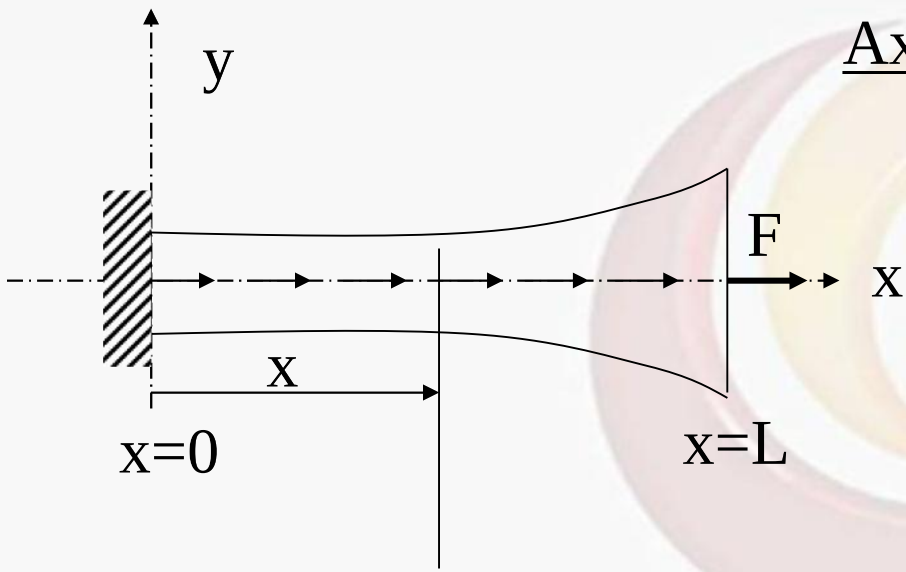
Shape functions in 1D

FINITE ELEMENT ANALYSIS

Lecture Objective:

- **Linear shape functions in 1D**
- **Quadratic and higher order shape functions**
- **Approximation of strains and stresses in an element**



Axially loaded elastic bar

A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\,dx - Fu(x = L)$$

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

Step 1. Assume a solution

$$w(x) = a_0 \varphi_o(x) + a_1 \varphi_1(x) + a_2 \varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "admissible" functions and a_o, a_1, etc are constants to be determined.

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw\ dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$

Step 3. Obtain the coefficients a_o, a_1, etc by setting

$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0 \ddot{}_{o}(x) + a_1 \ddot{}_{1}(x) + a_2 \ddot{}_{2}(x) + \ldots$$

Where the coefficients have been obtained from step 3

Need to find a systematic way of choosing the approximation functions.

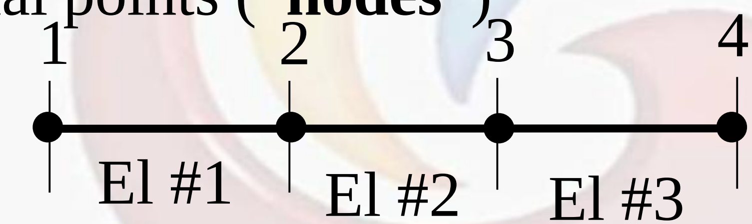
One idea: Choose polynomials!

$w(x) = a_0$ Is this good? (Is ‘1’ an “admissible” function?)

$w(x) = a_1 x$ Is this good? (Is ‘x’ an “admissible” function?)

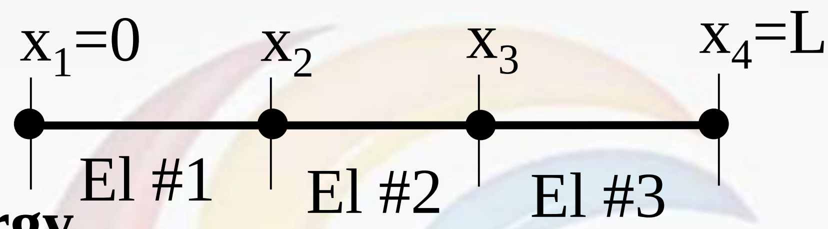
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$



Total potential energy

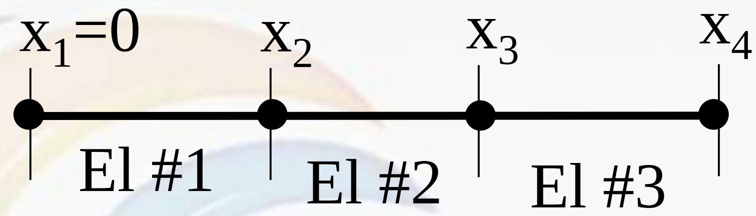
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\ dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

Recall that in the "**direct stiffness**" approach for a bar element, we derived the stiffness matrix of each element directly (See lecture on Trusses) using the following steps:

TASK 1: Approximate the displacement within each bar as a straight line

TASK 2: Approximate the strains and stresses and realize that a bar (with the approximation stated in Task 1) is exactly like a spring with k=EA/L

TASK 3: Use the principle of **force equilibrium** to generate the stiffness matrix

Questions:

- Illustrate the shape functions used for linear element.
- Derive the shape function for 1 d linear element.
- Illustrate the local and Global co-ordinate system

Text Book-

1. Finite Element Analysis by S.S bhavikatti six multicolour edition,2018.New age International publisher. ISBN: 678-26-74589-23-4.

2. A Textbook of Finite Element Analysis Formulation and Programming by D.K.mahraj,Edition 2019. Publisher Willey India ISBN : 978-93-88425-93-3.

Reference Book-

1. Finite element analysis ,Theory and application with Ansys by Moaveni ,2nd edition 2015 ,publisher Pearson, ISBN- 528-43-88435-9.

2. Finite element Analysis By David V. Hutton ,Publisher Elizabeth A. Jomes ,4th edition 2017. ISBN: 0-07-23-9536-2

THANK YOU