Major Project
On

IMPACT OF HUMAN ACTIVITIES ON THE PERSISTANCE OF EXTINCT SPECIES

Submitted in Partial Fulfilment of the Requirement for the Degree of

Master of Science

In

Mathematics

By

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(Established under Galgotias University Uttar Pradesh Act No. 14 of 2011)

School of Basic and Applied Science

CERTIFICATE

This is to Certify that Mr. Himanshu Upadhyay has carried out his project work entitled "**Impact of Human Activities on the Persistence of Extinct Species**" under my supervision. This work is fit for submission for the award of Master Degree in Mathematics.

The final project MSCM9999 is completed satisfactorily towards fulfilling the requirements for submission of final project in the IV^{th} Semester, 2020. The results obtained, in this project report have not been submitted in part or full, to any other university or institution for degree or diploma.

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CANDIDATE'S DECLARATION

I hereby declare that the dissertation entitled "**Impact of Human Activities on the Persistence of Extinct Species**" submitted by me in partial fulfillment for the degree of M.Sc in Mathematics to the division of Mathematics, School of Basic and Applied Science, Galgotias University, Greater Noida, Uttar Pradesh, India in my original work. It has not been submitted in part or full to this University of any other Universities for the award of diploma or degree.

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ABSTRACT

The environmental carrying capacity is usually assumed to be fixed quantity in the classical predator-prey population growth models. However, this assumption is not realistic as the environment generally varies with time. In a bid for greater realism, functional forms of carrying capacities have been widely applied to describe varying environments. Modeling carrying capacity as a state variable serves as another approach to capture the dynamical behavior between population and its environment. The proposed population model is based on the ratio-dependent models that have been utilized in the study of food chains. Using a simple non-linear system, the proposed model can be linked to an intra-guild predation model in which predator and prey share the same resource. Distinct from other models, we formulate the carrying capacity proportional to a biotic resource and both predator and prey species can directly alter the amount of resource available by interacting with it. Numerical analysis is presented to illustrate the systems dynamical behavior.

INTRODUCTION

SPECIES ARE, PART OF NATURE

THEIR FORMATION IS, ART OF NATURE

PROTECT SPECIES; OTHERWISE YOU WILL LOOSE YOUR EXISTENCE.

"Protecting the planet's precious biodiversity has never been more important. Every day, the obstacles to saving native species from extinction and preserving ecosystems are growing," said **Sean T.**

We must seek out with some opportunities to bring together data, science, and technology to help solve one of the scariest environmental challenges of our time, the mass extinction of untold numbers of species.

How large the species affected through human activities? And In what time period Specie might loss unfold? Among all, Humans is at the top for Habitat destruction is the leading cause of species extinction. Many of the times, the species of a given habitat which are found across large areas are represented in the smaller areas. So habitat loss initially causes little extinction, then many only as the last remnants of habitat are destroyed **[2].**

Extinction: A Natural phenomenon-

In today's time, around 90 percent of all the organism are not alive those have ever lived on the Earth. Although it may sounds cheesy but Humans have made it worse by accelerating natural extinction rates due to our roles in habitat loss, climatic change, invasive species, disease, overfishing and hunting. "We're losing whole suites of species that have distinct ecological roles to play," says Stuart Pimm[2], Professor of Conservation at Duke University. For instance, top predators such as sea otters and sharks have dwindled, throwing their ecosystems off-kilter. Dozens of new species go extinct every day, and scientists say that more than 20,000 plants and animals are on the brink of disappearing forever. A quarter of known mammal species is at risk of extinction. The main body that tracks species decline is the International Union for the Conservation of Nature. The organization evaluates species in the wild, and, along with data from a variety of sources, categorizes their vulnerability on its Red List of Threatened Species.

As the human population grows and continue to consume biotic resources at voracious rates, we are crowding out, poisoning and eating all other species into extinction. With the world population hitting 7 billion, the Center is marking this milestone by releasing a list of species in the United States facing extinction caused by the growing human population. The 10 species represent a range of geography, as well as species diversity but all are critically threatened by the effects of human population. Some, like the Florida panther and Mississippi gopher frog, are rapidly losing habitat as the human population expands. Others are seeing their habitat dangerously altered like the small flowering sand plain gerardia in New England or, like the Bluefin tuna, are buckling under the weight of massive over-fishing. WWF has identified 33 'Priority Places' that host some of the world's richest ecosystem, warning that up to half the species in these spots face extinction if climate change is left unchecked. Some of the example species are:

1. Marine turtles (Mediterranean)-Climate change threatens feeding patterns and breeding grounds.
2. Amphibians (Amazon Guiana's)-74% of amphibians could be threatened with local extinction if global temperature rise 4.5^{o}C.
3. African wild dog (Miombo woodlands) - Hooter days potentially mean shorter hunting periods and less food for these dogs.
4. African elephant (Coastal East Africa)-Hotter temperatures and less rain will have a dire effect on elephant numbers.
5. Amur leopards (Amur-Heilong)-Mammals, such as the rare Amur leopard could be forced to migrate in warmer conditions.
6. Rock wallaby (South-west Australia)-As their habitat becomes increasingly arid, rock wallabies are expected to migrate the coast.

The mathematical models based on dynamical system for the interaction of a species with biotic resources in the presence of human population are one of the classical applications of mathematics to biology. The development and use of analytic techniques and the growth of computer power have progressively improved our understanding of these types of models.

The ecosystem services approach to conservation tries to establish the value that society derives from the natural world such that the true cost of proposed development actions becomes apparent to decision makers. Species are an integral component of ecosystems, and the value they provide in terms of services should be a standard part of ecosystem assessments. Humans depend on biodiversity in myriad ways, yet species are being rapidly lost due to human activities. How humans contribute to species extinctions? We do so in six major ways and through synergistic combinations of these ways **[8].**

1. Habitat destruction, degradation, and fragmentation.

2. Introduction of non-native (alien) species.

3. Overexploitation of biotic resources.

4. Pollution.

5. Spread of diseases.

6. Climate Change and Anthropogenic Global Warming.

The evidence for accelerating species extinctions is highly compelling, and links to the influence of humans on extinctions can be readily identified. Unfortunately, many people reject the notion that humans have profoundly altered the Earth's ecosystems, and refuse to take responsibility for caring for God's creation.

