Electricity and Magnetism

Topic: Divergence and Curl of Vector Field

1. Divergence of Vector Field
2. Divergence Theorem
3. Curl of Vector
4. Laplacian Operator

Maxwell's Equations in Differential Form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \quad \mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \quad \mathbf{B} = \rho$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Divergence

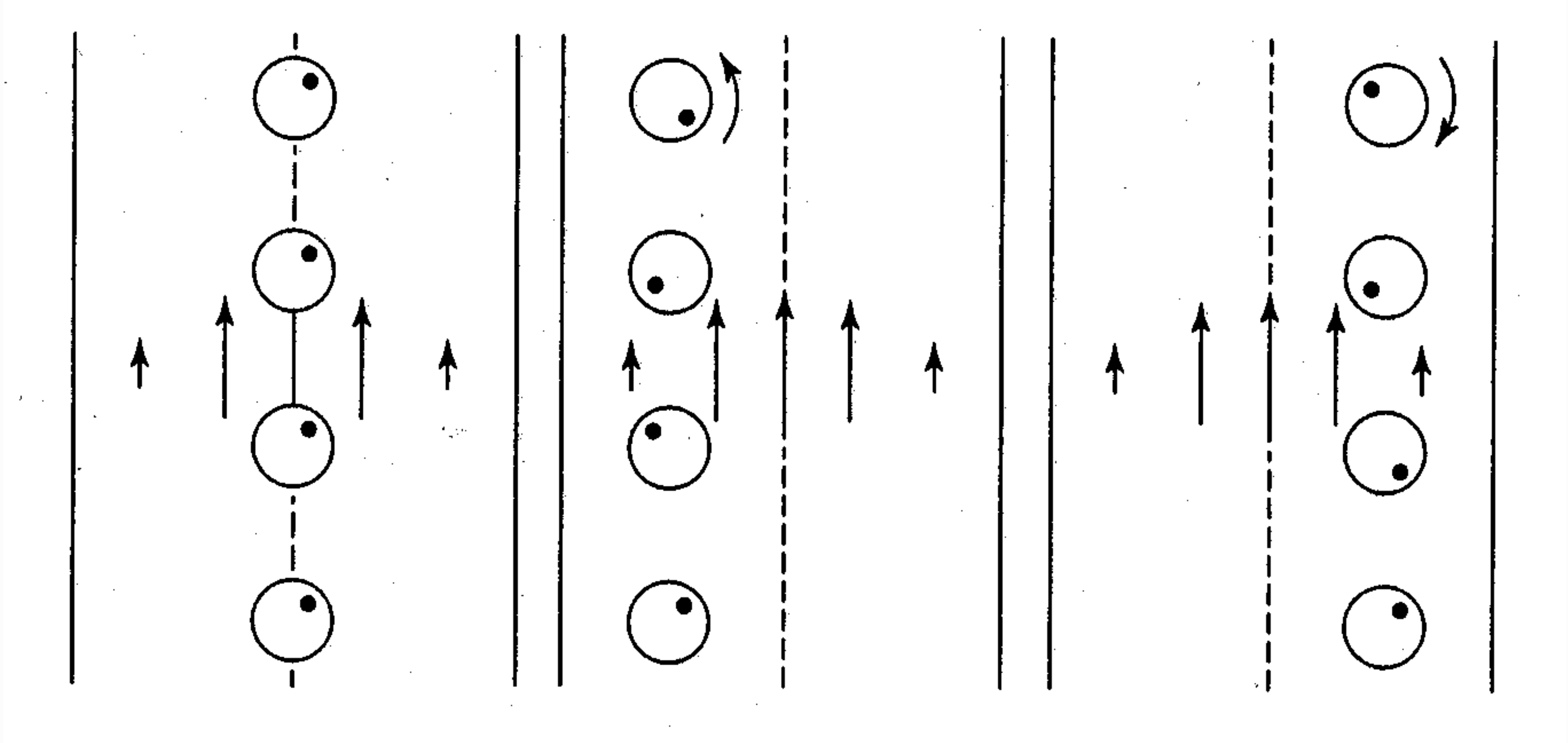
$$\nabla \quad \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Basic definition of curl

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0} \left[\frac{\int_C \mathbf{A} \quad d\mathbf{l}}{\Delta S} \right]_{\max} \mathbf{a}_n$$

$|\nabla \times \mathbf{A}|$ is the maximum value of circulation of **A** per unit area in the limit that the area shrinks to the point.

Direction of $\nabla \times \mathbf{A}$ is the direction of the normal vector to the area in the limit that the area shrinks to the point, and in the right-hand sense.



$$\mathbf{v} = \begin{cases} v_0 \dfrac{2x}{a} \mathbf{a}_z & \text{for } 0 < x < \dfrac{a}{2} \\ v_0 \left(2 - \dfrac{2x}{a} \right) \mathbf{a}_z & \text{for } \dfrac{a}{2} < x < a \end{cases}$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix} = -\frac{\partial v_z}{\partial x} \mathbf{a}_y = \begin{cases} -\dfrac{2v_0}{a} \mathbf{a}_y \\ \dfrac{2v_0}{a} \mathbf{a}_y \end{cases}$$

$$\therefore [\nabla \times \mathbf{v}]_y = \begin{cases} \text{negative for } 0 < x < \dfrac{a}{2} \\ \text{positive for } \dfrac{a}{2} < x < a \end{cases}$$

Two Useful Theorems:

Stokes' theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Divergence theorem

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dv$$

A useful identity

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla g \nabla \times \mathbf{A} = \frac{\partial}{\partial x}[\nabla \times \mathbf{A}]_x + \frac{\partial}{\partial y}[\nabla \times \mathbf{A}]_y + \frac{\partial}{\partial z}[\nabla \times \mathbf{A}]_z$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = 0$$