Algebraic Structures, Semigroup and Monoid

Binary Operation

- Let G be a set. A binary operation on G is a function that assigns each order pair of elements of G an element of G.

$$f : G \times G \to G$$

Remark : 'o' is a binary operation on G iff aob ∈ G.

Algebraic Structure

- A non empty set together with one or more than one binary operation is called algebraic structure.

Examples :-

1. (R,+, ·) is an algebraic structure.
2. (N, +) , (Z, +), (Q, +) are algebraic structures.

Semigroup

If S is a nonempty set and * be a binary operation on S, then the algebraic system {S, * } is called a semigroup , if the operation * is associative.

viz., if for any a,b,c∈S,

$$(a*b)*c=a*(b*c)$$

Since the characteristic property of a binary operation on S is the closure property, it is not necessary to mention it explicitly when algebraic systems are defined.

Example :-

if E is the set of positive even numbers, then (E, +) and (E, X) are semigroups.

Monoid

If a semigroup {M, * } has an identity element with respect to the operation * , then {M, * } is called a monoid.

viz., if for any a,b,c∈M

$$(a*b)*c=a*(b*c)$$

and if there exists an element e∈M such that for any a∈M, e∗a=a∗e=a, then the algebraic system {M, * } is called a monoid.

Example :-

1. if N is the set of natural numbers, then (N,+) and (N,X) are monoids with the identity elements 0 and 1 respectively.
2. The semigroups (E,+) and (E,X) are not monoids.

Group

A non empty set G together with an operation 'o' is called a group if the following conditions are satisfied :

1. Closure axiom, $\forall\ a, b \in G \Rightarrow aob \in G$.
2. Associative axiom, $aob\ oc = ao(boc)\ \forall\ a, b, c \in G$
3. Existence of identity, $\exists$ an element $e \in G$, called identity $aoe = eoa = a\ \forall\ a \in G$.
4. Existence of inverse, $a \in G$, $\exists\ a^{-1} \in G\ s.t\ a^{-1}oa = aoa^{-1} = e$ This a^{-1} is called inverse of a.

References:

1. Contemporary Abstract Algebra – Gallian
2. A Course in Abstract Algebra: Vijay K. Khanna and S. K. Bhambri