BOSE-EINSTEIN STATISTICS

Topics Covered:

- Bose-Einstein condensation

Consider a Bose-Einstein gas of N bosons. The most probable distribution of these bosons is given by the Bose-Einstein distribution law:

$$n_i = g_i / exp(\alpha+\beta\varepsilon_i)-1 \quad \text{..........(1)}$$

where, $\alpha=-\mu/kT$ and $\beta=1/kT$

Because, in general, the energies of the quantum states are very closely spaced, we can integrate equation (1) to get the total number of particles.

The density of states for spinless particles is given as:

$$\rho(\epsilon) = \frac{V}{4\pi^2}\frac{(2\,m)^{3/2}}{\hbar^3}\,\epsilon^{1/2},$$

$$N = \int_0^\infty \frac{\rho(\epsilon)}{e^{\beta\,(\epsilon-\mu)}-1}\,d\epsilon = \frac{V}{4\pi^2}\frac{(2\,m)^{3/2}}{\hbar^3}\int_0^\infty \frac{\epsilon^{1/2}}{e^{\beta\,(\epsilon-\mu)}-1}\,d\epsilon.$$

However, there is a significant flaw in this formulation as the ground-state, $\varepsilon=0$, has been left out [because $\rho(0)=0$]. Under ordinary circumstances, this omission does not matter. However, at very low temperatures, bosons tend to condense into the ground-state, and the occupation number of this state becomes very much larger than that of any other state. Under these circumstances, the ground-state must be included in the calculation.

We can overcome the previous difficulty in the following manner. Let there be N_0 bosons in the ground-state, and N_{ex} in the various excited states, so that

$$N = N_0 + N_{\rm ex}.$$

Because the ground-state is excluded from expression, the integral only gives the number of bosons in excited states. In other words,

$$N_{\rm ex} = \frac{V}{4\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\int_0^\infty \frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon.$$

Now, because the ground-state has zero energy:

$N_0 \approx 1/exp(-\mu/kT)-1$

We conclude that

$$-\frac{\mu}{kT} \simeq \ln\left(1+\frac{1}{N}\right) \simeq \frac{1}{N},$$

for large N . Hence, at very low temperatures, we can safely set exp(-μ/kT)= 1. Therefore,

$$N_{\rm ex} = \frac{2}{\sqrt{\pi}}\left(\frac{2\pi\, m\, kT}{h^2}\right)^{3/2} V \int_0^\infty \frac{x^{1/2}}{{\rm e}^x - 1}\, dx,$$

where $x=\varepsilon/kT$ and the value of the integral is

$\Gamma(3/2)\zeta(3/2)$ where, $\Gamma(3/2) = \sqrt{\pi}/2$ $\zeta(3/2) = 2.612$

Therefore,

$$N_{\mathrm{ex}} = \zeta(3/2)\, V \left(\frac{2\pi m k T}{h^2} \right)^{3/2} .$$

Bose temperature, $\mathbf{T_B}$, is defined as the temperature above which all the bosons are in excited states. Setting $\mathbf{N_{ex}=N}$ and $\mathbf{T=T_B}$ in the previous expression, we obtain

$$T_B = \frac{h^2}{2\pi m k} \left[\frac{N}{\zeta(3/2)\, V} \right]^{2/3} .$$

Moreover,

$$\frac{N_{ex}}{N} = \left(\frac{T}{T_B}\right)^{3/2} .$$

Thus, the fractional number of bosons in the ground-state is

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/2} .$$

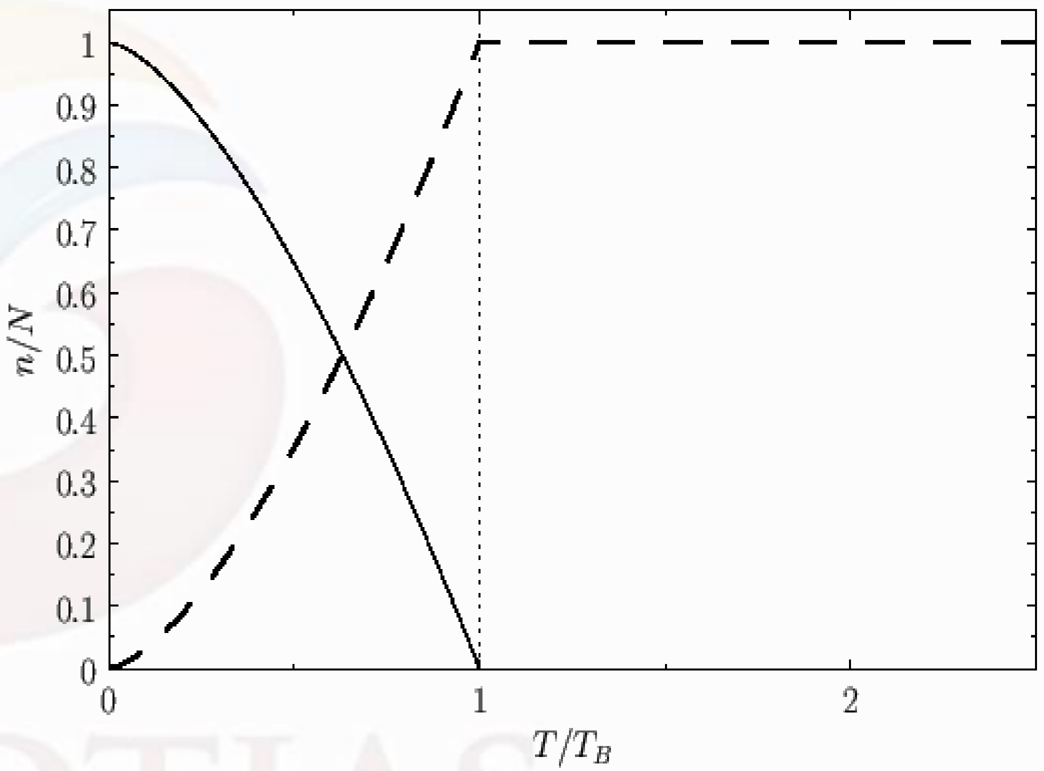

Figure: Variation with temperature of $\mathbf{N_0/N}$ (solid curve) and $\mathbf{N_{ex}/N}$ (dashed curve) for a boson gas.

REFERENCES

- utexas.edu
- Statistical Mechanics, Satya Prakash
- A Textbook of Statistical Mechanics, Prof Suresh Chandra, CBS Publishers
- Statistical Mechanics, R.K. Pathria, Butterworth Heinemann: 2nd Ed., 1996, Oxford University Press