1}{\delta s+s-\beta s+1}(\frac{1}{(n+1)^{\delta-\beta-1/r}})]$$

$$=O(n\xi\,(\frac{1}{n+1})\frac{1}{(n+1)^{-\frac{1}{r}-\beta}})$$

$$|I_2|=O(n\xi(\frac{1}{n+1})\,(n+1)^{\beta+1/r}). \qquad (5.1.3)$$

Using (5.1.2) and (5.1.3) in (5.1.1)

$$|T_N-f(x)|=O(n\xi\,(\frac{1}{n+1})\,(n+1)^{1/r+\beta}$$

$$\|T_N^{(E,s)}-f\|=O(n\xi(\frac{1}{n+1})(n+1)^{\beta+1/r}).$$

5.2 We shall prove the theorem 3.2 for conjugate series.

Proof: $\overline{s_n(f)}-\overline{f(x)}=\frac{1}{2\pi}\int_0^{\pi}\varphi(t)\frac{\cos(n+\frac{1}{2})}{\sin(\frac{t}{2})}$ dt

$$\overline{N^{p,q}}-\overline{f(x)}=\frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k\int_0^{\pi}\frac{\varphi(t)}{\sin\frac{t}{2}}\Big[\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}S^{k-v}\frac{\cos(n+\frac{1}{2})}{\sin(\frac{t}{2})}dt\Big]$$

$$=\Big[\int_0^{\frac{\pi}{(n+1)}}+\int_{\frac{\pi}{(n+1)}}^{\pi}\Big]\varphi(t)\,Q_n(t)\,dt$$

$$=I_1+I_2 \qquad (5.2.1)$$

$$|I_1|\le\int_0^{\frac{\pi}{(n+1)}}|\varphi(t)||Q_n(t)|dt$$

$$\le\int_0^{\frac{\pi}{(n+1)}}\Big|\frac{t\,\varphi(t)\operatorname{sinv}(\frac{t}{2})}{\xi(t)}\frac{\xi(t)Qn(t)}{t\operatorname{sinv}(\frac{t}{2})}\Big|\,dt$$

Apply Hölder's inequality

$$\le\Big[\int_0^{\frac{\pi}{(n+1)}}\Big(\frac{t\,\varphi(t)\operatorname{sinv}(\frac{t}{2})}{\xi(t)}\Big)^r dt\Big]^{1/r}\Big[\int_0^{\frac{\pi}{(n+1)}}\Big(\frac{\xi(t)Qn(t)}{t\operatorname{sinv}(\frac{t}{2})}\Big)^s dt\Big]^{1/s}$$

$$=O\Big(\frac{1}{n+1}\Big)\Big[\int_0^{\frac{\pi}{(n+1)}}\Big(\frac{\xi(t)Qn(t)}{t\operatorname{sinv}(\frac{t}{2})}\Big)^s dt\Big]^{1/s} \qquad \text{lemma (4.3)}$$

$$=O\Big(\frac{1}{n+1}\Big)\Big[\int_0^{\frac{\pi}{(n+1)}}\Big(\frac{1}{t^2}\frac{\xi(t)}{\operatorname{sinv}(\frac{t}{2})}\Big)^s dt\Big]^{1/s}$$

$$=O\Big(\frac{1}{n+1}\Big)\Big[\int_0^{\frac{\pi}{(n+1)}}\Big(\frac{1}{t^2}\frac{\xi(t)(\frac{t}{2})^v}{\operatorname{sinv}\left(\frac{t}{2}\right)(\frac{t}{2})^v}\Big)^s dt\Big]^{1/s}$$

$=O(\frac{1}{n+1})\ [\int_0^{\frac{\pi}{(n+1)}}(\frac{\xi(t)}{t^{2+v}})^s dt]^{1/s}$

Using mean value theorem

$=O((\frac{1}{n+1})\ \xi(\frac{\pi}{n+1}))\ [\int_\epsilon^{\frac{\pi}{(n+1)}}(\frac{1}{t^{2+v}})^s dt]^{1/s}$

$\xi(\frac{1}{n+1}) \leq \pi\xi(\frac{1}{n+1})$

$=O((\frac{1}{n+1})\ \pi\xi(\frac{1}{n+1})\ [\int_\epsilon^{\frac{\pi}{(n+1)}}(\frac{1}{t^{2s+vs}})dt]^{1/s})$

$=O((\frac{1}{n+1})\ \xi(\frac{1}{n+1})\ [(\frac{\pi}{n+1})^{-2s-vs+1} - (\epsilon)^{-2s-vs+1}]^{1/s})$

$=O((\frac{1}{n+1})\ \xi(\frac{1}{n+1})[(n+1)^{2+v-1/s})$

$=O((n+1)^{\delta}[\int_{\frac{\pi}{(n+1)}}^{\pi}(\frac{\xi(\frac{1}{y})}{y^{-\delta+1+v}})^s\frac{1}{y^2}dy\ ]^{1/s})$

$\xi(t)$ is a positive increasing so, $\frac{\xi(\frac{1}{y})}{\frac{1}{y}}$ is also a positive increasing function and using second mean value theorem

