Nonassociative rings satisfying the identities
a(bc) = b(ca) and (a, a, a) =0

A Project Report Submitted
in Partial Fulfillment of the Requirements
for the Degree of

MASTER OF SCIENCE
in
Mathematics

by
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to the
DEPARTMENT OF MATHEMATICS
GALGOTIAS UNIVERSITY
GREATER NOIDA - 201310, INDIA
MAY 2020

CERTIFICATE

This is to certify that the work contained in this report entitled "**Nonassociative rings satisfying the identities a(bc) = b(ca) and (a, a, a) =0**" submitted by Murari Roy (Roll No. 18SBAS2040005) to Department of Mathematics, Galgotias University,Greater Noida towards the requirement of the course MSCM9999 Project has been carrying out by him under my guidance and supervision.

The final project MSCM9999 is completed satisfactorily towards fulfilling the requirements for submission of final project in the 4th Semester, 2020. The results obtained, in this project report have not been submitted in part or full, to any other university or institution for degree or diploma.

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ACKNOWLEDGEMENT

Preparing a project is never a unilateral effort; I wish to acknowledgement the guidance and support of the professionals and academicians in bringing up a real picture of the concept from this project. It’s a matter of great pleasure for me to submit a project report on **“Nonassociative rings satisfying the identities $a(bc) = b(ca)$ and $(a, a, a) =0$”.**

I am indebted to **Dr. Dhabalendu Samanta** for his insightful guidance throughout the project. His unfailing enthusiasm and continuous guidance kept me motivated and encouraged throughout the project.

In addition, I extend my sincere gratitude to **Dr. Dhabalendu Samanta (H.O.D Of Mathematics) and Dr. VARSHA GAUTAM (Program Chair, Mathematics Department)** for landing their expertise and guidance. I am also thankful to all faculty members of division of mathematics whose motivation leads to the completion of this project work.

I am also full of sense and will likely to express our gratitude towards all those who helped us out during the different stages of our project and encouraged us with their ideas and views, which I am guiding and motivating factors in completion of the tedious task of project.

ABSTRACT

In [4], A. Behn, Ivan Correa and Irvin Roy Hentzel proved that the products of five elements are associative and commutative in flexible algebras (i.e. (a, b, a) = 0) satisfying the polynomial identity a(bc) = b(ca). In this project, we have studied various structure of nonassociative rings satisfying identities a(bc) = b(ca) and (a, a, a) =0. We have proved that the identities $a^2[b,c]-c(a,a,b)=0$ and $(a,b,c)(d(ef))=0$ hold in a nonassocitve rings satisfying the identity $a(bc)=b(ca)$. Also, we have proved that the Jordan identity $(a,b,aa)=0$ holds in a nonassocitve rings satisfying the identity $a(bc)=b(ca)$. Then we have studied third power associative, nonassociative rings R satisfying the identity $a(bc)=b(ca)$ and proved that R is power associative. Also, we have proved that the identities $a^n[a,b]=0$ and $a(a,b,c)+a(b,a,c)=c(a,a,b)$ hold in a third power associative, nonassociative rings satisfying the identity $a(bc)=b(ca)$.

1. **Introduction**

1.1 **Nonassociative rings**

An associative ring R is an additive abelian group in which a multiplication is defined, satisfying

$$(ab)c=a(bc) \qquad (1)$$

and

$$(a+b)c=ac+bc, \qquad c(a+b)=ca+cb \qquad (2)$$

for all a,b,c R. An algebra R over a field F is ring which is a vector space over F with $\alpha(ab)=(\alpha a)b=a(\alpha b)$ for all a,b,c ϵ R and α ϵ F. If (1) is not assumed to hold, R will be referred to as a nonassociative ring. Use of this term does not mean (1) fails to hold, but only that (1) is assumed to hold. When the condition (1) is not satisfied ,the ring is said to be not associative.

The first example of a nonassociative ring seems to have arisen in the middle of the last century in the form of the caylay numbers(octanions). Lie algebras, Jordan algebras, the octonions, and three-dimensional Euclidean space equipped with the cross product operation are nonassociative algebras.