The ongoing growth in human population and resource consumption is changing the planet in fundamental ways. One consequence of this growth is the loss of biodiversity, which is typically estimated either by the net movement of species towards higher categories of extinction risk or as the rate at which species are actually going extinct. By either measure, biodiversity loss is on the rise **[9].**

The major ''systematic drivers'' of modern species loss are changes in land use (habitat loss degradation and fragmentation), overexploitation, invasive species, disease, climate change (global warming) connected to increasing concentration of atmospheric carbon dioxide, and increases in nitrogen deposition. Mechanisms for prehistoric (caused by humans >200 years ago) extinctions are likely to have been similar: overhunting, introduced predators and diseases, and habitat destruction when early people first arrived in virgin landscapes.

ALL LIFE HAS A RIGHT TO BE HERE.

In this Dynamic and Unpredictable world, where man is destroying the Natural Environment and disturbing the habitat of species, it doesn't matter whether its Aerial, Terrestrial, Aquatic.

As per IUCN REDLIST report, More than 31,000 species are threatened with extinction which is 27% of the assessed species. Out of which 41% are Amphibians, 25% are Mammals, 34% are Conifers, 14% are Birds, 30% are Sharks & Rays, 33% are Reef Corals and 27% are selected Crustaceans.

Each time species goes extinct, the world around us unravels a bit. The consequences are profound, not just in those places and for those species but for all of us. These are tangible consequential losses, such as crop pollination and water purification, but also spiritual and cultural ones.

Although often obscured by the noise and rush of modern life, people retain deep emotional connections to the wild world. Wildlife and plants have inspired our mythologies, histories and languages and how we view the world. The presence of wildlife brings joy and enriches us all and plays a crucial role in our life and absence of each extinction makes our home a lonelier and colder place for us and future generations.

No group of animals has a higher rate of endangerment than amphibians. Scientists estimate that around 6,300 known species of amphibians are at risk of extinction **[6]**. The current amphibian extinction rate may range from 25,039 to 45,474 times the background extinction rate **[7]**.

As per 2009 report on the state of birds in the United States found that 31 percent of the 800 species in the country are of conservation concern **[8]**. Globally, Bird Life International estimates

that 12 percent of known 9,865 bird species are now considered threatened, with 192 species, or 2 percent, facing an "extremely high risk" of extinction in the wild — two more species than in 2008. Habitat loss and degradation have caused most of the bird declines, but the impacts of invasive species and capture by collectors play a big role, too.
All around the world water is used for the same reasons, to provide life and nourishment to plants, animals, and humans. Sadly, humans often pollute the water resources until they are unusable. This is an example of polluted water. Humans aren't the only one who suffers from polluted water. Many animals and aquatic species become ill or die from polluted water. The damming of rivers, the dumping of various pollutants, and invasive species make aquatic ecosystems most threatened on the planet; thus there are many fish species that are endangered in both freshwater and marine habitats.

The American Fisheries Society identified 700 species of freshwater or anadromous fish in North America as being imperiled, amounting to 39 percent of all such fish on the continent **[11]**. At least 82 fish species are imperiled in North American marine waters. Across the globe, 1,851 species of fish — 21 percent of all fish species evaluated — were deemed at risk of extinction by the IUCN in 2010, including more than a third of sharks and rays.
Invertebrates, from butterflies to mollusks to earthworms to corals, are vastly diverse and though no one knows just how many invertebrate species exist, they're estimated to account for about 97 percent of the total species of animals on Earth **[12]**. Out of the 1.3 million known invertebrate species, about 9,526 species are at risk of extinction which is 30 percent of the species, evaluated by IUCN. Freshwater invertebrates are severely threatened by water pollution, groundwater withdrawal, and water projects. In the ocean, reef-building corals are declining at an alarming rate: 2008's first-ever comprehensive global assessment of these animals revealed that a third of reef-building corals are threatened.

Perhaps one of the most striking elements of the present extinction crisis is the fact that the majority of our closest relatives — the primates — are severely endangered. About 90 percent of primates who lives in tropical forests are disappearing at fast rates. This group contains monkeys, lemurs, lorids, galagos, tarsiers, apes and humans too. Almost 50 percent of the world's primate species are at risk of extinction, IUCN estimates. Overall, the IUCN estimates that half the globe's 5,491 known mammals are declining in population and a fifth are clearly at risk of disappearing forever with no less than 1,131 mammals across the globe classified as endangered, threatened, or vulnerable. In addition to primates, marine mammals — including several species of whales, dolphins, and porpoises — are among those mammals slipping most quickly toward extinction.
By photosynthesis process, plants provide the oxygen we breathe and the food we eat and are thus the foundation of most life on Earth. They're also major sources of medicines in use today. Of the more than 300,000 known species of plants, the IUCN has evaluated only 12,914 species, finding that about 68 percent of evaluated plant species are threatened with extinction.

Unlike animals, plants can't move as their habitat is destroyed, making them particularly vulnerable to extinction. Indeed, one study found that habitat destruction leads to an "extinction debt," whereby plants that appear dominant will disappear over time because they aren't able to disperse to new habitat patches **[13]**. Global warming is likely to substantially exacerbate this problem. Already, scientists say, warming temperatures are causing quick and dramatic changes in the range and distribution of plants around the world. With plants making up the backbone of ecosystems and the base of the food chain, that's very bad news for all species, which depend on plants for food, shelter, and survival.

Globally, 21 percent of the total evaluated reptiles in the world are deemed endangered or vulnerable to extinction by the IUCN — 594 species — while in the United States, 32 reptile species are at risk, about 9 percent of the total. Island reptile species have been dealt the hardest blow, with at least 28 island reptiles having died out since 1600.

