Lecture-14

Solution of Volterra integral equation of second kind by Resolvent kernels

Find the resolvent kernel of the integral equation

$$y(x) = (1-2x-4x^2) + \int_0^x \{3+6(x-t)-4(x-t)^2\} y(t)dt.$$

solution

Here $f(x) = 1 - 2x - 4x^2$, $\lambda = 1$ and $K(x, t) = 3 + 6(x-t) - 4(x-t)^2$.

Let $K(x,t) = a_0(x) + a_1(x)(x-t) + a_2(x)\dfrac{(x-t)^2}{2!}$.

Comparing, we get

$a_0(x) = 3,\ a_1(x) = 6,\ a_2(x) = -8.$

Also,

$$R(x,t;\lambda) = \frac{d^3 g(x,t;\lambda)}{dx^3} = \frac{d^3 g(x,t;1)}{dx^3} \quad \text{as } \lambda = 1,$$

where $g(x, t;1)$ satisfies the differential equation

$$\frac{d^3 g}{dx^3} - \left\{3\frac{d^2 g}{dx^2} + 6\frac{dg}{dx} - 8g\right\} = 0$$

or $(D^3 - 3D^2 - 6D + 8)g = 0, \quad D \equiv d/dx.$ ...(2)

satisfying the conditions $g = 0,\ \dfrac{dg}{dx} = 0$ at $x = t$ and $\dfrac{d^2 g}{dx^2} = 1$ at $x = t$. ....(3)

The auxiliary equation of (2) is given by

$$m^3 - 3m^2 - 6m + 8 = 0,$$

$$m = 1, -2, 4.$$

Therefore

$$g = Ae^x + Be^{-2x} + Ce^{4x}$$

$$\frac{dg}{dx} = Ae^x - 2Be^{-2x} + 4Ce^{4x}$$

$$\frac{d^2g}{dx^2} = Ae^x + 4Be^{-2x} + 16Ce^{4x}$$

Also,

$$R(x,t;\lambda) = \frac{d^3 g(x,t;\lambda)}{dx^3} = \frac{d^3 g(x,t;1)}{dx^3} \quad \text{as } \lambda = 1,$$

where $g(x, t;1)$ satisfies the differential equation

$$\frac{d^3 g}{dx^3} - \left\{3\frac{d^2 g}{dx^2} + 6\frac{dg}{dx} - 8g\right\} = 0$$

or $\quad (D^3 - 3D^2 - 6D + 8)g = 0, \quad D \equiv d/dx.$ ...(2)

satisfying the conditions $g = 0, \ \frac{dg}{dx} = 0$ at $x = t$ and $\frac{d^2 g}{dx^2} = 1$ at $x = t.$ ....(3)

Using the given conditions on g , we have

$$g(x,t;1) = -\frac{1}{9}e^{x-t} + \frac{1}{18}e^{-2x+2t} + \frac{1}{18}e^{4x-4t}$$

Now, we differentiate it thrice w. r. t. x, we have

$$g'(x,t;1) = -\frac{1}{9}e^{x-t} - \frac{1}{9}e^{-2x+2t} + \frac{2}{9}e^{4x-4t}$$

$$g''(x,t;1) = -\frac{1}{9}e^{x-t} + \frac{2}{9}e^{-2x+2t} + \frac{8}{9}e^{4x-4t}$$

$$g'''(x,t;1) = -\frac{1}{9}e^{x-t} - \frac{4}{9}e^{-2x+2t} + \frac{32}{9}e^{4x-4t}.$$

Hence, the required solution is

$$y(x) = (1-2x-4x^2) + \int_0^x R(x,t;1)(1-2t-4t^2)\,dt.$$

$$= (1-2x-4x^2) + \int_0^x \left\{-\frac{1}{9}e^{x-t} - \frac{4}{9}e^{-2(x-t)} + \frac{32}{9}e^{4(x-t)}\right\}(1-2t-4t^2)\,dt.$$

Solving, we get $y(x) = e^x$.

Reference:

https://nptel.ac.in/courses/111/107/111107103/