1.2 Notations and Basic Identities

Associator : $(a,b,c)=(ab)c-a(bc)$

$$(R,R,R)=\{ \sum_{Finite} [a,b]:\ a,b\epsilon R\}$$

Commutator: $[a,b]=ab-ba$

$$[R,R]=\{\sum_{Finite} [a,b]:a,b\epsilon$$

We shall make use of the following identities:

$$[ab,c]+[bc,a]+[ca,b]=(a,b,c)+(b,c,a)+(c,a,b). \qquad (3)$$

$$[ab,a]+a[a,b]=(a,b,a). \qquad (4)$$

$$[ab,c]=a[b,c]+[a,c]b+(a,b,c)+(c,a,b)-(a,c,b). \qquad (5)$$

$$(wa,b,c)-(w,ab,c)+(w,a,bc)-w(a,b,c)-(w,a,b)c=0. \qquad (6)$$

The above identities hold in any nonassociative ring.

1.3 Nonassociative rings satisfying the identities a(bc) = b(ca) and (a,a,a)=0

In [1], M. Kleinfeld has proved that prime ring satisfying $a(bc) = b(ca)$with characteristic $\neq 2$ is commutative and associative. In [2], I. correa prove that every finite-dimensional flexible right-nilalgebra satisfying a(bc) = b(ca) is nilpotent. In [3], A. Behn and et. al. have proved that every semiprime cyclic ring is associative and commutative. In [4], A. Behn, Ivan Correa and Irvin Roy Hentzel proved that the products of five elements are associative and commutative in flexible algebras (possibly infinite-dimensional) satisfying the polynomial identity a(bc) = b(ca). In [5], D.

Samanta & I. R. Hentzel proved that nonassociative rings satisfying a(bc)= b(ca) and (a, a, b) = (b, a, a) with characteristic ≠2 are associative and commutative of degree five.

2. Proposed Problem

In this project, we have studied nonasociative rings satisfying the identities

$$a(bc) - b(ca) = 0 \quad (7)$$

and

$$(a, a, a) = 0 \quad (8)$$

Here, we have considered nonassociative rings satisfying stronger identities than the identity considered in [3] and thus all results obtained in [3] will hold in ring satisfying identities (7) and (8). On the other hand, we have considered nonassociative rings satisfying weaker identities than the identities considered in [4] and thus all results obtained in [4] may not hold in rings satisfying a(bc)=b(ca) and (a,a,a)=0. One of the major objective of this project is to prove some of the results obtained in [4], without assuming flexible law. Also, we are trying to prove some new results that were neither obtained in [3] nor obtained in [4] by studying nonassociative rings satisfying a(bc)=b(ca) and (a,a,a)=0.

3. Work Done

3. 1 Nonassociative rings satisfying the identity $\boldsymbol{a(bc) = b(ca)}$

Rings satisfying the identity $a(bc) - b(ca) = 0$have studied in [3]. In this section, we have proved identities that were not mentioned in [3].

Lemma 3.1.1 Let R be a nonassociative rings satisfying the identity (7). Then following identities hold in R:

(i) $a[a, b] = 0$;

(ii) $(a, b, a) = [ab, a]$;

(iii) $a^2[b, c] = c(a, a, b)$.

Proof. (i) Applying (7) three times, we get $a[a, b] = a(ab) - a(ba) = a(ab) - b(aa) = a(ab) - a(ab) = 0$.

(ii) From (5), we have$[ab, a] = a[b, a] + [a, a]b + (a, b, a) + (a, a, b) - (a, a, b)$. Using Lemma 3.1.1(i) we get $[ab, a] = (a, b, a)$.

(iii) Applying Lemma 3.1.1(i) we have $a^2[b,c] = -b[a^2,c] = -b\big((aa)c\big) + b\big(c(aa)\big)$.Using (7) thrice, we obtain $a^2[b,c] = -(aa)(cb) + c\big((aa)b\big) = -c\big(b(aa)\big) + c\big((aa)b\big) = -c\big(a(ab)\big) + c\big((aa)b\big) = c\big((aa)b - a(ab)\big) = c(a,a,b)$.