KILLING TIGERS IS THE GREED NOT THE NEED : SAVE TIGERS

According to the IUCN Red List of Threatened Species, the Panthera Tigris, is listed as 'Endangered'. Throughout central, eastern and southern Asia, tiger occurred once. However, in the past 100 years, the tiger has lost more than 93 percent of its historic range and now only survives in scattered populations in 13 countries, from India to Southeast Asia, and in Sumatra, China and the Russian Far East.

The Caspian, Javan and Bali Tigers are already extinct, while the South China Tiger has not been observed for many years. Poaching and illegal killing are the major threats to the species, to meet an illicit demand in high-value tiger body parts for the Oriental medicine market. Habitat loss and overhunting of tigers and their natural prey have also caused a reduction in distribution and, over the past century, tiger numbers have fallen from about 100,000 individuals to an estimated 3,500. Tigers are included on Appendix I of the Convention for International Trade in Endangered Species (CITES), prohibiting their international commercial trade. The future of this species also depends upon conserving and protecting large areas of suitable habitat with viable populations, while working with local communities to discourage retributive killings. The thirteen Tiger Range Countries came together in an unprecedented pledge to double the world's Tiger population by 2022, the next Year of the Tiger on the Asian lunar calendar, with a goal of achieving at least 6,000 Tigers. This figure was based on a baseline global population of 3,200, agreed upon at a preparatory workshop held in Kathmandu, Nepal in October 2009; 3,200 Tigers was the IUCN Red List population estimate at that time. 2,154 tigers was an updated version of the estimate of Tiger numbers in the global Tiger estimate published in the 2010 Red List assessment *(Walston et al. 2010a)***.** This was not a complete estimate of global Tiger numbers (for example, most Amur Tigers in Russia are found in unprotected areas), but justified because Tiger status outside the source sites is generally poor and poorly known.

IUCN Guidelines *(IUCN Standards and Petitions Subcommittee 2010)* define population as the number of mature individuals, defined as "individuals known, estimated or inferred to be capable of reproduction."

Survival rate of breeding adult females is a key parameter, with models suggesting population declines when mortality of breeding females rises over 15% *(Chapron et al. 2008).* Population declines in recent years have been most pronounced outside protected areas *(Walston et al. 2010b)*. The IUCN Guidelines advise that "mature individuals that will never produce new recruits should not be counted." Thus, for the purposes of the Red List assessment in 2010, the estimated population in the Source Sites was used as a proxy for the breeding population of adult Tigers.

HOW CRUEL COULD WE BE,

HOW BLIND? CAN'T WE SEE?

THE LORD OF JUNGLES, TO BE PRAYED.

THE WILDLIFE WEALTH, TO BE SAVED.

India is home to more than 70 percent of the tigers in the world. It contributes 2967 in numbers to the Tigers Population. If I go for the state-wise, then in the top three, Madhya Pradesh has the highest number of tigers-526 big cats followed by Karnataka with 524 big cats and Uttarakhand with 442.

As per the IUCN Status, this species comes in endangered category. Project Tiger was established in 1973 and till date, there are 50 Tiger reserves in India in which the Orang, Assam being a smallest core, among 50 nationally protected areas, it has the highest tiger density. Nagarjunsagar-Srisailam Tiger Reserve is the India's largest tiger reserve. In-principle approval has been accorded by the National Tiger Conservation Authority (NTCA) for creation of the three tiger reserves in Ratapani (MP), Sunadeba (Odisha), Guru Ghasidas (Chhattisgarh). The core areas enjoys the Legal status of national park or a sanctuary where the Buffer or Peripheral areas is a mix of forest and non-forest land/managed as a multiple use area. For providing central assistance to the tiger States for tiger conservation in designated tiger reserves. The National Tiger Conservation Authority (NTCA) is a statutory body of the Ministry with an overarching supervisory / coordination role performing functions as provided in the Wildlife (Protection) Act, 1972. Pugmarks& Camera are the double sampling methods for the Tiger Census. Initially, M-STrIPES (Monitoring System for Tigers-Intensive Protection and Ecological Status) were a desktop version but now it's also used as a Android phone-based application for collecting, archiving and analyzing data. Also, the Tiger Estimation exercise is the world's largest wildlife survey effort in terms of Coverage, Intensity of Sampling and Quantum of camera trapping.

Importance of Tiger censes.

Tigers need to be counted as they are indicator of the health of the ecosystem, are linked to water security of the area, assessment of co-varieties such as signs of tigers, prey density, human interface. At an interval of four years India conducts its All India Tiger Estimation. Four cycles off the estimation have already been completed in 2006, 2010, 2014 and 2018. These estimates showed of 1411, 1706, 2226 and 2967 tigers respectively. The Wildlife (Protection) Act, 1972 was amended in the year 2006. Tiger conservation was given Statutory backing and the newly created NTCA was mandated to carry out estimation of population of tigers and its natural prey species and assess status of their habitat.

Predator-Prey interaction is fundamental. The dynamics of food webs depend critically on their structures. It is important to understand what determines the occurrence of pair wise Predator-Prey interactions and by extension, the structure of food webs. The mathematical models based on dynamic system for the interaction of a species with biotic resources in the presence of human population are one of the classical applications of mathematics to biology. We will consider the three dynamical variables to study the impact of human activities on the persistence of a species with the concept of logistic growth of one kind.

HISTORY

There was a scientist named Lotka, who proposed a model named LOTKA VOLTERRA PREY-PREDATOR MODEL in the theory of Auto catalytic chemical reaction in 1910. Lotka extended this model to "organic Systems" in 1920 using a plant species and a herbivorous animal species as an example to analyze PREDATOR PREY interaction.

There were some assumptions made in the model...

1. The prey population find ample food at all time.
2. The food supply of the predator population depends entirely on the size of prey population.
3. The rate of change of population is proportional to its size.
4. There is no environment complexity.
5. The predator has limitless appetite.

