Lecture-12

Solution of Volterra integral equation of second kind by Resolvent kernels

Example: Find the resolvent kernel for the Volterra-type integral equation

$$y(x) = (1+x^2) + \int_0^x \left(\frac{1+x^2}{1+t^2}\right) y(t)\,dt.$$

and hence determine its solution.

The resolvent kernel for given integral equation

$$R(x,t;\lambda) = \left(\frac{1+x^2}{1+t^2}\right) e^{x-t}$$

$$K_1(x,t) = K(x,t) = \frac{1+x^2}{1+t^2}$$

$$K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$$

$$= \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2}\, dz = \left(\frac{1+x^2}{1+t^2}\right)\int_t^x dz = \frac{1+x^2}{1+t^2}(x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz = \int_t^x \frac{1+x^2}{1+z^2} \cdot \frac{1+z^2}{1+t^2}(z-t)\, dz$$

$$= \left(\frac{1+x^2}{1+t^2}\right)\int_t^x (z-t)\, dz = \left(\frac{1+x^2}{1+t^2}\right)\left\{\frac{(z-t)^2}{2}\right\}_t^x$$

$$= \left(\frac{1+x^2}{1+t^2}\right)\frac{(x-t)^2}{2!}$$

Thus,

$$K_n(x,t) = \left(\frac{1+x^2}{1+t^2}\right)\frac{(x-t)^{n-1}}{(n-1)!}$$

The solution of the Volterra integral equation of the second kind is given by

$$y(x) = f(x) + \lambda \int_a^x R(x,t;\lambda) f(t)\,dt$$

$$= (1+x^2) + \int_0^x \left(\frac{1+x^2}{1+t^2}\right) e^{x-t} (1+t^2)\,dt$$

$$= (1+x^2) + \int_0^x (1+x^2) e^{x-t}\,dt$$

Particular form II : Suppose that the kernel $K(x, t)$ is a polynomial of degree $(n - 1)$ in x, then we can represent it as

$$K(x,t) = b_0(t) + b_1(t)(t-x) + \ldots + \frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1}. \qquad \ldots(3)$$

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x,t;\lambda) = -\frac{1}{\lambda}\frac{d^n}{dt^n}h(x,t;\lambda)$$

where $h(x, t; \lambda)$ is a solution of the differential equation

$$\frac{d^n h}{dt^n}+\lambda\left[b_0(t)\frac{d^{n-1}h}{dt^{n-1}}+b_1(t)\frac{d^{n-2}h}{dt^{n-2}}+\ldots+b_{n-1}(t)h\right]=0,$$

satisfying the conditions

$$h\big|_{t=x}=\frac{dh}{dt}\bigg|_{t=x}=\ldots=\frac{d^{n-2}h}{dt^{n-2}}\bigg|_{t=x}=0;\ \frac{d^{n-1}h}{dt^{n-1}}\bigg|_{t=x}=1.$$

The required solution is given by

$$y(x)=f(x)+\int_a^x R(x,t;\lambda)f(t)dt.$$

Example: Consider

$$y(x) = (\cos x - x - 2) + \int_a^x (t - x) y(t) dt.$$

Reference:

https://nptel.ac.in/courses/111/107/111107103/