# UNIT II - DIVIDE-AND-CONQUER

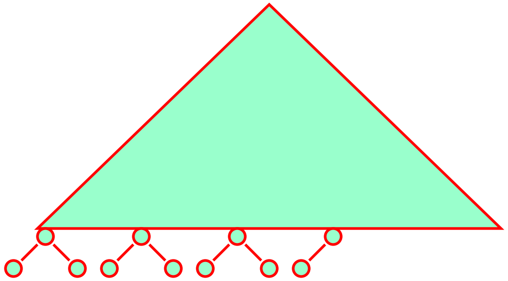
Divide and Conquer Methodology – Binary Search – Merge Sort – Quick Sort – Heap Sort – Multiplication of Large Integers – Strassen's Matrix Multiplication

# Heap and Heap Sort

**Definition:**

**A heap is a binary tree with keys at its nodes (one key per node) such that:**

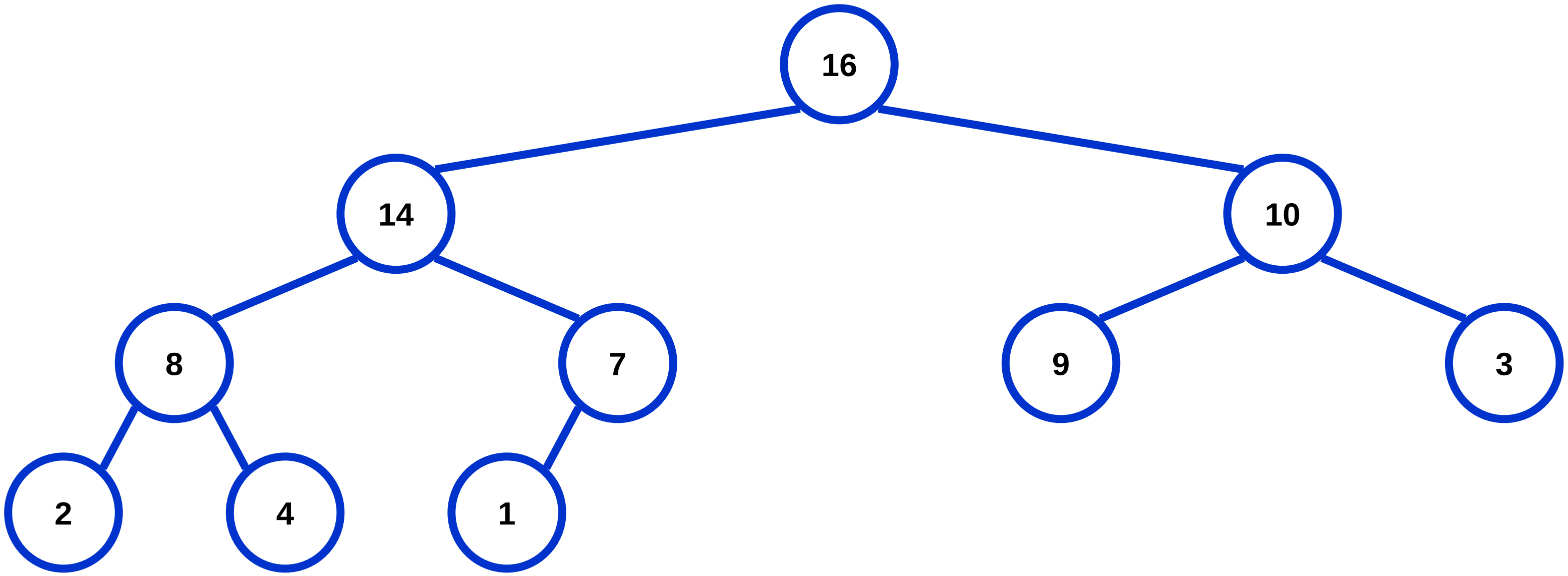
- **It is essentially complete, i.e., all its levels are full except possibly the last level, where only some rightmost keys may be missing**



