School of Mechanical Engineering

Course Code : BTME3061

Course Name: FINITE ELEMENT ANALYSIS

Finite Element Analysis- PLANAR ELEMENTS

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Program Name: B.Tech (ME)

- **Lecture Objective-**

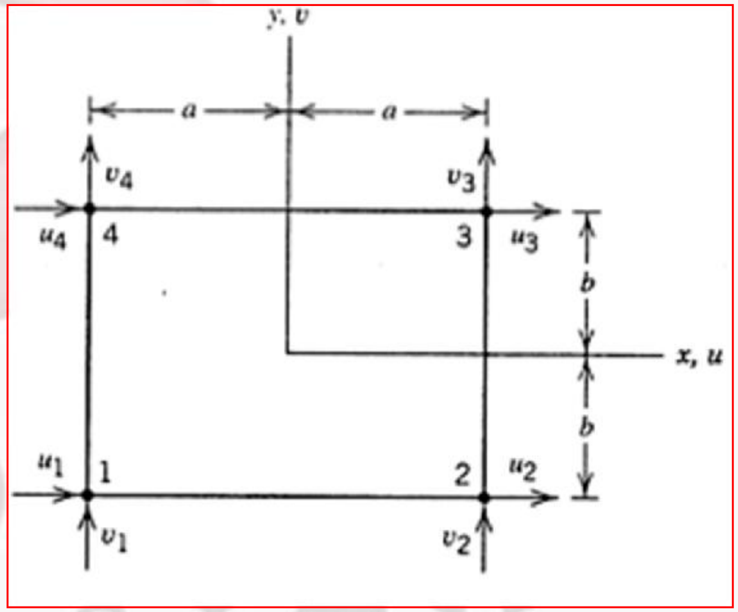
- Shape function and Strain displacement matrix for quadratic bilinear element.
- Shape function and Strain displacement matrix for quadratic quadrilateral element.

Bilinear Quadratic

- The Q4 element is a quadrilateral element that has four nodes. In terms of generalized coordinates, its displacement field is:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy \qquad (3.4\text{-}1)$$

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x \qquad (3.4\text{-}2)$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$



- Shape functions and strain-displacement matrix

$$N_1 = \frac{(a-x)(b-y)}{4ab} \qquad N_2 = \frac{(a+x)(b-y)}{4ab}$$
$$N_3 = \frac{(a+x)(b+y)}{4ab} \qquad N_4 = \frac{(a-x)(b+y)}{4ab} \qquad (3.4\text{-}3)$$

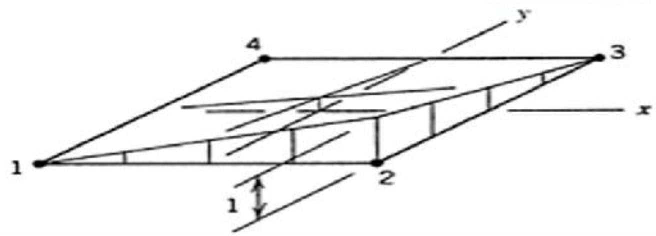
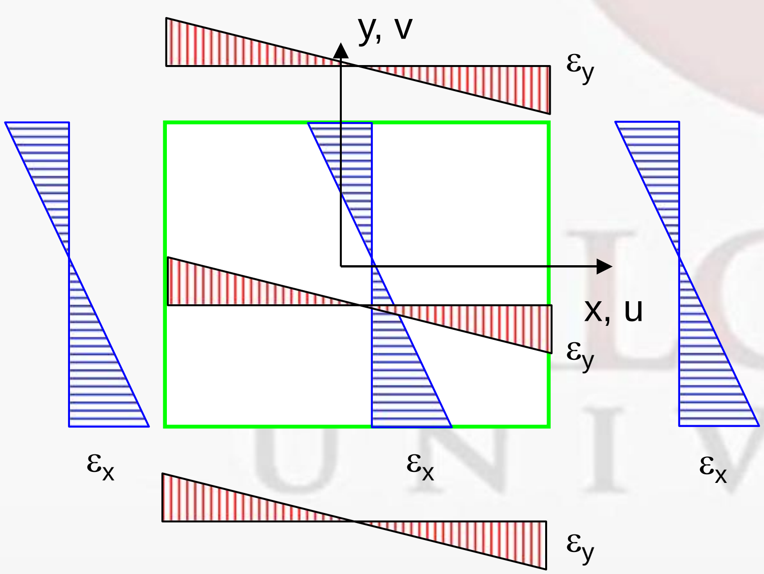


Fig. 3.4-3. Shape function N_2 of the bilinear quadrilateral. (For visualization only, imagine that displacement occurs normal to the xy plane.)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & \cdots \\ 0 & -(a-x) & 0 & -(a+x) & \cdots \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_4 \end{Bmatrix} \qquad (3.4\text{-}4)$$

- The element stiffness matrix is obtained the same way
- A big challenge with this element is that the displacement field has a bilinear approximation, which means that the strains vary linearly in the two directions. But, the linear variation does not change along the length of the element.