$=o((\frac{1}{n+1})\ \xi(\frac{1}{n+1})\ (n+1)^{2+v-1/s})$

$=O((\frac{1}{n+1})\ \xi(\frac{1}{n+1})\ (n+1)^{v+1+1/r})$

$I_1=O(\xi(\frac{1}{n+1})\ (n+1)^{v+1/r})$ (5.2.2)

$I_2 \leq \int_{\frac{\pi}{(n+1)}}^{\pi} |\varphi(t)||Q_n(t)|dt$

$\leq [\int_{\frac{\pi}{(n+1)}}^{\pi}(t^{-\delta}\frac{\varphi(t)\mathrm{sinv}(\frac{t}{2})}{\xi(t)})^r dt\ ]^{1/r}\ [\int_{\frac{\pi}{(n+1)}}^{\pi}(\frac{1}{t^{-\delta}}\frac{\xi(t)|Qn(t)|}{\mathrm{sinv}(\frac{t}{2})})^s dt]^{1/s}$

$=O((n+1)^{\delta}\,[\int_{\frac{\pi}{(n+1)}}^{\pi}(\frac{1}{t^{-\delta}}\frac{\xi(t)|Qn(t)|}{\sin v(\frac{t}{2})})^{s}dt]^{1/s})$

$=O((n+1)^{\delta}\,[\int_{\frac{\pi}{(n+1)}}^{\pi}(\frac{1}{t^{-\delta+1}}\frac{\xi(t)}{\sin v(\frac{t}{2})})^{s}dt]^{1/s})$ lemma(4.4)

$=O((n+1)^{\delta}[\int_{\frac{\pi}{(n+1)}}^{\pi}(\frac{\xi(t)}{t^{-\delta+1+v}})^{s}dt\,]^{1/s})$

Second mean value theorem.

$=O((n+1)^{\delta}\frac{\xi(\frac{\pi}{n+1})}{(\frac{\pi}{n+1})}\,[\int_{k}^{\frac{n+1}{\pi}}(\frac{dy}{y^{-vs+\delta s+2}})^{1/s})$

$=O((n+1)^{\delta}\,(n+1)\,\xi\left(\frac{1}{n+1}\right)\{\,[\frac{y^{-\delta s+vs-2+1}}{-\delta s+vs-2+1}]_{k}^{n+1/}\pi\}^{1/s})$

$=O((n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)[(n+1)^{-\delta+v-1/s}])$

$=O(\xi\left(\frac{1}{n+1}\right)(n+1)^{\delta+1-\delta+v-1/s})$

$I_2=O(\xi\left(\frac{1}{n+1}\right)(n+1)^{v+1/r}\frac{1}{r}+\frac{1}{s}=1.$ (5.2.3)

Using (5.2.2) and (5.2.3) in (5.2.1)

$|\overline{N^{p,q}}-\bar{f}|=O\{\,(n+1)^{v+1/r}\,\xi(\frac{1}{n+1})\}$

Now using the L_r –norm of a function, we get

$\|\,\overline{N^{p,q}}-\bar{f}\,\|_r=O((n+1)^{v+1/r}\xi(\frac{1}{n+1})\,).$

6. APPLICATION

We show the application in form of corollaries.

COROLLARY 6.1 If we take q=1and s=1 our summability reduced to (N,p)(E,1) summability.

If following Hardy

$$E_n^1 = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}S^k$$

tends to S, as n tends to infinite, then an infinite series $\sum_{n=0}^{\infty} u_n$ with the partial sums S_n is said to be summable (E,1) to the definite number S.

COROLLARY 6.2 The (C,1) transform of the (E,1) transform E_n^1 defines the (C,1)(E,1) transform of the partial sum S_n of the series $\sum u_n$.thus if

$$(CE)_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} E_n^1 \rightarrow S, \quad \text{as } n \rightarrow \infty$$

Where E_n^1denotes the (E,1) transform of S_n and S is a finite constant then the series $\sum u_n$ is said to be summable by (C,1)(E,1) mean or simple (C,1)(E,1) to S.

The (N,p_n) transform of the (E,1) transform E_n^1 defines the $(N,p_n)(E,1)$ transform of the partial sums S_n of the series $\sum u_n$.

COROLLARY 6.3 If we put q=0 then our theorems 3.1 and 3.2 reduce for (N,p)(E,s) product mean belonging to generalised $W(L_r,\xi(t))(r\geq 1)$-class.

.

7. CONCLUSION

Analysis and Approximation of signals are of great importance in science and engineering because a signal conveys the attribute of some physical phenomenon. Functions in $L_p(p\geq 1)$ spaces are assumed to be most appropriate for practical purposes; for example L_1,L_2 and L_∞ are of particular interest for engineers in digitization

Fourier methods are commonly used for signal analysis and system design in modern telecommunications, radar and image processing systems. The theory of classical Fourier analysis can be extended to discrete time signals and leads to many effective algorithms that can be directly implemented on general computers or special purpose digital signal processing devices.

Thus the study of error estimate of functions in various function spaces such as Lipschitz, Holder, Zygmund, Besov spaces etc using some summability means of trigonometric Fourier series also known as trigonometric Fourier approximation (TFA)in the literature has received a growing interest of investigators over the past few decades. The scientists and engineers use the properties of TFA in designing digital filters.

The problem of determining the order of best approximation plays a very important role in approximation theory .In this paper we compute the product summability (N, p, q) (E, s) for Fourier and its conjugate series.

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