**The key at each node is ≥ keys at its children**

# Heaps

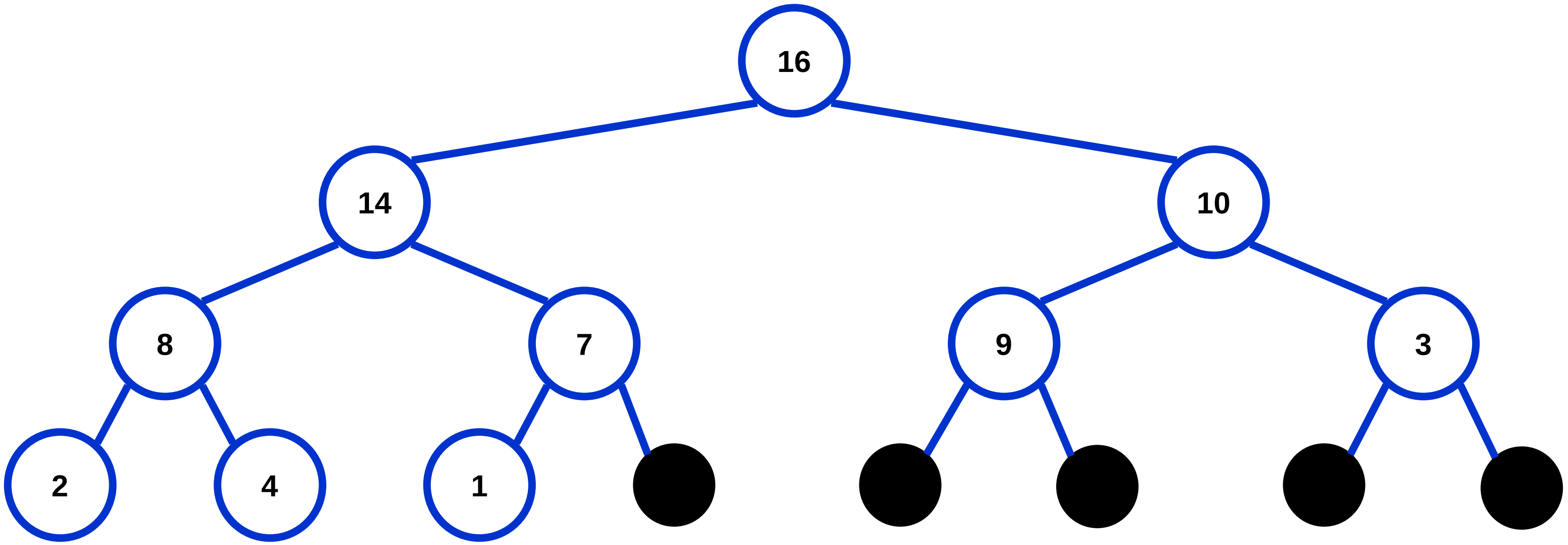
- A *heap* can be seen as a complete binary tree:




  - *What makes a binary tree complete?*
  - *Is the example above complete?*

# Heaps

- A *heap* can be seen as a complete binary tree:

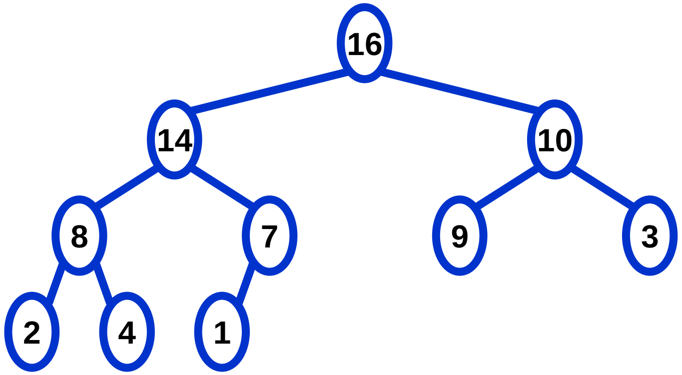



- The book calls them "nearly complete" binary trees; can think of unfilled slots as null pointers

# Heaps

- In practice, heaps are usually implemented as arrays:

*A* = | 16 | 14 | 10 | 8 | 7 | 9 | 3 | 2 | 4 | 1 | =



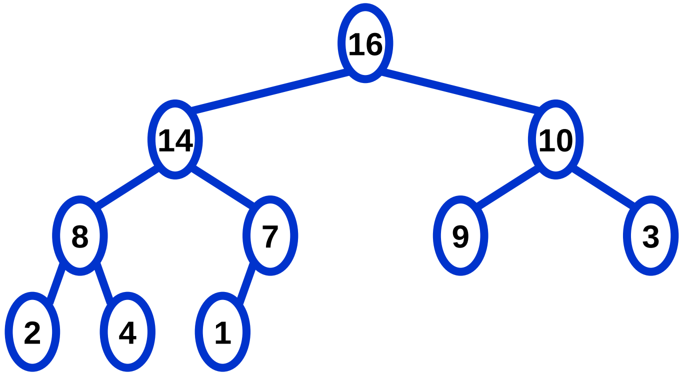
# Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[*i*]
  - The parent of node *i* is A[*i*/2] (note: integer divide)
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

*A* =

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 14 | 10 | 8 | 7 | 9 | 3 | 2 | 4 | 1 |

=



# Referencing Heap Elements

- So…

```
Parent(i) { return ⌊i/2⌋; }
Left(i) { return 2*i; }
right(i) { return 2*i + 1; }
```

- An aside: *How would you implement this most efficiently?*
  - Trick question, I was looking for "**`i << 1`**", etc.
  - But, any modern compiler is smart enough to do this for you (and it makes the code hard to follow)

# The Heap Property

- Heaps also satisfy the *heap property*:

  A[*Parent*(*i*)] ≥ A[*i*] for all nodes *i* > 1

  - In other words, the value of a node is at most the value of its parent
  - *Where is the largest element in a heap stored?*

# Heap Height

- Definitions:
  - The *height* of a node in the tree = the number of edges on the longest downward path to a leaf
  - The height of a tree = the height of its root
- *What is the height of an n-element heap? Why?*
- This is nice: basic heap operations take at most time proportional to the height of the heap