This Model consist of two population

1. **Prey Population**

$$\frac{dx}{dt} = \alpha x - \beta xy$$

2. **Predator Population**

$$\frac{dy}{dt} = \delta xy - \eta y$$

Where,

α is the growth rate,

β is the decay rate of prey population,

δ is the growth rate,

η is a decay rate of predator population.

To understand this concept more clearly we must know the kind of interaction between the species.

KINDS OF INTERACTION

- **Commensalism** – This is a relationship in which one organism gets benefits while other is getting no harmed nor helped.
  For example – Tree frog use leaves to hide.
- **Parasitism** – A relationship in which one organism get benefit while other is getting harmed.
  Example – Human and mosquito.
- **Mutualism** – A relationship in which both the organism gets benefit.
  For example – Flower need bees for pollination and bees need flower for nectar.

Since Lotka Volterra model deals with two variable but we are dealing with three dynamic variables to study the impact of human related activities on the persistence of extinct species.

MATHEMATICAL TOOLS

1. **Dynamical System-**
   A System in which a function describe the time dependence of a point in a geometrical space.

2. **Equilibrium-**
   A simplest possible solution to the dynamical system.

ANALYSIS OF NON-LINEAR DIFFERENTIAL EQUATION:

In general, rather than looking for the detailed solutions either in numerical or analytical form, one can seek the aspects of system behaviour. For example, one might ask whether there are equilibrium points and whether they are stable. In Nonlinear systems, in addition to equilibrium points, one might look for threshold effects. The approach therefore, includes characterizing in broad terms the critical aspects of the system behaviour.

An analysis of a nonlinear dynamical system may devote considerable attention to the characterization of the equilibrium points. In the linear systems, equilibrium points are basically the solutions to linear equations.

The nonlinear case is different in two essential respects:

1. First, since the equilibrium points are solutions, in this case, to nonlinear equations, finding such solutions is somewhat more of an accomplishment than in the linear case.

2. The equilibrium point distribution is potentially more complex in the nonlinear case than in the linear case.

Note: A system may have none, one or numbers of equilibrium points in virtually a spatial pattern in state space.

Thus characterization of equilibrium points is not only technically more difficult, it is a much broader question. Ultimately, interest centers not just on the existence of the equilibrium points but also on their stability properties.

STABILITY

Stability properties characterize how a system behaves if its state is initiated close to, but not precisely at a given equilibrium point.

1. If a system is initiated with the state as that of an

   equilibrium point, then it will never move.

2. When initiated close by, however, the state may remain close by, or it may move away.

Roughly speaking, an equilibrium point is stable whenever the system state is initiated near that point, the state remains near it, perhaps even tending towards the equilibrium point as time increases.

- Local Stability
- Global Stability

EQUILIBRIUM POINT

An equilibrium (or equilibrium point) of a dynamical system generated by a set of differential equations is a solution that does not change with time.

ROUTH-HURWITZ CRITERION

The local Stability behavior of equilibrium of the system can be determined by examining the sign of the eigen values of the matrix obtained by linearizing the system around the equilibrium point, known as Jacobian matrix evaluated at equilibrium point. To examine the sign of the eigen values of the Jacobian matrix of the system, the Routh-Hurwitz criterion is used. For the Routh-Hurwitz Criterion, we consider the following degree polynomial with real coefficient. Consider the n^{th}- degree polynomial with real coefficient

$$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \ldots\ldots\ldots..a_n$$

LYAPUNOV METHOD

Lyapunov's direct method is a technique for investigating stability of the equilibrium point, which generalizes how energy behaves in a physical system to an arbitrary dynamical system. The "Lyapunov function" plays the role of this energy function: it is minimum when the system is in equilibrium, it increases in value when the equilibrium is perturbed since such a perturbation is equivalent to injecting energy into the system, and if the system is stable the energy will not increase, indeed it usually decreases as the system return to equilibrium. Consider the system with an equilibrium point x_e .

Let N be a neighborhood of the equilibrium point $\frac{dV}{dt}$.

The Lyapunov function is defined as follows, V: N→ R

1. V is positive definite,
2. $\frac{dV}{dt} = \left(\frac{dV}{dx}\right)^T \dot{x} = \left(\frac{dV}{dx}\right)^T$ f(x) is a negative semi-definite.

Theorem –

If there exist a lyapunov function in neighborhood of an equilibrium point x_e for the system then the equilibrium point x_e is **locally stable**.

Theorem –

If there exists a lyapunov function in a neighborhood of the equilibrium point x_e for the system with the additional property that $\frac{dV}{dt}$ is negative definite then the equilibruim point x_e is **locally asymptotic stable**.

STABILITY OF EQUILIBRIUM POINT

- An equilibrium point x_e of the system is said to be **locally stable** if for a given $\exists\ \delta = \delta(\in, t_0) > 0$, such that any solution x(t) of the system satisfies $|x(t) - x_e| <\in$ whenever $|x_0 - x_e| < \delta$. And it is said to be uniformly stable if does not depend on t_0 .
- An equilibrium point of the system is said to be **asymptotic stable** if it is stable and, in addition, there exists $\delta_0 > 0$ such that whenever $|x_0 - x_e| < \delta_0$ then $x(t)$ as t_0 .
- An equilibrium point of the system is said to be **unstable** if it is not stable.

GLOBAL STABILITY

To discuss the global stability of an equilibrium point, we first define the region of attraction as follows:

- **Def** – The region of attraction Ω is the set of all points x in the domain of $f(x(t))$ such that the solution of system is defined for all $t \geq t_0$ and converges to an equilibrium point x_e as t tends to infinity.
- **Def**- The equilibrium point x_e of the system is said to be globally asymptotically stable if the region of attraction is the whole spaceR^{n}.
- **Theorem**- If there exists a lyapunov function define on a region of attraction Ω of the state space and containing equilibrium point x_e for the system such that $V(x)$ has a unique minimum at x_e w.r.t. all other points in Ω then the equilibrium point x_e is non-linearly stable.

