Properties of bonded linear transformations

By

Dr. Alok Tripathi
Department of Mathematics
Galgotias University

Theorem: Let $T: N \rightarrow N'$ be a linear transformation. Then T is bounded iff T maps bounded sets in N into bounded sets in N'.

Proof: Since T is bounded linear transformation.

$$\|T(x)\| \leq M\|x\|, \forall x \in N.$$

Let B be a bounded subset of N. Then

$$\|x\| \leq K, \forall x \in B.$$

From above we can say that

$$\|T(x)\| \leq MK, \forall x \in B.$$

It shows that T(B) is bounded in N'.

Conversely let T map bounded sets in N into bounded sets in N'. To prove that T is a bounded linear transformation, let us take the closed unit sphere $S_1[0]$ in N as a bounded set. By hypothesis, its image $T(S_1[0])$ must be bounded set in N'. Therefore there is a constant K such that

$$\|T(x)\| \le K, \forall\, x \in S_1[0].$$

Let x be any non zero vector in N. Then $\left(\frac{x}{\|x\|}\right) \in S_1[0]$ and so we get $\left\|T\left(\frac{x}{\|x\|}\right)\right\| \le K$ which implies $\|T(x)\| \le K\|x\|$.

Theorem: If N and N' are normed linear spaces and $T: N \to N'$, then following are equivalent.

(a) $\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in N, x \neq 0\right\}$

(b) $\|T\| = \sup\{\|T(x)\| : x \in N, \|x\| \leq 1\}$.

(c) $\|T\| = \sup\{\|T(x)\| : x \in N, \|x\| = 1\}$.

Proof:

If x satisfies (a), then we have $\frac{\|Tx\|}{\|x\|} = \left\|T\left(\frac{x}{\|x\|}\right)\right\|$.

Since $\left\|\frac{x}{\|x\|}\right\| = 1$, x satisfies (c) so that (a) $\leq$ (c). ...(1)

Since $\{x \in N : ||x|| = 1\} \subset \{x \in N : ||x|| \leq 1\}$. We get from the properties of supremum,

$$(c) \leq (b) \qquad ...(2)$$

From (1) and (2), we get

$$(a) \leq (c) \leq (b). \qquad ...(3)$$

If $||x|| \leq 1, x \neq 0$, then we get

$$||Tx|| \leq \frac{||Tx||}{||x||}$$ which implies on taking supremum (b) ≤ (a). ...(4)

From (3) and (4), we obtain (a) ≤ (c) ≤ (b) ≤ (a). Therefore (a), (b) and (c) are all equivalent. From these we see at once $||Tx|| \leq ||T|| \; ||x||$.

Theorem: Let N and N’ be normed linear spaces and let $T: N \to N'$ be bounded linear transformation of N into N’. If M is the kernel of T, then

(i) M is a closed subspace of N
(ii) T induces a natural transformation T’ of N/M onto N’ such that $||T'|| = ||T||$.

Proof: The kernel of an operator T is defined as $\ker(T) = \{x \in N: T(x) = 0\}$. The kernel of a linear operator is a subspace of N. If M contains all of its limit points then it will be closed.

Let x be a limit point of ker(T). Then there exist a sequence (x_n) in Ker(T) such that $x_n \to x$. Since T is continuous, $T(x_n) \to T(x)$. But $T(x_n) = 0, \forall n$ and so $T(x) = 0$. This proves that Ker(T) is closed.

By (i) M is a closed subspace of N and so N/M is a quotient space with the norm of a coset given by

$$||x + M|| = \inf\{||x + m|| : m \in M\}.$$

We shall define $T' : N/M \to N'$ by $T'(x + M) = T(x)$ and show that T' is linear and $||T'|| = ||T||$.

Let $x + M, y + M \in N/M$ and α, β be any two scalars.

Then $T'(\alpha(x+M)+\beta(y+M)) = T'(\alpha x + M + \beta y + M) = T'(\alpha x + \beta y + M)$.
But $T'(\alpha x + \beta y + M) = T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha T'(x+M) + \beta T'(y+M)$.
Therefore $T'(\alpha(x+M)+\beta(y+M)) = \alpha T'(x+M) + \beta T'(y+M)$.
This proves that T' is linear.
Further

$$\begin{aligned} \|T'\| &= \sup\{\|T'(x+M)\| : x \in N, \|x+M\| \le 1\} \\ &= \sup\{\|T(x)\| : x \in N, \inf\{\|x+m\| : m \in M\} \le 1\} \\ &= \sup\{\|T(x)\| : x \in N, m \in M, \|x+m\| \le 1\}. \end{aligned}$$

If $m \in M$, then we have $T(m) = 0$. Using this we get from the above step

$$\begin{aligned} \|T'\| &= \sup\{\|T(x)+T(m)\| : x \in N, m \in M, \|x+m\| \le 1\} \\ &= \sup\{\|T(x+m)\| : x \in N, m \in M, \|x+m\| \le 1\}. \end{aligned}$$

If $x \in N$, $m \in M$, then $x + m \in N$ and any vector x in N can be written as $x + 0$ where $0 \in M$.
Using this, we get

$$\|T'\| = \sup\{\|T(x)\| : x \in N, \|x\| \le 1\} = \|T\|.$$

Thus we have established $\|T'\| = \|T\|$.

Video Links:

1. https://youtu.be/TJcnVWj6jVg

Reference

- A first course in functional Analysis by D. Somasundaram

Thank You