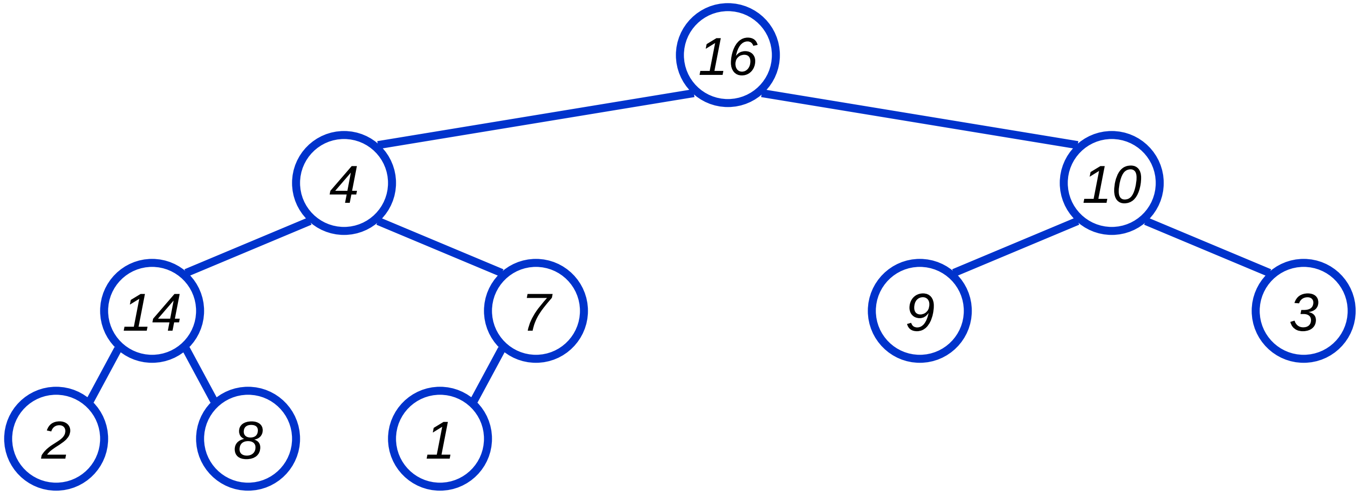
# Heap Operations: Heapify()

- **`Heapify()`**: maintain the heap property
  - Given: a node *i* in the heap with children *l* and *r*
  - Given: two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property
    - *What do you suppose will be the basic operation between i, l, and r?*

# Heap Operations: Heapify()

```
Heapify(A, i)
{
  l = Left(i); r = Right(i);
  if (l <= heap_size(A) && A[l] > A[i])
      largest = l;
  else
      largest = i;
  if (r <= heap_size(A) && A[r] > A[largest])
      largest = r;
  if (largest != i)
      Swap(A, i, largest);
      Heapify(A, largest);
}
```

# Heapify() Example




A =

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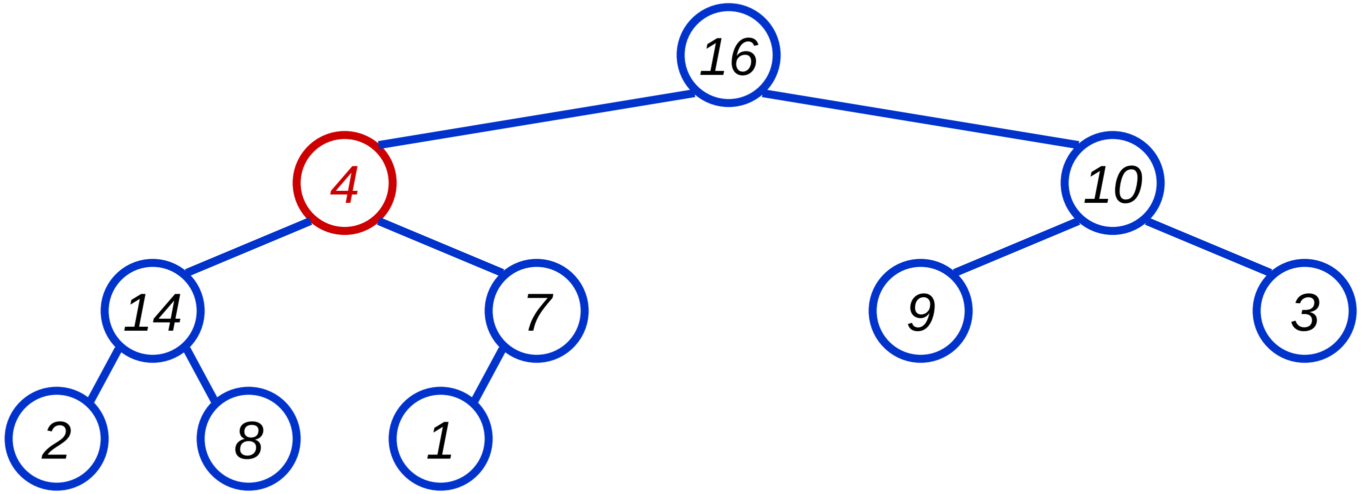


