Lecture-3

Classification of integral equations of Volterra and Fredholm types

SINGULAR INTEGRAL EQUATION. DEFINITION

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is known as *singular integral equation.* For example, the integral equations

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-t|} y(t)\, dt$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} y(t)\, dt, 0 < \alpha < 1$$

are singular integral equations.

Leibnitz's rule of integration under integral sign

Let $F(x, t)$ and $\frac{\partial F}{\partial x}$ be continuous functions of both x and t and let the first order derivatives of $G(x)$ and $H(x)$ be continuous.

Then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)dt = \int_{G(x)}^{H(x)} \frac{\partial F(x,t)}{\partial x}dt + F(x,H(x))\frac{dH}{dx} - F(x,G(x))\frac{dG}{dx} .$$

If H(x) and G(x) are constants then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)dt = \int_{G(x)}^{H(x)} \frac{\partial F(x,t)}{\partial x}dt .$$

Important result for converting a multiple integral into single integral

$$\int_{x_0}^{x} dx \int_{x_0}^{x} dx \ldots \int_{x_0}^{x} y(x)dx = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} y(t)dt \quad ,$$

where n is a positive integer.

SPECIAL KINDS OF KERNELS.

The following special cases of the kernel of an integral equation are of main interest

A kernel $K(x, t)$ is symmetric (or complex symmetric or Hermitian) if
$K(x, t) = K(t, x)$

where the bar donates the complex conjugate. A real kernel $K(x, t)$ is symmetric if $K(x, t) = K(t, x)$.

For example, $\sin(x + t)$, $\log(x\,t)$, $x^2t^2 + xt + 1$ etc. are all symmetric kernels. Again, $\sin(2x + 3t)$ and $x^2t^3 + 1$ are not symmetric kernels.

Again $K(x, t) = i(x - t)$ is a symmetric kernel.

Separable or degenerate kernel

A kernel $K(x, t)$ is called separable if it can be expressed as the sum of a finite number ofterms, each of which is the product of a function of x only and a function of t only, *i.e.*,

$$K(x,t) = \sum_{i=1}^{n} g_i(x)\; h_i(t).$$

INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE. DEFINITION.

Consider an integral equation in which the kernel $K(x, t)$ is dependent solely on the difference

$x - t$, *i.e.*,

$K(x, t) = K(x - t)$, ... (1)

where K is a certain function of one variable. Then integral equations

$$y(x) = f(x) + \lambda \int_a^x K(x-t)\, y(t)\, dt, \quad \dots (2)$$

$$y(x) = f(x) + \lambda \int_a^b K(x-t)\, y(t)\, dt \quad \dots (3)$$

are called integral equations of convolution type. $K(x - t)$ is called *difference kernel.*

Reference: https://nptel.ac.in/courses/111/107/111107103/

INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS by M.D. Raisinghania