WHAT IS MATHEMATICAL MODELING?

Mathematical model defines a formulation or an equation that expresses the essential characteristics of a physical system or process in mathematical terms. It is useful to study the effects of different components and prediction about behavior. These are basically of two types- Continuous model and Discrete model.

- Continuous model uses calculus and continuous function theory, integrals and differential equations. For example, these are used to model transition among infinite number of states and smooth transitions.
- Discrete models are based on logic and set theory. These are state transition functions and are used to model transition among a finite number of states.

In mathematical modeling, we translate those beliefs into the language of mathematics. This has many advantages

1. Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.
2. Mathematics is a language with well-defined rules for manipulations.
3. All the results that mathematicians have proved over hundreds of years are at our disposal.
4. Computers can be used to perform numerical calculations.

There is a large element of compromise in Mathematical Modeling. Hence the first level of compromise is to identify the most important parts of the system. These will be included in the model, the rest will be excluded. The second level of compromise concerns the amount of mathematical manipulation which is worthwhile. Although mathematics has the potential to prove general results **[26]**, these results depend critically on the form of equations used. Small changes in the structure of equations may require enormous changes in the mathematical methods. Using computers to handle the model equations may never lead to elegant results, but it is much more robust against alterations.

<u>Mathematical Modeling & its Objectives</u>

Mathematical modeling can be used for a number of different reasons. How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modeling is done. Some examples for range of expressions are:

1. Developing scientific understanding –
   Through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know);

2. Test the effect of changes in a system;

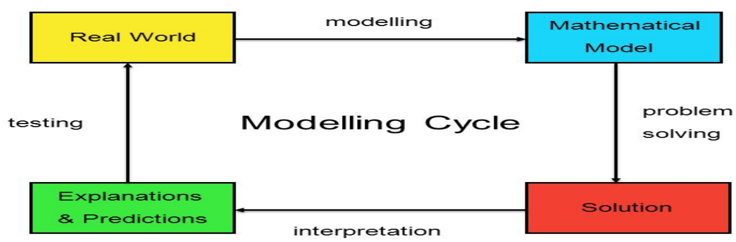
3. Aid decision making, including the tactical decisions by manager and strategic decisions by planners.

Also Mathematical Model:

- is indispensable in many applications
- is successful in many further applications
- gives precision and direction for problem solution
- enables a thorough understanding of the system modeled
- prepares the way for better design or control of a system
- allows the efficient use of modern computing capabilities **[27][28].**

HOW MATHEMATICAL MODELING SHOULD BE DONE?

It is helpful to divide up the process of modeling broadly into four categories namely, building, studying, testing and use. Although it might be nice to think that modeling projects progress smoothly from building through to use, this is hardly ever the case. In general, defects found at the studying and testing stages are corrected by returning to the building stage. Note that if any changes are made to the model, then the studying and testing stages must be repeated. A pictorial representation of potential routes through the stages of modeling is:



This process of repeated iteration is typical of modeling projects, and is one of the most useful aspects of modeling in terms of improving our understanding about how the system works.

We shall use this division of modeling activities to provide a structure for the rest of this course.

Learning about mathematical modeling is an important step from a theoretical mathematical training to an application-oriented mathematical expertise, and makes the student fit for mastering the challenges of our modern technological culture.

REASONS FOR SPECIES EXTINCTION

The first and the most important reason is **Human reproduction**. More human reproduction means More human population which implies more food and land required to survive which results in urbanization ,overexploitation of resources, overhunting and hence the pollution increases.

As human population grows and continue to consume biotic resources at voracious level, we are crowding out, poisoning and eating all other species into extinction.

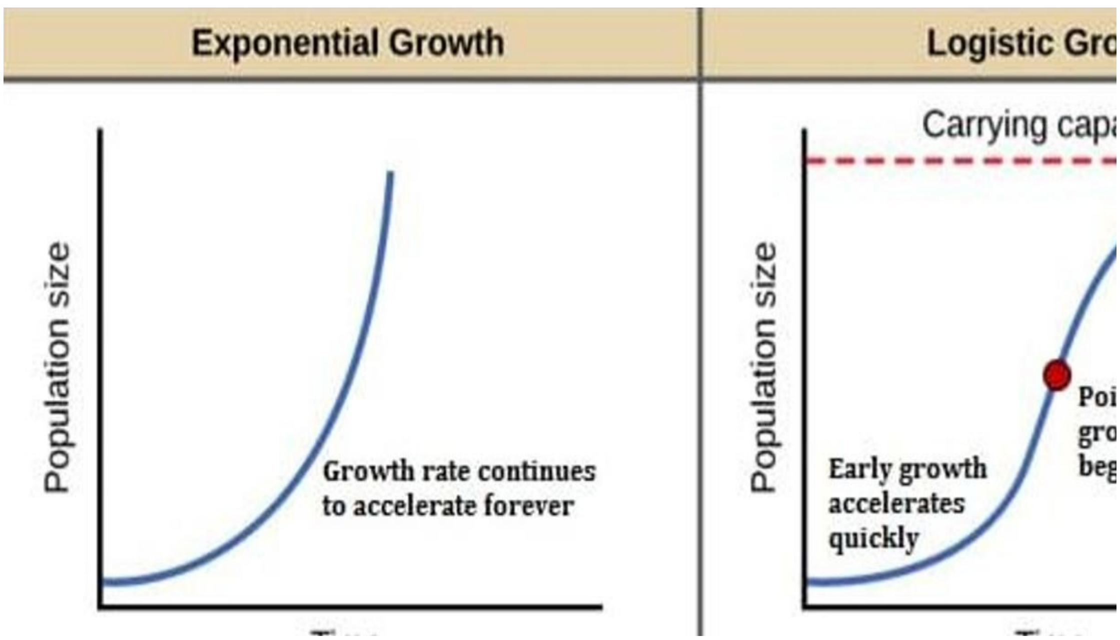
The 10 species represent a range of geography as well as species diversity but all are critically threatened by the effect of human population. For example Florida's Panther, Mississippi Gopher frog, polar bear and many others.