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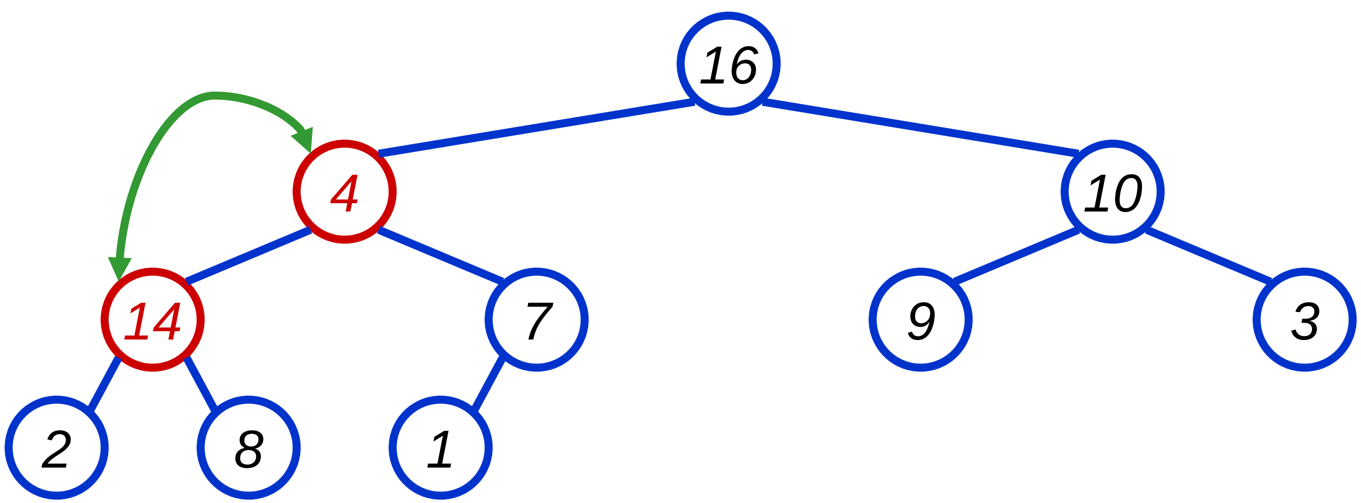


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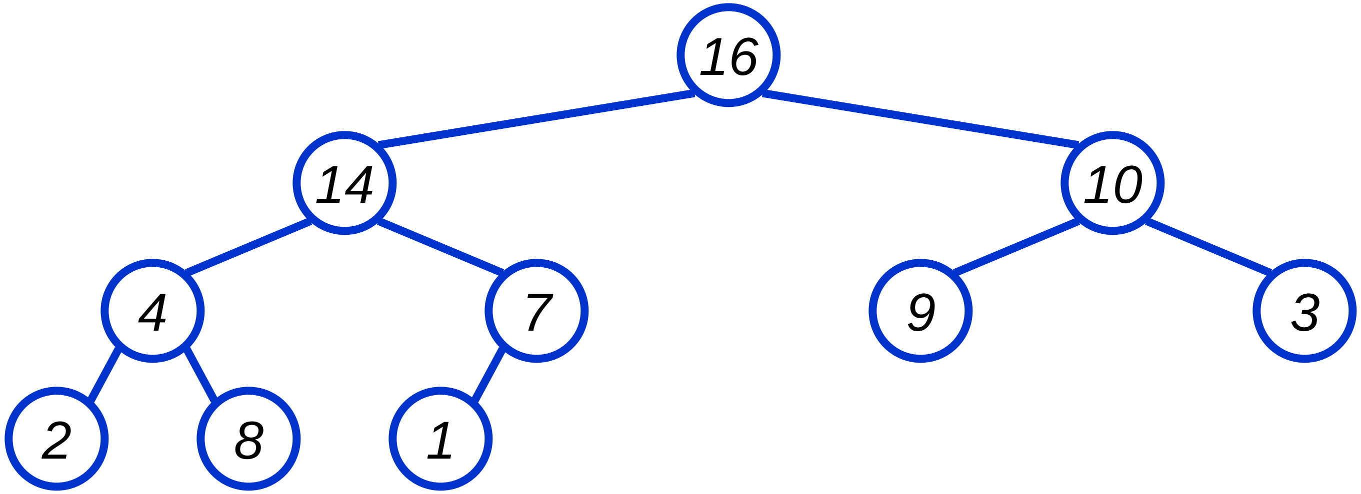


