

Lecture-12

Solution of Volterra integral equation of second kind by Resolvent kernels

Example: Find the resolvent kernel for the Volterra-type integral equation

$$y(x) = (1+x^2) + \int_0^x \left(\frac{1+x^2}{1+t^2} \right) y(t) dt.$$

and hence determine its solution.

The resolvent kernel for given integral equation

$$R(x,t;\lambda) = \left(\frac{1+x^2}{1+t^2} \right) e^{x-t}$$

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$$K_1(x,t) = K(x,t) = \frac{1+x^2}{1+t^2}$$

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t) dz$$
$$= \int_t^x \frac{1+x^2}{1+z^2} \frac{1+z^2}{1+t^2} dz = \left(\frac{1+x^2}{1+t^2}\right) \int_t^x dz = \frac{1+x^2}{1+t^2} (x-t)$$

$$K_3(x,t) = \int_t^x K(x,z) K_2(z,t) dz = \int_t^x \frac{1+x^2}{1+z^2} \frac{1+z^2}{1+t^2} (z-t) dz$$
$$= \left(\frac{1+x^2}{1+t^2}\right) \int_t^x (z-t) dz = \left(\frac{1+x^2}{1+t^2}\right) \left\{ \frac{(z-t)^2}{2} \right\}_t^x$$
$$= \left(\frac{1+x^2}{1+t^2}\right) \frac{(x-t)^2}{2!}$$

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Thus,

$$K_n(x,t) = \left(\frac{1+x^2}{1+t^2}\right) \frac{(x-t)^{n-1}}{(n-1)!}$$

The solution of the Volterra integral equation of the second kind is given by

$$\begin{aligned} y(x) &= f(x) + \lambda \int_a^x K(x,t;\lambda) f(t) dt \\ &= (1+x^2) + \int_0^x \left(\frac{1+x^2}{1+t^2}\right) e^{x-t} (1+t^2) dt \\ &= (1+x^2) + \int_0^x (1+x^2) e^{x-t} dt \end{aligned}$$



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Particular form II : Suppose that the kernel $K(x, t)$ is a polynomial of degree $(n - 1)$ in x , then we can represent it as

$$K(x, t) = b_0(t) + b_1(t)(t - x) + \dots + \frac{b_{n-1}(t)}{(n-1)!} (t - x)^{n-1}. \quad \dots(3)$$

Then the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x, t; \lambda) = -\frac{1}{\lambda} \frac{d^n}{dt^n} h(x, t; \lambda)$$

where $h(x, t; \lambda)$ is a solution of the differential equation



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$$\frac{d^n h}{dt^n} + \lambda \left[b_0(t) \frac{d^{n-1} h}{dt^{n-1}} + b_1(t) \frac{d^{n-2} h}{dt^{n-2}} + \dots + b_{n-1}(t) h \right] = 0,$$

satisfying the conditions

$$h \Big|_{t=x} = \frac{dh}{dt} \Big|_{t=x} = \dots = \frac{d^{n-2} h}{dt^{n-2}} \Big|_{t=x} = 0; \frac{d^{n-1} h}{dt^{n-1}} \Big|_{t=x} = 1.$$

The required solution is given by

$$y(x) = f(x) + \int_a^x R(x, t; \lambda) f(t) dt.$$



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Example: Consider

$$y(x) = (\cos x - x - 2) + \int_a^x (t - x)y(t)dt.$$

Reference:

<https://nptel.ac.in/courses/111/107/111107103/>

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