Following Lemma was proved in [3, lemma 1, page no 133]

Lemma 3.1.2 Let R be a ring satisfying the identity (7). Then following identities hold in R.

(i) $(ab)(cd) = (da)(cb)$;

(ii) $(ab)((cd)(ef)) = (da)((cb)(ef))$;

(iii) $(ab, cd, ef) = 0$;

(iv) $(a,b,cd)(ef) = 0$.

Lemma 3.1.3 Let R be a ring satisfying the identity (7). Then following identities hold in R:

(i) $(a,b,cd) + (b,c,ad) + (c,a,bd) = 0$;

(ii) $(a,b,cd) + (c,b,da) + (d,b,ac) = 0$.

Proof. (i) We have

$$(a,b,cd) + (b,c,ad) + (c,a,bd)$$
$$= (ab)(cd) - a\big(b(cd)\big) + (bc)(ad) - b\big(c(ad)\big) + (ca)(bd) - c\big(a(bd)\big).$$

Applying (7) twice, we obtain

$$(a,b,cd) + (b,c,ad) + (c,a,bd)$$
$$= (ab)(cd) - b\big((cd)a\big) + (bc)(ad) - c\big((ad)b\big) + (ca)(bd) - a((bd)c)$$
$$= (ab)(cd) - (cd)(ab) + (bc)(ad) - (ad)(bc) + (ca)(bd) - (bd)(ca).$$

Now using Lemma 3.1.2(i), we get

$$(a,b,cd) + (b,c,ad) + (c,a,bd)$$
$$= (ab)(cd) - (bc)(ad) + (bc)(ad) - (ca)(bd) + (ca)(bd) - (ab)(cd)$$
$$= 0.$$

(ii) We have

$$(a,b,cd) + (c,b,da) + (d,b,ac)$$
$$= (ab)(cd) - a\big(b(cd)\big) + (cb)(da) - c\big(b(da)\big) + (db)(ac) - d\big(b(ac)\big).$$

Applying (7) twice, we obtain

$$(a,b,cd) + (c,b,da) + (d,b,ac)$$
$$= (ab)(cd) - b\big((cd)a\big) + (cb)(da) - b\big((da)c\big) + (db)(ac) - b\big((ac)d\big)$$
$$= (ab)(cd) - (cd)(ab) + (cb)(da) - (da)(cb) + (db)(ac) - (ac)(db).$$

Now using Lemma 3.1.2(i) twice, we get

$$(a,b,cd) + (c,b,da) + (d,b,ac)$$

$= (ab)(cd) - (bc)(ad) + (cb)(da) - (bd)(ca) + (db)(ac) - (ba)(dc)$

$= (ab)(cd) - (db)(ac) + (cb)(da) - (ab)(cd) + (db)(ac) - (cb)(da).$

$= 0.$

Lemma 3.1.4 Let R be a ring satisfying the identity (7). Then following identities hold in R:

(i) $a(a,c,b) + a(b,c,a) + b(a,c,a) = 0;$

(ii) $a(c,a,b) + a(c,b,a) + b(c,a,a) = 0.$

Proof. (i) Applying (7) twice, we get

$$a(c,a,a) = a((ca)a) - a(c(aa)) = (ca)(aa) - c((aa)a) = (ca)(aa) - (aa)(ac).$$

Using Lemma 3.1.2(i), we obtain

$$a(c,a,a) = (ca)(aa) - (ca)(aa) = 0.$$

Linearization of $a(c,a,a) = 0$ gives the following identity

$$a(c,a,b) + a(c,b,a) + a(c,b,b) + b(c,a,a) + b(c,a,b) + b(c,b,a) = 0. \qquad (9)$$