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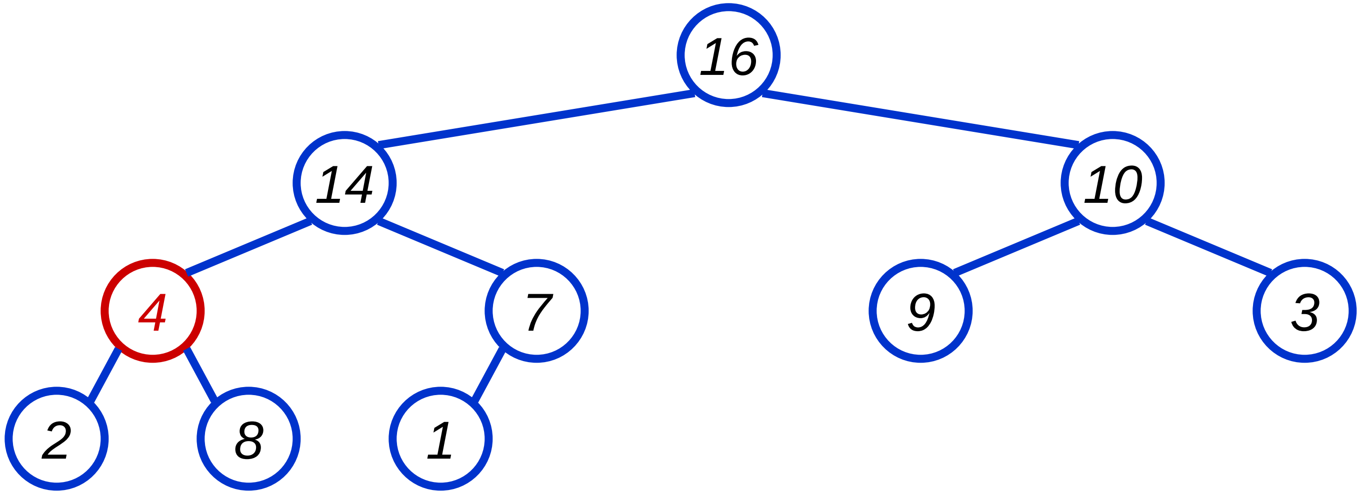


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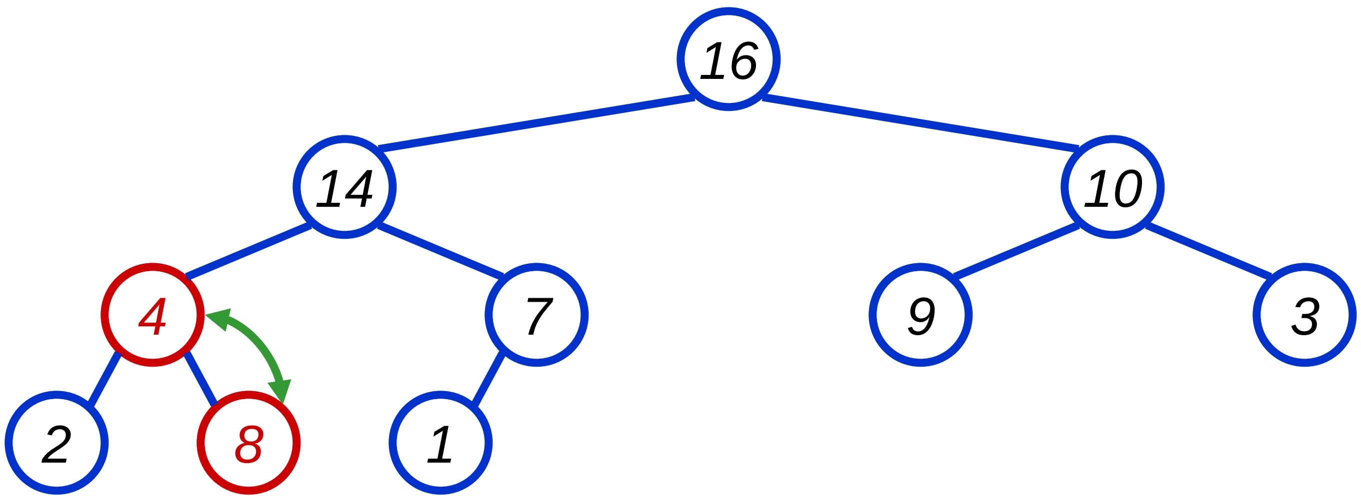


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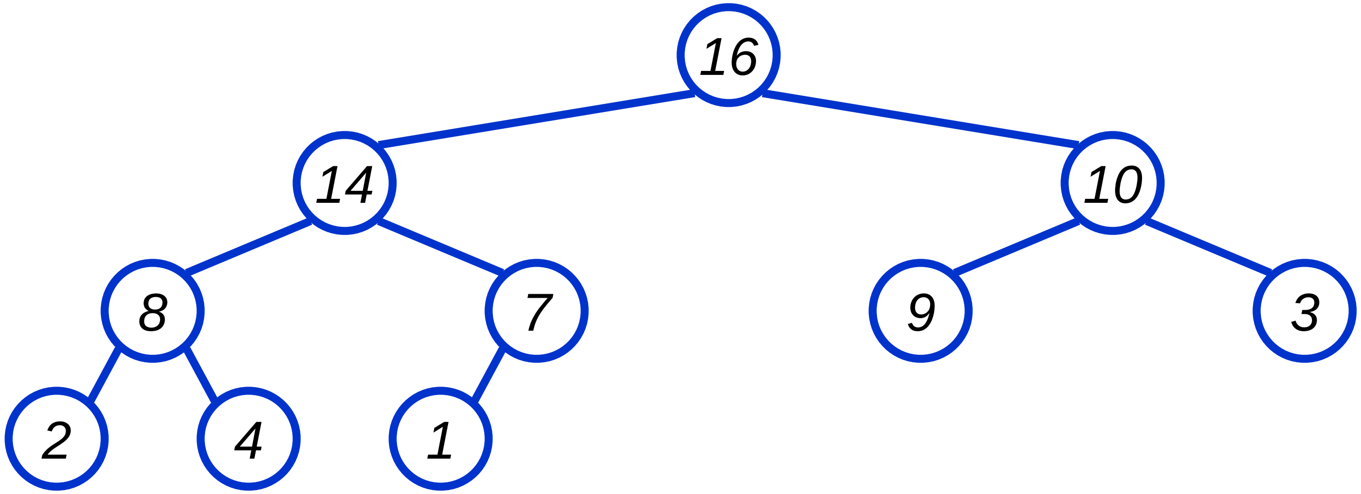