$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

ε_x varies with y but not with x
ε_y varies with x but not with y

- So, this element will struggle to model the behavior of a beam with moment varying along the length.
  - Inspite of the fact that it has linearly varying strains - it will struggle to model when M varies along the length.
- Another big challenge with this element is that the displacement functions force the edges to remain straight - no curving during deformation.

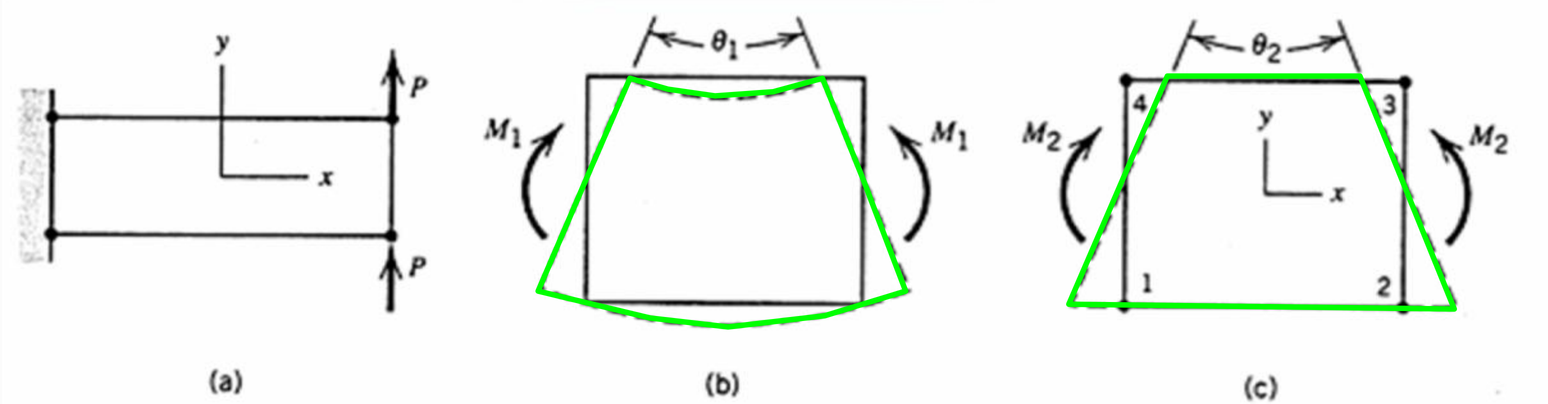
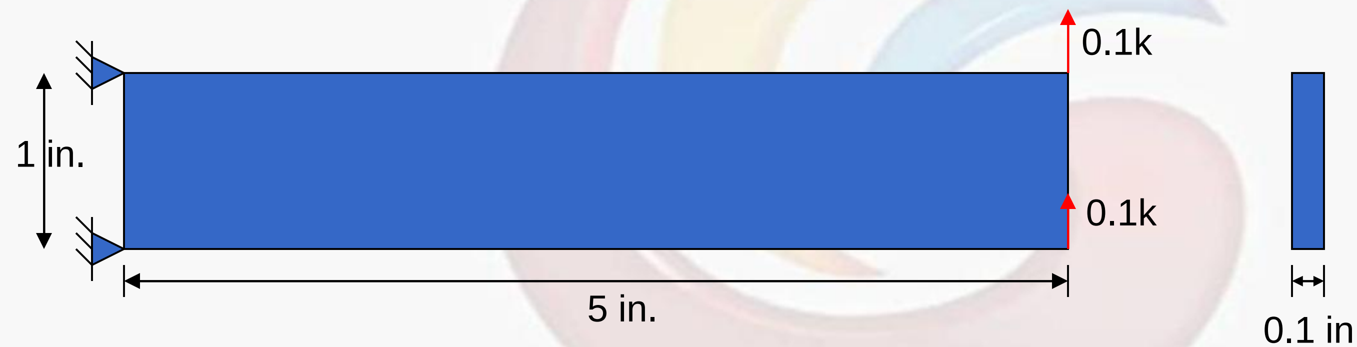


Fig. 3.4-2. (a) A one-element cantilever beam under transverse tip loading. (b) Correct deformation mode of a rectangular block in pure bending. (c) Deformation mode of the bilinear quadrilateral under bending load.

- The sides of the element remain straight - as a result the angle between the sides changes.
  - Even for the case of pure bending, the element will develop a change in angle between the sides - which corresponds to the development of a spurious shear stress.
  - The Q4 element will resist even pure bending by developing both normal and shear stresses. This makes it too stiff in bending.
- The element converges properly with mesh refinement and in most problems works better than the CST element.

