Electricity and Magnetism

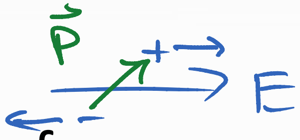
Topic Covered: Biot-Savart Law

Outline

- Magnetic force law
  - Magnetic field & Lorentz Force – force on two straight wires
  - Force between current elements – analog of Coulomb's law
  - Separation of field – Lorentz force & Biot-Savart law
- Examples
  - Straight wire
  - Circular loop
  - Current sheet

Magnetic field definition from torque – electrostatic analogy

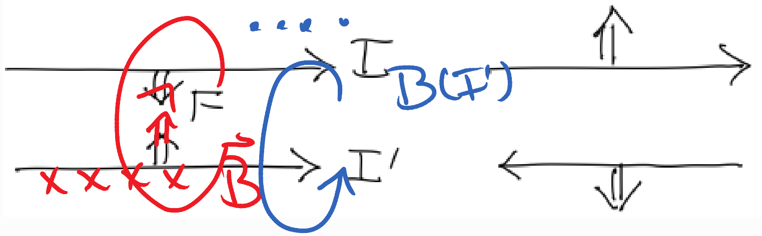
$$\vec{\tau} = \vec{p} \times \vec{E}$$



$$\tau = \vec{m} \times \vec{B}$$

Magnetic field from force on a straight wire – current element

$$\vec{F}/\ell = \left(\frac{\mu_0}{2\pi}\frac{I'}{d}\right) I = \vec{I} \times \vec{B}$$



Lorentz force for currents – TWO right hand rules – parity ?

$$d\vec{F} = I d\vec{l} \times \vec{B} = \frac{dq}{dt} d\vec{l} \times \vec{B} = q \vec{v} \times \vec{B}$$

Lorentz force for moving particles – or ANY current element

$$\vec{F} = \underbrace{q}_{\rho d\tau}(\vec{E} + \vec{v} \times \vec{B}) = (\rho \vec{E} + \vec{J} \times \vec{B})\, d\tau$$

$$\rho \vec{v} = \frac{q}{dx\,dy\,dz}\frac{dx}{dt} = \frac{q}{dy\,dz\,dt} = J$$

Relativistic version

$$K^{\nu} = J_{\mu} F^{\mu\nu}$$

Coulomb's Law

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\int_{\vec r}\int_{\vec r'} dq\, \underbrace{dq'\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} \equiv \int dq\, \vec{E}$$

Biot-Savart Law

$$\vec{E} \equiv \frac{1}{4\pi\varepsilon_0}\int_{\vec r'} \frac{dq'\,\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{\vec r'} \frac{dq}{\mathscr{r}}}_{V}$$

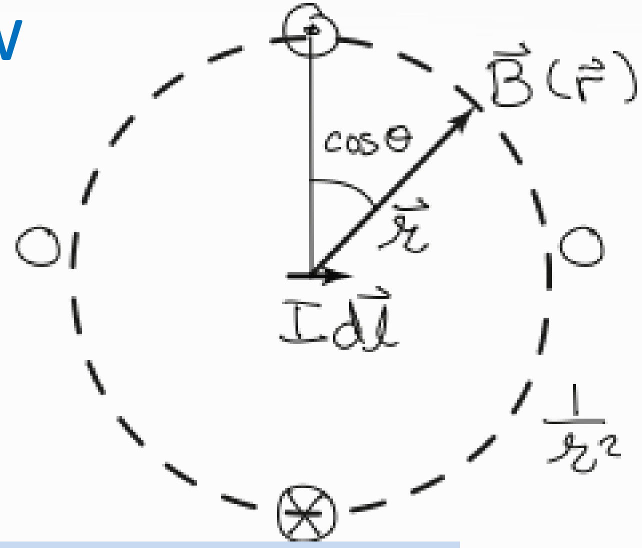
$$\vec{F}_m = \frac{-\mu_0}{4\pi}\oint_{\vec r}\oint_{\vec r'} \overbrace{I\vec{dl}}^{d\vec{q}}\cdot\overbrace{I'\vec{dl}'}^{d\vec{q}'}\,\underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} = \oint I'\vec{dl}\times\vec{B}$$

Consistency of separation of force into: I'dl' –> B –> I dl

A × (B × C) = B (A · C) − C (A · B)

$$\vec{F}_m = \oint I\vec{dl}\times\oint' \frac{\mu_0}{4\pi} I'\vec{dl}'\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\oint\oint' I'\vec{dl}'\,\underbrace{\left(I\vec{dl}\cdot -\nabla\frac{1}{\mathscr{r}}\right)}_{\oint I\, d\frac{1}{\mathscr{r}} = 0} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\left(I\vec{dl}\cdot I'\vec{dl}'\right)$$