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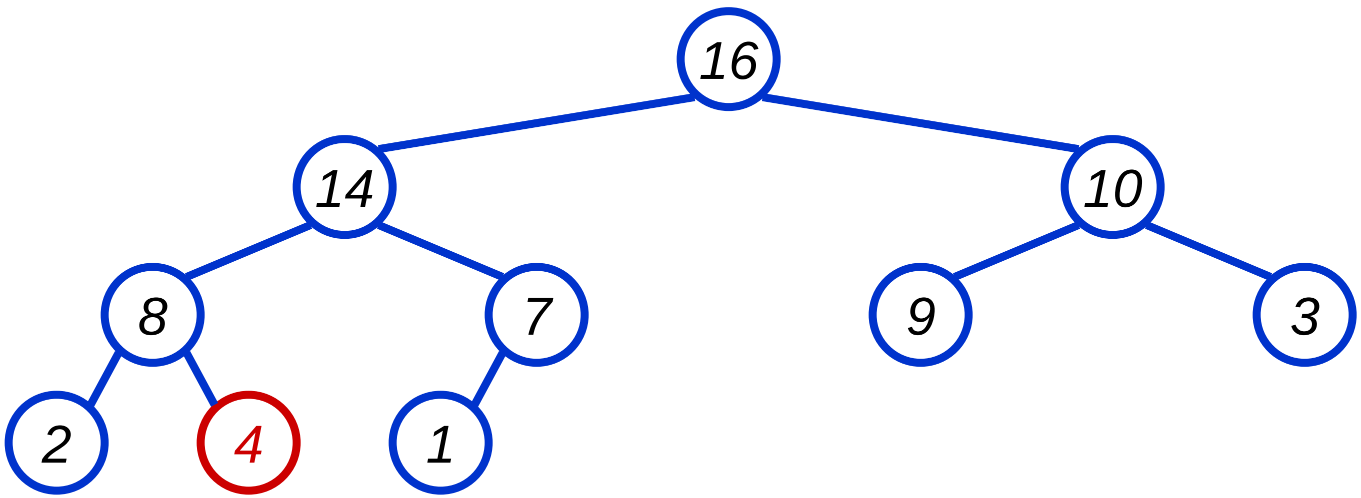


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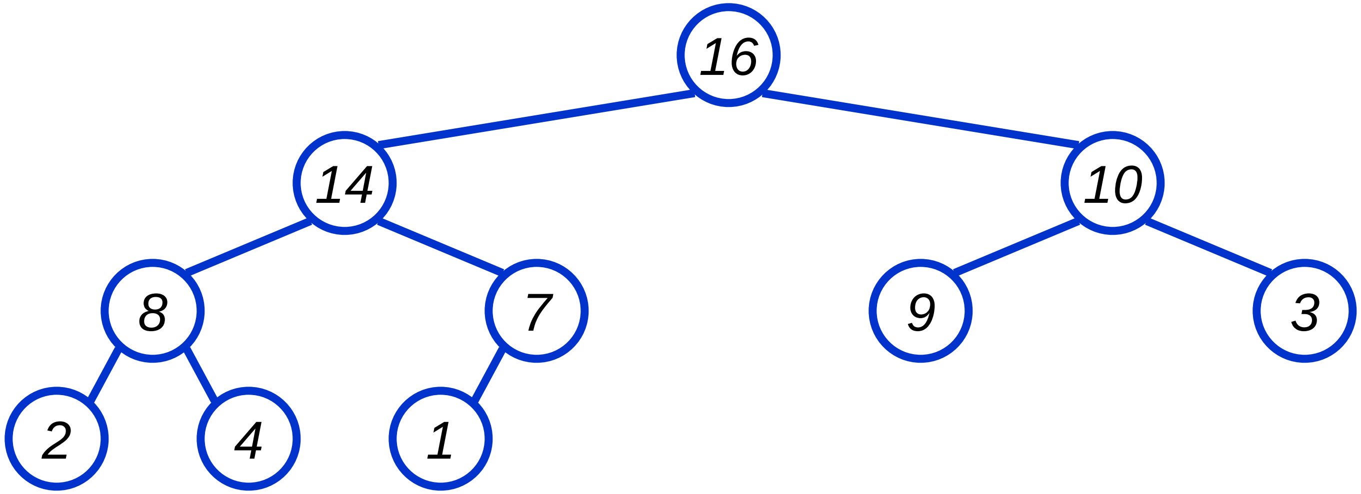


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# Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`**?
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

