Unit-Frobenius Method and Special Functions

Topic: Frobenius Method

Frobenius' Theorem

If $x = x_0$ is a regular singular point of given ODE, then there exists one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} C_n (x - x_0)^n = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Example : Frobenius' Method

- Because $x = 0$ is a regular singular point of

$$3xy'' + y' - y \tag{1}$$

we try to find a solution .
Now,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

Example Contd...

$$3xy'' + y' - y$$

$$= 3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= x^r\left[r(3r-2)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty}(n+r)(3n+3r-2)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=1}^{\infty}c_n x^n}_{k=n}\right]$$

$$= x^r\left[r(3r-2)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(3k+3r+1)c_{k+1} - c_k]x^k\right] = 0$$

Example Contd…

which implies $r(3r-2)c_0 = 0$

$$(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0,\ k = 0, 1, 2, \ldots$$

Since nothing is gained by taking $c_0 = 0$, then

$$r(3r-2) = 0 \qquad (2)$$

and

$$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)}, \qquad k = 0, 1, 2, \cdots \qquad (3)$$

From (2), $r = 0, 2/3$, when substituted into (3),

Example Contd…

$r_1 = 2/3,$

$$c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \qquad k = 0,1,2,\ldots \qquad (4)$$

$r_2 = 0,$

$$c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \qquad k = 0,1,2,\ldots \qquad (5)$$

Example Contd…

From (4)

$$c_2 = \frac{c_1}{8\cdot 2} = \frac{c_0}{2!\,5\cdot 8}$$

$$c_3 = \frac{c_2}{11\cdot 3} = \frac{c_0}{3!\,5\cdot 8\cdot 11}$$

$$c_4 = \frac{c_3}{14\cdot 4} = \frac{c_0}{4!\,5\cdot 8\cdot 11\cdot 14}$$

$$\vdots$$

$$c_n = \frac{c_0}{n!\,5\cdot 8\cdot 11\cdots(3n+2)}$$

From(5)

$$c_2 = \frac{c_1}{2\cdot 4} = \frac{c_0}{2!\,1\cdot 4}$$

$$c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{3!\,1\cdot 4\cdot 7}$$

$$c_4 = \frac{c_3}{4\cdot 10} = \frac{c_0}{4!\,1\cdot 4\cdot 7\cdot 10}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{n!\,1\cdot 4\cdot 7\cdots(3n-2)}$$

Example Contd…

These two series both contain the same multiple c_0. Omitting this term, we have

$$y_1(x) = x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,5\cdot 8\cdot 11\cdots(3n+2)}x^n\right] \quad (6)$$

$$y_2(x) = x^{0}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,1\cdot 4\cdot 7\cdots(3n-2)}x^n\right] \quad (7)$$

Example Contd…

By the ratio test, both (6) and (7) converges for all finite value of x, that is, $|x| < \infty$. Also, from the forms of (6) and (7), they are linearly independent. Thus the solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \quad 0 < x < \infty$$

Indicial Equation

- Equation (2) is called the ***indicial equation***, where the values of r are called the ***indicial roots,*** or ***exponents***.

- If $x = 0$ is a regular singular point of (1), then $p = xP$ and $q = x^2Q$ are analytic at $x = 0$.

Thus the power series expansions

$$p(x) = xP(x) = a_0 + a_1x + a_2x^2 + \ldots$$
$$q(x) = x^2Q(x) = b_0 + b_1x + b_2x^2 + \ldots \qquad (8)$$

are valid on intervals that have a positive radius of convergence.
By multiplying (2) by x^2, we have

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \qquad (9)$$

After some substitutions, we find the indicial equation, $r(r-1) + a_0r + b_0 = 0$ (10)

Example

Solve $2xy''+(1+x)y'+y=0$

Solution

Let $y=\sum_{n=0}^{\infty} c_n x^{n+r}$, then

$$2xy''+(1+x)y'+y$$

$$=2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}$$

$$=\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r}$$

Example Contd...

$$= x^r\left[r(2r-1)c_0x^{-1} + \underbrace{\sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_nx^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r+1)c_nx^n}_{k=n} \right]$$

$$= x^r\left[r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k \right]$$

which implies $r(2r-1) = 0$ (1)

$$(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k = 0, \quad k = 0, 1, 2, \cdots \quad (2)$$

Example Contd…

From (1), we have r_1 = ½ , r_2 = 0.
Foe r_1 = ½ , we divide by k + 3/2 in (2) to obtain

$$c_{k+1} = \frac{-c_k}{2(k+1)}, \quad k = 0, 1, 2, \cdots \quad (3)$$

Foe r_2 = 0 , (2) becomes

$$c_{k+1} = \frac{-c_k}{2k+1}, \quad k = 0, 1, 2, \cdots \quad (4)$$

Example Contd…

From (3)

$$c_1 = \frac{-c_0}{2 \cdot 1}$$

$$c_2 = \frac{-c_1}{2 \cdot 2} = \frac{c_0}{2^2 \cdot 2!}$$

$$c_3 = \frac{-c_2}{2 \cdot 3} = \frac{-c_0}{2^3 \cdot 3!}$$

$$c_4 = \frac{-c_3}{2 \cdot 4} = \frac{c_0}{2^4 \cdot 4!}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{2^n n!}$$

From (4)

$$c_1 = \frac{-c_0}{1}$$

$$c_2 = \frac{-c_1}{3} = \frac{c_0}{1 \cdot 3}$$

$$c_3 = \frac{-c_2}{5} = \frac{-c_0}{1 \cdot 3 \cdot 5}$$

$$c_4 = \frac{-c_3}{7} = \frac{c_0}{1 \cdot 3 \cdot 5 \cdot 7}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}$$

Example Contd…

Thus for $r_1 = ½$

$$y_1(x) = x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}$$

for $r_2 = 0$

$$y_2(x) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}x^n, \quad |x|<\infty$$

and on $(0, \infty)$, the solution is $y(x) = C_1y_1 + C_2y_2$.

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Thank You