Course Code : MSCM301

Course Name: Functional Analysis

Properties of bonded linear transformations By

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Theorem: Let $T: N \rightarrow N'$ be a linear transformation. Then T is bounded iff T maps bounded sets in N into bounded sets in N'.

Proof: Since T is bounded linear transformation. $||T(x)|| \le M ||x||, \forall x \in N.$ Let B be a bounded subset of N. Then $||x|| \le K, \forall x \in B.$ From above we can say that $||T(x)|| \le MK, \forall x \in B.$ It shows that T(B) is bounded in N'.

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Conversely let T map bounded sets in N into bounded sets in N'. To prove that T is a bounded linear transformation, let us take the closed unit sphere $S_1[0]$ in N as a bounded set. By hypothesis, its image $T(S_1[0])$ must be bounded set in N'. Therefore there is a constant K such that

 $\|T(x)\| \le K, \forall x \in S_1[0].$ Let x be any non zero vector in N. Then $\left(\frac{x}{\|x\|}\right) \in S_1[0]$ and so we get $\|T\left(\frac{x}{\|x\|}\right)\| \le K$ which implies $\|T(x)\| \le K\|x\|.$

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Theorem: If N and N' are normed linear spaces and $T: N \rightarrow N'$, then following are equivalent.

$$(a)||T|| = \sup\left\{\frac{||T(x)||}{||x||}: x \in N, x \neq 0\right\}$$

(b) $||T|| = \sup\{||T(x)||: x \in N, ||x|| \leq 1\}.$
(c) $||T|| = \sup\{||T(x)||: x \in N, ||x|| = 1\}.$

Proof: If x satisfies (a), then we have $\frac{||T_X||}{||x||} = \left\| T\left(\frac{x}{||x||}\right) \right\|$.

Since
$$\left\|\frac{x}{||x||}\right\| = 1$$
, x satisfies (c) so that (a) \leq (c).

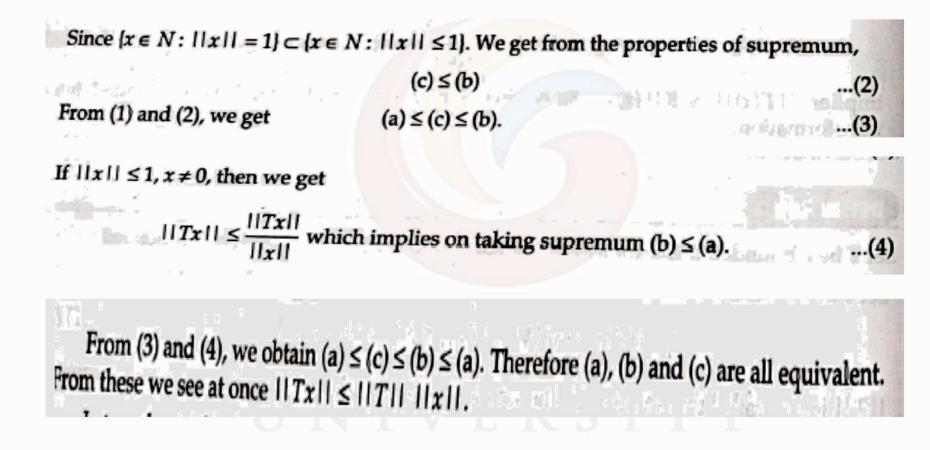
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Theorem: Let N and N' be normed linear spaces and let $T: N \rightarrow N'$ be bounded linear transformation of N into N'. If M is the kernel of T, then

(i) M is a closed subspace of N

(ii) T induces a natural transformation T' of N/M onto N' such that ||T'|| = ||T||.

Proof: The kernel of an operator T is defined as $ker(T) = \{x \in N: T(x) = 0\}$. The kernel of a linear operator is a subspace of N. If M contains all of its limit points then it will be closed.

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Let x be a limit point of ker(T). Then there exist a sequence (x_n) in Ker(T) such that $x_n \to x$. Since T is continuous, $T(x_n) \to T(x)$. But $T(x_n) = 0, \forall n$ and so T(x) =0. This proves that Ker(T) is closed.

By (i) M is a closed subspace of N and so N/M is a quotient space with the norm of a coset given by

 $||x + M|| = \inf \{||x + m|| : m \in M\}.$

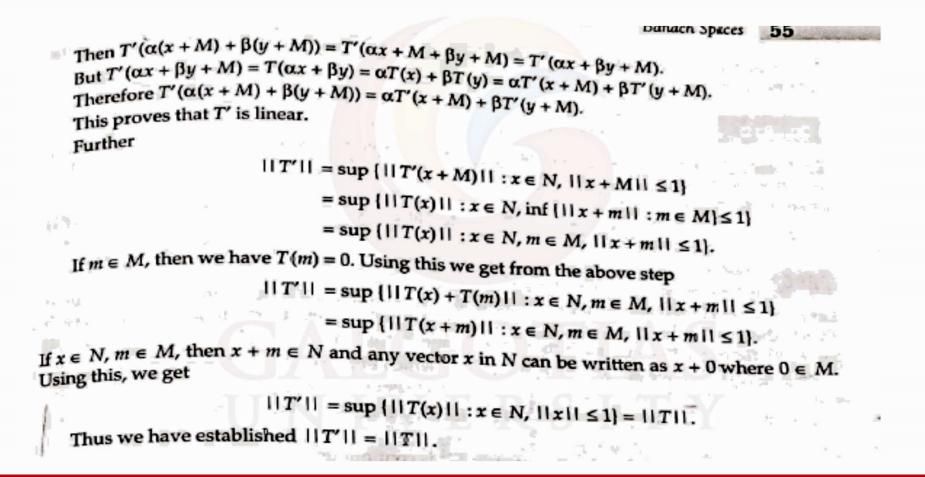
We shall define $T': N/M \rightarrow N'$ by T'(x + M) = T(x) and show that T' is linear and |T'|| = ||T||.

Let x + M, $y + M \in N/M$ and α , β be any two scalars.

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Video Links:

1. https://youtu.be/TJcnVWj6jVg



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Reference

A first course in functional Analysis by D. Somasundaram



Name of the Faculty: Dr. Alok Tripathi

Course Code :

Course Name: Calculus

Thank You

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