Replacing $a$by $-a$in (9) and adding the resulting identity with (9), we obtain

$$2a(c,a,b) + 2a(c,b,a) + 2b(c,a,a) = 0.$$

Since characteristic $\neq 2$, we have

$$2a(c,a,b) + 2a(c,b,a) + 2b(c,a,a) = 0.$$

(ii) Applying (7) twice, we have

$$a(c,a,a) = a((ca)a) - a(c(aa)) = (ca)(aa) - c((aa)a) = (ca)(aa) - (aa)(ac).$$

Using Lemma 3.1.2(i), we obtain

$$a(c,a,a) = (ca)(aa) - (ca)(aa) = 0.$$

Linearization of $a(c,a,a) = 0$ gives the following identity

$$a(c,a,b) + a(c,b,a) + a(c,b,b) + b(c,a,a) + b(c,a,b) + b(c,b,a) = 0. \qquad (10)$$

Replacing $a$by $-a$in (10) and adding the resulting identity with (10), we obtain

$$2a(c,a,b) + 2a(c,b,a) + 2b(c,a,a) = 0.$$

Since characteristic $\neq 2$, we have

$$a(c,a,b) + a(c,b,a) + b(c,a,a) = 0.$$

Theorem 3.1.5 Let R be a ring satisfying the identity (7). Then$(a,b,c)(d(ef)) = 0$.

Proof. We have

$$(a,b,c)(d(ef))$$

$$= ((ab)c - a(bc))(d(ef))$$

$$= ((ab)c)(d(ef)) - (a(bc))(d(ef)).$$

Applying Lemma 3.1.2(i), we obtain

$$(a,b,c)\big(d(ef)\big)$$

$$= \big((ef)(ab)\big)(dc) - (a(bc))(d(ef)).$$

Using (7) twice, we get

$$(a,b,c)\big(d(ef)\big)$$

$$= \big((ef)(ab)\big)(dc) - (b(ca))(d(ef))$$

$$= \big((ef)(ab)\big)(dc) - (c(ab))(d(ef)).$$

Applying Lemma 3.1.2(i) twice, we obtain

$$(a,b,c)\big(d(ef)\big)$$

$$= \big((ef)(ab)\big)(dc) - ((ef)c)(d(ab))$$

$$= \big((ef)(ab)\big)(dc) - ((ab)(ef))(dc).$$

Using (7), we get

$$(a,b,c)\big(d(ef)\big)$$

$$= \big((ef)(ab)\big)(dc) - (e(f(ab))(dc)$$

$$((ef)(ab) - e(f(ab)))(dc)$$

$$(e,f,ab)(dc)$$

Applying Lemma 3.1.2(iv), we obtain

$$(a,b,c)\big(d(ef)\big) = 0.$$

Lemma 3.1.6 Let R be a ring satisfying the identity (7). Then R satisfies the Jordan identity $(a,b,aa) = 0$.

Proof. Repeatedly applying (7), we have

$$(a,b,aa)$$

$$= (ab)(aa) - a(b(aa))$$

$$= a(a(ab)) - a(b(aa))$$

$$= a(a(ba)) - a(b(aa))$$

$$= a(b(aa)) - a(b(aa))$$

$$= 0.$$

3.2 Nonassociative rings satisfying the identity a(bc)=b(ca) and (a,a,a) = 0

In this section we will study nonassocitive rings satisfying the identities (7) and (8)

Theorem 3.2.1 Let R be ring satisfying the identities (7) and (8). Then R is power associative.

Proof: Define $a^{i+1} = a\,a^i$ for all i. For n = 1,2,3, there is nothing to prove. First we will prove that R is fourth power associative. That is, we have to prove, $a^3 a = a^4$ and $a^2 a^2 = a^4$.

