Boolean Algebra and De Morgan’s Theorems

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Introduction : Boolean Algebra

- George Boole developed an algebraic description for processes involving logical thought and reasoning.
  - Became known as Boolean Algebra
- Claude Shannon later demonstrated that Boolean Algebra could be used to describe switching circuits.
  - Switching circuits are circuits built from devices that switch between two states (e.g. 0 and 1).
  - Switching Algebra is a special case of Boolean Algebra in which all variables take on just two distinct values
- Boolean Algebra is a powerful tool for analyzing and designing logic circuits.

Boolean Expressions

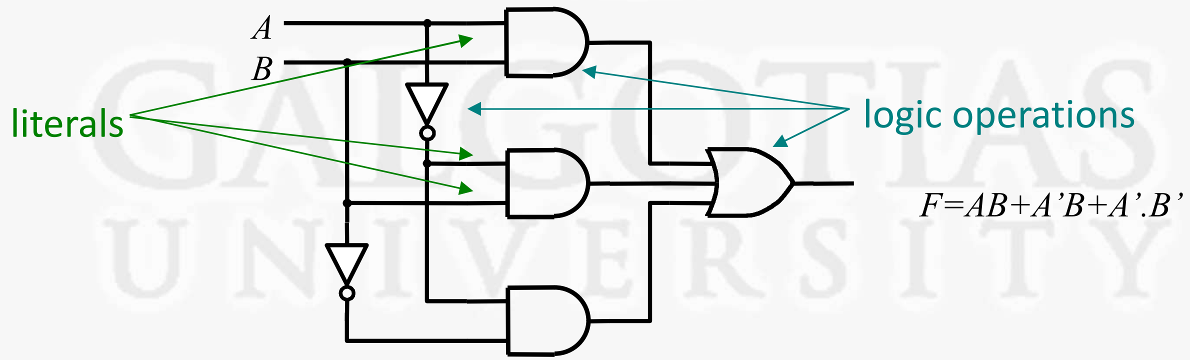
- Boolean expressions are composed of
  1- Literals variables and their complements
  2- Logical operations
  - Examples
- F = A.B'.C + A'.B.C' + A.B.C + A'.B'.C’

literals

logic operations

Boolean Expressions

- Boolean expressions are realized using a network (or combination) of logic gates.
  - Each logic gate implements one of the logic operations in the Boolean expression
  - Each input to a logic gate represents one of the literals in the Boolean expression



Duality Principle:

The duality principle states that a Boolean expression remains valid if operators OR and AND are interchanged and 1's and 0's in the expression are also interchanged.

In order to understand this principle, consider the Boolean Theorem 1 viz.

$A + 0 = A$

According to duality principle, this Boolean expression remains valid if OR function is replaced

by AND function and 0 by 1. In that case, the Boolean expression becomes :

$A \cdot 1 = A$

Basic Laws and Theorems

Commutative Law	A + B = B + A	A.B = B.A
Associative Law	A + (B + C) = (A + B) + C	A . (B . C) = (A . B) . C
Distributive Law	A.(B + C) = AB + AC	A + (B . C) = (A + B) . (A + C)
Null Elements	A + 1 = 1	A . 0 = 0
Identity	A + 0 = A	A . 1 = A
Idempotence	A + A = A	A . A = A
Complement	A + A' = 1	A . A' = 0
Involution	A'' = A	
Absorption (Covering)	A + AB = A	A . (A + B) = A
Simplification	A + A'B = A + B	A . (A' + B) = A . B
DeMorgan's Rule	(A + B)' = A'.B'	(A . B)' = A' + B'
Logic Adjacency (Combining)	AB + AB' = A	(A + B) . (A + B') = A
Consensus	AB + BC + A'C = AB + A'C	(A + B) . (B + C) . (A' + C) = (A + B) . (A' + C)

Idempotence

$$A + A = A$$

$$F = ABC + ABC' + ABC$$

$$F = ABC + ABC'$$

Note: terms can also be added using this theorem

$$A . A = A$$

$$G = (A' + B + C').(A + B' + C).(A + B' + C)$$

$$G = (A' + B + C') + (A + B' + C)$$

Note: terms can also be added using this theorem

Complement

$$A + A' = 1$$

$$F = ABC'D + ABCD$$

$$F = ABD.(C' + C)$$

$$F = ABD$$

$$A . A' = 0$$

$$G = (A + B + C + D).(A + B' + C + D)$$

$$G = (A + C + D) + (B . B')$$

$$G = A + C + D$$

Distributive Law

- $A.(B + C) = AB + AC$
- $A + (B.C) = (A + B).(A + C)$

- $F = WX.(Y + Z)$
- $F = WXY + WXZ$
- $F = WX + (Y.Z)$
- $F = (WX + Y).(WX + Z)$

- $G = B'.(AC + AD)$
- $G = AB'C + AB'D$
- $G = B' + (A.C.D)$
- $G = (B' + A).(B' + C).(B' + D)$

- $H = A.(W'X + WX' + YZ)$
- $H = AW'X + AWX' + AYZ$
- $H = A + ((W'X).(WX'))$
- $H = (A + W'X).(A + WX')$

Absorption (Covering)