The Red List report by International Union for conservation of nature found that 27% of more than 100000 species are at the risk of extinction uploaded in December 2018. As human population Increases, species population decreases.

As the human population increases, population of species decreases.

So now let's study **Population Growth model**,

1. Exponential growth population model
2. Logistic growth population model**[25]**



Exponential growth model is associated with the name of Thomas Robert Malthus (1766-1834) who first realised that any species can potentially increase in numbers proportional to their population **[17]**.

$$\frac{dy}{dx} = e^x$$

Logistic growth model is associated with the name of A.G. McKendrick for the growth of bacteria by considering the fact that its growth decreases as the density of bacteria increases **[18].**

$$\frac{dN}{dt} = r_N(1- \frac{N}{K})$$

Where,

N is population

r_N is the intrinsic growth rate

K is Carrying Capacity

The measurement of how the size of population changes over time is called population growth rate which depends upon population size, birth rate, death rate.

Exponential growth is a specific way that in which a quantity may increase overtime.

We also know that exponential growth surpasses the linear and cubic growth.

F(x) = e^x

F(x) = ax

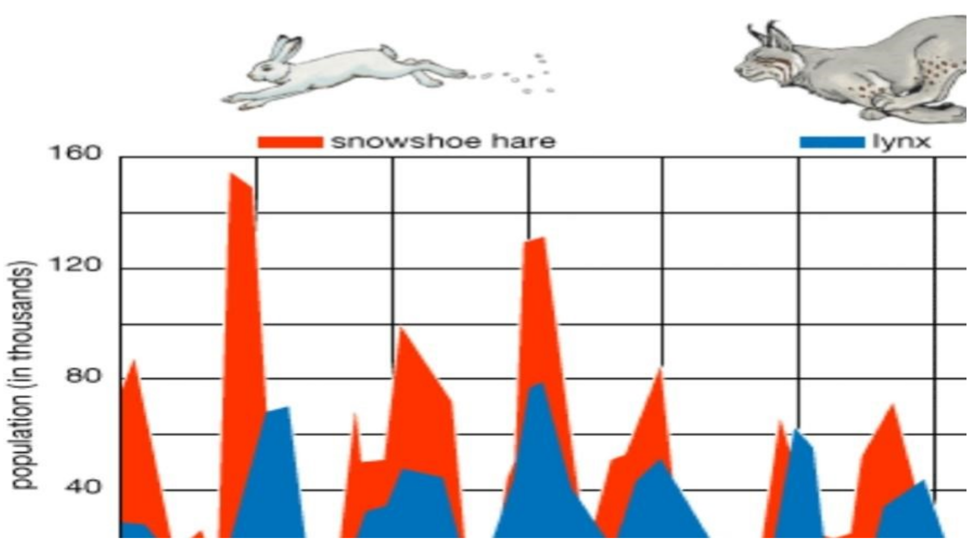
F(x) = ax^3

We will study logistic growth model rather than exponential. As carrying capacity is limited .

When the growth rate of population decreases as the number of individual increases, is called **Logistic population growth.**

Carrying capacity is the maximum number of individual the environment can support.

$$\frac{dN}{dt} = r_N \left(1- \frac{N}{K}\right)$$



PROPOSED PROBLEM

We have considered three dynamic variables to study the impact of human related activities on the persistence of species.

Some PARAMETERS are being used in our model. Suppose,

S (t) be the species living in the forest at time (t).

F (t) be the biotic resources present in a forest at time (t).

N (t) be the size of the human population living in a consider region at time (t).

Let r_S, r_F, r_N be the intrinsic growth rate of species population, forestry and human population respectively.

K_F and K_N be the carrying capacity of forest and human population respectively.

MATHEMATICAL FORMULATION

The mathematical formulation for our model is given by the following equations using **[19], [20] and [21].**

$$\frac{dS}{dt} = r_S S\left(1 - \frac{S}{\rho F}\right) + \beta_1 FS - \alpha_1 SN$$

$$\frac{dF}{dt} = r_F F\left(1 - \frac{F}{K_F}\right) - \alpha_2 FS - \alpha_3 FN$$

$$\frac{dN}{dt} = r_N N\left(1 - \frac{N}{K_N}\right) + \beta_2 FN$$

With initial conditions

$$S(0) = S_0 > 0,\ F(0) = F_0 > 0,\ N(0) = N_0 > 0$$

Where, First fraction of each equation represents the logistic growth while other terms represent interaction of species with decay or growth rate.

Rewriting mathematical formulation,

$$
\begin{aligned}
&\frac{dS}{dt} = (r_S + \beta_1 F - \alpha_1 N)S - \frac{r_S S^2}{\rho F}, \\
&\frac{dF}{dt} = (r_F - \alpha_2 S - \alpha_3 N)F - \frac{r_F}{K_N} F^2 \\
&\frac{dN}{dt} = (r_N + \beta_2 F)N - \frac{r_N}{K_N} N^2 \\
&or \\
&\frac{dS}{dt} = q(F,N)S(1 - \frac{S}{M(F,N)}), \\
&\frac{dF}{dt} = r(S,N)F(1 - \frac{F}{K(S,N)}), \\
&\frac{dN}{dt} = s(F)N(1 - \frac{N}{L(F)}).
\end{aligned}
$$

