Lecture-5

Conversion of boundary value problem into integral equations

Example: Transform the BVP $\frac{d^2y}{dx^2} + xy = 1$, $y(0) = y(1) = 0$ into an integral equation.

Solution: We have

$$\left(\frac{dy}{dx}\right)_0^x = \int_0^x 1\,dx - \int_0^x xy(x)dx \,.$$

or

$$y'(x) = x - \int_0^x xy(x)dx + c \,, \quad \text{(let } y'(0)=c\text{)}$$

Again integrating we get

$$y(x) = \frac{x^2}{2} - \int_0^x (x-t)ty(t)dt + cx \qquad \ldots(1)$$

Now, putting $x = 1$

$$c = \int_0^1 (1-t)ty(t)dt - \frac{1}{2} \quad .$$

Substituting the value of c in (1), we have

$$y(x) = \frac{x^2}{2} - \int_0^x (x-t)ty(t)dt + x\left(\int_0^1 (1-t)ty(t)dt - \frac{1}{2}\right),$$

$$y(x)=\frac{x^2}{2}-\int_0^x (x-t)ty(t)dt+x\left(\int_0^x (1-t)ty(t)dt+\int_x^1 (1-t)ty(t)dt\right)-\frac{1}{2}x$$

$$=\frac{x(x-1)}{2}+\int_0^1 K(x,t)y(t)dt.$$

where

$$K(x,t)=\begin{cases} t^2(1-x), & 0\le t<x \\ xt(1-t), & x\le t\le 1 \end{cases}.$$

Practice problem

1. Transform the BVP problem

$$\frac{d^2y}{dx^2}+y=x, y(0)=0, y'(1)=0$$

to an integral equation.

2. Transform the BVP problem

$$\frac{d^2y}{dx^2}+\lambda y=x, y(0)=0, y(1)=1.$$

to an integral equation.

Reference:

https://nptel.ac.in/courses/111/107/111107103/