Unit-Frobenius Method and Special Functions

Topic: The Solution of Legender's Equation

The Solution of Legender Equation

- Since $x = 0$ is an ordinary point of (2), we use

$$y = \sum_{n=0}^{\infty} c_n x^n$$

After substitutions and simplifications, we obtain

$$n(n+1)c_0 + 2c_2 = 0$$

$$(n-1)(n+2)c_1 + 6c_3 = 0$$

$$(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0$$

or in the following forms:

$$c_2 = -\frac{n(n+1)}{2!}c_0$$

$$c_3 = -\frac{(n-1)(n+2)}{3!}c_1$$

$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j\,, \quad j = 2, 3, 4, \cdots \qquad (25)$$

Using (25), at least $|x| < 1$, we obtain

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

$$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right] \quad (26)$$

Notices: If n is an even integer, the first series terminates, whereas y_2 is an infinite series.
If n is an odd integer, the series y_2 terminates with x^n.

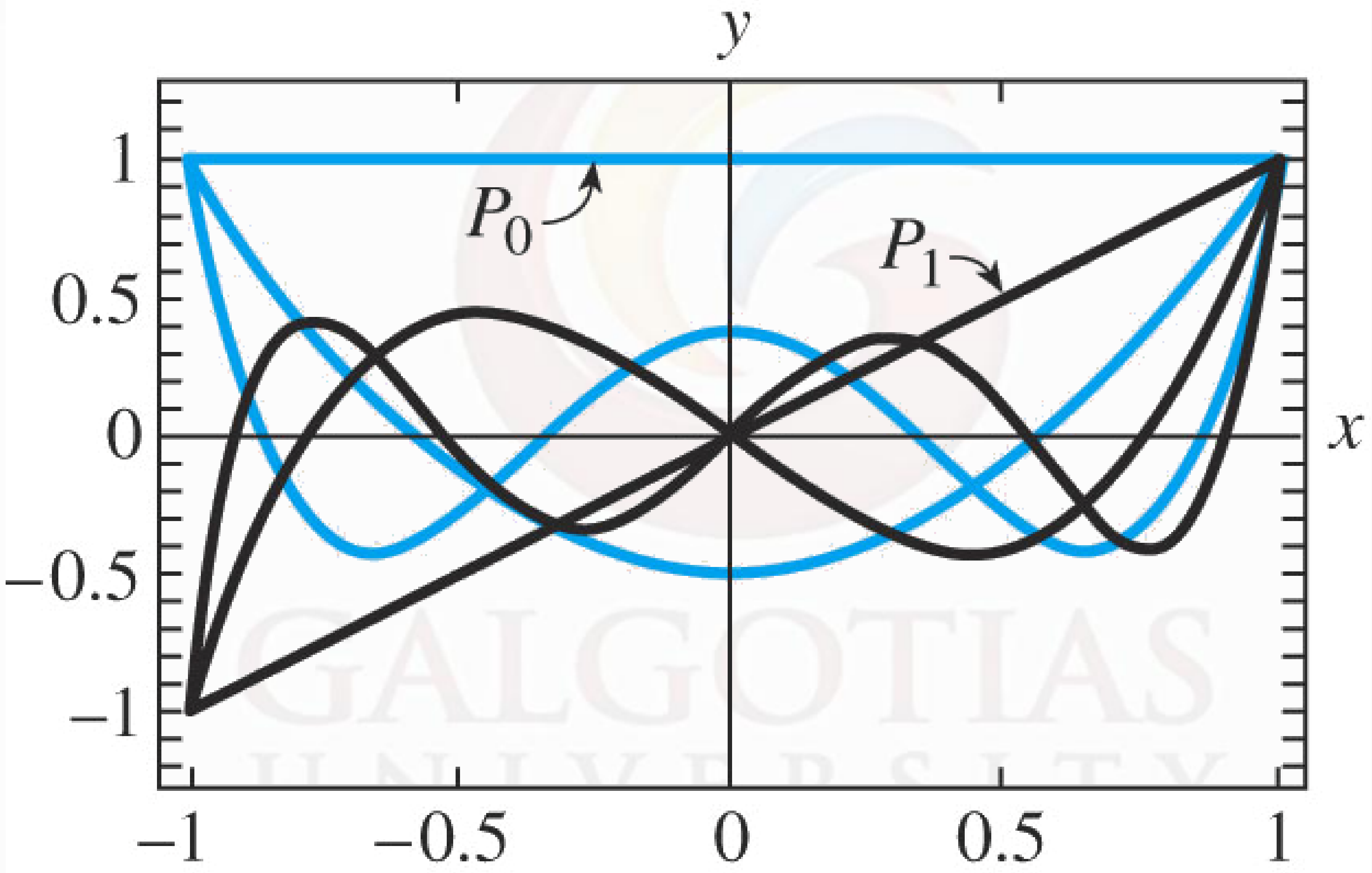
Legender Polynomials

- The following are *n*th order Legender polynomials:

$$P_0(x) = 1, \qquad P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3) - 3x \qquad (27)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

They are in turn the solutions of the DEs. See Fig 5.5

$$\begin{aligned}
n=0: &\quad (1-x^2)y''-2xy'=0 \\
n=1: &\quad (1-x^2)y''-2xy'+2y=0 \\
n=2: &\quad (1-x^2)y''-2xy'+6y=0 \\
n=3: &\quad (1-x^2)y''-2xy'+12y=0 \\
\vdots &\qquad\quad \vdots
\end{aligned} \tag{28}$$


y
1
0.5
0
−0.5
−1
P_0
P_1
x
−1
−0.5
0
0.5
1

Properties

- (1) $P_n(-x) = (-1)^n P_n(x)$
- (2) $P_n(1) = 1$
- (3) $P_n(-1) = (-1)^n$
- (4) $P_n(0) = 0,\ n \text{ odd}$
- (5) $P'_n(0) = 0,\ n \text{ even}$

Recurrence Relation

- Without proof, we have

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0 \qquad (29)$$

which is valid for k = 1, 2, 3, …

Another formula by differentiation to generate Legender polynomials is called the ***Rodrigues' formula:***

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \;\; n = 0, 1, 2, \ldots \qquad (30)$$

REFERENCES

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Thank You