- In this model, q(F,N), r(S,N), s(F) and M(F,N), K(S,N), L(F) are the modified growth rates and carrying capacity of S, F and N respectively with the properties,

$$\mathrm{q(0,0) = r_S} > 0,\ \frac{\partial q}{\partial F} > 0,\ \frac{\partial q}{\partial N} < 0, \text{ for F > 0 , N > 0,}$$

$$\mathrm{r(0,0) = r_F} > 0,\ \frac{\partial r}{\partial S} < 0,\ \frac{\partial r}{\partial N} < 0, \text{ for S > 0 , N > 0,}$$

$$\mathrm{s(0) = r_N} > 0,\ \frac{\partial s}{\partial F} > 0, \text{ for F > 0,}$$

$$\mathrm{M(0,N) = M(F_0,0)} > 0, \frac{\partial M}{\partial F} > 0, \frac{\partial q}{\partial N} < 0, \text{ for F > 0, N > 0,}$$

$$\mathrm{K(0,0) = K_F} > 0, \frac{\partial K}{\partial S} < 0, \frac{\partial K}{\partial N} < 0, \text{ for S > 0, N > 0,}$$

$$\mathrm{L(0) = K_N} > 0, \frac{\partial L}{\partial F} > 0, \text{ for F > 0.}$$

EQUILIBRIUM POINTS

To find out the Equilibrium points, we need to put the rates in model system equal to zero. Hence the model equations are:

$$r_S S\left(1-\frac{S}{\rho F}\right)+\beta_1 FS-\alpha_1 SN=0 \qquad (1)$$

$$r_F F\left(1-\frac{F}{K_F}\right)-\alpha_2 FS-\alpha_3 FN=0 \qquad (2)$$

$$r_N N\left(1-\frac{N}{K_N}\right)+\beta_2 FN=0 \qquad (3)$$

From (1), S=0 or $r_S\left(1-\frac{S}{\rho F}\right)+\beta_1 F-\alpha_1 N=0$ (4)

From (2), F=0 or $r_F\left(1-\frac{F}{K_F}\right)-\alpha_2 S-\alpha_3 N=0$ (5)

From (3), N=0 or $r_N\left(1-\frac{N}{K_N}\right)+\beta_2 F=0$ (6)

- For trivial case, S=0, N=0, F=0 provides the equilibrium point $E_1(0,0,0)$.
- For the case, S=0, N=0,

Using (5), $r_F\left(1-\frac{F}{K_F}\right)=0$, we have $K_F>F$,

which provides the equilibrium point $E_2(0,F_2,0)$ with a condition that $K_F>F$.

- Similarly, for the case, S=0, N=0 in (6),

$r_N\left(1-\frac{N}{K_N}\right)=0$, we have $K_N>N$,

which results in the existence of the equilibrium point $E_3(0,0,N_3)$.

No equilibrium points exists for the case N=0, F=0 because no species can exists without biotic resources.

- For the case S=0, $N\neq 0, F\neq 0$, from eq (5) and (6), we have

$$r_F\left(1-\frac{F}{K_F}\right)-\alpha_3 N=0 \qquad (7)$$

$$r_N\left(1-\frac{N}{K_N}\right)+\beta_2 F=0 \qquad (8)$$

From (7), $F=\left(\frac{r_F-\alpha_3.N}{r_F}\right)K_F$, (9) From (8), $N = \left(\frac{r_N+\beta_2.N}{r_N}\right) K_N$ (10)

Using (9) in (8). We have,

$$N_4 = \left(\frac{r_N + \beta_2 K_F}{\frac{r_N}{K_N} + \frac{\beta_2 K_F \alpha_3}{r_F}}\right)$$

$$F_4 = \left(\frac{r_F - \alpha_3 N_4}{r_F}\right) K_F$$

Provides the equilibrium point $E_4(0, F_4, N_4)$ exists only if

$r_F - \alpha_3 N_4 > 0 \Rightarrow N_4 < \frac{r_F}{\alpha_3}$.

- For the case N = 0, S $\neq$ 0, F $\neq$ 0, from eq (4) and (5), we have

$r_S \left(1 - \frac{S}{\rho F}\right) + \beta_1 F = 0$ (11)

$r_F \left(1 - \frac{F}{K_F}\right) - \alpha_2 S = 0$ (12)

From (12), $S = \frac{r_F}{\alpha_2}\left(1 - \frac{F}{K_F}\right)$ (13)

Using (13) in (11), we have

$$r_S \left(\rho F - \frac{r_F}{\alpha_2}\left(1 - \frac{F}{K_F}\right)\right) + \beta_1 \rho F^2 = 0 \Rightarrow f_1(F) = 0$$

$$\frac{df_1}{dF} = r_S \left(\rho + \frac{r_F}{\alpha_2 K_F}\right) + 2\beta_1 \rho F > 0$$

At F=0, $f_1(0) = -\frac{r_F}{\alpha_2} < 0,$

At F= K_F, $f_1(K_F) = r_S(\rho K_F + \beta_1 \rho K_F^2) > 0$

$$S_5 = \frac{r_F}{\alpha_2}\left(1 - \frac{F_5}{K_F}\right), F_5 < K_F$$

Hence the equilibrium point $E_5(S_5, F_5, 0)$ always exists, if F_5 is under carrying capacity.

- For the case N $\neq$ 0, S $\neq$ 0, F $\neq$ 0, From (10), (4) ⇒ $r_S\left(1-\frac{S}{\rho F}\right)+\beta_1 F-\alpha_1\left(\frac{r_N+\beta_{2.}N}{r_N}\right)K_N=0$ (14)

And $r_F\left(1-\frac{F}{K_F}\right)-\alpha_2 S-\frac{K_N\alpha_3}{r_N}(r_N+\beta_{2.}F)=0$ (15)

As for the growth of S and F,

$$r_S-\beta_1 F-\alpha_1 N>0$$

And $$r_F-\alpha_2 S-\alpha_3 N>0$$

We will get the positive value of S_5, F_5 from the (14) and (15) respectively. Hence the equilibrium point $E_6(S_6, F_6, N_6)$ always exists.