Biot-Savart Law



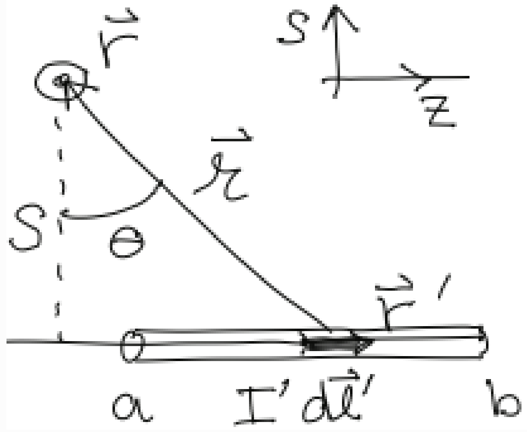
$$\vec{B} \equiv \frac{\mu_0}{4\pi}\oint_{\vec r'} \frac{I'\vec{dl}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Straight wire

$$\vec{B} = \frac{\mu_0}{4\pi}\int I'd\vec{l}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I'}{4\pi}\int_{z'=a}^{b} \frac{dz'\hat{z} \times (s\hat{s} + (z-z')\hat{z})}{(s^2 + (z-z')^2)^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{z'=a}^{b} \frac{s\,dz'\,\hat{\phi}}{(s^2+(z-z')^2)^{3/2}} = \frac{\mu_0 I'}{4\pi}\int \frac{s^2\sec^2\theta\,d\theta\,\hat{\phi}}{\underbrace{(s^2(1+\tan^2\theta))^{3/2}}_{s^3\sec^3\theta}}$$

$$= \frac{\mu_0 I'}{4\pi s}\int_{z'=a}^{b} \cos\theta\,d\theta\,\hat{\phi} = \frac{\mu_0 I'}{4\pi s}(\sin\theta_b - \sin\theta_a)\hat{\phi}$$



~ for an infinite wire: $\vec{B} = \frac{\mu_0 I'}{2\pi s}\hat{\phi}$

~ for a second parallel wire: $\vec{F} = \int_{\vec{r}} I d\vec{l} \times \vec{B} = -\frac{\mu_0}{2\pi}\frac{II'}{s}\hat{s}\,l$

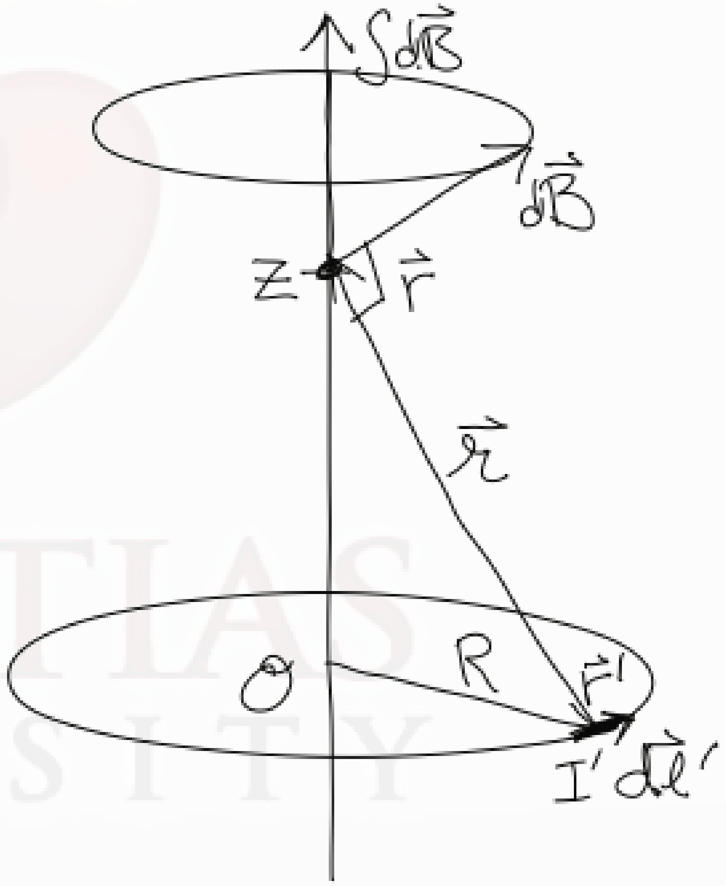
Dipole loop

$$\vec{B} = \frac{\mu_0}{4\pi}\oint I'd\vec{l}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I'Rd\phi'\hat{\phi} \times \frac{z\hat{z} - R\hat{s}'}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I'Rd\phi' \frac{z\hat{s}' + R\hat{z}}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I'Rd\phi' \frac{z(\cos\phi'\hat{x} + \sin\phi'\hat{y}) + R\hat{z}}{\mathscr{r}^3}$$