# Analyzing Heapify(): Formal

- Fixing up relationships between *i*, *l*, and *r* takes Θ(1) time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
  - Draw it
- Answer: 2*n*/3 (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

$$T(n) \leq T(2n/3) + \Theta(1)$$

# Analyzing Heapify(): Formal

- So we have

$$T(n) \leq T(2n/3) + \Theta(1)$$

- By case 2 of the Master Theorem,

$$T(n) = O(\lg n)$$

- Thus, **`Heapify()`** takes logarithmic time

# Heap Operations: BuildHeap()

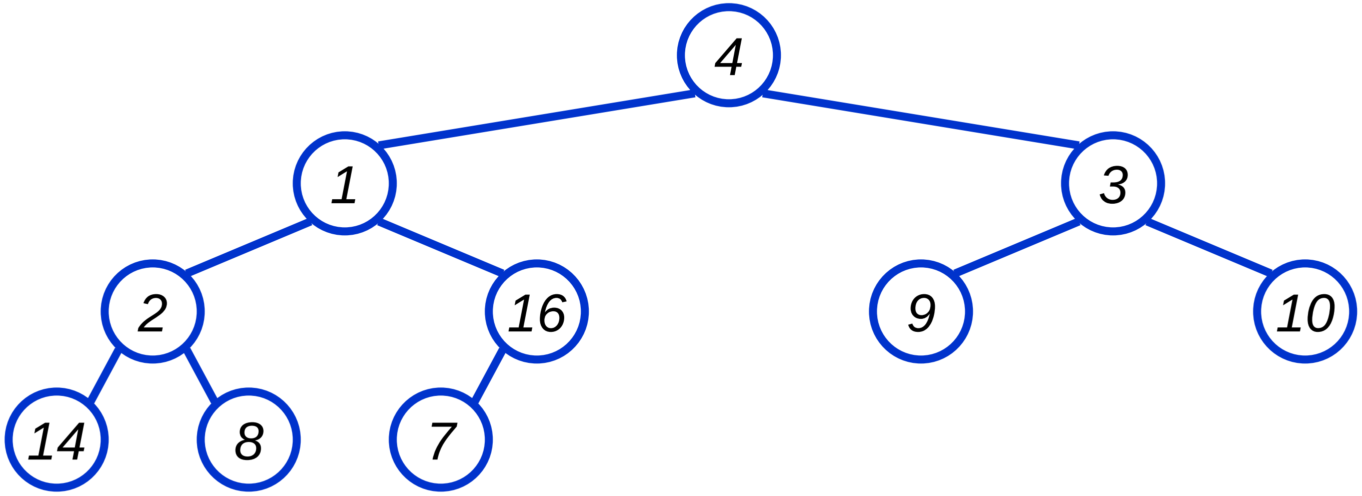
- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

# BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
  heap_size(A) = length(A);
  for (i = ⌊length[A]/2⌋ downto 1)
     Heapify(A, i);
}
```

# BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



# Analyzing BuildHeap()

- Each call to **`Heapify()`** takes O(lg *n*) time
- There are O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(*n* lg *n*)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(*n*)
  - *How can this be? Is there a flaw in the above reasoning?*

# Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O($h$) time where $h$ is the height of the subtree
  - $h$ = O(lg $m$), m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an $n$-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height $h$
- CLR 7.3 uses this fact to prove that **`BuildHeap()`** takes O($n$) time

# Heapsort

- Given **`BuildHeap()`**, an in-place sorting algorithm is easily constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

# Heapsort Algorithm