- A + AB = A
  - F = A'BC + A'
    - F = A'
- G = XYZ + XY'Z + X'Y'Z' + XZ
  - G = XYZ + XZ + X'Y'Z'
    - G = XZ + X'Y'Z'
  - H = D + DE + DEF
    - H = D
- A.(A + B) = A
  - F = A'.(A' + BC)
    - F = A'
- G = XZ.(XZ + Y + Y')
  - G = XZ.(XZ + Y)
    - G = XZ
- H = D.(D + E + EF)
  - H = D

Simplification

$$A + A'B = A + B$$

$$F = (XY + Z).(Y'W + Z'V') + (XY + Z)'$$

$$F = Y'W + Z'V' + (XY + Z)'$$

$$A.(A' + B) = A . B$$

$$G = (X + Y).((X + Y)' + (WZ))$$

$$G = (X + Y) . WZ$$

Logic Adjacency (Combining)

$$A.B + A.B' = A$$

$$F = (X + Y).(W'X'Z) + (X + Y).(W'X'Z)'$$

$$F = (X + Y)$$

$$(A + B).(A + B') = A$$

$$G = (XY + X'Z').(XY + (X'Z')')$$

$$G = XY$$

Summary

Single variable theorems

Theorem 1 :	$A + 0 = A$
Theorem 2 :	$A \cdot 1 = A$
Theorem 3 :	$A + \overline{A} = 1$
Theorem 4 :	$A \cdot \overline{A} = 0$
Theorem 5 :	$A + A = A$
Theorem 6 :	$A \cdot A = A$
Theorem 7 :	$A + 1 = 1$
Theorem 8 :	$A \cdot 0 = 0$
Theorem 9 :	$\overline{\overline{A}} = A$

Summary

Multivariable theorems: These theorems refer to the condition when more than one input to the logic gate are variable.

Theorem 10	:	$A + B = B + A$	*Commutative Law*
Theorem 11	:	$A \cdot B = B \cdot A$	
Theorem 12	:	$A + (B + C) = (A + B) + C$	*Associative Law*
Theorem 13	:	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$	
Theorem 14	:	$A \cdot (B + C) = A \cdot B + A \cdot C$	*Distributive Law*
Theorem 15	:	$(A + B) \cdot (C + D) = A \cdot C + B \cdot C + A \cdot D + B \cdot D$	
Theorem 16	:	$A + A \cdot B = A$	*De Morgan's Theorems*
Theorem 17	:	$(\overline{A + B}) = \overline{A} \cdot \overline{B}$	
Theorem 18	:	$(\overline{A \cdot B}) = \overline{A} + \overline{B}$	

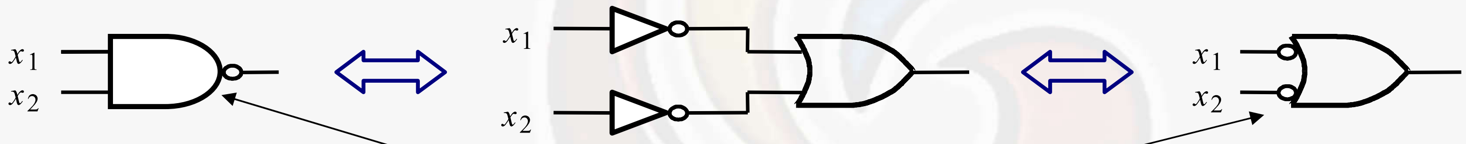
De Morgan’s Theorems

- Can be stated as follows:
  - The complement of the product (AND) is the sum (OR) of the complements.
    - (X.Y)' = X' + Y'
  - The complement of the sum (OR) is the product (AND) of the complements.
    - (X + Y)' = X' . Y'
- Easily generalized to n variables.
- Can be proven using a Truth table

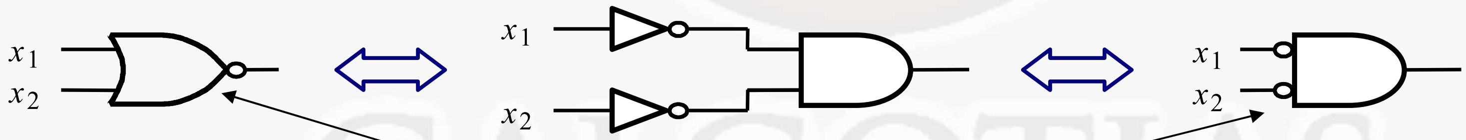
Proving De Morgan's Law

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0
			LHS		RHS	

De Morgan's Theorems



(a) $\overline{x_1 x_2} = \bar{x}_1 + \bar{x}_2$



(b) $\overline{x_1 + x_2} = \bar{x}_1 \bar{x}_2$

Example:

Simplify the expression:

$$X = \overline{A}\overline{B}C + A\overline{B}C + AB\overline{C} + ABC$$

Solution:

Step 1. Note that the first two terms have $\overline{B}C$ as common factors while the last two terms have AB as common factors.

$$\therefore \quad X = \overline{B}C\,(A + \overline{A}) + AB\,(C + \overline{C})$$

Step 2. $A + \overline{A} = 1$ and $C + \overline{C} = 1$ so that :

$$X = \overline{B}C \cdot 1 + AB \cdot 1$$

Step 3. Since $\overline{B}C \cdot 1 = \overline{B}C$ and $AB \cdot 1 = AB$ so that :

$$X = AB + \overline{B}C$$

Note that not only is the Boolean expression simplified, but so is the resultant logic circuit.

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- Modern Digital Electronics, R P Jain, 4th Ed., 2010, Tata McGraw Hill