EIGEN VALUES

To determine the Eigen values, first find out the Jacobian matrix from the equations:

$$r_S S\left(1-\frac{S}{\rho F}\right)+\beta_1 FS-\alpha_1 SN=0$$

$$r_F F\left(1-\frac{F}{K_F}\right)-\alpha_2 FS-\alpha_3 FN=0$$

$$r_N N\left(1-\frac{N}{K_N}\right)+\beta_2 FN=0$$

Then for the local stability we have, $J_0 =$

$$\begin{bmatrix} r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N & \beta_1 S & -\alpha_1 S \\ -\alpha_2 F & r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N & -\alpha_3 F \\ 0 & \beta_2 N & r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F \end{bmatrix}$$

For finding Eigen Values,

- Putting S=0, N=0, F=0 in J_0, we have

$$\begin{bmatrix} r_S & 0 & 0 \\ 0 & r_F & 0 \\ 0 & 0 & r_N \end{bmatrix}$$

  which is always unstable in the direction of S, F and N for having three positive eigen values as r_S , r_F and r_N respectively. Hence $E_1(0,0,0)$ is always unstable in the direction of S, F and N.

- Putting S=0, N=0, F=K_F in J_0, we have

$$\begin{bmatrix} r_S+\beta_1 K_F & 0 & 0 \\ -\alpha_2 K_F & -r_F & -\alpha_3 K_F \\ 0 & 0 & r_N \end{bmatrix}$$

then $(r_S+\beta_1 K_F)$ and r_N are the two positive Eigen values. Hence $E_2(0,F_2,0)$ is always unstable in the direction of N and S.

- Putting S=0, F=0, N=K_N in J_0,

$$\begin{bmatrix} r_S - \alpha_1 N & \beta_1 S & -\alpha_1 S \\ 0 & r_F - \alpha_3 N & 0 \\ 0 & \beta_2 N & r_N\left(1-\frac{2N}{K_N}\right) \end{bmatrix}$$

then $(r_S - \alpha_1 K_N) > 0$, $(r_F - \alpha_3 K_N) > 0$ are the two positive eigen values . Hence $E_3(0,0,N_3)$ is unstable in the direction of S and F.

- Putting S=0 in J_0,

$$\begin{bmatrix} r_S + \beta_1 F - \alpha_1 N & 0 & 0 \\ -\alpha_2 F & r_F\left(1-\frac{2F}{K_F}\right) - \alpha_3 N & -\alpha_3 F \\ 0 & \beta_2 N & r_N\left(1-\frac{2N}{K_N}\right) + \beta_2 F \end{bmatrix}$$

Then we have a cubic equation of the form

$L(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda^1 + b_3=0$

Where,

$$b_1 = r_S + \beta_1 F - \alpha_1 N + r_F\left(1-\frac{2F}{K_F}\right) - \alpha_3 N + r_N\left(1-\frac{2N}{K_N}\right) + \beta_2 F$$

$$b_2 = a_2 a_3 + \beta_2\alpha_3 FN + a_1 a_3 + a_1 a_2 + \alpha_2\beta_1 FS$$

$$b_3 = a_1(a_2 a_3 + \beta_2\alpha_3 FN) + \alpha_2 F(a_3\beta_1 S)$$

On applying the **Routh-Hurwitz Criterion**:

I. $|b_1| > 0$

$$\Rightarrow \left(r_S + \beta_1 F - \alpha_1 N + r_F\left(1-\frac{2F}{K_F}\right) - \alpha_3 N + r_N\left(1-\frac{2N}{K_N}\right) + \beta_2 F\right) > 0$$

II. $\begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} > 0 \Rightarrow (b_1 b_2 - b_3) > 0$

- Putting N=0 in J_0,

$$\begin{bmatrix} r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F & \beta_1 S & -\alpha_1 S \\ -\alpha_2 F & r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S & -\alpha_3 F \\ 0 & 0 & r_N+\beta_2 F \end{bmatrix}$$

Then $(r_N+\beta_2 F)>0$ is the Eigen value.
Hence $E_5(S_5, F_5, 0)$ is always unstable in the direction of N.

- For S $\neq 0$, F $\neq 0$,N $\neq 0$ in J_0, we have

$$\begin{bmatrix} r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N & \beta_1 S & -\alpha_1 S \\ -\alpha_2 F & r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N & -\alpha_3 F \\ 0 & \beta_2 N & r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F \end{bmatrix}$$

Then we have a cubic equation of the form

$L(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda^1 + b_3 = 0$

Where,

$$b_1 = r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N + r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N + r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F$$

$$b_2 = \left[r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F\right]\left[r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N\right]$$

$$+\beta_2\alpha_3 + \left[r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N\right]\left[r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F\right]$$

$$+\left[r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N\right]\left[r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N\right]$$

$$+\alpha_2\beta_1 FS$$

$$b_3 = a_1(a_2 a_3 + \beta_2\alpha_3 FN) + \alpha_2 F(a_3\beta_1 S)$$

On applying the **Routh-Hurwitz Criterion**:

I. $|b_1| > 0$

$$\Rightarrow \left(r_S\left(1-\frac{2S}{\rho F}\right)+\beta_1 F-\alpha_1 N+\ r_F\left(1-\frac{2F}{K_F}\right)-\alpha_2 S-\alpha_3 N+r_N\left(1-\frac{2N}{K_N}\right)+\beta_2 F\right)>0$$

II. $\begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} > 0 \Rightarrow (b_1 b_2 - b_3) > 0$

CONCLUSION

In the present work, a non-linear prey predator dynamical model has been proposed and analyzed. We have considered that the carrying capacity of the extinct species is proportional to the density of bio resource available in the region. The impact of human activities has also been considered in the modeling process. It is found that the model system has six non-negative equilibrium points, (i) the zero equilibrium (ii) species and bio-resource free equilibrium, (iii) species and Human population free equilibrium, (iv) species free equilibrium, (v) human population free equilibrium and (vi) the positive equilibrium.

Through the analysis of the model it has been found that all equilibrium points exist without any condition. Further, it has carried out that all the equilibrium points are unstable without any condition except the all positive equilibrium point. The all positive equilibrium point is locally stable under some conditions given by Routh-Hurwitz criteria.

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