Lecture-6

Conversion of integral equations into differential equations

Example: Derive the integral equation from the differential equation:

$y''(x) - \sin x\, y'(x) + e^x y(x) = x$, where $y(0) = 1$, $y'(0) = -1$. ...(1)

we may write the given differential equation as

$$y''(x) = y'(x)\sin x - e^x y(x) + x \;,$$

on integrating, we have

$$\int_0^x y''(x)dx = \int_0^x y'(x)\sin x dx - \int_0^x e^x y(x)dx + \left(\frac{x^2}{2}\right)_0^x$$

$$y'(x) = y(x)\sin x - \int_0^x \left(e^x + \cos x\right)y(x)dx + \frac{x^2}{2} - 1 \;.$$

Again integrating, we get

$$[y(x)]_0^x = \int_0^x y(x)\sin x\,dx - \int_0^x (x-t)(e^t+\cos t)y(t)dt + \left(\frac{x^3}{6}-x\right)_0^x$$

$$y(x) = \frac{x^3}{6} - x + 1 + \int_0^x \left[-(x-t)(e^t+\cos t)+\sin t\right]y(t)dt \qquad ...(2)$$

Which is a Volterra integral equation of second kind.

Now letus recover IVP from integral equation

Differentiating (2) w. r. t. x, we get

$$y'(x) = \frac{x^2}{2} - 1 - \int_0^x \left(e^t + \cos t\right)y(t)dt + y(x)\sin x$$

At $x = 0$, we obtain $y'(0) = -1$.

Again differentiating with respect to x

$$y''(x) = x - \left[\int_0^x \frac{\partial}{\partial x}\left\{(e^t + \cos t)y(t)\right\}dt + (e^x + \cos x)y(x)\frac{d}{dx}x - (e^0 + \cos 0)y(0)\frac{d}{dx}0 \right] + y(x)\cos x + y'(x)\sin x$$

$$\Rightarrow \; y''(x) - y'\sin x + e^x y = x \; .$$

Conversion of integral equation into BVP

Similarly, we can recover the BVP

$$y'' + x\,y = 1,\ y(0) = 0,\ y(1) = 1$$

from the corresponding Fredholm integral equation

$$y(x) = \frac{x(x+1)}{2} + \int_0^1 K(x,t)\,y(t)\,dt \qquad \ldots(1)$$

where

$$K(x,t) = \begin{cases} t^2(1-x), & 0 \le t < x \\ xt(1-t), & x \le t \le 1 \end{cases}.$$

Find $y'(x)$ and $y''(x)$ of equation (1) we will get the boundary value problem.

Reference:

https://nptel.ac.in/courses/111/107/111107103/