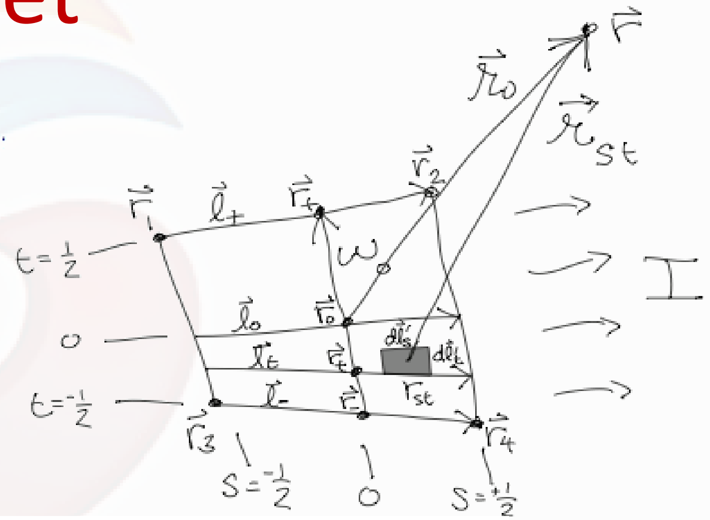
$$= \frac{\mu_0 I'}{4\pi} 2\pi \frac{R^2\hat{z}}{\mathscr{r}^3} = \frac{\mu_0 I' R^2 \hat{z}}{2(R^2+z^2)^{3/2}}$$



Current-sheet

$$\begin{cases} \vec{\ell}_+ = \vec{r}_2 - \vec{r}_1 & \vec{r}_+ = \frac{1}{2}(\vec{r}_2 + \vec{r}_1) \\ \vec{\ell}_- = \vec{r}_4 - \vec{r}_3 & \vec{r}_- = \frac{1}{2}(\vec{r}_4 + \vec{r}_2) \end{cases}$$

$$\begin{cases} \vec{\ell}_0 = \frac{1}{2}(\vec{\ell}_+ + \vec{\ell}_-) & \vec{r}_0 = \frac{1}{2}(\vec{r}_+ + \vec{r}_-) \\ \vec{u}_0 = (\vec{\ell}_+ - \vec{\ell}_-) & \vec{\omega}_0 = (\vec{r}_+ - \vec{r}_-) \end{cases}$$



$$\vec{r}'(s,t) \equiv \vec{r}_{st} = \vec{r}_t + s\vec{\ell}_t \qquad \text{where} \qquad \begin{aligned} \vec{r}_t &= \vec{r}_0 + t\,\vec{\omega}_0 \\ \vec{\ell}_t &= \vec{\ell}_0 + t\,\vec{u}_0 \end{aligned}$$

$$= \underbrace{\vec{r}_0}_{0^{th}\text{ order}} + \underbrace{s\vec{\ell}_0 + t\vec{\omega}_0}_{1^{st}\text{ order}} + \underbrace{st\,\vec{u}_0}_{2^{nd}\text{ order}}$$

$$d\vec{\ell}_s' = \frac{\partial \vec{r}_{st}}{\partial s}\,ds = (\vec{\ell}_0 + t\vec{u}_0)\,ds \qquad d\vec{\ell}_t' = s \leftrightarrow t = (\vec{\omega}_0 + s\hat{u}_0)\,dt$$

$$d\vec{a} = d\vec{\ell}_s' \times d\vec{\ell}_t' = (\vec{\ell}_0 \times \vec{\omega}_0 + (s\vec{\ell}_0 - t\vec{\omega}_0) \times \vec{u}_0)\,ds\,dt$$

$$\vec{K}da = \frac{dI}{d\omega_\perp}|d\vec{\omega} \times d\vec{\ell}|\,\hat{\ell} = \frac{dI}{d\omega_\perp}d\omega_\perp d\vec{\ell} = I\frac{dt}{\Delta t}\vec{\ell}_t\frac{ds}{\Delta s} = I(\vec{\ell}_0 + t\vec{u}_0)\,ds\,dt$$