Lecture-11

Solution of Volterra integral equation of second kind by Resolvent kernels

Neumann series

We have

$$y(x) = f(x) + \lambda \int_a^x K(x,t)\, y(t)\, dt.$$

As a zero order approximation to the required solution $y(x)$, let us take

$$y_0(x) = f(x).$$

Then the first order approximation is given by

$$y_1(x) = f(x) + \lambda \int_a^x K(x,t) y_0(t)\,dt$$

and

$$y_2(x) = f(x) + \lambda \int_a^x K(x,t) y_1(t)\,dt$$

$$y_2(x) = f(x) + \lambda \int_a K(x,z) y_1(z)\,dz$$

$$= f(x) + \lambda \int_a^x K(x,z) \left\{ f(z) + \lambda \int_a^z K(z,t) y_0(t)\,dt \right\} dz$$

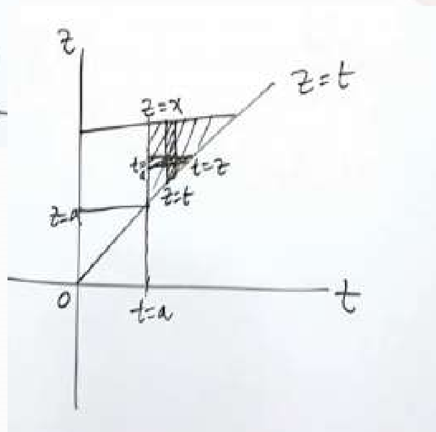
$$= f(x) + \lambda \int_a^x K(x,z) f(z)\,dz + \lambda^2 \int_a^x \left\{ \int_a^z K(x,z) K(z,t) y_0(t)\,dt \right\} dz$$

On changing the order of the integration, we have

$$y_2(x) = f(x) + \lambda \int_a^x K(x,z) f(z) dz + \lambda^2 \int_a^x \left\{ \int_t^x K(x,z) K(z,t) dz \right\} f(t) dt$$

$$\int_a^x \int_a^z K(x,z) K(z,t)\, y_0(t)\, dt\, dz$$

$$= \int_a^x \left(\int_t^x K(x,z) K(z,t) dz \right) y_0(t) dt$$



$$y_2(x) = f(x) + \lambda \int_a^x K(x,t) f(t)dt + \lambda^2 \int_a^x K_2(x,t) f(t)dt,$$

where

$$K_2(x,t) = \int_t^x K(x,z)K(z,t)dz$$

continuing this process

$$K_3(x,t) = \int_t^x K(x,z)K_2(z,t)dz$$

and in general

$$K_{n+1}(x,t) = \int_t^x K(x,z)K_n(z,t)dz, \quad \text{where } n = 1, 2, 3\ldots$$

and

$$K_1(x,t) = K(x,t)$$

The functions $K_n(x, t)$ are called **iterated kernels**.

By mathematical induction, we have

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^x K_m(x,t) f(t) dt \quad .$$

As $n \to \infty$, we get the Neumann series

$$y(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x,t) f(t) dt$$

$$= f(x) + \int_a^x \left\{ \sum_{m=1}^{\infty} \lambda^m K_m(x,t) \right\} f(t) dt \quad .$$

$$y(x) = f(x) + \lambda \int_a^x R(x,t;\lambda) f(t)\,dt \quad .$$

where

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \ ,$$

and is called the **resolvent kernel.**

Hence the solution of the given integral equation may be written as

$$y(x) = f(x) + \lambda \int_a^x R(x,t;\lambda) f(t)\,dt.$$

Reference:

https://nptel.ac.in/courses/111/107/111107103/