Lecture Notes

on

Greedy Algorithms:
Huffman Coding

July 2020
(Be safe and stay at home)

V. Greedy Algorithms

Greedy algorithms – Overview

- Algorithms for solving (optimization) problems typically go through a sequence of steps, with a set of choices at each step.

Huffman codes

- used for data compression, typically saving 20%–90%

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- Basic idea:
  represent often encountered characters by shorter (binary) codes

Huffman codes

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Char.	a	b	c	d	e	f
Freq.	45	13	12	16	9	5
3-bit fixed length code	000	001	010	011	100	101
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Huffman codes

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- Variable-length code saves 25%.

Huffman codes

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Huffman codes

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Huffman codes

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Huffman codes

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   - Decode:
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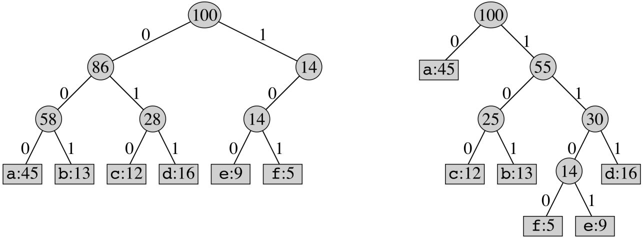
   (a) the (not-full-binary) tree corresponding to the fixed-legnth code,
   (b) the (full-binary) tree corresponding to the prefix code:



(a) (b)

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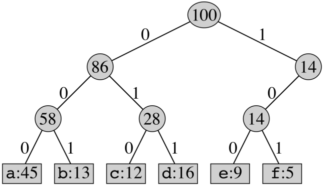

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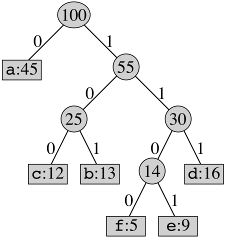

(b)

7. *Fact: an optimal code for a file is always represented by a full binary*

Huffman codes

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Then the number of bits ("**cost** of the tree/code T") required to encode the file

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$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c),$$

- A code T is **optimal** if $B(T)$ is minimal.

Huffman codes

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2. Beginning with $|C|$ leaves, performs a sequence of $|C| - 1$ "merging" operations to create T
3. "Merging" operation is *greedy:* the two with lowest frequencies are merged.

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- A min-priority queue supports the following operations:
  - Insert(S,x): inserts the element x into the set S, i.e., $S = S \cup \{x\}$.
  - Minimum(S): returns the element of S with the smallest "key".
  - ExtractMin(S): removes and returns the element of S with the smallest "key".
  - DecreaseKey(S,x,k): decreases the value of element x's key to the new value k, which is assumed to be at least as small as x's current key value.

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- A max-priority queue supports the operations: Insert(S, x), Maximum(S), ExtractMax(S), IncreaseKey(S, x, k).
- Section 6.5 describes a binary heap implementation.
  - Cost: let $n = |S|$, then
    - initialization building heap $= O(n)$
    - each heap operation $= O(\lg n)$

Huffman codes

- Pseudocode:

```
Huffmancode(C)
n = |C|
Q = C    // min-priority queue, keyed by freq attribute
for i = 1 to n-1
    allocate a new node z
    z_left = x = ExtractMin(Q)
    z_right = y = ExtractMin(Q)
    freq[z] = freq[x] + freq[y]
    Insert(Q,z)
endfor
return ExtractMin(Q)  // the root of the tree
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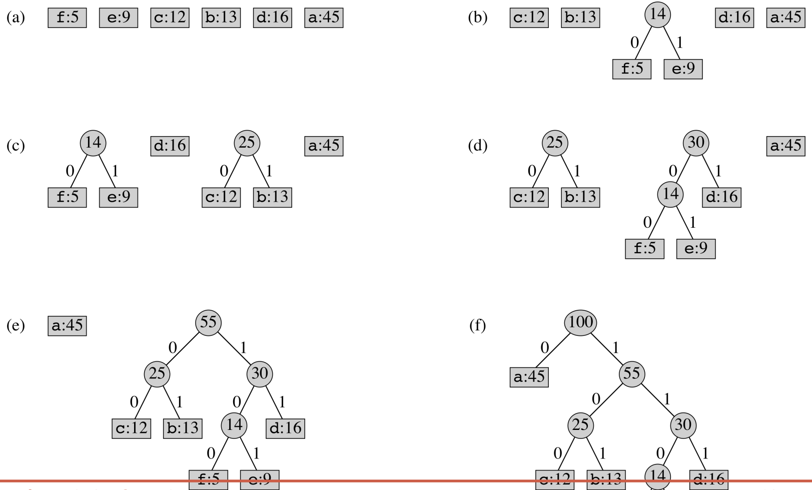
- Running time:

$$
\begin{aligned}
T(n) &= \underline{\text{init. Heap}} + \underline{(n-1)\text{ loop} \times \text{each Heap op.}} \\
&= O(n) + O(n \lg n) = O(n \lg n)
\end{aligned}
$$

Huffman codes

Example



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1. **The greedy-choice property**

   If $x, y \in C$ having the lowest frequencies, then there exists an optimal code T such that

   - $d_T(x) = d_T(y)$
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2. **The optimal substructure property**

   If $x, y \in C$ have the lowest frequencies, and let z be their parent. Then the tree

   $$T' = T - \{x, y\}$$

   represents an optimal prefix code for the alphabet

   $$C' = (C - \{x, y\}) \cup \{z\}.$$

Huffman codes

By the above two properties, after each greedy choice is made, we are left with an optimization problem of the same form as the original. By induction, we have

Theorem. Huffman code is an optimal prefix code.

Q & A?

Queries are welcome on slack channel for discussion