Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- **Theorem: (Standard matrix for a linear transformation)**

Let $T : R^n \rightarrow R^m$ be a linear trtansformation such that

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix who se n columns correspond to $T(e_i)$

$$A = [T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

Proof:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$$T \text{ is a L.T.} \Rightarrow T(\mathbf{v}) = T(v_1 e_1 + v_2 e_2 + \cdots + v_n e_n)$$
$$= T(v_1 e_1) + T(v_2 e_2) + \cdots + T(v_n e_n)$$
$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

Find the standard matrix for the L.T. $T : R^3 \to R^2$ define by $T(x, y, z) = (x - 2y, 2x + y)$

Sol:

Vector Notation	Matrix Notation
$T(e_1) = T(1, 0, 0) = (1, 2)$	$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$T(e_2) = T(0, 1, 0) = (-2, 1)$	$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
$T(e_3) = T(0, 0, 1) = (0, 0)$	$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = [T(e_1) \mid T(e_2) \mid T(e_3)]$$
$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

i.e. $T(x, y, z) = (x - 2y, 2x + y)$

- Note:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x - 2y + 0z \\ \leftarrow & 2x + 1y + 0z \end{matrix}$$

- Ex : (Finding a matrix relative to nonstandard bases)

Let $T: R^2 \rightarrow R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$[T(1, 2)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad [T(-1, 1)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = [[T(1, 2)]_{B'} \quad [T(1, 2)]_{B'}] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex :

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2,1) = 1(1,2) - 1(-1,1) \qquad B = \{(1,2), (-1,1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1,0) + 3(0,1) = (3,3) \qquad B' = \{(1,0), (0,1)\}$$

- Check:

$$T(2,1) = (2+1, 2(2)-1) = (3,3)$$

Reference : fac.ksu.edu.sa