We have $a^2 a^2 = a^2 (a\,a)$. Using (7),we get

$$a^2 a^2 = a\,(a\,a^2) = a\,a^3 = a^4. \qquad (11)$$

Now, we will prove $a^3 a = a^4$. From (6), we have

$$(a^2,a,a) - (a,a^2,a) + (a,a,a^2) = a(a,a,a) + (a,a,a)a.$$

Using (8), we get

$$(a^2,a,a) - (a,a^2,a) + (a,a,a^2) = 0.$$

This implies

$$(a^2,a,a) – (a,a^2,a) + (a\,a)a^2 - a(a\,a^2) = 0.$$

Using (8), we obtain

$$(a^2,a,a) – (a,a^2,a) + a^2 a^2 – a\,a^3 = 0.$$

Using (11), we get

$$(a^2,a,a) – (a,a^2,a) + a^4 - a^4 = 0.$$

This gives

$$(a^2,a,a) – (a,a^2,a) = 0.$$

That is , we have

$$(a^2,a,a) = (a,a^2,a). \qquad (12)$$

From (8), we get

$$(a^2,a,a) + (a,a,a^2) + (a,a^2,a) + (a^2,a,a) + (a,a,a^2) + (a,a^2,a) = 0.$$

Thus we have

$$2(a^2,a,a) + 2(a,a,a^2) + 2(a,a^2,a) = 0.$$

Since characteristic 2,we get

$$(a^2,a,a) + (a,a,a^2) +(a,a^2,a) = 0.$$

Using (12) and (7),we get

$$2(a^2,a,a) + (a^2 a^2) - a(aa^2) = 0.$$

Using (10) we get

$$2(a^2,a,a) + a^4 – a(a^3) = 0.$$

By definition, we get

$$2(a^2,a,a) + a^4 - a^4 = 0.$$

Thus we have

$$2(a^2,a,a) = 0.$$

Since characteristic 2,we get

$$(a^2,a,a) = 0.$$

Therefore, ,we obtain $(a^2a)a - a^2(aa) = 0$. This gives $a^3 a - a^2 a^2 = 0$. Therefore, we have $a^3 a = a^4$. Hence R is 4th power associative.

Let us assume that R is mth power associative. Then we have to prove that R is (m+1)th power associative.

Recall the definition $a a^m = a^{m+1}$.

Now to prove R is (m+1)th power associative, we have to prove $a^i a^j = a^{i+j}$ where $i+j = m+1$.

We have $a^{m-1} a^2 = a^{m-1} (a a)$. Using (7),we get $a^{m-1} a^2 = a(a a^{m-1})$. Using definition twice we get

$$a^{m-1} a^2 = a a^m = a^{m+1} \quad (13)$$

Now follow the same as in case of 4th power associative, we have

$$(a^{m-1},a,a) = 0.$$

This gives $(a^{m-1} a) a = a^{m-1} a^2$. Therefore, by assumption, we have $a^m a = a^{m-1} a^2$.

Using (13), we get $a^m a = a^{m+1}$. Thus we have $a^i a = a^{i+1}$ for all $i<m+1$.

By assumption, we have $a^i a^j = a^{i+j}$ for all $i+j = m$.

So, for all i, j such that $i+j = m+1$, from definition, we obtain $a^i . a^j = a^i (a a^{j-1})$. Using (7),we get $a^i . a^j = a (a^{j-1} a^i) = a (a^{i+j-1}) = a^{i+j-1+1} = a^{i+j}$. Thus R is (m+1) the power associative.

Hence by mathematical induction R is power associative. This complete the prove.

Theorem 3.2.2 Let R be ring satisfying the identities (7) and (8). Then the following identity $a^n[a,b] = 0$holds in R for all positive integer n.

Proof. Using Lemma 3.1.1(i), we have

$$a^2(a,b) = -a(a^2,b) = a(b,a^2)$$

$$a(b(aa)) - a((aa)b) = b((aa)a) - (aa)(ba)$$

$$b((aa)a) - b(a(aa))$$

Applying Theorem 3.2.1, we have $a^2[a,b] = ba^3 - ba^3 = 0$.

Now let us assume $a^n [a,b] = 0$ for $n = k$. Then we have to prove $a^{k+1} [a,b] = 0$.