Example Problem

- Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333 in^4$$

$$\pm = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60 ksi$$

$$£ = \frac{\pm}{E} = 0.00207$$

$$¢ = \frac{PL^3}{3EI} = \frac{0.2 \times 125}{3 \times 29000 \times 0.008333} = 0.0345 in.$$

Quadratic Quadrilateral Element

- The 8 noded quadratic quadrilateral element uses quadratic functions for the displacements

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$
$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2 \quad (3.5\text{-}1)$$

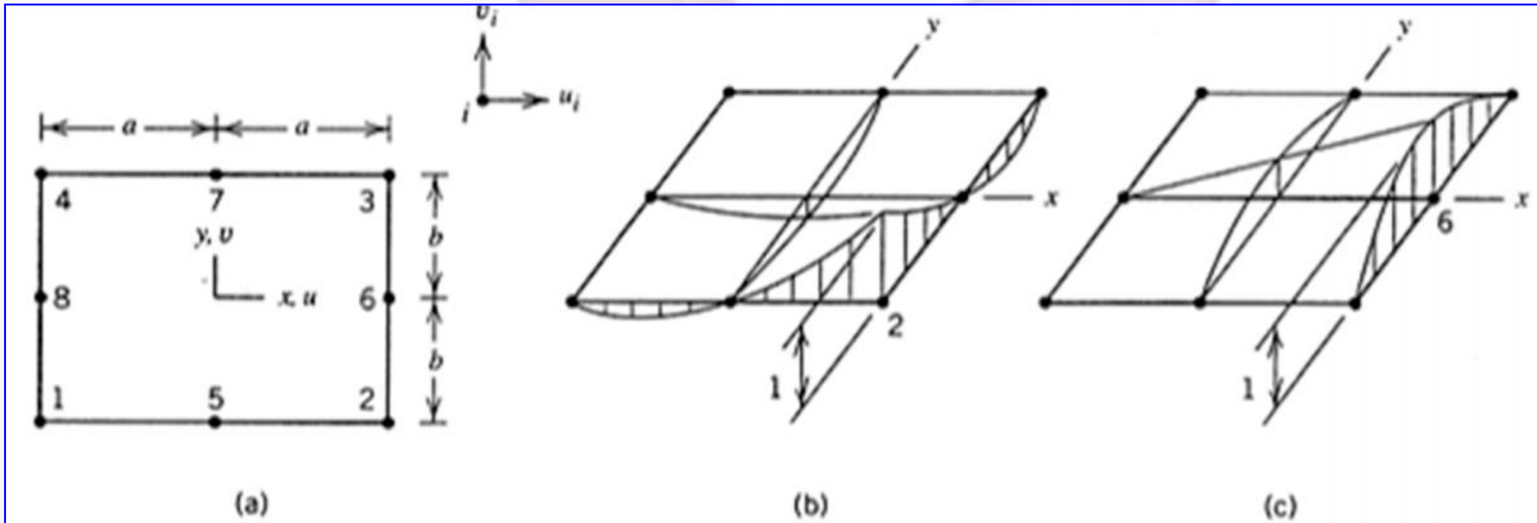


Fig. 3.5-1. (a) A quadratic quadrilateral. (b,c) Shape functions N_2 and N_6. (For visualization only, imagine that displacement occurs normal to the xy plane.)

- Shape function examples:

$$u = \sum N_i\, u_i \qquad v = \sum N_i\, v_i \tag{3.5-2}$$

where index i runs from 1 to 8, which explains the "8" in the name Q8. As examples, two of the eight shape functions are

$$\begin{aligned} N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta) - \tfrac{1}{4}(1+\xi)(1-\eta^2) \\ N_6 &= \tfrac{1}{2}(1+\xi)(1-\eta^2) \end{aligned} \tag{3.5-3}$$

- Strain distribution within the element

$$\begin{aligned} \varepsilon_x &= \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2 \\ \varepsilon_y &= \beta_{11} + \beta_{13} x + 2\beta_{14} y + \beta_{15} x^2 + 2\beta_{16} xy \\ \gamma_{xy} &= (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y \\ &\quad + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2 \end{aligned} \tag{3.5-4}$$

- Questions-
- 1. Explain the terms

  (i) Constant strain triangle (CST)

  (ii) Linear strain triangle(LST) and

  (iii) Quadratic strain triangles (QST).
- 2. Explain the term Cr-continuity.

- Text Book-

1. Finite Element Analysis by S.S bhavikatti six multicolour edition,2018.New age International publisher. ISBN: 678-26-74589-23-4.

2. A Textbook of Finite Element Analysis Formulation and Programming by D.K.mahraj,Edition 2019. Publisher Willey India ISBN : 978-93-88425-93-3.

- Reference Book-

1. Finite element analysis ,Theory and application with Ansys by Moaveni ,2nd edition 2015 ,publisher Pearson, ISBN- 528-43-88435-9.

2. Finite element Analysis By David V. Hutton ,Publisher Elizabeth A. Jomes ,4th edition 2017. ISBN: 0-07-23-9536-2

THANK YOU