```
Heapsort(A)
{
    BuildHeap(A);
    for (i = length(A) downto 2)
    {
        Swap(A[1], A[i]);
        heap_size(A) -= 1;
        Heapify(A, 1);
    }
}
```

# Analyzing Heapsort

- The call to **`BuildHeap()`** takes O($n$) time
- Each of the $n$ - 1 calls to **`Heapify()`** takes O(lg $n$) time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1)\, O(\lg n)$
  $= O(n) + O(n \lg n)$
  $= O(n \lg n)$

# Priority Queues

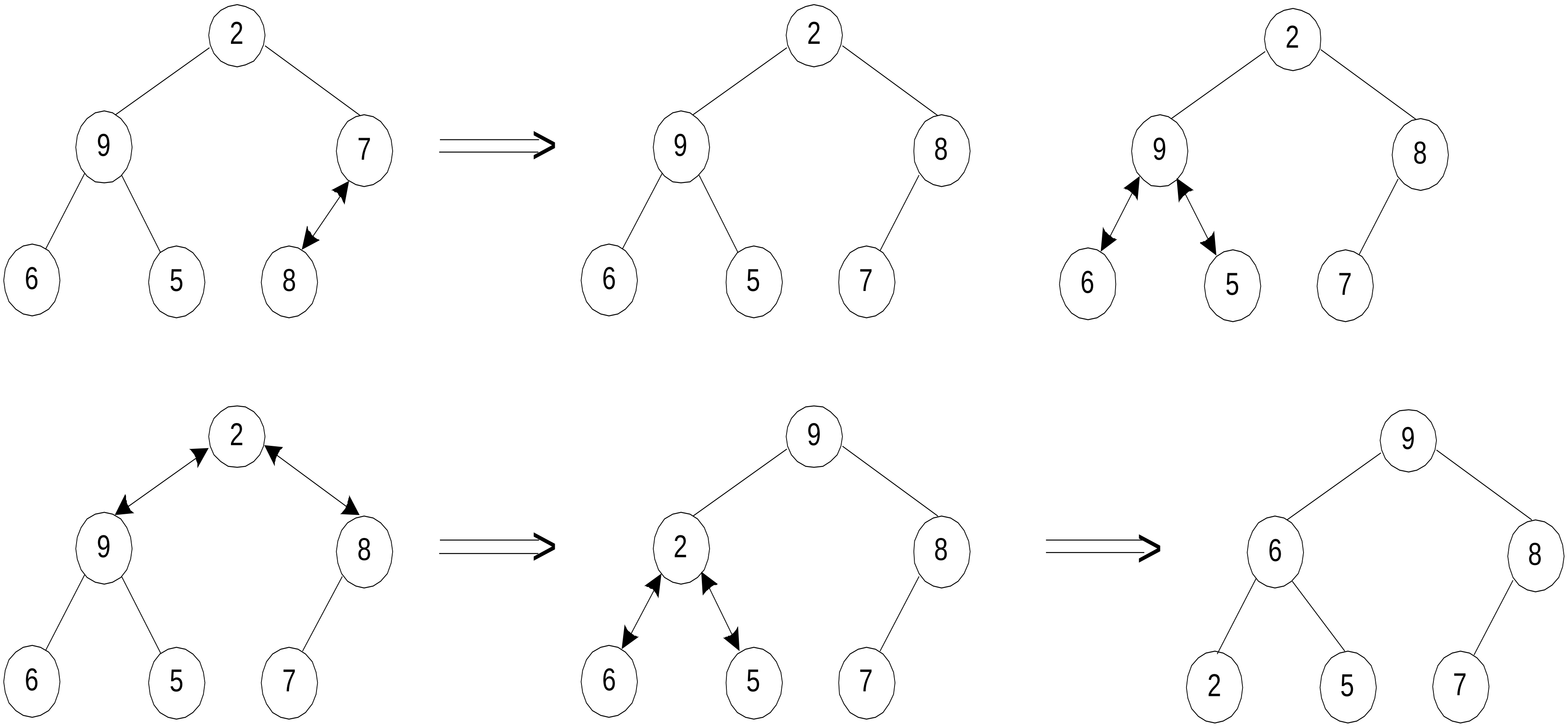
- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set *S* of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**
  - ***What might a priority queue be useful for?***

# Priority Queue Operations

- **Insert(S, x)** inserts the element x into set S
- **Maximum(S)** returns the element of S with the maximum key
- **ExtractMax(S)** removes and returns the element of S with the maximum key
- *How could we implement these operations using a heap?*

# Example of Heap Construction

## Construct a heap for the list 2, 9, 7, 6, 5, 8



# Example of Sorting by Heapsort

Sort the list 2, 9, 7, 6, 5, 8 by heapsort

**Stage 1 (heap construction)**

1 9 7 6 5 8
2 9 8 6 5 7
2 9 8 6 5 7
9 2 8 6 5 7
9 6 8 2 5 7

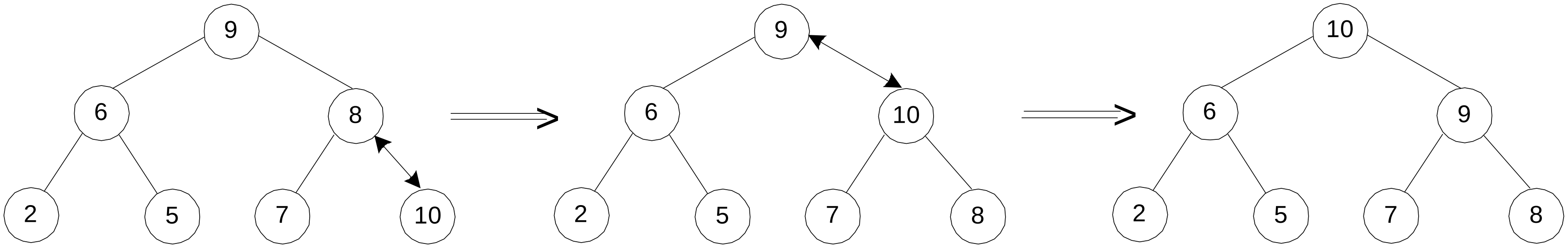
**Stage 2 (root/max removal)**

9 6 8 2 5 7
7 6 8 2 5 | 9
8 6 7 2 5 | 9
5 6 7 2 | 8 9
7 6 5 2 | 8 9
2 6 5 | 7 8 9
6 2 5 | 7 8 9
5 2 | 6 7 8 9
5 2 | 6 7 8 9
2 | 5 6 7 8 9

# Insertion of a New Element into a Heap

- Insert the new element at last position in heap.
- Compare it with its parent and, if it violates heap condition, exchange them
- Continue comparing the new element with nodes up the tree until the heap condition is satisfied

**Example: Insert key 10**




**Efficiency: O(log *n*)**

# Thank You