Now $a^{k+1} [a,b] = a^{k+1}(ab) - a^{k+1} (ba)$. Repeatedly using (7) and Theorem 3.2.1, we get $a^{k+1} [a,b] = a(b a^{k+1}) - b(a^{k+1} a) = b(a^{k+1} a) - b a^{k+2} = b a^{k+2} - b a^{k+2} = 0$.

Thus $a^n [a,b] = 0$ holds for $n=k+1$. Hence by mathematical induction, we have $a^n [a,b] = 0$ for all positive integer.

Lemma 3.2.3 Let R be ring satisfying the identities (7) and (8). Then the following identities hold in R:

(i) $(ab)[a,c] + (ba)[a,c] + a^2[b,c] = 0$;

(ii)$(a,a,b) + (a,b,a) + (b,a,a) = 0$.

Proof. (i) From Theorem 3.2.2 we have

$$a^2[a,b] = 0.$$

Linearization of this implies

$$(ab)[a,c] + (ba)[a,c] + b^2[a,c] + a^2[b,c] + (ab)[b,c] + (ba)[b,c] = 0. \quad (14)$$

Replacing $a$by $-a$in (14) and adding the resulting identity with (11), we obtain

$$2(ab)[a,c] + 2(ba)[a,c] + 2a^2[b,c] = 0.$$

Since characteristic 2,we have

$$(ab)[a,c + (ba)[a,c] + a^2[b,c] = 0.$$

(iii) By linearization of (8), we have

$$(a,b,c) + (b,c,a) + (c,a,b) + (a,c,b) + (c,b,a) + (b,a,c) = 0.$$

Putting $b = a \quad and \quad c = b$, we get

$$(a,a,b) + (a,b,a) + (b,a,a) + (a,b,a) + (b,a,a) + (a,a,b) = 0.$$

This implies

$$2(a,a,b) + 2(a,b,a) + 2(b,a,a) = 0.$$

Since characteristic 2,we have

$$(a,a,b) + (a,b,a) + (b,a,a) = 0.$$

Theorem 3.2.4 Let R be ring satisfying the identities (7) and (8). Then the following identity hold $a(a,b,c) + a(b,a,c) = c(a,a,b)$in R.

Proof. Repeatedly using (7) we have

$$a(b,b,a)$$
$$= a((bb)a) - a(b(ba))$$
$$= (bb)(aa) - a(b(ab))$$
$$= (bb)(aa) - a(a(bb)).$$

First using Lemma 3.1.1(i) and then using (7), we obtain

$$a(b,b,a)$$
$$= (bb)(aa) - a((bb)a)$$
$$(bb)(aa) - (bb)(aa)$$
$$0.$$

Now linearization of $a(b,b,a) = 0$implies

$$a(b,c,a) + a(c,b,a) = 0. \quad (15)$$

Linearization of (15) gives

$$a(b,c,d) + a(c,b,d) + d(b,c,a) + d(c,b,a) = 0. \quad (16)$$

Putting $d = c$in (16), we get

$$a(b,c,c) + a(c,b,c) + c(b,c,a) + c(c,b,a) = 0.$$

Using Lemma 3.2.3(ii), we have

$$-a(c,c,b)+c(b,c,a)+c(c,b,a)=0.$$

This gives

$$a(c,c,b)=c(b,c,a)+c(c,b,a).$$

Interchanging $a$and c, we obtain

$$c(a,a,b)=a(a,b,c)+a(b,a,c).$$

4. Conclusion

In this project, we have studied nonassociative rings satisfying $a(bc)=b(ca) and (a,a,a)=0$.In the section 3. 1, we have studied nonassocitve rings satisfying the identity $a(bc)=b(ca)$and proved that the identities $a^2(b,c)-c(a,a,b)=0$and $(a,b,c)(d(ef))=0$hold in R . Also, we have proved that the Jordan identity $(a,b,aa)=0$holds in R. In the section 3. 2, we have studied third power associative nonassociative rings satisfying the identity $a(bc)=b(ca)$and proved that R is power associative. Also, we have proved that the identities $a^n[a,b]=0$and $a(a,b,c)+a(b,a,c)=c(a,a,b)$ holds in R.

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